Gauge-gravity duality in low dimensions

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In loving memory of Roxie

Abstract

This thesis studies formal aspects of the gauge/gravity duality in the context of ensemble averaged holography. The former states that a (quantum)gravitational theory on a d-dimensional space can be identified with a quantum gauge theory on the boundary of this space—which is $(\mathrm{d}-1)$-dimensional. In ensemble average holography, the boundary theory is not a single theory but an ensemble of theories. That is to say, to match boundary quantities—such as partition functions—to bulk quantities we have to take an average over the ensemble of boundary theories. Here we focus on $\mathrm{d}=2$ and $\mathrm{d}=3$.

For $\mathrm{d}=2$, we study the established duality between Jackiw-Teitelboim (JT) gravity—a gravitational theory in two-dimensions which includes a scalar field—and random matrices. The latter can be seen as an ensemble of Hamiltonians on the one-dimensional boundary. We approach the problem from a topological gravity point of view in the sense that we express partition functions of JT gravity in terms of topological gravity correlators which obey the Korteweg-de Vries (KdV) hierarchy—a infinite system of differential equations. This approach is applied in particular to deformations of JT gravity, i.e. JT gravity with a more general potential, to map solutions of the KdV hierarchy to classes of JT deformations. This is explicitly achieved for a particular JT deformation, namely the one that introduces conical defects in the bulk manifold. We formulate generalisations to JT partition functions that when evaluated at specific values of their arguments give back the known JT partition functions and study their low temperature expansions with a focus on the ones that correspond to conical defects.

For $\mathrm{d}=3$, we focus on ensemble averages of Narain conformal field theories (CFTs) and their bulk duals. We start with a treatment of the former, by discussing how one can calculate ensemble averages of partition functions of such CFTs using the Siegel-Weil formula and generalise this to supersymmetric versions of Narain CFTs. Then, we introduce two classes of $\mathbb{Z}_2$ Narain orbifolds called "factorisable" and "non-factorisable" and calculate the ensemble averages of their partition functions as well. Next, we treat symmetric product orbifolds of Narain CFTs and calculate ensemble averages of partition functions and correlators of twists fields.

Having discussed the technical details of Narain CFTs, orbifolds and their ensemble averages, we move on to study their potential bulk duals. Starting with the known Narain-U(1) gravity correspondence, we propose a bulk dual for the ensemble average of the symmetric product orbifold of Narain CFTs which can reproduce parts of the boundary ensemble averaged partition functions and correlators. Also, we generalise this to the supersymmetric case (at least for the partition functions). Finally, we speculate on ensemble averages of products of arbitrary CFTs and exemplify these ideas using products of Narain CFTs and the "factorisable" and "non-factorisable" orbifolds.

Mainz, 29 August, 2024

Zusammenfassung

Diese Arbeit untersucht formale Aspekte der Gauge/Gravitations-Dualität im Kontext der ensemblegemittelten Holographie. Ersteres besagt, dass eine (Quanten-)Gravitationstheorie auf einem d-dimensionalen Raum mit einer Quantenfeldtheorie am Rand dieses Raums—welcher $(d-1)$-dimensional ist—identifiziert werden kann. In der ensemblegemittelten Holographie ist die Randtheorie keine einzelne Theorie, sondern ein Ensemble von Theorien. Das bedeutet, dass wir, um Randgrößen—wie z. B. Partitionsfunktionen—mit Bulk-Größen abzugleichen, einen Durchschnitt über das Ensemble der Randtheorien nehmen müssen. Hier konzentrieren wir uns auf die Fälle $d = 2$ und $d = 3$.

Für $d = 2$ untersuchen wir die etablierte Dualität zwischen Jackiw-Teitelboim (JT) Gravitation—eine Gravitationstheorie in zwei Dimensionen, die ein Skalarfeld beinhaltet—und Zufallsmatrizen. Letztere können als ein Ensemble von Hamiltonoperatoren auf dem eindimensionalen Rand angesehen werden. Wir nähern uns dem Problem aus der Sicht der topologischen Gravitation, indem wir Partitionsfunktionen der JT-Gravitation in Bezug auf Korrelationen der topologischen Gravitation ausdrücken, welche der Korteweg-de Vries (KdV) Hierarchie gehorchen—ein unendliches System von Differentialgleichungen. Dieser Ansatz wird insbesondere auf Deformationen der JT-Gravitation angewendet. D.h. JT-Gravitation mit einem allgemeineren Potential, so dass Lösungen der KdV-Hierarchie auf Klassen von JT-Deformationen abgebildet werden. Dies wird explizit für eine bestimmte JT-Deformation erreicht, nämlich die, die konische Defekte in der Bulk-Mannigfaltigkeit einführt. Wir formulieren Verallgemeinerungen zu JT-Partitionsfunktionen, die bei spezifischen Werten ihrer Argumente die bekannten JT-Partitionsfunktionen reproduzieren, und untersuchen deren Niedrigtemperaturentwicklungen insbesondere diejenigen, die konische Defekte haben.

Für $d = 3$ konzentrieren wir uns auf Ensemblemittelwerte von Narain-konformen Feldtheorien (CFTs) und deren Bulk-Dualitäten. Wir beginnen mit einer Behandlung der ersteren, indem wir diskutieren, wie man Ensemblemittelwerte von Partitionsfunktionen solcher CFTs mithilfe der Siegel-Weil-Formel berechnen kann und verallgemeinern diese auf supersymmetrische Versionen von Narain-CFTs. Anschließend führen wir zwei Klassen von $\mathbb{Z}_2$ Narain-Orbifolds ein, die als „faktorisierbar" und „nicht faktorisierbar" bezeichnet werden, und berechnen ebenfalls die Ensemblemittelwerte ihrer Partitionsfunktionen. Als nächstes behandeln wir symmetrische Produktorbifolds von Narain-CFTs und berechnen Ensemblemittelwerte von Partitionsfunktionen und Korrelationen von Twist-Feldern.

Nachdem wir die technischen Details von Narain-CFTs, Orbifolds und deren Ensemblemittelwerten besprochen haben, wenden wir uns deren potenziellen Bulk-Dualitäten zu. Beginnend mit der bekannten Narain-$U(1)$-Gravitationskorrespondenz schlagen wir ein duale Bulk-Formulierung für den Ensemblemittelwert der symmetrischen Produktorbifolds von Narain-CFTs vor, das Teile der ensemblegemittelten Rand-Partitionsfunktionen und deren Korrelationen reproduzieren kann. Außerdem verallgemeinern wir dies auf den supersymmetrischen Fall (zumindest für die Partitionsfunktionen).

Schließlich spekulieren wir über Ensemblemittelwerte von Produkten beliebiger CFTs und veranschaulichen diese Ideen anhand von Produkten von Narain-CFTs und den „faktorisierbaren" und „nicht faktorisierbaren" Orbifolds.

Mainz, 29 August, 2024

This thesis is based on the three following publications

1. Stefan Forste, Hans Jockers, Joshua Kames-King, Alexandros Kanargias, and Ida G. Zadeh.
   Deformations of JT gravity via topological gravity and applications.
   J. High Energ. Phys. 2021, 154 (2021). https://doi.org/10.1007/JHEP11(2021)154

2. Stefan Forste, Hans Jockers, Joshua Kames-King, and Alexandros Kanargias.
   Ensemble averages of $\mathbb{Z}_2$ orbifold classes of Narain CFTs.
   J. High Energ. Phys. 2024, 240 (2024). https://doi.org/10.1007/JHEP05(2024)240

3. Joshua Kames-King, Alexandros Kanargias, Bob Knighton, and Mykhaylo Usatyuk.
   The lion, the witch, and the wormhole: ensemble averaging the symmetric product orbifold.
   J. High Energ. Phys. 2024, 236 (2024). https://doi.org/10.1007/JHEP07(2024)236

They appear as [1, 2, 3] respectively in the main text.

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Contents

Chapter 1

Introduction

This thesis studies relations of ensemble averages of quantum theories to gravitational (or gravity-like) theories. Our motivation lies mainly in the exploration of the capabilities, and development of, the tools of theoretical physics. The topic can be viewed, in a sense, as studies of manifestations of the Holographic Principle in two and three spacetime dimensions. In this chapter, we give a broad introduction, which is not meant to be rigorous, in order to convey the ideas underlying and motivating this line of research.

An early encounter

We begin this with an analogy taken from one's early university mathematical education:

One of the most striking theorems that we learn in vector calculus is the divergence theorem. It loosely states that the integral over a volume $\mathcal{V}$ of the divergence $\nabla\cdot\mathbf{E}$ of a vector field $\mathbf{E}$ is equal to the surface integral of $\mathbf{E}$ on the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Thus, knowing the values of the vector field $\mathbf{E}$ on the boundary of the volume $\mathcal{V}$, we can deduce the value of a quantity defined inside $\mathcal{V}$. Put in symbols the theorem states that

$$\iiint_{\mathcal{V}} \nabla\cdot\mathbf{E}\,\mathrm{d}\mathcal{V} = \iint_{\partial\mathcal{V}} \mathbf{E}\cdot\mathrm{d}\mathbf{S}\,, \tag{1.1}$$

where $\mathrm{d}\mathcal{V}$ is the volume element in $\mathcal{V}$ and $\mathrm{d}\mathbf{S}$ is the normal (outward-pointing) surface element on $\partial\mathcal{V}$. Applying this theorem to electrodynamics (for example put $\mathbf{E}$ to be the electric field), we find Gauss' Law: the total electric charge Q_e enclosed in the volume $\mathcal{V}$ is proportional to the net flux flowing out of $\partial\mathcal{V}$ (see Fig. 1.1 for an illustration):

$$\frac{Q_e}{\varepsilon_0} = \iint_{\partial\mathcal{V}} \mathbf{E}\cdot\mathrm{d}\mathbf{S}\,. \tag{1.2}$$

Here ε_0 is the electric permittivity of the vacuum.

The reason of this example is to bring home the message that the boundary and its "degrees of freedom" are very important and can sometimes lead to simplifications in calculations. If we allow ourselves to be a little cavalier about this (perhaps a little misleading) analogy, we could say it is a baby version of the Holographic Principle, which we will make more precise shortly.

A profound encounter: general relativity and black holes

Einstein's general theory of relativity, is one of the most successful and revolutionary physical theories in the sense that it introduced a deep connection between geometry, spacetime and matter. With a concise equation it describes gravitational phenomena that range from what we experience everyday to cosmic scale processes. Einstein's

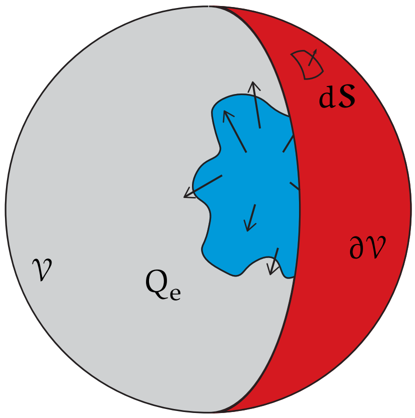

Figure 1.1: A charge Q_e configuration lying inside a volume $\mathcal{V}$. The electric flux out of the boundary gives the charge inside.

equation relates the curvature of spacetime to the energy momentum tensor of matter

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa\mathcal{T}_{\mu\nu} \,. \tag{1.3}$$

$\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are called the Ricci tensor and Ricci scalar of spacetime respectively, $g_{\mu\nu}$ is the metric and $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor of matter. The proportionality constant is $\kappa = \frac{8\pi G}{c^4}$, where G is Newton's constant and c the speed of light.

This theory correctly reproduces Newton's gravitational theory in the limit of slow moving objects in a weak stationary gravitational field, also called the "Newtonian Limit", see e.g. [4]. One of its first successes was the description of Mercury's deviation from an elliptical orbit and this theory has also everyday practical applications for example through the GPS system which wouldn't be possible without general relativity. It describes and predicts one of the most fascinating physical objects, namely black holes and is the basic theoretical framework with which we aim to describe the large scale structure of our universe.

Black holes are singular solutions to Einstein's equations (1.3). They have an "event horizon", a region of spacetime from which nothing can escape once it enters and are uniquely characterized—in four-dimensional and asymptotically flat spacetimes—by their mass M, their charge Q and their angular momentum **L** as measured by an observer at infinity [5, 6, 7, 8, 9].

Remarkably, their existence has been demonstrated in 2016 by the LIGO and Virgo collaborations via the observation of gravitational waves whose waveform matches that of the prediction by general relativity for the merging of two black holes [10]. In fact, there has been even more direct evidence of black holes using interferometry methods for the candidate black holes at the centre of galaxies M87 [11] and ours [12].

It turns out that black holes furnish another example, perhaps the most prominent one in physics, where the boundary plays a very important role and "knows" a lot about the interior. This observation is related to the thermodynamic description of black holes and in particular their entropy.

To be more specific, in the 1970's many profound similarities of black holes to thermodynamics where discovered. Specifically, Christodoulou [13] showed that for Kerr black holes (e.g. black holes that have angular momentum) there exists a quantity called "irreducible mass" m_{ir} that can not decrease. The irreducible mass m_{ir} is actually related to the surface area of the horizon A of the black hole [14] as

$$m_{ir}^2 = \frac{1}{16\pi} A \ .$$

These observations agree with the theorem of Hawking [15] that the event horizon area A can not decrease. In particular, when two black holes of event horizon areas A_1, A_2 collide, the resulting black hole from the merging has horizon area $A_{12} \geqslant A_1 + A_2$.

There is now an apparent similarity to the properties of the entropy S and the second law of thermodynamics. This was observed by Bekenstein who proposed that black holes have an entropy proportional to their horizon area [16, 17, 18]

$$S_{bh} \sim A \tag{1.4}$$

and suggested a generalised second law which states that the *sum of the black hole entropy and the ordinary entropy outside the black hole never decreases*. The proportionality constant in the entropy-area relation was fixed by Hawking to be 1/4 [19] and in appropriate units we have $S_{bh} = \frac{1}{4}A$. Remarkably, Bekenstein [20] also argued that black holes have the maximum entropy allowed by quantum mechanics and general relativity for their size and mass.

From the statistical derivation of thermodynamics, we know that the entropy is related to the number of microstates of the system that give rise to the same macrostate and that it typically scales with the volume of the system. Surprisingly, for a black hole, the entropy scales with the area of the horizon—which can be viewed as the boundary of the black hole. This raises an interesting puzzle: why does a property of the boundary determine the entropy of the whole system and how do we derive microscopically the entropy of a black hole? Such a derivation of the entropy formula for a black hole would shed much light on these questions. This has been achieved for some examples of black holes, see e.g. [21, 22].

Thus, we see once more that the boundary (in form of the horizon area) plays a very important role and in particular to the physics of black holes. This is a remarkable and rather unintuitive property and serves as a prominent sign in favour of the Holographic Principle.

The Holographic Principle

With the intuition we gained from the previous discussions, we can now guess what the Holographic Principle [23, 24] (see also [25]) is about. It states that a system enclosed in a volume can be described solely by degrees of freedom that live on the boundary of that volume. This can have profound implications for physical theories and as we will see when we talk about the Anti-de-Sitter/Conformal field theory (AdS/CFT) correspondence can lead to a definition of quantum gravity, namely a physical theory that puts general relativity and quantum mechanics in the same framework.

As we mentioned in the beginning, this thesis studies, incarnations of this principle in physics and more specifically in two and three spacetime dimensions. Before we get to the exact topics of this thesis, we continue with a short survey of the other cornerstone—alongside the general theory of relativity—of theoretical physics: quantum field theory. Then we move on to the AdS/CFT correspondence and string theory.

The triumphs of quantum field theory

Physical phenomena are described—as of today—in terms of four fundamental interactions: electromagnetism, gravity, strong and weak interaction. Quantum field theory, which, in the context of particle physics, incorporates both quantum mechanics and special relativity in one framework, describes to very high precision—via the standard model of particle physics—all the fundamental interactions except gravity.

The standard model

The standard model of particle physics is a quantum field theory and in particular a non-abelian gauge theory with gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, where SU stands for "special unitary" and $SU(3)_c$ corresponds to colour, $SU(2)_L$ to weak isospin and $U(1)_Y$ to weak hypercharge. It combines quantum chromodynamics (QCD) with electroweak theory [26, 27, 28]. The field content of the standard model is (below we omit the $U(1)_Y$ charges for simplicity)

1. Three generations of quarks and leptons (fermions)

$$Q^i = \begin{pmatrix} u^i \\ d^i \end{pmatrix}_L , \; L^i = \begin{pmatrix} e^i \\ \nu^i \end{pmatrix}_L , i = 1,2,3 , \tag{1.5}$$

that transform as doublets under $SU(2)_L$ and their right-handed partners u^i_R, d^i_R,e^i_R, $i = 1,2,3$ which are $SU(2)_L$ singlets. The fields $u^i_L, d^i_L, u^i_R, d^i_R$ transform also in the fundamental of $SU(3)_c$ while the rest are singlets.

2. The gauge bosons of the strong interaction (gluons) G stemming from the $SU(3)_c$ factor—eight in total.

3. The Higgs field, which is a doublet of $SU(2)_L$, a singlet of $SU(3)_c$.

At this stage, the model of course contains also the gauge bosons W^1, W^2, W^3 of the $SU(2)_L$ gauge symmetry and the gauge boson B of $U(1)_Y$ and all the aforementioned fermions are massless.

After spontaneous symmetry breaking these fermions, except for the neutrino, acquire a mass via the Higgs mechanism [29, 30, 31] and constitute the particles we observe today. Within the standard model described here, the neutrino remains massless as a result of the absence of a right-handed field. Upon appropriate redefinitions of the gauge bosons W^1, W^2, W^3, B, we are left with the remaining elementary particles

4. The (massive) gauge bosons of weak interactions $W^{\pm}, Z$.

5. The (massless) gauge boson of electromagnetism (the photon) A.

The standard model has a finite number of free parameters which can be fixed by experiment leading to concrete predictions. It is the most well tested and successful physical model. Experiments have found evidence (direct or indirect) for all the particles it includes with the most recent one being the Higgs particle which was discovered in 2012 [32, 33]. As an example, we state the well-known fact that the standard model predicts the electron magnetic moment to 1 part in 10^{12} [34].[1] For an introduction to quantum field theory and the standard model of particle physics see, e.g., [38, 39].

Quantum field theory in condensed matter physics

The success of quantum field theories does not lie only in particle physics. Although this is not the focus of this thesis we should mention that the methods of quantum field theory are widely used in condensed matter physics—also with great success. Here the energies are typically much less than the particle physics example.

Amongst other phenomena, quantum field theories are used to describe the Ising model, superconductivity and the quantum Hall effect. Ideas like the Higgs mechanism

[1]For precision calculation of the electron magnetic moment see, e.g., [35, 36, 37].

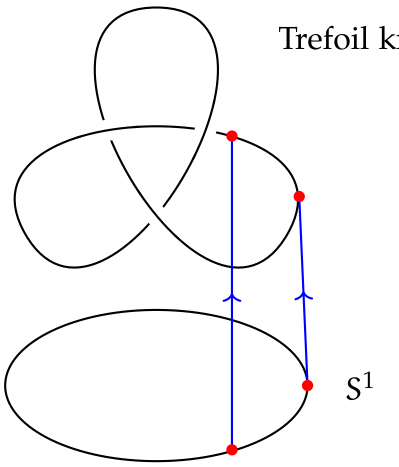

Figure 1.2: The map from the circle to the trefoil knot is depicted. Each red point on the circle is mapped to a point on the knot. We can make the mapping such that one revolution on the circle corresponds to one revolution on the knot.

and spontaneous symmetry breaking appear also in condensed matter physics. In 1963, Anderson proposed a mechanism [40]—the predecessor of the Higgs mechanism—that gives the photon a mass in plasma oscillations. This served as an explicit example of Schwinger's idea that gauge invariance does not necessarily imply a vanishing mass for the vector boson when it is strongly coupled to a current [41]. For an introduction to quantum field theory methods in condensed matter physics see, e.g., [42, 43] and for finite temperature field theory [44].

Quantum field theory and mathematics

Another very successful application of the methods of quantum field theories lies somewhat further from describing physical phenomena. Specifically, quantum field theories can be used to calculate quantities that arise from theories within physics but have very useful applications in mathematics.

Here, we illustrate this with a particular example, namely the relation of knot invariants and quantum field theory. Knots are very intriguing objects studied by mathematicians that show remarkable complexity even though their conceptual construction is intuitive. We can imagine a knot, for example, as the way we tie our shoes or use ropes to fasten things. More formally, a knot is a way to embed a circle S^1 into three-dimensional Euclidean space $\mathbb{R}^3$, see Fig. 1.2. Different knots are made by arbitrarily complicated crossing configurations. It turns out that we can assign certain polynomials to knots that fall in the more general category of knot "invariants".[2] A classic example is the so-called Jones polynomial [45, 46], the construction of which is dictated by the type of the knots' crossings.

The pioneering work of Witten [47], connected the Jones polynomial to Chern-Simons theory—a topological quantum (gauge) field theory. In particular, expectation values of certain quantities in the gauge theory called "Wilson loops" give precisely the aforementioned polynomial. This observation, opened up a new chapter in the relation between physics and mathematics. Calculations coming from physics, such as Chern-Simons theory, can lead to conjectures in mathematics and help to give insight.

[2]These invariants do not classify knots completely though. Two knots that have the same invariant could be topologically different, but different invariants correspond to topologically distinct knots.

A common framework

In the previous sections we argued that the standard model of particle physics (and quantum field theory) and general relativity are very successful physical theories and describe how we understand the world so far. Nevertheless there are observations that indicate that our understanding is not complete and that there is something beyond these theories. For example, the standard model does not explain neither why there is more matter than anti-matter nor neutrino oscillations. Furthermore, in general relativity occur singularities (e.g. black holes), something that suggests that there could be a theory beyond it that "cures" them. Another problem that combines the two is that galaxy rotation curves are not correctly reproduced by the established standard model and general relativity (this is usually attributed to the existence of dark matter). This list does not exhaust all the problems with the standard model and general relativity but it is not our goal to list the them all here.

However, we should address the elephant in the room. We have claimed that there are two very successful physical theories: general relativity—which is a classical theory—and the standard model—which is a quantum theory. The latter unites in a quantum field theory the strong, weak and electromagnetic forces. On the other hand, all matter couples to general relativity and it is natural to ask whether the latter enjoys a quantum description. Apart from unification, another hope would be that quantising gravity could smear its singularities and help us understand the microstates of black holes that give rise to their entropy.[3] In fact, general relativity can be treated as a quantum field theory (see, e.g., comments in [38] and the classic example of [49]). The problem is that it is a non-renormalizable quantum field theory, which means that if we want to use it for arbitrarily high energies we need a UV completion—this resembles the case of Fermi theory which serves as a good effective theory up to some scale. A theory that achieves this on a perturbative level and which we describe below is string theory.[4] Importantly, to deeper understand quantum aspects of general relativity, we would like to have a framework that goes beyond perturbation theory and provides a non-perturbative definition of string theory. Such a framework is provided by gauge/gravity duality and more specifically the AdS/CFT correspondence in string theory.

The present thesis studies gauge/gravity duality from a particular point of view. That is, ensemble-averaged holography in the context of the AdS/CFT correspondence. In the rest of this introduction, we explain the importance of this correspondence and motivate ensemble-averaged holography.

String theory and the AdS/CFT correspondence

The AdS/CFT correspondence [52, 53, 54] (see also [55]) was motivated from the point of view of string theory and thus we would like to begin with a quick review of the latter explaining how one starts studying perturbative string theory, the spectrum and effective field theories derived from strings. Then, we will see how non-perturbative notions, like branes, enter the picture and combine with perturbation theory to give rise to the AdS/CFT correspondence. More about these topics can be found in refs. [56, 57, 58, 59, 60, 61, 62, 63, 64, 65] , which we also follow alongside the original references.

[3] A nice analogy here comes from the atom whose collapse was "saved" by quantum mechanics. For this, and an introduction to quantum gravity see [48].

[4] Another approach could be that general relativity is non-perturbatively renormalizable. A way to study this scenario is asymptotic safety introduced by Weinberg [50], see e.g. [51].

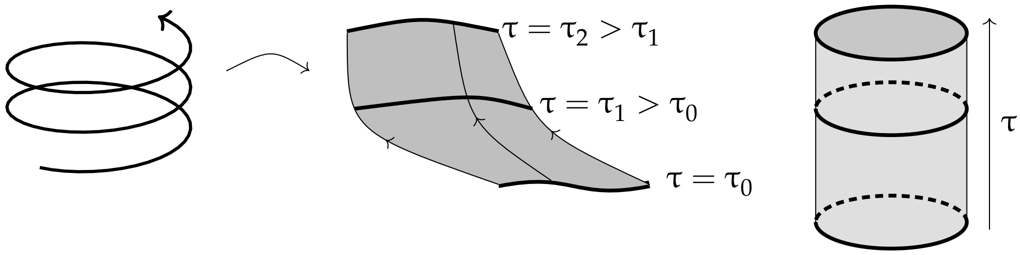

Figure 1.3: The world line of a point particle and the world-sheet of a string propagating in space. On the left, the point particle traces a spiral going upwards. On the right, an open string sweeps a "sheet" and a closed string a cylinder as the parameter τ evolves.

The action for a string

When we study the motion of point particle, we talk about the notion of "world-line". This is the curve that a particle traces in spacetime, see Fig. 1.3. As usual, we can study this system via the action principle, but the question is: what is the action for the free particle? The answer comes from the "natural" invariant quantity that we can define, namely the length of the world-line. Specifically, for a free point particle of mass $m > 0$ moving in D-dimensional Minkowksi spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}\,(-,+,+,...,+)$, the action reads

$$S_{pp} = -m \int_{\tau_0}^{\tau_1} d\tau \left[-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu} \right]^{1/2} . \tag{1.6}$$

Here τ is a parameter that parametrises the world-line and $X^\mu = X^\mu(\tau)$, $\mu = 0, 1, ..., D$ are maps from the world-line to the D-dimensional Minkowski target-space that describe the position of the particle. The proportionality constant $-m$, can be fixed by taking a non-relativistic limit.

For the motion of a one-dimensional object (a string) the generalization is immediate. Open strings trace a surface that is topologically a strip and closed strings trace a surface that looks like a cylinder. The action, called the Nambu-Goto action, is now given by the area of the surface that is swept by the string in spacetime called the world-sheet Σ

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_\Sigma d\tau d\sigma \left[-\det{}_{\kappa\lambda} \left(\frac{\partial X^\mu}{\partial \sigma^\kappa}\frac{\partial X^\nu}{\partial \sigma^\lambda}\eta_{\mu\nu} \right) \right]^{1/2} . \tag{1.7}$$

The world-sheet Σ is parametrised by two parameters $\sigma^1 = \tau \in (\tau_0, \tau_1)$ and $\sigma^2 = \sigma \in [0, \ell)$ and now $X^\mu = X^\mu(\tau, \sigma)$ are maps

$$X^\mu : \Sigma \to \mathbb{R}^{1,D-1} \;,\; (\tau, \sigma) \mapsto X^\mu(\tau, \sigma) . \tag{1.8}$$

The constant α', called the Regge slope, is related to the string tension T by

$$T = \frac{1}{2\pi\alpha'} , \tag{1.9}$$

as can be seen by a non-relativistic expansion of the Nambu-Goto action. The action (1.7) even though very intuitive, is very tricky to quantize given the presence of the square root which leads to non-locality. One way out of this problem is to use the Polyakov action which is classically equivalent to the Nambu-Goto. The Polyakov action, introduces

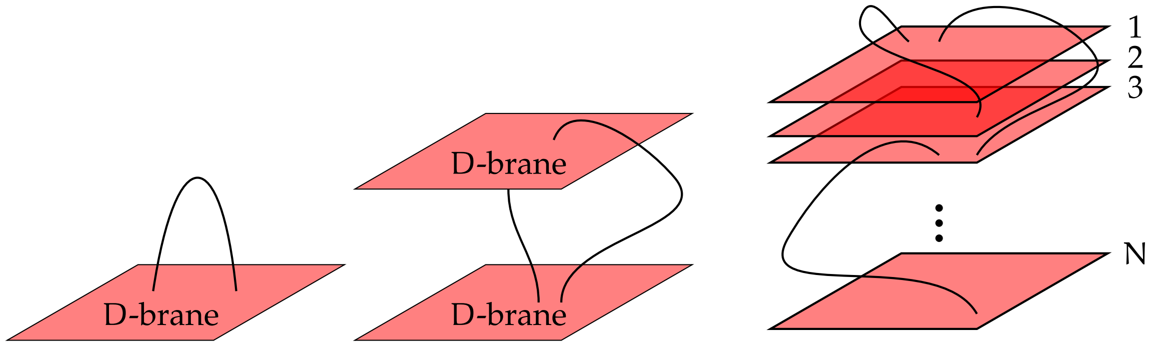

Figure 1.4: Open strings ending on D-branes. From left to right we see a string starting and ending on the same brane, strings ending on different branes and a "stack" of N branes where strings can end. The endpoints of the strings move freely on the D-branes.

an auxiliary metric on the world-sheet $\gamma_{\kappa\lambda}(\tau,\sigma)$ and reads

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{-\gamma}\gamma^{\kappa\lambda}\partial_\kappa X^\mu \partial_\lambda X^\nu \eta_{\mu\nu} \, , \tag{1.10}$$

where $\gamma = \det_{\rho\xi}\gamma_{\rho\xi}$. The quantisation of this action is now straightforward.[5]

Open, closed strings and spectrum

The kind of boundary conditions that the string obeys depend on whether it is an open or closed string. The latter obeys a periodicity condition and the former obeys Neumann and/or Dirichlet boundary conditions. Of particular interest is the case of open strings whose endpoints are restricted to move on a codimension p hypersurface of the target-space, or equivalently, Dirichlet boundary conditions. These break Poincaré symmetry but have great implications as we will see. We can have various scenarios for the endpoints. For instance, as depicted in Fig. 1.4, a string can start and end on a hypersurface (D-brane in the figure) or start on one D-brane and end on another. Ending on different branes introduces new labels on a string state called the Chan-Paton labels. For example, if we consider the rightmost case of Fig. 1.4 and the limit where the branes are coincident, the string state with no excitations can be written as

$$|a, b; \vec{p}\rangle \ , a, b = 1, 2, ..., N \, , \tag{1.11}$$

and $\vec{p}$ denotes the momentum of the string parallel to the branes. D-branes can be treated simply as surfaces where strings end in the case of small string coupling $g_s \ll 1$ and since their tension T_p scales like $\sim {}^1\!/_{g_s}$ they should be thought of in general as non-perturbative dynamical objects. In stronger coupling one should include the gravitational backreaction that they produce as extended objects.

Low energy effective action

Quantisation leads to the perturbative spectrum and string excitations give states with different masses and spins. In particular, taking the low energy limit, only the massless modes survive and give rise to a low energy effective action that can be used to describe the dynamics. To obtain such an effective action, one typically looks at string

[5]One can add to this action extra terms consistent with Poincaré invariance. Of particular importance is the topological term $\sim \chi(\Sigma)$, where χ is the Euler characteristic of the world-sheet—a number that depends on its topology. The proportionality constant is in fact eventually related to the string coupling.

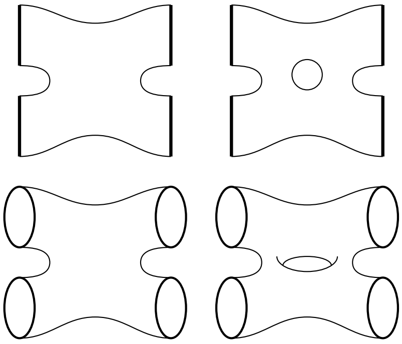

Figure 1.5: In the upper part we see two open strings joining and then splitting. In the lower part we see the same for closed strings. The world sheet can have different topologies and the order of g_s in the perturbative expansion depends on the number of handles.

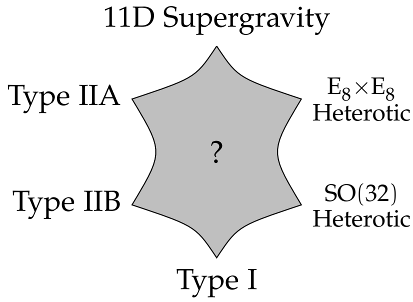

Figure 1.6: The five superstring theories and 11-dimensional supergravity viewed as limiting cases of a single theory, called M-theory.

interactions such as joining and splitting depicted in Fig 1.5. The effective action then describes the amplitudes for these interactions in the low energy limit.

Branes enter this picture nicely as they lead to gauge theories in the low energy limit. A classic example is that of IIB string theory in 10 dimensions, one of the five consistent (ten-dimensional) superstring theories, see Fig 1.6.

IIB strings and the AdS/CFT conjecture

Considering a stack of N coincident D3-branes (extended along three spacelike and one timelike direction) in IIB string theory in ten-dimensional Minkowski space and looking at the low energy limit of two different regimes of this theory, we can argue for the AdS/CFT conjecture. The first regime is that of weak coupling $g_s N \ll 1$ where the branes can be treated simply as surfaces where strings end and do not alter the Minkowski background.[6] In the low energy limit (an energy scale is given by the string length $\sim \sqrt{\alpha'}$), massless modes of open strings give an $\mathcal{N} = 4$ SU(N) Super-Yang-Mills (SYM) theory in four dimensions living on the world-volume of the brane and those of the closed strings give free ten-dimensional supergravity. The two theories are decoupled from each-other and the coupling constant of SYM, g_{YM}, is given in terms of the string coupling as $g_{YM}^2 \sim g_s$.

In the large $g_s N$ regime, the strings propagate in a background that is no longer flat ten-dimensional Minkowksi spacetime as one has to include the branes' back-reaction. The background is given by the IIB supergravity solution (we only write down the metric and omit the RR five-form)[7]

$$\begin{aligned} ds^2 &= \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} dx^n dx^n \\ L^4 &= 4\pi g_s N \alpha'^2 \ , \ r^2 = x^n x^n \ , \ \mu, \nu = 0,1,2,3 \ , \ n = 4,5,6,7,8,9 \ . \end{aligned} \tag{1.12}$$

For large $r \gg L$, the metric behaves asymptotically as ten-dimensional Minkowski

[6] Note that the effective coupling acquires a factor of N. This is related to the Chan-Paton labels.
[7] Strictly speaking, we also require small g_s so far.

IIB strings with N coincident D3-branes

$$
\begin{tikzcd}
\begin{matrix} g_s N \\ \text{small} \end{matrix} \arrow[r, leftrightarrow] \arrow[d, "\text{low energy limit}"] & \begin{matrix} g_s N \\ \text{large} \end{matrix} \arrow[d, "\text{low energy limit}"] \\
\begin{matrix} \text{free ten-dim. SUGRA} \\ \oplus \\ \text{four-dim. } SU(N)\ \mathcal{N}=4 \text{ SYM} \end{matrix} \arrow[r, leftrightarrow] & \begin{matrix} \text{free ten-dim. SUGRA} \\ \oplus \\ \text{IIB strings on } AdS_5 \times S^5 \end{matrix}
\end{tikzcd}
$$

Figure 1.7: Taking the low energy limit in two different regimes of IIB string theory with N coincident D3-branes gives two effective descriptions that match except for one sector. Identifying the brane descriptions in the two regimes, forces the identification of four-dim. $SU(N)$ $\mathcal{N} = 4$ SYM and IIB strings on $AdS_5 \times S^5$.

spacetime $\mathbb{R}^{1,9}$ and for $r \ll L$ it takes the $AdS_5 \times S^5$ form

$$ds^2 \sim \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \,, \tag{1.13}$$

where $d\Omega_5^2$ is the metric on S^5.

It turns out that the low energy effective description of this regime decouples in free ten-dimensional supergravity and IIB string theory on $AdS_5 \times S^5$ background. Now, assuming that the supergravity solution (1.12) and the description of branes in the small $g_s N$ regime as surfaces where strings end without backreaction describe the same object and that we can commute the low energy limit with the change in $g_s N$, we identify

$$\begin{aligned}&\text{four-dim. } SU(N)\ \mathcal{N} = 4 \text{ SYM with IIB strings on } AdS_5 \times S^5 \\ &\text{and parameters } g_{YM}^2 \sim g_s \,, \ \frac{L^4}{4\pi\alpha'^2} = g_s N \,.\end{aligned}$$

See Fig 1.7. This is the Maldacena, or AdS/CFT conjecture and in its stronger form it holds for any N and g_s. The $\mathcal{N} = 4$ part is a conformal field theory hence the “CFT”. Obviously, we have omitted and glossed over a great amount of details in view of length and the introduction’s purpose. For a deeper analysis of the AdS/CFT conjecture we refer to the aforementioned references.

The AdS/CFT conjecture is an example of the Holographic principle as the conformal field theory lives on the boundary of AdS_5.[8] The equivalence of these two theories relates a strongly coupled to a weakly coupled regime. In other words it enables us to calculate quantities of a strongly coupled theory using a weakly coupled one. To see this, note that the SYM theory is weakly coupled when the ’t Hooft coupling $g_{YM}^2 N$ (and g_{YM}) are small.[9] On the other hand, we can trust the gravity approximation to IIB strings when the AdS length is large compared to the string scale $L^4/\alpha'^2 \sim g_{YM}^2 N \gg 1$. These two regimes are excluding each other.

[8] Actually, four-dimensional Minkowski space with some added points at infinity is the boundary of AdS_5. An analogous statement holds also for higher dimensions.

[9] This implies a large N, which is the so-called planar limit of SYM [66].

Taking the conjecture seriously, enables us to define quantum gravity and string theory on asymptotically AdS spaces. This is done by identifying the bulk theory with all its excitations, including sums over geometries asymptotic to AdS, with a CFT that lives on the boundary of AdS. This is useful as we have much more better control over CFTs than a complete definition of string theory and for example non-perturbative effects can be studied via the dual theory.

On the computational level, there is a dictionary that relates observables in the bulk theory (e.g. IIB string theory) and observables on the boundary [54]. To illustrate, denote the bulk theory partition function with boundary condition that a scalar field φ approaches the value φ_0 at the asymptotic boundary as

$$Z_{\text{bulk}}(\varphi_0) \,. \tag{1.14}$$

Then the correspondence states that the field φ_0 acts a source for the CFT at the boundary and we have the relation

$$\langle e^{\int \varphi_0 \mathcal{O}} \rangle_{\text{CFT}} = Z_{\text{bulk}}(\varphi_0) \,. \tag{1.15}$$

In the (super)gravity limit, which means $g_s \ll 1$, $L^2/\alpha' \gg 1$, the bulk partition function $Z_{\text{bulk}}(\varphi_0)$ can be approximated by the exponential of the on-shell action

$$Z_{\text{bulk}}(\varphi_0) = e^{-S_{\text{supergravity}}(\varphi)} = \langle e^{\int \varphi_0 \mathcal{O}} \rangle_{\text{CFT}} \,. \tag{1.16}$$

Here we also have to take the analogous limit on the CFT side, namely large N and large $g_{\text{YM}}^2 N$. In this limit, the CFT is strongly coupled but we can compute quantities using the classical approximation to the bulk theory.

In the above limit, we have more control over string theory on the $AdS_5 \times S^5$ background and less control over the gauge theory—which becomes strongly coupled. Interestingly, there is a limit of the $AdS_5 \times S^5$ geometry where the string can be quantised and one has perturbative control over both sides of the duality. This is the plane-wave limit or Berenstein-Maldacena-Nastase (BMN) limit [67][10]. This is an example where one can go beyond flat space.

In eq. (1.16), there can be more than one saddle points. In particular there exist supergravity solutions called “wormholes” that raise puzzles on the AdS/CFT conjecture [69]. These solutions are connected geometries with multiple disconnected boundaries, see Fig 1.8. The conjecture then tells us that the CFT partition function should factorize as the boundary theory would live on a the union of the two boundaries. However, the bulk partition function does not necessarily factorize as the bulk manifold is connected. We will soon see how one can naturally circumvent the boundary factorization problem considering ensemble averages.

Ensembles and Holography

We have seen that the AdS/CFT conjecture, relates two specific theories: the bulk quantum gravity/string theory to the conformal field theory living on the boundary, the classic example being IIB string theory and $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills. These theories have typically high amount of symmetry, like supersymmetry, and it is important to find other not so exotic cases of holographic correspondences. Instances of such dualities come from ensemble-averaged holography. This is another kind of holographic correspondence, inspired by AdS/CFT, where the boundary theory is now an ensemble average over possible boundary CFTs or quantum field theories in general.

[10]The plane-wave can also be seen as a background of IIB string theory without taking a limit of the $AdS_5 \times S^5$ background. See, e.g. ref [68].

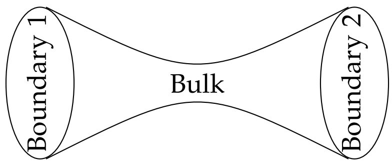

Figure 1.8: An example of the topology of a wormhole solution. The boundaries are disconnected but the whole manifold is connected through the bulk.

More precisely, consider a boundary CFT that depends on some moduli $\mathfrak{m}$. These take values in the moduli space $\mathcal{M}_{\text{CFT}}$ of the CFT. Given a measure $d\mu(\mathfrak{m})$, one can integrate over this moduli space and define averages of quantities of the CFT. For instance, the average $\langle Z \rangle$ of a partition function $Z(\mathfrak{m})$ is written schematically as

$$\langle Z \rangle = \int_{\mathcal{M}_{\text{CFT}}} d\mu(\mathfrak{m}) Z(\mathfrak{m}) \,. \tag{1.17}$$

To gain intuition, one could think of the parameters $\mathfrak{m}$ as radii and angles. For a scalar field compactified on a circle of radius R, the only modulus is the radius itself $\mathfrak{m} = R$ and taking the average would mean averaging over all possible radii.

The idea then is that averaged quantities match those of a bulk theory whose boundary is the space where each member of the ensemble lives. For the case of wormhole-like bulk geometries, the average introduces naturally (via the average) correlations between quantities. To illustrate this, consider a CFT on the union of two identical surfaces $\Sigma \cup \Sigma$. The partition function Z then factorises into the product of two partition functions on Σ

$$Z(\mathfrak{m}) = Z_\Sigma(\mathfrak{m}) Z_\Sigma(\mathfrak{m}) \,. \tag{1.18}$$

Taking the average

$$\langle Z \rangle = \int_{\mathcal{M}_{\text{CFT}}} d\mu \, Z_\Sigma(\mathfrak{m}) Z_\Sigma(\mathfrak{m}) \tag{1.19}$$

does not necessarily factorize as we are integrating and have assumed that both partition functions depend on the same moduli $\mathfrak{m}$.

Ensemble-averaged holography is rather unexpected from the usual AdS/CFT point of view. One the boundary side it performs an extra operation: averaging over moduli of the CFT. It would be interesting to explore further to what extend there is an analogous operation (such as coarse-graining) in the bulk theory and whether one can formulate ensemble-averaged holography in terms of usual gauge/gravity duality. See, e.g. [70, 71, 72].

Two main examples of ensemble average holography

Here we would like to briefly describe the two main examples in the literature for this kind of holography, namely that of Jackiw-Teitelboim (JT) gravity and matrix models [73], and that of Narain CFTs and "U(1)"-gravity [74, 75]. They concern two and three-dimensional bulk geometries respectively.

Two-dimensional bulk

The first case of an ensemble-averaged correspondence was given by Saad, Shenker and Stanford in [73]. JT gravity is a two dimensional model of gravity [76, 77, 78] that includes a scalar field, usually called "dilaton" that couples to the Ricci scalar $\mathcal{R}$. The action for a bulk manifold $\mathcal{M}$ reads

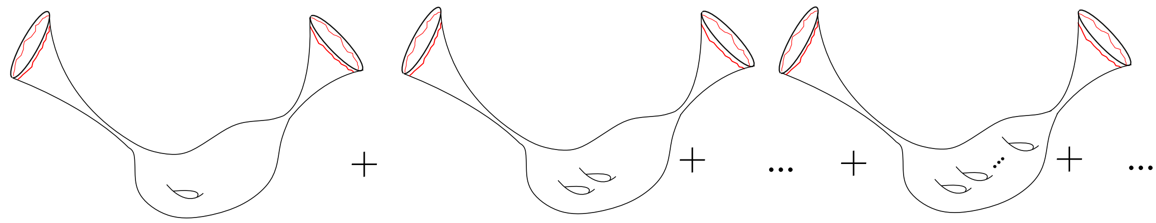

Figure 1.9: An example of the geometries considered in [73] for two asymptotic boundaries. The red lines depict a cutting of the boundary. The topological term mentioned in the text depends on the number of handles of the bulk.

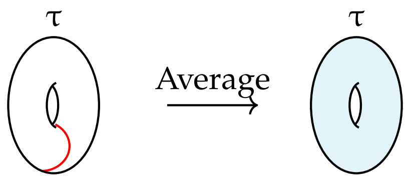

Figure 1.10: Making the red cycle of a two-dimensional torus of modular parameter τ contractible, we get a three-manifold called handlebody. Doing the same for other cycles of the boundary torus gives other, inequivalent handlebodies with the same boundary.

$$S_{JT} = -\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\phi\,(\mathcal{R}+2) + S_{\text{boundary}} + S_{\text{topological}}\,. \tag{1.20}$$

Where g is the determinant of the metric and ϕ is the scalar field. The S_{boundary} denotes the boundary action (analogous to the Gibbons-Hawking-York term in general relativity [79, 80]) and $S_{\text{topological}}$ is a topological term proportional to the Euler characteristic of the Riemann surface. We will be more explicit about these terms in the main text. The boundary in the setting of [73] is asymptotic and homeomorphic to a union of circles. The fact that the coupling of the scalar field is linear leads to it acting as a Lagrange multiplier: integrating it out enforces $\mathcal{R} = -2$. Hence the surface Σ has to be hyperbolic. The path integral, done carefully by [73], explicitly shows perturbative equivalence of correlation function of JT gravity to those of a random matrix model. The latter can be seen as an ensemble of quantum mechanical Hamiltonians living on the one-dimensional boundaries.

Three-dimensional bulk

One dimension higher, the first example was that of Maloney and Witten [74]—whose approach we mainly follow—and N. Afkhami-Jeddi et. al. [75]. There one starts from two-dimensional CFT and tries to construct a bulk theory to match the ensemble average. The boundary theory ensemble now consists of Narain CFTs [81, 82]. These CTFs can be thought of, for now, as generalizations of the free boson on the circle in more dimensions. We have D free bosons that are compactified on a D-dimensional torus T^D. The average over these CFTs is possible due to the Siegel-Weil formula [83, 84, 85, 86] and the dual holographic theory is given in terms of a $U(1)^{2D}$ Chern-Simons theory summed over handlebodies. The later are three-manifolds, constructed by “filling in” two-dimensional tori. In this sense the Narain CFT lives on the boundary two-dimensional torus of the handlebody (see Fig. 1.10).

This Thesis

In this work, we will explain with more technical details the JT/random matrix and Narain-$U(1)^{2D}$ holographic correspondences and show further developments made by the papers [1, 2, 3]—from which some parts of this thesis are freely taken. Among other things which will be explained, in these works we study the JT/random matrix correspondence from a topological gravity point of view, define generalisations of JT partition functions and study the low temperature limit of JT with conical singularities. Additionally, the Narain duality is extended to permutation orbifolds of the CFT and to supersymmetric Narain theories. Finally, we generalise the moduli space of S_2 permutation orbifolds using a particular $\mathbb{Z}_2$ orbifold construction and employ the Siegel-Narain formula to find explicit expressions for the averages of partition functions of the related generalised orbifolds.

Chapter 2

Jackiw-Teitelboim gravity via topological gravity

In this chapter, we initiate a more technical discussion. The goal here is to elucidate the work in [1]. To do this, we begin with a short but more explicit than in the Introduction exposition of the relation between JT gravity, moduli spaces of bordered Riemann surfaces and topological gravity.

2.1 JT gravity and its path integral.

As we mentioned in the Introduction, Jackiw-Teitelboim gravity is a theory in two spacetime dimensions that contains a scalar field ϕ coupled to gravity via the interaction $\sim \phi\mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar. For the description of JT and its path integral, we mainly follow [73].

The action

To begin with, we repeat the action of JT gravity (1.20), but now with more details

$$S_{\mathrm{JT}} = -\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\phi\,(\mathcal{R}+2) - \int_{\partial\mathcal{M}} \phi\sqrt{h}\,(K-1) - \frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\mathcal{R} + \int_{\partial\mathcal{M}} \sqrt{h}K\right]. \tag{2.1}$$

Some explanations are now in order:

- The first term $-\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\phi\,(\mathcal{R}+2)$ is the "bulk" JT action. It is an integral over the two-dimensional manifold $\mathcal{M}$ and contains the interaction of the scalar field ϕ with the Ricci scalar $\mathcal{R}$—$\sqrt{g}$ is the square root of the determinant g of the metric on $\mathcal{M}$. Due to the linear form of the interaction, it significantly simplifies the computation of the path integral as we will soon see.

- The second term $-\int_{\partial\mathcal{M}} \phi\sqrt{h}\,(K-1)$ is a boundary term serving the same purpose as the Gibbons-York-Hawking [79, 80] term in general relativity: it makes the variational problem well-defined in the presence of a boundary. K is the trace of the "extrinsic curvature"
$$K = g^{\mu\nu}\nabla_\mu n_\nu\,, \tag{2.2}$$
where n_ν is a vector normal to the boundary and ∇ denotes the covariant derivative. The part $\sim \phi\sqrt{h}$, where h is the determinant of the induced metric on the boundary, is there for "holographic renormalization" purposes in order to cancel

divergences that occur in the calculation of the boundary action. For more details on the boundary terms on general relativity see [87, 88, 89, 90].[1]

- The third and fourth terms combine to give a topological term via the Gauss-Bonnet theorem, which states that

$$\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\mathcal{R} + \int_{\partial\mathcal{M}} \sqrt{h}K = 2\pi\chi(\mathcal{M}) \ , \tag{2.3}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the surface $\mathcal{M}$. In terms of the genus g and number of boundaries n it is given by

$$\chi(\mathcal{M}) = 2 - 2g - n \ .^2 \tag{2.4}$$

This combines with the constant S_0 to give a weighing to the sum over topologies.

Boundary conditions and bulk manifold

So far, we didn't say much about the surface $\mathcal{M}$ on which JT gravity lives. $\mathcal{M}$ is orientable and has n asymptotic boundaries which we denote $\partial\mathcal{M}_i$ with $i = 1, 2, ..., n$. Each of them has the topology of a circle and has regularized length $\ell_1 = \beta_1/\epsilon, ..., \ell_n = \beta_n/\epsilon$ with ϵ to be taken to 0 in the end. Also, the induced metric on each boundary and the dilaton field are taken to be:

$$g_{uu}|_{\partial\mathcal{M}_i} = \frac{1}{\epsilon^2} \qquad \phi|_{\partial\mathcal{M}_i} \equiv \phi_b = \frac{\gamma}{\epsilon} = \frac{\gamma}{\beta_i}\cdot \ell_i \tag{2.5}$$

where u is a proper length coordinate along each boundary running from 0 to β_i. To illustrate, the manifold $\mathcal{M}$ looks like the ones in Fig. 1.9. The red curves there represent the closed curve along which we cut the surface to regulate the boundary length (like a cut-off). Given a fixed number of boundaries n, the path integral consists of a topological sum which essentially sums over the genera (number of handles) of $\mathcal{M}$. These contributions are denoted as

$$\langle Z(\beta_1)...Z(\beta_n)\rangle_{\text{conn.}} \simeq \sum_{g=0}^{\infty} e^{S_0(2-2g-n)} Z_{g,n}(\beta_1, ..., \beta_n) \ . \tag{2.6}$$

The exponential term $e^{S_0(2-2g-n)}$ comes from the topological term in the action and is fixed by the genus and the number of boundaries. We see that distinct topologies of Riemann surfaces are weighted by the action S_0 that relates to the gravitational coupling G_N as $G_N \sim 1/S_0$. Hence, the partition function is a non-perturbative expansion in the gravitational coupling G_N of JT gravity [73]. The symbol $\simeq$ is used to denote that this is perturbative expansion.

Performing the path integral

There is one crucial property of this model that makes it tractable. Namely, we can integrate out the scalar field ϕ, which acts as a Lagrange multiplier. This is done by rotating the contour to $\phi \mapsto i\phi$ and doing the ϕ part of the functional integral before integrating over metrics. This gives the constraint (arising as a functional form of the Dirac delta function)

$$\mathcal{R} + 2 = 0 \ . \tag{2.7}$$

[1] As an interesting historical side remark, we mention that Einstein had also considered boundary terms [91].

[2] We use the same letter for the genus and the determinant of the metric but there should be no confusion due to context.

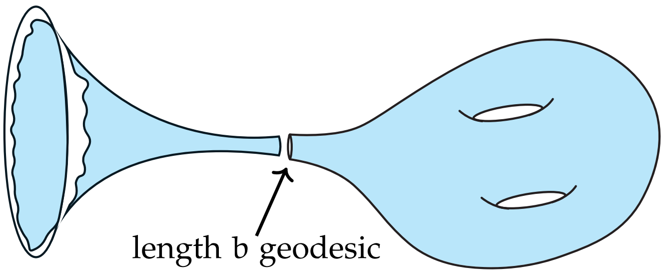

Figure 2.1: An example of genus two with one asymptotic boundary. To calculate the functional integral we can "cut" the surface along a closed geodesic of length b, do the functional integral over each piece and then glue back the surface by integrating over b. Figure inspired by [73].

This means that $\mathcal{M}$ has to be hyperbolic surface, i.e. it has constant negative curvature. To illustrate how the calculation continues, consider an example where $\mathcal{M}$ has two handles (so $g = 2$) and one asymptotic boundary like in Fig. 2.1. The quantity that corresponds to this surface is $Z_{2,1}(\beta_1)$. Schematically, the functional integral that has to be computed is

$$Z_{2,1}(\beta_1) = \int \mathcal{D}(\text{Moduli space of bulk}) \int \mathcal{D}(\text{shape of boundary}) e^{\int_{\partial\mathcal{M}} \phi_b \sqrt{h}(K-1)} \ . \tag{2.8}$$

This can be done via a cutting and gluing process. Specifically, we can choose a closed geodesic of minimal length (see Fig. 2.1) that separates $\mathcal{M}$ into a "trumpet" and a bordered Riemann surface with one geodesic boundary and two handles. We can do the functional integral on the trumpet and on the Riemann surface and then glue them back together by integrating over the twist τ around the cut (this goes from 0 to b) and the length of the geodesic b that goes from 0 to ∞. Denoting the "trumpet" contribution as $Z^{\text{trumpet}}(\beta_1, b)$ and the integral over the genus two Riemann surface as $V_{2,1}(b)$, we have

$$Z_{2,1}(\beta_1) = \int_0^\infty Z^{\text{trumpet}}(\beta_1, b) V_{2,1}(b) b db \ . \tag{2.9}$$

There is also a proportionality constant that can be fixed to 1. The factor of b comes from the integral over the twist $\int_0^b d\tau = b$.[3]

The trumpet integral

The quantity $Z^{\text{trumpet}}(\beta_1, b)$ is computed via the integral over the shape of the boundary in (2.8). It is in fact one-loop exact [92]. To calculate it, one has to evaluate the boundary action and integrate over the boundary shape. This can be done using the following metric for the trumpet geometry (see Fig. 2.2)

$$ds^2 = d\sigma^2 + \cosh^2\!\sigma d\tau^2 \ , \quad \tau \sim \tau + b \text{ and } \sigma \in [0, \infty) \ . \tag{2.10}$$

The integral over the boundary shape is essentially a functional integral over functions $\tau = \tau(u)$, where u is a proper length coordinate along the boundary.[4] The result is

$$Z^{\text{trumpet}}(\beta_1, b) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\gamma}{\beta_1}} e^{-\frac{\gamma}{2}\frac{b^2}{\beta_1}} \ . \tag{2.11}$$

There is however one more geometry that needs to be computed that we haven't mentioned so far, that of the disk.

[3] The twist τ denotes how much we rotate before we glue thee two surfaces. It can not be more than b as then we would be over-counting.

[4] The boundary shape is given by a curve $(\sigma(u), \tau(u))$.

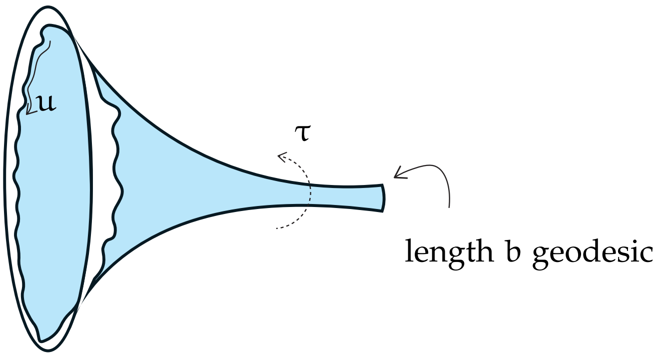

Figure 2.2: The trumpet geometry. Figure inspired by [73].

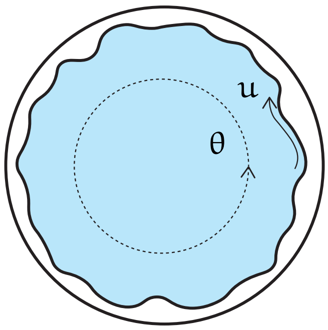

Figure 2.3: The disk geometry. Figure inspired by [73].

The disk

This occurs when the genus is zero and we have one boundary, see Fig. 2.3. Again, this is a one-loop exact computation [92] and is done in a similar way. Now the metric takes the form

$$ds^2 = d\rho^2 + \sinh^2\rho d\theta^2 \ , \quad \theta \sim \theta + 2\pi \text{ and } \rho \in [0, \infty) \ , \tag{2.12}$$

and the result is

$$Z^{\text{disk}}(\beta_1) \equiv Z_{0,1}(\beta_1) = e^{\frac{\gamma}{2\beta}(2\pi)^2} \left(\frac{\gamma}{\beta}\right)^{\frac{3}{2}} \frac{1}{\sqrt{2\pi}} \ . \tag{2.13}$$

The bulk moduli space and Mirzakhani's recursion

We should also explain the quantity that came from the bulk integration $V_{2,1}(b)$. When integrating out the dilaton, we are left with the boundary integral, which gives the trumpet (or disk), and an integration over bulk metrics. Since the exponential in the path integral of the bulk is equal to 1, what remains is just the volume of that moduli space, which we denoted as $V_{2,1}(b)$. Remarkably, these volumes can be all computed using Mirzakhani's recursion relation [93]. This equation, which we do not write for conciseness, enables one to compute all $V_{g,n}(b)$ recursively.[5] For more about bordered Riemann surfaces and their moduli space volumes, we refer to [93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104].

[5]Note that the constraint for a hyperbolic surface constrains g, n to satisfy $2 - 2g - n < 0$.

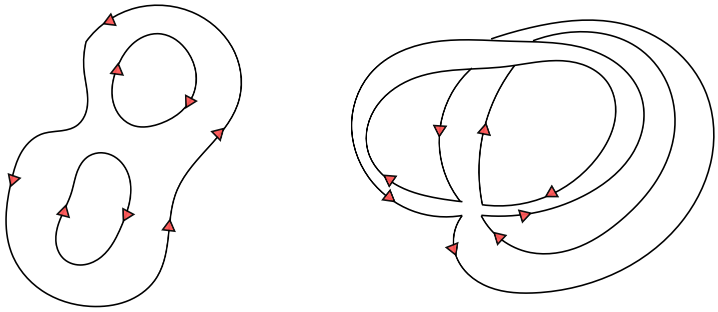

Figure 2.4: Examples of matrix model diagrams with a quartic vertex. The one on the left can be drawn on a genus zero surface and the second on a genus one.

The topological expansion formula

Combining all the above, we can write down a formula for any $Z_{g,n}(\beta_1, ..., \beta_n)$ by following the same procedure. Now we have n trumpets and a Riemann surface of genus g and n geodesic boundaries. Cutting and gluing gives

$$Z_{0,2}(\beta_1, \beta_2) = \int_0^\infty Z^{\text{trumpet}}(\beta_1, b) Z^{\text{trumpet}}(\beta_2, b) b db \tag{2.14}$$

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int_0^\infty db_1\ b_1 ... \int_0^\infty db_n\ b_n\ V_{g,n}(b_1, ..., b_n) Z^{\text{trumpet}}(\beta_1, b_1) ... Z^{\text{trumpet}}(\beta_n, b_n)\ , \tag{2.15}$$

where we wrote separately the special case of gluing two trumpets $Z_{0,2}(\beta_1, \beta_2)$–this contribution is called "double trumpet". The above are of course supplemented by the disk result given in (2.13). These two last cases of the trumpet and disk are special as $(g, n) = (0, 2),\ (0, 1)$ do not correspond to hyperbolic surfaces.

In the remarkable work [73], it was shown that the topological expansion of JT gravity we just described is equivalent to that of a matrix model. The later is essentially a zero-dimensional quantum field theory where the fields are square matrices. The path integral of such a theory is a multi-dimensional integral over the entries of the matrix. Obviously, different classes of matrices, such as Hermitian, unitary and so on, restrict the integration domains. Such theories have diagrammatic expansion (like the Feynman diagrams) but with ribbon graphs. These graphs consist of double lines roughly corresponding to the two indices of the matrices. Now, these diagrams can be given a topology in terms of what genus surface they can be drawn on without self intersections [66]. This leads to a genus expansion of matrix model observables, see Fig. 2.4. The statement of the correspondence between JT and matrix models is

$$\langle Z(\beta_1) ... Z(\beta_n) \rangle_{\text{conn.}} \hat{=} \langle \operatorname{tr} e^{-\beta_1 H} \ldots \operatorname{tr} e^{-\beta_n H} \rangle_{\text{MM}}\ . \tag{2.16}$$

Here the left hand side is the connected thermal partition function $Z(\beta_1, \ldots, \beta_n)$ of JT gravity for geometries with n asymptotic boundary components characterised by their inverse temperatures β_i, $i = 1, \ldots, n$. The right hand side is the corresponding correlator of the dual Hermitian matrix integral. Interestingly, these correlators enjoy an interpretation as observables in an ensemble of quantum mechanical systems whose random Hamiltonians H are given by Hermitian matrices H of the matrix model [73].[6] The proof of (2.16) makes use of the relation between topological recursion and the

[6]According to ref. [105], the intriguing appearance of an ensemble of quantum mechanical systems can also be argued for via the relationship of JT gravity to the Sachdev–Ye–Kitaev model.

moduli space volumes $V_{g,n}(b_1,...,b_n)$ shown in [106]. This duality is generalised in ref. [107], where extensions of JT gravity are associated to other matrix models [108, 109, 110]. For more about matrix models, topological recursion see [111, 112, 113, 114, 115, 107, 116, 117, 111, 118, 119, 120].

2.2 Deformations of JT gravity

Relevant to our work are also more general versions of JT gravity called deformations of JT. To that end, consider a more general bulk action (class of models) restricting to at most two derivatives [121, 122, 123, 124, 125] :

$$I_W = -\frac{1}{2}\int d^2x\sqrt{g}\left(D(\phi_0)\mathcal{R} + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi_0\partial_\beta\phi_0 + W_0(\phi_0)\right) \tag{2.17}$$

where $D(\phi_0), W_0(\phi_0)$ are functions of the scalar field ϕ_0 such $D(\phi_0) \neq 0$, $\frac{d}{d\phi_0}D(\phi_0) \neq 0$. This model has the most general local reparametrisation invariant action with terms of dimension $\leqslant 2$ [121] and can be brought to a form in which only one function is needed to describe it. This can be achieved by redefining the metric as:

$$\bar{g}_{\mu\nu} = e^{2\omega(\phi_0)}g_{\mu\nu} \tag{2.18}$$

with ω obeying the differential equation[7] :

$$-\frac{1}{4} + \frac{dD}{d\phi_0}\frac{d\omega}{d\phi_0} = 0 \tag{2.19}$$

and also by defining:

$$\phi \equiv D(\phi_0) \qquad W(\phi) \equiv e^{-2\omega}W_0 \tag{2.20}$$

the action becomes (omitting the bars):

$$I_W = -\frac{1}{2}\int d^2x\sqrt{g}\left(\phi\mathcal{R} + W(\phi)\right). \tag{2.21}$$

JT is obviously a special case of this model with $W_{JT}(\phi) = 2\phi$. To study perturbations away from this - or deformations of JT - we can write:

$$W(\phi) = W_{JT}(\phi) + U(\phi) = 2\phi + U(\phi) \tag{2.22}$$

A particularly interesting example of $U(\phi)$ is

$$U(\phi) = 2\epsilon e^{-(2\pi-\alpha)\phi}, \quad 0 < \alpha < \pi\,. \tag{2.23}$$

This potential does not affect the asymptotic boundary conditions and the gravitational path integral can be evaluated perturbatively in the coupling ϵ [126]. Carrying out the path integral over the scalar field ϕ at the perturbative order ϵ^k changes the constraint (2.7) to [127, 126, 128]

$$R(x) + 2 = 2\sum_{j=1}^{k}(2\pi-\alpha)\,\delta^{(2)}(x-x_j)\,, \tag{2.24}$$

[7]This constraint comes from partially integrating the ω part of the transformed Ricci scalar.

with a remaining integral of the positions $x_1, \dots, x_k$ over the Riemann surface $\mathcal{M}$.

