Analysis and numerical solution of the Peterlin viscoelastic model

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Mathematics is the language
in which God has written the universe.

Galileo Galilei

Abstract

Liquids and gasses form a vital part of nature. Many of these are complex fluids with non-Newtonian behaviour. We introduce a mathematical model describing the unsteady motion of an incompressible polymeric fluid. Each polymer molecule is treated as two beads connected by a spring. For the nonlinear spring force it is not possible to obtain a closed system of equations, unless we approximate the force law. The Peterlin approximation replaces the length of the spring by the length of the average spring. Consequently, the macroscopic dumbbell-based model for dilute polymer solutions is obtained. The model consists of the conservation of mass and momentum and time evolution of the symmetric positive definite conformation tensor, where the diffusive effects are taken into account. In two space dimensions we prove global in time existence of weak solutions. Assuming more regular data we show higher regularity and consequently uniqueness of the weak solution. For the Oseen-type Peterlin model we propose a linear pressure-stabilized characteristics finite element scheme. We derive the corresponding error estimates and we prove, for linear finite elements, the optimal first order accuracy. Theoretical error of the pressure-stabilized characteristic finite element scheme is confirmed by a series of numerical experiments.

Kurzfassung

Flüssigkeiten und Gase bilden einen wesentlich Bestandteil der Natur. Hier kommen vor allem komplexe Fluide vor, die nichtnewtonsches Verhalten aufweisen. Wir stellen ein mathematisches Modell vor, das die instationäre Bewegung eines inkompressiblen polymeren Fluide beschreibt. Ein Polymermolekül wird beschrieben durch zwei Kugeln, die durch eine Feder verbunden sind. Es ist nicht möglich, die nichtlineare Federkraft durch ein geschlossenes Gleichungssystem zu beschreiben, außer wir approximieren das Kraftgesetz. Die Peterlin-Näherung ersetzt die Länge der Feder durch die Durchschnittslänge. Auf diese Weise wird ein makroskopisches Modell für verdünnte Polymerlösungen auf der Grundlage des Hantelmodells hergeleitet. Das Modell besteht aus den Erhaltungsgleichung für Masse und Impuls sowie der zeitliche Evolutionsgleichung für den symmetrisch possitiv-definiten Konformationstensor, die Diffusionseffekte berücksichtigt. In zwei Raumdimensionen beweisen wir globale Existenz in der Zeit von schwachen Lösungen. Unter strengeren Glattheitsanforderungen an die Daten zeigen wir höhere Regularität und folglich Eindeutigkeit der schwachen Lösung. Wir stellen eine lineare druckstabilisierte Finite-Elemente-Charakteristiken-Methode für das Peterlinmodell vom Oseen-Typ. Wir leiten entsprechende Fehlerabschätzungen her und zeigen für lineare finite Elemente optimale Approximationsordnung erster Ordnung. Die theoretische Fehlerabschätzung der druckstabilisierte Finite-Elemente-Charakteristiken-Methode wird durch eine Reihe numerischer Experimente bestätigt.

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1 Introduction

Fluids are inevitable part of our everyday life. Modelling liquids and gases in different settings and under different conditions is an interesting and challenging task. As pointed out in [92], many biological, industrial or geological fluids can no longer be described by a linear relation between the stress and the deformation tensor. These complex fluids fall into a class of the so-called *non-Newtonian fluids*. One of the important features of this class is viscoelasticity. The resistance of viscous materials to the shear flow and strain is linear in time when a stress is applied, while the elastic materials strain when stretched and quickly return to their original state once the stress is removed. A *viscoelastic fluid* exhibits both the viscous and elastic behaviour when being deformed, see, e.g., [104] for more details.

Models of viscoelastic fluids

Recently, an increasing number of mathematicians has become interested in mathematical modelling and numerical simulation of complex fluids. As mentioned in [92], there exists already a rich body of literature on mathematical analysis and problem-suited numerical methods. Mathematical models consist of the conservation laws describing the conservation of mass and momentum. The stress tensor is typically written as a sum of viscous stress tensor depending linearly on the deformation tensor and the extra stress due to the viscoelastic contribution. In macroscopic models the latter is given by a complex constitutive equation to capture the corresponding viscoelastic properties. In order to describe the evolution of viscoelastic stress tensor various approaches can be used. Let us mention, for example, differential, integral models or micro-macro models based on the kinetic formulation of the probability distribution function.

Polymeric materials

Viscoelasticity is a typical property of *polymers*. A polymer is a large molecule or a chain of molecules whose structure is composed of multiple repeating units, cf. [35, 103]. There is a wide range of polymers, from synthetic plastic (polystyrene) to biopolymers (DNA and proteins). The long chain molecules, typical for polymeric liquids, are modelled as chains of beads and springs or beads and rods. The spring forces, stochastic forces and forces exerted by the surrounding fluid are responsible for the movement of molecules.

As already pointed out in [90], there are basically three different approaches how to model polymeric materials. Firstly, the *dilute solution theories* which consider the polymer molecule to be surrounded by a Newtonian fluid. In this case, the hydrodynamic drag forces are resulting in the interaction between the molecules and the flow. On the contrary, there are the network theories motivated by the theories of rubber elasticity. Molecules are linked together at junction points in a network. The interaction between the polymer molecule and the flow results from the motion of the junctions. Finally, the middle ground of the above two theories are the reptation theories, which visualize the molecule as slithering inside a tube formed by other polymer molecules. For details, see, e.g., [122], [124]. The history of molecular modelling can be found in [19] and the references therein.

The simplest model representing the dilute solution theories is the so-called *dumbbell model* consisting of two beads connected by a spring, see [90]. Considering the linear force law for the spring force $\mathbf{F}(\mathbf{R}) = H\mathbf{R}$, where $\mathbf{R}$ is the vector connecting the beads, we obtain the upper convected Maxwell model (UCM). The well-known Oldroyd-B model has the stress that is a linear superposition of UCM model and the Newtonian model.

For the nonlinear force, $\mathbf{F}(\mathbf{R}) = \gamma(|\mathbf{R}|^2)\mathbf{R}$, it is not possible to obtain a closed system of equations except for approximating the force law. The *Peterlin approximation* replaces this law by $\mathbf{F}(\mathbf{R}) = \gamma(\langle|\mathbf{R}|^2\rangle)\mathbf{R}$, i.e. the length of the spring in the spring function γ is replaced by the length of the average spring $\langle|\mathbf{R}|^2\rangle = \operatorname{tr}\mathbf{C}$. Consequently, we can derive the evolution equation for the conformation tensor $\mathbf{C}$, which is in a closed form, cf. [122].

Analytical results

Mathematical analysis of complex viscoelastic fluids is an active research area. In the literature we can find already various mathematical results dealing with the question of well-posedness of the viscoelastic flows and in particular with the Oldroyd-B model. In what follows, we list a few of these results, cf. [90, 91, 92].

Differential models

Well-known differential models are the Oldroyd-B or the Johnson-Segelman models, where an additional transport equation is added to describe the time evolution of the polymer stress tensor. Concerning the local in time existence results and global in time results for small data in the case of the Oldroyd-type models let us mention the classical results of Fernández-Cara, Guillén and Ortega [53] and of Guillopé and Saut [61]. Theoretical results for stationary generalized Oldroyd-B, power-law flows were published by Arada and Sequeira [6], see also [65] for further related results on the existence of strong solutions in exterior domains obtained by Hieber, Naito and Shibata.

Global existence of the solution to the Oldroyd-B equations is still an open question. For the case of the so-called co-rotational Oldroyd-B model, where the gradient of velocity $\nabla\mathbf{u}$ in the evolution equation for the elastic stress tensor is replaced by its anti-symmetric part $\frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T)$, the global existence result for fully two- and three-dimensional flow has been recently obtained by Lions and Masmoudi [89]. The goal is to get strong convergence of the elastic stress tensor. To this end, the authors introduce a new quantity that measures losses of compactness in nonlinear terms and apply DiPerna, Lions theory of renormalized solutions. Once the strong convergence for the elastic stress tensor is obtained one can clearly pass to limit in all nonlinear terms and deal with other terms as in the Navier-Stokes theory. Unfortunately, the proof cannot be extended easily to other Oldroyd-type fluids since a specific structure of the co-rotational model has been used here.

In the viscoelastic models the transport equation for the elastic stress tensor plays an important role. Bahouri and Chemin [11] proved a losing a priori estimate for the transport equation. Based on this theory, Chemin and Masmoudi [98] showed the blow-up criterion in two-dimensional situation. Recently, this result was improved by Lei, Masmoudi and Zhou [85]. Global existence of weak solutions for small data can be found, e.g., in [38]. Local existence of solutions and global existence of small solutions of some rate type fluids have been shown by Lin, Liu and Zhang in [87].

In the recent work [12] Barrett and Boyaval studied the so-called diffusive Oldroyd-B model both from numerical as well as analytical point of view. For two space dimensions they were able to prove the global existence of weak solutions. The diffusive Oldroyd-B model has been also studied by Constantin and Kliegl in [36] and the global regularity in two space dimensions has been proven. Let us point out that in standard derivations of bead-spring models the diffusive term in the equation for the elastic stress tensor is routinely omitted. As pointed out in [13, 43, 133] in the case of heterogeneous fluid velocity this diffusive term indeed appears in the Fokker-Planck equation and, consequently, also in the corresponding macroscopic equation for the elastic stress.

Micro-macro models

On the other hand, as already mentioned above, see also [90], complex viscoelastic fluids can be also modelled using the molecular description of the complex fluids, which yields the so-called micro-macro models. Here we couple the macroscopic equation for the conservation of mass and momentum with the Fokker-Planck equation arising from the kinetic approach. The Fokker-Planck equation is a nonlinear equation describing time evolution for the particle distribution. The (macroscopic) elastic stress tensor, appearing on the right-hand side of the momentum equation, is then obtained by an averaging process by means of the particle distribution function, cf. the Kramers expression. Indeed, the Oldroyd-B model can be obtained as an exact closure of the linear Fokker-Planck equation, see, e.g., [36].

Mathematical literature dealing with the analysis of such micro-macro viscoelastic models is growing quite rapidly, see, e.g., [15, 16, 37, 38, 43, 73, 74, 82, 83, 84, 87, 99, 100, 101, 121] and the references therein. For example, in [99, 100, 101] Masmoudi and collaborators combine the macroscopic fluid model with the so-called FENE (finitely extensible nonlinear elastic) model, which assumes that the interaction potential can be infinite at finite extension length. In [100] the global existence of weak solution for FENE dumbbell polymeric flows is proved. The proof is based on the control of the propagation of strong convergence of some well chosen quantity by studying a transport equation for its defect measure. Furthermore, in [101] the existence of global smooth solutions for a coupled micro-macro model for polymeric fluid in two space dimensions under the co-rotational assumption is obtained.

For the dilute polymers, using the kinetic model and having a diffusive term, the global existence of weak solutions has been proven by Barrett and Süli in [13]. In this paper the authors work with the FENE model in order to represent viscoelastic effects. Thus, the spring force $\mathbf{F}(\mathbf{R})$ is no more linear but given by such a nonlinear potential, see [13] for more details. In [14] analogous existence result for the Hookean-type kinetic model with a diffusive term has been presented.

Numerical solution

Concerning numerical simulations of the Oldroyd-type flows several numerical schemes have been derived, see, e.g., [2, 40, 50, 51, 76, 93, 97, 105, 143] and the references therein. Development of a stable and convergent numerical method for the Oldroyd-B equations, in particular, in the case when the elastic effects dominate, is still a challenging problem.

As already mentioned in [92], a major obstacle is the high Weissenberg number problem. The non-dimensional parameter We expresses a ratio of the relaxation time to a typical flow time scale. The so-called numerical blow-up is a widely known phenomenon in numerical simulations of the high Weissenberg number viscoelastic flows. It is anticipated that the blow-up has various reasons: influence of domain singularities, missing analytical results on the well-posedness of global weak solutions and numerical instabilities. The latter is a purely numerical phenomenon that arises due to the inadequacy of polynomial interpolations to approximate spatial exponential profiles, which is the case of the elastic stress tensor.

Fattal and Kupferman in [51] have proposed a new promising approach using the log-transformation of the conformation tensor. See also further related works [21, 31, 41] and the references therein. New energy dissipative numerical methods, based on either the finite element or the finite volume methods combined with the method of characteristics, have been recently proposed in [93], where both the diffusion and the logarithmic transformation of the elastic stress are considered.

Limiting case of the Oldroyd-B model

In our recent paper [92] we have considered a viscoelastic model that can be achieved as a limiting case of the Oldroyd-B model when the relaxation time goes to infinity. Note that this assumption is equivalent to the fact that the Weissenberg number We is set to infinity. The question of global in time existence of weak solutions of this model is still open. Local existence of strong solutions has been proven in [77], [87] as well as in [32].

As far as we know our paper has been the first contribution to the numerical analysis of this model. More interestingly, we would like to point out that we did not need any particular stabilization techniques for the high Weissenberg number problem. We have obtained stable and convergent results using suitable numerical approximations that are typically used in computational fluid dynamics. In particular, we combined the lowest order Taylor-Hood finite element discretization of the flow part (piecewise quadratic velocity and piecewise linear pressure) with either piecewise linear finite elements or finite volumes for the deformation gradient.

Structure of the thesis

The thesis aims to present the analysis and numerical solution of a particular viscoelastic fluid flow model based on the so-called Peterlin approximation. The first part is devoted to the existence and uniqueness study of global weak solutions. The error analysis of a numerical approximation supported by a series of simulations is presented in the second part.

The first part begins with introducing the notation and listing useful lemmas, theorems and inequalities in Chapter 2. The mathematical model for an incompressible viscoelastic fluid is presented in Chapter 3. The fluid flow is governed by the conservation of mass, momentum and the evolution equation for the conformation tensor, which is by definition symmetric and positive definite. We prove, in Section 3.2, that our model indeed preserves the positive definiteness of the conformation tensor. In the next two sections the formal energy estimates and the free energy estimate are derived. Chapter 4 deals with the existence and uniqueness of global in time weak solutions to our problem. The existence is shown by studying the Galerkin approximation, see Section 4.2. Provided the data are more regular, we show that the weak solution enjoys higher regularity. Consequently, uniqueness of global more regular weak solution is proven in Section 4.4. In Chapter 5 a generalized diffusive Peterlin model is considered. The analogous theoretical results on existence and uniqueness of global weak solution to the generalized model are shown in Sections 5.2 and 5.3.

In the second part of the thesis we study the viscoelastic model from the numerical point of view. Due to the complexity of such system we confined ourselves with the Oseen-type problem, where the convective terms are linearised. Chapter 6 contains the derivation of a semi-implicit pressure-stabilized characteristics finite element scheme. In Section 6.2 the first order approximation of the material derivative, based on the method of characteristic curves, is introduced. The spatial discretization is given by the finite element method. Since the continuous piecewise linear approximation of the velocity, pressure and the conformation tensor is considered, we employ a stabilization technique to satisfy the inf-sup condition, see Section 6.3. Finally, the existence of discrete solution is proven and the positive definiteness of the discrete conformation tensor is discussed. In Chapter 7 the error analysis for the proposed numerical scheme is presented. The first section of this chapter provides the stability result. The optimal first order rate of convergence is proven afterwards. Analogously as for the weak solution we are able to show the higher energy bounds of the discrete solution, see Section 7.4. We also introduce a fully implicit nonlinear version of the characteristics finite element scheme and compare both approaches. Finally, in Chapter 8, after a brief description of the scheme implementation, we demonstrate the accuracy of our semi-implicit linear scheme by a series of experiments. The first order convergence is confirmed, cf. Section 8.2. Moreover, a few results of the lid-driven cavity flow problem are presented in Section 8.3.

After conclusions and remarks on future goals, we recall some useful results for the study of evolution problems, see Appendix A. In Appendix B we briefly explain the mixed variational problems that lead to the inf-sup condition. Finally, a detailed transcript of the numerical scheme is presented in Appendix C.

2 Preliminaries

This chapter collects some useful preliminaries needed for the analysis and numerical solution of our viscoelastic fluid flow model. It is divided into three parts. Firstly, the notation and functional spaces are introduced. Afterwards, a list of theorems, lemmas and inequalities used throughout the thesis is included. At the end, the definition and properties of a symmetric positive definite matrix are given.

2.1. Notation and functional spaces

Let $d \geq 2$ denote the space dimension and let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ be a vector. Let u, v be scalar-valued functions, $\mathbf{u} = (u_1, \ldots, u_d)$, $\mathbf{v} = (v_1, \ldots, v_d)$ be vector-valued functions and $\mathbf{C} = \{C_{ij}\}$, $\mathbf{D} = \{D_{ij}\}$, $i, j = 1, \ldots, d$, be $d \times d$ tensor-valued functions. We will use the following notation

$$
\begin{aligned}
&(\nabla u)_i = \frac{\partial u}{\partial x_i}, && (\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}, && (\nabla \mathbf{C})_{ijk} = \frac{\partial C_{ij}}{\partial x_k},\\
&\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}, && (\Delta \mathbf{u})_i = \sum_{j=1}^d \frac{\partial^2 u_i}{\partial x_j^2}, && (\Delta \mathbf{C})_{ij} = \sum_{k=1}^d \frac{\partial^2 C_{ij}}{\partial x_k^2},\\
&\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^d u_i v_i, && \mathbf{C} : \mathbf{D} = \sum_{i,j=1}^d C_{ij} D_{ij}, && \nabla \mathbf{C} : \nabla \mathbf{D} = \sum_{i,j,k=1}^d \frac{\partial C_{ij}}{\partial x_k} \frac{\partial D_{ij}}{\partial x_k},\\
&\nabla u \cdot \mathbf{n} = \sum_{i=1}^d \frac{\partial u}{\partial x_i} n_i, && (\nabla \mathbf{u} \cdot \mathbf{n})_i = \nabla u_i \cdot \mathbf{n}, && (\nabla \mathbf{C} \cdot \mathbf{n})_{ij} = \nabla C_{ij} \cdot \mathbf{n}\\
&(\mathbf{v} \cdot \nabla) u = \sum_{i=1}^d v_i \frac{\partial u}{\partial x_i}, && ((\mathbf{v} \cdot \nabla)\mathbf{u})_i = \sum_{j=1}^d v_j \frac{\partial u_i}{\partial x_j}, && ((\mathbf{v} \cdot \nabla)\mathbf{C})_{ij} = \sum_{k=1}^d v_k \frac{\partial C_{ij}}{\partial x_k},\\
&\operatorname{div} \mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}, && (\operatorname{div} \mathbf{C})_i = \sum_{j=1}^d \frac{\partial C_{ji}}{\partial x_j},
\end{aligned}
$$

where $\mathbf{n} \in \mathbb{R}^d$ is the outer normal vector. We denote the *rate of strain tensor*, if you like the symmetric part of the gradient or the *deformation tensor*, by

$$
\mathrm{D}(\mathbf{u}) := \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}. \tag{2.1}
$$

The *material derivative*, for a given vector function $\mathbf{v}$, reads

$$\frac{\mathrm{D}u}{\mathrm{D}t} := \frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u, \quad \left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\right)_i := \frac{\partial u_i}{\partial t} + (\mathbf{v}\cdot\nabla)u_i, \quad \left(\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}\right)_{ij} := \frac{\partial C_{ij}}{\partial t} + (\mathbf{v}\cdot\nabla)C_{ij},$$

for a scalar, a vector and a tensor-valued function, respectively.

Let $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be the multiindex, $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Then the α-th partial derivative D^α is defined by

$$D^0 u := u \quad \text{and} \quad D^\alpha u := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}}, \quad \text{for} \quad |\alpha| \geq 1.$$

Further, let $\mathbf{C}^T$, $\operatorname{tr}\mathbf{C}$ and $\det\mathbf{C}$ denote the *transpose*, the *trace* and the *determinant* of the tensor $\mathbf{C}$, respectively. The above introduced notation

$$\mathbf{C} : \mathbf{D} := \sum_{i,j} C_{ij} D_{ij} = \operatorname{tr}(\mathbf{C}\mathbf{D}^T) = \operatorname{tr}(\mathbf{C}^T\mathbf{D}), \tag{2.2}$$

for $\mathbf{C}$, $\mathbf{D} \in \mathbb{R}^{m\times n}$, stands for the *Frobenius product.* It is the inner product of the vector space formed by $m \times n$ real matrices and it induces the *Frobenius norm*

$$\|\mathbf{D}\|_F := \left(\sum_{i,j} |d_{ij}|^2\right)^{1/2} = \operatorname{tr}(\mathbf{D}\mathbf{D}^T). \tag{2.3}$$

We will use the notation $\mathbf{D}^2 := \mathbf{D} : \mathbf{D}$.

Functional spaces

In what follows we assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with the Lipschitz-continuous boundary $\partial\Omega$, unless stated otherwise. $\bar{\Omega}$ denotes the closure of Ω. The definitions of some useful functional spaces follow.

The space $C^k(\bar{\Omega})$ [3]

For $k \geq 0$ we denote by $C^k(\bar{\Omega})$ the space of all k-times continuously differentiable functions in Ω. Then

$$C^\infty(\bar{\Omega}) := \bigcap_{k=0}^{\infty} C^k(\bar{\Omega})$$

stands for the space off all infinitely many times continuously differentiable functions in Ω. The space $C^k(\bar{\Omega})$ is equipped with the norm

$$\|u\|_{C^k} := \max_{0\leq|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha u| \quad \text{if} \quad k < \infty.$$

We abbreviate $C(\bar{\Omega}) := C^0(\bar{\Omega})$. Further, we denote

$$C_0^\infty(\bar{\Omega}) := \left\{u \,:\, u \in C^\infty(\bar{\Omega}) \text{ having a compact support in } \Omega\right\},$$
$$C_{0,div}^\infty(\bar{\Omega}) := \left\{\mathbf{u} \,:\, \mathbf{u} \in C_0^\infty(\bar{\Omega}),\ \operatorname{div}\mathbf{u} = 0\right\}.$$

The Lebesgue spaces $L^p(\Omega)$, $L^2_{div}(\Omega)$ and $L^2_0(\Omega)$ [3, 52, 120, 142]

For $1 \le p < \infty$ we denote by $L^p(\Omega)$ the space of all measurable functions whose p-th powers are Lebesgue-integrable in Ω. $L^\infty(\Omega)$ denotes the space of essentially bounded functions in Ω. The norms are given by

$$\|u\|_{L^p} := \left(\int_\Omega |u|^p \; dx \right)^{1/p}, \qquad \|u\|_{L^\infty} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

Let us note that $L^2(\Omega)$ is the Hilbert space with the scalar product

$$(u, v) := \int_\Omega u(x)v(x) \; dx \, .$$

Now, we introduce two typical functional spaces for fluid dynamics. Let us define

$$L^2_{div}(\Omega) := \overline{C^\infty_{0,div}(\bar{\Omega})}^{\|\cdot\|_{L^2}}, \quad L^2_0(\Omega) := \left\{ u \, : \, u \in L^2(\Omega), \int_\Omega u \; dx \, = 0 \right\}.$$

The Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $H^1_{0,div}(\Omega)$ and $H^m(\Omega)$ [3, 49, 142]

Let $H^1(\Omega) = W^{1,2}(\Omega)$ denote the following Sobolev space

$$H^1(\Omega) := \left\{ u \, : \, u \in L^2(\Omega), \, \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for } i = 1, \dots, d \right\},$$

where the partial derivatives are understood in the distributional sense. The corresponding scalar product and norm of this Hilbert space are given by

$$(u, v)_{H^1} := \int_\Omega uv + \nabla u \cdot \nabla v \; dx \, , \quad \|u\|_{H^1} := \left(\int_\Omega |u|^2 + |\nabla u|^2 \; dx \right)^{1/2},$$

respectively. We define other two spaces typical for the analysis of fluid flow problems, i.e.

$$H^1_0(\Omega) := \overline{C^\infty_0(\bar{\Omega})}^{\|\cdot\|_{H^1}} \quad \text{and} \quad H^1_{0,div}(\Omega) := \overline{C^\infty_{0,div}(\bar{\Omega})}^{\|\cdot\|_{H^1}}.$$

It holds that

$$\begin{aligned} H^1_0(\Omega) &= \left\{ u \in H^1(\Omega) \, : \, u \text{ is zero on the boundary} \right\}, \\ H^1_{0,div}(\Omega) &= \left\{ \mathbf{u} \in H^1(\Omega) \, : \, \operatorname{div} \mathbf{u} = 0, \; \mathbf{u} \text{ is zero on the boundary} \right\}. \end{aligned}$$

Since the above two spaces are closed subspaces of $H^1(\Omega)$, they are equipped with the same norm $\|\cdot\|_{H^1}$. Moreover, due to the Poincaré inequality, cf. [142], there is an equivalent norm given by

$$\|u\|_{H^1_{0,div}} = \|u\|_{H^1_0} := \|\nabla u\|_{L^2} \, .$$

Further, for each integer $m \ge 0$ the Sobolev space $H^m(\Omega) = W^{m,2}(\Omega)$ is defined by

$$H^m(\Omega) := \left\{ u \, : \, D^\alpha u \in L^2(\Omega), \; \forall \, \alpha \text{ such that } |\alpha| \le m \right\}$$

with the scalar product and norm

$$(u,v)_{H^m} := \sum_{0\le|\alpha|\le m} \int_\Omega D^\alpha u\, D^\alpha v\; dx\,, \quad \|u\|_{H^m} := \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^2}^2\right)^{1/2},$$

respectively. The space $H^m(\Omega)$ equipped with the above norm is the Hilbert space. We shall also make use of the semi-norm

$$|u|_m := \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^2}^2\right)^{1/2}.$$

The Bochner space $L^p(0,T;X(\Omega))$ [49, 120]

Let $1 \le p \le \infty$ and $X(\Omega)$ be a Banach space. Let $v(x,t)$ be a function defined on $\Omega \times (0,T)$ such that, for each $t \in (0,T)$ the function $v(t) := v(x,t)$ is measurable and it is an element of the space $X(\Omega)$. Then we denote for $1 \le p < \infty$

$$L^p(0,T;X(\Omega)) := \left\{ v \;:\; v(t) : (0,T) \to X(\Omega),\; \left(\int_0^T \|v(t)\|_X^p\; dt\right)^{1/p} < \infty \right\},$$

and for $p = \infty$

$$L^\infty(0,T;X(\Omega)) := \left\{ v \;:\; v(t) : (0,T) \to X(\Omega),\; \operatorname*{ess\,sup}_{t\in(0,T)} \|v(t)\|_X < \infty \right\}.$$

The norms are given by

$$\|v\|_{L^p(X)} := \left(\int_0^T \|v(t)\|_X^p\; dt\right)^{1/p}, \qquad \|v\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{t\in(0,T)} \|v(t)\|_X\,.$$

The space $H^1(0,T;X(\Omega))$ is defined by

$$H^1(0,T;X(\Omega)) := \left\{ v \in L^2(0,T;X(\Omega)) \;:\; \frac{\partial v}{\partial t} \in L^2(0,T;X(\Omega)) \right\},$$

where $\partial v/\partial t$ means the distributional derivative with values in $X(\Omega)$. An analogous definition holds for the space $H^j(0,T;X(\Omega))$, for an integer $j \ge 2$.

In a similar way we can define the space $C^k(0,T;X(\Omega))$.

The space Z^m [111]

For $m \in \mathbb{N}$ and t_0, $t_1 \in \mathbb{R}$ we introduce the space

$$Z^m(t^0,t^1) := \left\{ u \in H^j(t^0,t^1;H^{m-j}(\Omega)),\; j = 0,\dots,m \;:\; \|u\|_{Z^m(t^0,t^1)} < \infty \right\}, \tag{2.4}$$

where the norm is given by

$$\|u\|_{Z^m(t^0,t^1)} := \max_{j=0,\dots,m} \|u\|_{H^j(t^0,t^1;H^{m-j}(\Omega))}\,.$$

We define $Z^m := Z^m(0,T)$.

Let X be a real normed linear space. Then the dual space X^* is the space of all bounded linear functionals $f : X \to \mathbb{R}$ equipped with the norm

$$\|f\|_{X^*} := \sup_{\|u\|_X \neq 0} \frac{\langle f, u\rangle_{X^* \times X}}{\|u\|_X},$$

which makes it the Banach space.

Let X be a functional space. Then $\mathbf{u} \in X$ and $\mathbf{C} \in X$ mean that $u_i \in X$ and $C_{ij} \in X$, for $i, j = 1, \ldots d$, respectively. We might use $\mathbf{u} \in X^d$ or $\mathbf{C} \in X^{d\times d}$ to avoid confusion when necessary. However, for simplicity of the notation we do not distinguish, in the notation of functional spaces, between the spaces of scalar, vector or tensor functions.

The following notation shall be employed for the norms of the spaces introduced above

$$\begin{aligned}
\|\cdot\|_k &:= \|\cdot\|_{L^k(\Omega)} & \qquad \|\cdot\|_{L^k(X)} &:= \|\cdot\|_{L^k(0,T;X(\Omega))}\\
\|\cdot\|_0 &:= \|\cdot\|_{C(\Omega)} & \qquad \|\cdot\|_{H^k(X)} &:= \|\cdot\|_{H^k(0,T;X(\Omega))}\\
\|\cdot\|_{l,m} &:= \|\cdot\|_{W^{l,m}(\Omega)} & \qquad \|\cdot\|_{C^k(X)} &:= \|\cdot\|_{C^k([0,T];X(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathbf{v}, q)\|_{H^2\times H^1} &:= \left(\|\mathbf{v}\|^2_{2,2} + \|q\|^2_{1,2} \right)^{1/2}\\
\|(\mathbf{v}, q, \mathbf{D})\|_{H^2\times H^1\times H^2} &:= \left(\|\mathbf{v}\|^2_{2,2} + \|q\|^2_{1,2} + \|\mathbf{D}\|^2_{2,2} \right)^{1/2}\\
\|(\mathbf{v}, \mathbf{D})\|_{Z^2(t_0,t_1)} &:= \left(\|\mathbf{v}\|^2_{Z^2(t_0,t_1)} + \|\mathbf{D}\|^2_{Z^2(t_0,t_1)} \right)^{1/2}.
\end{aligned}$$

2.2. Useful theorems and inequalities

We give a brief overview of theorems, lemmas and inequalities needed for further analysis.

The Gauss-Green theorem [49, 52, 120]

Suppose $u \in C^1(\bar{\Omega})$. Then

$$\int_\Omega \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} u n_i \, dS\,,\ i = 1, \ldots, d.$$

For a vector field $\mathbf{u} \in C^1(\bar{\Omega})$ we get **the divergence (Gauss) theorem**, i.e.

$$\int_\Omega \operatorname{div} \mathbf{u}\, dx = \int_{\partial\Omega} \mathbf{u}\cdot\mathbf{n}\, dS\,,$$

where $\mathbf{n}$ is an outward normal vector. The Gauss-Green theorem applied on the scalar function $uv \in C^1(\bar{\Omega})$ yields **the Green formula**

$$\int_\Omega \frac{\partial u}{\partial x_i} v\, dx = -\int_\Omega u \frac{\partial v}{\partial x_i}\, dx + \int_{\partial\Omega} uvn_i\, dS\,,\ i = 1, \ldots, d.$$

Subject to the assumption that $\partial\Omega$ is Lipschitz-continuous the above formulae hold for u, $v \in H^1(\Omega)$ and $\mathbf{u} \in H^1(\Omega)$ as well.

The Gronwall lemma [49, 52, 120]

differential form : Let $f \in L^1(t_0, T)$ be a non-negative function, φ be a continuous function on $[t_0, T]$. If φ satisfies

$$\varphi'(t) \leq f(t)\varphi(t) \quad \forall\, t \in [t_0, T], \tag{2.5a}$$

then

$$\varphi(t) \leq \varphi(t_0) \exp\left\{\int_{t_0}^{t} f(s)\ ds\right\} \quad \forall\, t \in [t_0, T]. \tag{2.5b}$$

integral form: Let $f \in L^1(t_0, T)$ be a non-negative function, g and φ be continuous functions on $[t_0, T]$. If φ satisfies

$$\varphi(t) \leq g(t) + \int_{t_0}^{t} f(s)\varphi(s)\, ds \quad \forall\, t \in [t_0, T], \tag{2.6a}$$

then

$$\varphi(t) \leq g(t) + \int_{t_0}^{t} f(s)g(s) \exp\left\{\int_{s}^{t} f(\tau)\, d\tau\right\}\, ds \quad \forall\, t \in [t_0, T]. \tag{2.6b}$$

If moreover g is non-decreasing, then

$$\varphi(t) \leq g(t) \exp\left\{\int_{t_0}^{t} f(\tau)\, d\tau\right\} \quad \forall\, t \in [t_0, T]. \tag{2.6c}$$

The discrete Gronwall lemma [120, 141]

Let a_0 be a non-negative number, Δt be a positive number such that

$$\Delta t \leq \frac{1}{2a_0} \tag{2.7a}$$

for $a_0 \neq 0$. Further, let $\{x^n\}_{n\geq 0}$, $\{y^n\}_{n\geq 1}$, $\{b^n\}_{n\geq 1}$, $\{a_1^n\}_{n\geq 1}$ be some non-negative sequences. Suppose

$$\bar{D}_{\Delta t}\, x^n + y^n \leq a_0\, x^n + a_1^n\, x^{n-1} + b^n, \ \forall\, n \geq 1. \tag{2.7b}$$

Then for all $n \geq 1$ it holds that

$$x^n + \Delta t \sum_{i=1}^{n} y^i \leq \exp\left\{2a_0\, n\Delta t + \Delta t \sum_{i=1}^{n} a_1^i\right\} \left(x^0 + \Delta t \sum_{i=1}^{n} b^i\right). \tag{2.7c}$$

Let us note that if a positive number a_0^{n-1} appears in (2.7b) instead of a constant a_0, then condition (2.7a) induces the dependence of Δt on $n-1$.

Here $\bar{D}_{\Delta t}$ stands for the backward difference operator

$$\bar{D}_{\Delta t}\, x^n := \frac{x^n - x^{n-1}}{\Delta t}.$$

The Sobolev embeddings theorem [3, 49, 142]

In particular it holds

$$H^1(\Omega) \hookrightarrow C(\bar{\Omega}) \qquad \text{if } d = 1,$$

$$H^1(\Omega) \hookrightarrow L^p(\Omega),\ \forall p \in [1, \infty),\ H^2(\Omega) \hookrightarrow C(\bar{\Omega}) \qquad \text{if } d = 2.$$

The Hölder inequality [49, 52]

Let $u \in L^p(\Omega)$, $v \in L^q(\Omega)$. Assume $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$\int_\Omega |uv|\ dx\ \le \|u\|_p \|v\|_q .$$

Similarly, for x, $y \in \mathbb{R}^d$, we have

$$\sum_{i=1}^d |x_i y_i| \le \left(\sum_{i=1}^d |x_i|^p \right)^{1/p} \left(\sum_{i=1}^d |y_i|^q \right)^{1/q} .$$

For $p = q = 2$ it is sometimes called the Cauchy-Schwarz inequality. In general, let $1 \le p_1, \ldots, p_m \le \infty$ with $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$, and assume $u_k \in L^{p_k}(\Omega)$ for $k = 1, \ldots, m$. Then

$$\int_\Omega |u_1 \ldots u_m|\ dx\ \le \prod_{k=1}^m \|u_k\|_{p_k} .$$

The interpolation inequalities [80]

Let the dimension $d = 2$ and let the boundary $\partial\Omega$ be piecewise smooth. Then the following holds true

$$\|u\|_4 \le c \|u\|_2^{1/2} \|\nabla u\|_2^{1/2},\ u \in H_0^1(\Omega) \tag{2.8a}$$

$$\|v\|_4 \le c(\Omega) \left(\|v\|_2 + \|v\|_2^{1/2} \|\nabla v\|_2^{1/2} \right),\ v \in H^1(\Omega), \tag{2.8b}$$

and for $r > 2$

$$\|v\|_r \le c(\Omega) \left(\|v\|_2 + \|v\|_2^{2/r} \|\nabla v\|_2^{(r-2)/r} \right),\ v \in H^1(\Omega). \tag{2.8c}$$

The Korn inequality [46, 49]

There exists a positive constant c such that

$$\left(\|\mathrm{D}(u)\|_2^2 + \|u\|_2^2 \right)^{1/2} \ge c \|u\|_{1,2} \quad \forall\ u \in H^1(\Omega).$$

The norms $\|\mathrm{D}(\cdot)\|_2$ and $\|\cdot\|_{1,2}$ are equivalent in $H_0^1(\Omega)$.

The Young inequality [49]

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{and} \quad ab \le \epsilon a^p + c(\epsilon) b^q$$

for $a, b, \epsilon > 0$ and $c(\epsilon) = (\epsilon p)^{-q/p} q^{-1}$. For $p = q = 2$ it is sometimes called the Cauchy inequality.

2.3. Symmetric positive definite matrices

We give the definition of a symmetric and a positive definite matrix and we list all the important properties used later. Further, we introduce a matrix function. More precisely, we define the logarithm and the inverse of a symmetric positive definite matrix. For the details and proofs we refer to, e.g., [1, 67, 81, 119].

Symmetric and positive definite matrix

A matrix $\mathbf{D} \in \mathbb{R}^{d\times d}$ is **symmetric** if $\mathbf{D} = \mathbf{D}^T$, where $\mathbf{D}^T$ denotes the transpose of $\mathbf{D}$.
It holds that

- all its eigenvalues λ_i are real
- there exists an *orthogonal decomposition (eigendecomposition)* $\mathbf{D} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where $\mathbf{Q}$ is a real orthogonal matrix, its columns are the eigenvectors of $\mathbf{D}$, and $\mathbf{\Lambda}$ is a real diagonal matrix having the eigenvalues of $\mathbf{D}$ on its diagonal
- it is positive definite if and only if all its eigenvalues are positive.

A matrix $\mathbf{D} \in \mathbb{R}^{d\times d}$ is **positive definite** in $\mathbb{R}^d$ if $x^T\,\mathbf{D}\,x > 0\ \forall\, x \in \mathbb{R}^d$, $x \neq 0$.
It holds that

- it is not necessarily symmetric
- if $r > 0$ is a real number, then $r\mathbf{D}$ is positive definite
- $\det \mathbf{D} > 0$, it is invertible and its inverse is also positive definite
- the diagonal entries D_{ii} are real and positive, $\operatorname{tr} \mathbf{D} > 0$.

The determinant and the trace of $\mathbf{D}$ can be expressed as the product and the sum of its eigenvalues

$$\det \mathbf{D} = \prod_{i=1}^{d} \lambda_i \quad \text{and} \quad \operatorname{tr} \mathbf{D} = \sum_{i=1}^{d} \lambda_i,$$

respectively.

Let us denote the space of all *symmetric positive definite matrices in* $\mathbb{R}^{d\times d}$ by $\mathcal{S}^d_+$.

Norm equivalence

In what follows we will work with the positive definite matrices that are symmetric. We shall use their important property, i.e. the following norm equivalence.

Proposition 2.1. *(norm equivalence)*
Let $\mathbf{D} \in S^d_+$. Then the following holds true

$$\|\mathbf{D}\|^p_{L^p(L^p)} \leq \|\operatorname{tr} \mathbf{D}\|^p_{L^p(L^p)} \leq d^{p-1}\, \|\mathbf{D}\|^p_{L^p(L^p)}\,,\ p \geq 2. \tag{2.9a}$$

If $\mathbf{D} \in \mathbb{R}^{d\times d}$ is an arbitrary matrix we have

$$\|\operatorname{tr} \mathbf{D}\|^p_{L^p(L^p)} \leq d^{p-1}\, \|\mathbf{D}\|^p_{L^p(L^p)}\,,\ p \geq 2. \tag{2.9b}$$

Proof. Since $\mathbf{D} \in S^d_+$, each eigenvalue λ_i of $\mathbf{D}$ and also $\operatorname{tr} \mathbf{D}$ are positive. Moreover, the Frobenius product (2.2), the orthogonal decomposition of $\mathbf{D}$ and the cyclic property of trace yield

$$\sum_{i,j=1}^{d} |D_{ij}|^2 = \mathbf{D} : \mathbf{D} = \operatorname{tr}\Big((\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T)(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T)\Big) = \operatorname{tr} \boldsymbol{\Lambda}^2 = \sum_{i=1}^{d} \lambda_i^2,$$
$$\sum_{i,j=1}^{d} |D_{ij}|^p = \mathbf{D}^{p/2} : \mathbf{D}^{p/2} = \operatorname{tr}\Big((\mathbf{Q}\boldsymbol{\Lambda}^{p/2}\mathbf{Q}^T)(\mathbf{Q}\boldsymbol{\Lambda}^{p/2}\mathbf{Q}^T)\Big) = \operatorname{tr} \boldsymbol{\Lambda}^p = \sum_{i=1}^{d} \lambda_i^p, \qquad p > 2.$$

Then the following estimate for $p \geq 2$ holds true

$$\begin{aligned}\|\mathbf{D}\|^p_{L^p(L^p)} &= \int_0^T \int_\Omega \sum_{i,j=1}^{d} |D_{ij}|^p \; dx\, dt \; = \int_0^T \int_\Omega \sum_{i=1}^{d} \lambda_i^p \; dx\, dt \\ &\leq \int_0^T \int_\Omega \left(\sum_{i=1}^{d} \lambda_i\right)^p \; dx\, dt \; = \|\operatorname{tr} \mathbf{D}\|^p_{L^p(L^p)}.\end{aligned}$$

For an arbitrary matrix $\mathbf{D} \in \mathbb{R}^{d\times d}$ we have, by the discrete Hölder inequality,

$$\begin{aligned}\|\operatorname{tr} \mathbf{D}\|^p_{L^p(L^p)} &= \int_0^T \int_\Omega \left(\sum_{i=1}^{d} D_{ii}\right)^p \; dx\, dt \; \leq d^{p-1} \int_0^T \int_\Omega \sum_{i=1}^{d} |D_{ii}|^p \; dx\, dt \\ &\leq d^{p-1} \int_0^T \int_\Omega \sum_{i,j=1}^{d} |D_{ij}|^p \; dx\, dt \; = d^{p-1} \|\mathbf{D}\|^p_{L^p(L^p)}, \qquad p \geq 2.\end{aligned}$$

□

Matrix function

Let $\mathbf{D} \in \mathcal{S}^d_+$. Then a matrix function $f(\mathbf{D})$ is defined as

$$f(\mathbf{D}) := \mathbf{Q} f(\boldsymbol{\Lambda}) \mathbf{Q}^T, \quad \text{where } f(\boldsymbol{\Lambda}) := \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_d) \end{pmatrix}.$$

Here the function $f : (0, \infty) \to \mathbb{R}$ is defined on the spectrum of $\mathbf{D}$. In this thesis we will use the logarithm of a symmetric positive definite matrix $\mathbf{D}$ which is defined as

$$\ln \mathbf{D} := \mathbf{Q} \ln \boldsymbol{\Lambda} \mathbf{Q}^T, \tag{2.10}$$

where $\ln \boldsymbol{\Lambda}$ is a diagonal matrix with the logarithm of each entry of Λ on its diagonal. Although the orthogonal decomposition is not unique, $\ln \mathbf{D}$ given by (2.10) is uniquely defined. Secondly, we will need the inverse of $\mathbf{D}$ defined through the function $f(x) = 1/x$ as

$$\mathbf{D}^{-1} := \mathbf{Q} \frac{1}{\boldsymbol{\Lambda}} \mathbf{Q}^T, \tag{2.11}$$

where $1/\boldsymbol{\Lambda}$ is a diagonal matrix with the eigenvalues of the inverse matrix on its diagonal.

3 Mathematical model

Fluid flow models play an important role in physics, medicine, meteorology, biology, industry, astronomy and many other fields. In the present thesis we are interested in non-Newtonian viscoelastic fluids as many biological and industrial fluids do not satisfy the Newtonian assumption on linear dependence of the stress tensor on the deformation tensor. In the literature we can find different models that are used to describe various aspects of complex viscoelastic fluids. In what follows we present a model for some polymeric fluids, whose system of governing equations consists of the conservation of mass and momentum as well as of the equation for time evolution of the conformation tensor that represents the constitutive relation for elastic stress.

Conservation laws

We can describe the motion of every fluid by the balance laws, i.e. by the conservation of mass and momentum, and the balance of energy, if thermal effects are important. In what follows we assume a constant temperature. As pointed out in [122], compressibility of polymeric fluids is not often of importance. Thus the *conservation of mass* is formulated as

$$\operatorname{div} \mathbf{u} = 0,$$

where $\mathbf{u}$ is the velocity of the fluid. The density is also assumed to be constant. Then the *balance of momentum* reads

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \operatorname{div} \mathcal{T} - \nabla p,$$

where $\mathcal{T}$ is the extra stress tensor and p is the pressure. To complete the mathematical formulation, we need a constitutive law for $\mathcal{T}$.

Linear models

A linear dependence of the extra stress tensor on the deformation tensor yields the *Newtonian fluid* governed by

$$\mathcal{T} = 2\nu \mathrm{D}(\mathbf{u}),$$

where ν is a viscosity coefficient and $\mathrm{D}(\mathbf{u})$ is the strain rate tensor, cf. (2.1). Not only the current motion of the fluid, but also the history of the fluid is important for viscoelastic fluids. The Boltzmann theories of linear viscoelasticity express the linear dependence of the stress on the history of the motion in integral form. On the other hand, Maxwell's theories of linear viscoelasticity link the stress with the velocity gradient by an ordinary differential equation, see [122] and the references therein for more details.

Nonlinear models

There are different ways how to generalize linear models. If the extra stress is given by

$$\mathcal{T} = 2\mu\left(|\mathrm{D}(\mathbf{u})|^2\right)\mathrm{D}(\mathbf{u}),$$

where the viscosity μ is a nonlinear function of $|\mathrm{D}(\mathbf{u})|$, then it describes the *generalized Newtonian fluid*. The most used nonlinear generalization of Boltzmann's linear model is the K-BKZ model, see, e.g., [75, 122]. To give an example of the nonlinear modification of Maxwell's theories, we mention the UCM model, i.e. the *upper convected Maxwell model*, which is given by

$$\overset{\triangledown}{\mathcal{T}} + \lambda\mathcal{T} = 2\nu\mathrm{D}(\mathbf{u}),$$

where $\overset{\triangledown}{\mathcal{T}}$ denotes the so-called Oldroyd or upper convected derivative, i.e.

$$\overset{\triangledown}{\mathcal{T}} := \frac{\partial\mathcal{T}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathcal{T} - (\nabla\mathbf{u})\mathcal{T} - \mathcal{T}(\nabla\mathbf{u})^T,$$

which is objective (frame-indifferent), see [113]. The well-known *Oldroyd-B* model has the stress that is a linear superposition of the upper convected Maxwell model and the Newtonian model.

Kinetic models

Another popular way to describe complex fluids is based on a combination of molecular and continuum models resulting in the so-called micro-macro models. The Fokker-Planck equation, i.e. a nonlinear equation describing time evolution of the particle distribution, is coupled with the macroscopic equations for the conservation of mass and momentum. As already mentioned, the Oldroyd-B model can be obtained as an exact closure of the linear Fokker-Planck equation, see, e.g., [36].

One of the possible ways how to model the behaviour of polymeric materials are the dilute solution theories that treat each polymer molecule separately. As we have already mentioned, see also [90], the simplest model representing this approach is the so-called dumbbell model consisting of two beads connected by a spring. In order to derive a closed system of equations we have to approximate the nonlinear spring force $\mathbf{F}(\mathbf{R}) = \gamma(|\mathbf{R}|^2)\mathbf{R}$.

In this thesis we consider the Peterlin approximation that replaces this force law by $\mathbf{F}(\mathbf{R}) = \gamma(\langle|\mathbf{R}|^2\rangle)\mathbf{R}$, where $\langle|\mathbf{R}|^2\rangle = \operatorname{tr}\mathbf{C}$ is the length of the average spring. Consequently, we can derive the evolution equation for the conformation tensor $\mathbf{C} := \langle\mathbf{R}\mathbf{R}\rangle$ in a closed form, see [122]. Here the notation

$$\langle f\rangle := \int_{\mathbb{R}^3} f(\mathbf{R})\psi(\mathbf{R},x,t)\,d\mathbf{R},$$

where $\psi(\mathbf{R},x,t)$ is a probability density function, is used.

General dumbbell-inspired model for polymers

In the recent papers of Renardy [125, 127] the yield stress behaviour of non-Newtonian fluids has been investigated. In particular, it has been shown how complex yield stress behaviour can arise as a limiting case of nonlinear viscoelastic constitutive theories. More precisely, a constitutive model which has two contributions to the stress is investigated. One of them evolves on a "fast" time scale, while the second on a "slow" time scale. The Newtonian contribution is the one evolving on a fast time scale. The other is a viscoelastic contribution, where the relaxation time is let to tend to infinity, described in terms of the conformation tensor $\mathbf{C}$. In [125, 127] a general dumbbell-inspired model for dilute polymer solutions

$$\begin{aligned}&\mathbf{T} = \psi(\operatorname{tr}\mathbf{C})\,\mathbf{C}\\ &\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C} - (\nabla\mathbf{u})\mathbf{C} - \mathbf{C}(\nabla\mathbf{u})^T = \chi(\operatorname{tr}\mathbf{C})\,\mathbf{I} - \phi(\operatorname{tr}\mathbf{C})\mathbf{C}\end{aligned}$$

has been proposed. Here ψ, χ, ϕ are in principle arbitrary functions of $\operatorname{tr}\mathbf{C}$ yielding constitutive relations for a particular viscoelastic fluid. As pointed out in [127] we can view this model as a modification of the upper convected Maxwell model, in which the relaxation time and viscosity depend on $\operatorname{tr}\mathbf{C}$. For example, as mentioned in [90], for the well-known PEC model, see [144], we would have that ϕ, χ are linear functions of $\operatorname{tr}\mathbf{C}$ and ψ is a function of $(\operatorname{tr}\mathbf{C})^{-1}$.

Inspired by [122, 124, 125, 126, 127], let us consider the system of equations describing the motion of an incompressible polymeric fluid, the so-called *general diffusive Peterlin model*,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} + \operatorname{div}\mathbf{T} - \nabla p \tag{3.1a}$$

$$\operatorname{div}\mathbf{u} = 0 \tag{3.1b}$$

$$\mathbf{T} = \psi(\operatorname{tr}\mathbf{C})\,\mathbf{C} \tag{3.1c}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C} - (\nabla\mathbf{u})\mathbf{C} - \mathbf{C}(\nabla\mathbf{u})^T = \chi(\operatorname{tr}\mathbf{C})\,\mathbf{I} - \phi(\operatorname{tr}\mathbf{C})\mathbf{C} + \varepsilon\Delta\mathbf{C}. \tag{3.1d}$$

In the constitutive equation (3.1d) we have moreover taken into account the diffusive effects arising due to a heterogeneous velocity field. As pointed out by Barrett and Süli [14], see also [43, 133], this fact yields x-dissipative terms in the microscopic model, i.e. in the Fokker-Planck equation, and consequently in the corresponding macroscopic model, i.e. in the constitutive equation describing time evolution of the conformation tensor. In [43] a careful justification of the presence of the diffusive terms in the Fokker-Planck equations through the asymptotic analysis is presented. The diffusion coefficient ε is proportional to $(\ell/L)^2/We$, where L and ℓ are characteristic macroscopic and microscopic length scales and We is the Weissenberg number. It is a reference number characterizing viscoelastic property of the material. Estimates for $(\ell/L)^2$ presented in [18] show that $(\ell/L)^2$ is in the range of about 10^{-9} and 10^{-7}. As emphasized in [14] the model reduction by neglecting these small diffusive effects is mathematically counter-productive leading to a degenerate parabolic-hyperbolic system (3.1) with $\varepsilon = 0$. On the other hand, when the diffusive terms are taken into account the resulting system (3.1) remains parabolic.

3.1. The diffusive Peterlin model

The main aim of this thesis is to analyse a model for some complex polymeric fluids, where the Peterlin approximation together with diffusive effects are used in order to describe the evolution equation for the elastic conformation tensor.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $T > 0$. We consider the system of equations on $\Omega \times (0, T)$ describing the unsteady motion of an incompressible viscoelastic fluid

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} + \operatorname{div} \mathbf{T} - \nabla p \tag{3.2a}$$

$$\operatorname{div} \mathbf{u} = 0 \tag{3.2b}$$

$$\mathbf{T} = \operatorname{tr} \mathbf{C}\, \mathbf{C} \tag{3.2c}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{C} - (\nabla \mathbf{u})\mathbf{C} - \mathbf{C}(\nabla \mathbf{u})^T = \operatorname{tr} \mathbf{C}\, \mathbf{I} - (\operatorname{tr} \mathbf{C})^2 \mathbf{C} + \varepsilon \Delta \mathbf{C}. \tag{3.2d}$$

Here $\mathbf{u}(x,t) \in \mathbb{R}^2$ and $p(x,t) \in \mathbb{R}$ denote the velocity of fluid and the pressure, respectively, for all $(x,t) \in \Omega \times [0,T)$. The equation (3.2c) gives the expression of the elastic stress tensor $\mathbf{T}(x,t) \in \mathbb{R}^{2\times 2}$ by the conformation tensor $\mathbf{C}(x,t) \in \mathbb{R}^{2\times 2}$, which is symmetric and positive definite for all $(x,t) \in \Omega \times [0,T)$. We impose the homogeneous Dirichlet boundary condition on $\mathbf{u}$ and the no-flux boundary condition on $\mathbf{C}$ on the boundary $\partial\Omega \times (0,T)$

$$\left(\mathbf{u}, \frac{\partial \mathbf{C}}{\partial \mathbf{n}}\right) = (\mathbf{0}, \mathbf{0})\,. \tag{3.2e}$$

We also prescribe the initial condition for the velocity and the conformation tensor

$$(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0)\,, \tag{3.2f}$$

where $\mathbf{u}_0 \in \mathbb{R}^2$ and $\mathbf{C}_0 \in \mathbb{R}^{2\times 2}$ are the enough smooth initial data defined on Ω. Given constants ν and ε describe the *fluid viscosity* and the *elastic stress viscosity*, respectively.

The diffusive Peterlin model (3.2), see [90], is derived from the dumbbell-inspired model for dilute polymer solutions (3.1) assuming that χ, ψ are linear functions of $\operatorname{tr} \mathbf{C}$, and allowing a nonlinear behaviour through a quadratic function ϕ.

Following Barrett and Süli [13] we would like to emphasize that the diffusive term appearing in the evolution equation for the conformation tensor is not a regularizing term but rather an outcome of physical modelling. Let us point out that in the analysis presented below we only require $\varepsilon > 0$ and there is no assumption on the size of ε.

3.2. Positive definiteness

In this section we prove that the parabolic equation (3.2d) preserves the positive definiteness of the conformation tensor for all $(x,t) \in \Omega \times (0,T)$ provided the initial datum $\mathbf{C}_0$ is a positive definite tensor.

Firstly, we recall the definition of the uniformly parabolic differential operator and the consequence of the strong minimum principle for the parabolic equations which will be used later in the proof.

Definition 3.1. *(uniformly parabolic differential operator)*
Let L denote for each time t a second-order partial differential operator in the non-divergence form, i.e.

$$Lw := -\sum_{i,j=1}^{d} A^{ij}(x,t)w_{x_i x_j}(x,t) + \sum_{i=1}^{d} B^i(x,t)w_{x_i}(x,t) + C(x,t)w(x,t)$$

for given coefficients A^{ij}, B^i and C, $i,j = 1,\dots,d$. We say that $\partial_t + L$ is a uniformly parabolic operator if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{d} A^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x,t) \in \Omega \times (0,T)$, $\xi \in \mathbb{R}^d$.

The strong minimum (maximum) principle for parabolic equations can be found in, e.g., [49, 78]. The following lemma on positivity of solutions to the parabolic initial boundary value problem is proven in [42], see Lemma A.4.

Lemma 3.2. *(positivity of solution to parabolic IBVP)*
Let $\partial_t + L$ be a uniformly parabolic operator with everywhere bounded and continuous coefficients A^{ij}, B^i, C, and let w be smooth on $\bar{\Omega} \times [0,T]$ satisfying

$$\begin{aligned}
&\partial_t w + Lw \geq 0 && \text{in } \ \Omega \times (0,T) \\
&\frac{\partial w}{\partial n} = 0 && \text{on } \ \partial\Omega \times (0,T) \\
&w = w_0 && \text{on } \ \Omega \times \{0\}
\end{aligned}$$

with $w_0(x) \geq 0$ not vanishing everywhere. If the function C is bounded from below, i.e.

$$m = \inf_{\bar{\Omega}\times(0,T)} C(x,t) > -\infty,$$

then $w > 0$ on $\Omega \times (0,T)$.

Preserving positive definiteness of C

Now, we prove that the smooth solution of (3.2d) remains positive definite provided the initial datum is positive definite.

Lemma 3.3. *(positive definiteness of the conformation tensor)*
Let $\mathbf{C} \in C^1([0,T];C^2(\bar{\Omega}))$ be a solution to (3.2d) satisfying the no-flux boundary condition. Let $\mathbf{u} \in C^1([0,T];C^2(\bar{\Omega}))$ be a divergence free vector field. If the initial datum $\mathbf{C}_0$ is a symmetric positive definite matrix, then the solution $\mathbf{C}(x,t)$ remains so for all $x \in \Omega$ and all times $t \in (0,T)$.

Proof. Let us rewrite equation (3.2d) for each component separately

$$\begin{aligned}
&\frac{\partial C_{11}}{\partial t} + (\mathbf{u}\cdot\nabla) C_{11} - 2\frac{\partial u_1}{\partial x_1} C_{11} - \frac{\partial u_1}{\partial x_2}(C_{12}+C_{21}) = \operatorname{tr}\mathbf{C} - (\operatorname{tr}\mathbf{C})^2 C_{11} + \varepsilon\Delta C_{11}\\
&\frac{\partial C_{12}}{\partial t} + (\mathbf{u}\cdot\nabla) C_{12} - (\operatorname{div}\mathbf{u}) C_{12} - \frac{\partial u_1}{\partial x_2} C_{22} - \frac{\partial u_2}{\partial x_1} C_{11} = -(\operatorname{tr}\mathbf{C})^2 C_{12} + \varepsilon\Delta C_{12}\\
&\frac{\partial C_{21}}{\partial t} + (\mathbf{u}\cdot\nabla) C_{21} - (\operatorname{div}\mathbf{u}) C_{21} - \frac{\partial u_1}{\partial x_2} C_{22} - \frac{\partial u_2}{\partial x_1} C_{11} = -(\operatorname{tr}\mathbf{C})^2 C_{21} + \varepsilon\Delta C_{21}\\
&\frac{\partial C_{22}}{\partial t} + (\mathbf{u}\cdot\nabla) C_{22} - 2\frac{\partial u_2}{\partial x_2} C_{22} - \frac{\partial u_2}{\partial x_1}(C_{12}+C_{21}) = \operatorname{tr}\mathbf{C} - (\operatorname{tr}\mathbf{C})^2 C_{22} + \varepsilon\Delta C_{22}.
\end{aligned}$$

Obviously, C_{12} and C_{21} satisfy the same equation, thus the solution of (3.2d) is a symmetric tensor. Now, we show that the determinant and the trace of $\mathbf{C}$ remain positive for all $(x,t) \in \Omega \times (0,T)$ following the idea used in [36] for the two-dimensional regularized Oldroyd-B model. Let us define the functions

$$\begin{aligned}
a(x,t) &:= \frac{1}{2}\left(C_{11}(x,t) - C_{22}(x,t)\right),\\
b(x,t) &:= C_{12}(x,t) = C_{21}(x,t),\\
c(x,t) &:= C_{11}(x,t) + C_{22}(x,t) = \operatorname{tr}\mathbf{C}(x,t).
\end{aligned}$$

The conformation tensor $\mathbf{C}$ can be expressed in terms of a, b, c by

$$\mathbf{C} = \begin{pmatrix} \frac{c}{2}+a & b\\ b & \frac{c}{2}-a \end{pmatrix}$$

with the determinant

$$\det\mathbf{C} = \frac{c^2}{4} - a^2 - b^2 =: d. \tag{3.3}$$

Taking into account $\operatorname{div}\mathbf{u} = 0$, we rewrite equation (3.2d) in terms of a, b and c

$$\begin{aligned}
&\frac{\partial a}{\partial t} + (\mathbf{u}\cdot\nabla) a - c\lambda + \omega b = -c^2 a + \varepsilon\Delta a\\
&\frac{\partial b}{\partial t} + (\mathbf{u}\cdot\nabla) b - \omega a - c\mu = -c^2 b + \varepsilon\Delta b\\
&\frac{\partial c}{\partial t} + (\mathbf{u}\cdot\nabla) c - 4(\lambda a + \mu b) = 2c - c^3 + \varepsilon\Delta c,
\end{aligned}$$

where λ, μ and ω represent the rate of strain and the vorticity, respectively, i.e.

$$\lambda(x,t) := \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right), \quad \mu(x,t) := \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right), \quad \omega(x,t) := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

We define the function

$$\gamma(x,t) := c - 2\sqrt{a^2+b^2}. \tag{3.4}$$

Without loss of generality we assume that either $a \neq 0$ or $b \neq 0$. Otherwise the conformation tensor $\mathbf{C}$ would be a diagonal matrix with the same components $C_{11} = C_{22}$, and such diagonal matrix is already included in the pressure part of the stress tensor, c.f. [68], [94]. Thus the function γ, as a smooth composition of $C^1\left([0,T];C^2(\Omega)\right)$ functions, is continuously differentiable in time and twice in space on $\Omega \times (0,T)$. Then the corresponding equation for γ on $\Omega \times (0,T)$ reads

$$\begin{aligned}
\frac{\mathrm{D}\gamma}{\mathrm{D}t} &= \frac{\mathrm{D}\left(c - 2\sqrt{a^2+b^2}\right)}{\mathrm{D}t} = \frac{\mathrm{D}c}{\mathrm{D}t} - \frac{2a}{\sqrt{a^2+b^2}}\frac{\mathrm{D}a}{\mathrm{D}t} - \frac{2b}{\sqrt{a^2+b^2}}\frac{\mathrm{D}b}{\mathrm{D}t} = \\
&= 4(\lambda a + \mu b) + 2c - c^3 + \varepsilon\Delta c - \frac{2a}{\sqrt{a^2+b^2}}\left(c\lambda - \omega b - c^2 a + \varepsilon\Delta a\right) - \\
&- \frac{2b}{\sqrt{a^2+b^2}}\left(\omega a + c\mu - c^2 b + \varepsilon\Delta b\right) = \\
&= 4(\lambda a + \mu b) + 2c - c^3 + \varepsilon\Delta c - \frac{2a\,(c\lambda - \omega b) + 2b\,(\omega a + c\mu)}{\sqrt{a^2+b^2}} + \frac{2c^2\,(a^2+b^2)}{\sqrt{a^2+b^2}} - \\
&- \frac{2\varepsilon\,(a\Delta a + b\Delta b)}{\sqrt{a^2+b^2}} = \\
&= 4(\lambda a + \mu b) + 2\gamma + 4\sqrt{a^2+b^2} - c^3 - \frac{2c\,(\lambda a + \mu b)}{\sqrt{a^2+b^2}} + 2c^2\sqrt{a^2+b^2} + \\
&+ \varepsilon\Delta c - \frac{2\varepsilon\,(a\Delta a + b\Delta b)}{\sqrt{a^2+b^2}} = \\
&= -\left(c^2 + \frac{2(\lambda a + \mu b)}{\sqrt{a^2+b^2}} - 2\right)\gamma + 4\sqrt{a^2+b^2} + \varepsilon\Delta c - \frac{2\varepsilon\,(a\Delta a + b\Delta b)}{\sqrt{a^2+b^2}}.
\end{aligned} \tag{3.5}$$

The following equality

$$\begin{aligned}
\Delta\left(\sqrt{a^2+b^2}\right) &= \sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\left(\sqrt{a^2+b^2}\right) = \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\frac{a\frac{\partial a}{\partial x_i} + b\frac{\partial b}{\partial x_i}}{\sqrt{a^2+b^2}}\right) = \\
&= \frac{a\Delta a + b\Delta b}{\sqrt{a^2+b^2}} + \frac{|\nabla a|^2 + |\nabla b|^2}{\sqrt{a^2+b^2}} - \frac{\sum_{i=1}^{2}\left(a\frac{\partial a}{\partial x_i} + b\frac{\partial b}{\partial x_i}\right)^2}{\left(a^2+b^2\right)^{3/2}} = \\
&= \frac{a\Delta a + b\Delta b}{\sqrt{a^2+b^2}} + \frac{\sum_{i=1}^{2}\left(a\frac{\partial a}{\partial x_i} - b\frac{\partial b}{\partial x_i}\right)^2}{\left(a^2+b^2\right)^{3/2}}
\end{aligned}$$

yields

$$-\frac{a\Delta a + b\Delta b}{\sqrt{a^2+b^2}} \geq -\Delta\left(\sqrt{a^2+b^2}\right),$$

and consequently

$$\varepsilon\Delta c - \frac{2\varepsilon\,(a\Delta a + b\Delta b)}{\sqrt{a^2+b^2}} \geq \varepsilon\Delta c - \varepsilon\Delta\left(2\sqrt{a^2+b^2}\right) = \varepsilon\Delta\gamma. \tag{3.6}$$

Taking into account equation (3.5) and inequality (3.6) we can write

$$\frac{\partial\gamma}{\partial t} + (\mathbf{u}\cdot\nabla)\,\gamma + \left(c^2 + \frac{2(\lambda a + \mu b)}{\sqrt{a^2+b^2}} - 2\right)\gamma - \varepsilon\Delta\gamma \geq 4\sqrt{a^2+b^2} \geq 0. \tag{3.7}$$

According to Definition 3.1 inequality (3.7) can be seen on $\Omega \times (0,T)$ as

$$\partial_t \gamma + L\gamma \geq 0,$$

where $\partial_t + L$ is the uniformly parabolic operator with $\theta = \varepsilon$ and the coefficients

$$A^{ij} = 0 \text{ for } i \neq j \text{ and } A^{ii} = \varepsilon,\ B^i = u_i,\ C = c^2 + \frac{2(\lambda a + \mu b)}{\sqrt{a^2+b^2}} - 2,$$

that are continuous and bounded due to the regularity of $\mathbf{C}$ and $\mathbf{u}$. Moreover, there exists a constant m such that $C(x,t) \geq m$ on $\bar{\Omega} \times (0,T)$.

Further, the conformation tensor satisfies the no-flux boundary condition

$$\frac{\partial \mathbf{C}}{\partial \mathbf{n}} = \mathbf{0},\ \text{ i.e. } \sum_{k=1}^{2} \frac{\partial C_{ij}}{\partial x_k} n_k = 0 \text{ and thus } \frac{\partial C_{ij}}{\partial \mathbf{n}} = 0 \text{ for all } i,j = 1,2.$$

As a consequence we get

$$\begin{aligned}\frac{\partial \gamma}{\partial \mathbf{n}} &= \frac{\partial c}{\partial \mathbf{n}} - \frac{\partial\left(2\sqrt{a^2+b^2}\right)}{\partial \mathbf{n}} = \frac{\partial c}{\partial \mathbf{n}} - \frac{2a}{\sqrt{a^2+b^2}}\frac{\partial a}{\partial \mathbf{n}} - \frac{2b}{\sqrt{a^2+b^2}}\frac{\partial b}{\partial \mathbf{n}} = \\ &= \frac{\partial C_{11}}{\partial \mathbf{n}} + \frac{\partial C_{22}}{\partial \mathbf{n}} - \frac{1}{\sqrt{a^2+b^2}}\frac{\partial C_{11}}{\partial \mathbf{n}} + \frac{1}{\sqrt{a^2+b^2}}\frac{\partial C_{22}}{\partial \mathbf{n}} - \frac{2}{\sqrt{a^2+b^2}}\frac{\partial C_{12}}{\partial \mathbf{n}} = 0\end{aligned}$$

on $\partial\Omega \times (0,T)$.

The last step is to verify if the initial datum obeys $\gamma_0 = \gamma(x,0) \geq 0$. Since $\mathbf{C}_0$ is a positive definite matrix, its trace $c_0 = \operatorname{tr} \mathbf{C}_0$ and its determinant $d_0 = \det \mathbf{C}_0$ are positive. It follows from (3.3) that $c^2 = 4\left(a^2+b^2+d\right)$ and thus

$$\gamma_0 = 2\sqrt{(a_0)^2 + (b_0)^2 + d_0} - 2\sqrt{(a_0)^2+(b_0)^2} \geq 0.$$

Finally, Lemma 3.2 yields $\gamma(x,t) > 0$ for all $(x,t) \in \Omega \times (0,T)$, and by (3.4) we have $c > 2\sqrt{a^2+b^2}$. This implies $c > 0$ and $c^2 > 4(a^2+b^2)$. Hence $c = \operatorname{tr}\mathbf{C}$ and $d = \det \mathbf{C}$ are positive and consequently both eigenvalues are positive. Since $\mathbf{C}$ is symmetric, we can conclude that $\mathbf{C}(x,t)$ is a positive definite tensor for all $(x,t) \in \Omega \times (0,T)$. □

Remark 3.4. *The above result is true in any dimension. Indeed, one can show that many classical models for viscoelastic fluids preserve the positive definiteness of the conformation tensor, see, e.g., [70], [21]. Following their technique we can show that the positive definiteness is also preserved by the Peterlin model for $\varepsilon = 0$. On the other hand, the solution of the heat equation remains positive due to the maximum principle. Consequently, applying the operator-splitting approach we obtain the desired positive definiteness of the conformation tensor for the diffusive ($\varepsilon \neq 0$) Peterlin model.*

3.3. Formal energy estimates

The most difficult task in finding a priori estimates for the diffusive Peterlin model is to treat the nonlinear terms arising from the coupling of the equation for the velocity (3.2a) and the equation for the conformation tensor (3.2d).

There are two approaches to show the formal energy estimates in two space dimensions. Firstly, we present the way valid only for the two-dimensional case, which can be found in Section 2 of [90]. Secondly, the general approach for any dimension $d \geq 2$ is demonstrated.

Two-dimensional case

The relevant nonlinear terms in the diffusive Peterlin model can be controlled by the following identity.

Proposition 3.5.
Let $\mathbf{C} \in \mathbb{R}^{2\times 2}$ *be a symmetric tensor and let* $\mathbf{u} \in \mathbb{R}^2$ *be a solenoidal vector field. Then the following identity holds true*

$$\operatorname{tr}\mathbf{C}\,\mathbf{C} : \nabla\mathbf{u} = \frac{1}{2}\left[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\right] : \mathbf{C}. \tag{3.8}$$

Proof. Let us rewrite the left-hand side of (3.8), using the symmetry of $\mathbf{C}$, in the following way

$$\operatorname{tr}\mathbf{C}\,\mathbf{C} : \nabla\mathbf{u} = \operatorname{tr}\mathbf{C}\sum_{i,j=1}^{2} C_{ij}\frac{\partial u_i}{\partial x_j} = \sum_{i=1}^{2} C_{ii}^2\frac{\partial u_i}{\partial x_i} + \operatorname{tr}\mathbf{C}\,C_{12}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + C_{11}C_{22}\operatorname{div}\mathbf{u}.$$

The right-hand side of (3.8) can be rewritten in an analogous way

$$\begin{aligned}\frac{1}{2}\left[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\right] : \mathbf{C} &= \sum_{i,j,k=1}^{2}\frac{\partial u_i}{\partial x_k}C_{kj}C_{ij} + C_{ik}\frac{\partial u_j}{\partial x_k}C_{ij}\\ &= \sum_{i=1}^{2} C_{ii}^2\frac{\partial u_i}{\partial x_i} + \operatorname{tr}\mathbf{C}\,C_{12}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + C_{12}^2\operatorname{div}\mathbf{u}.\end{aligned}$$

Since $\operatorname{div}\mathbf{u} = 0$, we can conclude that identity (3.8) holds true. □

Now, let us derive the energy estimates. We multiply the momentum equation (3.2a) by $\mathbf{u}$, integrate over Ω and use the Green formula

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2\,dx - \underbrace{\frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}|\mathbf{u}|^2\,dx}_{=0} + \underbrace{\frac{1}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})|\mathbf{u}|^2\,dS}_{=0} =\\ &= -\nu\int_\Omega|\nabla\mathbf{u}|^2\,dx + \underbrace{\nu\int_{\partial\Omega}(\mathbf{n}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,dS}_{=0} - \int_\Omega\mathbf{T} : \nabla\mathbf{u}\,dx + \underbrace{\int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{T})\cdot\mathbf{u}^T\,dS}_{=0} +\\ &+ \underbrace{\int_\Omega\operatorname{div}\mathbf{u}\,p\,dx}_{=0} - \underbrace{\int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})p\,dS}_{=0}.\end{aligned}$$

The solenoidal velocity and the homogeneous Dirichlet boundary condition for $\mathbf{u}$ yield the following equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega|\mathbf{u}|^2\,dx = -\nu\int_\Omega|\nabla\mathbf{u}|^2\,dx - \int_\Omega\mathbf{T} : \nabla\mathbf{u}\,dx\,. \tag{3.9}$$

Further, we multiply (3.2d) by $\frac{1}{2}\mathbf{C}$ and integrate this equation. Using the Green formula we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}|^2 \, dx - \underbrace{\frac{1}{4}\int_\Omega \operatorname{div}\mathbf{u}\,\mathbf{C}:\mathbf{C}\, dx}_{=0} + \underbrace{\frac{1}{4}\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})\mathbf{C}:\mathbf{C}\, dS}_{=0} - \\
&- \frac{1}{2}\int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C}\, dx = \frac{1}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^2 \, dx - \frac{1}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^2\mathbf{C}:\mathbf{C}\, dx - \\
&- \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx + \underbrace{\frac{\varepsilon}{2}\int_{\partial\Omega} \mathbf{C}(\nabla\mathbf{C}\cdot\mathbf{n})\, dS}_{=0}\,.
\end{aligned}$$

Due to the divergence freedom of the velocity and the no-flux boundary condition for the conformation tensor we have

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}|^2 \, dx - \frac{1}{2}\int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C}\, dx = \\
&\qquad = \frac{1}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^2 \, dx - \frac{1}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^2\mathbf{C}:\mathbf{C}\, dx - \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx\,.
\end{aligned} \tag{3.10}$$

Summing up equations (3.9) and (3.10) yields the following equality

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}|^2 \, dx + \nu\int_\Omega |\nabla\mathbf{u}|^2 \, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx + \\
&\qquad + \frac{1}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^2\mathbf{C}:\mathbf{C}\, dx = \frac{1}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^2 \, dx - \\
&\qquad \underbrace{- \int_\Omega \operatorname{tr}\mathbf{C}\,\mathbf{C}:\nabla\mathbf{u}\, dx + \frac{1}{2}\int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C}\, dx}_{=0}\,.
\end{aligned}$$

Let us note that the nonlinear term arising from the divergence of the elastic stress tensor $\mathbf{T}$ in the momentum equation (3.2a) and the nonlinear term appearing in the Oldroyd derivative of the conformation tensor in equation (3.2d) are cancelled due to (3.8). We recall that $\mathbf{T}^2 = (\operatorname{tr}\mathbf{C}\,\mathbf{C}) : (\operatorname{tr}\mathbf{C}\,\mathbf{C}) = (\operatorname{tr}\mathbf{C})^2\,\mathbf{C}:\mathbf{C}$ and $(\operatorname{tr}\mathbf{C})^2 \leq 2\mathbf{C}^2$, see (2.9a). Hence we obtain the energy equality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4}\int_\Omega |\mathbf{C}|^2 \, dx\right) + \nu\int_\Omega |\nabla\mathbf{u}|^2 \, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx + \\
&\qquad + \frac{1}{2}\int_\Omega |\mathbf{T}|^2 \, dx = \int_\Omega |\mathbf{C}|^2 \, dx\,.
\end{aligned} \tag{3.11}$$

By the Gronwall inequality in differential form, cf. (2.5), for $f = 4$ and

$$\varphi(t) = \frac{1}{2}\int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4}\int_\Omega |\mathbf{C}|^2 \, dx\,,$$

we have,

$$\frac{1}{2}\int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4}\int_\Omega |\mathbf{C}|^2 \, dx \leq e^{4t}\left\{\frac{1}{2}\int_\Omega |\mathbf{u}_0|^2 \, dx + \frac{1}{4}\int_\Omega |\mathbf{C}_0|^2 \, dx\right\},\; t\in[0,T].$$

Therefore

$$\sup_{t\in[0,T]} \|\mathbf{u}\|_2^2 + \sup_{t\in[0,T]} \|\mathbf{C}\|_2^2 \leq c(T)\left(\|\mathbf{u}_0\|_2^2 + \|\mathbf{C}_0\|_2^2\right). \tag{3.12}$$

Consequently, by integrating (3.11) in time over $[0,T]$ we obtain

$$\nu \|\nabla\mathbf{u}\|_{L^2(L^2)}^2 + \frac{\varepsilon}{2}\|\nabla\mathbf{C}\|_{L^2(L^2)}^2 + \frac{1}{2}\|\mathbf{T}\|_{L^2(L^2)}^2 \leq c(T)\left(\|\mathbf{u}_0\|_2^2 + \|\mathbf{C}_0\|_2^2\right). \tag{3.13}$$

To conclude, inequalities (3.12) and (3.13) indicate that the following functional spaces are appropriate for the solution to the problem (3.2)

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1_{0,div}(\Omega)^2),\\
&\mathbf{C} \in L^\infty(0,T;L^2(\Omega)^{2\times2}) \cap L^2(0,T;H^1(\Omega)^{2\times2})\\
&\mathbf{T} \in L^2(0,T;L^2(\Omega)^{2\times2}).
\end{aligned}$$

Let us point out that the above a priori bounds together with the interpolation inequality (2.8b) yield $\mathbf{C} \in L^4(0,T;L^4(\Omega)^{2\times2})$. Indeed, it holds that

$$\begin{aligned}
\int_0^T \|\mathbf{C}\|_4^4 \, dt &\leq c\int_0^T \|\mathbf{C}\|_2^4 + \|\mathbf{C}\|_2^2\|\nabla\mathbf{C}\|_2^2 \, dt\\
&\leq c(T)\|\mathbf{C}\|_{L^\infty(L^2)}^4 + c\|\mathbf{C}\|_{L^\infty(L^2)}^2\|\mathbf{C}\|_{L^2(H^1)}^2.
\end{aligned}$$

The advantage of this approach is no need of the positive definiteness of the conformation tensor, since this property is lost at the level of weak solutions. That is the reason why we shall follow this idea in order to show the existence of a weak solution.

General case

For any dimension $d \geq 2$ the formal energy estimates can be found employing the positive definiteness of the conformation tensor. First, we introduce an analogous identity to (3.8) which holds for any symmetric tensor $\mathbf{C}$ and any $\mathbf{u}$ in any dimension.

Proposition 3.6.
Let $d \geq 2$, $\mathbf{C} \in \mathbb{R}^{d\times d}$ be a symmetric tensor and let $\mathbf{u} \in \mathbb{R}^d$. Then the following identity holds true

$$\operatorname{tr}\mathbf{C}\,\mathbf{C}:\nabla\mathbf{u} = \frac{1}{2}\operatorname{tr}\mathbf{C}\operatorname{tr}\left((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\right). \tag{3.14}$$

Proof. Let us recall that the trace of a matrix product does not depend on the order of matrices. Then by (2.2) and the symmetry of $\mathbf{C}$ we have

$$\mathbf{C}:\nabla\mathbf{u} = \operatorname{tr}\left(\mathbf{C}(\nabla\mathbf{u})^T\right) = \operatorname{tr}\left((\nabla\mathbf{u})\mathbf{C}\right),$$

and consequently

$$\operatorname{tr}\left((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\right) = 2\mathbf{C}:\nabla\mathbf{u}.$$

Then it is obvious that (3.14) holds true. □

Now, for any $d \geq 2$, we show the energy estimates using (3.14) and the positive definiteness of $\mathbf{C}$. The first step is exactly the same as in the two-dimensional case resulting in the energy equality for velocity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \, dx = -\nu \int_\Omega |\nabla \mathbf{u}|^2 \, dx - \int_\Omega \mathbf{T} : \nabla \mathbf{u} \, dx \,. \tag{3.15}$$

The second step is slightly different. We multiply equation (3.2d) by $\operatorname{tr}\mathbf{C}$, take half the trace of the multiplied equation and integrate

$$\begin{aligned} \frac{1}{2}\int_\Omega \operatorname{tr}\mathbf{C}\,\frac{\partial \operatorname{tr}\mathbf{C}}{\partial t}\, dx + \frac{1}{2}\int_\Omega \operatorname{tr}\mathbf{C}\,(\mathbf{u}\cdot\nabla)\operatorname{tr}\mathbf{C}\, dx - \frac{1}{2}\int_\Omega \operatorname{tr}\mathbf{C}\operatorname{tr}\Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big)\, dx = \\ = \frac{1}{2}\int_\Omega d|\operatorname{tr}\mathbf{C}|^2\, dx - \frac{1}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^4\, dx + \frac{\varepsilon}{2}\int_\Omega \operatorname{tr}\mathbf{C}\,\Delta \operatorname{tr}\mathbf{C}\, dx \,. \end{aligned}$$

Using the Green formula, the divergence freedom of velocity and the no-flux boundary condition for the conformation tensor we have

$$\begin{aligned} \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx - \underbrace{\frac{1}{4}\int_\Omega \operatorname{div}\mathbf{u}|\operatorname{tr}\mathbf{C}|^2\, dx}_{=0} + \underbrace{\frac{1}{4}\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})|\operatorname{tr}\mathbf{C}|^2\, dS}_{=0} - \\ - \frac{1}{2}\int_\Omega \operatorname{tr}\mathbf{C}\operatorname{tr}\Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big)\, dx = \frac{1}{2}\int_\Omega d|\operatorname{tr}\mathbf{C}|^2\, dx - \\ - \frac{1}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^4\, dx - \frac{\varepsilon}{4}\int_\Omega |\nabla \operatorname{tr}\mathbf{C}|^2\, dx + \underbrace{\frac{\varepsilon}{4}\int_{\partial\Omega} \operatorname{tr}\mathbf{C}\,(\nabla \operatorname{tr}\mathbf{C}\cdot\mathbf{n})\, dS}_{=0}, \end{aligned}$$

which now yields the energy equality for the trace of the conformation tensor

$$\begin{aligned} \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx - \frac{1}{2}\int_\Omega \operatorname{tr}\mathbf{C}\operatorname{tr}\Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big)\, dx = \\ = \frac{d}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx - \frac{1}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^4\, dx - \frac{\varepsilon}{4}\int_\Omega |\nabla \operatorname{tr}\mathbf{C}|^2\, dx \,. \end{aligned} \tag{3.16}$$

Thus, adding equations (3.15) and (3.16) together, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2\, dx + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx + \nu\int_\Omega |\nabla\mathbf{u}|^2\, dx + \frac{\varepsilon}{4}\int_\Omega |\nabla \operatorname{tr}\mathbf{C}|^2\, dx + \\ + \frac{1}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^4\, dx = \frac{d}{2}\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx - \\ \underbrace{- \int_\Omega \operatorname{tr}\mathbf{C}\,\mathbf{C} : \nabla\mathbf{u}\, dx + \frac{1}{2}\int_\Omega \operatorname{tr}\mathbf{C}\operatorname{tr}\Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big)\, dx}_{=0}, \end{aligned}$$

where the nonlinear terms arising from the coupling of equations (3.2a) and (3.2d) are now cancelled due to identity (3.14). Similarly as in the two-dimensional case we get, by the Gronwall inequality (2.5),

$$\frac{1}{2}\int_\Omega |\mathbf{u}|^2\, dx + \frac{1}{4}\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx \leq e^{2dt}\left\{\frac{1}{2}\int_\Omega |\mathbf{u}_0|^2\, dx + \frac{1}{4}\int_\Omega |\operatorname{tr}\mathbf{C}_0|^2\, dx\right\},\ t\in[0,T],$$

and consequently

$$\sup_{t\in[0,T]} \|\mathbf{u}\|_2^2 + \sup_{t\in[0,T]} \|\mathrm{tr}\,\mathbf{C}\|_2^2 + \nu \|\nabla \mathbf{u}\|^2_{L^2(L^2)} + \varepsilon \|\nabla \mathrm{tr}\,\mathbf{C}\|^2_{L^2(L^2)} + \|\mathrm{tr}\,\mathbf{C}\|^4_{L^4(L^4)} \le c(d,T)\left(\|\mathbf{u}_0\|_2^2 + \|\mathrm{tr}\,\mathbf{C}_0\|_2^2\right).$$

Therefore the proper functional spaces for the velocity and the trace of the conformation tensor are

$$\begin{aligned} &\mathbf{u} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;H^1_{0,div}(\Omega)^d) \\ &\mathrm{tr}\,\mathbf{C} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^4(0,T;L^4(\Omega)). \end{aligned} \tag{3.17a}$$

The positive definiteness of the conformation tensor $\mathbf{C}$, by the norm equivalence (2.9a), yields a priori bound

$$\mathbf{C} \in L^4(0,T;L^4(\Omega)^{d\times d}). \tag{3.17b}$$

In order to bound $\nabla\mathbf{C}$ in the proper norm we need one more step. Multiplying (3.2d) by $\mathbf{C}$, integrating by employing the Green formula, using $\mathrm{div}\,\mathbf{u} = 0$ and applying the boundary condition for $\mathbf{C}$ we get the energy equality for the conformation tensor

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega |\mathbf{C}|^2\,dx + \varepsilon \int_\Omega |\nabla \mathbf{C}|^2\,dx + \int_\Omega |\mathbf{T}|^2\,dx &= \\ = \int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\,dx + \int_\Omega \left[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\right] : \mathbf{C}\,dx\,. \end{aligned} \tag{3.18}$$

The problematic nonlinear term arising from the Oldroyd derivative can be now estimated, by the Hölder and the Young inequalities, as follows

$$\int_\Omega \left[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\right] : \mathbf{C}\,dx \le c\int_\Omega |\nabla\mathbf{u}|^2\,dx + c\int_\Omega |\mathbf{C}|^4\,dx\,.$$

Integrating equality (3.18) in time yields

$$\begin{aligned} &\frac{1}{2}\|\mathbf{C}\|_2^2 + \varepsilon\int_0^t\int_\Omega |\nabla\mathbf{C}|^2\,dx\,dt + \int_0^t\int_\Omega |\mathbf{T}|^2\,dx\,dt \\ &\quad\le \frac{1}{2}\|\mathbf{C}_0\|_2^2 + \int_0^t\int_\Omega |\mathbf{C}|^2\,dx\,dt + c\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,dx\,dt + c\int_0^t\int_\Omega |\mathbf{C}|^4\,dx\,dt\,. \end{aligned}$$

Then the integral form of the Gronwall inequality, cf. (2.6), for $f = 2$,

$$\begin{aligned} \varphi(t) &= \frac{1}{2}\|\mathbf{C}\|_2^2\,, \\ g(t) &= \frac{1}{2}\|\mathbf{C}_0\|_2^2 + c\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,dx\,dt + c\int_0^t\int_\Omega |\mathbf{C}|^4\,dx\,dt\,, \end{aligned}$$

yields

$$\sup_{t\in[0,T]} \|\mathbf{C}\|_2^2 \le c(T)\left(\|\mathbf{C}_0\|_2^2 + \|\nabla\mathbf{u}\|^2_{L^2(L^2)} + \|\mathbf{C}\|^4_{L^4(L^4)}\right), \tag{3.19}$$

and consequently

$$\varepsilon \|\nabla \mathbf{C}\|^2_{L^2(L^2)} + \|\mathbf{T}\|^2_{L^2(L^2)} \le c(T)\left(\|\mathbf{C}_0\|^2_2 + \|\nabla \mathbf{u}\|^2_{L^2(L^2)} + \|\mathbf{C}\|^4_{L^4(L^4)}\right). \tag{3.20}$$

Taking into account (3.19), (3.20) and the above bounds (3.17) we can conclude

$$\mathbf{C} \in L^\infty(0,T;L^2(\Omega)^{d\times d}) \cap L^2(0,T;H^1(\Omega)^{d\times d}), \qquad \mathbf{T} \in L^2(0,T;L^2(\Omega)^{d\times d}).$$

We have presented two different ways how to derive the formal a priori estimates for the diffusive Peterlin model. The final result for $d = 2$ is

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1_{0,div}(\Omega)^2)\\
&\mathbf{C} \in L^\infty(0,T;L^2(\Omega)^{2\times 2}) \cap L^2(0,T;H^1(\Omega)^{2\times 2}) \cap L^4(0,T;L^4(\Omega)^{2\times 2})\\
&\mathbf{T} \in L^2(0,T;L^2(\Omega)^{2\times 2}).
\end{aligned}$$

3.4. Free energy estimate

The free energy can be useful to study the long-time behaviour of solutions or to investigate the stability of numerical schemes. For the Oldroyd-B model in both the conformation tensor formulation and the logarithmic formulation the free energy is presented in [21]. The entropy methods used to characterize the long-time behaviour of some micro-macro models for polymeric fluids (the Hookean or the FENE dumbbell models) can be found in [74]. For macroscopic models (the Oldroyd-B or the FENE-P models) the a priori estimate based on the free energy is derived in [69]. We refer to [21, 69, 71, 74, 86] and the references therein for more details.

In what follows we present the so-called free energy estimate for the diffusive Peterlin model in two space dimensions. To this end we need the properties of the symmetric positive definite matrices comprised in Lemmas 3.7, 3.8 and 3.9 below. These results can be found in, e.g., [21], [132].

Lemma 3.7.
Let $\mathbf{D} \in \mathcal{S}^2_+$. Then

$$\operatorname{tr}\, \ln \mathbf{D} = \ln \det \mathbf{D} \tag{3.21a}$$

$$\operatorname{tr} \mathbf{D}\, \mathbf{D} - 2\ln \mathbf{D} - \mathbf{I} \textit{ is symmetric and } \operatorname{tr}\, (\operatorname{tr} \mathbf{D}\, \mathbf{D} - 2\ln \mathbf{D} - \mathbf{I}) \ge 0 \tag{3.21b}$$

$$\mathbf{D} + \mathbf{D}^{-1} - 2\mathbf{I} \textit{ is symmetric and } \operatorname{tr}\, \left(\mathbf{D} + \mathbf{D}^{-1} - 2\mathbf{I}\right) \ge 0. \tag{3.21c}$$

Proof. The logarithm of $\mathbf{D}$ is by (2.10) defined as $\ln \mathbf{D} = \mathbf{Q} \ln \mathbf{\Lambda} \mathbf{Q}^T$. Since $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and the trace is invariant under cyclic permutations, we can write

$$\operatorname{tr}\, \ln \mathbf{D} = \operatorname{tr}\, \left(\mathbf{Q} \ln \mathbf{\Lambda} \mathbf{Q}^T\right) = \operatorname{tr}\, \left(\mathbf{Q}^T \mathbf{Q} \ln \mathbf{\Lambda}\right) = \operatorname{tr}\, \ln \mathbf{\Lambda} = \ln \lambda_1 + \ln \lambda_2.$$

On the other hand

$$\ln \det \mathbf{D} = \ln(\lambda_1 \lambda_2) = \ln \lambda_1 + \ln \lambda_2,$$

since $\ln(xy) = \ln x + \ln y$ for all $x, y > 0$. Hence (3.21a) holds true.

If $\mathbf{D} = \mathbf{D}^T$ then $\ln \mathbf{D}$, $\mathbf{D}^{-1}$, and consequently $\operatorname{tr} \mathbf{D}\, \mathbf{D} - 2 \ln \mathbf{D} - \mathbf{I}$ and $\mathbf{D} + \mathbf{D}^{-1} - 2\mathbf{I}$ are symmetric matrices. Their eigenvalues read

$$\mu_i = (\lambda_1 + \lambda_2)\, \lambda_i - 2 \ln \lambda_i - 1, \qquad \xi_i = \lambda_i + \frac{1}{\lambda_i} - 2,$$

for $i = 1, 2$, respectively. Here $\lambda_i > 0$, $i = 1, 2$ are the eigenvalues of $\mathbf{D}$. Since $x - 1 \geq \ln x$ and $k \ln x = \ln x^k$ for $x > 0$, we have

$$\operatorname{tr}\,(\operatorname{tr} \mathbf{D}\, \mathbf{D} - 2 \ln \mathbf{D} - \mathbf{I}) = \mu_1 + \mu_2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 - \ln \lambda_1^2 - \ln \lambda_2^2 - 2 \geq 0.$$

Similarly, by the inequality $x + 1/x \geq 2$ for all $x > 0$ we get

$$\operatorname{tr}\left(\mathbf{D} + \mathbf{D}^{-1} - 2\mathbf{I}\right) = \xi_1 + \xi_2 = \sum_{i=1}^{2} \lambda_i + \frac{1}{\lambda_i} - 4 \geq 0.$$

This concludes the proof. $\square$

Let us note that in comparison with Lemma 1.1. in [21] the matrix in (3.21b) is slightly modified to fit our needs.

