

Strategies for Optimizing Performance and Resource Overhead in Quantum Networks

Dissertation zur Erlangung des Grades
„Doktor der Naturwissenschaften“

am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität in Mainz



eingereicht von

Subrahmanya Siddardha Chelluri

geboren in Visakhapatnam, Andhra Pradesh, India

Mainz, March 2026

Berichterstatter: Removed due to data privacy

Datum der mündlichen Prüfung: June 02, 2026

This work is licensed under a Creative Commons Attribution 4.0 International License
(CC BY 4.0).

<https://creativecommons.org/licenses/by/4.0/>

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन ।
मा कर्मफलहेतुर्भूर्मा ते सङ्गोऽस्त्वकर्मणि ॥

— Śrīmad Bhagavad Gītā, Chapter 2, Verse 47

Meaning: Do your duty with full effort, without attachment to the results or acting for reward, and do not fall into inaction.

This thesis is dedicated to

- My parents, for their unwavering encouragement, constant support, and for being truly active participants in this journey, from our decision to quit school to the submission of this thesis.
- My teachers, for not only showing me the beauty and joy of doing mathematics and physics but also for teaching me how to live with happiness and purpose.
- The two women in my life, my sister and my wife, for their constant support, care, and countless efforts behind the scenes that allowed me to stay grounded and focus on my PhD.

Zusammenfassung

Quantennetzwerke versprechen bedingungslos sichere Kommunikation und verteiltes Quantencomputing. Die Realisierung dieser Einsatzmöglichkeiten erfordert die zuverlässige Erzeugung und Aufrechterhaltung multipartiter Verschränkung über großräumige Netzwerke hinweg. Die Verteilung komplexer verschränkter Zustände auf viele Knoten wird jedoch zunehmend schwieriger aufgrund physikalischer Imperfektionen, architektonischer Einschränkungen und der wachsenden Anforderungen an Netzwerkressourcen. Diese Dissertation präsentiert praktische Methoden, die zwei zentrale Aspekte dieses umfassenden Problems adressieren. Erstens stellen wir graphentheoretische Protokolle vor, die GHZ-Zustände in Bell-Paar-Netzwerken ausschließlich mithilfe lokaler Operationen und klassischer Kommunikation erzeugen. Dadurch werden NP-schwere Teilprobleme umgangen und zugleich die Anzahl der benötigten Gatteroperationen sowie der Verbrauch von Bell-Paaren minimiert. Getestet an zufälligen Netzwerktopologien und validiert auf Quantenhardware von IBM bieten diese Methoden einen skalierbaren Weg zu verteiltem Quantencomputing und Mehrparteienkommunikation. Zweitens verbessern wir die Leistungsfähigkeit von Quantenrepeater-Netzwerken der zweiten Generation durch die Einbindung bosonischer Quantenfehlerkorrektur. Wir zeigen, dass Speicher mit kürzeren Kohärenzzeiten, wenn sie durch Fehlerkorrektur geschützt sind, unkodierte Speicher mit deutlich längeren Kohärenzzeiten übertreffen können. Darüber hinaus behandeln wir das Problem der optimalen Zuweisung von Speicherpuffern in Repeater-Netzwerken, indem wir Strategien präsentieren, die Wartezeiten reduzieren und dadurch die Effizienz der Verschränkungsverteilung verbessern. Insgesamt demonstrieren diese Ergebnisse praktische und effektive Methoden zur Leistungssteigerung und zum Voranbringen der Realisierung kurzfristig verfügbarer Quantennetzwerke.

Abstract

Quantum networks promise unconditionally secure communication and distributed quantum computing. Realizing these capabilities requires the reliable creation and maintenance of multipartite entanglement across large-scale networks. However, distributing complex entangled states among many nodes becomes increasingly challenging due to physical imperfections, architectural constraints, and the growing demands on network resources. This thesis presents practical methods that address two key facets of this broader problem. First, we introduce graph-theoretic protocols that generate GHZ states in Bell-pair networks using only local operations and classical communication, bypassing NP-hard subroutines while minimizing gate counts and Bell-pair consumption. Benchmarked on random network topologies and validated on IBM quantum hardware, these methods provide a scalable path to distributed quantum computing and multiparty communication. Second, we improve the performance of second-generation quantum repeater networks by incorporating bosonic quantum error correction. We show that memories with shorter coherence times, when protected by error correction, can outperform unencoded memories with significantly longer coherence times. We further address the problem of optimal memory-buffer allocation across repeater networks, introducing policies that reduce waiting times, thereby improving entanglement distribution efficiency. Overall, these results demonstrate practical and effective methods for improving performance and advancing the realization of near-term quantum networks.

List of publications

The main part of this dissertation consists of the following three papers:

- Paper I** S. Siddardha Chelluri, Sumeet Khatri and Peter van Loock,
“A resource- and computationally-efficient protocol for multipartite entanglement distribution in Bell-pair networks”,
arXiv:2412.04252, in review at Nature Physics Journal, Quantum Information (npj-QI).
- Paper II** S. Siddardha Chelluri, Stephan Schuster, Sumeet and Riccardo Roma,
“Shallow-depth GHZ state generation on NISQ devices”,
arXiv:2507.19145, in review at Quantum Science and Technology.
- Paper III** S. Siddardha Chelluri, Sanchar Sharma, Frank Schmidt, Silvia von Kusminskiy and Peter van Loock,
“Bosonic quantum error correction with microwave cavities for quantum repeaters”,
arXiv:2503.21569, in review at Phys. Rev. A.
- Note:** This thesis also includes an additional study beyond the papers listed above, briefly discussed in Chapter 4. As the corresponding paper is still in preparation, it is not included in the cumulative section, although it forms part of my PhD research. The details of the paper are as follows:
Lina Vandr , E. Shchukin, S. Siddardha Chelluri, and Peter van Loock,
“Effects of Memory Buffer Distribution on the Performance of Quantum Repeaters”, (paper in preparation).

Acknowledgments

Removed for privacy reasons.

Contents

1	Introduction	1
2	Quantum Information Basics	5
2.1	Qubit	5
2.2	Single-Qubit Operations	5
2.3	Composite Systems	7
2.4	Quantum Error Correction	12
2.5	Quantum Key Distribution	17
2.6	Quantum Repeaters	20
2.7	Graph States	23
2.8	Quantum Networks	26
2.9	Physical Realizations of Qubits	28
3	Classical Networks	33
3.1	Fundamentals of Graph Theory	33
3.2	Computational Complexity	36
3.3	Random Graphs and Network Models	37
4	Results	41
4.1	Generation of Entanglement in Quantum Networks (Paper I)	41
4.2	Efficient GHZ State Generation on NISQ Hardware (Paper II)	49
4.3	Bosonic Quantum Error Correction for Quantum Repeaters (Paper III)	54
4.4	Quantum Repeater Performance with Memory Buffer	59
5	Conclusion and Outlook	65
6	Bibliography	67
7	Publications	79

1 Introduction

The vision of quantum computing can be traced back to Richard Feynman’s pioneering insight in the early 1980s [1]. He recognized that since nature itself follows the rules of quantum mechanics, simulating quantum systems on classical computers incurs an exponential overhead, whereas a machine built on quantum principles could perform such tasks far more efficiently. This foundational idea not only established the motivation for quantum computation but also shaped the development of algorithms and hardware capable of exploiting quantum phenomena for information processing. Over the past several years, this vision has begun to materialize with the advent of noisy intermediate-scale quantum (NISQ) devices [2–6]. These early quantum processors have transformed quantum computing from a largely theoretical concept into a practical, though still evolving, technology. They are designed to harness unique quantum effects such as superposition and entanglement to tackle tasks that remain intractable for conventional machines. One of the most compelling applications is the simulation of complex quantum systems, such as large molecules and advanced materials. By performing these simulations more efficiently, quantum hardware holds great promise for breakthroughs in chemistry, material science, and pharmaceutical research, where understanding intricate quantum interactions is key to designing new substances and drugs. At the same time, a robust ecosystem of quantum software tools has emerged to support algorithm development, hardware access, and hybrid workflows. Platforms such as Qiskit [7], PennyLane [8], Cirq [9], etc, provide high-level programming interfaces for building and simulating quantum circuits, running them on real devices, and integrating quantum routines into classical pipelines. These tools also enable benchmarking and characterization of quantum processors, helping researchers understand device performance and guide the transition toward larger, fault-tolerant architectures.

Quantum algorithms offer further breakthroughs. Grover’s search algorithm [10] provides a quadratic speedup for unstructured search problems, showcasing how quantum approaches can enhance even more general computational tasks. In Shor’s seminal work [11], it was demonstrated that quantum computers can perform prime factorization exponentially faster than current classical computers. This is highly significant because, while multiplying large numbers is straightforward, determining their prime factors is computationally difficult, which forms the basis of many commonly used cryptographic systems, including RSA [12]. Consequently, the emergence of large-scale, fault-tolerant quantum computers would render many existing encryption methods insecure, making the development of quantum-resistant cryptographic protocols an urgent priority.

At the same time, the very same quantum principles that threaten classical encryption also open the door to new modes of secure communication. A key feature of quantum mechanics, the no-cloning theorem [13], ensures that unknown quantum states cannot be copied. This makes it possible to distribute cryptographic keys in a way that is fundamentally secure: any attempt at eavesdropping can be detected. Quantum key distribution (QKD) protocols use this principle to generate a shared secret key between two parties, which can then be used with a one-time pad to achieve perfectly secure communication [14, 15]. Unlike classical encryption, the security of QKD does not rely on the computational difficulty of a problem but on fundamental principles of quantum mechanics, such as the no-cloning theorem and the fact that measurement disturbs a quantum system. Any attempt to intercept or measure the quantum states used to produce the key will inevitably introduce detectable changes, allowing the communicating parties to

know if the key has been compromised. These ideas have given rise to the broader field of quantum communication, which studies how information can be transmitted, processed, and protected using quantum systems.

Apart from QKD, there is also a classical approach to securing information known as post-quantum cryptography (PQC) [16]. These are algorithms designed to be hard to break even for quantum computers, and they can be used as alternatives to RSA. One prominent approach in post-quantum cryptography is lattice-based cryptography, which relies on the hardness of problems in high-dimensional lattices and is believed to be resistant to attacks by both classical and quantum computers [17,18]. However, PQC has two main limitations. First, quantum algorithms are still evolving, so it is not guaranteed that these schemes will remain secure against future quantum attacks. Second, there is the “store now, decrypt later” threat, in which encrypted data is intercepted today and decrypted later when more powerful classical or quantum computers are available. Quantum key distribution (QKD) addresses both of these issues. Its security is not based on computational difficulty but on the fundamental laws of quantum mechanics, and because any measurement disturbs the quantum states, eavesdroppers cannot attempt brute-force methods.

Even though QKD offers stronger security due to the limitations of post-quantum cryptography, its practical implementation remains a major challenge. Modern communication networks rely on light transmitted through optical fibers because of its high speed and reliability. However, the intensity of light decays exponentially as it travels through fibers [19]. In classical systems, this signal can be amplified, but in quantum communication, amplification is not straightforward due to the very principles that make QKD secure: the no-cloning theorem and the fact that measurement disturbs quantum states. As a result, the secret key rate for point-to-point QKD links decreases exponentially with distance [20]. Several solutions have been proposed to address this limitation, including satellite-based communication [21–23] and, most prominently, the use of quantum repeaters [24].

