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Soldner, Einstein, Gravitational Light Deflection and Factors of Two

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The Newtonian value of 0.″84 for the gravitational deflection of a ray of light from a distant star, grazing the rim of the Sun, was derived already in 1801 by Johann Georg von Soldner. The same value was obtained by Albert Einstein in 1911 on the basis of the equivalence principle alone. Four years later, Einstein predicted twice that value on the basis of the full theory of general relativity, a value that was later confirmed by observation. A direct comparison of Soldner's and Einstein's works is obscured by a confluence of various factors of 2, arising both from different conventions and from printing errors.

1. Introduction

Einstein's 1915 prediction of gravitational light deflection[1] is one of the three classical tests of the general theory of relativity and was derived in one of the most celebrated papers ever published in the *Annalen*.[2] Famously, the relativistic value of a deflection angle of 1.7" for light rays grazing the rim of the Sun was confirmed by a British eclipse expedition by observing the position of stars near the Sun during a solar eclipse in 1919, and this spectacular success immediately gave rise to Einstein's world fame.[3] A few years earlier, in 1911, Einstein had predicted a deflection value of only half the size in another well-known *Annalen* paper.[4] The 1911 prediction was based on Einstein's equivalence principle alone and therefore did not take into account the curvature of space, unlike the later derivation of the full relativistic value. The half value coincides with the value one obtains if one considers the deflection of light in a Newtonian setting for a particle moving with the velocity of light in the gravitational field of the Sun.

Soon after Einstein's rise to fame, Philipp Lenard,[5] in an ill-minded attempt to belittle Einstein's success and to challenge his originality,[6] drew attention to a forgotten paper by Johann Georg von Soldner of 1801,[7,8] that had already given a Newtonian treatment of light deflection. To be sure, there is no indication that Einstein, or most everyone at the time, knew of Soldner's paper from more than a 100 years ago, much less that Einstein would have been directly influenced by it. Nevertheless, it is interesting to compare Soldner's treatment that was done in the context of Newtonian mechanics and a particle concept of light with Einstein's treatment on the basis of the equivalence principle, which was done using Huygens' wave theory. In a recent paper, Jean-Marie Ginoux has done just that in a careful comparison of the two calculations.[9] It turns out that a direct comparison of Soldner's and Einstein's treatments is rather tricky, not the least because of a confusing conflation of various factors of 2. The following note is largely based on Ref. [9].

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2. Factors of Two by Different Conventions

The most obvious difference between Soldner's and Einstein's derivations is the definition of the deflection angle. Soldner treated the problem by looking at a light ray that starts out from a point A at the surface of the deflecting body and goes out from A tangentially to the surface in the direction of AD, see **Figure 1**. The light ray then gets pulled toward the gravitating body and its path becomes the line AMQ. Hence, the deflection angle ω in Soldner's notation is the angle between the tangent to the surface of the deflecting body AD and the asymptotic direction of the deflected light ray RDB. It is therefore only half the value of the deflection angle for a terrestial observer, who would see a light ray emitted by a star and deflected by a gravitating body like the Sun.

A second difference between Soldner's and our modern conventions concerns the definition of the gravitational acceleration felt by a particle on the surface of the gravitating body at point A. Soldner called the acceleration of gravity at the surface of the body g, but asserted that the force by which a light ray at M is being pulled in the direction of MC is $2gr^{-2}$, the dimension of Soldner's g therefore being length^3 time^{-2}. Moreover, as Treder and Jackisch had pointed out some time ago,[10] the different definition of g reflects a different convention of defining acceleration followed in German physical literature in the 18th century. Soldner's definition reflects a convention which sets the acceleration a in terms of traversed space s and time t as $a = s/t^2$, writing Galileo's law of free fall as $s = a \cdot t^2$ rather than our modern definition of a as the second derivative of motion, which, of course, implies $s = (1/2)a \cdot t^2$.