We briefly illustrate this: The potential (2.23) appears in the path integral as an exponential of an integral over the Riemann surface $\mathcal{M}$. Expanding in ϵ, we get

$$e^{-\int_{\mathcal{M}} d^2y\sqrt{g(y)}2\epsilon e^{-(2\pi-\alpha)\phi(y)}} = 1 - 2\epsilon \int_{\mathcal{M}} d^2x\sqrt{g(x_1)}e^{-(2\pi-\alpha)\phi(x_1)} + \frac{(2\epsilon)^2}{2!}\int_{\mathcal{M}} d^2x_1\sqrt{g(x_1)}\int_{\mathcal{M}} d^2x_2\sqrt{g(x_2)}e^{-(2\pi-\alpha)\phi(x_1)}e^{-(2\pi-\alpha)\phi(x_2)} + \dots \quad (2.25)$$

To evaluate the path integral at order ε^m take the position integrations outside to get something proportional to [126]

$$\sim \frac{1}{m!}\prod_{i=1}^{m}\int d^2x_i\sqrt{g(x_i)}\int \mathcal{D}\phi\mathcal{D}g e^{\frac{1}{2}\int d^2y\sqrt{g(y)}\phi(y)\left(\mathcal{R}+2-(2\pi-\alpha)\sum_{i=1}^{m}\delta^{(2)}(y-x_i)\right)} \quad (2.26)$$

where by $\mathcal{D}g$, we denote the functional integral over the bulk metrics and the Dirac delta function is such that $\int d^2x\sqrt{g(x)}\delta^{(2)}(x-y) = 1$. It is now apparent that the constraint induced by integrating over the scalar field ϕ becomes eq. (2.24).

Thus, at the given perturbative order ϵ^k we get hyperbolic surfaces with k conical singularities with identification angles α. As a result, the path integral of JT gravity with the potential (2.23) can be perturbatively interpreted as a sum over all possible hyperbolic Riemann surfaces $\mathcal{M}$ with any number of conical singularities with identification angles α at arbitrary positions on $\mathcal{M}$.[8] Furthermore, we can interpret the deformation (2.23) as coupling JT gravity to a gas of defects characterized by the coupling constant ϵ and the identification angle α [128, 126]. The structure can readily be generalised to an arbitrary finite number (possibly even to an infinite number or to a continuous family) of defect species with individual couplings ϵ_j and identification angles α_j [128, 126], such that a more general class of deformations to JT gravity can be realised.

The partition function of JT gravity with defects

Performing the path integral of JT gravity interacting with a gas of defects is similar to the usual JT path integral we outlined in section 2.1. The relevant path integrals localise on the Weil–Petersson volumes of hyperbolic Riemann surfaces with geodesic boundary components and conical defects, folded with the path integral of the Schwarzian theory describing the one-dimensional action at the asymptotic boundaries [73]. Crucial to this, is the fact that the identification angle is $\alpha \in (0,\pi)$ such that one can find a minimal length closed geodesic in a homology class of a closed loop. This is important for being able to decompose the surface into more basic building blocks, for example for gluing a "trumpet" to a Riemann surface with a geodesic boundary and a number of conical singularities. For a single asymptotic boundary component and a r types of defect species, the resulting partition function reads [126, 128]

$$Z(\beta) = e^{S_0}Z^{\text{disk}}(\beta) + e^{S_0}\sum_{j=1}^{r}\epsilon_j Z^{\text{disk}}(\beta,\alpha_j) + \sum_{g,n=0}^{\infty} e^{(1-2g)S_0}\sum_{j_1,\dots,j_n=1}^{r}\frac{\epsilon_{j_1}\cdots\epsilon_{j_n}}{n!}\int_0^\infty db\, b\, Z^{\text{trumpet}}(\beta,b)V_{g,b,(\alpha_{j_1},\dots,\alpha_{j_n})}\,. \quad (2.27)$$

Here the parameters ϵ_j, $j = 1,\dots,r$, are the coupling constants to the r distinct defect

[8]We can think of a conical singularity as in Fig. 2.6. Also, note that the singular points are assumed to not collide (for more on this, see ref. [126]).

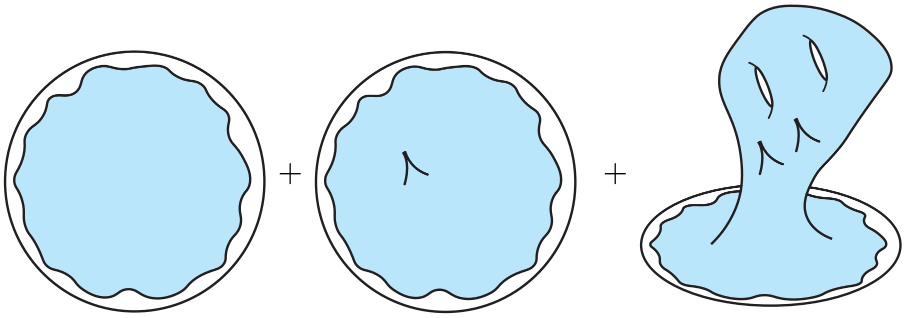

Figure 2.5: A heuristic illustration of the topological expansion in (2.27). The sharp edges denote conical singularities. The sum over these singularities is not only over their number but also their type. Accompanying that, we have sum over handles (genera).

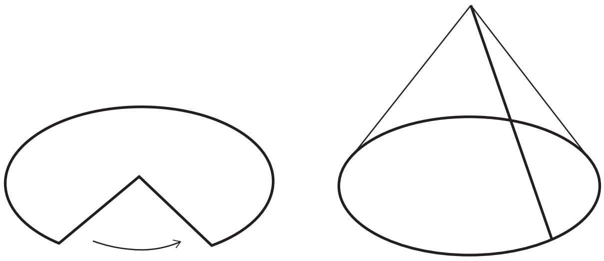

Figure 2.6: A deficit angle, or conical singularity, can be obtained by cutting a portion of a disk and the re-gluing it.

types that are characterised by the identification angles α_j of their associated conical singularities on the hyperbolic Riemann surfaces, see also fig. fig. 2.5. Furthermore, β is the inverse temperature attributed to the configurations of wiggles at the asymptotic boundary of the hyperbolic Riemann surfaces and $V_{g,b,(\alpha_{j_1},\dots,\alpha_{j_n})}$ denotes the volume of the moduli space of hyperbolic Riemann surfaces with a single geodesic boundary of length b and n conical singularities of type $(\alpha_{j_1},\dots,\alpha_{j_n})$. The first two terms in this expansion capture the contributions of disks with no conical singularities and a single conical singularity, respectively. The remaining topologies appear in the second line where by the sum over all genera and number of boundaries we mean that the cases of the disk and disk with one defect are excluded. This can also be seen as a convention for the "would-be" volumes that correspond to these cases. A new ingredient here is the contribution of the disk with one conical singularity [92, 127]

$$Z^{\text{disk}}(\beta,\alpha_j) = \frac{\gamma^{\frac{1}{2}} e^{\frac{\gamma\alpha_j^2}{2\beta}}}{(2\pi\beta)^{\frac{1}{2}}} \,. \tag{2.28}$$

For partition functions of more geodesic boundaries one proceeds in a similar way. We will write down partition functions with more boundaries in the next sections, when we express everything in terms of topological gravity.

The moduli space volumes with defects

In eq. (2.27), appeared the volumes of the moduli spaces of hyperbolic Riemann surfaces with geodesic boundaries and conical singularities. For such a Riemann surface of genus g with p geodesic boundaries of length $\vec{b} = (b_1, ..., b_p)$ and q conical singularities of magnitudes $\vec{\alpha} = (\alpha_1, ..., \alpha_q)$, denote these volumes as $V_{g,\vec{b},\vec{\alpha}}$. These can be obtained by replacing the lengths $b_{p+1}, ..., b_q$ of $V_{g,\vec{b},(b_{p+1},...,b_q)}$ with $(i\alpha_1, \dots, i\alpha_q)$ [129]. In other words, we have

$$V_{g,\vec{b},\vec{\alpha}} = V_{g,(b_1,...,b_p,i\alpha_1,...,i\alpha_q)} \, . \tag{2.29}$$

This is a very important result as it enables us to deduce the result of volumes with defects without any extra complicated calculation.

2.3 Relation to topological gravity

One can continue and calculate the volumes via recursion relations and compute correlators of JT gravity.[9] In this work, we are interested in taking another perspective into this problem, namely exploring JT gravity from the point of view of topological gravity, or equivalently intersection theory on the moduli space of stable curves. For the original duality of JT and matrix models this relation was established in [130, 131] but here we want to generalize this to include JT with defects and even more general objects that hopefully can be matched to arbitrary deformations of JT gravity. As we will see, tuning certain parameters in the topological gravity expansion of JT correlators maps the expansion to JT with defects or not. The idea is that changing these parameters in an arbitrary way could possibly map to an arbitrary deformation $U(\phi)$ of JT. While the work [1] was being completed, ref. [132] appeared, which has certain overlap with some of our discussions in sections 2.3 to 2.6.

2.3.1 Preliminaries

In this section we give an introduction to the relation between Weil-Petersson volumes of hyperbolic Riemann surfaces with geodesic boundaries and correlators of topological gravity. This is necessary also in order to introduce the used notation.

What is a moduli space

Before we go any further, we would like to give some intuition for what is meant by a moduli space for the reader that has not encountered such a notion before.

We can think of such a space as a way of describing objects that can be parametrised or classified in a certain way. For example, consider the space of all circles on a plane. To distinguish between two elements of this collection, we only need to know the radius of the circle. Hence the moduli space is $\mathbb{R}^+$, since the radius can be any positive number. In a sense, an element of the moduli space is a representative of some equivalence class. For the circle example, we considered all circles of the same radius as equivalent no matter the position of their centre: a circle of radius 1 at $(0, 0)$ is equivalent to a circle of radius 1 at (x, y) for every $x, y \in \mathbb{R}$. A more relevant example would be that of hyperbolic Riemann surfaces of a given genus g and n geodesic boundary components of lengths $b_1, ..., b_n$. The moduli space $\mathcal{M}_{g,n}(b_1, ..., b_n)$ of such a collection contains representative surfaces (like the unit circle centred at the origin in our previous example) of fixed genus and boundary lengths that are equivalent via certain kinds of deformations to other surfaces with the same genus and boundary lengths. The volume of this moduli space tells us how large this space is and gives a measure of "how many" these surfaces are.

[9] By correlators we mean quantities like (2.6).

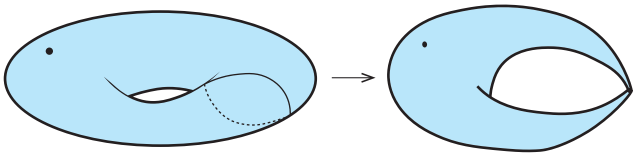

Figure 2.7: A handle degenerating to a nodal point in a genus $g = 1$ curve with $n = 1$ marked points.

Intersection numbers on the moduli space of stable curves

To establish a foundation and introduce the necessary notation, we begin by collecting some mathematical preliminaries on the Weil-Petersson volumes of hyperbolic Riemann surfaces with geodesic boundaries and conical singularities, viewed from the perspective of intersection theory on the moduli spaces of stable curves.

Let $\mathcal{M}_{g,n}$ represent the moduli space of smooth curves of genus g with n distinct marked points. This moduli space is not compact, as it excludes the limiting curve with a handle degenerating to a nodal point, see fig. 2.7, and the limit where two marked points collide. The Deligne-Mumford compactification, $\overline{\mathcal{M}}_{g,n}$, includes these limits as stable curves with nodal singularities and is a well defined moduli space. The complex dimensions of these moduli spaces are given by:

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n \, . \tag{2.30}$$

These spaces are defined only for pairs of g, n that give rise to stable curves: for genus $g \geqslant 2$ we can have any number of marked points $n \geqslant 0$, for $g = 1$ we have $n \geqslant 1$ and for $g = 0$ we need $n \geqslant 3$.[10]

The moduli space $\overline{\mathcal{M}}_{g,n}$ admits various cohomology classes which we can integrate. For us the relevant integrals are the intersection numbers (or correlators)

$$\left\langle \kappa_1^{\ell} \tau_{d_1} \dots \tau_{d_n} \right\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{\ell} \psi_1^{d_1} \dots \psi_n^{d_n} \, , \quad \ell, d_1, \dots, d_n \in \mathbb{Z}_{\geqslant 0} \, , \tag{2.31}$$

where κ_1 is the first Miller–Morita–Mumford class and ψ_i's are first Chern classes of specific lines bundles on $\overline{\mathcal{M}}_{g,n}$, of which we do not give a more detailed definition here for conciseness. Both κ_1 and the ψ_i's are elements of $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.[11] When integrating over a space of complex dimension $3g - 3 + n$ we have to make sure that the integrand is a top form. This way we get the selection rule

$$\left\langle \kappa_1^{\ell} \tau_{d_1} \dots \tau_{d_n} \right\rangle_{g,n} \neq 0 \quad \Rightarrow \quad \ell + d_1 + \dots + d_n = 3g - 3 + n \, . \tag{2.32}$$

Generating functions of the correlators

For the correlators defined in (2.31), we can define a generating functions [115]

$$F(\{t_k\}) = \sum_{g=0}^{+\infty} g_s^{2g} \left\langle e^{\sum_{d=0}^{\infty} t_d \tau_d} \right\rangle_g = \sum_{g=0}^{+\infty} g_s^{2g} \sum_{\{n_d\}} \left(\prod_{d=0}^{\infty} \frac{t_d^{n_d}}{n_d!} \right) \left\langle \tau_0^{n_0} \tau_1^{n_1} \dots \right\rangle_g \, , \tag{2.33}$$

[10]The curve with $(g, n) = (1, 0)$ is not stable as it has continuous automorphisms.
[11]$H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the 2nd cohomology group of $\overline{\mathcal{M}}_{g,n}$ with rational coefficients.

and

$$G(s,\{t_k\}) = \sum_{g=0}^{+\infty} g_s^{2g} \left\langle e^{s\kappa_1 + \sum_{d=0}^{\infty} t_d \tau_d} \right\rangle_g = \sum_{g=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{g_s^{2g} s^m}{m!} \sum_{\{n_d\}} \left(\prod_{d=0}^{\infty} \frac{t_d^{n_d}}{n_d!} \right) \left\langle \kappa_1^m \tau_0^{n_0} \tau_1^{n_1} \dots \right\rangle_g , \tag{2.34}$$

where g_s is a genus expansion parameter and $\{t_d\}$ are called couplings. The $n_0, n_1, \ldots$ are non-negative integers. These two generating functions are actually related [133, 134, 135]

$$G(s,\{t_k\}) = F(\{t_k + \gamma_k\}) , \qquad \gamma_0 = \gamma_1 = 0 , \quad \gamma_k = \frac{(-1)^k}{(k-1)!} s^{k-1} . \tag{2.35}$$

Relation to moduli space volumes and conical singularities

The κ_1 class is actually proportional to the Weil-Petersson Kähler (WP) form [136]

$$\omega_{WP} = 2\pi^2 \kappa_1 . \tag{2.36}$$

The WP form enables one to find a volume form and integrate over the moduli space. This is a crucial property and relates the generating functions F, G to the Weil–Petersson volumes V_g of the moduli space of genus g curves (for $g \geqslant 2$) without any marked points, i.e.

$$G(2\pi^2, \{t_k = 0\}) = \sum_{g=2}^{+\infty} g_s^{2g} \int_{\overline{\mathcal{M}}_{g,0}} e^{\omega_{WP}} = \sum_{g=2}^{+\infty} g_s^{2g} \int_{\overline{\mathcal{M}}_{g,0}} \mathrm{vol}_{WP} = \sum_{g=2}^{+\infty} g_s^{2g} V_g . \tag{2.37}$$

Where the second equality essentially picks up the top form. The integral then is an integral of the Weil–Petersson volume form over $\overline{\mathcal{M}}_{g,0}$, which is essentially the definition of the volume.

As shown by Mirzakhani [137] this idea can be extended and applied to moduli space volumes of hyperbolic Riemann surfaces with geodesic boundaries. Let $V_{g,\vec{b}}$ with $\vec{b} = (b_1, \ldots, b_n)$ denote the moduli space volume of hyperbolic Riemann surfaces of genus g and n geodesic boundaries of lengths $b_1, \ldots, b_n$. Mirzakhani showed that it can be calculated in terms of the intersection numbers we defined above

$$V_{g,\vec{b}} = \int_{\overline{\mathcal{M}}_{g,n}} e^{\omega_{WP} + \frac{1}{2}\sum_{\ell=1}^{n} b_\ell^2 \psi_\ell} = \left\langle e^{2\pi^2\kappa_1 + \frac{1}{2}\sum_{\ell=1}^{n} b_\ell^2 \psi_\ell} \right\rangle_{g,n} . \tag{2.38}$$

Using this expression, one can derive a similar relation as (2.37)

$$G(2\pi^2\lambda, \{t_k = \sum_{i=1}^{p} \frac{\lambda^k b_i^{2k}}{2^k k!} \delta_j\}) = \sum_g \frac{g_s^{2g}}{\lambda^3} \sum_{i_1,\ldots,i_p=0}^{+\infty} \left(\prod_{s=1}^{p} \frac{(\lambda\delta_s)^{i_s}}{i_s!} \right) \lambda^{3g} V_{g,(\underbrace{b_1,\ldots,b_1}_{i_1 \text{ times}},\ldots,\underbrace{b_p,\ldots,b_p}_{i_p \text{ times}})} . \tag{2.39}$$

Where the factor λ comes from a rescaling of cohomology classes.

Finally, as we saw earlier, we can turn a geodesic boundary into a conical singularity by essentially replacing a boundary's length with i times the magnitude of the identification angle α. From this information, we can deduce that the generating function for hyperbolic Riemann surfaces with boundary components of geodesic lengths $b_1, \ldots, b_p$ and conical singularities with identification angles $\alpha_1, \ldots, \alpha_q$ becomes in terms of the non-zero parameter λ

$$G(2\pi^2\lambda,\{t_k = \sum_{i=1}^{p} \frac{\lambda^k b_i^{2k}}{2^k k!}\delta_i + \sum_{j=1}^{q} \frac{\lambda^k(-\alpha_j^2)^k}{2^k k!}\epsilon_j\}) \tag{2.40}$$
$$= \sum_g \frac{g_s^{2g}}{\lambda^3} \sum_{\substack{i_1,\dots,i_p=0 \\ j_1,\dots,j_q=0}}^{+\infty} \left(\prod_{s=1}^{p} \frac{(\lambda\delta_s)^{i_s}}{i_s!} \prod_{t=1}^{q} \frac{(\lambda\epsilon_t)^{j_t}}{j_t!} \right) \lambda^{3g} V_{g,(\underbrace{b_1,\dots,b_1}_{i_1 \text{ times}},\dots,\underbrace{b_p,\dots,b_p}_{i_p \text{ times}}),(\underbrace{\alpha_1,\dots,\alpha_1}_{j_1 \text{ times}},\dots,\underbrace{\alpha_q,\dots,\alpha_q}_{j_q \text{ times}})} .$$

These formulas are a crucial first step in expressing the topological expansion of JT gravity with and without defects in terms of topological gravity correlators.

2.3.2 Formulating JT gravity with defects in terms of intersection theory

We saw in the previous paragraphs that choosing particular values for the couplings $\{t_k\}$'s in the generating functions (2.33) and (2.34) gave connection to either the moduli space volumes of hyperbolic Riemann surfaces with geodesic boundaries or the moduli space volumes of hyperbolic Riemann surfaces with conical singularities. Specifically, we have

$$\{t_k = \sum_{i=1}^{p} \frac{\lambda^k b_i^{2k}}{2^k k!}\delta_j\} \mapsto \text{ G generates } V_{g,\vec{b}} \tag{2.41}$$
$$\{t_k = \sum_{i=1}^{p} \frac{\lambda^k b_i^{2k}}{2^k k!}\delta_i + \sum_{j=1}^{q} \frac{\lambda^k(-\alpha_j^2)^k}{2^k k!}\epsilon_j\} \mapsto \text{ G generates } V_{g,\vec{b},\vec{\alpha}} \,. \tag{2.42}$$

Both of these choices for the couplings came essentially from the scalar potential in the JT action and the constraint it imposes once we integrate out the dilaton. Looking at eq. (2.22), the undeformed JT case corresponds to $U(\phi) = 0$ and the conical singularities case to $U(\phi) = 2\sum_i \epsilon_i e^{-(2\pi-\alpha_i)\phi}$. We refer to couplings $\{t_k\}$ that fulfil constraints coming from scalar potentials as **on-shell** couplings and otherwise **of-shell**.

This connection between the moduli space volumes and the generating functions (2.33) and (2.34) evaluated at on-shell couplings enables one to establish a direct link between the partition functions of JT gravity and correlation functions in topological gravity [130, 131]. In the following, we will extend this connection to JT gravity interacting with a gas of defects.

Using the selection rule (2.32) and eq. (2.38), we can verify again that the last term in eq. (2.27) **does not** contain terms with none or only one defect for genus zero. This is because for $n_b = 1, n_\alpha$ being the number o geodesic boundaries and defects respectively, the selection rule becomes

$$\ell + d_1 + \ldots + d_{n_b+n_\alpha} = 3g - 2 + n_\alpha \,, \tag{2.43}$$

hence, for genus zero, n_α has to be $n_\alpha \geqslant 2$.

Expressing the partition function in terms of topological gravity correlators

Rearranging the sum over defects in eq. (2.27), together with the relation $e^{-S_0} = \lambda^{\frac{3}{2}} g_s$ and after integrating over b, we obtain the JT partition function $Z(\beta)$

$$Z(\beta) = \frac{1}{\sqrt{2\pi} g_s} \left(\frac{\gamma}{\lambda\beta}\right)^{\frac{3}{2}} \left(e^{\frac{2\pi^2\gamma}{\beta}} + \frac{\beta}{\gamma} \sum_{j=1}^{r} \epsilon_j \, e^{\frac{\gamma\alpha_j^2}{2\beta}} + \sum_{\ell=0}^{+\infty} \left(\frac{\lambda\beta}{\gamma}\right)^{\ell+2} \frac{\partial}{\partial t_\ell} G(2\pi^2\lambda,\{t_k = \delta_k\}) \right) , \tag{2.44}$$

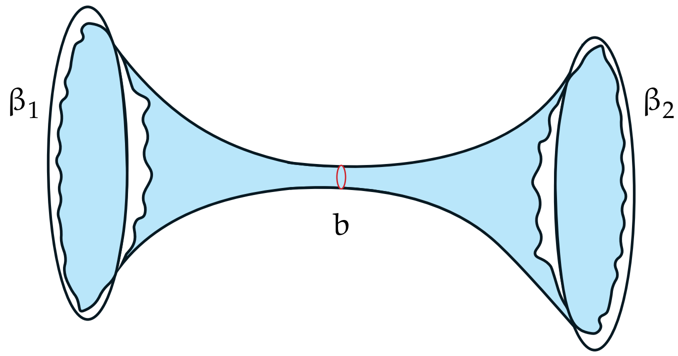

Figure 2.8: The "double trumpet" geometry. Two trumpet geometries are glued along a simple closed geodesic.

with

$$\delta_k = \sum_j \delta_{k,j} \,, \qquad \delta_{k,j} = \frac{\lambda^{k-1}(-\alpha_j^2)^k}{2^k k!}\epsilon_j \,. \tag{2.45}$$

Where in the last term in eq. (2.44) we first take the derivative and then fix the t_k values in G—this also appears later. Note that only the last term of the partition function $Z(\beta)$ given in eq. (2.44) is mapped to the topological correlators (2.31), whereas the first two terms associated to disk topologies capture the semi-classical contributions to the partition function in the presence of a gas of defects.[12] This can be seen from the discussion around eq.(2.43). We can group these contributions as

$$Z(\beta) = Z(\beta)^{\text{non-top.}} + Z(\beta)^{\text{top.}} \,. \tag{2.46}$$

Where "non-top." and "top." denote contributions not related and related to topological gravity correlators respectively. $Z(\beta)^{\text{top.}}$ contains the third term in eq. (2.44) and the rest belong to $Z(\beta)^{\text{non-top.}}$. A similar grouping of terms will be relevant for multiple boundary contributions to the path integral.

Generalization to multiple boundaries

So far we have seen how we can exploit the relation between the generating functions F, G of topological gravity correlators and the moduli space volumes in order to express the JT partition function in terms of these correlators. This can be generalised straightforwardly to multiple boundary contributions. First, we check which terms are going to be related to topological gravity correlators. The analogous constraint to eq. (2.43), becomes (we write only the right hand side as this is enough for the argument)

$$3g - 3 + n_b + n_\alpha \,. \tag{2.47}$$

For two boundaries $n_b = 2$, the **not allowed term** is only the pair $(g = 0, n_\alpha = 0)$. Hence the partition function for two boundaries $Z(\beta_1, \beta_2)$ becomes

$$Z(\beta_1, \beta_2) = Z(\beta_1, \beta_2)^{\text{non-top.}} + Z(\beta_1, \beta_2)^{\text{top.}} \,. \tag{2.48}$$

The "non-top." part correspond to the pair $(g = 0, n_\alpha = 0)$ which is called the "double trumpet" geometry and has the cylinder topology, see fig. 2.8. This is the same as eq. (2.14), which we repeat

$$Z_{0,2}(\beta_1, \beta_2) = \int_0^\infty Z^{\text{trumpet}}(\beta_1, b) Z^{\text{trumpet}}(\beta_2, b) b db \,. \tag{2.49}$$

[12]By "partition function", we mean the contribution to the path integral for a fixed number of boundary components. This number should be clear from the context.

This can be calculated explicitly [73]

$$Z_{0,2}(\beta_1, \beta_2) = \frac{\sqrt{\beta_1 \beta_2}}{2\pi\beta_1 + 2\pi\beta_2} \,. \tag{2.50}$$

Before we show the "top." contributions, we note that the partition functions with $m > 2$ boundaries $Z(\beta_1, \ldots, \beta_m)$ due to the constraint (2.32) receives only topological gravity contributions

$$Z(\beta_1, \ldots, \beta_m) = Z(\beta_1, \ldots, \beta_m)^{\text{top.}} \quad \text{for} \quad m > 2 \,. \tag{2.51}$$

Working similarly as before, using the relation between the volumes $V_{g,\vec{b},\vec{\alpha}}$, we find that

$$Z(\beta_1, \ldots, \beta_m)^{\text{top.}} = \frac{1}{g_s^2} \mathcal{B}(\beta_1) \cdots \mathcal{B}(\beta_m) G(2\pi^2\lambda, \{t_k = \delta_k\}) \quad \text{for} \quad m \geqslant 1 \,, \tag{2.52}$$

with δ_k as defined in eq. (2.45) and in terms of the differential operator

$$\mathcal{B}(\beta) = g_s \sqrt{\frac{\lambda\beta}{2\pi\gamma}} \sum_{\ell=0}^{+\infty} \left(\frac{\lambda\beta}{\gamma}\right)^{\ell} \frac{\partial}{\partial t_\ell} \,. \tag{2.53}$$

It is shown in ref. [131] that the differential operator $\mathcal{B}(\beta)$ creates an asymptotic boundary component at temperature β. It is universal in the sense that without any modifications it also creates asymptotic boundary components in the presence of defects. The operator $\mathcal{B}(\beta)$ as a function of β relates to the operator in ref. [138], which in the context of two-dimensional topological gravity creates in a surface a hole of specified boundary length. Therefore, we refer to $\mathcal{B}(\beta)$ as the boundary creation operator.

The obtained simple forms (2.44) and (2.52) of the partition function $Z(\beta)$ and its multi-boundary generalisations $Z(\beta_1, \ldots, \beta_m)$ in the presence of a gas of defects have a nice interpretation from the topological gravity perspective. The Weil–Petersson volumes (2.38) are computed with the Kähler class $2\pi^2\kappa_1$ on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ [137]. The generating function $G(2\pi^2\lambda, \{t_k\})$ now expresses these volumes (as functions of the scaling and genus expansion parameters λ and g_s) in terms of the shifted generating function $F(\{t_k + \gamma_k\})$ of topological gravity according to eq. (2.35). As explained in ref. [73, 130], JT gravity can be interpreted as topological gravity with non-vanishing background parameters $\{\gamma_k\}$. Including now a gas of defects (characterised by their couplings ϵ_j and identification angles α_j) further deforms the background couplings $\{\gamma_k\}$. The leading order contribution arises from single-defect interactions while the higher order corrections are due to multi-defect interactions. These order-by-order contributions can be viewed as a Taylor expansion about the JT gravity background parameters $\{\gamma_k\}$, which altogether sum up to the deformation $\{\gamma_k + \delta_k\}$. Thus, JT gravity interacting with a gas of defects yields yet other expansion point of the generating function $F(\{t_k\})$. It would be interesting to see if there are special expansion points that are singled out from the topological gravity point of view and whether any other expansion point can be mapped to a scalar potential. Schematically, we have

Standard JT gravity, $U(\phi) = 0 \mapsto G(2\pi^2\lambda, \{t_k\}) = F(\{t_k + \gamma_k\})$

JT gravity with conical singularities, $U(\phi) = 2\sum_i \epsilon_i e^{-(2\pi-\alpha_i)\phi} \mapsto$

$$G(2\pi^2\lambda, \{t_k + \sum_j \delta_{k,j}\}) = F(\{t_k + \gamma_k + \sum_j \delta_{k,j}\})$$

JT gravity with arbitrary scalar potential, $U(\phi) \mapsto$?

As in ref. [130], in the following we set the coupling γ and the scaling parameter λ to the convenient values

$$\lambda = \gamma = \frac{1}{2\pi^2} \,. \tag{2.54}$$

Then the boundary creation operator $\mathcal{B}(\beta)$ and the background parameters δ_k simplify to

$$\mathcal{B}(\beta) = g_s \sqrt{\frac{\beta}{2\pi}} \sum_{\ell=0}^{+\infty} \beta^\ell \frac{\partial}{\partial t_\ell} \,, \qquad \delta_k = \sum_j \left(-\frac{\alpha_j^2}{4\pi^2}\right)^k \frac{2\pi^2 \epsilon_j}{k!} \,, \tag{2.55}$$

and the partition functions become

$$\begin{aligned} Z(\beta) &= \frac{1}{\sqrt{2\pi} g_s \beta^{\frac{3}{2}}} \left(e^{\frac{1}{\beta}} + 2\pi^2 \beta \sum_{j=1}^{r} \epsilon_j \, e^{\frac{\alpha_j^2}{4\pi^2 \beta}} \right) + \frac{1}{g_s^2} \mathcal{B}(\beta) G(1, \{t_k = \delta_k\}) \,, \\ Z(\beta_1, \beta_2) &= \frac{\sqrt{\beta_1 \beta_2}}{2\pi\beta_1 + 2\pi\beta_2} + \frac{1}{g_s^2} \mathcal{B}(\beta_1)\mathcal{B}(\beta_2) G(1, \{t_k = \delta_k\}), \\ Z(\beta_1, \ldots, \beta_m) &= \frac{1}{g_s^2} \mathcal{B}(\beta_1) \cdots \mathcal{B}(\beta_m) G(1, \{t_k = \delta_k\}) \quad \text{for} \quad m \geqslant 3 \,, \end{aligned} \tag{2.56}$$

where the first two partition functions receive both non-topological and topological contributions.

The upshot

The main takeaway of this sections was that we were able to express the partition functions of JT gravity with conical singularities compactly in terms of the generating functions of topological gravity correlators extending the works of ref. [130]. We saw that plugging different values for the couplings t_k's corresponds to different JT gravity scenarios: with and without conical singularities. In our work [1], we formulate and generalize JT partition functions for arbitrary values of t_k's. This can be a first step in building a dictionary between scalar potential perturbations of JT gravity and values of the couplings t_k's. In the following, we explore these generalised partition functions by first introducing the KdV hierarchy—an infinite system of equations that enables us to calculate these generalised partition functions.

2.4 KdV Hierarchy and Off-Shell Partition Functions

An infinite sequence of differential equations

The so-called KdV (Korteweg–de Vries) hierarchy is a system of an infinite number of differential equations. A function $u = u(\{t_k\}),\ k = 1, 2, ...$ satisfies the KdV hierarchy if

$$\partial_k u = \partial_0 \mathcal{R}_{k+1}(u, \partial_0 u, \partial_0^2 u, \ldots) \quad \text{with} \quad \partial_k \equiv \frac{\partial}{\partial t_k} \,, \quad k = 0, 1, 2, 3, \ldots \,. \tag{2.57}$$

Such a function u is called a **tau-function** to the KdV hierarchy. Here $\mathcal{R}_k$, $k = 1, 2, 3, \ldots$, are the Gelfand–Dikii polynomials [139], which are polynomials in the derivatives $\partial_0^\ell u(\{t_k\})$, $\ell = 0, 1, 2, \ldots$ of $u(\{t_k\})$, and depend on the parameter g_s. Together with the condition $\mathcal{R}_k(\{\partial_0^\ell u \equiv 0\}) = 0$ they are defined with the initial polynomial $\mathcal{R}_1 = u$ recursively as [139]

$$\partial_0 \mathcal{R}_{k+1} = \frac{1}{2k+1} \left(2u \, (\partial_0 \mathcal{R}_k) + (\partial_0 u) \, \mathcal{R}_k + \frac{g_s^2}{4} \partial_0^3 \mathcal{R}_k \right) \,. \tag{2.58}$$

For example, calculating the first three polynomials we find

$$\mathcal{R}_1 = u\,, \quad \mathcal{R}_2 = \frac{u^2}{2} + \frac{g_s^2}{12}\partial_0^2 u\,, \quad \mathcal{R}_3 = \frac{u^3}{3!} + \frac{g_s^2}{24}\left(2u\partial_0^2 u + (\partial_0 u)^2\right) + \frac{g_s^4}{240}\partial_0^4 u\,. \tag{2.59}$$

The leading order term of the Gelfand–Dikii polynomials is given by

$$\mathcal{R}_k|_{g_s=0} = \frac{u^k}{k!}\,, \tag{2.60}$$

independent of any derivatives $\partial_0 u(\{t_k\})$, $\partial_0^2 u(\{t_k\})$, $\partial_0^3 u(\{t_k\})$, ...

Relation to topological gravity correlators

As conjectured by Edward Witten [115] and proven by Maxim Kontsevich [140] the function defined as

$$u(\{t_k\}) = \frac{\partial^2}{\partial t_0^2} F(\{t_k\}) \tag{2.61}$$

is a tau-function of the KdV hierarchy. In particular, it is a solution with "initial condition" $u(t_0, 0, ..., 0) = t_0$. Now, a crucial observation for our work is that the hierarchy depends only implicitly on the couplings t_k. Thus, we can construct an infinite family of solutions by simply shifting the argument of F.

Generating more general solutions and JT partition functions

Taking the above into consideration, we see that the function $v(\{t_k\}) = \partial_0^2 F(\{t_k + \Delta t_k\})$ is a tau function for any set of constants $\{\Delta t_k\}$. For instance, a tau function is realised by the generating function $G(s, \{t_k\})$ of Weil–Petersson volumes (see eq. 2.35) as it is related to F via a shift in the couplings $\{t_k\}$'s

$$G(1, \{t_k + \delta_k\}) = F(\{t_k + \gamma_k + \delta_k\})\,, \tag{2.62}$$

in terms of the constants $\Delta t_k = \gamma_k + \delta_k$, see eqs. (2.35) and (2.55). The particular tau function $u(\{t_k\})$ of topological gravity and hence the tau function $v(\{t_k\})$ with the shifted couplings obey the string equation [141]

$$\partial_0 u = 1 + \sum_{k=1}^{+\infty} t_k \partial_k u\,, \qquad \partial_0 v = 1 + \sum_{k=1}^{+\infty} (t_k + \Delta t_k)\, \partial_k v\,. \tag{2.63}$$

The string equation together with the KdV hierarchy determine unambiguously the tau functions $u(\{t_k\})$ and $v(\{t_k\})$ [115]. The string equation can be viewed as the initial condition specifying a unique solution to the KdV hierarchy. Hence, nothing stops us from generalising the topological parts of expressions (2.56) by not fixing the values of t_k. We call the **universal** partition functions the quantities defined as

$$\begin{aligned} Z(\{t_k\};\beta)^{\text{top.}} &= \frac{1}{g_s^2}\mathcal{B}(\beta)F(\{t_k\}) = \frac{1}{g_s^2}\mathcal{B}(\beta)F_0 + Z^{(g>0)}(\{t_k\};\beta)^{\text{top.}}\,, \\ Z(\{t_k\};\beta_1,\ldots,\beta_m)^{\text{top.}} &= \frac{1}{g_s^2}\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_m)F(\{t_k\}) \quad \text{for} \quad m>1\,. \end{aligned} \tag{2.64}$$

For general values of the couplings t_k, these partition functions are **off-shell**, i.e. they do not (a-priori) come from a scalar potential deformation of JT gravity. Plugging in the values $t_k = \gamma_k + \delta_k$ or $t_k = \gamma_k$ we get the **on-shell** partition functions, namely those corresponding to JT with and without defects. In the following, we will show how one can compute these universal partition functions and, in particular, express them in a way that they depend implicitly on the couplings t_k.

Evaluating the genus expansions of the generating functions

We begin with the genus expansions of $u(\{t_k\})$ and $F(\{t_k\})$ as from these we can deduce the genus expansion of partition functions. The former read

$$u(\{t_k\}) = \sum_{\ell=0}^{+\infty} g_s^{2\ell}\, u_\ell(\{t_k\}) \,, \qquad F(\{t_k\}) = \sum_{\ell=0}^{+\infty} g_s^{2\ell}\, F_\ell(\{t_k\}) \,. \tag{2.65}$$

These genus expansions are such that the relation $u(\{t_k\}) = \frac{\partial^2}{\partial t_0^2} F(\{t_k\})$ is obeyed, hence $u_g(\{t_k\}) = \frac{\partial^2}{\partial t_0^2} F_g(\{t_k\}),\ g = 0, 1, 2, \ldots$.

Now, using the selection rules (2.32) the genus g contribution to the generating function $F(\{t_k\})$ can be expanded as [142]

$$F_g(\{t_k\}) = \sum_{\substack{k_i \\ \sum_{i\geqslant 0}(i-1)k_i = 3g-3}} F_{g,\{k_s\}}(\{t_k\}) \prod_{j\geqslant 0} t_j^{k_j} \,, \tag{2.66}$$

where $F_{g,\{k_s\}}(\{t_k\}) = \langle \prod_{i\geqslant 0} \frac{\tau_i^{k_i}}{k_i} \rangle_g$. The brackets denote the topological gravity correlator at genus g and the marked points are determined by the selection rules. We can translate this to an ansatz for the function $u(\{t_k\})$ by taking the second t_0 derivative of (2.66).

Genus zero

Specialising to genus zero we have

$$u_0(\{t_k\}) = \sum_{N=0}^{+\infty} \sum_{\sum n_k = N} u_{0,\{n_k\}} t_0^{1-N+\sum k n_k} \left(t_1^{n_1} t_2^{n_2} \cdots\right) \,. \tag{2.67}$$

Where $n_k, k = 1, 2, \ldots$ are non-negative integer numbers. This ansatz together with the relation shown by [142]

$$u_0 - I_0(u_0, \{t_k\}) = 0 \,, \tag{2.68}$$

where $I_0(u_0, \{t_k\})$ is the $n = 0$ case of the functions defined as

$$I_n(u_0, \{t_k\}) = \sum_{k=0}^{+\infty} t_{k+n} \frac{u_0^k}{k!} \quad \text{for} \quad n = 0, 1, 2, \ldots \,, \tag{2.69}$$

enables one to calculate u_0 order-by-order. The first few terms in the formal expansion read

$$u_0(\{t_k\}) = t_0 + t_0 t_1 + \left(t_0 t_1^2 + \frac{1}{2} t_0^2 t_2\right) + \left(t_0 t_1^3 + \frac{3}{2} t_0^2 t_1 t_2 + \frac{1}{6} t_0^3 t_3\right) + \ldots \,. \tag{2.70}$$

By "order" we mean the sum of the products of the t_k subscripts times their exponents. For example the order of $\prod_i t_i^{k_i}$ is $\sum_i i k_i$. Eq. (2.70) includes terms up to order three. By integrating (2.68), one can actually find an expression for F_0[142]

$$F_0(u_0, \{t_k\}) = \frac{u_0^3}{3!} - \sum_{k=0}^{+\infty} t_k \frac{u_0^{k+2}}{(k+2)k!} + \frac{1}{2} \sum_{k=0}^{+\infty} \frac{u_0^{k+1}}{k+1} \sum_{n=0}^{k} \frac{t_n t_{k-n}}{n!(k-n)!} \,. \tag{2.71}$$

Hence, F_0 is calculated to the desired order by plugging (2.70) into (2.71).

Higher genus

For genus $g \geqslant 1$ the genus expansions can be expressed in terms of the functions defined in eq. 2.69. For this note that Itzykson and Zuber [142], establish that for $g > 1$, the genus expansion contributions F_g have the form

$$F_g = (1-I_1)^{g-1} \sum_{\sum_{k=2}^{3g-2}(k-1)\ell_k = 3g-3} f_{g,\{\ell_k\}} \left(\frac{I_2}{(1-I_1)^2}\right)^{\ell_2} \cdot \ldots \cdot \left(\frac{I_{3g-2}}{(1-I_1)^{3g-2}}\right)^{\ell_{3g-2}} , \quad (2.72)$$

in terms of the finitely many coefficients $f_{g,\{\ell_k\}}$ (with the subscript $\{\ell_k\} = \{\ell_2, \ell_3, \ldots\}$). Using properties of the functions $I_n(u_0, \{t_k\})$ (in particular properties of their ∂_0 derivatives), the genus expansion contributions for u_g for $g \geqslant 1$ take the form

$$u_g = (1-I_1)^{g-1} \sum_{\sum_{k=2}^{3g}(k-1)\ell_k = 3g-1} u_{g,\{\ell_k\}} \left(\frac{I_2}{(1-I_1)^2}\right)^{\ell_2} \cdot \ldots \cdot \left(\frac{I_{3g}}{(1-I_1)^{3g}}\right)^{\ell_{3g}} . \quad (2.73)$$

Inserting this ansatz into the KdV hierarchy (2.57) (recursively in the genus) determines unambiguously the numerical coefficients $u_{g,\{\ell_k\}}$.

Genus one and two

For instance up to genus $g = 2$ we arrive at

$$u_1 = \frac{1}{12}\left(\frac{I_2}{(1-I_1)^2}\right)^2 + \frac{1}{24}\frac{I_3}{(1-I_1)^3} , \quad (2.74)$$

$$u_2 = (1-I_1)\left(\frac{49 I_2^5}{288(1-I_1)^{10}} + \frac{11 I_3 I_2^3}{36(1-I_1)^9} + \frac{7 I_4 I_2^2}{96(1-I_1)^8} + \frac{109 I_3^2 I_2}{1152(1-I_1)^8} + \frac{I_5 I_2}{90(1-I_1)^7} + \frac{17 I_3 I_4}{960(1-I_1)^7} + \frac{I_6}{1152(1-I_1)^6}\right) . \quad (2.75)$$

At genus one $u_1(\{t_k\})$ integrates to

$$F_1 = -\frac{1}{24}\log(1-I_1) . \quad (2.76)$$

We can now use the expression for u_2 of eq. (2.75) to calculate F_2

$$F_2 = \frac{7}{1440}\frac{I_2^3}{(1-I_1)^5} + \frac{29}{5760}\frac{I_2\, I_3}{(1-I_1)^4} + \frac{1}{1152}\frac{I_4}{(1-I_1)^3} . \quad (2.77)$$

This method of [142] is very powerful as it allows one to calculate recursively the genus expansion of u, hence that of F and find the values of topological gravity correlators. The latter is done via solving (2.68) to the desired order, plugging it in the expression of F_g and comparing coefficients with the definition of F_g.

The genus expansion of universal partition functions

We saw that we can solve the KdV hierarchy in terms of the functions I_n. Expressing the universal partition functions (2.64) in this way, we arrive at expressions that depend only implicitly on the couplings t_k through $F_0(u_0(\{t_k\}), \{t_k\})$ and $I_n(u_0(\{t_k\}), \{t_k\})$, $n = 1, 2, 3, \ldots$. To do this we should act with the boundary creation operators $\mathcal{B}(\beta)$ on F, see (2.64). We find that—except for the leading genus zero contribution to the partition function with one asymptotic boundary—the universal partition functions are expressible in terms of the functions I_n, i.e. (2.64) take the form

$$\begin{aligned} Z(\{\mathcal{B}(\beta)F_0, I_n\};\beta)^{\text{top.}} &= \frac{1}{g_s^2}\mathcal{B}(\beta)F(\{t_k\}) = \frac{1}{g_s^2}\mathcal{B}(\beta)F_0 + Z^{(g>0)}(\{I_n\};\beta)^{\text{top.}}\ , \\ Z(\{I_n\};\beta_1,\ldots,\beta_m)^{\text{top.}} &= \frac{1}{g_s^2}\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_m)F(\{t_k\}) \quad \text{for} \quad m>1\ . \end{aligned} \tag{2.78}$$

In particular, the partition function with a single asymptotic boundary component enjoys the genus expansion

$$Z(\{\mathcal{B}(\beta)F_0, I_n\};\beta)^{\text{top.}} = \frac{1}{g_s^2}\mathcal{B}(\beta)F_0 + \sqrt{\frac{\beta}{2\pi}}e^{\beta I_0}\sum_{g=1}^{+\infty} g_s^{2g-1}(1-I_1)^{g-1}Z_g(\{I_n\},\beta)\ , \tag{2.79}$$

where

$$Z_1 = \frac{1}{24}\left(\frac{\beta}{1-I_1}+\frac{I_2}{(1-I_1)^2}\right)\ , \tag{2.80}$$

and for $g>0$

$$\begin{aligned} Z_g = \sum_{\sum_{k=2}^{3g-2}(k-1)\ell_k=3g-3} f_{g,\{\ell_k\}} \sum_{s=2}^{3g-2}\ell_s\left(\frac{1+2s}{3(1-I_1)}\left(\beta+\frac{I_2}{1-I_1}\right)+\frac{I_{s+1}}{I_s(1-I_1)}+\frac{\beta^s}{I_s}\right) \\ \cdot\left(\frac{I_2}{(1-I_1)^2}\right)^{\ell_2}\cdot\ldots\cdot\left(\frac{I_{3g-2}}{(1-I_1)^{3g-2}}\right)^{\ell_{3g-2}}\ , \end{aligned} \tag{2.81}$$

in terms of the constants $f_{g,\{\ell_k\}}$ defined in eq. (2.72). With eq. (2.80) and inserting the coefficients from (2.77) into Z_2 we find explicitly up to genus two

$$\begin{aligned} Z^{(g>0)}(\{I_n\};\beta)^{\text{top.}} &= \frac{g_s}{24}\sqrt{\frac{\beta}{2\pi}}e^{\beta I_0}\left(\frac{\beta}{1-I_1}+\frac{I_2}{(1-I_1)^2}\right) \\ &+ \frac{g_s^3}{5760}\sqrt{\frac{\beta}{2\pi}}e^{\beta I_0}\left(\frac{5\beta^4}{(1-I_1)^4}+\frac{29\beta^3I_2+29\beta^2I_3+15\beta I_4+5I_5}{(1-I_1)^5}\right. \\ &\quad \left.+\frac{84\beta^2I_2^2+116\beta I_3I_2+44I_4I_2+29I_3^2}{(1-I_1)^6}+\frac{20I_2^2(7\beta I_2+10I_3)}{(1-I_1)^7}+\frac{140I_2^4}{(1-I_1)^8}\right) \\ &\quad +\ldots\ . \end{aligned} \tag{2.82}$$

Similar formulas can be worked out for the universal partition functions with several asymptotic boundary components, namely

$$Z(\{I_n\};\beta_1,\ldots,\beta_m)^{\text{top.}} = \prod_{i=1}^{m}\left(e^{\beta_i I_0}\sqrt{\frac{\beta_i}{2\pi}}\right)\sum_{g=0}^{\infty} g_s^{2g+m-2}(1-I_1)^{g-1}Z_g(\{I_n\},\beta_1,\ldots,\beta_m)\ , \tag{2.83}$$

where

$$\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_m)F_g = \frac{g_s\sqrt{\beta_1\cdots\beta_m}}{(2\pi)^{\frac{m}{2}}}e^{(\beta_1+\ldots+\beta_m)I_0}(1-I_1)^{g-1}Z_g(\{I_n\},\beta_1,\ldots,\beta_m)\ . \tag{2.84}$$

In particular for two asymptotic boundary components the leading order contributions are given by

$$\begin{aligned} Z(\{I_n\};\beta_1,\beta_2) &= \frac{\sqrt{\beta_1\beta_2}}{2\pi\beta_1+2\pi\beta_2}e^{(\beta_1+\beta_2)I_0} \\ &+ \frac{g_s^2\sqrt{\beta_1\beta_2}}{48\pi}e^{(\beta_1+\beta_2)I_0}\left(\frac{\beta_1^2+\beta_1\beta_2+\beta_2^2}{(1-I_1)^2}+\frac{2(\beta_1+\beta_2)I_2+I_3}{(1-I_1)^3}+\frac{2I_2^2}{(1-I_1)^4}\right)+\ldots\ . \end{aligned} \tag{2.85}$$

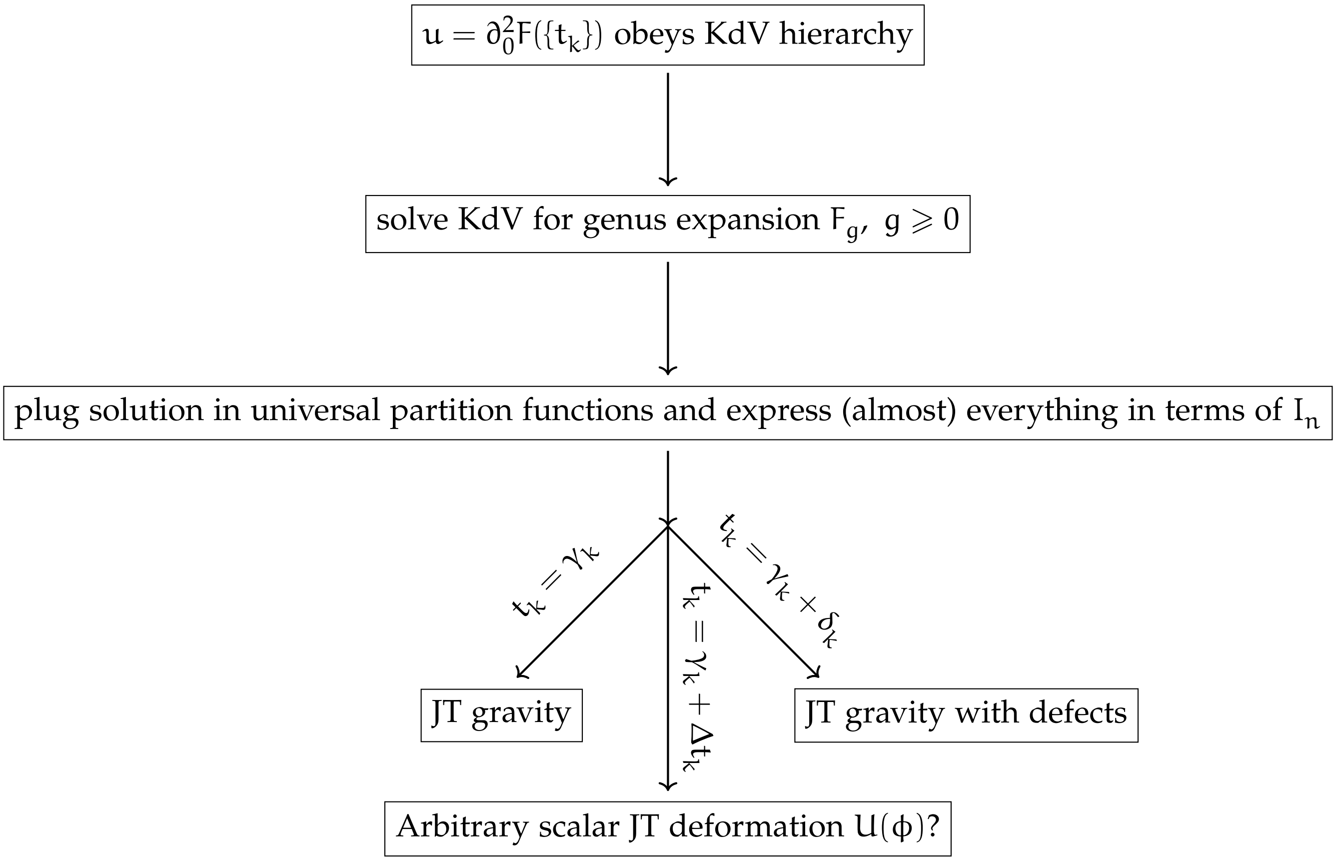

Figure 2.9: A logical diagram of this work. Of course the part that goes to JT gravity has been worked out in [130, 131]. We aim to generalise this picture by deforming with arbitrary couplings.

This expression includes the semi-classical contribution, see eq. (2.56).[13] Of course plugging in the correct values for the couplings t_k, we can find the partition function for JT with or without defects.

As is depicted in fig. 2.9, we solve the KdV hierarchy for arbitrary couplings t_k and ask what is the significance of particular values of these couplings. Can we map any deformation of the scalar potential to different values of the couplings? We have only answered this question when it comes to the deformation that corresponds to JT gravity with defects $\sim \epsilon e^{-(2\pi-\alpha)\phi}$ where the normal JT coupling values get shifted by δ_k. This deformation has the property that it does not alter the asymptotic behaviour of JT as it decays exponentially at large values of the scalar field.

The genus expansion parameter g_s

While the presented genus expansion in the coupling $g_s \sim e^{-1/G_N}$ is non-perturbative in the gravitational coupling G_N of JT gravity, it is perturbative in the dual matrix model formulation, where the expansion parameter g_s describes quantum fluctuations about the classical energy density of states [73, 143]. In fact the discussed partition functions $Z(\{I_n\}; \beta_1, \ldots, \beta_m)$ are divergent series in g_s due to the factorial growth $(2g)!$ of the con-

[13]This might not be so apparent—it comes from the fact that acting with two boundary creation operators on F_0 gives

$$\mathcal{B}(\beta_1)\mathcal{B}(\beta_2)F_0 = \frac{g_s^2\sqrt{\beta_1\beta_2}}{2\pi\beta_1 + 2\pi\beta_2}\left(e^{(\beta_1+\beta_2)I_0} - 1\right) .$$

The "-1" term combines with the semi-classical contribution of (2.56).

tributions at order g_s^{2g} [144, 73]. Therefore, the partition functions $Z(\{I_n\};\beta_1,\dots,\beta_m)$ are asymptotic series that require a non-perturbative completion arising from non-perturbative effects of the order e^{-1/g_s}. For further details on this issue and the possible emergence of non-perturbative instabilities, we refer the reader to ref. [73, 143] and the solutions proposed in refs. [143, 145, 146].

In the next section, we focus on the partition functions $Z(t_0,t_1;\beta_1,\dots,\beta_m)$ studied in refs. [130, 131], where we assign on-shell values to the couplings t_k, $k=2,3,4,\dots$, while keeping the first two couplings t_0 and t_1 off-shell [144].

2.5 Leading off-shell couplings

Fixing all couplings except the two leading ones: t_0, t_1

We specialise now to the case where the couplings $t_k,\ k=2,3,...$ are fixed to $\gamma_k+\delta_k$. The couplings t_0,t_1 remain arbitrary and the JT partition functions are defined as

$$Z(t_0,t_1;\beta_1,\dots,\beta_m)\equiv Z(\{t_0,t_1,t_{k\geqslant 2}=\gamma_k+\delta_k\};\beta_1,\dots,\beta_m)\ . \tag{2.86}$$

Of course, if we choose $t_0=\delta_0$ and $t_1=\delta_1$, we obtain the on-shell JT partition function. Similar definitions as (2.86) hold for u and F

$$u(t_0,t_1)=u(\{t_0,t_1,t_{k\geqslant 2}=\gamma_k+\delta_k\})\ ,\qquad F(t_0,t_1)=F(\{t_0,t_1,t_{k\geqslant 2}=\gamma_k+\delta_k\})\ , \tag{2.87}$$

with

$$u(t_0,t_1)=\partial_0^2F(t_0,t_1)\ . \tag{2.88}$$

Note that these functions can respectively be obtained from their universal expressions (2.78), (2.73), and (2.72) by inserting the on-shell values of the couplings $t_{k\geqslant 2}$ into the functions I_n. With t_0,t_1 arbitrary and the others fixed, we can deduce that the function I_1 depends only on t_1 and $u_0=I_0$ and that for $n\geqslant 2$, I_n are all series in u_0 with no explicit t_0,t_1 dependence—this is clear from their definition (2.69). Therefore, it is convenient to introduce new (formal) variables (y,t) given by [144, 130]

$$y=u_0\ ,\qquad t=1-I_1\ . \tag{2.89}$$

In other words, we can express everything in terms of the variables y,t. The reason of these considerations is however to compute JT partition functions and in particular the single boundary ones. We turn to this matter now.

Computing single boundary partition functions

Since the couplings $t_{k\geqslant 2}$ are taken on-shell we cannot obtain $Z(t_0,t_1;\beta)$ by acting with the boundary creation operator $\mathcal{B}(\beta)$ on the generating function $F(t_0,t_1)$ because the boundary operator $\mathcal{B}(\beta)$ contains derivatives with respect to those parameters that have been fixed to their on-shell values. Thus, either we compute $Z(t_0,t_1;\beta)$ from the universal partition function (2.78) or we determine a differential equation with $Z(t_0,t_1;\beta)$ as its solution. For the latter approach we follow the authors of ref. [130]. We also spare some of the details in view of compactness and refer the reader to [130, 1].

A recursion relation

Using the KdV hierarchy, essentially integrating eq. (2.57), one can write the following relation for the topological part of the single boundary partition function

$$\partial_0 Z^{\text{top.}}(t_0,t_1;\beta)=\frac{1}{g_s^2}\mathcal{B}(\beta)\partial_0F(\{t_k\})\bigg|_{\{t_{k\geqslant 2}=\gamma_k+\delta_k\}}=-\frac{1}{g_s\sqrt{2\pi\beta}}+W(t_0,t_1;\beta)\ , \tag{2.90}$$

with the definition

$$W(t_0, t_1; \beta) = \frac{1}{g_s\sqrt{2\pi\beta}} \sum_{\ell=0}^{+\infty} \beta^\ell \mathcal{R}_\ell \,, \tag{2.91}$$

in terms of the Gelfand–Dikii polynomials (2.59) and $\mathcal{R}_0 = 1$. It turns out that W enjoys a genus expansion of the form

$$W(y, t;\, \beta) = \frac{e^{\beta y}}{\sqrt{2\pi\beta}} \sum_{g=0}^{+\infty} g_s^{2g-1}\, W_g(y, t; \beta) \,, \tag{2.92}$$

where we express everything in terms of y, t. The functions $W_0, W_{g\geqslant 1}$ obey

$$W_0(y, t; \beta) = 1 \text{ and } W_g(y, t; \beta) = \sum_{k=2g}^{5g-1} W_{g,k}(y; \beta)\, t^{-k} \quad \text{for} \quad g \geqslant 1 \,. \tag{2.93}$$

In our work [1], we derived an explicit recursive relation for $W_{g,k}$. Namely, with $W_0(y, t; \beta) = 1$ and

$$W_{1,k} = \frac{\beta}{k} u_{1,k} + \frac{\beta^3}{24}\delta_{k,2} + \frac{1}{36}(3I_2\beta^2 + I_3\beta)\delta_{k,3} + \frac{\beta}{16} I_2^2 \delta_{k,4} \quad \text{for} \quad k = 2, 3, 4 \,, \tag{2.94}$$

we have, for genus $g \geqslant 2$

$$\begin{aligned} W_{g,k} = &\sum_{h=1}^{g-1} \sum_{n=2h}^{5h-1} \left(\frac{n}{k} I_2 u_{g-h,k-n-1} W_{h,n} + \frac{1}{k} u_{g-h,k-n} D_y W_{h,n} \right) + \frac{\beta}{k} u_{g,k} \\ &+ \frac{1}{12k} \Big[D_y^3 W_{g-1,k-2} + \Big(3(k-2) I_2 D_y^2 + (3k-8) I_3 D_y + (k-3) I_4 \Big) W_{g-1,k-3} \\ &\qquad + \Big(3(k^2 - 5k + 5) I_2^2 D_y + (k-4)(3k-5) I_2 I_3 \Big) W_{g-1,k-4} \\ &\qquad\qquad + (k-5)(k-3)(k-1) I_2^3 W_{g-1,k-5} \Big] \,, \end{aligned} \tag{2.95}$$

where we set $W_{h,n} \equiv 0$ for $n \notin \{2h, \ldots, 5h-1\}$ and $u_{h,n} \equiv 0$ for $n \notin \{2h+1, \ldots, 5h-1\}$. In particular, for genus two we readily compute

$$\begin{aligned} W_2(y, t; \beta) = &\frac{\beta}{5760} \left(\frac{5\beta^5}{t^4} + \frac{44\beta^4 I_2 + 58\beta^3 I_3 + 44\beta^2 I_4 + 20\beta I_5 + 5I_6}{t^5} \right. \\ &+ \frac{200\beta^3 I_2^2 + 400\beta^2 I_2 I_3 + 145\beta I_3^2 + 220\beta I_2 I_4 + 102 I_3 I_4 + 64 I_2 I_5}{t^6} \\ &\left. + \frac{5I_2\left(112\beta^2 I_2^2 + 240\beta I_3 I_2 + 84 I_4 I_2 + 109 I_3^2\right)}{t^7} + \frac{20 I_2^3 \left(49\beta I_2 + 88 I_3\right)}{t^8} + \frac{980 I_2^5}{t^9} \right) \,. \end{aligned} \tag{2.96}$$

Calculating the partition functions

We can now use the derived recursion relation to calculate JT partition functions. For this, we define the genus expansion[14]

$$Z(y, t; \beta) = \sum_{g=0}^{+\infty} g_s^{2g-1} \widetilde{Z}_g(y, t; \beta) \,. \tag{2.97}$$

[14]Note that the newly introduced contributions $\widetilde{Z}_g$ to the partition function differ from the definition of Z_g given in eq. (2.79) by a normalisation.

$g = 0$—the non-top. contributions

Recall that the non-top. part of the genus zero single boundary partition function, given in eq. (2.56), reads

$$\frac{1}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\left(e^{\frac{1}{\beta}} + 2\pi^2\beta\epsilon\, e^{\frac{\alpha^2}{4\pi^2\beta}}\right) , \tag{2.98}$$

where here we specialised to the case of a single type of defect. Note that this expression depends, by definition, on the on-shell values of t_0, t_1. It is computed by doing the path integral of JT on a disk with a single defect and is not written in terms of a differential operator acting on the generating function of intersection numbers, like the topological contributions to the single boundary partition function. We will now rewrite it in terms of y, t by identifying (t_0, t_1) with $(2\pi^2\epsilon, -\alpha^2\epsilon/2)$ and solving for t_0 and for t_1 the definitions (2.89):

$$t_0 = yt + \sum_{k=2}^{\infty} t_k \frac{y^k}{k(k-2)!} \qquad t_1 = 1 - t - \sum_{k=1}^{\infty} t_{k+1}\frac{y^k}{k!} . \tag{2.99}$$

The non-top. part then is equal to

$$\begin{aligned}\widetilde{Z}_0(y, t; \beta)^{\text{semi.}} = &\frac{t(1+y\beta)}{\sqrt{2\pi}\beta^{\frac{3}{2}}} + \frac{1}{\sqrt{2\pi\beta}}\sum_{k=2}^{+\infty}\frac{y^k(\gamma_k+\delta_k)}{k(k-2)!}\\ &+ \frac{1}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\sum_{k=2}^{+\infty}\frac{y^{k-1}(\gamma_k+\delta_k)}{(k-1)!} + \frac{1}{\sqrt{2\pi\beta}}\sum_{k=2}^{+\infty}\frac{\delta_k+\gamma_k}{(-\beta)^k} .\end{aligned} \tag{2.100}$$

We should emphasize that strictly speaking the non-top. part is a function of the on-shell values of t_0, t_1. Here, we rewrite these in terms of y, t (which again should hold for on-shell y, t values).