The basic idea of a quantum repeater is to break a long communication channel into shorter segments, create entanglement within each segment, and then link these segments through entanglement swapping so that distant parties become entangled [24, 25]. Numerous quantum repeater schemes have been proposed [26–29], employing components such as quantum memories and quantum error correction [30–33]. One of the most challenging aspects in this area is evaluating and comparing different repeater protocols. A common figure of merit is the number of secret bits transmitted per second. However, it has been shown that as the number of segments increases, the problem of calculating waiting times grows exponentially [26]. Various improvements, such as multiplexing and one-way repeater architectures, have also been suggested [30, 34–37]. In particular the use of quantum error correction is interesting. Quantum error correction (QEC) is a set of techniques that protect quantum information from noise and decoherence without directly measuring the encoded data [13, 38–42]. By spreading a single logical qubit across multiple physical qubits, QEC detects and corrects errors through carefully chosen measurements on ancillary qubits, leaving the logical state intact. It is a foundational requirement for building large-scale, fault-tolerant quantum computers and is also relevant in quantum communication, where it can improve the fidelity and rate of entanglement distribution over long distances.

A natural extension of quantum repeaters is the development of quantum networks [43, 44]. A quantum network is a system of interconnected nodes that can create, share, and process entangled quantum states across different locations. For communication tasks, quantum networks are still at an early stage of exploration, but their goal is to enable secure communication not just between two parties but among multiple parties. Beyond communication, both quantum repeaters and quantum networks have applications in other areas such as blind quantum computing [45, 46], distributed quantum computing [47–49], and

high-precision clock synchronization across distant locations [50, 51].

A key requirement for all these applications is the ability to create and maintain multipartite entanglement across distant nodes in a network. In this thesis, we focus broadly on entanglement distribution in both quantum repeaters and quantum networks for a variety of quantum information tasks. We begin by examining networks and the challenges of distributing entanglement within them. In Paper I [52], we present new practical algorithms for efficiently distributing entanglement in Bell-pair-based networks. These algorithms combine classical and quantum techniques and draw on results from computer science and graph theory. We also optimize them with respect to several figures of merit relevant for generating multipartite entanglement. Since quantum computing is expected to become modular, our results can be applied not only to quantum communication but also to quantum computation tasks.

Next, as the field moves toward large-scale, fault-tolerant quantum processors, benchmarking and validating quantum hardware becomes increasingly important. Techniques for characterizing quantum noise, gate fidelities, and overall circuit performance are essential for assessing progress and identifying pathways to error-corrected quantum computation. In this context, we implemented two different algorithms to generate multipartite entangled states on IBM quantum hardware and also carried out simulations that take into account the current noise and errors present in the devices. We have compared current state-of-the-art experimental results with futuristic simulations to identify the bottlenecks in present-day hardware and to determine which types of errors play the most prominent role. These results are presented in paper II [53].

Finally, we address the challenge of distributing entanglement in quantum repeaters. In this context, we address two main problems. First, we investigate how quantum error correction can be used to improve the secret key rates of repeaters, which is presented in Paper III [54]. Second, we explore how to optimally allocate memory buffers to enhance repeater performance. The results of this work are currently in preparation for publication [55].

The rest of the thesis is organized as follows. In chapter 2, we present the foundations of quantum information theory, introduce the theory behind quantum repeaters and networks, and discuss rate analysis. We then cover quantum error correction and examine bosonic quantum error correction in detail, with a focus on the Binomial code. In chapter 3, we introduce relevant mathematical tools from graph theory and briefly discuss random graphs and complexity classes. In chapter 4, the results of all the papers are summarized and discussed. The thesis concludes with a summary and outlook and a statement of my individual contributions to each paper.

2 Quantum Information Basics

2.1 Qubit

We begin our discussion of quantum information by introducing its most fundamental building block: the *qubit*. A quantum system with two distinguishable states is called a *qubit*. Its state is described by a normalized vector in a two-dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^2$. The computational basis vectors are denoted as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An arbitrary pure state of a qubit is a superposition

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \text{with } \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Density Operators

Having introduced qubits as vectors in a Hilbert space, we now turn to a more general and powerful way of describing quantum states. The description of a qubit using a state vector $|\psi\rangle$ applies when the system is prepared in a well-defined quantum state. Such states are called *pure states* and are represented mathematically by projectors of the form

$$\rho = |\psi\rangle\langle\psi|.$$

In many situations, however, we do not have complete information about the state of the system. For example, a qubit may be prepared in $|0\rangle$ with probability p and in $|1\rangle$ with probability $1 - p$. This type of classical uncertainty cannot be represented by a single state vector. Instead, the state of the system is described by a *mixed state*, represented by a density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

where the system is in $|\psi_i\rangle$ with probability p_i .

A density operator is therefore a positive semidefinite operator ρ on the qubit Hilbert space satisfying

$$\rho^\dagger = \rho, \quad \rho \succeq 0, \quad \text{Tr}(\rho) = 1.$$

Pure states form a special case of density operators, while mixed states capture classical randomness over quantum states. The density-operator formalism provides a general framework for describing qubit states in the presence of noise, uncertainty, or incomplete knowledge.

2.2 Single-Qubit Operations

The development of quantum technologies relies on understanding and controlling qubit dynamics. Therefore, let us now explore how qubit states change and how they can be controlled.

Single-Qubit Unitary Transformations

The evolution of a closed quantum system is described by a unitary operator U satisfying $U^\dagger U = \mathbb{I}$. For a single qubit, U acts on a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ as

$$|\psi'\rangle = U|\psi\rangle,$$

producing another normalized state in the same two-dimensional Hilbert space. Any single-qubit unitary can be written (up to a global phase) as

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos \frac{\theta}{2} & -e^{i\lambda} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & e^{i(\phi+\lambda)} \cos \frac{\theta}{2} \end{pmatrix},$$

where $\theta, \phi, \lambda \in \mathbb{R}$ are real parameters. This general form encompasses all physically realizable single-qubit operations, including rotations and phase shifts.

Pauli Operators and Their Action on Qubits

Pauli operators are among the most basic and widely used single-qubit transformations, and they serve as building blocks for more complex quantum operations. They form a fundamental set of single-qubit transformations:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Together with the identity $\mathbb{I}$, they form an orthogonal operator basis for all 2×2 matrices. Their physical effects on the computational basis states are:

$$\begin{aligned} X|0\rangle &= |1\rangle, & X|1\rangle &= |0\rangle, & (\text{bit flip}) \\ Z|0\rangle &= |0\rangle, & Z|1\rangle &= -|1\rangle, & (\text{phase flip}) \\ Y|0\rangle &= i|1\rangle, & Y|1\rangle &= -i|0\rangle, & (\text{bit and phase flip}). \end{aligned}$$

The Pauli matrices satisfy

$$X^2 = Y^2 = Z^2 = \mathbb{I}, \quad [X, Y] = 2iZ, \quad [Y, Z] = 2iX, \quad [Z, X] = 2iY,$$

and form the generators of the $SU(2)$ group, which describes all possible single-qubit rotations.

The Hadamard Gate

Another widely used single-qubit quantum gate is the Hadamard gate, denoted H , defined as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It transforms the computational basis states into equal superpositions:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The Hadamard thus connects the computational basis $\{|0\rangle, |1\rangle\}$ with the superposition basis $\{|+\rangle, |-\rangle\}$, where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

This operation is central to many quantum algorithms and protocols, as it enables coherent interference between computational paths.

Measurement

The final step of any quantum computation is to extract classical information from it, which brings us to the notion of measurement. Quantum measurement is represented by a set of operators $\{M_m\}$ satisfying $\sum_m M_m^\dagger M_m = \mathbb{I}$. Upon measuring a state ρ , outcome m occurs with probability $p_m = \text{Tr}(M_m \rho M_m^\dagger)$, and the post-measurement state becomes

$$\rho_m = \frac{M_m \rho M_m^\dagger}{p_m}.$$

A projective measurement in the computational basis corresponds to the projectors

$$M_0 = |0\rangle\langle 0|, \quad M_1 = |1\rangle\langle 1|,$$

yielding classical outcomes 0 and 1. For a pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the probabilities are $|\alpha|^2$ and $|\beta|^2$. For a general mixed state ρ , the probabilities are

$$p_0 = \text{Tr}(\rho|0\rangle\langle 0|), \quad p_1 = \text{Tr}(\rho|1\rangle\langle 1|).$$

2.3 Composite Systems

Up to now, we have focused on the behavior of individual qubits. However, the true power of quantum information arises when multiple qubits are combined to form larger systems. In such cases, new phenomena emerge that have no analogue in single-qubit physics, due to the interactions between two or more qubits.

For multiple qubits, the joint Hilbert space is given by the tensor product $\mathcal{H}^{\otimes n} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. The computational basis of an n -qubit register consists of the vectors

$$|x_1 x_2 \cdots x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle, \quad x_i \in \{0, 1\}.$$

Quantum correlations that cannot be written as tensor products of single-qubit states correspond to *entanglement*, an essential quantum-mechanical resource for quantum technologies like computation and communication.

Generally, for a composite system in a mixed state, the task of deciding whether a given density operator is entangled (inseparable) or separable is fully resolved in 2×2 and 2×3 dimensions via the PPT criterion [56, 57]; outside these dimensions, no universally applicable, efficiently checkable necessary-and-sufficient criterion is known (except for 1×1 mode Gaussian states), and one must rely on partial tests and entanglement witnesses.

Two-Qubit States and Tensor Products

To understand entanglement more concretely, we start with the simplest nontrivial composite system: two qubits. For a register of two qubits, the total Hilbert space is $\mathcal{H}^{\otimes 2} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. The computational basis is given by

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\},$$

where $|ab\rangle = |a\rangle \otimes |b\rangle$ for $a, b \in \{0, 1\}$. A general pure state of two qubits can be written as

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle, \quad \sum_{i,j} |\alpha_{ij}|^2 = 1.$$

A two-qubit state is said to be *separable* (or product) if it can be expressed as a tensor product of two single-qubit states,

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle,$$

otherwise it is *entangled*.

Bell States and Entanglement

Among all two-qubit states, the Bell states play a distinguished role. They serve both as canonical examples of maximal entanglement and as building blocks for many quantum protocols. A canonical orthonormal basis for the two-qubit Hilbert space is the *Bell basis*, consisting of the four maximally entangled states

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

These states satisfy $\text{Tr}_2(|\Phi^\pm\rangle\langle\Phi^\pm|) = \text{Tr}_2(|\Psi^\pm\rangle\langle\Psi^\pm|) = \frac{\mathbb{I}}{2}$, demonstrating maximal bipartite entanglement: each subsystem is completely mixed when traced over the other.

Entangling operations, such as the controlled-NOT (CNOT) gate, are required to generate such states from product inputs. For example,

$$|\Phi^+\rangle = (\text{CNOT})(H \otimes \mathbb{I})|00\rangle,$$

where H is the Hadamard gate.

GHZ States

Having explored entanglement for two qubits, we now extend our attention to correlations involving more qubits. When more than two qubits are involved, new forms of entanglement emerge that cannot be reduced to pairwise correlations.