Lemma 3.8. *(the Jacobi formula)*
For any symmetric positive definite matrix $\mathbf{D}(t) \in C^1([0, T))$ *we have, for all* $t \in [0, T)$,

$$\left(\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) : \mathbf{D}^{-1} = \operatorname{tr}\left(\mathbf{D}^{-1}\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}\, \ln \mathbf{D} \tag{3.22a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathbf{D}\right) : \mathbf{D} = \operatorname{tr}\left(\mathbf{D}\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathbf{D}\right) = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr} \mathbf{D}. \tag{3.22b}$$

Proof. The matrix $\mathbf{D} \in \mathcal{S}_+^d$ can be decomposed for all $t \in [0, T)$ as

$$\mathbf{D}(t) = \mathbf{Q}(t)\boldsymbol{\Lambda}(t)\mathbf{Q}(t)^T,$$

where the orthogonal matrix $\mathbf{Q}(t)$ and the diagonal matrix $\boldsymbol{\Lambda}(t)$ are continuously differentiable. Let us compute the derivatives

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\boldsymbol{\Lambda}\mathbf{Q}^T + \mathbf{Q}\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}t}\mathbf{Q}^T + \mathbf{Q}\boldsymbol{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t} \\
\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathbf{D} &= \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} \ln \boldsymbol{\Lambda}\mathbf{Q}^T + \mathbf{Q}\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}t}\boldsymbol{\Lambda}^{-1}\mathbf{Q}^T + \mathbf{Q} \ln \boldsymbol{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}.
\end{aligned}$$

Taking the trace we get

$$\begin{aligned}
\operatorname{tr}\left(\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) &= \operatorname{tr}\left(\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}t}\right) + \operatorname{tr}\left(\boldsymbol{\Lambda}\frac{\mathrm{d}\left(\mathbf{Q}^T\mathbf{Q}\right)}{\mathrm{d}t}\right) = \operatorname{tr}\left(\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}t}\right) \\
\operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t} \ln \mathbf{D}\right) &= \operatorname{tr}\left(\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}t}\boldsymbol{\Lambda}^{-1}\right) + \operatorname{tr}\left(\ln \boldsymbol{\Lambda}\frac{\mathrm{d}\left(\mathbf{Q}^T\mathbf{Q}\right)}{\mathrm{d}t}\right) = \operatorname{tr}\left(\frac{\mathrm{d} \ln \boldsymbol{\Lambda}}{\mathrm{d}t}\right),
\end{aligned}$$

since $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ for all times t. Thus, identity (2.2) and the cyclic property of trace yield

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{D}^{-1}\frac{d}{dt}\mathbf{D}\right) = \left(\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) : \mathbf{D}^{-1} &= \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\mathbf{\Lambda}\mathbf{Q}^T + \mathbf{Q}\frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}t}\mathbf{Q}^T + \mathbf{Q}\mathbf{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}\right) : \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^T \\
&= \operatorname{tr}\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\mathbf{Q}^T + \mathbf{Q}\frac{\mathrm{d}\ln\mathbf{\Lambda}}{\mathrm{d}t}\mathbf{Q}^T + \mathbf{Q}\mathbf{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}\mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^T\right) \\
&= \operatorname{tr}\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\mathbf{Q}^T + \frac{\mathrm{d}\ln\mathbf{\Lambda}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}\mathbf{Q}\right) \\
&= \operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{dt}}\ln\mathbf{D}\right) = \frac{\mathrm{d}}{\mathrm{dt}}\operatorname{tr}\,\ln\mathbf{D}.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{D}\frac{d}{dt}\ln\mathbf{D}\right) = \left(\frac{d}{dt}\ln\mathbf{D}\right) : \mathbf{D} &= \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\ln\mathbf{\Lambda}\mathbf{Q}^T + \mathbf{Q}\frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}t}\mathbf{\Lambda}^{-1}\mathbf{Q}^T + \mathbf{Q}\ln\mathbf{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}\right) : \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T \\
&= \operatorname{tr}\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\ln\mathbf{\Lambda}\mathbf{\Lambda}\mathbf{Q}^T + \mathbf{Q}\frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}t}\mathbf{Q}^T + \mathbf{Q}\ln\mathbf{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T\right) \\
&= \operatorname{tr}\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\ln\mathbf{\Lambda}\mathbf{\Lambda}\mathbf{Q}^T + \frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}t} + \ln\mathbf{\Lambda}\frac{\mathrm{d}\mathbf{Q}^T}{\mathrm{d}t}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T\right) \\
&= \operatorname{tr}\left(\frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{dt}}\operatorname{tr}\mathbf{D}.
\end{aligned}$$

$\square$

The following result and its proof can be found in [93, 132].

Lemma 3.9.
Let $\mathbf{B}$, $\mathbf{D} \in \mathcal{S}^d_+$ and let $f : (0,\infty) \to \mathbb{R}$ be a decreasing function. Then it holds that

$$(\mathbf{B} - \mathbf{D}) : (f(\mathbf{B}) - f(\mathbf{D})) \leq 0 \tag{3.23a}$$
$$\nabla\mathbf{B} : \nabla\mathbf{B}^{-1} \leq 0. \tag{3.23b}$$

Proof. Let us introduce the orthogonal decompositions of given matrices

$$\mathbf{B} = \mathbf{R}\mathbf{\Gamma}\mathbf{R}^T, \quad \mathbf{D} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T,$$

where $\mathbf{\Gamma} = \operatorname{diag}\{\gamma_i\}$, $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_i\}$, and γ_i, λ_i for $i = 1,\dots,d$ are the real positive eigenvalues of $\mathbf{B}$ and $\mathbf{D}$, respectively. Let $\mathbf{O} := \mathbf{Q}^T\mathbf{R}$, then the matrix $\mathbf{O}$ is also orthogonal and it holds that $\sum_{i=1}^{d} O_{ij}^2 = 1$ for $j = 1,\dots,d$ and $\sum_{j=1}^{d} O_{ij}^2 = 1$ for $i = 1,\dots,d$. Firstly, we compute

$$\begin{aligned}
\operatorname{tr}\,(\mathbf{B}\,f(\mathbf{D})) = \operatorname{tr}\,\left(\mathbf{R}\mathbf{\Gamma}\mathbf{R}^T\mathbf{Q}f\left(\mathbf{\Lambda}\right)\mathbf{Q}^T\right) &= \operatorname{tr}\,\left(\mathbf{O}\mathbf{\Gamma}\mathbf{O}^T f\left(\mathbf{\Lambda}\right)\right) = \mathbf{O}\mathbf{\Gamma} : f\left(\mathbf{\Lambda}\right)\mathbf{O} \\
&= \sum_{i,j=1}^{d}\left(\sum_{k=1}^{d} O_{ik}\mathbf{\Gamma}_{kj}\right)\left(\sum_{k=1}^{d} f(\mathbf{\Lambda})_{ik}O_{kj}\right) = \sum_{i,j=1}^{d} O_{ij}\gamma_j f(\lambda_i)O_{ij} \\
&= \sum_{i,j=1}^{d} O_{ij}^2\gamma_j f(\lambda_i),
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{B}\, f(\mathbf{B})\right) = \operatorname{tr}\left(\mathbf{R}\boldsymbol{\Gamma}\mathbf{R}^T\mathbf{R} f\left(\boldsymbol{\Gamma}\right)\mathbf{R}^T\right) = \operatorname{tr}\left(\boldsymbol{\Gamma} f\left(\boldsymbol{\Gamma}\right)\right) = \boldsymbol{\Gamma} : f\left(\boldsymbol{\Gamma}\right) = \sum_{j=1}^{d} \gamma_j\, f(\gamma_j) \\
= \sum_{j=1}^{d} \gamma_j\, f(\gamma_j) \sum_{i=1}^{d} O_{ij}^2.
\end{aligned} \tag{3.25}$$

Similarly, we get

$$\operatorname{tr}\left(\mathbf{D}\, f(\mathbf{B})\right) = \sum_{i,j=1}^{d} O_{ij}^2 f(\gamma_j)\lambda_i, \qquad \operatorname{tr}\left(\mathbf{D}\, f(\mathbf{D})\right) = \sum_{i=1}^{d} \lambda_i\, f(\lambda_i) \sum_{j=1}^{d} O_{ij}^2. \tag{3.26}$$

Inequality (3.23a) then follows. Indeed, the above equalities (3.24) - (3.26) and identity (2.2) yield

$$\begin{aligned}
(\mathbf{B} - \mathbf{D}) : \left(f(\mathbf{B}) - f(\mathbf{D})\right) &= \operatorname{tr}\left(\mathbf{B}\, f(\mathbf{B}) + \mathbf{D}\, f(\mathbf{D}) - \mathbf{B}\, f(\mathbf{D}) - \mathbf{D}\, f(\mathbf{B})\right) \\
&= \sum_{i,j=1}^{d} \gamma_j\, f(\gamma_j) O_{ij}^2 + \sum_{i,j=1}^{d} \lambda_i\, f(\lambda_i) O_{ij}^2 - \sum_{i,j=1}^{d} O_{ij}^2 \gamma_j f(\lambda_i) - \sum_{i,j=1}^{d} O_{ij}^2 f(\gamma_j)\lambda_i \\
&= \sum_{i,j=1}^{d} \left(\gamma_j - \lambda_i\right)\left(f(\gamma_j) - f(\lambda_i)\right) \leq 0,
\end{aligned}$$

where we have taken into account that f is decreasing.

Now, we show inequality (3.23b). Let us consider the function $f = 1/x$, for $x > 0$. Then, by (2.11), $f\left(\mathbf{B}\right) = \mathbf{B}^{-1}$ and we can write

$$\frac{\partial \mathbf{B}}{\partial x_i} : \frac{\partial \mathbf{B}^{-1}}{\partial x_i} = \frac{\partial \mathbf{B}}{\partial x_i} : \frac{\partial f\left(\mathbf{B}\right)}{\partial x_i} = \lim_{h \to 0} \frac{\mathbf{B}(x_i + h) - \mathbf{B}(x_i)}{h} : \frac{f\left(\mathbf{B}(x_i + h)\right) - f\left(\mathbf{B}(x_i)\right)}{h}.$$

By (3.23a) for $\mathbf{B} := \mathbf{B}(x_i + h)$ and $\mathbf{D} := \mathbf{B}(x_i)$ we obtain

$$\nabla \mathbf{B} : \nabla \mathbf{B}^{-1} = \sum_{i=1}^{d} \frac{\partial \mathbf{B}}{\partial x_i} : \frac{\partial \mathbf{B}^{-1}}{\partial x_i} \leq 0,$$

which concludes the proof. □

After preparing the necessary preliminaries, we are ready to derive the free energy estimate for the diffusive Peterlin viscoelastic model in the two-dimensional setting.

Free energy estimate for the diffusive Peterlin model

In what follows, we assume that $(\mathbf{u}, p, \mathbf{C})$ is a sufficiently smooth solution of (3.2) such that all subsequent computations are valid, e.g., the regularity from Lemma 3.3 is sufficient. First of all, let us define the free energy.

Definition 3.10. *(free energy)*
The free energy of the Peterlin viscoelastic fluid modelled by (3.2) *is defined as*

$$F(\mathbf{u}, \mathbf{C}) := \frac{1}{2} \int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4} \int_\Omega \operatorname{tr}\left(\mathbf{T} - 2 \ln \mathbf{C} - \mathbf{I}\right) dx\,. \tag{3.27}$$

Let us note, that the free energy is the sum of the kinetic and the entropic term. The kinetic energy $\frac{1}{2}\int_\Omega |\mathbf{u}|^2 \, dx$ is always non-negative. Due to Lemma 3.3 and (3.21b) the entropic term is well-defined and non-negative provided the initial datum $\mathbf{C}_0$ is a symmetric positive definite tensor.

Theorem 3.11. *(non-increasing free energy)*
Let $(\mathbf{u}, p, \mathbf{C})$ *be a smooth solution to* (3.2) *and let the initial datum* $\mathbf{C}_0$ *be a symmetric positive definite matrix. Then the free energy given by* (3.27) *satisfies*

$$\frac{d}{dt}F(\mathbf{u},\mathbf{C}) + \nu\,\|\nabla\mathbf{u}\|_2^2 + \frac{\varepsilon}{2}\,\|\nabla \mathrm{tr}\,\mathbf{C}\|_2^2 + \frac{1}{2}\int_\Omega \mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\Big(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}\Big)\, dx \le 0,$$

and consequently it is non-increasing in time.

Proof. Let us recall the energy equality (3.9) for velocity and the energy equality (3.16) for the trace of the conformation tensor obtained in the previous section, i.e.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \, dx = -\nu\int_\Omega |\nabla\mathbf{u}|^2 \, dx - \int_\Omega \mathbf{T} : \nabla\mathbf{u}\, dx \tag{3.28}$$

and

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathrm{tr}\,\mathbf{C}|^2 \, dx - \int_\Omega \mathrm{tr}\,\mathbf{C}\,\mathbf{C} : \nabla\mathbf{u}\, dx &= \\ = \int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\, dx - \frac{1}{2}\int_\Omega |\mathrm{tr}\,\mathbf{C}|^4 \, dx &- \frac{\varepsilon}{2}\int_\Omega |\nabla \mathrm{tr}\,\mathbf{C}|^2\, dx\,.\end{aligned} \tag{3.29}$$

Further, let us multiply equation (3.2d) for $\mathbf{C}$ by its inverse $\mathbf{C}^{-1}$, which is a symmetric positive definite matrix, and integrate using the Green formula to get

$$\begin{aligned}\int_\Omega \left(\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C}\right) : \mathbf{C}^{-1}\, dx - \int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C}^{-1}\, dx &+ \\ + \int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\mathbf{C} : \mathbf{C}^{-1}\, dx = \int_\Omega \mathrm{tr}\,\mathbf{C}\,\mathrm{tr}\,\mathbf{C}^{-1}\, dx &- \\ - \varepsilon\int_\Omega \nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\, dx + \underbrace{\varepsilon\int_{\partial\Omega} \mathbf{C}^{-1}\,(\nabla\mathbf{C}\cdot\mathbf{n})\, dS}_{=0}&,\end{aligned} \tag{3.30}$$

where the no-flux boundary condition for $\mathbf{C}$ is taken into account. The Frobenius inner product (2.2) gives us

$$\Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C}^{-1} = \mathrm{tr}\,\Big((\nabla\mathbf{u})\mathbf{C}\mathbf{C}^{-1} + \mathbf{C}(\nabla\mathbf{u})^T\mathbf{C}^{-1}\Big) = 2\mathrm{tr}\,(\nabla\mathbf{u}) = 2\mathrm{div}\,\mathbf{u} = 0,$$

and $\mathbf{C} : \mathbf{C}^{-1} = \mathrm{tr}\,\mathbf{I} = 2$. Employing the Jacobi formula (3.22a) we find

$$\int_\Omega \left(\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{C}\right) : \mathbf{C}^{-1}\, dx = \int_\Omega \left(\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right) \mathrm{tr}\,\ln\mathbf{C}\, dx\,.$$

Hence (3.30) multiplied by 1/2 becomes

$$\begin{aligned}\frac{1}{2}\int_\Omega \frac{\partial \mathrm{tr}\,\ln\mathbf{C}}{\partial t}\, dx + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathrm{tr}\,\ln\mathbf{C}\, dx + \int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\, dx &= \\ = \frac{1}{2}\int_\Omega \mathrm{tr}\,\mathbf{C}\,\mathrm{tr}\,\mathbf{C}^{-1}\, dx - \frac{\varepsilon}{2}\int_\Omega \nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\, dx\,.&\end{aligned} \tag{3.31}$$

We now combine "(3.28) + (3.29) - (3.31)" to get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\int_\Omega |\mathbf{u}|^2\,dx + \frac{1}{4}\int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\,dx - \frac{1}{2}\int_\Omega \mathrm{tr}\,\ln\mathbf{C}\,dx\right) + \nu\,\|\nabla\mathbf{u}\|_2^2 + \frac{\varepsilon}{2}\,\|\nabla\mathrm{tr}\,\mathbf{C}\|_2^2 + \\
+ \frac{1}{2}\int_\Omega (\mathrm{tr}\,\mathbf{C})^2\left((\mathrm{tr}\,\mathbf{C})^2 - 4 + \frac{\mathrm{tr}\,\mathbf{C}^{-1}}{\mathrm{tr}\,\mathbf{C}}\right) dx = \frac{\varepsilon}{2}\int_\Omega \nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\,dx\,.
\end{aligned}
\tag{3.32}
$$

Let us recall that the trace, the inverse and the trace of the inverse of the elastic stress tensor $\mathbf{T} = \mathrm{tr}\,\mathbf{C}\,\mathbf{C}$ are

$$
\mathrm{tr}\,\mathbf{T} = (\mathrm{tr}\,\mathbf{C})^2, \qquad \mathbf{T}^{-1} = \frac{1}{\mathrm{tr}\,\mathbf{C}}\mathbf{C}^{-1}, \qquad \mathrm{tr}\,\mathbf{T}^{-1} = \frac{\mathrm{tr}\,\mathbf{C}^{-1}}{\mathrm{tr}\,\mathbf{C}},
$$

respectively. Then we can rewrite (3.32) as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left(\int_\Omega \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{4}\mathrm{tr}\,\left(\mathbf{T} - 2\ln\mathbf{C} - \mathbf{I}\right)dx\right) + \nu\,\|\nabla\mathbf{u}\|_2^2 + \frac{\varepsilon}{2}\,\|\nabla\mathrm{tr}\,\mathbf{C}\|_2^2 + \\
+ \frac{1}{2}\int_\Omega \mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\left(\mathbf{T} - 2\mathbf{I} + \mathbf{T}^{-1}\right) dx = \frac{\varepsilon}{2}\int_\Omega \nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\,dx\,.
\end{aligned}
\tag{3.33}
$$

Due to Lemma 3.9 we have

$$
\frac{\varepsilon}{2}\int_\Omega \nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\,dx \le 0.
$$

Therefore (3.33) yields the inequality

$$
\frac{\mathrm{d}}{\mathrm{dt}}F(\mathbf{u},\mathbf{C}) + \nu\,\|\nabla\mathbf{u}\|_2^2 + \frac{\varepsilon}{2}\,\|\nabla\mathrm{tr}\,\mathbf{C}\|_2^2 + \frac{1}{2}\int_\Omega (\mathrm{tr}\,\mathbf{C})^2\mathrm{tr}\,\left(\mathbf{T} - 2\mathbf{I} + \mathbf{T}^{-1}\right) dx \le 0,
$$

which we wanted to prove. As $\mathrm{tr}\,\mathbf{T} = (\mathrm{tr}\,\mathbf{C})^2 > 0$ and (3.21c) holds, we have

$$
\nu\,\|\nabla\mathbf{u}\|_2^2 + \frac{\varepsilon}{2}\,\|\nabla\mathrm{tr}\,\mathbf{C}\|_2^2 + \frac{1}{2}\int_\Omega \mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\left(\mathbf{T} - 2\mathbf{I} + \mathbf{T}^{-1}\right) dx \ge 0.
$$

We can conclude that

$$
\frac{\mathrm{d}}{\mathrm{dt}}F(\mathbf{u},\mathbf{C}) \le 0
$$

and the free energy $F(\mathbf{u},\mathbf{C})$ given by (3.27) is non-increasing in time. □

In this chapter we have introduced the diffusive Peterlin model, for which the formal energy estimates have been shown. We have proven that the model preserves the positive definiteness of the conformation tensor for all $(x,t) \in \Omega \times (0,T)$. Further, the free energy estimate for smooth solutions was demonstrated. In Chapter 4 we shall define a weak solution to this problem and present the existence, regularity and uniqueness results.

4 Global weak solutions

This chapter deals with a global in time weak solution to the diffusive Peterlin model. After preparing some preliminaries in the first section, we prove in Section 4.2 the global existence of the weak solutions in two space dimensions by studying the Galerkin approximation, a priori estimates, compactness results and passage to the limit. Unfortunately the functional spaces obtained for the conformation tensor do not allow to obtain the uniqueness of the weak solution. Therefore, in Section 4.3, we show that the weak solution to this model enjoys higher regularity provided the data are more regular. Finally, uniqueness of the more regular weak solution is shown in the last section.

The content of this chapter is partially presented in Sections 3 - 4 of [90].

4.1. Preliminaries

Throughout the chapter we shall use the standard notation for the functional spaces in two space dimensions

$$V := H^1_{0,div}(\Omega)^2, \quad H := L^2_{div}(\Omega)^2, \quad W := H^1(\Omega)^{2\times 2}.$$

The space V is equipped with the norm $|||\cdot||| := \|\cdot\|_{H^1_0} = \|\nabla\cdot\|_2$. Let us recall that the scalar product on $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$, where

$$(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}\;dx\,,\;\; \mathbf{u},\mathbf{v}\in L^2(\Omega)^2, \quad (\mathbf{C},\mathbf{D}) \quad = \int_\Omega \mathbf{C}:\mathbf{D}\;dx\,,\;\; \mathbf{C},\mathbf{D}\in L^2(\Omega)^{2\times 2}.$$

Now, we introduce the multilinear forms and derive some useful properties in order to study the weak solution to the diffusive Peterlin model.

Lemma 4.1. *(multilinear forms)*
Let $\Omega\subset\mathbb{R}^2$ be a bounded domain. Then the forms

$$((\mathbf{u},\mathbf{v})) := \int_\Omega \nabla\mathbf{u}:\nabla\mathbf{v}\;dx \qquad \text{and} \qquad b\,(\mathbf{u},\mathbf{v},\mathbf{w}) := \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\;dx$$

are bilinear continuous on $V\times V$ and trilinear continuous on $V\times V\times V$, respectively. Secondly, the forms

$$((\mathbf{C},\mathbf{D})) := \int_\Omega \nabla\mathbf{C}:\nabla\mathbf{D}\;dx \qquad \text{and} \qquad B\,(\mathbf{u},\mathbf{C},\mathbf{D}) := \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{C}:\mathbf{D}\;dx$$

are bilinear continuous on $W\times W$ and trilinear continuous on $V\times W\times W$, respectively.

Sketch of proof. The first part of the above lemma is a standard result in the theory of the Navier-Stokes equations. For the proof see, e.g., [142]. In the second part the forms needed in the study of the equation for the conformation tensor (3.2d) are defined. Their continuity and multilinear properties can be easily showed as in [142]. □

Lemma 4.2. *(properties of trilinear forms)*
For any open set Ω it holds that

$$b(\mathbf{u},\mathbf{u},\mathbf{v}) = -b(\mathbf{u},\mathbf{v},\mathbf{u}) \quad \mathbf{u} \in V, \mathbf{v} \in H_0^1(\Omega)^2 \tag{4.1}$$
$$B(\mathbf{u},\mathbf{C},\mathbf{D}) = -B(\mathbf{u},\mathbf{D},\mathbf{C}) \quad \mathbf{u} \in V,\ \mathbf{C},\mathbf{D} \in W. \tag{4.2}$$

Proof. The assertion (4.1) is proven in [142]. Analogously, we prove (4.2). Firstly, let us note that

$$B(\mathbf{u},\mathbf{C},\mathbf{C}) = 0 \quad \mathbf{u} \in V,\ \mathbf{C} \in W. \tag{4.3}$$

It suffices to show (4.3) for $\mathbf{u} \in C^\infty_{0,div}(\Omega)$ and $\mathbf{C} \in C^\infty(\Omega)$, because of the classical approximation of $H^1(\Omega)$ by smooth functions. For such $\mathbf{u}$ and $\mathbf{C}$ we get

$$\begin{aligned} B(\mathbf{u},\mathbf{C},\mathbf{C}) &= \int_\Omega \sum_{i,j,k=1}^{2} u_k \frac{\partial C_{ij}}{\partial x_k} C_{ij}\ dx\ = \int_\Omega \frac{1}{2} \sum_{i,j,k=1}^{2} u_k \frac{\partial C_{ij}^2}{\partial x_k}\ dx \\ &= -\frac{1}{2}\int_\Omega \sum_{i,j,k=1}^{2} \frac{\partial u_k}{\partial x_k} C_{ij}^2\ dx\ + \frac{1}{2}\int_{\partial\Omega} \sum_{i,j,k=1}^{2} u_k n_k C_{ij}^2\ dS \\ &= -\frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}\,\mathbf{C}^2\ dx\ + \frac{1}{2}\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})\,\mathbf{C}^2\ dS\ = 0. \end{aligned}$$

By the trilinear property of B and (4.3) we have

$$\begin{aligned} B(\mathbf{u},\mathbf{C},\mathbf{D}) + B(\mathbf{u},\mathbf{D},\mathbf{C}) &= B(\mathbf{u},\mathbf{C},\mathbf{D}) + B(\mathbf{u},\mathbf{D},\mathbf{D}) + B(\mathbf{u},\mathbf{D},\mathbf{C}) + B(\mathbf{u},\mathbf{C},\mathbf{C}) \\ &= B(\mathbf{u},\mathbf{C}+\mathbf{D},\mathbf{C}+\mathbf{D}) = 0, \end{aligned}$$

which yields $B(\mathbf{u},\mathbf{C},\mathbf{D}) = -B(\mathbf{u},\mathbf{D},\mathbf{C})$. □

For further useful notions and results from functional analysis and theory of ordinary differential equations we refer to Appendix A.

4.2. Existence of weak solutions

The goal of this section is to show the existence of a global in time weak solution. Firstly, we define the weak solution to the diffusive Peterlin model and we introduce its Galerkin approximation. Then we derive the energy estimates and, due to the compactness result, we pass to the limit.

Definition 4.3. *(weak solution to the diffusive Peterlin model)*
Let $(\mathbf{u}_0, \mathbf{C}_0) \in H \times L^2(\Omega)^{2\times 2}$*. The couple*

$$(\mathbf{u}, \mathbf{C}) \in \Big[L^\infty(0,T;H) \cap L^2(0,T;V)\Big] \times \Big[L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W)\Big] \tag{4.4a}$$

is called a weak solution to the problem (3.2) *if*

$$(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0) \tag{4.4b}$$

and if it satisfies the weak formulation

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v}\, dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, dx + \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, dx = -\int_\Omega \operatorname{tr}\mathbf{C}\,\mathbf{C} : \nabla\mathbf{v}\, dx \tag{4.4c}$$

$$\begin{aligned} \int_\Omega \frac{\partial \mathbf{C}}{\partial t} : \mathbf{D}\, dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{C} : \mathbf{D}\, dx - \int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{D}\, dx + \\ + \varepsilon \int_\Omega \nabla\mathbf{C} : \nabla\mathbf{D}\, dx = \int_\Omega \operatorname{tr}\mathbf{C}\,\mathbf{I} : \mathbf{D}\, dx - \int_\Omega (\operatorname{tr}\mathbf{C})^2\mathbf{C} : \mathbf{D}\, dx \end{aligned} \tag{4.4d}$$

$$\forall\ (\mathbf{v}, \mathbf{D}) \in V \times W,\ \text{a.e. } t \in (0,T).$$

Using the notation from Section 4.1, the weak formulation (4.4c) - (4.4d) reads

$$\begin{aligned} &\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + b\,(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \nu\,((\mathbf{u}, \mathbf{v})) = -\,(\operatorname{tr}\mathbf{C}\,\mathbf{C}, \nabla\mathbf{v}) \\ &\left(\frac{\partial \mathbf{C}}{\partial t}, \mathbf{D}\right) + B\,(\mathbf{u}, \mathbf{C}, \mathbf{D}) - \Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T, \mathbf{D}\Big) + \varepsilon\,((\mathbf{C}, \mathbf{D})) = \\ &\qquad = (\operatorname{tr}\mathbf{C}\,\mathbf{I}, \mathbf{D}) - \Big((\operatorname{tr}\mathbf{C})^2\mathbf{C}, \mathbf{D}\Big). \end{aligned}$$

The main result of this section follows.

Theorem 4.4. *(existence of weak solutions)*
There exists a global in time weak solution to the Peterlin model (3.2) *given by* (4.4)*.*

Proof. Firstly, we define the Galerkin approximation of the weak solution.

Galerkin approximation

The spaces V, as a closed subspace of the separable Hilbert space $H^1(\Omega)^2$, and W are the separable Hilbert spaces. There exist the orthonormal countable bases such that

$$V = \overline{\operatorname{span}\{\mathbf{v}_i\}_{i=1}^\infty} \quad \text{and} \quad W = \overline{\operatorname{span}\{\mathbf{D}_i\}_{i=1}^\infty}.$$

We define the m-th approximate Galerkin solution of (4.4b) - (4.4d) as follows

$$(\mathbf{u}_m(t), \mathbf{C}_m(t)) = \left(\sum_{i=1}^m g_{im}(t)\mathbf{v}_i, \sum_{i=1}^m G_{im}(t)\mathbf{D}_i\right) \tag{4.5a}$$

$$(\mathbf{u}_m(0), \mathbf{C}_m(0)) = (\mathbf{u}_{0m}, \mathbf{C}_{0m}) \tag{4.5b}$$

$$(\mathbf{u}'_m(t), \mathbf{v}_j) + b\,(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j) + \nu\,((\mathbf{u}_m(t), \mathbf{v}_j)) = -\,(\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}_m(t), \nabla\mathbf{v}_j) \tag{4.5c}$$

$$\begin{aligned}(\mathbf{C}'_m(t), \mathbf{D}_j) + B\,(\mathbf{u}_m(t), \mathbf{C}_m(t), \mathbf{D}_j) - \Big((\nabla\mathbf{u}_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}_m(t))^T, \mathbf{D}_j\Big) + \\ + \varepsilon\,((\mathbf{C}_m(t), \mathbf{D}_j)) = (\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{I}, \mathbf{D}_j) - \Big((\mathrm{tr}\,\mathbf{C}_m(t))^2\mathbf{C}_m(t), \mathbf{D}_j\Big) \qquad (4.5\mathrm{d}) \\ \text{for} \quad j = 1, \dots, m, \ t \in [0, T].\end{aligned}$$

Functions $\mathbf{u}_{0m}$ and $\mathbf{C}_{0m}$ are the orthogonal projections in H of $\mathbf{u}_0$ and in $L^2(\Omega)$ of $\mathbf{C}_0$ on the spaces spanned by $\mathbf{v}_i$ and $\mathbf{D}_i$, respectively. Equations (4.5c) - (4.5d) form a nonlinear system of differential equations for the functions $g_{1m}, \dots, g_{mm}$ and $G_{1m}, \dots, G_{mm}$

$$\begin{aligned}\sum_{i=1}^{m} (\mathbf{v}_i, \mathbf{v}_j)\, g'_{im}(t) + \sum_{i,k=1}^{m} b\,(\mathbf{v}_i, \mathbf{v}_k, \mathbf{v}_j)\, g_{im}(t) g_{km}(t) + \nu \sum_{i=1}^{m} ((\mathbf{v}_i, \mathbf{v}_j))\, g_{im}(t) = \\ = -\sum_{i,k=1}^{m} (\mathrm{tr}\,\mathbf{D}_i\,\mathbf{D}_k, \nabla\mathbf{v}_j)\, G_{im}(t) G_{km}(t)\end{aligned} \tag{4.6a}$$

$$\begin{aligned}\sum_{i=1}^{m} (\mathbf{D}_i, \mathbf{D}_j) G'_{im}(t) + \sum_{i,k=1}^{m} B\,(\mathbf{v}_k, \mathbf{D}_i, \mathbf{D}_j)\, g_{km}(t) G_{im}(t) - \\ - \sum_{i,k=1}^{m} \Big((\nabla\mathbf{v}_k)\mathbf{D}_i + \mathbf{D}_i(\nabla\mathbf{v}_k)^T, \mathbf{D}_j\Big)\, G_{im}(t) g_{km}(t) + \varepsilon \sum_{i=1}^{m} ((\mathbf{D}_i, \mathbf{D}_j))\, G_{im}(t) = \\ = \sum_{i=1}^{m} (\mathrm{tr}\,\mathbf{D}_i\,\mathbf{I}, \mathbf{D}_j)\, G_{im}(t) - \sum_{i,k,l=1}^{m} (\mathrm{tr}\,\mathbf{D}_k\,\mathrm{tr}\,\mathbf{D}_l\,\mathbf{D}_i, \mathbf{D}_j)\, G_{im}(t) G_{km}(t) G_{lm}(t).\end{aligned} \tag{4.6b}$$

The initial condition (4.5b) is equivalent to the $2m$ scalar initial conditions

$$g_{im}(0) = \text{ the } i\text{-th component of } \mathbf{u}_{0m} \tag{4.7}$$

$$G_{im}(0) = \text{ the } i\text{-th component of } \mathbf{C}_{0m}. \tag{4.8}$$

The nonlinear differential system of equations (4.6) together with the initial conditions (4.8) has a maximal solution on the interval $[0, t_m]$, cf. Theorem A.6 in Appendix A. The following a priori estimates shows that $t_m = T$. This gives us the existence of the m-th Galerkin solution defined on the interval $[0, T]$.

A priori estimates

We multiply (4.5c) and (4.5d) by $g_{jm}(t)$ and $\frac{1}{2}G_{jm}(t)$, respectively, and take the sum for $j = 1, \dots, m$. Then we sum up the resulting equalities and employ identity (3.8), analogously as in Section 3.3, to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}_m(t)|^2\, dx + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}_m(t)|^2\, dx + \nu\int_\Omega |\nabla\mathbf{u}_m(t)|^2\, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}_m(t)|^2\, dx + \\ + \frac{1}{2}\int_\Omega |\mathrm{tr}\,\mathbf{C}_m(t)|^2\mathbf{C}_m(t)^2\, dx = \frac{1}{2}\int_\Omega |\mathrm{tr}\,\mathbf{C}_m(t)|^2\, dx\,,\end{aligned}$$

which, similarly as in Section 3.3, yields

$$(\mathbf{u}_m, \mathbf{C}_m) \in \Big[L^\infty(0, T; H) \cap L^2(0, T; V)\Big] \times \Big[L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W)\Big]\,.$$

More precisely, there exists a positive constant $k = k(\Omega, T, 1/\nu, 1/\varepsilon, \mathbf{u}_0, \mathbf{C}_0)$ independent of m such that

$$\|\mathbf{u}_m\|_{L^\infty(H)} + \|\mathbf{u}_m\|_{L^2(V)} + \|\mathbf{C}_m\|_{L^\infty(L^2)} + \|\mathbf{C}_m\|_{L^2(W)} \leq k. \tag{4.9}$$

Let us note, that also $\|\mathbf{C}_m\|_{L^4(L^4)} \leq k$, by the above bound and the interpolation inequality (2.8b). Estimate (4.9) gives us the weak and *-weak convergence of our approximate Galerkin sequence in the appropriate functional spaces. However, we need a strong convergence in order to pass to the limit in the nonlinear terms.

Compactness

For better readability we now omit the lower index m of the Galerkin approximation. We shall apply the Lions-Aubin lemma, cf. Lemma A.2, to get the compact embeddings. To this end, let us first define the operators

$$\begin{aligned}
&\mathcal{A} : V \to V^* && \langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle := ((\mathbf{u}, \mathbf{v}))\\
&\mathcal{B} : V \to V^* && \langle \mathcal{B}\mathbf{u}, \mathbf{v}\rangle := b(\mathbf{u}, \mathbf{u}, \mathbf{v})\\
&\mathcal{E} : L^2(\Omega)^{2\times 2} \to V^* && \langle \mathcal{E}\mathbf{T}, \mathbf{v}\rangle := (\mathbf{T}, \nabla\mathbf{v}).
\end{aligned}$$

Then (4.5c) can be rewritten in the following operator form

$$\mathbf{u}' = -\mathcal{E}\mathbf{T} - \mathcal{B}\mathbf{u} - \nu\mathcal{A}\mathbf{u}, \ \mathbf{u} \in V, \ \mathbf{T} \in L^2(\Omega)^{2\times 2}. \tag{4.10}$$

First, we have the standard estimate $|\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle| \leq |||\mathbf{u}||| |||\mathbf{v}|||$ and thus

$$\int_0^T \|\mathcal{A}\mathbf{u}\|_{V^*}^2 \, dt \leq \int_0^T |||\mathbf{u}|||^2 \, dt. \tag{4.11}$$

Identity (4.1) and the interpolation inequality (2.8a) yield

$$|\langle \mathcal{B}\mathbf{u}, \mathbf{v}\rangle| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq c\|\mathbf{u}\|_4 |||\mathbf{v}||| \|\mathbf{u}\|_4 \leq c\|\mathbf{u}\|_2 |||\mathbf{u}||| |||\mathbf{v}|||,$$

and consequently

$$\int_0^T \|\mathcal{B}\mathbf{u}\|_{V^*}^2 \, dt \leq c\int_0^T \|\mathbf{u}\|_2^2 |||\mathbf{u}|||^2 \, dt \leq c\|\mathbf{u}\|_{L^\infty(0,T;H)}^2 \int_0^T |||\mathbf{u}|||^2 \, dt. \tag{4.12}$$

Moreover, since $|\langle \mathcal{E}\mathbf{T}, \mathbf{v}\rangle| \leq \|\mathbf{T}\|_2 \|\mathbf{v}\|_2$, we get

$$\int_0^T \|\mathcal{E}\mathbf{T}\|_{V^*}^2 \, dt \leq \int_0^T \|\mathbf{T}\|_2^2 \, dt. \tag{4.13}$$

Finally, (4.10) - (4.13) imply $\mathbf{u}' \in L^2(0, T; V^*)$. Since the following compact and continuous embeddings hold

$$V \hookrightarrow\hookrightarrow L^4_{div}(\Omega) \hookrightarrow H \hookrightarrow V^*,$$

we can apply the Lions-Aubin lemma to get the compact embedding of the Galerkin sequence $\{\mathbf{u}_m\}_{m=1}^\infty$ into the space $L^2(0, T; L^4_{div}(\Omega))$.

We have a similar result for the conformation tensor $\mathbf{C}$. Let us define the operators

$$
\begin{aligned}
&\tilde{\mathcal{A}} : W \to W^* && \langle \tilde{\mathcal{A}}\mathbf{C}, \mathbf{D}\rangle := ((\mathbf{C}, \mathbf{D}))\\
&\tilde{\mathcal{B}} : V \times W \to W^* && \langle \tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C}), \mathbf{D}\rangle := B\,(\mathbf{u}, \mathbf{C}, \mathbf{D})\\
&\mathcal{O} : V \times W \to W^* && \langle \mathcal{O}(\mathbf{u}, \mathbf{C}), \mathbf{D}\rangle := \Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T, \mathbf{D}\Big)\\
&\mathcal{T} : W \to W^* && \langle \mathcal{T}\mathbf{C}, \mathbf{D}\rangle := \Big(\operatorname{tr}\mathbf{C}\,\mathbf{I} - (\operatorname{tr}\mathbf{C})^2\mathbf{C}, \mathbf{D}\Big)\,.
\end{aligned}
$$

Then equation (4.5d) can be rewritten in the operator form

$$
\mathbf{C}' = \mathcal{O}(\mathbf{u}, \mathbf{C}) - \mathcal{T}\mathbf{C} - \tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C}) - \varepsilon\tilde{\mathcal{A}}\mathbf{u},\ \ \mathbf{u} \in V,\ \ \mathbf{C} \in W. \tag{4.14}
$$

We have the analogous estimates of the operators $\tilde{\mathcal{A}}\mathbf{C}$ and $\tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C})$. Indeed, the estimate

$$
|\langle \tilde{\mathcal{A}}\mathbf{C}, \mathbf{D}\rangle| \le \|\nabla\mathbf{C}\|_2\,\|\nabla\mathbf{D}\|_2 \le c\,\|\mathbf{C}\|_W\,\|\mathbf{D}\|_W
$$

yields

$$
\int_0^T \left\|\tilde{\mathcal{A}}\mathbf{C}\right\|_{W^*}^2\,dt\ \le c\int_0^T \|\mathbf{C}\|_W^2\,dt\,. \tag{4.15}
$$

By the interpolation inequality (2.8a) and identity (4.2) we get

$$
|\langle \tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C}), \mathbf{D}\rangle| = |B\,(\mathbf{u}, \mathbf{D}, \mathbf{C})\,| \le c\,\|\mathbf{u}\|_4\,\|\nabla\mathbf{D}\|_2\,\|\mathbf{C}\|_4 \le c\,\|\mathbf{u}\|_2^{1/2}\,|||\mathbf{u}|||^{1/2}\,\|\mathbf{C}\|_4\,\|\mathbf{D}\|_W\,,
$$

which, by the Hölder inequality, gives us

$$
\begin{aligned}
\int_0^T \left\|\tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C})\right\|_{W^*}^2\,dt\ &\le c\int_0^T \|\mathbf{u}\|_2\,|||\mathbf{u}|||\,\|\mathbf{C}\|_4^2\,dt\\
&\le c\,\|\mathbf{u}\|_{L^\infty(0,T;H)}\int_0^T |||\mathbf{u}|||\,\|\mathbf{C}\|_4^2\,dt\\
&\le c\,\|\mathbf{u}\|_{L^\infty(0,T;H)}\,\|\mathbf{u}\|_{L^2(0,T;V)}\,\|\mathbf{C}\|_{L^4(0,T;L^4(\Omega))}^2\,.
\end{aligned} \tag{4.16}
$$

We find the estimates of the operators $\mathcal{O}(\mathbf{u}, \mathbf{C})$ and $\mathcal{T}\mathbf{C}$ in a similar way. It holds that

$$
\begin{aligned}
|\langle \mathcal{O}(\mathbf{u}, \mathbf{C}), \mathbf{D}\rangle| &\le c|||\mathbf{u}|||\,\|\mathbf{C}\|_4\,\|\mathbf{D}\|_4\\
|\langle \mathcal{T}\mathbf{C}, \mathbf{D}\rangle| &\le \|\operatorname{tr}\mathbf{C}\|_2\,\|\operatorname{tr}\mathbf{D}\|_2 + \|\operatorname{tr}\mathbf{C}\|_4^2\,\|\mathbf{C}\|_4\,\|\mathbf{D}\|_4\,.
\end{aligned}
$$

For the Hölder coefficients $p = 3/2$ and $q = 3$ we get

$$
\int_0^T \|\mathcal{O}(\mathbf{u}, \mathbf{C})\|_{W^*}^{4/3}\,dt\ \le c\int_0^T |||\mathbf{u}|||^{4/3}\,\|\mathbf{C}\|_4^{4/3}\,dt\ \le c\,\|\mathbf{u}\|_{L^2(0,T;V)}^{4/3}\,\|\mathbf{C}\|_{L^4(0,T;L^4(\Omega))}^{4/3} \tag{4.17}
$$

$$
\begin{aligned}
\int_0^T \|\mathcal{T}\mathbf{C}\|_{W^*}^{4/3}\,dt\ &\le c\int_0^T \|\operatorname{tr}\mathbf{C}\|_2^{4/3} + \|\operatorname{tr}\mathbf{C}\|_4^{8/3}\,\|\mathbf{C}\|_4^{4/3}\,dt\\
&\le c\,\|\operatorname{tr}\mathbf{C}\|_2^{4/3} + c\,\|\operatorname{tr}\mathbf{C}\|_4^{8/3}\,\|\mathbf{C}\|_4^{4/3}\,.
\end{aligned} \tag{4.18}
$$

By (4.15) - (4.18) we have $\tilde{\mathcal{A}}\mathbf{C}$, $\tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C}) \in L^2(0,T;W^*)$, $\mathcal{O}(\mathbf{u}, \mathbf{C})$, $\mathcal{T}\mathbf{C} \in L^{4/3}(0,T;W^*)$. Thus, by (4.14), we see that $\mathbf{C}' \in L^{4/3}(0,T;W^*)$. The embeddings

$$
W \hookrightarrow\hookrightarrow L^4(\Omega) \hookrightarrow W^*
$$

yield due to the Lions-Aubin lemma the compact embedding of $\{\mathbf{C}_m\}_{m=1}^\infty$ into the space $L^2(0,T;L^4(\Omega))$.

Passage to the limit

Due to the above a priori estimates and compact embeddings of the Galerkin approximate sequence $\{(\mathbf{u}_m, \mathbf{C}_m)\}_{m=1}^{\infty}$ there exists a subsequence, denoted for better readability again by $(\mathbf{u}_m, \mathbf{C}_m)$, such that

$$\mathbf{u}_m \rightharpoonup^* \mathbf{u} \text{ in } L^\infty(0,T;H) \tag{4.19a}$$
$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;V) \tag{4.19b}$$
$$\mathbf{u}_m \to \mathbf{u} \text{ in } L^2(0,T;L^4_{div}(\Omega)) \tag{4.19c}$$
$$\mathbf{C}_m \rightharpoonup^* \mathbf{C} \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{4.19d}$$
$$\mathbf{C}_m \rightharpoonup \mathbf{C} \text{ in } L^2(0,T;W) \tag{4.19e}$$
$$\mathbf{C}_m \to \mathbf{C} \text{ in } L^2(0,T;L^4(\Omega)). \tag{4.19f}$$

As a result we are able to pass to the limit with the Galerkin approximate solution (4.5) for $m \to \infty$. Let $\varphi \in C^1([0,T])$ be such that $\varphi(T) = 0$. We multiply (4.5c) and (4.5d) by $\varphi(t)$ and integrate by parts over $[0,T]$. Then the integration yields

$$\int_0^T (\mathbf{u}'_m(t), \mathbf{v}_j)\, \varphi(t)\ dt = -\int_0^T (\mathbf{u}_m(t), \mathbf{v}_j)\, \varphi'(t)\ dt + \Big[(\mathbf{u}_m(t), \mathbf{v}_j)\, \varphi(t) \Big]_0^T$$
$$\int_0^T (\mathbf{C}'_m(t), \mathbf{D}_j)\, \varphi(t)\ dt = -\int_0^T (\mathbf{C}_m(t), \mathbf{D}_j)\, \varphi'(t)\ dt + \Big[(\mathbf{C}_m(t), \mathbf{D}_j)\, \varphi(t) \Big]_0^T,$$

and consequently (4.5c) and (4.5d) become

$$\begin{aligned} -\int_0^T \Big(\mathbf{u}_m(t), \mathbf{v}_j\, \varphi'(t)\Big)\ dt + (\mathbf{u}_{0m}, \mathbf{v}_j)\, \varphi(0) + \nu \int_0^T \Big(\Big(\mathbf{u}_m(t), \mathbf{v}_j\, \varphi(t)\Big)\Big)\ dt + \\ + \int_0^T b\Big(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j\, \varphi(t)\Big)\ dt = -\int_0^T \int_\Omega \nabla(\mathbf{v}_j\, \varphi(t)) : \mathbf{T}_m(t)\ dx\, dt \end{aligned} \tag{4.20a}$$

$$\begin{aligned} &-\int_0^T \Big(\mathbf{C}_m(t), \mathbf{D}_j\, \varphi'(t)\Big)\ dt + (\mathbf{C}_{0m}, \mathbf{D}_j)\, \varphi(0) + \int_0^T B\Big(\mathbf{u}_m(t), \mathbf{C}_m(t), \mathbf{D}_j\, \varphi(t)\Big)\ dt - \\ &-\int_0^T \Big((\nabla \mathbf{u}_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla \mathbf{u}_m(t))^T, \mathbf{D}_j\, \varphi(t)\Big)\ dt + \varepsilon \int_0^T \Big(\Big(\mathbf{C}_m(t), \mathbf{D}_j\, \varphi(t)\Big)\Big)\ dt = \\ &= \int_0^T \Big(\operatorname{tr} \mathbf{C}_m(t)\, \mathbf{I} - (\operatorname{tr} \mathbf{C}_m(t))^2\, \mathbf{C}_m(t), \mathbf{D}_j\, \varphi(t)\Big)\ dt\,. \end{aligned} \tag{4.20b}$$

Passing to the limit in equation (4.20a)

By (4.19a) and (4.19b) we directly get

$$\int_0^T \Big(\mathbf{u}_m(t), \mathbf{v}_j\, \varphi'(t)\Big)\ dt \to \int_0^T \Big(\mathbf{u}(t), \mathbf{v}_j\, \varphi'(t)\Big)\ dt\,,$$
$$\int_0^T \Big(\Big(\mathbf{u}_m(t), \mathbf{v}_j\, \varphi(t)\Big)\Big)\ dt \to \int_0^T \Big(\Big(\mathbf{u}(t), \mathbf{v}_j\, \varphi(t)\Big)\Big)\ dt$$

as $m \to \infty$, respectively. Further, we pass to the limit with the trilinear form b, using identity (4.1) and the strong convergence (4.19c). Let us consider the difference

$$\int_0^T b\Big(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j\,\varphi(t)\Big)\; dt\; - \int_0^T b\Big(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}_j\,\varphi(t)\Big)\; dt\; =$$
$$= - \int_0^T b\Big(\mathbf{u}_m(t), \mathbf{v}_j\,\varphi(t), \mathbf{u}_m(t)\Big)\; dt\; + \int_0^T b\Big(\mathbf{u}(t), \mathbf{v}_j\,\varphi(t), \mathbf{u}(t)\Big)\; dt\; =$$
$$= \int_0^T b\Big(\mathbf{u}(t) - \mathbf{u}_m(t), \mathbf{v}_j\,\varphi(t), \mathbf{u}_m(t)\Big) + b\Big(\mathbf{u}(t), \mathbf{v}_j\,\varphi(t), \mathbf{u}(t) - \mathbf{u}_m(t)\Big)\; dt\,.$$

Then

$$\int_0^T b\Big(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j\,\varphi(t)\Big)\; dt\; - \int_0^T b\Big(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}_j\,\varphi(t)\Big)\; dt$$
$$\leq c \max_{t\in[0,T]} |\varphi(t)| \int_0^T \|\mathbf{u} - \mathbf{u}_m\|_4\, |||\mathbf{v}_j|||\, \|\mathbf{u}_m\|_4 + \|\mathbf{u}\|_4\, |||\mathbf{v}_j|||\, \|\mathbf{u} - \mathbf{u}_m\|_4\; dt$$
$$\leq c|||\mathbf{v}_j|||\Big(\, \|\mathbf{u}_m\|_{L^2(L^4)}\, \|\mathbf{u} - \mathbf{u}_m\|_{L^2(L^4)} + \|\mathbf{u}\|_{L^2(L^4)}\, \|\mathbf{u} - \mathbf{u}_m\|_{L^2(L^4)}\,\Big).$$

Since $\|\mathbf{u} - \mathbf{u}_m\|_{L^2(L^4)}$ goes to zero letting m to infinity, we can conclude that

$$\int_0^T b\Big(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j\,\varphi(t)\Big)\; dt\; \to \int_0^T b\Big(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}_j\,\varphi(t)\Big)\; dt\,.$$

Similarly, we first estimate the difference

$$\begin{aligned}
|\mathbf{T}_m(t) - \mathbf{T}(t)|^2 &= |\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}_m(t) - \mathrm{tr}\,\mathbf{C}\,\mathbf{C}(t)|^2\\
&= |\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}_m(t) - \mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}(t) + \mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}(t) - \mathrm{tr}\,\mathbf{C}(t)\,\mathbf{C}(t)|^2\\
&\leq |\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}_m(t) - \mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}(t)|^2 + |\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}(t) - \mathrm{tr}\,\mathbf{C}(t)\,\mathbf{C}(t)|^2\\
&= |\mathrm{tr}\,\mathbf{C}_m(t)|^2|\mathbf{C}_m(t) - \mathbf{C}(t)|^2 + |\mathrm{tr}\,\mathbf{C}_m(t) - \mathrm{tr}\,\mathbf{C}(t)|^2|\mathbf{C}(t)|^2.
\end{aligned}$$

Then by the Hölder inequality we get

$$\int_0^T \int_\Omega \nabla(\mathbf{v}_j\,\varphi(t)) : \mathbf{T}_m(t) - \nabla(\mathbf{v}_j\;\varphi(t)) : \mathbf{T}(t)\; dx\, dt$$
$$\leq \int_0^T \int_\Omega |\nabla(\mathbf{v}_j\,\varphi(t)) : (\mathbf{T}_m(t) - \mathbf{T}(t))|\; dx\, dt$$
$$\leq c \max_{t\in[0,T]} |\varphi(t)| |||\mathbf{v}_j||| \int_0^T \left(\int_\Omega |\mathbf{T}_m(t) - \mathbf{T}(t)|^2\; dx\right)^{1/2}\; dt$$
$$\leq c|||\mathbf{v}_j|||\Big(\, \|\mathrm{tr}\,\mathbf{C}_m\|_{L^2(L^4)}\, \|\mathbf{C}_m - \mathbf{C}\|_{L^2(L^4)} + \|\mathrm{tr}\,\mathbf{C}_m - \mathrm{tr}\,\mathbf{C}\|_{L^2(L^4)}\, \|\mathbf{C}\|_{L^2(L^4)}\,\Big),$$

which goes to zero due to the strong convergence (4.19f).

Passing to the limit in equation (4.20b)

Analogously as in equation (4.20a), we get by (4.19d) and (4.19e), the convergences

$$\int_0^T \Big(\mathbf{C}_m(t), \mathbf{D}_j\,\varphi'(t)\Big)\; dt\; \to \int_0^T \Big(\mathbf{C}(t), \mathbf{D}_j\;\varphi'(t)\Big)\; dt$$
$$\int_0^T \Big(\Big(\mathbf{C}_m(t), \mathbf{D}_j\,\varphi(t)\Big)\Big)\; dt\; \to \int_0^T \Big(\Big(\mathbf{C}(t), \mathbf{D}_j\,\varphi(t)\Big)\Big)\; dt\,,$$

respectively. Similarly, by identity (4.2) and the strong convergences (4.19c) and (4.19f) we can pass to the limit with the trilinear form B. Indeed,

$$\begin{aligned}
&\int_0^T B\Big(\mathbf{u}_m(t), \mathbf{C}_m(t), \mathbf{D}_j\, \varphi(t)\Big)\, dt - \int_0^T B\Big(\mathbf{u}(t), \mathbf{C}(t), \mathbf{D}_j\, \varphi(t)\Big)\, dt = \\
&= -\int_0^T B\Big(\mathbf{u}_m(t), \mathbf{D}_j\, \varphi(t), \mathbf{C}_m(t)\Big)\, dt + \int_0^T B\Big(\mathbf{u}(t), \mathbf{D}_j\, \varphi(t), \mathbf{C}(t)\Big)\, dt = \\
&= \int_0^T B\Big(\mathbf{u}(t) - \mathbf{u}_m(t), \mathbf{D}_j\, \varphi(t), \mathbf{C}_m(t)\Big) + B\Big(\mathbf{u}(t), \mathbf{D}_j\, \varphi(t), \mathbf{C}(t) - \mathbf{C}_m(t)\Big)\, dt\,,
\end{aligned}$$

and thus, by the Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^T B\Big(\mathbf{u}_m(t), \mathbf{C}_m(t), \mathbf{D}_j\, \varphi(t)\Big)\, dt - \int_0^T B\Big(\mathbf{u}(t), \mathbf{C}(t), \mathbf{D}_j\, \varphi(t)\Big)\, dt \\
&\le c \max_{t\in[0,T]} |\varphi(t)|\, \|\nabla \mathbf{D}_j\|_2 \int_0^T \|\mathbf{u}_m - \mathbf{u}\|_4\, \|\mathbf{C}_m\|_4 + \|\mathbf{u}\|_4\, \|\mathbf{C}_m - \mathbf{C}\|_4\, dt \\
&\le c\, \|\mathbf{D}_j\|_W \Big(\|\mathbf{u}_m - \mathbf{u}\|_{L^2(L^4)}\, \|\mathbf{C}_m\|_{L^2(L^4)} + \|\mathbf{u}\|_{L^2(L^4)}\, \|\mathbf{C}_m - \mathbf{C}\|_{L^2(L^4)}\Big)\,,
\end{aligned}$$

which goes to zero letting m to infinity. Further, we can write

$$\begin{aligned}
\nabla \mathbf{u}_m(t)\mathbf{C}_m(t) - \nabla \mathbf{u}(t)\mathbf{C}(t) &= \nabla \mathbf{u}_m(t)\Big(\mathbf{C}_m(t) - \mathbf{C}(t)\Big) + \nabla\Big(\mathbf{u}_m(t) - \mathbf{u}(t)\Big)\mathbf{C}(t) \\
\mathbf{C}_m(t)(\nabla \mathbf{u}_m(t))^T - \mathbf{C}(t)(\nabla \mathbf{u}(t))^T &= \Big(\mathbf{C}_m(t) - \mathbf{C}(t)\Big)(\nabla \mathbf{u}_m(t))^T + \mathbf{C}(t)\nabla\Big(\mathbf{u}_m(t) - \mathbf{u}(t)\Big)^T.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^T \Big(\nabla \mathbf{u}_m(t)\mathbf{C}_m(t) - \nabla \mathbf{u}(t)\mathbf{C}(t), \mathbf{D}_j\ \varphi(t)\Big)\, dt \\
&= c \max_{t\in[0,T]} |\varphi(t)|\, \|\mathbf{D}_j\|_4 \int_0^T |||\mathbf{u}_m|||\, \|\mathbf{C}_m - \mathbf{C}\|_4 + |||\mathbf{u}_m - \mathbf{u}|||\, \|\mathbf{C}\|_4\, dt \\
&\le c\, \|\mathbf{D}_j\|_W \Big(\, \|\mathbf{u}_m\|_{L^2(V)}\, \|\mathbf{C}_m - \mathbf{C}\|_{L^2(L^4)} + \|\mathbf{u}_m - \mathbf{u}\|_{L^2(L^4)}\, \|\mathbf{C}\|_{L^2(L^4)}\Big),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^T \Big(\mathbf{C}_m(t)(\nabla \mathbf{u}_m(t))^T - \mathbf{C}(t)(\nabla \mathbf{v}(t))^T, \mathbf{D}_j\ \varphi(t)\Big)\, dt \\
&\le c\, \|\mathbf{D}_j\|_W \Big(\, \|\mathbf{C}_m - \mathbf{C}\|_{L^2(L^4)}\, \|\mathbf{u}_m\|_{L^2(V)} + \|\mathbf{C}\|_{L^2(L^4)}\, \|\mathbf{u}_m - \mathbf{u}\|_{L^2(L^4)}\Big).
\end{aligned}$$

By (4.19c) and (4.19f), we thus have

$$\begin{aligned}
&\int_0^T \Big(\nabla \mathbf{u}_m(t)\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla \mathbf{u}_m(t))^T, \mathbf{D}_j\ \varphi(t)\Big)\, dt \\
&\qquad\qquad \to \int_0^T \Big(\nabla \mathbf{u}(t)\mathbf{C}(t) + \mathbf{C}(t)(\nabla \mathbf{u}(t))^T, \mathbf{D}_j\ \varphi(t)\Big)\, dt \ \text{ as } \ m \to \infty.
\end{aligned}$$

What remains is passing to the limit in the last integral in (4.20b). The first part is straightforward. By (4.19f)

$$\int_0^T \Big(\operatorname{tr} \mathbf{C}_m(t)\, \mathbf{I}, \mathbf{D}_j\, \varphi(t)\Big)\, dt \to \int_0^T \Big(\operatorname{tr} \mathbf{C}\, \mathbf{I}, \mathbf{D}_j\, \varphi(t)\Big)\, dt\,,$$

since

$$\int_0^T \Big(\mathrm{tr}\ (\mathbf{C}_m(t) - \mathbf{C}(t))\ \mathbf{I}, \mathbf{D}_j\, \varphi(t)\Big)\ dt\ \le c \max_{t\in[0,T]} |\varphi(t)|\ \|\mathrm{tr}\, \mathbf{D}_j\|_2\ \|\mathrm{tr}\ (\mathbf{C}_m - \mathbf{C})\|_{L^2(L^4)}\,.$$

To treat the second part of the last integral in (4.20b), we rewrite the difference

$$\begin{aligned}
(\mathrm{tr}\, \mathbf{C}_m(t))^2\, \mathbf{C}_m(t) - (\mathrm{tr}\, \mathbf{C}(t))^2\, \mathbf{C}(t) = \\
&= \mathrm{tr}\Big(\mathbf{C}_m(t) - \mathbf{C}(t)\Big)\, \mathrm{tr}\, \mathbf{C}(t)\, \mathbf{C}(t) + (\mathrm{tr}\, \mathbf{C}_m(t))^2 \Big(\mathbf{C}_m(t) - \mathbf{C}(t)\Big) + \\
&+ \mathrm{tr}\, \mathbf{C}_m(t)\, \mathrm{tr}\Big(\mathbf{C}_m(t) - \mathbf{C}(t)\Big)\, \mathbf{C}(t)
\end{aligned}$$

and estimate

$$\begin{aligned}
&\int_0^T \Big(\, (\mathrm{tr}\, \mathbf{C}_m(t))^2\, \mathbf{C}_m(t) - (\mathrm{tr}\, \mathbf{C}(t))^2\, \mathbf{C}(t), \mathbf{D}_j\, \varphi(t)\Big)\ dt \\
&\le c \max_{t\in[0,T]} |\varphi(t)|\ \|\mathbf{D}_j\|_4 \Bigg(\int_0^T \|\mathrm{tr}\, \mathbf{C}_m - \mathrm{tr}\, \mathbf{C})\|_4\ \|\mathrm{tr}\, \mathbf{C}\|_4\ \|\mathbf{C}\|_4\ dt + \\
&\qquad + \int_0^T \|\mathrm{tr}\, \mathbf{C}_m\|_4^2\ \|\mathbf{C}_m - \mathbf{C}\|_4\ dt + \\
&\qquad + \int_0^T \|\mathbf{C}_m\|_4\ \|\mathrm{tr}\, \mathbf{C}_m - \mathrm{tr}\, \mathbf{C}\|_4\ \|\mathbf{C}\|_4\ dt \Bigg) \\
&\le c\, \|\mathbf{D}_j\|_W \Bigg(\|\mathrm{tr}\ (\mathbf{C}_m - \mathbf{C})\|_{L^2(L^4)}\ \|\mathbf{C}\|_{L^4(L^4)}^2 + \|\mathrm{tr}\, \mathbf{C}_m\|_{L^4(L^4)}^2\ \|\mathbf{C} - \mathbf{C}_m\|_{L^2(L^4)} + \\
&\qquad + \|\mathrm{tr}\, \mathbf{C}_m\|_{L^4(L^4)}\ \|\mathrm{tr}\ (\mathbf{C}_m - \mathbf{C})\|_{L^2(L^4)}\ \|\mathbf{C}\|_{L^4(L^4)} \Bigg).
\end{aligned}$$

Due to (4.19f) we conclude

$$\int_0^T \Big(\, (\mathrm{tr}\, \mathbf{C}_m(t))^2\, \mathbf{C}_m(t), \mathbf{D}_j\, \varphi(t)\Big)\ dt\ \to \int_0^T \Big(\, (\mathrm{tr}\, \mathbf{C}(t))^2\, \mathbf{C}(t), \mathbf{D}_j\, \varphi(t)\Big)\ dt$$

letting m to infinity.

The above limits are true for each $\mathbf{v}_j$ and $\mathbf{D}_j$, and hence for all $\mathbf{v} \in V$ and all $\mathbf{D} \in W$, since $\{\mathbf{v}_j\}_{j=1}^\infty$ and $\{\mathbf{D}_j\}_{j=1}^\infty$ form the basis of V and W, respectively. The limiting process in the test functions is easy due to the linearity of all operators in the test functions. The limit of (4.20) is then

$$\begin{aligned}
&- \int_0^T \Big(\mathbf{u}(t), \mathbf{v}\, \varphi'(t)\Big)\ dt\ + (\mathbf{u}_0, \mathbf{u})\, \varphi(0) + \nu \int_0^T \Big(\Big(\mathbf{u}(t), \mathbf{v}\, \varphi(t)\Big)\Big)\ dt + \\
&+ \int_0^T b\Big(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}\, \varphi(t)\Big)\ dt\ = - \int_0^T \int_\Omega \nabla(\mathbf{u}\, \varphi(t)) : \mathbf{T}(t)\ dx\, dt
\end{aligned}$$

$$\begin{aligned}
&- \int_0^T \Big(\mathbf{C}(t), \mathbf{D}\, \varphi'(t)\Big)\ dt\ + (\mathbf{C}_0, \mathbf{D})\, \varphi(0) + \int_0^T B\Big(\mathbf{u}(t), \mathbf{C}(t), \mathbf{D}\, \varphi(t)\Big)\ dt + \\
&- \int_0^T \Big((\nabla \mathbf{u}(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla \mathbf{u}(t))^T, \mathbf{D}\, \varphi(t)\Big)\ dt\ + \varepsilon \int_0^T \Big(\Big(\mathbf{C}(t), \mathbf{D}\, \varphi(t)\Big)\Big)\ dt\ = \\
&= \int_0^T \Big(\mathrm{tr}\, \mathbf{C}(t)\, \mathbf{I} - (\mathrm{tr}\, \mathbf{C}(t))^2\, \mathbf{C}(t), \mathbf{D}\, \varphi(t)\Big)\ dt\,.
\end{aligned}$$

Initial condition

We shall first realise that the initial condition (4.4b) makes sense. Since the couple $(\mathbf{u}, \mathbf{C}) \in L^2(0,T;V) \times L^2(0,T;W)$ satisfies the weak formulation (4.4), we know that $\mathbf{u}$ is almost everywhere equal to a continuous function from $[0,T]$ to V^* and $\mathbf{C}$ is almost everywhere equal to a continuous function from $[0,T]$ to W^*, c.f. [52, 142] or Appendix A.

We multiply (4.4c) and (4.4d) by $\varphi(t)$ and integrate per partes in time. Then we could see by comparison with the limit equations, that

$$(\mathbf{u}_0 - \mathbf{u}(0), \mathbf{v})\, \varphi(0) = 0 \quad \text{and} \quad (\mathbf{C}_0 - \mathbf{C}(0), \mathbf{D})\, \varphi(0) = 0$$

for each $\mathbf{v}$, $\mathbf{D}$, φ of the corresponding type. We can choose φ such that $\varphi(0) = 1$, and therefore

$$(\mathbf{u}_0 - \mathbf{u}(0), \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in V \quad \text{and} \quad (\mathbf{C}_0 - \mathbf{C}(0), \mathbf{D}) = 0 \quad \forall\, \mathbf{D} \in W.$$

This implies that the initial condition (4.4b) is satisfied. □

4.3. Higher regularity

Let us point out that the following property needed for the uniqueness study

$$\mathbf{C} \in C([0,T]; L^2(\Omega)),\ 2\langle \mathbf{C}', \mathbf{C} \rangle = \frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{C}\|_2^2$$

is missing even in two space dimensions, since in the above existence result we only obtain that $\mathbf{C}' \in L^{4/3}(0,T;W^*)$. In order to obtain uniqueness of the weak solution we firstly investigate possible higher regularity of our weak solution.

Theorem 4.5. *(regularity of weak solutions)*
Let the domain Ω be of class C^2 and $(\mathbf{u}_0, \mathbf{C}_0) \in [H^2(\Omega)^2 \cap V] \times H^2(\Omega)^{2\times 2}$. Then the weak solution (4.4) satisfies additionally

$$\begin{aligned} &\mathbf{u}' \in L^\infty(0,T;H) \cap L^2(0,T;V), && \mathbf{C}' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W), \\ &\mathbf{u} \in L^\infty(0,T;H^2(\Omega)), && \mathbf{C} \in L^\infty(0,T;H^2(\Omega)). \end{aligned}$$

Proof. We return to the Galerkin approximation (4.5) used in the proof of the existence of weak solutions. We need to show that for this approximate solution there exists a positive constant K independent of m such that

$$\|\mathbf{u}_m'\|_{L^\infty(H)} + \|\mathbf{u}_m'\|_{L^2(V)} + \|\mathbf{C}_m'\|_{L^\infty(L^2)} + \|\mathbf{C}_m'\|_{L^2(W)} \le K.$$

More regular initial data

In order to obtain a regularity result we now assume that our basis functions $\{\mathbf{v}_i\}_{i=1}^\infty$, $\{\mathbf{D}_i\}_{i=1}^\infty$, are the eigenfunctions of the Stokes and the Laplace operator, respectively. See Section A.3 in Appendix A and the references therein for more details.

Since $\mathbf{u}_0 \in H^2(\Omega) \cap V$, $\mathbf{C}_0 \in H^2(\Omega)$, we can choose $\mathbf{u}_{0m}$ and $\mathbf{C}_{0m}$ as the orthogonal projections in $V \cap H^2(\Omega)$ of $\mathbf{u}_0$ onto the space spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and in $H^2(\Omega)$ of $\mathbf{C}_0$ onto the space spanned by $\mathbf{D}_1, \ldots, \mathbf{D}_m$, respectively. Then, for $\tilde{c} = \tilde{c}(\Omega) > 0$,

$$\|\mathbf{u}_{0m}\|_{2,2} \leq \tilde{c}\,\|\mathbf{u}_0\|_{2,2} \qquad \text{and} \qquad \mathbf{u}_{0m} \to \mathbf{u}_0 \text{ in } H^2(\Omega)^2, \text{ as } m \to \infty, \tag{4.21a}$$

$$\|\mathbf{C}_{0m}\|_{2,2} \leq \tilde{c}\,\|\mathbf{C}_0\|_{2,2} \qquad \text{and} \qquad \mathbf{C}_{0m} \to \mathbf{C}_0 \text{ in } H^2(\Omega)^{2\times 2}, \text{ as } m \to \infty, \tag{4.21b}$$

cf. Section A.3 or [95]. We multiply equations (4.5c) and (4.5d) for the Galerkin approximation by $g'_{jm}(t)$ and $G'_{jm}(t)$, respectively. We add the resulting equations for $j = 1, \ldots, m$. This gives us

$$\begin{aligned}
&\|\mathbf{u}'_m(t)\|_2^2 + b\left(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{u}'_m(t)\right) + \nu\left(\left(\mathbf{u}_m(t), \mathbf{u}'_m(t)\right)\right) = -\left(\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{C}_m(t), \nabla\mathbf{u}'_m(t)\right)\\
&\|\mathbf{C}'_m(t)\|_2^2 + B\left(\mathbf{u}_m(t), \mathbf{C}_m(t), \mathbf{C}'_m(t)\right) - \Big((\nabla\mathbf{u}_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}_m(t))^T, \mathbf{C}'_m(t)\Big) +\\
&\qquad + \varepsilon\left(\left(\mathbf{C}_m(t), \mathbf{C}'_m(t)\right)\right) = \Big(\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{I} - (\operatorname{tr}\mathbf{C}_m(t))^2\mathbf{C}_m(t), \mathbf{C}'_m(t)\Big).
\end{aligned}$$

In particular, at time $t = 0$, we get, by the Green formula,

$$\|\mathbf{u}'_m(0)\|_2^2 = -b\Big(\mathbf{u}_{0m}, \mathbf{u}_{0m}, \mathbf{u}'_m(0)\Big) + \nu\Big(\Delta\mathbf{u}_{0m}, \mathbf{u}_{0m}\Big) + \Big(\operatorname{div}\left(\operatorname{tr}\mathbf{C}_{0m}\,\mathbf{C}_{0m}\right), \mathbf{u}'_m(0)\Big).$$

Then the Hölder inequality implies

$$\|\mathbf{u}'_m(0)\|_2 \leq \|\mathcal{B}\mathbf{u}_{0m}\|_2 + \nu\,\|\Delta\mathbf{u}_{0m}\|_2 + \|\operatorname{div}\left(\operatorname{tr}\mathbf{C}_{0m}\,\mathbf{C}_{0m}\right)\|_2.$$

Using (4.21) we have the inequalities

$$\begin{aligned}
\|\Delta\mathbf{u}_{0m}\|_2 &\leq c\,\|\mathbf{u}_{0m}\|_{2,2} \leq c_0\,\|\mathbf{u}_0\|_{2,2}\\
\|\operatorname{div}\left(\operatorname{tr}\mathbf{C}_{0m}\,\mathbf{C}_{0m}\right)\|_2 &= \|\nabla\operatorname{tr}\mathbf{C}_{0m}\cdot\mathbf{C}_{0m} + \operatorname{tr}\mathbf{C}_{0m}\operatorname{div}\mathbf{C}_{0m}\|_2\\
&\leq c\,\|\operatorname{tr}\mathbf{C}_{0m}\|_{2,2}\,\|\mathbf{C}_{0m}\|_{2,2} \leq c\,\|\mathbf{C}_{0m}\|_{2,2}^2 \leq c_1\,\|\mathbf{C}_0\|_{2,2}^2.
\end{aligned}$$

For the trilinear form b it holds, by the Hölder inequality, that

$$b\left(\mathbf{u}, \mathbf{u}, \mathbf{v}\right) \leq c\,\|\mathbf{u}\|_4\,\|\nabla\mathbf{u}\|_4\,\|\mathbf{v}\|_2 \leq c|\!|\!|\mathbf{u}|\!|\!|\,\|\mathbf{u}\|_{2,2}\,\|\mathbf{v}\|_2,\ \ \mathbf{u} \in H^2(\Omega),\ \ \mathbf{v} \in L^2(\Omega),$$

and hence

$$\|\mathcal{B}_{\mathbf{u}_{0m}}\|_2 \leq c|\!|\!|\mathbf{u}_{0m}|\!|\!|\,\|\mathbf{u}_{0m}\|_{2,2} \leq c\,\|\mathbf{u}_{0m}\|_{2,2}^2 \leq c_2\,\|\mathbf{u}_0\|_{2,2}.$$

Thus we get

$$\|\mathbf{u}'_m(0)\|_2 \leq (\nu c_0 + c_2)\,\|\mathbf{u}_0\|_{2,2} + c_1\,\|\mathbf{C}_0\|_{2,2}^2 =: a_1, \tag{4.22}$$

which implies that $\mathbf{u}'_m(0)$ belongs to a bounded set in H.

Further, we have the analogous result for the initial value $\mathbf{C}'_m(0)$. By the same arguments as above we obtain

$$\begin{aligned}
\|\mathbf{C}'_m(0)\|_2^2 &= -B\Big(\mathbf{u}_{0m}, \mathbf{C}_{0m}, \mathbf{C}'_m(0)\Big) + \Big((\nabla\mathbf{u}_{0m})\mathbf{C}_{0m} + \mathbf{C}_{0m}(\nabla\mathbf{u}_{0m})^T, \mathbf{C}'_m(0)\Big) +\\
&\quad + \varepsilon\Big(\Delta\mathbf{C}_{0m}, \mathbf{C}'_m(0)\Big) + \Big(\operatorname{tr}\mathbf{C}_{0m}\,\mathbf{I} - (\operatorname{tr}\mathbf{C}_{0m})^2\mathbf{C}_{0m}, \mathbf{C}'_m(0)\Big)\\
\|\mathbf{C}'_m(0)\|_2 &\leq \|\mathcal{B}(\mathbf{u}_{0m}, \mathbf{C}_{0m})\|_2 + \Big\|(\nabla\mathbf{u}_{0m})\mathbf{C}_{0m} + \mathbf{C}_{0m}(\nabla\mathbf{u}_{0m})^T\Big\|_2 + \varepsilon\,\|\Delta\mathbf{C}_{0m}\|_2 +\\
&\quad + \Big\|\operatorname{tr}\mathbf{C}_{0m}\,\mathbf{I} - (\operatorname{tr}\mathbf{C}_{0m})^2\mathbf{C}_{0m}\Big\|_2.
\end{aligned}$$

Again, using (4.21) we get

$$\begin{aligned}
\|\Delta \mathbf{C}_{0m}\|_2 &\le c\,\|\mathbf{C}_{0m}\|_{2,2} \le c_3\,\|\mathbf{C}_0\|_{2,2}\\
\left\|(\nabla \mathbf{u}_{0m})\mathbf{C}_{0m} + \mathbf{C}_{0m}(\nabla \mathbf{u}_{0m})^T\right\|_2 &\le c\,\|\nabla \mathbf{u}_{0m}\|_4\,\|\mathbf{C}_{0m}\|_4 \le c\,\|\mathbf{u}_{0m}\|_{2,2}\,\|\mathbf{C}_{0m}\|_{2,2}\\
&\le c_4\,\|\mathbf{u}_0\|_{2,2}\,\|\mathbf{C}_0\|_2\\
\left\|\operatorname{tr}\mathbf{C}_{0m}\,\mathbf{I} - (\operatorname{tr}\mathbf{C}_{0m})^2\,\mathbf{C}_{0m}\right\|_2 &\le \sqrt{2}\,\|\operatorname{tr}\mathbf{C}_{0m}\|_2 + \|\operatorname{tr}\mathbf{C}_{0m}\|_6^3\\
&\le c\,\|\mathbf{C}_{0m}\|_{2,2} + c\,\|\mathbf{C}_{0m}\|_{2,2}^3 \le c_5\,\|\mathbf{C}_0\|_{2,2} + c_6\,\|\mathbf{C}_0\|_{2,2}^2 .
\end{aligned}$$

By the Hölder inequality we have for the trilinear form B the following estimate

$$B\,(\mathbf{u},\mathbf{C},\mathbf{D}) \le c\,\|\mathbf{u}\|_4\,\|\nabla\mathbf{C}\|_4\,\|\mathbf{v}\|_2 \le c\,\|\mathbf{u}\|_{1,2}\,\|\mathbf{C}\|_{2,2}\,\|\mathbf{D}\|_2\,,$$
$$\mathbf{u}\in H^2(\Omega),\ \mathbf{C}\in H^2(\Omega),\ \mathbf{D}\in L^2(\Omega),$$

and thus

$$\|\mathcal{B}(\mathbf{u}_{0m},\mathbf{C}_{0m})\|_2 \le c\,\|\mathbf{u}_{0m}\|_{2,2}\,\|\mathbf{C}_{0m}\|_{2,2} \le c_7\,\|\mathbf{u}_0\|_{2,2}\,\|\mathbf{C}_0\|_{2,2}\,.$$

Finally, we obtain

$$\|\mathbf{C}'_m(0)\|_2 \le (\varepsilon c_3 + c_5)\,\|\mathbf{C}_0\|_{2,2} + (c_4 + c_7)\,\|\mathbf{u}_0\|_{2,2}\,\|\mathbf{C}_0\|_{2,2} + c_6\,\|\mathbf{C}_0\|_{2,2}^3 =: a_2. \tag{4.23}$$

This implies that $\mathbf{C}'_m(0)$ belongs to a bounded set in $L^2(\Omega)$.

A priori estimate for $(\mathbf{u}'_m, \mathbf{C}'_m)$

Differentiating equations (4.5c) - (4.5d) for the Galerkin approximation in time we get

$$\begin{aligned}
&(\mathbf{u}''_m(t),\mathbf{v}_j) + \nu\,((\mathbf{u}'_m(t),\mathbf{v}_j)) + b\,(\mathbf{u}'_m(t),\mathbf{u}_m(t),\mathbf{v}_j) + b\,(\mathbf{u}_m(t),\mathbf{u}'_m(t),\mathbf{v}_j) =\\
&\qquad = -\,(\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t),\nabla\mathbf{v}_j) - (\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{C}'_m(t),\nabla\mathbf{v}_j)
\end{aligned}\tag{4.24a}$$

$$\begin{aligned}
&(\mathbf{C}''_m(t),\mathbf{D}_j) + B\,(\mathbf{u}'_m(t),\mathbf{C}_m(t),\mathbf{D}_j) + B\,(\mathbf{u}_m(t),\mathbf{C}'_m(t),\mathbf{D}_j) -\\
&\qquad - \Big((\nabla\mathbf{u}'_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}'_m(t))^T,\mathbf{D}_j\Big) -\\
&\qquad - \Big((\nabla\mathbf{u}_m(t))\mathbf{C}'_m(t) + \mathbf{C}'_m(t)(\nabla\mathbf{u}_m(t))^T,\mathbf{D}_j\Big) + \varepsilon\,((\mathbf{C}'_m(t),\mathbf{D}_j)) =\\
&\qquad = (\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{I},\mathbf{D}_j) - \Big((\operatorname{tr}\mathbf{C}_m(t))^2\,\mathbf{C}'_m(t) - 2\operatorname{tr}\mathbf{C}_m(t)\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t),\mathbf{D}_j\Big)\,,
\end{aligned}\tag{4.24b}$$

$$j = 1,\dots,m,\ t\in[0,T].$$

Now, we shall find the a priori estimates for $\mathbf{u}'_m$ and $\mathbf{C}'_m$. We multiply equations (4.24a) and (4.24b) by $g'_{jm}(t)$ and $G'_{jm}(t)$, respectively. Summing the resulting equations for $j = 1,\dots,m$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\mathbf{u}'_m(t)\|_2^2 + b\,(\mathbf{u}'_m(t),\mathbf{u}_m(t),\mathbf{u}'_m(t)) + \nu |\!|\!|\mathbf{u}'_m(t)|\!|\!|^2 =\\
&\qquad = -\,(\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t),\nabla\mathbf{u}'_m(t)) - (\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{C}'_m(t),\nabla\mathbf{u}'_m(t))\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\mathbf{C}'_m(t)\|_2^2 + B\,(\mathbf{u}'_m(t),\mathbf{C}_m(t),\mathbf{C}'_m(t)) - \Big((\nabla\mathbf{u}'_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}'_m(t))^T,\mathbf{C}'_m(t)\Big) -\\
&\qquad - \Big((\nabla\mathbf{u}_m(t))\mathbf{C}'_m(t) + \mathbf{C}'_m(t)(\nabla\mathbf{u}_m(t))^T,\mathbf{C}'_m(t)\Big) + \varepsilon\,\|\nabla\mathbf{C}'_m(t)\|_2^2 =\\
&\qquad = (\operatorname{tr}\mathbf{C}'_m(t),\operatorname{tr}\mathbf{C}'_m(t)) - \Big((\operatorname{tr}\mathbf{C}_m(t))^2\mathbf{C}'_m(t),\mathbf{C}'_m(t)\Big) -\\
&\qquad - 2\,(\operatorname{tr}\mathbf{C}_m(t)\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t),\mathbf{C}'_m(t))\,.
\end{aligned}$$

Then their sum yields

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\mathbf{u}_m'(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\mathbf{C}_m'(t)\|_2^2 + \nu|||\mathbf{u}_m'(t)|||^2 + \varepsilon\,\|\nabla\mathbf{C}_m'(t)\|_2^2 =& \\
= -b\Big(\mathbf{u}_m'(t), \mathbf{u}_m(t), \mathbf{u}_m'(t)\Big) - B\Big(\mathbf{u}_m'(t), \mathbf{C}_m(t), \mathbf{C}_m'(t)\Big)-& \\
- \Big(\mathrm{tr}\,\mathbf{C}_m'(t)\,\mathbf{C}_m(t), \nabla\mathbf{u}_m'(t)\Big) - \Big(\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}_m'(t), \nabla\mathbf{u}_m'(t)\Big)+& \\
+ \Big((\nabla\mathbf{u}_m'(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}_m'(t))^T, \mathbf{C}_m'(t)\Big)+& \\
+ \Big((\nabla\mathbf{u}_m(t))\mathbf{C}_m'(t) + \mathbf{C}_m'(t)(\nabla\mathbf{u}_m(t))^T, \mathbf{C}_m'(t)\Big) + \Big(\mathrm{tr}\,\mathbf{C}_m'(t), \mathrm{tr}\,\mathbf{C}_m'(t)\Big)-& \\
- \Big((\mathrm{tr}\,\mathbf{C}_m(t))^2\mathbf{C}_m'(t), \mathbf{C}_m'(t)\Big) - 2\Big(\mathrm{tr}\,\mathbf{C}_m(t)\,\mathrm{tr}\,\mathbf{C}_m'(t)\,\mathbf{C}_m(t), \mathbf{C}_m'(t)\Big).&
\end{aligned} \tag{4.25}$$

Further, we shall estimate the integrals on the right-hand side of (4.25). Using the Hölder and the Young inequalities, the interpolation inequality (2.8a) and identity (4.2) for B, we have

$$\begin{aligned}
b\Big(\mathbf{u}_m'(t), \mathbf{u}_m(t), \mathbf{u}_m'(t)\Big) &\le c|||\mathbf{u}_m(t)|||\,\|\mathbf{u}_m'(t)\|_4^2 \\
&\le c|||\mathbf{u}_m(t)|||\,|||\mathbf{u}_m'(t)|||\,\|\mathbf{u}_m'(t)\|_2 \\
&\le \frac{\nu}{10}|||\mathbf{u}_m'(t)|||^2 + \frac{c}{\nu}\,\|\mathbf{u}_m'(t)\|_2^2\,|||\mathbf{u}_m(t)|||^2 \qquad (4.26\text{a}) \\
B\Big(\mathbf{u}_m'(t), \mathbf{C}_m(t), \mathbf{C}_m'(t)\Big) &= -B\Big(\mathbf{u}_m'(t), \mathbf{C}_m'(t), \mathbf{C}_m(t)\Big) \\
&\le c\,\|\nabla\mathbf{C}_m'(t)\|_2\,\|\mathbf{u}_m'(t)\|_4\,\|\mathbf{C}_m(t)\|_4 \\
&\le \frac{\varepsilon}{14}\,\|\nabla\mathbf{C}_m'(t)\|_2^2 + \frac{c}{\varepsilon}|||\mathbf{u}_m'(t)|||\,\|\mathbf{u}_m'(t)\|_2\,\|\mathbf{C}_m(t)\|_4^2 \\
&\le \frac{\varepsilon}{14}\,\|\nabla\mathbf{C}_m'(t)\|_2^2 + \frac{\nu}{10}|||\mathbf{u}_m'(t)|||^2 + \frac{c}{\nu\varepsilon^2}\,\|\mathbf{u}_m'(t)\|_2^2\,\|\mathbf{C}_m(t)\|_4^4\,.
\end{aligned} \tag{4.26b}$$

For the nonlinear viscoelastic terms, by the Hölder, the Young and the interpolation inequality (2.8b), it holds that

$$\begin{aligned}
(\mathrm{tr}\,\mathbf{C}_m'(t)\,\mathbf{C}_m(t), \nabla\mathbf{u}_m'(t)) &\le c\,\|\mathrm{tr}\,\mathbf{C}_m'(t)\|_4\,\|\mathbf{C}_m(t)\|_4\,|||\mathbf{u}_m'(t)||| \\
&\le c\,\|\mathbf{C}_m'(t)\|_4\,\|\mathbf{C}_m(t)\|_4\,|||\mathbf{u}_m'(t)||| \\
&\le c\,\Big(\|\mathbf{C}_m'(t)\|_2^{1/2}\,\|\nabla\mathbf{C}_m'(t)\|_2^{1/2} + \|\mathbf{C}_m'(t)\|_2\Big)\,\|\mathbf{C}_m(t)\|_4\,|||\mathbf{u}_m'(t)||| \\
&\le \frac{\nu}{10}|||\mathbf{u}_m'(t)|||^2 + \frac{\varepsilon}{14}\,\|\nabla\mathbf{C}_m'(t)\|_2^2 + \\
&+ \frac{c}{\nu}\,\|\mathbf{C}_m'(t)\|_2^2\,\|\mathbf{C}_m(t)\|_4^2 + \frac{c}{\nu^2\varepsilon}\,\|\mathbf{C}_m'(t)\|_2^2\,\|\mathbf{C}_m(t)\|_4^4\,,
\end{aligned} \tag{4.26c}$$

and similarly

$$\begin{aligned}
(\mathrm{tr}\,\mathbf{C}_m(t)\,\mathbf{C}_m'(t), \nabla\mathbf{u}_m'(t)) &\le c\,\|\mathrm{tr}\,\mathbf{C}_m(t)\|_4\,\|\mathbf{C}_m'(t)\|_4\,|||\mathbf{u}_m'(t)||| \\
&\le c\,\|\mathbf{C}_m(t)\|_4\,\|\mathbf{C}_m'(t)\|_4\,|||\mathbf{u}_m'(t)||| \\
&\le \frac{\nu}{10}|||\mathbf{u}_m'(t)|||^2 + \frac{\varepsilon}{14}\,\|\nabla\mathbf{C}_m'(t)\|_2^2 + \\
&+ \frac{c}{\nu}\,\|\mathbf{C}_m'(t)\|_2^2\,\|\mathbf{C}_m(t)\|_4^2 + \frac{c}{\nu^2\varepsilon}\,\|\mathbf{C}_m'(t)\|_2^2\,\|\mathbf{C}_m(t)\|_4^4\,.
\end{aligned} \tag{4.26d}$$

The remaining elastic terms can be estimated in an analogous way, i.e.

$$
\begin{aligned}
&\Big((\nabla \mathbf{u}_m(t))\mathbf{C}'_m(t) + \mathbf{C}'_m(t)(\nabla \mathbf{u}_m(t))^T, \mathbf{C}'_m(t)\Big) \\
&\qquad \leq c\, \|\mathbf{C}'_m(t)\|_4^2\, |||\mathbf{u}_m(t)||| \\
&\qquad \leq c \Big(\|\mathbf{C}'_m(t)\|_2 \, \|\nabla \mathbf{C}'_m(t)\|_2 + \|\mathbf{C}'_m(t)\|_2^2 \Big)\, |||\mathbf{u}_m(t)||| \\
&\qquad \leq \frac{\varepsilon}{14}\, \|\nabla \mathbf{C}'_m(t)\|_2^2 + \frac{c}{\varepsilon}\, \|\mathbf{C}'_m(t)\|_2^2\, |||\mathbf{u}_m(t)|||^2 + c\, \|\mathbf{C}'_m(t)\|_2^2\, |||\mathbf{u}_m(t)|||
\end{aligned} \tag{4.26e}
$$

$$
\begin{aligned}
&\Big((\nabla \mathbf{u}'_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla \mathbf{u}'_m(t))^T, \mathbf{C}'_m(t)\Big) \\
&\qquad \leq c\, \|\mathbf{C}_m(t)\|_4\, \|\mathbf{C}'_m(t)\|_4\, |||\mathbf{u}'_m(t)||| \\
&\qquad \leq \frac{\nu}{10} |||\mathbf{u}'_m(t)|||^2 + \frac{\varepsilon}{14}\, \|\nabla \mathbf{C}'_m(t)\|_2^2 + \frac{c}{\nu}\, \|\mathbf{C}'_m(t)\|_2^2\, \|\mathbf{C}_m(t)\|_4^2 + \\
&\qquad + \frac{c}{\nu^2 \varepsilon}\, \|\mathbf{C}'_m(t)\|_2^2\, \|\mathbf{C}_m(t)\|_4^4 .
\end{aligned} \tag{4.26f}
$$

The last two terms are again estimated using the Hölder, the Young and the interpolation inequality (2.8b) as follows

$$
\begin{aligned}
\Big((\operatorname{tr} \mathbf{C}_m(t))^2 \mathbf{C}'_m(t), \mathbf{C}'_m(t)\Big) &\leq c\, \|\operatorname{tr} \mathbf{C}_m(t)\|_4^2\, \|\mathbf{C}'_m(t)\|_4^2 \\
&\leq c\, \|\mathbf{C}_m(t)\|_4^2 \Big(\|\mathbf{C}'_m(t)\|_2\, \|\nabla \mathbf{C}'_m(t)\|_2 + \|\mathbf{C}'_m(t)\|_2^2 \Big) \\
&\leq \frac{\varepsilon}{14}\, \|\nabla \mathbf{C}'_m(t)\|_2^2 + \frac{c}{\varepsilon}\, \|\mathbf{C}'_m(t)\|_2^2\, \|\mathbf{C}_m(t)\|_4^4 + \\
&\quad + c\, \|\mathbf{C}'_m(t)\|_2^2\, \|\mathbf{C}_m(t)\|_4^2 ,
\end{aligned}
$$

$$
\begin{aligned}
2 \left(\operatorname{tr} \mathbf{C}_m(t) \operatorname{tr} \mathbf{C}'_m(t)\, \mathbf{C}_m(t), \mathbf{C}'_m(t)\right) &\leq c\, \|\operatorname{tr} \mathbf{C}_m(t)\|_4\, \|\operatorname{tr} \mathbf{C}'_m(t)\|_4\, \|\mathbf{C}_m(t)\|_4\, \|\mathbf{C}'_m(t)\|_4 \\
&\leq c\, \|\operatorname{tr} \mathbf{C}_m(t)\|_4^2\, \|\mathbf{C}'_m(t)\|_4^2 \\
&\leq \frac{\varepsilon}{14}\, \|\nabla \mathbf{C}'_m(t)\|_2^2 + \frac{c}{\varepsilon}\, \|\mathbf{C}'_m(t)\|_2^2\, \|\mathbf{C}_m(t)\|_4^4 + \\
&\quad + c\, \|\mathbf{C}'_m(t)\|_2^2\, \|\mathbf{C}_m(t)\|_4^2 .
\end{aligned}
$$

Using these estimates we get from (4.25) that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\, \|\mathbf{u}'_m(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\, \|\mathbf{C}'_m(t)\|_2^2 + \frac{\nu}{2} |||\mathbf{u}'_m(t)|||^2 + \frac{\varepsilon}{2}\, \|\nabla \mathbf{C}'_m(t)\|_2^2 \leq \\
&\qquad \leq \|\mathbf{u}'_m(t)\|_2^2 \left(\frac{c}{\nu} |||\mathbf{u}_m(t)|||^2 + \frac{c}{\nu \varepsilon^2}\, \|\mathbf{C}_m(t)\|_4^4 \right) + \\
&+ \|\mathbf{C}'_m(t)\|_2^2 \left(\left(\frac{c}{\nu} + c \right) \|\mathbf{C}_m(t)\|_4^2 + \left(\frac{c}{\nu^2 \varepsilon} + \frac{c}{\varepsilon} \right) \|\mathbf{C}_m(t)\|_4^4 + \frac{c}{\varepsilon} |||\mathbf{u}_m(t)|||^2 + c |||\mathbf{u}_m(t)||| \right).
\end{aligned}
$$

Hence, the following inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}} \Big(\|\mathbf{u}'_m(t)\|_2^2 + \|\mathbf{C}'_m(t)\|_2^2 \Big) + \nu |||\mathbf{u}'_m(t)|||^2 + \varepsilon\, \|\nabla \mathbf{C}'_m(t)\|_2^2 \\
&\qquad\qquad \leq \Big(\|\mathbf{u}'_m(t)\|_2^2 + \|\mathbf{C}'_m(t)\|_2^2 \Big) \beta(t),
\end{aligned} \tag{4.27}
$$

where

$$\beta(t) := c(1/\nu, 1/\varepsilon)\bigg(\|\mathbf{C}_m(t)\|_4^4 + \|\mathbf{C}_m(t)\|_4^2 + |||\mathbf{u}_m(t)|||^2 + |||\mathbf{u}_m(t)||| \bigg)$$

is an integrable function, holds. The Gronwall inequality, employing (4.22) and (4.23), yields

$$\|\mathbf{u}_m'(s)\|_2^2 + \|\mathbf{C}_m'(s)\|_2^2 \leq \bigg(\|\mathbf{u}_m'(0)\|_2^2 + \|\mathbf{C}_m'(0)\|_2^2 \bigg) \exp\left\{ \int_0^s \beta(t) \right\},$$

i.e.

$$\sup_{t\in[0,T]} \|\mathbf{u}_m'(t)\|_2^2 + \sup_{t\in[0,T]} \|\mathbf{C}_m'(t)\|_2^2 \leq \Big(a_1^2 + a_2^2\Big) b.$$

Here the positive constant $b = b(T, 1/\nu, 1/\varepsilon, \|\mathbf{u}_m\|_{L^2(V)}, \|\mathbf{C}_m\|_{L^4(L^4)})$ is arising from the integral of $\beta(t)$. Then norms $\|\mathbf{u}_m\|_{L^2(V)}$ and $\|\mathbf{C}_m\|_{L^4(L^4)}$ are bounded by the constant k, see the first a priori estimate (4.9) for the Galerkin approximate solution. Consequently, we have the following a priori estimate

$$\|\mathbf{u}_m'\|_{L^\infty(H)} + \|\mathbf{C}_m'\|_{L^\infty(L^2)} \leq K, \tag{4.28}$$

and then by integrating (4.27) over $[0,T]$ we have, by (4.28), that also

$$\|\mathbf{u}_m'\|_{L^2(V)} + \|\mathbf{C}_m'\|_{L^2(W)} \leq K.$$

The constant $K = K(T, 1/\nu, 1/\varepsilon, \mathbf{u}_0, \mathbf{C}_0)$ is positive and dependent only on the data. We have shown the uniform a priori estimates for $(\mathbf{u}_m', \mathbf{C}_m')$, so we finally get that also the limit

$$(\mathbf{u}', \mathbf{C}') \in \Big[L^\infty(0,T;H) \cap L^2(0,T;V)\Big] \times \Big[L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W)\Big].$$

Higher a priori estimate for $(\mathbf{u}_m, \mathbf{C}_m)$

Now, in order to show that $\mathbf{u} \in L^\infty(0,T;H^2(\Omega))$ we consider the velocity equation (4.4c) in the following form

$$\nu\left((\mathbf{u}(t), \mathbf{v})\right) = (g(t), \mathbf{v}), \;\; \mathbf{v} \in V,$$

where $g(t) = -\mathbf{u}'(t) - \mathcal{B}\mathbf{u}(t) - \mathcal{E}\mathbf{T}(t)$. We already know that $\mathbf{u}' \in L^\infty(0,T;L^2(\Omega))$. By the Gronwall inequality and the above a priori estimates for $\mathbf{C}'$ we have $\mathbf{C} \in L^\infty(0,T;W)$. Then, since

$$|\langle \mathcal{E}\mathbf{T}(t), \mathbf{v}\rangle| = |(\operatorname{tr}\mathbf{C}(t)\,\mathbf{C}(t), \nabla\mathbf{v})| \leq c|||\mathbf{v}|||\, \|\mathbf{C}(t)\|_W^2,$$

we get that $\mathcal{E}\mathbf{T} \in L^\infty(0,T;L^2(\Omega))$. Further,

$$|b\left(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}\right)| \leq c\, \|\mathbf{u}(t)\|_4\, |||\mathbf{u}(t)|||\, \|\mathbf{v}\|_4 \leq c|||\mathbf{u}(t)|||^2\, \|\mathbf{v}\|_4, \tag{4.29}$$

thus $\mathcal{B}\mathbf{u} \in L^\infty(0,T;L^{4/3}(\Omega))$. Consequently,

$$g \in L^\infty(0,T;L^{4/3}(\Omega)). \tag{4.30}$$

Using the regularity result for the Stokes equation, cf. [142] or Section A.3, we have $\mathbf{u} \in L^\infty(0,T;W^{2,4/3})$. By the Sobolev embeddings theorem in two dimensions we have $W^{2,4/3}(\Omega) \hookrightarrow C(\Omega)$. Hence $\mathbf{u} \in L^\infty(0,T;L^\infty)$. Now, we can improve (4.30). We replace (4.29) by the inequality

$$|b\left(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}\right)| \le c\,\|\mathbf{u}\|_{L^\infty(L^\infty)}\,|||\mathbf{u}(t)|||\,\|\mathbf{v}\|_2\,,$$

and thus $g \in L^\infty(0,T;H)$. Again, the regularity of the solution to the Stokes equation yields $\mathbf{u} \in L^\infty(0,T;H^2(\Omega))$.