$g = 0$—the topological contributions

Here the rewriting in terms of y, t is more straightforward. The topological terms are written as in eq. (2.78) in terms of a boundary creation operator acting on the genus zero part of the generating function F. One can express the quantity $\mathcal{B}(\beta)F_0$ in terms of the variables y, t and immediately read of the topological contribution to the genus zero, one boundary partition function

$$\begin{aligned}\widetilde{Z}_0(y, t; \beta)^{\text{top.}} = &\frac{t}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\Big(e^{\beta y} -\\ &(1+y\beta)\Big) - \frac{e^{\beta y}}{\sqrt{2\pi\beta}}\sum_{k=2}^{+\infty}\frac{y^k(\gamma_k+\delta_k)}{k!} - \frac{1}{\sqrt{2\pi\beta}}\sum_{k=2}^{+\infty}\frac{y^k(\gamma_k+\delta_k)}{k(k-2)!}\\ &+ \frac{e^{\beta y}-1}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\sum_{k=2}^{+\infty}\frac{y^{k-1}(\gamma_k+\delta_k)}{(k-1)!} + \frac{1}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\sum_{k=0}^{+\infty}\sum_{\ell=2}^{+\infty}\frac{y^{k+\ell+1}\beta^{k+2}}{(k+\ell+1)!}\binom{k+\ell}{k}(\gamma_\ell+\delta_\ell) .\end{aligned} \tag{2.101}$$

Higher genera

Employing eq. (2.81), the genus expansion terms $\widetilde{Z}_g$ become

$$\widetilde{Z}_g(y, t; \beta) = \frac{e^{\beta y}}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\sum_{k=2g-1}^{5g-3} Z_{g,k}(y; \beta)t^{-k} \quad \text{for} \quad g \geqslant 1 . \tag{2.102}$$

Using the derived recursion relation for the $W_{g,k}$'s we can integrate (2.90) and find all $\widetilde{Z}_g$. The constants of integration are fixed by the structure of the genus expansion (2.102). Calculating, for example the genus one and two contributions we find expressions that agree with (2.80) and (2.82). For completeness we write them down: for genus one—in agreement with eq. (2.80)—the result

$$\widetilde{Z}_1(y,t;\beta) = \frac{e^{\beta y}}{24\sqrt{2\pi\beta}}\left(\frac{\beta^2}{t} + \frac{\beta I_2}{t^2}\right) , \tag{2.103}$$

whereas for genus two—in agreement with eq. (2.82)—we obtain

$$\begin{aligned}\widetilde{Z}_2(y,t;\beta) = \frac{\sqrt{\beta}\, e^{\beta y}}{5760\sqrt{2\pi}}\Bigg(&\frac{5\beta^4}{t^3} + \frac{29\beta^3 I_2 + 29\beta^2 I_3 + 15\beta I_4 + 5I_5}{t^4}\\ &+\frac{84\beta^2 I_2^2 + 116\beta I_3 I_2 + 44 I_4 I_2 + 29 I_3^2}{t^5} + \frac{20 I_2^2(7\beta I_2 + 10 I_3)}{t^6} + \frac{140\beta I_2^4}{t^7}\Bigg) .\end{aligned} \tag{2.104}$$

2.6 The density of eigenvalues

In this section we focus on writing the single boundary partition function as an integral of some function that can hopefully be given the interpretation of a density of eigenvalues. This is motivated as follows:

Thinking in terms of the matrix model/JT duality, the single boundary JT partition function is written in terms of matrix model correlators as (see e.g. eq. (2.16))

$$Z(\beta) = \langle \operatorname{tr} e^{-\beta H}\rangle_{MM} = \int dH \mu(H) \operatorname{tr} e^{-\beta H} . \tag{2.105}$$

Introducing a suitable measure $\mu(H)$ on the space of matrices H. Writing the trace tr as a sum over eigenvalues we have

$$Z(\beta) = \int dH \mu(H) \sum_i e^{-\beta\lambda_i} . \tag{2.106}$$

We can re-write this sum as $\int dE \sum_i \delta(E-\lambda_i) e^{-\beta E}$. The partition function is now (we exchange the matrix integral with the regular integral)

$$Z(\beta) = \int dE \rho(E) e^{-\beta E} , \quad \rho(E) \equiv \langle \sum_i \delta(E-\lambda_i)\rangle_{MM} . \tag{2.107}$$

Where $\delta(...)$ is the Dirac delta function. What is happening is that for a given matrix H the density of eigenvalues is obviously discrete (a matrix has a discrete set of eigenvalues). However here we are doing something different. Namely, we take an average over matrices H in some ensemble and thus the averaged density of eigenvalues $\rho(E)$ can, in principle, be a continuous function as it is the integral of a sum of Dirac delta functions (with a suitable measure).

Writing the JT single boundary partition function as in eq. (2.107), we can read of a density of eigenvalues. In other words we look for a $\rho = \rho(E; y, t)$ such that

$$Z(y,t;\beta) = \int dE\, e^{-\beta E} \rho(E; y, t) . \tag{2.108}$$

Aside: Schrödinger problem and non-perturbative effects

The partition function $Z(y,t;\beta)$ can be analysed from a different point of view, namely that of a Schrödinger problem [147, 148, 149]. Now $\rho(E;y,t)$ is the spectral density of the Hamilton operator $\mathcal{H} = \hbar^2\partial_0^2 + u(t_0,t_1)$ where $\hbar = g_s/\sqrt{2}$ and E are the eigenvalues of $\mathcal{H}$.[15] This formulation offers a framework for a non-perturbative description in the genus expansion g_s. However, since in our context the tau function $u(t_0,t_1)$ itself is only given as an asymptotic series in g_s, setting up the appropriate non-perturbatively exact Schrödinger problem is nevertheless a difficult task. This question has been discussed and analysed with numerical methods in refs. [143, 145, 146].

Leading order contribution

In this work, we focus on the leading, genus zero, contribution to the spectral density, i.e. we look for the integral representation

$$\widetilde{Z}_0(y,t;\beta) = \int dE\, e^{-\beta E}\rho_0(E;y,t)\,, \tag{2.109}$$

where $\rho_0(E;y,t)$ is called the genus zero spectral density. First we have to write down $\widetilde{Z}_0(y,t;\beta)$ in terms of the variables y,t. Recall that the genus zero part split in two contributions: the semi-classical/non-topological and the topological ones, which we computed in terms of y,t in eqs. (2.100) and (2.101). In [1], we showed that the spectral density is

$$\rho_0(E;y,t) = \frac{\sqrt{2}}{\pi}\sqrt{E+y}\,\big(t+J'(y)\big) - \frac{1}{\sqrt{2}\pi}\int_{-y}^{E} dv\frac{J'(-v)}{\sqrt{E-v}}\,. \tag{2.110}$$

Where the functions $J(y)$ is defined as

$$J(y) = \sum_{k=2}^{+\infty}\frac{y^k(\gamma_k+\delta_k)}{k!}\,, \tag{2.111}$$

and the integral in eq. (2.109) is taken from $E=-y$ to $E\to+\infty$. This result is obtained by rewriting $\widetilde{Z}_0$ as

$$\tilde{Z}_0 = \frac{e^{\beta y}}{\sqrt{2\pi}\beta^{\frac{3}{2}}}\big(t+J'(y)\big) - \frac{e^{\beta y}}{\sqrt{2\pi\beta}}J(y) + \sqrt{\frac{\beta}{2\pi}}\int_{-\infty}^{y} dv\, e^{v\beta}J(v)\,, \tag{2.112}$$

and assuming that the function $J(v)$ is continuously differentiable in the interval $(-\infty,y)$, and that the stated integral (for $\beta>0$) is finite.

In other words, we have

$$\widetilde{Z}_0(y,t;\beta) = \int_{-y}^{+\infty} dE\, e^{-\beta E}\rho_0(E;y,t)\,, \tag{2.113}$$

with ρ_0 given by (2.110). It is important that the function ρ_0 is non-negative in $E\in(-y,+\infty)$ in order to be given a spectral density interpretation. A sufficient condition for this is $J'(y)\geqslant -t$ and $J'(v)\geqslant 0$ for $v\in(-y,+\infty)$.[16]

[15] We use the terms "eigenvalue density" and "spectral density" interchangeably.

[16] Even if the spectral density were negative in some ranges of energy, one could hope that there is a mechanism such as a phase transition that renders it positive. See for example refs. [123, 150].

The semi-classical ground state

For energies E close to the negative coupling $-y$ the calculated energy density $\rho_0(E; y, t)$ behaves as

$$\rho_0(E; y, t) = \frac{\sqrt{2}t}{\pi}\sqrt{E+y} + \mathcal{O}(|E - E_0|^{\frac{3}{2}}) \ . \tag{2.114}$$

Therefore, we can interpret the negative coupling $-y$ as the (semi-classical) ground state energy of the dual system.[17] In particular, for JT gravity in the absence of defects the on-shell value of y becomes zero, and hence the ground state energy vanishes. Coupling JT gravity to a gas of defect, however, yields a non-vanishing on-shell value for y according to eqs. (2.55) and (2.89), which therefore results in a non-trivial shift of the ground state energy. This observation is in agreement with the results obtained in refs. [128, 126].

In practice

Let us now briefly see what one should do to apply this formulas to compute explicit results for the one defect type case. Recall the on-shell values of the couplings

$$t_k = \frac{(-1)^k}{(k-1)!} + \left(-\frac{\alpha^2}{4\pi^2}\right)^k \frac{2\pi^2\epsilon}{k!} \quad \text{for} \quad k \geqslant 2 \, , \tag{2.115}$$

and the t_0, t_1

$$(t_0, t_1)|_{\text{on-shell}} = 2\pi^2\epsilon \left(1, -\frac{\alpha^2}{4\pi^2}\right) \ . \tag{2.116}$$

From these, we have to deduce the on-shell values for the pair (y, t). To do so, recall the definition of y, t in eq. (2.89). Plugging in the on-shell value of t_0, t_1, we find the functional relations

$$\begin{aligned} 0 &= -\sqrt{y}\,\mathcal{J}_1(2\sqrt{y})|_{\text{on-shell}} + (2\pi^2\epsilon)\mathcal{J}_0\left(\frac{\alpha\sqrt{y}}{\pi}\right)\bigg|_{\text{on-shell}} \, , \\ t|_{\text{on-shell}} &= \mathcal{J}_0(2\sqrt{y})|_{\text{on-shell}} + (2\pi^2\epsilon)\frac{\alpha}{2\pi\sqrt{y}}\mathcal{J}_1\left(\frac{\alpha\sqrt{y}}{\pi}\right)\bigg|_{\text{on-shell}} \, , \end{aligned} \tag{2.117}$$

in terms of the Bessel functions $\mathcal{J}_\nu(x)$ of the first kind

$$\mathcal{J}_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)\,k!}\left(\frac{x^2}{4}\right)^k \, , \quad \mathcal{J}_{-n}(x) \equiv (-1)^n \mathcal{J}_n(x) \text{ for integer } n \, . \tag{2.118}$$

We can also show, by inserting the y, t on-shell values, that the genus zero density of states of JT with a single defect type can be written as

$$\rho_0(E; y, t) = \frac{1}{\sqrt{2}\,\pi}\int_{-y}^{E} dv \frac{\mathcal{I}_0(2\sqrt{v}) + (2\pi^2\epsilon)\frac{\alpha}{2\pi\sqrt{v}}\mathcal{I}_1\left(\frac{\alpha\sqrt{y}}{\pi}\right)}{\sqrt{E-v}} \tag{2.119}$$

with the modified Bessel functions defined as $\mathcal{I}_\nu(x) = i^{-\nu}\mathcal{J}_\nu(ix)$. This result is in agreement with refs. [126, 128].

[17] This corresponds also to the (semi-classical) ground state of the Schrödinger problem we mentioned in page 39.

Small ϵ

A first sanity check is that for $\epsilon \to 0$ we recover the result of ref. [130], namely $(y,t)|_{\text{on-shell}} = (0,1)$. Now, solving for small ϵ the relations (2.117), we find up to and including$\mathcal{O}(\epsilon^3)$

$$
\begin{aligned}
y|_{\text{on-shell}} =& 2\pi^2\epsilon + \pi^2\left(2\pi^2 - \alpha^2\right)\epsilon^2 + \frac{\pi^2(15\alpha^4 - 72\pi^2\alpha^2 + 80\pi^4)}{24}\epsilon^3 + \dots\,, \\
t|_{\text{on-shell}} =& 1 + \frac{\alpha^2 - 4\pi^2}{2}\epsilon - \frac{\alpha^4 - 8\pi^2\alpha^2 + 8\pi^4}{8}\epsilon^2 \\
&+ \frac{21\alpha^6 - 216\pi^2\alpha^4 + 576\pi^4\alpha^2 - 448\pi^6}{288}\epsilon^3 + \dots\,.
\end{aligned}
\tag{2.120}
$$

With these expansions we can read off the shift in the ground state which is $E_0 = -2\pi^2\epsilon + \mathcal{O}(\epsilon^2)$. Moreover, by plugging in the relation

$$
I_n(y) = \frac{(-1)^n}{(\sqrt{y})^{n-1}}\mathcal{J}_{n-1}(2\sqrt{y}) + (2\pi^2\epsilon)\left(-\frac{\alpha}{2\pi\sqrt{y}}\right)^n \mathcal{J}_n\left(\frac{\alpha\sqrt{y}}{\pi}\right) \quad \text{for} \quad n \geqslant 2\,, \tag{2.121}
$$

which is shown by inserting the on-shell values of the couplings in the functions I_n , $n \geqslant 2$, in eqs. (2.103) and (2.104) and then expanding in ϵ we can compute the genus one and genus two to any desired order in ϵ.

Generalising to multiple types of defect is straightforward. We only mention here that, for example, the ground state is shifted by $-2\pi^2\sum_j \epsilon_j$. So each defect type j contributes to the ground state energy in a similar way that depends on the coupling ϵ_j.

2.7 Low temperature

2.7.1 An expansion scheme for partition functions

We have seen so far an organisation of JT gravity's perturbation theory in terms of the genus (for a fixed number of boundaries). For example, for the single boundary partition functions, given the genus g, we have contributions in various powers of the inverse temperature β and defect couplings ϵ_j (for each type). An expansion in terms of temperature is more suited from a physics point of view, given the fact that for many problems we are interested in results up to a certain energy scale.

Single boundary

In [1], we adapt the low temperature expansion scheme of ref. [130] to the variables (y,t) defined in eq. (2.89) (this is done quite naturally). The scheme (or, alternatively, double scaling limit) for a single boundary is taken to be

$$
\beta \to +\infty \quad \text{with} \quad \frac{g_s\beta^{\frac{3}{2}}}{t} = \text{const.}\,, \; y\beta = \text{const.}\,. \tag{2.122}
$$

The asymptotic low temperature expansion is written as [130]

$$
Z(y,t;T) = \frac{T^{\frac{3}{2}}\, e^{\frac{g_s^2}{24t^2T^3} + \frac{y}{T}}}{\sqrt{2\pi}\, g_s}\mathcal{Z}_{y,t}(T) \quad \text{where} \quad \mathcal{Z}_{y,t}(T) = \sum_{\ell=0}^{+\infty} T^\ell z_\ell(y,t)\,, \tag{2.123}
$$

where the first few coefficient functions z_ℓ are calculated to be

$$z_0 = t\,, \quad z_1 = \left(1 + \frac{g_s^4}{240t^4T^6}\right) I_2\,,$$
$$z_2 = \left(\frac{7g_s^4}{240t^5T^6} + \frac{g_s^6}{576t^7T^9} + \frac{g_s^8}{57600t^9T^{12}}\right) I_2^2 + \left(-2 + \frac{g_s^2}{12t^2T^3} + \frac{g_s^4}{120t^4T^6} + \frac{g_s^6}{3360t^6T^9}\right) I_3\,. \tag{2.124}$$

This is done via a recursion relation [130]. Here we omit the details on computing these coefficients but we stress that this analysis of the low temperature limit is general in the sense that we can consider other on-shell values for the couplings (y, t) (and also for the couplings $t_k = \gamma_k + \delta_k$ appearing implicitly in the expansion (2.123)). In particular, if we consider small deviations from the on-shell values $(y, t) = (1, 0)$ (and small perturbations δ_k for $k \geqslant 2$) of pure JT gravity, we can study the low temperature expansions of deformations to pure JT gravity.

For instance, solving eq. (2.89) for small deformations δ_k, $k = 1, 2, 3, \ldots$, away from pure JT gravity yields for the coupling parameters (y, t) appearing in the above limit the expansion

$$\begin{aligned} y &= \delta_0 + \frac{1}{2}(2\delta_0\delta_1 - \delta_0^2) + \ldots\,, \\ t &= 1 - (\delta_0 + \delta_1) + (\delta_0^2 - \delta_0\delta_1 - \delta_0\delta_2) + \ldots\,, \end{aligned} \tag{2.125}$$

which at leading order for a single defect become $y = 2\pi^2\epsilon + \mathcal{O}(\epsilon^2)$ and $t = 1 + \mathcal{O}(\epsilon)$ (see eq. (2.120)).

Multiple boundaries

For partition functions of more boundaries, the scheme (2.122) generalises in a canonical way to [131]

$$\beta_i \to +\infty \quad \text{with} \quad \frac{g_s \beta_i^{\frac{3}{2}}}{t} = \text{const.}\,,\ y\beta_i = \text{const.} \quad \text{for} \quad i = 1, 2, ..., \#\text{boundaries}\,, \tag{2.126}$$

which yields for the low temperature limit of the partition function $Z(\beta_1, \beta_2)$ the result [131]

$$Z(y, t; \beta_1, \beta_2) = \frac{t\, e^{\frac{g_s^2(\beta_1+\beta_2)^3}{24t^2} + y(\beta_1+\beta_2)}}{\sqrt{2\pi}\, g_s (\beta_1 + \beta_2)^{\frac{3}{2}}} \operatorname{erf}\left(\frac{g_s}{2\sqrt{2}\,t}\sqrt{\beta_1\beta_2(\beta_1 + \beta_2)}\right) + \mathcal{O}(\beta_1^{-1}, \beta_2^{-1})\,. \tag{2.127}$$

Another expansion scheme

Taking leading order approximation of the limit (2.126) for JT with defects, i.e. $y = 2\pi^2\epsilon + \mathcal{O}(\epsilon^2)$ and $t = 1 + \mathcal{O}(\epsilon)$ (see eq. (2.120)), we deduce another expansion scheme for multiple boundary partition functions $Z(\beta_1, ..., \beta_m)$

$$\beta_i \to +\infty \quad \text{with} \quad g_s \beta_i^{3/2} = \text{const.}\,,\ \epsilon\beta_i = \text{const.} \quad \text{for all} \quad i = 1, \ldots, m\,, \tag{2.128}$$

with distinct inverse temperatures β_i for the individual boundary components.[18] The inverse temperatures of the boundary components are conveniently described in terms

[18]In the absence of defects the low temperature limit of the partition function $Z(\beta_1, \beta_2)$ was previously derived in ref. [131]. For the uniform limit $\beta \to +\infty$ with $\beta = \beta_1 = \ldots = \beta_m$ the low temperature limit of the partition functions together with defects has been first reported in ref. [151].

of the universal inverse temperature scale β and the dimensionless constants

$$\mathfrak{b}_i = \frac{\beta_i}{\beta} \,. \tag{2.129}$$

Then the above limit becomes $\beta \to +\infty$ for constant positive values $\mathfrak{b}_i$ while keeping $g_s\beta^{3/2}$ and $\epsilon\beta$ fixed. In the limit (2.128) (the topological part of) the partition function of eq. (2.56) becomes

$$\begin{aligned} Z(\beta_1,\ldots,\beta_m)^{\text{top.}} &= \frac{1}{g_s^2}\mathcal{B}(\beta_1)\cdots\mathcal{B}(\beta_m)G(1,\{t_k=\delta_k\}) \\ &= \sum_{g,n=0}^{+\infty} \frac{(g_s\beta^{\frac{3}{2}})^{2g-2+m}(\epsilon\beta)^n}{(2\pi)^{\frac{m}{2}}} \\ &\quad \cdot \sum_{\ell_1,\ldots,\ell_m=0}^{+\infty} \beta^{\ell_1+\ldots+\ell_m-m-n-3g+3}\, \mathfrak{b}_1^{\ell_1+\frac{1}{2}} \cdots \mathfrak{b}_m^{\ell_m+\frac{1}{2}} \partial_{\ell_1}\cdots\partial_{\ell_m} G_{g,m+n}(\{t_k\})\Big|_{t_k=\delta_k/\epsilon} \,, \end{aligned} \tag{2.130}$$

with the generating function $G(1,\{t_k\}) = \sum_{g,n} g_s^{2g} G_{g,n}(\{t_k\})$ decomposed into the contributions $G_{g,n}$ indexed by their genus g and their number of marked points n. Imposing now the selection rule (2.32) and inserting $\delta_0 = 2\pi^2\epsilon$, we arrive at

$$\begin{aligned} &Z(\beta_1,\ldots,\beta_m)^{\text{top.}} \\ &\quad = \sum_{g,n=0}^{+\infty} \frac{(g_s\beta^{\frac{3}{2}})^{2g-2+m}(2\pi^2\epsilon\beta)^n}{(2\pi)^{\frac{m}{2}}\, n!} \sum_{\ell_1,\ldots,\ell_m=0}^{+\infty} \mathfrak{b}_1^{\ell_1+\frac{1}{2}}\cdots\mathfrak{b}_m^{\ell_m+\frac{1}{2}} \left\langle \tau_0^n \tau_{\ell_1}\cdots\tau_{\ell_m} \right\rangle_g + \mathcal{O}(\beta^{-1}) \,, \end{aligned} \tag{2.131}$$

in terms of the non-vanishing correlators (2.31) on the moduli space of stable curves $\overline{\mathcal{M}}_{g,m+n}$ of genus g with $m+n$ marked points.[19] Following ref. [131] and using results of refs. [115, 152] we show in [1] that the single and double boundary partition functions of JT gravity with defects (single type) in the limit (2.128) become

$$Z(\beta) = \frac{e^{\frac{g_s^2}{24}\beta^3 + 2\pi^2\epsilon\beta}}{\sqrt{2\pi}\, g_s \beta^{\frac{3}{2}}} + \mathcal{O}(\beta^{-1}) \tag{2.132}$$

$$Z(\beta_1,\beta_2) = \frac{e^{\frac{g_s^2}{24}(\beta_1+\beta_2)^3 + 2\pi^2\epsilon(\beta_1+\beta_2)}}{\sqrt{2\pi}\, g_s(\beta_1+\beta_2)^{\frac{3}{2}}}\ \mathrm{erf}(2^{-3/2} g_s \sqrt{\beta_1\beta_2(\beta_1+\beta_2)}) \,. + \mathcal{O}(\beta^{-1}) \tag{2.133}$$

2.7.2 Phase Transition and Spectral Form Factor

Using the low temperature limit of the partition functions $Z(y,t;\beta_1,\beta_2)$ and $Z(y,t;\beta)$ of the previous section and applying numerical methods, we study two well-established and related phenomena, namely the phase transition [153, 154], which exchanges the dominance between the connected versus the disconnected geometries in the two boundary partition function, and the spectral form factor, which arises as a certain analytic continuation of the two-boundary partition function.[20] In particular, we analyse the dependence of these quantities in the presence of defects.

[19]The correction terms $\mathcal{O}(\beta^{-1})$ depend on the genus expansion parameter g_s and the coupling ϵ in such a way that in the double scaling limit (2.128) they approach zero at least with the rate $\sim 1/\beta$.

[20]The spectral form factor was first introduced in the AdS/CFT context in ref. [155].

Figure 2.10: The left figure shows a disconnected geometry—here illustrated in terms of two AdS_2 disks at genus zero—that dominates the spectral form factor at early times τ, whereas the right figure depicts a connected geometry with two boundaries—shown is the double trumpet contribution—that becomes dominant at late times τ.

Phase Transition

There are two types of geometries that contribute to the two-point function. On the one hand there are geometries with two disconnected components, each with a single boundary component, and on the other hand there are connected geometries with two boundary components, as illustrated in fig. 2.10 (where only the genus zero contributions are depicted for simplicity). At low temperatures we have according to eqs. (2.123) and (2.127) (in the chosen low temperatue expansion scheme) the following two quantities

$$\begin{aligned} Z(y,t;\beta)^2 &= \frac{e^{2y\beta} e^{\frac{g_s^2\beta^3}{12t^2}}}{2\pi g_s^2\beta^3} t^2 + \mathcal{O}(\beta^{-1}) \,, \\ Z(y,t;\beta,\beta) &= \frac{e^{2y\beta} e^{\frac{\beta^3 g_s^2}{3t^2}}}{4\sqrt{\pi}\beta^{3/2} g_s} t \operatorname{erf}\left(\frac{\beta^{3/2} g_s}{2t}\right) + \mathcal{O}(\beta^{-1}) \,. \end{aligned} \tag{2.134}$$

Independent of the specific choices for the on-shell values of the parameters (y,t), we can make some quite general comments. Taking the ratio of the two-point contributions in eq. (2.134), the dependence on the shift in energy given by y drops out (at leading order in the temperature). Hence, the phase transition (and as a consequence also the spectral form factor introduced later) is determined by the off-shell parameter t. Explicitly analysing the ratio of the two contributions (2.134) in the low temperature regime yields with the dimensionless constant $c := g_s\beta^{3/2}/t$ the dimensionless (numerical) critical value $c_{\text{crit.}}$ for the phase transition according to

$$\frac{Z(y,t;\beta,\beta)}{Z(y,t;\beta)^2} = 1 \;\Rightarrow\; \frac{1}{2}\sqrt{\pi} c e^{\frac{c^2}{4}} \operatorname{erf}\left(\frac{c}{2}\right) = 1 \;\Rightarrow\; c_{\text{crit.}} \approx \pm 1.24013 \,. \tag{2.135}$$

Let us now focus on JT gravity with defects. This means that we take (y,t) to their on-shell values (2.117) and that we work with the quantities in eq. (2.136), where the on-shell values of (y,t) are found numerically for a given set of ϵ and α, i.e.

$$\begin{aligned} Z(\beta)^2 &= \left.\frac{e^{2y\beta} e^{\frac{g_s^2\beta^3}{12t^2}}}{2\pi g_s^2\beta^3} t^2\right|_{y,t\text{ on-shell}} \,, \\ Z(\beta,\beta) &= \left.\frac{e^{2y\beta} e^{\frac{\beta^3 g_s^2}{3t^2}}}{4\sqrt{\pi}\beta^{3/2} g_s} t \operatorname{erf}\left(\frac{\beta^{3/2} g_s}{2t}\right)\right|_{y,t\text{ on-shell}} \,. \end{aligned} \tag{2.136}$$

Keeping the above in mind, we plot the connected and disconnected parts of the two-point function in Fig. 2.11. We can see the general behaviour of JT gravity in the absence

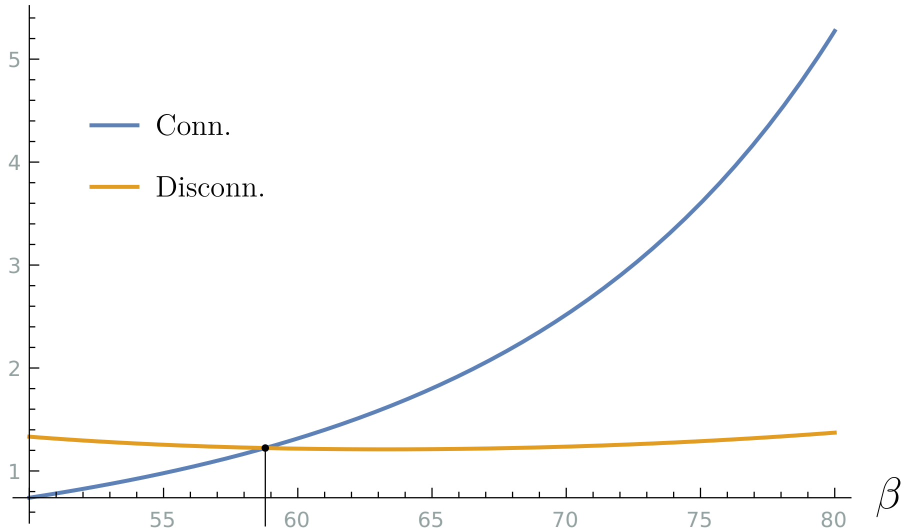

Figure 2.11: We plot the connected versus the disconnected geometry contributions of eq. (2.136). The identification angle α is fixed to $\alpha = \frac{\pi}{2}$, the defect amplitude is $\epsilon = 0.001$ and $g_s = 0.0027$. In the range of the plot we have a maximum of $\sim 4.2\%$ relative error, which measures the ratio of the terms ignored (order T^2) over the terms kept in the low temperature expansion.

of defects is reproduced: at high temperatures the disconnected geometry dominates, whereas for low temperatures the connected part constitutes the more dominant contribution [146, 131]. This is the two-dimensional instantiation of a Hawking-Page phase transition [156, 157]. However, we should also notice that, as shown in fig. 2.12, for larger ϵ, the phase transition occurs at a smaller value of β.

Spectral Form Factor

Now we come to the analysis of the spectral form factor $Z(\beta + i\tau, \beta - i\tau)$, which is a real function of the time τ defined via an analytic continuation of the two-point function $Z(\beta_1, \beta_2)$. The spectral form factor is essential in the analysis of quantum chaotic behaviour and plays an increasingly important role in the study of black hole physics [105]. For the case of JT gravity in the presence of defects the spectral form factor has not yet been analysed. The task is to understand the role of the parameter ϵ.

For large groups of systems obeying quite common assumptions (such as the eigenstate thermalisation hypothesis [158, 159]), one expects the spectral form factor to exhibit certain universal features. Early times are characterised by decay and hence a “slope", followed by a rise and hence a “ramp", and lastly at late times we encounter a “plateau" with fixed value given by the one-point function $Z(2\beta)$.[21] Let us define the normalised spectral form factor in the following manner

$$G(\beta, \tau) := \frac{Z(\beta + i\tau, \beta - i\tau)}{Z(2\beta)} = \mathrm{erf}\left(\frac{\beta^{3/2} g_s \sqrt{\frac{\tau^2}{\beta^2} + 1}}{2t} \right), \tag{2.137}$$

where we are normalising with respect to the contribution $Z(2\beta)$ as this sets the height of the plateau. Due to the low temperature dominance of the connected contribution

[21]The “plateau” cannot be obtained if the perturbative series is truncated at some finite g. To render an asymptotic series convergent non-perturbative contributions have to be taken into account [153]. In the zero temperature/zero coupling limit considered in ref. [130] and here the perturbative series converges.

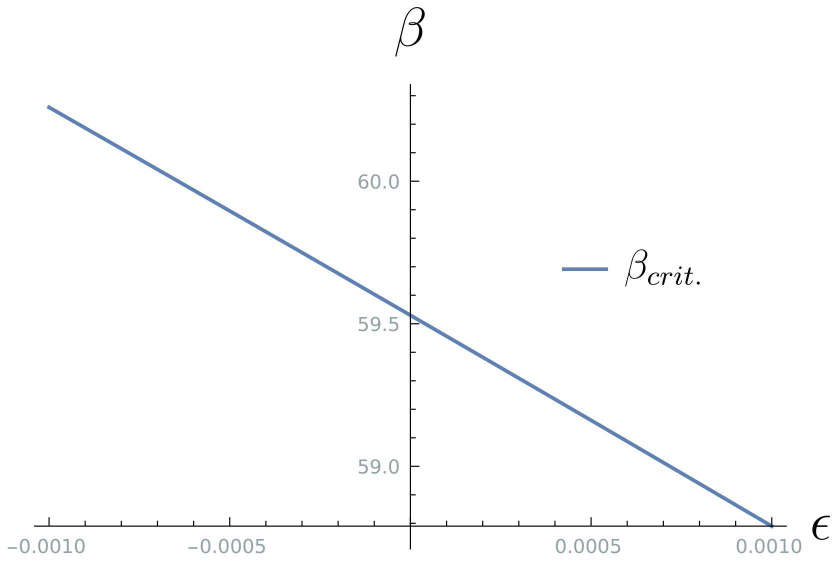

Figure 2.12: The phase transition temperature (the point for which $Z(\beta,\beta) = Z(\beta)^2$) as a function of the defect amplitude. The identification angle α is fixed to $\alpha = \frac{\pi}{2}$ and $g_s = 0.0027$.

as outlined above, we would expect late times to be dominated by connected contributions. A closer look at eq. (2.134) shows that this is guaranteed by the functional form of both expressions. We are only considering connected geometries in eq. (2.137) as we are mainly interested in the ramp and plateau behaviour. We want to reiterate some statements of refs. [73, 131], which help in understanding the importance of the corrections outlined in eq. (2.128). The $g = 0$ part of the two-boundary correlator only furnishes the "ramp" behaviour as shown in ref. [153]. We can see that the approximation (2.128) already allows for the creation of the plateau [131]. Furthermore, if we work in the limit (2.122) both the phase transition and spectral form factor become sensitive to the presence of defects.

We note that the transition from ramp to plateau now depends on ϵ. More specifically, for larger values of ϵ we can move it to earlier times, whereas negative values moves it to later times, which mirrors the behaviour discovered for the phase transition.

We may also consider changes in the identification angle α while keeping ϵ fixed for both the phase transition and the spectral form factor. While the dependence on α within the range $0 \leqslant \alpha < \pi$ can be studied straightforwardly with the methods presented here, it would be even more interesting to consider changes in α over the full range of identification angles. This could possibly be achieved by implementing the results of ref. [160].

2.8 Summary and outlook

We started this chapter by introducing JT gravity, its deformations and its path integral. Specifically, we described how the latter can be performed and linked to volumes of moduli spaces of bordered Riemann surfaces possibly including conical singularities. Then we showed the relation of the above to solutions of the KdV hierarchy and explained how one can use it to calculate the genus expansion. We have formulated this in a rather general way so one can adapt it to various values of $\{t_k\}$'s and research connections to other KdV solutions. We finally studied low temperature expansions schemes and applied them to check the dominance of connected versus disconnected

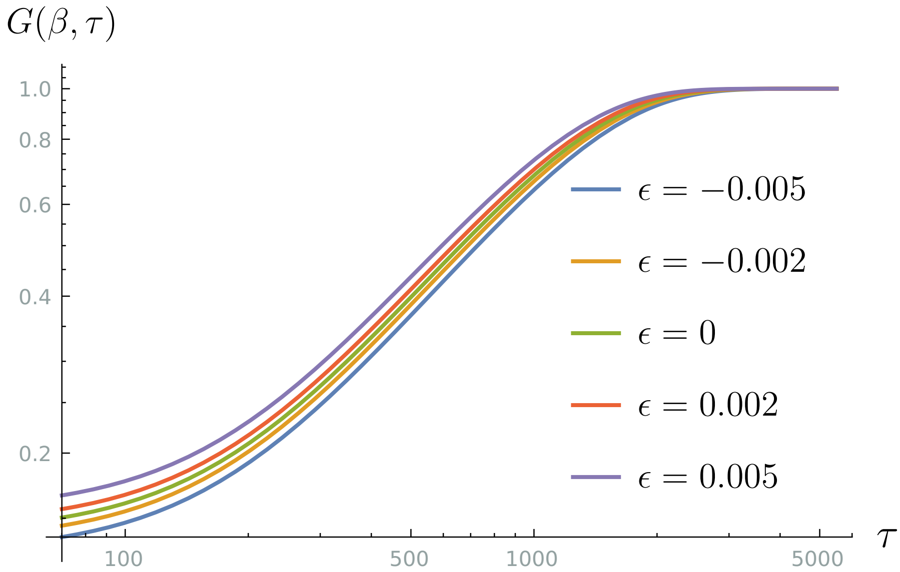

Figure 2.13: Shown is the spectral form factor for different values of ϵ with $g_s = \frac{1}{4 \cdot 180^{3/2}}$, $\beta = 180$, $\alpha = \pi/2$.

contributions to the two-boundary partition function and also calculate the spectral form factor.

KdV approach vs. JT deformations

We have analysed (thermal) partition function of JT gravity from the point of view of solutions to the KdV hierarchy inspired by the approach of ref. [130]. It is clear that such solutions can describe usual JT gravity (already from [130]) and, from our work, JT with a finite number of defect types. This method—namely mapping JT with and without defects to solutions to the KdV hierarchy—can be mapped to the approaches of refs. [128, 126] as there is a clear geometric way to go from the (perturbative) JT path integral to the KdV solutions via the relation of the later to moduli space volumes. However, this is not so immediate for more general scalar potentials that are not a-priori related to defects. It would be interesting to study how and if general scalar perturbations of JT gravity map to other solutions of the KdV hierarchy.

Significance of other KdV solutions

We know that other solutions (tau-functions) of the KdV hierarchy have two-dimensional gravitational interpretations. Namely, the Witten-Kontsevich tau-functions relates to two dimensional topological gravity [115, 140] and the Brézin–Gross–Witten tau-function describes JT supergravity [161]. Yet other tau-functions are discussed from the mathematical perspective in ref. [162]. One could try to study in a systematic way solutions to the KdV hierarchy and their possible relation to theories originating from physics.

Analytic and not asymptotic?

The genus expansion of JT is an asymptotic series in g_s—also when written in terms of the KdV hierarchy—that requires non-perturbative corrections to become analytic. It is a compelling albeit difficult goal to search for analytic rather asymptotic solutions to the KdV hierarchy. For JT this problem is studied in refs. [143, 146, 150], where the proposal for non-perturbative completion of KdV solutions of refs. [163, 164] is employed. Both the results of ref. [130] and our work furnish an easy and systematic access to higher genus contributions, such that modern resurgence techniques could come into play to address non-perturbative effects in this context (see e.g. [165]).

Low temperature and applications

Following refs. [130, 131], we calculate JT partition functions in a low temperature limit where non-perturbative corrections are suppressed.[22] We used these calculations to study the Hawking-Page-like phase transition that occurs between connected and disconnected contributions and the spectral form factor. We expect that the qualitative features of the phase transition and the spectral form factor do not change much but it is an important future direction to include higher order corrections to these quantities and study their behaviour. It would also be interesting to understand these phenomena from a three-dimensional perspective given that JT gravity with defects has been linked to three-dimensional gravity in ref. [128].

[22]This can be seen by the fact that the stated limits imply that both g_s and ϵ_j go to zero for low temperatures. This means that non-perturbative effects of the order e^{-1/g_s} and $e^{-1/|\epsilon_j|}$ also vanish at low temperature.

Chapter 3

One dimension higher: ensemble averaging two-dimensional CFTs

Having seen the rich story that is part of the JT gravity/matrix model duality, it is natural to ask: what happens in higher dimensions? And, in particular, in three bulk dimensions, which is the "obvious" next dimensionality to check. In chapter 2 we focused on the relation of JT partition functions to the KdV hierarchy, i.e. we did not talk much about the averaging process itself. However, given the relation of JT to matrix models, one could, in principle, talk in terms of averages of boundary Hamiltonians and KdV solutions.

In the following chapters, the focus will be more balanced. Our goal is to describe the works [2, 3] and in the process we will learn many things about (explicit) ensemble averages of two-dimensional conformal field theories, three-dimensional geometries and their relation. In this chapter we will discuss the ensemble averages and in chapter 4 we will continue with the latter.

For theoretical background discussions we mostly follow the references [74, 166, 167, 168, 169, 57, 170, 171, 172, 173]. Also further details of some calculations/derivations we demonstrate here can be found in these references. Finally, ref. [174] contains a pedagogical introduction to the symmetric product orbifold and its relation to string theory on $AdS_3 \times S^3 \times T^4$.

3.1 Narain theories and averaging

With start this discussion with the "trivial" case of a Narain theory (we will define this term more precisely later), namely the free boson CFT compactified on a circle of radius R.

The free boson

The simplest (conformal) field theory is perhaps that of a free boson. The free boson field, defined on the real plane $\mathbb{R}^2$, is a map

$$X : \mathbb{R}^2 \to \mathbb{R},\ x \mapsto X(x)\ . \tag{3.1}$$

The action of such a field reads

$$S_{FB}[X] = g \int_{\mathbb{R}^2} d^2x \partial_a X \partial^a X\ , \tag{3.2}$$

where the indices a are contracted with the metric on $\mathbb{R}^2$, and g is a normalisation constant.

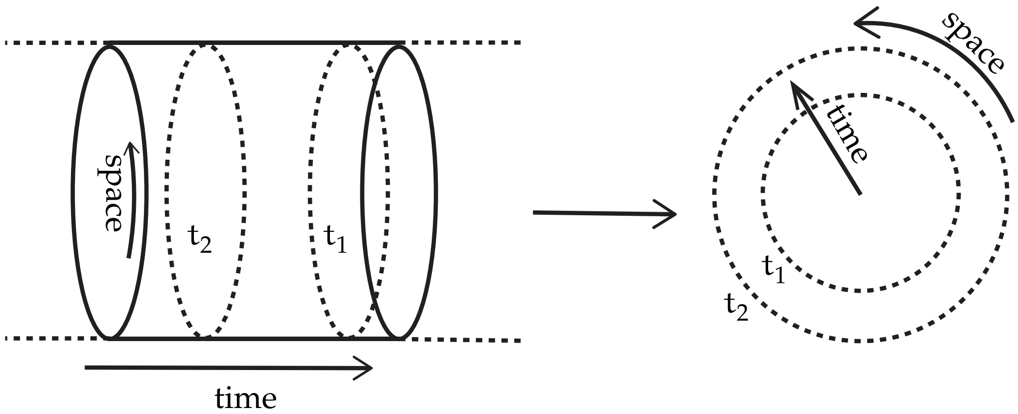

Figure 3.1: Mapping the (infinite) cylinder to the punctured complex plane. Constant time slices on the cylinder become circles on the punctured plane and space is an angular variable. The punctured complex plane is a convenient space for quantisation of a CFT in the framework of radial quantisation (the term “radial” essentially corresponds to the fact that constant times are circles).

In the context of string theory and more precisely closed string theory, this model naturally appears but instead of $\mathbb{R}^2$, the field X lives on an infinite cylinder. This is because the CFT is defined on the string world-sheet and the topology of a closed string's world-sheet is a cylinder. The latter can be viewed as the quotient of $\mathbb{R}^2$ modulo a one-dimensional lattice.[1] This topology is also convenient for canonical quantisation as it distinguishes a time direction. Mapping the cylinder to the punctured complex plane $\mathbb{C}\setminus\{0\} \simeq \mathbb{R}^2\setminus\{(0,0)\}$, one can perform radial quantisation and study the quantum theory of a free boson, see fig. 3.1.

This model, which a simple example of a sigma-model, can be also defined on a torus T^2. This can be done by identifying points on $\mathbb{R}^2$ that differ by a lattice vector for some two-dimensional lattice Λ_2. In other words $T^2 \simeq \mathbb{R}^2/\Lambda_2$. Alternatively, one can think of the torus as cutting a finite portion of the infinite cylinder and identifying its ends. Typically, putting a quantum field theory on a torus is done to calculate its partition function—a quantity that will be in our focus. Below we elaborate a little on the two-dimensional torus and its relation to lattices.

The torus T^2

We can define a (flat) torus by identifying points on the complex plane $\mathbb{C} \simeq \mathbb{R}^2$. Given two points $z, w \in \mathbb{C}$, this equivalence relation is

$$z \sim w + \lambda\,, \tag{3.3}$$

where λ lives on a two-dimensional lattice generated by two vectors λ_1 and λ_2. In order to have a non-vanishing angle between these two generating lattice vectors, we demand $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$. The two-dimensional lattice is defined as

$$\Lambda_2 = \{n\lambda_1 + m\lambda_2 | \, n, m \in \mathbb{Z}\}\ . \tag{3.4}$$

Without loss of generality, we can choose $\mathrm{Im}(\lambda_1) = 0$. The lattice Λ_2 looks something like fig. 3.2. Now, given the pair (λ_1, λ_2) that defines the lattice Λ_2, we notice that the pairs $(\lambda_1 + \lambda_2, \lambda_1)$ and $(\lambda_1 + \lambda_2, \lambda_2)$ give rise to the same lattice Λ_2. In fact all the pairs

[1] Similarly to constructing a cylinder from a piece of paper by gluing two of its ends.

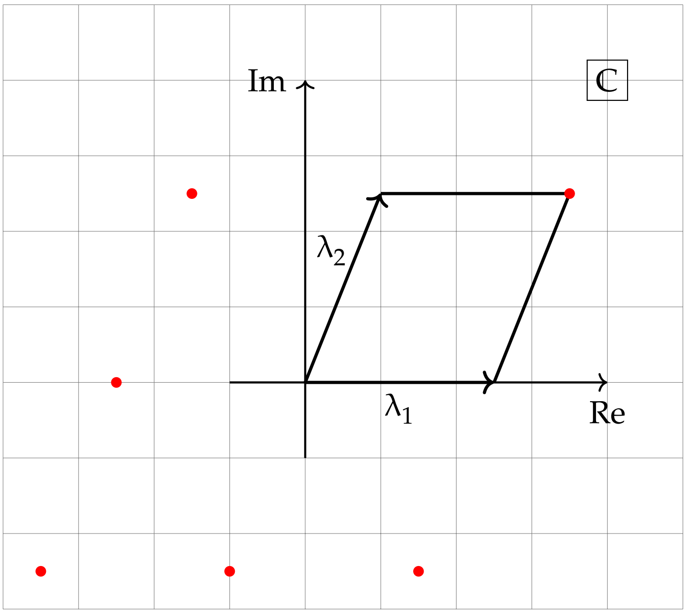

Figure 3.2: The two dimensional lattice Λ_2. Some points of the lattice are drawn (red). Any point on the lattice can be written as $\lambda = m\lambda_1 + n\lambda_2$. Identifying points on $\mathbb{C}$ via this lattice gives a two-dimensional (flat) torus.

(λ_1', λ_2') given as

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) , \tag{3.5}$$

generate the same lattice Λ_2.

Keeping this in mind, note that in conformal field theory, only the shape of the torus should matter and not its size.[2] A quantity that describes the torus' shape is the modular parameter τ defined as

$$\tau = \frac{\lambda_2}{\lambda_1} . \tag{3.6}$$

Under a transformation of the generating vectors like in eq. (3.5), the modular parameter transforms as

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d} . \tag{3.7}$$

Here, we see that the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

give the same new modular parameter τ'. Hence in eq. (3.7), we identify elements of $SL(2, \mathbb{Z})$ that differ by an overall sign. This means we take the quotient

$$PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2 , \tag{3.8}$$

with equivalence relation

$$A \sim -A , \quad A \in SL(2, \mathbb{Z}) . \tag{3.9}$$

[2]By shape we mean the relative size of the two cycles of the torus. The size corresponds to an overall scale.

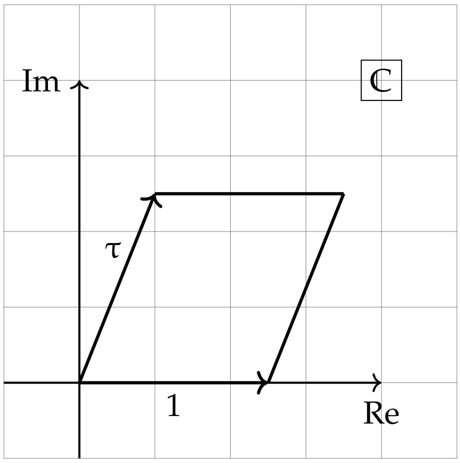

Figure 3.3: The shape of a two-dimensional torus is described by the modular parameter τ. Points on the upper half-plane are then identified by the action of the modular group Γ since they correspond to the same lattice. In other words, two apparently different modular parameters might actually describe the same torus shape if they are related by a modular transformation 3.7.

The group $\Gamma \equiv \mathrm{PSL}(2,\mathbb{Z})$, called the **modular group** of the torus, is generated by the transformations

$$S : \tau \mapsto -\frac{1}{\tau}, \quad T : \tau \mapsto \tau + 1 \,. \tag{3.10}$$

These generators have a matrix representation as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \,. \tag{3.11}$$

It is straightforward to show the relations $S^2 = I_{2\times2}$ and $(ST)^3 = I_{2\times2}$, where $I_{2\times2}$ is the 2×2 unit matrix. With these generators, the modular group can be written as

$$\Gamma \simeq \langle\, S, T \,|\, S^2 = I_{2\times2}, (ST)^3 = I_{2\times2} \rangle \,. \tag{3.12}$$

As we mentioned above, in conformal field theory we care about shapes and not scales, and the parameter τ is the relevant one to describes this property. To know the shape of a torus, we only need to know τ and it turns out that τ lives in the upper half-plane $\mathbb{H} = \{\mathbb{C} \,|\, \mathrm{Im}\,\tau > 0\}$. This can be shown by rescaling λ_1 and noting that we can interchange λ_1 and λ_2. A generic τ looks like in fig. 3.3.

The upper half-plane is a covering space of what is called the moduli space of τ. This moduli space $\mathcal{M}_\tau$, also called the moduli space of complex structures on the torus, is obtained via the quotient

$$\mathcal{M}_\tau \simeq \mathbb{H}/\Gamma \,. \tag{3.13}$$

To describe points that are distinct on the moduli space $\mathcal{M}_\tau$, we need a fundamental domain. This is a domain F_0 on $\mathbb{H}$ such that when we act on its points with Γ, we get the whole upper half-plane. The standard fundamental domain for this case is

$$F_0 = \left\{ \mathrm{Im}\,\tau > 0,\ -\frac{1}{2} \leqslant \mathrm{Re}\,\tau \leqslant 0 \text{ and } |\tau| \geqslant 1 \right\} \cup \left\{ \mathrm{Im}\,\tau > 0,\ 0 < \mathrm{Re}\,\tau < \frac{1}{2} \text{ and } |\tau| > 1 \right\} \,. \tag{3.14}$$

This is depicted in fig. 3.4.

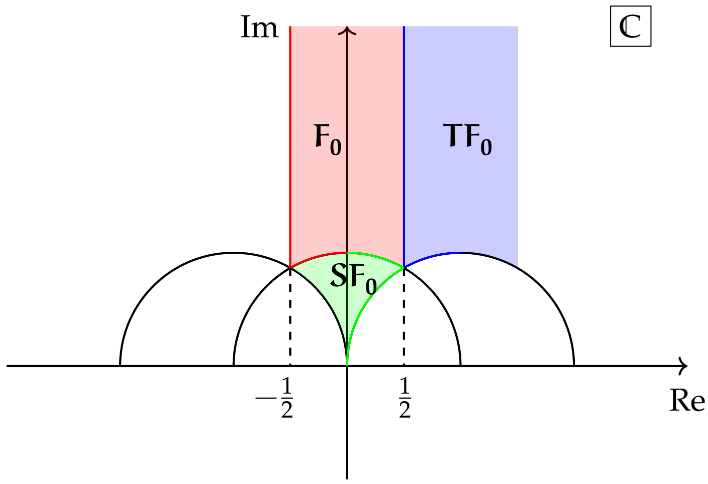

Figure 3.4: The (standard) fundamental domain F_0 for the complex structure moduli space $\mathcal{M}_\tau$ of the two-dimensional torus is drawn. We also depict some examples of domains that one gets when acting with the generators of Γ on F_0.

The partition function

The partition function of a quantum system with Hamiltonian H is given in terms of a Hilbert space trace of the operator $e^{-\beta H}$, where β is the inverse temperature. Doing such a calculation in quantum field theory usually leads to a functional integral over field configurations in Euclidean space where now fields are periodic in (Euclidean) time. The period is identified with β.

Starting from a CFT on the infinite cylinder, we want to make an analogous definition to the trace $\operatorname{tr} e^{-\beta H}$. To do so, note that the exponent of this expression is the period in time (β) times the generator of time translations H. Moving along the time direction β units makes a closed loop (because of the periodicity), so we can define the partition function for our CFT in a similar way.

Let's choose our cylinder to be of circumference 1 (meaning we identify points on $\mathbb{C}$ if they differ by 1), length $\operatorname{Im}\tau$ and choose the time direction to be the vertical one: $\operatorname{Im} z \,, z \in \mathbb{C}$. To get the partition function, we have to compactify the time direction. This can be done by cutting a finite piece of the cylinder and identifying its ends. The identification (gluing) can involve a twisting of one end. Translations along the vertical ("time") axis are implemented by the Hamiltonian H and the twisting by the momentum P. To get a closed cycle, we have to combine these two which amounts to moving along the τ direction of a torus. The partition function reads

$$Z(\tau) = \operatorname{tr}_{\mathcal{H}} e^{-2\pi\tau_2 H + 2\pi\tau_1 P} \,, \tag{3.15}$$

where $\mathcal{H}$ denotes the Hilbert space of the CFT. This expression can be written in terms of the Virasoro generators $L_0, \overline{L}_0$ and the variable $q = e^{2\pi i\tau}$ as

$$Z(\tau) = \operatorname{tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \,, \tag{3.16}$$

where c is the central charge of the CFT, which for a free boson equals 1, and $\overline{q}$ is the complex conjugate of q.

The Virasoro generators are very important in a conformal field theory as they organise the spectrum of the theory. Analogously to the harmonic oscillator in quantum mechanics, one starts with highest weight eigenstates of the operators L_0 and $\overline{L}_0$. These define the so called primary states (like the vacuum of the harmonic oscillator) from which we can construct the whole spectrum. This is done by acting with the operators $L_{-|n|}, \overline{L}_{-|m|}$ $n, m = 1, 2, \ldots$. The states created in this way are called descendant states.

It is clear now how to perform a trace like the one in eq. 3.16—one has to sum over all highest weight states and their respective descendants. Doing so for the free boson gives

$$Z_{FB}(\tau) = \frac{1}{\sqrt{\operatorname{Im}\tau}\,|\eta(\tau)|^2} \,. \tag{3.17}$$

Where $\eta(\tau)$ is the Dedekind eta function defined in Appendix A.1. The $\operatorname{Im}\tau$ part comes from integrating over the continuous spectrum of primary states that the free boson possesses. For the free boson on the circle that we will study next this becomes a sum. We see now explicitly that the partition function depends only on the shape of the torus τ. Using formulas found in the same Appendix, it is straightforward to show that the partition function is invariant under transformation of the modular parameter under the modular group Γ. This is of course to be expected in a conformal field theory as modular parameters connected by modular transformations give rise to equivalent tori (of the same shape).

What we described here is the operator approach to calculate the partition function. This calculation can also be done in the functional integral formalism where schematically we have

$$Z_{FB}(\tau) = \int \mathcal{D}X e^{-S_{FB}[X]} \,. \tag{3.18}$$

Here the field X is defined from the beginning on the torus and one integrates over all field configurations with periodicity conditions $X(x+\tau) = X(x), X(x+1) = X(x)$. Doing so involves expanding the field X in terms of eigenfunctions of the Laplacian and calculating the Laplacian's determinant. Eventually the two approaches, functional and operator, should give the same answer.

We started this whole discussion on CFTs and partition functions because we want in the end to talk about averaging over conformal field theories. For this reason we move now to another example.

The free boson on a circle

Consider now a free boson field that is compactified on a circle of radius R. We can implement this by considering the scalar field to be an angular variable, i.e. a map from the real numbers to the unit circle

$$X : \mathbb{R}^2 \to S^1,\ x \mapsto X(x) \,. \tag{3.19}$$

The space on which X takes values is called the target-space (here the target-space is the circle S^1). Since $X(x) \in S^1$ the points $X(x)$ and $X(x) + 2\pi n,\ n \in \mathbb{Z}$ are identified

$$X(x) \sim X(x) + 2\pi n,\ n \in \mathbb{Z} \,. \tag{3.20}$$

The action is similar to the free boson one but now the radius R enters

$$S_R[X] = \frac{R^2}{4\pi\alpha'} \int_{\mathbb{R}^2} d^2x \partial_a X \partial^a X \,. \tag{3.21}$$

The radius plays the role of a the target-space metric $G dX dX,\ G = R^2$, so we could also write the action as $\sim \frac{1}{4\pi\alpha'} \int G \partial_a X \partial^a X$.

Putting the CFT on a torus makes the boson field a map from T^2 to S^1

$$X : T^2 \sim_{\text{top.}} S^1 \times S^1 \to S^1,\ x \mapsto X(x) \,. \tag{3.22}$$

Here by $T^2 \sim_{\text{top.}} S^1 \times S^1$ we denote the topological equivalence of a T^2 to a product of two circles. This topological property of the two-dimensional torus leads to a characterisation of possible classes of maps X. This is a result of the double periodicity that X has now. Specifically, the field $X(x)$ is periodic under $x \mapsto x+1$ and $x \mapsto x+\tau$, for $x \in \mathbb{C}$ and τ being the modular parameter of the torus.[3] Since $X(x) \in S^1$, periodicity means that the field is mapped to itself modulo shifts of the form $2\pi n$, $n \in \mathbb{Z}$. Explicitly, the sectors are labelled by two integers m, n

$$X(x+\tau) = X(x) + 2\pi n \text{ and } X(x+1) = X(x) + 2\pi m \ . \tag{3.23}$$

The integer n is called the winding number and m the momentum. These names are more motivated when one studies this problem from the string theory point of view, or the CFT on the cylinder. There n is the periodicity along the circle of the cylinder and m corresponds to the discrete momenta due to the compact space that X takes values in (similarly to quantum mechanics where we get a discrete spectrum when we confine the system in finite volume).

The winding and momentum integers give rise to an infinite but discrete (in contrast to continuous for the free boson) set of primary states $|m, n\rangle$. Calculating the partition function now gives

$$Z_{S^1}(\tau; R) = \frac{1}{|\eta(\tau)|^2} \sum_{w,m\in\mathbb{Z}} e^{-2\pi i \tau_1 w m - \pi\tau_2\left(\frac{\alpha'}{R^2}w^2 + \frac{R^2}{\alpha'}m^2\right)} \ . \tag{3.24}$$

This partition function is modular invariant, as one can show using the Poisson resummation formula. The latter is useful for the proof of invariance under the S transformation. T invariance can be easily seen: a T transformation changes τ_1 to $\tau_1 + 1$ which is just a phase in expression (3.24).

Calculating the partition function in the functional integral formalism, one splits the field into a "classical" part $X_{cl}^{(m,n)}(x)$ and a periodic part $\widetilde{X}(x)$. The first one takes care of the conditions (3.23) and is given by

$$X_{cl}^{(m,n)}(x) = \frac{i\pi}{\operatorname{Im}\tau}\left(x(m\bar{\tau} - n) - \bar{x}(m\tau - n)\right) \ . \tag{3.25}$$

The summation over m, n in the partition functions above corresponds to summing over the sectors of this classical solution (after Poisson re-summation). The remaining functional integral over $\widetilde{X}$ gives the Dedekind eta part and involves a Laplacian determinant calculation.

Target-space (T)-duality of the circle partition function

By looking at the partition function (3.24), we can naively think that for each $R \in \mathbb{R}$ we get a different expression/theory. However, this is not the case as the following holds

$$Z_{S^1}(\tau; {}^{\alpha'}\!/_{R}) = Z_{S^1}(\tau; R) \ . \tag{3.26}$$

The radius $R^* = \sqrt{\alpha'}$, for which $\frac{\alpha'}{R^*} = R^*$ holds, is called the self-dual radius. This symmetry of the partition function means that we can restrict the radius to be in the set $\left[\sqrt{\alpha'}, +\infty\right)$ or $\left(0, \sqrt{\alpha'}\right]$.[4] We denote this moduli space $\mathcal{M}_{S^1}$.

[3] Note that we could also express these periodicity conditions in terms of torus lattice generators λ_1, λ_2. However, since only the modular parameter is relevant in a CFT, we can fix the torus lattice to be generated by τ and 1. Equivalently, we can choose, without loss of generality, $\lambda_1 = 1, \lambda_2 = \tau$.

[4] Strictly speaking, one should also check that T-duality is a true symmetry of the full conformal field

Averaging over the radius

We saw that the moduli space of the target-space radius R of the compactified boson is $\mathcal{M}_{S^1}$. Let's choose the interval $\left[\sqrt{\alpha'}, +\infty\right)$. We would like to average over these values to get an "averaged" partition function that will not depend on the radius R. To do so, we need a measure on $\mathcal{M}_{S^1}$ such that we can integrated over it. This is given in terms of a metric on $\mathcal{M}_{S^1}$ called the Zamolodchikov metric [175]. For the case of the circle, this metric is (up to normalisation)

$$ds^2_{\mathcal{M}_{S^1}} = \frac{dR^2}{R^2} \,. \tag{3.27}$$

This metric is calculated through two-point correlators of exactly marginal operators of the conformal field theory. These are operators that can be added to the action (or Lagrangian density) and do not ruin conformal invariance.[5] In other words, they deform the CFT keeping that symmetry and preserving the central charge.[6] Such operators for the free boson on the circle look like

$$\frac{R\delta R}{2\pi\alpha'}\partial_a X \partial^a X \,. \tag{3.28}$$

this can be seen by varying the action $S_R[X]$ with respect to the radius. Adding 3.28 to the Lagrangian density of $S_R[X]$ leads to $S_{R+\delta R}[X]$.

Thus, we can define an average of the partition function Z_{S^1} as

$$\langle Z_{S^1}(\tau)\rangle \equiv \frac{1}{\mathrm{Vol}(M_{S^1})}\int_{\mathcal{M}_{S^1}} d\mu(R) Z_{S^1}(\tau; R) = \frac{1}{\mathrm{Vol}(M_{S^1})}\int_{\sqrt{\alpha'}}^{+\infty} \frac{dR}{R} Z_{S^1}(\tau; R) \,. \tag{3.29}$$

Where $d\mu(R) = {}^{dR}/_{R}$ is the measure on the moduli space induced from the Zamolodchikov metric and $\mathrm{Vol}(M_{S^1})$ is the volume of the moduli space $\mathcal{M}_{S^1}$ calculated with the same measure. There is however a problem in this example: neither the integral of the partition function nor the volume converge! This can bee seen by noting that for large radii, the partition function scales linearly with the radius. For the volume this is defined as

$$\mathrm{Vol}(M_{S^1}) = \int_{\sqrt{\alpha'}}^{+\infty} \frac{dR}{R} \,, \tag{3.30}$$

and clearly does not converge.

We can regulate it via introducing a large-radius cut-off ${}^{\Lambda}/_{\sqrt{\alpha'}} >> 1$

$$\langle Z_{S^1}(\tau)\rangle_{\text{reg}} = \int_{\sqrt{\alpha'}}^{\Lambda} \frac{dR}{R} Z_{S^1}(\tau; R) \qquad \mathrm{Vol}(M_{S^1})_{\text{reg}} = \int_{\sqrt{\alpha'}}^{\Lambda} \frac{dR}{R} = \log \frac{\Lambda}{\sqrt{\alpha'}} \,. \tag{3.31}$$

Higher dimensional compactifications

We saw that for one compactified free boson (target-space S^1) both the moduli space volume and the integral of the partition function diverge. In principle, one could regulate this but now we want to show what happens in the examples that no regularisation

theory and not just the partition function. See for example ref. [57], where this is done in the context of string theory. For our purposes, we focus on the properties of the partition functions.

[5]Marginal operators are primary fields of the CFT with conformal dimension $(h, \bar{h}) = (1, 1)$.

[6]The central charge, usually denoted c, characterises the central extension of the Virasoro algebra that corresponds to the CFT. For a single boson compactified on the circle, it is equal to 1.

is needed. For this one needs at at least 3 compact bosons: this makes both the volume of the respective moduli spaces and the integral of the partition function finite.

To generalise the single compactified boson to D dimensions we have to essentially take a target-space which is a D-dimensional torus T^D. Such a torus can be constructed similarly as a two-dimensional torus but now with a D-dimensional lattice Λ_D. That is to say, $T^D \simeq \mathbb{R}^D/\Lambda_D$, where the quotient means that now points in $\mathbb{R}^D$ are identified if they are separated by a lattice vector.