A central example of multipartite entanglement is the *Greenberger–Horne–Zeilinger (GHZ) state* [58, 59], defined for n qubits as

$$|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}).$$

The GHZ state generalizes the Bell state $|\Phi^+\rangle$ to more than two qubits and exhibits maximal multipartite entanglement. Its reduced density matrices for any single qubit are maximally mixed, yet the global state is pure and displays strong correlations that have no classical analog. GHZ states play an important role in quantum communication protocols, such as quantum secret sharing and distributed consensus, and serve as benchmarks for demonstrating multipartite entanglement in experiments. With examples of two-qubit and multi-qubit entanglement in place, we now turn to the operations that create and manipulate such states.

Two-Qubit Unitary Operations

Just as single-qubit states evolve under single-qubit unitaries, two-qubit states evolve under two-qubit unitary operations. A general two-qubit gate is represented by a 4×4 unitary matrix U acting on $\mathcal{H}^{\otimes 2}$. The most important class of such gates are the *controlled* gates, which apply an operation on one qubit conditioned on the state of another.

The *controlled-NOT (CNOT) or CX* gate [60, 61] acts as

$$\text{CNOT}|a, b\rangle = |a, b \oplus a\rangle,$$

and has the matrix representation

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It flips the second (target) qubit if the first (control) qubit is in the state $|1\rangle$. CNOT, together with arbitrary single-qubit unitaries, forms a universal gate set: any n -qubit unitary can be approximated to arbitrary accuracy using these operations. Similarly, we define a *controlled-Z* (CZ) gate as applying a conditional phase (on the target qubit) rather than a bit flip: only the $|11\rangle$ component acquires a minus sign, while $|00\rangle, |01\rangle, |10\rangle$ are unchanged.

Clifford and Non-Clifford Gates

Quantum gates, which are unitary operations acting on one or more qubits, can be broadly grouped into two fundamental classes: *Clifford* gates and *non-Clifford* gates. This distinction is important because, although Clifford gates form a structured and well-behaved set, they are not sufficient on their own for universal quantum computation.

The set of *Pauli operators* on n qubits is defined as

$$\mathcal{P}_n = \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n},$$

and forms a group under matrix multiplication. The *Clifford group* $\mathcal{C}_n$ is the normalizer of the Pauli group inside the unitary group $U(2^n)$,

$$\mathcal{C}_n = \{U \in U(2^n) \mid U\mathcal{P}_n U^\dagger = \mathcal{P}_n\}.$$

This means that Clifford unitaries map Pauli operators to other Pauli operators under conjugation.

For single qubits, the Clifford group is generated by the Hadamard and phase gates,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

together with the two-qubit CNOT gate. These operations preserve the structure of stabilizer states and stabilizer codes, making them central to quantum error correction and fault-tolerant circuit design.

However, Clifford gates alone do not form a universal gate set. To perform universal quantum computation, at least one *non-Clifford* gate must be added. A standard choice is the T gate,

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

The combination of Clifford gates with a non-Clifford gate such as T yields universality, meaning that any unitary operation can be approximated to arbitrary accuracy using circuits built from these gates.

The divide between Clifford and non-Clifford gates also has significant computational consequences. The *Gottesman–Knill theorem* states that quantum circuits composed only of Clifford gates, along with preparation of stabilizer states and measurement in any stabilizer basis, can be efficiently simulated on a classical computer [62]. Therefore, non-Clifford gates are the essential ingredient that gives quantum computers computational power beyond what is efficiently achievable classically. This makes non-Clifford gates a key resource in quantum computation, and motivates techniques such as magic-state distillation to implement them fault-tolerantly.

Quantum State Teleportation

In the context of this thesis, let us consider an important primitive known as quantum state teleportation. Teleportation enables the transfer of an unknown quantum state from one spatially separated party to another, using only pre-shared entanglement and classical communication. The key insight is that no quantum information physically travels across the channel. Instead, the protocol consumes one maximally entangled Bell pair and two classical bits, allowing the receiver to reconstruct the exact state through local corrections.

Assume Alice wishes to transmit an unknown qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

to Bob. They share a Bell pair prepared in the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The joint initial state is

$$|\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)).$$

Alice applies a CNOT gate with the unknown qubit as control and her half of the Bell pair as target, followed by a Hadamard on the unknown qubit. The global state can be rewritten in the Bell basis as

$$\frac{1}{2}(|\Phi^+\rangle(\alpha|0\rangle + \beta|1\rangle) + |\Phi^-\rangle(\alpha|0\rangle - \beta|1\rangle) + |\Psi^+\rangle(\beta|0\rangle + \alpha|1\rangle) + |\Psi^-\rangle(\beta|0\rangle - \alpha|1\rangle)).$$

Alice measures her two qubits in the Bell basis, obtaining one of four classical outcomes. She transmits two classical bits to Bob. Conditioned on the received outcome, Bob performs the corresponding Pauli correction to recover the original state:

$$\text{Outcome } |\Phi^+\rangle \rightarrow I, \tag{2.1}$$

$$\text{Outcome } |\Phi^-\rangle \rightarrow Z, \tag{2.2}$$

$$\text{Outcome } |\Psi^+\rangle \rightarrow X, \tag{2.3}$$

$$\text{Outcome } |\Psi^-\rangle \rightarrow XZ. \tag{2.4}$$

after which his qubit becomes exactly $|\psi\rangle$. The original state at Alice's station is destroyed, ensuring consistency with the no-cloning principle. Teleportation thus provides a fundamental mechanism for long-distance quantum state transfer using only local operations and classical communication, a paradigm that appears repeatedly in distributed quantum information processing.

Quantum Circuits for Teleportation

Quantum circuits provide a convenient operational description of quantum information processing in terms of elementary gates, measurements, and classical control. In this formalism, qubits are represented by horizontal wires, unitary operations by gate symbols acting on one or more wires, and measurements by projective measurement boxes whose outcomes may be used to classically control subsequent operations.

Quantum state teleportation admits a particularly transparent circuit representation. The protocol involves three qubits. The first qubit encodes the unknown input state $|\psi\rangle$ held by Alice. The second and third qubits form a pre-shared Bell pair (The circuit to produce Bell-Pair is in the blue box in Figure 2.1), with the second qubit held by Alice and the third held by Bob.

A standard quantum circuit representation of the teleportation protocol is shown in Figure 2.1. The circuit proceeds as follows (excluding the first Bell-Pair generation). Alice applies a controlled NOT gate with the input qubit as control and her Bell pair qubit as target, followed by a Hadamard gate on the input qubit. She then measures both of her qubits in the computational basis. The two classical measurement outcomes are transmitted to Bob over a classical channel (double line in Figure 2.1). Bob applies Pauli corrections to his qubit conditioned on the received classical bits. After these corrections, Bob's qubit is in the state $|\psi\rangle$.

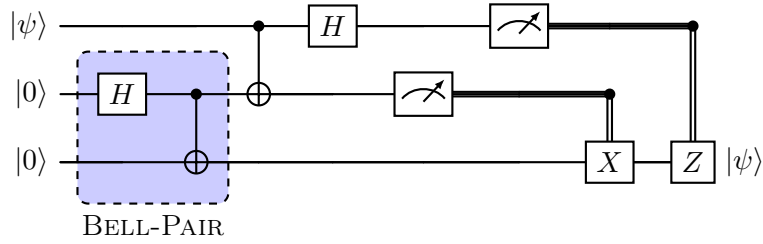


Figure 2.1: Quantum teleportation circuit with Bell measurement and feed-forward corrections.

Here the lower two qubits are first prepared in the maximally entangled Bell state $|\Phi^+\rangle$ using a Hadamard gate followed by a controlled NOT gate. The measurement outcomes of Alice's two qubits are represented by classical wires, which classically control Bob's Pauli X and Z corrections. If the first measurement outcome is 1, Bob applies an X gate. If the second measurement outcome is 1, Bob applies a Z gate. When both outcomes are zero, no correction is applied.

This circuit representation makes explicit that teleportation involves only local quantum operations, classical communication, and pre-shared entanglement. No quantum system carrying the state $|\psi\rangle$ traverses the spatial separation between Alice and Bob. The destruction of the original state by measurement guarantees consistency with the no-cloning principle, while the deterministic recovery at Bob's side highlights the power of entanglement-assisted communication.

2.4 Quantum Error Correction

Having established how quantum information is manipulated through gates and how quantum states evolve under noise, we now turn to the question of how such information can be protected from errors. Quantum error correction (QEC) provides a framework to protect quantum information from decoherence and operational imperfections arising during storage, transmission, or computation. Unlike classical error correction, quantum codes must preserve coherence between superposed states while obeying the no-cloning theorem, which forbids direct copying of quantum information.

Noise Models in Quantum Information

Before describing how errors can be corrected, it is essential to understand how they arise. In quantum information theory, *noise models* provide a mathematical description of how quantum states evolve under the influence of environmental interactions, leading to decoherence and dissipation. These processes are formally represented as *quantum channels*, i.e., completely positive trace-preserving (CPTP) maps acting on the system's density operator ρ [13].

A general quantum channel $\mathcal{E}$ can be expressed in Kraus form as

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \quad (2.5)$$

where the Kraus operators $\{E_i\}$ satisfy the completeness relation

$$\sum_i E_i^\dagger E_i = I. \quad (2.6)$$

Different physical mechanisms correspond to distinct sets of Kraus operators. We discuss here the *depolarizing*, and *amplitude damping* noise models. These examples illustrate two fundamental types of quantum noise: complete randomization of information and dissipation of energy, respectively.

Amplitude Damping Channel

We begin with a noise process that captures energy loss, one of the most common error mechanisms in physical qubit implementations. The *amplitude damping* (AD) channel describes the process by which a qubit decays from its excited state $|1\rangle$ to its ground state $|0\rangle$ with probability γ , representing spontaneous emission or energy dissipation into the environment.

The Kraus operators for the AD channel are given by

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad (2.7)$$

where $0 \leq \gamma \leq 1$ is the damping probability.

For a general single-qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \quad (2.8)$$

the evolution under amplitude damping is given by

$$\mathcal{E}_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \quad (2.9)$$

$$= \begin{pmatrix} \rho_{00} + \gamma \rho_{11} & \sqrt{1-\gamma} \rho_{01} \\ \sqrt{1-\gamma} \rho_{10} & (1-\gamma) \rho_{11} \end{pmatrix}. \quad (2.10)$$

This transformation shows that the population of the excited state ρ_{11} decays exponentially to the ground state with rate γ , while the off-diagonal coherences are attenuated by a factor of $\sqrt{1-\gamma}$, leading to a loss of quantum superposition.

Depolarizing Channel

In contrast to amplitude damping, the next model represents noise that does not prefer any direction or basis. The *depolarizing channel* represents a type of quantum noise where the qubit loses all information and becomes maximally mixed with probability p . It models situations in which the system is subjected to random, isotropic errors, such as uncontrolled interactions with the environment.