Now, we shall show $\mathbf{C} \in L^\infty(0,T;H^2(\Omega))$. We can rewrite the weak formulation for the conformation tensor (4.4d) in the following operator form

$$-\,\varepsilon\left((\mathbf{C}(t),\mathbf{D})\right) = \left(G(t),\mathbf{D}\right),\ \mathbf{D} \in H^1(\Omega),$$

where $G(t) = -\mathbf{C}'(t) - \tilde{\mathcal{B}}(\mathbf{u}(t),\mathbf{C}(t)) + \mathcal{O}\left(\mathbf{u}(t),\mathbf{C}(t)\right) + \mathcal{T}\mathbf{C}(t)$. The following inequality

$$\begin{aligned}
|\langle \mathcal{T}\mathbf{C}(t),\mathbf{D}\rangle| &= \left|\Big(\operatorname{tr}\mathbf{C}(t)\,\mathbf{I} - \left(\operatorname{tr}\mathbf{C}(t)\right)^2\mathbf{C}(t),\mathbf{D}\Big)\right| \\
&\le c\Bigg(\ \|\operatorname{tr}\mathbf{C}(t)\|_2\,\|\operatorname{tr}\mathbf{D}\|_2 + \|\operatorname{tr}\mathbf{C}(t)\|_6^3\,\|\mathbf{D}\|_2\Bigg) \\
&\le c\Bigg(\ \|\operatorname{tr}\mathbf{C}(t)\|_W + \|\operatorname{tr}\mathbf{C}(t)\|_W^3\Bigg)\,\|\mathbf{D}\|_2
\end{aligned}$$

implies $\mathcal{T}\mathbf{C} \in L^\infty(0,T;L^2(\Omega))$, since we know $\mathbf{C} \in L^\infty(0,T;W)$. Further, we can write

$$|B\left(\mathbf{u}(t),\mathbf{C}(t),\mathbf{D}\right)| \le c\,\|\mathbf{u}\|_{L^\infty(L^\infty)}\,\|\mathbf{C}(t)\|_W\,\|\mathbf{D}\|_2\,, \tag{4.31}$$

which leads to $\mathcal{B}(\mathbf{u},\mathbf{C}) \in L^\infty(0,T;L^2(\Omega))$. Similarly, we get $\mathcal{O}(\mathbf{u},\mathbf{C}) \in L^\infty(0,T;L^{4/3}(\Omega))$, by the estimate

$$\begin{aligned}
|\langle \mathcal{O}(\mathbf{u}(t),\mathbf{C}(t)),\mathbf{D}\rangle| &= \left|\Big((\nabla\mathbf{u}(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla\mathbf{u}(t))^T,\mathbf{D}\Big)\right| \\
&\le c\,\|\mathbf{C}(t)\|_W\,|||\mathbf{u}(t)|||\,\|\mathbf{D}\|_4\,.
\end{aligned} \tag{4.32}$$

Thus, we have

$$G(t) \in L^\infty(0,T;L^{4/3}(\Omega)). \tag{4.33}$$

Employing the regularity result for the solution to the Laplace equation we obtain $\mathbf{C} \in L^\infty(0,T;W^{2,4/3})$. The embedding $W^{2,4/3}(\Omega) \hookrightarrow C(\Omega)$ yields $\mathbf{C} \in L^\infty(0,T;L^\infty)$. Thus, we can improve (4.33) and replace (4.32) by the inequality

$$|\langle \mathcal{O}(\mathbf{u}(t),\mathbf{C}(t)),\mathbf{D}\rangle| \le c_3\,\|\mathbf{C}\|_{L^\infty(L^\infty)}\,|||\mathbf{u}(t)|||\,\|\mathbf{D}\|_2\,.$$

Finally, we get $\mathcal{O}(\mathbf{u},\mathbf{C}) \in L^\infty(0,T;L^2(\Omega))$ and $G(t) \in L^\infty(0,T;L^2(\Omega))$. Hence, the regularity of the solution to the Laplace equation gives us $\mathbf{C} \in L^\infty(0,T;H^2(\Omega))$. $\square$

4.4. Uniqueness of regular weak solution

In this section we study the question of uniqueness of the regular weak solution.

Theorem 4.6. *(uniqueness of regular weak solution)*
Let the domain Ω be of class C^2 and let the data $(\mathbf{u}_0, \mathbf{C}_0) \in [H^2(\Omega)^2 \cap V] \times H^2(\Omega)^{2\times 2}$. Then the weak solution of (3.2) is unique.

Proof. Let us recall, see Theorem 4.5, that for more regular data satisfying the assumptions of Theorem 4.6 we have also more regular weak solution, i.e.

$$\begin{aligned} \mathbf{u}' &\in L^\infty(0,T;H) \cap L^2(0,T;V), & \mathbf{C}' &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W), \\ \mathbf{u} &\in L^\infty(0,T;H^2(\Omega)), & \mathbf{C} &\in L^\infty(0,T;H^2(\Omega)). \end{aligned} \tag{4.34}$$

Let us assume $(\mathbf{u}_1, \mathbf{C}_1)$, $(\mathbf{u}_2, \mathbf{C}_2)$ are two different weak solutions satisfying the same initial condition and let us denote $(\mathbf{u}, \mathbf{C}) := (\mathbf{u}_1 - \mathbf{u}_2, \mathbf{C}_1 - \mathbf{C}_2)$. We test the weak solution at a.e. t with $\mathbf{u}(t)$ and $\mathbf{C}(t)$, respectively, and we add the resulting equations together. Then the difference $(\mathbf{u}, \mathbf{C})$ satisfies the following equality

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{u}(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{C}(t)\|_2^2 + \nu |||\mathbf{u}(t)|||^2 + \varepsilon \|\nabla \mathbf{C}(t)\|_2^2 = \\ = -b\Big(\mathbf{u}(t), \mathbf{u}_1(t), \mathbf{u}(t)\Big) - B\Big(\mathbf{u}(t), \mathbf{C}_1(t), \mathbf{C}(t)\Big) + \\ + \Big((\nabla \mathbf{u}_2(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla \mathbf{u}_2(t))^T, \mathbf{C}(t)\Big) + \\ + \Big((\nabla \mathbf{u}(t))\mathbf{C}_1(t) + \mathbf{C}_1(t)(\nabla \mathbf{u}(t))^T, \mathbf{C}(t)\Big) - \\ - \Big(\mathrm{tr}\,\mathbf{C}(t)\,\mathbf{C}_1(t), \nabla \mathbf{u}(t)\Big) - \Big(\mathrm{tr}\,\mathbf{C}_2(t)\,\mathbf{C}(t), \nabla \mathbf{u}(t)\Big) + \Big(\mathrm{tr}\,\mathbf{C}(t), \mathrm{tr}\,\mathbf{C}(t)\Big) - \\ - \Big(\mathrm{tr}\,\mathbf{C}(t)\,\mathrm{tr}\,\mathbf{C}_1(t)\,\mathbf{C}_1(t), \mathbf{C}(t)\Big) - \Big(\mathrm{tr}\,\mathbf{C}_2(t)\,\mathrm{tr}\,\mathbf{C}(t)\,\mathbf{C}_1(t), \mathbf{C}(t)\Big) - \\ - \Big((\mathrm{tr}\,\mathbf{C}_2(t))^2 \mathbf{C}(t), \mathbf{C}(t)\Big). \end{aligned} \tag{4.35}$$

To apply the Gronwall inequality for (4.35), we need to estimate each term on the right-hand side of (4.35). Let us firstly show the estimates of the trilinear terms. Applying the Hölder, the Young and the interpolation inequality (2.8a) with identity (4.2) we get

$$\begin{aligned} b\Big(\mathbf{u}(t), \mathbf{u}_1(t), \mathbf{u}(t)\Big) &\le c \|\mathbf{u}(t)\|_4^2 |||\mathbf{u}_1(t)||| \\ &\le c \|\mathbf{u}(t)\|_2 |||\mathbf{u}(t)||| |||\mathbf{u}_1(t)||| \\ &\le \frac{\nu}{10} |||\mathbf{u}(t)|||^2 + \frac{c}{\nu} \|\mathbf{u}(t)\|_2^2 |||\mathbf{u}_1(t)|||^2 \end{aligned} \tag{4.36a}$$

$$\begin{aligned} -B\Big(\mathbf{u}(t), \mathbf{C}_1(t), \mathbf{C}(t)\Big) &= B\Big(\mathbf{u}(t), \mathbf{C}(t), \mathbf{C}_1(t)\Big) \\ &\le c \|\mathbf{u}(t)\|_4 \|\nabla \mathbf{C}(t)\|_2 \|\mathbf{C}_1(t)\|_4 \\ &\le c \|\mathbf{u}(t)\|_2^{1/2} |||\mathbf{u}(t)|||^{1/2} \|\nabla \mathbf{C}(t)\|_2 \|\mathbf{C}_1(t)\|_4 \\ &\le \frac{\varepsilon}{16} \|\nabla \mathbf{C}(t)\|_2^2 + \frac{\nu}{10} |||\mathbf{u}(t)|||^2 + \frac{c}{\nu\varepsilon^2} \|\mathbf{u}(t)\|_2^2 \|\mathbf{C}_1(t)\|_4^4. \end{aligned} \tag{4.36b}$$

Further, the nonlinear elastic terms from (4.35) can be estimated by the Hölder, the Young and the interpolation inequality (2.8) as follows

$$\begin{aligned}\Big((\nabla \mathbf{u}_2(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla \mathbf{u}_2(t))^T, \mathbf{C}(t)\Big) &\le c|||\mathbf{u}_2(t)|||\, \|\mathbf{C}(t)\|_4^2 \\ &\le \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \frac{c}{\varepsilon}\|\mathbf{C}(t)\|_2^2\, |||\mathbf{u}_2(t)|||^2 + \\ &+ c\,\|\mathbf{C}(t)\|_2^2\, |||\mathbf{u}_2(t)||| \end{aligned} \tag{4.36c}$$

$$\begin{aligned}\Big((\nabla \mathbf{u}(t))\mathbf{C}_1(t) + \mathbf{C}_1(t)(\nabla \mathbf{u}(t))^T, \mathbf{C}(t)\Big) &\le c|||\mathbf{u}(t)|||\, \|\mathbf{C}_1(t)\|_4\, \|\mathbf{C}(t)\|_4 \\ &\le \frac{\nu}{10}|||\mathbf{u}(t)|||^2 + \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \\ &+ \frac{c}{\nu^2\varepsilon}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_1(t)\|_4^4 + \frac{c}{\nu}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_1(t)\|_4^2 , \end{aligned} \tag{4.36d}$$

and analogously

$$\begin{aligned}\Big(\mathrm{tr}\,\mathbf{C}(t)\,\mathbf{C}_1(t), \nabla\mathbf{u}(t)\Big) &\le |||\mathbf{u}(t)|||\, \|\mathbf{C}_1(t)\|_4\, \|\mathrm{tr}\,\mathbf{C}(t)\|_4 \\ &\le \frac{\nu}{10}|||\mathbf{u}(t)|||^2 + \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \\ &+ \frac{c}{\nu^2\varepsilon}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_1(t)\|_4^4 + \frac{c}{\nu}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_1(t)\|_4^2 \end{aligned} \tag{4.36e}$$

$$\begin{aligned}\Big(\mathrm{tr}\,\mathbf{C}_2(t)\,\mathbf{C}(t), \nabla\mathbf{u}(t)\Big) &\le |||\mathbf{u}(t)|||\, \|\mathrm{tr}\,\mathbf{C}_2(t)\|_4\, \|\mathbf{C}(t)\|_4 \\ &\le \frac{\nu}{10}|||\mathbf{u}(t)|||^2 + \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \\ &+ \frac{c}{\nu^2\varepsilon}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_2(t)\|_4^4 + \frac{c}{\nu}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_2(t)\|_4^2 . \end{aligned} \tag{4.36f}$$

The estimate of the next integral is straightforward

$$\Big(\mathrm{tr}\,\mathbf{C}(t), \mathrm{tr}\,\mathbf{C}(t)\Big) = \|\mathrm{tr}\,\mathbf{C}(t)\|_2^2 \le 2\,\|\mathbf{C}(t)\|_2^2 .$$

For the estimates of the last three integrals on the right-hand side of (4.35) we proceed in the same manner as above

$$\begin{aligned}\Big(\mathrm{tr}\,\mathbf{C}(t)\,\mathrm{tr}\,\mathbf{C}_1(t)\,\mathbf{C}_1(t), \mathbf{C}(t)\Big) &\le c\,\|\mathbf{C}_1(t)\|_4^2\, \|\mathbf{C}(t)\|_4^2 \\ &\le c\,\|\mathbf{C}_1(t)\|_4^2\, \|\mathbf{C}(t)\|_2\, \|\nabla\mathbf{C}(t)\|_2 + c\,\|\mathbf{C}_1(t)\|_4^2\, \|\mathbf{C}(t)\|_2^2 \\ &\le \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \frac{c}{\varepsilon}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_1(t)\|_4^4 + c\,\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_1(t)\|_4^2 \end{aligned}$$

$$\begin{aligned}\Big(\mathrm{tr}\,\mathbf{C}_2(t)\mathrm{tr}\,\mathbf{C}(t)\mathbf{C}_1(t), \mathbf{C}(t)\Big) &\le c\,\|\mathrm{tr}\,\mathbf{C}_2(t)\|_4\, \|\mathbf{C}_1(t)\|_4\, \|\mathbf{C}(t)\|_4^2 \\ &\le \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \frac{c}{\varepsilon}\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_2(t)\|_4^2\, \|\mathbf{C}_1(t)\|_4^2 + \\ &+ c\,\|\mathbf{C}(t)\|_2^2\, \|\mathbf{C}_2(t)\|_4\, \|\mathbf{C}_1(t)\|_4 , \end{aligned}$$

and finally

$$\Big((\operatorname{tr}\mathbf{C}_2(t))^2\mathbf{C}(t),\mathbf{C}(t)\Big) \leq \|\operatorname{tr}\mathbf{C}_2(t)\|_4^2\,\|\mathbf{C}(t)\|_4^2$$
$$\leq \frac{\varepsilon}{16}\|\nabla\mathbf{C}(t)\|_2^2 + \frac{c}{\varepsilon}\|\mathbf{C}(t)\|_2^2\|\operatorname{tr}\mathbf{C}_2(t)\|_4^4 + c\,\|\mathbf{C}(t)\|_2^2\|\operatorname{tr}\mathbf{C}_2(t)\|_4^2 .$$

Putting all the above estimates together we obtain, from (4.35), the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{u}(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{C}(t)\|_2^2 + \frac{\nu}{2}|||\mathbf{u}(t)|||^2 + \frac{\varepsilon}{2}\|\nabla\mathbf{C}(t)\|_2^2 \leq$$
$$\leq \|\mathbf{u}(t)\|_2^2\left(\frac{c}{\nu}|||\mathbf{u}_1(t)|||^2 + \frac{c}{\nu\varepsilon^2}\|\mathbf{C}_1(t)\|_4^4\right) +$$
$$+\|\mathbf{C}(t)\|_2^2\Bigg(2+\frac{c}{\varepsilon}\left(\|\mathbf{C}_1(t)\|_4^4 + \|\mathbf{C}_2(t)\|_4^4 + \|\mathbf{C}_2(t)\|_4^2\|\mathbf{C}_1(t)\|_4^2 + |||\mathbf{u}_2(t)|||^2\right) +$$
$$+\frac{c}{\nu^2\varepsilon}\left(\|\mathbf{C}_1(t)\|_4^4 + \|\mathbf{C}_2(t)\|_4^4\right) +$$
$$+c\left(\|\mathbf{C}_1(t)\|_4^2 + \|\mathbf{C}_2(t)\|_4 + \|\mathbf{C}_2(t)\|_4\|\mathbf{C}_1(t)\|_4 + |||\mathbf{u}_2(t)|||\right)\Bigg),$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\mathbf{u}(t)\|_2^2 + \|\mathbf{C}(t)\|_2^2\right) + \nu|||\mathbf{u}(t)|||^2 + \varepsilon\|\nabla\mathbf{C}(t)\|_2^2 \leq \left(\|\mathbf{u}(t)\|_2^2 + \|\mathbf{C}(t)\|_2^2\right)\delta(t), \tag{4.37}$$

where

$$\delta(t) := c(1/\nu,1/\varepsilon)\Bigg(1 + |||\mathbf{u}_1(t)|||^2 + |||\mathbf{u}_2(t)|||^2 + |||\mathbf{u}_2(t)||| + \|\mathbf{C}_2(t)\|_4^4 + \|\mathbf{C}_2(t)\|_4^2 +$$
$$+\|\mathbf{C}_1(t)\|_4^4 + \|\mathbf{C}_1(t)\|_4^2\Bigg)$$

is an integrable function. Analogously as in the proof of Theorem 4.4 the Gronwall inequality yields

$$\|\mathbf{u}(s)\|_2^2 + \|\mathbf{C}(s)\|_2^2 \leq \left(\|\mathbf{u}(0)\|_2^2 + \|\mathbf{C}(0)\|_2^2\right)\underbrace{\exp\left\{\int_0^s \delta(t)\right\}}_{\leq D},$$

where D is a positive constant dependent on $1/\nu$, $1/\varepsilon$, $\|\mathbf{u}_1\|_{L^2(V)}$, $\|\mathbf{u}_2\|_{L^2(V)}$, $\|\mathbf{C}_2\|_{L^4(L^4)}$ and $\|\mathbf{C}_1\|_{L^4(L^4)}$. Since the two solutions $(\mathbf{u}_1,\mathbf{C}_1)$ and $(\mathbf{u}_2,\mathbf{C}_2)$ satisfy the same initial condition, we have $(\mathbf{u}(0),\mathbf{C}(0)) = (\mathbf{0},\mathbf{0})$. Consequently, the inequality

$$\|\mathbf{u}(s)\|_2^2 + \|\mathbf{C}(s)\|_2^2 \leq 0$$

implies uniqueness of the weak solution.

□

In the third chapter, we have proven, in two space dimensions, global in time existence of the weak solutions to the Peterlin viscoelastic model describing time evolution of complex viscoelastic fluids. In order to get uniqueness of the weak solution, we have first showed, for more regular data, the higher regularity of the weak solution.

The analytical result on global in time existence and uniqueness of (more regular) weak solution is an important prerequisite in order to study convergence and error estimates of a suitable finite element approximation of the diffusive Peterlin model. We propose a numerical scheme in Chapter 6, followed by the error analysis in Chapter 7. However, before introducing the numerical approximation, we investigate another interesting question - a generalization of analytical results for more general constitutive model.

5 Generalized model

We are interested in the existence and uniqueness result for a weak solution of the general diffusive Peterlin model (3.1). Nonetheless, using the same mathematical techniques as in the previous chapter, we present the analytical results for a generalized model (3.2) with $\psi(s) = s$. To this end, the elastic stress tensor $\mathbf{T}$ is assumed to be given by (3.2c), while the constitutive equation for the conformation tensor $\mathbf{C}$ is assumed to be (3.1d). It means we shall study the following *generalized diffusive Peterlin model*

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} + \operatorname{div} \mathbf{T} - \nabla p \tag{5.1a}$$

$$\operatorname{div} \mathbf{u} = 0 \tag{5.1b}$$

$$\mathbf{T} = \operatorname{tr} \mathbf{C}\, \mathbf{C} \tag{5.1c}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{C} - (\nabla \mathbf{u})\mathbf{C} - \mathbf{C}(\nabla \mathbf{u})^T = \chi(\operatorname{tr} \mathbf{C})\, \mathbf{I} - \phi(\operatorname{tr} \mathbf{C})\mathbf{C} + \varepsilon \Delta \mathbf{C} \tag{5.1d}$$

with the same boundary and initial conditions

$$\left(\mathbf{u}, \frac{\partial \mathbf{C}}{\partial \mathbf{n}}\right) = (\mathbf{0}, \mathbf{0})\,, \qquad (\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0)\,. \tag{5.1e}$$

The existence of a weak solution will be proven under the assumptions that

(A1) χ and ϕ are continuous functions defined on $\mathbb{R}$

(A2) the polynomial growth conditions

$$A_1 |s|^\alpha \le \phi(s) \le A_2 |s|^\alpha$$
$$\chi(s) \le B s^\beta$$

for $\alpha \ge 0$, $\beta > 0$ and some constants $0 < A_1 < A_2$, $B > 0$ are satisfied.

In order to show the higher regularity result we will moreover assume that

(A3) χ and ϕ are C^1-functions on $\mathbb{R}$

(A4) the derivatives χ', ϕ' satisfy the following growth conditions

$$\tilde{A}_1 |s|^{\alpha-1} \le \phi'(s) \le \tilde{A}_2 |s|^{\alpha-1}$$
$$\chi'(s) \le \tilde{B} s^{\beta-1},$$

for some constants $0 < \tilde{A}_1 < \tilde{A}_2$, $\tilde{B} > 0$.

The existence, regularity and uniqueness results presented in the previous chapter can be extended for the cases

$$0 \le \alpha \le 2,\ 1 \le \beta < \alpha + 1,$$

where if $\alpha = 0$ then $\beta = 1$. We would like to point out, that the case $\alpha = 2, \beta = 1$, i.e. the diffusive Peterlin model (3.2), is the limiting case.

Before showing the formal energy estimates, let us first state a straightforward consequence of the Vitali theorem, see Lemma 2.4. in [96].

Lemma 5.1.
Let $M \subset \mathbb{R}^d$ be measurable and bounded. Let the sequence $\{f_m\}_{m\in\mathbb{N}}$ be uniformly bounded in $L^q(M)$ for a $q > 1$. Finally, let $f_m \to f$ a.e. in M for some $f \in L^q(M)$. Then

$$\int_M f_m \to \int_M f.$$

This convergence result is important for passing to the limit with the Galerkin approximation in the generalized terms in the constitutive equation for the conformation tensor.

5.1. Formal energy estimates

Assuming **(A1) - (A2)** we present the formal energy estimates for model (5.1). Our aim is to derive the exponents α and β for which it makes sense to define a weak solution. The energy equality (3.9) holds true unchanged

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \, dx = -\nu\int_\Omega |\nabla\mathbf{u}|^2 \, dx - \int_\Omega \mathbf{T} : \nabla\mathbf{u} \, dx \,. \tag{5.2}$$

Similarly as in Section 3.3 we multiply (5.1d) by $\frac{1}{2}\mathbf{C}$ and integrate this equation using the Green formula, keeping in mind the divergence freedom of the velocity and the boundary conditions. We obtain

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}|^2 \, dx - \frac{1}{2}\int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C} \, dx &= \\ = \frac{1}{2}\int_\Omega \chi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C} \, dx - \frac{1}{2}\int_\Omega \phi(\mathrm{tr}\,\mathbf{C})\mathbf{C} : \mathbf{C} \, dx - \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx \,.\end{aligned} \tag{5.3}$$

The sum of equations (5.2) and (5.3), by identity (3.8) for the symmetric tensor $\mathbf{C}$ and the solenoidal vector field $\mathbf{u}$ in two space dimensions, reads

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}|^2 \, dx + \nu\int_\Omega |\nabla\mathbf{u}|^2 \, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx + \\ + \frac{1}{2}\int_\Omega \phi(\mathrm{tr}\,\mathbf{C})\mathbf{C} : \mathbf{C} \, dx = \frac{1}{2}\int_\Omega \chi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C} \, dx \,.\end{aligned}$$

Employing the polynomial growth conditions assumed in **(A2)** we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{C}|^2 \, dx + \nu\int_\Omega |\nabla\mathbf{u}|^2 \, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\mathbf{C}|^2 \, dx + \\ + \frac{A_1}{4}\int_\Omega |\mathrm{tr}\,\mathbf{C}|^{\alpha+2} \, dx \le \frac{B}{2}\int_\Omega (\mathrm{tr}\,\mathbf{C})^{\beta+1} \, dx \,.\end{aligned} \tag{5.4}$$

Firstly, we find the exponents for which the energy inequality (5.4) indicates the appropriate functional spaces for further analysis.

- Obviously, for $\beta = \alpha + 1$ we need $A_1 > 2B$.
- For $\beta \leq 1$, by the Sobolev embeddings theorem and inequality (2.9b), we get

$$\frac{B}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^{\beta+1}\, dx \;\leq c\int_\Omega |\operatorname{tr}\mathbf{C}|^2\, dx \;\leq 2c\int_\Omega |\mathbf{C}|^2\, dx\,.$$

  Then we can treat inequality (5.4) analogously as (3.11).

- For $\alpha > 0$ we need $\alpha+2 > \beta+1$ in order to apply the Young inequality on the integral on the right-hand side of (5.4) such that

$$\frac{B}{2}\int_\Omega (\operatorname{tr}\mathbf{C})^{\beta+1}\, dx \;\leq \delta\int_\Omega (\operatorname{tr}\mathbf{C})^{\alpha+2}\, dx \;+\frac{c}{\delta}$$

  for $0 < \delta < A_1/4$.

From now on we suppose that

$$\begin{aligned}&\alpha > 0: \;\; \beta < \alpha+1 \text{ or } \beta = \alpha+1 \text{ and } A_1 > 2B\\ &\alpha = 0: \;\; \beta \leq 1.\end{aligned}$$

Then the Gronwall inequality yields the desired a priori bounds for the velocity and the conformation tensor

$$\mathbf{u} \in L^\infty(0,T;H)\cap L^2(0,T;V), \quad \mathbf{C}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W).$$

Let us note that by the interpolation inequalities (2.8b) and (2.8c) together with the above a priori estimates we get $\mathbf{C}\in L^4(0,T;L^4(\Omega))$ and $\mathbf{C}\in L^{2r/(r-2)}(0,T;L^r(\Omega))$ for $r > 2$, respectively. In addition, for $\alpha > 2$, we have an important a priori bound for the trace of the conformation tensor

$$\operatorname{tr}\mathbf{C}\in L^{\alpha+2}(0,T;L^{\alpha+2}(\Omega)).$$

In what follows, we will work with the best bound for $\operatorname{tr}\mathbf{C}$ that we can obtain, and with the appropriate bound of $\mathbf{C}$, i.e.

$$\operatorname{tr}\mathbf{C}\in L^4(0,T;L^4(\Omega)), \qquad \mathbf{C}\in L^4(0,T;L^4(\Omega)) \qquad \text{for } \alpha\in[0,2], \tag{5.5a}$$

$$\operatorname{tr}\mathbf{C}\in L^{\alpha+2}(0,T;L^{\alpha+2}(\Omega)), \quad \mathbf{C}\in L^{2r/(r-2)}(0,T;L^r(\Omega)) \quad \text{for } \alpha > 2,\ r > 2. \tag{5.5b}$$

Our further goal is to get bounds on $\chi(\operatorname{tr}\mathbf{C})$ and $\phi(\operatorname{tr}\mathbf{C})\mathbf{C}$ in $L^q(0,T;L^q(\Omega))$, $q > 1$. By the assumption **(A2)** we directly get

$$\int_0^T\int_\Omega |\chi(\operatorname{tr}\mathbf{C})|^q\, dx\, dt \;\leq B^q\int_0^T\int_\Omega |\operatorname{tr}\mathbf{C}|^{\beta q}\, dx\, dt\,. \tag{5.6}$$

- If $\alpha\in[0,2]$ then $\beta \leq 3$. We see, from (5.5a) and (5.6), that $\chi(\operatorname{tr}\mathbf{C})\in L^q(0,T;L^q(\Omega))$ for $1 < q \leq 4/\beta$.
- Analogously, due to (5.5b) and (5.6), we get that $\chi(\operatorname{tr}\mathbf{C})$ is bounded in $L^q(0,T;L^q(\Omega))$ for $1 < q \leq \alpha+2/\beta$, since for $\alpha > 2$ we have $\beta \leq \alpha+1$.

We proceed with the similar discussion for $\phi(\operatorname{tr}\mathbf{C})\mathbf{C}$. The growth assumptions **(A2)** yield

$$\int_0^T\int_\Omega |\phi(\operatorname{tr}\mathbf{C})\mathbf{C}|^q \, dx\, dt \le A_2^q \int_0^T\int_\Omega |\operatorname{tr}\mathbf{C}|^{\alpha q}|\mathbf{C}|^q \, dx\, dt\,. \tag{5.7}$$

According to (5.5), we apply the Hölder inequality to bound the term by the proper norms of $\operatorname{tr}\mathbf{C}$ and $\mathbf{C}$.

- For $\alpha \in [0,2]$ we have $\phi(\operatorname{tr}\mathbf{C})\mathbf{C} \in L^q(0,T;L^q(\Omega))$ for $1 < q < 4/\alpha+1$. Indeed, for such $q < 4/\alpha+1 < 4/\alpha$ we choose the Hölder coefficients

  $$p_1 = \frac{4}{\alpha q}, \qquad p_2 = \frac{4}{4-\alpha q}$$

  to estimate (5.7) as follows

  $$\begin{aligned}\int_0^T\int_\Omega |\phi(\operatorname{tr}\mathbf{C})\mathbf{C}|^q \, dx\, dt &\le c\int_0^T \|\operatorname{tr}\mathbf{C}\|_4^{\alpha q}\,\|\mathbf{C}\|_{4q/(4-\alpha q)}^q \, dt\\ &\le c\,\|\operatorname{tr}\mathbf{C}\|_{L^4(L^4)}^{\alpha q}\,\|\mathbf{C}\|_{L^s(L^s)}^q\,,\end{aligned}$$

  where $s = 4q/4-\alpha q < 4$, and thus the Sobolev embeddings finally yields

  $$\int_0^T\int_\Omega |\phi(\operatorname{tr}\mathbf{C})\mathbf{C}|^q \, dx\, dt \le c\,\|\operatorname{tr}\mathbf{C}\|_{L^4(L^4)}^{\alpha q}\,\|\mathbf{C}\|_{L^4(L^4)}^q\,.$$

- Let $\alpha > 2$. For $1 < q < \alpha+2/\alpha$ the Hölder coefficients

  $$p_1 = \frac{\alpha+2}{\alpha q}, \qquad p_2 = \frac{\alpha+2}{\alpha+2-\alpha q}$$

  analogously yield

  $$\begin{aligned}\int_0^T\int_\Omega |\phi(\operatorname{tr}\mathbf{C})\mathbf{C}|^q \, dx\, dt &\le c\int_0^T \|\operatorname{tr}\mathbf{C}\|_{\alpha+2}^{\alpha q}\,\|\mathbf{C}\|_{(\alpha+2)q/(\alpha+2-\alpha q)}^q \, dt\\ &\le c\,\|\operatorname{tr}\mathbf{C}\|_{L^{\alpha+2}(L^{\alpha+2})}^{\alpha q}\,\|\mathbf{C}\|_{L^s(L^s)}^q\,,\end{aligned}$$

  where now $s = (\alpha+2)q/\alpha+2-\alpha q$. We need $s \le 4$, i.e. $q \le 4(\alpha+2)/5\alpha+2$. Since q should be grater than 1, we require $\alpha < 6$. Hence, for $1 < q < 4(\alpha+2)/5\alpha+2$ the function $\phi(\operatorname{tr}\mathbf{C})\mathbf{C}$ is bounded in $L^q(0,T;L^q(\Omega))$.

To conclude, for the coefficients α and β satisfying

$$\begin{aligned}&0 < \alpha < 6: \ \beta < \alpha+1 \text{ or } \beta = \alpha+1 \text{ and } A_1 > 2B\\ &\alpha = 0: \ \beta \le 1,\end{aligned}$$

we shall define the weak solution to the generalized diffusive Peterlin model and show its existence by the same mathematical techniques and arguments as in Section 4.2.

5.2. Existence of weak solutions

We introduce the weak solution to the generalized problem (5.1) and show that, for the exponents α and β derived in the previous section, the solutions exist.

Theorem 5.2. *(existence of weak solutions to the generalized model)*
Let the assumptions **(A1)** - **(A2)** *on* χ, ϕ *be satisfied and let* $(\mathbf{u}_0, \mathbf{C}_0) \in H \times L^2(\Omega)^{2\times 2}$ *be given. Let one of the following assumptions be satisfied:*

$$\begin{aligned} & 0 < \alpha < 6,\ 0 < \beta < \alpha + 1 \ \textit{or}\ \beta = \alpha + 1 \ \textit{with}\ A_1 > 2B \\ \textit{or} \quad & \alpha = 0 \ \textit{and}\ 0 < \beta \leq 1. \end{aligned} \tag{5.8}$$

Then there exists a couple

$$(\mathbf{u}, \mathbf{C}) \in \Big[L^\infty(0,T;H) \cap L^2(0,T;V)\Big] \times \Big[L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W)\Big] \tag{5.9a}$$

satisfying the initial condition $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0)$ *and the weak formulation*

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v}\ dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dx + \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\ dx = -\int_\Omega \operatorname{tr}\mathbf{C}\,\mathbf{C} : \nabla\mathbf{v}\ dx \tag{5.9b}$$

$$\begin{aligned} \int_\Omega \frac{\partial \mathbf{C}}{\partial t} : \mathbf{D}\ dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{C} : \mathbf{D}\ dx - \int_\Omega \Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{D}\ dx + \\ + \varepsilon \int_\Omega \nabla\mathbf{C} : \nabla\mathbf{D}\ dx = \int_\Omega \chi(\operatorname{tr}\mathbf{C})\mathbf{I} : \mathbf{D}\ dx - \int_\Omega \phi(\operatorname{tr}\mathbf{C})\mathbf{C} : \mathbf{D}\ dx \end{aligned} \tag{5.9c}$$

$$\forall\ \mathbf{v} \in V,\ \forall\ \mathbf{D} \in W,\ \text{ a.e. } t \in (0,T).$$

The couple $(\mathbf{u}, \mathbf{C})$ *is called the weak solution to the problem* (5.1)*.*

Proof. The proof is analogous to the proof of Theorem 4.4. We will only concentrate on the generalized terms in equation (5.9c) for the conformation tensor.

Galerkin approximation

The m-th approximate Galerkin solution now reads

$$(\mathbf{u}_m(t), \mathbf{C}_m(t)) = \left(\sum_{i=1}^m g_{im}(t)\mathbf{v}_i, \sum_{i=1}^m G_{im}(t)\mathbf{D}_i\right) \tag{5.10a}$$

$$(\mathbf{u}_m(0), \mathbf{C}_m(0)) = (\mathbf{u}_{0m}, \mathbf{C}_{0m}) \tag{5.10b}$$

$$(\mathbf{u}_m'(t), \mathbf{v}_j) + b\,(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}_j) + \nu\,((\mathbf{u}_m(t), \mathbf{v}_j)) = -\,(\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{C}_m(t), \nabla\mathbf{v}_j) \tag{5.10c}$$

$$\begin{aligned} (\mathbf{C}_m'(t), \mathbf{D}_j) + B\,(\mathbf{u}_m(t), \mathbf{C}_m(t), \mathbf{D}_j) - \Big((\nabla\mathbf{u}_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}_m(t))^T, \mathbf{D}_j\Big) + \\ + \varepsilon\,((\mathbf{C}_m(t), \mathbf{D}_j)) = (\chi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{I}, \mathbf{D}_j) - (\phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}_m(t), \mathbf{D}_j) \end{aligned} \tag{5.10d}$$

$$j = 1, \dots, m,\ t \in [0,T].$$

We recall that $\{\mathbf{v}_j\}_{j=1}^{\infty}$ and $\{\mathbf{D}_j\}_{j=1}^{\infty}$ are the countable orthonormal bases of the spaces $V = H^1_{0,div}(\Omega)^2$ and $W = H^1(\Omega)^{2\times 2}$, respectively. Further, $\mathbf{u}_{0m}$ and $\mathbf{C}_{0m}$ are the orthogonal projections in H of $\mathbf{u}_0$ and in $L^2(\Omega)$ of $\mathbf{C}_0$ on the spaces spanned by $\mathbf{v}_j$ and $\mathbf{D}_j$. The nonlinear system of differential equations together with the initial conditions (and the following a priori bounds) gives us the solution $(\mathbf{u}_m, \mathbf{C}_m)$ defined on interval $[0,T]$.

A priori estimates

Repeating the formal energy estimates for the m-th Galerkin approximation and using the interpolation inequalities (2.8b) and (2.8c) we obtain

$$
\begin{aligned}
&\mathbf{u}_m \in L^\infty(0,T;H) \cap L^2(0,T;V)\\
&\mathbf{C}_m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W) \cap L^4(0,T;L^4(\Omega))\\
&\mathbf{T}_m \in L^2(0,T;L^2(\Omega))\\
&\operatorname{tr}\mathbf{C}_m \in L^4(0,T;L^4(\Omega)),\ \alpha \in [0,2]\\
&\operatorname{tr}\mathbf{C}_m \in L^{\alpha+2}(0,T;L^{\alpha+2}(\Omega)),\ \alpha > 2
\end{aligned}
\tag{5.11}
$$

Finally, we have shown that there exists a positive constant $k = k\,(T, 1/\nu, 1/\varepsilon, \mathbf{u}_0, \mathbf{C}_0, \alpha, \beta)$ such that

$$\|\mathbf{u}_m\|_{L^\infty(H)} + \|\mathbf{u}_m\|_{L^2(V)} + \|\mathbf{C}_m\|_{L^\infty(L^2)} + \|\mathbf{C}_m\|_{L^2(W)} + \|\operatorname{tr}\mathbf{C}_m\|_{L^{\alpha+2}(L^{\alpha+2})} \le k.$$

Let us recall that depending on α we work with the following a priori bounds for $\operatorname{tr}\mathbf{C}_m$ and $\mathbf{C}_m$

$$\operatorname{tr}\mathbf{C}_m \in L^4(0,T;L^4(\Omega)), \qquad \mathbf{C}_m \in L^4(0,T;L^4(\Omega)) \qquad \text{for } \alpha \in [0,2], \tag{5.12a}$$

$$\operatorname{tr}\mathbf{C}_m \in L^{\alpha+2}(0,T;L^{\alpha+2}(\Omega)), \quad \mathbf{C}_m \in L^{2r/(r-2)}(0,T;L^r(\Omega)) \quad \text{for } \alpha > 2,\ r > 2. \tag{5.12b}$$

Compactness

In order to pass to the limit with the Galerkin solution we need, besides Lemma 5.1, the strong convergence. The compact embedding for $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is obtained as in the proof of Theorem 4.4, by the Lions-Aubin-Simon lemma, cf. Lemma A.2 in Appendix A. We again omit the index m for better readability. Let us recall the definition of the operators

$$
\begin{aligned}
&\tilde{\mathcal{A}} : W \to W^* && \langle \tilde{\mathcal{A}}\mathbf{C}, \mathbf{D}\rangle := ((\mathbf{C}, \mathbf{D}))\\
&\tilde{\mathcal{B}} : V \times W \to W^* && \langle \tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C}), \mathbf{D}\rangle := B\,(\mathbf{u}, \mathbf{C}, \mathbf{D})\\
&\mathcal{O} : V \times W \to W^* && \langle \mathcal{O}(\mathbf{u}, \mathbf{C}), \mathbf{D}\rangle := \Big((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T, \mathbf{D}\Big)
\end{aligned}
$$

and let us define

$$
\begin{aligned}
&\mathcal{T}_1 : W \to W^* && \langle \mathcal{T}_1\mathbf{C}, \mathbf{D}\rangle := (\chi(\operatorname{tr}\mathbf{C})\mathbf{I}, \mathbf{D})\\
&\mathcal{T}_2 : W \to W^* && \langle \mathcal{T}_2\mathbf{C}, \mathbf{D}\rangle := (\phi(\operatorname{tr}\mathbf{C})\mathbf{C}, \mathbf{D})\,.
\end{aligned}
$$

Then (5.10d) can be rewritten in the following operator form

$$\mathbf{C}' = \mathcal{O}(\mathbf{u}, \mathbf{C}) + \mathcal{T}_1\mathbf{C} - \mathcal{T}_2\mathbf{C} - \tilde{\mathcal{B}}(\mathbf{u}, \mathbf{C}) - \varepsilon\tilde{\mathcal{A}}\mathbf{u},\ \ \mathbf{u} \in V,\ \ \mathbf{C} \in W.$$

We already know, see (4.15) - (4.17), that

$$\tilde{\mathcal{A}}\mathbf{C},\, \tilde{\mathcal{B}}(\mathbf{u},\mathbf{C}) \in L^2(0,T;W^*) \ \text{ and } \ \mathcal{O}(\mathbf{u},\mathbf{C}) \in L^{4/3}(0,T;W^*). \tag{5.13}$$

Employing the growth conditions **(A2)** we show that $\mathcal{T}_1\mathbf{C}$ and $\mathcal{T}_2\mathbf{C}$ are bounded in $L^l(0,T;W^*)$ for a $l \geq 1$.

Estimates of the operator $\mathcal{T}_1\mathbf{C}$

The exponents α and β are assumed to be as in (5.8). For $\beta = 0$, by the growth assumption **(A2)** on the function χ, we get

$$\begin{aligned} B\int_0^T\int_\Omega \operatorname{tr}\mathbf{D}\; dx\, dt \ &= B\int_0^T \|\operatorname{tr}\mathbf{D}\|_1\; dt \\ \int_0^T \|\mathcal{T}_1\mathbf{C}\|_{W^*}^{l_1}\; dt \ &\leq c. \end{aligned}$$

Consequently, $\mathcal{T}_1\mathbf{C} \in L^{l_1}(0,T;W^*)$ for any $l_1 \geq 1$. Further, for $\beta > 0$ we consider two cases with respect to α separately:

- $0 \leq \alpha \leq 2$
  From (5.8) we know $\beta \leq 3$. Then the Hölder inequality for $p = {}^4\!/\!_\beta$ and $q = {}^4\!/\!_{4-\beta}$ yields

  $$\int_0^T\int_\Omega \chi(\operatorname{tr}\mathbf{C}(t))\operatorname{tr}\mathbf{D}\; dx\, dt \ \leq c\int_0^T \|\chi(\operatorname{tr}\mathbf{C}(t))\|_{4/\beta}\, \|\operatorname{tr}\mathbf{D}\|_{4/(4-\beta)}\; dt\,.$$

  By the Sobolev embeddings theorem it holds that $\|\operatorname{tr}\mathbf{D}\|_{4/(4-\beta)} \leq c\,\|\operatorname{tr}\mathbf{D}\|_W$. Then

  $$\begin{aligned} \int_0^T \|\mathcal{T}_1\mathbf{C}\|_{W^*}^{l_2}\; dt \ &\leq c\int_0^T \|\chi(\operatorname{tr}\mathbf{C}(t))\|_{4/\beta}^{l_2}\; dt \\ &\leq c\int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_4^{l_2\beta}\; dt\,. \end{aligned}$$

  It means there exists a coefficient l_2 such that $1 \leq l_2 \leq {}^4\!/\!_\beta$ and $\mathcal{T}_1\mathbf{C} \in L^{l_2}(0,T;W^*)$.
- $2 < \alpha < 6$
  Analogously, since $\beta \leq \alpha+1$, we choose $p = {}^{\alpha+2}\!/\!_\beta > 1$ and $q = {}^{\alpha+2}\!/\!_{\alpha+2-\beta} > 1$. The Hölder inequality gives us

  $$\int_0^T\int_\Omega \chi(\operatorname{tr}\mathbf{C}(t))\operatorname{tr}\mathbf{D}\; dx\, dt \ \leq c\int_0^T \|\chi(\operatorname{tr}\mathbf{C}(t))\|_{(\alpha+2)/\beta}\, \|\operatorname{tr}\mathbf{D}\|_{(\alpha+2)/(\alpha+2-\beta)}\; dt\,.$$

  Similarly, by the Sobolev embedding $W \hookrightarrow L^{\frac{\alpha+2}{\alpha+2-\beta}}(\Omega)$, we can write

  $$\begin{aligned} \int_0^T \|\mathcal{T}_1\mathbf{C}\|_{W^*}^{l_3}\; dt \ &\leq c\int_0^T \|\chi(\operatorname{tr}\mathbf{C}(t))\|_{(\alpha+2)/\beta}^{l_3}\; dt \\ &\leq c\int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_{\alpha+2}^{l_3\beta}\; dt\,. \end{aligned}$$

  Thus there exists a coefficient $1 \leq l_3 \leq {}^{\alpha+2}\!/\!_\beta$ such that $\mathcal{T}_1\mathbf{C} \in L^{l_3}(0,T;W^*)$.

Estimates of the operator $\mathcal{T}_2\mathbf{C}$

Let us now repeat a similar procedure for the operator $\mathcal{T}_2\mathbf{C}$. By the Hölder inequality, the Sobolev embeddings theorem and assumptions (5.8) on the coefficients α and β we get the analogous results. Firstly, for $\alpha = 0$ we have

$$A_1 \int_0^T \int_\Omega \mathbf{C}(t) : \mathbf{D}\; dx\, dt \;= A_1 \int_0^T \|\mathbf{C}(t)\|_2 \, \|\mathbf{D}\|_2 \; dt\,.$$

Then for any $l_4 \geq 1$ we get

$$\int_0^T \|\mathcal{T}_1\mathbf{C}\|_{W^*}^{l_4} \; dt \;\leq A_1 \int_0^T \|\mathbf{C}(t)\|_2^{l_4} \; dt \;\leq c\, \|\mathbf{C}\|_{L^\infty(L^2)}^{l_4}\,,$$

and hence $\mathcal{T}_2\mathbf{C} \in L^{l_4}(0,T;W^*)$ as desired. Again, in order to work with the appropriate energy estimate for $\operatorname{tr}\mathbf{C}_m$ we consider two cases with respect to α separately:

- $0 < \alpha \leq 2$
  Let us consider the Hölder inequality with the coefficients $s = {}^4\!/\!_\alpha$, $r = 4$ and $z = {}^4\!/\!_{3-\alpha}$ to get

  $$\int_0^T \int_\Omega \phi(\operatorname{tr}\mathbf{C}(t))\mathbf{C}(t) : \mathbf{D}\; dx\, dt \;\leq c \int_0^T \|\phi(\operatorname{tr}\mathbf{C}(t))\|_{4/\alpha} \, \|\mathbf{C}(t)\|_4 \, \|\mathbf{D}\|_{4/(3-\alpha)} \; dt\,.$$

  It holds that $\|\mathbf{D}\|_{4/(3-\alpha)} \leq c\, \|\mathbf{D}\|_W$. Further, for some Hölder coefficients p, q we have

  $$\begin{aligned}\int_0^T \|\mathcal{T}_2\mathbf{C}\|_{W^*}^{l_5} \; dt \;&\leq c \int_0^T \|\phi(\operatorname{tr}\mathbf{C}(t))\|_{4/\alpha}^{l_5} \, \|\mathbf{C}(t)\|_4^{l_5} \; dt \\ &\leq c \int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_4^{l_5\alpha} \, \|\mathbf{C}(t)\|_4^{l_5} \; dt \\ &\leq c \left(\int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_4^{l_5 \alpha\, p} \; dt \right)^{1/p} \left(\int_0^T \|\mathbf{C}(t)\|_4^{l_5\, q} \; dt \right)^{1/q}.\end{aligned}$$

  Since $\mathbf{C}_m \in L^4(0,T;L^4(\Omega))$ and $\operatorname{tr}\mathbf{C}_m \in L^4(0,T;L^4(\Omega))$, we choose the coefficient l_5 and the Hölder coefficients $p, q > 1$ as

  $$1 < l_5 = \frac{4}{1+\alpha} < 4, \qquad p = \frac{1+\alpha}{\alpha}, \qquad q = 1 + \alpha,$$

  in order to get

  $$\int_0^T \|\mathcal{T}_2\mathbf{C}\|_{W^*}^{l_5} \; dt \;\leq c \left(\int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_4^4 \; dt \right)^{\alpha/(1+\alpha)} \left(\int_0^T \|\mathbf{C}(t)\|_4^4 \; dt \right)^{1/(1+\alpha)}.$$

  Hence, there exists a coefficient $l_5 \in (1,4)$ such that $\mathcal{T}_2\mathbf{C} \in L^{l_5}(0,T;W^*)$.

- $2 < \alpha < 6$
  Since $\alpha < 6$, we can choose the Hölder coefficients s, r, z as

  $$s = \frac{\alpha+2}{\alpha}, \qquad r = \frac{2(\alpha+2)}{\alpha-2}, \qquad z = \frac{2(\alpha+2)}{6-\alpha}.$$

Then

$$\int_0^T \int_\Omega \phi(\operatorname{tr}\mathbf{C}(t))\mathbf{C}(t) : \mathbf{D}\; dx\, dt \;\le c \int_0^T \|\phi(\operatorname{tr}\mathbf{C}(t))\|_s \,\|\mathbf{C}(t)\|_r \,\|\mathbf{D}\|_z \; dt\,.$$

Due to the Sobolev embeddings theorem we consequently get

$$\begin{aligned}
\int_0^T \|\mathcal{T}_2\mathbf{C}\|_{W^*}^{l_6}\; dt \;&\le c \int_0^T \|\phi(\operatorname{tr}\mathbf{C}(t))\|_{(\alpha+2)/\alpha}^{l_6}\,\|\mathbf{C}(t)\|_r^{l_6}\; dt \\
&\le c \int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_{\alpha+2}^{l_6\alpha}\,\|\mathbf{C}(t)\|_r^{l_6}\; dt \\
&\le c \left(\int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_{\alpha+2}^{l_6\alpha\, p}\; dt\right)^{1/p} \left(\int_0^T \|\mathbf{C}(t)\|_r^{l_6\, q}\; dt\right)^{1/q},
\end{aligned}$$

for some $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We know $\operatorname{tr}\mathbf{C}_m \in L^{\alpha+2}(0,T;L^{\alpha+2}(\Omega))$, and for

$$r = \frac{2(\alpha+2)}{\alpha-2} > 2$$

we have $\mathbf{C} \in L^{2r/(r-2)}(0,T;L^r(\Omega))$. Let $l_6 = 1$. Hence, we choose

$$p = \frac{\alpha+2}{\alpha} \qquad \text{and} \qquad q = \frac{2r}{r-2} = \frac{\alpha+2}{2}$$

to conclude that

$$\int_0^T \|\mathcal{T}_2\mathbf{C}\|_{W^*}\; dt \;\le c \left(\int_0^T \|\operatorname{tr}\mathbf{C}(t)\|_{\alpha+2}^{\alpha+2}\; dt\right)^{\alpha/(\alpha+2)} \left(\int_0^T \|\mathbf{C}(t)\|_r^{2r/(r-2)}\; dt\right)^{2/(\alpha+2)},$$

and that $\mathcal{T}_2\mathbf{C}$ is bounded in $L^1(0,T;W^*)$.

We have shown that

$$\mathcal{T}_1\mathbf{C} \in L^{\hat{l}}(0,T;W^*),\; \hat{l} = \min_{i=1,2,3} l_i > 1, \qquad \mathcal{T}_2\mathbf{C} \in L^1(0,T;W^*), \tag{5.14}$$

for the coefficients α and β satisfying assumptions (5.8). Then (5.13) and (5.14) imply $\mathbf{C}' \in L^1(0,T;W^*)$. The embeddings

$$W \hookrightarrow\hookrightarrow L^q(\Omega) \hookrightarrow W^*$$

for $1 \le q < \infty$ yield, due to the Lions-Aubin-Simon lemma, the compact embedding of $\{\mathbf{C}_m\}_{m=1}^\infty$ into the space $L^2(0,T;L^q(\Omega))$. Consequently, there exists a subsequence, denoted again by $(\mathbf{u}_m, \mathbf{C}_m)$, of the Galerkin approximate sequence such that the convergences (4.19) hold true. Moreover,

$$\mathbf{C}_m \to \mathbf{C} \;\text{ in }\; L^2(0,T;L^q(\Omega)),\; 1 \le q < \infty. \tag{5.15}$$

Passage to the limit and the initial condition

We only concentrate on the limit in the generalized terms in equation (5.10d). All the other terms in (5.10) are treated using (4.19) as in the proof of Theorem 4.4. In order to show that

$$\int_0^T \int_\Omega \chi(\operatorname{tr} \mathbf{C}_m)\mathbf{I} : \mathbf{D}\,\varphi(t)\;dx\,dt \;\to \int_0^T \int_\Omega \chi(\operatorname{tr} \mathbf{C})\mathbf{I} : \mathbf{D}\,\varphi(t)\;dx\,dt \tag{5.16a}$$

$$\int_0^T \int_\Omega \phi(\operatorname{tr} \mathbf{C}_m)\mathbf{C} : \mathbf{D}\,\varphi(t)\;dx\,dt \;\to \int_0^T \int_\Omega \phi(\operatorname{tr} \mathbf{C})\mathbf{C} : \mathbf{D}\,\varphi(t)\;dx\,dt\,, \tag{5.16b}$$

for m letting to infinity, we employ Lemma 5.1.

Passing to the limit in χ-term

For $\beta = 0$ the limit is trivial. Let $\beta > 0$. By (5.15) we know that $\operatorname{tr} \mathbf{C}_m \to \operatorname{tr} \mathbf{C}$ in $L^2(0,T;L^q(\Omega))$, $1 \le q < \infty$. Then there exists a subsequence (without changing the notation) such that $\operatorname{tr} \mathbf{C}_m \to \operatorname{tr} \mathbf{C}$ a.e. in $\Omega \times (0,T)$. Since χ is continuous, we have also $\chi(\operatorname{tr} \mathbf{C}_m) \to \chi(\operatorname{tr} \mathbf{C})$ a.e. in $\Omega \times (0,T)$. Moreover, there exists $p > 1$ such that

$$\int_0^T \int_\Omega |\chi(\operatorname{tr} \mathbf{C}_m)\mathbf{I} : \mathbf{D}\varphi(t)|^p\;dx\,dt\;\le C,$$

where C is a constant independent of m. In order to show the existence of the coefficient p we consider the cases for $\alpha \in [0,2]$ and $\alpha \in (2,6)$ separately:

- $0 \le \alpha \le 2$
  Let $1 < p_1 < {}^4\!/\!_\beta$ and let us consider the Hölder coefficients
  $$s = \frac{4}{p_1\beta}, \qquad\qquad z = \frac{4}{4 - p_1\beta}.$$
  Then
  $$\begin{aligned}\int_0^T \int_\Omega |\chi(\operatorname{tr} \mathbf{C}_m)\mathbf{I} : \mathbf{D}\varphi(t)|^{p_1}\;dx\,dt\;&\le c \max_{t\in[0,T]} |\varphi(t)|^{p_1} \int_0^T \int_\Omega |\operatorname{tr} \mathbf{C}_m|^{p_1\beta}|\operatorname{tr} \mathbf{D}|^{p_1}\;dx\,dt\\ &\le c\,\|\operatorname{tr} \mathbf{D}\|^{p_1}_{p_1 z} \int_0^T \|\operatorname{tr} \mathbf{C}_m\|_4^{p_1\beta}\;dt\end{aligned}$$
  is bounded by a constant independent of m, due to (5.12a).
- $2 < \alpha < 6$
  We choose
  $$s = \frac{\alpha+2}{p_2\beta}, \qquad\qquad z = \frac{\alpha+2}{\alpha+2-p_2\beta}$$
  such that $s, z > 1$ and $\frac{1}{s} + \frac{1}{z} = 1$ provided
  $$1 < p_2 < \frac{\alpha+2}{\beta}.$$
  Then it holds that
  $$\begin{aligned}\int_0^T \int_\Omega |\chi(\operatorname{tr} \mathbf{C}_m)\mathbf{I} : \mathbf{D}\varphi(t)|^{p_2}\;dx\,dt\;&\le c \max_{t\in[0,T]} |\varphi(t)|^{p_2} \int_0^T \int_\Omega |\operatorname{tr} \mathbf{C}_m|^{p_2\beta}|\operatorname{tr} \mathbf{D}|^{p_2}\;dx\,dt\\ &\le c\,\|\operatorname{tr} \mathbf{D}\|^{p_2}_{p_2 z} \int_0^T \|\operatorname{tr} \mathbf{C}_m\|_{\alpha+2}^{p_2\beta}\;dt\,.\end{aligned}$$
  Due to (5.12b) the above term is bounded by a constant C independent of m.

Let $p \in \{p_1, p_2\}$ depend on the coefficient α. We introduce the sequence $\{f_m\}_{m=1}^{\infty}$ with

$$f_m := \chi(\operatorname{tr}\mathbf{C}_m)\mathbf{I} : \mathbf{D}\varphi(t) \in L^p(0,T;L^p(\Omega)).$$

Further, let $f := \chi(\operatorname{tr}\mathbf{C})\mathbf{I} : \mathbf{D}\varphi(t) \in L^p(0,T;L^p(\Omega))$. It holds that $f_m \to f$ a.e. in $\Omega \times (0,T)$. We have just shown that $\{f_m\}_{m=1}^{\infty}$ is uniformly bounded in $L^p(0,T;L^p(\Omega))$ for $p > 1$. Lemma 5.1 yields the limit (5.16a) for m approaching infinity, i.e.

$$\int_0^T \int_\Omega \chi(\operatorname{tr}\mathbf{C}_m)\mathbf{I} : \mathbf{D}\,\varphi(t)\;dx\,dt \;\to \int_0^T \int_\Omega \chi(\operatorname{tr}\mathbf{C})\mathbf{I} : \mathbf{D}\,\varphi(t)\;dx\,dt\,.$$

Passing to the limit in ϕ-term

We only show the limiting process for $\alpha > 0$ since for $\alpha = 0$ the limit is trivial. As already mentioned, due to (5.15) there exists a subsequence (without changing the notation) $\operatorname{tr}\mathbf{C}_m \to \operatorname{tr}\mathbf{C}$ a.e. in $\Omega \times (0,T)$. The same argument yields a subsequence (without changing the notation) of $\{\mathbf{C}_m\}_{m=1}^{\infty}$ such that $\mathbf{C}_m \to \mathbf{C}$ a.e. in $\Omega \times (0,T)$. By the continuity of ϕ we also have $\phi(\operatorname{tr}\mathbf{C}_m) \to \phi(\operatorname{tr}\mathbf{C})$ a.e. in $\Omega \times (0,T)$. Finally, we obtain $\phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m \to \phi(\operatorname{tr}\mathbf{C})\mathbf{C}$ a.e. in $\Omega \times (0,T)$. Further, we show that there exists a constant C independent of m and a coefficient $r > 1$ such that

$$\int_0^T \int_\Omega |\phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m : \mathbf{D}\varphi(t)|^r\;dx\,dt\;\le C.$$

We consider two cases with respect to α separately:

- $0 \le \alpha \le 2$

  Let us consider a coefficient r_1 such that

  $$1 < r_1 < \frac{4}{\alpha+1} < \frac{4}{\alpha} < 4.$$

  Then, by the Hölder inequalities for

  $$s = \frac{4}{r_1\alpha} > 1, \qquad u = \frac{4}{r_1} > 1, \qquad z = \frac{4}{4 - r_1\alpha - r_1} > 1, \qquad \frac{1}{s} + \frac{1}{u} + \frac{1}{z} = 1,$$

  in space and for

  $$v = \frac{4}{r_1\alpha} > 1, \qquad\qquad w = \frac{4}{r_1} > 1, \qquad\qquad \frac{1}{v} + \frac{1}{w} \le 1$$

  in time, we get

  $$\begin{aligned}
  &\int_0^T \int_\Omega |\phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m : \mathbf{D}\varphi(t)|^{r_1}\;dx\,dt \\
  &\qquad \le c \max_{t\in[0,T]} |\varphi(t)|^{r_1} \int_0^T \int_\Omega |\operatorname{tr}\mathbf{C}_m|^{r_1\alpha} |\mathbf{C}_m|^{r_1} |\mathbf{D}|^{r_1}\;dx\,dt \\
  &\qquad \le c\,\|\mathbf{D}\|_{r_1 z}^{r_1} \int_0^T \|\operatorname{tr}\mathbf{C}_m\|_4^{r_1\alpha}\,\|\mathbf{C}_m\|_4^{r_1}\;dt \\
  &\qquad \le c\,\|\mathbf{D}\|_{r_1 z}^{r_1} \left(\int_0^T \|\operatorname{tr}\mathbf{C}_m\|_4^{r_1\alpha v}\;dt\right)^{1/v} \left(\int_0^T \|\mathbf{C}_m\|_4^{r_1 w}\;dt\right)^{1/w} \\
  &\qquad \le c\,\|\mathbf{D}\|_W^{r_1}\,\|\operatorname{tr}\mathbf{C}_m\|_{L^4(L^4)}^{(r_1\alpha)/4}\,\|\mathbf{C}_m\|_{L^4(L^4)}^{r_1/4}\,.
  \end{aligned}$$

  The a priori bounds (5.12a) yield the desired estimate independently of m.

- $2 < \alpha < 6$

  Let us assume that r_2 and l satisfy the inequalities
  $$1 < r_2 < \frac{\alpha+2}{\alpha} < 2 < \frac{r_2(\alpha+2)}{\alpha+2-r_2\alpha} < l.$$
  Then, we can choose the Hölder coefficients as
  $$s = \frac{\alpha+2}{r_2\alpha}, \qquad u = \frac{l}{r_2}, \qquad z = \frac{l(\alpha+2)}{l(\alpha+2) - r_2\alpha l - r_2(\alpha+2)},$$
  and
  $$v = \frac{\alpha+2}{r_2\alpha}, \qquad w = \frac{\alpha+2}{\alpha+2-r_2\alpha}$$
  for the estimates of integrals in space and time, respectively. We can write
  $$\begin{aligned}\int_0^T \int_\Omega &|\phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m : \mathbf{D}\varphi(t)|^{r_2}\ dx\,dt \\ &\le c \max_{t\in[0,T]} |\varphi(t)|^{r_2} \int_0^T \int_\Omega |\operatorname{tr}\mathbf{C}_m|^{r_2\alpha} |\mathbf{C}_m|^{r_2} |\mathbf{D}|^{r_2}\ dx\,dt \\ &\le c\,\|\mathbf{D}\|_{r_2 z}^{r_2} \int_0^T \|\operatorname{tr}\mathbf{C}_m\|_{\alpha+2}^{r_2\alpha}\, \|\mathbf{C}_m\|_l^{r_2}\ dt \\ &\le c\,\|\mathbf{D}\|_{r_2 z}^{r_2} \left(\int_0^T \|\operatorname{tr}\mathbf{C}_m\|_{\alpha+2}^{\alpha+2}\ dt \right)^{(r_2\alpha)/(\alpha+2)} \left(\int_0^T \|\mathbf{C}_m\|_l^{r_2 w}\ dt \right)^{1/w}.\end{aligned}$$
  Due to (5.12b) we choose
  $$w = \frac{2l}{r_2(l-2)},$$
  and consequently we obtain
  $$\begin{aligned}\int_0^T \int_\Omega &|\phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m : \mathbf{D}\varphi(t)|^{r_2}\ dx\,dt \\ &\le c\,\|\mathbf{D}\|_{r_2 z}^{r_2} \left(\int_0^T \|\operatorname{tr}\mathbf{C}_m\|_{\alpha+2}^{\alpha+2}\ dt \right)^{\frac{r_2\alpha}{\alpha+2}} \left(\int_0^T \|\mathbf{C}_m\|_l^{\frac{2l}{l-2}}\ dt \right)^{\frac{r_2(l-2)}{2l}}\end{aligned}$$
  as wanted. The equality
  $$w = \frac{2l}{(l-2)r_2} = \frac{\alpha+2}{\alpha+2-r_2\alpha} \qquad \text{yields} \qquad r_2 = \frac{2l(\alpha+2)}{(\alpha+2)(l-2)+2l\alpha}.$$
  Since we require that
  $$l > \frac{r_2(\alpha+2)}{\alpha+2-r_2\alpha} = \frac{2l}{l-2},$$
  the condition $l > 4$ has to be satisfied. In order to have $r_2 > 1$ we need
  $$l < \frac{2(\alpha+2)}{\alpha-2}.$$
  These two inequalities for l yield again the condition $\alpha < 6$.

Let now $r \in \{r_1, r_2\}$ depend on α. We define

$$g_m := \phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m : \mathbf{D}\varphi(t) \qquad \text{and} \qquad g := \phi(\operatorname{tr}\mathbf{C})\mathbf{C} : \mathbf{D}\varphi(t),$$

where g_m, $g \in L^r(0,T;L^r(\Omega))$. It holds that $g_m \to g$ a.e. in $\Omega \times (0,T)$ and $\{g_m\}_{m=1}^{\infty}$ is uniformly bounded in $L^r(0,T;L^r(\Omega))$ for $r > 1$. Again, by Lemma 5.1 we have the limit (5.16b) for m tending to infinity, i.e.

$$\int_0^T \int_\Omega \phi(\operatorname{tr}\mathbf{C}_m)\mathbf{C}_m : \mathbf{D}\,\varphi(t)\;dx\,dt \to \int_0^T \int_\Omega \phi(\operatorname{tr}\mathbf{C})\mathbf{C} : \mathbf{D}\,\varphi(t)\;dx\,dt\,.$$

Finally, after the limiting process, we see that the limit of the Galerkin approximate solution satisfies the weak formulation

$$\begin{aligned}
-\int_0^T \Big(\mathbf{u}(t), \mathbf{v}\,\varphi'(t)\Big)\;dt\; + (\mathbf{u}_0, \mathbf{v})\,\varphi(0) + \nu \int_0^T \Big(\Big(\mathbf{u}(t), \mathbf{v}\,\varphi(t)\Big)\Big)\;dt\, + \\
+ \int_0^T b\Big(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}\,\varphi(t)\Big)\;dt\; = -\int_0^T \int_\Omega \operatorname{tr}\mathbf{C}(t)\,\mathbf{C}(t) : \nabla\mathbf{v}\,\varphi(t)\;dx\,dt
\end{aligned}$$

$$\begin{aligned}
-\int_0^T \Big(\mathbf{C}(t), \mathbf{D}\,\varphi'(t)\Big)\;dt\; + (\mathbf{C}_0, \mathbf{D})\,\varphi(0) + \varepsilon \int_0^T \Big(\Big(\mathbf{C}(t), \mathbf{D}\,\varphi(t)\Big)\Big)\;dt\, + \\
+ \int_0^T B\Big(\mathbf{u}(t), \mathbf{C}(t), \mathbf{D}\,\varphi(t)\Big)\;dt\; - \int_0^T \Big((\nabla\mathbf{u}(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla\mathbf{u}(t))^T, \mathbf{D}\,\varphi(t)\Big)\;dt\; = \\
= \int_0^T \Big(\chi(\operatorname{tr}\mathbf{C}(t))\mathbf{I} - \phi(\operatorname{tr}\mathbf{C}(t))\mathbf{C}(t), \mathbf{D}\,\varphi(t)\Big)\;dt
\end{aligned}$$

for any $(\mathbf{v}, \mathbf{D}) \in V \times W$ and any $\varphi \in C^1([0,T])$, $\varphi(T) = 0$. The initial data is satisfied by the same arguments as in the proof of existence of the weak solution to the diffusive Peterlin model, see Section 4.2. □

5.3. Higher regularity and uniqueness

In what follows we show the higher regularity, and consequently, uniqueness of the weak solution to the generalized diffusive Peterlin model.

Higher regularity

Theorem 5.3. *(regularity of weak solutions to the generalized model)*
Let the assumptions **(A1)** - **(A4)** *be satisfied. Let* Ω *be of class* C^2 *and the initial data* $(\mathbf{u}_0, \mathbf{C}_0) \in [H^2(\Omega)^2 \cap V] \times H^2(\Omega)^{2\times 2}$. *Further, let one of the following assumptions be satisfied:*

$$\begin{aligned}
&0 < \alpha \le 2,\; 1 \le \beta < \alpha + 1 \;\textit{or}\; \beta = \alpha + 1 \;\textit{with}\; A_1 > 2B, \\
\textit{or} \quad &\alpha = 0 \;\textit{and}\; \beta = 1
\end{aligned} \tag{5.17}$$

Then the weak solution (5.9) *satisfies additionally*

$$\begin{aligned}
&\mathbf{u}' \in L^\infty(0,T;H) \cap L^2(0,T;V), \qquad && \mathbf{C}' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W), \\
&\mathbf{u} \in L^\infty(0,T;H^2(\Omega)), && \mathbf{C} \in L^\infty(0,T;H^2(\Omega)).
\end{aligned}$$

Proof. Let us note that the assumptions (5.17) on the exponents α and β are contained in the assumptions (5.8) required for the existence of weak solutions. Hence, the weak solutions to the generalized diffusive Peterlin model exist, see Theorem 5.2. Here we moreover need α to be less than or equal to 2. Since the proof is the analogue of the proof of Theorem 4.5, we again only concentrate on the generalized terms in the equation for the conformation tensor (5.1d). We shall find a positive constant K independent of m such that

$$\|\mathbf{u}'_m\|_{L^\infty(H)} + \|\mathbf{u}'_m\|_{L^2(V)} + \|\mathbf{C}'_m\|_{L^\infty(L^2)} + \|\mathbf{C}'_m\|_{L^2(W)} \leq K.$$

More regular initial data

The basis functions $\{\mathbf{v}_i\}_{i=1}^\infty$, $\{\mathbf{D}_i\}_{i=1}^\infty$ of V and W, respectively, are again taken as the eigenfunctions of the Stokes and the Laplace operator, respectively, and the properties (4.21) of $(\mathbf{u}_{0m}, \mathbf{C}_{0m})$ hold true. We still have $\mathbf{u}'_m(0)$ belonging to a bounded set in H, since (4.22) holds true, i.e.

$$\|\mathbf{u}'_m(0)\|_2 \leq (\nu c_0 + c_2)\|\mathbf{u}_0\|_{2,2} + c_1 \|\mathbf{C}_0\|_{2,2}^2 =: a_1. \tag{5.18}$$

In order to get $\mathbf{C}'(0)$ bounded in $L^2(\Omega)$, we need to estimate the right-hand side of the equality obtained by multiplying equation (5.10d) by $G'_{jm}(t)$, summing for $j = 1, \ldots, m$, and applying the Green formula at $t = 0$

$$\begin{aligned} \|\mathbf{C}'_m(0)\|_2^2 = &- B\Big(\mathbf{u}_{0m}, \mathbf{C}_{0m}, \mathbf{C}'_m(0)\Big) + \varepsilon\Big(\Delta\mathbf{C}_{0m}, \mathbf{C}'_m(0)\Big) + \\ &+ \Big((\nabla\mathbf{u}_{0m})\mathbf{C}_{0m} + \mathbf{C}_{0m}(\nabla\mathbf{u}_{0m})^T, \mathbf{C}'_m(0)\Big) + \\ &+ \Big(\chi(\operatorname{tr}\mathbf{C}_{0m})\mathbf{I} - \phi(\operatorname{tr}\mathbf{C}_{0m})\mathbf{C}_{0m}, \mathbf{C}'_m(0)\Big). \end{aligned}$$

The Hölder inequality yields

$$\begin{aligned} \|\mathbf{C}'_m(0)\|_2 \leq{}& \|\mathcal{B}(\mathbf{u}_{0m}, \mathbf{C}_{0m})\|_2 + \varepsilon\|\Delta\mathbf{C}_{0m}\|_2 + \left\|(\nabla\mathbf{u}_{0m})\mathbf{C}_{0m} + \mathbf{C}_{0m}(\nabla\mathbf{u}_{0m})^T\right\|_2 + \\ &+ \|\chi(\operatorname{tr}\mathbf{C}_{0m})\|_2 + \|\phi(\operatorname{tr}\mathbf{C}_{0m})\mathbf{C}_{0m}\|_2. \end{aligned}$$

Again, using (4.21) we get

$$\begin{aligned} \|\mathcal{B}(\mathbf{u}_{0m}, \mathbf{C}_{0m})\|_2 &\leq c_3\|\mathbf{u}_0\|_{2,2}\|\mathbf{C}_0\|_{2,2} \\ \|\Delta\mathbf{C}_{0m}\|_2 &\leq c_4\|\mathbf{C}_0\|_{2,2} \\ \left\|(\nabla\mathbf{u}_{0m})\mathbf{C}_{0m} + \mathbf{C}_{0m}(\nabla\mathbf{u}_{0m})^T\right\|_2 &\leq c_5\|\mathbf{u}_0\|_{2,2}\|\mathbf{C}_0\|_{2,2}. \end{aligned}$$

Analogously, we can estimate the generalized terms

$$\begin{aligned} \|\chi(\operatorname{tr}\mathbf{C}_{0m})\|_2 &\leq B\left\|\operatorname{tr}\mathbf{C}_{0m}^\beta\right\|_2 = B\|\operatorname{tr}\mathbf{C}_{0m}\|_{2\beta}^\beta \leq c_6\|\mathbf{C}_0\|_{2,2}^\beta \\ \|\phi(\operatorname{tr}\mathbf{C}_{0m})\mathbf{C}_{0m}\|_2 &\leq A_2\|\operatorname{tr}\mathbf{C}_{0m}^\alpha\mathbf{C}_{0m}\|_2 \leq c\|\mathbf{C}_{0m}\|_{2\alpha+2}^{\alpha+1} \leq c_7\|\mathbf{C}_0\|_{2,2}^{\alpha+1}. \end{aligned}$$

Finally, we obtain

$$\|\mathbf{C}'_m(0)\|_2 \leq (c_3 + c_5)\|\mathbf{u}_0\|_{2,2}\|\mathbf{C}_0\|_{2,2} + \varepsilon c_4\|\mathbf{C}_0\|_{2,2} + c_6\|\mathbf{C}_0\|_{2,2}^\beta + c_7\|\mathbf{C}_0\|_{2,2}^{\alpha+1} =: a_2. \tag{5.19}$$

A priori estimate for $(\mathbf{u}'_m, \mathbf{C}'_m)$

The time derivative of equations (5.10) for the Galerkin approximation reads

$$\begin{aligned}
(\mathbf{u}''_m(t), \mathbf{w}_j) &+ \nu\,((\mathbf{u}'_m(t), \mathbf{w}_j)) + b\,(\mathbf{u}'_m(t), \mathbf{u}_m(t), \mathbf{w}_j) + b\,(\mathbf{u}_m(t), \mathbf{u}'_m(t), \mathbf{w}_j) = \\
&= -\,(\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t), \nabla\mathbf{w}_j) - (\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{C}'_m(t), \nabla\mathbf{w}_j) \qquad (5.20a)\\
(\mathbf{C}''_m(t), \mathbf{D}_j) &+ \varepsilon\,((\mathbf{C}'_m(t), \mathbf{D}_j)) + B\,(\mathbf{u}'_m(t), \mathbf{C}_m(t), \mathbf{D}_j) + B\,(\mathbf{u}_m(t), \mathbf{C}'_m(t), \mathbf{D}_j) - \\
&- \Big((\nabla\mathbf{u}'_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}'_m(t))^T, \mathbf{D}_j\Big) - \\
&- \Big((\nabla\mathbf{u}_m(t))\mathbf{C}'_m(t) + \mathbf{C}'_m(t)(\nabla\mathbf{u}_m(t))^T, \mathbf{D}_j\Big) = \\
&= (\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{I}, \mathbf{D}_j) - \\
&- (\phi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t) - \phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}'_m(t), \mathbf{D}_j)\,, \qquad (5.20b)\\
&\qquad j = 1, \dots, m,\ t \in [0, T].
\end{aligned}$$

Now, we show the a priori estimates for $(\mathbf{u}'_m, \mathbf{C}'_m)$. We multiply equations (5.20a) and (5.20b) by $g'_{jm}(t)$ and $G'_{jm}(t)$, respectively. We take the sum for $j = 1, \dots, m$ and then we add the resulting equations together to obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\mathbf{u}'_m(t)\|_2^2 &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\mathbf{C}'_m(t)\|_2^2 + \nu|\!|\!|\mathbf{u}'_m(t)|\!|\!|^2 + \varepsilon\,\|\nabla\mathbf{C}'_m(t)\|_2^2 = \\
&= -b\Big(\mathbf{u}'_m(t), \mathbf{u}_m(t), \mathbf{u}'_m(t)\Big) - B\Big(\mathbf{u}'_m(t), \mathbf{C}_m(t), \mathbf{C}'_m(t)\Big) - \\
&- \Big(\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t), \nabla\mathbf{u}'_m(t)\Big) - \Big(\operatorname{tr}\mathbf{C}_m(t)\,\mathbf{C}'_m(t), \nabla\mathbf{u}'_m(t)\Big) + \\
&+ \Big((\nabla\mathbf{u}'_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{u}'_m(t))^T, \mathbf{C}'_m(t)\Big) + \qquad (5.21)\\
&+ \Big((\nabla\mathbf{u}_m(t))\mathbf{C}'_m(t) + \mathbf{C}'_m(t)(\nabla\mathbf{u}_m(t))^T, \mathbf{C}'_m(t)\Big) + \\
&+ \Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{I}, \mathbf{C}'_m(t)\Big) - \\
&- \Big(\phi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t) - \phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}'_m(t), \mathbf{C}'_m(t)\Big).
\end{aligned}$$

We recall the estimates (4.26) of the integrals on the right-hand side of (5.21) from Section 4.3. Here we only estimate the last three integrals for $\alpha \in [0, 2]$. This can be easily done, since the limiting case for $\alpha = 2$ has been already shown in the proof of Theorem 4.5.

Estimates of χ'*-term*

For $\beta = 1$ we simply have

$$\Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{I}, \mathbf{C}'_m(t)\Big) \le \int_\Omega \tilde{B}|\operatorname{tr}\mathbf{C}'_m(t)|^2\,dx = \tilde{B}\,\|\operatorname{tr}\mathbf{C}'_m(t)\|_2^2\,.$$

Let now $\beta > 1$. It holds that $1 < \beta \le \alpha + 1$. Then for the Hölder coefficients

$$p = \frac{4}{\beta - 1} > 1, \qquad\qquad q = \frac{4}{5 - \beta} > 1$$

we can write

$$\begin{aligned}
\Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{I}, \mathbf{C}'_m(t)\Big) &\le \tilde{B}\int_\Omega (\operatorname{tr}\mathbf{C}_m(t))^{\beta-1}|\operatorname{tr}\mathbf{C}'_m(t)|^2\,dx \\
&\le \tilde{B}\,\|\operatorname{tr}\mathbf{C}_m(t)\|_4^{\beta-1}\,\|\operatorname{tr}\mathbf{C}'_m(t)\|_{8/(5-\beta)}^2\,.
\end{aligned}$$

The interpolation inequality (2.8c) for $r = {}^8\!/_{5-\beta} > 2$ yields

$$\begin{aligned}
&\Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}_m'(t)\,\mathbf{I}, \mathbf{C}_m'(t)\Big) \\
&\qquad \le c\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{\beta-1}\Big(\left\|\mathbf{C}_m'(t)\right\|_2^2 + \left\|\mathbf{C}_m'(t)\right\|_2^{4/r}\left\|\nabla\mathbf{C}_m'(t)\right\|_2^{2(r-2)/2}\Big) \\
&\qquad \le c\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{\beta-1}\left\|\mathbf{C}_m'(t)\right\|_2^2 + \frac{\varepsilon}{12}\left\|\nabla\mathbf{C}_m'(t)\right\|_2^2 + \frac{c}{\varepsilon}\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{r(\beta-1)/2}\left\|\mathbf{C}_m'(t)\right\|_2^2.
\end{aligned}$$

Since $\beta - 1 < 4$, the norm $\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{\beta-1}$ is integrable in time. Similarly, $\beta \le 3$ implies

$$\frac{r(\beta-1)}{2} = \frac{4(\beta-1)}{5-\beta} \le 4,$$

i.e. the norm $\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{r(\beta-1)/2}$ is also integrable in time.

Estimates of ϕ-term

Let $\alpha = 0$. Then

$$\Big(\phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}_m'(t), \mathbf{C}_m'(t)\Big) \le A_2\int_\Omega \left|\mathbf{C}_m'(t)\right|^2\,dx = A_2\left\|\mathbf{C}_m'(t)\right\|_2^2.$$

Now, we consider the case $\alpha \in (0,2]$. We take the Hölder coefficients

$$p = \frac{4}{\alpha} > 1, \qquad\qquad q = \frac{4}{4-\alpha} > 1,$$

in order to get

$$\begin{aligned}
\Big(\phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}_m'(t), \mathbf{C}_m'(t)\Big) &\le A_2\int_\Omega \left|\operatorname{tr}\mathbf{C}_m(t)\right|^\alpha\left|\mathbf{C}_m'(t)\right|^2\,dx \\
&\le c\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^\alpha\left\|\mathbf{C}_m'(t)\right\|_{8/(4-\alpha)}^2.
\end{aligned}$$

Due to the continuous embedding $L^4(\Omega) \hookrightarrow L^{\frac{8}{4-\alpha}}(\Omega)$ and the interpolation inequality (2.8b) we can write

$$\begin{aligned}
\Big(\phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}_m'(t), \mathbf{C}_m'(t)\Big) &\le c\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^\alpha\Big(\left\|\mathbf{C}_m'(t)\right\|_2^2 + \left\|\mathbf{C}_m'(t)\right\|_2\left\|\nabla\mathbf{C}_m'(t)\right\|_2\Big) \\
&\le \frac{\varepsilon}{12}\left\|\nabla\mathbf{C}_m'(t)\right\|_2^2 + c\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^\alpha\left\|\mathbf{C}_m'(t)\right\|_2^2 + \\
&\quad + \frac{c}{\varepsilon}\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{2\alpha}\left\|\mathbf{C}_m'(t)\right\|_2^2.
\end{aligned}$$

Obviously, the norms $\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^\alpha$ and $\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{2\alpha}$ are integrable in time.

Estimates of ϕ'-term

The estimate is trivial for $\alpha = 0$. Indeed, $\phi(\operatorname{tr}\mathbf{C}_m(t))$ is then equal to a constant $A \in (A_1, A_2)$. Hence the derivative is 0 and the integral vanishes. The convenient choice of the Hölder coefficients is, for $\alpha \in (0,2]$,

$$p = \frac{4}{\alpha-1}, \qquad\qquad q = \frac{4}{3-\alpha}, \qquad\qquad r = 2.$$

Thus, the Hölder inequality together with the interpolation inequality (2.8b) and the embedding $L^4(\Omega) \hookrightarrow L^{\frac{4}{3-\alpha}}(\Omega)$ yield

$$
\begin{aligned}
&\Big(\phi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t),\mathbf{C}'_m(t)\Big)\\
&\qquad\le \int_\Omega \tilde{A}_2(\operatorname{tr}\mathbf{C}_m(t))^{\alpha-1}\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t):\mathbf{C}'_m(t)\;dx\\
&\qquad\le \tilde{A}_2\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{\alpha-1}\left\|\mathbf{C}_m(t)\right\|_{4/(3-\alpha)}\left\|\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}'_m(t)\right\|_2\\
&\qquad\le c\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_4^{\alpha-1}\left\|\mathbf{C}_m(t)\right\|_{4/(3-\alpha)}\left\|\mathbf{C}'_m(t)\right\|_4^2\\
&\qquad\le c\left\|\mathbf{C}_m(t)\right\|_4^{\alpha}\left\|\mathbf{C}'_m(t)\right\|_2^2+\frac{\varepsilon}{12}\left\|\nabla\mathbf{C}'_m(t)\right\|_2^2+\frac{c}{\varepsilon}\left\|\mathbf{C}_m(t)\right\|_4^{2\alpha}\left\|\mathbf{C}'_m(t)\right\|_2^2.
\end{aligned}
$$

Taking into account the estimates of all the integrals on the right-hand side of (5.21) we get the inequality

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left(\left\|\mathbf{u}'_m(t)\right\|_2^2+\left\|\mathbf{C}'_m(t)\right\|_2^2\right)&+\nu\left\|\!\left|\mathbf{u}'_m(t)\right|\!\right\|^2+\varepsilon\left\|\nabla\mathbf{C}'_m(t)\right\|_2^2\\
&\le\left(\left\|\mathbf{u}'_m(t)\right\|_2^2+\left\|\mathbf{C}'_m(t)\right\|_2^2\right)\beta(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta(t):=c(1/\nu,1/\varepsilon)\Big(1&+\left\|\mathbf{C}_m(t)\right\|_4^4+\left\|\mathbf{C}_m(t)\right\|_4^2+\left\|\!\left|\mathbf{u}_m(t)\right|\!\right\|^2+\left\|\!\left|\mathbf{u}_m(t)\right|\!\right\|+\\
&+\left\|\mathbf{C}_m(t)\right\|_4^{\alpha}+\left\|\mathbf{C}_m(t)\right\|_4^{2\alpha}\Big)
\end{aligned}
$$

is for $\alpha\in[0,2]$ an integrable function. The Gronwall inequality, using (5.18) and (5.19), yields

$$
\begin{aligned}
\left\|\mathbf{u}'_m(s)\right\|_2^2+\left\|\mathbf{C}'_m(s)\right\|_2^2&\le\left(\left\|\mathbf{u}'_m(0)\right\|_2^2+\left\|\mathbf{C}'_m(0)\right\|_2^2\right)\exp\left\{\int_0^s\beta(t)\right\}\\
&\le\left(a_1^2+a_2^2\right)\tilde{b},
\end{aligned}
$$

where $\tilde{b}=\tilde{b}(1/\nu,1/\varepsilon,\left\|\mathbf{u}_m\right\|_{L^2(V)},\left\|\mathbf{C}_m\right\|_{L^4(L^4)})$ is a positive constant. Let us recall that the norms contained in the constant $\tilde{b}$ are bounded independently of m by the constant k, see (5.11). Similarly as in the proof of Theorem 4.5, we get a priori estimate for the time derivative of the Galerkin approximation

$$
\left\|\mathbf{u}'_m\right\|_{L^\infty(H)}+\left\|\mathbf{u}'_m\right\|_{L^2(V)}+\left\|\mathbf{C}'_m\right\|_{L^\infty(L^2)}+\left\|\mathbf{C}'_m\right\|_{L^2(W)}\le K,
$$

where $K=K(T,1/\nu,1/\varepsilon,\mathbf{u}_0,\mathbf{C}_0,\alpha,\beta)$ is a positive constant dependent only on the data. We have shown the uniform a priori estimates for $(\mathbf{u}'_m,\mathbf{C}'_m)$, hence the limit also satisfies

$$
(\mathbf{u}',\mathbf{C}')\in\Big[L^\infty(0,T;H)\cap L^2(0,T;V)\Big]\times\Big[L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W)\Big].
$$

We refer to Section 4.3 where it has been shown that $\mathbf{u}\in L^\infty(0,T;H^2(\Omega))$. Here we only show that $\mathcal{T}\mathbf{C}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, since this is enough to get

$$
\mathbf{C}\in L^\infty(0,T;H^2(\Omega)).
$$

Indeed, let us recall equation (5.9c) in the following operator form

$$-\varepsilon\left((\mathbf{C}(t),\mathbf{D})\right)=(G(t),\mathbf{D})\,,\ \mathbf{D}\in H^1(\Omega),$$

where

$$G(t)=-\mathbf{C}'(t)-\tilde{\mathcal{B}}(\mathbf{u}(t),\mathbf{C}(t))+\mathcal{O}\left(\mathbf{u}(t),\mathbf{C}(t)\right)+\mathcal{T}\mathbf{C}(t).$$

By (4.31) and (4.32) from Section 4.3 we already have

$$\mathcal{B}(\mathbf{u},\mathbf{C})\in L^\infty(0,T;L^2(\Omega))\qquad\text{and}\qquad\mathcal{O}(\mathbf{u},\mathbf{C})\in L^\infty(0,T;L^{4/3}(\Omega)).$$

Since $\mathbf{C}\in L^\infty(0,T;W)$, the following inequality

$$\begin{aligned}|\langle\mathcal{T}\mathbf{C},\mathbf{D}\rangle|&=|(\chi(\operatorname{tr}\mathbf{C}(t))\mathbf{I}-\phi\left(\operatorname{tr}\mathbf{C}(t)\right)\mathbf{C}(t),\mathbf{D})|\\&\le B\left\|\operatorname{tr}\mathbf{C}(t)\right\|_{2\beta}^{\beta}\left\|\operatorname{tr}\mathbf{D}\right\|_2+A_2\left\|\operatorname{tr}\mathbf{C}(t)\right\|_{2\alpha+2}^{\alpha+1}\left\|\mathbf{D}\right\|_2\\&\le c\Bigg(\left\|\mathbf{C}(t)\right\|_W^{\beta}+\left\|\mathbf{C}(t)\right\|_W^{\alpha+1}\Bigg)\left\|\mathbf{D}\right\|_2\end{aligned}$$

implies, due to $W\hookrightarrow L^{2\beta}(\Omega)$ and $W\hookrightarrow L^{2\alpha+2}(\Omega)$, that $\mathcal{T}\mathbf{C}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Thus, we have $G(t)\in L^\infty(0,T;L^{4/3}(\Omega))$ and by the same arguments as in the proof of Theorem 4.5 we can conclude that $\mathbf{C}\in L^\infty(0,T;H^2(\Omega))$. □

The case $\alpha>2$

Let us note that the existence of the weak solutions to the generalized diffusive Peterlin model has been proven for $\alpha\in[0,6)$. However, the higher regularity result has been shown only for $\alpha\in[0,2]$. We shall demonstrate what happens for $\alpha>2$ when proving the higher regularity of the weak solutions (5.9).