To implement this, we can take D angular fields that are identified under

$$X^i \sim X^i + 2\pi, \quad i = 1, 2, ..., D \ , \tag{3.32}$$

and parametrise the target space geometry in terms of a metric G. The generalisation of the action (3.21) reads

$$S_{G,B}[X] = \frac{1}{4\pi\alpha'} \int_{\mathbb{R}^2} d^2x \left(G_{mn} \delta^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n + i B_{mn} \epsilon^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \right) \ . \tag{3.33}$$

Here $\delta^{\alpha\beta}$ is the Kronecker delta, $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol and G_{mn} is symmetric, positive-definite matrix. We have introduced a new ingredient that is part of compact scalars at dimensions $D > 1$, namely the B-field. This is a two form field which appears in a total derivative term in the action and does not alter the equations of motion. It affects the zero modes of the fields X^i, hence also the partition function. In string theory context it can be interpreted as a background field (the same interpretation can be given to the metric G). These background fields are vacuum expectation values of fields in the string spectrum. Here, we focus on the partition function of this sigma-model which reads

$$Z_{T^D}(\tau; G, B) = \frac{1}{|\eta(\tau)|^{2D}} \Theta_{H(G,B)}(0, 0, \tau) = \frac{1}{|\eta(\tau)|^{2D}} \Theta_D(m, \tau) \ , \tag{3.34}$$

in terms of the Siegel-Narain theta function defined in Appendix A.2. The sums that appear in these theta functions generalise the sum in the circle partition function (3.24). These conformal field theories are also called **Narain** CFTs.

Ensemble averaging over toroidal CFTs

In order to calculate the ensemble averages $\langle Z_{T^D}(\tau) \rangle$ of the conformal field theory partition function in terms of the world-sheet modular parameter τ, it is necessary to first determine the moduli space for the family of conformal field theories together with a measure. Making an analogy to the circle case, we need the values of the parameters G_{mn}, B_{mn} that give rise to inequivalent theories and a measure on that parameter space.

For the ensemble of Narain conformal field theories on a target-space torus T^D, the moduli space $\mathcal{M}_{T^D}$ is well-known and realized by the homogeneous space

$$\mathcal{M}_{T^D} \simeq O(D, D, \mathbb{Z}) \backslash O(D, D, \mathbb{R}) / O(D, \mathbb{R}) \times O(D, \mathbb{R}) \ . \tag{3.35}$$

A measure $d\mu(m)$ is obtained from the Zamolodchikov metric of a member $m \in \mathcal{M}_{T^D}$ in the ensemble of Narain theories. This metric is given by[7]

$$ds^2 = G^{mp} G^{nq} \left(dG_{mn} dG_{pq} + dB_{mn} dB_{pq} \right) \ . \tag{3.36}$$

[7] It is calculated by computing the two-point function of the exactly marginal operator $\mathcal{O} \sim \delta G_{mn} \delta^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n + i \delta B_{mn} \epsilon^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n$. This metric is equivalently the Haar measure of $O(D, D; \mathbb{R})$ descended to the quotient.

We will not derive (3.35) but we give some insight. The partition function of a Narain CFT (3.34) can be written in terms of sum over an even self-dual lattice of signature (D, D), $\Gamma_{D,D}$ as

$$Z_{T^D}(\tau; G, B) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(\vec{p}_L, \vec{p}_R) \in \Gamma_{D,D}} \overline{q}^{\frac{1}{2}\vec{p}_L \cdot \vec{p}_L} q^{\frac{1}{2}\vec{p}_R \cdot \vec{p}_R} , \tag{3.37}$$

where $\vec{p}_L, \vec{p}_R$ are called the left and right moving momenta respectively. They are given by (see e.g. [172] Chapter 4)

$$p^i_{L/R} = \frac{G^{ij}}{\sqrt{2}} \left(\sqrt{\alpha'} m_j + \frac{B_{jk} \mp G_{jk}}{\sqrt{\alpha'}} n_k \right) , \quad m, n \in \mathbb{Z}^D \tag{3.38}$$

$$\vec{p}_{L/R} \cdot \vec{p}_{L/R} = p^i_{L/R} G_{ij} p^j_{L/R} . \tag{3.39}$$

Now, different lattices $\Gamma_{D,D}$ give different Narain theories. These lattices can be generated by acting on a given lattice with $O(D, D)$ transformations. However rotating the left or the right moving momenta with an element of $O(D)$ gives the same theory. Also, the analogue of target-space duality on the circle (which takes $R \to \alpha'/R$) is given by the group $O(D, D; \mathbb{Z})$. Thus moding out by the last two symmetries we get the moduli space (3.35).

Observing that the moduli dependence of the partition function $Z_{T^D}(\tau; m)$ is entirely captured by its proportional Siegel–Narain Theta function $\Theta(m, \tau)$, calculating the ensemble average of the partition function amounts to averaging the Siegel–Narain Theta function $\Theta(m, \tau)$ [74, 75]. This average is determined by the Siegel–Weil formula [83, 84, 85, 86], which yields

$$\int_{\mathcal{M}_{T^D}} d\mu(m)\, \Theta(m, \tau) = \frac{E_{D/2}(\tau)}{(\mathrm{Im}\,\tau)^{D/2}} \quad \text{for} \quad D \geqslant 3 .^8 \tag{3.40}$$

Here $E_{D/2}(\tau)$ is the real analytic Eisenstein function, which is a modular function with respect to the modular parameter τ of the modular group $PSL(2, \mathbb{Z})$. It is defined as

$$E_s(\tau) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{\mathrm{Im}(\tau)^s}{|c\tau + d|^{2s}} , \tag{3.41}$$

where in the summation (c, d) denotes the greatest common divisor of the integers c and d. The real analytic Eisenstein series $E_s(\tau)$ is a modular function in τ that is defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The real analytic Eisenstein series can also be written in the following way, which will be very useful in the bulk interpretation of ensemble averages

$$E_s(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} (\mathrm{Im}\,\gamma \cdot \tau)^s . \tag{3.42}$$

Here Γ_∞ is the subset of the modular group which leaves invariant the imaginary part $\mathrm{Im}\,\tau$, and $\Gamma_\infty \backslash SL(2, \mathbb{Z})$ is the left quotient.[9] By $\gamma \cdot \tau$ we denote the action of the element γ on the modular parameter τ. The sum over modular images can be represented by

[8] For a nice derivation of this formula, via a differential equation, see ref. [74].

[9] The subgroup Γ_∞ consists of all $SL(2, \mathbb{Z})$ matrices of the form $\begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix}$. Two matrices $\gamma, \gamma' \in SL(2, \mathbb{Z})$ are considered equivalent in $\Gamma_\infty \backslash SL(2, \mathbb{Z})$ if $\gamma = h \cdot \gamma'$ for some $h \in \Gamma_\infty$. The sum in (3.42) includes one matrix from each equivalence class in $\Gamma_\infty \backslash SL(2, \mathbb{Z})$.

matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ with coprime $(c,d) = 1$. In this way we get the definition in eq. (3.41). For more details on the real Eisenstein series see for instance refs. [176, 177].

The average of the partition function $Z_{T^D}(\tau; G, B)$ can be written now as

$$\langle Z_{T^D}(\tau)\rangle = \frac{E_{D/2}(\tau)}{(\operatorname{Im}\tau)^{D/2}\,|\eta(\tau)|^{2D}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}\,. \tag{3.43}$$

To show this one can use that the product $(\operatorname{Im}\tau)^{D/2}\,|\eta(\tau)|^{2D}$ is invariant under $SL(2,\mathbb{Z})$ transformations.

Adding supersymmetry to the Narain ensemble

Making the sigma model (3.33) supersymmetric is straightforward. We add D free fermions, one partner for each bosonic field X^I. The fermions ψ^I are world-sheet spinors with spacetime indices (the spinor indices are suppressed). The supersymmetric action reads

$$S_{G,B}[X,\psi] = \frac{1}{4\pi\alpha'}\int_{\mathbb{R}^2} d^2x\left(G_{mn}\left(\delta^{\alpha\beta}\partial_\alpha X^m\partial_\beta X^n + \overline{\psi}^I\left(\gamma^\mu\partial_\mu\right)\psi^J\right) + iB_{mn}\epsilon^{\alpha\beta}\partial_\alpha X^m\partial_\beta X^n\right)\,. \tag{3.44}$$

Where γ^μ, $\mu = 1,2$ are the γ-matrices in two dimensions that satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2I_{2\times2}\delta^{\mu\nu}$, and $\overline{\psi}^I = C\psi^I$ with C the charge conjugation matrix. The latter obeys $C = -C^\top$ and $C\gamma^\mu C^{-1} = -\left(\gamma^\mu\right)^\top$.

The action (3.44) is invariant under the transformations

$$\delta X^I = \overline{\epsilon}\psi^I\,, \qquad \delta\psi^I = \left(\gamma^\mu\partial_\mu\right)X^I\epsilon\,, \tag{3.45}$$

where ϵ is a constant spinor that parametrises the supersymmetric transformations. Compactifying the fields X^I on T^D follows in a similar way as before and this does not affect supersymmetry as $\delta\psi^I$ is given only in terms of the derivative of X^I. In this way the fermionic and bosonic parts decouple in the partition function as we will see.

The partition function of supersymmetric Narain theories

Putting the above theory on the torus, we have to choose boundary conditions for the fermions. The torus has two non-trivial cycles (up to homotopy): 1 and τ.[10] The boundary conditions are labelled by two half-integers α, β:

$$\psi^I(x+\tau) = e^{2\pi i\alpha}\psi^I(x)\,, \qquad \psi^I(x+1) = e^{2\pi i\beta}\psi^I(x)\,. \tag{3.46}$$

This means that going around the $1,\tau$ cycles, we pick up a sign. The choice of boundary conditions is called a choice of **spin structure**.[11] The partition function for a such a choice labelled by α, β is given by (see e.g. refs. [57, 178])

$$Z_{T^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau; m) = Z_{\text{ferm}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)^D Z_{T^D}(\tau; m) = \frac{\left|\theta\begin{bmatrix}1/2+\beta\\ 1/2+\alpha\end{bmatrix}\right|^D}{|\eta(\tau)|^D}\frac{\Theta_D(m,\tau)}{|\eta(\tau)|^{2D}}\,, \tag{3.47}$$

[10] A cycle is a closed curved on the torus.

[11] The torus has 2 non-contractible cycles hence 2 possible choices of spin structure. For a Riemann surface of genus g this becomes 2^{2g} choices: two for every non-contractible cycle.

where we see that the partition function splits into a fermionic and bosonic part and we have defined

$$Z_{\text{ferm}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) = \frac{\left| \theta \begin{bmatrix} 1/2 + \beta \\ 1/2 + \alpha \end{bmatrix} \right|}{|\eta(\tau)|} \,. \tag{3.48}$$

The fermionic partition function has a very important modular property, namely

$$Z_{\text{ferm}} \begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix} \left(\frac{a\tau + b}{c\tau + d} \right) = Z_{\text{ferm}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) \,, \tag{3.49}$$

when acted upon by an element of the modular group. Also, the average of the full supersymmetric partition function follows easily from the non-supersymmetric case as the moduli dependence is all in the bosonic part

$$\int_{\mathcal{M}_{T^D}} d\mu(m)\, Z_{T^D} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau; m) = Z_{\text{ferm}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau)^D \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{2D}} \,. \tag{3.50}$$

3.2 Narain $\mathbb{Z}_2$ orbifolds and averaging

A toy example of an orbifold

In this section, we will explore Narain $\mathbb{Z}_2$ orbifolds and their ensemble averages. First, we talk about what an orbifold of a CFT is by sketching a basic example, namely the $\mathbb{Z}_2$ orbifold of the single compactified boson on a circle.

When the target-space (here the circle S^1) has a discrete symmetry, we can mod out this symmetry on the target-space. The resulting space is called an orbifold. Let the target-space be a Riemannian manifold (M, g), where g is a Riemannian metric. If this manifold has an isometry group G, we can take the quotient

$$\mathcal{O} = M/G \,. \tag{3.51}$$

Now the elements $x \in M$ and $G \cdot x$ are identified

$$x \sim \lambda \cdot x \,, \; \lambda \in G \,. \tag{3.52}$$

In other words, x is identified with its G-orbit. Acting with G on M can have fixed points (points that remain invariant under G) and this leads to singular points (or loci in general) in the orbifold $\mathcal{O}$.

Focusing on the free boson on the circle, the manifold is obviously S^1 and for G we choose the $\mathbb{Z}_2$ action (recall also eq. (3.20))

$$\iota : X(x) \mapsto -X(x) \,. \tag{3.53}$$

We can also visualise this procedure on the level of the manifold S^1. Identifying $X(x)$ with $-X(x)$ the circle becomes a line segment, see fig. 3.5. The points 0 and πR are invariant under the $\mathbb{Z}_2$ action. This is because $0 \mapsto -0 = 0$ and $\pi R \mapsto -\pi R \sim -\pi R + 2\pi R = \pi R$.

Starting with the CFT on the cylinder the Hilbert space of the orbifold splits into the untwisted and twisted Hilbert spaces $\mathcal{H}_+$, $\mathcal{H}_-$. The first corresponds to the usual boundary conditions on the cylinder $X(x+1) = X(x)$ and the twisted one to the twisted boundary conditions resulting from the orbifolding $X(x+1) = -X(x)$. The partition function now in the operator formalism reads

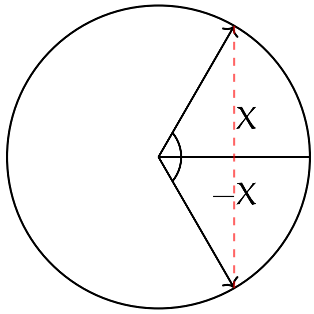

Figure 3.5: Identifying the field X with −X leaves out only the $[0, \pi]$ segment. The points $0, \pi$ are fixed points of the $\mathbb{Z}_2$ action.

$$Z_{S^1/\mathbb{Z}_2}(\tau; R) = \mathrm{tr}_{\mathcal{H}_+} \frac{1+\iota}{2} q^{L_0^{(+)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(+)} - \frac{c}{24}} + \mathrm{tr}_{\mathcal{H}_-} \frac{1+\iota}{2} q^{L_0^{(-)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(-)} - \frac{c}{24}} . \tag{3.54}$$

Where we have also labelled the Virasoro operators with $\pm$ depending on the sector and the operator $\frac{1+\iota}{2}$ projects onto $\mathbb{Z}_2$ invariant states. We label each contribution to the trace as

$$\begin{aligned}
Z^{(+,+)}_{S^1/\mathbb{Z}_2}(\tau; R) &= \mathrm{tr}_{\mathcal{H}_+} q^{L_0^{(+)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(+)} - \frac{c}{24}} = Z_{S^1}(\tau; R) , && \text{untwisted-no insertion} \\
Z^{(+,-)}_{S^1/\mathbb{Z}_2}(\tau) &= \mathrm{tr}_{\mathcal{H}_+} \iota q^{L_0^{(+)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(+)} - \frac{c}{24}} = 2 \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right| , && \text{untwisted-insertion} \\
Z^{(-,+)}_{S^1/\mathbb{Z}_2}(\tau) &= \mathrm{tr}_{\mathcal{H}_-} q^{L_0^{(-)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(-)} - \frac{c}{24}} = 2 \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right| , && \text{twisted-no insertion} \\
Z^{(-,-)}_{S^1/\mathbb{Z}_2}(\tau) &= \mathrm{tr}_{\mathcal{H}_-} \iota q^{L_0^{(-)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(-)} - \frac{c}{24}} = 2 \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right| , && \text{twisted-insertion} .
\end{aligned} \tag{3.55}$$

The twisted-insertion has no radius dependence because the operator ι restricts the trace only to the state $|0, 0\rangle$. The twisted sector contributions restrict the theory to the fixed points of the orbifold $0, \pi R$ hence the factor of 2 and the absence of radius dependence.

From the functional integral point of view with the theory on the torus, the orbifolding introduces new possible boundary conditions for the field X when we go around the non-trivial cycles (up to homotopy) of the torus. Namely, we can label by a pair $(r, s),\ r, s = +, -$ depending on which cycle gets a minus sign

$$\begin{aligned}
(+,+):&\ X(x + k \cdot 1 + \ell \cdot \tau) = X(x) ,\ k, \ell \in \mathbb{Z} \\
(+,-):&\ X(x + k \cdot 1 + \ell \cdot \tau) = (-1)^{\ell} X(x) ,\ k, \ell \in \mathbb{Z} \\
(-,+):&\ X(x + k \cdot 1 + \ell \cdot \tau) = (-1)^{k} X(x) ,\ k, \ell \in \mathbb{Z} \\
(-,-):&\ X(x + k \cdot 1 + \ell \cdot \tau) = (-1)^{k+\ell} X(x) ,\ k, \ell \in \mathbb{Z} .
\end{aligned} \tag{3.56}$$

When we calculate the partition function, we sum over all possible boundary conditions and the path integral formally looks like

$$Z_{S^1/\mathbb{Z}_2}(\tau; R) = \frac{1}{2} \sum_{a,b=+,-} \int_{(a,b)} \mathcal{D}X e^{-S[X]} . \tag{3.57}$$

Where $\int_{(a,b)}$ means that the path integral is done with the field X obeying (a, b) boundary conditions. Of course these have to be combined with possible winding and momentum sectors. The normalisation factor $2 = |\mathbb{Z}_2|$ is there to avoid over-counting.

The full partition functions reads

$$Z_{S^1/\mathbb{Z}_2}(\tau;R) = \frac{1}{2}\left(Z_{S^1}(\tau;R) + 2\left[\left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|\right]\right) . \tag{3.58}$$

As we can see also from Appendix A.1, the part of the partition function with the $\theta_r(\tau),\ r = 2,3,4$ is modular invariant by itself as the theta functions transform into each other via modular transformations.

More generally on orbifolds

The above structure is generalised for abelian orbifold groups G of finite order $|G|$. Now, we get a twisted sector for each element $p \in G$ and the projection operator is

$$P_G = \frac{1}{|G|}\sum_{p\in G} p . \tag{3.59}$$

Twisted sectors correspond to boundary conditions on the cylinder $X(x+1) = p\cdot X(x)$. In the $\mathbb{Z}_2$ example above there was just one group element (other than the identity) and the group action on X gave a minus sign. The partition function is a sum over all twisted sectors, with corresponding Hilbert spaces $\mathcal{H}_p$

$$Z_{\mathcal{M}/G}(\tau) = \sum_{p\in G}\frac{1}{|G|}\sum_{g\in G}\mathrm{tr}_{\mathcal{H}_p}\left(g q^{L_0^{(p)}-\frac{c}{24}}\bar{q}^{\bar{L}_0^{(p)}-\frac{c}{24}}\right) = \frac{1}{|G|}\sum_{p,g\in G} Z^{(p,g)}(\tau) .^{12} \tag{3.60}$$

Under modular transformations the contributions transform as

$S:\ Z^{(p,g)}(\tau) \mapsto Z^{(g,p)}(\tau)$, because it exchanges the torus cycles

$T:\ Z^{(p,g)}(\tau) \mapsto Z^{(p,pg)}(\tau)$, because it shifts the insertion-direction by a twist-direction.

From the path integral point of view we have to sum over all possible boundary conditions when the field goes around a non-trivial cycle of the torus, which for now we denote by Σ. This is equivalent to saying that we have to sum over G-bundles over Σ. To do so, we have to choose a monodromy for each element of the fundamental group $\pi(G)$ of G.[13] This can be seen as a group homomorphism $\rho : \pi_1(\Sigma) \to G$. The contributions $Z^{(p,g)}(\tau)$ correspond to the monodromies around the two non-trivial cycles of the torus Σ. Specifically,

$$Z^{(p,g)}(\tau) \text{ corresponds to } X(x+1) = p\cdot X(x),\ X(x+\tau) = g\cdot X(x) . \tag{3.61}$$

The partition function is formally written as

$$Z_{\mathcal{M}/G}(\tau) = \frac{1}{|G|}\sum_{\rho\in \mathrm{Hom}(\pi_1(\Sigma),G)}\int_\rho \mathcal{D}X e^{-S[X]} , \tag{3.62}$$

where $\int_\rho$ means that the field X, when transported around $\gamma \in \pi_1(\Sigma)$ it goes back to $\rho(\gamma)\cdot X$. Since $\pi_1(\Sigma) = \mathbb{Z}\oplus\mathbb{Z}$ for the torus, we get a sum like eq. (3.60). Note that the

[12] For non-abelian groups the sum is restricted to p, g that commute to ensure well-definiteness of boundary conditions. We will see such an example is section 3.3.

[13] Let γ be an element of $\pi_1(G)$. Choosing a monodromy amounts to picking an element $g_\gamma \in G$ such that when the field is transported around γ the element g_γ acts on it: $X(\gamma\cdot x) = g_\gamma \cdot X(x)$. For a nice discussion on this see, e.g. ref [168].

fact that ρ is a group homomorphism ensures that p, g commute. This has to do with the fact that π_1 of the two-torus is abelian, hence from

$$\rho(\gamma_1 \cdot \gamma_2) = \rho(\gamma_2 \cdot \gamma_1) \ , \ \forall \gamma_1, \gamma_2 \in \pi_1(\Sigma) \tag{3.63}$$

we get $\rho(\gamma_1) \cdot \rho(\gamma_2) = \rho(\gamma_2) \cdot \rho(\gamma_1)$. In this example where the orbifold group is also abelian this is a not so surprising statement but as we will see later, for non-abelian G, the fact that ρ is a group homomorphism and π_1 abelian for the two-torus makes the sum be over commuting elements of G.

It can happen that $Z^{(p,g)}(\tau)$ form sectors in the sense that they transform into each other in groups without mixing. Then the partition function includes phases $\epsilon_{p,g}$ called discrete torsion [179, 180]

$$Z_{\mathcal{M}/G}(\tau) = \frac{1}{|G|} \sum_{p,g \in G} \epsilon_{p,g} Z^{(p,g)}(\tau) \ . \tag{3.64}$$

These are fixed, sometimes not completely, by demanding modular invariance.

The same orbifold in higher dimensions

The circle orbifold we described above can be easily generalised to a T^D target-space since every D-dimensional torus has the $\mathbb{Z}_2$ symmetry where all coordinates are multiplied by a $-$. What will change is of course that instead of a circle partition function we have a T^D partition function and the number of fixed points which is 2^D

$$Z_{T^D/\mathbb{Z}_2}(\tau; m) = \frac{1}{2}\left(Z_{T^D}(\tau; m) + 2^D \left[\left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|^D + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right|^D + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|^D \right] \right) \ . \tag{3.65}$$

The moduli space and averaging

Since the $\mathbb{Z}_2$ symmetry is there for any T^D, or any lattice Λ_D, the moduli space of the orbifold is the same as the non-orbifolded theory. Normalising the volume of the moduli space to 1 (take $D > 2$), we have [166]

$$\langle Z_{T^D/\mathbb{Z}_2}(\tau) \rangle = \frac{1}{2}\left(\langle Z_{T^D}(\tau) \rangle + 2^D \left[\left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|^D + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right|^D + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|^D \right] \right) \ . \tag{3.66}$$

Considering other orbifolds could change this statement and the moduli space can change. For example, ensembles of $\mathbb{Z}_N$, $N = 2, 3, ...$ were considered in ref. [166]. There not every torus has a $\mathbb{Z}_N$ symmetry and this restricts the space of possible target-spaces. While in ref. [166] the studied $\mathbb{Z}_N$ orbifolds act uniformly on all torus directions, in [2] we focus on $\mathbb{Z}_2$ orbifolds of toroidal conformal field theories whose geometric action on their target space tori is more general. These more general classes of $\mathbb{Z}_2$ orbifold theories fall also into the ensembles considered in the interesting work [181], which discusses ensembles of theories resulting from Narain lattices with arbitrary signatures and orbifolds thereof from a more general but less geometric point of view.

3.2.1 Two-dimensional factorizable and non–factorisable Toroidal $\mathbb{Z}_2$ Orbifold CFTs

In this section, we elaborate on the toroidal orbifolds studied in our work [2]. We start with a two-dimensional target-space and then generalise to D dimensions. The two-dimensional ensemble averaging is not as rigorous because the averaged quantities need to be regularised. However, this gives valuable insight into the higher dimensional case.

Two-dimensional toroidal target-spaces

Before going in to the details of the orbifolds, we define some quantities for the two dimensional CFT of two free bosons on T^2. In this case, it is convenient to express the target-space also in terms of a complex structure. For this, we write the T^2 as

$$T^2 \simeq \mathbb{C}/(\mathbb{Z}+u\mathbb{Z}) , \tag{3.67}$$

in terms of the complex structure parameter u in the upper half-plane $\mathcal{H}$, i.e., $\operatorname{Im} u > 0$. $\mathbb{Z}+u\mathbb{Z}$ denotes the lattice generated by 1 and u.

For the target-space, it is also convenient to introduce the complexified Kähler modulus which combines the volume modulus k of the torus and the anti-symmetric B-field

$$t = \frac{1}{\alpha'}\left(B + ik\right) := b + i\kappa .^{14} \tag{3.68}$$

The quantities b, κ are dimensionless and the modulus t lives also on the upper half-plane $\mathbb{H}$ (since the volume is positive). The space of T^2 CFTs is now described in terms of the two parameters u, t.

The moduli space of T^2 conformal field theories

When we discussed the CFTs with target T^D, we saw that the moduli space is given by the space (3.35). For the case $D = 2$, this can be shown to be isomorphic to (see, e.g. [57])

$$\mathcal{M}_{T^2} = \left(\mathbb{H}/\mathrm{PSL}\left(2,\mathbb{Z}\right) \times \mathbb{H}/\,\mathrm{PSL}\left(2,\mathbb{Z}\right)\right) / \left(\mathbb{Z}_2 \times \mathbb{Z}_2\right) . \tag{3.69}$$

As we explained in section 3.1, the quotient $\mathbb{H}/\mathrm{PSL}\left(2,\mathbb{Z}\right)$ is the fundamental domain F_0 of the complex structure moduli space of a two-dimensional torus. So the moduli space of a CFT with target T^2 is two copies of F_0 modded out by a discrete group action. One $\mathbb{Z}_2$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts by exchanging t with u and the other by mapping $t \mapsto -\bar{t}, u \mapsto -\bar{u}$. These are of course also symmetries of the partition function $Z_{T^2}(\tau; m)$ (recall m denotes a choice of u, t). In particular, from eq. (3.69), we see that the theory is invariant under modular transformations acting individually either on u and/or t. This symmetry is not to be confused with the modular invariance of the base space, i.e. the torus where the CFT is defined on. We now define the T^2 orbifolds that we work with and then we will see how this moduli space is modified (reduced to a locus of) for these orbifolds.

Orbifolds of T^2 CFTs

These are generated by the $\mathbb{Z}_2$ involution of the torus $\iota_{\mathbb{Z}_2} : T^2 \to T^2$, i.e., $\iota_{\mathbb{Z}_2}^2 = \mathbf{1}$, which induces a $\mathbb{Z}_2$ action on the two-form B-field via pullback $\iota_{\mathbb{Z}_2}^*$.

We do not consider fixed-point free involutions $\iota_{\mathbb{Z}_2}$—referred to as shift orbifolds in ref. [182]—because the associated free $\mathbb{Z}_2$ action simply yields another toroidal CFT of T^2 (with adjusted background parameters u and t).

[14] The volume (Kähler) modulus of a torus is proportional to its volume. This is the area of the parallelogram that defines the torus on the plane. We need this quantity because the partition function depends both on the shape and volume of the target T^2. The complexified Kähler modulus is a nice way to pack the information of the B-field and the volume.

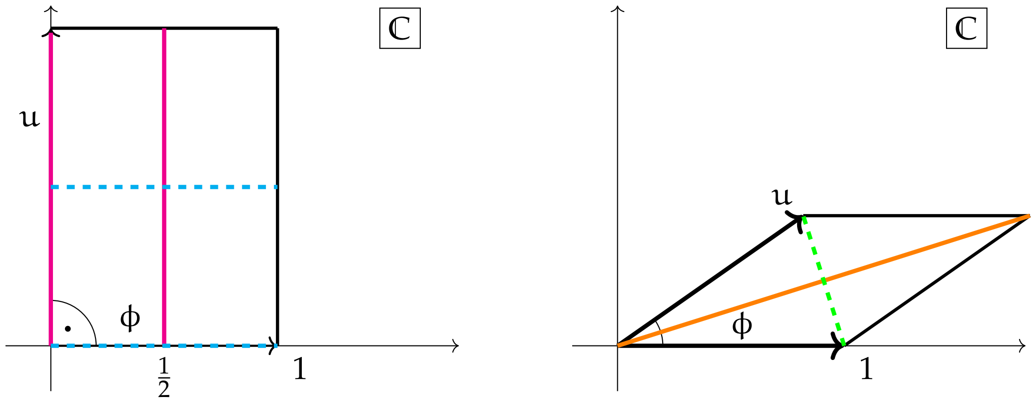

Figure 3.6: Factorizable and non-factorizable lattices. On the left, the two fixed point loci of the reflection along each lattice generator are depicted with solid magenta and dashed cyan lines. On the right, the symmetry axes along the diagonal lines are depicted with solid orange and dashed green lines.

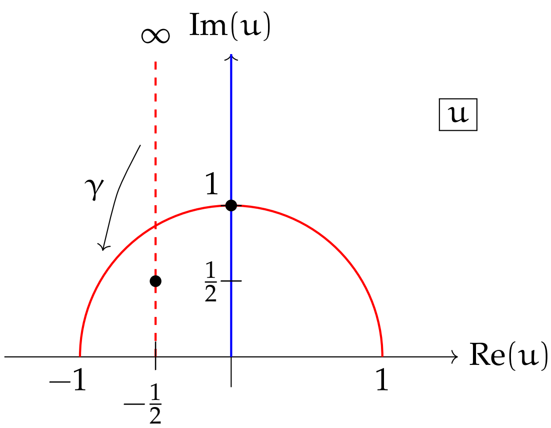

Figure 3.7: Complex structure plane. The blue line and the solid red arc correspond, respectively, to the factorizable and non-factorizable lattice. The dashed red line is mapped to the red arc by the modular transformation $\gamma(\tilde{u}) = {}^{\tilde{u}}/_{\tilde{u}+1}$, where $\tilde{u}$ is on the $\mathrm{Re}\,u = -{}^1/_2$ half-line (see item 2 on page 66). The two black dots are mapped to each other under γ. The point $\tilde{u} \to \infty$ is mapped to 1 and the point $\tilde{u} \to -{}^1/_2$ to -1.

Involutions $\iota_{\mathbb{Z}_2}$ with non-trivial fixed points arise from reflections about points and lines, which must respectively be symmetry points and symmetry axes of the corresponding torus T^2. As the lattice point $z \in \mathbb{Z} + u\mathbb{Z}$ is always a symmetry point of reflection with respect to the origin in T^2, the associated toroidal $\mathbb{Z}_2$ orbifold is well-defined for any choice of moduli u and t. This is the case for the circle and T^D orbifolds we studied in section 3.2. Such orbifold conformal field theories and their ensembles are studied in detail in ref. [166], and we will not further analyse them here.

We focus on involutions relating to reflections along a line in T^2. The line has to be a symmetry axis of the torus in order for this to be possible and this requirement imposes constraints on the moduli u, t of the target-space. We distinguish between classes of symmetry axes of two-dimensional tori

1. Reflections along an axis associated to a lattice generator of $\mathbb{Z} + u\mathbb{Z}$: Factorisable tori.

   This amounts to reflecting along the bounding edge of a primitive cell of the torus lattice. For this reflection to be a symmetry, the primitive cell has to be a rectangle (see left of fig. 3.6). We can reflect with respect to either the vertical or the horizontal axis (magenta and cyan respectively). In both cases we get two fixed point loci. This construction constraints the complex structure u to be purely imaginary $\mathrm{Re}u = 0$. Note that on the level of $\mathcal{M}_{T^2}$, the sets $\mathrm{Re}u \in [1, \infty)$ and $\mathrm{Re}u \in (0, 1]$ are equivalent as they are related by an S transformation (see eq. (3.69)).

2. Reflections along an axis associated to a lattice vector: Non-factorisable tori.

   This vector can be the sum or the difference of the generators of the lattice $\mathbb{Z} + u\mathbb{Z}$ (see right of fig. 3.6). For this reflection to be a symmetry, the primitive cell of the torus lattice ought to be a rhombus. This construction imposes that u is a pure phase $|u| = 1$. Note that target space S transformations identify the angles ϕ with $\pi - \phi$ by sending $e^{i\phi}$ to $-e^{-i\phi}$. This restricts, on the level of $\mathcal{M}_{T^2}$, ϕ to $(0, \pi/2]$.

   Actually, $\mathrm{Re}\,\tilde{u} = -\frac{1}{2}$ also realizes non-factorizable tori because the lattice generators $\tilde{u}$ and $\tilde{u} + 1$ form a rhombus in this case. These two sides of the rhombus are exchanged by a reflection along the imaginary axis. This alternative description is often used in the literature and it is related to $|u| = 1$ via the modular transformation $u = \tilde{u}/\tilde{u}+1$ in complex structure moduli space (see Fig. 3.7).

The induced action on the B-field

The involution $\iota_{\mathbb{Z}_2}$ for the reflections of either factorizable tori or non-factorizable tori induces a $\mathbb{Z}_2$-action on the B-field (via the pull-back $\iota^*_{\mathbb{Z}_2}$), which maps $b = \mathrm{Re}\,t$ to $-b$. Due to the periodicity of the B-field, $b \sim b + 1$, there are two possible invariant choices for the background value of b, namely $b = 0$ or $b = \frac{1}{2}$.[15] Hence, with $\mathrm{Re}\,t = 0$ and $\mathrm{Re}\,t = \frac{1}{2}$ there are two possible classes of background values for the complexified Kähler modulus t, which admit the discussed $\mathbb{Z}_2$ orbifold action. The fact that there are two possibilities for the Kähler modulus is not a coincidence as mirror symmetry of T^2 maps the factorizable torus in complex structure moduli space to a configuration with vanishing B-field in the complexified Kähler moduli space, and the non-factorizable torus in complex structure to a half-integral B-field in Kähler moduli space, and vice versa.

The moduli space of the orbifolds

The moduli space $\mathcal{M}_{T^2/\mathbb{Z}_2}$ of the family of toroidal $\mathbb{Z}_2$ orbifold conformal field theories associated to the involution $\iota_{\mathbb{Z}_2}$ is the subspace of the moduli space $\mathcal{M}_{T^2}$, which parametrizes those two-dimensional tori $T^2_{(u,t)}$ that admit the involution $\iota_{\mathbb{Z}_2}$ as a $\mathbb{Z}_2$ symmetry, i.e.,

$$\mathcal{M}_{T^2/\mathbb{Z}_2} = \left\{ (u,t) \in \mathcal{M}_{T^2} \,\middle|\, \exists\, \iota_{\mathbb{Z}_2} \text{ on } T^2_{(u,t)} \right\} \subset \mathcal{M}_{T^2} \,. \tag{3.70}$$

Here $T^2_{(u,t)}$ denotes for a given $(u,t) \in \mathcal{M}_{T^2}$ the two-dimensional torus with complex structure u and Kähler structure t.

Factorisable $\mathbb{Z}_2$ orbifolds with vanishing B-field

We focus on T^2 factorisable orbifolds with vanishing B-field. As explained above, the primitive cell of T^2 for these orbifolds is rectangular. The involution $\iota_{\mathbb{Z}_2}$ acts by reflecting along the imaginary axis of the complex plane $\mathbb{C}$. Note that reflection along the real axis forms another involution of such a factorizable torus. However, the orbifold theories arising from either one of these involutions are equivalent because upon rotating and

[15]The periodicity is given by the T transformation of the moduli space (3.69).

conformally rescaling the lattice $\mathbb{Z}+u\mathbb{Z}$ of the two-dimensional torus to the conformally equivalent lattice $\mathbb{Z}+u^{-1}\mathbb{Z}$ exchanges these two involutions and the conformal field theory on $\mathbb{Z}+u^{-1}\mathbb{Z}$ is equivalent to the one on $\mathbb{Z}+u\mathbb{Z}$. This follows from target space $\mathrm{PSL}(2,\mathbb{Z})$ invariance.

The complex and volume moduli are purely imaginary and we denote them as $u = ic,\ t = i\kappa$. These moduli live on the fundamental domain F_0, hence $1 \leqslant c < +\infty$, $1 \leqslant \kappa < +\infty$. One of the $\mathbb{Z}_2$ in the quotient (3.69) is trivial in this case as it takes u, t to themselves. The other $\mathbb{Z}_2$ exchanges the two. The moduli space reads

$$\mathcal{M}_{T^2/\mathbb{Z}_2} \simeq [1,+\infty)^2/\mathbb{Z}_2 \subset \mathcal{M}_{T^2} \ . \tag{3.71}$$

Where $\mathbb{Z}_2$ is the exchange of u, t.

We can parametrise the target T^2 as a Cartesian product of two circles of radii R_1, R_2(one corresponding to the real part and the other to the imaginary part). The lattice that corresponds to this torus is generated by $e_1 = (R_1, 0)$ and $e_2 = (0, R_2)$. From these we can calculate the complex structure and volume modulus in terms of the radii

$$\kappa = \frac{R_1 R_2}{\alpha'} \ , \qquad c = \frac{R_2}{R_1} \ . \tag{3.72}$$

It is evident that κ is proportional to the volume (area) of the rectangle with edges e_1, e_2 and the complex structure measure the relative length of the edges.

This orbifold factorises as

$$T^2/\mathbb{Z}_2 \simeq S^1/\mathbb{Z}_2 \times S^1 \ , \tag{3.73}$$

where the first and second factors on the right hand side are parametrized in terms of the real and imaginary part axes, respectively. Denoting z an element of the universal covering space of T^2, $\mathbb{C}$, this orbifold acts as

$$\iota_{\mathbb{Z}_2} : z = \mathrm{Re}z + i\mathrm{Im}z \mapsto z' = -\mathrm{Re}z + i\mathrm{Im}z \ . \tag{3.74}$$

The partition function then splits into a product of a circle partition function (corresponding to the S^1) and a circle orbifold part (corresponding to the $S^1/\mathbb{Z}_2$)

$$Z_{T^2_{\text{fac}}/\mathbb{Z}_2}(\tau; c, \kappa) = Z_{S^1/\mathbb{Z}_2}(\tau; R_1(c,\kappa))\, Z_{S^1}(\tau; R_2(c,\kappa)) \ . \tag{3.75}$$

Here, we have expressed the partition function in terms of the complex structure and Kähler modulus of the target-space c, κ. Note that this partition function is invariant under the exchange of κ and c which corresponds to T-duality on R_1. Also, it is invariant under T-duality of R_2 which corresponds to the transformation $c \mapsto 1/\kappa,\ \kappa \mapsto 1/c$. The simultaneous R_1, R_2 T-duality is the map $c \mapsto 1/c, \kappa \mapsto 1/\kappa$.

Ensembles of two-dimensional orbifolds

The moduli space $\mathcal{M}_{T^2}$ of the conformal field theory of two free bosons with a two-dimensional torus T^2 as its target-space may be locally parameterized in terms of the complex structure modulus u and the complexified Kähler modulus t. The two-point correlators of the exactly marginal operators define the Weil–Petersson metric on the moduli space $\mathcal{M}_{T^2}$ [175], which for the toroidal conformal field theory is locally a product of two two-dimensional hyperbolic spaces with the metric (up to a constant prefactor)

$$ds^2 = \frac{du\, d\bar{u}}{(\mathrm{Im}\, u)^2} + \frac{dt\, d\bar{t}}{(\mathrm{Im}\, t)^2} \ . \tag{3.76}$$

This comes from the fact that the moduli space of T^2 compactifications is locally a product of two copies of the fundamental domain, see eq. (3.69). One may explicitly arrive at the above expression (up to normalization) by calculating the Zamolodchikov metric (3.36) and plugging in the metric and the B-field in terms of the complex structure and complexified Kähler moduli u, t.[16] The moduli spaces $\mathcal{M}_{T^2/\mathbb{Z}_2}$ of the analyzed $\mathbb{Z}_2$ orbifold conformal field theories are subspaces of $\mathcal{M}_{T^2}$. The Weil–Petersson metric of the moduli space $\mathcal{M}_{T^2/\mathbb{Z}_2}$ is the induced metric from the metric (3.76), because the exactly marginal operators in the untwisted sector of the orbifold theories have the same two-point correlation functions as in the unorbifolded theory.

Ensembles of factorizable $\mathbb{Z}_2$ orbifold with vanishing B-field

The parameters of this $\mathbb{Z}_2$ orbifold are the two real moduli (c, κ) which we defined two paragraphs ago. A fundamental domain of the moduli (c, κ) reads

$$(c, \kappa) \in [1, \infty) \times [1, \infty) \,.^{17} \tag{3.77}$$

The measure of the moduli induced from eq. (3.76) becomes

$$ds^2 = \left(\frac{dc}{c}\right)^2 + \left(\frac{d\kappa}{\kappa}\right)^2 \,. \tag{3.78}$$

The volume of the moduli space of this $\mathbb{Z}_2$ orbifold theory is logarithmically divergent. Note that for this class of conformal field theories, the ensemble average over the entire moduli space is divergent as well [74]. Nevertheless, we can still study ensemble averages over measurable subsets of the moduli space by, for instance, regularizing the integral with a large dimensionless cut off λ for large (and small) values of the moduli c and κ, like in eq. (3.31). Using the partition function (3.75), we arrive at

$$\begin{aligned}\left\langle Z_{T^2_{\text{fac}}/\mathbb{Z}_2}(\tau)\right\rangle_{\text{reg}} = \frac{1}{V_{\text{reg}}}\int_{1/\lambda}^{\lambda}\frac{dc}{c}\int_{1/\lambda}^{\lambda}\frac{d\kappa}{\kappa}\bigg\{&\frac{1}{2}Z_{S^1}(\tau;R_1(c,\kappa))\,Z_{S^1}(\tau;R_2(c,\kappa))+ \\ &\left(\left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|+\left|\frac{\eta(\tau)}{\theta_3(\tau)}\right|+\left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|\right)Z_{S^1}(\tau;R_2(c,\kappa))\bigg\}\,,\end{aligned} \tag{3.79}$$

in terms of the regularized moduli space volume

$$V_{\text{reg}} = \int_{1/\lambda}^{\lambda}\frac{dc}{c}\int_{1/\lambda}^{\lambda}\frac{d\kappa}{\kappa} = 8\left(\text{Vol}(M_{S^1})_{\text{reg}}\right)^2$$

defined similarly as in eq. (3.31). Here we have integrated over a four-fold cover of the moduli space and divided by the respective volume. Using the regularised circle ensemble average of the same equation, we express the regularized ensemble average $\left\langle Z_{T^2_{\text{fac}}/\mathbb{Z}_2}(\tau)\right\rangle_{\text{reg}}$ as [2]:

$$\left\langle Z_{T^2_{\text{fac}}/\mathbb{Z}_2}(\tau)\right\rangle_{\text{reg}} = \frac{1}{2}\left\langle Z_{S^1}(\tau)\right\rangle^2_{\text{reg}} + \left(\left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|+\left|\frac{\eta(\tau)}{\theta_3(\tau)}\right|+\left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|\right)\left\langle Z_{S^1}(\tau)\right\rangle_{\text{reg}} \,. \tag{3.80}$$

The first summand is simply the (regularized) ensemble average of the tensor product of two circular conformal field theories, whereas the second contribution comes from the circular fixed-point loci of the $\mathbb{Z}_2$ orbifold. The regularised averaged are defined as in eq. (3.31) with suitable redefinitions of the cut offs.

[16]See for example chapter 10 in ref.[57] for an explicit relation between u, t and the metric and B-field of the T^2 target-space.

[17]In principle, we should divide out by the exchange of c and κ but this will not make any difference as it only leads to extra overall numerical factors that are absorbed in the normalisation.

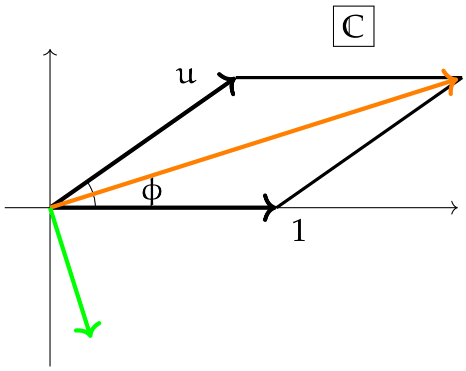

Figure 3.8: The lattice spanned by the orange and green vectors yields a double cover of the non-factorizable torus lattice with complex structure modulus u.

Non-factorisable $\mathbb{Z}_2$ orbifolds with vanishing B-field

Let us now focus on the case of non-factorisable $\mathbb{Z}_2$ orbifolds with $B = 0$. As mentioned in page 66, such tori can be described with a complex structure modulus u that is a pure phase, i.e., $u = e^{i\phi}$ with $\phi \in \mathbb{R}$. We consider the $\mathbb{Z}_2$ orbifold corresponding to the involution $\iota_{\mathbb{Z}_2}$ that reflects the points on the torus along the diagonal of the primitive cell, which is the line through the origin and the lattice points $u + 1$ (c.f., the solid orange diagonal in the right panel of Fig. 3.6).[18] This family of $\mathbb{Z}_2$ orbifold conformal field theories is parametrised by the complex structure and Kähler moduli of the form

$$u = e^{i\phi}\,, \qquad t = i\kappa\,, \qquad \phi \in (0, \tfrac{\pi}{2}]\,, \qquad 1 \leqslant \kappa < \infty\,, \tag{3.81}$$

which corresponds to the moduli space

$$\mathcal{M}_{T^2/\mathbb{Z}_2} \simeq \big((0, \tfrac{\pi}{2}] \times [1, +\infty)\big) \subset \mathcal{M}_{T^2}\,. \tag{3.82}$$

Note that for non-factorizable tori the lattice points 1 and $u = e^{i\phi}$ yield two circles S^1 of equal radii R and hence equal circumferences $2\pi R$. We can think of this torus as being generated by the lattice spanned by the vectors $e_2 = R(\cos\phi, \sin\phi)$ and $e_1 = R(1, 0)$. Their radii R relate to the real angular complex structure parameter ϕ and the real Kähler modulus κ as

$$\kappa = \tfrac{1}{\alpha'} R^2 \sin\phi\,. \tag{3.83}$$

Again, this is proportional to the area of the primitive cell. Furthermore, these two circles form representatives of the homology classes generating $H_1(T^2, \mathbb{Z})$, which get exchanged by the involution $\iota_{\mathbb{Z}_2}$. As a consequence the sum of these two homology cycles yields an invariant homology class with respect to the involution $\iota_{\mathbb{Z}_2}$. This invariant class can be represented by the diagonal circle of T^2, which is the fixed-point locus of the involution.

There are various ways to calculate the partition function. For instance, one can view the orbifold as a shift orbifold [182] of a factorisable torus target space or use the methods of [183]. Perhaps the quickest is to calculate the untwisted sector traces and

[18]The $\mathbb{Z}_2$ orbifold theory corresponding to the other involution of the non-factorizable torus—arising from the reflection along the diagonal through the points 1 and u (the dashed green line in Fig. 3.6)—is equivalent to the $\mathbb{Z}_2$ orbifold theory attributed to the involution of the first type. This can explicitly be seen by noting that the torus associated to the lattice $-\mathbb{Z} + e^{i\phi}\mathbb{Z} \simeq e^{i\phi}(\mathbb{Z} + e^{i(\pi-\phi)}\mathbb{Z})$ is equivalent to the rotated lattice $\mathbb{Z} + e^{i(\pi-\phi)}\mathbb{Z}$. As this particular conformal transformation exchanges the two described involutions, the associated toroidal $\mathbb{Z}_2$ orbifold conformal field theories are equivalent.

then act with modular transformations to deduce the rest contributions.[19] Denoting the contributions with $(\pm,\pm)$ like in page 61, we write the partition function as

$$Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa) = \frac{1}{2}\left(Z^{(++)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa) + Z^{(+-)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa)\right) + \frac{1}{2}\left(Z^{(-+)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa) + Z^{(--)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa)\right) . \tag{3.84}$$

Were we make explicit in the argument the dependence on the angle ϕ and the Kähler modulus κ. To calculate the $(+,+)$ contribution, we plug in the metric calculated from the target-space lattice into expression (A.10). The metric reads

$$G = R^2 \begin{pmatrix} 1 & \cos\phi \\ \cos\phi & 1 \end{pmatrix} . \tag{3.85}$$

The $(+,+)$ contribution can then be massaged into the following expression

$$\begin{aligned}\frac{1}{2}Z^{(++)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa) &= \frac{1}{4}\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}(2\tau;\tfrac{R_1}{\sqrt{2}})\, Z_{S^1}(2\tau;\tfrac{R_2}{\sqrt{2}}) \\ &+ \frac{1}{4}\left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_1}{\sqrt{2}})\, Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_2}{\sqrt{2}}) + \frac{1}{4}\left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}(\tfrac{\tau+1}{2};\tfrac{R_1}{\sqrt{2}})\, Z_{S^1}(\tfrac{\tau+1}{2};\tfrac{R_2}{\sqrt{2}}) \\ &- \frac{1}{2}Z_{S^1}(\tau;R_1)\, Z_{S^1}(\tau;\tfrac{R_2}{2}) - \frac{1}{2}Z_{S^1}(\tau;\tfrac{R_1}{2})\, Z_{S^1}(\tau;R_2) ,\end{aligned} \tag{3.86}$$

where

$$R_1 = \sqrt{2\alpha'\kappa\cot\frac{\phi}{2}} , \qquad R_2 = \sqrt{2\alpha'\kappa\tan\frac{\phi}{2}} . \tag{3.87}$$

R_1 corresponds to the orange segment of Fig. 3.8 and R_2 to the green segment of the same figure.[20]

This is a rather tedious manipulation done in ref.[2]. One can do this by first splitting the sum over momenta and windings into sectors (even and odd integers). When all are even it is easy to show that one gets the 2τ contribution above. Then by acting with modular transformations one can make an educated guess about the result and prove it by analysing the rest sectors.

For the contributions $(+,-),(-,-),(-,+)$, we calculate the trace in the untwisted sector with insertion $Z^{(+-)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa)$. This is essentially done by restricting to $\mathbb{Z}_2$ eigenstates in the trace of the $(+,+)$ and gives

$$Z^{(+-)}_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa) = \frac{1}{2}Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_1}{\sqrt{2}}) . \tag{3.88}$$

Note that this depends only on the R_1 length, which is the fixed locus of the orbifold. the full partition function reads

[19] When the CFT has discrete torsion this is more subtle as the group action might have separately closed orbits.

[20] R_1 and R_2 can be also written as $\frac{R_1}{\sqrt{2}} = R\sqrt{(1+\cos\phi)}$ and $\frac{R_2}{\sqrt{2}} = R\sqrt{(1-\cos\phi)}$.

$$
\begin{aligned}
Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa) &= \frac{1}{4}\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}(2\tau;\tfrac{R_1}{\sqrt{2}})\, Z_{S^1}(2\tau;\tfrac{R_2}{\sqrt{2}}) \\
&+ \frac{1}{4}\left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_1}{\sqrt{2}})\, Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_2}{\sqrt{2}}) + \frac{1}{4}\left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2 Z_{S^1}(\tfrac{\tau+1}{2};\tfrac{R_1}{\sqrt{2}})\, Z_{S^1}(\tfrac{\tau+1}{2};\tfrac{R_2}{\sqrt{2}}) \\
&- \frac{1}{2} Z_{S^1}(\tau;R_1)\, Z_{S^1}(\tau;\tfrac{R_2}{2}) - \frac{1}{2} Z_{S^1}(\tau;\tfrac{R_1}{2})\, Z_{S^1}(\tau;R_2) \\
&+ \frac{1}{2} Z_{S^1}(2\tau;\tfrac{R_1}{\sqrt{2}}) + \frac{1}{2} Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_1}{\sqrt{2}}) + \frac{1}{2} Z_{S^1}(\tfrac{\tau+1}{2};\tfrac{R_1}{\sqrt{2}})\,,
\end{aligned} \tag{3.89}
$$

where the last two terms of the last line are calculated from (3.88) via modular transformations.

The case $\pi/2$

For $\phi = \pi/2$, we see that the target-space torus looks like the factorisable case (see fig. 3.6). In particular, the radii R_1, R_2 become equal. However the orbifolds are not the same because now the $\mathbb{Z}_2$ action exchanges these two equal circles. In the terminology of ref. [3] the conformal field theory simplifies to the $S_2 \simeq \mathbb{Z}_2$ orbifold conformal field theory of the product of two circles. Namely, for $\frac{1}{\sqrt{2}}R_1 = \frac{1}{\sqrt{2}}R_2$ the partition function (3.89) becomes

$$
\begin{aligned}
&Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;0,\tfrac{R_1^2}{2\alpha'}) = \\
&\qquad \frac{1}{2} Z_{S^1}(\tau;\tfrac{R_1}{\sqrt{2}})^2 + \frac{1}{2}\left(Z_{S^1}(2\tau;\tfrac{R_1}{\sqrt{2}}) + Z_{S^1}(\tfrac{\tau}{2};\tfrac{R_1}{\sqrt{2}}) + Z_{S^1}(\tfrac{\tau+1}{2};\tfrac{R_1}{\sqrt{2}})\right)\,,
\end{aligned} \tag{3.90}
$$

which is indeed in agreement with the partition function of the S_2 symmetric orbifold conformal field theories studied ref. [3], where the S_2 permutes the two equally sized target space circles S^1. In a sense the non-factorisable $\mathbb{Z}_2$ orbifold we defined is a generalisation of the S_2 orbifold studied in ref. [3]. In later sections we will also make analogous statements for higher dimensions.

Ensembles of non-factorisable $\mathbb{Z}_2$ orbifolds with vanishing B-field

The moduli space of this class of toroidal $\mathbb{Z}_2$ orbifold is parametrized by the angular complex structure modulus ϕ and the real Kähler modulus κ in the range (3.81). From eq. (3.76) we arrive at the induced moduli space metric

$$
ds^2 = \left(\frac{d\phi}{\sin\phi}\right)^2 + \left(\frac{d\kappa}{\kappa}\right)^2\,. \tag{3.91}
$$

The volume of the moduli space exhibits logarithmic divergences as ϕ and κ approach zero and $+\infty$, respectively. Thus, we define the regularized ensemble average

$$
\left\langle Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle_{\text{reg}} = \frac{1}{V_{\text{reg}}}\int_{\delta}^{\pi-\delta}\frac{d\phi}{\sin\phi}\int_{\frac{1}{\lambda}}^{\lambda}\frac{d\kappa}{\kappa}\, Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau;\phi,\kappa)\,. \tag{3.92}
$$

Here the ensemble average is regularized by introducing a small positive angle δ for the angular variable ϕ and a large value λ (independent from the previously defined λ) for the Kähler modulus κ. We integrate over a 4-fold cover of the moduli space (3.81), which is normalized by

$$
V_{\text{reg}} = \int_{\delta}^{\pi-\delta}\frac{d\phi}{\sin\phi}\int_{\frac{1}{\lambda}}^{\lambda}\frac{d\kappa}{\kappa}\,. \tag{3.93}
$$

Upon expressing the moduli in terms of R_1 and R_2 defined in eq. (3.87), we obtain for the regularized ensemble average

$$\begin{aligned}&\left\langle Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle_{\text{reg}}\\ &= \frac{1}{4\left(\text{Vol}(M_{S^1})_{\text{reg}}\right)^2}\int_{\sqrt{\alpha'}/\Lambda_1}^{\Lambda_1/\sqrt{\alpha'}}\frac{dR_1}{R_1}\int_{\sqrt{\alpha'}/\Lambda_2}^{\Lambda_2/\sqrt{\alpha'}}\frac{dR_2}{R_2}\, Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}\left(\tau; 2\arctan\left(\tfrac{R_2}{R_1}\right), \tfrac{1}{2\alpha'}R_1R_2\right),\end{aligned} \tag{3.94}$$

where the large cut-offs Λ_1 and Λ_2 arise from the transformation the regulators δ and λ in eq. (3.92) and the normalization factor is given by $V_{\text{reg}} = 8\left(\text{Vol}_{S^1,\text{reg}}\right)^2$. Expressed in terms of the circular ensemble average (3.31), this (regularized) ensemble average becomes

$$\begin{aligned}&\left\langle Z_{T^2_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle_{\text{reg}} =\\ &= \frac{1}{4}\left(\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2\langle Z_{S^1}(2\tau)\rangle^2_{\text{reg}} + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2\left\langle Z_{S^1}(\tfrac{\tau}{2})\right\rangle^2_{\text{reg}} + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2\left\langle Z_{S^1}(\tfrac{\tau+1}{2})\right\rangle^2_{\text{reg}}\right)\\ &\qquad - \langle Z_{S^1}(\tau)\rangle^2_{\text{reg}} + \frac{1}{2}\left(\langle Z_{S^1}(2\tau)\rangle_{\text{reg}} + \left\langle Z_{S^1}(\tfrac{\tau}{2})\right\rangle_{\text{reg}} + \left\langle Z_{S^1}(\tfrac{\tau+1}{2})\right\rangle_{\text{reg}}\right).\end{aligned} \tag{3.95}$$

Note that the individual terms assembled in brackets in eq. (3.95) form modular invariant combinations with respect to the modular group acting on the world-sheet modular parameter τ.

Partition functions in terms of Siegel-Narain theta functions

Before moving on to higher dimensional generalisations of the above constructions, we write the untwisted no-insertion contributions to the factorisable and non-factorisable T^2 orbifolds in terms of the Siegel-Narain theta functions of Appendix A.2. The resulting expressions will be particularly useful when we compute the average partition function over the moduli space for higher dimensional target spaces in section 3.2.2, following the methods of ref. [184].

For the factorizable $\iota_{\mathbb{Z}_2}$ involution, the contribution from the untwisted sector with no insertions to eq. (3.75) is of the form:

$$Z^{(+,+)}_{T^2_{\text{fac}}/\mathbb{Z}_2}(\tau; c, \kappa) = \frac{1}{|\eta(\tau)|^4}\,\Theta_{H(g)}(0,0,\tau)\,\Theta_{H(\tilde{g})}(0,0,\tau) \tag{3.96}$$

where $g, \tilde{g}$ are the metrics of the two circles.

For the non-factorizable involution (3.89), we find (this is also explained in Appendix A.3 in the context of higher dimensional target tori)

$$Z^{(+,+)}_{T^2_{\text{non-fac}}}(\tau; \phi, \kappa) = \frac{1}{|\eta(\tau)|^4}\sum_{\Delta\in\{0,1\}^2}\Theta_h(0, \tfrac{1}{2}\Delta, 2\tau)\,\Theta_{\tilde{h}}(0, \tfrac{1}{2}\Delta, 2\tau)\,. \tag{3.97}$$

Where $h, \tilde{h}$ are also defined in the same Appendix and depend on $g/2$ and $\tilde{g}/2$. In fact, one can show that the above expressions are modular invariant using the properties of the twisted theta functions under modular transformations. We shall generalise these formulas in section 3.2.2, where we study higher dimensional toroidal target spaces.

3.2.2 Higher-dimensional factorizable and non–factorisable Toroidal $\mathbb{Z}_2$ Orbifold CFTs

Having seen the basic constructions of factorisable and non-factorisable $\mathbb{Z}_2$ orbifolds on T^2, we now move on the higher dimensional analogues [2]. This also enables us to obtain finite ensemble averages and finite moduli space volumes of the toroidal orbifold conformal field theories to be defined. More specifically, we now consider D free bosons for $D \geqslant 6$ with periodic boundary conditions parametrizing the target space torus T^D. The $\mathbb{Z}_2$ orbifold action is again characterized by an involution $\iota_{\mathbb{Z}_2} : T^D \to T^D$ together with the induced action via the pull-back $\iota^*_{\mathbb{Z}_2}$ acting on the toroidal metric and the two-form B-field. Note that the orbifolds that we define and study here are not a complete classification of all possible $\mathbb{Z}_2$ toroidal orbifolds—this goes beyond the scope of this work but is an interesting problem to try and tackle.

We generalise the class of toroidal $\mathbb{Z}_2$ orbifolds studied in ref. [166], where the involution $\iota_{\mathbb{Z}_2}$ acted on all directions of the target-space torus.

Factorisable $\mathbb{Z}_2$ orbifolds and their ensemble averages

Recall that in the T^2 case, the factorisable orbifold corresponded to $T^2 \simeq S^1 \times S^1$ (as a Riemannian manifold) and $\mathbb{Z}_2$ acted on one of the S^1s.[21] We extend this to $D = \ell + m$, $\ell, m \geqslant 3$ dimensions by considering factorisable tori $T^D \simeq T^\ell \times T^m$ equipped with a block diagonal metric and block diagonal B-field. The $\mathbb{Z}_2$ involution $\iota_{\mathbb{Z}_2}$ is defined as

$$\iota_{\mathbb{Z}_2} : \ (x, y) \mapsto (-x, y) \ , \tag{3.98}$$

where (x, y) are the coordinates on the universal covering space $\mathbb{R}^\ell \times \mathbb{R}^m$ of the torus $T^\ell \times T^m$. Moreover, the block-diagonal metric and the block-diagonal B-field are invariant with respect to the pull-back $\iota^*_{\mathbb{Z}_2}$.[22] We denote the toroidal $\mathbb{Z}_2$ orbifold resulting from the involution $\iota_{\mathbb{Z}_2}$ by

$$T^D_{\text{fac}}/\mathbb{Z}_2 \simeq T^\ell/\mathbb{Z}_2 \times T^m \ . \tag{3.99}$$

The partition function factorises into a product of a toroidal CFT partition function and a toroidal $\mathbb{Z}_2$ orbifold partition function of the kind studied in ref. [166]

$$Z_{T^D_{\text{fac}}/\mathbb{Z}_2}(\tau; \mathfrak{l}, \mathfrak{m}) = Z_{T^\ell/\mathbb{Z}_2}(\tau; \mathfrak{l}) Z_{T^m}(\tau; \mathfrak{m}) \ , \tag{3.100}$$

were $\mathfrak{l}, \mathfrak{m}$ denote the moduli (metric and B-field) of T^ℓ, T^m respectively and $Z_{T^m}, Z_{T^\ell/\mathbb{Z}_2}$ were defined in eqs. (3.34) and (3.65) respectively.

This factorisation turns out to be a global property of the moduli space $\mathcal{M}_{T^D_{\text{fac}}/\mathbb{Z}_2}$ of these orbifolds which satisfies [2]

$$\mathcal{M}_{T^D_{\text{fac}}/\mathbb{Z}_2} = \mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m} \ . \tag{3.101}$$

$\mathcal{M}_{T^d}$ denotes the moduli space of a T^d CFT and was defined in (3.35). Note that the involution $\iota_{\mathbb{Z}_2}$ reflects all directions on T^ℓ hence $\mathcal{M}_{T^\ell} \simeq \mathcal{M}_{T^\ell/\mathbb{Z}_2}$ (as in ref. [166]).

---

[21] Topologically, any torus of arbitrary dimension factorizes into tori of smaller dimension, e.g., $T^D \simeq_{\text{top.}} T^\ell \times T^m$ for $D = \ell + m$. However, as a Riemannian manifold it only factorizes in this way when the metric is block diagonal.

[22] In the two-dimensional case, the analogue of block-diagonal B-field is B=0.