For a single qubit, the depolarizing channel can be expressed in Kraus form as

$$\mathcal{E}_{\text{dep}}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z), \quad (2.11)$$

where $0 \leq p \leq 1$ is the depolarization probability, and X, Y, Z are the standard Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.12)$$

Equivalently, using Kraus operators, the channel can be written as

$$\mathcal{E}_{\text{dep}}(\rho) = \sum_{i=0}^3 E_i \rho E_i^\dagger, \quad (2.13)$$

with

$$E_0 = \sqrt{1-p}I, \quad E_1 = \sqrt{\frac{p}{3}}X, \quad E_2 = \sqrt{\frac{p}{3}}Y, \quad E_3 = \sqrt{\frac{p}{3}}Z. \quad (2.14)$$

For an arbitrary single-qubit state

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \quad (2.15)$$

the action of the depolarizing channel yields

$$\mathcal{E}_{\text{dep}}(\rho) = \begin{pmatrix} (1 - \frac{2p}{3})\rho_{00} + \frac{p}{3} & (1 - \frac{4p}{3})\rho_{01} \\ (1 - \frac{4p}{3})\rho_{10} & (1 - \frac{2p}{3})\rho_{11} + \frac{p}{3} \end{pmatrix}. \quad (2.16)$$

This representation clearly shows that the depolarizing channel uniformly contracts the Bloch vector towards the origin, reducing both population and coherence information, and in the limit $p \rightarrow 1$ completely transforms any state into the maximally mixed state $\rho \rightarrow I/2$.

Quantum Error Correction Theory

With noise models established, we can now address how quantum information can be protected against such errors. The central idea of quantum error correction is to protect the quantum information stored in a few *logical qubits* by encoding it into the collective state of a larger number of *physical qubits*. Logical qubits represent the abstract information we wish to preserve, while physical qubits are the actual carriers of that information subject to noise and decoherence.

A quantum error-correcting *code* $\mathcal{C}$ is defined as a 2^k -dimensional subspace of the full n -qubit Hilbert space $\mathcal{H}^{\otimes n}$. It encodes k logical qubits into n physical qubits. The orthonormal basis vectors spanning $\mathcal{C}$ are called *codewords*, and their superpositions represent the valid logical states of the encoded system.

The encoding process is described by a linear isometry

$$\mathcal{E}_{\text{enc}} : \mathcal{H}^{\otimes k} \longrightarrow \mathcal{H}^{\otimes n},$$

which maps an arbitrary logical state $|\psi\rangle = \sum_i \alpha_i |i\rangle$ to its encoded form

$$|\psi_L\rangle = \mathcal{E}_{\text{enc}}(|\psi\rangle) = \sum_i \alpha_i |i_L\rangle,$$

where $\{|i_L\rangle\}$ is an orthonormal basis for the code $\mathcal{C}$.

After encoding, the system interacts with its environment, and errors occur on the physical qubits. As discussed earlier, such processes are described by Kraus operators $\{E_i\}$ acting on $\mathcal{H}^{\otimes n}$. These operators map the code $\mathcal{C}$ to a collection of subspaces $\{E_i\mathcal{C}\}$, referred to as *error spaces*.

The identification of which error has occurred is achieved through a measurement known as *syndrome detection*. This measurement distinguishes between the different error spaces $\{E_i\mathcal{C}\}$ but leaves the encoded logical state within each subspace unaffected. The outcome of the syndrome measurement specifies the error class while preserving the coherence of the logical information.

Once the syndrome has been obtained, a suitable *recovery operation* $\mathcal{R}$ is applied to reverse the corresponding error. The goal is to ensure that there exists a recovery map satisfying

$$(\mathcal{R} \circ \mathcal{E})(\rho_L) = \rho_L, \quad \forall \rho_L \in \mathcal{C},$$

where $\mathcal{E}$ denotes the noise process acting on the encoded state.

The conditions under which such a recovery operation exists are formalized by the Knill–Laflamme criteria, discussed next.

Knill–Laflamme conditions.

Let $\{E_i\}$ be a set of possible errors. The code $\mathcal{C}$ can correct all errors from this set $\{E_i\}$ if and only if, for all codewords $|\psi_a\rangle, |\psi_b\rangle \in \mathcal{C}$ and all E_i, E_j ,

$$\langle \psi_a | E_i^\dagger E_j | \psi_b \rangle = C_{ij} \delta_{ab}, \quad (2.17)$$

where C_{ij} are the elements of a Hermitian matrix depending only on the errors, not on the logical states.

Equation (2.17) can be understood as comprising two distinct requirements:

1. Error orthogonality:

$$\langle \psi_a | E_i^\dagger E_j | \psi_b \rangle = 0 \quad \text{for } a \neq b.$$

This ensures that different logical codewords affected by the same or different errors remain orthogonal. Hence, measurement of an error syndrome reveals the error without collapsing superpositions of logical states.

2. Error indistinguishability within the code:

$$\langle \psi_a | E_i^\dagger E_j | \psi_a \rangle = C_{ij} \quad \text{independent of } a.$$

This guarantees that all logical codewords are affected identically by a given error, so that the recovery operation can restore any encoded state without dependence on its logical content.

When both conditions are satisfied, the action of any correctable error E_i maps the code space $\mathcal{C}$ to a subspace orthogonal to those produced by other errors, forming a set of mutually orthogonal *error spaces*. The recovery operation can then identify the corresponding subspace and invert the error unitarily, restoring the state to the original code space.

Thus, a quantum error-correcting code is characterized by the choice of the code subspace $\mathcal{C}$ and a set of errors $\{E_i\}$ satisfying Eq. (2.17), ensuring that logical information is preserved despite physical noise.

Binomial code

The *Binomial code* [63] is a family of bosonic quantum error-correcting codes that protect a logical qubit against a finite number of photon loss, photon gain, and dephasing errors by encoding it into a finite superposition of Fock states. The code was introduced as a physically realizable alternative to continuous-variable encodings such as Cat codes [64, 65] and the Gottesman–Kitaev–Preskill (GKP) code [66], but unlike Cat or GKP codes, Binomial codes are finite superposition of Fock states.

For integers $L \geq 0$, $G \geq 0$, and $D \geq 0$, the Binomial code is designed to correct up to L photon losses, G photon gains, and D dephasing errors. The logical codewords are constructed from Fock states spaced by

$$S + 1 = L + G + 1,$$

and are defined as

$$|0_L\rangle = \frac{1}{\sqrt{2^N}} \sum_{\substack{p=0 \\ p \text{ even}}}^{N+1} \sqrt{\binom{N+1}{p}} |p(S+1)\rangle, \quad |1_L\rangle = \frac{1}{\sqrt{2^N}} \sum_{\substack{p=0 \\ p \text{ odd}}}^{N+1} \sqrt{\binom{N+1}{p}} |p(S+1)\rangle, \quad (2.18)$$

where

$$N = \max\{L, G, 2D\}.$$

The Binomial code has the same average photon number between the two logical states. The separation of the occupied Fock states by $S + 1$ photons ensures that photon loss and gain errors up to order L and G , respectively, map the code space into orthogonal subspaces labeled by the photon number modulo $S + 1$. Measurement of this generalized photon-number parity provides the error syndrome. The Binomial code satisfies the Knill–Laflamme conditions [13]

$$\langle i_L | E_m^\dagger E_n | j_L \rangle = C_{mn} \delta_{ij}$$

for all error operators

$$E \in \{a^\ell, (a^\dagger)^g, \hat{n}^d\}, \quad \ell \leq L, \quad g \leq G, \quad d \leq D,$$

and therefore exactly corrects this entire discrete error set.

Approximate quantum error correction under photon loss

In a realistic physical setting, photon loss occurs continuously in time. The dynamics of a lossy bosonic mode with decay rate κ is described by the Lindblad master equation [67]

$$\dot{\rho} = \kappa \left(a \rho a^\dagger - \frac{1}{2} \{a^\dagger a, \rho\} \right).$$

Over a finite time interval δt , the corresponding quantum channel can be written in Kraus form as

$$\rho(\delta t) = \sum_{\ell=0}^{\infty} E_\ell \rho(0) E_\ell^\dagger,$$

with Kraus operators

$$E_\ell = \sqrt{\frac{(1 - e^{-\kappa\delta t})^\ell}{\ell!}} e^{-\frac{1}{2}\kappa\delta t \hat{n}} a^\ell, \quad \ell = 0, 1, 2, \dots$$

Exact correction of all such errors is impossible, but the probability of higher-order errors scales as powers of $\kappa\delta t$.

A Binomial code that corrects up to L photon losses satisfies the quantum error-correction conditions for all Kraus operators E_ℓ with $\ell \leq L$ to order $(\kappa\delta t)^L$. Importantly, this includes not only photon jump errors a^ℓ but also the no-jump evolution

$$E_0 = e^{-\frac{1}{2}\kappa\delta t \hat{n}},$$

which deterministically distorts the state by suppressing higher Fock components.

To first order in $\kappa\delta t$, the no-jump evolution causes leakage from the logical subspace into a known orthogonal error subspace. Because the Binomial code equalizes photon-number moments between logical states, this leakage is independent of the encoded quantum information and can be coherently reversed by a conditional unitary operation. When combined with generalized parity measurements that detect photon loss events, the recovery map restores the logical state with an error of order $(\kappa\delta t)^2$.

More generally, a Binomial code correcting L photon losses implements approximate quantum error correction for the continuous-time amplitude damping channel with accuracy

$$\mathcal{O}((\kappa\delta t)^{L+1}),$$

provided error detection and recovery are performed at time intervals δt . This establishes the Binomial code as an approximate quantum error-correcting code for realistic bosonic dissipation, beyond the idealized discrete-error model [63].

2.5 Quantum Key Distribution

With the groundbreaking algorithm proposed by Peter Shor [11], which demonstrates that currently used cryptographic systems can be compromised by sufficiently powerful quantum computers, the urgency to develop cryptographic schemes resistant even to quantum attacks has become paramount. In addition to post-quantum cryptographic methods, the growing concern over “harvest now, decrypt later” strategies and the unpredictable advancement of quantum algorithms highlight the importance of exploring alternative approaches. Interestingly, quantum mechanics itself provides a potential solution to the security challenges it introduces. Quantum communication, in particular, arises from one of the fundamental principles of quantum theory: the no-cloning theorem.

No-Cloning Theorem

The no-cloning theorem states that it is impossible to clone an unknown quantum state. This fundamental result ensures that an adversary cannot copy quantum information without disturbing it, and it forms the conceptual backbone of all QKD security proofs.

Theorem 1 (No Cloning Theorem). *There is no unitary operator U on a bipartite Hilbert space $\mathcal{H} \otimes \mathcal{H}$ and no fixed “blank” state $|s\rangle \in \mathcal{H}$ such that*

$$U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for all normalized states $|\psi\rangle \in \mathcal{H}$.