Estimates of χ'-term

The estimate of χ'-term is straightforward. Firstly, let $\beta=\alpha+1$ and $2B<A_1$. Then

$$\begin{aligned}\Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}_m'(t)\,\mathbf{I},\mathbf{C}_m'(t)\Big)&\le\int_\Omega\tilde{B}(\operatorname{tr}\mathbf{C}_m(t))^{\beta-1}|\operatorname{tr}\mathbf{C}_m'(t)|^2\;dx\\&=\tilde{B}\left\|(\operatorname{tr}\mathbf{C}_m(t))^{\alpha/2}\operatorname{tr}\mathbf{C}_m'(t)\right\|_2^2.\end{aligned}$$

Secondly, for $1<\beta<\alpha+1$ we choose the Hölder coefficients

$$p=\frac{\alpha+2}{\beta-1}>1\qquad\qquad q=\frac{\alpha+2}{\alpha+3-\beta}>1$$

in order to get

$$\begin{aligned}\Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}_m'(t)\,\mathbf{I},\mathbf{C}_m'(t)\Big)&\le\int_\Omega\tilde{B}(\operatorname{tr}\mathbf{C}_m(t))^{\beta-1}(\operatorname{tr}\mathbf{C}_m'(t))^2\;dx\\&\le\tilde{B}\left\|\operatorname{tr}\mathbf{C}_m(t)\right\|_{\alpha+2}^{\beta-1}\left\|\operatorname{tr}\mathbf{C}_m'(t)\right\|_{2(\alpha+2)/(\alpha+3-\beta)}^2.\end{aligned}$$

By the interpolation inequality (2.8c) for

$$r=\frac{2(\alpha+2)}{\alpha+3-\beta}>2$$

we obtain

$$\begin{aligned}
&\Big(\chi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{I},\mathbf{C}'_m(t)\Big)\\
&\qquad\le c\,\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{\beta-1}\Big(\|\mathbf{C}'_m(t)\|_2^2+\|\mathbf{C}'_m(t)\|_2^{4/r}\,\|\nabla\mathbf{C}'_m(t)\|_2^{2(r-2)/2}\Big)\\
&\qquad\le c\,\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{\beta-1}\,\|\mathbf{C}'_m(t)\|_2^2+\frac{\varepsilon}{12}\,\|\nabla\mathbf{C}'_m(t)\|_2^2+\frac{c}{\varepsilon}\,\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{r(\beta-1)/2}\,\|\mathbf{C}'_m(t)\|_2^2\,.
\end{aligned}$$

We now work with $1<\beta<\alpha+1$ for $\alpha\in(2,6)$. Thus $\beta-1<\alpha+2$ and $\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{\beta-1}$ is integrable in time. Since also $2\beta<\alpha+4$ we get that

$$\frac{r(\beta-1)}{2}=\frac{(\alpha+2)(\beta-1)}{\alpha+3-\beta}<\alpha+2.$$

Consequently, $\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{r(\beta-1)/2}$ is also integrable in time.

Estimates of ϕ-term

By estimating ϕ-term we shall take into account the proper sign of the integral. Indeed, we do the estimates from below

$$\begin{aligned}
\Big(\phi(\operatorname{tr}\mathbf{C}_m(t))\mathbf{C}'_m(t),\mathbf{C}'_m(t)\Big)&\ge\frac{A_1}{2}\int_\Omega|\operatorname{tr}\mathbf{C}_m(t)|^\alpha|\operatorname{tr}\mathbf{C}'_m(t)|^2\,dx\\
&=\frac{A_1}{2}\left\|(\operatorname{tr}\mathbf{C}_m(t))^{\alpha/2}\operatorname{tr}\mathbf{C}'_m(t)\right\|_2^2.
\end{aligned}$$

Our aim is to get a suitable inequality in order to apply the Gronwall lemma analogously as in the case $\alpha\in[0,2]$. Up to now, all the estimates of the integrals on the right-hand side of (5.21) (except the ϕ'-term) would yield

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Bigg(\|\mathbf{u}'_m(t)\|_2^2+\|\mathbf{C}'_m(t)\|_2^2\Bigg)&+\frac{\nu}{2}|\!|\!|\mathbf{u}'_m(t)|\!|\!|^2+\frac{\varepsilon}{2}\,\|\nabla\mathbf{C}'_m(t)\|_2^2+\\
&+\left(\frac{A_1}{2}-\tilde{B}\right)\left\|(\operatorname{tr}\mathbf{C}_m(t))^{\alpha/2}\operatorname{tr}\mathbf{C}'_m(t)\right\|_2^2\\
&\le\frac{1}{2}\Bigg(\|\mathbf{u}'_m(t)\|_2^2+\|\mathbf{C}'_m(t)\|_2^2\Bigg)\beta(t)
\end{aligned}\tag{5.22}$$

with $\beta(t)$ integrable in time. We would only require $A_1>2\tilde{B}$. What remains is to show that the estimates of ϕ'-term taking into account all the previous bounds. We shall see the term, from where the constraint $\alpha\le 2$ arises.

Estimates of ϕ'-term

For $r>1$ and $s>2$ it holds that

$$\frac{1}{2}+\frac{\alpha-2}{2(\alpha+2)}+\frac{1}{r}+\frac{1}{s}=1,$$

which, by the Hölder, the Young and the interpolation inequalities, allows us to write

$$\begin{aligned}
&\Big(\phi'(\operatorname{tr}\mathbf{C}_m(t))\operatorname{tr}\mathbf{C}'_m(t)\,\mathbf{C}_m(t),\mathbf{C}'_m(t)\Big)\\
&\qquad\le c\left\|(\operatorname{tr}\mathbf{C}_m(t))^{\alpha/2}\operatorname{tr}\mathbf{C}'_m(t)\right\|_2\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{(\alpha-2)/2}\|\mathbf{C}_m(t)\|_r\,\|\mathbf{C}'_m(t)\|_s\\
&\qquad\le \delta\left\|(\operatorname{tr}\mathbf{C}_m(t))^{\alpha/2}\operatorname{tr}\mathbf{C}'_m(t)\right\|_2^2+\frac{c}{\delta}\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{\alpha-2}\|\mathbf{C}_m(t)\|_r^2\,\|\mathbf{C}'_m(t)\|_s^2\\
&\qquad\le \delta\left\|(\operatorname{tr}\mathbf{C}_m(t))^{\alpha/2}\operatorname{tr}\mathbf{C}'_m(t)\right\|_2^2+\frac{c}{\delta}\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{\alpha-2}\|\mathbf{C}_m(t)\|_r^2\,\|\mathbf{C}'_m(t)\|_2^2+\\
&\qquad+\frac{\varepsilon}{12}\|\nabla\mathbf{C}'_m(t)\|_2^2+\frac{c}{\delta^2\varepsilon}\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{s(\alpha-2)/2}\|\mathbf{C}_m(t)\|_r^s\,\|\mathbf{C}'_m(t)\|_2^2,
\end{aligned}$$

for $0<\delta<A_1/2-\tilde{B}$. If we chose, for instance,

$$r=\frac{2(\alpha+2)}{\alpha-2}>2\qquad\qquad s=\frac{2(\alpha+2)}{6-\alpha}>2\qquad\qquad \frac{1}{r}+\frac{1}{s}=\frac{2}{\alpha+2}$$

then $\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{\alpha-2}\|\mathbf{C}_m(t)\|_r^2$ is integrable in time. However,

$$\|\operatorname{tr}\mathbf{C}_m(t)\|_{\alpha+2}^{s(\alpha-2)/2}\|\mathbf{C}_m(t)\|_r^s$$

would not be integrable, unless $\alpha\le 2$.

Uniqueness of regular weak solution

Having the regularity result for the weak solution to the generalized diffusive Peterlin model, we are now able to prove that the more regular weak solution from Theorem 5.3 is unique.

Theorem 5.4. *(uniqueness of regular weak solution to the generalized model)*
Let the assumptions of Theorem 5.3 be satisfied. Then the weak solution (5.9) *is unique.*

Proof. We derive, analogously as (4.35), the equality satisfied by the difference $(\mathbf{u},\mathbf{C})=(\mathbf{u}_1-\mathbf{u}_2,\mathbf{C}_1-\mathbf{C}_2)$, where $(\mathbf{u}_1,\mathbf{C}_1)$, $(\mathbf{u}_2,\mathbf{C}_2)$ are two different weak solutions (5.9) satisfying the same initial condition, i.e.

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{u}(t)\|_2^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{C}(t)\|_2^2+\nu|\!|\!|\mathbf{u}(t)|\!|\!|^2+\varepsilon\|\nabla\mathbf{C}(t)\|_2^2=&\\
=-b\Big(\mathbf{u}(t),\mathbf{u}_1(t),\mathbf{u}(t)\Big)-B\Big(\mathbf{u}(t),\mathbf{C}_1(t),\mathbf{C}(t)\Big)+&\\
+\Big((\nabla\mathbf{u}_2(t))\mathbf{C}(t)+\mathbf{C}(t)(\nabla\mathbf{u}_2(t))^T,\mathbf{C}(t)\Big)+&\\
+\Big((\nabla\mathbf{u}(t))\mathbf{C}_1(t)+\mathbf{C}_1(t)(\nabla\mathbf{u}(t))^T,\mathbf{C}(t)\Big)-&\\
-\Big(\operatorname{tr}\mathbf{C}(t)\,\mathbf{C}_1(t),\nabla\mathbf{v}(t)\Big)-\Big(\operatorname{tr}\mathbf{C}_2(t)\,\mathbf{C}(t),\nabla\mathbf{v}(t)\Big)+&\\
+\Big(\,(\chi(\operatorname{tr}\mathbf{C}_1(t))-\chi(\operatorname{tr}\mathbf{C}_2(t)))\,\mathbf{I},\mathbf{C}(t)\Big)-&\\
-\Big(\phi(\operatorname{tr}\mathbf{C}_1(t))\mathbf{C}_1(t)-\phi(\operatorname{tr}\mathbf{C}_2(t))\mathbf{C}_2(t),\mathbf{C}(t)\Big).&
\end{aligned}\tag{5.23}$$

For the estimates of the integrals on the right-hand side of the above equation see (4.36). We again estimate only the last two, coming from the generalization of the diffusive

Peterlin model. Let $(x,t) \in \Omega \times (0,T)$ be fixed. Then $a := \operatorname{tr} \mathbf{C}_1(x,t)$ and $b := \operatorname{tr} \mathbf{C}_2(x,t)$ are real numbers. Since χ and ϕ are continuously differentiable functions, we get for an $s \in (0,1)$ and $z_1 = (1-s)\operatorname{tr} \mathbf{C}_1(x,t) + s \operatorname{tr} \mathbf{C}_2(x,t))$,

$$\chi(\operatorname{tr} \mathbf{C}_1(t)) - \chi(\operatorname{tr} \mathbf{C}_2(t)) = \chi'(z_1)(\operatorname{tr} \mathbf{C}_1(x,t) - \operatorname{tr} \mathbf{C}_2(x,t)),$$

and analogously

$$\phi(\operatorname{tr} \mathbf{C}_1(x,t)) - \phi(\operatorname{tr} \mathbf{C}_2(x,t)) = \phi'(z_2)\,(\operatorname{tr} \mathbf{C}_1(x,t) - \operatorname{tr} \mathbf{C}_2(x,t))$$

for a.e. $x \in \Omega$ and a.e. $t \in (0,T)$.

Estimates of χ-term

Let $\beta > 1$. It holds that

$$p = \frac{\alpha+2}{\beta-1} > 1 \qquad q = \frac{\alpha+2}{\alpha+3-\beta} > 1 \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Then, by the Hölder inequality, we get

$$\begin{aligned}
&\Big(\chi(\operatorname{tr} \mathbf{C}_1(t)) - \chi(\operatorname{tr} \mathbf{C}_2(t)), \operatorname{tr} \mathbf{C}(t)\Big) \\
&\qquad \le \int_\Omega |\chi'((1-s)\operatorname{tr} \mathbf{C}_1(x,t) + s \operatorname{tr} \mathbf{C}_2(x,t)))||\operatorname{tr} \mathbf{C}(x,t)|^2 \, dx \\
&\qquad \le \tilde{B} \int_\Omega |(1-s)\operatorname{tr} \mathbf{C}_1(x,t) + s \operatorname{tr} \mathbf{C}_2(x,t)|^{\beta-1} |\operatorname{tr} \mathbf{C}(x,t)|^2 \, dx \\
&\qquad \le c \int_\Omega \Big(|\operatorname{tr} \mathbf{C}_1(x,t)|^{\beta-1} + |\operatorname{tr} \mathbf{C}_2(x,t)|^{\beta-1}\Big) |\operatorname{tr} \mathbf{C}(x,t)|^2 \, dx \\
&\qquad \le c \, \|\operatorname{tr} \mathbf{C}(t)\|^2_{2(\alpha+2)/(\alpha+3-\beta)} \Big(\|\operatorname{tr} \mathbf{C}_1(t)\|^{\beta-1}_{\alpha+2} + \|\operatorname{tr} \mathbf{C}_2(t)\|^{\beta-1}_{\alpha+2}\Big).
\end{aligned}$$

The interpolation inequality (2.8c) for

$$r = \frac{2(\alpha+2)}{\alpha+3-\beta} > 2$$

then yields

$$\begin{aligned}
&\Big(\chi(\operatorname{tr} \mathbf{C}_1(t)) - \chi(\operatorname{tr} \mathbf{C}_2(t)), \operatorname{tr} \mathbf{C}(t)\Big) \\
&\qquad \le c \, \|\mathbf{C}(t)\|_2^2 \Big(\|\operatorname{tr} \mathbf{C}_1(t)\|^{\beta-1}_{\alpha+2} + \|\operatorname{tr} \mathbf{C}_2(t)\|^{\beta-1}_{\alpha+2}\Big) + \frac{\varepsilon}{12} \, \|\nabla \mathbf{C}(t)\|_2^2 + \\
&\qquad + \frac{c}{\varepsilon} \, \|\mathbf{C}(t)\|_2^2 \Big(\|\operatorname{tr} \mathbf{C}_1(t)\|^{r(\beta-1)/2}_{\alpha+2} + \|\operatorname{tr} \mathbf{C}_2(t)\|^{r(\beta-1)/2}_{\alpha+2}\Big),
\end{aligned}$$

where we have also used the continuous embedding of W into $L^{\alpha+2}(\Omega)$ and the Hölder inequality with the coefficients $r/2$ and $r/r-2$. As we have already shown, for $\alpha \in [0,2]$ it holds that

$$\frac{r(\beta-1)}{2} \le \alpha + 2,$$

and hence the norms $\|\operatorname{tr} \mathbf{C}_1(t)\|^{r(\beta-1)/2}_{\alpha+2}$ and $\|\operatorname{tr} \mathbf{C}_2(t)\|^{r(\beta-1)/2}_{\alpha+2}$ are integrable in time.

Estimates of ϕ-terms

We can write

$$\phi(\operatorname{tr}\mathbf{C}_1(t))\mathbf{C}_1(t) - \phi(\operatorname{tr}\mathbf{C}_2(t))\mathbf{C}_2(t) = \phi(\operatorname{tr}\mathbf{C}_1(t))\mathbf{C}(t) + \Big(\phi(\operatorname{tr}\mathbf{C}_1(t)) - \phi(\operatorname{tr}\mathbf{C}_2(t))\Big)\mathbf{C}_2(t).$$

The estimate of the first integral is trivial for $\alpha = 0$. For $\alpha > 0$ we get

$$\begin{aligned}\Big(\phi(\operatorname{tr}\mathbf{C}_1(t))\mathbf{C}(t), \mathbf{C}(t)\Big) &\le A_2 \int_\Omega (\operatorname{tr}\mathbf{C}_1(t))^\alpha \mathbf{C}(t) : \mathbf{C}(t)\ dx \\ &\le c\,\|\mathbf{C}(t)\|_4^2\,\|\operatorname{tr}\mathbf{C}_1(t)\|_{2\alpha}^\alpha \\ &\le c\,\|\mathbf{C}(t)\|_2^2\,\|\operatorname{tr}\mathbf{C}_1(t)\|_4^\alpha + \frac{\varepsilon}{12}\,\|\nabla\mathbf{C}(t)\|_2^2 + \frac{c}{\varepsilon}\,\|\mathbf{C}(t)\|_2^2\,\|\operatorname{tr}\mathbf{C}_1(t)\|_4^{2\alpha},\end{aligned}$$

where the Hölder and the interpolation inequality (2.8b) were used.

Let us estimate the second ϕ-term, analogously as χ-term. For $\alpha = 0$ the estimate is trivial. Let us consider the case $\alpha \in (0, 2]$. It holds that

$$p = \frac{4}{\alpha - 1} > 1 \qquad q = 2 > 1 \qquad s = \frac{4}{3 - \alpha} > 1 \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{q} = 1,$$

which allows us to write

$$\begin{aligned}&\Big(\Big(\phi(\operatorname{tr}\mathbf{C}_1(t)) - \phi(\operatorname{tr}\mathbf{C}_2(t))\Big)\mathbf{C}_2(t), \mathbf{C}(t)\Big) \\ &\quad\le \int_\Omega |\phi'((1-s)\operatorname{tr}\mathbf{C}_1(x,t) + s\operatorname{tr}\mathbf{C}_2(x,t)))||\operatorname{tr}\mathbf{C}(x,t)||\mathbf{C}_2(x,t)||\mathbf{C}(x,t)|\ dx \\ &\quad\le \tilde{A}_2 \int_\Omega |(1-s)\operatorname{tr}\mathbf{C}_1(x,t) + s\operatorname{tr}\mathbf{C}_2(x,t)|^{\alpha-1}|\mathbf{C}_2(x,t)||\mathbf{C}(x,t)|^2\ dx \\ &\quad\le c\int_\Omega \Big(|\operatorname{tr}\mathbf{C}_1(x,t)|^{\alpha-1} + |\operatorname{tr}\mathbf{C}_2(x,t)|^{\alpha-1}\Big)\,|\mathbf{C}_2(x,t)||\mathbf{C}(x,t)|^2\ dx \\ &\quad\le c\,\|\mathbf{C}(t)\|_4^2\,\|\mathbf{C}_2(t)\|_{4/(3-\alpha)}\Big(\|\operatorname{tr}\mathbf{C}_1(t)\|_4^{\alpha-1} + \|\operatorname{tr}\mathbf{C}_2(t)\|_4^{\alpha-1}\Big) \\ &\quad\le c\,\|\mathbf{C}(t)\|_2^2\,\|\mathbf{C}_2(t)\|_{4/(3-\alpha)}\Big(\|\operatorname{tr}\mathbf{C}_1(t)\|_4^{\alpha-1} + \|\operatorname{tr}\mathbf{C}_2(t)\|_4^{\alpha-1}\Big) + \\ &\quad+ \frac{c}{\varepsilon}\,\|\mathbf{C}(t)\|_2^2\,\|\mathbf{C}_2(t)\|_{4/(3-\alpha)}^2\Big(\|\operatorname{tr}\mathbf{C}_1(t)\|_4^{2\alpha-2} + \|\operatorname{tr}\mathbf{C}_2(t)\|_4^{2\alpha-2}\Big) + \\ &\quad+ \frac{\varepsilon}{12}\,\|\nabla\mathbf{C}(t)\|_2^2,\end{aligned}$$

where again the interpolation inequality (2.8b) and the Hölder inequality were used. It holds that $\|\operatorname{tr}\mathbf{C}\|_{4/(3-\alpha)} \le c\,\|\operatorname{tr}\mathbf{C}\|_4$. Putting the above estimates together we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{u}(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{C}(t)\|_2^2 + \frac{\nu}{2}\|\mathbf{u}(t)\|^2 + \frac{\varepsilon}{2}\,\|\nabla\mathbf{C}(t)\|_2^2 \le& \\ \le \Big(\frac{1}{2}\,\|\mathbf{u}(t)\|_2^2 + \frac{1}{2}\,\|\mathbf{C}(t)\|_2^2\Big)\delta(t),&\end{aligned} \tag{5.24}$$

where now

$$\delta(t) := c(1/\nu, 1/\varepsilon)\Bigg(|||\mathbf{u}_1(t)|||^2 + |||\mathbf{u}_2(t)|||^2 + |||\mathbf{u}_2(t)||| + \|\mathbf{C}_1(t)\|_4^4 + \|\mathbf{C}_1(t)\|_4^2 +$$

$$+ \|\operatorname{tr}\mathbf{C}_1\|_{\alpha+2}^{(\beta-1)(\alpha+2)/(\alpha+3-\beta)} + \|\operatorname{tr}\mathbf{C}_2\|_{\alpha+2}^{(\beta-1)(\alpha+2)/(\alpha+3-\beta)} +$$

$$+ \|\mathbf{C}_2(t)\|_4 \left(\|\operatorname{tr}\mathbf{C}_1(t)\|_4^{\alpha-1} + \|\operatorname{tr}\mathbf{C}_2(t)\|_4^{\alpha-1}\right) + \|\operatorname{tr}\mathbf{C}_1\|_{\alpha+2}^{\beta-1} + \|\operatorname{tr}\mathbf{C}_2\|_{\alpha+2}^{\beta-1} +$$

$$+ \|\mathbf{C}_2(t)\|_4^2 \left(\|\operatorname{tr}\mathbf{C}_1(t)\|_4^{2\alpha-2} + \|\operatorname{tr}\mathbf{C}_2(t)\|_4^{2\alpha-2}\right) + \|\operatorname{tr}\mathbf{C}_1\|_4^{\alpha} + \|\operatorname{tr}\mathbf{C}_1\|_4^{2\alpha} \Bigg).$$

Function $\delta(t)$ is integrable in time. Thus, the Gronwall inequality yields

$$\|\mathbf{u}(s)\|_2^2 + \|\mathbf{C}(s)\|_2^2 \le \left(\|\mathbf{u}(0)\|_2^2 + \|\mathbf{C}(0)\|_2^2 \right) \exp\left\{ \int_0^s \delta(t) \right\}.$$

The same arguments as in the proof of Theorem 4.6 yield uniqueness of the regular weak solution $(\mathbf{u}, \mathbf{C})$ to the generalized diffusive Peterlin model.

□

6 Numerical approximation

This chapter aims to study an application of the pressure-stabilized characteristics finite element method for the diffusive Peterlin model. This method is a combination of the method of characteristic curves and the Brezzi-Pitkäranta stabilization for conforming linear finite elements. It has been applied successfully for the Navier-Stokes and the Oseen equations in the recent works [111, 112], where the optimal error estimates have been proven theoretically and confirmed by a series of numerical experiments. In what follows we generalize and extend the error analysis of the so-called stabilized Lagrange-Galerkin method for the Peterlin viscoelastic fluid.

In the first section the diffusive Oseen-type Peterlin model is introduced. The method of characteristics, followed by the discretization of the material derivative, is explained in Section 6.2. The pressure-stabilized finite element method is described in the next section. Afterwards, in Section 6.4, a semi-implicit linear numerical scheme is derived and the existence of the discrete solution is studied.

6.1. The Oseen-type Peterlin model

Our aim is to present the numerical analysis of the pressure-stabilized characteristics finite element method for the diffusive Peterlin model (3.2). We concentrate on the main difficulties arising from the coupling of the Navier-Stokes equations with the additional parabolic equation for the conformation tensor. As a starting point of the analysis we confined ourselves with the Oseen-type problem.

To this end let $\mathbf{w} : \Omega \times [0, T] \to \mathbb{R}^2$ be a given velocity that shall be used to linearise the convective terms in the model (3.2). We recall that the material derivative then reads

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla.$$

We note that the Laplacian of a solenoidal velocity $\mathbf{u}$ can be rewritten as the divergence of the symmetric part of the gradient $\mathrm{D}(\mathbf{u})$. Indeed,

$$\operatorname{div}(2\mathrm{D}(\mathbf{u})) = \operatorname{div}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \begin{pmatrix} \Delta u_1 + \partial_{x_1}(\operatorname{div} \mathbf{u}) \\ \Delta u_2 + \partial_{x_2}(\operatorname{div} \mathbf{u}) \end{pmatrix} = \Delta \mathbf{u}.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz-continuous boundary and let $T > 0$. Then the *Oseen-type Peterlin model* reads

$$\begin{aligned}
\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \operatorname{div}\left(2\nu \mathrm{D}(\mathbf{u})\right) + \nabla p = \operatorname{div}\left(\operatorname{tr}\mathbf{C}\,\mathbf{C}\right) + \mathbf{f} \\
\operatorname{div}\mathbf{u} = 0 \\
\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t} - \varepsilon\Delta\mathbf{C} = (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T - \left(\operatorname{tr}\mathbf{C}\right)^2\mathbf{C} + \operatorname{tr}\mathbf{C}\,\mathbf{I} + \mathbf{F}.
\end{aligned} \tag{6.1a}$$

It is equipped with the boundary and initial conditions

$$\left(\mathbf{u}, \frac{\partial\mathbf{C}}{\partial\mathbf{n}}\right) = (\mathbf{0}, \mathbf{0}), \qquad (\mathbf{u}(0), \mathbf{C}(0)) = \left(\mathbf{u}^0, \mathbf{C}^0\right), \tag{6.1b}$$

respectively, for some initial data $(\mathbf{u}^0, \mathbf{C}^0) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ defined on Ω. Let us note that we moreover consider some given external forces $(\mathbf{f}, \mathbf{F}) : \Omega \times (0, T) \to \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$. This will be useful in Section 8.2 in order to study the experimental order of convergence numerically.

From now on we use the following notation for the classical functional spaces

$$V := H_0^1(\Omega)^2, \; Q := L_0^2(\Omega), \; W := H^1(\Omega)^{2\times 2}.$$

Remark 6.1. *In Chapters 4 and 5 we have used V to denote the Sobolev space $H^1_{0,div}(\Omega)^2$.*

In order to present the weak formulation of the problem (6.1) we define the bilinear forms a_u on $V \times V$, b on $V \times Q$, $\mathcal{A}$ on $(V \times Q) \times (V \times Q)$ and a_c on $W \times W$ by

$$\begin{aligned}
a_u(\mathbf{u}, \mathbf{v}) &:= 2\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\mathbf{v})\, dx & b(\mathbf{u}, q) &:= -\int_\Omega \operatorname{div}\mathbf{u}\; q\; dx \\
\mathcal{A}\left((\mathbf{u}, p), (\mathbf{v}, q)\right) &:= \nu\, a_u(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, q) + b(\mathbf{v}, p) & a_c(\mathbf{C}, \mathbf{D}) &:= \int_\Omega \nabla\mathbf{C} : \nabla\mathbf{D}\; dx\,,
\end{aligned} \tag{6.2}$$

respectively. Further, we assume the following hypothesis holds.

Hypothesis 1. *(given velocity)*
The function $\mathbf{w}$ satisfies $\mathbf{w} \in C([0, T]; W_0^{1,\infty}(\Omega)^2)$.

The weak formulation of the Oseen-type Peterlin model (6.1) follows.

Definition 6.2. *(weak formulation of the Oseen-type model)*
Let $(\mathbf{u}^0, \mathbf{C}^0) \in L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ and $(\mathbf{f}, \mathbf{F}) \in L^2(0, T; L^2(\Omega)^2) \times L^2(0, T; L^2(\Omega)^{2\times 2})$ be given. Find $(\mathbf{u}, p, \mathbf{C}) : (0, T) \to V \times Q \times W$ such that for $t \in (0, T)$,

$$\begin{aligned}
\left(\frac{\mathrm{D}\mathbf{u}(t)}{\mathrm{D}t}, \mathbf{v}\right) + \mathcal{A}\left((\mathbf{u}, p)(t), (\mathbf{v}, q)\right) &= -\left(\operatorname{tr}\mathbf{C}(t)\,\mathbf{C}(t), \nabla\mathbf{v}\right) + (\mathbf{f}(t), \mathbf{v}) \\
\left(\frac{\mathrm{D}\mathbf{C}(t)}{\mathrm{D}t}, \mathbf{D}\right) + \varepsilon\, a_c(\mathbf{C}(t), \mathbf{D}) &= \left((\nabla\mathbf{u}(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla\mathbf{u}(t))^T, \mathbf{D}\right) - \\
&\quad - \left(\left(\operatorname{tr}\mathbf{C}(t)\right)^2\mathbf{C}(t), \mathbf{D}\right) + \left(\operatorname{tr}\mathbf{C}(t)\,\mathbf{I}, \mathbf{D}\right) + (\mathbf{F}(t), \mathbf{D}) \\
&\qquad \forall\,(\mathbf{v}, q, \mathbf{D}) \in V \times Q \times W
\end{aligned} \tag{6.3a}$$

and the initial condition

$$(\mathbf{u}(0), \mathbf{C}(0)) = \left(\mathbf{u}^0, \mathbf{C}^0\right) \tag{6.3b}$$

is satisfied.

6.2. Time discretization

We consider an equidistant discretization of the time interval $[0, T]$. Let Δt denote the *time increment* and let $t^n := n\Delta t$ for $n = 0, \dots, N_T$ be the *time step*. Here $N_T := T/\Delta t$ is the *number of time steps*. For $n = 1, \dots, N_T$ it holds that $\Delta t = t^n - t^{n-1}$.

6.2.1. Method of characteristics

The basic idea of the method of characteristic curves is to follow the trajectory of the particle. It has been widely used for numerical treatment of the convection-diffusion equation as well as of the Navier-Stokes equations. As pointed out in [5], employing the *Lagrangian description* of the flow, the method identifies each mesh point with a fluid particle at time t^0 and follows it at subsequent times $t^1, \dots, t^n$ as the flow goes on. Then the solution of a given problem is obtained by solving the diffusion operator on the new mesh. The advantages of this approach over the conventional Eulerian schemes are the numerical stability and absence of the convective term that often leads to numerical difficulties. However, this approach might cause the deformation of the initial mesh and consequently the lost of accuracy of the numerical approximation.

To avoid the problems related to mesh deformation we consider the simplest version of the method of characteristics that *follows the particle backward in time*. More precisely, it seeks the position of the particle at time t^{n-1} that will reach the mesh point at time t^n. It still has common advantages of the characteristic methods, i.e. robust stability with little numerical diffusion and symmetry of the resulting matrix of the system of algebraic equations.

Finite difference or finite element methods have been combined with either the first or the second order approximation of the material derivative based on the method of characteristic curves to derive some numerical schemes for the flow problems, see, e.g., [5, 20, 45, 107, 109, 110, 111, 112, 116, 117, 118, 129, 140].

Approximation of the material derivative

Let $X(t; x, t^n)$ denote the position at time t of a fluid particle that reaches the point $x \in \Omega$ at time t^n. Then the trajectory $X(\cdot; x, t^n) : (0, T) \to \mathbb{R}^2$ of such particle, for a point $x \in \Omega$ and an integer n, $1 \le n \le N_T$, satisfies the ordinary differential equation

$$\begin{aligned} X'(t) &= \mathbf{w}(X, t) \ \text{ in } (t^{n-1}, t^n) \\ X(t^n) &= x, \end{aligned} \tag{6.4}$$

whose solution is indeed the characteristic curve of the operator $\frac{\mathrm{D}}{\mathrm{D}t}$.

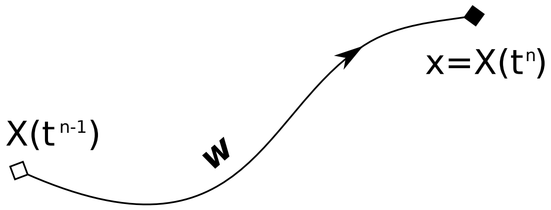


Figure 6.1.: Trajectory of a fluid particle

The assumptions of Picard's existence theorem, cf. Theorem A.6 in Appendix A, are satisfied due to Hypothesis 1. Hence, there exists a unique solution to the initial value problem (6.4). Let us now approximate this solution as

$$\begin{aligned} X(t^{n-1};x,t^n) &= x - \int_{t^{n-1}}^{t^n} \mathbf{w}(X(t;x,t^n),t)\,dt \\ &\approx x - \mathbf{w}(X(t^n;x,t^n),t^n)\Delta t = x - \mathbf{w}(x,t^n)\Delta t. \end{aligned} \tag{6.5}$$

Consequently, we define a mapping X_1^n by

$$X_1^n(x) := x - \mathbf{w}^n(x)\Delta t. \tag{6.6}$$

The backward difference approximation of the total derivative of a smooth function g at (x,t^n), $n \geq 1$ is given by

$$\frac{\mathrm{d}g}{\mathrm{d}t}(X(t;x,t^n))_{|t=t^n} = \frac{g(X(t^n;x,t^n),t^n) - g(X(t^{n-1};x,t^n),t^{n-1})}{\Delta t} + O(\Delta t). \tag{6.7}$$

For the Oseen-type problem the characteristics method takes into account that

$$\frac{\partial g}{\partial t} + (\mathbf{w}\cdot\nabla)g$$

can be written as the total derivative of g in the direction of a given velocity $\mathbf{w}$. Therefore, in the interval (t^{n-1},t^n), it holds that

$$\frac{\mathrm{D}g(X(t),t)}{\mathrm{D}t} = \frac{\mathrm{d}g(X(t),t)}{\mathrm{d}t}. \tag{6.8}$$

A combination of (6.5), (6.7) and (6.8) yields the *first order* approximation of the material derivative at a point (x,t^n), $1 \leq n \leq N_T$, i.e.

$$\begin{aligned} \frac{\mathrm{D}g}{\mathrm{D}t}(x,t^n) &= \frac{\mathrm{d}g}{\mathrm{d}t}(X(t;x,t^n))_{|t=t^n} \\ &= \frac{g^n(x) - (g^{n-1}\circ X_1^n)(x)}{\Delta t} + O(\Delta t). \end{aligned} \tag{6.9}$$

Let us note that we have used the notation

$$g^n(x) := g(x,t^n), \qquad \mathbf{w}^n(x) := \mathbf{w}(x,t^n), \qquad (g^{n-1}\circ X_1^n)(x) := g(X_1^n(x),t^{n-1}).$$

The point $X_1^n(x)$ is called the *upwind point* of x with respect to $\mathbf{w}^n(x)$. The next proposition, see, e.g., [129, 140], gives a sufficient condition to guarantee that all upwind points are in Ω.

Proposition 6.3. *(upwind points)*
Under Hypothesis 1 and the inequality

$$\Delta t < \frac{1}{\|\mathbf{w}\|_{C(W^{1,\infty})}}, \tag{6.10}$$

it holds that, for any $n = 0,\dots,N_T$,

$$X_1^n(\Omega) = \Omega.$$

Proof. Firstly, we show that $X_1^n(\Omega) \subset \Omega$. Let us define $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Since the given velocity $\mathbf{w}$ vanishes on the boundary, it holds, for $x \in \Omega$, that

$$|x - X_1^n(x)| = |\mathbf{w}^n(x)|\Delta t \le \delta(x)\Delta t \, \|\mathbf{w}\|_{C(W^{1,\infty})} < \delta(x), \tag{6.11}$$

which implies that $X_1^n(x) \in \Omega$. Moreover, we get $X_1^n(\partial\Omega) = \partial\Omega$. Secondly, the mapping $X_1^n : \Omega \to \Omega$ is injective. Indeed, let $x, y \in \Omega$, $x \neq y$. Then it holds that

$$\begin{aligned} |X_1^n(x) - X_1^n(y)| &= |x - y - \Delta t(\mathbf{w}^n(x) - \mathbf{w}^n(y))| \\ &\ge \Big| |x - y| - \Delta t |\mathbf{w}^n(x) - \mathbf{w}^n(y)| \Big| \\ &\ge |x - y| - \Delta t \, \|\mathbf{w}\|_{C(W^{1,\infty})} |x - y| > 0. \end{aligned}$$

Finally, we prove the surjectivity by contradiction. To this end we assume

$$\exists\, \tilde{y} \in \Omega : X_1^n(\Omega) \neq \tilde{y} \quad \forall\, x \in \Omega. \tag{6.12}$$

Let Ω_1 be an open set such that $\Omega \subset \Omega_1$ and $\operatorname{dist}(\mathbb{R}^2 \backslash \Omega_1, \Omega) > 2e$, where $e := \sup_{x \in \Omega} \delta(x)$. Further, we define a mapping $\mu : \Omega_1 \to \Omega_1$ by

$$\mu(x) = \begin{cases} x, & x \in \Omega_1 \backslash \Omega \\ X_1^n(x), & x \in \Omega. \end{cases}$$

Since $\mathbf{w}$ is continuous, the mapping μ is also continuous on Ω_1. Then the set B given by

$$B := \Big\{ \phi(x) : x \in B(\tilde{y}, 2e) \Big\},$$

where $B(\tilde{y}, 2e)$ is a closed ball with the center $\tilde{y}$ and the radius $2e$, is simply connected. By (6.11) and the definition of μ we get, for $x \in \Omega_1$, that

$$|\mu(x) - x| < e.$$

Thus $\tilde{y} \in B$ which contradicts the assumption (6.12). Hence the mapping X_1^n is a bijection and we conclude $X_1^n(\Omega) = \Omega$. $\square$

We fix Δt_0 satisfying the condition (6.10) and

$$\det\left(\frac{\partial X_1^n(x)}{\partial x} \right) \ge \frac{1}{2}, \quad \forall\, n \in \{0, \ldots, N_T\}, \quad \forall\, \Delta t \in (0, \Delta t_0]. \tag{6.13}$$

From now on we assume that

$$\Delta t \in (0, \Delta t_0]. \tag{6.14}$$

6.3. Spatial discretization

There are different approaches to treat the partial differential equations numerically. We are interested in the so-called *ansatz methods* which are characterized by prescribing an approximate solution in a certain form, see [60]. In general, one needs to determine the coefficients in a linear combination of a set of chosen functions. For instance, the collocation method requires the differential equation to be satisfied at a specified discrete set of points. However, it is more popular to use methods based on the weak formulation of the problem. Since the pointwise validity of a differential equation is typically not available, the ansatz functions can be less smooth than in the collocation method.

Galerkin method is a technique to discretize the variational problem. Instead of solving the problem in an infinite-dimensional space S, one chooses a finite-dimensional space $S_h = \text{ span } \{\varphi_j(x)\}_{j=1}^M$. Then the *discrete solution* can be expressed as

$$g_h(x) = \sum_{j=1}^{M} g_j \, \varphi_j(x).$$

The choice of ansatz and test functions is crucial for the Galerkin methods as it strongly influence whether the implementation of the method is effective or not. One of the popular variants of this method is the finite element method where splines are used as ansatz functions.

6.3.1. Finite element method

The finite element method is a specific process of constructing some finite-dimensional spaces S_h whose basis functions have small support. This process has three typical features. Specifically, the decomposition of the domain into geometrically simple subdomains, the definition of a local space of ansatz functions and the fulfilment of some inter-subdomain conditions that guarantee some useful properties of S_h. There is an extensive literature on the finite element method and we refer to, e.g., [26, 33, 58, 60, 117, 120].

In what follows we shall briefly describe the three aspects of the finite element method in the two-dimensional case.

1) Triangulation

We assume Ω is a polygonal domain. It can be completely covered by the union of a finite number of disjoint quadrilaterals or triangles K_i, i.e.

$$\bar{\Omega} = \bigcup_{i=1}^{N_E} K_i, \qquad \text{and} \qquad \text{int } K_i \cap \text{int } K_j = \emptyset \text{ if } i \neq j. \tag{6.15}$$

The K is called *element*, its sides and vertices are called *edges* and *nodal points*, respectively. The natural numbers N_E and N_P denote the *number of elements* and the *number of nodal points*, respectively.

For any element K let ρ_K be the supremum of the diameters of the inscribed circles in K. We define the *diameter of an element* K and the representative length of the mesh, i.e. the *mesh size* by

$$h_K := \sup_{x,y \in K} \|x - y\|, \qquad h := \max_{K \in \mathcal{T}_h} h_K,$$

respectively. From now on the mesh size h satisfies $h \in (0, h_0)$ for a fixed $h_0 > 0$.

Let the *triangulation* $\mathcal{T}_h := \{K_i\}_{i=1}^{N_E}$ be given by the subdomains K_i satisfying (6.15). We say $\mathcal{T}_h$ is an *admissible* triangulation if there are no hanging nodes, i.e. if for each pair of subdomains K_i and K_j exactly one of the following four cases occurs

- $K_i \cap K_j$ is a complete edge of both K_i and K_j
- $K_i \cap K_j$ is a nodal point of the triangulation
- $K_i \cap K_j = \emptyset$
- $K_i = K_j$.

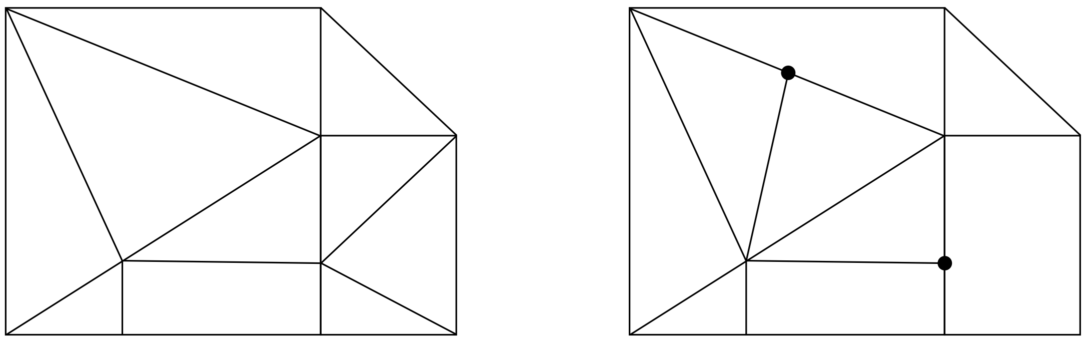


Figure 6.2.: An admissible triangulation (left) and hanging nodes (right)

A family of triangulations $\{\mathcal{T}_h\}_{h \searrow 0}$ of Ω is called a *regular family* if there exists a uniform constant σ_1 independent of the mesh size such that

$$\frac{h_K}{\rho_K} \leq \sigma_1 \quad \forall\, K \in \mathcal{T}_h,\ \forall\, h. \tag{6.16}$$

The family of triangulations satisfying (6.16) is sometimes called in the literature shape-regular or quasi-uniform. Further, we say that a family $\{\mathcal{T}_h\}_{h \searrow 0}$ satisfies the *inverse assumption* if there exists a positive constant σ_2 independent of h such that

$$\frac{h}{h_K} \leq \sigma_2 \quad \forall\, K \in \mathcal{T}_h,\ \forall\, h. \tag{6.17}$$

Remark 6.4. *Concerning the assumption that Ω is a polygonal domain, let us make a note on the "variational crime", see [66, 135]. This term was introduced by Strang in 1970's, cf. [136, 137]. To commit a variational crime means to use some approximations instead of solving the exact Galerkin problem. This might violate the validity of the abstract Galerkin error analysis. For instance, if Ω has a smooth curved boundary then it can not be triangulated exactly, which yields $S_h \not\subset S$. Further, in practice, it is not possible to compute the integrals in the discrete formulation exactly. This also leads to the variational crime, as the numerical integration is used. Nevertheless, real numerical computations converge, and therefore one does not worry about these approximations. Strang pointed this out and he quantified the errors caused by these crimes in his work, cf. [24]. See also [26] where Chapter 10 is entirely about the variational crimes.*

2) Shape functions and degrees of freedom

A *finite element* (K, Π_K, Σ_K) is characterised by the geometry of the *subdomain* K, by the local space Π_K of ansatz functions and by the set Σ_K of linear functionals on Π_K.

The typical choice for Π_K is the space of k-th order polynomials defined on K, denoted by $\mathcal{P}_k(K)$. The first reason for choosing polynomials as ansatz functions is a simple computation of the coefficients of the resulting algebraic system of equations, since the piecewise polynomials as ansatz and test functions lead to a sparse coefficient matrix. The second reason is that the interpolation properties of polynomials are the key to all convergence results.

The set Σ_K, whose elements are called *degrees of freedom*, is also of importance. We note that the number of degrees of freedom has to be equal to the dimension d_K of the space Π_K. Let $l_{K,i} : C^s(K) \to \mathbb{R}$, $i = 1, \ldots, d_K$, be linear and continuous functionals that are linearly independent. There are different types of functionals which can be considered in the finite element method. If only the point evaluation of a function is involved in Σ_K then the finite element is called a *Lagrange element*. Other possibilities to choose the degrees of freedom are the point evaluation of the first partial derivative, the normal derivative on the edge or the integral mean values on K or its edges. The smoothness parameter s has to be chosen such that $l_{K,i}$ are continuous. For the Lagrange element $s = 0$ is sufficient.

A finite element is said to be *unisolvent* if any $p \in \Pi_K$ is uniquely determined by the degrees of freedom of the element. It means that for each $c = (c_1, \ldots, c_{d_K}) \in \mathbb{R}^{d_K}$ there is exactly one element in Π_K such that c_i is the image of i-th functional. If we choose the unit Cartesian vectors for c then it follows that a set $\{\varphi_{K,i}\}_{i=1}^{d_K}$ exists and $\varphi_{K,i} \in \Pi_K$ such that

$$l_{K,i}(\varphi_{K,i}) = \delta_{ij}, \; i, j = 1, \ldots, d_K,$$

where δ_{ij} is the Kronecker delta. Consequently, $\{\varphi_{K,i}\}_{i=1}^{d_K}$ forms a basis of Π_K that is called a local basis. The functions $\varphi_{K,i}$ are called *shape functions*.

From now on we shall work with the vertex-oriented Lagrange finite elements whose degrees of freedom are the nodal values of the function. Let us define a global basis of the *finite element space* S_h. If we take the *set of degrees of freedom of the finite element space* as

$$\Sigma_h := \{l_{h,j} : 1 \leq j \leq N_P\}$$

then the basis functions φ_j, $j = 1, \ldots, N_P$ of the finite element space are defined by

$$\varphi_j \in S_h, \qquad l_{h,i}(\varphi_j) = \delta_{i,j}, \; i = 1, \ldots, N_P.$$

These basis functions are derived from the shape functions of the finite elements. Indeed, for $j \in \{1, \ldots, N_p\}$, let $l_{h,j} \in \Sigma_h$ and let P_j be the associated nodal point. Further, let $\{K_k\}_{k \in J(P_j)}$ be the set of all finite elements of $\mathcal{T}_h$ that have P_j as one of their nodes. For each $k \in J(P_j)$, let $\varphi_{K_k,j}$ be the shape function of the element K_k associated with the restriction of $l_{h,j}$ to K_k. Then the function $\varphi_j \in S_h$ defined by

$$\varphi_j = \begin{cases} \varphi_{K_k,j} & \text{over } K_k, \; k \in J(P_j) \\ 0 & \text{elsewhere} \end{cases} \tag{6.18}$$

is the basis function of the space S_h associated with the degree of freedom $l_{h,j}$, i.e. in this case with the nodal point P_j.

With any function $g \in C(\Omega)$ we associate the function given by

$$\Pi_h g := \sum_{j=1}^{N_P} l_{h,j}(g)\varphi_j = \sum_{j=1}^{N_P} g(P_j)\varphi_j \in S_h, \tag{6.19}$$

which is the co-called *Lagrange interpolation operator.* For $1 \leq j \leq N_P$, it holds that

$$l_{h,j}(\Pi_{S_h} g) = l_{h,j}(g) = g(P_j).$$

3) Conforming elements

We say the element is *conforming* if $S_h \subset S$. This guarantees that some properties of S are valid on S_h. Let us provide a sufficient condition to guarantee the $H^1(\Omega)$-conformity of the finite element.

Lemma 6.5.
Let $g : \bar{\Omega} \to \mathbb{R}$ be a function satisfying $g_{|K_i} \in C^1(K_i)$ for $i = 1, \ldots, N_E$. Then

$$g \in C(\bar{\Omega}) \Rightarrow g \in H^1(\Omega).$$

For the proof we refer to, e.g., Lemma 4.1. in [60].
We have introduced the basic notions and described the features of the finite element method. Finally, let us note that a family of finite elements is said to be *affine* if all its finite elements are affine-equivalent to a single finite element $(\tilde{K}, \tilde{\Pi}_K, \tilde{\Sigma}_K)$ which is called a *reference finite element*, cf., e.g., [33].

Finite element approximation of the Oseen-type Peterlin model

We consider an affine family of unisolvent conforming linear vertex-oriented Lagrange finite elements whose admissible family of triangulations $\mathcal{T}_h$ is regular and satisfies the inverse assumption.

In other words, we choose the space of *linear* functions as the local space of ansatz functions on K, i.e. $\Pi_K = \mathcal{P}_1(K)$ with the dimension $d_K = 3$. In order to ensure that the finite elements are conforming in $H^1(\Omega)$ the *continuous* piecewise linear approximation is considered. The functionals in Σ_K are taken as the point values at the nodes of the element K, i.e. for $p \in \Pi_K$ we have $l_{K,i}(p) = p(P_j)$. Finally, the global basis is given by (6.18). Now, we give the definition of the discrete functional spaces for the approximation of the velocity, pressure and the conformation tensor that solve the weak formulation (6.3) of the Oseen-type Peterlin model.

Let the discrete functional spaces X_h, M_h, Y_h and V_h, Q_h, W_h be defined by

$$X_h := \left\{ \mathbf{v}_h \in C(\bar{\Omega})^2 : \ \mathbf{v}_{h_{|K}} \in \mathcal{P}_1(K)^2, \forall K \in \mathcal{T}_h \right\}, \qquad V_h := X_h \cap V \tag{6.20a}$$

$$M_h := \left\{ q_h \in C(\bar{\Omega}) : \ q_{h_{|K}} \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h \right\}, \qquad Q_h := M_h \cap Q \tag{6.20b}$$

$$Y_h := \left\{ \mathbf{D}_h \in C_{sym}(\bar{\Omega})^{2\times 2} : \ \mathbf{D}_{h_{|K}} \in \mathcal{P}_1(K)^{2\times 2}_{sym}, \forall K \in \mathcal{T}_h \right\}, \quad W_h := Y_h \cap W, \tag{6.20c}$$

respectively. Due to Lemma 6.5 it holds that $X_h \times M_h \times Y_h \subset H^1(\Omega)^2 \times H^1(\Omega) \times H^1(\Omega)^{2\times 2}$. Let us note that $\mathbf{v}_h \in V_h$ is zero on the boundary $\partial\Omega$. Further, W_h is a discrete space of symmetric matrix functions. We seek the discrete solution of the velocity, pressure and the conformation tensor in the finite element spaces V_h, Q_h and W_h, respectively.

6.3.2. Pressure stabilization

The Galerkin approximation of certain fluid flow models leads to a saddle point problem whose well-posedness in the framework of mixed variational formulations is established due to Brezzi [27] and Babuška [9, 10]. The discrete stability conditions important for the construction of reliable numerical schemes arise. One of the conditions is the well-known *inf-sup condition*, or the Ladyzhenskaya-Babuška-Brezzi condition, cf. [8, 79, 106]. For the finite element method it is important to choose the finite-dimensional space with good stability and approximation properties. There is a brief view on saddle point problems via theory of mixed variational formulations in Appendix B. It aims to explain the purpose of stability conditions for the proper choice of the finite element spaces.

It is well-known that $\mathcal{P}_2/\mathcal{P}_0$ or the Taylor-Hood element $\mathcal{P}_2/\mathcal{P}_1$ belong to the group of *inf-sup stable* Lagrange finite elements. However, the number of degrees of freedom is much higher than for $\mathcal{P}_1/\mathcal{P}_1$ element. This element, on the other hand, *does not* satisfy the convectional inf-sup condition, cf., e.g. [58] or Appendix B,

$$\inf_{q_h \in Q_h} \sup_{\mathbf{u}_h \in V_h} \frac{b(\mathbf{u}_h, q_h)}{\|\mathbf{u}_h\|_V \, \|q_h\|_Q} \geq \beta. \tag{6.21}$$

Therefore a stabilization technique has to be considered. In this case we employ the *Brezzi-Pitkäranta stabilization method*, cf. [29]. As mentioned in [111], other stabilization techniques as the Galerkin least square method or the pressure stabilizing Petrov Galerkin method, cf. [23, 54, 56], lead to higher number of degrees of freedom and moreover, combined with the characteristics method some of them lose the symmetry of the system matrix.

Pressure-stabilized bilinear form

Let δ_0 be a positive constant and $(.,.)_K$ be the $L^2(K)^2$ inner product. We define the bilinear form $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ by

$$\begin{aligned} \mathcal{A}_h\left((\mathbf{u}, p), (\mathbf{v}, q)\right) &:= \mathcal{A}\left((\mathbf{u}, p), (\mathbf{v}, q)\right) - \mathcal{S}_h(p, q) \\ &= \nu \, a_u(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, q) + b(\mathbf{v}, p) - \mathcal{S}_h(p, q), \end{aligned} \tag{6.22}$$

where the bilinear form $\mathcal{S}_h$ defined on $H^1(\Omega) \times H^1(\Omega)$ is given by

$$\mathcal{S}_h(p, q) := \delta_0 \sum_{K \in \mathcal{T}_h} h_K^2 \, (\nabla p, \nabla q)_K. \tag{6.23}$$

If the finite element spaces V_h, Q_h are given by (6.20a) and (6.20b), respectively, then one can show that the above defined stabilized bilinear form $\mathcal{A}_h$ satisfies the so-called *generalized inf-sup condition.*

Lemma 6.6. *(generalized inf-sup condition)*
There exists a positive constant γ, independent of h and ν, such that

$$\inf_{(\mathbf{u}_h, p_h) \in V_h \times Q_h} \sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{\mathcal{A}_h\left((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\right)}{\|(\mathbf{u}_h, p_h)\|_{V \times Q} \, \|(\mathbf{v}_h, q_h)\|_{V \times Q}} \geq \gamma. \tag{6.24}$$

The complete proof can be found in [56] and the references therein.

6.4. The semi-implicit linear scheme

In order to approximate the solution of the Oseen-type Peterlin model (6.1) numerically, we now combine

- semi-implicit discretization in time
- first order approximation of the material derivative by the characteristics method (6.9)
- continuous piecewise linear finite element approximation (6.20)
- pressure-stabilization (6.22).

Consequently, the definition of the discrete variational formulation follows.

Definition 6.7. *(pressure-stabilized characteristics finite element scheme)*
Let $\{(\mathbf{f}_h^n, \mathbf{F}_h^n)\}_{n=1}^{N_T} \subset L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ *and the initial data* $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ *be given. We seek* $\{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ *such that, for* $n = 1, \dots, N_T$,

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\Big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h) \Big) = -\Big(\mathrm{tr}\, \mathbf{C}_h^n \; \mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h\Big) + \Big(\mathbf{f}_h^n, \mathbf{v}_h\Big) \tag{6.25a}$$

$$\begin{aligned}\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon\, a_c\Big(\mathbf{C}_h^n, \mathbf{D}_h\Big) &= \Big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla \mathbf{u}_h^n)^T, \mathbf{D}_h\Big) - \\ &- \Big((\mathrm{tr}\, \mathbf{C}_h^{n-1})^2 \mathbf{C}_h^n, \mathbf{D}_h\Big) + \Big(\mathrm{tr}\, \mathbf{C}_h^{n-1}\, \mathbf{I}, \mathbf{D}_h\Big) + \Big(\mathbf{F}_h^n, \mathbf{D}_h\Big)\end{aligned} \tag{6.25b}$$

$$\forall\ (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h.$$

Before presenting the existence results for the solution of the above semi-implicit scheme let us introduce the useful notation

$$(\mathbf{f}_h, \mathbf{F}_h) := \Big\{ (\mathbf{f}_h^n, \mathbf{F}_h^n) \Big\}_{n=1}^{N_T}, \qquad (\mathbf{u}_h, p_h, \mathbf{C}_h) := \Big\{ (\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n) \Big\}_{n=1}^{N_T},$$

and the semi-norm associated to the bilinear form $\mathcal{S}_h$, i.e.

$$|q|_h := \left(\sum_{K \in \mathcal{T}_h} h_K^2 (\nabla q, \nabla q)_K \right)^{1/2}, \ q \in H^1(\Omega). \tag{6.26}$$

In what follows we show the existence and uniqueness of the discrete solution. Moreover, employing the implicit function theorem we prove local in time existence of the positive definite discrete conformation tensor $\mathbf{C}_h^n$. To this end, we first rewrite the system of equations (6.25) in a matrix form.

Matrix form of the numerical scheme

Expanding the solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$ in terms of the appropriate basis functions we indeed get a linear algebraic system of equations with the square matrix

$$\Big(\mathbb{X}_1 + \Delta t\, \mathbb{X}_2 + \Delta t\, \mathbb{Z}^{n-1}\Big). \tag{6.27}$$

Here the matrices $\mathbb{X}_1$ and $\mathbb{X}_2$ correspond to $(\mathbf{u}_h^n, \mathbf{v}_h)$, $(\mathbf{C}_h^n, \mathbf{D}_h)$ and $\mathcal{A}_h((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h))$, $\varepsilon\, a_c(\mathbf{C}_h^n, \mathbf{D}_h)$, respectively. The matrix $\mathbb{Z}^{n-1}$ depends on the solution from the previous time step as it arises from the semi-implicit terms $((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla \mathbf{u}_h^n)^T, \mathbf{D}_h)$, $((\mathrm{tr}\, \mathbf{C}_h^{n-1})^2 \mathbf{C}_h^n, \mathbf{D}_h)$ and $(\mathrm{tr}\, \mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h)$.

We denote by $\mathbf{y}^n$ the vector whose entries corresponds to the nodal values of the discrete solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$. For the given $(\mathbf{f}_h^n, \mathbf{F}_h^n)$ and $\mathbf{y}^{n-1}$, the scheme (6.25) is equivalent to finding the vector $\mathbf{y}^n$ satisfying

$$\left(\mathbb{X}_1 + \Delta t\, \mathbb{X}_2 + \Delta t\, \mathbb{Z}^{n-1}\right) \mathbf{y}^{\mathbf{n}} = \mathbf{r}^{n-1}(\Delta t) + \Delta t\, \tilde{\mathbf{r}}^{n-1}. \tag{6.28}$$

The right-hand side vector $\mathbf{r}^{n-1}$ depends on Δt as it corresponds to $(\mathbf{u}_h^{n-1} \circ X_1^n, \mathbf{v}_h)$, $(\mathbf{C}_h^{n-1} \circ X_1^n, \mathbf{D}_h)$. The precise definition of the matrices and vectors can be found in Appendix C. We note that, for N_P denoting the number of nodal points, the system (6.28) corresponds to $6N_P$ linear equations with $6N_P$ unknowns, to be precise, $2N_P$, N_P and $3N_P$ unknowns for the velocity, pressure and for the symmetric conformation tensor, respectively.

Proposition 6.8. *(existence of discrete solution)*
Assume Hypothesis 1. Let $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ be given with $\mathbf{C}_h^0$ being a symmetric matrix. Then for any time increment Δt satisfying (6.14) *there exists a unique solution $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ to the scheme* (6.25) *or equivalently of the system* (6.28) *with $\mathbf{C}_h^n$ being a symmetric matrix for $n = 1, \ldots, N_T$.*

Proof. We show that the matrix

$$\mathbb{X} := \mathbb{X}_1 + \Delta t\, \mathbb{X}_2 + \Delta t\, \mathbb{Z}^{n-1}$$

is invertible for all Δt and for all $n = 1, \ldots, N_T$. It is known that a square matrix is invertible, or non-singular, if and only if the corresponding homogeneous system does not have any non-trivial solutions. Let us assume the right-hand side of the system (6.28) is zero vector. Then, following the idea of formal energy estimates based on Proposition 3.5, we test the corresponding part of the scheme (6.25) by $\left(\mathbf{u}_h^n, -p_h^n, \frac{1}{2}\mathrm{tr}\, \mathbf{C}_h^n\, \mathbf{I}\right)$ to get

$$\begin{aligned} \|\mathbf{u}_h^n\|_2^2 + \nu\, \Delta t\, \|\mathrm{D}(\mathbf{u}_h^n)\|_2^2 + \delta_0\, \Delta t |p_h^n|_h^2 + \Delta t \left(\mathrm{tr}\, \mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla \mathbf{u}_h^n\right) &= 0 \\ \frac{1}{2}\, \|\mathrm{tr}\, \mathbf{C}_h^n\|_2^2 + \frac{\varepsilon}{2}\, \Delta t\, \|\nabla \mathrm{tr}\, \mathbf{C}_h^n\|_2^2 - \Delta t \left(\nabla \mathbf{u}_h^n : \mathbf{C}_h^{n-1}, \mathrm{tr}\, \mathbf{C}_h^n\right) + \frac{1}{2} \left\|\mathrm{tr}\, \mathbf{C}_h^{n-1}\ \mathrm{tr}\, \mathbf{C}_h^n\right\|_2^2 &= 0, \end{aligned}$$

which in the sum yields

$$\begin{aligned} \|\mathbf{u}_h^n\|_2^2 + \nu\, \Delta t\, \|\mathrm{D}(\mathbf{u}_h^n)\|_2^2 + \delta_0\, \Delta t |p_h^n|_h^2 + \frac{1}{2}\, \|\mathrm{tr}\, \mathbf{C}_h^n\|_2^2 + \frac{\varepsilon}{2}\, \Delta t\, \|\nabla \mathrm{tr}\, \mathbf{C}_h^n\|_2^2 + \\ + \frac{1}{2} \left\|\mathrm{tr}\, \mathbf{C}_h^{n-1}\ \mathrm{tr}\, \mathbf{C}_h^n\right\|_2^2 = 0. \end{aligned}$$

Thus $\mathbf{u}_h^n = \mathbf{0}$, $p_h^n = 0$ and $\mathrm{tr}\, \mathbf{C}_h^n = 0$. Further, we use the test function $(\mathbf{0}, 0, \mathbf{C}_h^n)$ to obtain

$$\|\mathbf{C}_h^n\|_2^2 + \varepsilon\, \Delta t\, \|\nabla \mathbf{C}_h^n\|_2^2 - \Delta t \left((\nabla \mathbf{u}_h^n)\mathbf{C}_h^n + \mathbf{C}_h^n(\nabla \mathbf{u}_h^n)^T, \mathbf{C}_h^n\right) + \left\|\mathrm{tr}\, \mathbf{C}_h^{n-1}\ \mathbf{C}_h^n\right\|_2^2 = 0.$$

This equality, using the fact that $\mathbf{u}_h^n = \mathbf{0}$, implies $\mathbf{C}_h^n = \mathbf{0}$. We have shown that the zero vector is the unique solution to the linear homogeneous system of equations corresponding to the matrix $\mathbb{X}$. Thus the matrix $\mathbb{X}$ is invertible and the existence of unique solution to (6.28) follows immediately. Finally, we note that the second equation of (6.25) preserves the symmetry of the conformation tensor, i.e. for a given symmetric matrix $\mathbf{C}_h^{n-1}$ the solution $\mathbf{C}_h^n$ is again a symmetric matrix. This concludes the proof. $\square$

The second existence result is based on the implicit function theorem. For enough small Δt the positive definiteness of the discrete conformation tensor $\mathbf{C}_h^n$ is proven.

Proposition 6.9. *(local in time existence preserving positive definiteness)* *Assume Hypothesis 1. Let* $(\mathbf{f}_h^n, \mathbf{F}_h^n) \in L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ *and* $(\mathbf{u}_h^{n-1}, \mathbf{C}_h^{n-1}) \in V_h \times Q_h$ *be given with* $\mathbf{C}_h^{n-1}$ *being a symmetric positive definite tensor. Then there exists a constant* $c_{n-1}(\mathbf{u}_h^{n-1}, \mathbf{C}_h^{n-1}) > 0$ *such that for all* $\Delta t \in (0, c_{n-1})$ *satisfying the condition* (6.14) *there exists a unique solution* $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n) \in V_h \times Q_h \times W_h$ *to the scheme* (6.25) *with* $\mathbf{C}_h^n$ *being a symmetric positive definite tensor.*

Proof. Let us recall that we denote by $\mathbf{y}^n \in \mathbb{R}^{6N_P}$ the vector consisting of the nodal values of the solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$ of our scheme. For given $\mathbf{u}_h^{n-1}$ the equation

$$b(\mathbf{u}_h^{n-1}, q_h) - \mathcal{S}_h(p_h^{n-1}, q_h) = 0, \ \forall\, q_h \in Q_h, \tag{6.29}$$

has a unique solution p_h^{n-1} as it is equivalent to an algebraic system of equations with an invertible matrix, see Appendix C. Thus, for given vector $(\mathbf{u}_h^{n-1}, p_h^{n-1}, \mathbf{C}_h^{n-1})$ we get the known vector $\mathbf{y}^{n-1}$. We remind $\mathcal{S}_+^2$ denotes the subset of $\mathbb{R}^{2\times 2}$ that contains all symmetric positive definite tensors. It is an open set in $\mathbb{R}^{2\times 2}$. Now, let us denote by $\mathcal{S}_+^* := \mathcal{S}_+^*(\mathbb{R}^{6N_P})$ the set of all vectors corresponding to the nodal values of the velocity, pressure and symmetric positive definite conformation tensor. The subset $\mathcal{S}_+^*$ is open in $\mathbb{R}^{6N_P}$. Further, we define a mapping $\mathbf{L} : \mathbb{R}_0^+ \times \mathcal{S}_+^* \to \mathbb{R}^{6N_P}$ by

$$\mathbf{L}(\Delta t, \mathbf{y}) := \left(\mathbb{X}_1 + \Delta t\, \mathbb{X}_2 + \Delta t\, \mathbb{Z}^{n-1}\right) \mathbf{y} - \mathbf{r}^{n-1}(\Delta t) - \Delta t\, \tilde{\mathbf{r}}^{n-1}.$$

Finding a solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$ of the scheme (6.25) is then equivalent to finding a zero of $\mathbf{L}$ for any given vector $\mathbf{y}^{n-1}$. The mapping $\mathbf{L}$ is continuously differentiable with respect to Δt and $\mathbf{y}$. Obviously, it holds that

$$\mathbf{D_y}\, \mathbf{L}(\Delta t, \mathbf{y}) = \left(\mathbb{X}_1 + \Delta t\, \mathbb{X}_2 + \Delta t\, \mathbb{Z}^{n-1}\right).$$

Let us recall that the dependence of $\mathbf{r}^{n-1}$ on Δt is only related to the computation of the backward flow such that $\mathbf{r}^{n-1}(0) = \mathbf{y}^{n-1}$. Therefore, at the point $(0, \mathbf{y}^{n-1})$, it holds that

$$\mathbf{L}(0, \mathbf{y}^{n-1}) = \mathbf{0} \qquad \text{and} \qquad \mathbf{D_y}\, \mathbf{L}(0, \mathbf{y}^{n-1}) = \mathbb{X}_1. \tag{6.30}$$

The matrix $\mathbb{X}_1$ is an invertible matrix, since it is a block lower triangular matrix with the non-singular matrices on its diagonal, see again Appendix C. Due to the implicit function theorem and (6.30) there exists a neighbourhood $[0, c^{n-1}) \times U(\mathbf{y}^{n-1})$ of $(0, \mathbf{y}^{n-1})$ in $\mathbb{R}_0^+ \times \mathcal{S}_+^*$ and a continuously differentiable function $R : [0, c_{n-1}) \to U(\mathbf{y}^{n-1})$, such that for all $0 \le \Delta t < c_{n-1}$

$$R(\Delta t) = \mathbf{y} \iff \mathbf{L}(\Delta t, R(\Delta t)) = \mathbf{0}.$$

The positive constant c_{n-1} depends on $\mathbf{y}^{n-1}, \nu, \varepsilon$, and on the triangulation $\mathcal{T}_h$ of the domain. $\square$

The above result imposes a limitation on the time increment which depends on time-dependent data.

Remark 6.10. *The proof of Proposition 6.9 follows the idea from [21], where the finite element approximation of the Oldroyd-B model is studied. Roughly speaking, the discontinuous $\mathcal{P}_0$-approximation of the conformation tensor allows the authors to find a discrete free energy estimate. Then the global in time existence of the approximate solution preserving positive definiteness of the discrete conformation tensor can be shown.*

Here we consider the continuous piecewise linear approximation of the conformation tensor and thus $(\mathbf{C}_h^n)^{-1} \notin W_h$. Therefore we are not able to derive the discrete free energy estimate unless modifying the scheme by employing a projection onto the discontinuous piecewise constant finite element space.

The positive definiteness of the discrete conformation tensor is an important prerequisite in order to perform the stability and error analyses for the semi-implicit linear scheme. We can conclude that

- the positive definiteness of the conformation tensor for the diffusive Peterlin model is preserved at the continuous level, cf. Lemma 3.3
- the linear numerical scheme (6.25) enjoys unique global solution, cf. Proposition 6.8
- for enough small Δt there exists a unique solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$ with a symmetric positive definite matrix $\mathbf{C}_h^n$, cf. Proposition 6.9
- all numerical experiments, for enough small Δt, retain the positive definiteness of $\mathbf{C}_h$, see Chapter 8.

For further analysis we therefore consider apparently natural assumption.

Hypothesis 2. *(positive definiteness of discrete conformation tensor)*
There exists a constant $c_0(\mathbf{u}_h^0, \mathbf{C}_h^0) > 0$ such that for all $\Delta t \in (0, c_0)$ satisfying (6.14) there exists a unique solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ with $\mathbf{C}_h^n$ being a symmetric positive definite tensor for all $n = 0, \ldots, N_T$.

Our extensive numerical tests indicate that the upper bound c_0 of the time increment Δt seems to be independent of the numerical solution.

7 Stability and error analysis

In this chapter we present the numerical analysis of the diffusive Oseen-type Peterlin model approximated by the pressure-stabilized characteristics finite element method. Firstly, the weak formulation and the semi-implicit linear scheme, introduced in the previous chapter, are recalled and the preliminary results, needed for the stability and error analyses of the problem, are included. In Section 7.2, the stability of the numerical scheme, by showing a priori estimates of the discrete solution, is studied. Afterwards, in Section 7.3, the optimal first order accuracy of the discretization scheme is proven. Moreover, assuming more regular data, the higher energy bounds are shown in Section 7.4. Finally, a comparison with an implicit nonlinear scheme is provided.

7.1. Preliminaries

In the entire chapter we assume Hypotheses 1 and 2 hold true. It means we suppose that $\mathbf{w} \in C([0,T]; W_0^{1,\infty}(\Omega))$, and the symmetric discrete conformation tensor is positive definite. In addition, we assume condition (6.14) on the time increment Δt is satisfied, i.e. $\Delta t \in (0, \Delta t_0]$ for $\Delta t_0 > 0$ satisfying

$$\Delta t_0 < \frac{1}{\|\mathbf{w}\|_{C(W^{1,\infty})}} \quad \text{and} \quad \det\left(\frac{\partial X_1^n(x)}{\partial x}\right) \geq \frac{1}{2}, \quad \forall\, n \in \{0, \ldots, N_T\}, \quad \forall\, \Delta t \in (0, \Delta t_0].$$

We introduce the discrete norms and semi-norms that shall be used in this chapter. Let X be a functional space. Then we define

$$\|g\|_{l^\infty(X)} := \max_{n=0,\ldots,N_T} \|g^n\|_X\,, \qquad \|g\|_{l^k(X)} := \left(\Delta t \sum_{n=1}^{N_T} \|g^n\|_X^k\right)^{1/k},$$

$$|q^n|_h := \left(\sum_{K \in \mathcal{T}_h} h_K^2\, \|\nabla q^n\|_{2,K}^2\right)^{1/2}, \qquad |q|_{l^2(|\cdot|_h)} := \left(\Delta t \sum_{n=1}^{N_T} |q^n|_h^2\right)^{1/2}.$$

The notation $\|\cdot\|_{2,K}$ stands for the L^2-norm over the triangle K. Similarly, we might use different norms over K or its edge ∂K, i.e. $\|\cdot\|_{\cdot,K}$ and $\|\cdot\|_{\cdot,\partial K}$, respectively.

Now, we remind the weak formulation (6.3) and the stabilized Lagrange-Galerkin scheme (6.25) for the diffusive Oseen-type Peterlin model derived in the previous chapter.

Weak solution

Let the external forces and the initial data

$$(\mathbf{f},\mathbf{F}) \in L^2(0,T;L^2(\Omega)^2)\times L^2(0,T;L^2(\Omega)^{2\times 2}),\ (\mathbf{u}^0,\mathbf{C}^0)\in L^2(\Omega)^2\times L^2(\Omega)^{2\times 2},$$

respectively, be given, with $\mathbf{C}^0$ being a symmetric positive definite tensor. The triple $(\mathbf{u},p,\mathbf{C}) : (0,T) \to V\times Q\times W$ is called a weak solution to the Oseen-type Peterlin model if $(\mathbf{u}(0),\mathbf{C}(0)) = (\mathbf{u}^0,\mathbf{C}^0)$ and if it satisfies for $t\in(0,T)$

$$\left(\frac{\mathrm{D}\mathbf{u}(t)}{\mathrm{D}t},\mathbf{v}\right) + \mathcal{A}\Big(\,(\mathbf{u},p)\,(t),(\mathbf{v},q)\,\Big) = -\Big(\operatorname{tr}\mathbf{C}(t)\,\mathbf{C}(t),\nabla\mathbf{v}\Big) + \Big(\mathbf{f}(t),\mathbf{v}\Big) \tag{7.1a}$$

$$\begin{aligned}\left(\frac{\mathrm{D}\mathbf{C}(t)}{\mathrm{D}t},\mathbf{D}\right) + \varepsilon\, a_c\Big(\mathbf{C}(t),\mathbf{D}\Big) &= \Big((\nabla\mathbf{u}(t))\mathbf{C}(t)+\mathbf{C}(t)(\nabla\mathbf{u}(t))^T,\mathbf{D}\Big) - \\ &- \Big((\operatorname{tr}\mathbf{C}(t))^2\,\mathbf{C}(t),\mathbf{D}\Big) + \Big(\operatorname{tr}\mathbf{C}(t)\,\mathbf{I},\mathbf{D}\Big) + \Big(\mathbf{F}(t),\mathbf{D}\Big) \qquad (7.1\text{b})\\ &\qquad\qquad \forall\ (\mathbf{v},q,\mathbf{D})\in V\times Q\times W.\end{aligned}$$

We recall that $V = H_0^1(\Omega)^2$, $Q = L_0^2(\Omega)$ and $W = H^1(\Omega)^{2\times 2}$ are the functional spaces. The bilinear form $\mathcal{A}$, cf. (6.2), is given by

$$\mathcal{A}\Big((\mathbf{u},p)(t),(\mathbf{v},q)\Big) = \nu\, a_u(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),q) + b(\mathbf{v},p(t)).$$

Remark 7.1. *The weak formulation* (7.1) *for* $\mathbf{w}=\mathbf{u}$ *is equivalent to the weak formulation* (5.9) *of the diffusive Peterlin model, cf. [142].*

Discretization scheme

Analogously, let

$$(\mathbf{f}_h,\mathbf{F}_h)\in l^2(L^2(\Omega)^2)\times l^2(L^2(\Omega)^{2\times 2}),\ (\mathbf{u}_h^0,\mathbf{C}_h^0)\in V_h\times W_h$$

be given. The matrix $\mathbf{C}_h^0$ is also assumed to be symmetric positive definite. We seek the discrete solution $(\mathbf{u}_h,p_h,\mathbf{C}_h)\subset V_h\times Q_h\times W_h$ that satisfies for $n=1,\dots,N_T$,

$$\left(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right) + \mathcal{A}_h\Big(\,(\mathbf{u}_h^n,p_h^n)\,,(\mathbf{v}_h,q_h)\,\Big) = -\,\Big(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1},\nabla\mathbf{v}_h\Big) + \Big(\mathbf{f}_h^n,\mathbf{v}_h\Big) \tag{7.2a}$$

$$\begin{aligned}\left(\frac{\mathbf{C}_h^n-\mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{D}_h\right) + \varepsilon\, a_c\Big(\mathbf{C}_h^n,\mathbf{D}_h\Big) &= \Big((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1}+\mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T,\mathbf{D}_h\Big) - \\ &- \Big((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n,\mathbf{D}_h\Big) + \Big(\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{I},\mathbf{D}_h\Big) + \Big(\mathbf{F}_h^n,\mathbf{D}_h\Big) \qquad (7.2\text{b})\\ &\qquad\qquad \forall\ (\mathbf{v}_h,q_h,\mathbf{D}_h)\in V_h\times Q_h\times W_h.\end{aligned}$$

Let us recall that the conforming linear Lagrange elements are used to approximate the velocity, pressure as well as the symmetric conformation tensor. We refer to (6.20) for the definition of the finite element spaces V_h, Q_h and W_h. See (6.22) and (6.23) to remind the stabilized form

$$\mathcal{A}_h\Big((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\Big) = \nu\, a_u(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{u}_h,q_h) + b(\mathbf{v}_h,p_h) - \mathcal{S}_h(p_h,q_h).$$

Constants

We use c to represent a generic positive constant independent of the discretization parameters h and Δt. The constants with a particular meaning in this chapter are

$$\begin{aligned} c_0 &:= c\left(\mathbf{u}^0, \mathbf{C}^0\right) \quad c_1 := c\left(\|\mathbf{w}\|_{C(W^{1,\infty})}\right) \\ c_f &:= c\left(\mathbf{f}_h, \mathbf{F}_h\right) \quad c_{11} := c\left(\|\mathbf{w}\|_{W^{1,\infty}(W^{1,\infty})}\right) \end{aligned} \tag{7.3a}$$

- *stability result*

$$\tilde{c} := \exp\left\{c_1 T\right\}, \quad c_1^s := \tilde{c}\, c\left(T^{1/4}, \varepsilon^{-1/4}\right), \quad c_2^s := \tilde{c}\, c\left(T^{1/2}, {}^1\!/\!\sqrt{\nu}, {}^1\!/\!\sqrt{\varepsilon}\right) \tag{7.3b}$$

- *error estimates*

$$\begin{aligned} &c_i := c\left(\mathbf{u}_h^0, \mathbf{C}_h^0\right), \; c_2 := c_2\left(\|\mathbf{C}\|_{H^1(H^2)}\right), \; c_3 := c_3\left(\|\nabla \mathbf{u}\|_{H^1(H^1)}\right) \\ &\bar{c} := \exp\left\{c\left(\tilde{c}, c_i, c_f, T, {}^1\!/\nu, {}^1\!/\varepsilon, {}^1\!/\nu^2\varepsilon, \|\mathbf{w}\|_{C(W^{1,\infty})}, \|(\mathbf{u}, \mathbf{C})\|_{H^1(H^2\times H^2)}\right)\right\} \\ &c_1^e := \bar{c}\, c\left(T^{1/2}, \nu, \sqrt{\nu}, {}^1\!/\!\sqrt{\nu}, {}^\nu\!/\!\sqrt{\varepsilon}, \sqrt{\varepsilon}, \|\mathbf{w}\|_{C(W^{1,\infty})}, \|(\mathbf{u}, \mathbf{C})\|_{H^1(H^2\times H^2)}\right) \\ &c_2^e := c_1^e\, c\left(c_1, c_2, c_3, c_i, c_f, T^{1/2}, \sqrt{\varepsilon}, {}^1\!/\!\sqrt{\varepsilon}, {}^1\!/\!\sqrt{\nu}\right) \end{aligned} \tag{7.3c}$$

- *higher a priori estimates*

$$\begin{aligned} &\hat{c} := \exp\left\{c\left(c_1, c_0, c_i, c_f, \tilde{c}, T, {}^1\!/\nu, {}^1\!/\varepsilon, {}^1\!/\nu^2\varepsilon\right)\right\} \\ &c_1^h := \hat{c}\, c\left(h_0, c_0, c_1, c_{11}, c_i, c_f, \tilde{c}, T^{1/2}, \nu, \sqrt{\nu}, {}^1\!/\!\sqrt{\nu}, {}^\varepsilon\!/\!\sqrt{\nu}, \varepsilon, {}^1\!/\!\sqrt{\varepsilon}\right) \\ &c_2^h := c\left(c_1, c_i, c_f, \tilde{c}, c_1^h, T, T^{1/2}, \sqrt{\nu}, {}^1\!/\!\sqrt{\nu}, {}^1\!/\!\sqrt{\varepsilon}, {}^{\sqrt{\nu}}\!/\!\sqrt{\varepsilon}\right) \\ &c_3^h := c\left(c_1, c_i, c_f, \tilde{c}, c_1^h, T^{1/2}, {}^1\!/\!\sqrt{\nu}, {}^1\!/\!\sqrt{\varepsilon}, \sqrt{\varepsilon}\right). \end{aligned} \tag{7.3d}$$

In what follows, we list the preliminary results needed for the analysis of the pressure-stabilized characteristics finite element method applied on the Oseen-type Peterlin model.

Lemma 7.2. *(upwind point estimates)*
Assume Hypothesis 1. Let the condition on the time increment (6.14) *be satisfied, and let* $p \in \{2, 2\times 2\}$. *Then, for the mapping* $X_1^n : \Omega \to \Omega$ *given by* (6.6) *and for any* $n \in \{0, \dots, N_T\}$, *it holds that*

$$\|\mathbf{g} \circ X_1^n\|_2 \le (1 + c_1 \Delta t)\, \|\mathbf{g}\|_2\,, \qquad \forall\, \mathbf{g} \in L^2(\Omega)^p \tag{7.4a}$$

$$\|\mathbf{g} - \mathbf{g} \circ X_1^n\|_2 \le c_1 \Delta t\, \|\mathbf{g}\|_{1,2}\,, \qquad \forall\, \mathbf{g} \in H^1(\Omega)^p. \tag{7.4b}$$

Moreover, if $\mathbf{w} \in W^{1,\infty}(0,T; W_0^{1,\infty}(\Omega)^2)$ *then, for any* $n \in \{1, \dots, N_T\}$, *it holds that*

$$\left\|\mathbf{g} \circ X_1^{n-1} - \mathbf{g} \circ X_1^n\right\|_2 \le c_{11} (\Delta t)^2\, \|\mathbf{g}\|_{1,2}\,, \qquad \forall\, \mathbf{g} \in H^1(\Omega)^p. \tag{7.4c}$$

Proof. First, we give the proof of (7.4a), cf. Lemma 1 in [129]. Let J be the Jacobian of the transformation from x to $y = X_1^n(x)$. Then we can rewrite the left-hand side of (7.4a) as follows

$$\|\mathbf{g} \circ X_1^n\|_2^2 = \int_\Omega |\mathbf{g}(X_1^n(x))|^2 \; dx \; = \int_\Omega |\mathbf{g}(y)|^2 |J^{-1}|\, dy.$$

For the Jacobian of the transformation $y = x - \mathbf{w}^n(x)\Delta t$ it holds that

$$\begin{aligned} J = \det\left(\frac{\partial y}{\partial x}\right) &= \det\begin{pmatrix} 1 - \frac{\partial w_1^n(x)}{\partial x_1}\Delta t & -\frac{\partial w_1^n(x)}{\partial x_2}\Delta t \\ -\frac{\partial w_2^n(x)}{\partial x_1}\Delta t & 1 - \frac{\partial w_2^n(x)}{\partial x_2}\Delta t \end{pmatrix} \\ &= 1 - \operatorname{div}\mathbf{w}^n(x)\Delta t + \frac{\partial w_1^n(x)}{\partial x_1}\frac{\partial w_2^n(x)}{\partial x_2}(\Delta t)^2 - \frac{\partial w_1^n(x)}{\partial x_2}\frac{\partial w_2^n(x)}{\partial x_1}(\Delta t)^2. \end{aligned}$$

Hence, $|J - 1| \le c_1\Delta t$. Since the condition (6.14) assumes Δt to be small enough such that $J \ge 1/2$, it holds that

$$\frac{1}{2} \le 1 - c_1\Delta t \le J,$$

and consequently,

$$J^{-1} = \frac{1}{J} \le \frac{1}{1 - c_1\Delta t} \le 1 + 2c_1\Delta t,$$

which concludes the proof of (7.4a).

Next, we follow the proof of Lemma 6 in [111]. Let any $n \in \{0, \dots, N_T\}$ be fixed. Let us now define

$$y(x, s) := x - s\mathbf{w}^n(x)\Delta t$$

for $s \in [0, 1]$, and let $J_s := \det(\partial y/\partial x)$ be the Jacobian. It holds that $J_s \ge 1/2$ for all $s \in [0, 1]$. Further,

$$\mathbf{g}(x) - \mathbf{g} \circ X_1^n(x) = \Big[\mathbf{g}(y(x, s))\Big]_{s=1}^0 = \Delta t \int_0^1 (\mathbf{w}^n(x) \cdot \nabla)\mathbf{g}(y(x, s))\, ds.$$

The Hölder inequality and the transformation from x to y together with the evaluation of the Jacobian J_s yield inequality (7.4b). Indeed,

$$\begin{aligned} \|\mathbf{g}(x) - \mathbf{g} \circ X_1^n(x)\|_2^2 &\le (\Delta t)^2 \int_\Omega \int_0^1 |(\mathbf{w}^n(x) \cdot \nabla)\mathbf{g}(y(x, s))|^2\, ds\, dx \\ &\le 2c_1(\Delta t)^2 \int_0^1 \|\nabla \mathbf{g}\|_2^2\, ds \\ &\le c_1(\Delta t)^2\, \|\mathbf{g}\|_{1,2}^2\,. \end{aligned}$$

Finally we prove, in a similar way, the last inequality (7.4c). For $n \in \{1, \dots, N_T\}$ and $s \in [0, 1]$ we define

$$y(x, \mathbf{w}(x, t(s))) := x - \mathbf{w}(x, t^{n-1} + s\Delta t)\Delta t.$$

Once more $\tilde{J}_s := \det(\partial y/\partial x)$ is the Jacobian. Let us note that again $\tilde{J}_s \ge 1/2$ for all $s \in [0, 1]$. Analogously as above, we can write

$$\begin{aligned} \mathbf{g} \circ X_1^{n-1} - \mathbf{g} \circ X_1^n &= \Big[\mathbf{g}(y(x, \mathbf{w}(x, t(s))))\Big]_{s=1}^0 \\ &= -\int_0^1 \frac{\partial \mathbf{g}(y(x, \mathbf{w}(x, t(s))))}{\partial s}\, ds \\ &= (\Delta t)^2 \int_0^1 \left(\frac{\partial \mathbf{w}}{\partial t} \cdot \nabla\right) \mathbf{g}(y(x, \mathbf{w}(x, t(s))))\, ds. \end{aligned}$$

Similarly, changing the variable from x to y we obtain

$$\begin{aligned}\left\|\mathbf{g}\circ X_1^{n-1}-\mathbf{g}\circ X_1^{n}\right\|_2^2 &\le (\Delta t)^4\int_\Omega\int_0^1\left|\left(\frac{\partial \mathbf{w}}{\partial t}\cdot\nabla\right)\mathbf{g}(y(x,\mathbf{w}(x,t(s))))\right|^2 ds\, dx\\ &\le 2c_{11}(\Delta t)^4\int_0^1\|\nabla\mathbf{g}\|_2^2\, ds\\ &\le c_{11}(\Delta t)^4\,\|\mathbf{g}\|_{1,2}^2\,,\end{aligned}$$

which concludes the proof. □

We also refer to [112] and the references therein for general inequalities of type (7.4). For the higher a priori estimates of the discrete solution we shall estimate some integrals over edges ∂K of an element K. Let us prepare the following lemmas.

Lemma 7.3. *(multiplicative trace inequality)*
Let $\{\mathcal{T}_h\}_{h\searrow 0}$, $h\in(0,h_0)$, be a family of regular triangulations $\mathcal{T}_h=\{K_i\}_{i=1}^{N_E}$ of Ω. Then there exists a constant $\gamma_1>0$ independent of h, K such that for $K\in\mathcal{T}_h$, $g\in H^1(K)$, it holds that

$$\|g\|_{2,\partial K}^2\le\gamma_1\left(\|g\|_{2,K}\,|g|_{1,2,K}+h_K^{-1}\,\|g\|_{2,K}^2\right). \tag{7.5}$$

For the proof see, e.g., [44, 92]. The so-called inverse inequality, cf., e.g., [33], follows immediately.

Lemma 7.4. *(inverse inequality)*
For a regular affine family $\{\mathcal{T}_h\}_{h\searrow 0}$ of triangulations satisfying the inverse assumption and for $S_h\subset H^1(\Omega)$ there exists a constant $C=C(\sigma_1,\sigma_2)>0$ such that for all $g_h\in S_h$ it holds that

$$\|\nabla g_h\|_{2,K}\le Ch^{-1}\,\|g_h\|_{2,K}\,. \tag{7.6}$$

Here σ_1, σ_2 are the constants of (6.16) *and* (6.17)*, respectively.*

Remark 7.5. *We note that we work with a suitable family of triangulations and moreover, our finite element spaces are H^1-conforming, i.e. $V_h\subset H^1(\Omega)$, $Q_h\subset H^1(\Omega)$, and $W_h\subset H^1(\Omega)$. Thus, the inverse inequality can be applied on $\mathbf{u}_h\in V_h$, $p_h\in Q_h$ as well as on $\mathbf{C}_h\in W_h$.*

In the proof of the higher energy estimates we shall employ a straightforward consequence of the above two lemmas.

Corollary 7.6.
Under the assumptions of Lemma 7.4 there exists a constant $C=C(\sigma_1,\sigma_2,\gamma_1)>0$ such that for $K\in\mathcal{T}_h$, $h\in(0,h_0)$ and for all $\mathbf{D}_h\in W_h$, it holds that

$$\|\mathbf{D}_h\|_{2,\partial K}\le C\,h^{-1/2}\,\|\mathbf{D}_h\|_{2,K}\,. \tag{7.7}$$

Next lemma recalls the standard linear interpolation operators and the corresponding interpolation errors. For more details we refer to, e.g., [26, 33, 34].

Lemma 7.7. *(interpolation errors)*
Let $\{\mathcal{T}_h\}_{h\searrow 0}$ be a regular family of triangulations. There exist linear operators

$$\Pi_h : H^2(\Omega) \to M_h, \qquad\qquad \Pi_h^C : L_0^2(\Omega) \to Q_h.$$

Here Π_h denotes the Lagrange interpolation operator, cf. (6.19), *and Π_h^C stands for the Clément interpolation operator. Let*

$$\Pi_h : H^2(\Omega)^2 \to X_h, \qquad\qquad \Pi_h : H^2(\Omega)^{2\times 2} \to Y_h$$

be the Lagrange interpolation operators for vector-valued and matrix-valued functions, respectively. Then, for some positive constants γ_2, γ_3 and γ_4 independent of h it holds that

$$\|\Pi_h \mathbf{v} - \mathbf{v}\|_{1,2} \le \gamma_2\, h\, \|\mathbf{v}\|_{2,2}\,, \qquad \forall\, \mathbf{v} \in H^2(\Omega)^2 \tag{7.8a}$$

$$\|\Pi_h \mathbf{D} - \mathbf{D}\|_{1,2} \le \gamma_3\, h\, \|\mathbf{D}\|_{2,2}\,, \qquad \forall\, \mathbf{D} \in H^2(\Omega)^{2\times 2} \tag{7.8b}$$

$$\left\|\Pi_h^C q - q\right\|_2 \le \gamma_4\, h\, \|q\|_{1,2}\,, \qquad \forall\, q \in L_0^2(\Omega) \cap H^1(\Omega). \tag{7.8c}$$

Remark 7.8. *Let us recall that M_h is the finite element space of continuous piecewise linear functions. It holds that $X_h = M_h^2$ and $Y_h = M_h^{2\times 2}$. Then the Lagrange interpolation operator is defined for vector and matrix functions by*

$$\begin{aligned} &\Pi_h : H^2(\Omega)^2 \to X_h && (\Pi_h \mathbf{v})_i := \Pi_h v_i,\ \mathbf{v} \in H^2(\Omega)^2 \\ &\Pi_h : H^2(\Omega)^{2\times 2} \to Y_h && (\Pi_h \mathbf{D})_{ij} := \Pi_h D_{ij},\ \mathbf{D} \in H^2(\Omega)^{2\times 2}, \end{aligned}$$

for $i, j = 1, 2$, respectively. We use the same notation for the Lagrange interpolation operator for the scalar, vector and matrix-valued functions.