Soldner's derivation then proceeds by identifying the light ray's path as a hyperbolic conic section, and his result was written for the tangent of ω as

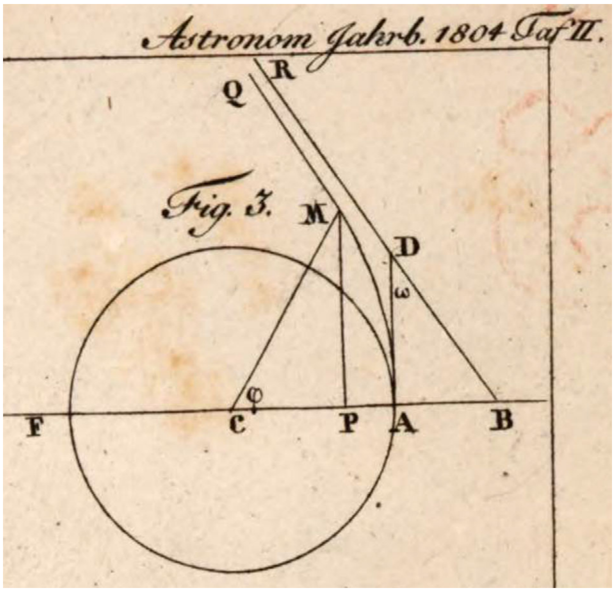


Figure 1. The Figure associated with Soldner's article.[7] Reproduced with permission from copy of the *Bayerische Staatsbibliothek*, Signatur: Eph. astr. 23-1804. urn:nbn:de:bvb:12-bsb105383333-5, p.281

$$\tan g\,\omega = \frac{2g}{v\sqrt{v^2 - 4g}}, \tag{1}$$

where v is the velocity of light at point A. In his derivation, Soldner expressed lengths in units of the radius Δ of the attracting body, effectively setting $\Delta = 1$. Reintroducing Δ into his formula, we can write the right hand side of Equation (1) as $2g/\Delta^2 v^2 \sqrt{1 - 4g/\Delta^2 v^2}$. Observing that in his notation g is given by $KM/2$, where K is the gravitational constant and M the mass of the attracting body, we find his angle of deflection to be[5]

$$\tan g\,\omega = \frac{KM}{\Delta^2 v^2} \cdot \frac{1}{\sqrt{1 - \frac{2KM}{v^2 \Delta^2}}}. \tag{2}$$

Already the combination of the two different definitions of ω and g has given rise to confusion in the reception of Soldner's paper, as is witnessed, for example, by a copy of the relevant issue of the *Astronomisches Jahrbuch* held by the *Bayerische Staatsbibliothek* in Munich. This copy was used for its digitization program and can now be inspected online.[11] As **Figure 2** shows, one reader at some point corrected a factor of 2 in Equation (1) with pencil. Another reader then rescinded that earlier correction with a blue pen.

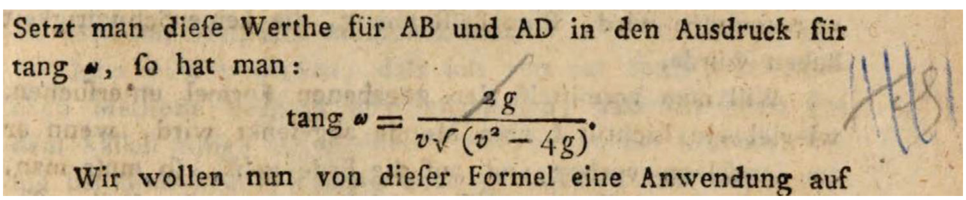
In the context of his own derivation, Soldner's result is correct as it stands. Incidentally, it deviates from a Newtonian derivation of gravitational light deflection that would be seen by a terrestrial observer in the minus sign appearing in the square root expression.[12] Soldner takes the velocity of light to be the terrestrial vacuum value at point A, implying that the velocity of the light ray would slow down in the asymptotic outgoing directions. A Newtonian derivation rather would have the value of the velocity of light take on its vacuum value at asymptotic infinity and have it increased at point A. However, that effect is minuscule given the relative size of g and v^2.