Proof. Assume, for contradiction, that a unitary U and a fixed state $|s\rangle$ exist with

$$U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle \quad \text{for all } |\psi\rangle \in \mathcal{H}.$$

Pick two arbitrary normalized states $|\psi\rangle$ and $|\phi\rangle$. Unitarity preserves inner products, so

$$\begin{aligned} \langle \psi \otimes s | \phi \otimes s \rangle &= \langle U(\psi \otimes s) | U(\phi \otimes s) \rangle \\ &= \langle \psi \otimes \psi | \phi \otimes \phi \rangle. \end{aligned}$$

Evaluate both sides. The left side factorizes:

$$\langle \psi \otimes s | \phi \otimes s \rangle = \langle \psi | \phi \rangle \langle s | s \rangle = \langle \psi | \phi \rangle,$$

since $|s\rangle$ is normalized. The right side factorizes as well:

$$\langle \psi \otimes \psi | \phi \otimes \phi \rangle = \langle \psi | \phi \rangle \langle \psi | \phi \rangle = (\langle \psi | \phi \rangle)^2.$$

Thus we obtain

$$\langle \psi | \phi \rangle = (\langle \psi | \phi \rangle)^2.$$

Let $z = \langle \psi | \phi \rangle$. This equation becomes $z = z^2$, so $z(1 - z) = 0$. Hence $z = 0$ or $z = 1$. That means any two states in $\mathcal{H}$ are either orthogonal or identical. This contradicts the existence of distinct nonorthogonal states in any Hilbert space of dimension at least two. Therefore no such universal unitary U and blank state $|s\rangle$ exist. $\square$

The no-cloning theorem directly motivates the first quantum cryptographic protocols, since it implies that any attempt by an eavesdropper to copy or measure quantum information necessarily introduces detectable disturbances.

The BB84 Protocol

The first application of quantum mechanics to communication was proposed by Bennett and Brassard, who introduced the quantum key distribution (QKD) protocol, now commonly known as BB84 [14]. This protocol can subsequently be combined with the one-time pad (OTP) scheme to enable perfectly secure communication that is theoretically impossible to hack.

Let Alice and Bob be two parties who wish to communicate privately and the eavesdropper is referred to as Eve. We assume that Alice and Bob use a public, authenticated channel for communication, i.e. Eve can listen to messages, but she cannot manipulate them. This is an important assumption as otherwise Eve can easily break the encryption.

1. **Preparation:** Alice randomly chooses a bit $b_i \in \{0, 1\}$ and a basis $\beta_i \in \{Z, X\}$ for each qubit. She prepares:

$$\begin{cases} |0\rangle_Z, |1\rangle_Z & \text{if } \beta_i = Z, \\ |+\rangle_X = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle_X = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & \text{if } \beta_i = X. \end{cases}$$

2. **Transmission:** Alice sends all prepared qubits $\{|\psi_i\rangle\}$ to Bob through a quantum channel.
3. **Measurement:** For each received qubit, Bob independently chooses a random basis $\beta'_i \in \{Z, X\}$ and measures the qubit in that basis, obtaining a bit b'_i .
4. **Sifting:** Alice and Bob publicly announce their chosen bases $\{\beta_i\}$ and $\{\beta'_i\}$ over a classical channel. They keep only the bits for which $\beta_i = \beta'_i$. The resulting correlated bit string is the *sifted key*.
5. **Error Estimation:** They publicly compare a random subset of the sifted key to estimate the quantum bit error rate (QBER). If the error rate exceeds a threshold, they abort; otherwise, they proceed.
6. **Post-Processing:** Alice and Bob perform *information reconciliation* and *privacy amplification* over a classical authenticated channel to obtain the final shared secret key.

Security and Error Rates

The security of the BB84 protocol originates from Alice's use of two non-commuting bases, the Z - and X -eigenbases, for encoding classical bits into quantum states. These bases are mutually unbiased, meaning that a measurement in one basis provides no information about a state prepared in the other. This property ensures that any eavesdropper, Eve, who intercepts and measures the transmitted qubits, cannot gain information without disturbing them.

If Eve performs an intercept-resend attack, she must guess which basis Alice used. She guesses the correct basis with probability $1/2$ and then obtains the correct bit without causing any disturbance. However, when she guesses the wrong basis, which also happens with probability $1/2$, her measurement result is completely random, and the qubit she resends to Bob is disturbed. Consequently, when Bob measures such a disturbed qubit in the correct basis, his outcome differs from Alice's with probability $1/2$, which overall leads to an disagreement (error) rate of 25% in their shared bits.

In the protocol, Bob measures each received qubit in a randomly chosen basis, and after the transmission, Alice and Bob publicly reveal their bases and keep only the results where they coincide. For these matching instances, they obtain the *sifted key*. The discrepancies

between Alice’s and Bob’s bits in the sifted key define the *quantum bit error rate* (QBER), which quantifies the fraction of bits that disagree between them. Since the errors can differ depending on the measurement basis, one distinguishes between

$$e_Z = \text{error rate in the } Z\text{-basis}, \quad e_X = \text{error rate in the } X\text{-basis}. \quad (2.19)$$

These quantities represent the observed error probabilities that may arise from both imperfections in the setup and possible eavesdropping. The amount of uncertainty or information loss associated with these errors is characterized by the binary entropy function

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x), \quad (2.20)$$

which gives the information content of a binary random variable with bias x . Using these definitions, the *secret key fraction*, that is, the fraction of sifted bits that can be distilled into a provably secure key after information reconciliation and privacy amplification, is given by

$$r_{\text{sec}} = 1 - h(e_Z) - h(e_X). \quad (2.21)$$

Finally, the *secret key rate* quantifies the rate at which secure bits are generated per unit time and is obtained by multiplying the raw key rate R_{raw} (the rate of successfully transmitted and detected qubits) by the secret key fraction:

$$R_{\text{sec}} = R_{\text{raw}} r_{\text{sec}} = R_{\text{raw}} [1 - h(e_Z) - h(e_X)]. \quad (2.22)$$

Authentication and Practical Considerations

It is crucial that the classical communication channel between Alice and Bob be authenticated; otherwise, Eve could launch a *man-in-the-middle* attack, impersonating Bob to Alice and Alice to Bob. In such a case, Eve could independently execute the BB84 protocol with each party, establishing two separate secret keys—one shared with Alice and one with Bob—allowing her to decrypt, read, and even alter messages between them undetected.

To prevent this, Alice and Bob must begin with a short pre-shared authentication key or use a classical authentication scheme whose computational assumptions are trusted to hold for the short duration of the key exchange. This initial secret is used only to confirm each other’s identities, not to generate the key itself, and it ensures that all subsequent messages exchanged over the classical channel are genuine.

Extensions and Variants of BB84

While in the original BB84 protocol Alice chooses each basis with equal probability (50% Z , 50% X), this is not strictly necessary. In practice, using one basis more frequently—typically the Z -basis—can improve efficiency because most of the final key is derived from measurements in that basis, while a smaller fraction of X -basis measurements suffices for estimating Eve’s potential information via the error rate. This approach, known as *biased basis choice*, increases the raw key rate without compromising security, provided the estimation in the less-used basis remains statistically reliable [68].

Furthermore, several generalizations of BB84 employ three mutually unbiased bases (X , Y , and Z) [69], or other higher-dimensional encoding schemes, which can enhance robustness against specific types of noise or allow for more secure implementations under practical device imperfections.

2.6 Quantum Repeaters

Transmission loss in an optical fibre channel imposes a fundamental limit on the performance of point-to-point quantum key distribution (QKD). In a standard single-mode telecom fibre the attenuation coefficient is typically $\alpha \approx 0.2 \text{ dB/km}$, so that the channel transmissivity decays as

$$\eta(L) = 10^{-\alpha L/10}$$

with distance L , leading to an exponential suppression of the raw key rate. A milestone result by Stefano Pirandola et al. [20] derived the point-to-point secret-key capacity bound (often called the PLOB bound) for a pure-loss bosonic channel:

$$K \leq -\log_2(1 - \eta) \approx 1.44\eta$$

bits per channel use in the asymptotic limit. Thus, point-to-point links are not practical for long distances and future quantum internet [43]. Satellite-based QKD offers very long reach and potential global connectivity, though it comes with weather dependence, limited duty cycles, and pointing/tracking challenges [21, 23, 70, 71].

An alternative route toward global quantum communication is to introduce intermediate stations that divide the total distance into shorter segments, establish entanglement within each segment, and subsequently connect them through entanglement swapping. These stations, known as *quantum repeaters*, employ quantum memories and local operations to overcome the repeater-less exponential loss scaling, enabling a polynomial dependence of the communication rate on distance [24]. Since entanglement does not need to be generated simultaneously across all segments, the scaling of the total communication time depends on the segment length rather than the total distance.

Working of a Quantum Repeater

The physical structure of a memory-based quantum repeater is shown in Figure 2.2.

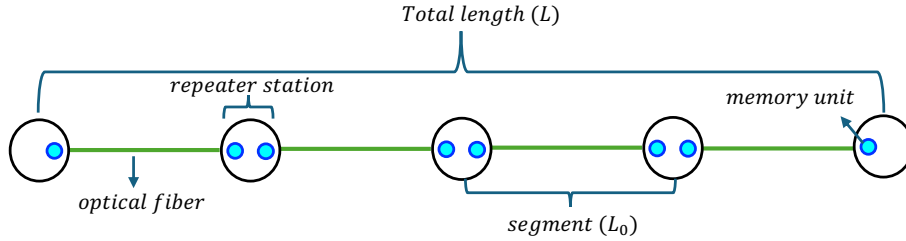


Figure 2.2: Physical structure of a four-segment quantum repeater.

The total channel length is divided into shorter segments, with quantum memories placed at the ends of each segment to store entanglement. The objective is to establish entanglement between the memories at the two ends of the total distance. This is accomplished through two fundamental building blocks: *entanglement generation* and *entanglement swapping*.

Entanglement generation is a probabilistic process that occurs within each segment [72]. One common approach creates atom–photon entanglement locally, with atoms as memory qubits and photons as flying qubits, and sends the photons toward a middle station. There, a beam splitter and coincidence detection project the remote memories into an entangled state. Alternatively, an entangled-photon source at the segment center can generate photon pairs sent toward the memories. In both cases, successful entanglement is heralded, defining an elementary time step determined by the segment length and light-travel time L_0/c , as illustrated in Figure 2.3.

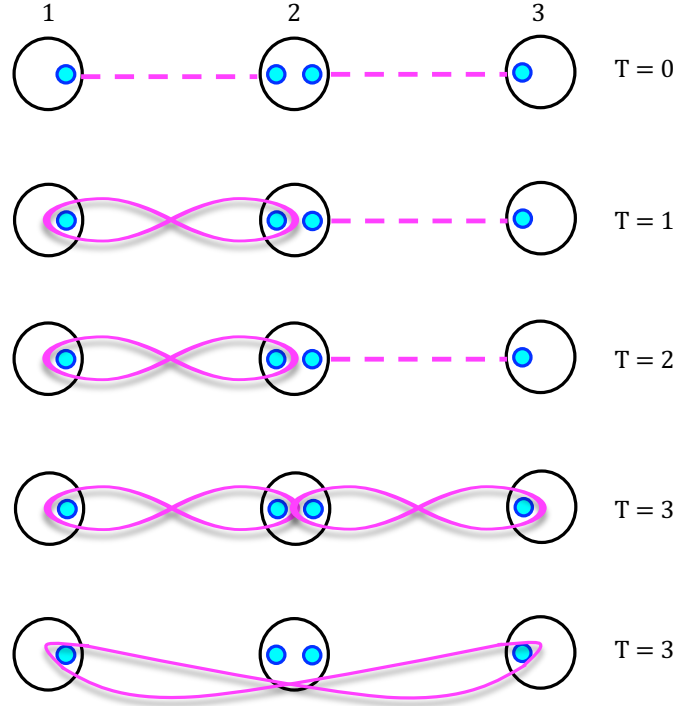


Figure 2.3: Time evolution of a two segment quantum repeater.