The interpolation operator Π_h has the following properties.

Proposition 7.9.
Let $\Pi_h : H^2(\Omega)^{2\times 2} \to Y_h$ be the Lagrange interpolation operator. Then it holds that

(i) $\|\Pi_h \mathbf{D}\|_0 \le c\, \|\Pi_h \mathbf{D}\|_{2,2} \le c\, \|\mathbf{D}\|_{2,2}\,,\ \mathbf{D} \in H^2(\Omega)^{2\times 2}$,

(ii) if $\mathbf{D} \in H^2(\Omega)^{2\times 2}$ is a positive definite tensor, then $\Pi_h \mathbf{D} \in Y_h$ is also positive definite.

Proof. The first inequality in (*i*) holds due to the Sobolev embeddings theorem. The second inequality follows from the error of the interpolation operator, cf. [33]. Indeed, it holds that

$$\|\Pi_h \mathbf{D}\|_{2,2} \le \|\Pi_h \mathbf{D} - \mathbf{D}\|_{2,2} + \|\mathbf{D}\|_{2,2} \le \left(\tilde{\gamma}_3\, h^0 + 1\right) \|\mathbf{D}\|_{2,2} \le c\, \|\mathbf{D}\|_{2,2}\,.$$

The proof of (*ii*) follows. For any scalar function g it holds that if $g > 0$ then $\Pi_h g > 0$. For a positive definite tensor $\mathbf{D}$ we have $x^T\, \mathbf{D}\, x > 0$, see Section 3.2. Thus $\Pi_h(x^T\, \mathbf{D}\, x) > 0$. The linearity of the operator implies

$$0 < \Pi_h(x^T\, \mathbf{D}\, x) = \Pi_h \left(\sum_{i,j=1}^{2} x_i D_{ij} x_j \right) = \sum_{i,j=1}^{2} x_i\, \Pi_h D_{ij}\, x_j = x^T\, \Pi_h \mathbf{D}\, x,$$

which concludes the proof. □

In order to study the accuracy of the discrete scheme we also need to define the so-called Stokes projection.

Definition 7.10. *(Stokes projection)*
For $(\mathbf{u}, p) \in V \times Q$ *we define the Stokes projection* $(\hat{\mathbf{u}}_h, \hat{p}_h) \in V_h \times Q_h$ *by*

$$\mathcal{A}_h\Big((\hat{\mathbf{u}}_h, \hat{p}_h), (\mathbf{v}_h, q_h)\Big) = \mathcal{A}\Big((\mathbf{u}, p), (\mathbf{v}_h, q_h)\Big), \qquad \forall\, (\mathbf{v}_h, q_h) \in V_h \times Q_h, \tag{7.9}$$

or equivalently

$$\nu\, a_u(\hat{\mathbf{u}}_h, \mathbf{v}_h) + b(\hat{\mathbf{u}}_h, q_h) + b(\mathbf{v}_h, \hat{p}_h) - \mathcal{S}_h\,(\hat{p}_h, q_h) = \nu\, a_u(\mathbf{u}, \mathbf{v}_h) + b(\mathbf{u}, q_h) + b(\mathbf{v}_h, p).$$

It is possible to show the first order accuracy of the Stokes projection defined above. To this end, we need the following result obtained by standard scaling argument, cf. [56].

Proposition 7.11.
Let $\{\mathcal{T}_h\}_{h\searrow 0}$ *be a regular family of triangulations satisfying the inverse assumption* (6.17)*. Then there exists a positive constant* γ_4 *independent of* h *such that*

$$|q_h|_h \le \gamma_4 \, \|q_h\|_2 \,, \; \forall\, q_h \in Q_h.$$

Proof. By definition (6.26) of the semi-norm $|.|_h$ and by the inverse inequality (7.6), we have

$$|q_h|_h^2 = \sum_{K \in \mathcal{T}_h} h_K^2 \, \|\nabla q_h\|_{2,K}^2 \le C^2 \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{h^2} \, \|q_h\|_{2,K}^2 \le \gamma_4 \, \|q_h\|_2^2 \,,$$

since $h_K \le h \le \sigma_2\, h_K$. ☐

Employing the generalized inf-sup condition (6.24) and the interpolation errors (7.8), the error of the Stokes projection can be shown. The result follows, cf., e.g., [28, 111].

Proposition 7.12. *(error of the Stokes projection)*
Suppose $(\mathbf{u}, p) \in (V \cap H^2(\Omega)^2) \times (Q \cap H^1(\Omega))$*. Let* $(\hat{\mathbf{u}}_h, \hat{p}_h) \in V_h \times Q_h$ *be the Stokes projection of* $(\mathbf{u}, p)$ *defined by* (7.9)*. Then, there exists a positive constant* $C = C(\nu)$ *such that*

$$\|\mathbf{u} - \hat{\mathbf{u}}_h\|_V \,, \; \|p - \hat{p}_h\|_Q \,, \; |p - \hat{p}_h|_h \le C\, h \, \|(\mathbf{u}, p)\|_{H^2 \times H^1} \,.$$

Proof. Let Π_h and Π_h^C be the Lagrange and the Clément interpolation operator, respectively. Then Lemma 6.6 together with the definition of the Stokes projection (7.9) allows us to write

$$\begin{aligned}
&\left\|\Big(\hat{\mathbf{u}}_h - \Pi_h \mathbf{u}, \hat{p}_h - \Pi_h^C p\Big)\right\|_{V \times Q} \\
&\qquad \le \frac{1}{\gamma} \sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{\mathcal{A}_h\left((\hat{\mathbf{u}}_h - \Pi_h \mathbf{u}, \hat{p}_h - \Pi_h^C p), (\mathbf{v}_h, q_h)\right)}{\|(\mathbf{v}_h, q_h)\|_{V \times Q}} \\
&\qquad = \frac{1}{\gamma} \sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{\mathcal{A}_h\left((\mathbf{u} - \Pi_h \mathbf{u}, p - \Pi_h^C p), (\mathbf{v}_h, q_h)\right) + \mathcal{S}_h\,(p, q_h)}{\|(\mathbf{v}_h, q_h)\|_{V \times Q}} \\
&\qquad \le c\left(\nu\, \|\mathrm{D}(\mathbf{u} - \Pi_h \mathbf{u})\|_2 + \|\mathbf{u} - \Pi_h \mathbf{u}\|_V + \left\|p - \Pi_h^C p\right\|_2 + |p - \Pi_h^C p|_h + |p|_h\right).
\end{aligned}$$

Consequently, Proposition 7.11 and the interpolation error of the Lagrange and the Clément operators, cf. (7.8a), (7.8c), yield

$$\begin{aligned}\left\|\left(\hat{\mathbf{u}}_h - \Pi_h \mathbf{u}, \hat{p}_h - \Pi_h^C p\right)\right\|_{V\times Q} &\leq c(\nu)\left(\left\|\left(\mathbf{u} - \Pi_h \mathbf{u}, p - \Pi_h^C p\right)\right\|_{V\times Q} + |p|_h\right)\\ &\leq c(\nu) h \left\|(\mathbf{u}, p)\right\|_{H^2\times H^1}. \end{aligned} \tag{7.10}$$

The error of the Stokes projection can be rewritten as

$$\begin{aligned}\left\|(\mathbf{u} - \hat{\mathbf{u}}_h, p - \hat{p}_h)\right\|_{V\times Q} &= \left\|(\mathbf{u} - \Pi_h\mathbf{u} + \Pi_h\mathbf{u} - \hat{\mathbf{u}}_h, p - \Pi_h^C + \Pi_h^C - \hat{p}_h)\right\|_{V\times Q}\\ &\leq \left\|(\mathbf{u} - \Pi_h\mathbf{u}, p - \Pi_h^C p)\right\|_{V\times Q} + \left\|(\Pi_h\mathbf{u} - \hat{\mathbf{u}}_h, \Pi_h^C p - \hat{p}_h)\right\|_{V\times Q},\end{aligned}$$

where the first norm on the right-hand side is nothing but the interpolation error given by (7.8a) and (7.8c). The second term is bounded by (7.10). Consequently,

$$\left\|(\mathbf{u} - \hat{\mathbf{u}}_h, p - \hat{p}_h)\right\|_{V\times Q} \leq c(\nu) h \left\|(\mathbf{u}, p)\right\|_{H^2\times H^1},$$

which concludes the proof. □

After preparing the useful preliminary results we proceed with the stability and error analyses of the semi-implicit linear scheme for the Oseen-type Peterlin model.

7.2. Stability of the scheme

The first main result of this chapter concerns the stability of the numerical scheme. This section demonstrates the natural a priori estimates of the discrete solution that, in fact, imply the stability of the scheme.

Theorem 7.13. *(stability result)*
Suppose Hypotheses 1 and 2 together with condition (6.14) *hold true. Let the external forces* $(\mathbf{f}_h, \mathbf{F}_h) \in l^2(L^2(\Omega)^2) \times l^2(L^2(\Omega)^{2\times 2})$ *and the initial data* $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ *be given,* $\mathbf{C}_h^0$ *being a symmetric positive definite matrix. Let* $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ *be the solution of the semi-implicit linear scheme* (7.2)*. Then there exist some constants* $\tilde{c}$ *and* c_2^s *of* (7.3b) *such that*

$$\begin{aligned}&\left\|\mathbf{u}_h\right\|_{l^\infty(L^2)}, \sqrt{\nu}\left\|D(\mathbf{u}_h)\right\|_{l^2(L^2)}, |p_h|_{l^2(|\cdot|_h)}\\ &\qquad\leq \tilde{c}\Bigg(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\operatorname{tr}\mathbf{C}_h^0\right\|_2 + \left\|\mathbf{f}_h\right\|_{l^2(L^2)} + \left\|\operatorname{tr}\mathbf{F}_h\right\|_{l^2(L^2)}\Bigg)\\ &\left\|\mathbf{C}_h\right\|_{l^\infty(L^2)}, \sqrt{\varepsilon}\left\|\nabla\mathbf{C}_h\right\|_{l^2(L^2)}, \left\|\operatorname{tr}\mathbf{C}_h^{-1}\mathbf{C}_h\right\|_{l^2(L^2)}\\ &\qquad\leq c_2^s\Bigg(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\mathbf{C}_h^0\right\|_2 + \left\|\mathbf{u}_h^0\right\|_2^2 + \left\|\mathbf{C}_h^0\right\|_4^2 +\\ &\qquad\qquad + \left\|\mathbf{f}_h\right\|_{l^2(L^2)} + \left\|\mathbf{F}_h\right\|_{l^2(L^2)} + \left\|\mathbf{f}_h\right\|_{l^2(L^2)}^2 + \left\|\operatorname{tr}\mathbf{F}_h\right\|_{l^2(L^2)}^2\Bigg).\end{aligned}$$

Proof. The proof follows the idea of the formal energy estimates from Section 3.3. Hence, due to the positive definiteness of the discrete solution $\mathbf{C}_h^n$ we find the a priori bounds in two steps.

Step 1

We find the energy estimates for the approximation of the velocity, pressure and the trace of the conformation tensor. Let $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ be the solution of the scheme (7.2). Substituting

$$\left(\mathbf{u}_h^n, -p_h^n, \frac{1}{2}\operatorname{tr}\mathbf{C}_h^n\mathbf{I}\right) \in V_h \times Q_h \times W_h$$

into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in equations (7.2), we get

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{u}_h^n\right) + \mathcal{A}_h\Big(\left(\mathbf{u}_h^n, p_h^n\right), \left(\mathbf{u}_h^n, -p_h^n\right)\Big) = -\left(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla\mathbf{u}_h^n\right) + \left(\mathbf{f}_h^n, \mathbf{u}_h^n\right) \tag{7.11a}$$

$$\begin{aligned}
&\left(\frac{\operatorname{tr}\mathbf{C}_h^n - \operatorname{tr}\left(\mathbf{C}_h^{n-1}\circ X_1^n\right)}{2\Delta t}, \operatorname{tr}\mathbf{C}_h^n\right) + \frac{\varepsilon}{2}a_c\left(\mathbf{C}_h^n, \operatorname{tr}\mathbf{C}_h^n\ \mathbf{I}\right) = \\
&\quad = \left(\operatorname{tr}\mathbf{C}_h^{n-1}, \operatorname{tr}\mathbf{C}_h^n\right) + \frac{1}{2}\left(\operatorname{tr}\mathbf{F}_h^n, \operatorname{tr}\mathbf{C}_h^n\right) + \\
&\quad + \frac{1}{2}\left(\operatorname{tr}\left[(\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T\right], \operatorname{tr}\mathbf{C}_h^n\right) - \frac{1}{2}\left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\operatorname{tr}\mathbf{C}_h^n, \operatorname{tr}\mathbf{C}_h^n\right).
\end{aligned} \tag{7.11b}$$

By inequality (7.4a) from Lemma 7.2 we obtain

$$\begin{aligned}
\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{u}_h^n\right) &= \frac{1}{\Delta t}\left\{\frac{1}{2}\left(\|\mathbf{u}_h^n\|_2^2 - \left\|\mathbf{u}_h^{n-1}\circ X_1^n\right\|_2^2\right) + \frac{1}{2}\left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n\right\|_2^2\right\} \\
&\geq \bar{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{u}_h^n\|_2^2\right) - c_1\left\|\mathbf{u}_h^{n-1}\right\|_2^2,
\end{aligned} \tag{7.12a}$$

and in the same way,

$$\left(\frac{\operatorname{tr}\mathbf{C}_h^n - \operatorname{tr}\left(\mathbf{C}_h^{n-1}\circ X_1^n\right)}{2\Delta t}, \operatorname{tr}\mathbf{C}_h^n\right) \geq \bar{D}_{\Delta t}\left(\frac{1}{4}\|\operatorname{tr}\mathbf{C}_h^n\|_2^2\right) - c_1\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\right\|_2^2. \tag{7.12b}$$

The evaluation of the other terms in (7.11) read

$$\begin{aligned}
\mathcal{A}_h\Big(\left(\mathbf{u}_h^n, p_h^n\right), \left(\mathbf{u}_h^n, -p_h^n\right)\Big) &= \nu\, a_u\left(\mathbf{u}_h^n, \mathbf{u}_h^n\right) + b(\mathbf{u}_h^n, -p_h^n) + b(\mathbf{u}_h^n, p_h^n) - \mathcal{S}_h(p_h^n, -p_h^n) \\
&= 2\nu\,\|\mathrm{D}(\mathbf{u}_h^n)\|_2^2 + \delta_0|p_h^n|_h^2 &(7.12c)\\
\frac{\varepsilon}{2}a_c\left(\mathbf{C}_h^n, \operatorname{tr}\mathbf{C}_h^n\ \mathbf{I}\right) &= \frac{\varepsilon}{2}\|\nabla\operatorname{tr}\mathbf{C}_h^n\|_2^2 &(7.12d)\\
\frac{1}{2}\left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\operatorname{tr}\mathbf{C}_h^n, \operatorname{tr}\mathbf{C}_h^n\right) &= \frac{1}{2}\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^n\right\|_2^2, &(7.12e)
\end{aligned}$$

and using the Hölder and the Young inequalities,

$$\Big(\mathrm{tr}\,\mathbf{C}_h^{n-1}\,,\mathrm{tr}\,\mathbf{C}_h^n\Big) \le \beta_1 \left\|\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 + \frac{1}{4\beta_1}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_2^2 \tag{7.12f}$$

$$\Big(\mathbf{f}_h^n, \mathbf{u}_h^n\Big) \le \beta_2 \left\|\mathbf{u}_h^n\right\|_2^2 + \frac{1}{4\beta_2}\left\|\mathbf{f}_h^n\right\|_2^2 \tag{7.12g}$$

$$\frac{1}{2}\Big(\mathrm{tr}\,\mathbf{F}_h^n, \mathrm{tr}\,\mathbf{C}_h^n\Big) \le \beta_1 \left\|\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 + \frac{1}{8\beta_1}\left\|\mathrm{tr}\,\mathbf{F}_h^n\right\|_2^2, \tag{7.12h}$$

for appropriate $\beta_i > 0$, $i = 1, 2$. Moreover, identity (3.14) for $\mathbf{u} = \mathbf{u}_h^n$, $\mathbf{C} = \mathbf{C}_h^{n-1}$ yields

$$\Big(\mathrm{tr}\,\mathbf{C}_h^n\;\mathbf{C}_h^{n-1}, \nabla\mathbf{u}_h^n\Big) - \frac{1}{2}\Big(\mathrm{tr}\,\Big[(\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T\Big], \mathrm{tr}\,\mathbf{C}_h^n\Big) = 0. \tag{7.12i}$$

We sum up (7.11a) and (7.11b). Taking into account all the estimates (7.12) we obtain, for all $n \ge 1$,

$$\begin{aligned}
\bar{D}_{\Delta t}\left(\frac{1}{2}\left\|\mathbf{u}_h^n\right\|_2^2 + \frac{1}{4}\left\|\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2\right) + 2\nu\left\|\mathrm{D}(\mathbf{u}_h^n)\right\|_2^2 + \delta_0|p_h^n|_h^2 + {}&\\
+\frac{\varepsilon}{2}\left\|\nabla\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 + \frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 &\le\\
\le c_1\left(\left\|\mathbf{u}_h^{n-1}\right\|_2^2 + \left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_2^2\right) + \frac{1}{4\beta_1}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_2^2 &+\\
+\beta_2\left\|\mathbf{u}_h^n\right\|_2^2 + 2\beta_1\left\|\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 + \frac{1}{4\beta_2}\left\|\mathbf{f}_h^n\right\|_2^2 + \frac{1}{8\beta_1}\left\|\mathrm{tr}\,\mathbf{F}_h^n\right\|_2^2&.
\end{aligned}$$

The above inequality, employing the notation from the discrete Gronwall lemma (2.7), can be rewritten in the form

$$\bar{D}_{\Delta t}\,x^n + y^n \le a_0\,x^n + a_1\,x^{n-1} + b^n,$$

where

$$\begin{aligned}
x^n &= \frac{1}{2}\left\|\mathbf{u}_h^n\right\|_2^2 + \frac{1}{4}\left\|\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2,\; a_0 = 2\beta_2 + 8\beta_1,\; a_1 = c_1 + \frac{1}{\beta_1}\\
y^n &= 2\nu\left\|\mathrm{D}(\mathbf{u}_h^n)\right\|_2^2 + \delta_0|p_h^n|_h^2 + \frac{\varepsilon}{2}\left\|\nabla\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 + \frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2\\
b^n &= \frac{1}{4\beta_2}\left\|\mathbf{f}_h^n\right\|_2^2 + \frac{1}{8\beta_1}\left\|\mathrm{tr}\,\mathbf{F}_h^n\right\|_2^2.
\end{aligned}$$

Therefore, choosing the constants β_1 and β_2 such that $\Delta t \le {}^1\!/_{2a_0} = {}^1\!/_{(4\beta_2+16\beta_1)}$, we obtain, by the discrete Gronwall inequality,

$$\begin{aligned}
&\left\|\mathbf{u}_h^n\right\|_2^2 + \frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^n\right\|_2^2 + \Delta t\sum_{i=1}^n\left(4\nu\left\|\mathrm{D}(\mathbf{u}_h^i)\right\|_2^2 + 2\delta_0|p_h^i|_h^2 + \varepsilon\left\|\nabla\mathrm{tr}\,\mathbf{C}_h^i\right\|_2^2 + \left\|\mathrm{tr}\,\mathbf{C}_h^{i-1}\mathrm{tr}\,\mathbf{C}_h^i\right\|_2^2\right)\\
&\qquad\le \mathrm{e}^{c_1\,n\Delta t}\left(\left\|\mathbf{u}_h^0\right\|_2^2 + \frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_2^2 + c\Delta t\sum_{i=1}^n\left(\left\|\mathbf{f}_h^i\right\|_2^2 + \left\|\mathrm{tr}\,\mathbf{F}_h^i\right\|_2^2\right)\right),\; n \ge 1.
\end{aligned}$$

Finally, the a priori bounds for the approximation of the velocity, pressure and the trace of the conformation tensor follow

$$\begin{aligned}
&\|\mathbf{u}_h\|_{l^\infty(L^2)}\,, \sqrt{\nu}\,\|\mathrm{D}(\mathbf{u}_h)\|_{l^2(L^2)}\,,\ |p_h|_{l^2(|.|_h)},\\
&\|\mathrm{tr}\,\mathbf{C}_h\|_{l^\infty(L^2)}\,, \sqrt{\varepsilon}\,\|\nabla \mathrm{tr}\,\mathbf{C}_h\|_{l^2(L^2)}\,,\ \left\|\mathrm{tr}\,\mathbf{C}_h^{\cdot-1}\mathrm{tr}\,\mathbf{C}_h\right\|_{l^2(L^2)}\\
&\qquad \le \tilde{c}\left(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_2 + \|\mathbf{f}_h\|_{l^2(L^2)} + \|\mathrm{tr}\,\mathbf{F}_h\|_{l^2(L^2)}\right). \qquad (7.13a)
\end{aligned}$$

Using the interpolation inequality (2.8b) we get

$$\|\mathrm{tr}\,\mathbf{C}_h\|_{l^4(L^4)} \le \tilde{c}\,c\left(T+\frac{1}{\varepsilon}\right)^{1/4}\left(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_2 + \|\mathbf{f}_h\|_{l^2(L^2)} + \|\mathrm{tr}\,\mathbf{F}_h\|_{l^2(L^2)}\right).$$

Then the positive definiteness of $\mathbf{C}_h^n$ implies, by the norm equivalence (2.9a), that

$$\|\mathbf{C}_h\|_{l^4(L^4)} \le c_1^s\left(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_2 + \|\mathbf{f}_h\|_{l^2(L^2)} + \|\mathrm{tr}\,\mathbf{F}_h\|_{l^2(L^2)}\right). \qquad (7.13b)$$

Step 2

To show the analogous energy estimates for the discrete conformation tensor $\mathbf{C}_h$ we employ the above a priori bounds (7.13). By substituting $\mathbf{C}_h^n \in W_h$ into $\mathbf{D}_h$ in equation (7.2b) we get

$$\begin{aligned}
\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{C}_h^n\right) + \varepsilon\, a_c\Big(\mathbf{C}_h^n, \mathbf{C}_h^n\Big) &= \Big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla \mathbf{u}_h^n)^T, \mathbf{C}_h^n\Big) -\\
&- \Big((\mathrm{tr}\,\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \mathbf{C}_h^n\Big) + \Big(\mathrm{tr}\,\mathbf{C}_h^{n-1}, \mathrm{tr}\,\mathbf{C}_h^n\Big) + \Big(\mathbf{F}_h^n, \mathbf{C}_h^n\Big).
\end{aligned}$$

The terms of the latter equation shall be treated similarly as in the first step. The analogous estimates of (7.12b), (7.12d) - (7.12f), (7.12h) with the appropriate constant $\beta_3 > 0$ read

$$\begin{aligned}
\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X_1^n}{2\Delta t}, \mathbf{C}_h^n\right) &\ge \bar{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{C}_h^n\|_2^2\right) - c_1\left\|\mathbf{C}_h^{n-1}\right\|_2^2\\
\varepsilon\, a_c\Big(\mathbf{C}_h^n, \mathbf{C}_h^n\Big) &= \varepsilon\,\|\nabla \mathbf{C}_h^n\|_2^2\\
\Big((\mathrm{tr}\,\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \mathbf{C}_h^n\Big) &= \left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathbf{C}_h^n\right\|_2^2\\
\Big(\mathrm{tr}\,\mathbf{C}_h^{n-1}, \mathrm{tr}\,\mathbf{C}_h^n\Big) &\le \beta_3\,\|\mathrm{tr}\,\mathbf{C}_h^n\|_2^2 + \frac{1}{4\beta_3}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_2^2\\
&\le 2\beta_3\,\|\mathbf{C}_h^n\|_2^2 + \frac{1}{2\beta_3}\left\|\mathbf{C}_h^{n-1}\right\|_2^2\\
\Big(\mathbf{F}_h^n, \mathbf{C}_h^n\Big) &\le \beta_3\,\|\mathbf{C}_h^n\|_2^2 + \frac{1}{4\beta_3}\|\mathbf{F}_h^n\|_2^2.
\end{aligned}$$

Further, the term coupling the velocity and the conformation tensor can be now handled due to (7.13). The Hölder and the Young inequalities yield

$$\Big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla \mathbf{u}_h^n)^T, \mathbf{C}_h^n\Big) \le \frac{c}{2}\|\nabla \mathbf{u}_h^n\|_2^2 + \frac{c}{4}\left\|\mathbf{C}_h^{n-1}\right\|_4^4 + \frac{c}{4}\|\mathbf{C}_h^n\|_4^4.$$

Combining all the estimates above we get, for all $n \geq 1$, the inequality

$$
\begin{aligned}
\bar{D}_{\Delta t}\left(\frac{1}{2}\left\|\mathbf{C}_h^n\right\|_2^2\right)+\varepsilon\left\|\nabla \mathbf{C}_h^n\right\|_2^2+\left\|\operatorname{tr} \mathbf{C}_h^{n-1} \mathbf{C}_h^n\right\|_2^2 \leq & \\
\leq c_1\left\|\mathbf{C}_h^{n-1}\right\|_2^2 &+\frac{1}{2 \beta_3}\left\|\mathbf{C}_h^{n-1}\right\|_2^2+c\left\|\mathbf{C}_h^{n-1}\right\|_4^4+ \\
&+3 \beta_3\left\|\mathbf{C}_h^n\right\|_2^2+c\left\|\mathbf{C}_h^n\right\|_4^4+c\left\|\nabla \mathbf{u}_h^n\right\|_2^2+\frac{1}{4 \beta_3}\left\|\mathbf{F}_h^n\right\|_2^2,
\end{aligned}
$$

that again has the form (2.7b). Consequently, by employing the discrete Gronwall inequality with the constant β_3 satisfying $\Delta t \leq 1/12\beta_3$, we obtain

$$
\begin{aligned}
\frac{1}{2}\left\|\mathbf{C}_h^n\right\|_2^2 &+\Delta t \sum_{i=1}^n\left(\varepsilon\left\|\nabla \mathbf{C}_h^i\right\|_2^2+\left\|\operatorname{tr} \mathbf{C}_h^{i-1} \mathbf{C}_h^i\right\|_2^2\right) \leq \\
&\leq \mathrm{e}^{c_1 n \Delta t}\left(\frac{1}{2}\left\|\mathbf{C}_h^0\right\|_2^2+c \Delta t \sum_{i=1}^n\left(\left\|\nabla \mathbf{u}_h^i\right\|_2^2+\left\|\mathbf{C}_h^i\right\|_4^4+\left\|\mathbf{C}_h^{i-1}\right\|_4^4+\left\|\mathbf{F}_h^i\right\|_2^2\right)\right)
\end{aligned}
$$

for all $n \geq 1$. This means

$$
\begin{aligned}
\frac{1}{2}\left\|\mathbf{C}_h\right\|_{l^{\infty}(L^2)}^2 &+\varepsilon\left\|\nabla \mathbf{C}_h\right\|_{l^2(L^2)}^2+\left\|\operatorname{tr} \mathbf{C}_h^{\cdot-1} \mathbf{C}_h\right\|_{l^2(L^2)}^2 \leq \\
&\leq \mathrm{e}^{c_1 T}\left(\left\|\mathbf{C}_h^0\right\|_2^2+\left\|\nabla \mathbf{u}_h\right\|_{l^2(L^2)}^2+\left\|\mathbf{C}_h\right\|_{l^4(L^4)}^4+\Delta t\left\|\mathbf{C}_h^0\right\|_4^4+\left\|\mathbf{F}_h\right\|_{l^2(L^2)}^2\right).
\end{aligned}
\tag{7.14}
$$

The Korn inequality and a priori bound (7.13a) imply

$$
\left\|\nabla \mathbf{u}_h\right\|_{l^2(L^2)}^2 \leq c(1/\nu) \mathrm{e}^{2 c_1 T}\left(\left\|\mathbf{u}_h^0\right\|_2+\left\|\operatorname{tr} \mathbf{C}_h^0\right\|_2+\left\|\mathbf{f}_h\right\|_{l^2(L^2)}+\left\|\operatorname{tr} \mathbf{F}_h\right\|_{l^2(L^2)}\right)^2 . \tag{7.15}
$$

Similarly, a priori bound (7.13b) yields

$$
\left\|\mathbf{C}_h\right\|_{l^4(L^4)}^4 \leq\left(c_1^s\right)^4\left(\left\|\mathbf{u}_h^0\right\|_2+\left\|\operatorname{tr} \mathbf{C}_h^0\right\|_2+\left\|\mathbf{f}_h\right\|_{l^2(L^2)}+\left\|\operatorname{tr} \mathbf{F}_h\right\|_{l^2(L^2)}\right)^4 . \tag{7.16}
$$

Therefore, combining (7.14) - (7.16), we finally obtain the a priori bound on the discrete conformation tensor

$$
\begin{aligned}
\left\|\mathbf{C}_h\right\|_{l^{\infty}(L^2)}, \sqrt{\varepsilon}\left\|\nabla \mathbf{C}_h\right\|_{l^2(L^2)}, &\left\|\operatorname{tr} \mathbf{C}_h^{\cdot-1} \mathbf{C}_h\right\|_{l^2(L^2)} \\
\leq c_2^s\Bigg(&\left\|\mathbf{u}_h^0\right\|_2+\left\|\mathbf{C}_h^0\right\|_2+\left\|\mathbf{u}_h^0\right\|_2^2+\left\|\mathbf{C}_h^0\right\|_4^2+ \\
&+\left\|\mathbf{f}_h\right\|_{l^2(L^2)}+\left\|\mathbf{f}_h\right\|_{l^2(L^2)}^2+\left\|\mathbf{F}_h\right\|_{l^2(L^2)}+\left\|\operatorname{tr} \mathbf{F}_h\right\|_{l^2(L^2)}^2\Bigg).
\end{aligned}
\tag{7.17}
$$

This concludes the proof of Theorem 7.13. □

7.3. Error estimates

This section aims to present the error analysis of the pressure-stabilized characteristics finite element approximation of the Oseen-type Peterlin model. The discrete scheme has been derived using the first order approximation of the material derivative and employing the conforming linear finite element approximation, cf. Chapter 6. The optimal first order convergence shall be proven.

In order to prove the accuracy of our numerical scheme we need to assume more regular weak solution.

Hypothesis 3. *(regularity assumptions)*
The solution $(\mathbf{u}, p, \mathbf{C})$ *to the problem* (7.1) *satisfies*

$$\begin{aligned}(\mathbf{u}, p) &\in \left[Z^2 \cap H^1(0,T; V \cap H^2(\Omega)^2)\right] \times H^1(0,T; Q \cap H^1(\Omega)) \\ \mathbf{C} &\in Z^2 \cap H^1(0,T; H^2(\Omega)^{2\times 2}).\end{aligned} \tag{7.18}$$

The main result on the error estimates follows.

Theorem 7.14. *(error estimates)*
Let us assume Hypotheses 1, 2 and 3. Let condition (6.14) *on the time increment* Δt *be satisfied, and let* $(\mathbf{f}_h, \mathbf{F}_h) = (\mathbf{f}, \mathbf{F})$. *Further, let* $\mathbf{u}_h^0 \in V_h$ *be the first component of the Stokes projection of* $(\mathbf{u}^0, 0)$, *and let* $\mathbf{C}_h^0 \in W_h$ *be the Lagrange interpolation of a symmetric positive definite tensor* $\mathbf{C}^0$. *Then, for a time increment satisfying additionally*

$$\Delta t \le \frac{1}{c\left(1/\nu, 1/\varepsilon, \|(\mathbf{u}, \mathbf{C})\|_{H^1(H^2\times H^2)}\right)},$$

there exist some constants c_1^e *and* c_2^e *of* (7.3), *independent of the discretization parameters, such that*

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(L^2)}, \sqrt{\nu}\,\|D(\mathbf{u} - \mathbf{u}_h)\|_{l^2(L^2)}, |p - p_h|_{l^2(|.|_h)}, \|\mathbf{C} - \mathbf{C}_h\|_{l^\infty(L^2)}& \\ \le c_1^e \left\{ h\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \right\}& \\ \sqrt{\varepsilon}\,\|\nabla(\mathbf{C} - \mathbf{C}_h)\|_{l^2(L^2)} \le c_2^e \left\{ h\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \right\}&,\end{aligned}$$

where $(\mathbf{u}, p, \mathbf{C})$ *is the solution to the weak formulation* (7.1) *of the Oseen-type Peterlin problem and* $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ *is the solution of the semi-implicit linear pressure-stabilized characteristics finite element scheme* (7.2).

Proof of Theorem 7.14

The proof is divided into three parts. We start by introducing the error and its suitable decomposition. In the second part we derive the equations satisfied by errors. The last part contains the error estimates that shall be obtained in two steps.

Decomposition of error

We begin the proof by the definition of the approximation error. Let

$$(\mathbf{u}_h, p_h, \mathbf{C}_h) := \left\{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\right\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$$

be the discrete solution of scheme (7.2) and let

$$\left\{\,(\mathbf{u}, p, \mathbf{C})\,(t);\; t \in (0,T)\right\} \subset V \times Q \times W$$

be the solution to the weak formulation (7.1). Further,

$$(\hat{\mathbf{u}}_h, \hat{p}_h)\,(t) \in V_h \times Q_h$$

is the Stokes projection of $(\mathbf{u}, p)\,(t) \in H^2(\Omega)^2 \times H^1(\Omega)$ defined by (7.9) and

$$\Pi_h \mathbf{C}(t) \in W_h$$

is the Lagrange interpolation of $\mathbf{C}(t) \in H^2(\Omega)^{2\times 2}$.

At time t^n, we define the *error* e^n of the approximation of the velocity, pressure and of the conformation tensor by

$$e_u^n := \mathbf{u}^n - \mathbf{u}_h^n, \quad e_p^n := p^n - p_h^n, \quad e_c^n := \mathbf{C}^n - \mathbf{C}_h^n, \tag{7.19}$$

respectively. Now, using the Stokes projection of $(\mathbf{u}, p)$ and the Lagrange interpolation of $\mathbf{C}$, we decompose the errors into two parts

$$\begin{aligned} e_u^n &= (\mathbf{u}^n - \hat{\mathbf{u}}_h^n) + (\hat{\mathbf{u}}_h^n - \mathbf{u}_h^n) &&=: \eta_{u,h}^n + \delta_{u,h}^n \\ e_p^n &= (p^n - \hat{p}_h^n) + (\hat{p}_h^n - p_h^n) &&=: \eta_{p,h}^n + \delta_{p,h}^n \\ e_c^n &= (\mathbf{C}^n - \Pi_h \mathbf{C}^n) + (\Pi_h \mathbf{C}^n - \mathbf{C}_h^n) &&=: \eta_{c,h}^n + \delta_{c,h}^n. \end{aligned} \tag{7.20}$$

The first one stands for either the error of the Stokes projection, $\eta_{u,h}^n$, $\eta_{p,h}^n$, or the interpolation error, $\eta_{c,h}^n$. These terms are easily evaluated due to Proposition 7.12 and Lemma 7.7. In the course of the proof, we shall use the following estimates

$$\left\|\eta_{u,h}^n\right\|_{1,2}^2 + \left\|\eta_{c,h}^n\right\|_{1,2}^2 \le c(\nu^2) h^2 \left\|(\mathbf{u}^n, p^n)\right\|_{H^2 \times H^1}^2 + \gamma_3\, h^2 \left\|\mathbf{C}^n\right\|_{H^2}^2 \tag{7.21a}$$

$$\begin{aligned} \|\eta_{u,h}\|_{H^1(t^{n-1},t^n;L^2)}^2 + \|\eta_{c,h}\|_{H^1(t^{n-1},t^n;L^2)}^2 \le{}& c(\nu^2) h^2 \left\|(\mathbf{u}, p)\right\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + \\ &+ \gamma_3\, h^2 \left\|\mathbf{C}\right\|_{H^1(t^{n-1},t^n;H^2)}^2 . \end{aligned} \tag{7.21b}$$

Since, in particular, it holds that $\|e\| \le \|\delta\| + \|\eta\|$, it is enough to acquire the estimates of δ-terms to bound the errors e_u, e_p, e_c. The second parts of the errors, $\delta_{u,h}^n$, $\delta_{p,h}^n$, and $\delta_{c,h}^n$ shall be determined from the system of equations corresponding to the Oseen-type Peterlin model. More precisely, in the next part of the proof, we derive the equations satisfied by errors. Afterwards, by choosing a proper test function, we get the equations for δ-terms.

Equations satisfied by errors

Let us consider the weak formulation (7.1) at time step t^n with the Stokes projection and with $(\mathbf{v}_h, q_h, \mathbf{D}_h) \in V \times Q \times W$ as the test function

$$
\begin{aligned}
&\left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t}, \mathbf{v}_h\right) + \mathcal{A}_h\Big((\hat{\mathbf{u}}_h^n, \hat{p}_h^n), (\mathbf{v}_h, q_h) \Big) = -\Big(\mathrm{tr}\,\mathbf{C}^n\,\mathbf{C}^n, \nabla \mathbf{v}_h\Big) \\
&\left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t}, \mathbf{D}_h\right) + \varepsilon\, a_c\Big(\mathbf{C}^n, \mathbf{D}_h\Big) = \Big((\nabla\mathbf{u}^n)\mathbf{C}^n + \mathbf{C}^n(\nabla\mathbf{u}^n)^T, \mathbf{D}_h\Big) - \\
&\qquad\qquad - \Big((\mathrm{tr}\,\mathbf{C}^n)^2\,\mathbf{C}^n, \mathbf{D}_h\Big) + \Big(\mathrm{tr}\,\mathbf{C}^n\,\mathbf{I}, \mathbf{D}_h\Big).
\end{aligned}
$$

We recall equations (7.2) satisfied by the discrete solution at time step t^n, i.e.

$$
\begin{aligned}
&\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\Big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h) \Big) = -\Big(\mathrm{tr}\,\mathbf{C}_h^n\;\mathbf{C}_h^{n-1}, \nabla \mathbf{v}_h\Big) \\
&\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon\, a_c\Big(\mathbf{C}_h^n, \mathbf{D}_h\Big) = \Big((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T, \mathbf{D}_h\Big) - \\
&\qquad\qquad - \Big((\mathrm{tr}\,\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \mathbf{D}_h\Big) + \Big(\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathbf{I}, \mathbf{D}_h\Big).
\end{aligned}
$$

The difference of the above pairs of equations yields the error equations for the velocity and pressure

$$
\begin{aligned}
&\left(\frac{e_u^n - e_u^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \\
&\quad + \nu\, a_u\left(\delta_{u,h}^n, \mathbf{v}_h\right) + b\left(\delta_{u,h}^n, q_h\right) + b\left(\mathbf{v}_h, \delta_{p,h}^n\right) - \mathcal{S}_h\left(\delta_{p,h}^n, q_h\right) = \\
&\quad = -\left(\mathrm{tr}\,e_c^n\,\mathbf{C}^{n-1} + \mathrm{tr}\,\mathbf{C}_h^n\,e_c^{n-1}, \nabla\mathbf{v}_h\right) - \left(\mathrm{tr}\,\mathbf{C}^n\left(\mathbf{C}^n - \mathbf{C}^{n-1}\right), \nabla\mathbf{v}_h\right),
\end{aligned}
\tag{7.22}
$$

and for the conformation tensor as well

$$
\begin{aligned}
&\left(\frac{e_c^n - e_c^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon\, a_c\Big(e_c^n, \mathbf{D}_h\Big) = \\
&\quad = \Big((\nabla e_u^n)\mathbf{C}^{n-1} + \mathbf{C}^{n-1}(\nabla e_u^n)^T + (\nabla\mathbf{u}_h^n)e_c^{n-1} + e_c^{n-1}(\nabla\mathbf{u}_h^n), \mathbf{D}_h\Big) + \\
&\quad + \Big((\nabla\mathbf{u}^n)\left(\mathbf{C}^n - \mathbf{C}^{n-1}\right) + \left(\mathbf{C}^n - \mathbf{C}^{n-1}\right)(\nabla\mathbf{u}^n)^T, \mathbf{D}_h\Big) - \\
&\quad - \Big(\Big((\mathrm{tr}\,\mathbf{C}^n)^2 - (\mathrm{tr}\,\mathbf{C}^{n-1})^2\Big)\,\mathbf{C}^n, \mathbf{D}_h\Big) - \\
&\quad - \Big((\mathrm{tr}\,\mathbf{C}^{n-1})^2\mathbf{C}^n - (\mathrm{tr}\,\mathbf{C}_h^{n-1})^2\,\mathbf{C}_h^n, \mathbf{D}_h\Big) + \Big((\mathrm{tr}\,\mathbf{C}^n - \mathrm{tr}\,\mathbf{C}_h^{n-1})\mathbf{I}, \mathbf{D}_h\Big).
\end{aligned}
\tag{7.23}
$$

Let us substitute

$$
\left(\delta_{u,h}^n, -\delta_{p,h}^n, \delta_{c,h}^n\right) \in V_h \times Q_h \times W_h
$$

into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in both error equations. Then in (7.22) we get

$$
b\left(\delta_{u,h}^n, -\delta_{p,h}^n\right) + b\left(\delta_{u,h}^n, \delta_{p,h}^n\right) = 0.
$$

We add the resulting equations together, in order to handle the coupling between the equations for the velocity and for the conformation tensor. Then, by (7.20) we get

$$
\begin{aligned}
&\underbrace{\left(\frac{\delta_{u,h}^n - \delta_{u,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n\right) + \left(\frac{\delta_{c,h}^n - \delta_{c,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{c,h}^n\right)}_{L_1} + \\
&+ \underbrace{\nu\, a_u\left(\delta_{u,h}^n, \delta_{u,h}^n\right) - \mathcal{S}_h\left(\delta_{p,h}^n, -\delta_{p,h}^n\right) + \varepsilon\, a_c\left(\delta_{c,h}^n, \delta_{c,h}^n\right)}_{L_2} = \\
= &- \underbrace{\varepsilon\, a_c\left(\eta_{c,h}^n, \delta_{c,h}^n\right)}_{I_1} - \\
&- \underbrace{\left(\operatorname{tr} e_c^n\, \mathbf{C}^{n-1} + \operatorname{tr} \mathbf{C}_h^n\, e_c^{n-1}, \nabla \delta_{u,h}^n\right)}_{I_2} - \underbrace{\left(\operatorname{tr} \mathbf{C}^n \left(\mathbf{C}^n - \mathbf{C}^{n-1}\right), \nabla \delta_{u,h}^n\right)}_{I_3} + \\
&+ \underbrace{\left((\nabla e_u^n)\mathbf{C}^{n-1} + \mathbf{C}^{n-1}(\nabla e_u^n)^T + (\nabla \mathbf{u}_h^n) e_c^{n-1} + e_c^{n-1}(\nabla \mathbf{u}_h^n), \delta_{c,h}^n\right)}_{I_4} + \\
&+ \underbrace{\left((\nabla \mathbf{u}^n)\left(\mathbf{C}^n - \mathbf{C}^{n-1}\right) + \left(\mathbf{C}^n - \mathbf{C}^{n-1}\right)(\nabla \mathbf{u}^n)^T, \delta_{c,h}^n\right)}_{I_5} + \\
&+ \underbrace{\left((\operatorname{tr} \mathbf{C}^n - \operatorname{tr} \mathbf{C}_h^{n-1})\mathbf{I}, \delta_{c,h}^n\right)}_{I_6} - \underbrace{\left(\left((\operatorname{tr} \mathbf{C}^n)^2 - \left(\operatorname{tr} \mathbf{C}^{n-1}\right)^2\right) \mathbf{C}^n, \delta_{c,h}^n\right)}_{I_7} - \\
&- \underbrace{\left((\operatorname{tr} \mathbf{C}^{n-1})^2 \mathbf{C}^n - (\operatorname{tr} \mathbf{C}_h^{n-1})^2\, \mathbf{C}_h^n, \delta_{c,h}^n\right)}_{I_8} - \\
&- \underbrace{\left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n\right)}_{I_9} - \underbrace{\left(\frac{\eta_{u,h}^n - \eta_{u,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n\right)}_{I_{10}} - \\
&- \underbrace{\left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1} \circ X_1^n}{\Delta t}, \delta_{c,h}^n\right)}_{I_{11}} - \underbrace{\left(\frac{\eta_{c,h}^n - \eta_{c,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{c,h}^n\right)}_{I_{12}}.
\end{aligned}
\tag{7.24}
$$

We denote the right-hand and the left-hand sides of (7.24) by

$$
I := \sum_{i=1}^{12} I_i
$$

and $L := L_1 + L_2$, respectively. The last part of the proof contains the estimates of integrals L_1, L_2 and I_i, $i = 1, \ldots, 12$.

Estimates of errors

Using the error equation (7.24) we are able to find the proper bounds of δ-terms, which, in combination with (7.21), will conclude the proof of Theorem 7.14. However, the bound on $\|\nabla \delta_{c,h}\|_{l^2(L^2)}$ will be determined afterwards using the results from the first step. The reason for two-step proof is explained at the end, see Remark 7.15.

Step 1

Let us firstly bound the left-hand side of (7.24) from below. Analogously as for (7.12a) in the proof of Theorem 7.13, we get by the Hölder, the Young inequalities and by (7.4a),

$$
\begin{aligned}
L_1 &= \left(\frac{\delta_{u,h}^n - \delta_{u,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n \right) + \left(\frac{\delta_{c,h}^n - \delta_{c,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{c,h}^n \right) \\
&\geq \bar{D}_{\Delta t}\left(\frac{1}{2} \left\| \delta_{u,h}^n \right\|_2^2 + \frac{1}{2} \left\| \delta_{c,h}^n \right\|_2^2 \right) - c_1 \left(\left\| \delta_{u,h}^{n-1} \right\|_2^2 + \left\| \delta_{c,h}^{n-1} \right\|_2^2 \right).
\end{aligned}
$$

Obviously, it holds that

$$
\begin{aligned}
L_2 &= \nu\, a_u \left(\delta_{u,h}^n, \delta_{u,h}^n \right) - \mathcal{S}_h \left(\delta_{p,h}^n, -\delta_{p,h}^n \right) + \varepsilon\, a_c \left(\delta_{c,h}^n, \delta_{c,h}^n \right) \\
&= 2\nu \left\| \mathrm{D}(\delta_{u,h}^n) \right\|_2^2 + \delta_0 |\delta_{p,h}^n|_h^2 + \varepsilon \left\| \nabla \delta_{c,h}^n \right\|_2^2 .
\end{aligned}
$$

The lower bound of the left-hand side then becomes

$$
\boxed{
\begin{aligned}
L \geq \bar{D}_{\Delta t}&\left(\frac{1}{2} \left\| \delta_{u,h}^n \right\|_2^2 + \frac{1}{2} \left\| \delta_{c,h}^n \right\|_2^2 \right) + 2\nu \left\| \mathrm{D}(\delta_{u,h}^n) \right\|_2^2 + \delta_0 |\delta_{p,h}^n|_h^2 + \varepsilon \left\| \nabla \delta_{c,h}^n \right\|_2^2 - \\
&- c_1 \left(\left\| \delta_{u,h}^{n-1} \right\|_2^2 + \left\| \delta_{c,h}^{n-1} \right\|_2^2 \right).
\end{aligned}
}
\tag{7.25}
$$

We proceed with calculations for the right-hand side of error equation (7.24). We shall estimate the integrals I_i, $i = 1, \ldots, 12$, separately. Our aim is to find a proper evaluation of these terms such that the discrete Gronwall lemma (2.7) can be applied, resulting in the desired estimates of δ-terms. In general, we employ the Hölder, the Young, the Korn inequalities, the interpolation inequalities (2.8) and the norm equivalence (2.9). The positive constants $\alpha_1 = c(\nu)$, $\alpha_2 = c(\varepsilon)$ and $\alpha_3 = c(\varepsilon)$ arising from the Young inequalities will be specified later. Let us begin with the simplest integral. The Hölder and the Young inequalities directly yield

$$
\boxed{ -I_1 \leq \frac{\varepsilon}{6} \left\| \nabla \delta_{c,h}^n \right\|_2^2 + \frac{3\varepsilon}{2} \left\| \nabla \eta_{c,h}^n \right\|_2^2 . }
$$

Further, we consider the terms arising from the divergence of the elastic stress tensor and from the upper convected derivative, i.e. integrals I_2 - I_5. For the first two terms we can write

$$
\begin{aligned}
-I_2 - I_3 &= -\left(\operatorname{tr} e_c^n\, \mathbf{C}^{n-1} + \operatorname{tr} \mathbf{C}_h^n\, e_c^{n-1}, \nabla \delta_{u,h}^n \right) - \left(\operatorname{tr} \mathbf{C}^n \left(\mathbf{C}^n - \mathbf{C}^{n-1} \right), \nabla \delta_{u,h}^n \right) \\
&= -\left(\operatorname{tr} \mathbf{C}^n \left(\mathbf{C}^n - \mathbf{C}^{n-1} \right), \nabla \delta_{u,h}^n \right) - \left(\operatorname{tr} \eta_{c,h}^n\ \mathbf{C}^{n-1}, \nabla \delta_{u,h}^n \right) - \\
&\quad - \left(\operatorname{tr} \Pi_h \mathbf{C}^n\, \eta_{c,h}^{n-1}, \nabla \delta_{u,h}^n \right) - \left(\operatorname{tr} \delta_{c,h}^n\ \Pi_h \mathbf{C}^{n-1}, \nabla \delta_{u,h}^n \right) - \left(\operatorname{tr} \mathbf{C}_h^n\, \delta_{c,h}^{n-1}, \nabla \delta_{u,h}^n \right).
\end{aligned}
$$

The resulting integrals are estimated, by the Korn, the Hölder, the Young inequalities and (2.9a), as follows

$$
\begin{aligned}
\left(\operatorname{tr}\mathbf{C}^n\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right),\nabla\delta^n_{u,h}\right) &\le \left\|\nabla\delta^n_{u,h}\right\|_2\|\operatorname{tr}\mathbf{C}^n\|_0\left\|\mathbf{C}^n-\mathbf{C}^{n-1}\right\|_2\\
&\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\frac{c}{\alpha_1}\left\|\mathbf{C}^n-\mathbf{C}^{n-1}\right\|_2^2\|\mathbf{C}^n\|_0^2\\
&\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\frac{c_2}{\alpha_1}\Delta t\,\|\mathbf{C}\|^2_{H^1(t^{n-1},t^n;L^2)} \qquad (7.26a)\\
\left(\operatorname{tr}\eta^n_{c,h}\;\mathbf{C}^{n-1},\nabla\delta^n_{u,h}\right) &\le \left\|\nabla\delta^n_{u,h}\right\|_2\left\|\operatorname{tr}\eta^n_{c,h}\;\right\|_2\left\|\mathbf{C}^{n-1}\right\|_0\\
&\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\eta^n_{c,h}\right\|_2^2 \qquad (7.26b)\\
\left(\operatorname{tr}\Pi_h\mathbf{C}^n\,\eta^{n-1}_{c,h},\nabla\delta^n_{u,h}\right) &\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\eta^{n-1}_{c,h}\right\|_2^2 \qquad (7.26c)\\
\left(\operatorname{tr}\delta^n_{c,h}\;\Pi_h\mathbf{C}^{n-1},\nabla\delta^n_{u,h}\right) &\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\delta^n_{c,h}\right\|_2^2. \qquad (7.26d)
\end{aligned}
$$

The last term, which induces the necessity of two-step proof, is determined by

$$
\begin{aligned}
(T_1)\quad &\left(\operatorname{tr}\mathbf{C}^n_h\;\delta^{n-1}_{c,h},\nabla\delta^n_{u,h}\right)\\
&\le \left\|\nabla\delta^n_{u,h}\right\|_2\|\operatorname{tr}\mathbf{C}^n_h\|_4\left\|\delta^{n-1}_{c,h}\right\|_4\\
&\le c\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2\|\operatorname{tr}\mathbf{C}^n_h\|_4\left(\left\|\delta^{n-1}_{c,h}\right\|_2+\left\|\delta^{n-1}_{c,h}\right\|_2^{1/2}\left\|\nabla\delta^{n-1}_{c,h}\right\|_2^{1/2}\right)\\
&\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\alpha_3\left\|\nabla\delta^{n-1}_{c,h}\right\|_2^2+\frac{c}{\alpha_1^2\alpha_3}\|\operatorname{tr}\mathbf{C}^n_h\|_4^4\left\|\delta^{n-1}_{c,h}\right\|_2^2+\frac{c}{\alpha_1}\|\operatorname{tr}\mathbf{C}^n_h\|_4^2\left\|\delta^{n-1}_{c,h}\right\|_2^2\\
&\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\alpha_3\left\|\nabla\delta^{n-1}_{c,h}\right\|_2^2+\frac{c}{\alpha_1^2\alpha_3}\|\mathbf{C}^n_h\|_4^4\left\|\delta^{n-1}_{c,h}\right\|_2^2+\frac{c}{\alpha_1}\|\mathbf{C}^n_h\|_4^2\left\|\delta^{n-1}_{c,h}\right\|_2^2,
\end{aligned}
\tag{7.27}
$$

where also the interpolation inequality (2.8b) was used. The analogous term arises from the following reformulation of the other two integrals

$$
\begin{aligned}
I_4+I_5 &= \left((\nabla e^n_u)\mathbf{C}^{n-1}+\mathbf{C}^{n-1}(\nabla e^n_u)^T+(\nabla\mathbf{u}^n_h)e^{n-1}_c+e^{n-1}_c(\nabla\mathbf{u}^n_h)^T,\delta^n_{c,h}\right)+\\
&+\left((\nabla\mathbf{u}^n)\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right)+\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right)(\nabla\mathbf{u}^n)^T,\delta^n_{c,h}\right)=\\
&=\left((\nabla\eta^n_{u,h})\mathbf{C}^{n-1}+\mathbf{C}^{n-1}(\nabla\eta^n_{u,h})^T,\delta^n_{c,h}\right)+\left((\nabla\hat{\mathbf{u}}^n_h)\eta^{n-1}_{c,h}+\eta^{n-1}_{c,h}(\nabla\hat{\mathbf{u}}^n_h)^T,\delta^n_{c,h}\right)+\\
&+\left((\nabla\delta^n_{u,h})\Pi_h\mathbf{C}^{n-1}+\Pi_h\mathbf{C}^{n-1}(\nabla\delta^n_{u,h})^T,\delta^n_{c,h}\right)+\left((\nabla\hat{\mathbf{u}}^n_h)\delta^{n-1}_{c,h}+\delta^{n-1}_{c,h}(\nabla\hat{\mathbf{u}}^n_h)^T,\delta^n_{c,h}\right)-\\
&-\left((\nabla\delta^n_{u,h})\delta^{n-1}_{c,h}+\delta^{n-1}_{c,h}(\nabla\delta^n_{u,h})^T,\Pi_h\mathbf{C}^n\right)+\left((\nabla\delta^n_{u,h})\delta^{n-1}_{c,h}+\delta^{n-1}_{c,h}(\nabla\delta^n_{u,h})^T,\mathbf{C}^n_h\right)+\\
&+\left((\nabla\mathbf{u}^n)\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right)+\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right)(\nabla\mathbf{u}^n)^T,\delta^n_{c,h}\right).
\end{aligned}
$$

Here the problematic term can be estimated in the same way, i.e.

$$
\begin{aligned}
(T_2)\quad &\left((\nabla\delta^n_{u,h})\delta^{n-1}_{c,h}+\delta^{n-1}_{c,h}(\nabla\delta^n_{u,h})^T,\mathbf{C}^n_h\right)\\
&\le \alpha_1\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2+\alpha_3\left\|\nabla\delta^{n-1}_{c,h}\right\|_2^2+\frac{c}{\alpha_1^2\alpha_3}\|\mathbf{C}^n_h\|_4^4\left\|\delta^{n-1}_{c,h}\right\|_2^2+\frac{c}{\alpha_1}\|\mathbf{C}^n_h\|_4^2\left\|\delta^{n-1}_{c,h}\right\|_2^2.
\end{aligned}
\tag{7.28}
$$

The estimates of other integrals of sum I_4+I_5 are included for completeness. Analogously as above, by the Korn, the Hölder, the Young inequalities and the interpolation inequality (2.8b), we get

$$\left((\nabla\eta_{u,h}^n)\mathbf{C}^{n-1}+\mathbf{C}^{n-1}(\nabla\eta_{u,h}^n)^T,\delta_{c,h}^n\right)\le c\left\|\delta_{c,h}^n\right\|_2^2+c_2\left\|\nabla\eta_{u,h}^n\right\|_2^2 \tag{7.29a}$$

$$\left((\nabla\hat{\mathbf{u}}_h^n)\eta_{c,h}^{n-1}+\eta_{c,h}^{n-1}(\nabla\hat{\mathbf{u}}_h^n)^T,\delta_{c,h}^n\right)\le\beta\left\|\delta_{c,h}^n\right\|_2^2+\frac{c_3}{\beta}\left\|\eta_{c,h}^{n-1}\right\|_4^2 \tag{7.29b}$$

$$\left((\nabla\delta_{u,h}^n)\Pi_h\mathbf{C}^{n-1}+\Pi_h\mathbf{C}^{n-1}(\nabla\delta_{u,h}^n)^T,\delta_{c,h}^n\right)\le\alpha_1\left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\delta_{c,h}^n\right\|_2^2 \tag{7.29c}$$

$$\begin{aligned}
&\left((\nabla\hat{\mathbf{u}}_h^n)\delta_{c,h}^{n-1}+\delta_{c,h}^{n-1}(\nabla\hat{\mathbf{u}}_h^n)^T,\delta_{c,h}^n\right)\\
&\qquad\le\left\|\nabla\hat{\mathbf{u}}_h^n\right\|_4\left\|\delta_{c,h}^{n-1}\right\|_2\left\|\delta_{c,h}^n\right\|_4\\
&\qquad\le\frac{c}{2}\left\|\delta_{c,h}^{n-1}\right\|_2^2+\frac{c}{2}\left\|\delta_{c,h}^n\right\|_4^2\left\|\nabla\hat{\mathbf{u}}_h^n\right\|_4^2\\
&\qquad\le c\left\|\delta_{c,h}^{n-1}\right\|_2^2+\alpha_2\left\|\nabla\delta_{c,h}^n\right\|_2^2+\frac{c}{\alpha_2}\left\|\nabla\hat{\mathbf{u}}_h^n\right\|_4^4\left\|\delta_{c,h}^n\right\|_2^2+c\left\|\nabla\hat{\mathbf{u}}_h^n\right\|_4^2\left\|\delta_{c,h}^n\right\|_2^2\\
&\qquad\le c\left\|\delta_{c,h}^{n-1}\right\|_2^2+\alpha_2\left\|\nabla\delta_{c,h}^n\right\|_2^2+\frac{c_3}{\alpha_2}\left\|\delta_{c,h}^n\right\|_2^2+c_3\left\|\delta_{c,h}^n\right\|_2^2
\end{aligned} \tag{7.29d}$$

$$\left((\nabla\delta_{u,h}^n)\delta_{c,h}^{n-1}+\delta_{c,h}^{n-1}(\nabla\delta_{u,h}^n)^T,\Pi_h\mathbf{C}^n\right)\le\alpha_1\left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\delta_{c,h}^{n-1}\right\|_2^2 \tag{7.29e}$$

$$\begin{aligned}
&\left((\nabla\mathbf{u}^n)\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right)+\left(\mathbf{C}^n-\mathbf{C}^{n-1}\right)(\nabla\mathbf{u}^n)^T,\delta_{c,h}^n\right)\\
&\qquad\le c\left\|\nabla\mathbf{u}^n\right\|_4\left\|\delta_{c,h}^n\right\|_4\left\|\mathbf{C}^n-\mathbf{C}^{n-1}\right\|_2\\
&\qquad\le\frac{c}{2}\left\|\delta_{c,h}^n\right\|_4^2\left\|\nabla\mathbf{u}^n\right\|_4^2+\frac{c}{2}\left\|\mathbf{C}^n-\mathbf{C}^{n-1}\right\|_2^2\\
&\qquad\le\alpha_2\left\|\nabla\delta_{c,h}^n\right\|_2^2+c_3\left\|\delta_{c,h}^n\right\|_2^2+\frac{c_3}{\alpha_2}\left\|\delta_{c,h}^n\right\|_2^2+c\Delta t\left\|\mathbf{C}\right\|_{H^1(t^{n-1},t^n;L^2)}^2.
\end{aligned} \tag{7.29f}$$

Finally, we can conclude

$$\boxed{\begin{aligned}
&I_2+I_3+I_4+I_5\\
&\qquad\le 8\alpha_1\left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2+2\alpha_2\left\|\nabla\delta_{c,h}^n\right\|_2^2+2\alpha_3\left\|\nabla\delta_{c,h}^{n-1}\right\|_2^2+\\
&\qquad+\left(\frac{c_2}{\alpha_1}+\frac{c}{\alpha_1}\left\|\mathbf{C}_h^n\right\|_4^2+\frac{c}{\alpha_1^2\alpha_3}\left\|\mathbf{C}_h^n\right\|_4^4+c\right)\left\|\delta_{c,h}^{n-1}\right\|_2^2+\left(\frac{c_2}{\alpha_1}+\frac{c_3}{\alpha_2}+c_3\right)\left\|\delta_{c,h}^n\right\|_2^2+\\
&\qquad+c_2\left\|\nabla\eta_{u,h}^n\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\eta_{c,h}^n\right\|_2^2+\frac{c_2}{\alpha_1}\left\|\eta_{c,h}^{n-1}\right\|_2^2+c_3\left\|\eta_{c,h}^{n-1}\right\|_4^2+\\
&\qquad+\frac{c_2}{\alpha_1}\Delta t\left\|\mathbf{C}\right\|_{H^1(t^{n-1},t^n;L^2)}^2.
\end{aligned}}$$

The next two terms on the right-hand side of equation (7.24) can be rewritten as

$$\begin{aligned}
I_6 - I_7 &= \Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}_h^{n-1})\mathbf{I}, \delta_{c,h}^n\Big) - \Big(\Big((\operatorname{tr}\mathbf{C}^n)^2 - (\operatorname{tr}\mathbf{C}^{n-1})^2\Big)\mathbf{C}^n, \delta_{c,h}^n\Big) \\
&= \Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1}), \operatorname{tr}\delta_{c,h}^n\Big) + \Big((\operatorname{tr}\mathbf{C}^{n-1} - \operatorname{tr}\mathbf{C}_h^{n-1}), \operatorname{tr}\delta_{c,h}^n\Big) - \\
&- \Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1})\operatorname{tr}\mathbf{C}^n\,\mathbf{C}^n, \delta_{c,h}^n\Big) - \Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1})\operatorname{tr}\mathbf{C}^{n-1}\,\mathbf{C}^n, \delta_{c,h}^n\Big).
\end{aligned}$$

Then the four estimates read

$$\begin{aligned}
\Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1}), \operatorname{tr}\delta_{c,h}^n\Big) &\le \left\|\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1}\right\|_2 \left\|\operatorname{tr}\delta_{c,h}^n\right\|_2 \\
&\le c\left\|\operatorname{tr}\delta_{c,h}^n\right\|_2^2 + c\left\|\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1}\right\|_2^2 \\
&\le c\left\|\delta_{c,h}^n\right\|_2^2 + c\Delta t\left\|\mathbf{C}\right\|_{H^1(t^{n-1},t^n;L^2)}^2
\end{aligned}$$

$$\begin{aligned}
\Big((\operatorname{tr}\mathbf{C}^{n-1} - \operatorname{tr}\mathbf{C}_h^{n-1}), \operatorname{tr}\delta_{c,h}^n\Big) &\le \left\|\operatorname{tr}\mathbf{C}^{n-1} - \operatorname{tr}\mathbf{C}_h^{n-1}\right\|_2 \left\|\operatorname{tr}\delta_{c,h}^n\right\|_2 \\
&\le c\left\|\operatorname{tr}\delta_{c,h}^n\right\|_2^2 + c\left\|\operatorname{tr}\mathbf{C}^{n-1} - \operatorname{tr}\mathbf{C}_h^{n-1}\right\|_2^2 \\
&\le c\left\|\delta_{c,h}^n\right\|_2^2 + c\left\|\delta_{c,h}^{n-1}\right\|_2^2 + c\left\|\eta_{c,h}^{n-1}\right\|_2^2
\end{aligned}$$

$$\begin{aligned}
\Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1})\operatorname{tr}\mathbf{C}^n\,\mathbf{C}^n, \delta_{c,h}^n\Big) &\le c\left\|\mathbf{C}^n\right\|_0^2 \left\|\delta_{c,h}^n\right\|_2 \left\|\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1}\right\|_2 \\
&\le c\left\|\delta_{c,h}^n\right\|_2^2 + c_2\Delta t\left\|\mathbf{C}\right\|_{H^1(t^n-t^{n-1};L^2)}^2 \\
\Big((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}^{n-1})\operatorname{tr}\mathbf{C}^{n-1}\,\mathbf{C}^n, \delta_{c,h}^n\Big) &\le c\left\|\delta_{c,h}^n\right\|_2^2 + c_2\Delta t\left\|\mathbf{C}\right\|_{H^1(t^n-t^{n-1};L^2)}^2,
\end{aligned}$$

resulting in

$$\boxed{I_6 - I_7 \le c\left\|\delta_{c,h}^n\right\|_2^2 + c\left\|\delta_{c,h}^{n-1}\right\|_2^2 + c\left\|\eta_{c,h}^{n-1}\right\|_2^2 + c_2\Delta t\left\|\mathbf{C}\right\|_{H^1(t^{n-1},t^n;L^2)}^2.}$$

The proper sign of integral I_8, i.e. a positive sign on the left-hand side of (7.24), allows us to treat the next nonlinearity of the model. We can rewrite this term as follows

$$\begin{aligned}
I_8 &= \Big((\operatorname{tr}\mathbf{C}^{n-1})^2\mathbf{C}^n - (\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \delta_{c,h}^n\Big) \\
&= 2\Big(\operatorname{tr}\delta_{c,h}^{n-1}\ \operatorname{tr}\Pi_h\mathbf{C}^{n-1}\,\Pi_h\mathbf{C}^n, \delta_{c,h}^n\Big) + \Big(\operatorname{tr}\eta_{c,h}^{n-1}\ \operatorname{tr}\Pi_h\mathbf{C}^{n-1}\,\mathbf{C}^n, \delta_{c,h}^n\Big) + \\
&+ \Big(\operatorname{tr}\mathbf{C}^{n-1}\operatorname{tr}\eta_{c,h}^{n-1}\ \mathbf{C}^n, \delta_{c,h}^n\Big) - 2\Big(\operatorname{tr}\delta_{c,h}^{n-1}\ \operatorname{tr}\Pi_h\mathbf{C}^{n-1}\,\delta_{c,h}^n, \delta_{c,h}^n\Big) - \\
&+ \Big((\operatorname{tr}\Pi_h\mathbf{C}^{n-1})^2\eta_{c,h}^n, \delta_{c,h}^n\Big) + \Big((\operatorname{tr}\Pi_h\mathbf{C}^{n-1})^2\delta_{c,h}^n, \delta_{c,h}^n\Big) + \\
&+ \Big((\operatorname{tr}\delta_{c,h}^{n-1})^2\delta_{c,h}^n, \delta_{c,h}^n\Big) - \Big((\operatorname{tr}\delta_{c,h}^{n-1})^2\Pi_h\mathbf{C}^n, \delta_{c,h}^n\Big).
\end{aligned}$$

Here the key integrals are

$$\begin{aligned}
\Big((\operatorname{tr}\delta_{c,h}^{n-1})^2\delta_{c,h}^n, \delta_{c,h}^n\Big) &= \left\|\operatorname{tr}\delta_{c,h}^{n-1}\ \delta_{c,h}^n\right\|_2^2 \\
\Big(\operatorname{tr}\delta_{c,h}^{n-1}\ \operatorname{tr}\Pi_h\mathbf{C}^{n-1}\,\delta_{c,h}^n, \delta_{c,h}^n\Big) &\le \frac{1}{4}\left\|\operatorname{tr}\delta_{c,h}^{n-1}\ \delta_{c,h}^n\right\|_2^2 + c\left\|\delta_{c,h}^n\right\|_2^2\left\|\Pi_h\mathbf{C}^{n-1}\right\|_0^2 \\
\Big((\operatorname{tr}\delta_{c,h}^{n-1})^2\Pi_h\mathbf{C}^n, \delta_{c,h}^n\Big) &\le \frac{1}{4}\left\|\operatorname{tr}\delta_{c,h}^{n-1}\ \delta_{c,h}^n\right\|_2^2 + \left\|\operatorname{tr}\delta_{c,h}^{n-1}\right\|_2^2\left\|\Pi_h\mathbf{C}^n\right\|_0^2,
\end{aligned}$$

and the rest is bounded analogously as all the integrals above

$$
\begin{aligned}
\left(\operatorname{tr}\mathbf{C}^{n-1}\operatorname{tr}\eta_{c,h}^{n-1}\ \mathbf{C}^n,\delta_{c,h}^n\right) &\le c\left\|\delta_{c,h}^n\right\|_2^2+c_2\left\|\eta_{c,h}^{n-1}\right\|_2^2\\
2\left(\operatorname{tr}\delta_{c,h}^{n-1}\ \operatorname{tr}\Pi_h\mathbf{C}^{n-1}\,\Pi_h\mathbf{C}^n,\delta_{c,h}^n\right) &\le c\left\|\delta_{c,h}^n\right\|_2^2+c_2\left\|\delta_{c,h}^{n-1}\right\|_2^2\\
\left(\operatorname{tr}\eta_{c,h}^{n-1}\ \operatorname{tr}\Pi_h\mathbf{C}^{n-1}\,\mathbf{C}^n,\delta_{c,h}^n\right) &\le c\left\|\delta_{c,h}^n\right\|_2^2+c_2\left\|\eta_{c,h}^{n-1}\right\|_2^2\\
\left((\operatorname{tr}\Pi_h\mathbf{C}^{n-1})^2\eta_{c,h}^n,\delta_{c,h}^n\right) &\le c\left\|\delta_{c,h}^n\right\|_2^2+c_2\left\|\eta_{c,h}^n\right\|_2^2\\
\left((\operatorname{tr}\Pi_h\mathbf{C}^{n-1})^2\delta_{c,h}^n,\delta_{c,h}^n\right) &\le c_2\left\|\delta_{c,h}^n\right\|_2^2.
\end{aligned}
$$

The final estimate of I_8, that is placed on the left-hand side of (7.24), is

$$
\boxed{I_8\ge\frac{1}{2}\left\|\operatorname{tr}\delta_{c,h}^{n-1}\ \delta_{c,h}^n\right\|_2^2-c_2\left\|\eta_{c,h}^n\right\|_2^2-c_2\left\|\eta_{c,h}^{n-1}\right\|_2^2-c_2\left\|\delta_{c,h}^{n-1}\right\|_2^2-c_2\left\|\delta_{c,h}^n\right\|_2^2.}
$$

What remains is to deal with the integrals I_9 - I_{12} arising from the characteristics part of the numerical scheme. Once more, by the Hölder and the Young inequalities, we have

$$
-I_9=-\left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t}-\frac{\mathbf{u}^n-\mathbf{u}^{n-1}\circ X_1^n}{\Delta t},\delta_{u,h}^n\right)\le\frac{1}{2}\left\|\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t}-\frac{\mathbf{u}^n-\mathbf{u}^{n-1}\circ X_1^n}{\Delta t}\right\|_2^2+\frac{1}{2}\left\|\delta_{u,h}^n\right\|_2^2.
$$

Similarly as in the proof of Lemma 7.2, following [111], let us define

$$
y(s):=x-(1-s)\mathbf{w}^n(x)\Delta t,\qquad t(s):=t^{n-1}+s\Delta t,\qquad s\in[0,1],
$$

and let $J_s:=\det(\partial y/\partial x)$ be the Jacobian of the transformation from x to y. Obviously,

$$
\begin{aligned}
\frac{\mathbf{u}^n-\mathbf{u}^{n-1}\circ X_1^n}{\Delta t}&=\frac{\mathbf{u}(x,t^n)-\mathbf{u}(x-\mathbf{w}^n(x)\Delta t,t^{n-1})}{\Delta t}=\frac{1}{\Delta t}\Big[\mathbf{u}(y(s),t(s))\Big]_{s=0}^1\\
&=\int_0^1\left(\frac{\partial}{\partial t}+(\mathbf{w}^n(x)\cdot\nabla)\right)\mathbf{u}(y(s),t(s))\,ds.
\end{aligned}
\tag{7.30}
$$

Then we can write

$$
\begin{aligned}
\left\|\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t}-\frac{\mathbf{u}^n-\mathbf{u}^{n-1}\circ X_1^n}{\Delta t}\right\|_2^2&=\left\|\frac{\mathrm{D}\mathbf{u}(\cdot,t^n)}{\mathrm{D}t}-\int_0^1\frac{\mathrm{D}\mathbf{u}(y(\cdot,s),t(s))}{\mathrm{D}t}\,ds\right\|_2^2\\
&=\int_\Omega\left|\int_0^1\int_s^1\frac{\mathrm{D}^2\mathbf{u}(y(x,s_1),t(s_1))}{\mathrm{D}t^2}\,ds_1\,ds\right|^2\,dx\\
&\le\int_0^1\int_s^1\left\|\frac{\mathrm{D}^2\mathbf{u}(y(\cdot,s_1),t(s_1))}{\mathrm{D}t^2}\right\|_2^2\,ds_1\,ds.
\end{aligned}
$$

By transforming x to y and s_1 to t we get

$$\begin{aligned}
\int_0^1 \int_s^1 \left\| \frac{\mathrm{D}^2\mathbf{u}(y(\cdot,s_1),t(s_1))}{\mathrm{D}t^2} \right\|_2^2 ds_1\, ds &\le \sqrt{2}\Delta t \int_0^1 \int_{t^{n-1}+s\Delta t}^{t^n} \left\| \frac{\mathrm{D}^2\mathbf{u}(\cdot,t)}{\mathrm{D}t^2} \right\|_2^2 dt\, ds \\
&\le c\Delta t \int_{t^{n-1}}^{t^n} \left\| \left(\frac{\partial}{\partial t} + (\mathbf{w}^n(\cdot)\cdot\nabla) \right)^2 \mathbf{u}(\cdot,t) \right\|_2^2 dt \\
&\le c_1 \Delta t \left\| \mathbf{u} \right\|_{Z^2(t^{n-1},t^n)}^2 .
\end{aligned}$$

We recall (2.4) for the definition of the functional space Z^2, i.e.

$$Z^2(t^{n-1},t^n) := \left\{ u \in H^j(t^{n-1},t^n;H^{2-j}(\Omega)) \;:\; j=0,1,2,\ \|u\|_{Z^2(t^{n-1},t^n)} < \infty \right\}.$$

To treat the integral I_{10} we proceed in the same way. Therefore we obtain

$$-I_{10} = -\left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1}\circ X_1^n}{\Delta t}, \delta_{c,h}^n \right) \le c_1 \Delta t \left\| \mathbf{C} \right\|_{Z^2(t^{n-1},t^n)}^2 + c \left\| \delta_{c,h}^n \right\|_2^2 .$$

To finish the evaluation of the right-hand side of (7.24) we need to bound the last two integrals that also contain the upwind point $X_1^n(x)$. Obviously,

$$\begin{aligned}
-I_{11} = -\left(\frac{\eta_{u,h}^n - \eta_{u,h}^{n-1}\circ X_1^n}{\Delta t}, \delta_{u,h}^n \right) &= -\left(\bar{D}_{\Delta t}\eta_{u,h}^n + \frac{\eta_{u,h}^{n-1} - \eta_{u,h}^{n-1}\circ X_1^n}{\Delta t}, \delta_{u,h}^n \right) \\
&\le \frac{1}{2} \left\| \bar{D}_{\Delta t}\eta_{u,h}^n \right\|_2^2 + \frac{1}{2} \left\| \frac{\eta_{u,h}^{n-1} - \eta_{u,h}^{n-1}\circ X_1^n}{\Delta t} \right\|_2^2 + \left\| \delta_{u,h}^n \right\|_2^2 .
\end{aligned}$$

Similarly as for (7.30), we get

$$\begin{aligned}
\left\| \bar{D}_{\Delta t}\eta_{u,h}^n \right\|_2^2 &= \frac{1}{(\Delta t)^2} \int_\Omega \left| \int_{t^{n-1}}^{t^n} \frac{\partial \eta_{u,h}(x,t)}{\partial t}\, dt \right|^2 dx \\
&\le \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \left\| \frac{\partial \eta_{u,h}}{\partial t} \right\|_2^2 dt \\
&\le \frac{1}{\Delta t} \left\| \eta_{u,h} \right\|_{H^1(t^{n-1},t^n;L^2(\Omega))}^2 ,
\end{aligned} \tag{7.31}$$

since we have the following equality

$$\bar{D}_{\Delta t}\eta_{u,h}^n = \frac{\eta_{u,h}(x,t^n) - \eta_{u,h}(x,t^{n-1})}{\Delta t} = \left[\frac{\partial \eta_{u,h}(x,t)}{\partial t} \right]_{t^{n-1}}^{t^n} . \tag{7.32}$$

Further, (7.4b) yields

$$\frac{1}{(\Delta t)^2} \left\| \eta_{u,h}^{n-1} - \eta_{u,h}^{n-1}\circ X_1^n \right\|_2^2 \le c_1 \left\| \eta_{u,h}^{n-1} \right\|_{1,2}^2 . \tag{7.33}$$

Hence, combining (7.31) - (7.33), we get

$$-I_{11} \le \left\| \delta_{u,h}^n \right\|_2^2 + \frac{1}{\Delta t} \left\| \eta_{u,h} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 + c_1 \left\| \eta_{u,h}^{n-1} \right\|_{1,2}^2 .$$

The last integral on the right-hand side of (7.24) is determined in the same way as I_{11} above. Thus, we obtain

$$-I_{12} = \left(\frac{\eta_{c,h}^{n} - \eta_{c,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{c,h}^n \right) \leq \left\| \delta_{c,h}^n \right\|_2^2 + \frac{1}{\Delta t} \left\| \eta_{c,h} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 + c_1 \left\| \eta_{c,h}^{n-1} \right\|_{1,2}^2 .$$

In summary, estimates of the integrals appearing due to the discretization of the material derivative give

$$\boxed{\begin{aligned} I_9 + I_{10} + I_{11} + I_{12} & \\ &\leq c \left\| \delta_{u,h}^n \right\|_2^2 + c \left\| \delta_{c,h}^n \right\|_2^2 + \\ &+ c_1 \left\| \eta_{u,h}^{n-1} \right\|_{1,2}^2 + c_1 \left\| \eta_{c,h}^{n-1} \right\|_{1,2}^2 + \frac{c}{\Delta t} \left\| \eta_{u,h} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 + \frac{c}{\Delta t} \left\| \eta_{c,h} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 + \\ &+ c_1 \Delta t \left\| \mathbf{u} \right\|_{Z^2(t^{n-1},t^n)}^2 + c_1 \Delta t \left\| \mathbf{C} \right\|_{Z^2(t^{n-1},t^n)}^2 . \end{aligned}}$$

We note the constants c_1, and c_2, c_3 of (7.3) depend on the norms of $\mathbf{w}$ and $(\mathbf{u}, p, \mathbf{C})$, respectively. Choosing the constants $\alpha_1 = \nu/8$, $\alpha_2 = \varepsilon/6$ and $\alpha_3 = \varepsilon/4$, and taking into account all the above bounds of integrals I_i, $i = 1, \ldots, 12$, we arrive at

$$\boxed{\begin{aligned} I \leq\ & \nu \left\| \mathrm{D}(\delta_{u,h}^n) \right\|_2^2 + \frac{\varepsilon}{2} \left\| \nabla \delta_{c,h}^n \right\|_2^2 + \frac{\varepsilon}{2} \left\| \nabla \delta_{c,h}^{n-1} \right\|_2^2 + c \left\| \delta_{u,h}^n \right\|_2^2 + c_1 \left\| \delta_{u,h}^{n-1} \right\|_2^2 + \\ &+ \left(\frac{c_2}{\nu} + \frac{c_3}{\varepsilon} + c_3 + c_2 + c \right) \left\| \delta_{c,h}^n \right\|_2^2 + \\ &+ \left(\frac{c_2}{\nu} + \frac{c}{\nu} \left\| \mathbf{C}_h^n \right\|_4^2 + \frac{c}{\nu^2 \varepsilon} \left\| \mathbf{C}_h^n \right\|_4^4 + c_1 + c_2 + c \right) \left\| \delta_{c,h}^{n-1} \right\|_2^2 + \\ &+ c_1 \left\| \eta_{u,h}^{n-1} \right\|_{1,2}^2 + c_2 \left\| \nabla \eta_{u,h}^n \right\|_2^2 + c_1 \left\| \eta_{c,h}^{n-1} \right\|_{1,2}^2 + \frac{3\varepsilon}{2} \left\| \nabla \eta_{c,h}^n \right\|_2^2 + \\ &+ \left(c_2 + \frac{c_2}{\nu} \right) \left\| \eta_{c,h}^n \right\|_2^2 + \left(c + c_2 + \frac{c_2}{\nu} \right) \left\| \eta_{c,h}^{n-1} \right\|_2^2 + c_3 \left\| \eta_{c,h}^{n-1} \right\|_4^2 + \\ &+ \frac{c}{\Delta t} \left\| \eta_{u,h} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 + \frac{c}{\Delta t} \left\| \eta_{c,h} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 + \\ &+ c_1 \Delta t \left\| \mathbf{u} \right\|_{Z^2(t^{n-1},t^n)}^2 + c_1 \Delta t \left\| \mathbf{C} \right\|_{Z^2(t^{n-1},t^n)}^2 + \\ &+ \left(c_2 + \frac{c_2}{\nu} \right) \Delta t \left\| \mathbf{C} \right\|_{H^1(t^{n-1},t^n;L^2)}^2 . \end{aligned} \qquad (7.34)}$$

The η-terms appearing in (7.34) are, as mentioned before, estimated by (7.21).