Therefore, the resulting ensemble average of this class of $\mathbb{Z}_2$ orbifold conformal field theories readily becomes

$$\begin{aligned}\left\langle Z_{T^D_{\text{fac}}/\mathbb{Z}_2}(\tau)\right\rangle &= \int_{\mathcal{M}_{T^\ell}\times\mathcal{M}_{T^m}} d\mu(\mathfrak{l},\mathfrak{m})\, Z_{T^D_{\text{fac}}/\mathbb{Z}_2}(\tau;\mathfrak{l},\mathfrak{m}) \\ &= \left(\int_{\mathcal{M}_{T^\ell}} d\mu(\mathfrak{l})\, Z_{T^\ell/\mathbb{Z}_2}(\tau;\mathfrak{l})\right)\left(\int_{\mathcal{M}_{T^m}} d\mu(\mathfrak{m})\, Z_{T^m}(\tau;\mathfrak{m}))\right) \\ &= \left\langle Z_{T^\ell/\mathbb{Z}_2}(\tau)\right\rangle \left\langle Z_{T^m}(\tau)\right\rangle \ ,\end{aligned} \tag{3.102}$$

and factorizes into a product of ensemble averages. Here $d\mu(\mathfrak{l},\mathfrak{m})$ is the measure of the moduli space $\mathcal{M}_{T^D_{\text{fac}}/\mathbb{Z}_2}$. It factors into the measures $d\mu(\mathfrak{l})$ and $d\mu(\mathfrak{m})$ for the moduli spaces of toroidal conformal field theories, which (for arbitrary d-dimensional tori) is normalized to

$$\text{Vol}_{T^d} = \int_{\mathcal{M}_{T^d}} d\mu(\mathfrak{d}) = 1 \ . \tag{3.103}$$

The ensemble average of a d-dimensional toroidal conformal field theory is calculated as in eq. (3.43) and we repeat it for convenience

$$\left\langle Z_{T^d}(\tau)\right\rangle = \int_{\mathcal{M}_{T^d}} d\mu(\mathfrak{d}) Z_{T^d}(\tau;\mathfrak{d}) = \frac{E_{d/2}(\tau)}{\text{Im}(\tau)^{\frac{d}{2}}\,|\eta(\tau)|^{2d}} \ , \quad d \geqslant 3 \ . \tag{3.104}$$

As a result and together with eq. (3.100) the ensemble average of the partition function (3.102) becomes

$$\begin{aligned}\left\langle Z_{T^D_{\text{fac}}/\mathbb{Z}_2}(\tau)\right\rangle = \frac{1}{2}&\frac{E_{\ell/2}(\tau)E_{m/2}(\tau)}{\text{Im}(\tau)^{\frac{\ell+m}{2}}\,|\eta(\tau)|^{2(\ell+m)}} \\ &+2^{\ell-1}\left[\left|\frac{\eta(\tau)}{\theta_2(\tau)}\right|^\ell + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right|^\ell + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|^\ell\right]\frac{E_{m/2}(\tau)}{\text{Im}(\tau)^{\frac{m}{2}}\,|\eta(\tau)|^{2m}} \ .\end{aligned} \tag{3.105}$$

Non-factorisable $\mathbb{Z}_2$ orbifolds and their ensemble averages

Here we will give a generalisation to the non-factorisable $\mathbb{Z}_2$ orbifolds studied in section 3.2.1. We consider a target-space which is a $D = 2\ell$-dimensional torus $T^{2\ell}$, with $\ell \geqslant 3$.[23] This target-space can be realised by a 2ℓ-dimensional lattice $\lambda_{2\ell}$ as

$$T^{2\ell} \simeq \mathbb{R}^{2\ell}/\Lambda_{2\ell} \ , \quad \Lambda_{2\ell} = \langle\!\langle\, s_1,\ldots,s_{2\ell}\,\rangle\!\rangle \ . \tag{3.106}$$

Where s_i, $i = 1,2,...,2\ell$ are generators of the lattice. The $\mathbb{Z}_2$ involution $\iota_{\mathbb{Z}_2}$ now exchanges the first ℓ generators s_i, $i = 1,2,...,\ell$ with the second ℓ generators s_i, $i = \ell+1,...,2\ell$. Explicitly, the involution $\iota_{\mathbb{Z}_2}$ is given by the lattice automorphisms

$$\iota_{\mathbb{Z}_2} :\ s_A \mapsto \begin{cases} s_{A+\ell} & \text{for } A \leqslant \ell \ , \\ s_{A-\ell} & \text{for } A > \ell \ . \end{cases} \tag{3.107}$$

In order for the involution $\iota_{\mathbb{Z}_2}$ to realize a $\mathbb{Z}_2$ symmetry on the associated toroidal conformal field theory, we require that the flat toroidal target space metric G and the background B-field B are invariant with respect to this geometric $\mathbb{Z}_2$ action, i.e.,

$$\iota^*_{\mathbb{Z}_2} G = G \ , \qquad \iota^*_{\mathbb{Z}_2} B = B \ . \tag{3.108}$$

[23]The reason we focus on $\ell \geqslant 3$ is again to have finite volumes of moduli spaces and averages of partition functions.

The metric G and B-field can be induced from the metric and B-field on the universal covering space $\mathbb{R}^{2\ell}$. Calling the latter $G_{u.c.}, B_{u.c.}$, the components of the metric G and B-field B of the $T^{2\ell}$ CFT can be computed by evaluating these on the generators s_A

$$G_{AB} = G_{u.c.}(s_A, s_B) \ , \qquad B_{AB} = B_{u.c.}(s_A, s_B) \ , \ A, B = 1, 2, ..., 2\ell \ . \tag{3.109}$$

Equation (3.107) then implies that

$$G_{ab} = G_{a+\ell, b+\ell} \ , \quad G_{a+\ell, b} = G_{a, b+\ell} \quad \text{for } a, b = 1, \ldots, \ell \ . \tag{3.110}$$

and

$$B_{ab} = B_{a+\ell, b+\ell} \ , \quad B_{a+\ell, b} = B_{a, b+\ell} \quad \text{for } a, b = 1, \ldots, \ell \ . \tag{3.111}$$

Which in turn imply that G, B take the matrix form

$$G = \begin{pmatrix} \mathfrak{G} & \widetilde{\mathfrak{G}} \\ \widetilde{\mathfrak{G}} & \mathfrak{G} \end{pmatrix} \ , \qquad B = \begin{pmatrix} \mathfrak{B} & \widetilde{\mathfrak{B}} \\ \widetilde{\mathfrak{B}} & \mathfrak{B} \end{pmatrix} \ , \tag{3.112}$$

where $\mathfrak{G}$, $\widetilde{\mathfrak{G}}$ are symmetric $\ell \times \ell$ and $\mathfrak{B}, \widetilde{\mathfrak{B}}$ are $\ell \times \ell$ skew-symmetric matrices.

When the off-diagonal blocks are zero, the $T^{2\ell}$ torus takes the factorised form $T^\ell \times T^\ell$, where each T^ℓ factor comes equipped with the same metric and B-field $\mathfrak{G}, \mathfrak{B}$. As the $\mathbb{Z}_2$ orbifold exchanges the two tori, the non-factorizable toroidal $\mathbb{Z}_2$ conformal field theory simplifies to the S_2 symmetric orbifold conformal field theory arising from the product of two tori, as studied in ref. [3]. This case is the higher-dimensional analogue of eq. (3.90). It corresponds to putting the tori T^ℓ at "90 degrees with each other".

In the following, we often refer to the constructed torus $T^{2\ell}$ with the metric (3.110) and the B-field (3.111) as the non-factorizable torus $T^{2\ell}_{\text{non-fac}}$, and we denote the $\mathbb{Z}_2$ orbifold associated to the involution $\iota_{\mathbb{Z}_2}$ of the non-factorizable torus by $T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2$.

A covering space of the target-space torus

It turns out that defining a 2^ℓ-fold cover $\widetilde{T}^{2\ell}$ of the target-space $T^{2\ell}$ is very convenient for the calculation of the partition function's average. This cover is a factorisable torus $\widetilde{T}^{2\ell} \simeq \widetilde{T}^\ell \times \widetilde{T}^\ell$ with a block-diagonal metric and B-field. This is most easily seen by its construction:

The relevant covering torus $\widetilde{T}^{2\ell} \simeq \mathbb{R}^{2\ell}/\widetilde{\Lambda}_{2\ell}$ is described in terms of the sublattice $\widetilde{\Lambda}_{2\ell} \subset \Lambda_{2\ell}$ of index 2^ℓ given by

$$\begin{aligned} &\widetilde{\Lambda}_{2\ell} = \langle\!\langle\, e_1, \ldots, e_\ell, f_1, \ldots, f_\ell \,\rangle\!\rangle \ , \\ &e_a = s_a - s_{a+\ell} \ , \quad f_a = s_a + s_{a+\ell} \ , \quad a = 1, \ldots, \ell \ . \end{aligned} \tag{3.113}$$

In terms of these generators, the metric $\widetilde{G}$ and B-field $\widetilde{B}$ of $\widetilde{T}^{2\ell}$ becomes block-diagonal with blocks $g, \widetilde{g}$ and $b, \widetilde{b}$. This is because

$$0 = G_{u.c.}(e_a, f_b) = G_{u.c.}(e_a, f_b) \ , \quad 0 = B_{u.c.}(e_a, f_b) = B_{u.c.}(e_a, f_b) \ , \ a, b = 1, \cdots, \ell \ . \tag{3.114}$$

Moreover, we have

$$\tilde{g}_{ab} = 2(G_{ab} - G_{a,b+\ell}) \ , \quad g_{ab} = 2(G_{ab} + G_{a,b+\ell}) \tag{3.115}$$

and

$$\tilde{b}_{ab} = 2(B_{ab} - B_{a,b+\ell}) \ , \quad b_{ab} = 2(B_{ab} + B_{a,b+\ell}) \ . \tag{3.116}$$

In terms of the metric and B-field $\ell \times \ell$-blocks g, $\tilde{g}$, b and $\tilde{b}$, we readily express the metric G and B of the non-factorizable torus $T^{2\ell}$ as

$$G = \frac{1}{4}\begin{pmatrix} g + \tilde{g} & g - \tilde{g} \\ g - \tilde{g} & g + \tilde{g} \end{pmatrix} \ , \qquad B = \frac{1}{4}\begin{pmatrix} b + \tilde{b} & b - \tilde{b} \\ b - \tilde{b} & b + \tilde{b} \end{pmatrix} \ , \tag{3.117}$$

where the exhibited block structure arises in terms of the basis (3.106).

The partition function of non-factorisable $\mathbb{Z}_2$ orbifolds

As in section 3.2, we write the partition function as

$$Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} = \frac{1}{2}\left(Z^{(++)}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} + Z^{(+-)}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} + Z^{(-+)}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} + Z^{(--)}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} \right) . \tag{3.118}$$

The first contribution, $(+,+)$, corresponds to the untwisted-no insertion part of the partition function. It is calculated by inserting the metric and B-fields of eq. (3.117) into the matrix

$$H(G,B) = \begin{pmatrix} \frac{\alpha'}{2}G^{-1} & \frac{1}{2}G^{-1}B \\ -\frac{1}{2}BG^{-1} & \frac{1}{2\alpha'}\left(G - BG^{-1}B\right) \end{pmatrix} , \tag{3.119}$$

(see also Appendix A.2 and A.3) and is given by

$$Z_{T^{2\ell}_{\text{non-fac}}}(\tau; G, B) = \frac{1}{|\eta(\tau)|^{4\ell}} \sum_{\Delta \in \{0,1\}^{2\ell}} \Theta_h(0, \tfrac{1}{2}\Delta, 2\tau)\, \Theta_{\tilde{h}}(0, \tfrac{1}{2}\Delta, 2\tau) . \tag{3.120}$$

Here the Siegel–Narain theta functions are defined with respect to the $2\ell \times 2\ell$ positive definite matrices

$$h \equiv H(\tfrac{g}{2}, \tfrac{b}{2}) , \qquad \tilde{h} \equiv H(\tfrac{\tilde{g}}{2}, \tfrac{\tilde{b}}{2}) , \tag{3.121}$$

that are determined via the matrix relation (3.119) in terms of the (rescaled) $\ell \times \ell$ matrices $g, \tilde{g}, b, \tilde{b}$. The modular parameter of these Siegel–Narain theta functions appearing in eq. (3.120) is 2τ as opposed to τ in a T^D toroidal partition function (3.34). As a result the modular invariance of this expression is not immediately manifest but can be shown using properties of Siegel-Narain theta functions. Note that the partition function $Z_{T^{2\ell}_{\text{non-fac}}}(\tau; G, B)$ as given in eq. (3.120) is a finite sum of products of two Siegel–Narain theta functions, which only depend on g, b, and $\tilde{g}$, $\tilde{b}$, respectively. This is a consequence of the fact that the partition function $Z_{T^{2\ell}_{\text{non-fac}}}$ can alternatively be obtained from a shift orbifold of the 2^ℓ-fold covering torus $\tilde{T}^\ell \times \tilde{T}^\ell$, which is factorizable.

For the rest contributions, we compute the $(+,-)$ by inserting the non-trivial $\mathbb{Z}_2$ element in the trace and find [2]

$$Z^{(+-)}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} = Z_{T^\ell}(2\tau; \tfrac{g}{2}, \tfrac{b}{2}) . \tag{3.122}$$

Note that, due to the insertion of the generator of the $\mathbb{Z}_2$ orbifold group, the contribution $Z^{(+-)}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}$ to the partition function depends only on the moduli g, b, which are the moduli of the fixed-point locus of the involution $\iota_{\mathbb{Z}_2}$.

We can then deduce the $(-,+), (-,-)$ contributions by acting with modular transformations on the $(+,-)$. The full partition function reads

$$\begin{aligned} Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2} = &\frac{1}{2}\frac{1}{|\eta(\tau)|^{4\ell}} \sum_{\Delta \in \{0,1\}^{2\ell}} \Theta_h(0, \tfrac{1}{2}\Delta, 2\tau)\, \Theta_{\tilde{h}}(0, \tfrac{1}{2}\Delta, 2\tau) \\ &+ \frac{1}{2}\left(Z_{T^\ell}(2\tau; \tfrac{g}{2}, \tfrac{b}{2}) + Z_{T^\ell}(\tfrac{\tau}{2}; \tfrac{g}{2}, \tfrac{b}{2}) + Z_{T^\ell}(\tfrac{\tau+1}{2}; \tfrac{g}{2}, \tfrac{b}{2}) \right) , \end{aligned} \tag{3.123}$$

where both the first and the second line are modular invariant contributions by themselves.

Ensembles of non-factorisable orbifolds

To determine the ensemble average of the non-factorizable toroidal $\mathbb{Z}_2$ orbifold partition function $Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}$, we first discuss the structure of its moduli space $\mathcal{M}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}$. The metric on the moduli space $\mathcal{M}_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}$ is the Zamolodchikov metric restricted to the non-factorizable tori $T^{2\ell}$ that are invariant with respect to the action of the involution $\iota_{\mathbb{Z}_2}$.

The Zamolodchikov metric of the torus $T^{2\ell}$ is given by (3.36), which becomes in terms of the matrix (3.119)

$$\begin{aligned} ds^2_{T^{2\ell}} &= \operatorname{tr}\left(G^{-1}dG\, G^{-1}dG - G^{-1}dB\, G^{-1}dB\right) \\ &= \frac{1}{2}\operatorname{tr}\left(H^{-1}dH\, H^{-1}dH\right) = -\frac{1}{2}\operatorname{tr}\left(dH dH^{-1}\right) . \end{aligned} \tag{3.124}$$

Here the positive definite $4\ell \times 4\ell$ matrix H is given in terms of the metric G and the B-field B according to eq. (3.119). Restricting the Zamolodchikov metric to the non-factorizable tori $T^{2\ell}_{\text{non-fac}}$, we insert eq. (3.117) and get

$$\begin{aligned} ds^2_{T^{2\ell}_{\text{non-fac}}} &= \operatorname{tr}\left(g^{-1}dg\, g^{-1}dg + g^{-1}db\, g^{-1}db\right) + \operatorname{tr}\left(\tilde{g}^{-1}d\tilde{g}\, \tilde{g}^{-1}d\tilde{g} + \tilde{g}^{-1}d\tilde{b}\, \tilde{g}^{-1}d\tilde{b}\right) \\ &= \frac{1}{2}\operatorname{tr}\left(h^{-1}dh\, h^{-1}dh\right) + \frac{1}{2}\operatorname{tr}\left(\tilde{h}^{-1}d\tilde{h}\, \tilde{h}^{-1}d\tilde{h}\right) . \end{aligned} \tag{3.125}$$

Thus the metric factorizes locally over the moduli space of non-factorizable tori $T^{2\ell}_{\text{non-fac}}$ into the two positive definite parts h and $\tilde{h}$.

To average, we need to know about the global structure of the moduli space. For this, we can analyse the latter in terms of the so-called majorants of the symmetric $2\ell \times 2\ell$ matrix of (ℓ, ℓ) signature Ω [83, 84]. Doing so, reveals that the moduli space of $\mathbb{Z}_2$, non-factorisable $T^{2\ell}$ CFTs is [2]

$$\mathcal{M}_{T^{2\ell}_{\text{non-fac}}} \simeq \mathcal{M}_{T^{\ell}} \times \mathcal{M}_{T^{\ell}} . \tag{3.126}$$

We give more details on this analysis in Appendix A.4.

It is important to underline that our construction of the moduli space $\mathcal{M}_{T^{2\ell}_{\text{non-fac}}}$ does not cover all possible conformal field theories that can be constructed from $\mathbb{Z}_2$ orbifolds associated to the involution $\iota_{\mathbb{Z}_2}$ acting on non-factorizable tori $T^{2\ell}_{\text{non-fact}}/\mathbb{Z}_2$ as defined in eq. (3.107). On top of the B-field B entering the matrix H in eq. (3.119), which obeys the relations (3.111), there are additional discrete choices for the B-field that are invariant with respect to a $\mathbb{Z}_2$ symmetry once the discrete transformations $O(2\ell, 2\ell, \mathbb{Z})$ are taken into account. In our treatment, we only consider B-field configurations that are invariant under the involution $\iota_{\mathbb{Z}_2}$ without taking into account such discrete transformations.

The family of S_2-symmetric orbifold conformal field theory studied in ref. [3] corresponds to the points in moduli space $\mathcal{M}_{T^{2\ell}_{\text{non-fac}}}$, where the equivalence classes of the matrices h and $\tilde{h}$ are equal. That is to say, the moduli space $\mathcal{M}_{T^\ell \times T^\ell/S_2}$ of the S_2-symmetric orbifold conformal field theory is the diagonal submoduli space

$$\mathcal{M}_{T^\ell \times T^\ell/S_2} \simeq \left\{ (\mathfrak{m}, \tilde{\mathfrak{m}}) \in \mathcal{M}_{T^{2\ell}_{\text{non-fac}}} \,\middle|\, \mathfrak{m} = \tilde{\mathfrak{m}} \right\} , \tag{3.127}$$

where $\mathfrak{m}$ and $\tilde{\mathfrak{m}}$ are equivalence classes of h and $\tilde{h}$.

The Zamolodchikov metric of the moduli space $\mathcal{M}_{T^N}$ of a toroidal conformal field theory yields the measure $d\mu(\mathfrak{m}_H)$, which upon integrating over the moduli space (3.35), for some $N \geqslant 3$, we normalize as

$$\int_{\mathcal{M}_{T^N}} d\mu(m_H) = 1 \ . \tag{3.128}$$

To calculate the ensemble averages of the partition functions derived in the previous paragraphs, it is necessary to average the Siegel–Narain theta functions $\Theta_H(0, \frac{1}{2}\Delta, \tau)$ defined in eq. (A.14) over the moduli space $\mathcal{M}_{T^N}$ as

$$\left\langle \Theta_H(0, \tfrac{1}{2}\Delta, x) \right\rangle = \int_{\mathcal{M}_{T^N}} d\mu(m_H)\, \Theta_H(0, \tfrac{1}{2}\Delta, x) \ . \tag{3.129}$$

This computation is detailed in refs. [84, 184]—see also Appendix A.2. Here we quote the result for $N \geqslant 3$ and $\Delta \in \{0,1\}^{2N}$

$$\left\langle \Theta_H(0, \tfrac{1}{2}\Delta, x) \right\rangle = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \dfrac{1}{|c\,x + d|^N} & \text{for } 0 = \Delta \in \{0,1\}^{2N} \ , \\ \dfrac{1}{2} \displaystyle\sum_{\substack{c \in \mathbb{Z}, d \in 2\mathbb{Z} \\ (c,d)=1}} \dfrac{(-1)^{\frac{d}{2}\Delta^T \Omega \Delta}}{|c\,x + d|^N} & \text{for } 0 \neq \Delta \in \{0,1\}^{2N} \ . \end{cases} \tag{3.130}$$

For the average of these Siegel–Narain theta functions there are three distinct cases. The tuple Δ is either zero or non-zero. In the latter case we distinguish between $\Delta^T \Omega \Delta$ being even or odd. Therefore, we define

$$\begin{aligned} \left\langle \Theta_H^{(0)}(x) \right\rangle &:= \langle \Theta_H(0,0,x) \rangle \ , \\ \left\langle \Theta_H^{(+)}(x) \right\rangle &:= \langle \Theta_H(0, \tfrac{1}{2}\Delta, x) \rangle \quad \text{for } \Delta^T \Omega \Delta \text{ even, } \Delta \neq 0 \ , \\ \left\langle \Theta_H^{(-)}(x) \right\rangle &:= \langle \Theta_H(0, \tfrac{1}{2}\Delta, x) \rangle \quad \text{for } \Delta^T \Omega \Delta \text{ odd} \ . \end{aligned} \tag{3.131}$$

Using eq. (3.130) and the definition for the real analytic Eisenstein series (3.41), we find for the average

$$\left\langle \Theta_H^{(0)}(x) \right\rangle = \frac{E_{N/2}(x)}{\operatorname{Im}(\tau)^{\frac{N}{2}}} \ . \tag{3.132}$$

Inserting the identities (A.41) into eq. (3.130) we arrive for the remaining averages at

$$\begin{aligned} \left\langle \Theta_H^{(+)}(x) \right\rangle &= \frac{1}{2^N - 1} \left(\frac{E_{N/2}(\frac{x}{2})}{\operatorname{Im}(\frac{x}{2})^{\frac{N}{2}}} - \frac{E_{N/2}(x)}{\operatorname{Im}(x)^{\frac{N}{2}}} \right) \ , \\ \left\langle \Theta_H^{(-)}(x) \right\rangle &= \frac{2}{2^N(2^N - 1)} \frac{E_{N/2}(\frac{x}{4})}{\operatorname{Im}(\frac{x}{4})^{\frac{N}{2}}} - \frac{2^N + 2}{2^N(2^N - 1)} \frac{E_{N/2}(\frac{x}{2})}{\operatorname{Im}(\frac{x}{2})^{\frac{N}{2}}} + \frac{1}{2^N - 1} \frac{E_{N/2}(x)}{\operatorname{Im}(x)^{\frac{N}{2}}} \ . \end{aligned} \tag{3.133}$$

With these averaged Siegel–Narain theta functions at hand, we can now determine the ensemble average of the partition function $Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}$ of the $\mathbb{Z}_2$ orbifold toroidal conformal field theories based on non-factorizable tori $T^{2\ell}_{\text{non-fac}}$. We recall that the measure (3.125) and the moduli space of this ensemble of conformal field theories factorizes, which becomes manifest once we parametrize the moduli space by the matrices h and $\widetilde{h}$ of eq. (3.121). Moreover, the partition function $Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau; h, \widetilde{h})$ of eq. (3.123) is a sum of products of terms, whose factors are Siegel–Narain theta functions that depend on the two respective matrices h and $\widetilde{h}$. Thus, the ensemble average factors over these sums of products as well, and we obtain

$$\begin{aligned}\left\langle Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle &= \int_{\mathcal{M}_{T^\ell}} d\mu(m_h)\int_{\mathcal{M}_{T^\ell}} d\mu(m_{\widetilde{h}})\ Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau;h,\widetilde{h})\\ &= \frac{1}{2}\frac{1}{|\eta(\tau)|^{4\ell}}\sum_{\Delta\in\{0,1\}^{2\ell}}\left\langle\Theta_h(0,\tfrac{1}{2}\Delta,2\tau)\right\rangle\ \left\langle\Theta_{\widetilde{h}}(0,\tfrac{1}{2}\Delta,2\tau)\right\rangle\\ &\quad+\frac{1}{2}\Big(\left\langle Z_{T^\ell}(2\tau;h)\right\rangle+\left\langle Z_{T^\ell}(\tfrac{\tau}{2};h)\right\rangle+\left\langle Z_{T^\ell}(\tfrac{\tau+1}{2};h)\right\rangle\Big)\ .\end{aligned} \tag{3.134}$$

The sum over $\Delta\in\{0,1\}^{2\ell}$ splits into the contribution $\Delta=0$, $(2^\ell-1)(2^{\ell-1}+1)$ summands with $\Delta\neq 0$ and $\Delta^T\Omega\Delta$ even, and $2^{\ell-1}(2^\ell-1)$ summands with $\Delta^T\Omega\Delta$ odd. Thus, inserting the definitions (3.131) and carrying out the sum over $\Delta\in\{0,1\}^{2\ell}$ we arrive at

$$\begin{aligned}\left\langle Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle &= \frac{1}{2}\frac{1}{|\eta(\tau)|^{4\ell}}\Bigg(\left\langle\Theta_H^{(0)}(2\tau)\right\rangle^2\\ &\quad+(2^\ell-1)(2^{\ell-1}+1)\left\langle\Theta_H^{(+)}(2\tau)\right\rangle^2+2^{\ell-1}(2^\ell-1)\left\langle\Theta_H^{(-)}(2\tau)\right\rangle^2\Bigg)\\ &\quad+\frac{1}{2}\Big(\left\langle Z_{T^\ell}(2\tau;h)\right\rangle+\left\langle Z_{T^\ell}(\tfrac{\tau}{2};h)\right\rangle+\left\langle Z_{T^\ell}(\tfrac{\tau+1}{2};h)\right\rangle\Big)\ .\end{aligned} \tag{3.135}$$

Finally, we express the determined average in terms of real analytic Eisenstein series using eqs. (3.133). In order to bring the final result into a manifest modular invariant form, we apply the identify

$$E_s(\tfrac{x+1}{2})=\frac{1+2^{2s-1}}{2^{s-1}}E_s(x)-E_s(2x)-E_s(\tfrac{x}{2})\ . \tag{3.136}$$

which is derived from the Fourier decomposition of the real analytic Eisenstein series in terms of the Hecke eigenmodes with respect to the Hecke operator T_2, for details, see Appendix A.5. Putting everything together, we arrive at our main result of this subsection, which is the manifest modular invariant ensemble average

$$\begin{aligned}&\left\langle Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle\\ &=\frac{1}{2\,|\eta(\tau)|^{4\ell}\,\mathrm{Im}(\tau)^\ell\,(2^\ell-1)}\Bigg(\left(E_{\ell/2}(2\tau)^2+E_{\ell/2}\left(\tfrac{\tau}{2}\right)^2+E_{\ell/2}\left(\tfrac{\tau+1}{2}\right)^2\right)\\ &\qquad\qquad-\left(1+2^{1-\ell}\right)E_{\ell/2}(\tau)^2\Bigg)\\ &+\frac{1}{2}\left(\frac{E_{\ell/2}(2\tau)}{\mathrm{Im}(2\tau)^{\frac{\ell}{2}}\,|\eta(2\tau)|^{2\ell}}+\frac{E_{\ell/2}\left(\frac{\tau}{2}\right)}{\mathrm{Im}(\frac{\tau}{2})^{\frac{\ell}{2}}\left|\eta\left(\frac{\tau}{2}\right)\right|^{2\ell}}+\frac{E_{\ell/2}\left(\frac{\tau+1}{2}\right)}{\mathrm{Im}(\frac{\tau+1}{2})^{\frac{\ell}{2}}\left|\eta\left(\frac{\tau+1}{2}\right)\right|^{2\ell}}\right)\ .\end{aligned} \tag{3.137}$$

Here the expressions inside the brackets in the first and third lines are modular invariant as a consequence of the Lemma 2 in Appendix A.5. Using formulas in Appendix A.1 and eq. (3.104), we can also write the averaged partition function as

$$\begin{aligned}&\left\langle Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau)\right\rangle\\ &=\frac{1}{2^{\ell+1}\,(2^\ell-1)}\left(\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^{2\ell}\langle Z_{T^\ell}(2\tau)\rangle^2+\left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^{2\ell}\langle Z_{T^\ell}(\tfrac{\tau}{2})\rangle^2+\left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^{2\ell}\langle Z_{T^\ell}(\tfrac{\tau+1}{2})\rangle^2\right)\\ &\quad-\frac{1+2^{1-\ell}}{2\,(2^\ell-1)}\langle Z_{T^\ell}(\tau)\rangle^2+\frac{1}{2}\left(\langle Z_{T^\ell}(2\tau)\rangle^2+\langle Z_{T^\ell}(\tfrac{\tau}{2})\rangle^2+\langle Z_{T^\ell}(\tfrac{\tau+1}{2})\rangle^2\right)\ .\end{aligned} \tag{3.138}$$

We observe that this expression for the ensemble average is consistent with the lower-dimensional regularized ensemble average stated in eq. (3.95).

Finally, notice that the ensemble average over the submoduli space $\mathcal{M}_{T^\ell \times T^\ell / S_2}$ defined in eq. (3.127) becomes

$$\begin{aligned} \left\langle Z_{T^{2\ell}_{\text{non-fac}}/\mathbb{Z}_2}(\tau) \right\rangle = &\frac{1}{2}\frac{1}{|\eta(\tau)|^{4\ell}} \left\langle \Theta_h(0,0,\tau)\Theta_h(0,0,\tau) \right\rangle \\ &+ \frac{1}{2}\Big(\left\langle Z_{T^\ell}(2\tau;h) \right\rangle + \left\langle Z_{T^\ell}(\tfrac{\tau}{2};h) \right\rangle + \left\langle Z_{T^\ell}(\tfrac{\tau+1}{2};h) \right\rangle \Big)\,, \end{aligned} \tag{3.139}$$

which is the ensemble average of the partition function for the product of two equal tori T^ℓ orbifolded by the permutation S_2 as calculated in ref. [3].

3.3 Symmetric product orbifolds of Narain theories and averaging

In this section, we will focus on symmetric product orbifolds of toroidal CFTs and their ensemble averages which were the focus of [3]. Ensemble averages of symmetric product orbifolds were first considered in ref. [72].

Orbifolds again

The symmetric product orbifold is an orbifold with group the permutation group of N elements S_N. This group is non-abelian for $N > 2$ and we need to take this into account in formulas such as (3.60) and (3.64) for the torus partition function. Specifically, compatibility of $X(x+1) = p \cdot X(x)$, and $X(x+\tau) = g \cdot X(x)$ demands that p, g commute, i.e. $[p, g] = 0$, and the partition function is written as

$$Z_{\mathcal{M}/G} = \frac{1}{|G|} \sum_{\substack{p,g \in G \\ [p,g]=0}} Z^{(p,g)}(\tau) \tag{3.140}$$

and similarly when discrete torsion is turned on.

In the path integral formalism, we saw around eq. (3.62) that summing over group homomorphisms ρ from $\pi_1(\Sigma = T^2)$ to G ensures that we are considering commuting elements of G.

Before we define permutation orbifolds, we write the functional form of an orbifold partition function (3.62) in a different way to gain more intuition. For this start with eq. (3.62) with Σ a genus $g = 1$ Riemann surface[24]

$$Z_{\mathcal{M}/G}(\Sigma) = \frac{1}{|G|} \sum_{\rho \in \mathrm{Hom}(\pi_1(\Sigma), G)} \int_\rho \mathcal{D}X e^{-S[X]}\,. \tag{3.141}$$

Observe that an element $\rho'' \in \mathrm{Hom}(\pi_1(\Sigma), G)$ that is conjugate to ρ, i.e. $\rho'' = (\rho')^{-1}\rho\rho'$, gives the same contribution to the path integral. This can be shown by a re-definition of the fields $X \mapsto \rho' X$. Hence, we can write

$$Z_{\mathcal{M}/G}(\Sigma) = \frac{1}{|G|} \sum_{\rho \in \mathrm{Hom}(\pi_1(\Sigma), G)/G} |G \cdot \rho| \int_\rho \mathcal{D}X e^{-S[X]}\,, \tag{3.142}$$

[24]This can be generalised immediately to connected Riemann surfaces of genus $g > 1$.

where in $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, G acts via conjugation and $|G \cdot \rho|$ is the order of the orbit of ρ under G. We can re-write this in terms of the order of the stabiliser $\mathrm{Stab}(\rho)$ of ρ under the G-action using the orbit stabiliser-theorem as

$$Z_{\mathcal{M}/G}(\Sigma) = \sum_{\rho \in \mathrm{Hom}(\pi_1(\Sigma), G)/G} \frac{1}{|\mathrm{Stab}(\rho)|} \int_\rho \mathcal{D}X e^{-S[X]} \,. \tag{3.143}$$

As a sanity check, note that when $\rho = e_G$, the identity in G, $|\mathrm{Stab}(\rho)| = |G|$. Also, for any central element, the order of the stabiliser is the order of the group and this is also consistent with the fact that for $G = \mathbb{Z}_2$ we only got factors of $1/2$.

The analogue statement in the operator approach is that the Hilbert spaces $\mathcal{H}_g, \mathcal{H}_h$ corresponding to conjugate elements $g, h^{-1}gh$, $g, h \in G$ are equivalent. The Hilbert space of the CFT can be written as

$$\mathcal{H} = \bigoplus_{g \in C} \mathrm{Inv}_{\mathrm{Stab}(g)} \left(\mathcal{H}_g\right) \,, \tag{3.144}$$

where $\mathrm{Stab}(g)$ denotes the subgroup of G consisting of elements commuting with g and $\mathrm{Inv}_{\mathrm{Stab}(g)} \left(\mathcal{H}_g\right)$ means that we project onto $\mathrm{Stab}(g)$-invariant states in the g-twisted Hilbert space. The sum $g \in C$ is a sum over representatives of the conjugacy classes of G. The partition function on a genus $g = 1$ surface can then be written in the operator language as the following trace

$$Z_{\mathcal{M}/G}(\Sigma) = \sum_{g \in C} \mathrm{tr}_{\mathcal{H}_g} \left(P^g q^{L_0^{(p)} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{(p)} - \frac{c}{24}} \right) \,, \qquad P^g = \frac{1}{|\mathrm{Stab}(g)|} \sum_{h \in \mathrm{Stab}(g)} h \tag{3.145}$$

Here P^g projects onto $\mathrm{Stab}(g)$-invariant states in the g-twisted Hilbert space. Using that $|\mathrm{Stab}(g)| \left|C_g\right| = |G|$, we can show that this expression is equivalent to eq. (3.140). For an illuminating discussion on these matters see refs. [168, 174].

Permutation orbifolds

Consider a "seed" conformal field theory $\mathcal{C}$ on a genus g Riemann surface Σ and denote its fields as Φ. The latter can be a collection of fields, e.g. in the T^D CFT, Φ would be the collection of $X^I, I = 1, 2..., D$. We can construct a new CFT from $\mathcal{C}$ by taking the N^{th} tensor power of $\mathcal{C}$

$$\mathcal{C}^{\otimes N} := \underbrace{\mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{\text{N times}} \,. \tag{3.146}$$

This construction is in particular useful when we want to construct a CFT with large central charge. The fields of $\mathcal{C}^{\otimes N}$ are now N-tuples $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_N)$ of the fields of $\mathcal{C}$, where the subscripts denote which copy of $\mathcal{C}$ each of Φ_i lives in. If $\mathcal{C}$ comes from an action $S_{\mathcal{C}}[\Phi]$, then $\mathcal{C}^{\otimes N}$ amounts to summing N copies of this action

$$S_{\mathcal{C}^{\otimes N}}[\boldsymbol{\Phi}] = \sum_{i=1,2,\ldots,N} S_{\mathcal{C}}[\Phi_i] \,. \tag{3.147}$$

Even if this is not the case, since we take the N^{th} tensor power of the CFT $\mathcal{C}$, the resulting Hilbert space is the N^{th} tensor product of the Hilbert space of $\mathcal{C}$.

There is now an obvious symmetry of $\mathcal{C}^{\otimes N}$, namely that of permuting the fields belonging to each copy. Specifically, for any subgroup Ω of S_N, we can define a group

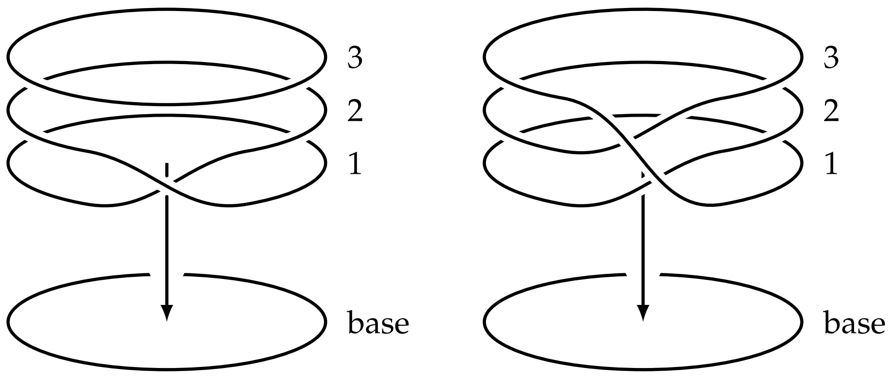

Figure 3.9: Twisted boundary conditions on the circle as 3-fold covering spaces. Left: the fields $\{\Phi_1, \Phi_2, \Phi_3\}$ satisfy twisted boundary conditions $\Phi_1(2\pi) = \Phi_2(0)$, while Φ_3 is single-valued. Right: The fields satisfy boundary conditions $\Phi_1(2\pi) = \Phi_2(0)$, $\Phi_2(2\pi) = \Phi_3(0)$, and $\Phi_3(2\pi) = \Phi_1(0)$.

action on the fields $\mathbf{\Phi}$ as

$$\omega \cdot \mathbf{\Phi} = \left(\Phi_{\omega(1)}, ..., \Phi_{\omega(N)}\right) \ , \quad \omega \in \Omega \ . \tag{3.148}$$

We can orbifold by this symmetry, to construct a new CFT, namely the permutation orbifold $\mathcal{C} \wr \Omega := \mathcal{C}^{\otimes N}/\Omega$ of $\mathcal{C}$.[25] Taking $\Omega = S_N$ gives rise to the symmetric product orbifold of $\mathcal{C}$, denoted also as $\mathrm{Sym}^N(\mathcal{C})$. This is the orbifolds that we focus on in this work but many statements can be generalised to subgroups of S_N.

Partition functions of $\mathrm{Sym}^N(\mathcal{C})$ orbifolds

As we saw earlier, computing orbifold partition functions amounts to summing over all (twisted) boundary conditions. Adapting eq. (3.140) to $G = S_N$ and specialising to Σ being a Riemann surface of genus $g = 1$ (flat torus), we have

$$Z_{\mathrm{Sym}^N(\mathcal{C})}(\Sigma) = \frac{1}{N!} \sum_{\substack{\omega_1, \omega_\tau \in S_N \\ [\omega_1, \omega_\tau] = 0}} Z^{(\omega_1, \omega_\tau)}(\tau) \ . \tag{3.149}$$

Here we have denoted as subscripts in the group elements the cycles that correspond to the monodromies of the fields. These are

$$\mathbf{\Phi}(x+1) = \omega_1 \cdot \mathbf{\Phi}(x) \ , \qquad \mathbf{\Phi}(x+\tau) = \omega_\tau \cdot \mathbf{\Phi}(x) \ . \tag{3.150}$$

To evaluate the partition function, we can apply a standard method that is used in permutation orbifolds. Namely, we consider covering spaces $\widetilde{\Sigma}$ of Σ on which the fields are singe-valued [185, 186]. To do this we take N copies of Σ and let Φ_i live there. Then we glue these copies together according to the boundary conditions ω_1, ω_τ to obtain an N-fold covering of Σ. This can be visualised for example in the case where Σ is a circle, see fig. 3.9. The contribution to the partition function with monodromies ω_1, ω_τ is given by the partition function of the seed CFT $\mathcal{C}$ on the covering space $\widetilde{\Sigma}_{\omega_1,\omega_\tau}$

$$Z^{(\omega_1, \omega_\tau)}(\tau) = Z_{\mathcal{C}}(\widetilde{\Sigma}_{\omega_1,\omega_\tau}) \ . \tag{3.151}$$

Now, as we saw before, elements in $\mathrm{Hom}(\pi_1(\Sigma), S_N)$ that are related by conjugation, give rise to the same contribution in the partition functions and they also correspond

[25] The symbol $\wr$ denotes the wreath product and it is often used for the permutation orbifolds. See, for example, [185].

to topologically equivalent covering spaces $\widetilde{\Sigma}$. Using similar arguments as in page 80, we can write the partition function as a sum over (topologically inequivalent) covering spaces $\widetilde{\Sigma}$ of Σ

$$Z_{\mathrm{Sym}^N(\mathcal{C})}(\Sigma) = \sum_{\widetilde{\Sigma}\to\Sigma} \frac{Z_{\mathcal{C}}(\widetilde{\Sigma})}{\left|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)\right|} \ , \tag{3.152}$$

where $\left|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)\right|$ is the order of the group of deck transformations of the covering $\widetilde{\Sigma}$—these are automorphisms of the covering $\widetilde{\Sigma}\to\Sigma$. This corresponds to $\mathrm{Stab}(\rho)$ in eq. (3.143) and can be seen by using the correspondence between $\mathrm{Hom}(\pi_1(\Sigma), S_N)$ and covering spaces of Σ.

Since Σ is a torus, the covering spaces $\widetilde{\Sigma}$ are going to be disjoint unions of tori. This can be seen for example from the Riemann-Hurwitz formula and by the fact that the coverings $\widetilde{\Sigma}\to\Sigma$ are unramified.[26] More precisely we have that

$$\widetilde{\Sigma} = \Sigma_1 \sqcup ... \sqcup \Sigma_n \to \Sigma \ , \ n = 1, 2, ..., N \ , \tag{3.153}$$

where the Σ_i are tori with modular parameters τ_i not necessarily equal to the modular parameter τ of Σ. The partition function of the CFT $\mathcal{C}$ (or any CFT) on a surface such as (3.153) can be written as

$$Z_{\mathcal{C}}(\widetilde{\Sigma}) = Z_{\mathcal{C}}\left(\sqcup_{i=1,...,n}\Sigma_i\right) = \prod_{i=1}^{n} Z_{\mathcal{C}}(\Sigma_i) \ . \tag{3.154}$$

Hence we see that in order to calculate the torus partition function of $\mathrm{Sym}^N(\mathcal{C})$ we only need to know the torus partition function of the seed theory $\mathcal{C}$. This is not the case when Σ is of higher genus and there, in order to calculate the partition function, we need to know the partition function of $\mathcal{C}$ on many different genera.

An example of $\mathbf{Sym}^N(\mathcal{C})$

We can demonstrate how calculations of partition functions go with a simple example, namely by studying the $N = 2$ case. We can easily write down eq. (3.149) as S_2 is abelian and only has two elements (written in cycle notation): the identity $e \equiv (1)$ and the permutation of two elements $\omega \equiv (12)$. The partition function reads

$$Z_{\mathrm{Sym}^2(\mathcal{C})}(\tau) = \frac{1}{2}\left(Z^{(e,e)}(\tau) + Z^{(\omega,e)}(\tau) + Z^{(e,\omega)}(\tau) + Z^{(\omega,\omega)}(\tau)\right) \ . \tag{3.155}$$

Now we can make contact with the covering space picture (3.152). Take the contribution $Z^{(e,\omega)}(\tau)$ for example. This corresponds to field configurations $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ that satisfy $\Phi_1(x+\tau) = \Phi_2(x)$. On a torus of twice the modular parameter 2τ, which double covers the τ torus, this is single valued and the contribution is written as

$$Z^{(e,\omega)}(\tau) = Z_{\mathcal{C}}(2\tau) \ . \tag{3.156}$$

For the $Z^{(\omega,e)}(\tau)$ we exchange the τ and 1 cycles to get a torus with cycles τ and 2 which is equivalent to $\tau/2$ and 1. For $Z^{(\omega,\omega)}(\tau)$ we construct a covering torus with cycles $\tau+1$, $1-\tau$ which is equivalent to cycles $(\tau+1)/(1-\tau), 1 \sim (\tau+1)/2, 1$, where in the last step we performed a TST modular transformation. Hence, we have

[26] A ramified covering is a covering in which there are points on the base space (here that would be Σ) at which sheets of the covering come together. An unramified covering does not have such points.

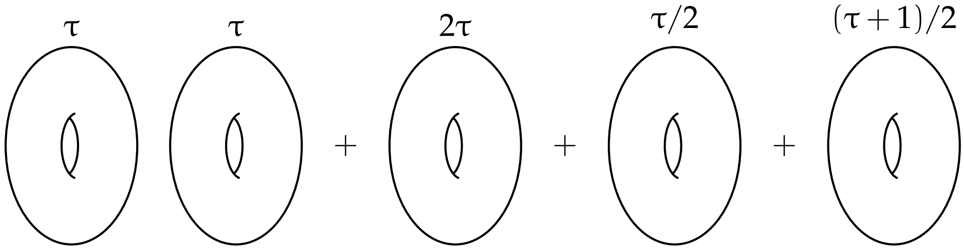

Figure 3.10: All covering space geometries contributing to the S_2 partition function. The Riemann-Hurwitz formula guarantees that these covering spaces are also tori.

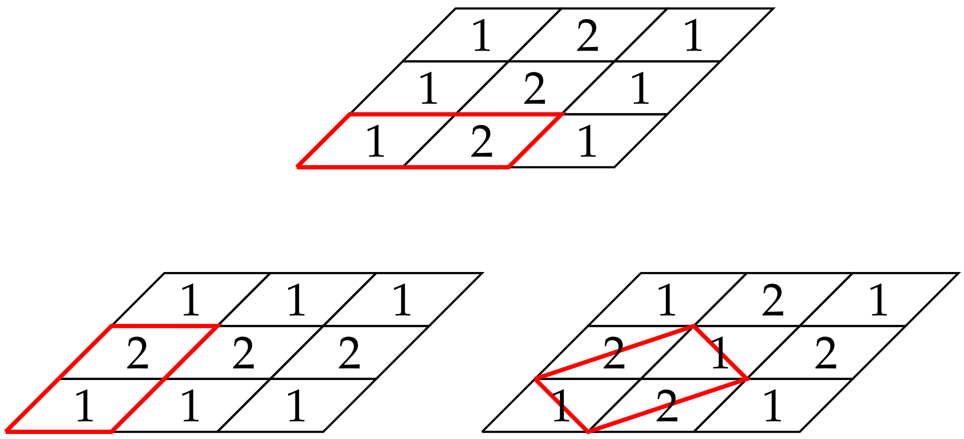

Figure 3.11: The connected covering spaces for the $N = 2$ symmetric orbifold on the torus. Individual cells represent the base torus (with modular parameter τ), and the numbers label which copy of the seed theory lives on which sheet of the base torus. The permutations ω_1, ω_τ prescribe how to stitch together the copies of the seed theory onto the covering space. The covering space itself is the fundamental domain for which the arrangement of labels 1, 2 is periodic (shown in red). The fundamental domains in the above examples have modular parameter $\tau/2$, 2τ, and $(\tau+1)/(1-\tau) \sim (\tau+1)/2$, respectively.

$$Z^{(\omega,e)}(\tau) = Z_{\mathcal{C}}(\tfrac{\tau}{2}) \,, \qquad Z^{(\omega,\omega)}(\tau) = Z_{\mathcal{C}}(\tfrac{\tau+1}{2}) \,. \tag{3.157}$$

The (e, e) contribution is just $Z_{\mathcal{C}}(\tau)^2$ (the τ torus is double covered by two copies of the same torus) and the partition function reads (see also fig. 3.10)

$$Z_{\mathrm{Sym}^2(\mathcal{C})}(\tau) = \frac{1}{2}\left(Z_{\mathcal{C}}(\tau)^2 + Z_{\mathcal{C}}(2\tau) + Z_{\mathcal{C}}(\tfrac{\tau}{2}) + Z_{\mathcal{C}}(\tfrac{\tau+1}{2})\right) \,. \tag{3.158}$$

Each term in this sum can be seen as the partition function of the seed theory $\mathcal{C}$ evaluated on a covering space $\widetilde{\Sigma} \to \Sigma$, see Figure 3.11. The sum of the last three terms is modular invariant given that $Z_{\mathcal{C}}(\tau)$ is, see Lemma 2 in Appendix A.5. As we go higher in N more and more terms are included in the partition function. For instance the S_3 case requires 18 pairs of commuting permutations and the partition function reads

$$\begin{aligned} Z_{\mathrm{Sym}^3(\mathcal{C})}(\tau) =& \frac{1}{6} Z_{\mathcal{C}}(\tau)^3 + \frac{1}{2} Z_{\mathcal{C}}(\tau) Z_{\mathcal{C}}(2\tau) + \frac{1}{2} Z_{\mathcal{C}}(\tau) Z_{\mathcal{C}}(\tfrac{\tau}{2}) + \frac{1}{2} Z_{\mathcal{C}}(\tau) Z_{\mathcal{C}}(\tfrac{\tau+1}{2}) \\ &+ \frac{1}{3} Z_{\mathcal{C}}(3\tau) + \frac{1}{3} Z_{\mathcal{C}}(\tfrac{\tau}{3}) + \frac{1}{3} Z_{\mathcal{C}}(\tfrac{\tau+1}{3}) + \frac{1}{3} Z_{\mathcal{C}}(\tfrac{\tau+2}{3}) \,. \end{aligned} \tag{3.159}$$

All of these terms can be interpreted as three-fold coverings of the τ torus by tori.

The partition function for all N

We can define a "grand canonical" partition function $\mathfrak{Z}(p,\tau)$ for the grand canonical ensemble of symmetric product orbifolds

$$\mathrm{Sym}(X) := \bigoplus_{N=0}^{\infty} (\mathcal{C}/S_N)\,, \tag{3.160}$$

by introducing a chemical potential p to keep track of N

$$\mathfrak{Z}(p,\tau) = \sum_{N=0}^{\infty} p^N Z_{\mathcal{C}/S_N}(\tau)\,. \tag{3.161}$$

In the case of the torus, it can be shown [187, 188] that the grand canonical partition function has a simple expression in terms of Hecke operators, namely

$$\mathfrak{Z}(p,\tau) = \exp\left(\sum_{k=1}^{\infty} p^k T_k Z_{\mathcal{C}}(\tau)\right)\,, \tag{3.162}$$

where the k^{th} Hecke operator is given by

$$T_k Z_{\mathcal{C}}(\tau) = \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z_{\mathcal{C}}\left(\frac{a\tau+b}{d}\right)\,. \tag{3.163}$$

The Hecke operators sum over all connected k-fold covering spaces of the original torus. The integers a, d indicate how many times the cover wraps the cycles of the torus and the integer b denotes a Dehn twist. Expanding the exponential (3.162) includes disconnected covering spaces as well. The coefficient of p^N gives the partition function of $\mathrm{Sym}^N(\mathcal{C})$, which can be written as

$$Z_{\mathrm{Sym}^N(\mathcal{C})}(\tau) = \sum_{\text{partitions of } N}\prod_{k=1}^{N}\frac{1}{N_k!}\left(T_k Z_{\mathcal{C}}(\tau)\right)^{N_k}\,, \tag{3.164}$$

where partitions of N sums over N_k such that $\sum_{k=1}^{N} kN_k = N$.

Finally, we list an equivalent definition of the Hecke operators in terms of modular transformations. Let M_k denote the space of 2×2 integer matrices with determinant k, and let $\Gamma = SL(2,\mathbb{Z})$ denote the modular group. The coset $\Gamma\backslash M_k$ is given by

$$\Gamma\backslash M_k = \left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\,\middle|\, ad = k,\quad b = 0,\ldots,d-1\right\}\,. \tag{3.165}$$

Thus, we can write

$$T_k Z_{\mathcal{C}}(\tau) = \frac{1}{k}\sum_{\gamma\in SL(2,\mathbb{Z})\backslash M_k} Z_{\mathcal{C}}(\gamma\cdot\tau)\,, \tag{3.166}$$

where 2×2 matrices are taken to act on τ in the usual way, i.e.

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot\tau = \frac{a\tau+b}{c\tau+d}\,. \tag{3.167}$$

Introducing fermionic degrees of freedom

In eq. (3.46), we introduced spin structures for fermionic degrees of freedom ψ. Now consider these ψ's being the fields of a fermionic CFT $\mathcal{C}$ of which we wish to take the symmetric product orbifold. The partition function is indexed by the choice of spin structure as

$$Z_{\mathcal{C}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) \equiv Z_{\mathcal{C},\vec{e}}(\tau)\,, \tag{3.168}$$

where $\vec{e} = (\alpha\,\beta)^{\mathsf{T}}$ is a column vector.

Considering the symmetric product of $\mathcal{C}$, a formula similar to (3.162) can be given which takes into account spin structures. We now must sum over permutations of the N fermions such that $\psi(x+\tau) = e^{2\pi i\alpha}\omega_{\tau}\cdot\psi(x)$ and $\psi(x+1) = e^{2\pi i\beta}\omega_1\cdot\psi(x+1)$. The effect of this is that the spin structure on the covering space can differ from the base space. The end result is given by

$$\mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau) = \exp\left(\sum_{k=1}^{\infty} p^k \mathcal{T}_k Z_{\mathcal{C}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\right)\,, \tag{3.169}$$

where the fermionic Hecke operator $\mathcal{T}_k$ acts as

$$\mathcal{T}_k Z_{\mathcal{C}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z_{\mathcal{C}}\begin{bmatrix}a\alpha+b\beta\\ d\beta\end{bmatrix}\left(\frac{a\tau+b}{d}\right)\,. \tag{3.170}$$

Here, we understand the parameters of the spin structure to be added modulo 2. The spin structure $(a\alpha+b\beta, d\beta)$ is the spin structure (α,β) pulled back to the covering torus. Note that if we take $\gamma \in \Gamma\backslash M_k$, then the spin structure on the covering space is written as

$$\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix} = \gamma\cdot\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\,, \tag{3.171}$$

and so, writing the spin structure as a column vector $\vec{e} = (\alpha\,\beta)^{\mathsf{T}}$, we can write the Hecke operator compactly as

$$\mathcal{T}_k Z_{\mathcal{C},\vec{e}} = \frac{1}{k}\sum_{\gamma\in\Gamma\backslash M_k} Z_{\mathcal{C},\gamma\cdot\vec{e}}(\gamma\cdot\tau)\,. \tag{3.172}$$

Ensemble averages of symmetric product orbifold partition functions

We now specialise to toroidal CFTs $\mathcal{C} = T^D$ and examine the ensemble averages, over only the Narain moduli, of partition functions of $\mathrm{Sym}^N(T^D)$.[27] Before doing so however, we should introduce the Siegel-Weil formula for higher genus—the generalisation of eq. (3.40). This is relevant when averaging partition functions of CFTs on, possibly not connected, Riemann surfaces of genus $g > 1$, like the first term of eq. (3.158). The partition function $Z_{T^D}(\mathfrak{m},\Omega)$ of a T^D CFT on a genus g Riemann surface Σ_g now depends on the moduli $\mathfrak{m} = G, B$ and the period matrix Ω of Σ_g. The latter is the higher genus analogue of the modular parameter τ, see Appendix A.6. We will not need the explicit expression of the partition function, so we write down directly the formula for the average

[27]The symmetric product orbifolds can be embedded in a larger moduli space than just the Narain part. One such example is the D1/D5 system compactified on T^4 [189], whose moduli space contains the 'orbifold point' $\mathrm{Sym}^N(\mathbb{T}^4)$ along with its Narain moduli, as well as four exactly marginal operators in the twist-2 sector. In this work, we only focus on averaging over the Narain part [3].

$$\langle Z_{T^D}(\Omega)\rangle = \frac{E_{D/2}(\Omega)}{(\det \mathrm{Im}\,\Omega)^{D/2}\left|\det'\,\bar{\partial}\right|^D}\,. \tag{3.173}$$

The determinant $\det'\,\bar{\partial}$ appearing in (3.173) is of the operator $\bar{\partial}$ on Σ_g omitting the zero-modes and generalises the Dedekind-eta contribution in eq. (3.43). We have introduced the higher genus generalization of the Eisenstein series

$$E_s(\Omega) = \sum_{\Gamma_0}\left(\det \mathrm{Im}\,\Omega_{\Gamma_0}\right)^s\,, \tag{3.174}$$

with Ω_{Γ_0} being the period matrix defined with respect to what is known as a Lagrangian sublattice Γ_0 of $H_1(\Sigma_g,\mathbb{Z})$, see Appendix A.6. This sum can also be written in terms of the Siegel modular group $Sp(2g,\mathbb{Z})$ action on the period matrix Ω of Σ_g as

$$E_s(\Omega) = \sum_{\gamma\in P\backslash Sp(2g,\mathbb{Z})}(\det \mathrm{Im}\gamma\cdot\Omega)^s\;. \tag{3.175}$$

Here $P = \left\{\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\in Sp(2g,\mathbb{Z})\right\}$ is the Siegel parabolic subgroup and leaves $\det \mathrm{Im}\Omega$ invariant. The action of $\gamma = \begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(2g,\mathbb{Z})$ is

$$\Omega\mapsto\gamma\cdot\Omega = (A\Omega+B)(C\Omega+D)^{-1}\;. \tag{3.176}$$

The Siegel modular group generalises the modular group $SL(2,\mathbb{Z})$ and P generalises Γ_∞. The combination $(\det \mathrm{Im}\Omega)^{D/2}\left|\det'\bar{\partial}\right|^D$ is invariant under the $Sp(2g,\mathbb{Z})$ action and, denoting $\left|\gamma\cdot\det'\bar{\partial}\right|^D$ the result of acting on $\left|\det'\bar{\partial}\right|^D$ with $\gamma\in Sp(2g,\mathbb{Z})$, the average can be written as

$$\langle Z_{T^D}(\Omega)\rangle = \sum_{\gamma\in P\backslash Sp(2g,\mathbb{Z})}\frac{1}{\left|\gamma\cdot\det'\bar{\partial}\right|^D}\,. \tag{3.177}$$

Equivalently, we can write this as a sum over Lagrangian sublattices $\Gamma_0\in H_1(\Sigma_g,\mathbb{Z})$ as

$$\langle Z_{T^D}(\Omega)\rangle = \sum_{\Gamma_0}\frac{1}{\left|\det'_{\Gamma_0}\bar{\partial}\right|^D}\,. \tag{3.178}$$

Where we have now labelled the determinant with the Lagrangian sublattice it corresponds to. In chapter 4, we will see that this summand can be given a bulk interpretation [74, 75]. For the sake of brevity, we have omitted some details such as properly defining the Siegel modular group, for more details on the genus g Eisenstein series and its relation to the ensemble average see e.g. ref.[74].

For disconnected Riemann surfaces, suppose we have a product of n partition functions on associated Riemann surfaces of genus g_i with period matrices Ω_i. We can form a matrix Ω which is the direct sum of the period matrices of the respective Riemann surfaces

$$\Omega = \bigoplus_{i=1}^{n}\Omega_i. \tag{3.179}$$

The ensemble average over disconnected Riemann surfaces is then given by the following generalization of the Siegel-Weil formula

$$\langle Z_{T^D}(\Omega_1)\dots Z_{T^D}(\Omega_n)\rangle = \frac{E_{D/2}(\Omega)}{\prod_{i=1}^{n}(\det \mathrm{Im}\,\Omega_i)^{D/2}\left|\det'\,\bar{\partial}_{\Sigma_{g_i}}\right|^D}\,, \tag{3.180}$$

where in the above Ω is no longer the period matrix of a single Riemann surface, but a direct sum of period matrices of disconnected Riemann surfaces.

The average might diverge

We have seen in earlier sections, that the ensemble average of the genus one partition function of the $D = 1, 2$ toroidal CFTs diverges. This can also happen in higher genera and specifically the average of the T^D partition function converges when $D - 1 > g$. For example for $g = 1$, we need $D > 2$, in agreement to what we saw before. Outside this parameter region, the ensemble average and the Eisenstein series (3.174) diverge. For the case of $n > 1$ disconnected boundaries we have

$$\langle Z_{T^D}(\Omega_1) \ldots Z_{T^D}(\Omega_n) \rangle < \infty, \qquad D - 1 > \sum_{i=1}^{n} g_i. \tag{3.181}$$

With these considerations in mind, we treat the relation (3.173) as a formal identity throughout this work, which we consider to hold even if both sides formally diverge. Since we will work with generic values of D, we can always choose D large enough such that a given expression converges. Alternatively, given that the Eisenstein series (3.174) admits an analytic continuation in D and can thus be defined for $D - 1 \leqslant g$, we can take (3.173) to define the average over the Narain moduli space when the integral itself does not converge. The interpretation as an average over Narain moduli in the sense of an integral over moduli parameters can be given only in the convergent region, see also comments in ref. [74]

An example: the double torus average

The average of products of (torus) partition functions appears a lot in the Narain average of $\mathrm{Sym}^N(T^D)$ as can be seen for example from eqs. (3.158) and (3.159). This situation corresponds to a period matrix that is the direct sum of period matrices, each being just the modular parameter of each torus

$$\Omega_i = \tau_i\,, \qquad \Omega = \mathrm{diag}(\tau_1, ..., \tau_n)\,. \tag{3.182}$$

We work out explicitly a specific example show what kind of contributions one gets from averaging such products. Specifically, consider the case with two boundaries $n = 2$ where each torus has the same modular parameter τ, so $\Omega = \mathrm{diag}(\tau, \tau)$. Applying (3.174) and (3.180) to this case, we obtain

$$\langle Z_{T^D}(\tau) Z_{T^D}(\tau) \rangle = \frac{1}{\mathrm{Im}(\tau)^D |\eta(\tau)|^{4D}} \sum_{\Gamma_0 \subset H_1(\Sigma \sqcup \Sigma, \mathbb{Z})} (\det \mathrm{Im}(\Omega_{\Gamma_0}))^{D/2}. \tag{3.183}$$

Where we have used that $\left|\det{}' \bar{\partial}\right| = |\eta(\tau)|^2$ on the torus. In the above $\Gamma_0 \subset H_1(\Sigma \sqcup \Sigma, \mathbb{Z})$ is a sum over possible contractible cycles on the two tori. This sum contains a set of contribution that give the disconnected average $\langle Z_{T^D}(\tau) \rangle^2$, in addition to contributions that do not factorise.

We now explain how to see the contribution of the disconnected average $\langle Z_{T^D}(\tau) \rangle^2$ in the sum. Let $A^{(1)}, A^{(2)}$ be the A-cycles of the two tori, while $B^{(1)}, B^{(2)}$ are their B-cycles. Take the contractible cycles specified by Γ_0 to be given by independent modular transformations γ_i of the $A^{(i)}$ cycles on the two respective torii. This corresponds to a choice of Γ_0 and Ω_{Γ_0} given by

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}} \left(\gamma_1(A^{(1)}), \gamma_2(A^{(2)}) \right), \qquad \Omega_{\Gamma_0} = \begin{pmatrix} \gamma_1 \cdot \tau & 0 \\ 0 & \gamma_2 \cdot \tau \end{pmatrix}. \tag{3.184}$$

The above choice of Γ_0 is decomposable[28] and amounts to picking all possible choices of contractible cycles on the two tori independently. For details on how to obtain Ω_{Γ_0}, see Appendix A.6. This choice immediately gives the following contribution to the average

$$\langle Z_{T^D}(\tau) Z_{T^D}(\tau)\rangle \supset \frac{1}{\mathrm{Im}(\tau)^D |\eta(\tau)|^{4D}} \sum_{\gamma_1,\gamma_2 \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \mathrm{Im}(\gamma_1 \cdot \tau)^{\frac{D}{2}} \, \mathrm{Im}(\gamma_2 \cdot \tau)^{\frac{D}{2}} \,. \tag{3.185}$$

By comparing to equation (3.43) we notice that this is the disconnected contribution squared $\langle Z_{T^D}(m,\tau)\rangle^2$. Contributions that do not factorise is such a way arise from other choices for Γ_0, an example of which is given by

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}} \left(A^{(1)} + A^{(2)}, B^{(1)} - B^{(2)} \right) . \tag{3.186}$$

We will examine this case in greater detail later when we talk about bulk duals, but we can see from now that this will correspond to some wormhole-like geometry due to its non-factorising property. To summarize, the average over products of partition functions contains disconnected contributions which can be identified with special choices of Γ_0, alongside contributions that do not factorise, which correspond to more non-trivial choices of contractible cycles.