Entanglement swapping extends entanglement between distant nodes. A Bell-state measurement on one memory from each of two independently entangled pairs projects the remaining memories into an entangled state, even though they never interacted directly [73]. This process can be repeated across repeater stations, allowing long-distance entanglement to be distributed step by step without requiring additional time per swap. The example in Figure 2.3 demonstrates the probabilistic entanglement distribution over two segments and the time steps required until successful swapping.

Repeater Generations and Performance Metrics

Quantum repeaters are commonly classified into three generations based on the methods used to overcome photon loss and operational errors. The *first generation* relies on entanglement swapping and purification with two-way classical communication across the entire channel length, limiting communication rates due to delays [25, 29, 74]. The *second generation* introduces quantum error correction on the memories to handle operational errors and storage while retaining heralded entanglement generation, significantly improving the rate. The *third generation* employs full quantum error correction to address photon loss and operational errors simultaneously, allowing one-way near-deterministic entanglement distribution, though it requires advanced photonic engineering and fault-tolerant operations.

A key figure of merit is the average time to generate a single entangled pair between extreme nodes [26], which in our simple example is $3L_0/c$. The inverse defines the *raw entanglement generation rate*, while the *secret key rate* quantifies the number of secure bits transmitted per second, typically evaluated using the BB84 protocol (see Equation 2.22). Theoretical evaluation of the entanglement generation rate is central to quantum repeater research and depends on the number of segments, memory properties, and error-correction strategies.

The work presented in **Paper III** of this thesis focuses on second-generation quantum

repeaters. In particular, we provide a theoretical analysis of the secret key rates for a quantum repeater system that employs a bosonic quantum error-correcting code. The memories are assumed to be microwave cavities.

2.7 Graph States

Graph states are a class of highly entangled multi-qubit states that play a central role in quantum information science [75]. They form a bridge between the mathematical framework of graph theory and the physical realization of multi-qubit entanglement. A graph state can be directly associated with a mathematical graph, where the structure of the graph determines the pattern of entanglement among qubits.

Definition. A graph is defined as $G = (V, E)$, where V is the set of vertices (or nodes) and $E \subseteq V \times V$ is the set of edges connecting pairs of vertices. To each vertex $v_i \in V$, we associate a qubit, and the edges $(v_i, v_j) \in E$ represent entanglement between qubits i and j .

Construction of a graph state. Given a graph $G = (V, E)$ with $n = |V|$ vertices, the corresponding quantum graph state $|G\rangle$ can be constructed in two steps:

1. **Initialization:** Prepare each qubit $i \in V$ in the state

$$|+\rangle_i = \frac{1}{\sqrt{2}}(|0\rangle_i + |1\rangle_i).$$

The total initial state of the n qubit system is therefore

$$|+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle.$$

2. **Entanglement creation:** For every edge $(i, j) \in E$, apply a controlled- Z (CZ) gate between qubits i and j . The resulting graph state is

$$|G\rangle = \prod_{(i,j) \in E} CZ_{ij} |+\rangle^{\otimes n}.$$

This procedure generates a highly entangled multi-qubit state whose structure mirrors the connectivity of the underlying graph. Examples of different graph states are illustrated in Fig. 2.4.

Examples.

- **Two-qubit graph (Bell pair):** For a graph with two vertices $V = \{1, 2\}$ and one edge $E = \{(1, 2)\}$,

$$|G\rangle = CZ_{12} |+\rangle_1 |+\rangle_2 = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle),$$

which is equivalent (up to a Hadamard) to the maximally entangled Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

- **Three-qubit graph (GHZ state):** For three vertices $V = \{1, 2, 3\}$ with edges $E = \{(1, 2), (1, 3)\}$,

$$|G\rangle = CZ_{12} CZ_{13} |+\rangle_1 |+\rangle_2 |+\rangle_3 = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle),$$

which can be locally transformed into the GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Well-known special cases such as GHZ states and cluster states serve as essential resources in measurement-based quantum computation (MBQC), quantum communication, and quantum error correction.

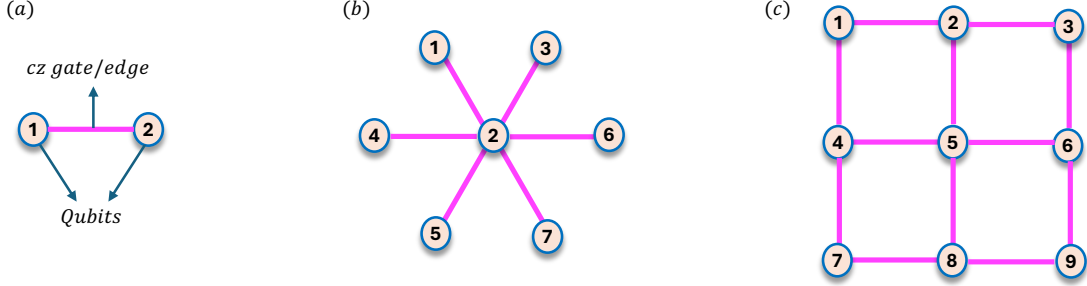


Figure 2.4: Example quantum graph states

Basic Operations on Graph States

Given a graph state $|G\rangle$ associated with a graph $G = (V, E)$, local single-qubit operations and measurements play a crucial role in manipulating and transforming its entanglement structure. In particular, they are used for quantum error correction, measurement based quantum computing (MQBC), quantum communications and sensing. These transformations correspond to well-defined graph operations, allowing one to describe the dynamics of quantum states directly in terms of their underlying graphs.

Local operations and equivalence. Graph states exhibit equivalence relations under local operations acting independently on each qubit. Of particular importance are local Clifford (LC) and local unitary (LU) equivalences, which preserve the entanglement structure of the state.

- **Local Clifford (LC) equivalence and local complementation:** Two graph states $|G\rangle$ and $|G'\rangle$ are said to be LC-equivalent if there exists a unitary of the form

$$U = C_1 \otimes C_2 \otimes \cdots \otimes C_n,$$

where each C_i is a single-qubit Clifford operator, such that $|G'\rangle = U|G\rangle$.

For graph states, LC transformations admit a simple graph-theoretic description in terms of *local complementations*. Specifically, a local complementation τ_i at a vertex i is defined as follows: it acts on the neighborhood $N(i)$ by replacing the induced subgraph with its complement. That is, for every pair of vertices $u, v \in N(i)$, the edge (u, v) is present in $\tau_i(G)$ if and only if it was absent in G , while all other edges remain unchanged.

On the quantum level, for each τ_i there exists a corresponding local Clifford unitary U_i (acting on qubit i and its neighbors) such that

$$U_i |G\rangle = |\tau_i(G)\rangle.$$

Consequently, two graph states are LC-equivalent if and only if their graphs are related by a finite sequence of local complementations.

- **Local unitary (LU) equivalence:** More generally, two graph states are LU-equivalent if they are related by arbitrary single-qubit unitaries,

$$U = U_1 \otimes U_2 \otimes \cdots \otimes U_n.$$

LC-equivalence is a special case of LU-equivalence. While LC-equivalence implies LU-equivalence, the converse does not hold in general for larger systems. Nevertheless, LC-equivalence provides a powerful and tractable framework for classifying graph states due to its direct correspondence with graph transformations.

Pauli basis measurements. Local measurements on individual qubits map one graph state to another on a reduced set of vertices. The effect of such measurements can be entirely described in terms of local complementations and vertex deletions. Up to local Pauli corrections, the transformations are as follows:

- **Measurement in the Z basis:** Measuring qubit i in the Z basis removes the vertex i and all its incident edges from the graph:

$$G \longrightarrow G - \{i\}.$$

The resulting state is a graph state on the remaining vertices.

- **Measurement in the Y basis:** Measurement in the Y basis is equivalent, up to local Clifford corrections, to performing a local complementation at vertex i followed by deletion of that vertex:

$$G \longrightarrow \tau_i(G) - \{i\}.$$

- **Measurement in the X basis:** For an X -basis measurement on vertex i , one first selects a neighbor $b \in N(i)$, performs a local complementation at b , then at i , and again at b , after which the vertex i is deleted:

$$G \longrightarrow \tau_b(\tau_i(\tau_b(G))) - \{i\}.$$

An important problem in quantum information theory, and one that is relevant to this thesis, concerns transformations between graph states. Graph states provide a combinatorial description of multipartite entanglement, where the underlying graph encodes the structure of correlations between qubits. Transformations of the graph therefore correspond to transformations of the entanglement structure of the quantum state. A central question is to determine when two graph states represent the same entanglement up to local operations. In particular, one asks which graph states can be transformed into one another using only local unitary (LU) operations, which do not change the amount or type of entanglement but merely correspond to local changes of basis [76, 77]. For example, the Greenberger–Horne–Zeilinger (GHZ) state is LU-equivalent to a complete graph state. Understanding such equivalences is important both conceptually and practically. Conceptually, it leads to a classification of multipartite entanglement into equivalence classes. Practically, it allows one to identify when different graph states are interchangeable for tasks such as measurement-based quantum computation, quantum error correction, and quantum communication.

2.8 Quantum Networks

Graph states have a wide range of applications in quantum communication and quantum computing. Generating them in a nonlocal manner is particularly relevant for tasks such as distributed quantum computing and multiparty quantum communication in a future quantum internet [43, 78, 79]. The physical infrastructure required to create such highly entangled, nonlocal quantum states is a quantum network [80]. In a quantum network, each node represents a quantum device capable of storing and processing qubits, while each edge corresponds to a quantum channel, such as an optical fiber link, connecting these devices. This framework establishes a natural bridge between quantum information theory and network science, enabling the description of complex entangled systems using familiar network theoretic concepts. An example of a quantum network is shown in Figure 4.1(a). The repeater chain (Figure 2.2) can also be viewed as a simpler instance of a linear quantum network.

When the nodes of the network are physically separated by large distances, direct multi-qubit operations between them are infeasible due to the absence of a shared quantum interaction and the extremely low communication throughput imposed by photon loss and noise. For this reason, the practically viable strategy is to first generate and distribute Bell pairs across the elementary links of the network, since Bell state distribution is the most robust and experimentally accessible nonlocal primitive. These elementary Bell pairs serve as entanglement resources that can later be combined to create more complex multipartite entangled states. Because the classical communication latency across long distances (a few hundred kilometers) is not negligible, it is advantageous to restrict the nonlocal processing to operations that require only local quantum gates at each node together with classical message exchange. This excludes coherent nonlocal operations such as gate teleportation from the real-time processing loop and places the focus on local operations and classical communication as the core mechanism for entanglement manipulation.