Let us denote

$$\begin{aligned} a_0 &= c + c_2 + c_3 + \frac{c_2}{\nu} + \frac{c_3}{\varepsilon} \\ a_1^n &= c + c_1 + c_2 + \frac{c_2}{\nu} + \frac{c}{\nu^2\varepsilon} \left\|\mathbf{C}_h^n\right\|_4^4 + \frac{c}{\nu} \left\|\mathbf{C}_h^n\right\|_4^2 \\ a_2 &= c(\nu^2) \left(c + c_1 + c_2 + c_3 + \varepsilon + \frac{c_2}{\nu} + \frac{c}{\varepsilon} \right) \\ a_3 &= c + c_1 + c_2 + \frac{c_2}{\nu}. \end{aligned}$$

Then, by (7.25) and (7.34), the equation (7.24) yields for all $n \geq 1$,

$$\begin{aligned} &\bar{D}_{\Delta t}\left(\frac{1}{2} \left\|\delta_{u,h}^n\right\|_2^2 + \frac{1}{2} \left\|\delta_{c,h}^n\right\|_2^2 \right) + \nu \left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2 + \frac{\varepsilon}{2} \left\|\nabla \delta_{c,h}^n\right\|_2^2 + \delta_0 |\delta_{p,h}^n|_h^2 + \\ &\qquad + \frac{1}{2} \left\|\mathrm{tr}\, \delta_{c,h}^{n-1} \ \delta_{c,h}^n\right\|_2^2 \leq \\ &\leq a_0 \left(\frac{1}{2} \left\|\delta_{u,h}^n\right\|_2^2 + \frac{1}{2} \left\|\delta_{c,h}^n\right\|_2^2 \right) + a_1^n \left(\frac{1}{2} \left\|\delta_{u,h}^{n-1}\right\|_2^2 + \frac{1}{2} \left\|\delta_{c,h}^{n-1}\right\|_2^2 \right) + \frac{\varepsilon}{2} \left\|\nabla \delta_{c,h}^{n-1}\right\|_2^2 + \\ &+ \frac{c(\nu^2)\, h^2}{\Delta t} \left\|(\mathbf{u}, p, \mathbf{C})\right\|_{H^1(t^{n-1},t^n;H^2\times H^1\times H^2)}^2 + a_2\, h^2 \left\|(\mathbf{u}, p, \mathbf{C})\right\|_{H^1(H^2\times H^1\times H^2)}^2 + \\ &+ a_3\, \Delta t \left\|(\mathbf{u}, \mathbf{C})\right\|_{Z^2(t^{n-1},t^n)}^2 . \end{aligned} \tag{7.35}$$

This inequality can be seen as

$$\bar{D}_{\Delta t}\, \tilde{x}^n + y^n + s^n \leq a_0\, x^n + a_1^n\, x^n + s^{n-1} + \tilde{b}^n + z, \tag{7.36}$$

where the subsequent notation has been employed

$$\begin{aligned} \tilde{x}^n &= \frac{1}{2} \left\|\delta_{u,h}^n\right\|_2^2 + \frac{1}{2} \left\|\delta_{c,h}^n\right\|_2^2 \\ y^n &= \nu \left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2 + \delta_0 |\delta_{p,h}^n|_h^2 + \frac{1}{2} \left\|\mathrm{tr}\, \delta_{c,h}^{n-1} \ \delta_{c,h}^n\right\|_2^2 \\ s^n &= \frac{\varepsilon}{2} \left\|\nabla \delta_{c,h}^n\right\|_2^2 \\ \tilde{b}^n &= \frac{c(\nu^2)\, h^2}{\Delta t} \left\|(\mathbf{u}, p, \mathbf{C})\right\|_{H^1(t^{n-1},t^n;H^2\times H^1\times H^2)}^2 + a_3\, \Delta t \left\|(\mathbf{u}, \mathbf{C})\right\|_{Z^2(t^{n-1},t^n)}^2 \\ z &= a_2\, h^2 \left\|(\mathbf{u}, p, \mathbf{C})\right\|_{H^1(H^2\times H^1\times H^2)}^2 . \end{aligned}$$

Let us mention that (7.36) is a special case of (2.7b) for $x^n = \tilde{x}^n + \Delta t s^n$ and $b^n = \tilde{b}^n + z$. Finally, the discrete Gronwall lemma for (7.35) yields the inequality

$$\begin{aligned} &\frac{1}{2} \left\|\delta_{u,h}^n\right\|_2^2 + \frac{1}{2} \left\|\delta_{c,h}^n\right\|_2^2 + \Delta t \frac{\varepsilon}{2} \left\|\nabla \delta_{c,h}^n\right\|_2^2 + \\ &\qquad + \Delta t \sum_{i=1}^{n} \left(\nu \left\|\mathrm{D}(\delta_{u,h}^i)\right\|_2^2 + \delta_0 |\delta_{p,h}^i|_h^2 + \frac{1}{2} \left\|\mathrm{tr}\, \delta_{c,h}^{i-1}\, \delta_{c,h}^i\right\|_2^2 \right) \leq \\ &\qquad \leq K_1 \Bigg\{ \frac{1}{2} \left\|\delta_{u,h}^0\right\|_2^2 + \frac{1}{2} \left\|\delta_{c,h}^0\right\|_2^2 + \Delta t \frac{\varepsilon}{2} \left\|\nabla \delta_{c,h}^0\right\|_2^2 + a_3 (\Delta t)^2 \left\|(\mathbf{u}, \mathbf{C})\right\|_{Z^2(0,T)}^2 + \\ &\qquad\qquad + h^2 \left(c(\nu^2) + a_2\, n \Delta t \right) \left\|(\mathbf{u}, p, \mathbf{C})\right\|_{H^1(0,T;H^2\times H^1\times H^2)}^2 \Bigg\}, \qquad n \geq 1, \end{aligned}$$

where $\Delta t \leq 1/2a_0$, and

$$K_1 := \exp\left\{2n\Delta t\, a_0 + \Delta t \sum_{i=1}^{n} a_1^i\right\} \leq \exp\left\{c\left(c_1, c_2, c_3, c_1^s, T, 1/\nu, 1/\varepsilon\right)\right\}$$

is a constant independent of h and Δt. Let us note that there is another constant independent of the discretization parameters, i.e. $a_3+c(\nu^2)+a_2\, T \leq c\,(c_1, c_2, c_3, T, \nu^2, \varepsilon, 1/\nu, \nu^2/\varepsilon)$. Further, it holds that

$$\delta_{u,h}^0 = \hat{\mathbf{u}}_h^0 - \mathbf{u}_h^0 = \mathbf{0}, \qquad \delta_{c,h}^0 = \Pi_h \mathbf{C}^0 - \mathbf{C}_h^0 = \mathbf{0}, \tag{7.37}$$

since $(\mathbf{u}_h^0, p_h^0)$ is the Stokes projection of $(\mathbf{u}^0, 0)$ and $\mathbf{C}_h^0 = \Pi_h \mathbf{C}^0$. Consequently, we can conclude that

$$\begin{aligned} &\|\delta_{u,h}\|_{l^\infty(L^2)}, \sqrt{\nu}\,\|\mathrm{D}(\delta_{u,h})\|_{l^2(L^2)}, \ |\delta_{p,h}|_{l^2(|.|_h)}, \ \|\delta_{c,h}\|_{l^\infty(L^2)}, \\ &\qquad \leq c_1^e \left\{ h\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1,\times H^2)} + \Delta t\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \right\}, \end{aligned} \tag{7.38}$$

where c_1^e is a constant given in (7.3c). In order to determine the errors e_u, e_p and e_c in the corresponding norms, we now combine the obtained bounds (7.38) of δ-terms with the bounds (7.21) of η-terms. This yields

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(L^2)}, \sqrt{\nu}\,\|\mathrm{D}(\mathbf{u} - \mathbf{u}_h)\|_{l^2(L^2)}, \ |p - p_h|_{l^2(|.|_h)}, \ \|\mathbf{C} - \mathbf{C}_h\|_{l^\infty(L^2)} \\ &\qquad \leq c_1^e \left\{ h\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \right\}. \end{aligned} \tag{7.39}$$

Let us note that the condition on the time increment arising form the discrete Gronwall lemma, i.e. $\Delta t \leq 1/2a_0$, implies

$$\Delta t \leq c\left(1/\nu, 1/\varepsilon, \|(\mathbf{u}, \mathbf{C})\|_{H^1(H^2\times H^2)}\right)^{-1}.$$

Step 2

To complete the proof we need to bound $\nabla \delta_{c,h}$ in $l^2(L^2)$-norm. To this end it is enough to consider the error equation for the conformation tensor (7.23). Using the test function $\delta_{c,h}^n \in W_h$ and (7.20) we get

$$\begin{aligned} &\left(\frac{\delta_{c,h}^n - \delta_{c,h}^{n-1}\circ X_1^n}{\Delta t}, \delta_{c,h}^n\right) + \varepsilon\, a_c\left(\delta_{c,h}^n, \delta_{c,h}^n\right) = -\varepsilon\, a_c\left(\eta_{c,h}^n, \delta_{c,h}^n\right) + \\ &\quad + \left((\nabla e_u^n)\mathbf{C}^{n-1} + \mathbf{C}^{n-1}(\nabla e_u^n)^T + (\nabla \mathbf{u}_h^n) e_c^{n-1} + e_c^{n-1}(\nabla \mathbf{u}_h^n), \delta_{c,h}^n\right) + \\ &\quad + \left((\nabla \mathbf{u}^n)\left(\mathbf{C}^n - \mathbf{C}^{n-1}\right) + \left(\mathbf{C}^n - \mathbf{C}^{n-1}\right)(\nabla \mathbf{u}^n)^T, \delta_{c,h}^n\right) + \\ &\quad + \left((\mathrm{tr}\,\mathbf{C}^n - \mathrm{tr}\,\mathbf{C}_h^{n-1})\mathbf{I}, \delta_{c,h}^n\right) - \left(\left((\mathrm{tr}\,\mathbf{C}^n)^2 - \left(\mathrm{tr}\,\mathbf{C}^{n-1}\right)^2\right)\mathbf{C}^n, \delta_{c,h}^n\right) - \\ &\quad - \left((\mathrm{tr}\,\mathbf{C}^{n-1})^2\mathbf{C}^n - (\mathrm{tr}\,\mathbf{C}_h^{n-1})^2\,\mathbf{C}_h^n, \delta_{c,h}^n\right) - \\ &\quad - \left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1}\circ X_1^n}{\Delta t}, \delta_{c,h}^n\right) - \left(\frac{\eta_{c,h}^n - \eta_{c,h}^{n-1}\circ X_1^n}{\Delta t}, \delta_{c,h}^n\right). \end{aligned} \tag{7.40}$$

We estimate the left-hand side and integrals I_k, $k \in \{1,5,6,7,8,10,12\}$, see (7.24), in the same way as in the first step of the proof. The only difference is the treatment of the problematic semi-implicit term $(T2)$, that is a part of the integral I_4. Now, we can write

$$
\begin{aligned}
(T2)\quad \Big((\nabla\delta^n_{u,h})\delta^{n-1}_{c,h} + \delta^{n-1}_{c,h}(\nabla\delta^n_{u,h})^T, \mathbf{C}^n_h\Big) &= \Big((\nabla\delta^n_{u,h})\delta^{n-1}_{c,h} + \delta^{n-1}_{c,h}(\nabla\delta^n_{u,h})^T, \Pi_h\mathbf{C}^n\Big) + \\
&+ \Big((\nabla\delta^n_{u,h})\mathbf{C}^{n-1}_h + \mathbf{C}^{n-1}_h(\nabla\delta^n_{u,h})^T, \delta^n_{c,h}\Big) - \\
&- \Big((\nabla\delta^n_{u,h})\Pi_h\mathbf{C}^{n-1} + \Pi_h\mathbf{C}^{n-1}(\nabla\delta^n_{u,h})^T, \delta^n_{c,h}\Big)\,.
\end{aligned}
$$

Thus, the integral I_4 leads to

$$
\begin{aligned}
I_4 = &\Big((\nabla\eta^n_{u,h})\mathbf{C}^{n-1} + \mathbf{C}^{n-1}(\nabla\eta^n_{u,h})^T, \delta^n_{c,h}\Big) + \Big((\nabla\hat{\mathbf{u}}^n_h)\eta^{n-1}_{c,h} + \eta^{n-1}_{c,h}(\nabla\hat{\mathbf{u}}^n_h)^T, \delta^n_{c,h}\Big) + \\
&+ \Big((\nabla\hat{\mathbf{u}}^n_h)\delta^{n-1}_{c,h} + \delta^{n-1}_{c,h}(\nabla\hat{\mathbf{u}}^n_h)^T, \delta^n_{c,h}\Big) + \Big((\nabla\delta^n_{u,h})\mathbf{C}^{n-1}_h + \mathbf{C}^{n-1}_h(\nabla\delta^n_{u,h})^T, \delta^n_{c,h}\Big)\,.
\end{aligned}
$$

The only term that has not yet been estimated reads

$$
\begin{aligned}
\Big((\nabla\delta^n_{u,h})\mathbf{C}^{n-1}_h + \mathbf{C}^{n-1}_h(\nabla\delta^n_{u,h})^T, \delta^n_{c,h}\Big) &\le c\left\|\nabla\delta^n_{u,h}\right\|_2^2 + \alpha_2\left\|\nabla\delta^n_{c,h}\right\|_2^2 + \\
&+ \frac{c}{\alpha_2}\left\|\delta^n_{c,h}\right\|_2^2\left\|\mathbf{C}^{n-1}_h\right\|_4^4 + c\left\|\delta^n_{c,h}\right\|_2^2\left\|\mathbf{C}^{n-1}_h\right\|_4^2\,.
\end{aligned}
$$

We choose $\alpha_2 = \varepsilon/9$ and combine the estimates of all the terms in (7.40). Consequently, we get the inequality

$$
\begin{aligned}
\bar{D}_{\Delta t}&\left(\left\|\delta^n_{c,h}\right\|_2^2\right) + \varepsilon\left\|\nabla\delta^n_{c,h}\right\|_2^2 + \left\|\operatorname{tr}\delta^{n-1}_{c,h}\;\;\delta^n_{c,h}\right\|_2^2 \le \\
&\le a_1\left\|\delta^{n-1}_{c,h}\right\|_2^2 + a_0^{n-1}\left\|\delta_{c,h}\right\|^2_{l^\infty(L^2)} + c\left\|\nabla\delta^n_{u,h}\right\|_2^2 + \frac{c\,h^2}{\Delta t}\left\|\mathbf{C}\right\|^2_{H^1(t^{n-1},t^n;H^2)} + \\
&+ a_2\,h^2\left\|(\mathbf{u},p,\mathbf{C})\right\|^2_{H^1(H^2\times H^1\times H^2)} + a_3\Delta t\left\|\mathbf{C}\right\|^2_{Z^2(t^{n-1},t^n)}\,, \qquad n\ge 1,
\end{aligned}
\tag{7.41}
$$

or equivalently,

$$
\bar{D}_{\Delta t}\,x^n + y^n \le a_1\,x^{n-1} + b^n + z.
$$

where now the coefficients read

$$
\begin{aligned}
a_0^{n-1} &= c + c_2 + c_3 + \frac{c_3}{\varepsilon} + \frac{c}{\varepsilon}\left\|\mathbf{C}^{n-1}_h\right\|_4^4 + c\left\|\mathbf{C}^{n-1}_h\right\|_4^2 \\
a_1 &= c + c_1 + c_2 \\
a_2 &= c + c_1 + c_2 + c_3 + \varepsilon \\
a_3 &= c + c_1 + c_2.
\end{aligned}
$$

Here we have used the following notation

$$
\begin{aligned}
x^n &= \left\|\delta^n_{c,h}\right\|_2^2 \\
y^n &= \varepsilon\left\|\nabla\delta^n_{c,h}\right\|_2^2 \\
b^n &= a_0^{n-1}\left\|\delta_{c,h}\right\|^2_{l^\infty(L^2)} + c\left\|\nabla\delta^n_{u,h}\right\|_2^2 + \frac{c\,h^2}{\Delta t}\left\|\mathbf{C}\right\|^2_{H^1(t^{n-1},t^n;H^2)} + a_3\Delta t\left\|\mathbf{C}\right\|^2_{Z^2(t^{n-1},t^n)} \\
z &= a_2\,h^2\left\|(\mathbf{u},p,\mathbf{C})\right\|^2_{H^1(H^2\times H^1\times H^2)}\,.
\end{aligned}
$$

Applying the discrete Gronwall inequality (2.7) on (7.41) we have

$$
\begin{aligned}
&\left\|\delta_{c,h}^n\right\|_2^2 + \varepsilon \sum_{i=1}^n \left\|\nabla \delta_{c,h}^i\right\|_2^2 \\
&\le K_2 \Bigg\{ \left\|\delta_{c,h}^0\right\|_2^2 + \left(n\,\Delta t\left(c + c_2 + c_3 + \left\|\mathbf{C}_h^0\right\|_4^4\right) + \|\mathbf{C}_h\|_{l^4(L^4)}^4\right) \|\delta_{c,h}\|_{l^\infty(L^2)}^2 + \\
&+ c\,\|\nabla \delta_{u,h}\|_{l^2(L^2)}^2 + (c + a_2 T)\, h^2 \,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1,\times H^2)}^2 + a_3 (\Delta t)^2 \,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)}^2 \Bigg\}.
\end{aligned}
$$

Here the constant $K_2 \le \exp\left\{T(c_1 + c_2)\right\}$ is again independent of h and Δt. Let us note that no condition on Δt has arose, since there is no x^n on the right-hand side of (7.41), due to (7.38). Moreover, we apply the estimates for $\|\delta_{c,h}\|_{l^\infty(L^2)}^2$ and $\|\nabla \delta_{u,h}\|_{l^2(L^2)}^2$ that yield

$$
\begin{aligned}
&\|\delta_{c,h}\|_{l^\infty(L^2)}^2 + \varepsilon\,\|\nabla \delta_{c,h}\|_{l^2(L^2)}^2 \\
&\le K_2 \Bigg\{ (c_1^e)^2 \left(T\left\|\mathbf{C}_h^0\right\|_4^4 + \|\mathbf{C}_h\|_{l^4(L^4)}^4 + \frac{c}{\nu} + c + a_2\,T\right) h^2\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1,\times H^2)}^2 + \\
&+ (c_1^e)^2 \left(T\left\|\mathbf{C}_h^0\right\|_4^4 + \|\mathbf{C}_h\|_{l^4(L^4)}^4 + \frac{c}{\nu} + a_3\right) (\Delta t)^2\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)}^3 \Bigg\},
\end{aligned}
\tag{7.42}
$$

where (7.37) was also taken into account. We recall the a priori bound (7.13b) for $\|\mathbf{C}_h\|_{l^4(L^4)}$. Consequently, we have for a constant c_2^e, cf. (7.3c),

$$
\sqrt{\varepsilon}\,\|\nabla \delta_{c,h}\|_{l^2(L^2)} \le c_2^e \Big\{ h\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \Big\}. \tag{7.43}
$$

Analogously as in the first step, we combine the above bound on $\nabla \delta_{c,h}$ with the bound (7.21a) of $\nabla \eta_{c,h}$ to get the desired error estimate

$$
\sqrt{\varepsilon}\,\|\nabla(\mathbf{C} - \mathbf{C}_h)\|_{l^2(L^2)} \le c_2^e \Big\{ h\,\|(\mathbf{u}, p, \mathbf{C})\|_{H^1(H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \Big\}. \tag{7.44}
$$

□

Remark 7.15. *We realize that integrals I_2 - I_5, that involve the two semi-implicit terms (T_1) and (T_2), could be treated in such a way that the proof might be done in only one step. Let us explain why we have used two steps to prove the result of Theorem 7.14. Our aim is to keep Δt independent of the time step t^n, if possible. Since the discrete Gronwall lemma requires the time increment to satisfy $\Delta t \le {}^1\!/_{2a_0}$, cf. (2.7), we were forced to perform the evaluation of integrals in (7.24) such that the constant a_0 was independent of the discrete solution. However, the two semi-implicit integrals, (T_1) and (T_2) forced the norm of $\nabla \delta_{c,h}^{n-1}$ to appear on the right-hand side of inequality (7.35), cf. (7.27) and (7.28). This has led to the term s^{n-1} in (7.36). Therefore, the discrete Gronwall lemma applied on (7.35), or equivalently on (7.36), could not determine the bound of $\|\nabla \delta_{c,h}\|_{l^2(L^2)}$. The price we had to pay for having Δt independent of the time step was the need of the second step to finish the proof of Theorem 7.14.*

Proof of Theorem 7.14 in one step

As we have already pointed out, the result of Theorem 7.14 can be achieved in one step. Basically, we shall handle the error equation (7.24) such that the discrete Gronwall inequality would directly yield not only the error estimates (7.39) but also (7.44). We will only concentrate on integrals I_2 - I_5 that contain the two semi-implicit integrals $(T1)$ and $(T2)$. Our aim is to get rid of the term $s^{n-1} = \left\|\nabla\delta_{c,h}^{n-1}\right\|_2^2$ on the right-hand side of inequality (7.35). We note once again, that this simplification implies the dependence of the time increment Δt on the discrete solution from the previous time steps, cf. (7.49).

Estimates of errors

We consider the error equation (7.24). The left-hand side L is estimated from below as before, see (7.25). Integrals I_1 and I_i, $i = 6, \ldots, 12$ are estimated as in the first step of the above two-step proof of Theorem 7.14. In what follows we provide another evaluation of integrals I_k, $k = 2, 3, 4, 5$, which contain the two problematic terms, cf. (7.27), (7.28). To avoid the norm of $\nabla\delta_{c,h}^{n-1}$ we rewrite term (T_1) as

$$
\begin{aligned}
(T_1)\quad -\left(\operatorname{tr}\mathbf{C}_h^n\ \delta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right) &= \left(\operatorname{tr}\delta_{c,h}^n\ \ \delta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right) - \left(\Pi_h\mathbf{C}^n\,\delta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right)\\
&= \left(\operatorname{tr}\delta_{c,h}^n\ \ \Pi_h\mathbf{C}^{n-1}, \nabla\delta_{u,h}^n\right) - \left(\operatorname{tr}\delta_{c,h}^n\ \ \mathbf{C}_h^{n-1}, \nabla\delta_{u,h}^n\right) -\\
&\quad - \left(\Pi_h\mathbf{C}^n\,\delta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right),
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
-I_2 - I_3 = &-\left(\operatorname{tr}\mathbf{C}^n\left(\mathbf{C}^n - \mathbf{C}^{n-1}\right), \nabla\delta_{u,h}^n\right) - \left(\operatorname{tr}\eta_{c,h}^n\ \ \mathbf{C}^{n-1}, \nabla\delta_{u,h}^n\right) -\\
&-\left(\operatorname{tr}\Pi_h\mathbf{C}^n\,\eta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right) - \left(\operatorname{tr}\delta_{c,h}^n\ \ \mathbf{C}_h^{n-1}, \nabla\delta_{u,h}^n\right) - \left(\Pi_h\mathbf{C}^n\,\delta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right).
\end{aligned}
\tag{7.45}
$$

We proceed analogously with integral (T_2), i.e. with the sum $I_4 + I_5$ of integrals from (7.24). Term (T_2) can be rewritten as

$$
\begin{aligned}
(T_2)\quad \left((\nabla\delta_{u,h}^n)\delta_{c,h}^{n-1} + \delta_{c,h}^{n-1}(\nabla\delta_{u,h}^n)^T, \mathbf{C}_h^n\right) &= \left((\nabla\delta_{u,h}^n)\delta_{c,h}^{n-1} + \delta_{c,h}^{n-1}(\nabla\delta_{u,h}^n)^T, \Pi_h\mathbf{C}^n\right) +\\
&\quad - \left((\nabla\delta_{u,h}^n)\Pi_h\mathbf{C}^{n-1} + \Pi_h\mathbf{C}^{n-1}(\nabla\delta_{u,h}^n)^T, \delta_{c,h}^n\right) +\\
&\quad + \left((\nabla\delta_{u,h}^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\delta_{u,h}^n)^T, \delta_{c,h}^n\right),
\end{aligned}
$$

which means that the sum $I_4 + I_5$ yields

$$
\begin{aligned}
I_4 + I_5 = &\left((\nabla\eta_{u,h}^n)\mathbf{C}^{n-1} + \mathbf{C}^{n-1}(\nabla\eta_{u,h}^n)^T, \delta_{c,h}^n\right) + \left((\nabla\hat{\mathbf{u}}_h^n)\eta_{c,h}^{n-1} + \eta_{c,h}^{n-1}(\nabla\hat{\mathbf{u}}_h^n)^T, \delta_{c,h}^n\right) +\\
&+ \left((\nabla\delta_{u,h}^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\delta_{u,h}^n)^T, \delta_{c,h}^n\right) + \left((\nabla\hat{\mathbf{u}}_h^n)\delta_{c,h}^{n-1} + \delta_{c,h}^{n-1}(\nabla\hat{\mathbf{u}}_h^n)^T, \delta_{c,h}^n\right) -\\
&+ \left((\nabla\mathbf{u}^n)\left(\mathbf{C}^n - \mathbf{C}^{n-1}\right) + \left(\mathbf{C}^n - \mathbf{C}^{n-1}\right)(\nabla\mathbf{u}^n)^T, \delta_{c,h}^n\right).
\end{aligned}
\tag{7.46}
$$

We recall (7.26a) - (7.26c) for the bounds of the first three terms in (7.45), while for the last one we get, analogously as for (7.26d),

$$
\left(\Pi_h\mathbf{C}^n\,\delta_{c,h}^{n-1}, \nabla\delta_{u,h}^n\right) \le \alpha_1\left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2 + \frac{c_2}{\nu}\left\|\delta_{c,h}^{n-1}\right\|_2^2.
$$

Further, we use (7.29a), (7.29b), (7.29d) and (7.29f) for the estimates of the four integrals in (7.46). What remains is to show how the two new integrals are determined. The fourth term in (7.45) yields

$$\begin{aligned}\left(\operatorname{tr}\delta^n_{c,h}\;\mathbf{C}^{n-1}_h, \nabla\delta^n_{u,h}\right) &\le \left\|\nabla\delta^n_{u,h}\right\|_2 \left\|\operatorname{tr}\delta^n_{c,h}\;\right\|_4 \left\|\mathbf{C}^{n-1}_h\right\|_4\\ &\le \alpha_1 \left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2 + \alpha_2 \left\|\nabla\delta^n_{c,h}\right\|_2^2 + \frac{c}{\alpha_1^2\alpha_2}\left\|\mathbf{C}^{n-1}_h\right\|_4^2 \left\|\delta^n_{c,h}\right\|_2^2 +\\ &\quad + \frac{c}{\alpha_1}\left\|\mathbf{C}^{n-1}_h\right\|_4^2 \left\|\delta^n_{c,h}\right\|_2^2 .\end{aligned}$$

Similarly, we have for the third integral in (7.46),

$$\begin{aligned}&\left((\nabla\delta^n_{u,h})\mathbf{C}^{n-1}_h + \mathbf{C}^{n-1}_h(\nabla\delta^n_{u,h})^T, \delta^n_{c,h}\right)\\ &\qquad\le \alpha_1 \left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2 + \alpha_2 \left\|\nabla\delta^n_{c,h}\right\|_2^2 + \frac{c}{\alpha_1^2\alpha_2}\left\|\mathbf{C}^{n-1}_h\right\|_4^2 \left\|\delta^n_{c,h}\right\|_2^2 + \frac{c}{\alpha_1}\left\|\mathbf{C}^{n-1}_h\right\|_4^2 \left\|\delta^n_{c,h}\right\|_2^2 .\end{aligned}$$

At the end, for suitably chosen constants $\alpha_1 = c(\nu)$ and $\alpha_2 = c(\varepsilon)$, we arrive at the inequality

$$\begin{aligned}&\bar{D}_{\Delta t}\Big(\frac{1}{2}\left\|\delta^n_{u,h}\right\|_2^2 + \frac{1}{2}\left\|\delta^n_{c,h}\right\|_2^2\Big) + \nu\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2 + \delta_0|\delta^n_{p,h}|_h^2 +\\ &\qquad\qquad + \frac{\varepsilon}{2}\left\|\nabla\delta^n_{c,h}\right\|_2^2 + \frac{1}{2}\left\|\operatorname{tr}\delta^{n-1}_{c,h}\;\delta^n_{c,h}\right\|_2^2 \le\\ &\le a_0^{n-1}\left(\frac{1}{2}\left\|\delta^n_{u,h}\right\|_2^2 + \frac{1}{2}\left\|\delta^n_{c,h}\right\|_2^2\right) + a_1\left(\frac{1}{2}\left\|\delta^{n-1}_{u,h}\right\|_2^2 + \frac{1}{2}\left\|\delta^{n-1}_{c,h}\right\|_2^2\right) +\\ &+ \frac{c(\nu^2)\,h^2}{\Delta t}\left\|(\mathbf{u},p,\mathbf{C})\right\|^2_{H^1(t^{n-1},t^n;H^2\times H^1\times H^2)} + a_2\,h^2\left\|(\mathbf{u},p,\mathbf{C})\right\|^2_{H^1(H^2\times H^1\times H^2)} +\\ &+ a_3\,\Delta t\left\|(\mathbf{u},\mathbf{C})\right\|^2_{Z^2(t^{n-1},t^n)},\end{aligned}\tag{7.47}$$

where the coefficients read

$$\begin{aligned}a_0^{n-1} &= c + c_2 + c_3 + \frac{c_2}{\nu} + \frac{c_3}{\varepsilon} + \frac{c}{\nu^2\varepsilon}\left\|\mathbf{C}^{n-1}_h\right\|_4^4 + \frac{c}{\nu}\left\|\mathbf{C}^{n-1}_h\right\|_4^2\\ a_1 &= c + c_1 + c_2 + \frac{c_2}{\nu}\\ a_2 &= c(\nu^2)\left(c + c_1 + c_2 + c_3 + \frac{c_2}{\nu} + \frac{c}{\varepsilon}\right)\\ a_3 &= c + c_1 + c_2 + \frac{c_2}{\nu}.\end{aligned}$$

The difference compared to (7.35) is in the first two coefficients. In this case, a_0^{n-1} depends on the solution from the previous time step and a_1 is constant. On the contrary, in inequality (7.35) the coefficient a_0 is constant and a_1^n depends on the solution from the current time step. Employing the notation

$$\begin{aligned}x^n &= \frac{1}{2}\left\|\delta^n_{u,h}\right\|_2^2 + \frac{1}{2}\left\|\delta^n_{c,h}\right\|_2^2\\ y^n &= \nu\left\|\mathrm{D}(\delta^n_{u,h})\right\|_2^2 + \delta_0|\delta^n_{p,h}|_h^2 + \frac{1}{2}\left\|\operatorname{tr}\delta^{n-1}_{c,h}\;\delta^n_{c,h}\right\|_2^2\\ \tilde{b}^n &= \frac{c(\nu^2)\,h^2}{\Delta t}\left\|(\mathbf{u},p,\mathbf{C})\right\|^2_{H^1(t^{n-1},t^n;H^2\times H^1\times H^2)} + a_3\,\Delta t\left\|(\mathbf{u},\mathbf{C})\right\|^2_{Z^2(t^{n-1},t^n)}\\ z &= a_2\,h^2\left\|(\mathbf{u},p,\mathbf{C})\right\|^2_{H^1(H^2\times H^1\times H^2)},\end{aligned}$$

inequality (7.47) attains the form

$$\bar{D}_{\Delta t}\, x^n + y^n \leq a_0^{n-1}\, x^n + a_1\, x^n + \tilde{b}^n + z.$$

There is no term s^{n-1} as desired. Once again, combining the bounds (7.21) of η-terms with the discrete Gronwall lemma, the final result on the error of the approximation follows

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{l^\infty(L^2)}, \sqrt{\nu}\,\|\mathrm{D}(\mathbf{u}-\mathbf{u}_h)\|_{l^2(L^2)}, |p-p_h|_{l^2(|\cdot|_h)},\\ &\|\mathbf{C}-\mathbf{C}_h\|_{l^\infty(L^2)}, \sqrt{\varepsilon}\,\|\nabla(\mathbf{C}-\mathbf{C}_h)\|_{l^2(L^2)}\\ &\qquad\leq c_1^e\left\{h\,\|(\mathbf{u},p,\mathbf{C})\|_{H^1(0,T;H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u},\mathbf{C})\|_{Z^2(0,T)}\right\}. \end{aligned} \tag{7.48}$$

We emphasize that condition (2.7a) on Δt induces the dependence of the time increment on the discrete solution form the previous time step, i.e.

$$\Delta t \leq c\left(c_1, c_2, c_3, 1/\nu, 1/\varepsilon, 1/(\nu^2\varepsilon), \left\|\mathbf{C}_h^{n-1}\right\|_4\right)^{-1}. \tag{7.49}$$

□

7.4. Higher a priori estimates

This section presents the higher energy estimates of the discrete solution of the semi-implicit linear scheme (7.2). Basically, the proper norms of the discrete time derivatives $\bar{D}_{\Delta t}\mathbf{u}_h$ and $\bar{D}_{\Delta t}\mathbf{C}_h$ shall be bounded independently of the discretization parameters.

To this end we assume more regular data. More precisely, we suppose the following three hypotheses hold.

Hypothesis 4. *(more regular given velocity)*
The function $\mathbf{w}$ *satisfies* $\mathbf{w} \in W^{1,\infty}(0,T; W_0^{1,\infty}(\Omega)^2)$.

Hypothesis 5. *(more regular initial data)*
It holds that $(\mathbf{u}^0, \mathbf{C}^0) \in [H^2(\Omega)^2 \cap V] \times H^3(\Omega)^{2\times 2}$ *with* $\operatorname{div}\mathbf{u}^0 = 0$.

Hypothesis 6. *(more regular external force)*
It holds that

$$(\mathbf{f}_h, \mathbf{F}_h) \in l^\infty(L^2(\Omega)^2) \times l^\infty(L^2(\Omega)^{2\times 2}),\ (\bar{D}_{\Delta t}\mathbf{f}_h, \bar{D}_{\Delta t}\mathbf{F}_h) \in l^2(L^2(\Omega)^2) \times l^2(L^2(\Omega)^{2\times 2}).$$

Let us recall the backward difference operator given by

$$\bar{D}_{\Delta t}\, x^n := \frac{x^n - x^{n-1}}{\Delta t}.$$

We can now formulate the main result of this section.

Theorem 7.16. *(higher a priori estimates for the semi-implicit scheme)*
Suppose Hypotheses 2, 4, 5 and 6 hold. We assume condition (6.14) *on* Δt *is satisfied. Further, we assume that time increment is enough small and depends on the discrete solution from the previous time step, cf.* (7.74)*. Let* $(\mathbf{u}_h^0, p_h^0) \in V_h \times Q_h$ *be the Stokes projection of* $(\mathbf{u}^0, 0)$ *and* $\mathbf{C}_h^0 \in W_h$ *be the Lagrange interpolation of* $\mathbf{C}^0$*. Then for the solution* $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ *of the scheme* (7.2) *there exist constants* c_1^h*,* c_2^h *and* c_3^h *of* (7.3d) *such that*

$$\begin{aligned}
&(i) && \left\|\bar{D}_{\Delta t}\mathbf{u}_h\right\|_{l^\infty(L^2)}, \ \sqrt{\nu}\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h)\right\|_{l^2(L^2)}, \ \left\|\bar{D}_{\Delta t}\mathbf{C}_h\right\|_{l^\infty(L^2)}, \\
& && \qquad |\bar{D}_{\Delta t}p_h|_{l^2(|.|_h)}, \ \left\|\mathrm{tr}\,\mathbf{C}_h^{-1}\bar{D}_{\Delta t}\mathbf{C}_h\right\|_{l^2(L^2)} \le c_1^h \\
&(ii) && \|p_h\|_{l^2(L^2)} \le c_2^h \\
&(iii) && \left\|\bar{D}_{\Delta t}\mathbf{C}_h\right\|_{l^2(L^2)}, \ \sqrt{\varepsilon}\,\|\nabla\mathbf{C}_h\|_{l^\infty(L^2)} \le c_3^h.
\end{aligned}$$

The result of Theorem 7.16 is divided into three parts. The bounds in (ii) and (iii) are obtained due to the estimates from part (i) combined with the first a priori estimates from the previous section. The proof of the first part requires the result of the subsequent lemma. The complete proof of Theorem 7.16 comes directly after the proof of this partial result.

Lemma 7.17.
Under the assumptions of Theorem 7.16 it holds that

$$\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 \le C, \tag{7.50}$$

where $C = C\left(h_0^2, c_0, c_1, c_i, c_f, \tilde{c}, T, \nu^2, \nu, 1/\nu, \varepsilon^2/\nu, \varepsilon^2, 1/\varepsilon\right)$ *is a positive constant independent of the discretization parameters.*

Proof. Firstly, let us note that $p_h^0 \in Q_h$ from the Stokes projection $(\mathbf{u}_h^0, p_h^0)$ of $(\mathbf{u}^0, 0)$ satisfies the equality

$$b\left(\mathbf{u}_h^0, q_h\right) - \mathcal{S}_h\left(p_h^0, q_h\right) = 0, \ \forall\, q_h \in Q_h. \tag{7.51}$$

Indeed, by the definition (7.9) we have

$$\nu\, a_u\left(\mathbf{u}^0, \mathbf{v}_h\right) + b\left(\mathbf{u}^0, q_h\right) = \nu\, a_u\left(\mathbf{u}_h^0, \mathbf{v}_h\right) + b\left(\mathbf{u}_h^0, q_h\right) + b\left(\mathbf{v}_h, p_h^0\right) - \mathcal{S}_h\left(p_h^0, q_h\right)$$

for all $(\mathbf{v}_h, q_h) \in V_h \times Q_h$. Let $\mathbf{v}_h = \mathbf{0}$. Then

$$0 = b\left(\mathbf{u}^0, q_h\right) = b\left(\mathbf{u}_h^0, q_h\right) - \mathcal{S}_h\left(p_h^0, q_h\right), \ \forall\, q_h \in Q_h,$$

since $\mathbf{u}^0$ is assumed to be solenoidal. Subject to equality (7.51) we eliminate the bilinear form b from the numerical scheme. Indeed, equation (7.2a) for $\mathbf{v}_h = \mathbf{0} \in V_h$ together with (7.51) yield, for $n = 0, \ldots, N_T$,

$$b\Big(\mathbf{u}_h^n, q_h\Big) - \mathcal{S}_h\Big(p_h^n, q_h\Big) = 0, \ \forall q_h \in Q_h.$$

That means, the identity

$$b\left(\bar{D}_{\Delta t}\mathbf{u}_h^n, q_h\right) - \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^n, q_h\right) = 0, \ \forall q_h \in Q_h, \qquad n = 1, \ldots, N_T, \tag{7.52}$$

holds true. Substituting $(\bar{D}_{\Delta t}\mathbf{u}_h^n, 0)$ into $(\mathbf{v}_h, q_h)$ in (7.2a) and using the identity (7.52) with $q_h = p_h^n$, we have, for $n = 1, \ldots, N_T$,

$$\begin{aligned}\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) + \nu\, a_u\left(\mathbf{u}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^n, p_h^n\right) +& \\ + \left(\mathrm{tr}\,\mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right) = \left(\mathbf{f}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right).&\end{aligned} \tag{7.53}$$

Not surprisingly the proof of Lemma 7.17 is done in two steps. In order to treat the nonlinear terms arising from the upper convected derivative of the conformation tensor and from the divergence of the elastic stress tensor we first need some bounds for the discrete solution of the velocity and the trace of the conformation tensor.

Step 1

Let $n = 1$. We consider the equation (7.53) derived above

$$\begin{aligned}\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0\circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \nu\, a_u\left(\mathbf{u}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^1\right) +& \\ + \left(\mathrm{tr}\,\mathbf{C}_h^1\ \mathbf{C}_h^0, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^1\right) = \left(\mathbf{f}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right).&\end{aligned} \tag{7.54}$$

Similarly, we substitute $\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}$ into $\mathbf{D}_h$ in (7.2b) and multiply the equation by one half to get

$$\begin{aligned}\frac{1}{2}\left(\frac{\mathrm{tr}\,\mathbf{C}_h^1 - \mathrm{tr}\,\mathbf{C}_h^0\circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) + \frac{\varepsilon}{2}\, a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) +& \\ + \frac{1}{2}\left((\mathrm{tr}\,\mathbf{C}_h^0)^2\mathrm{tr}\,\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) =& \\ = \left(\mathrm{tr}\,\mathbf{C}_h^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) + \left(\nabla\mathbf{u}_h^1 : \mathbf{C}_h^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) + \frac{1}{2}\left(\mathrm{tr}\,\mathbf{F}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right).&\end{aligned} \tag{7.55}$$

The sum of equations (7.54) and (7.55) reads

$$\begin{aligned}\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0\circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \frac{1}{2}\left(\frac{\mathrm{tr}\,\mathbf{C}_h^1 - \mathrm{tr}\,\mathbf{C}_h^0\circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) +& \\ + \nu\, a_u\left(\mathbf{u}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^1\right) +& \\ + \frac{\varepsilon}{2}\, a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) + \frac{1}{2}\left((\mathrm{tr}\,\mathbf{C}_h^0)^2\mathrm{tr}\,\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) =& \\ = -\left(\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{C}_h^0, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \left(\nabla\mathbf{u}_h^1 : \mathbf{C}_h^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) +& \\ + \left(\mathrm{tr}\,\mathbf{C}_h^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) + \left(\mathbf{f}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \frac{1}{2}\left(\mathrm{tr}\,\mathbf{F}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right).&\end{aligned} \tag{7.56}$$

We shall find the appropriate estimates of the integrals in (7.56). The first two terms on the left-hand side are estimated using the Hölder, the Young inequalities and (7.4b) as

$$\begin{aligned}
\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0 \circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) &= \left(\bar{D}_{\Delta t}\mathbf{u}_h^1 + \frac{\mathbf{u}_h^0 - \mathbf{u}_h^0 \circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) \\
&= \left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \frac{1}{\Delta t}\left(\mathbf{u}_h^0 - \mathbf{u}_h^0 \circ X_1^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) \\
&\geq \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 - c_1\left\|\mathbf{u}_h^0\right\|_{1,2}^2 \\
\frac{1}{2}\left(\frac{\mathrm{tr}\,\mathbf{C}_h^1 - \mathrm{tr}\,\mathbf{C}_h^0 \circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) &\geq \frac{1}{4}\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_2^2 - c_1\left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_{1,2}^2 .
\end{aligned}$$

The next two integrals on the left-hand side of (7.56) can be rewritten as

$$\begin{aligned}
\nu\, a_u\left(\mathbf{u}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^1\right) &= \nu\Delta t\, a_u\left(\bar{D}_{\Delta t}\mathbf{u}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \nu\, a_u\left(\mathbf{u}_h^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \\
&\quad + \Delta t\, \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, \bar{D}_{\Delta t}p_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^0\right) \\
&= 2\nu\Delta t\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^1)\right\|_2^2 + \delta_0\Delta t|\bar{D}_{\Delta t}p_h^1|_h^2 + \\
&\quad + \nu\, a_u\left(\mathbf{u}_h^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^0\right) .
\end{aligned} \tag{7.57}$$

We know that $(\mathbf{u}_h^0, p_h^0)$ is the Stokes projection of $(\mathbf{u}^0, 0)$. Hence, by equalities (7.51), (7.52) and by (7.9) we get

$$\begin{aligned}
\nu\, a_u\left(\mathbf{u}_h^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^0\right) &= \nu\, a_u\left(\mathbf{u}_h^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + b(\bar{D}_{\Delta t}\mathbf{u}_h^1, p_h^0) + \\
&\quad + b(\mathbf{u}_h^0, \bar{D}_{\Delta t}p_h^1) - \mathcal{S}_h\left(p_h^0, \bar{D}_{\Delta t}p_h^1\right) = \\
&= \nu\, a_u\left(\mathbf{u}^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right),
\end{aligned}$$

where the divergence freedom of the initial velocity was taken into account. As a consequence, equality (7.57) leads to

$$\begin{aligned}
\nu\, a_u\left(\mathbf{u}_h^1, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^1, p_h^1\right) &= 2\nu\Delta t\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^1)\right\|_2^2 + \delta_0\Delta t|\bar{D}_{\Delta t}p_h^1|_h^2 + \\
&\quad + \nu\, a_u\left(\mathbf{u}^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) .
\end{aligned}$$

Further, since we assume $\mathbf{u}^0 \in V \cap H^2(\Omega)^2$ we can write

$$\begin{aligned}
\nu\, a_u\left(\mathbf{u}^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) &= 2\nu \sum_{K\in\mathcal{T}_h}\int_K \mathrm{D}(\mathbf{u}^0) : \mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^1) \\
&= -2\nu \sum_{K\in\mathcal{T}_h}\int_K \Delta\mathbf{u}^0 : \bar{D}_{\Delta t}\mathbf{u}_h^1 + \underbrace{\left(\bar{D}_{\Delta t}\mathbf{u}_h^1 \cdot \nabla\right)\left(\nabla\cdot\mathbf{u}^0\right)}_{=0} + \\
&\quad \underbrace{+ 2\nu \sum_{\partial K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}\cdot\nabla\mathbf{u}^0\cdot\bar{D}_{\Delta t}\mathbf{u}_h^1 + \left(\bar{D}_{\Delta t}\mathbf{u}_h^1\cdot\nabla\right)\mathbf{u}^0\cdot\mathbf{n}}_{=0} \\
&= -2\nu\left(\Delta\mathbf{u}^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right),
\end{aligned}$$

which is now easily estimated

$$\nu\, a_u\left(\mathbf{u}^0, \bar{D}_{\Delta t}\mathbf{u}_h^1\right) \leq \frac{1}{12}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + c\nu^2\left\|\Delta\mathbf{u}^0\right\|_2^2.$$

In order to estimate the bilinear form a_c in (7.56) we rewrite this term as

$$a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) = \Delta t\, a_c\left(\bar{D}_{\Delta t}\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) + a_c\left(\mathbf{C}_h^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right).$$

Since $\nabla\mathbf{C}_h^0$ is not continuous on Ω, the integrals over ∂K would not vanish and the integration by parts would not lead to the desired estimate. Therefore, we make use of the error of the Lagrange interpolation operator. It means we write

$$\begin{aligned} a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) = \Delta t\left\|\nabla\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_2^2 + a_c\left(\mathbf{C}^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) + \\ + a_c\left(\mathbf{C}_h^0 - \mathbf{C}^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right). \end{aligned} \tag{7.58}$$

Analogously as for a_u we have

$$a_c\left(\mathbf{C}^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) = -\left(\Delta\mathbf{C}^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right).$$

For the last term on the right-hand side of (7.58) we get

$$\begin{aligned} a_c\left(\mathbf{C}_h^0 - \mathbf{C}^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) &= \sum_{K\in\mathcal{T}_h}\int_K \nabla(\mathbf{C}_h^0 - \mathbf{C}^0) : \nabla\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I} \\ &= -\sum_{K\in\mathcal{T}_h}\int_K(\underbrace{\Delta\mathbf{C}_h^0}_{=0} - \Delta\mathbf{C}^0) : \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I} + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\mathbf{n}\cdot\nabla(\mathbf{C}_h^0 - \mathbf{C}^0)\right)\cdot\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I} \\ &= \left(\Delta\mathbf{C}^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\mathbf{n}\cdot\nabla(\mathbf{C}_h^0 - \mathbf{C}^0)\right)\cdot\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}. \end{aligned}$$

Then equality (7.58) yields

$$a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}\right) = \Delta t\left\|\nabla\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_2^2 + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\mathbf{n}\cdot\nabla(\mathbf{C}_h^0 - \mathbf{C}^0)\right)\cdot\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I}.$$

Due to the interpolation error (7.8), the consequence of the multiplicative trace inequality (7.5) and the trace inequality (7.6), i.e. (7.7), we are able to bound the integrals over the edges ∂K in an appropriate way. Indeed,

$$\begin{aligned} \int_{\partial K}\left(\mathbf{n}\cdot\nabla(\mathbf{C}_h^0 - \mathbf{C}^0)\right)\cdot\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\,\mathbf{I} &\leq c\left\|\mathrm{tr}\,\mathbf{C}_h^0 - \mathrm{tr}\,\mathbf{C}^0\right\|_{1,2,\partial K}\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_{2,\partial K} \\ &\leq c\,h_K\left\|\mathrm{tr}\,\mathbf{C}^0\right\|_{2,2,\partial K} h_K^{-1/2}\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_{2,K} \\ &\leq c\,h_K^{1/2}\left\|\mathbf{C}^0\right\|_{2,2,\partial K}\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_{2,K}. \end{aligned}$$

We assume $\mathbf{C}^0 \in H^3(\Omega)^{2\times 2}$. Then the multiplicative trace inequality, cf. Lemma 7.3, applied for $\|\mathbf{C}^0\|_{2,2,\partial K}$ yields

$$\left\|\mathbf{C}^0\right\|_{2,2,\partial K}^2 \leq c\left(\left\|\mathbf{C}^0\right\|_{2,2,K}\left|\mathbf{C}^0\right|_{3,2,K} + h_K^{-1}\left\|\mathbf{C}^0\right\|_{2,2,K}^2\right).$$

Consequently, by the Young inequality we get for a $\alpha > 0$,

$$h_K^{1/2} \left\| \mathbf{C}^0 \right\|_{2,2,\partial K} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_{2,K} \leq \frac{c}{\alpha} \left(h_K \left\| \mathbf{C}^0 \right\|_{2,2,K} |\mathbf{C}^0|_{3,2,K} + \left\| \mathbf{C}^0 \right\|_{2,2,K}^2 \right) + \\ + \alpha \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_{2,K}^2 .$$

This results in

$$\sum_{K \in \mathcal{T}_h} c\, h_K^{1/2} \left\| \mathbf{C}^0 \right\|_{2,2,\partial K} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_{2,K} \leq \frac{c}{\alpha} \sum_{K \in \mathcal{T}_h} h_K \left\| \mathbf{C}^0 \right\|_{3,2,K}^2 + \left\| \mathbf{C}^0 \right\|_{2,2,K}^2 + \\ + \alpha \sum_{K \in \mathcal{T}_h} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_{2,K}^2 .$$

Finally, we get for appropriate α

$$\frac{\varepsilon}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\mathbf{n} \cdot \nabla (\mathbf{C}_h^0 - \mathbf{C}^0) \right) \cdot \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \, \mathbf{I} \leq c\, \varepsilon^2 \, (1 + h) \left\| \mathbf{C}^0 \right\|_{3,2}^2 + \frac{1}{40} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_2^2 . \tag{7.59}$$

Using the Hölder and the Young inequalities we get for the last integral on the left-hand side of (7.56) that

$$\frac{1}{2} \left((\mathrm{tr}\, \mathbf{C}_h^0)^2 \mathrm{tr}\, \mathbf{C}_h^1, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) \leq \frac{1}{40} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_2^2 + c \left\| \mathrm{tr}\, \mathbf{C}_h^1 \right\|_2^2 \left\| \mathrm{tr}\, \mathbf{C}_h^0 \right\|_0^4 .$$

We proceed with the estimate of the integrals on the right-hand side. Employing the identity

$$- \left(\mathrm{tr}\, \mathbf{C}_h^1 \, \mathbf{C}_h^0, \nabla \bar{D}_{\Delta t} \mathbf{u}_h^1 \right) = \left(\bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \, \mathbf{C}_h^0, \nabla \mathbf{u}_h^0 - \nabla \mathbf{u}_h^1 \right) - \left(\mathrm{tr}\, \mathbf{C}_h^0 \, \mathbf{C}_h^0, \nabla \bar{D}_{\Delta t} \mathbf{u}_h^1 \right)$$

we rewrite the right-hand side of (7.56) as

$$\begin{aligned} &\left(\mathrm{tr}\, \mathbf{C}_h^0, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) - \left(\mathrm{tr}\, \mathbf{C}_h^1 \, \mathbf{C}_h^0, \nabla \bar{D}_{\Delta t} \mathbf{u}_h^1 \right) + \left(\nabla \mathbf{u}_h^1 : \mathbf{C}_h^0, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) + \\ &\qquad + \left(\mathbf{f}_h^1, \bar{D}_{\Delta t} \mathbf{u}_h^1 \right) + \frac{1}{2} \left(\mathrm{tr}\, \mathbf{F}_h^1, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) = \\ &= \left(\mathrm{tr}\, \mathbf{C}_h^0, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) - \left(\mathrm{tr}\, \mathbf{C}_h^0 \, \mathbf{C}_h^0, \nabla \bar{D}_{\Delta t} \mathbf{u}_h^1 \right) + \left(\bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \, \mathbf{C}_h^0, \nabla \mathbf{u}_h^0 \right) + \\ &\qquad + \left(\mathbf{f}_h^1, \bar{D}_{\Delta t} \mathbf{u}_h^1 \right) + \frac{1}{2} \left(\mathrm{tr}\, \mathbf{F}_h^1, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) . \end{aligned}$$

Now, we shall estimate the resulting integrals. We apply the Hölder and the Young inequalities to get

$$\begin{aligned} \left(\mathrm{tr}\, \mathbf{C}_h^0, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) &\leq \frac{1}{40} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_2^2 + c \left\| \mathrm{tr}\, \mathbf{C}_h^0 \right\|_2^2 \\ \left(\bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \, \mathbf{C}_h^0, \nabla \mathbf{u}_h^0 \right) &\leq \frac{1}{40} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_2^2 + c \left\| \nabla \mathbf{u}_h^0 \right\|_2^2 \left\| \mathbf{C}_h^0 \right\|_0^2 \\ \left(\mathbf{f}_h^1, \bar{D}_{\Delta t} \mathbf{u}_h^1 \right) &\leq \frac{1}{12} \left\| \bar{D}_{\Delta t} \mathbf{u}_h^1 \right\|_2^2 + c \left\| \mathbf{f}_h^1 \right\|_2^2 \\ \frac{1}{2} \left(\mathrm{tr}\, \mathbf{F}_h^1, \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right) &\leq \frac{1}{40} \left\| \bar{D}_{\Delta t} \mathrm{tr}\, \mathbf{C}_h^1 \right\|_2^2 + c \left\| \mathrm{tr}\, \mathbf{F}_h^1 \right\|_2^2 . \end{aligned}$$

The last integral is treated analogously as the bilinear forms a_u and a_c, i.e. applying the integration by parts. However, $\operatorname{tr}\mathbf{C}_h^0\mathbf{C}_h^0$ is continuous on Ω, and hence, the integrals over inner edges vanish. Due to the homogeneous Dirichlet boundary condition of $\mathbf{u}_h$ we obtain directly

$$
\begin{aligned}
\left(\operatorname{tr}\mathbf{C}_h^0\,\mathbf{C}_h^0, \nabla \bar{D}_{\Delta t}\mathbf{u}_h^1\right) &= \sum_{K\in\mathcal{T}_h}\int_K \operatorname{tr}\mathbf{C}_h^0\,\mathbf{C}_h^0 : \nabla \bar{D}_{\Delta t}\mathbf{u}_h^1 \\
&= -\sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}\left(\operatorname{tr}\mathbf{C}_h^0\,\mathbf{C}_h^0\right)\cdot \bar{D}_{\Delta t}\mathbf{u}_h^1 + \underbrace{\sum_{\partial K\in\mathcal{T}_h}\int_{\partial K}\operatorname{tr}\mathbf{C}_h^0\,\mathbf{C}_h^0\cdot \bar{D}_{\Delta t}\mathbf{u}_h^1\cdot\mathbf{n}}_{=0} \\
&= -\left(\operatorname{div}\left(\operatorname{tr}\mathbf{C}_h^0\,\mathbf{C}_h^0\right), \bar{D}_{\Delta t}\mathbf{u}_h^1\right) \\
&\le \frac{1}{12}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + c\left\|\nabla\mathbf{C}_h^0\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2 .
\end{aligned}
$$

Taking into account all the estimates of both the left-hand and the right-hand sides of (7.56) we arrive at the inequality

$$
\begin{aligned}
&\frac{1}{4}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \frac{1}{8}\left\|\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^1\right\|_2^2 + 2\nu\Delta t\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^1)\right\|_2^2 + \delta_0\Delta t|\bar{D}_{\Delta t}p_h^1|_h^2 + \frac{\varepsilon}{2}\Delta t\left\|\nabla\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^1\right\|_2^2 \\
&\qquad\le c_1\left\|\mathbf{u}_h^0\right\|_{1,2}^2 + c_1\left\|\operatorname{tr}\mathbf{C}_h^0\right\|_{1,2}^2 + c\left\|\operatorname{tr}\mathbf{C}_h^0\right\|_2^2 + c\,\nu^2\left\|\Delta\mathbf{u}^0\right\|_2^2 + \\
&\qquad\quad + c\,\varepsilon^2\left(h_0^2+1\right)\left\|\mathbf{C}^0\right\|_{3,2}^2 + c\left\|\nabla\mathbf{u}_h^0\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2 + c\left\|\nabla\mathbf{C}_h^0\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2 + \\
&\qquad\quad + c\left\|\operatorname{tr}\mathbf{C}_h^1\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^4 + c\left\|\mathbf{f}_h^1\right\|_2^2 + c\left\|\operatorname{tr}\mathbf{F}_h^1\right\|_2^2 .
\end{aligned}
$$

Obviously, by (7.13), the right-hand side of the above inequality can be seen as a constant dependent only on the data. Indeed,

$$
\begin{aligned}
&\frac{1}{4}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \frac{1}{8}\left\|\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^1\right\|_2^2 + 2\nu\Delta t\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^1)\right\|_2^2 + \delta_0\Delta t|\bar{D}_{\Delta t}p_h^1|_h^2 + \frac{\varepsilon}{2}\Delta t\left\|\nabla\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^1\right\|_2^2 \\
&\le c\left(h_0^2, c_1, \nu^2, \varepsilon^2, \left\|\mathbf{u}^0\right\|_{2,2}^2, \left\|\mathbf{C}^0\right\|_{3,2}^2, \|\operatorname{tr}\mathbf{C}_h\|_{l^\infty(L^2)}^2, \|\mathbf{f}_h\|_{l^\infty(L^2)}^2, \|\mathbf{F}_h\|_{l^\infty(L^2)}^2\right) \\
&\le c\left(h_0^2, c_0, c_1, c_i, c_f, \tilde{c}, \nu^2, \varepsilon^2\right) =: C_1
\end{aligned}
\tag{7.60}
$$

Let us recall the property of the Lagrange interpolation, see (i) in Proposition 7.9, that allows us to write

$$
\left\|\mathbf{C}_h^0\right\|_{1,2}^2 \le c\left\|\mathbf{C}^0\right\|_{1,2}^2, \qquad \left\|\mathbf{C}_h^0\right\|_0^2 \le c\left\|\mathbf{C}^0\right\|_0^2 \le c\left\|\mathbf{C}^0\right\|_{2,2}^2 . \tag{7.61}
$$

Step 2

The second step is to find the analogous estimates for $\mathbf{C}_h^1$. To this end, we consider equation (7.2b) with $\bar{D}_{\Delta t}\mathbf{C}_h^1$ as the test function

$$
\begin{aligned}
&\left(\frac{\mathbf{C}_h^1 - \mathbf{C}_h^0\circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) + \varepsilon\, a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) + \left((\operatorname{tr}\mathbf{C}_h^0)^2\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) = \\
&\quad = \left(\operatorname{tr}\mathbf{C}_h^0, \bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^1\right) + \left((\nabla\mathbf{u}_h^1)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\mathbf{u}_h^1)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) + \left(\mathbf{F}_h^1, \bar{D}_{\Delta t}\mathbf{C}_h^1\right).
\end{aligned}
\tag{7.62}
$$

Once more, we need to estimate all the integrals in (7.62). Similarly as in the first step, we have, by (7.4b),

$$\left(\frac{\mathbf{C}_h^1 - \mathbf{C}_h^0 \circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) \geq \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 - c_1\left\|\mathbf{C}_h^0\right\|_{1,2}^2.$$

Analogously, the error of the Lagrange interpolation operator and inequalities (7.5) and (7.7) yield

$$\begin{aligned}
\varepsilon\, a_c\left(\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) &= \varepsilon\,\Delta t\, a_c\left(\bar{D}_{\Delta t}\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) + \varepsilon\, a_c\left(\mathbf{C}_h^0, \bar{D}_{\Delta t}\mathbf{C}_h^1\right)\\
&= \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \varepsilon\, a_c\left(\mathbf{C}^0, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) + \varepsilon\, a_c\left(\mathbf{C}_h^0 - \mathbf{C}^0, \bar{D}_{\Delta t}\mathbf{C}_h^1\right)\\
&\geq \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 - \frac{1}{24}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 - c\varepsilon^2\left(h_0^2+1\right)\left\|\mathbf{C}^0\right\|_{3,2}^2.
\end{aligned}$$

See the calculations (7.58) - (7.59) for the details. The Hölder and the Young inequalities give the estimate of the last integral on the left-hand side of (7.62). Indeed, we have

$$\left((\mathrm{tr}\,\mathbf{C}_h^0)^2\mathbf{C}_h^1, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) \leq \frac{1}{24}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + c\left\|\mathbf{C}_h^1\right\|_2^2\left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_0^4.$$

In the same manner, we estimate the first and the third integral on the right-hand side

$$\begin{aligned}
\left(\mathrm{tr}\,\mathbf{C}_h^0, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right) &\leq \frac{1}{24}\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^1\right\|_2^2 + c\left\|\mathrm{tr}\,\mathbf{C}_h^0\right\|_2^2\\
\left(\mathbf{F}_h^1, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) &\leq \frac{1}{24}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + c\left\|\mathbf{F}_h^1\right\|_2^2.
\end{aligned}$$

Further, we rewrite the last integral in an appropriate way

$$\begin{aligned}
&\left((\nabla\mathbf{u}_h^1)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\mathbf{u}_h^1)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) =\\
&= \left((\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1)^T, \mathbf{C}_h^1 - \mathbf{C}_h^0\right) + \left((\nabla\mathbf{u}_h^0)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\mathbf{u}_h^0)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right)\\
&= \Delta t\left((\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) + \left((\nabla\mathbf{u}_h^0)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\mathbf{u}_h^0)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right),
\end{aligned}$$

and estimate the resulting integrals

$$\begin{aligned}
\Delta t\left((\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) &\leq c\Delta t\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2\left\|\mathbf{C}_h^0\right\|_0\\
&\leq \frac{1}{24}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + c\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2,
\end{aligned}$$

where we have used that $(\Delta t)^2 < \Delta t$ for $\Delta t < 1$, and

$$\left((\nabla\mathbf{u}_h^0)\mathbf{C}_h^0 + \mathbf{C}_h^0(\nabla\mathbf{u}_h^0)^T, \bar{D}_{\Delta t}\mathbf{C}_h^1\right) \leq \frac{1}{24}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + c\left\|\nabla\mathbf{u}_h^0\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2.$$

Finally, the latter estimates of the terms in (7.62) yield

$$\begin{aligned}
&\frac{1}{4}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 \leq\\
&\qquad \leq c_1\left\|\mathbf{C}_h^0\right\|_{1,2}^2 + c\left\|\mathbf{C}_h^0\right\|_2^2 + c\,\varepsilon^2\left(h_0^2+1\right)\left\|\mathbf{C}^0\right\|_{3,2}^2 + c\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2 +\\
&\qquad + c\left\|\mathbf{C}_h^1\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^4 + c\left\|\nabla\mathbf{u}_h^0\right\|_2^2\left\|\mathbf{C}_h^0\right\|_0^2 + c\left\|\mathbf{F}_h^1\right\|_2^2.
\end{aligned}$$

A priori estimates (7.17), and (7.60) from the first step of this proof together with (7.61) and the Korn inequality yield

$$
\begin{aligned}
\frac{1}{4}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 &+ \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 \\
&\le c\left(h_0^2, c_1, \varepsilon^2, C_1/\nu, \left\|\mathbf{u}_h^0\right\|_{1,2}^2, \left\|\mathbf{C}^0\right\|_{3,2}^2, \|\mathbf{C}_h\|_{l^\infty(L^2)}^2, \|\mathbf{F}_h\|_{l^\infty(L^2)}^2\right) \\
&\le c\left(h_0^2, c_0, c_1, c_i, c_f, (c_2^s)^2, C_1/\nu, \varepsilon^2\right) =: C_2.
\end{aligned}
\tag{7.63}
$$

Combining (7.60) and (7.63), we finally obtain the desired estimate

$$
\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 \le C_1 + C_2 =: C,
\tag{7.64}
$$

where $C = C\left(h_0^2, c_0, c_1, c_i, c_f, \tilde{c}, T, \nu^2, \nu, 1/\nu, \varepsilon^2/\nu, \varepsilon^2, 1/\varepsilon\right)$.

□

The proof of Theorem 7.16 follows. In order to show the higher energy estimates we shall perform an analogous procedure as we did for the Galerkin approximations to get more regular weak solution, cf. Section 4.3. Basically, we shall derive the discrete formulation in time and test it with the appropriate discrete time derivatives of its solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$.

Proof of Theorem 7.16 (i)

Let us recall that equality (7.52) allows us to rewrite equation (7.2a) as (7.53), i.e. for $n = 1, \dots, N_T$ we have

$$
\begin{aligned}
\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) &+ \nu\, a_u\left(\mathbf{u}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^n, p_h^n\right) + \\
&+ \left(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right) = \left(\mathbf{f}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right).
\end{aligned}
\tag{7.65}
$$

Similarly, by substituting $(\bar{D}_{\Delta t}\mathbf{u}_h^{n+1}, 0)$ into $(\mathbf{v}_h, q_h)$ in equation (7.2a) and employing equality (7.52) with $q_h = p_h^{n-1}$, for $n = 2, \dots, N_T$, we obtain

$$
\begin{aligned}
\left(\frac{\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2}\circ X_1^{n-1}}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) &+ \nu\, a_u\left(\mathbf{u}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) + \mathcal{S}_h\left(\bar{D}_{\Delta t}p_h^n, p_h^{n-1}\right) + \\
&+ \left(\operatorname{tr}\mathbf{C}_h^{n-1}\ \mathbf{C}_h^{n-2}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right) = \left(\mathbf{f}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right).
\end{aligned}
\tag{7.66}
$$

Let us multiply both (7.65) and (7.66) by $1/\Delta t$ and take the difference of the resulting equations. Hence, we have for $n = 2, \dots, N_T$

$$
\begin{aligned}
&\left(\frac{\bar{D}_{\Delta t}\mathbf{u}_h^n - \bar{D}_{\Delta t}\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) + \frac{1}{\Delta t}\left(\frac{\mathbf{u}_h^{n-2}\circ X_1^{n-1} - \mathbf{u}_h^{n-2}\circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) + \\
&\quad + 2\nu\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h^n)\right\|_2^2 + \delta_0|\bar{D}_{\Delta t}p_h^n|_h^2 = \left(\bar{D}_{\Delta t}\mathbf{f}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) - \\
&\quad - \left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right) - \left(\operatorname{tr}\mathbf{C}_h^{n-1}\ \bar{D}_{\Delta t}\mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right).
\end{aligned}
\tag{7.67}
$$

Now, we repeat the same procedure for equation (7.2b) with the test function $\bar{D}_{\Delta t}\mathbf{C}_h^n$. For $n = 1, \ldots, N_T$, we obtain

$$\begin{aligned}
&\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \varepsilon\, a_c\left(\mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) = \\
&\quad = \left((\nabla \mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla \mathbf{u}_h^n)^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) - \left(\left(\operatorname{tr} \mathbf{C}_h^{n-1}\right)^2 \mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \\
&\quad + \left(\operatorname{tr} \mathbf{C}_h^{n-1}\, \mathbf{I}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \left(\mathbf{F}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right).
\end{aligned} \tag{7.68}$$

Analogously, we can write for $n = 2, \ldots, N_T$

$$\begin{aligned}
&\left(\frac{\mathbf{C}_h^{n-1} - \mathbf{C}_h^{n-2} \circ X_1^{n-1}}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \varepsilon\, a_c\left(\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) = \\
&\quad = \left((\nabla \mathbf{u}_h^{n-1})\mathbf{C}_h^{n-2} + \mathbf{C}_h^{n-2}(\nabla \mathbf{u}_h^{n-1})^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) - \left(\left(\operatorname{tr} \mathbf{C}_h^{n-2}\right)^2 \mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \\
&\quad + \left(\operatorname{tr} \mathbf{C}_h^{n-2}\, \mathbf{I}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \left(\mathbf{F}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right).
\end{aligned} \tag{7.69}$$

Taking the difference of equations (7.68) and (7.69) multiplied by $1/\Delta t$ we get for $n = 2, \ldots, N_T$

$$\begin{aligned}
&\left(\frac{\bar{D}_{\Delta t}\mathbf{C}_h^n - \bar{D}_{\Delta t}\mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \frac{1}{\Delta t}\left(\frac{\mathbf{C}_h^{n-2} \circ X_1^{n-1} - \mathbf{C}_h^{n-2} \circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \\
&\qquad + \varepsilon \left\|\nabla \bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \frac{1}{\Delta t}\left((\operatorname{tr} \mathbf{C}_h^{n-1})^2\mathbf{C}_h^n - (\operatorname{tr} \mathbf{C}_h^{n-2})^2\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) = \\
&\qquad = \left((\nabla \bar{D}_{\Delta t}\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla \bar{D}_{\Delta t}\mathbf{u}_h^n)^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \\
&\qquad + \left((\nabla \mathbf{u}_h^{n-1})\bar{D}_{\Delta t}\mathbf{C}_h^{n-1} + \bar{D}_{\Delta t}\mathbf{C}_h^{n-1}(\nabla \mathbf{u}_h^{n-1})^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \\
&\qquad + \left(\bar{D}_{\Delta t}\operatorname{tr} \mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\operatorname{tr} \mathbf{C}_h^n\right) + \left(\bar{D}_{\Delta t}\mathbf{F}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right).
\end{aligned} \tag{7.70}$$

The estimates of the terms on the left-hand sides of (7.67) and (7.70) follow. The first pair of integrals is bounded due to (7.4b). Indeed, the following inequalities hold true

$$\begin{aligned}
&\left(\frac{\bar{D}_{\Delta t}\mathbf{u}_h^n - \bar{D}_{\Delta t}\mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) = \\
&\qquad = \frac{1}{2\Delta t}\left(\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 - \left\|\bar{D}_{\Delta t}\mathbf{u}_h^{n-1} \circ X_1^n\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{u}_h^n - \bar{D}_{\Delta t}\mathbf{u}_h^{n-1} \circ X_1^n\right\|_2^2\right) \\
&\qquad \geq \frac{1}{2\Delta t}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 - \frac{1 + c_1\Delta t}{2\Delta t}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^{n-1}\right\|_2^2 \\
&\qquad \geq \bar{D}_{\Delta t}\left(\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2\right) - c_1\left\|\bar{D}_{\Delta t}\mathbf{u}_h^{n-1}\right\|_2^2,
\end{aligned}$$

and analogously,

$$\left(\frac{\bar{D}_{\Delta t}\mathbf{C}_h^n - \bar{D}_{\Delta t}\mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) \geq \bar{D}_{\Delta t}\left(\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\right) - c_1\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2.$$

The next pair of integrals is treated in a similar way. By (7.4c), we get the estimates

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\frac{\mathbf{u}_h^{n-2}\circ X_1^{n-1}-\mathbf{u}_h^{n-2}\circ X_1^{n}}{\Delta t}, \bar{D}_{\Delta t}\mathbf{u}_h^n\right)\\
&\qquad\qquad\geq -\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2-\frac{1}{2(\Delta t)^4}\left\|\mathbf{u}_h^{n-2}\circ X_1^{n-1}-\mathbf{u}_h^{n-2}\circ X_1^{n}\right\|_2^2\\
&\qquad\qquad\geq -\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2-c_{11}\left\|\mathbf{u}_h^{n-2}\right\|_{1,2}^2,
\end{aligned}$$

and also

$$\frac{1}{\Delta t}\left(\frac{\mathbf{C}_h^{n-2}\circ X_1^{n-1}-\mathbf{C}_h^{n-2}\circ X_1^{n}}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)\geq -\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2-c_{11}\left\|\mathbf{C}_h^{n-2}\right\|_{1,2}^2.$$

Further, the following inequality holds true

$$\begin{aligned}
&\frac{1}{\Delta t}\left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n-(\operatorname{tr}\mathbf{C}_h^{n-2})^2\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)=\\
&=\left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\bar{D}_{\Delta t}\mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)+\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)+\\
&\qquad\qquad+\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^{n-2}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)\\
&\geq\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\;\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2+\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)+\\
&\qquad\qquad+\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^{n-2}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right).
\end{aligned}$$

The sum of equations (7.67) and (7.70), taking into account the above estimates of their left-hand sides, yields

$$\begin{aligned}
&\bar{D}_{\Delta t}\left(\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2+\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\right)+2\nu\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^n)\right\|_2^2+\delta_0|\bar{D}_{\Delta t}p_h^n|_h^2+\\
&\qquad\qquad+\varepsilon\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2+\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\leq\\
&\leq c_1\left(\left\|\bar{D}_{\Delta t}\mathbf{u}_h^{n-1}\right\|_2^2+\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2\right)+c_{11}\left(\left\|\mathbf{u}_h^{n-2}\right\|_{1,2}+\left\|\mathbf{C}_h^{n-2}\right\|_{1,2}\right)+\\
&+\frac{1}{2}\left(\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2+\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2\right)+\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^n\right)-\\
&-\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^n\;\mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right)-\left(\operatorname{tr}\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right)+\\
&+\left((\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n)\mathbf{C}_h^{n-1}+\mathbf{C}_h^{n-1}(\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n)^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)+\\
&+\left((\nabla\mathbf{u}_h^{n-1})\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}+\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^{n-1})^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)-\\
&-\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^{n-2}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)-\left(\bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^{n-1}\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right)+\\
&+\left(\bar{D}_{\Delta t}\mathbf{f}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right)+\left(\bar{D}_{\Delta t}\mathbf{F}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right).
\end{aligned}\tag{7.71}$$

Let us estimate the integrals on the right-hand side of (7.71) separately. We employ the Hölder, the Young inequalities, Proposition 2.1 and the interpolation inequality (2.8b).

The simplest estimates are

$$\left(\bar{D}_{\Delta t}\mathbf{f}_h^n, \bar{D}_{\Delta t}\mathbf{u}_h^n\right) \leq \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{f}_h^n\right\|_2 \tag{7.72a}$$

$$\left(\bar{D}_{\Delta t}\mathbf{F}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) \leq \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{F}_h^n\right\|_2 \tag{7.72b}$$

$$\begin{aligned}\left(\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^n\right) &\leq \left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^n\right\|_2 \\ &\leq \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 .\end{aligned} \tag{7.72c}$$

For the terms arising from the divergence of the elastic stress tensor we get

$$\begin{aligned}\left(\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right) &\leq \left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2 \left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^n\right\|_4 \left\|\mathbf{C}_h^{n-1}\right\|_4 \\ &\leq \frac{\nu}{3}\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{c}{\nu}\left(\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2 \left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\right)\left\|\mathbf{C}_h^{n-1}\right\|_4^2 \\ &\leq \frac{\nu}{3}\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{\varepsilon}{6}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \frac{c}{\nu^2\varepsilon}\left\|\mathbf{C}_h^{n-1}\right\|_4^4 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \\ &+ \frac{c}{\nu}\left\|\mathbf{C}_h^{n-1}\right\|_4^2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 ,\end{aligned} \tag{7.72d}$$

and similarly

$$\begin{aligned}\left(\mathrm{tr}\,\mathbf{C}_h^{n-1}\ \bar{D}_{\Delta t}\mathbf{C}_h^{n-1}, \nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right) &\leq \frac{\nu}{3}\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{\varepsilon}{8}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 + \\ &+ \frac{c}{\nu^2\varepsilon}\left\|\mathbf{C}_h^{n-1}\right\|_4^4 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 + \frac{c}{\nu}\left\|\mathbf{C}_h^{n-1}\right\|_4^2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 .\end{aligned} \tag{7.72e}$$

The analogous procedure works for the terms that appear due to the Oldroyd derivative of the conformation tensor. Indeed,

$$\begin{aligned}\left((\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n)^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) &\leq c\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2 \left\|\mathbf{C}_h^{n-1}\right\|_4 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_4 \\ &\leq \frac{\nu}{3}\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{\varepsilon}{6}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \frac{c}{\nu^2\varepsilon}\left\|\mathbf{C}_h^{n-1}\right\|_4^4 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \\ &+ \frac{c}{\nu}\left\|\mathbf{C}_h^{n-1}\right\|_4^2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 ,\end{aligned} \tag{7.72f}$$

and by (2.8b) applied twice,

$$\begin{aligned}\left((\nabla\mathbf{u}_h^{n-1})\bar{D}_{\Delta t}\mathbf{C}_h^{n-1} + \bar{D}_{\Delta t}\mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^{n-1})^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) &\leq c\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_4 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_4 \\ &\leq c\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^{1/2} \left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^{1/2} \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^{1/2} \left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^{1/2} + \\ &+ c\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^{1/2} \left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^{1/2} \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2 + \\ &+ c\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^{1/2} \left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^{1/2} + \\ &+ c\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2 \\ &\leq \frac{\varepsilon}{6}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \frac{\varepsilon}{8}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 + \frac{c}{\varepsilon}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \left(\frac{c}{\varepsilon} + \frac{c}{2}\right)\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 + \\ &+ \left(\frac{c}{\varepsilon} + c\right)\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \left(\frac{c}{\varepsilon} + \frac{c}{2}\right)\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2 \left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2 .\end{aligned} \tag{7.72g}$$

The estimates of the last two integrals in (7.71) read

$$
\begin{aligned}
\left(\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathbf{C}_h^{n-1},\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right) &\leq \left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right\|_2\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_4\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_4\\
&\leq \frac{1}{4}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right\|_2^2+\frac{\varepsilon}{8}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2+\frac{c}{\varepsilon}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2\left\|\mathbf{C}_h^{n-1}\right\|_4^4+\\
&+c\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2\left\|\mathbf{C}_h^{n-1}\right\|_4^2
\end{aligned}
\tag{7.72h}
$$

$$
\begin{aligned}
\left(\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\mathrm{tr}\,\mathbf{C}_h^{n-2}\,\mathbf{C}_h^{n-1},\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right) &\leq \left\|\mathbf{C}_h^{n-1}\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right\|_2\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_4\left\|\mathrm{tr}\,\mathbf{C}_h^{n-2}\right\|_4\\
&\leq \left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right\|_2\left\|\bar{D}_{\Delta t}\mathrm{tr}\,\mathbf{C}_h^{n-1}\right\|_4\left\|\mathrm{tr}\,\mathbf{C}_h^{n-2}\right\|_4\\
&\leq \frac{1}{4}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^{n}\right\|_2^2+\frac{\varepsilon}{8}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2+\frac{c}{\varepsilon}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2\left\|\mathbf{C}_h^{n-2}\right\|_4^4+\\
&+c\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2\left\|\mathbf{C}_h^{n-2}\right\|_4^2.
\end{aligned}
\tag{7.72i}
$$

Our goal is to employ the discrete Gronwall lemma. In fact, the inequality (7.71) together with (7.72) becomes

$$
\begin{aligned}
&\bar{D}_{\Delta t}\left(\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2+\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\right)+\nu\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^n)\right\|_2^2+\delta_0|\bar{D}_{\Delta t}p_h^n|_h^2+\\
&\qquad+\frac{\varepsilon}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2+\frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\leq\\
&\leq\frac{\varepsilon}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2+\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2+c_1\left\|\bar{D}_{\Delta t}\mathbf{u}_h^{n-1}\right\|_2^2+\\
&+\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\left(2+\frac{c}{\varepsilon}+\left(\frac{c}{\varepsilon}+c\right)\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2+\frac{c}{\nu^2\varepsilon}\left\|\mathbf{C}_h^{n-1}\right\|_4^4+\frac{c}{\nu}\left\|\mathbf{C}_h^{n-1}\right\|_4^2\right)+\\
&+\left\|\bar{D}_{\Delta t}\mathbf{C}_h^{n-1}\right\|_2^2\left(1+c+c_1+\frac{c}{\varepsilon}+\left(\frac{c}{\varepsilon}+c\right)\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2+\frac{c}{\nu^2\varepsilon}\left\|\mathbf{C}_h^{n-1}\right\|_4^4+\frac{c}{\nu}\left\|\mathbf{C}_h^{n-1}\right\|_4^2+\right.\\
&\qquad\left.+\frac{c}{\varepsilon}\left\|\mathbf{C}_h^{n-1}\right\|_4^4+c\left\|\mathbf{C}_h^{n-1}\right\|_4^2+\frac{c}{\varepsilon}\left\|\mathbf{C}_h^{n-2}\right\|_4^4+c\left\|\mathbf{C}_h^{n-2}\right\|_4^2\right)+\\
&+\frac{1}{2}\left(\left\|\bar{D}_{\Delta t}\mathbf{f}_h^n\right\|_2^2+\left\|\bar{D}_{\Delta t}\mathbf{F}_h^n\right\|_2^2\right)+c_{11}\left(\left\|\mathbf{u}_h^{n-2}\right\|_{1,2}^2+\left\|\mathbf{C}_h^{n-2}\right\|_{1,2}^2\right).
\end{aligned}
$$

If we employ the notation

$$
\begin{aligned}
x^n &= \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2+\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2, \qquad & s^n &= \frac{\varepsilon}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2,\\
y^n &= \nu\left\|\mathrm{D}(\bar{D}_{\Delta t}\mathbf{u}_h^n)\right\|_2^2+\delta_0|\bar{D}_{\Delta t}p_h^n|_h^2+\frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2, & &\\
d^n &= \frac{1}{2}\left(\left\|\bar{D}_{\Delta t}\mathbf{f}_h^n\right\|_2^2+\left\|\bar{D}_{\Delta t}\mathbf{F}_h^n\right\|_2^2\right), & b^{n-2} &= c_{11}\left(\left\|\mathbf{u}_h^{n-2}\right\|_{1,2}^2+\left\|\mathbf{C}_h^{n-2}\right\|_{1,2}^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
a_0^{n-1} &= c(1/\nu,1/\varepsilon,1/(\nu^2\varepsilon))\left(\left\|\mathbf{C}_h^{n-1}\right\|_4^4+\left\|\mathbf{C}_h^{n-1}\right\|_4^2+\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2\right)\\
a_1^{n-1} &= \tilde{c}(c_1,1/\nu,1/\varepsilon,1/(\nu^2\varepsilon))\left(\left\|\mathbf{C}_h^{n-1}\right\|_4^4+\left\|\mathbf{C}_h^{n-1}\right\|_4^2+\left\|\mathbf{C}_h^{n-2}\right\|_4^4+\left\|\mathbf{C}_h^{n-2}\right\|_4^2+\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2\right),
\end{aligned}
$$

then we deal with the inequality of the form

$$\bar{D}_{\Delta t}x^n + y^n + s^n \le a_0^{n-1}x^n + a_1^{n-1}x^{n-1} + s^{n-1} + b^{n-2} + d^n, \tag{7.73}$$

for some non-negative sequences $\{a_0^{n-1}\}$, $\{a_1^{n-1}\}$, $\{s^n\}$, $\{b^{n-2}\}$, and $\{d^n\}$. Thus the discrete Gronwall lemma can be applied. We note that the coefficient a_0^{n-1} depends on the discrete solution from the previous time step. Therefore condition (2.7a) yields an analogous condition on Δt as (7.49) in the one-step proof of the error estimates, i.e.

$$\Delta t \le c\left(1/\nu, 1/\varepsilon, 1/(\nu^2\varepsilon), \left\|\mathbf{C}_h^{n-1}\right\|_4, \left\|\nabla \mathbf{u}_h^{n-1}\right\|_2\right)^{-1}. \tag{7.74}$$

By the discrete Gronwall lemma, for $n = 2, \ldots, N_T$, we then have

$$\begin{aligned}
&\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \Delta t\frac{\varepsilon}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \\
&\qquad + \Delta t\sum_{k=2}^{n}\left(\nu\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h^k)\right\|_2^2 + \delta_0|\bar{D}_{\Delta t}p_h^k|_h^2 + \frac{1}{2}\left\|\mathrm{tr}\,\mathbf{C}_h^{k-1}\bar{D}_{\Delta t}\mathbf{C}_h^k\right\|_2^2\right) \le \\
&\le K_1\left(\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \Delta t\frac{\varepsilon}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \Delta t\sum_{k=0}^{n-2}b^k + \Delta t\sum_{k=2}^{n}d^k\right),
\end{aligned}$$

where

$$\begin{aligned}
\Delta t\sum_{k=0}^{n-2}b^k + \Delta t\sum_{k=2}^{n}d^k \le c_{11}&\left(\Delta t\left\|\mathbf{u}_h^0\right\|_{1,2}^2 + \Delta t\left\|\mathbf{C}_h^0\right\|_{1,2}^2 + \|\mathbf{u}_h\|_{l^2(H^1)}^2 + \|\mathbf{C}_h\|_{l^2(H^1)}^2\right) + \\
&+ \frac{1}{2}\left(\left\|\bar{D}_{\Delta t}\mathbf{f}_h\right\|_{l^2(L^2)}^2 + \left\|\bar{D}_{\Delta t}\mathbf{F}_h\right\|_{l^2(L^2)}^2\right).
\end{aligned}$$

Let us denote

$$K_1 := \exp\left\{\Delta t\sum_{k=1}^{n-1}a_1^k + 2\Delta t\sum_{k=1}^{n-1}a_0^k\right\}.$$

By the first a priori estimates (7.13) we get

$$\begin{aligned}
K_1 &\le \exp\left\{c\left(c_1, T, 1/\nu, 1/\varepsilon, 1/(\nu^2\varepsilon), \left\|\mathbf{C}_h^0\right\|_4^4, \|\mathbf{C}_h\|_4^4, \|\nabla\mathbf{u}_h\|_2^2\right)\right\} \\
&\le \exp\left\{c\left(c_1, c_0, c_i, c_f, \tilde{c}, T, 1/\nu, 1/\varepsilon, 1/(\nu^2\varepsilon)\right)\right\}.
\end{aligned}$$

Then we write

$$\begin{aligned}
&\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \Delta t\,\varepsilon\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \\
&\qquad + \Delta t\sum_{k=2}^{n}\left(2\nu\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h^k)\right\|_2^2 + 2\delta_0|\bar{D}_{\Delta t}p_h^k|_h^2 + \left\|\mathrm{tr}\,\mathbf{C}_h^{k-1}\bar{D}_{\Delta t}\mathbf{C}_h^k\right\|_2^2\right) \le \\
&\le \hat{c}\Bigg(\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \Delta t\,\varepsilon\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{f}_h\right\|_{l^2(L^2)}^2 + \left\|\bar{D}_{\Delta t}\mathbf{F}_h\right\|_{l^2(L^2)}^2 + \\
&\qquad + c_{11}T\left\|\mathbf{u}_h^0\right\|_{1,2}^2 + c_{11}T\left\|\mathbf{C}_h^0\right\|_{1,2}^2 + c_{11}\|\mathbf{u}_h\|_{l^2(H^1)}^2 + c_{11}\|\mathbf{C}_h\|_{l^2(H^1)}^2\Bigg).
\end{aligned}$$

The norms $\|\mathbf{u}_h\|^2_{l^2(H^1)}$ and $\|\mathbf{C}_h\|^2_{l^2(H^1)}$ can be bounded by (7.13). If we were able to bound the term

$$\left\|\bar{D}_{\Delta t}\mathbf{u}_h^1\right\|_2^2 + \left\|\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2 + \varepsilon\,\Delta t\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^1\right\|_2^2$$

by a constant dependent only on the data, we would conclude the first part of the proof of Theorem 7.16. Here we recall Lemma 7.17, which indeed bounds this term by a constant $C = C\left(h_0^2, c_0, c_1, c_i, c_f, \tilde{c}, T, \nu^2, \nu, 1/\nu, \varepsilon^2/\nu, \varepsilon^2, 1/\varepsilon\right)$. Hence, for $n = 2, \dots, N_T$, the final inequality arises

$$\begin{aligned}
&\frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \Delta t\frac{\varepsilon}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \\
&\qquad + \Delta t\sum_{k=2}^{n}\left(\nu\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h^k)\right\|_2^2 + \delta_0|\bar{D}_{\Delta t}p_h^k|_h^2 + \left\|\mathrm{tr}\,\mathbf{C}_h^{k-1}\bar{D}_{\Delta t}\mathbf{C}_h^k\right\|_2^2\right)\\
&\qquad \leq \hat{c}\,c\left(h_0^2, c_0, c_1, c_{11}, c_i, c_f, \tilde{c}, T, \nu^2, \nu, 1/\nu, \varepsilon^2/\nu, \varepsilon^2, 1/\varepsilon\right),
\end{aligned}$$

which combined with (7.50) gives the desired bound

$$\begin{aligned}
&\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_{l^\infty(L^2)},\ \sqrt{\nu}\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h^n)\right\|_{l^2(L^2)},\ \left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_{l^\infty(L^2)},\\
&\qquad\qquad |\bar{D}_{\Delta t}p_h|_{l^2(|\cdot|_h)},\ \left\|\mathrm{tr}\,\mathbf{C}_h^{\cdot-1}\bar{D}_{\Delta t}\mathbf{C}_h\right\|_{l^2(L^2)} \leq c_1^h,
\end{aligned} \tag{7.75}$$

for a constant c_1^h of (7.3d). □

Remark 7.18. *We note that the energy estimate* $\left\|D(\bar{D}_{\Delta t}\mathbf{u}_h^n)\right\|_{l^2(L^2)}$ *from* (7.75) *implies the bound of* $\nabla\mathbf{u}_h$ *in* $l^\infty(L^2)$. *Indeed, by the Hölder, the Young and the Korn inequalities, we have*

$$\begin{aligned}
\|\nabla\mathbf{u}_h^n\|_2^2 - \left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2 &= \int_\Omega |\nabla\mathbf{u}_h^n|^2 - |\nabla\mathbf{u}_h^{n-1}|^2\,dx\\
&\leq \left(\int_\Omega |\nabla\mathbf{u}_h^n - \nabla\mathbf{u}_h^{n-1}|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla\mathbf{u}_h^n + \nabla\mathbf{u}_h^{n-1}|^2\,dx\right)^{1/2}\\
&\leq \left\|\nabla\mathbf{u}_h^n - \nabla\mathbf{u}_h^{n-1}\right\|_2\left(\|\nabla\mathbf{u}_h^n\|_2 + \left\|\nabla\mathbf{u}_h^{n-1}\right\|_2\right)
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\bar{D}_{\Delta t}\left(\|\nabla\mathbf{u}_h^n\|_2^2\right) &= \frac{\|\nabla\mathbf{u}_h^n\|_2^2 - \left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2}{\Delta t}\\
&\leq \frac{\left\|\nabla\mathbf{u}_h^n - \nabla\mathbf{u}_h^{n-1}\right\|_2\|\nabla\mathbf{u}_h^n\|_2}{\Delta t} + \frac{\left\|\nabla\mathbf{u}_h^n - \nabla\mathbf{u}_h^{n-1}\right\|_2\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2}{\Delta t}\\
&\leq \frac{\left\|\nabla\mathbf{u}_h^n - \nabla\mathbf{u}_h^{n-1}\right\|_2^2}{2(\Delta t)^2} + \frac{1}{2}\|\nabla\mathbf{u}_h^n\|_2^2 + \frac{1}{2}\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2\\
&= \frac{1}{2}\left\|\nabla\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2^2 + \frac{1}{2}\|\nabla\mathbf{u}_h^n\|_2^2 + \frac{1}{2}\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2^2.
\end{aligned}$$

Then the discrete Gronwall lemma, applying the Korn inequality, yields

$$\|\nabla\mathbf{u}_h\|_{l^\infty(L^2)}^2 \leq \mathrm{e}^{3/2\,T}\,c(T)\left(\left\|\nabla\mathbf{u}_h^0\right\|_2^2 + \|\nabla\mathbf{u}_h\|_{l^2(L^2)}^2 + \left\|D(\bar{D}_{\Delta t}\mathbf{u}_h)\right\|_{l^2(L^2)}^2\right),$$

where the right-hand side is bounded independently of the discretization parameters by (7.13) *and* (7.75).

Let us note that this is a crucial bound in order to move towards the error analysis of the fully nonlinear Peterlin model with the nonlinear advective terms.

Proof of Theorem 7.16 (ii)

Having the a priori estimate for the discrete time derivative of the velocity, we can finally show a bound of the discrete pressure p_h in $l^2(L^2)$-norm. The generalized inf-sup condition (6.24) yields

$$\begin{aligned}\|p_h^n\|_2 &\le \sqrt{\nu}\,\|(\mathbf{u}_h^n,p_h^n)\|_{V\times Q}\\ &\le \frac{\sqrt{\nu}}{\gamma}\sup_{(\mathbf{v}_h,q_h)\in V_h\times Q_h}\frac{\mathcal{A}_h\Big((\mathbf{u}_h^n,p_h^n),(\mathbf{v}_h,q_h)\Big)}{\|(\mathbf{v}_h,q_h)\|_{V\times Q}}\\ &= \frac{\sqrt{\nu}}{\gamma}\sup_{(\mathbf{v}_h,q_h)\in V_h\times Q_h}\frac{-\left(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right)-\left(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1},\nabla\mathbf{v}_h\right)+\left(\mathbf{f}_h^n,\mathbf{v}_h\right)}{\|(\mathbf{v}_h,q_h)\|_{V\times Q}}.\end{aligned}$$

It holds that

$$\begin{aligned}-\left(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right) &= -\left(\frac{\mathbf{u}_h^n\pm\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right)\\ &= -\left(\bar{D}_{\Delta t}\mathbf{u}_h^n,\mathbf{v}_h\right)-\left(\frac{\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right)\\ &\le \left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2\|\mathbf{v}_h\|_2+\frac{1}{\Delta t}\left\|\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-1}\circ X_1^n\right\|_2\|\mathbf{v}_h\|_2,\end{aligned}$$

and consequently, by (7.4b), we get

$$-\left(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t},\mathbf{v}_h\right)\le c\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2\|\mathbf{v}_h\|_V+c_1\left\|\mathbf{u}_h^{n-1}\right\|_V\|\mathbf{v}_h\|_V.$$

Further, we have the straightforward estimate $(\mathbf{f}_h^n,\mathbf{v}_h)\le c\,\|\mathbf{f}_h^n\|_2\,\|\mathbf{v}_h\|_V$ and

$$\begin{aligned}-\left(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1},\nabla\mathbf{v}_h\right) &\le c\,\|\operatorname{tr}\mathbf{C}_h^n\|_4\left\|\mathbf{C}_h^{n-1}\right\|_4\|\mathbf{v}_h\|_V\\ &\le c\,\|\operatorname{tr}\mathbf{C}_h^n\|_4^2\,\|\mathbf{v}_h\|_V+c\left\|\mathbf{C}_h^{n-1}\right\|_4^2\|\mathbf{v}_h\|_V.\end{aligned}$$

Hence, for the L^2-norm of p_h^n we get

$$\|p_h^n\|_2\le\frac{\sqrt{\nu}}{\gamma}\left\{c\left\|\bar{D}_{\Delta t}\mathbf{u}_h^n\right\|_2+c_1\left\|\nabla\mathbf{u}_h^{n-1}\right\|_2+c\,\|\operatorname{tr}\mathbf{C}_h^n\|_4^2+c\left\|\mathbf{C}_h^{n-1}\right\|_4^2+c\,\|\mathbf{f}_h^n\|_2\right\}.$$

The first and the second a priori estimates, cf. (7.13) and (7.75), imply

$$\begin{aligned}\|p_h\|_{l^2(L^2)}^2 &\le c(\nu)\left\{T\left\|\bar{D}_{\Delta t}\mathbf{u}_h\right\|_{l^\infty(L^2)}^2+\Delta t\sum_{i=1}^{N_T}\left(c_1\left\|\nabla\mathbf{u}_h^{i-1}\right\|_2^2+\left\|\operatorname{tr}\mathbf{C}_h^i\right\|_4^4+\left\|\mathbf{C}_h^{i-1}\right\|_4^4+\left\|\mathbf{f}_h^i\right\|_2^2\right)\right\}\\ &\le c(c_1,T,\nu)\Big\{\left\|\nabla\mathbf{u}_h^0\right\|_2^2+\left\|\mathbf{C}_h^0\right\|_4^4+\left\|\bar{D}_{\Delta t}\mathbf{u}_h\right\|_{l^\infty(L^2)}^2+\\ &\qquad+\|\nabla\mathbf{u}_h\|_{l^2(L^2)}^2+\|\mathbf{C}_h\|_{l^4(L^4)}^4+\|\mathbf{f}_h\|_{l^2(L^2)}^2\Big\}\\ &\le c(c_1,c_i,c_f,\tilde{c},(c_1^h)^2,T,T^2,\nu,1/\nu,1/\varepsilon,\nu/\varepsilon).\end{aligned}$$

□

Proof of Theorem 7.16 (iii)

Let us test equation (7.2b) with $\bar{D}_{\Delta t}\mathbf{C}_h^n$ to get, for $n = 1, \dots, N_T$,

$$\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \varepsilon\, a_c\left(\mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) =$$
$$= \left(\operatorname{tr}\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^n\right) + \left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) + \left(\mathbf{F}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right). \tag{7.76}$$

Let us estimate the integrals in (7.76). By (7.4b), we again have

$$\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^1}{\Delta t}, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) \geq \frac{1}{2}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 - c_1\left\|\mathbf{C}_h^{n-1}\right\|_{1,2}^2.$$

Further, the Hölder and the Young inequalities yield

$$\begin{aligned}\varepsilon\, a_c\left(\mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) &= \frac{\varepsilon}{\Delta t}\, a_c\left(\mathbf{C}_h^n, \mathbf{C}_h^n - \mathbf{C}_h^{n-1}\right) = \frac{\varepsilon}{\Delta t}\left\|\nabla\mathbf{C}_h^n\right\|_2^2 - \frac{\varepsilon}{\Delta t} a_c\left(\mathbf{C}_h^n, \mathbf{C}_h^{n-1}\right)\\ &\geq \bar{D}_{\Delta t}\left(\frac{\varepsilon}{2}\left\|\nabla\mathbf{C}_h^n\right\|_2^2\right).\end{aligned}$$

The two semi-implicit terms in (7.76) can be now estimated by the Hölder, the Young inequalities and by (2.9). Indeed, the bound on $\left\|\operatorname{tr}\mathbf{C}_h^{\cdot-1}\bar{D}_{\Delta t}\mathbf{C}_h\right\|_{l^2(L^2)}$ from part (i) of the proof, cf. (7.75), allows us to treat these two terms. We write

$$\begin{aligned}\left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) &\leq c\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2\left\|\mathbf{C}_h^n\right\|_4\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\right\|_4\\ &\leq \frac{c}{2}\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \frac{c}{4}\left\|\mathbf{C}_h^n\right\|_4^4 + \frac{c}{4}\left\|\mathbf{C}_h^{n-1}\right\|_4^4,\end{aligned}$$

and in the same way

$$\begin{aligned}\left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) &\leq c\left\|\nabla\mathbf{u}_h^n\right\|_2\left\|\mathbf{C}_h^{n-1}\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2\\ &\leq \frac{c}{2}\left\|\nabla\mathbf{u}_h^n\right\|_2^2 + \frac{c}{2}\left\|\mathbf{C}_h^{n-1}\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2\\ &\leq c\left\|\nabla\mathbf{u}_h^n\right\|_2^2 + c\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2.\end{aligned} \tag{7.77}$$

Let us note that since $\mathbf{C}_h$ is assumed to be symmetric and positive definite, we were able to apply the inequality $\mathbf{D}^2 \leq (\operatorname{tr}\mathbf{D})^2$ valid for $\mathbf{D} \in \mathcal{S}_+^d$ to get (7.77). The last two estimates are straightforward, i.e.

$$\begin{aligned}\left(\operatorname{tr}\mathbf{C}_h^{n-1}, \bar{D}_{\Delta t}\operatorname{tr}\mathbf{C}_h^n\right) &\leq c\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + c\left\|\mathbf{C}_h^{n-1}\right\|_2^2\\ \left(\mathbf{F}_h^n, \bar{D}_{\Delta t}\mathbf{C}_h^n\right) &\leq \frac{1}{4}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + c\left\|\mathbf{F}_h^n\right\|_2^2.\end{aligned}$$

Since we aim to apply the discrete Gronwall lemma, similarly as in the proof of part (i), we have focused on preparing a proper form of the inequality. The latter estimates have transformed equality (7.76) to the inequality, for $n \geq 1$,

$$\begin{aligned}\frac{1}{4}\left\|\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + \bar{D}_{\Delta t}\left(\frac{\varepsilon}{2}\left\|\nabla\mathbf{C}_h^n\right\|_2^2\right) &\leq c_1\left\|\mathbf{C}_h^{n-1}\right\|_{1,2}^2 + c\left\|\mathbf{C}_h^{n-1}\right\|_2^2 + c\left\|\mathbf{C}_h^{n-1}\right\|_4^4 + c\left\|\mathbf{C}_h^n\right\|_4^4 +\\ &+ c\left\|\nabla\mathbf{u}_h^n\right\|_2^2 + c\left\|\operatorname{tr}\mathbf{C}_h^{n-1}\,\bar{D}_{\Delta t}\mathbf{C}_h^n\right\|_2^2 + c\left\|\mathbf{F}_h^n\right\|_2^2.\end{aligned}$$

It has the form

$$\bar{D}_{\Delta t} x^n + y^n \le b^{n-1} + d^n, \tag{7.78}$$

where

$$\begin{aligned}
x^n &= \frac{\varepsilon}{2} \left\| \nabla \mathbf{C}_h^n \right\|_2^2, \qquad & d^n &= c \left\| \nabla \mathbf{u}_h^n \right\|_2^2 + c \left\| \mathbf{C}_h^n \right\|_4^4 + c \left\| \operatorname{tr} \mathbf{C}_h^{n-1} \bar{D}_{\Delta t} \mathbf{C}_h^n \right\|_2^2 + c \left\| \mathbf{F}_h^n \right\|_2^2, \\
y^n &= \frac{1}{4} \left\| \bar{D}_{\Delta t} \mathbf{C}_h^n \right\|_2^2, & b^{n-1} &= c_1 \left\| \mathbf{C}_h^{n-1} \right\|_{1,2}^2 + c \left\| \mathbf{C}_h^{n-1} \right\|_2^2 + c \left\| \mathbf{C}_h^{n-1} \right\|_4^4.
\end{aligned}$$

The discrete Gronwall lemma (2.7) finally yields

$$\begin{aligned}
&\frac{1}{4} \sum_{k=1}^{n} \left\| \bar{D}_{\Delta t} \mathbf{C}_h^k \right\|_2^2 + \frac{\varepsilon}{2} \left\| \nabla \mathbf{C}_h^k \right\|_2^2 \\
&\qquad \le \frac{\varepsilon}{2} \left\| \nabla \mathbf{C}_h^0 \right\|_2^2 + \sum_{k=1}^{n} c_1 \left\| \mathbf{C}_h^{k-1} \right\|_{1,2}^2 + c \left\| \mathbf{C}_h^{k-1} \right\|_2^2 + c \left\| \mathbf{C}_h^{k-1} \right\|_4^4 + c \left\| \mathbf{C}_h^k \right\|_4^4 \\
&\qquad\quad + \sum_{k=1}^{n} c \left\| \nabla \mathbf{u}_h^k \right\|_2^2 + c \left\| \operatorname{tr} \mathbf{C}_h^{k-1} \bar{D}_{\Delta t} \mathbf{C}_h^k \right\|_2^2 + c \left\| \mathbf{F}_h^k \right\|_2^2.
\end{aligned}$$

It means

$$\begin{aligned}
&\left\| \bar{D}_{\Delta t} \mathbf{C}_h \right\|_{l^2(L^2)}, \ \sqrt{\varepsilon} \left\| \nabla \mathbf{C}_h \right\|_{l^\infty(L^2)} \\
&\quad \le c \left(c_1, T^{1/2}, \sqrt{\varepsilon}, \left\| \mathbf{C}_h^0 \right\|_{1,2}, \left\| \mathbf{C}_h \right\|_{l^2(H^1)}, \left\| \nabla \mathbf{u}_h \right\|_{l^2(L^2)}, \left\| \operatorname{tr} \mathbf{C}_h^{\cdot-1} \bar{D}_{\Delta t} \mathbf{C}_h \right\|_{l^2(L^2)}, \left\| \mathbf{F}_h \right\|_{l^2(L^2)} \right) \\
&\quad \le c \left(c_1, c_i, c_f, \tilde{c}, c_1^h, T^{1/2}, \sqrt{\varepsilon}, 1/\sqrt{\nu}, 1/\sqrt{\varepsilon} \right).
\end{aligned}$$

Employing the first a priori estimates (7.13), (7.17) and the higher bounds (7.75), we conclude

$$\left\| \bar{D}_{\Delta t} \mathbf{C}_h \right\|_{l^2(L^2)}, \ \sqrt{\varepsilon} \left\| \nabla \mathbf{C}_h \right\|_{l^\infty(L^2)} \le c_3^h$$

for a constant c_3^h of (7.3d). □

7.5. The implicit nonlinear scheme

The aim of this section is to present yet another pressure-stabilized characteristics finite element method for the Oseen-type Peterlin viscoelastic equations (6.1), namely the fully implicit nonlinear scheme proposed by Tabata in [91]. For both schemes, the semi-implicit as well as the fully implicit scheme, the stability and the first order of convergence can be shown. In the following we discuss the assumptions and some details of the proofs.