Averaging $\mathrm{Sym}^N(T^D)$ partition functions

It is instructive to apply all of the above to the simplest case, namely $N = 2$. The torus partition function of the $\mathrm{Sym}^N(T^D)$ orbifolds reads (apply $\mathcal{C} = T^D$ to eq. (3.158))

$$Z_{\mathrm{Sym}^2(T^D)}(\tau; m) = \frac{1}{2} Z_{T^D}(\tau; m)^2 + \frac{1}{2} Z_{T^D}(2\tau; m) + \frac{1}{2} Z_{T^D}(\tfrac{\tau}{2}; m) + \frac{1}{2} Z_{T^D}(\tfrac{\tau+1}{2}; m) . \tag{3.187}$$

We split the partition function into "connected" and "disconnected" parts, where connectedness refers to whether the covering space is connected or not

$$Z_{\mathrm{Sym}^2(T^D)}(\tau; m) = Z_{\mathrm{Sym}^2(T^D),\mathrm{conn.}}(\tau; m) + Z_{\mathrm{Sym}^2(T^D),\mathrm{dis.}}(\tau; m) . \tag{3.188}$$

In terms of Siegel-Narain theta functions these are written as

$$\begin{aligned} Z_{\mathrm{Sym}^2(T^D),\mathrm{conn.}}(\tau; m) &= \frac{1}{2} \frac{\Theta_D(m, 2\tau)}{|\eta(2\tau)|^{2D}} + \frac{1}{2} \frac{\Theta_D(m, \frac{\tau}{2})}{|\eta(\frac{\tau}{2})|^{2D}} + \frac{1}{2} \frac{\Theta_D(m, \frac{\tau+1}{2})}{|\eta(\frac{\tau+1}{2})|^{2D}} , \\ Z_{\mathrm{Sym}^2(T^D),\mathrm{dis.}}(\tau; m) &= \frac{1}{2} \left(\frac{\Theta_D(m,\tau)}{|\eta(\tau)|^{2D}} \right)^2 . \end{aligned} \tag{3.189}$$

These correspond to contributions to the symmetric orbifold partition function from double-covers of the torus which are connected and disconnected, respectively. Using the Siegel-Weil formula, we can readily average the connected part as

$$\langle Z_{\mathrm{Sym}^2(T^D),\mathrm{conn.}}(\tau)\rangle = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \left(\frac{1}{|\eta(\gamma \cdot 2\tau)|^{2D}} + \frac{1}{|\eta(\gamma \cdot (\frac{\tau}{2}))|^{2D}} + \frac{1}{|\eta(\gamma \cdot (\frac{\tau+1}{2}))\,|^{2D}} \right) . \tag{3.190}$$

This can be also written as (see eq.(3.166))

[28] Intuitively, a decomposable Γ_0 amounts to picking independent contractible cycles on all surfaces[74].

$$
\begin{aligned}
\langle Z_{\mathrm{Sym}^2(T^D),\mathrm{conn.}}(\tau)\rangle &= \frac{1}{2} \sum_{\gamma' \in SL(2,\mathbb{Z})\backslash M_2} \langle Z_{T^D}(\gamma' \cdot \tau)\rangle \\
&= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \sum_{\gamma' \in SL(2,\mathbb{Z})\backslash M_2} \frac{1}{|\eta(\gamma \cdot \gamma' \cdot \tau)|^{2D}} \,.
\end{aligned} \tag{3.191}
$$

This way of writing will be very useful when we match this quantity to a bulk calculation.

The average for the disconnected part was worked out in page 88, and requires the Siegel-Weil formula for disconnected surfaces (3.180). The period matrix Ω of the disjoint union $\Sigma \sqcup \Sigma$ is given by $\Omega = \mathrm{diag}(\tau, \tau)$, and following the rules of [74], we can write the average of the disconnected component as a sum over Lagrangian sublattices Γ_0 of the total homology lattice $H_1(\Sigma \sqcup \Sigma, \mathbb{Z}) \cong H_1(\Sigma, \mathbb{Z}) \oplus H_1(\Sigma, \mathbb{Z})$, namely

$$
\langle Z_{\mathrm{Sym}^2(T^D),\mathrm{dis.}}(\tau)\rangle = \frac{1}{2\,\mathrm{Im}(\tau)^D |\eta(\tau)|^{4D}} \sum_{\Gamma_0 \subset H_1(\Sigma \sqcup \Sigma, \mathbb{Z})} (\det \mathrm{Im}(\Omega_{\Gamma_0}))^{D/2} \,, \tag{3.192}
$$

where Ω_{Γ_0} is the period matrix Ω evaluated on the sublattice Γ_0 as explained in Appendix A.6. We will see later, in the bulk discussion, that each Lagrangian sublattice Γ_0 is to be associated with a bulk geometry with boundary $\Sigma \sqcup \Sigma$ such that the boundary cycles Γ_0 are contractible in the bulk. Choices of Γ_0 correspond to either "wormhole" geometries, or completely disconnected bulk geometries.

Averaging partition functions for general N

The partition function of $\mathrm{Sym}^N(T^D)$ is given by eq. (3.164)

$$
Z_{\mathrm{Sym}^N(T^D)}(\tau; m) = \sum_{\text{partitions of } N} \prod_{k=1}^{N} \frac{1}{N_k!} \left(T_k Z_{T^D}(\tau; m)\right)^{N_k} \,. \tag{3.193}
$$

Using the Siegel-Weil formula (3.180), we can take the average

$$
\langle Z_{\mathrm{Sym}^N(T^D)}(\tau)\rangle = \sum_{\text{partitions of } N} \langle \prod_{k=1}^{N} \frac{1}{N_k!} \left(T_k Z_{T^D}(\tau)\right)^{N_k} \rangle \,. \tag{3.194}
$$

This is a quite complicated formula especially because of contributions of averages of products of T^D partition functions. We can again split this average into "connected" and "disconnected" contributions. The former include only contributions from connected covering spaces and using eq. (3.163) can be written as

$$
Z_{\mathrm{Sym}^N(T^D),\mathrm{conn.}}(\tau; m) = T_N Z_{T^D}(\tau; m) = \frac{1}{N} \sum_{ad=N} \sum_{b=0}^{d-1} Z_{T^D}\left(\tfrac{a\tau+b}{d}; m\right) \,. \tag{3.195}
$$

The average of this can be calculated as

$$
\begin{aligned}
\langle Z_{\mathrm{Sym}^N(T^D),\mathrm{conn.}}(\tau)\rangle &= \frac{1}{N} \sum_{ad=N} \sum_{b=0}^{d-1} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \frac{1}{\left|\eta\left(\frac{a\tau+b}{d}\right)\right|^{2D}} \\
&= \frac{1}{N} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \sum_{\gamma' \in SL(2,\mathbb{Z})\backslash M_N} \frac{1}{|\eta(\gamma \cdot \gamma' \cdot \tau)|^{2D}} \,.
\end{aligned} \tag{3.196}
$$

Where again we wrote the average as in eq. (3.191).

The "disconnected" contribution $Z_{\mathrm{Sym}^2(T^D),\mathrm{dis.}}(\tau;m)$ is more difficult to write out explicitly, but it would contain sums over multiple copies of partition functions with different modular parameters. As an explicit example, from (3.159) we see that for $N=3$ the "disconnected" piece would be given by

$$\begin{aligned}Z_{\mathrm{Sym}^2(T^D),\mathrm{dis.}}(\tau;m) = \frac{1}{6}Z_{T^D}(\tau;m)^3 + \frac{1}{2}Z_{T^D}(\tau;m)Z_{T^D}(2\tau;m)\\ + \frac{1}{2}Z_{T^D}(\tau;m)Z_{T^D}(\tfrac{\tau}{2};m) + \frac{1}{2}Z_{T^D}(\tau;m)Z_{T^D}(\tfrac{\tau+1}{2};m).\end{aligned} \tag{3.197}$$

We can again apply the Siegel-Weil formula for disconnected surfaces (3.180) to perform the average over the "disconnected" partition function. This will again contain multiple contributions that are factorised or not depending on the choice of Lagrangian sublattices.

Adding supersymmetry

Around page 59, we saw what happens when we introduce fermions to a T^D CFT and wrote down the partition function for a given spin structure. We can also consider the symmetric product orbifold of this supersymmetric theory, the only modification being the inclusion of the fermion partition function. It is easier to directly consider the grand canonical partition function in equation (3.169) which we showed can be written in terms of supersymmetric Hecke operators $\mathcal{T}_k$

$$\mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau) = \exp\left(\sum_{k=1}^{\infty} p^k \mathcal{T}_k Z_{T^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau;m)\right). \tag{3.198}$$

We can average the above partition function in the same way we carried out the bosonic average. Since the fermions do not depend on the moduli they factor out of the average. It is convenient to consider the average of the connected part of the partition function

$$\begin{aligned}\int_{\mathcal{M}_D} d\mu \log \mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau) &= \sum_{k=1}^{\infty} p^k \left\langle \mathcal{T}_k Z_{T^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\right\rangle\\ &= \sum_{k=1}^{\infty}\frac{p^k}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z_{\mathrm{ferm}}\begin{bmatrix}a\alpha+b\beta\\ d\beta\end{bmatrix}\left(\frac{a\tau+b}{d}\right)^D\left\langle Z_{T^D}\left(\frac{a\tau+b}{d}\right)\right\rangle,\end{aligned} \tag{3.199}$$

where we have split it up into the bosonic part which the average acts on, and the fermionic part. We will again reproduce the above average from a supersymmetric bulk theory. The above average can be calculated as in eq. (3.196).

3.4 Symmetric product orbifold correlation functions and ensemble averages

So far we have only seen how to calculate partition functions of, mainly toroidal, CFTs and their orbifolds. For the case of $\mathrm{Sym}^N(T^D)$, we now also discuss the calculation and ensemble averages (again over Narain moduli) of correlation functions.[29] We focus on correlators on the sphere $\mathbb{CP}^1$ and, after a general introduction, we will put $N=2$.

[29]Ensemble averages of toroidal orbifold correlation functions were studied also in [166] in the context of $\mathbb{Z}_N$ orbifolds.

Being an orbifold theory, the symmetric product orbifold $\mathrm{Sym}^N(\mathcal{C})$ of a CFT $\mathcal{C}$ contains non-local operators called twist fields. These implement monodromies of the fields of the theory. Given two points on the sphere z, ζ, we denote, as in [190], by $e^{2\pi i}z + \zeta$ the monodromy around ζ. Also, recall that the fields of the tensor product $\mathcal{C}^{\otimes N}$ are collected in the N-tuple $\mathbf{\Phi} = (\Phi_1, ..., \Phi_N)$. Then, a twist operator σ_ω associated to a permutation $\omega \in S_N$ is defined by the monodromy relation

$$\mathbf{\Phi}(e^{2\pi i}z + \zeta)\,\sigma_\omega(\zeta) = (\omega \cdot \mathbf{\Phi})(z + \zeta)\,\sigma_\omega(\zeta)\,, \tag{3.200}$$

where

$$\omega \cdot \mathbf{\Phi} = (\Phi_{\omega(1)}, \ldots, \Phi_{\omega(N)})\,. \tag{3.201}$$

On their own, twist fields σ_ω are not gauge-invariant. We can see this as follows. Consider the monodromy of an element $(\rho \cdot \mathbf{\Phi})(z)$ around the twist field $\sigma_\omega(\zeta)$. The twist field acts on $\mathbf{\Phi}(z)$:

$$(\rho \cdot \mathbf{\Phi})\,(e^{2\pi i}z + \zeta)\sigma_\omega(\zeta) = (\rho \cdot \omega \cdot \mathbf{\Phi})\,(z + \zeta)\sigma_\omega(\zeta) = \Big(\rho \cdot \omega \cdot \rho^{-1} \cdot (\rho \cdot \mathbf{\Phi})\Big)\,(z + \zeta)\sigma_\omega(\zeta)\,. \tag{3.202}$$

From this we can infer the action of the twist fields on the field $(\rho \cdot \mathbf{\Phi})\,(z)$, which lies on the same gauge slice as $\mathbf{\Phi}(z)$:

$$(\rho \cdot \mathbf{\Phi})\,(e^{2\pi i}z + \zeta)\sigma_\omega(\zeta) = (\rho \cdot \mathbf{\Phi})\,(e^{2\pi i}z + \zeta)\sigma_{\rho\omega\rho^{-1}}(\zeta)\,. \tag{3.203}$$

Where on the RHS the twist field acts on $(\rho \cdot \mathbf{\Phi})$ and on the LHS on $\mathbf{\Phi}(z)$. Hence under an overall permutation $\rho \in S_N$, twist fields transform as

$$\sigma_\omega \to \sigma_{\rho\omega\rho^{-1}}\,. \tag{3.204}$$

From a twist field σ_ω, we can construct a gauge-invariant twist field $\sigma_{[\omega]}$ by

$$\sigma_{[\omega]} = \mathcal{N}_{[\omega]} \sum_{\rho \in [\omega]} \sigma_\rho\,, \tag{3.205}$$

where $\mathcal{N}_{[\omega]} = 1/\sqrt{|[\omega]|}$ is a normalization factor so that $\sigma_{[\omega]}$ is canonically normalized.[30] Note that $\sigma_{[\omega]}$ only depends on the conjugacy class of ω in S_N. A generic correlator of gauge-invariant twist fields thus takes the form

$$\left\langle \prod_{i=1}^{n} \sigma_{[\omega_i]}(x_i) \right\rangle = \prod_{i=1}^{n} \left(\mathcal{N}_{[\omega_i]}\right) \sum_{\rho_1, ..., \rho_n \in S_N} \left\langle \prod_{i=1}^{n} \sigma_{\rho_i \omega_i \rho_i^{-1}}(x_i) \right\rangle\,. \tag{3.206}$$

That is, the correlators of gauge-invariant twist fields can be expressed purely in terms of an appropriate sum over correlators of the 'pure' twist fields σ_ω. From now on, we calculate only the correlators of pure twist fields σ_ω, keeping in mind that we should sum over conjugacy classes to obtain a gauge-invariant result. Expression (3.206) will in general have disconnected contributions, meaning contributions for which some terms of the right hand side factorize. Here we focus on the connected part of the correlators. This can be done by looking at correlators for which the group elements that appear in the twist fields generate a transitive subgroup of the permutation group acting on the elements of $\{1, 2, ..., N\}$ that appear in the correlator (see e.g. [190]).

[30] Here, by 'canonically normalized' we mean that the two-point function satisfies $\langle \sigma_{[\omega]}(x_1)\sigma_{[\omega]}(x_2)\rangle = 1/(x_1 - x_2)^{2h(\omega))}$, where $h(\omega)$ is the conformal weight of the twist field.

We calculate the sphere correlation functions

$$\langle\sigma_{\omega_1}(x_1)...\sigma_{\omega_n}(x_n)\rangle_{\mathbb{CP}^1} \equiv \langle\sigma_{\omega_1}(x_1)...\sigma_{\omega_n}(x_n)\rangle \ , \ n = 1, 2, ... \tag{3.207}$$

using the method of refs. [191, 192]. The idea is similar to the calculation of partition functions where one goes to the covering space of the base torus. Here the twist fields implement twisted boundary conditions of the fields of $\mathcal{C}$ on the punctured sphere $\mathbb{CP}^1 \setminus \{x_1, ..., x_n\}$. Consequently, we need coverings $\Gamma : \widetilde{\Sigma} \to \mathbb{CP}^1$, that are now ramified, such that the fields are single-valued on $\widetilde{\Sigma}$. Pulling back the fields to the covering picks up a conformal anomaly that is proportional to the central charge of the CFT. The resulting correlation function takes the form

$$\langle\sigma_{\omega_1}(x_1)\cdots\sigma_{\omega_n}(x_n)\rangle = e^{-S_L[\phi_\Gamma]} Z_{\mathcal{C}}(\widetilde{\Sigma}) , \tag{3.208}$$

where $Z_{\mathcal{C}}(\widetilde{\Sigma})$ is the partition function of the seed CFT $\mathcal{C}$ on $\widetilde{\Sigma}$ and S_L is the Liouville action given by

$$S_L[\phi] = \frac{c}{48\pi}\int_\Sigma d^2z\sqrt{g}\,(-\phi\Delta\phi + R\phi) \ , \tag{3.209}$$

where c is the central charge of the seed CFT $\mathcal{C}$, R is the scalar curvature on Σ, and Δ is the Laplacian on Σ.[31] The conformal anomaly is found by evaluating this action on the scalar

$$\phi_\Gamma = \log|\partial\Gamma|^2 . \tag{3.210}$$

This corresponds to a metric on the covering space

$$ds^2 = e^{\phi_\Gamma} dz\, d\bar{z} \tag{3.211}$$

where $z, \bar{z}$ are local coordinates on the covering space. In eq. (3.209), g is the fiducial metric $dzd\bar{z}$—the Weyl-transformation of the pullback via Γ of the metric on $\mathbb{CP}^1$.

Specialising to $\mathcal{C} = T^D$, we can express the correlation functions of twist fields as

$$\langle\sigma_{\omega_1}(x_1)\cdots\sigma_{\omega_n}(x_n)\rangle = e^{-S_L[\phi_\Gamma]} Z_{T^D}(\widetilde{\Sigma}; m) \ . \tag{3.212}$$

Here $Z_{T^D}(\Sigma; m)$ is the toroidal CFT partition function on $\widetilde{\Sigma}$ and m denote again the moduli of G, B of the CFT. This can be now immediately averaged over the Narain moduli m as the only moduli dependence lies in the partition function

$$\int_{\mathcal{M}_D} d\mu(m)\,\langle\sigma_{\omega_1}(x_1)\ldots\sigma_{\omega_n}(x_n)\rangle = e^{-S_L[\phi_\Gamma]}\int_{\mathcal{M}_D} d\mu(m)\, Z(\widetilde{\Sigma}; m) . \tag{3.213}$$

The genus $\tilde{g}$ of the Riemann surface $\widetilde{\Sigma}$ on which we have to calculate the partition function, is given by the Riemann-Hurwitz formula in terms of the genus g of the base space, the number M of distinct elements of $\{1, 2..., N\}$ that appear in the correlator and the lengths w_j, $j = 1, 2, ..., n$ of the cycles of ω_is as

$$\tilde{g} - 1 = M(g-1) + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1) \ . \tag{3.214}$$

Here we focus on sphere correlators, $g = 0$ and thus

$$\tilde{g} = 1 - M + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1) . \tag{3.215}$$

[31] In our conventions, we have $\sqrt{g}\,\Delta\phi = \partial(\sqrt{g}\,\bar{\partial}\phi) + \bar{\partial}(\sqrt{g}\,\partial\phi)$, where $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$.

Four-point correlation functions

We focus now on four-point correlators of $\mathrm{Sym}^2(T^D)$. S_2 is abelian, hence each element of the group is its one conjugacy class and a gauge-invariant correlator is

$$\langle \sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle\,, \tag{3.216}$$

where we used the $SL(2,\mathbb{C})$ isometry of the sphere to fix three points to $0, 1, \infty$. This correlator is also connected as the permutation (12) acts in a transitive way on the elements $\{1, 2\}$. The genus of the covering space $\widetilde{\Sigma}$ is $\tilde{g} = 1$ and we denote its modular parameter as τ. The Liouville action and covering map can be computed explicitly [186] and we find

$$\langle \sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle = \left(\frac{1}{2^{2c/3}|u(1-u)|^{c/12}}\right)\left(\frac{\Theta_D(m,\tau)}{|\eta(\tau)|^{2D}}\right)\,. \tag{3.217}$$

The average of the four-point function of $\sigma_{(1\,2)}$ is computed using the Siegel-Weil formula, and we find

$$\begin{aligned}\int_{\mathcal{M}_D} & d\mu(m)\ \langle \sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle \\ &= \frac{1}{2^{2c/3}|u(1-u)|^{c/12}} \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}\,,\end{aligned} \tag{3.218}$$

the central charge c is just the dimension D of the torus target.[32] τ admits a closed-form expression in terms of hypergeometric functions

$$\tau = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2},1:u)}{{}_2F_1(\frac{1}{2},\frac{1}{2},1:1-u)}\,. \tag{3.219}$$

For higher point correlators, inserting an even number $2g + 2$ twist operators the calculation is reduced to the partition function of T^D on a genus-g hyperelliptic curve $\widetilde{\Sigma}_g$

$$\langle \sigma_{(1\,2)}(x_1)\cdots\sigma_{(1\,2)}(x_{2g+2})\rangle = e^{-S_L[\Phi_\Gamma]} Z(\Sigma_g)\,, \tag{3.220}$$

This can be ensemble averaged again using the Siegel-Weil formula.

We end this chapter we a short summary.

3.5 Summary

The main goal of this chapter was to introduce two-dimensional conformal field theories and the ensemble averages. Specifically, the focus was on Narain theories and their orbifolds.

We started by discussing the free boson and the free boson compactified on a circle (both not orbifolded yet) to set the stage. Along the way, we introduced important topics such as the two-dimensional torus, the fundamental domain and, of course, partition

[32] We keep the central charge c arbitrary since, for example, for Narain theories with supersymmetry, the central charge is instead $c = 3D/2$.

functions. Having seen that the ensemble average of the compactified boson's partition function diverges, we moved on to higher dimensional theories. Now the target space is a D-dimensional torus T^D. The partition functions and moduli spaces of these theories were written down and the ensemble average was performed via the Siegel-Weil formula. We introduced the Eisenstein series and saw that the average of the partition function converges for $D \geqslant 3$. Next, we showed how this generalises to T^D theories with supersymmetry and discovered that the ensemble average is almost identical to the non-supersymmetric case due to the nice modular properties and no moduli dependence of the fermionic partition function.

We continued by introducing orbifolds of Narain theories, again starting with a simple example—the $\mathbb{Z}_2$ orbifold of the compactified boson. The partition function was written and a discussion was given on orbifolds in general. We ended this subsection with a generalisation of this orbifold to higher dimensions and the ensemble average of the partition function. Again, since the new contributions to the partition function due to the orbifold do not depend on toroidal moduli and since the moduli space is the same as the non orbifolded theory, the ensemble average boils down to doing the T^D average.

Next, we moved on to the orbifolds studied in ref. [2]. These split into two classes: factorisable and non-factorisable. We started by discussing the T^2 target space example where we defined carefully the orbifolds, their moduli spaces and partition functions and (regulated) ensemble averages. Then, we generalised this to higher dimensions and were able to write down the ensemble averaged partition functions in terms of a finite sum of Eisenstein series.

The last two subsections of this chapter were dedicated to the orbifolds studied in ref. [3]: symmetric product orbifolds. We defined these orbifolds and wrote their partition functions using the covering space method. We saw how these are written in terms of Hecke operators and their supersymmetric generalisations. Then, we calculated their ensemble averages over the toroidal moduli in terms of modular sums, or equivalently, Lagrangian sublattices of the first homology of the Riemann surface the CFT is defined on. Finally, we saw how ensemble averaging can be performed for correlators of symmetric product orbifolds.

Chapter 4

The bulk duals of ensemble averaged conformal field theories

In this chapter, we want to see how the formulas we derived in chapter 3 can be interpreted holographically. To this end, we start with the original correspondence studied in refs. [74, 75] to set the stage and continue with our works [2, 3]. Here we also follow refs. [193, 194, 195].

4.1 The Narain-$U(1)^{2D}$ correspondence

Recall that the torus partition function of a T^D Narain CFT at a point $\mathfrak{m} \in \mathcal{M}_{T^D}$ is given in terms of the Siegel-Narain theta function as in equation (3.34)

$$Z_{T^D}(\tau; \mathfrak{m}) = \frac{1}{|\eta(\tau)|^{2D}} \Theta_D(\mathfrak{m}, \tau) .$$

We wrote down the ensemble average in eq. (3.43) using the Siegel-Weil formula

$$\langle Z_{T^D}(\tau) \rangle = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{2D}} .$$

This modular sum, also called "Poincaré series", over $\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})$ images of the complex structure τ can be interpreted as a sum over three-dimensional geometries [74, 75]. Specifically, from a holographic point of view, we expect the conformal boundary of the bulk to be a $g = 1$ surface Σ (since this is where the CFT lives), which is topologically $S^1 \times S^1$. We can choose one of those circles to correspond to "time" and one to "space", like we did for the CFT partition function. Now, making the spatial cycle contractible (filling it in) we get a bulk geometry, typically called "thermal AdS", of the form $M \simeq D_2 \times S^1$, where D_2 is a disk. These kinds of geometries are called handlebodies and can be endowed with a hyperbolic metric. All other handlebodies, with the same conformal boundary, are found by making other boundary cycles contractible in the bulk. These other cycles are given in terms of the τ-cycle via a modular transformation $\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})$ (since elements in Γ_∞ do not give a new geometry). For instance, making the "time" cycle contractible gives the so-called Euclidean BTZ black hole and is related to thermal AdS via an S transformation of the τ-cycle. Hence the $\Gamma_\infty \backslash SL(2, \mathbb{Z})$ modular sum above can be viewed as sum over handlebodies—or three-dimensional, hyperbolic manifolds with conformal boundary Σ. We can think of these as the space inside an orientable surface embedded in $\mathbb{R}^3$, see fig. 4.1.

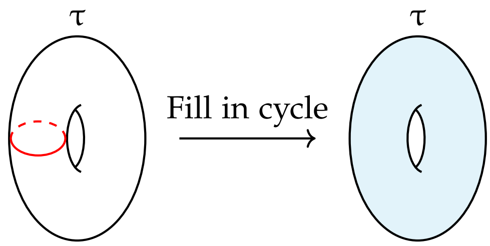

Figure 4.1: Filling in a cycle of the conformal boundary torus, we get a handlebody.

We saw a way in which the sum can be viewed as a sum over bulk geometries but how is the summand related to a bulk theory? To answer this, note that the boundary CFT has a $U(1)^D \times U(1)^D$ current algebra hence one expects that the bulk has a $U(1)^D \times U(1)^D$ gauge symmetry.[1] Specifically, by the standard holographic dictionary, these currents are dual to gauge fields in the bulk.

Indeed, the vacuum character $\chi_0^{U(1)}(\tau)$, i.e. the trace $\mathrm{tr}_{\mathcal{H}_0} q^{L_0 - \frac{c}{24}}$ where $\mathcal{H}_0$ is the Hilbert space built by the highest weight state $|0\rangle$, corresponding to a $U(1)$ current algebra is

$$\chi_0^{U(1)}(\tau) = \frac{1}{\eta(\tau)} \tag{4.1}$$

and hence the summand of the modular sum can be written as

$$\frac{1}{|\eta(\tau)|^{2D}} = \chi_0^{U(1)}(\tau)\overline{\chi_0^{U(1)}(\tau)}\,. \tag{4.2}$$

This is the vacuum character of $U(1)^D \times U(1)^D$ and refs. [74, 75] relate this to a $U(1)^D \times U(1)^D$ Chern-Simons theory on a handlebody M with $\partial M = \Sigma$ (M and Σ are oriented with compatible orientations). We can think of this Chern-Simons theory in terms of the action

$$S_{CS} = i\sum_{i=1}^{D}\int_M \left(A^i \wedge dA^i - B^i \wedge dB^i\right) - \frac{1}{2}\int_{\partial M} d^2z\sqrt{g}g^{ab}\left(A^i_a A^i_b + B^i_a B^i_b\right)\,, \tag{4.3}$$

where in the above we have included the proper boundary term with boundary metric g_{ab}, which corresponds to a choice of boundary Riemann surface. Also, A^i and B^i are independent gauge fields corresponding to the two copies of $U(1)^D$ (so the i's are gauge indices), dual to the boundary currents $\partial X^I, \bar{\partial} X^I$. A choice of boundary conditions that make the variational problem well defined are given by asymptotically fixing $A_{\bar{z}} = 0$ and $B_z = 0$, see [196, 197, 198].[2] The vacuum character that appears in the average is then identified with the one-loop (exact) partition function of this $U(1)^D \times U(1)^D$ Chern-Simons theory. This is a one-loop determinant expanded around the trivial flat connection and can be written in terms of $\det' \Delta_0$ and $\det' \Delta_1$, the regularised Laplacian determinants on hyperbolic three-space $\mathbb{H}^3$ acting on spin 0 and spin 1 fields respec-

[1] The current algebra is the algebra of the Laurent modes of the currents (chiral fields of conformal dimension $h = 1$) of the CFT. For example for the boson X on a circle, we have the currents ∂X, $\bar{\partial} X$. See e.g. [171].

[2] It turns out that the only bulk configurations that contribute also have $A_z = B_{\bar{z}} = 0$ on the boundary. This can be seen by noticing that the holonomy of A, B around the contractible cycle must vanish since the connection is flat [198].

tively, as (see also [199, 200])

$$Z_{\mathrm{CS}}^{\mathrm{U}(1)^{2\mathrm{D}}}(\tau) = \left(\frac{(\det{}'\Delta_0)^{3/2}}{(\det{}'\Delta_1)^{1/2}}\right)^{\mathrm{D}} = \frac{1}{|\eta(\tau)|^{2\mathrm{D}}}. \tag{4.4}$$

Where we wrote $\mathrm{U}(1)^{2\mathrm{D}}$ instead of $\mathrm{U}(1)^{\mathrm{D}} \times \mathrm{U}(1)^{\mathrm{D}}$ for ease of notation. This is a perturbative calculation and the statement that the bulk theory is a $\mathrm{U}(1)^{\mathrm{D}} \times \mathrm{U}(1)^{\mathrm{D}}$ Chern-Simons is a rather subtle one. For a discussion of subtleties, such as the compactness or not of the gauge group, see refs.[74, 75].[3] We will not care much about these subtleties in this work and take the 'seed' bulk partition function to be given by eq. (4.4). The logic on defining the bulk dual lies more towards defining it via the result of the average. This will be further motivated soon when we talk about disconnected boundaries.

The modular sum that appears in the ensemble average of the T^{D} CFT partition function on Σ can now be interpreted as a sum over handlebodies with boundary Σ of the one-loop partition function of the $\mathrm{U}(1)^{\mathrm{D}} \times \mathrm{U}(1)^{\mathrm{D}}$ Chern-Simons theory we just described. Schematically, we have

$$\langle Z_{\mathrm{T}^{\mathrm{D}}}(\tau)\rangle = \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2\mathrm{D}}} = \sum_{\substack{\text{handlebodies } M\\ \text{with } \partial M=\Sigma}} Z_{\mathrm{CS}}^{\mathrm{U}(1)^{2\mathrm{D}}}(M)\,. \tag{4.5}$$

The modular sum is a sum over Lagrangian sublattices of the boundary torus' homology $H_1(\Sigma,\mathbb{Z})$ and the γ in each term specifies which boundary cycle becomes contractible in the bulk. Writing $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ as below eq. (3.42), the cycle that becomes contractible is $c\tau + d$.

Note that in this case where the boundary is of genus one, $H_1(\Sigma,\mathbb{Z})$ can be viewed as a lattice $\mathbb{Z}^2$. Then any combination $c\tau+d$, with coprime c, d is a Lagrangian sublattice and the sum over coprime c, d is naturally identified with a sum over Lagrangian sublattices of $H_1(\Sigma,\mathbb{Z})$, see also [74].

More generally, given a connected Riemann surface Σ_g of genus g, there is a one-to-one correspondence between Lagrangian sublattices of $H_1(\Sigma_g,\mathbb{Z})$ and handlebodies with conformal boundary Σ_g. Specifically, like in the genus one case, a choice Γ_0 of a Lagrangian sublattice dictates which cycles of Σ_g become contractible in the bulk. Note however that to the same Γ_0 one can associate also other (infinitely many) three-dimensional manifolds but at least for handlebodies there is a 1-1 correspondence.

When the CFT is defined on Σ_g, the average of the partition is given by eq. (3.178) and using a similar calculation, the summand $1/\left|\det{}'_{\Gamma_0}\bar\partial\right|^{\mathrm{D}}$ appearing in the same equation can be viewed as one-loop determinant of a $\mathrm{U}(1)^{\mathrm{D}} \times \mathrm{U}(1)^{\mathrm{D}}$ Chern-Simons theory on the handlebody, with conformal boundary the genus g surface Σ_g of the CFT, that corresponds to the Lagrangian sublattice Γ_0. Summing over Lagrangian sublattices then sums over all handlebodies with boundary Σ_g.

Note that the sum over geometries does not include all possible bulks and is restricted to handlebodies. Naively one would take all bulk geometries but the Siegel-Weil formula essentially dictates which ones to keep. It tells us to sum over Lagrangian sublattices of the boundary first homology; a sum which in the case that the boundary is connected corresponds to a sum over handlebodies. In other words

[3] For a definition of the bulk theory as a limit of microscopically well-defined theories see [71].

$$\Sigma_g \text{ is connected } \rightarrow \sum_{\Gamma_0 \in H_1(\Sigma_g, \mathbb{Z})} = \sum_{\substack{\text{handlebodies } M \\ \text{with } \partial M = \Sigma_g}} . \tag{4.6}$$

In the case where the CFT is on a disconnected Riemann surface $\Sigma = \Sigma_{g_1} \sqcup ... \sqcup \Sigma_{g_n}$ (where Σ_{g_i} is a Riemann surface of genus g_i) the bulk picture is more complicated but the average given by (3.180), which we rewrite for convenience, can still be given a bulk interpretation

$$\langle Z_{\mathbb{T}^D}(m, \Omega_1) \dots Z_{\mathbb{T}^D}(m, \Omega_n) \rangle = \frac{\sum_{\Gamma_0} \left(\det \operatorname{Im} \Omega_{\Gamma_0} \right)^{\frac{D}{2}}}{\prod_{i=1}^{n} (\det \operatorname{Im} \Omega_i)^{\frac{D}{2}} \left| \det' \bar{\partial}_{\Sigma_{g_i}} \right|^D} . \tag{4.7}$$

Each Lagrangian sublattice Γ_0 in the sum again corresponds to a bulk manifold with certain asymptotic cycles contractible in the interior. This sum includes both disconnected handlebody contributions whose each connected component appeared when averaging a single partition function, as well as new wormhole contributions where the bulk manifold connects multiple disconnected boundaries. A crucial difference is that there is no apparent way—like with the handlebodies—to choose a bulk manifold given a Lagrangian sublattice.

For an independent bulk computation of (4.7) we should evaluate the Chern-Simons path integral on the bulk manifold specified by Γ_0. For wormhole geometries there are again infinitely many bulk manifolds with the same contractible cycles specified by a particular Γ_0, and it is unclear which one is picked out by the Narain average. Furthermore, we are left with the problem of evaluating the Chern-Simons partition function on the given wormhole geometry, for which we know of no general results, but see comments in [74]. We will forgo these issues and assume the bulk theory is directly defined by (4.7).

It seems that in this bulk gravitational-like theory the "bulk manifolds" are Lagrangian sublattices of $H_1(\Sigma, \mathbb{Z})$ and more information about what the bulk is is not accessible to it. This theory is also called U(1) gravity [75].

4.2 The bulk dual of the symmetric product orbifold

We are now ready to describe the bulk dual proposal of ref. [3]. For the original bulk construction of refs.[74, 75], we saw that the currents on the boundary correspond to gauge fields in the bulk. Denoting these currents by $J^I, \bar{J}^I$, we have the correspondence

$$J^I = \partial X^I \rightarrow A^I , \quad \bar{J}^I = \bar{\partial} X^I \rightarrow B^I , \tag{4.8}$$

where A, B are independent gauge fields. The gauge group is dictated by the current algebra $U(1)^D \times U(1)^D$ and the bulk action (omitting boundary terms) reads

$$S = \int_M (A \wedge dA - B \wedge dB) , \tag{4.9}$$

with an implicit summation of gauge indices (see eq. (4.3)).

We would like to extend this now to symmetric product orbifolds T^D/S_N. In this case, the boundary theory has N copies of a T^D conformal field theory and a "natural" guess is (again omitting suitable boundary terms)

$$S = \sum_{i=1}^{N} \int_M \left(A_{(i)} \wedge dA_{(i)} - B_{(i)} \wedge dB_{(i)} \right) . \tag{4.10}$$

Where again the $U(1)^D$ indices are implicit. The boundary CFT has N copies of $U(1)^D \times U(1)^D$ currents which translates to N copies of the bulk $U(1)^D \times U(1)^D$ gauge symmetry

$$A_{(i)} \to A_{(i)} + d\lambda^A_{(i)} , \quad B_{(i)} \to B_{(i)} + d\lambda^B_{(i)} . \tag{4.11}$$

But there is more than that since on the boundary we quotient (orbifold) by S_N. Under the S_N action, which is a symmetry of the boundary CFT, the currents are permuted

$$J^I \to J^{\omega(I)}, \bar{J}^I \to \bar{J}^{\omega(I)} , \quad \omega \in S_N . \tag{4.12}$$

This translates to a symmetry of the bulk Chern-Simons theory that acts as

$$A_{(i)} \to A_{(\omega(i))}, B_{(i)} \to B_{(\omega(i))} , \quad \omega \in S_N . \tag{4.13}$$

Since we are orbifolding by the S_N symmetry, we expect to have a S_N gauge symmetry in the bulk. The full gauge group of the bulk dual of the T^D/S_N permutation orbifold should then be the group generated by combinations of the $U(1)^D \times U(1)^D$ transformations (4.11) and the permutations (4.13). The group generated by these two transformations is a semidirect product of $(U(1)^D \times U(1)^D)^N$ with the permutation group S_N, and in the mathematical literature is denoted by the wreath product[4]

$$U(1)^D \times U(1)^D \wr S_N . \tag{4.14}$$

Armed with the above discussion, a natural duality to propose would be:

$$\begin{gathered}\text{Narain-ensemble of } T^D/S_N \text{ orbifolds} \\ \Longleftrightarrow \\ U(1)^D \times U(1)^D \wr S_N \text{ Chern-Simons coupled to topological gravity}\end{gathered} \tag{4.15}$$

Where by "coupled to topological gravity" we mean the sum over geometries. Although this gauge group is not discrete, it contains a discrete factor[5] of S_N. Thus, the above theory will exhibit behaviors universal to all discrete gauge theories. One of these is the existence of "twist operators" in the bulk, which in three-dimensional discrete gauge theories take the form of one-dimensional vortices [201, 202]. We now explain the bulk partition function of this theory and how gauging by S_N introduces new features such as vortices.

The bulk partition function

By similar arguments as in the CFT orbifold discussion in page 80, the partition function of the $U(1)^D \times U(1)^D \wr S_N$ theory will include gauge field configurations with non-trivial S_N monodromies around non-contractible loops of the three-manifold M on which the gauge theory is defined. Specifically, denoting by $\mathbf{\Phi}$, the field content of the gauge theory and by ω, p an element of S_N and the three-manifold M respectively we have the identification

$$\mathbf{\Phi}(p) \sim \omega \cdot \mathbf{\Phi}(p) , \text{ for all } p \in M . \tag{4.16}$$

[4]The wreath product is just mathematical notation meaning the gauge group which is generated by composing (4.11) and (4.13).

[5]Indeed, $U(1)^D \times U(1)^D \wr S_N$ is equivalent to $(U(1)^D \times U(1)^D)^N \times S_N$ as a topological space (but not as a group).

Hence, we allow for field configurations that when transported around a loop γ transform as

$$\boldsymbol{\Phi}(\gamma \cdot p) = \phi(\gamma) \cdot \boldsymbol{\Phi}(p)\,, \tag{4.17}$$

where $\boldsymbol{\Phi}(\gamma \cdot p)$ is shorthand for the value of $\boldsymbol{\Phi}$ after being transported along γ and $\phi(\gamma) \in S_N$. Since $\boldsymbol{\Phi}$ and $\phi(\gamma) \cdot \boldsymbol{\Phi}$ are physically equivalent, the above should be a perfectly allowed configuration in the path integral. As in the CFT discussion, we can take the maps ϕ to live in the space

$$\mathrm{Hom}(\pi_1(M), S_N)/S_N\,, \tag{4.18}$$

the moduli space of flat S_N-bundles on M. The quotient by S_N acts as conjugation and comes from the fact that conjugating ϕ by an element of S_N represents the same physical field configuration. A set of twisted boundary conditions $\phi : \pi_1(M) \to S_N$ can be identified with a covering space $\widetilde{M} \to M$ and the partition function of the $U(1)^D \times U(1)^D \wr S_N$ gauge theory can be written as a sum over these coverings

$$Z_{U(1)^{2D} \wr S_N}(M) = \sum_{\substack{\widetilde{M}\to M \\ \text{degree } N}} \frac{Z_{U(1)^{2D}}(\widetilde{M})}{|\mathrm{Aut}(\widetilde{M} \to M)|}\,. \tag{4.19}$$

What we have done is to take N copies of a gauge theory on $U(1)^D \times U(1)^D$, and then to gauge the S_N permutation symmetry of the resulting product theory. This is equivalent to defining a gauge theory with gauge group $U(1)^D \times U(1)^D \wr S_N$, and the arguments for computing path integrals in symmetric orbifold theories that we used in earlier sections carries over, and we arrive at (4.19), in direct analogy to the logic that allowed us to derive equation (3.152).[6]

Note that so far we have included twisted boundary conditions around non-contractible cycles of M. We will need to introduce such boundary conditions also around contractible cycles but to do so we have to include vortices in the gauge theory.

Including vortices

An interesting property of gauge theories with discrete gauge groups is the presence of vortices. These are codimension 2 objects that implement twisted boundary conditions to the gauge fields that are transported around them. They are the three-dimensional analogues of the twist fields we saw in page 92.

On a more formal level, a vortex is a codimension-2 sublocus L of a three-manifold M which carries charge $[\omega]$, where $[\omega]$ is some conjugacy class of elements of the discrete gauge group S_N.[7] We can formally write a vortex operator $\mathcal{V}_{[\omega],L}$ of charge $[\omega]$ associated to a sublocus L as the operator

$$\mathcal{V}_{[\omega],L} = \sum_{\sigma\in[\omega]} \mathcal{V}_{\sigma,L}\,, \tag{4.20}$$

[6]In fact, given any TQFT Z, not just gauge theories, one can construct a 'symmetric product' theory by formally averaging Z over covering spaces, see [203]. Such TQFTs are useful, for example, in computing generalizations of Hurwitz numbers, see also [204].

[7]We assign a charge to $\mathcal{V}$ in terms of conjugacy classes of S_N because otherwise the vortex $\mathcal{V}$ would not be a gauge-invariant object. This is because gauge transformations would act on a charge ω as $h\omega h^{-1}$. This generically changes the group element ω, but leaves it in the conjugacy class. This is analogous to our discussion of twist fields around page 92. For abelian groups like those considered in [166] this problem does not arise.

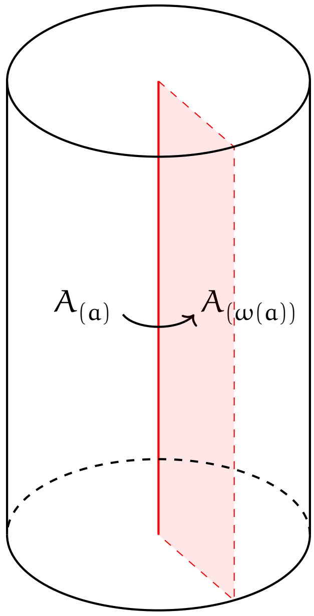

Figure 4.2: The effect of including a vortex associated to a permutation ω in the bulk dual of T^D/S_N.

where we sum over permutations σ in the conjugacy class $[\omega]$, which makes the operator gauge invariant. In discrete gauge theories the vortex operator cannot typically be represented in terms of fundamental fields of the theory. Its action is implemented by imposing twisted boundary conditions on the fields as they are transported around the vortex: transporting the fields A_I, B_I around $\mathcal{V}_{\sigma,L}$ enforces the twisted boundary condition $A_{(i)} \to A_{(\sigma(i))}$ and $B_{(i)} \to B_{(\sigma(i))}$.

To calculate the partition function of $U(1)^D \times U(1)^D \wr S_N$ in the presence of a charge $[\omega]$ vortex L, we sum over all homomorphisms $\phi : \pi_1(M \setminus L) \to S_N$ up to conjugation. This is done alongside the requirement that given a generator $\ell \in \pi_1(M \setminus L)$ of the fundamental group of $M \setminus L$ that winds once around L, its evaluation under ϕ is an element of $[\omega]$. In other words $\phi(\ell) \in [\omega]$. Now conjugacy classes of S_N are labelled by cycle types of permutations and say that $[\omega]$ is the conjugacy class of cycle type $w_1, ..., w_k$ where

$$w_1 + ... + w_k = N \,. \tag{4.21}$$

Then a homomorphism $\phi : \pi_1(M \setminus L) \to S_N$ with $\phi(\ell) \in [\pi]$ defines a covering space of M which is branched over L with branching structure given by the cycle type $w_1, \ldots, w_k$. In the language of Figure 4.2, this branched covering space $\widetilde{M} \to M$ is the N-fold cover such that the gauge fields are single-valued.

The partition function in the presence of a charge $[\omega]$ vortex is given by

$$Z_{U(1)^{2D}\wr S_N}(M; L, [\omega]) = \sum_{\substack{\widetilde{M}\to M \\ \text{branched over L}}} \frac{Z_{U(1)^{2D}}(\widetilde{M})}{|\mathrm{Aut}(\widetilde{M} \to M)|}\,, \tag{4.22}$$

in direct analogy to eq. (4.19). Note that this construction sums over boundary conditions in a consistent way, i.e. going around the same set of cycles in different orders gives the same boundary condition. This is also like a three-dimensional version of the covering space method used to calculate correlation function of twist fields in two-dimensional CFT, see page 93. Indeed, in the holographic picture these vortices will be dual to twist field insertions on the boundary.

Let us now do some calculations to make things more explicit.

The N = 2 example

Here we consider the $U(1)^D \times U(1)^D \wr S_N$ Chern-Simons theory for $N = 2$ on a handlebody (solid torus) M with boundary a Riemann surface Σ of genus 1. By the previous discussions, to find the partition function without vortex, we are instructed to sum over degree 2 coverings $\widetilde{M}$ of M. The fundamental group of M is $\pi_1(M) \simeq \mathbb{Z}$ and there are precisely two covering spaces: the one that consists of two copies of M and the handlebody with boundary modular parameter 2τ. For the partition function of $U(1)^D \times U(1)^D$ Chern-Simons theory on a handlebody with boundary modular parameter we use expression (4.4) and hence we have

$$Z_{U(1)^{2D}\wr S_2}(M) = \frac{1}{2}\frac{1}{|\eta(\tau)|^{4D}} + \frac{1}{2}\frac{1}{|\eta(2\tau)|^{2D}}\,, \tag{4.23}$$

where the factors of two come from the automorphism factors of the covering spaces.

Now, we can also introduce a nontrivial vortex. Let L run along the non-contractible cycle of M. The only nontrivial conjugacy class of S_2 is $[(1\,2)]$, and there are precisely two covering spaces of M branched over L with that structure: a handlebody with modular parameter $\tau/2$ and a handlebody with modular parameter $(\tau+1)/2$. Thus, the the partition function of the $U(1)^D \times U(1)^D \wr S_2$ Chern-Simons theory on M with vortex L is given by

$$Z_{U(1)^{2D}\wr S_2}(M;L) = \frac{1}{2}\frac{1}{\left|\eta(\frac{\tau}{2})\right|^{2D}} + \frac{1}{2}\frac{1}{\left|\eta(\frac{\tau+1}{2})\right|^{2D}}\,. \tag{4.24}$$

In a theory of quantum gravity it is natural to sum over bulk manifolds with fixed asymptotic boundary, which in the case of the handlebody M is given by a sum over modular images of the boundary torus. The natural partition function after coupling $U(1)^D \times U(1)^D \wr S_2$ Chern-Simons theory to topological gravity (summing over geometries) is then given by

$$Z_{\mathrm{Bulk}} = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}\left(\underbrace{\frac{1}{|\eta(\gamma\cdot\tau)|^{4D}}}_{\text{disconnected}} + \frac{1}{|\eta(2\gamma\cdot\tau)|^{2D}} + \underbrace{\frac{1}{\left|\eta(\frac{\gamma\cdot\tau}{2})\right|^{2D}} + \frac{1}{\left|\eta(\frac{\gamma\cdot\tau+1}{2})\right|^{2D}}}_{\text{vortex}}\right)\,. \tag{4.25}$$

In the above we have identified the term associated with a disconnected covering space of the torus, as well as the contributions arising from including the vortex. The last three terms represent covering spaces $\widetilde{M}$ which are connected, the last two of which are branched over the vortex L. All of the covering spaces, both with and without vortex, are shown in Figure 4.3. Performing deck transformations to the branched coverings has fixed loci. These are identified with the vortex.

Comparing to the symmetric orbifold

We would like to compare the above bulk calculation to the Narain-averaged symmetric orbifold result, which takes the form given in eqs. (3.190) to (3.192)

$$\begin{aligned}\langle Z_{\mathrm{Sym}^2(T^D)}(\tau)\rangle = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}&\left(\frac{1}{|\eta(\gamma\cdot(2\tau))|^{2D}} + \frac{1}{\left|\eta(\gamma\cdot(\frac{\tau}{2}))\right|^{2D}} + \frac{1}{\left|\eta(\gamma\cdot(\frac{\tau+1}{2}))\right|^{2D}}\right)\\ &+\frac{1}{2}\langle Z_{T^D}(\tau)^2\rangle\,,\end{aligned} \tag{4.26}$$

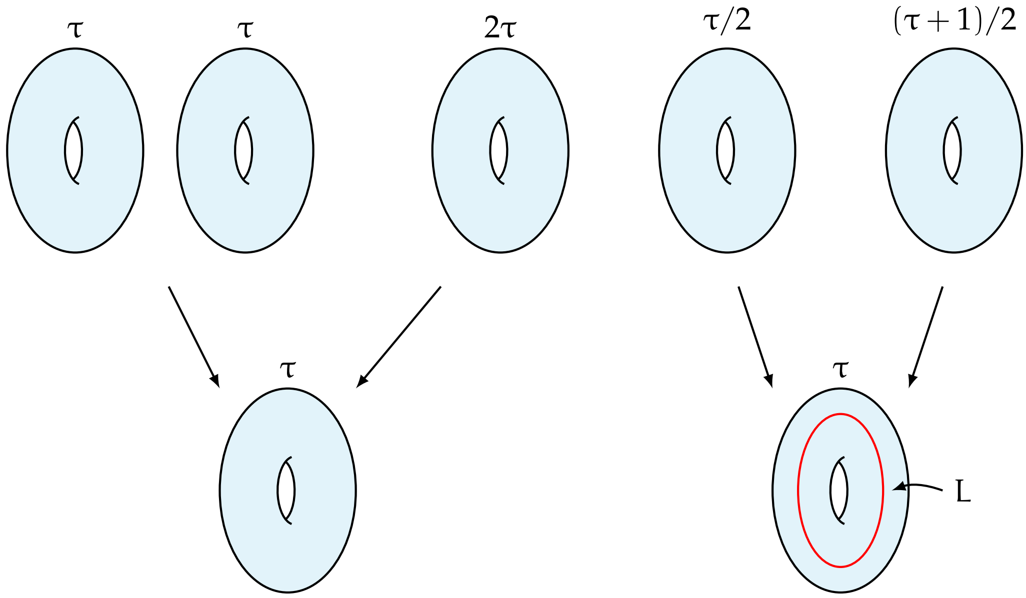

Figure 4.3: The four double covering spaces of a solid torus M without a vortex (on the left) and with a vortex (on the right). The connected covering spaces are also solid tori with modified modular parameters, while the disconnected covering space is simply $M \sqcup M$. The vortex corresponds to the fixed locus under deck transformations of the branched coverings.

where the second line is the disconnected part of the partition function. Comparing the connected parts of (4.25) and (4.26), we see that the modular parameters in the sum do not quite match. However, it is an algebraic fact that these two sums actually coincide,[8] i.e.

$$\begin{aligned}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}&\left(\frac{1}{|\eta(2\gamma\cdot\tau)|^{2D}}+\frac{1}{\left|\eta(\frac{\gamma\cdot\tau}{2})\right|^{2D}}+\frac{1}{\left|\eta(\frac{\gamma\cdot\tau+1}{2})\right|^{2D}}\right)\\&=\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}\left(\frac{1}{|\eta(\gamma\cdot(2\tau))|^{2D}}+\frac{1}{\left|\eta(\gamma\cdot(\frac{\tau}{2}))\right|^{2D}}+\frac{1}{\left|\eta(\gamma\cdot(\frac{\tau+1}{2}))\right|^{2D}}\right).\end{aligned} \tag{4.27}$$

or equivalently

$$\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}\sum_{\gamma'\in SL(2,\mathbb{Z})\backslash M_2}\frac{1}{|\eta(\gamma\cdot\gamma'\cdot\tau)|^{2D}}=\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})}\sum_{\gamma'\in SL(2,\mathbb{Z})\backslash M_2}\frac{1}{|\eta(\gamma'\cdot\gamma\cdot\tau)|^{2D}}, \tag{4.28}$$

Thus, the connected part of the $U(1)^D \times U(1)^D \wr S_2$ Chern-Simons theory coupled to topological gravity precisely reproduces the Narain average of the connected part of the symmetric orbifold theory T^D/S_2. As in [74, 75] we sum over only handlebodies with conformal boundary Σ, the genus one surface of the CFT. Note with this construction, we are able to write the CFT average as a partition function of a bulk theory on a manifold with a single boundary. This is made possible by the covering space method in the Chern-Simons calculation. Similarly, in this way we can reproduce the factors of 1/2 that appear in the average as coming from automorphism factors of the covering spaces of the bulk.

[8] See, for example, Theorem 6.9 of [205].

The disconnected contribution

Recall that the disconnected part of the $\mathrm{Sym}^N(T^D)$ partition function average is given by equation (3.192)

$$\langle Z_{\mathrm{Sym}^2(T^D),\mathrm{dis.}}(\tau)\rangle = \frac{1}{2\,\mathrm{Im}(\tau)^D|\eta(\tau)|^{4D}} \sum_{\Gamma_0\subset H_1(\Sigma\sqcup\Sigma,\mathbb{Z})} (\det \mathrm{Im}(\Omega_{\Gamma_0}))^{D/2} . \tag{4.29}$$

This corresponds to the disconnected Siegel-Weil formula (3.180) for period matrix

$$\Omega = \mathrm{diag}(\tau,\tau) , \tag{4.30}$$

which is the period matrix of the double cover $\Sigma\sqcup\Sigma$ of the boundary Σ. The sum is over Lagrangian sublattices of $H_1(\Sigma\sqcup\Sigma,\mathbb{Z})$ and we can interpret this sum as a sum over bulk manifolds $\widetilde{M}$ with boundary $\Sigma\sqcup\Sigma$ such that the sublattice Γ_0 is contractible in $\widetilde{M}$.

Now, the disconnected part of the $U(1)^D\times U(1)^D \wr S_N$ bulk theory that we propose is only a subset of the above sum

$$Z_{\mathrm{Bulk,\,dis.}} = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{4D}} = \frac{1}{2|\eta(\tau)|^{4D}\,\mathrm{Im}(\tau)^D}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \mathrm{Im}(\gamma\cdot\tau)^D , \tag{4.31}$$

where we have used the fact that $|\eta(\tau)|^4\,\mathrm{Im}(\tau)$ is modular invariant. To see this, pick the Lagrangian sublattice

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(\gamma(A^{(1)}),\gamma(A^{(2)})\right) , \tag{4.32}$$

where $A^{(1)}, A^{(2)}$ are the A cycles of the two boundaries and $\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})$ a modular transformation acting on the boundary cycles. The contribution to the determinant in the average from this configuration is

$$(\det \mathrm{Im}\,\Omega_{\Gamma_0})^{D/2} = \mathrm{Im}(\gamma\cdot\tau)^D , \tag{4.33}$$

and hence it exactly reproduces the result of the bulk. This works nicely as this sublattice corresponds to manifolds $\widetilde{M}\simeq M\sqcup M$, where M have boundaries with modular parameter $\gamma\cdot\tau$. These are exactly the covering spaces that appeared in the disconnected part of the bulk partition function. See figures 4.4,4.5.

Other contributions?

This is not all there is though. The average contains many more contributions that with the given bulk theory are not reproduced. However there might be instances where including more complicated bulk vortex configurations one can reproduce more terms from the average. Such an example we would like to demonstrate now.

The ensemble average (4.29) contains the Lagrangian sublattice (see also eq.(3.186))

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(A^{(1)}+A^{(2)}, B^{(1)}-B^{(2)}\right) , \tag{4.34}$$

which geometrically corresponds to a bulk manifold $\widetilde{M}\cong\Sigma\times I$. This manifold can actually be incorporated in the bulk calculation if we include a two vortex configuration as shown in the bottom right of Figure 4.6. However note that a Chern-Simons calculation on $\Sigma\times I$ has not been carried out and matched to the expected boundary answer. The two vortices in the figure individually act on the fields by the swap $[(1\,2)]$. The branched covering space of this vortex configuration has two boundaries, since one does not pick up a monodromy upon being transported along a cycle at the boundary, and the topology of the covering space is indeed $\Sigma\times I$ (see Figure 4.6). One can also come up with

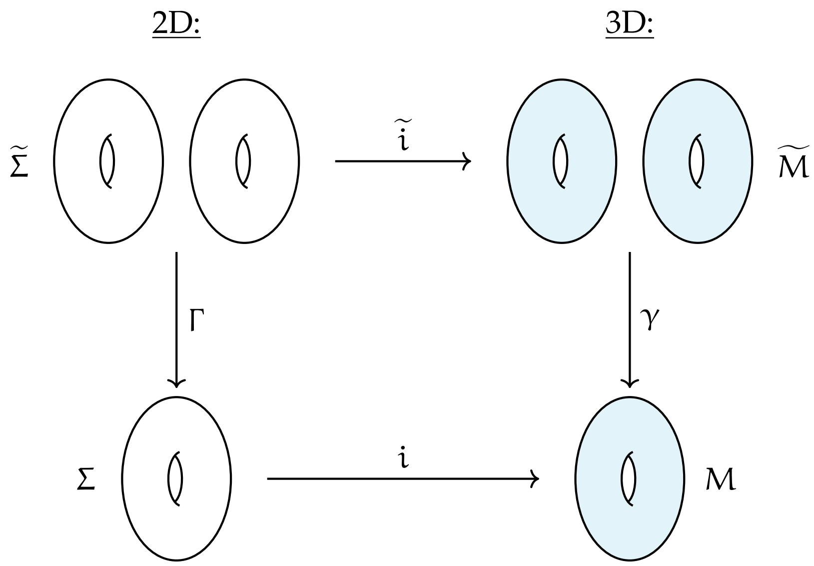

Figure 4.4: Starting from the bottom left, we have Σ where the $\mathrm{Sym}^N(T^D)$ lives. Calculating the partition function (in this case the disconnected part) amounts to going to the covering space which here is $\widetilde{\Sigma} \simeq \Sigma \sqcup \Sigma$. Ensemble averaging this, "fills" in the cycles of this covering space to create $\widetilde{M}$ which here is two copies of M. $\widetilde{M}$ is a covering space of a bulk manifold with boundary Σ and in this case we can give a bulk interpretation to M. $\iota, \widetilde{\iota}$ denote inclusion and Γ, γ covering maps. Other Lagrangian sublattices might fill in the cycles in such a way that $\widetilde{M}$ is connected and still has two boundaries. These are wormhole-like geometries and the bulk interpretation is not always clear.

stranger covering spaces which are not topologically $\Sigma \times I$ by, for example, applying Dehn twists to the vortex in Figure 4.6.

The symmetric product orbifold and the bulk theory are summing over geometries in two different ways. On the one hand the symmetric product orbifold sums over covering spaces $\widetilde{\Sigma}$ and then sums over fillings (Lagrangian sublattices)

$$\sum_{\widetilde{\Sigma}\to\Sigma}\sum_{\partial\widetilde{M}=\widetilde{\Sigma}} \frac{Z_{\mathrm{bulk}}(\widetilde{M})}{\left|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)\right|}\,, \tag{4.35}$$

where the sum over covers comes from the partition function of $\mathrm{Sym}^N(T^D)$ and the sum over bulks $\widetilde{M}$ with boundary $\widetilde{\Sigma}$ comes from the Siegel-Weil formula for the average.

On the other hand, the bulk theory first sums over fillings (Lagrangian sublattices) of M (with boundary Σ) and the sums over covering spaces (possible branched) $\widetilde{M}$ of M. Schematically

$$\sum_{\widetilde{M}\to M}\sum_{\partial M=\Sigma} \frac{Z_{\mathrm{bulk}}(\widetilde{M})}{\left|\mathrm{Aut}(\widetilde{M}\to M)\right|} \tag{4.36}$$

Demanding that these two sums are equal has little chance of working as we would like to interpret the manifolds $\widetilde{M}$ created by filling in cycles of $\widetilde{\Sigma}$ as covers of a bulk manifold M with boundary Σ and this is not always possible: there are many ways to

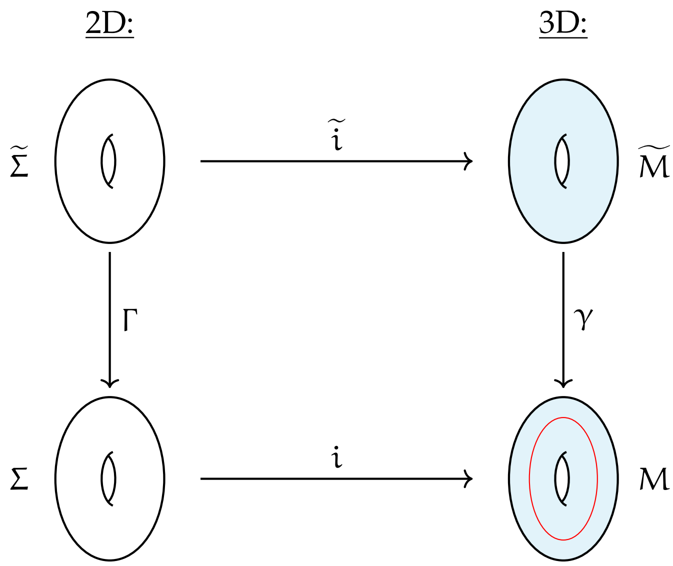

Figure 4.5: Similarly as in fig.4.4, for the connected parts the CFT partition function instructs us to consider the connected covering space $\widetilde{\Sigma}$. Ensemble averaging fills in the cycles of this cover to give a handlebody with conformal boundary $\widetilde{\Sigma}$. This $\widetilde{M}$ is in turn now also a cover, this time possibly branched of a locus L, of M. The latter can be given a holographic interpretation as a bulk manifold with conformal boundary Σ.

fill $\widetilde{\Sigma}$ that do not fulfil this condition. In other words, the diagram

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, hook, "\widetilde{i}"] \arrow[d, "\Gamma"] & \widetilde{M} \\ \Sigma & \end{tikzcd} \tag{4.37}$$

does not always have a completion of the form

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, hook, "\widetilde{i}"] \arrow[d, "\Gamma"] & \widetilde{M} \arrow[d, "\Gamma"] \\ \Sigma \arrow[r, hook, "i"] & M \end{tikzcd} \tag{4.38}$$

where $\Gamma : \widetilde{M} \to M$ is a (branched) covering map. When there is such a completion, there are instances, like in the $\Sigma \times I$ case in fig. 4.6, that an independent Chern-Simons calculation is non-existing. In these cases, we take the bulk theory to be defined by the result of the average, like below eq. (4.7). It is an interesting problem to reproduce the expected bulk result from a true bulk calculation.[9]

General N

For general N, we have the gauge group $U(1)^D \times U(1)^D \wr S_N$. For this we can define a grand canonical partition function for the bulk Chern-Simons theory on a solid torus M with conformal boundary a genus one surface Σ of modular parameter τ including a

[9]Even though we have not discussed this so far, we mention that when Σ is of higher genus and hyperbolic, one could in principle use the methods of ref.[199] to calculate the one-loop determinant of the Chern-Simons theory.