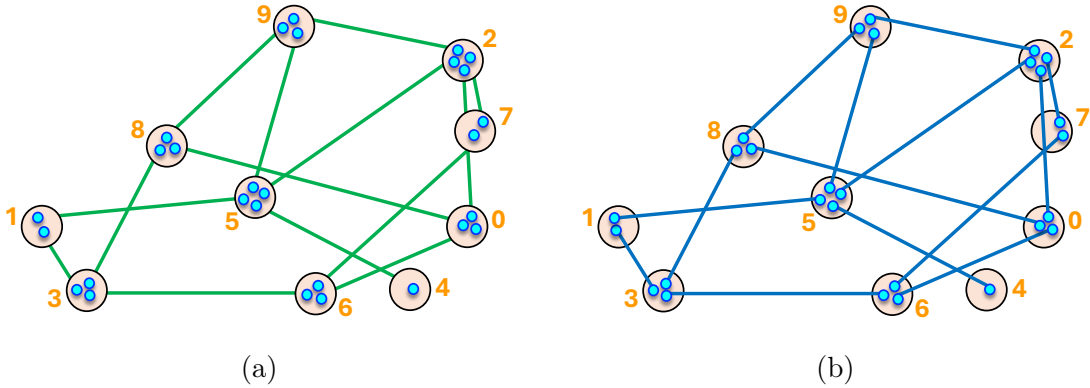


Figure 2.5: Example of a quantum network and the Bell pairs generated on its physical links. (a) Network topology where orange circles are nodes, green lines are physical connections, and blue circular discs are qubits at each node. (b) Established Bell pairs between connected nodes.

A key feature of quantum networks is therefore the use of Bell pairs, typically the maximally entangled state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad (2.23)$$

as the fundamental nonlocal resource. More complex entangled states are created from these primitive pairs through a sequence of local operations and classical communication (LOCC) [81]. Entanglement is a nonclassical correlation structure that cannot be generated

by LOCC alone, and the LOCC framework restricts each party to quantum operations on its own subsystem augmented with classical message exchange. LOCC operations cannot increase entanglement, which means that all nonlocal quantum resources must originate from previously distributed Bell pairs. In this picture, the quantum network functions as a distributed entanglement processing system where initial Bell pairs are transformed, via LOCC, into the graph states required for quantum communication and distributed quantum computing tasks.

The Bell pairs generated on the network topology presented in Figure 4.1(a) are depicted in Figure 4.1(b). Using these Bell pairs, one can generate more complex multipartite entangled states through a variety of established protocols, all of which consider the number of Bell pairs and the time required as the primary figures of merit. The problem of optimally distributing entanglement from these Bell pairs is analyzed in detail in this thesis, with the results presented in Papers I and II.

2.9 Physical Realizations of Qubits

A qubit is an abstract two-dimensional quantum system, but in practice it must be encoded in a controllable physical degree of freedom whose dynamics can be manipulated and measured with high fidelity. Many physical platforms have been developed for this purpose, each with different advantages in terms of coherence, scalability, and controllability. However, we discuss two widely used implementations relevant to this thesis; photonic qubits [82–84] and cavity-based qubits in circuit quantum electrodynamics (cQED) [85–88]. Photons provide excellent carriers of quantum information for long-distance transmission due to their weak interaction with the environment, while cavity-based systems enable strong light–matter interactions suitable for local quantum processing and high-fidelity control. This section summarizes the physical principles underlying these two platforms, focusing on how qubits are encoded, generated, manipulated, and measured, as well as the dominant sources of decoherence.

Photonic Qubits

Photonic qubits encode quantum information in degrees of freedom of single photons. Because photons interact only weakly with their environment, they can propagate over long distances with minimal decoherence, making them ideal for quantum communication.

Physical System and Encoding. The electromagnetic field is quantized into discrete modes described by bosonic annihilation and creation operators a and $a^\dagger$. The number states $|n\rangle$ satisfy

$$a^\dagger a |n\rangle = n |n\rangle. \quad (2.24)$$

Photonic qubits are typically encoded in a two-dimensional subspace of these optical modes. Several encodings are commonly used.

A widely used example is polarization encoding,

$$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle, \quad (2.25)$$

where $|H\rangle$ and $|V\rangle$ represent orthogonal horizontal and vertical polarization states.

Another common approach is dual-rail encoding, in which a single photon occupies one of two spatial modes,

$$|0_L\rangle = |1\rangle_a |0\rangle_b, \quad |1_L\rangle = |0\rangle_a |1\rangle_b. \quad (2.26)$$

Time-bin encoding is also widely used for long-distance communication. In this scheme, a qubit is defined using early and late temporal modes,

$$|\psi\rangle = \alpha|1_{t_0}, 0_{t_1}\rangle + \beta|0_{t_0}, 1_{t_1}\rangle. \quad (2.27)$$

This encoding is particularly robust in optical fibers since it is insensitive to polarization drift.

Photon Generation. Single photons can be generated using several physical mechanisms. One widely used process is spontaneous parametric down-conversion (SPDC) in nonlinear crystals [89], where a pump photon is converted into a pair of lower-energy photons. The interaction Hamiltonian can be written as

$$H_{\text{SPDC}} = \hbar\kappa \left(a_s^\dagger a_i^\dagger + a_s a_i \right), \quad (2.28)$$

where $a_s^\dagger$ and $a_i^\dagger$ create signal and idler photons. Detection of one photon heralds the presence of the other, providing a probabilistic source of single photons. This procedure can also be used to produce entangled photon pairs [90].

Operations and Control. Operations on photonic qubits are typically implemented using linear optical elements. Beamsplitters, phase shifters, and waveplates perform unitary transformations on optical modes [91].

For example, a beamsplitter acting on two modes a and b is generated by the Hamiltonian

$$H_{\text{BS}} = \hbar\theta(a^\dagger b + b^\dagger a), \quad (2.29)$$

leading to the mode transformation

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \quad (2.30)$$

More generally, a linear optical network transforms modes according to

$$a'_i = \sum_j U_{ij} a_j, \quad (2.31)$$

where U is a unitary matrix implemented by an interferometer. Integrated photonic circuits can realize large interferometric networks capable of performing complex multi-mode transformations.

Measurement. Measurement of photonic qubits is performed using single-photon detectors. An ideal photon-number measurement is described by projectors

$$\Pi_n = |n\rangle\langle n|. \quad (2.32)$$

In practice, detectors such as avalanche photodiodes (APDs) or superconducting nanowire single-photon detectors (SNSPDs) measure photon arrival events with high efficiency.

Polarization qubits are measured using polarizing beamsplitters followed by photon detection, while time-bin qubits are measured using interferometers that recombine temporal modes before detection.

Loss and Decoherence. The dominant error mechanism for photonic qubits is photon loss during propagation. In optical fibers, attenuation leads to an exponential decrease in transmission probability with distance [92]. This process can be modeled using a beam-splitter interaction between the signal mode and an environmental vacuum mode.

Additional imperfections include detector inefficiency, dark counts, and mode mismatch in interferometric networks.

Microwave Cavity Qubits (Circuit Quantum Electrodynamics)

Microwave cavity qubits arise from the interaction between superconducting artificial atoms and quantized electromagnetic modes confined within microwave resonators. This architecture, known as circuit quantum electrodynamics (cQED) [85, 93], enables strong light-matter coupling and precise quantum control [86].

Physical System and Encoding. A superconducting qubit behaves as an effective two-level system with Hamiltonian

$$H_q = \frac{\hbar\omega_q}{2}\sigma_z. \quad (2.33)$$

The microwave cavity supports quantized electromagnetic modes described by

$$H_{\text{cav}} = \hbar\omega_c a^\dagger a. \quad (2.34)$$

The qubit can be encoded either in the energy levels of the artificial atom or in photon states of the cavity mode.

State Preparation. Qubit states are typically prepared using resonant microwave pulses that drive coherent rotations between the ground and excited states. A drive field with amplitude Ω produces the Hamiltonian

$$H_{\text{drive}} = \frac{\hbar\Omega}{2}(\sigma_+ e^{-i\omega_d t} + \sigma_- e^{i\omega_d t}), \quad (2.35)$$

allowing arbitrary single-qubit rotations.

Light–Matter Interaction. The interaction between the qubit and the cavity mode is described by the Jaynes–Cummings Hamiltonian [94, 95],

$$H_{\text{JC}} = \hbar\omega_c a^\dagger a + \frac{\hbar\omega_q}{2}\sigma_z + \hbar g(a\sigma_+ + a^\dagger\sigma_-). \quad (2.36)$$

Here g is the vacuum Rabi coupling strength, which characterizes the coherent exchange of excitations between the qubit and the cavity.

Dispersive Operations. When the detuning $\Delta = \omega_q - \omega_c$ satisfies $|\Delta| \gg g$, the system enters the dispersive regime. A perturbative expansion yields the effective Hamiltonian

$$H_{\text{disp}} = \hbar(\omega_c + \chi\sigma_z)a^\dagger a + \frac{\hbar}{2}(\omega_q + \chi)\sigma_z, \quad (2.37)$$

where

$$\chi = \frac{g^2}{\Delta}. \quad (2.38)$$

This interaction enables qubit readout and effective qubit–qubit coupling mediated by virtual photons in the cavity.

Measurement. Qubit measurement is typically performed dispersively by probing the cavity transmission. Because the cavity frequency depends on the qubit state, the output field carries information about the qubit state.

Using input–output theory, the output field satisfies

$$a_{\text{out}}(t) = a_{\text{in}}(t) + \sqrt{\kappa} a(t), \quad (2.39)$$

where κ is the cavity decay rate.

Dissipation and Decoherence. Open-system dynamics are described using a Lindblad master equation [67],

$$\dot{\rho} = -\frac{i}{\hbar}[H_{\text{JC}}, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right). \quad (2.40)$$

Typical dissipation channels include

- qubit relaxation $L_1 = \sqrt{\gamma} \sigma_-$,
- qubit dephasing $L_\phi = \sqrt{\gamma_\phi} \sigma_z$,
- cavity photon loss $L_c = \sqrt{\kappa} a$.

These processes limit coherence times and gate fidelities in superconducting quantum circuits.

Photonic qubits and microwave cavity qubits therefore represent complementary approaches to quantum information processing. Photons provide robust carriers for long-distance quantum communication, while cavity-based systems enable strong coherent interactions and high-fidelity control required for local quantum processing.

3 Classical Networks

As discussed in the previous chapter, networks naturally play a central role in distributing quantum entanglement across multiple parties. To fully understand the results presented in this thesis, it is therefore essential to examine the theory of networks in detail. Networks provide a unifying framework for describing and analyzing systems composed of interconnected elements and arise across a wide range of disciplines, including physics, biology, computer science, and communication engineering [96, 97].

This chapter introduces the mathematical and computational foundations of networks, beginning with the fundamentals of graph theory [98–100]. We then explore computational complexity in network problems and discuss probabilistic network models that capture the structure of real-world systems. These tools form the basis for understanding network behavior, designing algorithms for network optimization, and analyzing the robustness and efficiency of complex interconnected systems.

3.1 Fundamentals of Graph Theory

Graph theory provides a mathematical framework for describing networks. A *graph* $G = (V, E)$ consists of a finite set of *vertices* (or nodes) $V = \{v_1, v_2, \dots, v_n\}$ and a set of *edges* $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ that connect pairs of vertices. Figure 3.1 shows a simple example of a graph.