Note, that there is only one difference, besides the discretization in time, in the definition of the implicit nonlinear pressure-stabilized characteristics finite element scheme compared to the semi-implicit linear scheme (6.25). That is an extra term

$$\left(\operatorname{div} \mathbf{u}_h^n (\mathbf{C}_h^n)^{\#}, \mathbf{D}_h \right) \tag{7.79}$$

added to the equation for the conformation tensor. The notation $(\mathbf{D})^{\#}$ stands for the *adjugate* of a symmetric matrix $\mathbf{D} \in \mathbb{R}^{2\times 2}$, i.e.

$$(\mathbf{D})^{\#} := \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}.$$

Let us provide the full definition of the implicit nonlinear scheme, see [91].

Definition 7.19. *(implicit nonlinear scheme)*
Let $(\mathbf{f}_h, \mathbf{F}_h) \in l^2(L^2(\Omega)^2) \times l^2(L^2(\Omega)^{2\times 2})$ *and the initial data* $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ *be given. Find the triple* $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ *such that for* $n = 1, \ldots, N_T$,

$$\begin{aligned}
&\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\Big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)\Big) = -\Big(\operatorname{tr} \mathbf{C}_h^n\, \mathbf{C}_h^n, \nabla \mathbf{v}_h\Big) + \Big(\mathbf{f}_h^n, \mathbf{v}_h\Big) \\
&\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon\, a_c\Big(\mathbf{C}_h^n, \mathbf{D}_h\Big) = \Big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^n + \mathbf{C}_h^n(\nabla \mathbf{u}_h^n)^T, \mathbf{D}_h\Big) + \\
&\quad + \Big(\operatorname{div} \mathbf{u}_h^n (\mathbf{C}_h^n)^{\#}, \mathbf{D}_h\Big) - \Big((\operatorname{tr} \mathbf{C}_h^n)^2\, \mathbf{C}_h^n, \mathbf{D}_h\Big) + \Big(\operatorname{tr} \mathbf{C}_h^n\, \mathbf{I}, \mathbf{D}_h\Big) + \Big(\mathbf{F}_h^n, \mathbf{D}_h\Big) \\
&\qquad\qquad\qquad\qquad \forall\ (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h.
\end{aligned} \tag{7.80}$$

Before we present the results on the stability and the accuracy of the above scheme, we present the existence and uniqueness result for its discrete solution, see [91].

Proposition 7.20. *(existence result for the implicit nonlinear scheme)*
Suppose Hypothesis 1 holds. Let the external forces $(\mathbf{f}_h, \mathbf{F}_h) \in l^2(L^2(\Omega)^2) \times l^2(L^2(\Omega)^{2\times 2})$ *and the initial data* $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ *be given. Then*

- *for every mesh size* $h > 0$ *and every* $0 < \Delta t \leq \min\{\Delta t_0, 1/5\}$ *there exists a solution to the implicit nonlinear scheme* (7.80).
- *for all* $h > 0$ *there exists* $\Delta t_1 > 0$ *such that for all* $0 < \Delta t \leq \min\{\Delta t_0, \Delta t_1\}$ *there exists a unique solution* $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ *to the implicit nonlinear scheme* (7.80).

Sketch of proof. The existence of the solution is obtained by the consequence of the Brouwer fixed point theorem, cf. Lemma 1.4. in [142]. The uniqueness follows from the Gronwall inequality using the a priori estimates together with the inverse inequality $\|\mathbf{C}\|_\infty \leq ch^{-1} \|\mathbf{C}\|_2$. □

The main advantage of the implicit scheme is that it *does not* require the positive definiteness of the discrete conformation tensor for the stability and error analyses. Let us present the key lemma, an analogue to Lemma 3.8, to treat the terms in the nonlinear scheme arising from the coupling of the equation for the velocity and the equation for the conformation tensor.

Lemma 7.21. [91]
Let $\mathbf{u} \in \mathbb{R}^2$ *and* $\mathbf{D} \in \mathbb{R}^{2\times 2}$ *be a symmetric matrix. Then it holds that*

$$(\operatorname{tr} \mathbf{D}\, \mathbf{D}, \nabla \mathbf{u}) - \frac{1}{2}\Big((\nabla \mathbf{u})\mathbf{D} + \mathbf{D}(\nabla \mathbf{u})^T, \mathbf{D}\Big) - \frac{1}{2}\Big(\operatorname{div} \mathbf{u}(\mathbf{D})^{\#}, \mathbf{D}\Big) = 0 \tag{7.81}$$

We omit the proof as it can be done by direct calculation. This lemma also explains why it was necessary to add the term (7.79) to the equation for the conformation tensor. In what follows we present the stability and accuracy results for the implicit nonlinear scheme.

Firstly, we cite the stability result, cf. [91], whose proof is done in one step, similarly as in the case of the formal energy estimates in the two-dimensional case in Section 3.3.

Theorem 7.22. *(stability result for the implicit nonlinear scheme)*
Suppose Hypothesis 1 holds. Let the external forces $(\mathbf{f}_h, \mathbf{F}_h) \in l^2(L^2(\Omega))^2 \times l^2(L^2(\Omega))^{2\times 2}$ *and the initial data* $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ *be given. Let* $0 < \Delta t \leq \min\{\Delta t_0, \Delta t_1\}$. *Then for the solution* $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ *of the implicit nonlinear scheme* (7.80) *there exists a constant* $\tilde{c}$ *of* (7.3b) *such that*

$$\begin{aligned}
&\|\mathbf{u}_h\|_{l^\infty(L^2)}\,, \sqrt{\nu}\,\|D(\mathbf{u}_h)\|_{l^2(L^2)}\,,\ |p_h|_{l^2(|\cdot|_h)},\\
&\|\mathbf{C}_h\|_{l^\infty(L^2)}\,, \sqrt{\varepsilon}\,\|\nabla \mathbf{C}_h\|_{l^2(L^2)}\,,\ \|\operatorname{tr}\mathbf{C}_h\mathbf{C}_h\|_{l^2(L^2)}\\
&\qquad\qquad \leq \tilde{c}\Big(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\mathbf{C}_h^0\right\|_2 + \|\mathbf{f}_h\|_{l^2(L^2)} + \|\mathbf{F}_h\|_{l^2(L^2)}\Big).
\end{aligned}$$

Proof. We employ Lemma 7.21 to show a priori estimates for the approximate solution of the nonlinear scheme in one step. Let $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ be the solution of the nonlinear scheme (7.80). Substituting

$$\left(\mathbf{u}_h^n, -p_h^n, \frac{1}{2}\,\mathbf{C}_h^n\right) \in V_h \times Q_h \times W_h$$

into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in the scheme (7.80), we obtain

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{u}_h^n\right) + \mathcal{A}_h\Big(\left(\mathbf{u}_h^n, p_h^n\right), \left(\mathbf{u}_h^n, -p_h^n\right)\Big) = -\Big(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^n, \nabla\mathbf{u}_h^n\Big) + \Big(\mathbf{f}_h^n, \mathbf{u}_h^n\Big) \tag{7.82}$$

$$\begin{aligned}
\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X_1^n}{2\Delta t}, \mathbf{C}_h^n\right) + \frac{\varepsilon}{2}\,a_c\Big(\mathbf{C}_h^n, \mathbf{C}_h^n\Big) &= \frac{1}{2}\,\Big((\nabla\mathbf{u}_h^n)\mathbf{C}_h^n + \mathbf{C}_h^n(\nabla\mathbf{u}_h^n)^T, \mathbf{C}_h^n\Big) + \\
&+ \frac{1}{2}\,\Big(\operatorname{div}\mathbf{u}_h^n(\mathbf{C}_h^n)^\#, \mathbf{C}_h^n\Big) + \frac{1}{2}\Big(\operatorname{tr}\mathbf{C}_h^n\,, \operatorname{tr}\mathbf{C}_h^n\Big) - \\
&- \frac{1}{2}\,\Big((\operatorname{tr}\mathbf{C}_h^n)^2\,\mathbf{C}_h^n, \mathbf{C}_h^n\Big) + \frac{1}{2}\Big(\mathbf{F}_h^n, \mathbf{C}_h^n\Big).
\end{aligned} \tag{7.83}$$

In order to get rid of the nonlinear terms arising due to the divergence of the elastic stress tensor and the upper convected derivative of the conformation tensor we add equations (7.82) and (7.83) together. Indeed, identity (7.81) from Lemma 7.21 for $\mathbf{u} = \mathbf{u}_h^n$ and $\mathbf{D} = \mathbf{C}_h^n$ yields

$$\frac{1}{2}\,\Big((\nabla\mathbf{u}_h^n)\mathbf{C}_h^n + \mathbf{C}_h^n(\nabla\mathbf{u}_h^n)^T, \mathbf{C}_h^n\Big) + \frac{1}{2}\,\Big(\operatorname{div}\mathbf{u}_h^n(\mathbf{C}_h^n)^\#, \mathbf{C}_h^n\Big) - \Big(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^n, \nabla\mathbf{u}_h^n\Big) = 0.$$

Thus, the sum of (7.82) and (7.83) reads

$$
\begin{aligned}
\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{u}_h^n\right) &+ \left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X_1^n}{2\Delta t}, \mathbf{C}_h^n\right) + \\
&+ \mathcal{A}_h\Big((\mathbf{u}_h^n, p_h^n), (\mathbf{u}_h^n, -p_h^n)\Big) + \frac{\varepsilon}{2}\, a_c\Big(\mathbf{C}_h^n, \mathbf{C}_h^n\Big) = \\
&= \frac{1}{2}\Big(\operatorname{tr}\mathbf{C}_h^n, \operatorname{tr}\mathbf{C}_h^n\Big) - \frac{1}{2}\Big((\operatorname{tr}\mathbf{C}_h^n)^2\,\mathbf{C}_h^n, \mathbf{C}_h^n\Big) + \\
&+ \Big(\mathbf{f}_h^n, \mathbf{u}_h^n\Big) + \frac{1}{2}\Big(\mathbf{F}_h^n, \mathbf{C}_h^n\Big).
\end{aligned}
\tag{7.84}
$$

Using the analogous estimates as (7.12a) - (7.12d), (7.12g) - (7.12h) for some constants β_i, $i = 1, 2$, and the following two equalities

$$
\begin{aligned}
\frac{1}{2}\Big((\operatorname{tr}\mathbf{C}_h^n)^2\,\mathbf{C}_h^n, \mathbf{C}_h^n\Big) &= \frac{1}{2}\int_\Omega (\operatorname{tr}\mathbf{C}_h^n)^2\,(\mathbf{C}_h^n : \mathbf{C}_h^n)\, dx = \frac{1}{2}\left\|\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^n\right\|_2^2 \\
\frac{1}{2}\Big(\operatorname{tr}\mathbf{C}_h^n, \operatorname{tr}\mathbf{C}_h^n\Big) &= \frac{1}{2}\left\|\operatorname{tr}\mathbf{C}_h^n\right\|_2^2 \le \|\mathbf{C}_h^n\|_2^2,
\end{aligned}
$$

we finally obtain the inequality for all $n \ge 1$

$$
\begin{aligned}
\bar{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{u}_h^n\|_2^2 + \frac{1}{2}\|\mathbf{C}_h^n\|_2^2\right) &+ 2\nu\,\|\mathrm{D}(\mathbf{u}_h^n)\|_2^2 + \delta_0|p_h^n|_h^2 + \frac{\varepsilon}{2}\|\nabla\mathbf{C}_h^n\|_2^2 + \\
&+ \frac{1}{2}\left\|\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^n\right\|_2^2 \le \\
\le c_1\left(\left\|\mathbf{u}_h^{n-1}\right\|_2^2 + \left\|\mathbf{C}_h^{n-1}\right\|_2^2\right) &+ \beta_2\|\mathbf{u}_h^n\|_2^2 + (1+\beta_1)\|\mathbf{C}_h^n\|_2^2 + \frac{1}{4\beta_2}\|\mathbf{f}_h^n\|_2^2 + \frac{1}{8\beta_1}\|\mathbf{F}_h^n\|_2^2 .
\end{aligned}
$$

The discrete Gronwall lemma (2.7) yields, for β_1, β_2 satisfying $\Delta t \le 1/_{4+4\beta_2+4\beta_1}$,

$$
\begin{aligned}
&\|\mathbf{u}_h\|_{l^\infty(L^2)}, \sqrt{\nu}\,\|\nabla\mathbf{u}_h\|_{l^2(L^2)}, |p_h|_{l^2(|.|_h)}, \\
&\|\mathbf{C}_h\|_{l^\infty(L^2)}, \sqrt{\varepsilon}\,\|\nabla\mathbf{C}_h\|_{l^2(L^2)}, \|\operatorname{tr}\mathbf{C}_h\,\mathbf{C}_h\|_{l^2(L^2)} \\
&\qquad\qquad \le \mathrm{e}^{c(c_1,T)}\left(\left\|\mathbf{u}_h^0\right\|_2 + \left\|\mathbf{C}_h^0\right\|_2 + \|\mathbf{f}_h\|_{l^2(L^2)} + \|\mathbf{F}_h\|_{l^2(L^2)}\right).
\end{aligned}
$$

This concludes the proof of Theorem 7.22. □

Obviously, the above proof is simpler than the proof of Theorem 7.13 in the sense that we do not need two steps to show the stability result. Similarly, the error estimates for the implicit nonlinear scheme (7.80) are shown at once, without imposing the dependence of the time increment on the discrete solution, cf. Theorem 7.23 below. We shall not give the complete proof from [91]. We only present an important part that makes the difference compared to the proof of Theorem 7.13.

Theorem 7.23. *(error estimates for the implicit nonlinear scheme)*
Suppose Hypotheses 1 and 3 hold. Let us assume $(\mathbf{f}_h, \mathbf{F}_h) = (\mathbf{f}, \mathbf{F})$. *Let* $(\mathbf{u}, p, \mathbf{C})$ *be the solution to the weak formulation of the Oseen-type Peterlin model* (7.1). *Further, let* $\mathbf{u}_h^0$ *be the first component of the Stokes projection of* $(\mathbf{u}^0, 0)$ *by* (7.9) *and* $\mathbf{C}_h^0$ *be the Lagrange interpolation of* $\mathbf{C}^0$. *Let* $0 < \Delta t \le \min\{\Delta t_0, \Delta t_1\}$ *and additionally*

$$\Delta t \le \frac{1}{c(1/\nu, 1/\varepsilon, \|(\mathbf{u}, \mathbf{C})\|_{H^1(0,T;H^2\times H^2)})}.$$

Then for the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ *of the implicit nonlinear scheme* (7.80) *there exists a constant* c^e *such that*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(L^2)},\ \sqrt{\nu}\, \|D((\mathbf{u} - \mathbf{u}_h))\|_{l^2(L^2)},\ |p - p_h|_{l^2(|.|_h)},\\
&\|\mathbf{C} - \mathbf{C}_h\|_{l^\infty(L^2)},\ \sqrt{\varepsilon}\, \|\nabla (\mathbf{C} - \mathbf{C}_h)\|_{l^2(L^2)}\\
&\qquad\qquad \le c^e \left\{ h\, \|(\mathbf{u}, p, \mathbf{C})\|_{H^1(0,T;H^2\times H^1\times H^2)} + \Delta t\, \|(\mathbf{u}, \mathbf{C})\|_{Z^2(0,T)} \right\}.
\end{aligned}$$

Remark 7.24. *Assuming* $\mathbf{u} \in Z^2 \cap H^1(0, T; V \cap W^{1,\infty}(\Omega)^2)$, *the above error estimates can be shown for* $\varepsilon = 0$ *with the constant* c^e *independent of* ε.

Sketch of proof. We can derive the equations satisfied by the errors, analogously as in the proof of Theorem 7.14. Now, we use the test function

$$\left(\delta_{u,h}^n, -\delta_{p,h}^n, \frac{1}{2}\delta_{c,h}^n\right) \in V_h \times Q_h \times W_h,$$

since we again aim to employ Lemma 7.21. The error equation thus reads

$$\begin{aligned}
&\left(\frac{\delta_{u,h}^n - \delta_{u,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n\right) + \left(\frac{\delta_{c,h}^n - \delta_{c,h}^{n-1} \circ X_1^n}{2\Delta t}, \delta_{c,h}^n\right) +\\
&\quad + \nu\, a_u\left(\delta_{u,h}^n, \delta_{u,h}^n\right) + \frac{\varepsilon}{2}\, a_c\left(\delta_{c,h}^n, \delta_{c,h}^n\right) - \mathcal{S}_h\left(\delta_{p,h}^n, -\delta_{p,h}^n\right) =\\
&\quad = -\frac{\varepsilon}{2}\, a_c\left(\eta_{c,h}^n, \delta_{c,h}^n\right) - \left(\operatorname{tr}\mathbf{C}^n\, \mathbf{C}^n - \operatorname{tr}\mathbf{C}_h^n\, \mathbf{C}_h^n, \nabla\delta_{u,h}^n\right) - \frac{1}{2}\left(\operatorname{div}\mathbf{u}_h^n (\mathbf{C}_h^n)^\#, \delta_{c,h}^n\right) +\\
&\quad + \frac{1}{2}\left((\nabla\mathbf{u}^n)\mathbf{C}^n + \mathbf{C}^n(\nabla\mathbf{u}^n)^T - (\nabla\mathbf{u}_h^n)\mathbf{C}_h^n - \mathbf{C}_h^n(\nabla\mathbf{u}_h^n)^T, \delta_{c,h}^n\right) -\\
&\quad - \frac{1}{2}\left((\operatorname{tr}\mathbf{C}^n)^2\mathbf{C}^n - (\operatorname{tr}\mathbf{C}_h^n)^2\, \mathbf{C}_h^n, \delta_{c,h}^n\right) + \frac{1}{2}\left((\operatorname{tr}\mathbf{C}^n - \operatorname{tr}\mathbf{C}_h^n)\mathbf{I}, \delta_{c,h}^n\right) -\\
&\quad - \left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n\right) - \left(\frac{\eta_{u,h}^n - \eta_{u,h}^{n-1} \circ X_1^n}{\Delta t}, \delta_{u,h}^n\right) -\\
&\quad - \frac{1}{2}\left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1} \circ X_1^n}{\Delta t}, \delta_{c,h}^n\right) - \left(\frac{\eta_{c,h}^n - \eta_{c,h}^{n-1} \circ X_1^n}{2\Delta t}, \delta_{c,h}^n\right).
\end{aligned} \tag{7.85}$$

Here we only concentrate on the coupling of the two equations in our model.

Let us rewrite the three corresponding integrals in appropriate way

$$
\begin{aligned}
&-\Big(\operatorname{tr}\mathbf{C}^n\,\mathbf{C}^n-\operatorname{tr}\mathbf{C}_h^n\,\mathbf{C}_h^n,\nabla\delta_{u,h}^n\Big)+\\
&+\frac{1}{2}\Big((\nabla\mathbf{u}^n)\mathbf{C}^n+\mathbf{C}^n(\nabla\mathbf{u}^n)^T-(\nabla\mathbf{u}_h^n)\mathbf{C}_h^n-\mathbf{C}_h^n(\nabla\mathbf{u}_h^n)^T,\delta_{c,h}^n\Big)+\\
&+\underbrace{\frac{1}{2}\Big(\operatorname{div}\mathbf{u}^n(\mathbf{C}^n)^\#,\delta_{c,h}^n\Big)}_{=0}-\frac{1}{2}\Big(\operatorname{div}\mathbf{u}_h^n(\mathbf{C}_h^n)^\#,\delta_{c,h}^n\Big)=\\
=&-\Big(\operatorname{tr}\eta_{c,h}^n\;\mathbf{C}^n,\nabla\delta_{u,h}^n\Big)+\underline{\Big(\operatorname{tr}\delta_{c,h}^n\;\;\delta_{c,h}^n,\nabla\delta_{u,h}^n\Big)}-\Big(\operatorname{tr}\delta_{c,h}^n\;\;\Pi_h\mathbf{C}^n,\nabla\delta_{u,h}^n\Big)-\\
&-\Big(\operatorname{tr}\Pi_h\mathbf{C}^n\eta_{c,h}^n,\nabla\delta_{u,h}^n\Big)+\Big(\operatorname{tr}\Pi_h\mathbf{C}^n\delta_{c,h}^n,\nabla\delta_{u,h}^n\Big)+\\
&+\frac{1}{2}\Big((\nabla\eta_{u,h}^n)\mathbf{C}^n+\mathbf{C}^n(\nabla\eta_{u,h}^n)^T,\delta_{c,h}^n\Big)+\frac{1}{2}\Big((\nabla\delta_{u,h}^n)\Pi_h\mathbf{C}^n+\Pi_h\mathbf{C}^n(\nabla\delta_{u,h}^n)^T,\delta_{c,h}^n\Big)+\\
&+\frac{1}{2}\Big((\nabla\hat{\mathbf{u}}_h^n)\eta_{c,h}^n+\eta_{c,h}^n(\nabla\hat{\mathbf{u}}_h^n)^T,\delta_{c,h}^n\Big)-\underline{\frac{1}{2}\Big((\nabla\delta_{u,h}^n)\delta_{c,h}^n+\delta_{c,h}^n(\nabla\delta_{u,h}^n)^T,\delta_{c,h}^n\Big)}+\\
&+\frac{1}{2}\Big((\nabla\hat{\mathbf{u}}_h^n)\delta_{c,h}^n+\delta_{c,h}^n(\nabla\hat{\mathbf{u}}_h^n)^T,\delta_{c,h}^n\Big)-\\
&-\underline{\frac{1}{2}\Big(\operatorname{div}\delta_{u,h}^n(\delta_{c,h}^n)^\#,\delta_{c,h}^n\Big)}+\frac{1}{2}\Big(\operatorname{div}\delta_{u,h}^n(\Pi_h\mathbf{C}^n)^\#,\delta_{c,h}^n\Big)+\frac{1}{2}\Big(\operatorname{div}\hat{\mathbf{u}}_h^n(\delta_{c,h}^n)^\#,\delta_{c,h}^n\Big)+\\
&+\frac{1}{2}\Big(\operatorname{div}\eta_{u,h}^n(\mathbf{C}^n)^\#,\delta_{c,h}^n\Big)+\frac{1}{2}\Big(\operatorname{div}\hat{\mathbf{u}}_h^n(\eta_{c,h}^n)^\#,\delta_{c,h}^n\Big),
\end{aligned}
$$

where, by Lemma 7.21, we have

$$
\Big(\operatorname{tr}\delta_{c,h}^n\;\;\delta_{c,h}^n,\nabla\delta_{u,h}^n\Big)-\frac{1}{2}\Big((\nabla\delta_{u,h}^n)\delta_{c,h}^n+\delta_{c,h}^n(\nabla\delta_{u,h}^n)^T,\delta_{c,h}^n\Big)-\frac{1}{2}\Big(\operatorname{div}\delta_{u,h}^n(\delta_{c,h}^n)^\#,\delta_{c,h}^n\Big)=0.
$$

The resulting integrals are then easily estimated. We omit all other estimates, since they are done in the same manner as for the error equation for the linear scheme, see proof of Theorem 7.14. Finally, we can write for all $n\geq 1$,

$$
\begin{aligned}
&\bar{D}_{\Delta t}\Big(\frac{1}{2}\left\|\delta_{u,h}^n\right\|_2^2+\frac{1}{4}\left\|\delta_{c,h}^n\right\|_2^2\Big)+\nu\left\|\mathrm{D}(\delta_{u,h}^n)\right\|_2^2+\delta_0|\delta_{p,h}^n|_h^2+\frac{\varepsilon}{4}\left\|\nabla\delta_{c,h}^n\right\|_2^2+\\
&\qquad\qquad+\frac{1}{4}\left\|\operatorname{tr}\delta_{c,h}^n\;\;\delta_{c,h}^n\right\|_2^2\leq\\
&\leq a_0\left(\frac{1}{2}\left\|\delta_{u,h}^n\right\|_2^2+\frac{1}{4}\left\|\delta_{c,h}^n\right\|_2^2\right)+c_1\left(\frac{1}{2}\left\|\delta_{u,h}^{n-1}\right\|_2^2+\frac{1}{4}\left\|\delta_{c,h}^{n-1}\right\|_2^2\right)+\\
&+\frac{c(\nu^2)h^2}{\Delta t}\left\|(\mathbf{u},p,\mathbf{C})\right\|_{H^1(t^{n-1},t^n;H^2\times H^1\times H^2)}^2+a_2h^2\left\|(\mathbf{u},p,\mathbf{C})\right\|_{H^1(H^2\times H^1\times H^2)}^2+\\
&+c_1\Delta t\left\|(\mathbf{u},\mathbf{C})\right\|_{Z^2(t^{n-1},t^n)}^2,
\end{aligned}
$$

where analogously as in the linear case we use the following notation

$$
\begin{aligned}
a_0&=c_2+c_3+\frac{c_2}{\nu}+\frac{c_3}{\varepsilon}\\
a_2&=c(\nu^2)\left(c_1+c_2+c_3+\frac{c_2}{\nu}+\frac{c}{\varepsilon}\right).
\end{aligned}
\tag{7.86}
$$

For $\Delta t \leq 1/2a_0$, the discrete Gronwall lemma (2.7) together with (7.37) and (7.21) yield

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_{l^\infty(L^2)}, \sqrt{\nu}\,\|\mathrm{D}(\mathbf{u}-\mathbf{u}_h)\|_{l^2(L^2)}, |p-p_h|_{l^2(|.|_h)},\\
&\|\mathbf{C}-\mathbf{C}_h\|_{l^\infty(L^2)}, \sqrt{\varepsilon}\,\|\nabla(\mathbf{C}-\mathbf{C}_h)\|_{l^2(L^2)}\\
&\qquad\qquad\qquad \leq c^e\left\{h\,\|(\mathbf{u},p,\mathbf{C})\|_{H^1(0,T;H^2\times H^1\times H^2)} + \Delta t\,\|(\mathbf{u},\mathbf{C})\|_{Z^2(0,T)}\right\},
\end{aligned}$$

where

$$c^e \leq \exp\left\{c\left(c_1,c_2,c_3,T,1/\nu,1/\varepsilon\right)\right\}c\left(c_1,c_2,c_3,\sqrt{\nu},\nu/\sqrt{\varepsilon}\right)$$

is a constant independent of h and Δt. We note that condition (2.7a) only imposes the dependence of the time increment on the data, see the coefficient (7.86). □

In general, theoretical error estimates for implicit numerical schemes can be treated in an easier way then for the corresponding semi-implicit or explicit schemes. This is true in our case as well. The stability and error estimates for the implicit scheme can be obtained using less steps. Moreover, the positive definiteness of the conformation tensor is not required.

On the other hand, the linear semi-implicit schemes are preferred for numerical simulations as they, in particular, require less computational time. For the comparison of the experimental error analysis of the two above schemes we refer to Subsection 8.2.3. The importance of the positive definiteness of the conformation tensor for the convergence of iterations is also discussed there.

In this chapter the stability and error analyses for the semi-implicit pressure-stabilized characteristics finite element scheme have been presented. The stability result for the linear numerical scheme approximating the Oseen-type diffusive Peterlin model has been shown. Similarly as for the theoretical results for the diffusive Peterlin model, cf. Chapter 4, the a priori estimates and the higher energy bounds have been derived. The optimal first order accuracy of the semi-implicit scheme has been proven. Finally, the linear semi-implicit scheme has been compared to the fully implicit nonlinear scheme.

8 Numerical experiments

The stability of the linear scheme and optimal convergence rate have been proven in the previous chapter. In order to demonstrate the validity of theoretical error estimates we perform the experimental error analysis. Moreover, we consider the lid-driven cavity problem, a well-known benchmark problem in the computational fluid dynamics.

This chapter contains three sections. Firstly, the implementation of the scheme and the algorithm are described. In Section 8.2 the experimental order of convergence, for several choices of the viscosity coefficients ν and ε, is computed. Finally, results for the lid-driven cavity flow problem are presented.

8.1. Implementation of the scheme

In order to realize our simulations we need to develop an algorithm to be implemented in a computer programme. To apply the pressure-stabilized characteristics finite element scheme for the Oseen-type Peterlin model we have modified the C code designed by Notsu [108]. This code has been successfully applied for a stabilized Lagrange-Galerkin approximation of the Oseen equations and the Navier-Stokes equations, cf. [111, 112].

We shall briefly describe the main features of our implementation. More precisely, we will in short explain the mesh generation, the numerical integration together with an element search algorithm and the treatment of the coupled system of equations. Finally, we give the algorithm that has been implemented to perform the series of numerical experiments.

8.1.1. Mesh

The computational domain is set to be a unit square $(0,1)^2$. We denote by N the *division number* of each side of the domain, and by $h := 1/N$ the *representative length of a mesh.* All finite elements $K_k \in \mathcal{T}_h$, $k = 1, \ldots, N_E$ are *triangles.* For the mesh generation we use FreeFEM++ [64].

The number of elements N_E, the number of nodal points N_P, the number of edges that form the boundary of the domain N_B, and the number of degrees of freedom DOF, can be found in the table below, together with the figure depicting a triangulation of the domain with $N = 16$.

N	N_E	N_P	N_B	DOF
16	594	330	64	990
32	2426	1278	128	3834
64	9716	4987	256	14961
128	39132	19823	512	59469

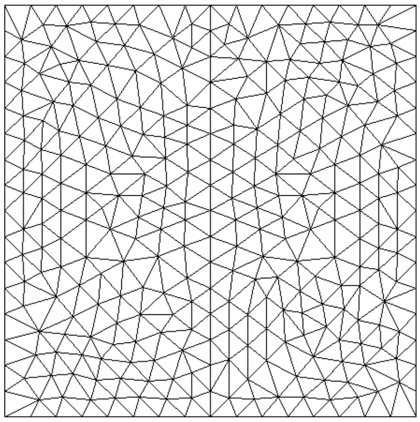


Table & Figure 8.2: Mesh parameters and triangulation of the domain with $N = 16$

Reference triangle

To solve a scheme based on the Galerkin method basically means to compute some integrals over the domain Ω, that is completely covered by the union of finite elements. In other words, we need to compute a given integral over each element $K \in \mathcal{T}_h$. In fact, we employ the transformation to the reference finite element $\tilde{K}$, where the integrals are easily evaluated.

We can rewrite a discrete function as

$$g_h(x) = \sum_{j=1}^{N_P} g_h(P_j)\varphi_j(x) =: \sum_{j=1}^{N_P} g_j \varphi_j(x),$$

where φ_j are the basis functions of the finite element space and P_j are the nodal points. Then, for the integral of g_h over a triangle K we have

$$\int_K g_h(x)\,dx = \sum_{j=1}^{N_P} g_j \int_K \varphi_j(x) = \sum_{j=1}^{N_P} g_j \int_{\tilde{K}} \tilde{\varphi}_j(\xi)\,|\mathcal{J}|\,d\xi,$$

where $\tilde{\varphi}_j(\xi)$ is a transformation of the basis function $\varphi_j(x)$ to the reference triangle, and $\mathcal{J} := \det(\partial x/\partial \xi)$ is the Jacobian of this transformation.

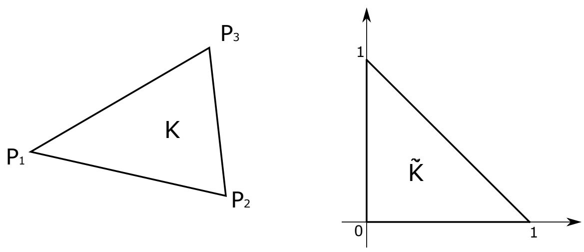


Figure 8.3.: A general triangle K (left) and the reference triangle $\tilde{K}$ (right)

In this work we consider the continuous piecewise linear finite element approximation. Thus, the shape functions of $\mathcal{P}_1(K)$ are the barycentric coordinates on the triangle K, and the basis functions on the reference triangle $\tilde{K}$ are

$$\varphi_{\tilde{K},1} = \xi_1, \qquad \varphi_{\tilde{K},2} = \xi_2, \qquad \varphi_{\tilde{K},3} = 1 - \xi_1 - \xi_2.$$

Then the transformation between any triangle $K \in \mathcal{T}_h$ and the reference triangle $\tilde{K}$ reads

$$\begin{aligned} x_1 &= x_1^1 + (x_1^2 - x_1^1)\xi_1 + (x_1^3 - x_1^1)\xi_2 \\ x_2 &= x_2^1 + (x_2^2 - x_2^1)\xi_1 + (x_2^3 - x_2^1)\xi_2, \end{aligned}$$

where $P_1 = (x_1^1, x_2^1)$, $P_2 = (x_1^2, x_2^2)$, and $P_3 = (x_1^3, x_2^3)$ are the vertices of the triangle K. Let $|K|$ be the area of the element. Then the absolute value of the Jacobian becomes

$$|\mathcal{J}| = \left|\det\left(\frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)}\right)\right| = \left|(x_1^2 - x_1^1)(x_2^3 - x_2^1) - (x_1^3 - x_1^1)(x_2^3 - x_2^1)\right| = 2|K|.$$

In order to compute the integrals in the semi-implicit linear scheme (6.25) we have used the following formulae valid on the reference triangle in two space dimensions

$$\int_{\tilde{K}} \varphi_i(\xi)\varphi_j(\xi) = \begin{cases} 1/12 & \text{if } i = j \\ 1/24 & \text{if } i \neq j \end{cases}$$

$$\int_{\tilde{K}} \varphi_i(\xi)\varphi_j(\xi)\varphi_k(\xi) = \begin{cases} 1/20 & \text{if } i = j = k \\ 1/60 & \text{if } i = j \neq k \text{ or } i = k \neq j \text{ or } j = k \neq i \\ 1/120 & \text{if } i \neq j \neq k. \end{cases}$$

Remark 8.1. *Let d be the space dimension, and let $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \ldots + \alpha_{d+1}$. Then for any $K \in \mathcal{T}_h$ and its barycentric coordinates $(\lambda_1, \ldots, \lambda_{d+1})$ it holds that*

$$\int_K \lambda_1^{\alpha_1}(x)\lambda_2^{\alpha_2}(x)\ldots\lambda_{d+1}^{\alpha_{d+1}}(x)\,dx = \frac{\alpha_1!\,\alpha_2!\,\ldots\alpha_{d+1}!\,d!}{(|\alpha| + d)!}.$$

8.1.2. Numerical integration formula

The method of characteristics used to discretize the material derivative yields the integrals of the form

$$\int_K \mathbf{u}_h^{n-1} \circ X_1^n(x)\,\mathbf{v}_h(x)\ dx, \qquad \int_K \mathbf{C}_h^{n-1} \circ X_1^n(x)\,\mathbf{D}_h(x)\ dx, \tag{8.1}$$

which need a special treatment. For the integration of the composite functions on triangles K we apply the *numerical integration formula of degree 5.* Here we briefly explain the main idea. For the details we refer to [108, 138, 139] and the references therein.

For a function $g \in C(K)$ we define its integral over K and a general numerical integration formula by

$$I[g, K] := \int_K g\,dx, \qquad I_h[g, K] := \sum_{i=1}^{M} g(a_i)\omega_i,$$

respectively. Here a_i denotes the i-th point and ω_i stands for the corresponding weight. We define the error of the numerical integration

$$E_h[g, K] := I[g, K] - I_h[g, K].$$

Then the accuracy of the formula is expressed by the degree of the error given by

$$\deg(E_h) := \sup\Big\{k \in \mathbb{N}_0 \;:\; E_h[g,K] = 0,\ \forall\, g \in \mathcal{P}_k(K)\Big\}.$$

We say that the numerical integration formula is of degree 5 when $\deg(E_h) = 5$.

Two-dimensional case [138]

- number of points in a triangle $M = 7$
- weights

$$\omega_1 = \frac{9}{40}, \qquad \omega_i = \frac{155-\sqrt{15}}{1200},\ i = 2,3,4, \qquad \omega_i = \frac{155+\sqrt{15}}{1200},\ i = 5,6,7$$

- points given in the barycentric coordinates, cf. Section C.1 in Appendix C,

$$a_i : \begin{cases} (\lambda_1,\lambda_2,\lambda_3) \in (t,t;t) & i = 1 \\ (\lambda_1,\lambda_2,\lambda_3) \in (p,p;q) & i = 2,3,4 \\ (\lambda_1,\lambda_2,\lambda_3) \in (r,r;s) & i = 5,6,7 \end{cases}$$

for

$$t = \frac{1}{3} \qquad p = \frac{6-\sqrt{15}}{21} \qquad q = \frac{9+2\sqrt{15}}{21} \qquad r = \frac{6+\sqrt{15}}{21} \qquad s = \frac{9-2\sqrt{15}}{21}.$$

Here $(j_1, j_2; j_3)$ denotes the set of all permutations of (j_1, j_2, j_3).

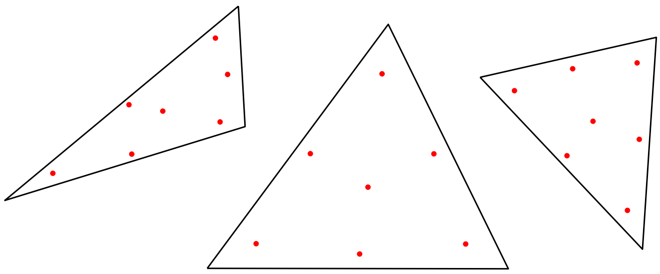


Figure 8.4.: Points for numerical integration of degree 5 on different triangles

Element search algorithm

We have already shown how the integrals (8.1) containing a composite functions are computed. However, there is another difficulty that has to be handled. If we denote

$$g := \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n(x) \qquad \text{or} \qquad g := \mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n(x),$$

then the numerical integration formula yields, for the integral of g over an element K, the sum $I_h[g,K] = \sum_{i=1}^{M} g(a_i)\omega_i$, that has to be evaluated. The points a_i are always in K, however the points $g(a_i)$ do not have to lie in the same element K, since they contain the upwind point $X_1^n(a_i)$ of $a_i \in K$. Therefore, in order to compute $I_h[g,K]$, we need to determine the proper index of the element in which the point $g(a_i)$ lies.

To this end, let us consider, for any element K_k, $k \in \{1, \dots, N_E\}$, the barycentric coordinates $(\lambda_1^k, \lambda_2^k, \lambda_3^k)$, see Appendix C. One of their properties which allows us to find the proper element that contains the point $g(a_i)$ reads:
$x^* \in K_k$ if and only if

$$\lambda_i^k(x^*) \geq 0, \ \forall\, i \in \{1, 2, 3\}. \tag{E_k}$$

The simplest approach is to go through all the elements and stop if the condition (E_k) is satisfied. This easy to code algorithm, however, needs a lot of computational time.

We employ the algorithm proposed by Notsu [108]. This approach needs the data $\{m_i^k\}_{i=1}^3$ for all elements K_k, $k = 1, \dots, N_E$. The m_i^k are the indices of the neighbouring elements of the element K_k. We set $m_i^k = -1$ if there is no neighbouring element, i.e. if at least one of the edges of the element K_k corresponds to the boundary of the domain.

Element search algorithm [108]

given $k_0 \in \{1, \dots, N_E\}$
while $k_0 \in \{1, \dots, N_E\}$ **do**
    if (E_{k_0}) is satisfied **then**
        set $k = k_0$ and break
        for $i = 1, 2, 3$ **do**
            if $\lambda_{k_0,i} < 0$ and $m_i^{k_0} \neq -1$ **then**
                set $k_1 = m_i^{k_0}$ and break
            end if
        end for
        set $k_0 = k_1$
    end if
end while
return k

8.1.3. Algorithm for the semi-implicit scheme

The semi-implicit time discretization indicates that the matrix of the system of equations at the current time step t^n depends on the solution from the old time step t^{n-1}, cf. (6.27). It means that solving the whole system at once might be costly and ineffective, as the matrix would have to be recomputed at every time step. Therefore, we handle the proposed semi-implicit linear scheme (6.25) as a coupled system of the Navier-Stokes equations with three convection-diffusion equations. Then we can make use of suitable properties of constant system matrices, such as symmetry and positive definiteness. This allows us to employ effective iterative solvers, such as the conjugate gradient (CG) or the minimal residual (MINRES) methods.

The CG method generates a sequence of conjugate or orthogonal vectors. These vectors are the residuals of the iterative solutions. This method is extremely effective when the coefficient matrix is symmetric and positive definite, since only a limited number of vectors needs to be stored. The MINRES method is a computational alternative for CG method in the case when the system matrix is symmetric and possibly indefinite, cf., e.g., [17, 63, 130].

The Navier-Stokes part yields a small-size symmetric matrix of the system of equations for the coefficients of $\mathbf{u}_h^n$ and p_h^n. Thus the MINRES method is applied. On the other hand, the convection-diffusion part results in three systems of equations for the coefficients of $C_{11,h}^n$, $C_{12,h}^n = C_{21,h}^n$ and $C_{22,h}^n$, whose system matrices are small-size symmetric, and moreover, positive definite. Hence the CG method is employed.

In order to implement the coupling of the Navier-Stokes part with the conformation tensor part we employ an inner iteration procedure.

Inner iteration scheme

For given $(\mathbf{u}_h^{n-1}, p_h^{n-1}, \mathbf{C}_h^{n-1})$ we find a solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$ to the semi-implicit scheme (6.25) as a limit of the series $\left\{\left(\mathbf{u}_h^{n,\ell}, p_h^{n,\ell}, \mathbf{C}_h^{n,\ell}\right)\right\}_{\ell=0}^{\infty}$, i.e.

$$(\mathbf{u}_h^{n-1}, p_h^{n-1}, \mathbf{C}_h^{n-1}) \equiv (\mathbf{u}_h^{n,0}, p_h^{n,0}, \mathbf{C}_h^{n,0}) \to (\mathbf{u}_h^{n,1}, p_h^{n,1}, \mathbf{C}_h^{n,1}) \to (\mathbf{u}_h^{n,2}, p_h^{n,2}, \mathbf{C}_h^{n,2}) \to \\ \cdots \to (\mathbf{u}_h^{n,\infty}, p_h^{n,\infty}, \mathbf{C}_h^{n,\infty}) \equiv (\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n).$$

In the real computations we employ a stopping criterion

$$E_{in} := \frac{1}{\Delta t}\left(\frac{\left\|\mathbf{u}_h^{n,\ell} - \mathbf{u}_h^{n,\ell-1}\right\|_2 + \left\|p_h^{n,\ell} - p_h^{n,\ell-1}\right\|_2}{\|\mathbf{u}_h^n\|_2 + \|p_h^n\|_2} + \frac{\left\|\mathbf{C}_h^{n,\ell} - \mathbf{C}_h^{n,\ell-1}\right\|_2}{\|\mathbf{C}_h^n\|_2}\right) < tol.$$

The tolerance is typically set to $tol = 10^{-2}$. In some cases smaller tolerance is chosen. We set the maximal allowed number of inner iterations to 20.

The algorithm that has been implemented in C code to solve the semi-implicit pressure-stabilized characteristics finite element method follows.

Algorithm 1 PSChFE scheme for the Peterlin viscoelastic model

for $n = 0, \dots, N_T - 1$ **do**
given $\left(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n\right)$ set $\left(\mathbf{u}_h^{n,0}, p_h^{n,0}, \mathbf{C}_h^{n,0}\right) = \left(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n\right)$
for $\ell = 1, \dots, 20$ **do**
solve

$$\left(\frac{\mathbf{u}_h^{n,\ell} - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\left((\mathbf{u}_h^{n,\ell}, p_h^{n,\ell}), (\mathbf{v}_h, q_h)\right) = \\ = -\left(\operatorname{tr}\mathbf{C}_h^{n,\ell-1}\mathbf{C}_h^{n-1}, \nabla\mathbf{v}_h\right) + (\mathbf{f}_h^n, \mathbf{v}_h)$$

$$\left(\frac{\mathbf{C}_h^{n,\ell} - \mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + a_c(\mathbf{C}_h^{n,\ell}, \mathbf{D}_h) = \\ = \left((\nabla\mathbf{u}_h^{n,\ell-1})\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^{n,\ell-1})^T, \mathbf{D}_h\right) + \\ + \left(\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{I} - ((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^{n,\ell-1}, \mathbf{D}_h\right) + (\mathbf{F}_h^n, \mathbf{D}_h)$$

if $\mathrm{E}_{in} < tol$ **then**
break
end if
end for
update $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{C}_h^{n+1}) = \left(\mathbf{u}_h^{n,\ell}, p_h^{n,\ell}, \mathbf{C}_h^{n,\ell}\right)$
end for

8.2. Experimental order of convergence

The aim of this section is to confirm theoretical convergence rate by a series of numerical experiments. We remind that the first order error estimates for the semi-implicit linear scheme approximating the Oseen-type Peterlin model have been proven, see Theorem 7.14 in Section 7.3.

Let us recall that, for the sake of computing the experimental order of convergence, we have considered some external forces in the Oseen-type Peterlin model (6.1). Consequently, we have its *analytical solution*. We set

$$\mu(x,t) := \frac{\sqrt{3}}{2\pi}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\}.$$

Then the functions

$$\begin{aligned}
\mathbf{u}(x,t) &:= \left(-\frac{\partial\mu}{\partial x_2}(x,t),\ \frac{\partial\mu}{\partial x_1}(x,t)\right), \quad p(x,t) := \sin\{\pi(x_1+2x_2+t)\},\\
C_{11}(x,t) &:= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+t)\}+1,\\
C_{22}(x,t) &:= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_2+t)\}+1,\\
C_{12}(x,t) &:= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\} \quad (= C_{21}(x,t)),
\end{aligned} \tag{8.2}$$

with the corresponding forces $(\mathbf{f},\mathbf{F})$, represent the exact solution to the Oseen-type Peterlin model (6.1). We choose $\mathbf{u}$ as the given velocity $\mathbf{w}$.

Let $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ be the solution of the numerical scheme (6.25) and let $(\mathbf{u}, p, \mathbf{C})$ be the analytical solution (8.2). We define the *relative errors* by

$$e_u := \frac{\|\mathbf{u}_h - \Pi_h\mathbf{u}\|}{\|\Pi_h\mathbf{u}\|} \qquad e_p := \frac{\|p_h - \Pi_h p\|}{\|\Pi_h p\|} \qquad e_c := \frac{\|\mathbf{C}_h - \Pi_h\mathbf{C}\|}{\|\Pi_h\mathbf{C}\|}.$$

Let g_h be a numerical approximation of a function g dependent on the discretization parameter h. The convergence $g_h \to g$ for $h \to 0$ can be quantify by

$$\|g_h - g\| \le ch^\alpha,$$

for a maximal α, which is called a *convergence order*. This gives us a possibility to verify the numerical scheme. If the solution g is known and if two approximations g_h and $g_{h'}$ are computed, we can evaluate α as

$$\alpha \approx EOC(h,h') := \frac{\log_2\left(\|g_h-\Pi_h g\|/\|g_{h'}-\Pi_{h'}g\|\right)}{\log_2\left(h/h'\right)},$$

subject to the assumption that the error is $\|g_h - g\| \approx ch^\alpha$. In our case the two subsequent discretization parameters are h and $h/2$. Thus the *experimental order of convergence* becomes

$$EOC(h, h/2) = \log_2\left(\frac{\|g_h - \Pi_h g\|}{\|g_{h/2} - \Pi_{h/2} g\|}\right).$$

8.2.1. Problem setting

As already mentioned the computational domain is a unit square $(0,1)^2$. The final time for all the computations in this section is $T = 0.5$. We fix $\delta_0 = 1$ in (6.23). Unless stated otherwise, we take the time increment $\Delta t = h/2$ for $N = 16, 32, 64, 128$. The initial data $(\mathbf{u}_h^0, \mathbf{C}_h^0)$ are the first component of the Stokes projection of $(\mathbf{u}(0), 0)$ given by (7.9) and the Lagrange interpolation (6.19) of $\mathbf{C}(0)$, respectively. To investigate convergence order numerically we consider several test cases for different values of ν and ε:

(1) $\nu = 10.0$, $\varepsilon = 10^{-k}$, $k = 0, 1, 2, 3, 4$

(2) $\nu = 1.0$, $\varepsilon = 10^{-k}$, $k = 0, 1, 2, 3, 4$

(3) $\nu = 0.1$, $\varepsilon = 10^{-k}$, $k = 1, 2, 3, 4$

(4) $\nu = 0.01$, $\varepsilon = 10^{-k}$, $k = 2, 3, 4$

(5) $\varepsilon = 0$, $\nu = 10^{-k}$, $k = 0, 1, 2, 3, 4$.

The elastic stress viscosity is typically smaller than the fluid viscosity. Therefore we have taken the values of ε to be less than or equal to ν. Let us note that we work with $\varepsilon > 0$. The theoretical results on stability and error estimates also require the elastic stress viscosity to be positive. Nevertheless, the proposed numerical scheme (6.25) yields the first order convergence even for $\varepsilon = 0$, see the test case **(5)**, where different values of ν for $\varepsilon = 0$ are considered.

8.2.2. Numerical results

In what follows we present tables with the relative errors of the approximation and the experimental order of convergence (EOC) for different meshes. In addition, the maximal number of inner iterations for each mesh is given. Besides every table there is a figure depicting the relative errors versus the discretization parameter h in a logarithmic scale. The slope between two subsequent values of h represents the EOC. The time increment has been taken as $\Delta t \approx h/2$ in order to keep the same orders of both Δt and h analogously to the theoretical result. In most of the test cases we shall observe better than the first order convergence in both space and time.

Positive definiteness of the conformation tensor

Theoretical results for the linear scheme require positive definiteness of the discrete conformation tensor, cf. Theorems 7.13 and 7.14. We control this important property in all subsequent computations. The smallness of time increment in order to have $\mathbf{C}_h$ positive definite depends on the mesh size h, the viscosities ν and ε and on the initial data. Nevertheless, the upper bound c_0 of the time increment, cf. Hypothesis 2, seems to be independent of the numerical solution in all test cases.

It has turned out that for the case $\nu = 0.01$, $\varepsilon = 0.001$ the time increment $\Delta t = h/2$ is not sufficiently small for $N = 16, 32$. For other two cases $\nu = 0.01$ and $\varepsilon = 10^{-k}$, $k = 4, \infty$ it is not enough small even for $N = 64$. That is the reason why we present, for these three pairs of viscosities, the results for the time increment $\Delta t = h/4$, which is satisfactory to ensure this property, see Tables & Figures 8.20, 8.21 and 8.25 below. Let us note that $\Delta t = 1/256$ is also a convenient choice of time increment.

(1) The case $\nu = 10.0$ and $\varepsilon = 10^{-k}$, $k = 0, 1, 2, 3, 4$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	5.38e-02	–	4.06e-02	–
1/32	2.11e-02	1.35	1.61e-02	1.33
1/64	9.15e-03	1.20	7.14e-03	1.18
1/128	4.74e-03	0.95	3.65e-03	0.97
h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	3.43e-02	–	1.29e-02	–
1/32	9.82e-03	1.80	5.65e-03	1.19
1/64	3.11e-03	1.66	2.65e-03	1.09
1/128	9.41e-04	1.72	1.33e-03	1.00
h	$e_p\ l^2(L^2)$	EOC		l_{max}
1/16	8.90e-01	–		4
1/32	6.00e-01	0.57		4
1/64	3.09e-01	0.96		4
1/128	1.13e-01	1.45		4

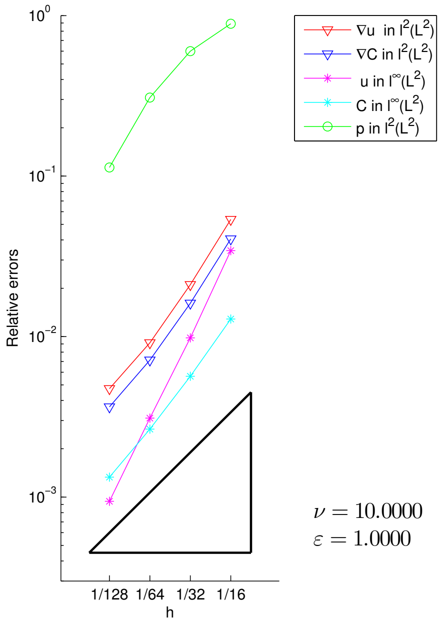


Table & Figure 8.5: Errors and EOC: $\nu = 10.0$, $\varepsilon = 1.0$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	5.36e-02	–	1.06e-01	–
1/32	2.11e-02	1.35	3.86e-02	1.46
1/64	9.15e-03	1.20	1.77e-02	1.13
1/128	4.74e-03	0.95	8.57e-03	1.04
h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	3.41e-02	–	1.88e-02	–
1/32	9.79e-03	1.80	7.51e-03	1.32
1/64	3.11e-03	1.65	3.37e-03	1.16
1/128	9.45e-04	1.72	1.58e-03	1.09
h	$e_p\ l^2(L^2)$	EOC		l_{max}
1/16	8.91e-01	–		4
1/32	6.02e-01	0.57		4
1/64	3.10e-01	0.96		4
1/128	1.14e-01	1.45		4

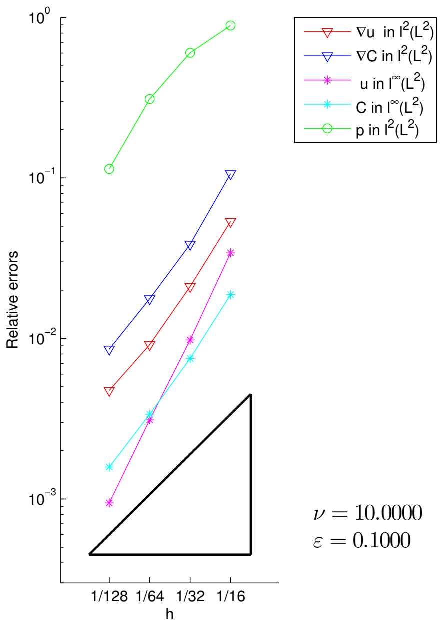


Table & Figure 8.6: Errors and EOC: $\nu = 10.0$, $\varepsilon = 0.1$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.34e-02	–	3.51e-01	–
1/32	2.10e-02	1.34	1.16e-01	1.60
1/64	9.15e-03	1.20	4.19e-02	1.47
1/128	4.74e-03	0.95	1.78e-02	1.23
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	3.40e-02	–	3.20e-02	–
1/32	9.76e-03	1.80	1.08e-02	1.57
1/64	3.11e-03	1.65	4.46e-03	1.28
1/128	9.50e-04	1.71	1.99e-03	1.16
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	8.92e-01	–		5
1/32	6.04e-01	0.56		4
1/64	3.11e-01	0.95		4
1/128	1.14e-01	1.45		4

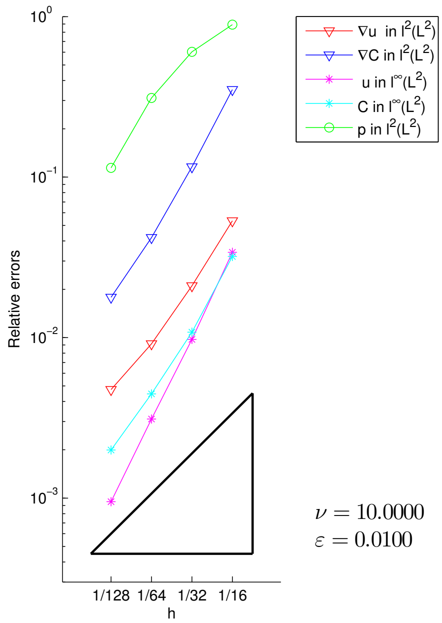


Table & Figure 8.7: Errors and EOC: $\nu = 10.0$, $\varepsilon = 0.01$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.33e-02	–	7.89e-01	–
1/32	2.10e-02	1.34	3.76e-01	1.07
1/64	9.15e-03	1.20	1.43e-01	1.39
1/128	4.74e-03	0.95	4.82e-02	1.57
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	3.39e-02	–	4.28e-02	–
1/32	9.75e-03	1.80	1.35e-02	1.66
1/64	3.11e-03	1.65	5.10e-03	1.41
1/128	9.50e-04	1.71	2.22e-03	1.20
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	8.92e-01	–		5
1/32	6.04e-01	0.56		4
1/64	3.12e-01	0.95		4
1/128	1.14e-01	1.45		4

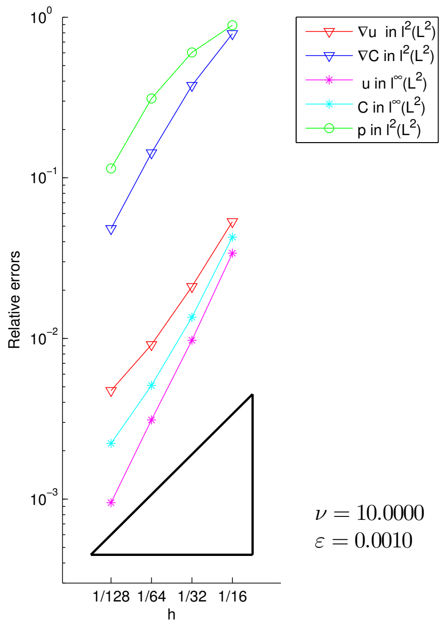


Table & Figure 8.8: Errors and EOC: $\nu = 10.0$, $\varepsilon = 0.001$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.33e-02	–	1.05e+00	–
1/32	2.10e-02	1.34	7.17e-01	0.54
1/64	9.15e-03	1.20	4.51e-01	0.67
1/128	4.74e-03	0.95	1.94e-01	1.22
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	3.39e-02	–	4.68e-02	–
1/32	9.75e-03	1.80	1.60e-02	1.55
1/64	3.11e-03	1.65	5.82e-03	1.46
1/128	9.50e-04	1.71	2.34e-03	1.31
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	8.92e-01	–		5
1/32	6.04e-01	0.56		4
1/64	3.12e-01	0.95		4
1/128	1.14e-01	1.45		4

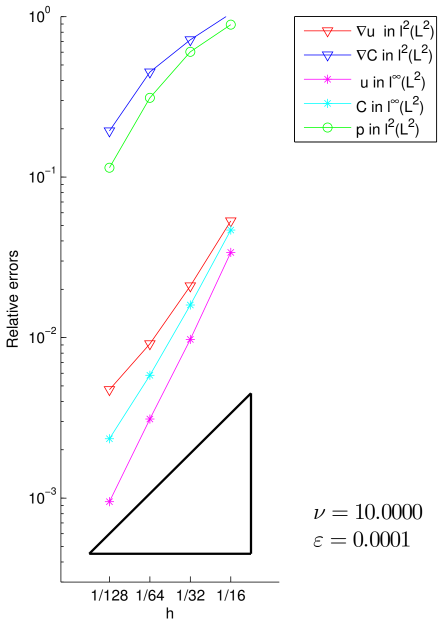


Table & Figure 8.9: Errors and EOC: $\nu = 10.0$, $\varepsilon = 0.0001$

(2) The case $\nu = 1.0$ and $\varepsilon = 10^{-k}$, $k = 0, 1, 2, 3, 4$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	6.21e-02	–	4.21e-02	–
1/32	2.43e-02	1.35	1.68e-02	1.32
1/64	9.97e-03	1.29	7.24e-03	1.21
1/128	4.92e-03	1.02	3.65e-03	0.99
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	4.69e-02	–	1.38e-02	–
1/32	1.51e-02	1.63	6.06e-03	1.19
1/64	4.70e-03	1.69	2.76e-03	1.13
1/128	1.57e-03	1.58	1.37e-03	1.02
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	4.43e-01	–		4
1/32	1.78e-01	1.32		4
1/64	6.21e-02	1.52		4
1/128	2.10e-02	1.57		4

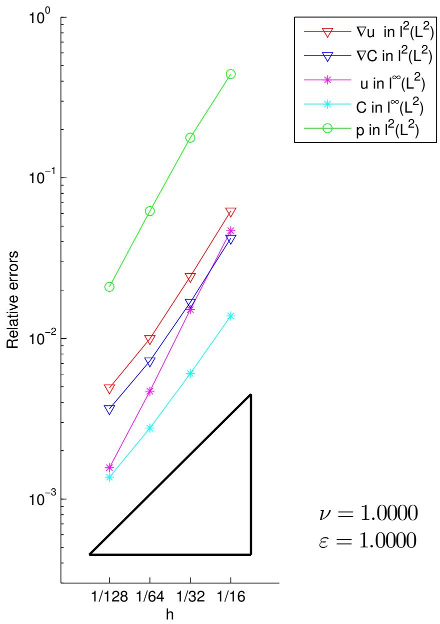


Table & Figure 8.10: Errors and EOC: $\nu = 1.0$, $\varepsilon = 1.0$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	6.03e-02	–	1.27e-01	–
1/32	2.40e-02	1.33	4.80e-02	1.41
1/64	9.95e-03	1.27	1.92e-02	1.32
1/128	4.93e-03	1.01	8.57e-03	1.16
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	4.46e-02	–	2.27e-02	–
1/32	1.45e-02	1.62	9.13e-03	1.31
1/64	4.56e-03	1.67	3.75e-03	1.28
1/128	1.52e-03	1.58	1.68e-03	1.15
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	4.64e-01	–		5
1/32	1.90e-01	1.29		4
1/64	6.59e-02	1.52		4
1/128	2.17e-02	1.60		4

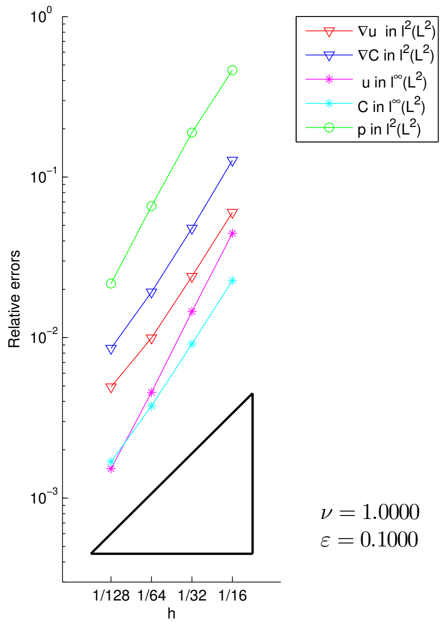


Table & Figure 8.11: Errors and EOC: $\nu = 1.0$, $\varepsilon = 0.1$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.84e-02	–	3.69e-01	–
1/32	2.37e-02	1.30	1.31e-01	1.50
1/64	9.91e-03	1.26	4.60e-02	1.50
1/128	4.93e-03	1.01	1.81e-02	1.35
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	4.43e-02	–	3.79e-02	–
1/32	1.42e-02	1.65	1.41e-02	1.43
1/64	4.51e-03	1.65	5.24e-03	1.43
1/128	1.51e-03	1.58	2.16e-03	1.28
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	4.76e-01	–		5
1/32	1.98e-01	1.26		4
1/64	6.92e-02	1.52		4
1/128	2.26e-02	1.62		4

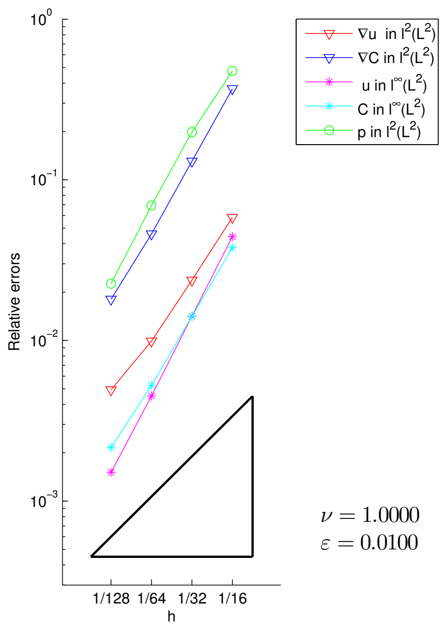


Table & Figure 8.12: Errors and EOC: $\nu = 1.0$, $\varepsilon = 0.01$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.80e-02	–	7.87e-01	–
1/32	2.36e-02	1.30	3.74e-01	1.07
1/64	9.88e-03	1.26	1.44e-01	1.38
1/128	4.92e-03	1.01	4.91e-02	1.55
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	4.43e-02	–	4.75e-02	–
1/32	1.41e-02	1.65	1.68e-02	1.50
1/64	4.51e-03	1.65	5.97e-03	1.49
1/128	1.51e-03	1.57	2.40e-03	1.32
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	4.79e-01	–		5
1/32	2.00e-01	1.26		4
1/64	7.03e-02	1.51		4
1/128	2.30e-02	1.61		4

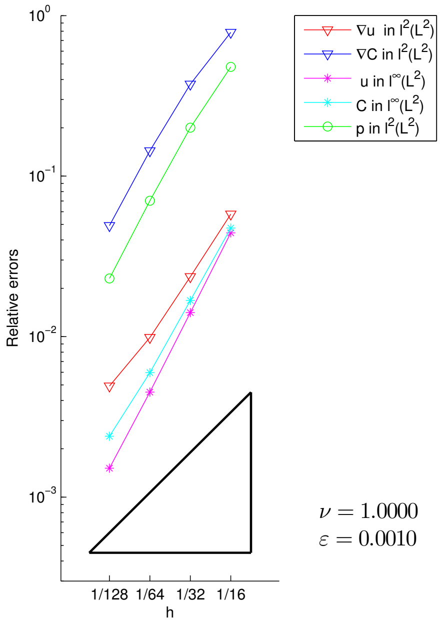


Table & Figure 8.13: Errors and EOC: $\nu = 1.0$, $\varepsilon = 0.001$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.80e-02	–	1.04e+00	–
1/32	2.36e-02	1.30	7.01e-01	0.56
1/64	9.87e-03	1.26	4.40e-01	0.67
1/128	4.91e-03	1.01	1.91e-01	1.21
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	4.43e-02	–	5.09e-02	–
1/32	1.41e-02	1.65	1.87e-02	1.45
1/64	4.51e-03	1.65	6.56e-03	1.51
1/128	1.51e-03	1.57	2.51e-03	1.38
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	4.79e-01	–		5
1/32	2.01e-01	1.26		4
1/64	7.05e-02	1.51		4
1/128	2.32e-02	1.61		4

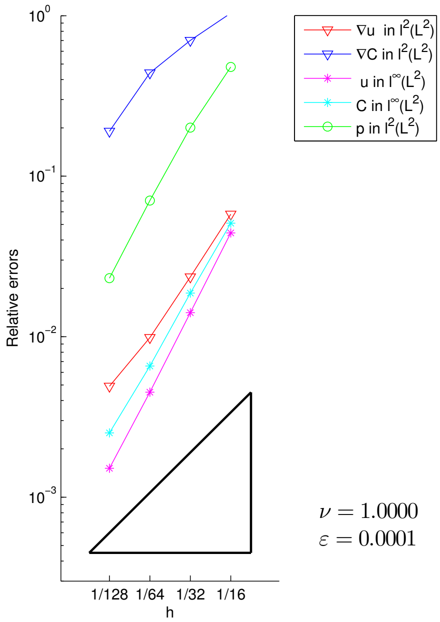


Table & Figure 8.14: Errors and EOC: $\nu = 1.0$, $\varepsilon = 0.0001$

(3) The case $\nu = 0.1$ and $\varepsilon = 10^{-k}$, $k = 1, 2, 3, 4$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	7.85e-02	–	1.65e-01	–
1/32	3.13e-02	1.33	5.89e-02	1.48
1/64	1.31e-02	1.25	2.30e-02	1.36
1/128	6.36e-03	1.05	1.02e-02	1.17
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	6.14e-02	–	2.88e-02	–
1/32	2.21e-02	1.48	1.14e-02	1.33
1/64	8.97e-03	1.30	4.91e-03	1.22
1/128	4.07e-03	1.14	2.30e-03	1.09
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	2.03e-01	–		5
1/32	7.14e-02	1.51		4
1/64	2.68e-02	1.41		4
1/128	1.11e-02	1.27		4

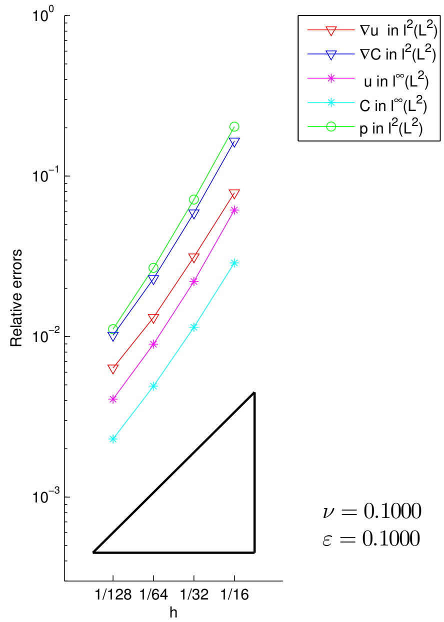


Table & Figure 8.15: Errors and EOC: $\nu = 0.1$, $\varepsilon = 0.1$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	7.13e-02	–	3.84e-01	–
1/32	2.90e-02	1.30	1.44e-01	1.41
1/64	1.23e-02	1.23	5.23e-02	1.47
1/128	6.03e-03	1.03	2.04e-02	1.36
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	5.94e-02	–	4.50e-02	–
1/32	1.95e-02	1.61	1.71e-02	1.40
1/64	7.72e-03	1.34	6.78e-03	1.33
1/128	3.41e-03	1.18	3.01e-03	1.17
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	2.42e-01	–		5
1/32	8.69e-02	1.48		4
1/64	3.12e-02	1.48		4
1/128	1.22e-02	1.35		4

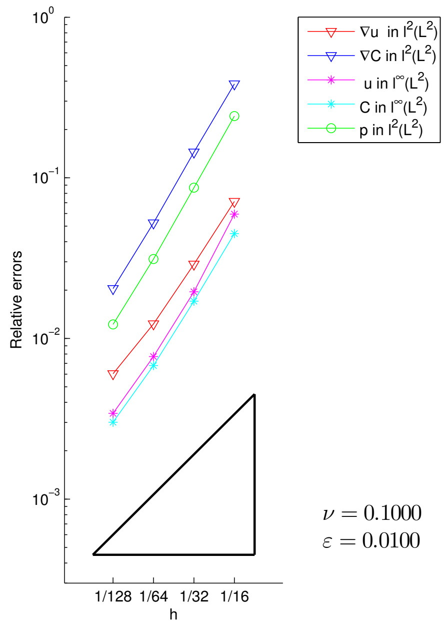


Table & Figure 8.16: Errors and EOC: $\nu = 0.1$, $\varepsilon = 0.01$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	7.16e-02	–	7.30e-01	–
1/32	2.88e-02	1.31	3.42e-01	1.09
1/64	1.21e-02	1.26	1.41e-01	1.28
1/128	5.93e-03	1.03	5.22e-02	1.44
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	5.93e-02	–	5.17e-02	–
1/32	1.95e-02	1.61	1.94e-02	1.42
1/64	7.65e-03	1.35	7.55e-03	1.36
1/128	3.35e-03	1.19	3.28e-03	1.20
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	2.52e-01	–		5
1/32	9.19e-02	1.45		4
1/64	3.33e-02	1.47		4
1/128	1.29e-02	1.37		4

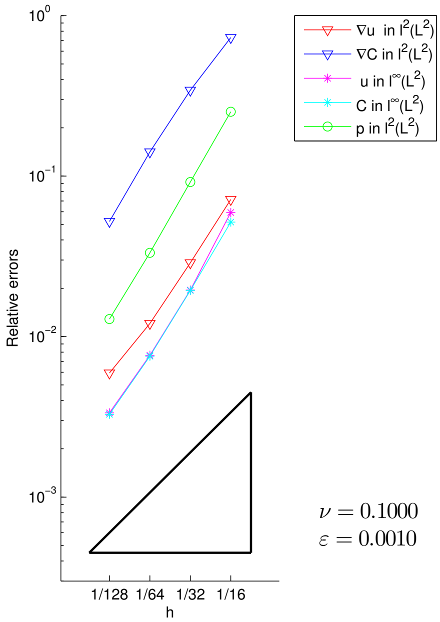


Table & Figure 8.17: Errors and EOC: $\nu = 0.1$, $\varepsilon = 0.001$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	7.22e-02	–	9.49e-01	–
1/32	2.93e-02	1.30	6.16e-01	0.62
1/64	1.23e-02	1.26	3.87e-01	0.67
1/128	5.92e-03	1.05	1.76e-01	1.14
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	5.93e-02	–	5.39e-02	–
1/32	1.95e-02	1.60	2.05e-02	1.40
1/64	7.66e-03	1.35	7.92e-03	1.37
1/128	3.35e-03	1.19	3.37e-03	1.23
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	2.53e-01	–		5
1/32	9.28e-02	1.45		4
1/64	3.38e-02	1.46		4
1/128	1.32e-02	1.36		4

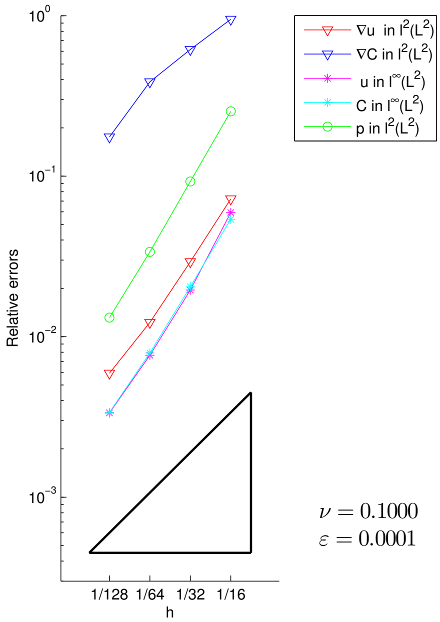


Table & Figure 8.18: Errors and EOC: $\nu = 0.1$, $\varepsilon = 0.0001$

(4) The case $\nu = 0.01$ and $\varepsilon = 10^{-k}$, $k = 2, 3, 4$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	9.27e-02	–	3.98e-01	–
1/32	3.73e-02	1.31	1.55e-01	1.36
1/64	1.65e-02	1.18	5.87e-02	1.40
1/128	8.16e-03	1.02	2.37e-02	1.31

h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	6.60e-02	–	4.70e-02	–
1/32	2.27e-02	1.54	1.85e-02	1.34
1/64	9.78e-03	1.22	7.79e-03	1.25
1/128	4.59e-03	1.09	3.60e-03	1.11

h	$e_p\ l^2(L^2)$	EOC	l_{max}
1/16	2.06e-01	–	7
1/32	7.41e-02	1.47	4
1/64	2.78e-02	1.41	4
1/128	1.16e-02	1.27	4

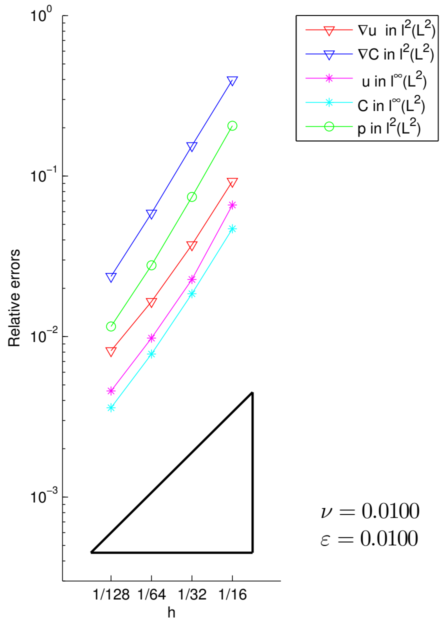


Table & Figure 8.19: Errors and EOC: $\nu = 0.01$, $\varepsilon = 0.01$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	8.71e-02	–	6.47e-01	–
1/32	3.44e-02	1.34	3.00e-01	1.11
1/64	1.37e-02	1.32	1.32e-01	1.18
1/128	6.75e-03	1.03	5.29e-02	1.32

h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	5.91e-02	–	4.57e-02	–
1/32	1.66e-02	1.83	1.59e-02	1.53
1/64	5.88e-03	1.50	5.57e-03	1.51
1/128	2.40e-03	1.30	2.18e-03	1.35

h	$e_p\ l^2(L^2)$	EOC	l_{max}
1/16	2.03e-01	–	5
1/32	7.01e-02	1.54	4
1/64	2.37e-02	1.57	4
1/128	8.31e-03	1.51	3

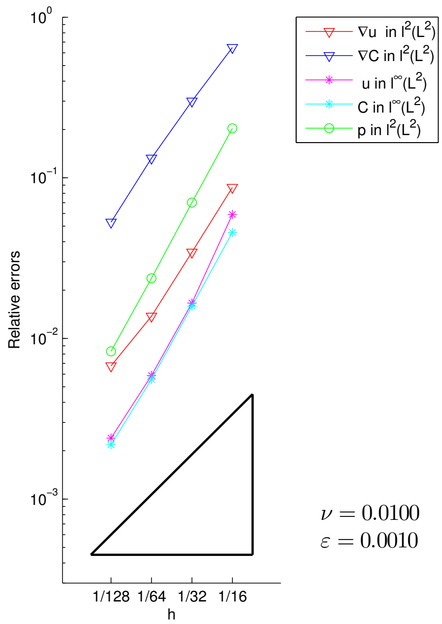


Table & Figure 8.20: Errors and EOC: $\nu = 0.01$, $\varepsilon = 0.001$, $dt = h/4$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	9.04e-02	–	8.39e-01	–
1/32	3.74e-02	1.27	5.32e-01	0.66
1/64	1.54e-02	1.28	3.34e-01	0.67
1/128	6.96e-03	1.14	1.62e-01	1.05
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	5.91e-02	–	4.75e-02	–
1/32	1.67e-02	1.82	1.67e-02	1.51
1/64	5.91e-03	1.50	5.90e-03	1.50
1/128	2.40e-03	1.30	2.29e-03	1.37
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	2.05e-01	–		5
1/32	7.13e-02	1.52		4
1/64	2.45e-02	1.54		4
1/128	8.80e-03	1.48		3

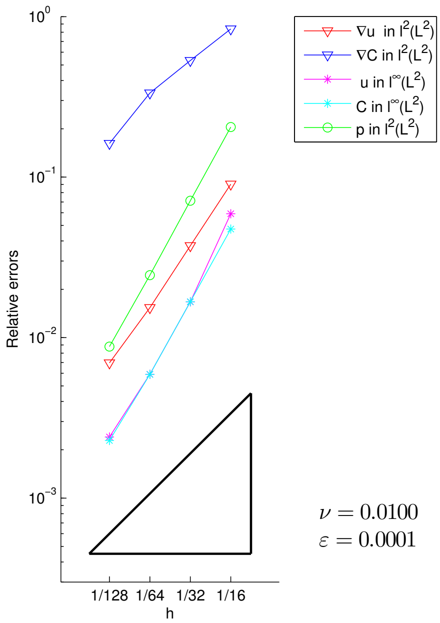


Table & Figure 8.21: Errors and EOC: $\nu = 0.01$, $\varepsilon = 0.0001$, $dt = h/4$

(5) The case $\varepsilon = 0$ and $\nu = 10^{-k}$, $k = -1, 0, 1, 2$

h	∇e_u $l^2(L^2)$	EOC	∇e_c $l^2(L^2)$	EOC
1/16	5.33e-02	–	1.09e+00	–
1/32	2.10e-02	1.34	8.34e-01	0.39
1/64	9.15e-03	1.20	7.35e-01	0.18
1/128	4.74e-03	0.95	5.58e-01	0.40
h	e_u $l^\infty(L^2)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	3.39e-02	–	4.75e-02	–
1/32	9.75e-03	1.80	1.69e-02	1.49
1/64	3.11e-03	1.65	6.78e-03	1.32
1/128	9.50e-04	1.71	2.81e-03	1.27
h	e_p $l^2(L^2)$	EOC		l_{max}
1/16	8.92e-01	–		5
1/32	6.04e-01	0.56		4
1/64	3.12e-01	0.95		4
1/128	1.14e-01	1.45		4

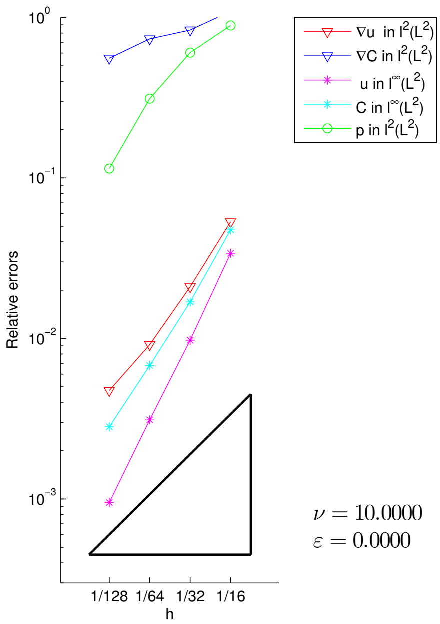


Table & Figure 8.22: Errors and EOC: $\varepsilon = 0$, $\nu = 10.0$

h	$\nabla e_u \ l^2(L^2)$	EOC	$\nabla e_c \ l^2(L^2)$	EOC
1/16	5.80e-02	–	1.08e+00	–
1/32	2.36e-02	1.30	8.15e-01	0.41
1/64	9.87e-03	1.26	7.16e-01	0.19
1/128	4.91e-03	1.01	5.44e-01	0.40
h	$e_u \ l^\infty(L^2)$	EOC	$e_c \ l^\infty(L^2)$	EOC
1/16	4.43e-02	–	5.15e-02	–
1/32	1.41e-02	1.65	1.94e-02	1.41
1/64	4.51e-03	1.65	7.35e-03	1.40
1/128	1.52e-03	1.57	2.92e-03	1.33
h	$e_p \ l^2(L^2)$	EOC		l_{max}
1/16	4.80e-01	–		5
1/32	2.01e-01	1.26		4
1/64	7.05e-02	1.51		4
1/128	2.32e-02	1.61		4

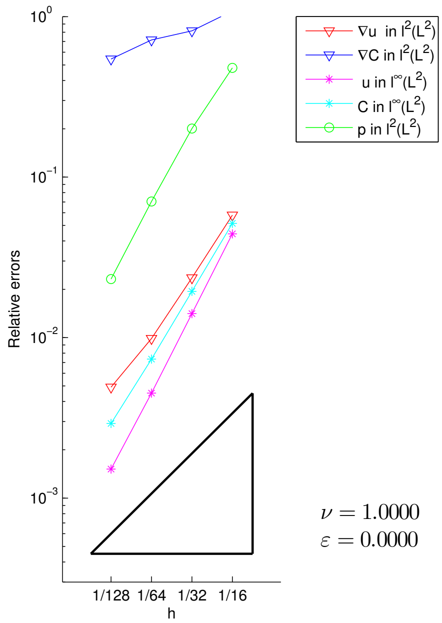


Table & Figure 8.23: Errors and EOC: $\varepsilon = 0$, $\nu = 1.0$

h	$\nabla e_u \ l^2(L^2)$	EOC	$\nabla e_c \ l^2(L^2)$	EOC
1/16	7.23e-02	–	9.90e-01	–
1/32	2.95e-02	1.29	7.16e-01	0.47
1/64	1.25e-02	1.24	6.25e-01	0.20
1/128	6.03e-03	1.05	4.81e-01	0.38
h	$e_u \ l^\infty(L^2)$	EOC	$e_c \ l^\infty(L^2)$	EOC
1/16	5.93e-02	–	5.44e-02	–
1/32	1.95e-02	1.60	2.09e-02	1.38
1/64	7.66e-03	1.35	8.36e-03	1.32
1/128	3.35e-03	1.19	3.58e-03	1.22
h	$e_p \ l^2(L^2)$	EOC		l_{max}
1/16	2.54e-01	–		5
1/32	9.29e-02	1.45		4
1/64	3.39e-02	1.46		4
1/128	1.32e-02	1.36		4

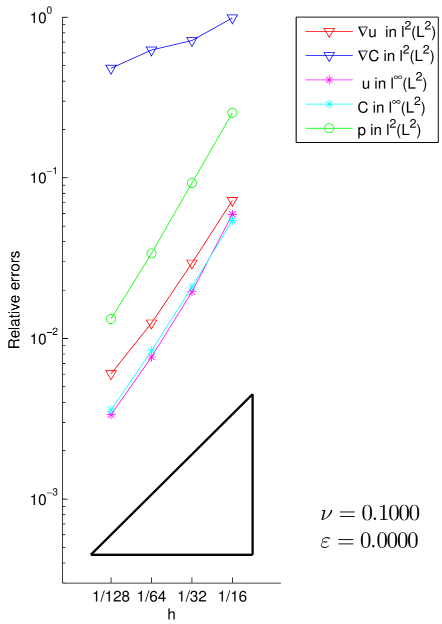


Table & Figure 8.24: Errors and EOC: $\varepsilon = 0$, $\nu = 0.1$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	9.09e-02	–	8.75e-01	–
1/32	3.83e-02	1.25	6.20e-01	0.50
1/64	1.68e-02	1.19	5.51e-01	0.17
1/128	8.00e-03	1.07	4.26e-01	0.37
h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	5.91e-02	–	4.78e-02	–
1/32	1.67e-02	1.82	1.71e-02	1.49
1/64	5.92e-03	1.50	6.30e-03	1.44
1/128	2.41e-03	1.30	2.49e-03	1.34
h	$e_p\ l^2(L^2)$	EOC		l_{max}
1/16	2.05e-01	–		5
1/32	7.14e-02	1.52		4
1/64	2.46e-02	1.54		4
1/128	8.94e-03	1.46		3

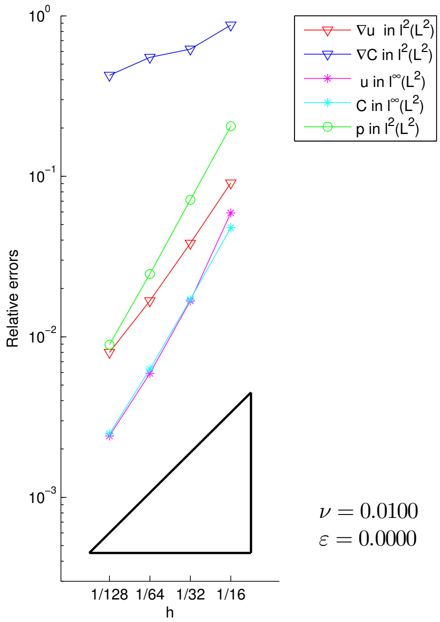


Table & Figure 8.25: Errors and EOC: $\varepsilon = 0$, $\nu = 0.01$, $dt = h/4$

Let us note that in the case **(5)** we have observed that the EOC for ∇e_c gets worse as the diffusion in the equation for the conformation tensor is missing for $\varepsilon = 0$.

8.2.3. Comparison with the implicit scheme

In order to demonstrate the efficiency of the semi-implicit linear pressure stabilized characteristics finite element scheme (6.25) derived in Chapter 6, we now compare the above numerical results with the results for the implicit nonlinear scheme (7.80) proposed by Tabata in [91]. The implicit scheme is solved by an analogous algorithm to Algorithm 1, subject to the same parameters and convergence criteria.

Number of inner iterations

In general, the inner iteration for the nonlinear scheme **(N)** requires more steps to converge than the inner iteration of the linear scheme **(L)**. Table 8.22 below captures the maximal number of inner iterations l_{max} for three different choices of viscosities.

(a) $\nu = 1.0$, $\varepsilon = 0.01$

h	**(L)**	**(N)**
1/16	5	7
1/32	4	6
1/64	4	5
1/128	4	4

(b) $\nu = 0.1$, $\varepsilon = 0.01$

h	**(L)**	**(N)**
1/16	5	11
1/32	4	6
1/64	4	5
1/128	4	4

(c) $\nu = 1.0$, $\varepsilon = 0$

h	**(L)**	**(N)**
1/16	5	8
1/32	4	6
1/64	4	5
1/128	4	4

Table 8.22.: Maximal number of inner iterations

Size of time increment

The advantage of the nonlinear scheme is that the positive definiteness of the discrete conformation tensor is not necessary to prove the stability and optimal error estimates, cf. Theorems 7.22 and 7.23. However, we have observed that the simulations, where the positive definiteness of the conformation tensor was not preserved, *did not converge*. This again indicates the importance of this physical property, which has been therefore controlled in all test cases for both schemes.

We have noticed that, for a particular choice of parameters, the nonlinear scheme requires smaller time increment in order to keep the positive definiteness of the discrete conformation tensor. More precisely, for $\varepsilon = \nu = 0.01$ in the linear case, the time increment $\Delta t = h/2$ is sufficiently small for all meshes in the sense that the inner iterations converge and the positive definiteness of $\mathbf{C}_h$ is preserved. However, the nonlinear scheme, for the same vicosities, needs smaller time increments for coarser meshes, i.e. $\Delta t = h/8$ for $N = 16$ and $\Delta t = h/4$ for $N = 32$.

As we have already mentioned in the linear case, the small viscosities might lead to the lost of the positive definiteness of the conformation tensor in a numerical approximation. For instance, for the choice $\nu = 0.01$ and $\varepsilon = 0.0001$, for $\Delta t = h/2$, we lose this physical feature and consequently the convergence at time $t = 0.41, 0.27, 0.20$, for $N = 16, 32, 64$, respectively, breaks down. Let us point out that this is consistent with the theoretical result for the linear scheme, cf. Proposition 6.9. The implicit function theorem, used to prove local in time existence of the positive definite conformation tensor as the solution to the linear scheme (6.25), requires the time increment to be small enough. Nevertheless, for $\Delta t = h/4$ the solution $\mathbf{C}_h$ is indeed positive definite for all N and we get slightly better than the first order convergence, see Table & Figure 8.21. We have compared this test case with the nonlinear scheme as well. Similarly, for $\Delta t = h/2$ and $N = 16, 32, 64$ the nonlinear scheme does not converge. Moreover, for meshes given by the division numbers 16 and 32 the solution is not convergent even for $\Delta t = h/4$. The relative errors and EOC for the time increments as in Table 8.23b are shown below, see Table & Figure 8.27.

(a) $\nu = \varepsilon = 0.01$

N	**(L)**	**(N)**
16	h/2	h/8
32	h/2	h/4
64	h/2	h/2
128	h/2	h/2

(b) $\nu = 0.01$, $\varepsilon = 0.0001$

N	**(L)**	**(N)**
16	h/4	h/8
32	h/4	h/8
64	h/4	h/4
128	h/2	h/2

Table 8.23.: Smallness of the time increment

For better illustration we give the results of the fully implicit nonlinear scheme for two of the above named cases, i.e.

(1) $\nu = 1.0$, $\varepsilon = 0.01$

(2) $\nu = 0.01$, $\varepsilon = 0.0001$.