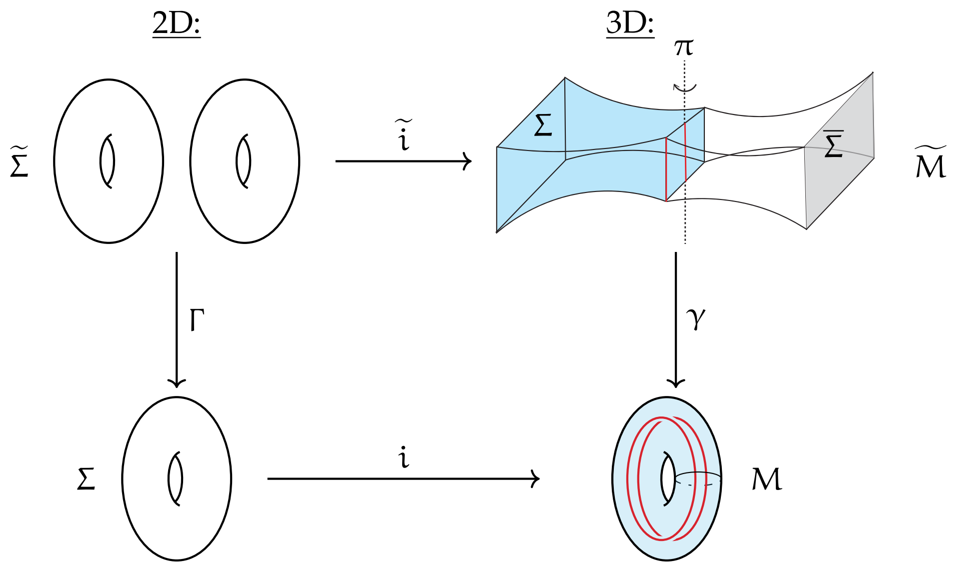

Figure 4.6: The covering space $\widetilde{\Sigma} \simeq \Sigma \sqcup \Sigma$ of Σ is filled in such a way that the topology of the resulting manifold $\widetilde{M}$ is an interval times Σ: $\widetilde{M} \simeq \Sigma \times I$. This is denoted on the upper right corner by the whole picture (the light blue and red in this picture are the bulk manifold M). One side is denoted by $\overline{\Sigma}$ to stress that the orientation of the two boundaries of $\widetilde{M}$ have to be compatible. The three-manifold $\widetilde{M} = \Sigma \times I$ has an orientation-preserving involution given by rotating the geometry around its center axis by angle π. The fixed points of this involution are the two circles shown in red, which run through the bulk. The quotient $\widetilde{M}/\mathbb{Z}_2$ (light blue) is a solid torus with two vortices (red) running through the non-contractible cycle. This is another example of a branched covering.

vortex locus L (kept implicit here) in a similar way as for the symmetric product orbifold in eq. (3.161)

$$\mathfrak{Z}_{CS}(M,p) = \sum_{N=0}^{\infty} p^N Z_{U(1)^{2D} \wr S_N}(M) \,. \tag{4.39}$$

We let the vortex L in the definition of the bulk partition function take any charge (dictated be a conjugacy class of S_N). That is, the $U(1)^D \times U(1)^D \wr S_N$ partition function is computed by summing over all degree N covering spaces of M branched along L with any allowed branching structure. Since $\pi_1(M\backslash L) \cong \pi_1(\partial M)$ (i.e. $M\backslash L$ retracts onto ∂M) the covering spaces of M branched over L are in one-to-one correspondence with the (unbranched) covering spaces of the boundary torus. This means the combinatorial counting of bulk and boundary covering spaces matches. The grand canonical partition function includes both connected and disconnected covering spaces, and through standard combinatorial arguments it is given by the exponential of the connected covering spaces

$$\mathfrak{Z}_{CS}(M,p) = \exp\left(\sum_{n=1}^{\infty} p^n T_n Z_{U(1)^{2D}}(\tau)\right) \,. \tag{4.40}$$

Indeed, the connected covering spaces are simply handlebodies whose boundaries are N-fold covering spaces of the boundary torus of M. From the above we can extract a formula for the partition function of the $U(1)^{2D} \wr S_N$ Chern-Simons theory with a vortex along the non-contractible cycle by keeping all terms with a power of p^N which is

$$Z_{U(1)^{2D} \wr S_N}(M) = \sum_{\text{paritions of } N} \prod_{k=1}^{N} \frac{1}{N_k!}\left(T_k Z_{U(1)^{2D}}(\tau)\right)^{N_k} , \tag{4.41}$$

where the partitions of N are $\sum_{k=1}^{N} kN_k = N$. This can be compared to the boundary partition function given in equation (3.164). The connected part of the partition function is given by

$$Z_{U(1)^{2D}\wr S_N,\text{conn.}}(M) = T_N Z_{U(1)^{2D}}(\tau)\,. \tag{4.42}$$

Which can also be obtained by recalling that the Hecke operator T_N sums over all connected covering spaces of the original torus.[10] To obtain the bulk partition function we additionally need to sum over all bulk handlebodies M, which is implemented by the sum over modular images.

We claim that the averaged free energy of the grand canonical symmetric orbifold exactly equals the free energy of the grand canonical Chern-Simons theory, summed over all solid tori M, i.e.

$$\int_{\mathcal{M}_D} d\mu \,\log \mathfrak{Z}(\tau,p) = \sum_{M} \log \mathfrak{Z}_{CS}(M,p)\,. \tag{4.43}$$

This is equivalent to stating that the connected covering space contribution to the symmetric orbifold is exactly reproduced by the bulk Chern-Simons theory. To check this claim, note that the averaged symmetric orbifold free energy takes the form

$$\begin{aligned}\int_{\mathcal{M}_D} d\mu \,\log \mathfrak{Z}(\tau,p) &= \sum_{n=1}^{\infty} p^n \left\langle T_n Z_{T^D}(\tau)\right\rangle \\ &= \sum_{n=1}^{\infty} \frac{p^n}{n} \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in SL(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma\cdot\gamma'\cdot\tau)|^{2D}}\,,\end{aligned} \tag{4.44}$$

where M_n is the set of all 2×2 integer matrices with determinant n and we have used the definition (3.166) of the n^{th} Hecke operator. Now, the sum over geometries in the Chern-Simons free energy is implemented by a sum over modular images of the boundary torus. We have

$$\begin{aligned}\sum_{M} \log \mathfrak{Z}_{CS}(M,p) &= \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{n=1}^{\infty} p^n T_n Z_{U(1)^{2D}}(\gamma\cdot\tau) \\ &= \sum_{n=1}^{\infty} \frac{p^n}{n} \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in SL(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma'\cdot\gamma\cdot\tau)|^{2D}}\,,\end{aligned} \tag{4.45}$$

where we have again used the definition of the Hecke operator T_n (see also footnote 10). Equations (4.44) and (4.45) appear to yield different results, since the summand of one includes the modular parameter $\gamma\cdot\gamma'\cdot\tau$, while the other includes $\gamma'\cdot\gamma\cdot\tau$. However, it turns out[11] that the sums (4.44) and (4.45) are composed of all the same terms, simply shuffled around, like in the $N=2$ case in page 105. That is,

$$\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in SL(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma\cdot\gamma'\cdot\tau)|^{2D}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in SL(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma'\cdot\gamma\cdot\tau)|^{2D}}\,. \tag{4.46}$$

This proves the claim of (4.43).

[10] Note that here we take the operators T_k to act on $Z_{U(1)^{2D}}(\tau)$ as in eq. (3.163) even though $Z_{U(1)^{2D}}(\tau)$ is not a modular invariant quantity. This is a slight abuse of notation as usually Hecke operators include a prefactor in their sum that depends on the modular weight of the function.

[11] See again Theorem 6.9 of [205].

Since the free energy of the grand canonical partition function (either in the case of the symmetric orbifold or of Chern-Simons) computes the connected contribution, we have the following result:

> The ensemble average of the connected part of the T^D/S_N orbifold torus partition function is equal to the connected part of the $U(1)^D \times U(1)^D \wr S_N$ Chern-Simons partition function, summed over all handlebodies bounded by the CFT torus for all N.

$$\left\langle Z_{\mathbb{T}^D/S_N,\text{conn.}}(\tau)\right\rangle = Z_{\text{Bulk, conn.}}. \tag{4.47}$$

All the connected covering spaces of the bulk M arise as "filled in" versions of the coverings $\widetilde{\Sigma}$ of Σ where the CFT lives.

As we have seen in the $N = 2$ example, however, the disconnected parts of the two theories cannot so easily be matched. For these, we have to discard the connected contributions from eq. (4.41). This is done by dropping the N_N term because when $N_N \neq 0$ then the other N_k, $k = 1, ..., N-1$ are zero and the result gives the connected contribution. Taking this into account, we have

$$Z_{\text{Bulk, dis.}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\text{parititons of N}} \prod_{k=1}^{N-1} \frac{1}{N_k!}\left(T_k Z_{U(1)^{2D}}(\gamma\cdot\tau)\right)^{N_k}, \tag{4.48}$$

where now we sum over disconnected partitions of N by imposing $\sum_{k=1}^{N-1} kN_k = N$. Similarly, the contribution of disconnected covering spaces to the boundary ensemble average is given by (3.164)

$$\left\langle Z_{T^D\wr S_N,\text{dis.}}(\tau)\right\rangle = \sum_{\text{parititons of N}} \left\langle \prod_{k=1}^{N-1} \frac{1}{N_k!}\left(T_k Z_{T^D}(\tau)\right)^{N_k}\right\rangle, \tag{4.49}$$

where we must use the disconnected Siegel-Weil formula (3.180) to evaluate the average. One contribution that we can easily match is the direct generalisation of the one we identified in the $N = 2$ case, namely the one with Lagrangian sublattice

$$\Gamma_0 = \text{Span}_{\mathbb{Z}}(\gamma(A^{(1)}), ..., \gamma(A^{(N)})) \quad \gamma\in\Gamma_\infty\setminus SL(2,\mathbb{Z})\ . \tag{4.50}$$

This corresponds to $N_1 = N$ in the above sums and $A^{(i)}$, $i = 1, ..., N$ are the A cycles of the N boundaries (Γ_0 is a sublattice of $H_1(\underbrace{\Sigma\sqcup ...\sqcup\Sigma}_{\text{N times}}, \mathbb{Z})$ in this case). From the bulk point of view this comes from the covering space $\widetilde{M}$ of M that is N copies of M. In [3] it is claimed that actually the whole disconnected part of the bulk is contained in the disconnected part of the average.

Higher genus and non-handlebody contributions

Here we briefly mention what happens to the proposed duality when considering CFTs on higher genus $g > 1$ Riemann surfaces Σ_g.

Putting $\text{Sym}^N(T^D)$ on Σ_g, the connected part of the partition function is given by

$$Z_{T^D/S_N,\text{conn.}}(\Sigma_g; m) = \frac{1}{N}\sum_{\widetilde{\Sigma}\to\Sigma_g} Z_{T^D}(\widetilde{\Sigma}; m)\,, \tag{4.51}$$

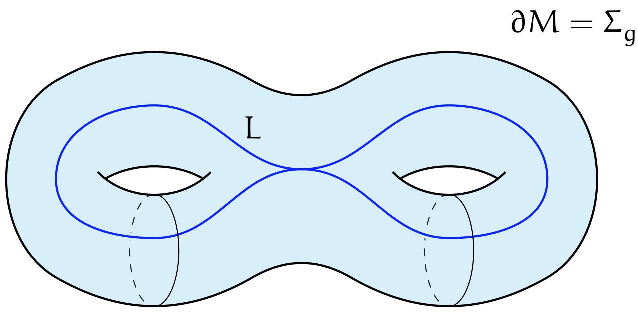

Figure 4.7: A handlebody M is a singular foliation of its boundary Σ_g. The leaves of the foliation become singular at the 'center' L of M, which roughly resembles Σ_g with the contractible cycles collapsed to points. In the Chern-Simons theory dual to the Narain-averaged symmetric orbifold, we treat L as the locus of permutation gauge vortices.

where the sum is over all connected unramified covering surfaces $\widetilde{\Sigma} \to \Sigma_g$ of degree N and $\mathfrak{m}$ denotes the moduli of the T^D CFT. As in the torus case this sums over all consistent boundary conditions (monodromies) around cycles of Σ_g. The Riemann-Hurwitz theorem tells us the genus of such a covering

$$g' = N(g-1)+1\,. \tag{4.52}$$

Now, from [74], we know that the ensemble average of $Z_{T^D}(\widetilde{\Sigma};\mathfrak{m})$ is given as a sum of the $U(1)^D \times U(1)^D$ Chern-Simons partition function on handlebodies whose boundaries are the covering surfaces $\widetilde{\Sigma}_{g'}$:

$$\left\langle Z_{T^D/S_N,\mathrm{conn.}}(\mathfrak{m},\Sigma_g)\right\rangle = \frac{1}{N}\sum_{\widetilde{\Sigma}_{g'}\to\Sigma_g}\;\sum_{\partial\widetilde{M}=\widetilde{\Sigma}_{g'}} Z_{U(1)^{2D}}(\widetilde{M})\,. \tag{4.53}$$

From a bulk point of view now, let M be a handlebody bounded by Σ_g and L be the codimension 2 locus shown in Figure 4.7. The fundamental group $\pi_1(M\backslash L)$ is isomorphic to the fundamental group of the boundary of M, i.e.

$$\pi_1(M\setminus L) \cong \pi_1(\Sigma_g)\,. \tag{4.54}$$

Since covering spaces are classified by choices of consistent twisted boundary conditions around different cycles, that is homomorphisms $\phi : \pi_1(\Sigma_g) \to S_N$, we have that covering spaces of M branched over L have the same structure as covering spaces of Σ_g.[12] Furthermore, the covering spaces $\widetilde{M}$ will also be handlebodies. There is one subtlety regarding the singular locus L. Earlier we demanded that vortex operators were defined along a dimension one submanifold of M, but L is not a manifold so it is not obvious in what sense a vortex operator can be associated to L. However, it remains perfectly consistent to impose twisted boundary conditions on the bulk Chern-Simons fields as they travel around L. Since $\pi_1(\Sigma_g) \cong \pi_1(M\backslash L)$ a choice of twisted boundary conditions on Σ_g descends to a consistent set of monodromies for the gauge fields as they travel around different cycles of L, and we define our bulk theory by demanding such monodromies.

Coupling our Chern-Simons theory to topological gravity results in summing over

[12]The isomorphism $\pi_1(M\setminus L) \cong \pi_1(\Sigma_g)$ is due to the fact that a handlebody can be foliated by copies of its boundary, up to a singular locus given by L. That is, $M\setminus L$ is homeomorphic to $\Sigma_g \times [0,1)$, see [197].

all handlebodies M with boundary Σ_g. Summarizing, we have the bulk contribution

$$\sum_{\partial M=\Sigma_g} Z_{\mathrm{CS,conn.}}(M) = \frac{1}{N} \sum_{\partial M=\Sigma_g} \sum_{\substack{\widetilde{M}\to M \\ \text{branched over } L}} Z_{U(1)^{2D}}(\widetilde{M})\,. \tag{4.55}$$

It seems that there are obstructions in matching expressions (4.53) and (4.55) even though they are the connected parts, which for the torus case we saw matched [3].

Another interesting fact is that for $g > 1$, the average of the disconnected part of the $\mathrm{Sym}^2(T^D)$ (take $N = 2$ for simplicity) contains Lagrangian sublattices $\Gamma_0 \subset H_1(\Sigma_g \sqcup \Sigma_g, \mathbb{Z})$ that correspond to three-manifolds $\widetilde{M} \simeq \Sigma_g \times I$. These are hyperbolic and for special choices of the complex structure of Σ_g, we can associated to $\widetilde{M}$ a bulk manifold M that is smooth and hyperbolic with boundary Σ_g and has some nice properties.[13] Namely, the non-contractible cycles of Σ_g do not become contractible when viewed as cycles of M and there are non-contractible cycles of M that are not visible from Σ_g. This is why M can be both smooth and have a connected double cover $\widetilde{M}$ with disconnected boundary.

By the standard U(1)-gravity dictionary, we should be able to associate the contribution of the aforementioned sublattice in the averaged partition function to the path integral of $U(1)^D \times U(1)^D$ Chern-Simons theory on the wormhole geometry $\widetilde{M}$.[14] This, in turn, should be reproduced by the $U(1)^D \times U(1)^D \wr S_2$ partition function on M with a nontrivial monodromy around the 'internal' generator(s) of $\pi_1(M)$ (i.e. the generator(s) of $\pi_1(M)$ which are not inherited from $\pi_1(\Sigma_g)$).

Since U(1) gravity with a connected boundary (i.e. the bulk dual of the non-orbifolded Narain ensemble) includes only handlebodies in its sum over geometries.[15] It would be thus be interesting to explore further in what sense the ensemble average of T^D/S_N CFTs includes more generic bulk geometries which are not visible in U(1) gravity, but which are generally expected to be included in a more general theory of three-dimensional quantum gravity (for example, semiclassical gravity). See also [206, 195].

4.2.1 Bulk interpretation of ensemble averaged correlators

So far we have analysed how to interpret from a bulk point of view ensemble averages of toroidal T^D CFTs and $\mathrm{Sym}^N(T^D)$ partition functions. Now we would like to go a little beyond this and study bulk interpretations of ensemble averages of $\mathrm{Sym}^N(T^D)$ correlators, which were studied in section 3.4.

Recall that the n-point correlator on the sphere $\mathbb{CP}^1$

$$\langle \sigma_{\omega_1}(x_1) \cdots \sigma_{\omega_n}(x_n) \rangle$$

is calculated by going to a covering space $\widetilde{\Sigma}$ of $\mathbb{CP}^1$ that is branched over the points $x_1, ..., x_n$ with branching ratio determined by the permutations $\omega_i \in S_N$. The result of the correlator is given in terms of the partition function of T^D on the covering $\widetilde{\Sigma}$ (there is

[13] Recall in the genus one case we associated to $\Sigma \times I$ a bulk manifold that contains two vortices, hence singular.

[14] Since it is hyperbolic, perhaps one could do it using the methods of [199].

[15] Strictly speaking, U(1) gravity only classifies bulk geometries by their associated Lagrangian sublattice, which for connected boundaries are in one-to-one correspondence with handlebodies.

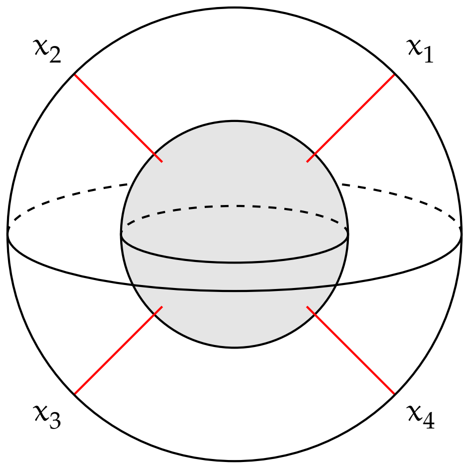

Figure 4.8: The bulk geometry dual to the four-point function of twist fields in the symmetric orbifold. In principle, the vortices can branch and tangle in complicated ways in the bulk.

prefactor independent of the T^D moduli which will be dealt with later, see eq. (3.212)). Taking the ensemble average of this instructs one to sum over bulk handlebodies $\widetilde{M}$ created by filling in cycles of $\widetilde{\Sigma}$ as specified by Lagrangian sublattices of $H_1(\widetilde{\Sigma}, \mathbb{Z})$.

From a holographic point of view, we think of $\mathbb{CP}^1$ as the conformal boundary of hyperbolic three-space $\mathbb{H}^3$.[16] The insertion of the twist fields on this boundary is expected to be dual to a configuration of vortices that start from these points continue into the bulk (which is $\mathbb{H}^3$) and end on another point at the boundary where a twist field is inserted, see figure 4.8. On the level of the bulk calculation this should correspond to going to a covering space $\widetilde{M}$ of $\mathbb{H}^3$ which is branched over the locus of the vortices—like for the partition functions with vortices we saw in the previous section. Given that the vortices end on the insertion points $x_1, ..., x_n$ a branched covering $\widetilde{M}$ will induce a branched covering $\widetilde{\Sigma}$ of $\mathbb{CP}^1$. However as in the case for partition functions, given a branched cover $\widetilde{\Sigma}$ of $\mathbb{CP}^1$ it is not always the case that a given three-manifold $\widetilde{M}$ with boundary $\widetilde{\Sigma}$ is a branched cover of $\mathbb{H}^3$. For the example we consider next this will not be a problem though. Also, for the rest of this discussion we focus on $\mathrm{Sym}^2(T^D)$—it is possible to consider higher N but we leave this to future work.

Four point functions

We consider the symmetric orbifold T^D/S_2, for which there is a unique twist operator $\sigma_{(1\,2)}$, since it is abelian. We consider the correlation functions of the form

$$\langle \sigma_{(1\,2)}(x_1) \cdots \sigma_{(1\,2)}(x_4) \rangle \,. \tag{4.56}$$

We calculated these in section 3.4. Just as in [166], we will find that the Narain averages of these correlators are indeed reproduced by a sum over bulk vortex configurations in $U(1)^D \times U(1)^D \wr S_2$ Chern-Simons theory, for which the vortices are constrained to be arranged in rational tangles.[17] As such, we are able to reproduce the averaged

[16] Any hyperbolic three-manifold can be written as a quotient $\mathbb{H}^3/G$, where G is a discrete subgroup of $PSL(2, \mathbb{C})$. These groups are called Kleinian and a particular example of these, the Schottky groups, give rise to handlebodies. For a nice summary see ref. [72].

[17] Our analysis differs slightly from that of [166], in that they consider orbifolds of the form $T^D/\mathbb{Z}_n$, which are qualitatively different from symmetric orbifolds. As such, their bulk theory is of the form $U(1)^D \times U(1)^D \rtimes \mathbb{Z}_n$.

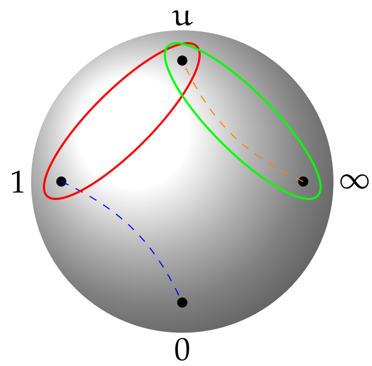

Figure 4.9: An illustration of the (non)contractibility of loops encircling points on the sphere. The green loop is contractible when continued in the bulk whereas the red loop is not (it has to cross the orange and blue vortices that live inside the sphere).

symmetric orbifold correlation function via a bulk Chern-Simons calculation.[18] Recall that the ensemble average of the four point function is given by eq. (3.218)

$$\begin{aligned}\int_{\mathcal{M}_D} d\mu(m)\ \langle\sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle \\ = \frac{1}{2^{2c/3}|u(1-u)|^{c/12}} \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}\,,\end{aligned} \tag{4.57}$$

As in [166], we can interpret the sum over modular images in (3.218) in terms of $U(1)^D\times U(1)^D \wr S_2$ Chern-Simons theory in the following way. Consider the CFT sphere on which we calculate the correlation functions to be the boundary of the ball $\mathbb{H}^3$. We extend each twist field $\sigma_{(1\,2)}$ as a vortex in the bulk which meets the boundary at the point x_i. Each vortex in the bulk implements a monodromy $A_{(1)} \to A_{(2)}$ in the bulk gauge field. Since a vortex cannot just end at one point, we need to join pairs of boundary points by vortices. Let us choose for the moment a vortex which joins the point at $x=0$ with the point at $x=1$ and another which joins $x=u$ with $x=\infty$. The two strands are in principle allowed to cross and link in the bulk in an arbitrary fashion, so long as they do not cross. For example, the 'trivial' configuration $\mathcal{T}_0$ on the left of Figure 4.10 connects 0 to 1 and u to ∞ in the simplest way possible—with no crossings in the bulk. The bulk geometry found by taking the double branched cover of the ball over the vortex $\mathcal{T}_0$ has the property that the cycle generated by a loop encircling u and ∞ is contractible in the bulk. For an illustration, see Figure 4.9. Indeed, as was noted in [166], the branched cover of the geometry in Figure 4.9 is a handlebody with torus boundary.[19]

In [166], the sum over modular images in equation (4.57) was argued to arise from a sum over vortex configurations in the bulk which are topologically 'rational tangles'. A rational tangle is a vortex configuration which is obtained by applying successive exchanges (braidings) of the points $0,1,u,\infty$ on the trivial tangle $\mathcal{T}_0$. For example, the three tangles shown in Figure 4.10 are the trivial tangle $\mathcal{T}_0$, the tangle obtained by starting with $\mathcal{T}_0$ and braiding the ends at $1,u$ around each other twice, and the tangle

[18] We emphasize that for $N>2$, the analysis does not work out so cleanly, and not every term in the averaged correlation functions will be reproducible by a bulk Chern-Simons calculation on a classical background.

[19] This can be seen by "cutting open" the hyperbolic ball along branch cuts associated to the vortices and gluing a second copy along the same branch cuts, see Figure 2 of [166].

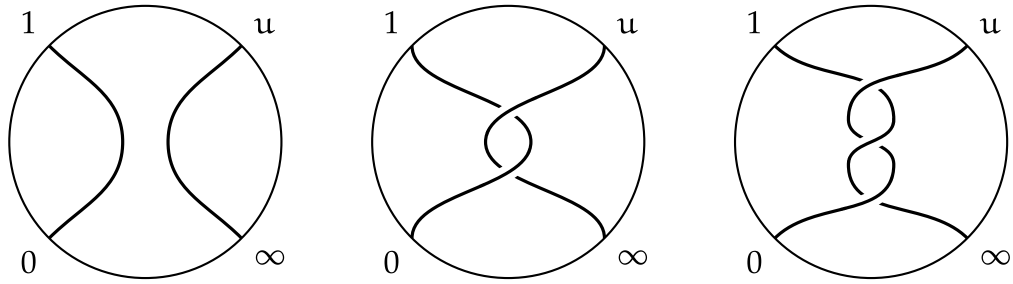

Figure 4.10: Some rational tangles contributing to the four-point function of twist fields in the averaged $T^D \wr S_2$ orbifold.

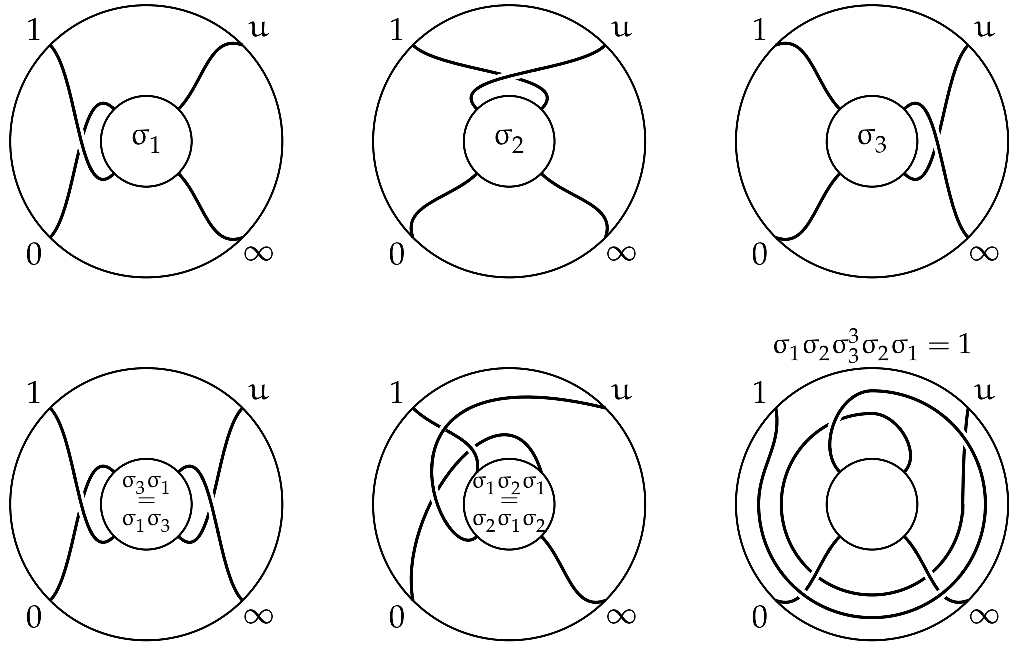

Figure 4.11: The action of the braid group of points on the sphere $B_4(S^2)$. The two circles depict cross sections of two-spheres. Inside the smaller two-sphere the strands could be arbitrarily tangled. The first row shows the action of the generators, the second depicts some of the relations that these generators satisfy. In the last drawing, the strands can be "untangled": the one that connects to 1 from the front of the inner S^2 and the one connected to u behind the inner S^2.

obtained from $\mathcal{T}_0$ by braiding the ends at $u, 1$ three times. Note that the direction of swapping matters.

Mathematically, a rational tangle is obtained from the trivial tangle $\mathcal{T}_0$ in the following way: Let $B_4(S^2)$ be the braid group of points on the sphere. There is a natural action of $B_4(S^2)$ on the ends of the vortices at $x = 0$, $x = 1$, $x = u$, and $x = \infty$. A basis of generators for B_4 are the 'standard braids' $\sigma_1, \sigma_2, \sigma_3$ which swap the pairs $(0, 1)$, $(1, u)$ and (u, ∞) respectively (with specified orientation) as in Figure 4.11. All rational tangles can be seen as the action of an element of $B_4(S^2)$ on the trivial tangle $\mathcal{T}_0$.

The braid group $B_4(S^2)$ is generated by $\sigma_1, \sigma_2, \sigma_3$, which satisfy the following relations:

$$\begin{gathered}\sigma_1\sigma_3 = \sigma_3\sigma_1\,, \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\,, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \\ \sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_1 = 1\,.\end{gathered} \tag{4.58}$$

The last of these relations is specific to the braid group on the sphere. Note that the braid group does not act on rational tangles faithfully. In particular, the combination $\sigma_1\sigma_3^{-1}$ acts trivially on any rational tangle since it corresponds to a reflection about the

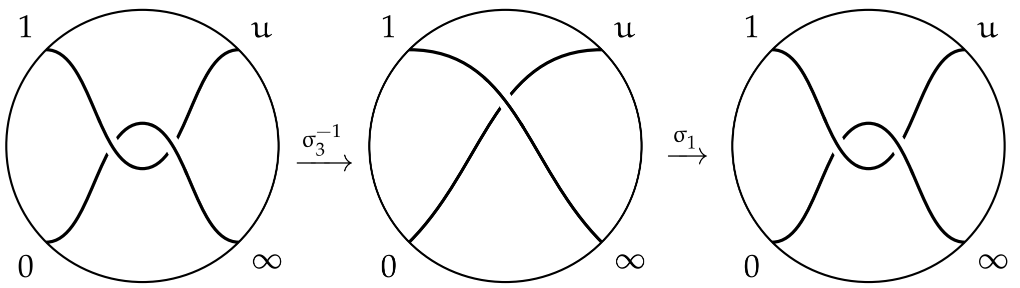

Figure 4.12: The action of $\sigma_1\sigma_3^{-1}$ on the tangle obtained by acting with σ_3^2. We see that this corresponds to the same tangle.

East-West axis as in Figure 4.12, see [207].

Let $\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle$ be the normal subgroup of $B_4(S^2)$ generated by $\sigma_1\sigma_3^{-1}$.[20] Since this is a normal subgroup, the quotient $B_4(S^2)/\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle$ is a group which acts in a well-defined manner on the space of tangles. This quotient effectively imposes $\sigma_1 \sim \sigma_3$ in the defining relations (4.58) of the braid group. It turns out that this quotient is nothing more than the modular group, i.e.

$$B_4(S^2)/\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle \cong \mathrm{PSL}(2,\mathbb{Z})\,. \tag{4.59}$$

The isomorphism is seen by the direct matrix identification

$$\sigma_1,\sigma_3 \to \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\,,\quad \sigma_2 = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\,. \tag{4.60}$$

These correspond to the T and ST^{-1} elements in $\mathrm{PSL}(2,\mathbb{Z})$. Finally, note that not even $B_4(S^2)/\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle$ acts faithfully on the space of rational tangles. This is because $\sigma_1\cdot\mathcal{T}_0 = \mathcal{T}_0$. Thus, the set of operations which acts faithfully on the set of rational tangles is

$$(B_4(S^2)/\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle)/\langle\sigma_1\rangle \cong \Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})\,. \tag{4.61}$$

Thus, the sum over rational tangles produces a sum over the coset $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$. This sum also precisely reproduces the sum over Lagrangian sublattices, since each rational tangle admits a combination of the cycles shown in Figure 4.9 which becomes contractible in the bulk. Furthermore, the branched double cover of each rational 2-tangle is a handlebody with torus boundary.[21] In other words the modular sum

$$\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}\,, \tag{4.62}$$

can be viewed as the partition function of the Chern-Simons bulk theory on $\mathbb{H}^3$ including rational tangle vortex configurations ending (and starting) on four points on the boundary. The modular sum sums over the possible rational tangle configurations and each term in the sum is the partition function for a given vortex configuration, calculated by going to the covering space which is a handlebody.

[20] That is, $\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle$ is the smallest normal subgroup of $B_4(S^2)$ which contains $\sigma_1\sigma_3^{-1}$.

[21] This follows either from the fact that the double cover of $\mathcal{T}_0$ is a torus handlebody and the fact that all rational 2-tangles are homeomorphic to $\mathcal{T}_0$. Intuitively, the action of the braid group on $\mathcal{T}_0$ can be thought of as implementing Dehn surgery on the double cover of $\mathcal{T}_0$. In the mathematics literature, this is often referred to as the 'Montesinos Trick' [208].

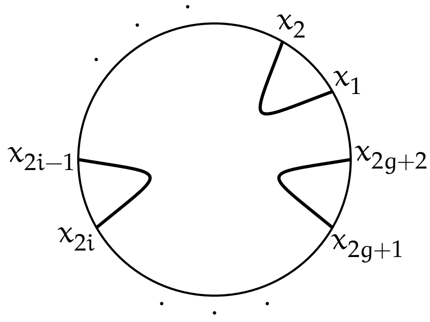

Figure 4.13: The trivial tangle $\mathcal{T}_0$.

Higher-point correlators

Recall that the $n = 2g+2$-point correlator of $\mathrm{Sym}^2(T^D)$ is written as in eq. (3.220)

$$\langle \sigma_{(1\,2)}(x_1) \cdots \sigma_{(1\,2)}(x_{2g+2}) \rangle = e^{-S_L[\Phi_\Gamma]} Z(\widetilde{\Sigma}_g)\,, \tag{4.63}$$

where we pick an even number of points to get a non-vanishing result and g is a non-negative positive integer. The calculation boils down again to the partition function of T^D on the covering $\widetilde{\Sigma}_g$—a genus g curve.

Ensemble averaging gives

$$\int_{\mathcal{M}_D} d\mu\ \langle \sigma_{(1\,2)}(x_1) \cdots \sigma_{(1\,2)}(x_{2g+2}) \rangle = e^{-S_L[\Phi_\Gamma]} \sum_{\gamma \in P\backslash \mathrm{Sp}(2g,\mathbb{Z})} Z_{\mathrm{CS}}^{U(1)^{2D}}(\gamma\cdot\Omega)\,, \tag{4.64}$$

where Ω is the period matrix of Σ_g, and $Z_{\mathrm{CS}}(\gamma\cdot\Omega)$ is shorthand for the expression

$$Z_{\mathrm{CS}}^{U(1)^{2D}}(\gamma\cdot\Omega) = \frac{(\det \mathrm{Im}(\gamma\cdot\Omega))^{D/2}}{(\det \mathrm{Im}(\Omega))^{D/2} |\det{}' \bar{\partial}_{\widetilde{\Sigma}_g}|^D}\,, \tag{4.65}$$

which is the $U(1)^D \times U(1)^D$ Chern-Simons partition function on a handlebody bounded by $\widetilde{\Sigma}_g$ [74]. The sum over modular images of Ω under the action of $P\backslash\mathrm{Sp}(2g,\mathbb{Z})$ defines a sum over different inequivalent handlebodies with boundary $\widetilde{\Sigma}_g$.

Again, one can show that rational tangle configurations in the bulk theory reproduce a handlebody in eq. (4.64), i.e. they admit branched covers that are handlebodies (whose boundary is $\widetilde{\Sigma}_g$). To see how a rational tangle like this corresponds to a handlebody consider the $\mathcal{T}_0$ trivial tangle in figure 4.13. We can define it by connecting "successive" points x_{2i-1} to x_{2i} as in figure 4.13. The double cover of the ball $\mathbb{H}^3$ branched over this locus will be a handlebody with boundary $\widetilde{\Sigma}_g$ such that the A cycles of $\widetilde{\Sigma}_g$ are contractible in the bulk (we can do this by choosing a suitable homology basis for $\widetilde{\Sigma}_g$, see figure 4.14).

By fixing one of the boundary points, we get all rational tangles by acting with the braid group $B_{2g+1}(S^2)$. One can show that this induces a $\mathrm{Sp}(2g,\mathbb{Z})$ action on the period matrix Ω of $\widetilde{\Sigma}_g$ and the resulting rational tangles all have double covers that are handlebodies whose boundary is $\widetilde{\Sigma}_g$ [3].

What we argued about so far is that considering rational tangle configurations of vortices in the bulk gives covering spaces which are handlebodies hence corresponds to a term in the sum (4.64). However, it turns out that for correlators with $n \geqslant 8$ (or genus $g \geqslant 3$) not every element in (4.64) corresponds to a rational tangle of vortices in the bulk

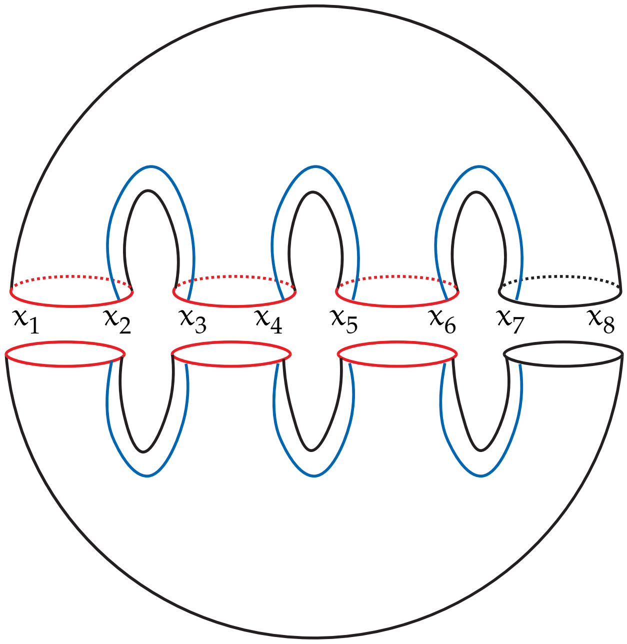

Figure 4.14: The homology basis of the genus g surface $\widetilde{\Sigma}_g$. The red and blue curves represent the A and B cycles respectively. Making the A cycles contractible gives the handlebody that double covers the rational tangle configuration of figure 4.13. It can be constructed by "cutting open" the hyperbolic ball along branch cuts associated to the vortices and then gluing a second copy—like for the four-point correlators. See also figure 4.15.

Cher-Simons theory [3]. One way to see this is that the action induced on the period matrix of $\widetilde{\Sigma}_g$, which comes from a representation of the braid group on $Sp(2g, \mathbb{Z})$, does not give the whole $Sp(2g, \mathbb{Z})$ but only a subgroup [209].

The conformal anomaly

So far we showed that there are cases where the bulk $U(1)^D \times U(1)^D \wr S_N$ Chern-Simons theory reproduces the ensemble average of T^D/S_N correlators but only focused on the modular sum (the Eisenstein series) and said nothing about the prefactor $e^{-S_L[\phi_\Gamma]}$ that appears in the average as a result of the conformal anomaly. On the level of the conformal field theory, the prefactor came from the conformal anomaly that arises from a Weyl-rescaling of the pullback metric on the covering space $\widetilde{\Sigma}$.

In a similar way, considering the bulk Chern-Simons partition function on M (here M was $\mathbb{H}^3$) in the presence of a vortex locus L one goes to the branched covering $\widetilde{M}$. Now, the path integral is not completely independent of the choice of metric and in particular depends on the boundary metric. The latter is induced via a pullback of the covering map in the same way as before. In fact it is the same as in the CFT case because the branched covering map of the boundary is induced from the branched covering of the bulk. Performing a Weyl transformation leads to a conformal anomaly also in the Chern-Simons path integral that matches the one from the CFT. For more details on this see [3] and references therein.

4.3 Adding supersymmetry

In previous sections, in particular in pages 59 and 91, we saw the ensemble averages of partition functions of supersymmetric T^D CFTs and their symmetric product orbifolds.

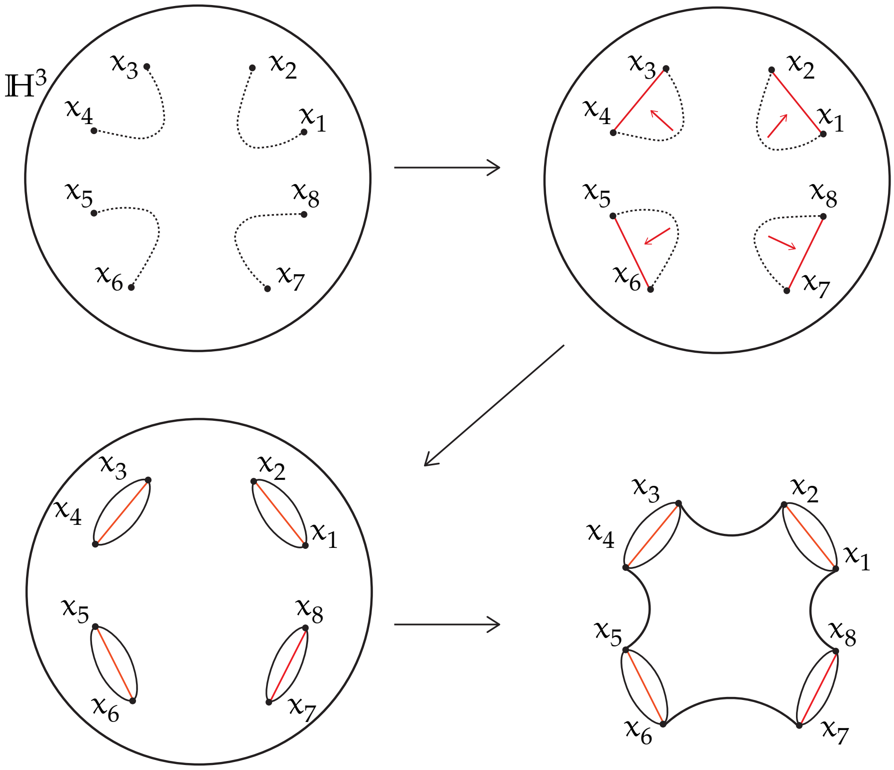

Figure 4.15: A heuristic picture on how to construct the double cover of a rational tangle configuration. The twist field insertions at points x_i continue into the bulk as vortices. Bringing the vortices close to the surface, cutting along a surface that surrounds them and gluing to this a second copy of itself gives the double cover—which is a handlebody with boundary $\widetilde{\Sigma}_g$. This is like figure 2 of ref. [166].

Let us recall two important formulas:

Firstly, equation (3.50), gives the ensemble average of a supersymmetric T^D CFT with spin structure α, β

$$\int_{\mathcal{M}_{T^D}} d\mu(m)\, Z_{T^D} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau; m) = Z_{\text{ferm}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau)^D \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{2D}} \,.$$

We will see that this can be recovered from the partition function of a $\mathcal{N} = (1,1)$ Chern-Simons theory, summed over geometries. The boundary fermions in the Chern-Simons theory have specified spin structure on the boundary, namely the same spin structure specified for the Narain CFT. As explained around equation (4.5), each element of the sum over geometries is specified by a choice (c, d) of coprime integers such that the cycle $c\tau + d$ becomes contractible in the bulk. Because the fermion picks up a sign of $(-1)^{2\alpha}$ around the τ-cycle and $(-1)^{2\beta}$ around the 1-cycle, the monodromy around the contractible cycle is given by $(-1)^{2(c\alpha+d\beta)}$. This contradicts the intuition that the boundary fermions must satisfy antiperiodic boundary conditions around a contractible bulk cycle. This only occurs if $c\alpha + d\beta$ is half-integer. To be concrete, let us take the boundary theory to be in the NS sector, i.e. $\alpha = \beta = 1/2$. Then in the bulk geometry associated to (c, d), the boundary fermion is periodic when $c + d$ is even, and antiperiodic when $c + d$ is odd. We can break up the sum over geometries into two sums:

$$\underbrace{Z_{\text{ferm}}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\tau)^D \sum_{\substack{(c,d)=1\\ c+d\ \text{odd}}} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}}_{\text{'good' boundary conditions}} + \underbrace{Z_{\text{ferm}}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\tau)^D \sum_{\substack{(c,d)=1\\ c+d\ \text{even}}} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}}_{\text{'bad' boundary conditions}}\,. \tag{4.66}$$

The 'good' boundary conditions correspond to the fermions being anti-periodic around the contractible cycle, while the bad correspond to the opposite. We give an explanation of this from the bulk perspective in the next subsection.

Secondly, equation (3.199)

$$\begin{aligned}\int_{\mathcal{M}_D} d\mu \log \mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau) &= \sum_{k=1}^{\infty} p^k \left\langle \mathcal{T}_k Z_{T^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\right\rangle \\ &= \sum_{k=1}^{\infty}\frac{p^k}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z_{\text{ferm}}\begin{bmatrix}a\alpha+b\beta\\ d\beta\end{bmatrix}\left(\frac{a\tau+b}{d}\right)^D\left\langle Z_{T^D}\left(\frac{a\tau+b}{d}\right)\right\rangle,\end{aligned} \tag{4.67}$$

we will see in this section that this can also be given a bulk interpretation. We move on now to describe the bulk dual including supersymmetry.

The bulk dual

The natural candidate bulk theory is a supersymmetric version of Chern-Simons. We will now briefly explain this supersymmetric theory, leaving our conventions and additional details to appendix A.7. In [210] supersymmetric Chern-Simons in flat Minkowski space was considered in the presence of a boundary. The boundary broke half the supersymmetry down to $\mathcal{N}=(1,0)$. In the conventions of [210] the flat space action is given by

$$S_{CS}^{\mathcal{N}=(1,0)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \bar{\lambda}\lambda) - \frac{1}{2}\int_{\partial M} d^2x\sqrt{h}(h^{mn}A_mA_n + \bar{\chi}_-\gamma^m\partial_m\chi_-). \tag{4.68}$$

In the above λ,χ are Majorana fermions and the notation $\chi_\pm$ means we project the fermion onto its top/bottom component respectively. The boundary is located at $x^3=0$ and the coordinates on the boundary are given by (x^1,x^2) indexed by the label m, the boundary metric is h and γ are the gamma matrices.

Due to the presence of the boundary, the action is not invariant under the most general supersymmetry transformation. However, it is invariant under half of the supersymmetry transformations $\mathcal{N}=(1,0)$.[22] These transformations are given by

$$\begin{aligned}\delta A_\mu &= (\bar{\lambda}\gamma_\mu\epsilon_+) + (\bar{\epsilon}_+\partial_\mu\chi_-),\\ \delta\lambda_a &= -\epsilon^{\mu\nu\rho}(\gamma_\rho\epsilon_+)_a\partial_\mu A_\nu,\\ \delta\chi_- &= (\gamma^\mu\epsilon_+)A_\mu = (\gamma^m\epsilon_+)A_m.\end{aligned} \tag{4.69}$$

Where ϵ_+ is a two component spinor projected onto only its top component. In the path integral we must choose boundary conditions that satisfy the variational principle and that are left invariant under the above supersymmetry transformations. It was found

[22] The action is invariant under these transformations without having to impose any boundary conditions on the fields. This approach was advocated as "supersymmetry without boundary conditions" in [210].

in [210, 211] that one such choice of boundary conditions is given by fixing given by $A_- = 0$ and $2\gamma^2\lambda_+ + \partial_-\chi_- = 0$, where we have defined $A_\pm = A_1 \pm A_2$ and $\partial_\pm = \partial_1 \pm \partial_2$. The boundary action in terms of these fields is given by[23]

$$S_{\text{bdy}}^{\mathcal{N}=(1,0)} = \frac{1}{2}\int_{\partial M} d^2x(A_+A_- + \overline{\chi}_-\gamma^2\partial_-\chi_-). \tag{4.70}$$

The second boundary condition we fixed, relating χ_- and λ_+, guarantees that our boundary condition for the gauge field is invariant under the ϵ_+ supersymmetry transformation $\delta A_- = 0$. Similarly, the other boundary condition $2\gamma^2\lambda_+ + \partial_-\chi_- = 0$ is also invariant under ϵ_+ transformations. We don't need to set any additional boundary conditions because the second term in the boundary action (4.70) varies into an equation of motion for χ_- on the boundary. One of the interesting features of this theory is that there is a dynamical boundary fermion χ_-, decoupled from the other fields, without any corresponding bulk action.

Similarly, there is a $\mathcal{N} = (0, 1)$ supersymmetric Chern-Simons theory invariant under ϵ_- transformations, given by the action

$$S_{\text{CS}}^{\mathcal{N}=(0,1)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \overline{\lambda}\lambda) + \frac{1}{2}\int_{\partial M} d^2x\sqrt{g}(g^{mn}A_mA_n + \overline{\chi}_+\gamma^m\partial_m\chi_+). \tag{4.71}$$

In the above λ, χ_+ are again Majorana fermions, except now χ has been projected onto the top component. The boundary action can be re-written as

$$S_{\text{bdy}}^{\mathcal{N}=(0,1)} = \frac{1}{2}\int_{\partial M} d^2x(-A_+A_- + \overline{\chi}_+\gamma^2\partial_+\chi_+). \tag{4.72}$$

A consistent choice of boundary conditions is given by $A_+ = 0$ and $\left(-2\gamma^2\lambda_- + \partial_+\chi_+\right) = 0$. We can consider the combined action

$$S = S_{\text{CS}}^{\mathcal{N}=(1,0)} - S_{\text{CS}}^{\mathcal{N}=(0,1)}, \tag{4.73}$$

where in total the bulk theory has $\mathcal{N} = (1, 1)$ supersymmetry with each half realized independently by one of two theories. The total action will depend on two independent Chern-Simons fields A, B, two independent auxiliary fields λ_1, λ_2 and two, fermions $\chi_\pm$ which have been projected onto the top/bottom component and effectively function as single component fermions. When considering the partition function of the total theory it will factorize into a contribution from the independent Chern-Simons fields, the two copies of the auxiliary fields λ_i, and the boundary fermions χ_-, χ_+. Since we are integrating over the auxiliary fields λ_i there is no restriction on the field configurations the boundary fermions take.

After analytically continuing to Euclidean signature and defining the theory on a bulk handlebody with an asymptotic boundary torus the full partition function is given by the product of contributions of: $U(1) \times U(1)$ Chern-Simons, a holomorphic and anti-holomorphic two-dimensional free fermion, and an overall normalization given by integrating over the auxiliary λ_i. Dropping the normalization given by the auxiliary fields gives the partition function

$$Z_{\text{ferm}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)Z_{\text{CS}}(\tau) = \left|\det\overline{\partial}_{\alpha,\beta}\right|\frac{1}{|\eta(\tau)|^2}. \tag{4.74}$$

where the first factor comes from the Chern-Simons contribution and the second comes from the free fermions.

[23]For this one uses the identity $\gamma^1\chi_- = -\gamma^2\chi_-$.

We take D copies of the above theory and perform a summation over all bulk handlebodies. The choice of asymptotic boundary conditions fixes the spin structures α, β around two particular boundary cycles $\tau, 1$. When summing over bulk handlebodies the standard prescription is to only include handlebodies which can inherit the spin structure specified at the asymptotic boundary [193]. That is, if the cycle $c\tau + d1$ is contractible then it must be true that the fermions are anti-periodic around that cycle. However, in our case the fermions reduce to a boundary term, and so we do not have such a constraint since the spin structure does not need to be extended to the entire handlebody. Summing over all handlebodies, taking into account that the fermions give identical contributions due to (3.49), we find

$$Z_{\text{Bulk}}(\tau) = Z_{\text{ferm}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau)^{D} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{2D}} . \tag{4.75}$$

We see that the bulk supersymmetric Chern-Simons theory precisely reproduce the boundary ensemble average.

Let us now explain how the above is modified when we consider the symmetric product orbifold of the supersymmetric Narain theories. We are again interested in implementing twisted boundary conditions along the contractible and non-contractible cycles for both the gauge fields and the fermions.

For the non-contractible cycles, recall that for the gauge fields of $U(1)^D \times U(1)^D \wr S_N$, around page 102, we summed over twisted boundary conditions around those cycles.[24] For the fermions the situation is similar and they transform as $\psi_{(i)} \to e^{2\pi i \alpha} \psi_{(\omega(i))}$, where $\omega \in S_N$.

For the contractible cycle, we can again use a vortex operator $\mathcal{V}$ to implement twisted boundary conditions for the fermions. We implement the action of the operator by specifying the monodromies it implements on the fermions, namely $\psi_{(i)} \to e^{2\pi i \beta} \psi_{(\omega(i))}$ as we travel around the contractible cycle. Summing over all possible choices of vortices, similar to the gauge theory, gives a summation over all twisted boundary conditions along the contractible cycle.

Combining these two ingredients together we find that a summation over twisted boundary conditions (bundles) and vortices again implements the twisted boundary conditions necessary for the symmetric product orbifold. Since the fermions and gauge fields both transform in the adjoint of the discrete gauge group they acquire the same monodromy $A_{(i)} \to A_{(\omega(i))}$ and $\psi_{(i)} \to e^{2\pi i \alpha} \psi_{(\omega(i))}$ as they travel around a vortex. Since the fields acquire the same monodromies it immediately follows that, identical to the bosonic case, performing a summation over bundles and vortices implements a summation over all degree N covering spaces of the base torus.

If we consider the grand canonical partition function of the supersymmetric Chern-Simons theory on handlebody M, since we are summing over covering spaces, we find that it is again given by the exponential of connected covers

$$\mathfrak{Z}_{\text{SCS}}(M, p) = \exp\left(\sum_{k=1}^{\infty} p^k \mathcal{T}_k Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) \right) . \tag{4.76}$$

Comparing to the boundary answer in equation (3.199), we immediately have that the connected covering space contributions match between the bulk and the boundary the-

[24] This amounts to summing over non-trivial S_N bundles.

ories after summing over handlebodies M

$$\int_{\mathcal{M}_D} d\mu \log \mathfrak{Z} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, p) = \sum_M \log \mathfrak{Z}_{SCS}(M, p) . \tag{4.77}$$

4.4 Lessons from ensemble averaging Narain $\mathbb{Z}_2$ orbifolds

In this section, we shift our focus to the ensemble averages of the kind of orbifolds we studied in section 3.2. Before discussing specifically about the Narain $\mathbb{Z}_2$ orbifolds we saw there, we start with a more general discussion regarding ensemble averages of two-dimensional CFTs arising from products of other CFTs [2].

4.4.1 Products of conformal field theories

We start by considering two families of two-dimensional CFTs: CFT_a , $a = 1, 2$, with moduli spaces $\mathcal{M}_a$. A point in these moduli spaces is denoted by m_a. For example, if the CFTs are toroidal of the type we studied in section 3.1, a point in the moduli space is a choice of metric and B-field G, B—or equivalently a matrix $H(G, B)$ (see e.g. appendix A.2). Furthermore, assume that the moduli spaces $\mathcal{M}_a$ have fixed finite dimensions $d_a = \mathrm{dim}\mathcal{M}_a$ and are of finite volume

$$\mathrm{Vol}(\mathcal{M}_a) = \int_{\mathcal{M}_a} d\mu(m_a) \, , \quad a = 1, 2 \, , \tag{4.78}$$

with respect to the measure $d\mu(m_a)$ induced from the Zamolodchikov metrics of the conformal field theories CFT_a. Then we can define the ensemble averages

$$\langle Z_a(\tau) \rangle = \frac{1}{\mathrm{Vol}(\mathcal{M}_a)} \int_{\mathcal{M}_a} d\mu(m_a) \, Z_a(\tau; m_a) \, , \tag{4.79}$$

which we assume to be also finite. The fact that the dimension of the moduli space $\mathcal{M}_a$ is d_a means that the CFT has d_a exactly marginal operators that are exactly marginal at any point in the moduli space and there is no obstruction in deforming the theories with respect to these operators at higher orders. The CFT might possess primary operators that are exactly marginal only for specific values of m_a. In this case a new stratum of families of CFTs can be attached to $\mathcal{M}_a$ at these points.[25]

Let us also assume that both ensemble averages of the partition functions $\langle Z_a(\tau) \rangle$ can be interpreted holographically in terms of three-dimensional bulk theories on three-dimensional spaces $M_3^{(a)}$, $a = 1, 2$ with toroidal boundaries $\partial M_3^{(a)} = T_\tau^2$, where τ is the modular parameter (complex structure) of the boundary torus. Schematically, we have

$$\langle Z_a(\tau) \rangle = \int \mathcal{D}[g, \phi_a] \, e^{-S_a[g, \phi_a]} \, . \tag{4.80}$$

Here the RHS is a functional integral over bulk field configurations ϕ_a with specified boundary conditions, bulk metrics that on the boundary give rise to T_τ^2, which possibly includes sums over topologies. For example, as we saw in chapter 4, in refs. [74, 75], the path integral over the metric g localizes to a sum over hyperbolic geometries M_3 with asymptotic boundary conditions $\partial M_3 = T_\tau^2$ with complex structure modulus τ

[25]This can happen for example for the compactified boson on a circle—see e.g. chapter 8.5 of [169].

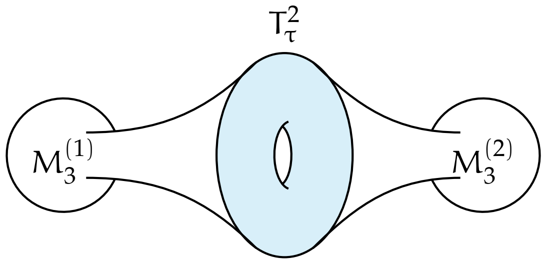

Figure 4.16: Bulk manifolds $M_3^{(1)}, M_3^{(2)}$ sharing the same boundary.

(handlebodies). In the context of the ensemble averages of higher genus partition functions of symmetric product orbifolds considered in ref. [3], we saw in the same chapter that smooth bulk geometries that are not handlebodies become relevant (see also refs. [206, 195]).

Now we consider the product of the two two-dimensional conformal field theories $\mathrm{CFT}_{1\otimes 2} \equiv \mathrm{CFT}_1 \times \mathrm{CFT}_2$, which by construction again yields a family of unitary conformal field theories parametrized by $(\mathfrak{m}_1, \mathfrak{m}_2) \in \mathcal{M}_1 \times \mathcal{M}_2$. Since the Zamolodchikov metric and hence the moduli space measure also factorize, the ensemble average of the family of product conformal field theories is just the product of the averages of its factors, i.e.,

$$\left\langle Z_{1\otimes 2}(\tau)\right\rangle = \langle Z_1(\tau)\rangle \, \langle Z_2(\tau)\rangle \ . \tag{4.81}$$

From the dual holographic perspective the observed factorization of the partition function poses a puzzle, as the partition function of a possible holographic dual three-dimensional bulk theory is also required to factorize. Assuming that a holographic description exists at all, we discuss in the following scenarios for possible bulk interpretations of such ensemble averages:

- Since the ensemble average $\left\langle Z_{1\otimes 2}(\tau)\right\rangle$ factorizes, a possible interpretation for a three-dimensional dual description is obtained in terms of the three-dimensional action $S_{1\otimes 2} = S_1[g_1, \phi_1] + S_2[g_2, \phi_2]$. In this setup the two metrics g_1 and g_2 are distinct on the two three-spaces $M_3^{(1)}$ and $M_3^{(2)}$, and both three-spaces $M_3^{(a)}$, $a = 1, 2$, should have a common asymptotic toroidal boundary $T^2_\tau = \partial M_3^{(1)} = \partial M_3^{(2)}$—as depicted in Fig. 4.16—on which the two metrics g_1 and g_2 coincide asymptotically. Then by construction the holographic correspondence becomes

$$\begin{aligned} \left\langle Z_{1\otimes 2}(\tau)\right\rangle &= \int \mathcal{D}[g_1, g_2, \phi_1, \phi_2]\, e^{-S_1[g_1,\phi_1]-S_2[g_2,\phi_2]} \\ &= \int \mathcal{D}[g_1, \phi_1]\, e^{-S_1[g_1,\phi_1]} \int \mathcal{D}[g_2, \phi_2]\, e^{-S_2[g_2,\phi_2]} \\ &= \langle Z_1(\tau)\rangle \, \langle Z_2(\tau)\rangle \ . \end{aligned} \tag{4.82}$$

Thus, the holographic dual arises from two distinct three-dimensional bulk theories that are glued together at a common asymptotic boundary. This idea also appeared in [3] when considering particular Lagrangian sublattices in the average of the disconnected part of the T^D/S_N orbifold. One example is the Lagrangian sublattice (see e.g. page 106)

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(A^{(1)}, B^{(2)}\right) \ . \tag{4.83}$$

This sublattice corresponds to a three-manifold which is a disjoint union of thermal AdS_3 and the Euclidean BTZ black hole (one corresponds to the A (τ) cycle being contractible and the other to the B (1) cycle being contractible). A way to include this in the bulk path integral is to allow different gauge field to live on independent bulk manifolds with the same asymptotic boundary.[26]

- The product conformal field theory $\mathrm{CFT}_{1\otimes 2}$ could be a (d_1+d_2)-dimensional subspace $\mathcal{M}_1 \times \mathcal{M}_2$ of a higher-dimensional moduli space $\mathcal{M}_{\text{total}}$, i.e, $\mathcal{M}_1 \times \mathcal{M}_2 \subset \mathcal{M}_{\text{total}}$ with $\dim \mathcal{M}_{\text{total}} > d_1 + d_2$. This can occur if the product CFT has additional marginal operators that parametrise directions in $\mathcal{M}_{\text{total}}$ that move into a direction normal to $\mathcal{M}_1 \times \mathcal{M}_2$.

  Generically, it is not expected that the larger moduli space $\mathcal{M}_{\text{total}}$ exhibits a product structure (unless enforced by symmetry). Therefore, the ensemble average of the partition function $Z_{\text{total}}(\tau; m_{\text{total}})$ of the total family of conformal field theories $\mathrm{CFT}_{\text{total}}$ does not exhibit the product structure any longer

  $$\langle Z_{\text{total}}(\tau)\rangle = \frac{1}{\mathrm{Vol}(\mathcal{M}_{\text{total}})} \int_{\mathcal{M}_{\text{total}}} d\mu(m_{\text{total}})\, Z_{\text{total}}(\tau; m_{\text{total}}) \,, \tag{4.84}$$

  where we assume that the volume of $\mathcal{M}_{\text{total}}$ is finite and that the integral over the partition function $\mathcal{M}_{\text{total}}$ converges.

  Since the dimension of the moduli space $\mathcal{M}_{\text{total}}$ is higher than the subspace $\mathcal{M}_1 \times \mathcal{M}_2$, the contribution to the ensemble average arising from this subspace has measure zero. Its contribution is therefore not relevant for the total ensemble average. As a result a holographic dual formulation for the conformal field theory $\mathrm{CFT}_{\text{total}}$ does not need to reflect a product structure any longer, and the average $\langle Z_{\text{total}}(\tau)\rangle$ can possibly arise from a conventional three-dimensional bulk theory on a three-dimensional space M_3 with an asymptotic boundary component T^2_τ.

- If the product of conformal field theories does not give rise to additional exactly marginal operators, the products of conformal field theories could be part of a larger ensemble, in which the product moduli space $\mathcal{M}_1 \times \mathcal{M}_2$ arises as a connected component. If this connected component has measure zero in this larger moduli space $\mathcal{M}_{\text{total}}$, then it will not contribute to an ensemble average over the whole moduli space $\mathcal{M}_{\text{total}}$. Additionally, the latter does not necessarily need to obey a product structure.

  Such a scenario is conceivable, if for instance the ensemble $\mathcal{M}_{\text{total}}$ of conformal field theories consists of different strata of different dimensions.

Before we move on to our examples from section 3.2, we comment briefly on the symmetric product orbifold. This is not exactly a product of CFTs but is created by an orbifold of a product of CFTs. The ensemble averaging we performed for this orbifold in previous sections focused on averaging only over the "diagonal" moduli space, i.e. the one where all the seed theories are described by the same moduli (see eq. (3.127)). This is in a sense a choice because, as we explained in section 3.2, at least for the $N = 2$ case the symmetric product orbifold can be deformed into the non-factorisable $\mathbb{Z}_2$ orbifold, and including these deformations would lead to the average of the latter. We saw in eq. (3.138) a "simple" closed expression for the non-factorisable orbifold average. In contrast, the average of the symmetric product orbifold seams more complicated. This could perhaps be a result of integrating over a slice of the whole moduli space and it would be interesting to see its manifestations for S_N (and $\mathbb{Z}_N$), $N > 2$ generalisations of the non-factorisable $\mathbb{Z}_2$ orbifold. Moreover, the symmetric product orbifold (for specific

[26]See also the discussion in Section 4.2 of [193], and also Section 3.1 of [195].

seed theories) contains even more moduli independent of the Narain ones. This is another potentially interesting problem to look at and see how the "enlarging" of the moduli space affects the averages and potential bulk interpretations.

Ensemble averages of products of toroidal CFTs

Take as CFT_1 a T^ℓ CFT and as CFT_2 a T^m CFT and denote the product as $\mathrm{CFT}_{T^\ell \otimes T^m}$. The ensemble average over the product moduli space $\mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m}$ yields[27]

$$\left\langle Z_{T^\ell \otimes T^m}(\tau) \right\rangle = \frac{E_{\ell/2}(\tau) E_{m/2}(\tau)}{\mathrm{Im}(\tau)^{\frac{\ell+m}{2}} |\eta(\tau)|^{2(\ell+m)}} = \langle Z_{T^\ell}(\tau) \rangle \langle Z_{T^m}(\tau) \rangle \ . \tag{4.85}$$

However this product theory contains additional exactly marginal operators that can deform it away from the product locus $\mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m}$. In fact, including these deformations, the moduli space becomes $\mathcal{M}_{T^{\ell+m}}$ into which $\mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m}$ naturally embeds. This yields the discussed holographic interpretation of refs. [74, 75].

Ensemble Average of Factorizable Toroidal $\mathbb{Z}_2$ Orbifold CFTs:

The ensemble average of the partition function of the family of factorizable toroidal $\mathbb{Z}_2$ orbifold conformal field theories studied in subsection 3.2 factorizes as established in eq. (3.102) because the moduli space $\mathcal{M}_{T^D/\mathbb{Z}_2}$ factorizes as $\mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m}$ (see eq. (3.101)).