3. Factors of Two by the Printer's Devil

So far, so good. What made things a bit more confusing now, is that the latter part of Soldner's paper does contain a printing error of a factor of 2, which should have been there but was omitted. Soldner's discussion of the problem was done without specifying the deflecting body. Only at the end of his paper, he discussed possible observational consequences and specified to the case of a light ray deflected by the Sun. It is at this point, that he quoted a numerical value for the deflection angle for a light ray grazing the rim of the Sun, which is stated to be $\omega = 0.''84$. Clearly, with his definition of ω, the observed angle for a terrestrial observer would then be twice this value, and hence would coincide with the full relativistic value, predicted by Einstein in 1915.

Lenard[5] had already pointed out that Soldner's quoted value is wrong, writing that "in reality" it should be "according to his formula $2\omega = \beta = 0.''84$, which seems to agree with experience, so far as it goes today" ("während in Wirklichkeit nach seiner Formel $2\omega = \beta = 0.84''$, was mit der Erfahrung, so weit dieselbe heute geht, auch zu stimmen scheint, ..."). Lenard had an interest in Soldner giving a value of $0.''84$ since he had made it clear before that in his understanding the British eclipse results would favor a Newtonian value rather than the full relativistic one.

Concerning the mistake, Treder and Jackisch agreed with Lenard and emphasized that Soldner here "fell a victim to the printer's devil." They also found handwritten corrections (conjectured to have been made perhaps by the editor Bode himself) in the copy of the *Jahrbuch* available to them. They wrote: "[...] Soldner's paper contains two misprints, to which Lenard rightly called attention. In the present exemplar of "Astronomisches Jahrbuch" of the Berliner Sternwarte of 1804 those two misprints have for generations been corrected by hand (it is unknown if by Bode himself). Both misprints consist in that now a factor 2 is missing although the deduction and the context do require it. So on page 170, for example, of Bode's Jahrbuch a factor 2 should precede the angle; but, as the context proves, this is a mere misprint."[10] In fact, inspection of the relevant copy which is now


Setzt man dieſe Werthe für AB und AD in den Ausdruck für
tang ω, ſo hat man:

$$\tan g\,\omega = \frac{2g}{v\sqrt{(v^2 - 4g)}}.$$

Wir wollen nun von dieſer Formel eine Anwendung auf

Figure 2. Soldner's final result on [7, p. 169]. Reproduced with permission from copy of the *Bayerische Staatsbibiliothek*, Signatur: Eph. astr. 23-1804. urn:nbn:de:bvb:12-bsb105383333-5, p.179.

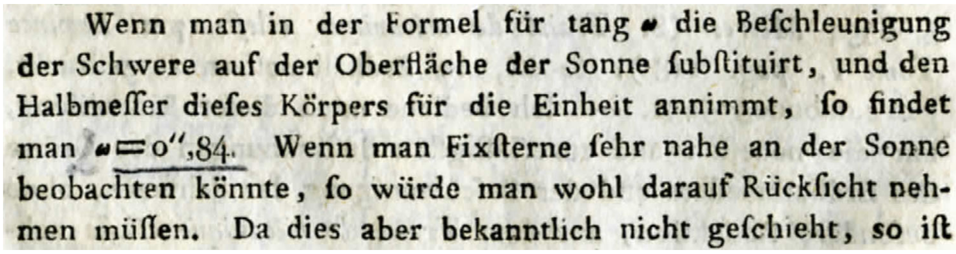

Wenn man in der Formel für tang ω die Beſchleunigung der Schwere auf der Oberfläche der Sonne ſubſtituirt, und den Halbmeſſer dieſes Körpers für die Einheit annimmt, ſo findet man ω = o″,84. Wenn man Fixſterne ſehr nahe an der Sonne beobachten könnte, ſo würde man wohl darauf Rückſicht nehmen müſſen. Da dies aber bekanntlich nicht geſchieht, so iſt

Figure 3. Detail from p. 170 of Soldner's article [7, p. 170]. Reproduced with permission from the copy of the *Leibniz-Institut für Astrophysik*, Potsdam.

held by the library of the *Leibniz-Institut für Astrophysik* in Potsdam shows a correction of the relevant numerical value given for ω on p. 170 of Soldner's article, see **Figure 3**.