(N1) The case $\nu = 1.0$ and $\varepsilon = 0.01$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	5.68e-02	–	4.02e-01	–
1/32	2.32e-02	1.29	1.41e-01	1.51
1/64	9.75e-03	1.25	4.59e-02	1.62
1/128	4.87e-03	1.00	1.52e-02	1.59
h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	4.31e-02	–	4.16e-02	–
1/32	1.36e-02	1.67	1.57e-02	1.41
1/64	4.26e-03	1.67	5.35e-03	1.55
1/128	1.40e-03	1.61	1.95e-03	1.46
h	$e_p\ l^2(L^2)$	EOC		l_{max}
1/16	4.80e-01	–		7
1/32	2.00e-01	1.26		6
1/64	6.79e-02	1.56		5
1/128	2.06e-02	1.72		4

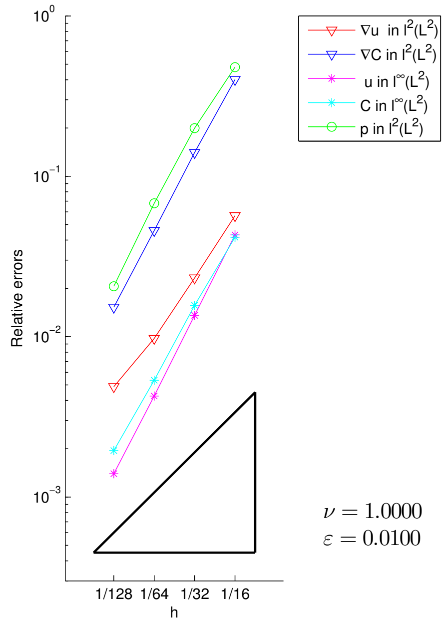


Table & Figure 8.26: **(N1)** Errors and EOC: $\nu = 1.0$, $\varepsilon = 0.01$

(N2) The case $\nu = 0.01$, and $\varepsilon = 0.0001$

h	$\nabla e_u\ l^2(L^2)$	EOC	$\nabla e_c\ l^2(L^2)$	EOC
1/16	9.07e-02	–	9.66e-01	–
1/32	3.86e-02	1.23	6.33e-01	0.61
1/64	1.54e-02	1.32	3.65e-01	0.79
1/128	7.27e-03	1.09	1.68e-01	1.12
h	$e_u\ l^\infty(L^2)$	EOC	$e_c\ l^\infty(L^2)$	EOC
1/16	5.58e-02	–	5.32e-02	–
1/32	1.49e-02	1.91	1.81e-02	1.55
1/64	5.38e-03	1.47	6.30e-03	1.52
1/128	3.79e-03	0.50	3.48e-03	0.85
h	$e_p\ l^2(L^2)$	EOC		l_{max}
1/16	2.48e-01	–		5
1/32	8.57e-02	1.53		4
1/64	2.76e-02	1.63		4
1/128	9.88e-03	1.48		4

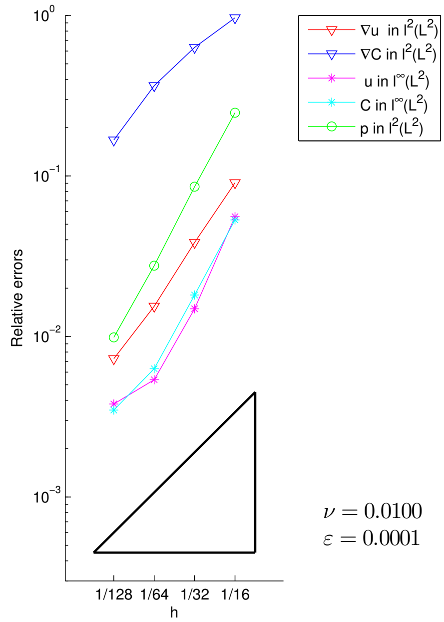


Table & Figure 8.27: **(N2)** Errors and EOC: $\nu = 0.01$, $\varepsilon = 0.0001$

We can conclude that the advantage of the semi-implicit linear scheme compared to the implicit nonlinear scheme is the shorter computational time. On the other hand, the nonlinear scheme does not require positive definiteness property in order to prove theoretical error estimates.

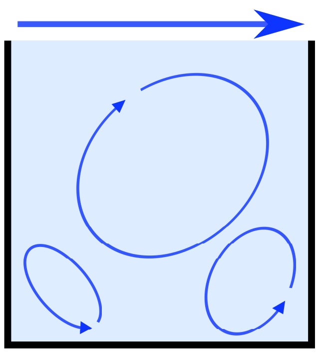
8.3. Lid-driven cavity flow problem

The lid-driven cavity flow problem has been the focus of interest in the computational fluid dynamic field since the pioneering paper by Burggraff [30] in 1966. It has been widely used to test new codes and to evaluate numerical methods. Despite its simple geometry and simple boundary conditions, the problem retains all the flow physics with counter rotating vortices appearing at the corners of the cavity. It is one of the benchmark problems for viscous incompressible fluid flow. The Newtonian fluid for the increasing Reynolds number has been mostly studied, cf., e.g., [48, 57] and references therein. Theoretical and experimental results for the lid-driven cavity flow of viscoelastic fluids can also be found in literature, see, e.g., [59, 115, 123].

In this section, we study a viscoelastic fluid flow inside a square cavity whose upper lid is moving with non-zero tangential velocity. To avoid the discontinuous velocities on the cavity boundary, we prescribe a polynomial velocity profile vanishing at the corners. We choose different values of the viscosities ν and ε to illustrate the efficiency of our numerical scheme (6.25).

8.3.1. Problem setting

The computational domain is again a unit square $(0,1)^2$. We execute computations up to a final time $T = 5$ for the mesh given by the division number $N = 128$. Further, we fix the time increment to be $\Delta t = h/2$ and once more the constant $\delta_0 = 1$ in (6.23). There are no external forces, i.e. $(\mathbf{f}, \mathbf{F}) = (\mathbf{0}, \mathbf{0})$. We solve the Navier-Stokes-type problem, $\mathbf{w} = \mathbf{u}_h^{n-1}$. The boundary conditions read



$$\mathbf{u}(x_1, 1, t) = (16x_1^2(1-x_1)^2, 0) \qquad \frac{\partial \mathbf{C}}{\partial \mathbf{n}}(x_1, x_2, t) = \mathbf{0} \quad (x_1, x_2) \in \partial(0,1)^2.$$

$$\mathbf{u} = \mathbf{0} \quad \text{else}$$

Test case (A)
Initial conditions:

$$\mathbf{u}(x_1, x_2, 0) : -\nu\Delta\mathbf{u} + \nabla p = \mathbf{0}$$
$$\mathbf{C}(x_1, x_2, 0) = \mathbf{I}$$
$$(x_1, x_2) \in [0,1]^2$$

Viscosities:

(A1) $\nu = 1.0$, $\varepsilon = 10^{-k}$, $k = 2, 3, 4$

(A2) $\nu = 0.1$, $\varepsilon = 10^{-k}$, $k = 2, 3$

(A3) $\nu = 0.01$, $\varepsilon = 0.0001$

(A4) $\varepsilon = 0$, $\nu = 10^{-k}$, $k = 0, 1, 2$

Test case (B)
Initial conditions:

$$\mathbf{u}(x_1, x_2, 0) = \mathbf{0}$$
$$\mathbf{C}(x_1, x_2, 0) = \frac{\sqrt{2}}{2}\mathbf{I}$$
$$(x_1, x_2) \in [0,1]^2$$

Viscosities:

(B1) $\nu = 1.0$, $\varepsilon = 0.01$

(B2) $\nu = 0.1$, $\varepsilon = 0.001$

(B3) $\nu = 0.01$, $\varepsilon = 0.0001$

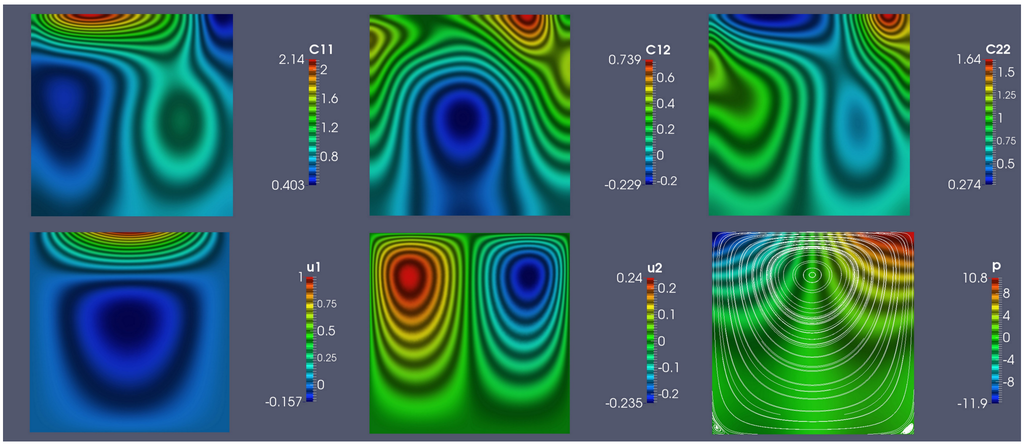
(B4) $\varepsilon = 0$, $\nu = 1.0$

8.3.2. Numerical results

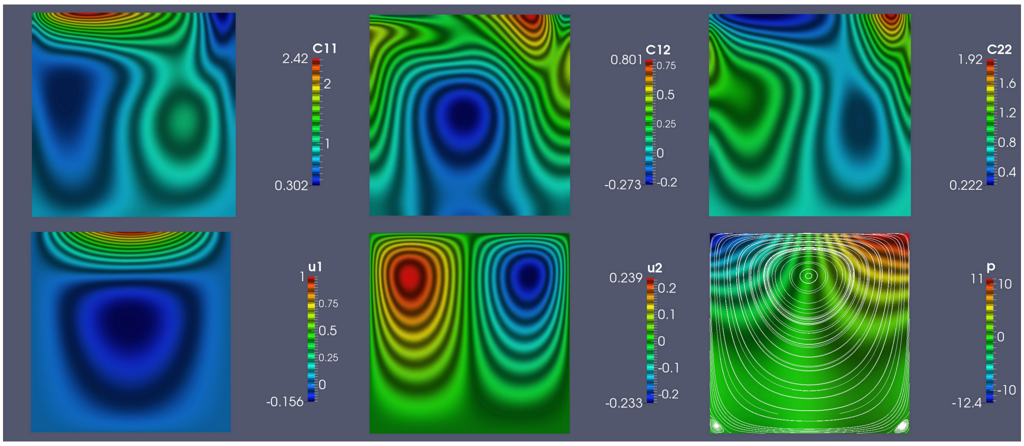
Figures presented below depict the solutions of the cavity flow problem at final time. Precisely, we plot each component of the velocity and the conformation tensor, separately. On the last picture we present the pressure with streamlines of the velocity.

(A1) The case $\nu = 1.0$, $\varepsilon = 10^{-k}$, $k = 2, 3, 4$

(a) $\varepsilon = 0.01$



(b) $\varepsilon = 0.001$



(c) $\varepsilon = 0.0001$

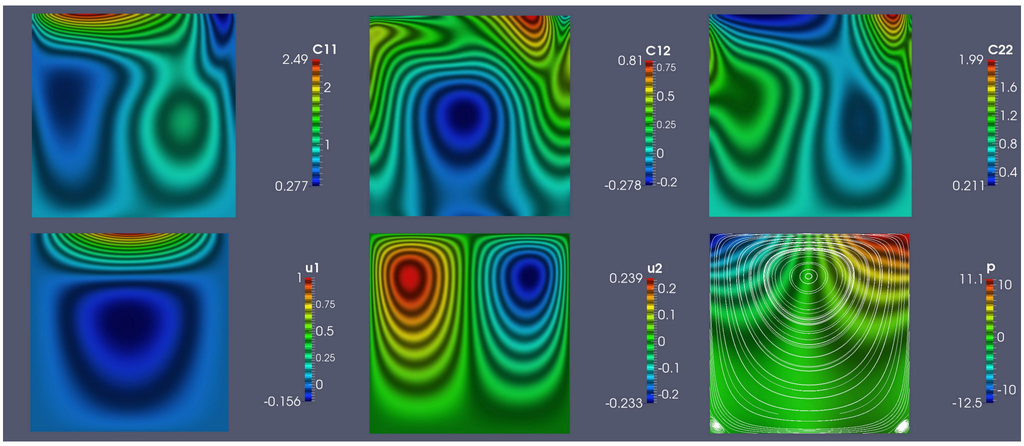
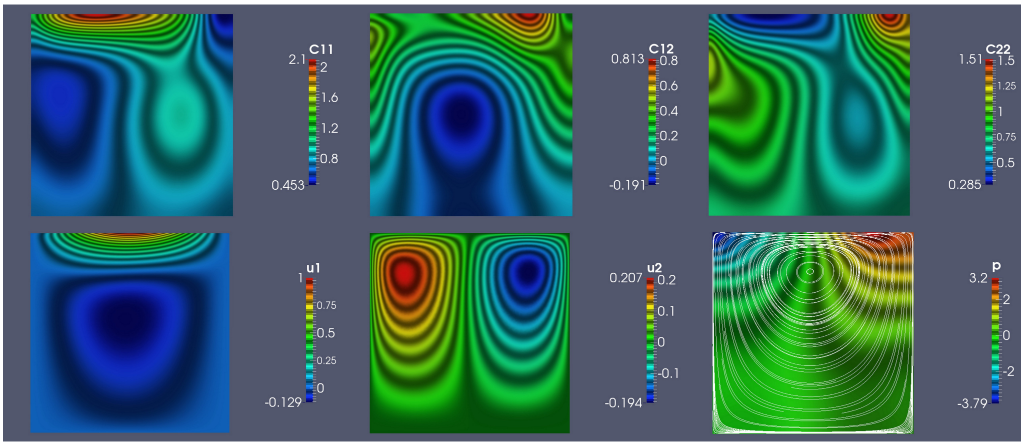


Figure 8.29.: **(A1)** Cavity flow: $\nu = 1.0$, $\varepsilon = 10^{-k}$, $k = 2, 3, 4$

(A2) The case $\nu = 0.1$, $\varepsilon = 10^{-k}$, $k = 2, 3$

(a) $\varepsilon = 0.01$



(b) $\varepsilon = 0.001$

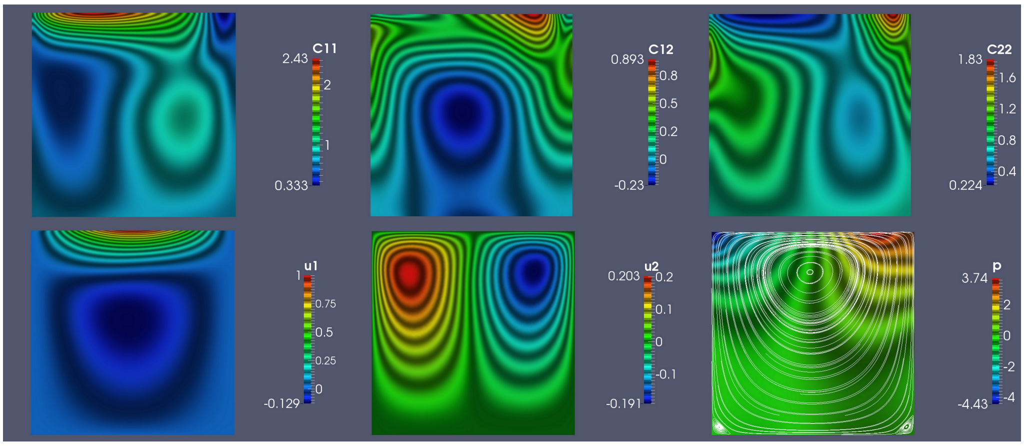


Figure 8.30.: **(A2)** Cavity flow: $\nu = 0.1$, $\varepsilon = 10^{-k}$, $k = 2, 3$

(A3) The case $\nu = 0.01$, $\varepsilon = 0.0001$

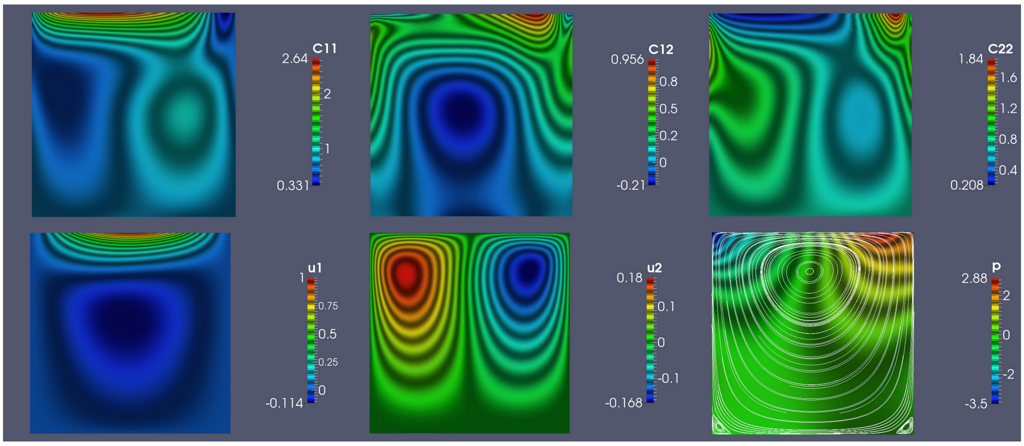
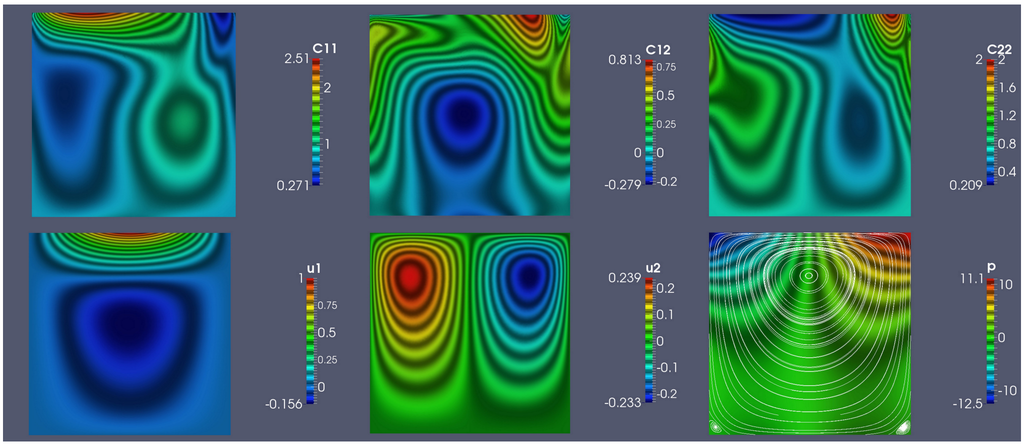


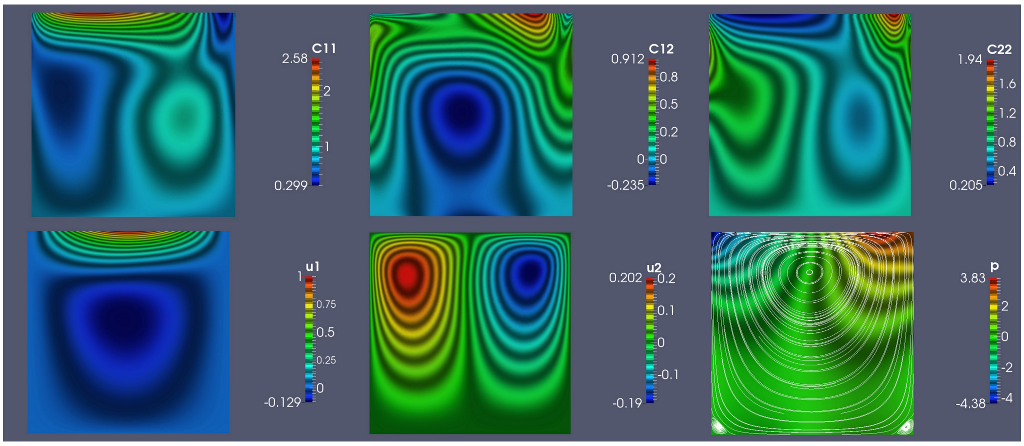
Figure 8.31.: **(A3)** Cavity flow: $\nu = 0.01$, $\varepsilon = 0.0001$

(A4) The case $\varepsilon = 0$, $\nu = 10^{-k}$, $k = 0, 1, 2$

(a) $\nu = 1.0$



(b) $\nu = 0.1$



(c) $\nu = 0.01$

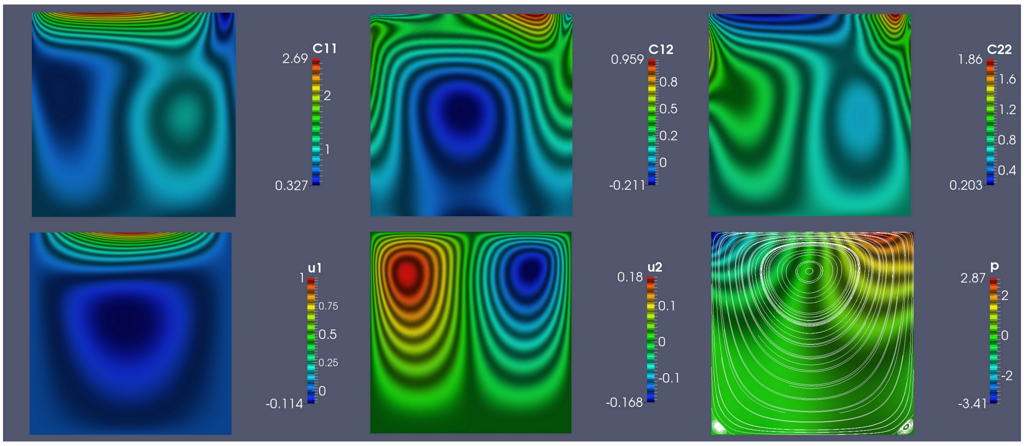


Figure 8.32.: **(A4)** Cavity flow: $\varepsilon = 0$, $\nu = 10^{-k}$, $k = 0, 1, 2$

Case (B)

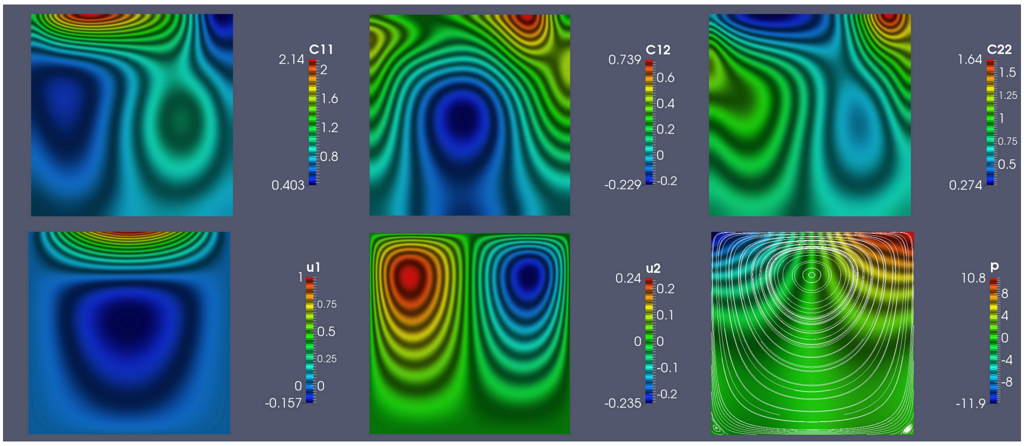


Figure 8.33.: **(B1)** Cavity flow: $\nu = 1.0$, $\varepsilon = 0.01$

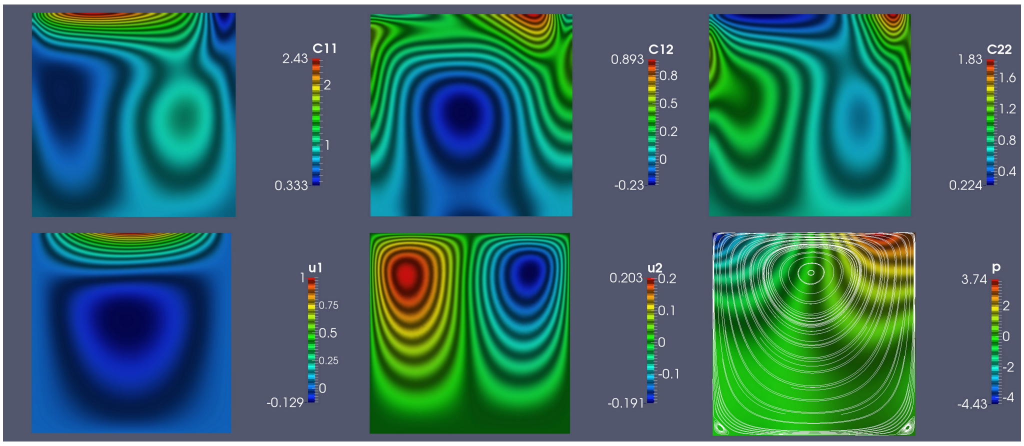


Figure 8.34.: **(B2)** Cavity flow: $\nu = 0.1$, $\varepsilon = 0.001$

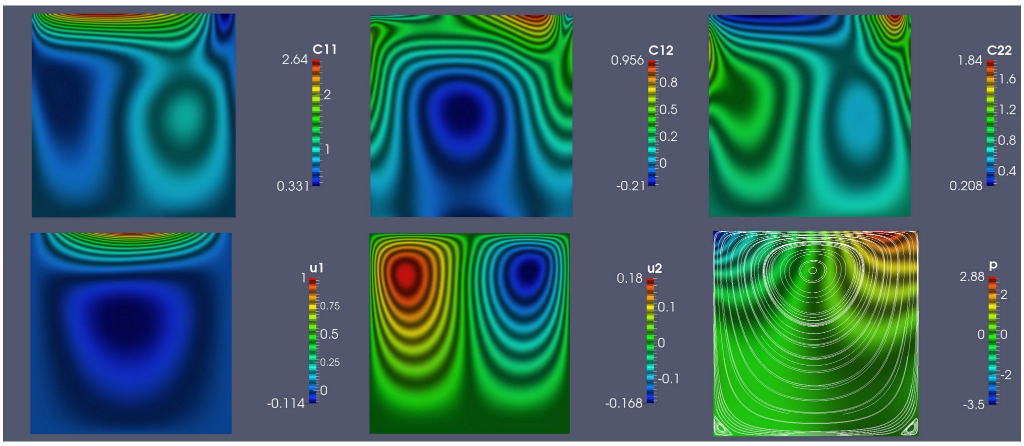


Figure 8.35.: **(B3)** Cavity flow: $\nu = 0.01$, $\varepsilon = 0.0001$

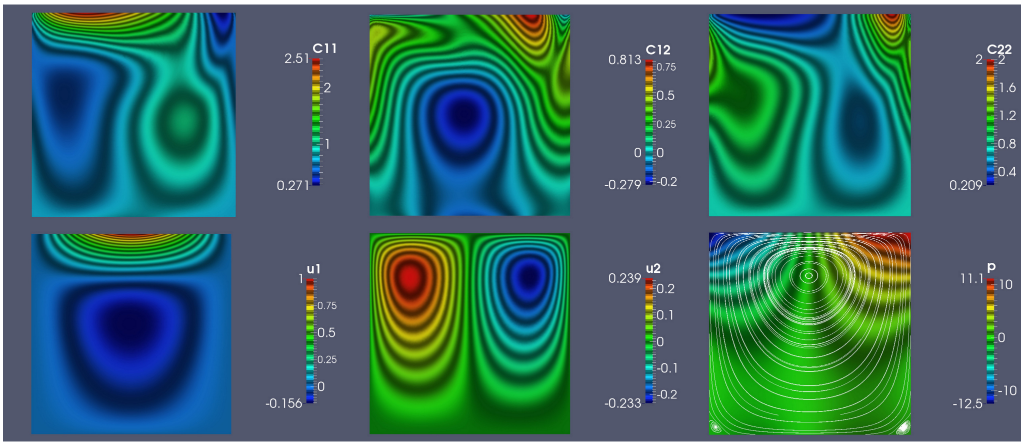


Figure 8.36.: **(B4)** Cavity flow: $\varepsilon = 0$, $\nu = 1.0$

We have clearly seen, that all important features of the physical solution, such as large center vortex and small corner vertices are captured very well. Interestingly, the choice of initial data and different parameter settings have only small influence on the qualitative behaviour of the solution. We note that the numerical solution of the conformation tensor is positive definite in all the above test cases for the lid-driven cavity flow problem.

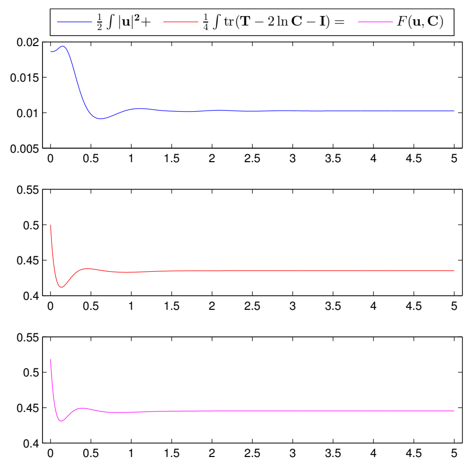
Free energy

In Section 3.4 we have shown the free energy estimate for the diffusive Peterlin model (3.2). Let us recall that the free energy is given by

$$F(\mathbf{u}, \mathbf{C}) = \frac{1}{2} \int_\Omega |\mathbf{u}|^2 \, dx + \frac{1}{4} \int_\Omega \mathrm{tr}\,(\mathbf{T} - 2 \ln \mathbf{C} - \mathbf{I}) \, dx \, . \tag{8.3}$$

For the smooth solution to the problem (3.2) equipped with the homogeneous Dirichlet and no-flux boundary conditions for the velocity and conformation tensor, respectively, it holds that $F(\mathbf{u}, \mathbf{C})$ is non-increasing in time, cf. Theorem 3.11.

The velocity in the lid-driven cavity flow problem *does not* satisfy the homogeneous Dirichlet boundary condition. Nevertheless, we have studied the time behaviour of the free energy (8.3) computed for the above test cases of the lid-driven cavity flow problem. We have observed that it is not only bounded, but indeed non-increasing up to initial oscillation caused by the incompatibility of the initial data. Several test cases demonstrate that the initial oscillation depend on the smallness of ν. In what follows we present a few graphs depicting the *kinetic energy*, the *entropy* and their sum, i.e. the *free energy*.



(A) Free energy: $\nu = 0.01$, $\varepsilon = 0.0001$

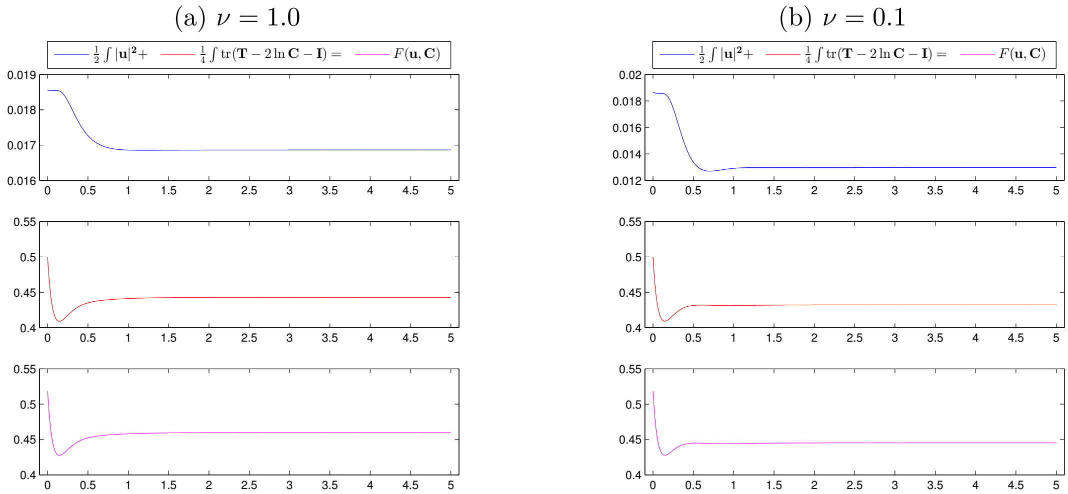


Figure 8.38.: **(A)** Free energy: $\varepsilon = 0.01$, $\nu = 10^{-k}$, $k = 0, 1$

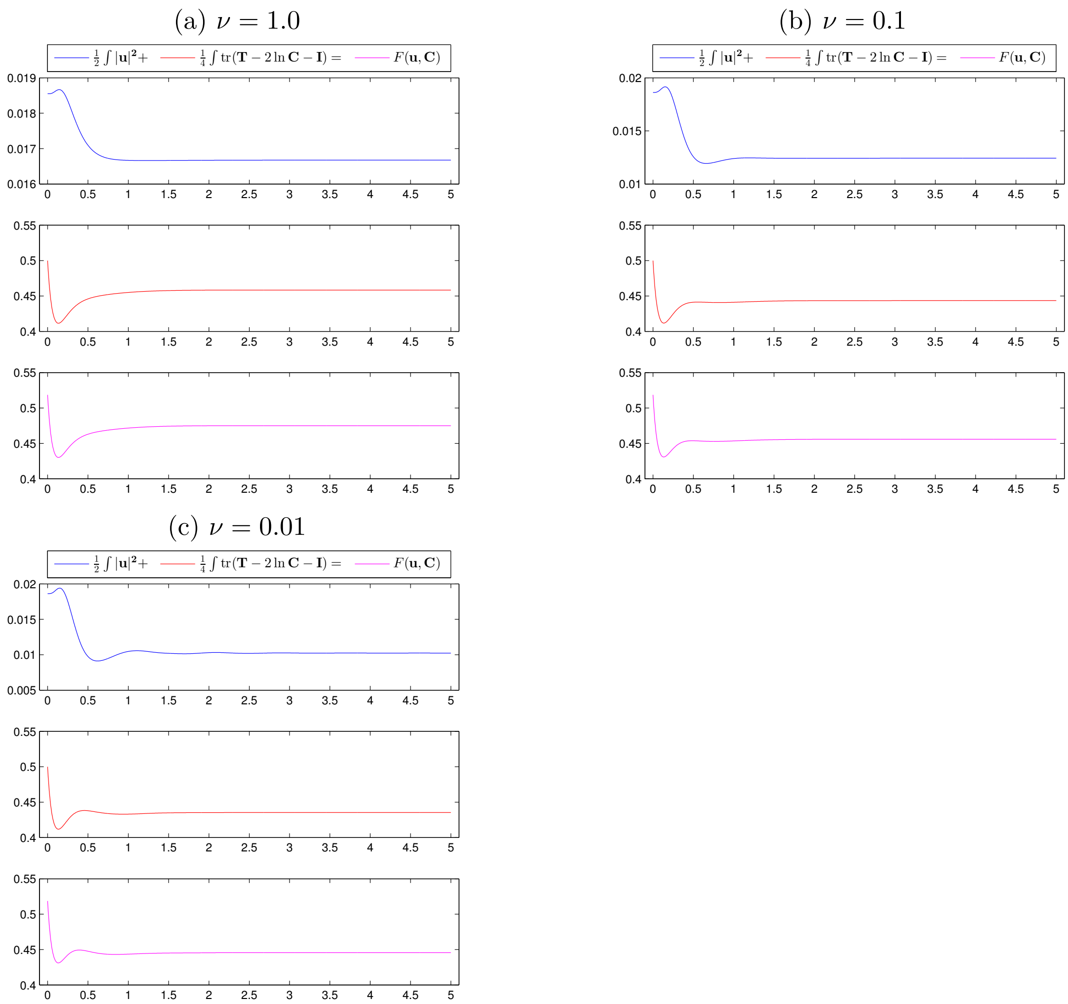


Figure 8.39.: **(A)** Free energy: $\varepsilon = 0$, $\nu = 10^{-k}$, $k = 1, 2$

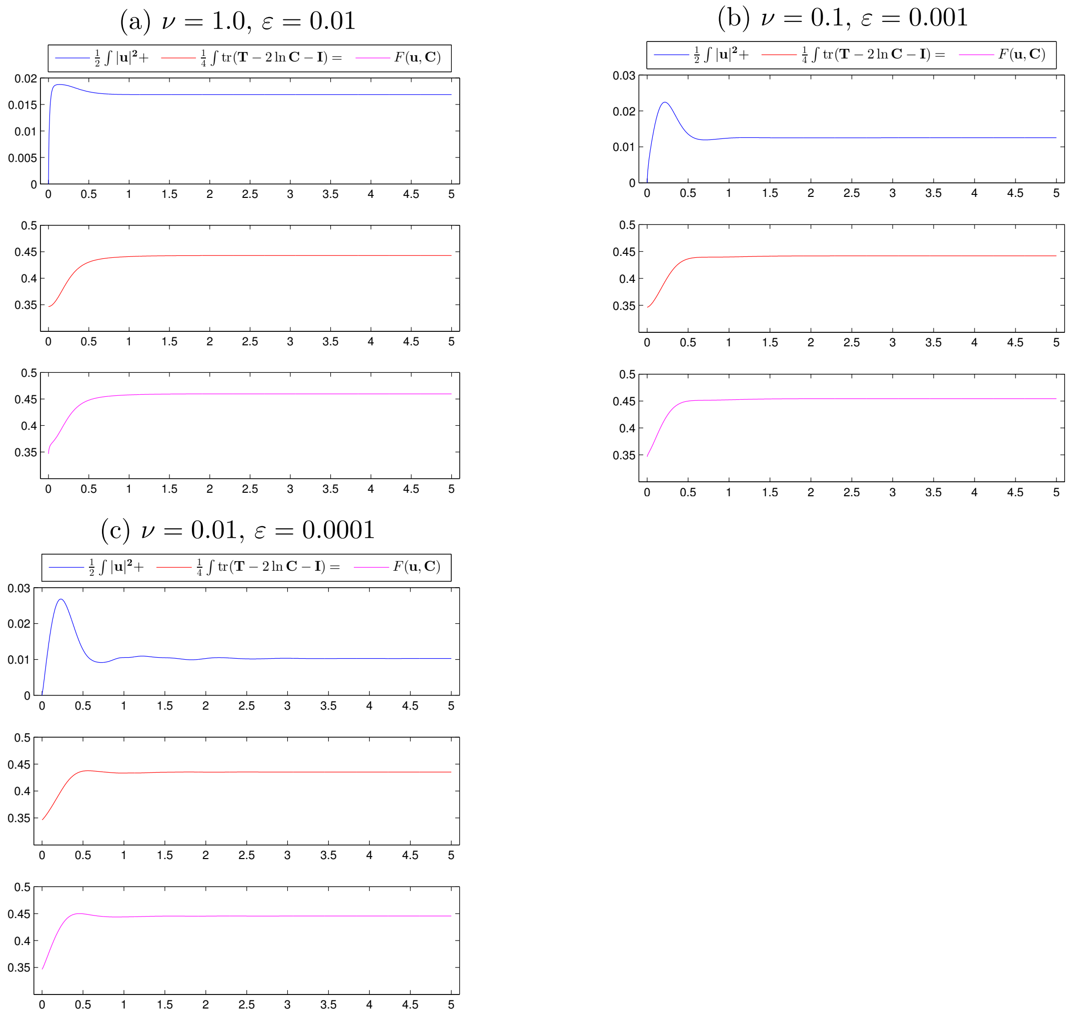


Figure 8.40.: **(B)** Free energy

We note that the two test cases **(A)** and **(B)** differed in the initial data.

We have presented extensive numerical simulations for the semi-implicit linear pressure-stabilized characteristics finite element scheme (6.25) derived in Chapter 6. Firstly, the implementation and the algorithm of the scheme have been described. In Section 8.2 several test cases were considered in order to confirm theoretical error estimates. In most of the cases the experimental order of convergence was superlinear, i.e. better than the first order.

It is believed that the lost of the positive definiteness is one of the reasons of the so-called high Weissenberg number problem for the Oldroyd-B model. We have observed that this physical property is an important feature that also influenced the convergence of the numerical schemes tested in this work. The fact that the time increment required to preserve the positive definiteness of the discrete conformation tensor has to be small, shows the consistency with the theory. It means, Δt depends on the mesh size and viscosities, but its upper bound seems to be independent of the discrete solution.

It has turned out that for the real computations the proposed semi-implicit linear scheme is in several cases more efficient than the fully implicit nonlinear scheme. The comparison has shown that, in general, the semi-implicit scheme needs less inner iterations and allows a bigger time increment in order to get satisfactory results.

Finally, several results for the lid-driven cavity flow problem have been presented. Moreover, graphs of the free energy of the Peterlin viscoelastic fluid as defined in Section 3.4 have been included and the behaviour of the discrete free energy has been discussed.

The figures in this thesis were produced using MATLAB [102], ParaView [4] and Inkscape [72].

Conclusions

Fluid flow is an interesting and challenging problem from both the analytical as well as numerical point of view. There is an endless demand for mathematical models that could describe the real behaviour and properties of fluids, and at the same time, for numerical schemes that could satisfactory solve these complex models.

In this thesis we have proposed a macroscopic dumbbell-based model for incompressible complex fluids, the Peterlin viscoelastic model, for which both the theoretical and numerical results have been obtained.

The next point of our interest is the molecular level of complex fluids. The micro structure of polymeric materials strongly influences their properties. It is thus interesting to study the coupling of the macroscopic conservation laws with the Fokker-Planck equation.

Firstly, we have proven global in time existence of weak solutions in two space dimensions for both the diffusive Peterlin model (3.2) and the generalized diffusive Peterlin model (5.1), see Theorem 4.4 and Theorem 5.2, respectively. In order to prove uniqueness of the weak solution to (3.2), cf. Theorem 4.6, we have firstly shown in Theorem 4.5 the higher regularity of the weak solution for more regular data. We have proceed analogously in the case of model (5.1), see Theorems 5.3 and 5.4.

The global existence of weak solutions in the three-dimensional case as well as the existence of strong solutions in both dimensions remain open questions. An important prerequisite to find the strong solutions is the positive definiteness of the conformation tensor proven in Lemma 3.3. Another interesting point for future study is the analysis of the general diffusive Peterlin model (3.1).

Secondly, the numerical solution of the proposed viscoelastic model has been investigated. The linear semi-implicit pressure-stabilized characteristics finite element scheme for the Oseen-type Peterlin model (6.1) has been proposed, see Definition 6.7. In Theorems 7.13 and 7.14 we have presented stability and optimal first order error estimates, respectively. The higher energy bounds, for more regular data, have been obtained in Theorem 7.16, analogously as in the continuous case.

In future, our goal is to study the convergence in the norms corresponding to the higher a priori estimates. Further, we would like to extend the numerical results for the fully nonlinear diffusive Peterlin model, i.e. with the nonlinear convective terms.

Let us note that the theoretical results for the linear scheme have been obtained subject to Hypothesis 2 on the positive definiteness of the discrete solution of the conformation tensor. The local in time existence of such solution has been obtained in Proposition 6.9.

We aim to prove this property globally in time. One possible approach is to study the discrete free energy. For a smooth solution of the diffusive Peterlin model there is indeed the non-increasing free energy, see Theorem 3.11. If we were able to show that the analogous discrete free energy, containing the logarithm of the conformation tensor, was bounded for all time steps, then the implicit function theorem would yield the desired result.

Finally, theoretical convergence order for the Oseen-type Peterlin model has been confirmed by a series of numerical experiments, see Section 8.2. In Section 8.3 we have tested our scheme for the lid-driven cavity flow problem, a well-known benchmark problem in the computational fluid dynamics. We would like to point out that the numerical experiments in Section 8.3 have been performed for a fully nonlinear model with the nonlinear convective terms.

In future, it might be interesting to consider the contraction flow with different boundary conditions, that play an important role for the viscoelastic fluid flows.

A Evolution problems

This chapter aims to provide useful results of the functional analysis that have been used in order to prove the existence and uniqueness of the weak solution to the diffusive Peterlin model, see Chapter 4. Firstly, an abstract setting for evolution problems is described. In the second section the regularity and properties of the eigenfunctions of the Stokes and the Laplace operators are discussed. Finally, a general existence theory for the solutions to the systems of ordinary differential equations is included.

A.1. Weak solutions

We give the definition of the so-called evolution triple together with lemmas useful to analyse weak solutions to some evolution problems. For more details and proofs we refer to, e.g., [52, 128, 142].

Evolution triple

The definition of the evolution triple, sometimes also called Gelfand's triple, follows.

Definition A.1. *(evolution triple)*
Let X be a real reflexive Banach space and Y be a real Hilbert space. Let us assume that X is continuously and densely contained in Y. Then the triple (X, Y, X^*) is called the evolution triple.

If (X, Y, X^*) is an evolution triple, then for every element $f \in Y$ there exists an element $F \in X^*$ such that $\langle F, \cdot \rangle = (f, \cdot)_Y$. Since the space X is dense in Y, the opposite is also true. This means that every element $f \in Y$ is determined by an element $F \in X^*$. Thus $Y \subset X^*$ and the embedding is continuous. Moreover, by the Riesz representation theorem, cf. [3], we can identify Y and Y^*, and we arrive at the inclusions

$$X \subset Y \equiv Y^* \subset X^*,$$

where each space is dense in the following one and the injections are continuous. As a consequence, the scalar product in Y of $f \in Y$ and $u \in X$ is the same as the duality between X^* and X

$$\langle f, u \rangle_{X^* \times X} = (f, u)_Y. \tag{A.1}$$

A typical example of the evolution triple can be found in the theory of the Navier-Stokes equations, where the standard functional spaces are $V := H^1_{0,div}(\Omega)^2$, $H := L^2_{div}(\Omega)^2$. They indeed form the evolution triple (V, H, V^*). To study the equation for the conformation tensor in the diffusive Peterlin model, we work with the spaces $W := H^1(\Omega)^{2\times 2}$ and $L^2(\Omega)^{2\times 2}$, which again assemble the evolution triple $(W, L^2(\Omega), W^*)$.

Compactness

We employ the Lions-Aubin-Simon lemma, cf. [22, 128, 131, 134], in order to get the compactness and consequently the strong convergence, which allow us to pass to the limit with the nonlinear terms in the Galerkin approximation, see the proof of existence of a weak solution, i.e. proof of Theorem 4.4.

Lemma A.2. *(Lions-Aubin-Simon lemma)*
Let X_0, X and X_1 be Banach spaces such that X_0, X_1 are reflexive and the following embedding holds

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1.$$

Let $T > 0$ be a fixed finite number. For two numbers $1 \le \alpha_0, \alpha_1 \le \infty$ we define the space

$$Y := \left\{u \in L^{\alpha_0}(0,T;X_0),\, u' \in L^{\alpha_1}(0,T;X_1)\right\}$$

equipped with the norm

$$\|u\|_Y := \|u\|_{L^{\alpha_0}(X_0)} + \|u\|_{L^{\alpha_1}(X_1)},$$

which makes it the Banach space. Then
(i) *if $\alpha_0 < \infty$ then embedding of Y in $L^{\alpha_0}(0,T;X)$ is compact.*
(ii) *if $\alpha_0 = \infty$ and if $\alpha_1 > 1$, the embedding of Y in $C([0,T];X)$ is compact.*

Remark A.3. *The result (i) of the above lemma, for finite numbers $\alpha_0, \alpha_1 > 1$, is known as the Lions-Aubin lemma, cf. [7, 52, 95, 142].*

Let us point out that our evolution triples satisfy the assumptions on the functional spaces of the above Lions-Aubin-Simon lemma.

To find the appropriate functional space for the time derivative of the Galerkin approximation, we keep in mind the following standard result, see [52, 142], which also assures that the initial condition for the weak solution makes sense.

Lemma A.4.
Let X be a given Banach space and let u, g be two functions in $L^1(0,T;X)$. Then the following three conditions are equivalent

(i) *u is a.e. equal to a primitive function of g,*

$$u(t) = \xi + \int_0^t g(s)\, ds, \qquad \xi \in X,\ a.e.\ t \in [0,T]$$

(ii) *for each test function $\phi \in C_0^\infty(\Omega)$ it holds that* $\int_0^T u(t)\phi(t)\, dt = -\int_0^T g(t)\phi(t)$

(iii) *for each $\eta \in X^*$,* $\frac{\mathrm{d}}{\mathrm{dt}}\langle u, \eta\rangle = \langle g, \eta\rangle$ *in the scalar distributional sense on $(0,T)$.*

If the assertions (i) - (iii) are satisfied, u is a.e. equal to a continuous function from $[0,T]$ into X.

Passage to the limit

Let us consider a sequence $\{u_m\}_{m=1}^{\infty}$ uniformly bounded in $L^\infty(0,T;Y) \cap L^2(0,T;X)$. Then there exists a subsequence $\{u_{m'}\}_{m'=1}^{\infty}$, which converges in the *-weak topology of $L^\infty(0,T;Y)$ to an element $u \in L^\infty(0,T;Y)$, i.e.

$$\int_0^T (u_{m'}(t) - u(t), v(t))\ dt \to 0, \quad \forall\ v \in L^1(0,T;Y),$$

and there exists a $L^2(0,T;X)$-weakly convergent subsequence, still denoted by $u_{m'}$, to an element $u^* \in L^2(0,T;X)$

$$\int_0^T \langle u_{m'}(t) - u^*(t), v(t)\rangle\ dt \to 0, \quad \forall\ v \in L^2(0,T;X^*).$$

By (A.1) we see that

$$\int_0^T (u_{m'}(t), v(t))\ dt \to \int_0^T (u^*(t), v(t))\ dt\,, \quad \forall\ v \in L^2(0,T;Y),$$

and combined with the *-weak convergence in $L^\infty(0,T;Y)$ we get

$$\int_0^T (u_{m'}(t) - u^*(t), v(t))\ dt\ = 0$$

for every $v \in L^2(0,T;Y)$. Thus

$$u = u^* \in L^\infty(0,T;Y) \cap L^2(0,T;X).$$

Uniqueness study

The key lemma for the uniqueness study follows. It is a particular case of a general theorem of interpolation of [88], which can be found in, e.g, [52, 128, 142].

Lemma A.5.
Let (X,Y,X^*) be an evolution trile. If a function u belongs to $L^2(0,T;X)$ and its derivative u' to $L^2(0,T;X^*)$, then u is almost everywhere equal to a function continuous from $[0,T]$ into Y. Further, the functions

$$t \to \|u(t)\|_Y^2\,,\ t \to \langle u(t), u'(t)\rangle$$

are both integrable on $[0,T]$ and the following equality holds, in the distributional sense, on $(0,T)$

$$\frac{\mathrm{d}}{\mathrm{dt}}\, \|u\|_Y^2 = 2\langle u, u'\rangle.$$

A.2. Ordinary differential equations

The Galerkin approximation of a weak solution usually yields the initial value problem for the nonlinear system of differential equations. We present an existence result for systems of ordinary differential equations that guarantees the existence of the Galerkin approximate solution, cf. [114].

Theorem A.6.
We consider the nonlinear system of ordinary differential equations together with the initial condition

$$\begin{aligned} y' &= f(t, y) \\ y(0) &= y^0, \end{aligned} \tag{A.2}$$

where $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $y^0 : \mathbb{R}^d \to \mathbb{R}^d$ are given. Then the following three assertions hold true

(*i*) ***Picard-Lindelöf theorem:***

Let f be continuous and Lipschitz-continuous in y.
Then there exists a unique $y \in C^1([0,T])$ that solves (A.2).

(*ii*) ***Peano theorem:***

Let f be continuous and bounded.
Then there exists at least one $y \in C^1([0,T])$ that solves (A.2).

(*iii*) *Let f be continuous.*
Then (A.2) *has at least one local solution. That is there exists a subinterval $[0,T']$ with $0 < T' \leq T$ and at least one $y \in C^1([0,T'])$ that solves* (A.2) *for $t \in [0,T']$.*

A.3. The Stokes and Laplace operators

The regularity and eigenfunctions of the Stokes and Laplace operator are employed to prove the higher regularity of the weak solution to the diffusive Peterlin model, see Theorem 4.5. Let us provide the classical results for completeness.

Regularity of solutions

The straightforward consequence of the classical elliptic regularity, cf. [49, 142], follows.

Proposition A.7. *(regularity of solutions to the Poisson problem)*
Let Ω be an open bounded set of class C^r, $r = \max\{m+2, 2\}$ for an integer $m > 0$. Let $\mathbf{u} \in W^{1,2}(\Omega)^d$ be the weak solution of the Poisson equation equipped with the Neumann boundary condition

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} \quad \textit{in } \Omega \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} &= \mathbf{0} \quad \textit{on } \partial\Omega. \end{aligned}$$

If $\mathbf{f} \in W^{m,2}(\Omega)^d$*, then* $\mathbf{u} \in W^{m+2,2}(\Omega)^d$ *and there exists a constant* $C = C(m, \Omega)$ *such that*

$$\|\mathbf{u}\|_{m+2,2} \leq C\Big\{ \|\mathbf{f}\|_{m,2} + \|\mathbf{u}\|_2 \Big\}.$$

An analogous result is true for the Stokes operator, see, e.g., [142].

Proposition A.8. *(regularity of solutions to the Stokes problem)*
Let Ω *be an open bounded set of class* C^r*,* $r = \max\{m+2, 2\}$ *for an integer* $m > 0$*. Let us assume that* $\mathbf{u} \in W^{2,\alpha}(\Omega)^d$*,* $p \in W^{1,\alpha}(\Omega)$*,* $1 < \alpha < \infty$ *is the solution to the generalized Stokes problem*

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \textit{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \textit{in } \Omega \\ \mathbf{u} &= \mathbf{0} \quad \textit{on } \partial\Omega. \end{aligned}$$

If $\mathbf{f} \in W^{m,\alpha}(\Omega)^d$*, then*

$$\mathbf{u} \in W^{m+2,\alpha}(\Omega)^d, \;\; p \in W^{m+1,\alpha}(\Omega).$$

Moreover, there exists a constant $C = C(\alpha, m, \nu, \Omega)$ *such that*

$$\|\mathbf{u}\|_{m+2,\alpha} + \|p\|_{m+1,\alpha} \leq C\Big\{ \|\mathbf{f}\|_{m,\alpha} + d_\alpha \|\mathbf{u}\|_\alpha \Big\},$$

where $d_\alpha = 0$ *for* $\alpha \geq 2$*,* $d_\alpha = 1$ *for* $1 < \alpha < 2$*.*

Eigenfunctions as basis functions

In order to obtain H^2-convergences of the projections of the initial data on the finite-dimensional subspaces, cf. (4.21), we use the eigenvalues of the Stokes and the Laplace operators as the basis functions of the spaces V and W, respectively. We employ the following theorem, see, e.g., [95, 142].

Theorem A.9. *(eigenvectors of the Stokes operator)*
There exists a countable set $\{\lambda_r\}_{r=1}^\infty \subset \mathbb{R}$ *and a corresponding family of eigenvectors* $\{\omega^r\}_{r=1}^\infty \subset V$ *solving*

$$((\omega^r, \mathbf{v})) = \lambda_r (\omega^r, \mathbf{v}), \;\; \mathbf{v} \in V$$

such that $\{\omega^r\}_{r=1}^\infty$ *forms a basis of* V *and*

(i) $\quad (\omega^r, \omega^s) = \delta_{rs}, \;\; \forall r, s \in \mathbb{N}$

(ii) $\quad 1 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r \to \infty$ *as* $r \to \infty$

(iii) $\quad \left(\left(\dfrac{\omega^r}{\sqrt{\lambda_r}}, \dfrac{\omega^s}{\sqrt{\lambda_s}}\right)\right) = \delta_{rs}, \;\; \forall r, s \in \mathbb{N}.$

Moreover, defining $H^m := span\{\omega^1, \ldots, \omega^m\}$*, and the operator*

$$P^m(\mathbf{v}) := \sum_{i=1}^m \left(\mathbf{v}, \omega^i\right) \omega^i : V \to H^m,$$

we get

$$\|P^m\|_{\mathcal{L}(V,V)} \leq 1, \;\; \|P^m\|_{\mathcal{L}(V^*,V^*)} \leq 1. \tag{A.3}$$

Proof of continuity of P^m. Let $\mathbf{v} \in V$. Then

$$\begin{aligned}\|P^m(\mathbf{v})\|_{1,2}^2 &= \sum_{i=1}^{m} \left(\mathbf{v}, \omega^i\right)^2 \left(\left(\omega^i, \omega^i\right)\right) = \sum_{i=1}^{m} \frac{\left(\mathbf{v}, \omega^i\right)^2}{(\lambda^i)^2} \left(\left(\omega^i, \omega^i\right)\right) \\ &\leq \sum_{i=1}^{m} \left(\mathbf{v}, \frac{\omega^i}{\sqrt{\lambda^i}}\right)^2 \left(\left(\omega^i, \omega^i\right)\right) \left(\left(\omega^i, \omega^i\right)\right) \leq \|\mathbf{v}\|_{1,2}^2 .\end{aligned}$$

The operator P^m is selfadjoint, thus (A.3) holds true. $\square$

We note that $V = H^1_{0,div}(\Omega)^2$ and $\mathcal{L}(V, V)$ denotes the space of linear operators from the space V to V. For the complete proof of even more general result of Theorem A.9 we refer to Theorem 4.11 in [95].

The analogous result is true for the eigenvalues and eigenvectors of the Laplace operator associated with either the Dirichlet or the Neumann boundary conditions, see, e.g., [39].

B The inf-sup condition

We briefly explain the meaning of the inf-sup condition also known as the Ladyzhenskaya-Babuška-Brezzi condition (LBB) for the well-posedness of a saddle point problem. For illustration the Stokes problem is presented. At the end, we give an example of unstable and stable finite element. For more details we refer to, e.g., [8, 9, 10, 27, 47, 55, 58, 62, 79].

B.1. Mixed variational problem

Let V and Q be separable Hilbert spaces, $a : V \times V \to \mathbb{R}$, $b : V \times Q \to \mathbb{R}$ be given bilinear forms and $l : V \to \mathbb{R}$, $j : Q \to \mathbb{R}$ be some prescribed linear functionals. We introduce a mixed variational problem.

Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} a(u,v)+b(p,v) &= l(v), \ \forall\, v \in V \\ b(u,q) &= j(q), \ \forall\, q \in Q. \end{aligned} \tag{B.1}$$

It is equivalent to the system of operator equations

$$\begin{aligned} Au+B'p &= l \ \in V^* \\ Bu &= j \ \in Q^*, \end{aligned} \tag{B.2}$$

where $A : V \to V^*$, $B : V \to Q^*$ and $B' : Q \to V^*$ are corresponding linear operators

$$\langle Au, v\rangle_{V^*\times V} = a(u,v), \qquad \langle Bu, q\rangle_{Q^*\times Q} = b(u,q) = \langle u, B'q\rangle_{V\times V^*}.$$

The operator B' is called the adjoint operator of B. Let us note that the kernel of the operator B is given by

$$N(B) := \{v \in V : Bv = 0\} = \{v \in V : b(v,q) = 0, \ \forall\, q \in Q\}.$$

We look for necessary and sufficient conditions such that for each $(l, j) \in V^* \times Q^*$ the problem (B.1) has a unique solution. In other words, such that the combined operator

$$J : V \times Q \to V^* \times Q^* \qquad J(u,p) = (Au + B'p, Bu)$$

is an isomorphism between the solution and data space. Obviously, the operator J is associated to the form

$$A\left((u,p),(v,q)\right) = a(u,v) + b(u,q) + b(v,p).$$

An answer to the question of well-posedness of the mixed variational problem (B.1) and of the system (B.2) is given by Brezzi [27], see also [9, 10] which in turn is an application of the Babuška theory for finite element methods [8]. Ladyzhenskaya in [79] proved the analogous condition for the continuous case.

Theorem B.1. *(Brezzi's splitting theorem)*
Assume that the bilinear forms a and b are continuous i.e.

$$a(u,v) \le C_a \|u\|_V \|v\|_V\,, \qquad \forall\, u,v \in V \tag{B.3a}$$
$$b(u,q) \le C_b \|u\|_V \|q\|_Q\,, \qquad \forall\, u \in V, q \in Q, \tag{B.3b}$$

with some constants C_a, $C_b > 0$. Further, we assume that for some α, $\beta > 0$ the conditions

$$a(u,u) \ge \alpha \|u\|_V^2\,, \qquad \forall\, u \in N(B) \tag{B.4a}$$
$$\sup_{u\in V} \frac{b(u,q)}{\|u\|_V} \ge \beta \|q\|_Q\,, \qquad \forall\, q \in Q \tag{B.4b}$$

are satisfied. Then for any given $(l,j) \in V^* \times Q^*$ the problem (B.1) *has a unique solution and it holds that*

$$\|u\|_V + \|p\|_Q \le C\left(\|l\|_{V^*} + \|j\|_{Q^*}\right),$$

where C depends only on C_a, C_b, α, β.

The continuity of a and b implies that A, B and B' are bounded linear operators. Due to (B.4) the operator J is boundedly invertible. More precisely, (B.4a) ensures that A is invertible on the kernel $N(B)$. For a being symmetric and non-negative, for instance in the case of the Stokes problem, this condition is necessary. To conclude, the assumptions of Brezzi's theorem indeed guarantee that the operator J is an isomorphism.

Galerkin approximation

Let V_h and Q_h be finite-dimensional closed subspaces of V and Q, respectively, and let $a : V_h \times V_h \to \mathbb{R}$, $b : V_h \times Q_h \to \mathbb{R}$ be given bilinear forms and $l : V_h \to \mathbb{R}$, $j : Q_h \to \mathbb{R}$ be some prescribed linear functionals. The Galerkin approximation of a mixed variational problem (B.1) follows.

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(u_h,v_h) + b(p_h,v_h) &= l(v_h),\ \forall\, v_h \in V_h \\ b(u_h,q_h) &= j(v_h),\ \forall\, q_h \in Q_h. \end{aligned} \tag{B.5}$$

Analogously, there is an equivalent operator formulation

$$\begin{aligned} A_h u_h + B_h' p_h &= l_h \ \in V_h^* \\ B_h u_h &= j_h \ \in Q^*, \end{aligned} \tag{B.6}$$

where again $A_h : V_h \to V_h^*$, $B_h : V_h \to Q_h^*$ and $B_h' : Q_h \to V_h^*$ are corresponding linear operators. We recall the kernel of the operator B_h is

$$N(B_h) := \{v_h \in V_h : B_h v_h = 0\} = \{v_h \in V_h : b(v_h, q_h) = 0 \,\forall\, q_h \in Q_h\}.$$

We are interested in finding the sufficient conditions on V_h and Q_h under which the problem (B.5) has a unique solution. The straightforward application of Brezzi's theorem yields the following result.

Theorem B.2.
Assume the discrete stability conditions

$$a(u_h, u_h) \geq \alpha_h \|u_h\|_{V_h}^2 , \qquad \forall\, u_h \in N(B_h) \tag{B.7a}$$

$$\sup_{u_h \in V} \frac{b(u_h, q_h)}{\|u_h\|_V} \geq \beta_h \|q_h\|_Q , \qquad \forall\, q_h \in Q_h \tag{B.7b}$$

are satisfied for some α_h, $\beta_h > 0$. Then there exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ to the problem (B.5) *satisfying*

$$\|u_h\|_V + \|p_h\|_Q \leq C_h \left(\|l\|_{V^*} + \|j\|_{Q^*} \right) ,$$

where C_h depends on C_a, C_b, α_h and β_h of (B.3) *and* (B.7).

The discrete stability conditions (B.7) are sufficient for the well-posedness of the discrete scheme. We note that for the symmetric and non-negative bilinear form a it is even the necessary condition for the existence of a unique solution of the discrete problem.

Error of the approximation

As we have already mentioned the choice of the discrete spaces for the Galerkin method is crucial for its implementation. These discrete spaces define the Galerkin method (B.5) completely and moreover, they indicate the error of the approximation. See [26, 27] for the proof of the following result.

Theorem B.3.
Let (B.3) *and* (B.7) *be satisfied. Then*

$$\|u - u_h\|_V + \|p - p_h\|_Q \leq C_h \left\{ \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right\} ,$$

where C_h depends only on the constants C_a, C_b, α_h, β_h of (B.3) *and* (B.7).

In this work we are interested in a finite element approximation. Concerning the above discussion it is important to select the discrete spaces with good approximation properties, cf. Theorem B.3, and also with good stability properties, such that (B.7) holds with α_h, $\beta_h \geq c > 0$.

B.2. Stokes problem

Let us give an example of a mixed variational formulation for a flow problem together with its Galerkin approximation.

The Stokes problem

Find the velocity $\mathbf{u}$ and pressure p satisfying

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned} \tag{B.8}$$

on Ω, and the homogeneous Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$. This problem indeed leads to a mixed variational formulation of the form (B.1). The appropriate spaces, bilinear forms and linear functionals are $V = H_0^1(\Omega)^2$, $Q = L_0^2(\Omega)$,

$$a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u},\nabla\mathbf{v}), \qquad b(\mathbf{u},q) = -(\operatorname{div}\mathbf{u},q), \qquad l(\mathbf{v}) = (\mathbf{f},\mathbf{v}), \qquad j(q) = 0,$$

respectively. Let us note, that we can rewrite the variational formulation of the Stokes problem, using the notation

$$B\left((\mathbf{u},p),(\mathbf{v},q)\right) := \nu a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},q) + b(\mathbf{v},p), \qquad F(\mathbf{v},q) := l(v)$$

as follows.

Given $\mathbf{f} \in H^{-1}(\Omega)^2$ find $(\mathbf{u},p) \in V \times Q$ such that

$$B\left((\mathbf{u},p);(\mathbf{v},q)\right) = F(\mathbf{v},q), \qquad \forall\,(\mathbf{v},q) \in V \times Q. \tag{B.9}$$

The assumptions (B.3) and (B.4a) are easily verified. The following lemma due to Nečas [106] and Bramble [25] guarantees that the condition (B.4b) is satisfied.

Lemma B.4.
There exists a constant $\beta = \beta(\Omega)$ such that

$$\sup_{\mathbf{u}\in H_0^1(\Omega)^2} \frac{(\operatorname{div}\mathbf{u},q)}{\|\mathbf{u}\|_{H_0^1(\Omega)^2}} \geq \beta\,\|q\|_{L_0^2(\Omega)}\,, \qquad \forall\,q \in L_0^2(\Omega).$$

Hence, due to Theorem B.1 the problem (B.9) is well-posed, i.e. there exists a constant β^* such that

$$\sup_{(\mathbf{v},q)\in V\times Q} \frac{B\left((\mathbf{u},p),(\mathbf{v},q)\right)}{\|\mathbf{v}\|_V + \|q\|_Q} \geq \beta^*\left(\|\mathbf{u}\|_V + \|p\|_Q\right) \qquad \forall(\mathbf{u},p) \in V \times Q$$

or equivalently

$$\inf_{(\mathbf{u},p)\in V\times Q}\ \sup_{(\mathbf{v},q)\in V\times Q} \frac{B\left((\mathbf{u},p),(\mathbf{v},q)\right)}{(\|\mathbf{v}\|_V + \|q\|_Q)(\|\mathbf{u}\|_V + \|p\|_Q)} \geq \beta^*. \tag{B.10}$$

It is a simple consequence of the Poincaré inequality and the stability condition (B.4b).

Galerkin approximation

We consider the Galerkin approximation of a mixed variational formulation of (B.8). Find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that

$$B\left((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\right) = F(\mathbf{v}_h, q_h), \qquad \forall\, (\mathbf{v}_h, q_h) \in V_h \times Q_h.$$

We already know that (B.3) holds. The form a is elliptic on the whole space V, thus the first discrete stability condition (B.7a) is satisfied by its validity on the continuous level. The only condition which has to be taken into account in order to choose the proper discrete spaces is the second discrete stability condition (B.7b). It can be rewritten as

$$\inf_{q_h \in Q_h} \sup_{\mathbf{u}_h \in V_h} \frac{b(\mathbf{u}_h, q_h)}{\|\mathbf{u}_h\|_V \|q_h\|_Q} \geq \beta \tag{B.11}$$

which indicates the name *inf-sup condition.* Since the same steps as for the continuous case are used to prove that (B.10) is valid also on the discrete level, we have the following result due to Brezzi [27] and Babuška [9], see also [55].

Proposition B.5.
If the finite element spaces V_h and Q_h satisfy the discrete inf-sup condition (B.7b) *or* (B.11)*, then the inequality*

$$\sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{B\left((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\right)}{\|\mathbf{v}_h\|_V + \|q_h\|_Q} \geq \beta^* \left(\|\mathbf{u}_h\|_V + \|p_h\|_Q\right) \quad \forall (\mathbf{u}_h, p_h) \in V_h \times Q_h$$

is valid.

It is worth mentioning, that if Proposition B.5 is valid then the method will converge optimally.

Remark B.6. *We note that for simplicity the value 0 has been every time excluded from the denominators of* inf *and* sup*.*

Unstable and stable finite elements

We say that a finite element is stable, or inf-sup stable, if the corresponding discrete spaces V_h and Q_h satisfy the discrete inf-sup condition. We give an example of both the unstable and stable finite element, see [47].

$\mathcal{P}_1/\mathcal{P}_0$ element

The simplest non-trivial conforming choice of the spaces for the Stokes problem is

$$V_h = H_0^1(\Omega)^2 \cap \mathcal{P}_1(\mathcal{T}_h)^2, \qquad Q_h = L_0^2(\Omega) \cap \mathcal{P}_0(\mathcal{T}_h).$$

However, the discrete inf-sup condition (B.11) is not satisfied in non-trivial cases. Indeed, the dimension of the space V_h is dim $V_h = 2n_i$, where n_i is the number of inner vertices. Similarly, dim $Q_h = N_E - 1$, where N_E is the number of elements. Let us recall the identity

$$N_E = 2n_i + n_b - 2,$$

where n_b is the number of vertices at the boundary. Thus dim $Q_h = 2n_i + n_b - 3$. For any non-trivial triangular mesh $n_b > 3$ and hence the dimension of Q_h is always bigger than the dimension of V_h. Therefore the operator B_h is not surjective and the operator B_h' is not injective. This means the pressure is not uniquely determined.

$\mathcal{P}_2/\mathcal{P}_0$ element

To satisfy the discrete inf-sup condition we need to increase the dimension of the discrete space V_h. For the spaces

$$V_h = H_0^1(\Omega)^2 \cap \mathcal{P}_2(\mathcal{T}_h)^2, \qquad Q_h = L_0^2(\Omega) \cap \mathcal{P}_0(\mathcal{T}_h)$$

it can be shown that the discrete stability condition (B.7b) is satisfied with a constant $\beta_h = \gamma\beta > 0$ where the constant γ is related to the mesh regularity.

C Numerical scheme in details

This chapter contains further details on the numerical scheme derived in Chapter 6. We introduce firstly the barycentric coordinates defined on a triangle. They are important for the numerical simulation of the Oseen-type Peterlin model as they form the shape functions of the linear Lagrange finite element, see Chapter 8. Secondly, the matrix form of the semi-implicit pressure-stabilized characteristics finite element scheme (6.25) is derived.

C.1. Barycentric coordinates

Let $P_i = (x_1^i, x_2^i)$, $i = 1, 2, 3$, be the nodal points of a triangle K. Every point inside this triangle can be written as a unique convex combination of its three vertices. It means, for each $x = (x_1, x_2) \in \bar{K}$ there is a unique sequence of three numbers λ_1, λ_2, $\lambda_3 \in [0, 1]$ such that

$$\sum_{i=1}^{3} \lambda_i = 1, \qquad \text{and} \qquad x = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3.$$

Obviously, $\lambda_i(P_j) = \delta_{ij}$, $i, j = 1, 2, 3$, where δ_{ij} is the Kronecker delta. The barycentric coordinates $(\lambda_1(x), \lambda_2(x), \lambda_3(x))$ of a point $x \in \bar{K}$ are defined as a solution to the linear system of three equations

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ x_1^1 & x_1^2 & x_1^3 \\ x_2^1 & x_2^2 & x_2^3 \end{pmatrix}}_{\mathbb{A}_K} \begin{pmatrix} \lambda_1(x) \\ \lambda_2(x) \\ \lambda_3(x) \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}.$$

Cramer's rule yields

$$\lambda_i(x) := \frac{|\det \mathbb{A}_K^i(x)|}{|\det \mathbb{A}_K|}, \; i = 1, 2, 3,$$

where

$$\mathbb{A}_K^1(x) = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_1^2 & x_1^3 \\ x_2 & x_2^2 & x_2^3 \end{pmatrix}, \quad \mathbb{A}_K^2(x) = \begin{pmatrix} 1 & 1 & 1 \\ x_1^1 & x_1 & x_1^3 \\ x_2^1 & x_2 & x_2^3 \end{pmatrix}, \quad \mathbb{A}_K^3(x) = \begin{pmatrix} 1 & 1 & 1 \\ x_1^1 & x_1^2 & x_1 \\ x_2^1 & x_2^2 & x_2 \end{pmatrix}.$$

We give also a geometrical interpretation of the barycentric coordinates. For each point $x \in K$ there is the so-called *barycentric splitting* of K into three subdomains such that

$$K = \bigcup_{i=1}^{3} K^i(x).$$

The areas of the triangles K, $K^i(x)$, $i = 1, 2, 3$, are then given by

$$|K| = \frac{1}{2}|\det \mathbb{A}_K| > 0, \qquad |K^i(x)| = \frac{1}{2}|\det \mathbb{A}_K^i(x)|,$$

respectively. Therefore a barycentric coordinate $\lambda_i(x)$ is the ratio of the area of the corresponding $K^i(x)$ and the area of K, i.e.

$$\lambda_i(x) = \frac{|K^i(x)|}{|K|},\ i = 1, 2, 3.$$

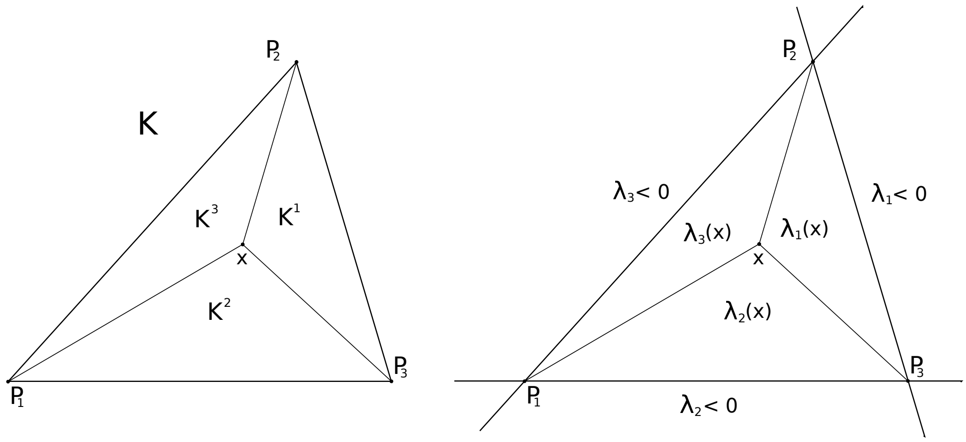


Figure C.1.: Barycentric splitting (left) and signs of barycentric coordinates (right)

Let us note that every point $x \in \bar{K}$ is uniquely determined by two arbitrarily chosen barycentric coordinates. Moreover, if $x \notin \bar{K}$ at least one and at most two barycentric coordinates are negative while $\lambda_1 + \lambda_2 + \lambda_3 = 1$, see Figure C.1.

C.2. Matrix form of the semi-implicit scheme

In this work we consider the continuous $\mathcal{P}_1/\mathcal{P}_1/\mathcal{P}_1$ finite element approximation of the velocity, pressure and conformation tensor. We denote by φ_i, $i = 1, \ldots, N_P$, the basis function of the finite element space M_h, cf. (6.20b). We shall choose the canonical basis of the discrete finite-dimensional spaces $X_h = M_h^2$ and $Y_h = M_h^{2\times 2}$, i.e. we have

$$X_h = \operatorname{span}\{\boldsymbol{\varphi}_{1i}, \boldsymbol{\varphi}_{2i}\}_{i=1}^{N_P}, \quad M_h = \operatorname{span}\{\varphi_i\}_{i=1}^{N_P}, \quad Y_h = \operatorname{span}\{\boldsymbol{\phi}_{1i}, \boldsymbol{\phi}_{2i}, \boldsymbol{\phi}_{3i}, \boldsymbol{\phi}_{4i}\}_{i=1}^{N_P},$$

for

$$\boldsymbol{\varphi}_{1i} := (\varphi_i(x), 0), \qquad \boldsymbol{\varphi}_{2i} := (0, \varphi_i(x))$$
$$\boldsymbol{\phi}_{1i} := \begin{pmatrix} \varphi_i(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\phi}_{2i} := \begin{pmatrix} 0 & \varphi_i(x) \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\phi}_{3i} := \begin{pmatrix} 0 & 0 \\ \varphi_i(x) & 0 \end{pmatrix}, \quad \boldsymbol{\phi}_{4i} := \begin{pmatrix} 0 & 0 \\ 0 & \varphi_i(x) \end{pmatrix}.$$

For the definition of the finite element spaces V_h, Q_h and W_h we refer to (6.20). We seek the discrete solution of the semi-implicit linear scheme (6.25), at the current time step t^n, in the form

$$
\begin{aligned}
\mathbf{u}_h^n &= \left(\sum_{j=1}^{N_P} a_j^n \varphi_j(x), \quad \sum_{j=1}^{N_P} b_j^n \varphi_j(x) \right), \qquad p_h^n = \sum_{j=1}^{N_P} p_j^n \varphi_j(x), \\
\mathbf{C}_h^n &= \begin{pmatrix} \sum_{j=1}^{N_P} A_j^n \varphi_j(x) & \sum_{j=1}^{N_P} B_j^n \varphi_j(x) \\ \sum_{j=1}^{N_P} B_j^n \varphi_j(x) & \sum_{j=1}^{N_P} D_j^n \varphi_j(x) \end{pmatrix},
\end{aligned}
\tag{C.1}
$$

where the unknown variables are $a_i, b_i, p_i, A_i, B_i, D_i$, $i = 1, \ldots, N_P$, i.e. the nodal values of the two components of velocity, the nodal values of pressure and the nodal values of the three components of the symmetric conformation tensor. We recall N_P is the number of nodal points and thus the dimension of the finite element spaces. Using (C.1) we can rewrite the discrete scheme (6.25) as the linear algebraic system of $6N_P$ equations with $6N_P$ unknowns. The system reads, for $i = 1, \ldots, N_P$, as follows

$$
\begin{aligned}
&\sum_{j=1}^{N_P} a_j^n M_{ji} + \nu \Delta t\, a_j^n A_{ji} - \Delta t\, a_j^n B_{ji}^1 + \Delta t \left(A_j^n + D_j^n \right) Q_{ji}^1 = \sum_{j=1}^{N_P} a_j^{n-1} (M_{ji}^1)^* + \Delta t\, r_{f1,i} \\
&\sum_{j=1}^{N_P} b_j^n M_{ji} + \nu \Delta t\, b_j^n A_{ji} - \Delta t\, b_j^n B_{ji}^2 + \Delta t \left(A_j^n + D_j^n \right) Q_{ji}^2 = \sum_{j=1}^{N_P} b_j^{n-1} (M_{ji}^1)^* + \Delta t\, r_{f2,i} \\
&- \sum_{j=1}^{N_P} a_j^n B_{ij}^1 - b_j^n B_{ij}^2 - p_j^n S_{ji} = 0 \\
&\sum_{j=1}^{N_P} A_j^n M_{ji} + \varepsilon \Delta t\, A_j^n \tilde{A}_{ji} - 2\Delta t\, a_j^n Q_{ij}^1 + \Delta t\, A_j^n Y_{ji} = \Delta t(\hat{r}_i + r_{F1,i}) + \sum_{j=1}^{N_P} A_j^{n-1} (M_{ji})^* \\
&\sum_{j=1}^{N_P} B_j^n M_{ji} + \varepsilon \Delta t\, B_j^n \tilde{A}_{ji} - \Delta t\, a_j^n Q_{ij}^1 - \Delta t\, b_j^n Q_{ij}^2 + \Delta t\, B_j^n Y_{ji} = \sum_{j=1}^{N_P} B_j^{n-1} (M_{ji})^* + \Delta t\, r_{F2,i} \\
&\sum_{j=1}^{N_P} D_j^n M_{ji} + \varepsilon \Delta t\, D_j^n \tilde{A}_{ji} - 2\Delta t\, b_j^n Q_{ij}^2 + \Delta t\, D_j^n Y_{ji} = \Delta t(\hat{r}_i + r_{F3,i}) + \sum_{j=1}^{N_P} D_j^{n-1} (M_{ji})^*.
\end{aligned}
\tag{C.2}
$$

The precise definition of the matrices and vectors appearing in the above system of equations follows. The constant small-size matrices, for $i, j = 1, \ldots, N_P$, are given by

$$
\begin{aligned}
M_{ij} &= \sum_{l=1}^{N_E} \int_{K_l} \varphi_i(x) \varphi_j(x)\, dx, & A_{ij} &= \sum_{l=1}^{N_E} 2 \int_{K_l} \mathrm{D}(\varphi_i(x)) \cdot \mathrm{D}(\varphi_j(x))\, dx, \\
S_{ij} &= \delta_0 \sum_{l=1}^{N_E} h_{K_l}^2 \int_{K_l} \nabla \varphi_i(x) \cdot \nabla \varphi_j(x), & \tilde{A}_{ij} &= \sum_{l=1}^{N_E} \int_{K_l} \nabla \varphi_i(x) \cdot \nabla \varphi_j(x)\, dx, \\
B_{ij}^1 &= \sum_{l=1}^{N_E} \int_{K_l} \frac{\partial \varphi_i(x)}{\partial x_1} \varphi_j(x)\, dx, & B_{ij}^2 &= \sum_{l=1}^{N_E} \int_{K_l} \frac{\partial \varphi_i(x)}{\partial x_2} \varphi_j(x)\, dx.
\end{aligned}
$$

We define

$$\mathbb{M}_u + \nu\Delta t\,\mathbb{A}_u := \begin{pmatrix} \mathbb{M} + \nu\Delta t\,\mathbb{A} & \mathbf{0} \\ \mathbf{0} & \mathbb{M} + \nu\Delta t\,\mathbb{A} \end{pmatrix} \in \mathbb{R}^{2N_P\times 2N_P}$$

$$\mathbb{M}_c + \varepsilon\Delta t\,\tilde{\mathbb{A}}_c := \begin{pmatrix} \mathbb{M} + \varepsilon\Delta t\,\tilde{\mathbb{A}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{M} + \varepsilon\Delta t\,\tilde{\mathbb{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{M} + \varepsilon\Delta t\,\tilde{\mathbb{A}} \end{pmatrix} \in \mathbb{R}^{3N_P\times 3N_P}$$

$$\mathbb{B} := \begin{pmatrix} \mathbb{B}^1 \\ \mathbb{B}^2 \end{pmatrix} \in \mathbb{R}^{2N_P\times N_P}.$$

The notation $\mathbb{M}^*$ expresses the following dependence of the mass matrix $\mathbb{M}$ on the time increment

$$(M_{ij})^* = \sum_{l=1}^{N_E} \int_{K_l} \varphi_i(x)\varphi_j(x - \mathbf{w}^n(x)\Delta t)\, dx. \tag{C.3}$$

Further, the matrices arising from the semi-implicit terms, and thus dependent on the old time step, are given by

$$O^1_{kji} = \sum_{l=1}^{N_E} \int_{K_l} \varphi_k(x)\varphi_j(x)\frac{\partial\varphi_i(x)}{\partial x_1}\, dx, \qquad O^2_{kji} = \sum_{l=1}^{N_E} \int_{K_l} \varphi_k(x)\varphi_j(x)\frac{\partial\varphi_i(x)}{\partial x_2}\, dx,$$

$$Q^1_{ji} = \sum_{k=1}^{N_P} A^{n-1}_k O^1_{kji} + B^{n-1}_k O^2_{kji}, \qquad Q^2_{ji} = \sum_{k=1}^{N_P} B^{n-1}_k O^1_{kji} + D^{n-1}_k O^2_{kji},$$

$$Y_{ji} = \sum_{l=1}^{N_E} \int_{K_l} \left(\sum_{k=1}^{N_p} \left(A^{n-1}_k + D^{n-1}_k\right) \varphi_k(x) \right)^2 \varphi_j(x)\varphi_i(x)\, dx.$$

Similarly as above, we define

$$\mathbb{Q}^{n-1}_c := \begin{pmatrix} \mathbb{Q}^1 & \mathbf{0} & \mathbb{Q}^1 \\ \mathbb{Q}^2 & \mathbf{0} & \mathbb{Q}^2 \end{pmatrix} \in \mathbb{R}^{2N_P\times 3N_P}$$

$$\mathbb{Q}^{n-1}_u := \begin{pmatrix} 2(\mathbb{Q}^1)^T & \mathbf{0} \\ (\mathbb{Q}^1)^T & (\mathbb{Q}^2)^T \\ \mathbf{0} & 2(\mathbb{Q}^2)^T \end{pmatrix} \in \mathbb{R}^{3N_P\times 2N_P}$$

$$\mathbb{Y}^{n-1}_c := \begin{pmatrix} \mathbb{Y} \\ \mathbb{Y} \\ \mathbb{Y} \end{pmatrix} \in \mathbb{R}^{3N_P\times N_P}.$$

The constant vectors $\mathbf{r}_f = (\mathbf{r}_{f1}, \mathbf{r}_{f2})^T$ and $\mathbf{r}_F = (\mathbf{r}_{F1}, \mathbf{r}_{F2}, \mathbf{r}_{F3})^T$ are known as they arise from the given external forces, i.e. $\mathbf{r}_f = \mathbb{M}\,\mathbf{f}^n_h \in \mathbb{R}^{2N_P}$ and $\mathbf{r}_F = \mathbb{M}\,\mathbf{F}^n_h \in \mathbb{R}^{3N_P}$. The right-hand side vector that depends on the old time step reads

$$\hat{\mathbf{r}}^{n-1} := \begin{pmatrix} \hat{\mathbf{r}} \\ \mathbf{0} \\ \hat{\mathbf{r}} \end{pmatrix}, \quad \text{where } \hat{r}_i = \sum_{l=1}^{N_E} \int_{K_l} \sum_{k=1}^{N_p} \left(A^{n-1}_k + D^{n-1}_k\right) M_{ki}.$$

We denote by $\tilde{\mathbf{r}}^{n-1}$, the following combination of the vectors $\mathbf{r}_f$, $\mathbf{r}_F$ and $\hat{\mathbf{r}}^{n-1}$

$$\tilde{\mathbf{r}}^{n-1} := \begin{pmatrix} \mathbf{r}_f \\ \mathbf{0} \\ \mathbf{r}_F + \hat{\mathbf{r}}^{n-1} \end{pmatrix} \in \mathbb{R}^{6N_P}.$$

Finally, the terms arising from the characteristic part of the numerical scheme are gathered in the vector

$$\mathbf{r}^{n-1}(\Delta t) := \begin{pmatrix} \mathbb{M}^*\mathbf{a}^{n-1} \\ \mathbb{M}^*\mathbf{b}^{n-1} \\ \mathbf{0} \\ \mathbb{M}^*\mathbf{A}^{n-1} \\ \mathbb{M}^*\mathbf{B}^{n-1} \\ \mathbb{M}^*\mathbf{D}^{n-1} \end{pmatrix} \in \mathbb{R}^{6N_P},$$

where the dependence on the time increment is related to the upwind point given by $X_1^n(x) = x - \mathbf{w}^n(x)\Delta t$, i.e. for $\Delta t = 0$ it holds that $\mathbb{M}^* = \mathbb{M}$, cf. (C.3). We can conclude that the matrix of the system (C.2) can be written as

$$\begin{pmatrix} \mathbb{M}_u + \nu\Delta t\,\mathbb{A}_u & -\Delta t\,\mathbb{B} & \Delta t\,\mathbb{Q}_c^{n-1} \\ -\mathbb{B}^T & -\mathbb{S} & \mathbf{0} \\ -\Delta t\,\mathbb{Q}_u^{n-1} & \mathbf{0} & \mathbb{M}_c + \varepsilon\Delta t\,\tilde{\mathbb{A}}_c + \Delta t\,\mathbb{Y}_c^{n-1} \end{pmatrix} \in \mathbb{R}^{6N_P \times 6N_P}$$

and the right-hand side of (C.2) as

$$\begin{pmatrix} \mathbb{M}_u^*\mathbf{u}^{n-1} + \Delta t\,\mathbf{r}_f \\ \mathbf{0} \\ \mathbb{M}_c^*\mathbf{C}^{n-1} + \Delta t\,\mathbf{r}_F + \Delta t\,\hat{\mathbf{r}}^{n-1} \end{pmatrix} = \mathbf{r}^{n-1}(\Delta t) + \Delta t\,\tilde{\mathbf{r}}^{n-1} \in \mathbb{R}^{6N_P}.$$

Let us rewrite scheme (6.25) in order to depict the corresponding matrices and vectors. We have

$$\begin{aligned} &\underbrace{(\mathbf{u}_h^n, \mathbf{v}_h)}_{\mathbb{M}_u} + \nu\Delta t\, \underbrace{a_u(\mathbf{u}_h^n, \mathbf{v}_h)}_{\mathbb{A}_u} + \Delta t\, \underbrace{b(\mathbf{v}_h, p_h^n)}_{\mathbb{B}} + \Delta t\, \underbrace{\left(\operatorname{tr}\mathbf{C}_h^n\ \mathbf{C}_h^{n-1}, \nabla\mathbf{u}_h\right)}_{\mathbb{Q}_c^{n-1}} - \\ &\qquad + \Delta t\, \underbrace{b(\mathbf{u}_h^n, q_h)}_{\mathbb{B}^T} - \Delta t\, \underbrace{\mathcal{S}_h(p_h^n, q_h)}_{\mathbb{S}} = \underbrace{\left(\mathbf{u}_h^{n-1} \circ X_1^n, \mathbf{u}_h\right)}_{\mathbf{r}^{n-1}(\Delta t)} + \Delta t\, \underbrace{(\mathbf{f}_h^n, \mathbf{v}_h)}_{\mathbf{r}_f}. \end{aligned}$$

$$\begin{aligned} &\underbrace{(\mathbf{C}_h^n, \mathbf{D}_h)}_{\mathbb{M}_c} + \varepsilon\Delta t\, \underbrace{a_c(\mathbf{C}_h^n, \mathbf{D}_h)}_{\tilde{\mathbb{A}}_c} - \Delta t\, \underbrace{\left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1} + \mathbf{C}_h^{n-1}(\nabla\mathbf{u}_h^n)^T, \mathbf{D}_h\right)}_{\mathbb{Q}_u^{n-1}} + \\ &\qquad + \Delta t\, \underbrace{\left(\left(\operatorname{tr}\mathbf{C}_h^{n-1}\right)^2 \mathbf{C}_h^n, \mathbf{D}_h\right)}_{\mathbb{Y}_c^{n-1}} = \underbrace{\left(\mathbf{C}_h^{n-1} \circ X_1^n, \mathbf{D}_h\right)}_{\mathbf{r}^{n-1}(\Delta t)} + \Delta t\, \underbrace{\left(\operatorname{tr}\mathbf{C}_h^{n-1}\ \mathbf{I}, \mathbf{D}_h\right)}_{\hat{\mathbf{r}}^{n-1}} + \\ &\qquad\qquad + \Delta t\, \underbrace{(\mathbf{F}_h^n, \mathbf{D}_h)}_{\mathbf{r}_F}. \end{aligned}$$

At the and we give a short transcript of the matrix form of the system (C.2) that has been used in Section 6.4, i.e.

$$\left(\mathbb{X}_1 + \Delta t\, \mathbb{X}_2 + \Delta t\, \mathbb{Z}^{n-1}\right) \mathbf{y}^n = \mathbf{r}^{n-1}(\Delta t) + \Delta t\, \tilde{\mathbf{r}}^{n-1},$$

where $\mathbf{y}^n := (\mathbf{a}^n, \mathbf{b}^n, \mathbf{p}^n, \mathbf{A}^n, \mathbf{B}^n, \mathbf{D}^n)^T \in \mathbb{R}^{6N_P}$ is the vector of the nodal values of the discrete solution $(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)$ and the following notation is used

$$\mathbb{X}_1 := \begin{pmatrix} \mathbb{M}_u & \mathbf{0} & \mathbf{0} \\ -\mathbb{B}^T & -\mathbb{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{M}_c \end{pmatrix}, \qquad \mathbb{X}_2 := \begin{pmatrix} \nu\mathbb{A}_u & -\mathbb{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varepsilon\tilde{\mathbb{A}}_c \end{pmatrix}.$$

Let us note, that in the proof of Proposition 6.9 we use the above notation because we need $\mathbf{L}(0, \mathbf{y}^{n-1}) = \mathbf{0}$ and $\mathbf{D}_{\mathbf{y}}\, \mathbf{L}(0, \mathbf{y}^{n-1}) = \mathbb{X}_1$ to be invertible, cf. (6.30). Other possibility is to denote

$$\mathbb{X}_1 := \begin{pmatrix} \mathbb{M}_u & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{M}_c \end{pmatrix}, \qquad \mathbb{X}_2 := \begin{pmatrix} \nu\mathbb{A}_u & -\mathbb{B} & \mathbf{0} \\ -\mathbb{B}^T & -\mathbb{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varepsilon\tilde{\mathbb{A}}_c \end{pmatrix}.$$

The difference in the above pair of notations is nothing but the multiplication of the third equation in the system (C.2) by $\Delta t \neq 0$.

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