As opposed to the moduli space of the product conformal field theory $\mathrm{CFT}_{T^\ell \otimes T^m}$ discussed in the previous paragraph, for generic points $(m_1, m_2) \in \mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m}$ the family of conformal field theories $\mathrm{CFT}_{T^\ell/\mathbb{Z}_2 \otimes T^m}$ does not have any primary fields with conformal dimension $(h, \bar{h}) = (1, 1)$ that deform it away from this moduli space (this is basically a result of the orbifold which distinguishes one direction). Therefore, the moduli space $\mathcal{M}_{T^\ell} \times \mathcal{M}_{T^m}$ is not a lower-dimensional slice of a higher-dimensional embedding moduli space.

Hence, the moduli space of the product conformal field theory $\mathrm{CFT}_{T^\ell/\mathbb{Z}_2 \otimes T^m}$ does not naturally extend to a larger family of conformal field theories. However, the factorized ensemble average $\left\langle Z_{T^\ell/\mathbb{Z}_2 \otimes T^m}(\tau) \right\rangle = \left\langle Z_{T^\ell/\mathbb{Z}_2}(\tau) \right\rangle \langle Z_{T^m}(\tau) \rangle$ of the partition function can still be obtained from the holographic duals of the conformal field theory factors $\mathrm{CFT}_{T^\ell/\mathbb{Z}_2}$ and CFT_{T^m} along the lines of eq. (4.82). The three-dimensional holographic duals of these two factors of families of conformal field theories are proposed in refs. [74, 75] and in ref. [166], respectively.

Ensemble Average of Non-Factorizable Toroidal $\mathbb{Z}_2$ Orbifold CFTs:

The moduli space $\mathcal{M}_{T^{2\ell}/\mathbb{Z}_2}$ for the family of conformal field theories resulting from non-factorizable toroidal $\mathbb{Z}_2$ orbifold conformal field theories studied in subsection 3.2.2 again factorizes (see eq. (A.40)). However, the partition function $Z_{T^{2\ell}_{\text{non-fac}}}$ does not factorize, and hence the ensemble average is also not of a factorized form. Nevertheless, the resulting ensemble average (3.137) becomes a finite sum of products of real analytic Eisenstein series. It would be interesting to propose a holographic dual, which possibly consists of non-trivial topological sectors, similarly as the vortex sectors considered in refs. [166, 3].

[27] Products of real analytic Eisenstein series from products of Narain conformal field theories or from subloci associated to such Narain products are also discussed in ref. [212].

4.5 Summary and outlook

In this final chapter, the goal was to give a holographic interpretation to the formulas we derived in chapter 3. We started with the established correspondence between ensembles of Narain CFTs and U(1)-gravity of refs. [74, 75] and then proposed a bulk dual to ensembles of symmetric product orbifolds as in [3]. We saw that we can reproduce, or give a holographic interpretation, to terms in the ensemble average of partition functions via a suitable Chern-Simons-like theory with S_N gauge symmetry and in particular we matched the connected contributions of the bulk and the average. Also we saw that more complicated vortex configurations have a chance to reproduce more terms in the average, like the double vortex of figure 4.6.

For the symmetric product orbifold we studied ensemble averages of correlators as well. Focusing on S_2, these were given a holographic interpretation in terms of a sum of bulk vortex configurations that are rational tangles. Each of these configurations has a handlebody covering space hence the matching to the average (which can be written as a sum over handlebodies). Moreover, we noted that for higher point functions (eight-point and higher) this interpretation is obstructed. We ended the discussion of ensemble averages of correlators by matching the conformal anomaly contributions in average to the one in the bulk. After the correlators, we proposed a supersymmetric bulk dual to the established U(1)-gravity and generalised this to a bulk dual to supersymmetric symmetric product orbifolds.

This chapter ended with a general discussion as in ref. [2] of CFTs that are products of other CFTs and the implications that this has to potential bulk duals. We specialised this discussion to U(1)-gravity and the orbifolds studied in [2].

We end this chapter with some interesting questions for the future

Holographic interpretation

For the ensemble of factorizable $\mathbb{Z}_2$ orbifold toroidal conformal field theories studied in [2], we propose a possible holographic interpretation in terms of previously discussed Chern–Simons theories on bulk manifolds that share a common boundary. As mentioned earlier this idea also appears in [3] for a certain class of Lagrangian sublattices. It would be important to quantitatively further check such a proposal and to discuss its implications.

Formulating a holographic dual for the non-factorizable case of ref. [2] seems even more challenging. While the moduli space still factors for non-factorizable $\mathbb{Z}_2$ orbifold toroidal conformal field theories, we have not put forward a proposal for a possible dual bulk theory. However, the derived analytic expression for the ensemble average of the partition function in terms of products of Eisenstein series suggests that the S_2 permutation symmetry might play an important role in the bulk theory as well. Alternatively, we can view the non-factorizable toroidal $\mathbb{Z}_2$ orbifolds as arising from a shift orbifold (see for instance ref. [182]) of the factorizable toroidal $\mathbb{Z}_2$ orbifolds. This perspective might also shed light on a possible holographic bulk interpretation in the future.

Other orbifolds, discrete torsion and supersymmetry

One can think of generalizing the construction of $\mathbb{Z}_2$ factorisable and non-factorisable toroidal orbifolds by considering $S_N, \mathbb{Z}_N$, $N = 3, 4, \ldots$, or other discrete groups such as $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_N \times \mathbb{Z}_M$. In particular, the latter cases are interesting because they admit discrete torsion (see e.g. eq. (3.64)). On the level of the partition function of orbifolds of conformal field theories, discrete torsion amounts to assigning suitable group-theoretic phase factors to its various orbifold sectors such that partition

$$
\begin{tikzcd}
\text{2d} & \text{3d} \\
\mathrm{Sym}^N\left(T^4\right) \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, "\text{Average}"] & \text{Tensionless String} \arrow[d, "\text{Average}"] \\
\int d\mu \, \mathrm{Sym}^N\left(T^4\right) \arrow[r, leftrightarrow, "\text{duality}"] & ?
\end{tikzcd}
$$

Figure 4.17: Ensemble averaging $\mathrm{Sym}^N(T^4)$ could serve as an attempt to ensemble average an exact duality.

function is modular invariant [57, 179, 180]. A concrete example for an orbifold conformal field theory with discrete torison is given by orbifold toroidal conformal field theories of the type [180]

$$\frac{T^D \times T^D \times T^D}{\mathbb{Z}_2 \times \mathbb{Z}_2} \, . \tag{4.86}$$

It would be interesting to study such classes of orbifolds with discrete torsion from the scope of our current work.

Another future direction is to add supersymmetry to the same orbifolds along the lines of what we did for the symmetric product [3].

Deformations and other moduli

As mentioned previously, symmetric product orbifold theories $\mathrm{Sym}^N(T^D)$ live in a larger moduli space than that of their Narain seed theory, and specifically for seed theories of central charge $c \leqslant 6$, symmetric product orbifolds admit marginal deformations by twist fields, which break the orbifold structure. One special such example is the D1/D5 system compactified on T^4 [189], whose moduli space contains the 'orbifold point' $\mathrm{Sym}^N(T^4)$ along with its Narain moduli, as well as four exactly marginal operators in the twist-2 sector. It would be interesting to consider an average over the full 20-dimensional moduli space of the D1/D5 CFT or the average over only the Narain moduli of a deformed CFT.

Vortices

We saw that certain contributions to the average can be viewed as bulk partition functions on bulk geometries with vortices. In particular, we saw that adding more than one vortex has the potential of reproducing more complicated terms of the average. An interesting problem would be to work out more vortex contributions and see if the can matched to terms in the average. Or the way around, write down Lagrangian sublattices and try to reproduce their contributions from bulk vortices.

A "stringy" embedding of ensemble–averaging

Part of the motivation to study ensemble averages of the symmetric product orbifold comes from the duality between the "tensionless string"[28] and (the supersymmetric version of) $\mathrm{Sym}^N(T^4)$ at large N. In particular we were interested to understand whether a sum over geometries emerges from the ensemble average, see also figure 4.17.

[28]Type IIB string theory on $AdS_3 \times S^3 \times T^4$ with one unit of pure NS-NS flux [213, 214, 215, 216, 72, 217, 218, 219, 220, 221, 222].

We were partially successful in doing so in the sense that we managed to give a holographic interpretation in terms of a sum over geometries with vortices to the ensemble average of the connected part of the $\mathrm{Sym}^N(T^4)$ partition function[29]

$$\langle T_k Z_{T^4}(\tau) \rangle \ , \tag{4.87}$$

or, equivalently, to the logarithm of the grand canonical partition function (3.162)

$$\langle \log \mathfrak{Z}(p, \tau) \rangle = \langle \sum_{k=1}^{\infty} p^k T_k Z_{T^4}(\tau) \rangle \ . \tag{4.88}$$

The grand canonical partition function $\mathfrak{Z}$ of $\mathrm{Sym}^N(T^4)$ is a natural partition function for the tensionless string [72] and holographically the connected covering spaces appearing in the exponent correspond to (single) strings propagating in AdS_3 which wind the boundary k times. Expanding the exponent includes contributions from disconnected strings as well.

In this sense, we were able to give an ensemble averaged interpretation to a single string propagating in AdS_3 in terms of a sum over handlebodies and vortices. Averaging over multiple strings corresponds to averaging $\mathfrak{Z}$ itself and not its logarithm. For this, we got more geometries whose holographic interpretation was not so clear. A subtlety that arises here is that even though the average of $\log \mathfrak{Z}$ converges in the $\mathrm{Sym}^N(T^4)$ case, the average of $\mathfrak{Z}$ contains divergent terms[30] to which one could still try however to give a bulk interpretation.

[29] We ignore spin structures in what follows.

[30] As we saw earlier we need $D > g + 1$ for convergence.

Chapter 5

Summary

In this thesis, we studied ensemble-averaged holography in two and three spacetime dimensions. We started, in chapter 2, with the former in the context of the Jackiw-Teitelboim (JT)/ random matrix correspondence established in ref. [73] with the goal to describe our work in ref. [1]. To begin with, we saw how the path integral of JT gravity and deformed JT gravity can be written in terms of moduli space volumes of bordered Riemann surfaces with or without conical singularities (see for example eqs.(2.15) and (2.27)). Following refs. [130, 131], we extended their formulation and wrote the partition functions of JT with conical singularities in terms of topological gravity correlators (eq. (2.44)). In doing so, we defined generalised JT partition functions that reproduce the previously discussed ones for particular values of parameters (eq. (2.56)). To calculate these partition functions order by order in the genus expansion we employed the Korteweg-de Vries (KdV) hierarchy with which we derived, in section 2.5, a recursion relation. Using the genus zero part of the generalised single-boundary partition function, we derive a density of eigenvalues (2.119) for a potential random matrix dual that, in the case of conical singularities, reproduces the results of refs. [126, 128]. In section 2.7, we adapted the low temperature scheme of ref. [130] and studied the ratio between connected and disconnected contributions to the partition function with two boundaries in JT with conical singularities illustrating how it changes with the defect magnitude. Finally, we demonstrated how the defect affects the spectral form factor and noticed that the transition from ramp to plateau happens at earlier times for larger defect magnitudes. In section 2.8, we gave a summary and outlook for this work.

In chapters 3 and 4, we moved on to one dimension higher and the goal was to illustrate our work in refs. [2, 3]. In chapter 3 we focused on calculating ensemble averages of two-dimensional conformal field theories (CFTs). We initiated this discussion with some basic examples to set the stage and define the Narain CFTs. We showed how the partition functions of such CFTs are written in terms of Siegel-Narain partition functions and calculated their ensemble averages, as in [74, 75]. We moved on to show how this generalises to supersymmetric Narain theories in eq. (3.50). In section 3.2 we started the discussion on orbifolds of Narain orbifolds. After a general discussion on orbifolds and averaging we defined factorisable and non-factorisable $\mathbb{Z}_2$ orbifolds for two-dimensional toroidal target spaces. We found their moduli spaces and calculated (regulated) ensemble averages of partition functions, for example see eqs.(3.80) and (3.95). Following that we generalised these orbifold constructions to higher dimensions. Again we performed the ensemble average for partition functions after studying the moduli space structure for each case. For the factorisable case, the average is eq.(3.105) and for the non-factorisable is given in eq. (3.138) (and in eq. (3.137) in terms of Eisenstein series). Especially for the non-factorisable case it was important to express the partition function (the untwisted no insertion part) in terms of Siegel-Narain functions as in (3.120).

In section 3.3, we turned our focus to symmetric product orbifolds of Narain CFTs. We saw how one can calculate partition functions using the covering space method and applied the Siegel-Weil formula to calculate their ensemble averages over the Narain moduli. For the connected parts we can give rather explicit answers, see eqs. (3.196), while the disconnected parts are more complicated. After showing how this works for the supersymmetric case in eq. (3.199), we discussed correlation functions in the context of symmetric product orbifolds. In particular, we looked at correlators of twist fields on the sphere, which are again calculated using a covering space method—only now the covering is ramified. The ensemble average can be done with the Siegel-Weil formula as the result for the correlator is given in terms of the partition function on the covering space, see eq. (3.218). We gave a summary of this chapter also in section 3.5.

Following that, in chapter 4, we explored bulk duals of the above ensemble averaged CFTs. After explaining the case of refs. [74, 75], we defined a candidate bulk dual for the ensemble averaged symmetric product orbifold in terms of a $\mathrm{U}(1)^{\mathrm{D}} \times \mathrm{U}(1)^{\mathrm{D}} \wr S_{\mathrm{N}}$ Chern-Simons theory summed over geometries. We saw that the connected contributions between bulk and boundary partition functions match and were able to also reproduce other contributions (but not all) of the boundary average via vortices in the bulk. We observed that there are terms in the ensemble average that can be potentially interpreted as bulk contributions from manifolds that are not handlebodies—in contrast to the usual $\mathrm{U}(1)$-gravity that contains only handlebodies in the ensemble average of the (single boundary) partition function.

We continued with an interesting interpretation of ensemble averages of correlation functions of twist fields in terms of rational tangle vortex configurations in the bulk, in a similar manner as ref. [166]. This was possible because the three-manifolds with these vortices have covering spaces that are handlebodies and the ensemble average of correlators is written in terms of a sum over handlebodies. This was demonstrated for four-point twist field correlators on the sphere in S_2 and was argued that for correlators of a certain degree and higher such an interpretation is most likely obstructed. Moreover, we defined a bulk dual to the supersymmetric extension of ensembles of Narain CFTs in terms of a Chern-Simons theory with $\mathcal{N} = (1, 1)$ supersymmetry and showed how this can be done for the symmetric product orbifold as well.

Finally, we discussed ensemble averages of products of arbitrary CFTs and speculated about possible properties of the bulk duals (if any) depending on properties of the CFT. For example, we saw that the moduli space of a CFT that is a product of two Narain CFTs naturally embeds into the larger moduli space in which the target space is a torus of dimension the sum of the dimensions of the two tori. Averaging over the product locus seems to require a bulk interpretation in terms of disconnected bulk manifolds that share a common boundary. However, averaging over the full moduli space leads to the usual $\mathrm{U}(1)$-gravity interpretation. The factorisable orbifold we defined in earlier sections has a moduli space that does not naturally embed into another one hence averaging over the product locus in this case is a canonical choice. For the non-factorisable $\mathbb{Z}_2$ orbifold we don't give a bulk dual but it is interesting to contrast this to the S_2 symmetric product orbifold bulk construction as the latter orbifold lives on a subslice of the former. We ended with a summary and outlook in section 4.5.

It is interesting to see, on a formal level, how rich the study of holography can be in the context of ensemble averages. For the case of two-dimensions the theories we studied connect to areas such as integrability, via the KdV hierarchy, random matrices and the theory of Riemann surfaces and their moduli spaces. In three-dimensions, we encountered connections to number theory via the Eisenstein series and the Siegel-Weil formula. Also, the notions of covering spaces and three-dimensional manifolds were of paramount importance. All this came from ensemble averaging relatively "simple" systems such as matrices and Narain CFT and it would be very interesting to look for other examples where ensemble averaging can be performed and discover what this can teach us about holography. Finally, ensemble averaging could teach something about coarse-graining theories. In systems where the fully fledged description seems complicated, an ensemble-averaged version might be able to give us insight via a simplified model.

Appendix A

Appendix

A.1 Theta functions and their transformations

Here we collect some useful definitions and relations related to theta functions and the Dedekind eta function. Theta functions with characteristics α, β are defined as

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \sum_{n\in\mathbb{Z}} \exp\left(i\pi(n+\alpha)^2\tau + 2\pi i(n+\alpha)(z+\beta) \right) . \tag{A.1}$$

Here $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$ is the modular parameter of the genus one Riemann surface that is defined in the upper half plane $\mathbb{H} = \{x + iy |\, y > 0; x, y \in \mathbb{R}\}$, and z is a point on this Riemann surface.

We are particularly interested in the theta functions $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) \equiv \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\tau)$. The following definitions appear in partition functions in the main text

$$\theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\tau) = \theta_1(\tau) , \quad \theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau) = \theta_2(\tau) , \quad \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) = \theta_3(\tau) , \quad \theta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau) = \theta_4(\tau) . \tag{A.2}$$

Note these theta functions transform under modular transformations as:

$$\begin{aligned} \theta_2(\tau+1) &= e^{\frac{i\pi}{4}}\theta_2(\tau) , & \theta_2\left(-\tfrac{1}{\tau}\right) &= \sqrt{-i\tau}\theta_4(\tau) \\ \theta_3(\tau+1) &= \theta_4(\tau) , & \theta_3\left(-\tfrac{1}{\tau}\right) &= \sqrt{-i\tau}\theta_3(\tau) \\ \theta_4(\tau+1) &= \theta_3(\tau) , & \theta_4\left(-\tfrac{1}{\tau}\right) &= \sqrt{-i\tau}\theta_2(\tau) \\ \eta(\tau+1) &= e^{\frac{i\pi}{12}}\eta(\tau) , & \eta\left(-\tfrac{1}{\tau}\right) &= \sqrt{-i\tau}\eta(\tau) \end{aligned} \tag{A.3}$$

The Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) , \qquad q = e^{2\pi i\tau} . \tag{A.4}$$

The following identities are useful:

$$\frac{\theta_2(\tau)}{\eta(\tau)} = \frac{2\eta(2\tau)^2}{\eta(\tau)^2} , \qquad \frac{\theta_4(\tau)}{\eta(\tau)} = \frac{\eta(\tau/2)^2}{\eta(\tau)^2} , \qquad \frac{\theta_3(\tau)}{\eta(\tau)} = \frac{\eta((\tau+1)/2)^2}{e^{\pi i/12}\eta(\tau)^2} \tag{A.5}$$

Starting with the first identity in eq. (A.5), we can prove the others by modular transformations, namely

$$\frac{\theta_2(\tau)}{\eta(\tau)} = \frac{2\eta(2\tau)^2}{\eta(\tau)^2} \xrightarrow{S} \frac{\theta_4(\tau)}{\eta(\tau)} = \frac{2\eta(-2/\tau)^2}{(-i\tau)\eta(\tau)^2} = \frac{2(-i\tau/2)\eta(\tau/2)^2}{(-i\tau)\eta(\tau)^2} = \frac{\eta(\tau/2)^2}{\eta(\tau)^2} , \tag{A.6}$$

$$\frac{\theta_4(\tau)}{\eta(\tau)} = \frac{\eta(\tau/2)^2}{\eta(\tau)^2} \xrightarrow{T} \frac{\theta_3(\tau)}{\eta(\tau)} = \frac{\eta((\tau+1)/2)^2}{e^{\pi i/12}\eta(\tau)^2} . \tag{A.7}$$

A.2 Partition Functions and Siegel–Narain Theta Functions

Toroidal Partition Functions and Theta Functions

Consider a two-dimensional CFT on a Riemann surface of genus $g = 1$ and target-space a N-dimensional torus T^N. The moduli of this theory are the metric G_{AB} and the anti-symmetric B-field B_{AB}. We sometimes denote these collectively as

$$\mathfrak{m} = \{G_{MN}, B_{MN}\} . \tag{A.8}$$

The partition function is

$$Z_{T^N}(\tau;\mathfrak{m}) = \frac{1}{|\eta(\tau)|^{2N}} \sum_{M,W\in\mathbb{Z}^N} \exp\left(F(\tau;\mathfrak{m})\right) \tag{A.9}$$

with

$$\begin{aligned} F(\tau;\mathfrak{m}) = 2\pi i\tau_1 M_A W^A - \alpha'\pi\tau_2\Big(&M_A G^{AB} M_B + \frac{1}{(\alpha')^2} W^A G_{AB} W^B - \\ &- \frac{2}{\alpha'} W^A B_A{}^B M_B - \frac{1}{(\alpha')^2} W^A B_A{}^B B_{BC} W^C\Big) . \end{aligned} \tag{A.10}$$

Siegel-Narain theta functions

In order to calculate efficiently the ensemble average of conformal field theories arising from toroidal target-spaces as developed in ref. [184], it is convenient to express the partition functions in terms of the Siegel–Narain theta functions [223, 84, 184]. Let Ω be a symmetric $2N \times 2N$ matrix of signature (N, N), such that 2Ω has integral entries and even entries on the diagonal.[1] Moreover, let H be a symmetric positive definite real $2N \times 2N$ matrix obeying

$$H\Omega^{-1}H = \Omega . \tag{A.11}$$

Then the Siegel–Narain theta functions in terms of Ω and H are defined as [223, 84, 184]

$$\Theta_{H,\Omega}(a, b, \tau) = \sum_{m\in\mathbb{Z}^{2N}} e^{-2\pi\,\mathrm{Im}(\tau)\,(m+b)^T H(m+b) + 2\pi i\,\mathrm{Re}(\tau)\,(m+b)^T\Omega(m+b) - 4\pi i\,\mathrm{Re}(\tau)\,a^T\Omega(m+\frac{1}{2}b)} . \tag{A.12}$$

with the twist vectors $a, b \in \mathbb{R}^{2N}$ and the modular parameter τ in the upper half-plane $\mathcal{H}$. Note that the positive definiteness of the matrix H ensures that the summation over $m \in \mathbb{Z}^{2N}$ converges in the definition of the theta function $\Theta_{H,\Omega}$. The Siegel–Narain theta functions of this work are all defined with respect to the pairing

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{1}_{N\times N} \\ \mathbf{1}_{N\times N} & 0 \end{pmatrix} . \tag{A.13}$$

Hence, for ease of notation we only refer to the positive definite $2N \times 2N$ matrix H in the expression for the Siegel–Narain theta functions, i.e.,

$$\Theta_H(a, b, \tau) \equiv \Theta_{H,\Omega}(a, b, \tau) . \tag{A.14}$$

[1] In refs. [223, 84], Siegel constructs theta functions for symmetric non-degenerate pairings Ω with arbitary signature (r, s).

In terms of the Siegel–Narain theta functions defined in eq. (A.14) the partition function $Z_{T^{2N}}(\tau; G, B)$ of the toroidal conformal field theory with the target-space torus T^{2N} becomes (see, e.g., ref. [57])

$$Z_{T^{2N}}(\tau; G, B) = \frac{1}{|\eta(\tau)|^{4N}} \Theta_{H(G,B)}(0, 0, \tau) , \tag{A.15}$$

with the real positive definite $4N \times 4N$-matrix $H(G, B)$ (which obeys the relation (A.11))

$$H(G, B) = \begin{pmatrix} \frac{\alpha'}{2} G^{-1} & \frac{1}{2} G^{-1} B \\ -\frac{1}{2} B G^{-1} & \frac{1}{2\alpha'} \left(G - B G^{-1} B\right) \end{pmatrix} , \tag{A.16}$$

with inverse

$$H^{-1}(G, B) = \begin{pmatrix} \frac{2}{\alpha'} \left(G - B G^{-1} B\right) & -2 B G^{-1} \\ 2 G^{-1} B & 2\alpha' G^{-1} \end{pmatrix} . \tag{A.17}$$

For convenience, we also define (for the T^N case)

$$\Theta_{H(G,B)}(0, 0, \tau) \equiv \Theta_N(\mathfrak{m}, \tau) . \tag{A.18}$$

For example eq. (A.15) is written as $1/|\eta(\tau)|^{4N} \times \Theta_{2N}(0, 0, \tau)$.

Average of Theta Functions

For the average of Siegel–Narain theta functions (A.14) with rational characteristic $b \in \mathbb{Q}^{2N}$ over the moduli space $\mathcal{M}_{T^N}$ of $2N \times 2N$ matrices H, one gets [184]

$$\langle \Theta_H(0, b, x) \rangle = \int_{\mathcal{M}_{T^N}} d\mathfrak{m}_H \, \Theta_H(0, b, x) = \sum_{(c,d)=1, c \geqslant 0} \frac{\gamma_{d \cdot b}}{|c\tau + d|^N} \, e^{2\pi i d c^* \, b^\mu \Omega_{\mu\nu} b^\nu} , \tag{A.19}$$

where (c, d) denotes the common greatest divisor of the integers c and d and the $2N \times 2N$-matrix Ω is given in eq. (A.13). The symbol γ_v for any $v \in \mathbb{R}^{2N}$ is defined as

$$\gamma_v = \begin{cases} 1 & \text{for } v \in \mathbb{Z}^{2N} , \\ 0 & \text{else} . \end{cases} \tag{A.20}$$

The integer c^* is part of a Bézout pair (c^*, d^*) obeying $c\, c^* + d\, d^* = (c, d) = 1$, which exists for any coprime integers c and d by Bézout's Lemma. Note that the average (A.19) is well-defined for any choice of Bézout's pair (c^*, d^*). For more details on this formula, see ref. [184].

A.3 Details on Non-Factorizable Tori

To calculate the $(+, +)$ partition function contribution to the non-factorisable, we need the quantity from (A.10), $F(\tau; \mathfrak{m})$ evaluated for the metric and B-field from eq. (3.117) (hence $\mathfrak{m}$ depends on $g, \widetilde{g}, b, \widetilde{b}$). To do so, first split the momentum and winding modes M, W into two 2ℓ-dimensional vectors, like so

$$M_A W^A = (m_1, .., m_\ell, \widetilde{m}_1, ..., \widetilde{m}_\ell) \begin{pmatrix} w^1 \\ \vdots \\ \widetilde{w}^1 \\ \vdots \\ \widetilde{w}^\ell \end{pmatrix} = m_a w^a + \widetilde{m}_a \widetilde{w}^a \tag{A.21}$$

and define the quantities

$$\mathfrak{r}_{\pm,a} = m_a \pm \widetilde{m}_a \;, \qquad \mathfrak{l}^a_\pm = w^a \pm \widetilde{w}^a \;. \tag{A.22}$$

These definitions, together with the identities

$$\left(m^\top, \widetilde{m}^\top\right) \begin{pmatrix} A+\widetilde{A} & A-\widetilde{A} \\ A-\widetilde{A} & A+\widetilde{A} \end{pmatrix} \begin{pmatrix} m \\ \widetilde{m} \end{pmatrix} = \left(m+\widetilde{m}\right)^\top A \left(m+\widetilde{m}\right) + \left(m-\widetilde{m}\right)^\top \widetilde{A} \left(m-\widetilde{m}\right) \tag{A.23}$$

$$\left(w^\top, \widetilde{w}^\top\right) \begin{pmatrix} A+\widetilde{A} & A-\widetilde{A} \\ A-\widetilde{A} & A+\widetilde{A} \end{pmatrix} \begin{pmatrix} m \\ \widetilde{m} \end{pmatrix} = \left(w+\widetilde{w}\right)^\top A \left(m+\widetilde{m}\right) + \left(w-\widetilde{w}\right)^\top \widetilde{A} \left(m-\widetilde{m}\right) \tag{A.24}$$

where $A, \widetilde{A}$ are $\ell \times \ell$ matrices, enable us to write the partition function in a nice form. Essentially these identities rely on the fact that

$$\left(P^{-1}\right)^\top \begin{pmatrix} A+\widetilde{A} & A-\widetilde{A} \\ A-\widetilde{A} & A+\widetilde{A} \end{pmatrix} P^{-1} = \begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix} \;, \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \;. \tag{A.25}$$

We get

$$F(\tau; m(g,b;\widetilde{g},\widetilde{b})) = F_+(\tau; m(g,b)) + F_-(\tau; m(\widetilde{g},\widetilde{b})) \;, \tag{A.26}$$

with

$$\begin{aligned} F_+(\tau;m) = \pi i \tau_1 \left(\mathfrak{r}_{+,a}\mathfrak{l}^a_+\right) - \alpha'\pi\tau_2\Big(\mathfrak{r}_{+,a}g^{ab}\mathfrak{r}_{+,b} &+ \frac{1}{4\left(\alpha'\right)^2}\left(\mathfrak{l}^a_+ g_{ab}\mathfrak{l}^b_+\right) \\ &- \frac{1}{\alpha'}\Big(\mathfrak{l}^a_+ b_a{}^b \mathfrak{r}_{+,b}\Big) - \frac{1}{4\left(\alpha'\right)^2}\left(\mathfrak{l}^a_+ (b)^2_{ab}\mathfrak{l}^b_+\right)\Big) \end{aligned} \tag{A.27}$$

and

$$\begin{aligned} F_-(\tau;m) = \pi i \tau_1 \left(\mathfrak{r}_{-,a}\mathfrak{l}^a_-\right) - \alpha'\pi\tau_2\Big(\mathfrak{r}_{-,a}\widetilde{g}^{ab}\mathfrak{r}_{-,b} &+ \frac{1}{4\left(\alpha'\right)^2}\left(\mathfrak{l}^a_- \widetilde{g}_{ab}\mathfrak{l}^b_-\right) \\ &- \frac{1}{\alpha'}\Big(\mathfrak{l}^a_- \widetilde{b}_a{}^b \mathfrak{r}_{-,b}\Big) - \frac{1}{4\left(\alpha'\right)^2}\left(\mathfrak{l}^a_- (\widetilde{b})^2_{ab}\mathfrak{l}^b_-\right)\Big) \;. \end{aligned} \tag{A.28}$$

Here, the dependence of the moduli m is on g, b or $\widetilde{g}, \widetilde{b}$. The upshot is that $F(\tau; m)$ splits into the sum of two quantities, one of which depends only on g, b and the other only on $\widetilde{g}, \widetilde{b}$ (as far as target space moduli are concerned). Note that $\mathfrak{r}_\pm, \mathfrak{l}_\pm$ are vectors in $\mathbb{Z}^\ell$ and take even or odd values in a correlated way. This means

$$\mathfrak{r}_\pm = 2r_\pm + p \;, \qquad \mathfrak{l}_\pm = 2l_\pm + q \tag{A.29}$$

are vectors in $\mathbb{Z}^\ell$ and $p, q \in \{0,1\}^\ell$. Plugging (A.29) into (A.27) and (A.28), we obtain (written in matrix notation)

$$\begin{aligned} F_+(\tau;m) = 2\pi i(2\tau_1)\left(r_+ + \frac{p}{2}\right)^\top\left(l_+ + \frac{q}{2}\right) &- \alpha'\pi(2\tau_2)2\Big(\left(r_+ + \frac{p}{2}\right)^\top g^{-1}\left(r_+ + \frac{p}{2}\right) \\ + \frac{1}{4\left(\alpha'\right)^2}\left(l_+ + \frac{q}{2}\right)^\top g\left(l_+ + \frac{q}{2}\right) &- \frac{1}{\alpha'}\left(l_+ + \frac{q}{2}\right)^\top bg^{-1}\left(r_+ + \frac{p}{2}\right) \\ &- \frac{1}{4\left(\alpha'\right)^2}\left(l_+ + \frac{q}{2}\right)^\top bg^{-1}b\left(l_+ + \frac{q}{2}\right)\Big) \end{aligned} \tag{A.30}$$

and similarly for $F_-(\tau;\mathfrak{m})$. The partition function can be written as

$$Z_{T^{2\ell}_{\text{non-fac}}}(\tau;G,B) = \frac{1}{|\eta(\tau)|^{4\ell}} \sum_{\Delta\in\{0,1\}^{2\ell}} \Theta_{h}(0,\tfrac{1}{2}\Delta,2\tau)\ \Theta_{\widetilde{h}}(0,\tfrac{1}{2}\Delta,2\tau)\ . \tag{A.31}$$

With

$$h \equiv H(\tfrac{g}{2},\tfrac{b}{2})\ , \qquad \widetilde{h} \equiv H(\tfrac{\widetilde{g}}{2},\tfrac{\widetilde{b}}{2})\ , \tag{A.32}$$

that are determined via the matrix relation (A.16) in terms of the (rescaled) $\ell\times\ell$ matrices $g,\widetilde{g},b,\widetilde{b}$.

A.4 The global structure of the moduli space of non-factorisable orbifolds

Here we analyse the global structure of the moduli space of non-factorizable tori $T^{2\ell}_{\text{non-fac}}$ following [2]. To set the stage, we first describe the moduli space $\mathcal{M}_{T^N}$ of conformal field theories for generic target space tori T^N with the positive definite $2N\times 2N$-matrices H obeying eq. (A.11) [83, 84]. Consider the 2N-dimensional lattice Γ with the even self-dual pairing 2Ω given in eq. (A.13). The symmetric matrix Ω is a non-degenerate bilinear form of signature (N,N) on the 2N-dimensional real vector space $V=\Gamma\otimes_{\mathbb{Z}}\mathbb{R}$. Let W_+ be a N-dimensional subvector space of V, such that the restriction $\Omega|_{W_+}$ is positive definite. Note that the choice of W_+ is not unique and W_+ is called a majorant of Ω. Furthermore, let W_- be the N-dimensional orthogonal complement $W_- = \{x\in V\,|\,\Omega(x,W_+)=0\}$. Then the vector space V decomposes into the direct sum

$$V = W_+\oplus W_-\ . \tag{A.33}$$

Due to the signature of Ω the restriction $\Omega|_{W_-}$ is negative definite. From this decomposition we obtain on V the positive definite symmetric bilinear form

$$H(u,v) := \Omega(u_+,v_+) - \Omega(u_-,v_-)\ , \tag{A.34}$$

with $u=u_++u_-$, $v=v_++v_-$, where $u_+,v_+\in W_+$ and $u_-,v_-\in W_-$. Note that H obeys the relation (A.11), which is equivalent to $\Omega^{-1}H - H^{-1}\Omega = 0$ and to

$$(\Omega^{-1}+H^{-1})(\Omega - H) = 0\ . \tag{A.35}$$

Conversely, given a positive symmetric matrix H obeying this matrix relation, the kernel of the second factor $\Omega - H$ defines the subvector space W_+ and hence the decomposition (A.33) associated to the positive definite symmetric pairing H.

The symmetric form Ω with signature (N,N) is invariant with respect to the indefinite orthogonal group $O(N,N,\mathbb{R})$ acting on the vector space V, namely $\Lambda^T\Omega\Lambda=\Omega$ for any $\Lambda\in O(N,N,\mathbb{R})$. However, the transformation on the vector space $V\mapsto\Lambda\cdot V$ acts non-trivially on the decomposition (A.33), and hence on the space of positive symmetric bilinear from H. Conversely, Witt's theorem ensures that the group $O(N,N,\mathbb{R})$ acts transitively on the space of positive definite symmetric $2\ell\times 2\ell$ bilinear forms H obeying eq. (A.35). The stabilizer subgroup preserving the decomposition (A.33) is $O(N,\mathbb{R})\times O(N,\mathbb{R})$. Therefore, we find altogether that the moduli space of majorants $\mathcal{M}^{(N)}_{\text{Maj}}$ of Ω reads

$$\mathcal{M}^{(N)}_{\text{Maj}} \simeq \frac{O(N,N,\mathbb{R})}{O(N,\mathbb{R})\times O(N,\mathbb{R})}\ . \tag{A.36}$$

As the moduli space of majorants yields a choice of metric G and B-field B according to eq. (A.16), it also parametrizes toroidal conformal field theories with target space T^N. However, two majorants that are related by a lattice automorphism $O(N, N, \mathbb{Z})$ of Γ yield equivalent toroidal conformal field theories. Therefore, we arrive at the well-known result, see, e.g., ref. [57], that the moduli space $\mathcal{M}_{T^N}$ is given by

$$\mathcal{M}_{T^N} \simeq \frac{\mathcal{M}_{\text{Maj}}^{(N)}}{O(N,N,\mathbb{Z})} \simeq O(N,N,\mathbb{Z})\Big\backslash O(N,N,\mathbb{R})\Big/ O(N,\mathbb{R}) \times O(N,\mathbb{R}) \ . \tag{A.37}$$

Now we are ready to discuss the global structure of the moduli space $\mathcal{M}_{T^{2\ell}_{\text{non-fac}}}$ of conformal field theories arising from non-factorizable target space tori $T^{2\ell}_{\text{non-fac}}$. We parametrize the moduli space $\mathcal{M}_{T^{2\ell}_{\text{non-fac}}}$ in terms of majorants that admit the $\mathbb{Z}_2$ orbifold action. The involution $\iota_{\mathbb{Z}_2}$ acting on the 2ℓ-dimensional lattice $\Lambda_{2\ell}$ induces a $\mathbb{Z}_2$-action $\tilde{\iota}_{\mathbb{Z}_2}$ on the 4ℓ-dimensional lattice $\Gamma \simeq \Lambda_{2\ell} \oplus \Lambda^*_{2\ell}$. By construction, the involution $\tilde{\iota}_{\mathbb{Z}_2}$ leaves the non-degenerate bilinear form Ω of signature $(2\ell, 2\ell)$ invariant, i.e., $\tilde{\iota}^*_{\mathbb{Z}_2}\Omega = \Omega$. Furthermore, the vector space $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ decomposes as $V = V^{(+)} \oplus V^{(-)}$, where $V^{(\pm)}$ are the ± 1 eigenspaces with respect to the involution $\tilde{\iota}_{\mathbb{Z}_2}$. It is straightforward to check that the non-degenerate bilinear form Ω of signature $(2\ell, 2\ell)$ restricts on $V^{(\pm)}$ to two non-degenerate bilinear forms $\Omega|_{V^{(\pm)}}$ both of signature (ℓ, ℓ). As a result the majorants compatible with the involution $\tilde{\iota}_{\mathbb{Z}_2}$ split as

$$V = W_+^{(+)} \oplus W_+^{(-)} \oplus W_-^{(+)} \oplus W_-^{(-)} \ , \tag{A.38}$$

where $W_\pm^{(+)} \oplus W_\pm^{(-)} = W_\pm$ and $W_+^{(\pm)} \oplus W_-^{(\pm)} = V^{(\pm)}$. The moduli spaces $\mathcal{M}^{(2\ell)}_{\text{Maj},\mathbb{Z}_2}$ of such $\mathbb{Z}_2$-equivariant majorants is parametrized by transformations that not only preserves the bilinear form Ω but also its two restrictions $\Omega|_{V^{(\pm)}}$ individually modulo those transformations that preserve the direct sum decomposition (A.38). Therefore, we arrive at

$$\mathcal{M}^{(2\ell)}_{\text{Maj},\mathbb{Z}_2} \simeq \frac{O(\ell,\ell,\mathbb{R}) \times O(\ell,\ell,\mathbb{R})}{O(\ell,\mathbb{R}) \times O(\ell,\mathbb{R}) \times O(\ell,\mathbb{R}) \times O(\ell,\mathbb{R})} \simeq \mathcal{M}^{(\ell)}_{\text{Maj}} \times \mathcal{M}^{(\ell)}_{\text{Maj}} \ . \tag{A.39}$$

As before, in order to describe the moduli space of conformal field theories of non-factorizable target space tori $T^{2\ell}_{\text{non-fac}}$, we need to further divide by those lattice automorphisms of Γ that are compatible with the involution $\tilde{\iota}_{\mathbb{Z}_2}$. These are realized by the discrete group $O(\ell,\ell,\mathbb{Z}) \times O(\ell,\ell,\mathbb{Z})$. Thus, the moduli space of the conformal field theories with non-factorizable tori $T^{2\ell}_{\text{non-fac}}$ as target spaces becomes

$$\begin{aligned} \mathcal{M}_{T^{2\ell}_{\text{non-fac}}} &\simeq \frac{\mathcal{M}^{(2\ell)}_{\text{Maj},\mathbb{Z}_2}}{O(\ell,\ell,\mathbb{Z}) \times O(\ell,\ell,\mathbb{Z})} \simeq O(\ell,\ell,\mathbb{Z})^{\times 2}\Big\backslash O(\ell,\ell,\mathbb{R})^{\times 2}\Big/ O(\ell,\mathbb{R})^{\times 4} \\ &\simeq \mathcal{M}_{T^\ell} \times \mathcal{M}_{T^\ell} \ . \end{aligned} \tag{A.40}$$

The two factors $\mathcal{M}_{T^\ell}$ of this moduli space are parametrised by the positive definite bilinear forms h and $\widetilde{h}$ explicitly given in eq. (A.32). Note that the arguments of the bilinear forms h and $\widetilde{h}$ in terms of the metric g and the B-field b and the metric $\widetilde{g}$ and the B-field $\widetilde{b}$ are rescaled by a factor $\frac{1}{2}$, which reflects the fact that these $\ell \times \ell$ blocks parametrize the diagonal tori $\widetilde{T}^\ell \times \widetilde{T}^\ell$ of the non-factorizable torus $T^{2\ell}_{\text{non-fac}}$ corresponding to the sublattice $\widetilde{\Lambda}_{2\ell}$ of index 2^ℓ.

A.5 Real Analytic Eisenstein Series

Real Analytic Eisenstein Identities

In the calculation of the ensemble average (3.130) we use for the real analytic Eisenstein series the identities

$$\begin{aligned}
\frac{1}{2}\sum_{\substack{c\in\mathbb{Z},d\in 2\mathbb{Z}\\(c,d)=1}}\frac{1}{|c\,x+d|^N} &= \frac{1}{2^N-1}\left(\frac{E_{N/2}(\frac{x}{2})}{\mathrm{Im}(\frac{x}{2})^{\frac{N}{2}}}-\frac{E_{N/2}(x)}{\mathrm{Im}(x)^{\frac{N}{2}}}\right),\\
\frac{1}{2}\sum_{\substack{c\in\mathbb{Z},d\in 2\mathbb{Z}\\(c,d)=1}}\frac{(-1)^{\frac{d}{2}}}{|c\,x+d|^N} &= \frac{1}{2^N-1}\left(\frac{2}{2^N}\frac{E_{N/2}(\frac{x}{4})}{\mathrm{Im}(\frac{x}{4})^{\frac{N}{2}}}-\frac{2^N+2}{2^N}\frac{E_{N/2}(\frac{x}{2})}{\mathrm{Im}(\frac{x}{2})^{\frac{N}{2}}}+\frac{E_{N/2}(x)}{\mathrm{Im}(x)^{\frac{N}{2}}}\right).
\end{aligned} \tag{A.41}$$

where in the sum the symbol (c,d) denotes the greatest common divisor of c and d, i.e, $(c,d)=1$ says that c and d are coprime. To show these identities we start with a useful lemma:

Lemma 1. *Let α be any positive integer. Then for any integers c and d the following two conditions are equivalent:*

(i) $(c,2^\alpha d)=1$, *(ii)* c *odd and* $(c,d)=1$.

Proof. If c and $2^\alpha d$ are coprime, then both c, 2^α and c, d are coprime. Hence, c odd and $(c,d)=1$. Conversely, if c is odd then c, 2^α are coprime. As c and d are coprime as well, altogether c, $2^\alpha d$ must be coprime and hence $(c,2^\alpha d)=1$. □

We now show the first real analytic Eisenstein identity (A.41) explicitly, and we start with the calculation

$$\begin{aligned}
\sum_{\substack{(c,d)=1\\ d\text{ even}}}|c\cdot x+d|^{-N} &= \sum_{(c,2d)=1}|c\cdot x+2d|^{-N} = 2^{-N}\sum_{\substack{(c,d)=1\\ c\text{ odd}}}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}\\
&= 2^{-N}\sum_{(c,d)=1}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}-2^{-N}\sum_{\substack{(c,d)=1\\ c\text{ even}}}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}\\
&= 2^{-N}\sum_{(c,d)=1}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}-2^{-N}\sum_{(2c,d)=1}|c\cdot x+d|^{-N}\\
&= 2^{-N}\sum_{(c,d)=1}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}-2^{-N}\sum_{\substack{(c,d)=1\\ d\text{ odd}}}|c\cdot x+d|^{-N}\\
&= 2^{-N}\sum_{(c,d)=1}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}-2^{-N}\sum_{(c,d)=1}|c\cdot x+d|^{-N}\\
&\qquad +2^{-N}\sum_{\substack{(c,d)=1\\ d\text{ even}}}|c\cdot x+d|^{-N},
\end{aligned} \tag{A.42}$$

where the summations are manipulated time and again using Lemma 1. Solving in this expression for $\sum_{\substack{(c,d)=1\\ d\text{ even}}}|c\cdot x+d|^{-N}$ yields

$$\sum_{\substack{(c,d)=1\\ d\text{ even}}}|c\cdot x+d|^{-N} = \frac{1}{2^N-1}\left(\sum_{(c,d)=1}\left|c\cdot\tfrac{x}{2}+d\right|^{-N}-\sum_{(c,d)=1}|c\cdot x+d|^{-N}\right). \tag{A.43}$$

Inserting the definition of the real analytic Eisenstein series (3.41), we arrive at the first identity (A.41).

For the second identity (A.41) our derivation is similar but a bit more tedious, because we first split the sum over d into positive and negative contributions. This can be achieved by introducing an auxiliary summation index d' for the even integers d, which discriminates between the positive and negative part by setting $d = 4d'$ and $d = 2(2d'+1)$. After splitting the sum in this way, we perform a similar calculation as in eq. (A.42) to obtain the second identity (A.41).

Hecke Operators and Modularity

The real analytic Eisenstein series $E_s(x)$ are eigenfunctions of the Hecke operators. This means

$$T_j E_s(x) := \frac{1}{\sqrt{j}} \sum_{\substack{ad=j, d>0 \\ 0\leqslant b\leqslant d-1}} E_s\left(\tfrac{ax+b}{d}\right) = \frac{\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}} E_s(x)\,, \tag{A.44}$$

where $\sigma_n(x) = \sum_{d|x} d^n$ is the sum of positive divisor function. We have

$$T_2 E_s(x) = \frac{1}{\sqrt{2}}\left(E_s(2x) + E_s(\tfrac{x}{2}) + E_s(\tfrac{x+1}{2})\right) \tag{A.45}$$

$$\sigma_{2s-1}(2) = 1 + 2^{2s-1}\,. \tag{A.46}$$

Let us finally also state a useful lemma, which we use in the main text to identify manifest modular invariant combinations of real analytic Eisenstein series:

Lemma 2. *Let $f(x)$ be a modular invariant function $f(x)$ with respect to the modular group* PSL$(2,\mathbb{Z})$*, which acts on the argument x by Möbius transformations. Then the function $g(x)$, given by*

$$g(x) = f(2x) + f(\tfrac{x}{2}) + f(\tfrac{x+1}{2})\,, \tag{A.47}$$

is modular invariant.

Proof. The modular group PSL$(2,\mathbb{Z})$ is generated by the standard generators T and S that map x to $x+1$ and x to $-\frac{1}{x}$, respectively. For the generator T we calculate $g(x+1) = f(2x+2) + f(\frac{x+1}{2}) + f(\frac{x+2}{2}) = g(x)$ because $f(2x+2) = f(2x)$ and $f(\frac{x+2}{2}) = f(\frac{x}{2})$ by the modularity of f. For the generator S we find

$$g(-\tfrac{1}{x}) = f(-\tfrac{2}{x}) + f(-\tfrac{1}{2x}) + f(\tfrac{x-1}{2x}) = f(\tfrac{x}{2}) + f(2x) + f(-\tfrac{2x}{x-1})\,,$$

where for the second equal sign the modularity of the function f is again used. Furthermore, by modularity of f, we have for the last term in this equation

$$f(-\tfrac{2x}{x-1}) = f(-\tfrac{2}{x-1} - 2) = f(-\tfrac{2}{x-1}) = f(\tfrac{x-1}{2}) = f(\tfrac{x+1}{2})\,,$$

which demonstrates altogether that $g(-\frac{1}{x}) = g(x)$. Thus, $g(x)$ is invariant with respect to both generators T and S, and hence is a modular invariant function. $\square$

A.6 Lagrangian sublattices

The notion of Lagrangian sublattices appears in the Siegel-Weil formula and for this we introduce them here.

A Riemann surface Σ_g of genus g has 2g canonical cycles $A_i, B_i,\ i = 1, 2, ..., g$ which generate its first homology group $H_1(\Sigma_g) \simeq \underbrace{\mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{2g}$. We can choose a basis of g differentials ϖ_i that fulfil the following conditions

$$\oint_{A_i} \varpi_j = \delta_{ij} \,. \tag{A.48}$$

Then, the period matrix Ω of Σ_g is defined to be

$$\Omega_{ij} = \oint_{B_i} \varpi_j \,. \tag{A.49}$$

For example, for Σ_1 we can choose $\Omega = \tau$, the modular parameter of the torus.

Given any two cycles $\gamma, \gamma' \in H_1(\Sigma_g)$, the intersection number $\langle \gamma, \gamma' \rangle \in \mathbb{Z}$ counts how many times the two cycles meet. A Lagrangian sublattice $\Gamma_0 \subset H_1(\Sigma_g, \mathbb{Z})$ is a primitive subgroup of $H_1(\Sigma_g, \mathbb{Z})$ generated by g cycles $\widetilde{A}_i$ that do not intersect, i.e. $\langle \widetilde{A}_i, \widetilde{A}_j \rangle = 0,\ i \neq j$. We have to also choose a dual pair of cycles $\widetilde{B}_i$ that are again non-intersecting but obey $\langle \widetilde{A}_i, \widetilde{B}_j \rangle = \delta_{ij}$. The differentials ϖ_i give rise to new differentials $\widetilde{\varpi}_i$ that satisfy

$$\oint_{\widetilde{A}_i} \widetilde{\varpi}_j = \delta_{ij} \,. \tag{A.50}$$

The period matrix Ω_{Γ_0} associated to the Lagrangian sublattice Γ_0 is defined to be

$$\left(\Omega_{\Gamma_0}\right)_{ij} = \oint_{\widetilde{B}_i} \widetilde{\varpi}_j \,. \tag{A.51}$$

In the case of multiple disconnected Riemann surfaces the first homology group is given by a direct sum of the homology groups. For two surfaces of genera g_1 and g_2 we have $H_1(\Sigma_{g_1} \sqcup \Sigma_{g_2}, \mathbb{Z}) \cong H_1(\Sigma_{g_1}, \mathbb{Z}) \oplus H_1(\Sigma_{g_2}, \mathbb{Z})$. A Lagrangian sublattice is then a group $\Gamma_0 \subset H_1(\Sigma_{g_1} \sqcup \Sigma_{g_2}, \mathbb{Z})$ generated by $g_1 + g_2$ cycles that have zero mutual intersection numbers. The period matrix associated to Γ_0 is defined in an identical way to the case of a single surface, and generalizes to any number of disconnected surfaces. The new ingredient with disconnected surfaces is that the cycles $\tilde{A}_i$ that define Γ_0 can now be linear combinations of cycles on disconnected surfaces.

A.7 Supersymmetric Chern-Simons

In this section we include additional details on our conventions and on $\mathcal{N} = 1$ supersymmetric U(1) Chern Simons in the presence of a boundary, closely following [210, 211]. We first discuss our conventions for Lorentzian signature supersymmetry. The gamma matrices satisfy the standard algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \gamma^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho, \tag{A.52}$$

where the first of the right equations is a definition for $\gamma^{\mu\nu}$ and they are explicitly given by

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.53}$$

The metric takes the form $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, and the spacetime coordinates are $x^\mu = (x^1, x^2, x^3)$ where x^1 is the time component, while the boundary is located at

$x^3 = 0$. We will use indices m, n to indicate components restricted to the boundary $x^m = (x^1, x^2)$. All spinors considered are Majorana $\bar{\lambda}^a \equiv C^{ab}\lambda_b$ where $C = -C^T$ is the charge conjugation matrix. Spinor indices are contracted top right to bottom left: $\bar{\lambda}\chi = \lambda^a \chi_a$, $\bar{\lambda}\gamma^\mu\chi = \lambda^a (\gamma^\mu)_a{}^b \chi_b$, where the gamma matrices implicitly have the index structure $(\gamma^\mu)_a{}^b$. Spinor indices a, b are raised and lowered with the antisymmetric charge conjugation matrix C: $\lambda^a = C^{ab}\lambda_b$, which is defined through $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$. We introduce projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma^3)$ and define fields $\chi_\pm$ with $P_\pm \chi = \chi_\pm$, which projects onto the top or bottom component of the spinor respectively.

$\mathcal{N} = (1, 0)$ **Lorentzian Chern Simons**

We first discuss the case of $\mathcal{N} = (1, 0)$ supersymmetric U(1) Chern Simons in the presence of a boundary in Minkowski space. The action can be constructed through the use of spinor superfields,[2] including appropriate boundary terms it is given by [210, 211]

$$S_{CS}^{\mathcal{N}=(1,0)} = \int_M d^3x (\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \bar{\lambda}\lambda) - \frac{1}{2} \int_{\partial M} d^2x \sqrt{h} (h^{mn} A_m A_n + \bar{\chi}_- \gamma^m \partial_m \chi_-). \quad \text{(A.54)}$$

In the above λ, χ are Majorana fermions, with $\chi_- = P_- \chi$ being a purely boundary fermion. The dynamical boundary fermion χ_- is unusual, and is required to cancel the boundary terms produced by the susy variation. We also have the boundary metric h_{mn}. The full susy variations under ϵ are given by [210]

$$\begin{aligned} \delta A_\mu &= -(\bar{\epsilon}\gamma_\mu \lambda) + (\bar{\epsilon}\partial_\mu \chi), \\ \delta \lambda_a &= -\epsilon^{\mu\nu\rho} (\gamma_\rho \epsilon)_a \partial_\mu A_\nu, \\ \delta \chi_a &= (\gamma^\mu \epsilon)_a A_\mu. \end{aligned} \quad \text{(A.55)}$$

However, the action (A.54) is only invariant up to a boundary term under a general variation (see (A.72)). For the full action to be invariant we must restrict to variations $\epsilon_+ = P_+ \epsilon$, which means the supersymmetry is broken down to $\mathcal{N} = (1, 0)$ in the presence of the boundary. This is true even without imposing any boundary conditions on the fields. The explicit susy variations under ϵ_+ are

$$\begin{aligned} \delta A_\mu &= -(\bar{\epsilon}_+ \gamma_\mu \lambda) + (\bar{\epsilon}_+ \partial_\mu \chi_-), \\ \delta \lambda_a &= -\epsilon^{\mu\nu\rho} (\gamma_\rho \epsilon_+)_a \partial_\mu A_\nu, \\ \delta \chi_- &= (\gamma^m \epsilon_+) A_m, \end{aligned} \quad \text{(A.56)}$$

and the action is invariant under this subset of transformations. Using the above, we can obtain the following variations

$$\delta A_- = \bar{\epsilon}_+ \left(2\gamma^2 \lambda_+ + \partial_- \chi_- \right), \quad \text{(A.57)}$$

$$\delta \left(2\gamma^2 \lambda_+ + \partial_- \chi_- \right) = \gamma^2 \epsilon_+ \partial_+ A_-. \quad \text{(A.58)}$$

where we have defined the notation $A_\pm = A_1 \pm A_2$, $\partial_\pm = \partial_1 \pm \partial_2$ which is distinct from the spinor $\pm$ projection notation and used $\gamma^1 \lambda_+ = \gamma^2 \lambda_+$. For a good variational principle, the boundary conditions we impose are $A_- = 0$ and $2\gamma^2 \lambda_+ + \partial_- \chi_- = 0$ [210, 211]. We see that the ϵ_+ transformations leave these boundary conditions invariant.

[2] The spinor superfield formalism includes an additional complex scalar that is the superpartner of χ, but it does not end up contributing to the action. We exclude it below, but see [210].

The boundary kinetic term for χ_- does not need to be cancelled for a good variational principle since it gives the equations of motion. In the path integral with these boundary conditions the integral over χ_- is effectively not constrained since we also integrate over λ_+, thus χ_- is essentially a free dynamical one-component boundary fermion. The integral over λ will give some overall normalization constant since it is non-dynamical.

Similarly, we can construct an action with $\mathcal{N} = (0,1)$ invariant under ϵ_- transformations by modifying the boundary term

$$S_{CS}^{\mathcal{N}=(0,1)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \bar{\lambda}\lambda) + \frac{1}{2}\int_{\partial M} d^2x\sqrt{g}(g^{mn}A_mA_n + \bar{\chi}_+\gamma^m\partial_m\chi_+), \quad (A.59)$$

where λ, χ_+ are again Majorana fermions, and χ_+ has been projected onto its top component. In this case the variations under ϵ_- are given by

$$\begin{aligned} \delta A_\mu &= -(\bar{\epsilon}_-\gamma_\mu\lambda) + (\bar{\epsilon}_-\partial_\mu\chi_+), \\ \delta\lambda_a &= -\epsilon^{\mu\nu\rho}(\gamma_\rho\epsilon_-)_a\partial_\mu A_\nu, \\ \delta\chi_+ &= (\gamma^m\epsilon_-)A_m. \end{aligned} \quad (A.60)$$

Which immediately gives us the variations

$$\delta A_+ = \bar{\epsilon}_-\left(-2\gamma^2\lambda_- + \partial_+\chi_+\right), \quad (A.61)$$

$$\delta\left(-2\gamma^2\lambda_- + \partial_+\chi_+\right) = -\gamma^2\epsilon_-\partial_-A_+. \quad (A.62)$$

Proper boundary conditions in this case correspond to $A_+ = 0$ and $-2\gamma^2\lambda_- + \partial_+\chi_+ = 0$ which are both preserved under ϵ_- transformations. In the main text we defined a total theory given by the difference of the above actions

$$S = S_{CS}^{\mathcal{N}=(1,0)} - S_{CS}^{\mathcal{N}=(0,1)}, \quad (A.63)$$

so that the bulk theory has the full $\mathcal{N} = (1,1)$ supersymmetry realized by different sectors. Since $\bar{\chi}_-\gamma^m\partial_m\chi_+ = \bar{\chi}_+\gamma^m\partial_m\chi_- = 0$ the boundary fermion term for the above action can be rewritten in a simple form

$$\int_{\partial M} d^2x\bar{\chi}\gamma^m\partial_m\chi, \quad (A.64)$$

which makes it clear that we have a dynamical two-dimensional free fermion on the boundary. The gauge fields do not interact with the fermions so the full path integral will simply be a product of the Chern-Simons contribution and the fermion contribution. In each case, the boundary fermions $\chi_\pm$ are projected onto the top/bottom component and function as single component spinors, so individually their partition functions will contribute a determinant of either a holomorphic or anti-holomorphic derivative after analytic continuation to Euclidean signature.

There is one additional subtlety to address, the above theories were defined on a flat background. However, for our purposes we are interested in supersymmetric Chern-Simons on a manifold with an asymptotic boundary torus. In general such manifolds do not admit flat metrics, so we need to consider the theory on a curved background [224]. While Chern-Simons is itself background independent, the supersymmetric version depends on a choice of background metric g through the quadratic bulk fermion term. Ignoring the boundary term, the action is of the form

$$S = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \sqrt{g}\,\bar{\lambda}\lambda). \quad (A.65)$$

However, the metric dependence is quite mild since the fermion is non-dynamical, and it produces an overall normalization for the partition function. Nevertheless, to preserve supersymmetry we need to choose a bulk metric g that satisfies the killing spinor equations. The end result is that the supersymmetry transformations will mildly depend on the background metric [224]. In the case of bulk handlebodies we can choose g to be given by the corresponding AdS_3 metric [225] and supersymmetry will be preserved, but for more general three-manifolds that appear when considering wormhole geometries little is known.

A.7.1 Details of the variations.

In this subsection we include additional details[3] on the variation of the supersymmetic Chern-Simons action in equation (A.54). For convenience we split the action into a bulk and boundary piece

$$S = \int_M d^3x \left(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \bar{\lambda}\lambda\right), \tag{A.66}$$

$$S_\partial = -\frac{1}{2}\int_{\partial M} d^2x \left(A^m A_m + \bar{\chi}_- \gamma^m \partial_m \chi_-\right). \tag{A.67}$$

Where we used $h_{mn} = \mathrm{diag}\,(-1, 1)$, where Latin indices m, n are again boundary indices and take values in $\{1, 2\}$. Varying with the full supersymmetric variations (A.55) we find

$$\delta S = 2\int_M d^3x \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \left(-\bar{\epsilon}\gamma_\mu \lambda + \bar{\epsilon}\partial_\mu \chi\right) + \int_{\partial M} d^2x \epsilon^{3mn} A_m \left(-\bar{\epsilon}\gamma_n \lambda + \bar{\epsilon}\partial_n \chi\right) \tag{A.68}$$

$$+2\int_M d^3x \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \left(\bar{\epsilon}\gamma_\mu \lambda\right). \tag{A.69}$$

The second line cancels the first term, and we can rewrite the remaining bulk term as a boundary term to obtain

$$\delta S = -\int_{\partial M} d^2x \epsilon^{3nm} A_n \left(\bar{\epsilon}\gamma_m \lambda + \bar{\epsilon}\partial_m \chi\right) \; . \tag{A.70}$$

The variation of the boundary term gives

$$\delta S_\partial = \int_{\partial M} d^2x \left(A^m \left(\bar{\epsilon}\gamma_m \lambda - \bar{\epsilon}\partial_m \chi\right) + \left(\partial_m \bar{\chi}_- \epsilon A^m - \partial_m \bar{\chi}_- \epsilon^{m\mu\rho} \gamma_\rho \epsilon A_\mu\right)\right). \tag{A.71}$$

The variation of the full action (A.54) is thus a total boundary term

$$\begin{aligned} \delta S_{\mathrm{CS}}^{\mathcal{N}=(1,0)} = & \int_{\partial M} d^2x \Big(-\epsilon^{3nm} A_n \left(\bar{\epsilon}\partial_m \chi + \bar{\epsilon}\gamma_m \lambda\right) + A^m \left(\partial_m \bar{\chi}_- \epsilon\right) - \epsilon^{m\mu\nu} A_\mu \left(\partial_m \bar{\chi}_- \gamma_\nu \epsilon\right)\Big) \\ & + \int_{\partial M} d^2x A^m \left(\bar{\epsilon}\gamma_m \lambda - \bar{\epsilon}\partial_m \chi\right) \; . \end{aligned} \tag{A.72}$$

Now we specialize to $P_+\epsilon = \epsilon_+$ variations. We immediately have the following cancellations. The third term cancels the last term since $\bar{\chi}_-\epsilon_+ = \bar{\epsilon}_+\chi$, while the first term is cancelled by fourth term since in the fourth term only $\bar{\chi}_-\gamma_3\epsilon_+ = \bar{\chi}_-\epsilon_+$ survives. Finally, the second term cancels the fifth term since we have $\epsilon^{3nm}\gamma_m = \gamma^3\gamma^n$ and $\bar{\epsilon}_\pm\gamma^3 = \mp\bar{\epsilon}_\pm$

[3] Some useful identities include $\bar{\lambda}\chi = \bar{\chi}\lambda$, $\bar{\lambda}\gamma_\mu\epsilon = -\bar{\epsilon}\gamma_\mu\lambda$, and since the spinors are Majorana we have $\bar{\lambda}^a\chi_a = C^{ab}\lambda_b\chi_a$.

due to the factor of the charge conjugation matrix C in $\bar{\epsilon}$. Note again that no special choice of boundary conditions was needed to make the action invariant under ϵ_+ variations.

The variation of the $\mathcal{N} = (0,1)$ supersymmetric Chern-Simons action under ϵ_- works similarly, and we obtain

$$\begin{aligned}\delta S_{CS}^{\mathcal{N}=(0,1)} =& \int_{\partial M} d^2x \left(-\epsilon^{3nm} A_n \left(\bar{\epsilon}\partial_m \chi + \bar{\epsilon}\gamma_m \lambda\right) - A^m \left(\partial_m \bar{\chi}_+ \epsilon\right) + \epsilon^{m\mu\nu} A_\mu \left(\partial_m \bar{\chi}_+ \gamma_\nu \epsilon\right)\right) \\ &+ \int_{\partial M} d^2x A^m \left(-\bar{\epsilon}\gamma_m \lambda + \bar{\epsilon}\partial_m \chi\right). \end{aligned} \tag{A.73}$$

Specializing to $P_- \epsilon = \epsilon_-$ variations we again find that all the boundary terms cancel in a similar way as in (A.72).

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