Indeed, Soldner's paper states very clearly that the relation does require that additional factor of 2. Immediately, in the passage before, he wrote: "If one were to investigate by means of the given formula how much the moon would deviate a light ray when it goes by the moon and comes to the earth, then one must, after substituting the corresponding magnitudes and taking the radius of the moon for unity, double the value found through the formula, because a light ray, which goes by the moon and comes to the earth describes two arms of a hyperbola."[8]

However, Soldner's own numerical evaluation of his equation is not so readily confirmed by a modern reader. He gives all numerical values in units of the radius of the deflecting body, which, as Lenard already observed[5], is not "conducive to a transparent numerical evaluation." Moreover, he gives the velocity of light in units of meters per decimal seconds. Let us refer to a decimal second by s_{10}. Soldner cites a value of 564".8 s_{10} for the time needed to get from the Sun to Earth. With $100\,000\, s_{10} = 86\,400\, s$, and a rough estimate of the astronomical unit of 1.5×10^{11}m, this gives a reasonable value for the velocity of light close to its current vacuum value of $\approx 3 \cdot 10^8$m s^{-1}. But then he quotes a value for the velocity of light v of 15.562 Earth radii per decimal second. Having given the radius of the Earth explicitly as 6 369 514 m, this clearly implies a value for the velocity of light that is a factor of 2.6 or so off the correct value of some 40.7 Earth radii per decimal second, an unexplained numerical inconsistency. For the gravitational acceleration g, Soldner then quotes from Laplace's *Traité de mecanique*,[13] published in the year VII of the French revolution, that is, in 1798, a value of 3.66394×10^2 m s^{-2} which, again, he gives using units of decimal second, and which translates to accord with our modern value of gravitational acceleration at the surface of the Earth g_{Earth} as

$$g = 3.66394\,\frac{m}{s_{10}^2} = \frac{3.66394}{0.864^2}\,\frac{m}{s^2} = 4.908\,\frac{m}{s^2} = \frac{1}{2}9.816\,\frac{m}{s^2} = \frac{1}{2}g_{\mathrm{Earth}}, \tag{3}$$

confirming the different definition of gravitational acceleration with our present convention. Plugging in his values for g and v for the deflection at the surface of the Earth, Soldner obtained a value for $\omega = 0.''0009798$, which does come out numerically but which, as Trumpler[14] already observed, is not the correct Newtonian value for the deflecting angle of the Earth. Nevertheless, plugging in the numbers for the radius and gravitational acceleration of the Sun, his equation does give a value of $2\omega = 0''.84$, as was quoted by Soldner, if we correct the typo displayed in Figure 3.

Incidentally, Einstein's 1916 *Annalen* paper was also marred by a printer's error. On its last page, Einstein asserted that "a light ray going past the Sun undergoes a deflection of 1.7″,[2] but the relevant Equation (74) gave the angle of deflection wrong by a factor of 2, that is, as $\kappa M/4\pi\Delta$ with $\kappa = 8\pi K/c^2$, where K is the gravitational constant, M the Sun's mass and Δ its radius. When asked about this error in 1920 by Carl Runge, Einstein wrote in response: "You are completely right with your correction. This error does indeed originally appear in my *Annalen* paper. It has already been rectified, however, in Teubner's collected edition *Das Relativitätsprinzip*."[15] Indeed, inspection of the reprint of Einstein's *Annalen* paper in the third edition of ref. [16] from 1920 reveals that its Equation (74) (as well as the associated Equation (70a)) has been emended, and there the correct value of $\kappa M/2\pi\Delta$ was given.

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Conflict of Interest

The author declares no conflict of interest.

Keywords

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