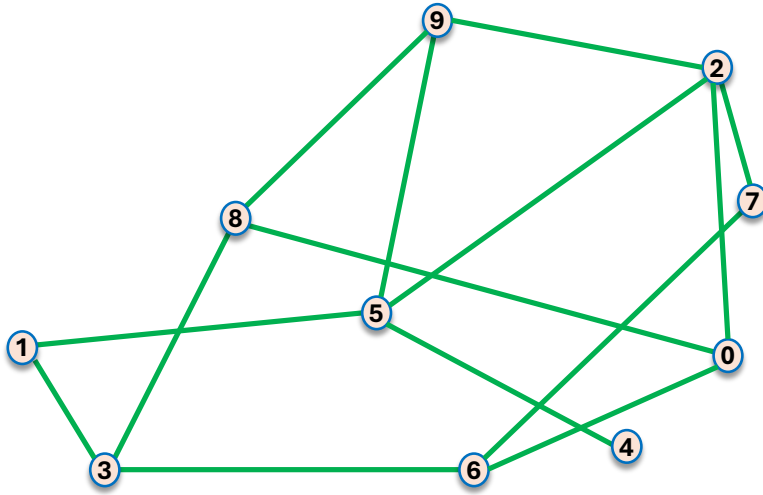


Figure 3.1: An example of a Graph. The orange circles denote the nodes or vertices of the network and the green lines the edges.

Undirected and Directed Graphs

If each edge $\{u, v\} \in E$ has no inherent direction, the graph is called *undirected*. In this case, $\{u, v\}$ and $\{v, u\}$ represent the same edge. If each edge has an assigned direction, the graph is called *directed* (or a *digraph*), and edges are written as ordered pairs (u, v) ,

meaning a connection from u to v . These two types of graphs form the basis for many network models studied in both theoretical and applied contexts.

Weighted Graphs

In many applications, edges carry a numerical value or *weight* $w(u, v)$, representing quantities such as distance, cost, or capacity. A *weighted graph* is therefore defined as a triple $G = (V, E, w)$, where $w : E \rightarrow \mathbb{R}^+$ assigns a positive real number to each edge. The *weight of a path* is the sum of its constituent edge weights. Weighted graphs are particularly important when modeling real-world networks where connections are not merely present or absent but have different strengths or costs.

Tree Graphs

A *tree* is a connected, undirected graph that contains no cycles. Equivalently, a tree is a graph in which any two vertices are connected by exactly one path. Trees are fundamental structures in both theoretical and applied network analysis, appearing in hierarchical organization, routing, and spanning structures.

Some important properties of trees include:

- A tree with n vertices has exactly $n - 1$ edges.
- Adding any edge to a tree creates a cycle.
- Removing any edge disconnects the graph.

A *rooted tree* designates one vertex as the root, from which a parent-child hierarchy is defined. Rooted trees are particularly useful for representing hierarchical networks, decision processes, or search structures.

Example: Consider a network of 5 nodes, with the following connections:

$$E = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}\}.$$

This graph is a tree since it is connected and contains no cycles. Node 1 can be chosen as the root, making nodes 2 and 3 its children, while nodes 4 and 5 are children of node 3.

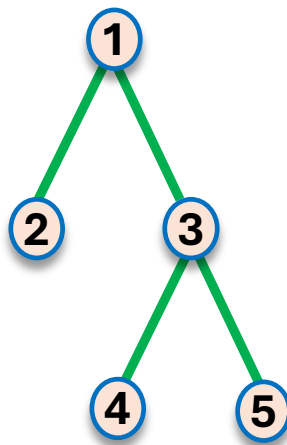


Figure 3.2: A simple tree graph with 5 nodes and 4 edges. Node 1 is the root.

Adjacency Representation

A graph can be represented in various equivalent forms. The *adjacency matrix* A of a simple graph G with n vertices is an $n \times n$ matrix defined by

$$A_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For weighted graphs, $A_{ij} = w(v_i, v_j)$ if an edge exists, and $A_{ij} = 0$ otherwise. Alternatively, one may use an *adjacency list*, which for each vertex v_i records the set of its neighboring vertices. These representations form the foundation for computational approaches to network analysis.

Paths, Connectivity, and Subgraphs

A *path* in G is a sequence of vertices $(v_1, v_2, \dots, v_k)$ such that each consecutive pair is connected by an edge. If every vertex is reachable from every other vertex through some path, the graph is said to be *connected*. A *subgraph* $G' = (V', E')$ is a graph whose vertex and edge sets satisfy $V' \subseteq V$ and $E' \subseteq E$. These concepts of paths, connectivity, and subgraphs are essential for defining optimization problems on graphs, which we discuss next.

Graph Traversal: BFS and DFS

Graph traversal algorithms are essential for exploring the structure of a graph, finding paths, and analyzing connectivity. Two fundamental traversal strategies are *Breadth-First Search (BFS)* and *Depth-First Search (DFS)* [96].

Breadth-First Search (BFS)

BFS explores a graph layer by layer, starting from a given *source vertex*. It first visits all neighbors of the source, then all neighbors of those neighbors, and so on. BFS is particularly useful for finding the shortest path (in terms of number of edges) between vertices in an unweighted graph.

Algorithm:

1. Initialize a queue with the source vertex and mark it as visited.
2. While the queue is not empty:
 - a) Dequeue a vertex v from the front of the queue.
 - b) For each unvisited neighbor u of v , mark u as visited and enqueue it.

Example: Consider the tree in Figure 3.2 with root node 1. BFS traversal starting from node 1 visits the nodes in the order:

$$1 \rightarrow 2, 3 \rightarrow 4, 5.$$

Depth-First Search (DFS)

DFS explores a graph by following a path as far as possible before backtracking. It uses a stack (either explicitly or via recursion) to remember vertices whose neighbors are yet to be explored. DFS is useful for detecting cycles, connected components, and for topological sorting in directed acyclic graphs.

Algorithm:

1. Start at a source vertex and mark it as visited.
2. Recursively visit each unvisited neighbor.
3. Backtrack when no unvisited neighbors remain.

Example: For the same tree in Figure 3.2, a DFS traversal starting from node 1 could visit the nodes in the order:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5.$$

Comparison: BFS explores in concentric layers and guarantees the shortest path in unweighted graphs, whereas DFS dives deep along a path and may explore longer paths first. Both algorithms have a time complexity of $\mathcal{O}(|V| + |E|)$ for a graph with $|V|$ vertices and $|E|$ edges.

3.2 Computational Complexity

Computational complexity theory is concerned with understanding the fundamental resources required to solve computational problems [101–103]. In practical applications, such as optimizing the distribution of entanglement in quantum networks, one must define a figure of merit to quantify what is being optimized, for example the number of Bell pairs used or the total time required. By studying the computational complexity of these problems, we can categorize them into different classes, which provides insight into which tasks can be efficiently solved, which require significant computational effort, and which are unlikely to admit efficient algorithms.

P, NP, and NP-Hard Problems

Mathematicians formalized these ideas and began classifying problems into a hierarchy according to the computational resources required to solve them. The class **P** consists of problems that can be solved in polynomial time by a deterministic algorithm. The class **NP** consists of problems for which a proposed solution can be *verified* in polynomial time, even if finding that solution may be computationally difficult.

A problem is said to be *NP-hard* if every problem in **NP** can be reduced to it in polynomial time, meaning that an efficient solution to this problem would yield efficient solutions to all **NP** problems. If an NP-hard problem also belongs to **NP**, it is called *NP-complete*. In 1972, Richard Karp famously identified 21 combinatorial problems that are NP-complete, demonstrating that a wide variety of computational tasks share this intrinsic difficulty [104]. Determining whether **P** = **NP** remains one of the most important open questions in computer science, with profound implications for optimization, cryptography, and algorithm design.

The graph-theoretic problems discussed in this thesis provide relevant examples of NP-hard tasks, highlighting the computational challenges inherent in optimizing network structures and entanglement distribution in quantum networks. In the context of this thesis, two NP-hard graph-theoretic problems are of particular interest: the *minimal Steiner tree problem* and the *minimum dominating set problem*.

The Minimum Steiner Tree Problem

The *Minimum Steiner Tree* (MST) problem is a classic example of an NP-hard combinatorial optimization problem [104]. Given a weighted undirected graph $G = (V, E, w)$ and a subset of vertices $T \subseteq V$ called *terminals*, the goal is to find a connected subgraph $G' = (V', E')$ of minimal total edge weight such that $T \subseteq V' \subseteq V$ and all vertices in T are connected within G' . Vertices in $V' \setminus T$ are called *Steiner vertices* and may be included to reduce the overall cost.

Formally, the problem is to minimize

$$W(G') = \sum_{\{u,v\} \in E'} w(u,v)$$

subject to the connectivity constraint on T . This problem generalizes the shortest path and minimum spanning tree problems, which can both be solved in polynomial time, but the inclusion of Steiner vertices makes the problem NP-hard. Understanding MST also motivates the study of heuristic and quantum approaches for large-scale network optimization.

The Minimum Dominating Set Problem

Another important NP-hard problem in graph theory is the *Minimum Dominating Set* (MDS) problem [104]. Given a graph $G = (V, E)$, a subset of vertices $D \subseteq V$ is called a *dominating set* if every vertex in the graph is either in D or adjacent to at least one vertex in D . Formally, D is dominating if

$$\forall v \in V, \quad v \in D \text{ or } \exists u \in D \text{ such that } \{u, v\} \in E.$$

The goal of the MDS problem is to find a dominating set of the smallest possible size.

$$D^* = \arg \min_{D \subseteq V} |D| \quad \text{subject to } D \text{ is dominating.}$$

Intuitively, the dominating set represents a minimal collection of “control” or “monitoring” vertices such that all other vertices are within one edge of some vertex in D . This concept naturally connects to random graph models and network analysis, where the structure of connectivity influences the difficulty and strategies for finding minimal dominating sets.

3.3 Random Graphs and Network Models

In many physical, biological, and communication systems, the connectivity between entities is not fixed but arises from probabilistic or generative processes [96, 97]. *Random graph models* provide a framework to study such systems statistically, capturing universal properties such as degree distributions, clustering, and path lengths that emerge in complex networks. Random networks also offer a useful test bed to evaluate the performance of new algorithms, allowing one to assess how well optimization strategies perform under different network topologies. Furthermore, when designing quantum networks and a future quantum internet, it is natural to consider the construction of these networks using existing communication infrastructure and resources, which often exhibits irregular or heterogeneous connectivity.

In this thesis, we focus on three representative random network models. The *Erdős-Rényi model* serves as a general-purpose benchmark for network behavior and connectivity [105, 106]. The *Barabási-Albert model* captures the scale-free structure observed in

many real-world networks, including the classical internet, making it particularly relevant for studying quantum network protocols over heterogeneous infrastructures [97, 107]. The *Waxman model* incorporates spatial constraints and link-dependent transmission loss, providing a natural framework for modeling optical fiber networks and the effects of distance-dependent attenuation in quantum communication [108]. Subsequent subsections describe the mathematical formulation of each of these models in detail.

Erdős–Rényi Model

The simplest and most widely studied random graph model is the *Erdős–Rényi (ER)* model. Given n vertices, each possible edge between two distinct vertices is included independently with probability p . This defines the random graph $G(n, p)$, where

$$\mathbb{P}[\{u, v\} \in E] = p, \quad \forall u, v \in V, u \neq v.$$

The ER model exhibits a sharp phase transition in connectivity: as p increases beyond a critical value $p_c \approx \frac{\ln n}{n}$, the graph almost surely transitions from a collection of small disconnected components to one that contains a single giant connected component. The expected vertex degree is $\langle k \rangle = p(n-1)$, and the degree distribution follows a binomial law that approaches a Poisson distribution in the large- n limit.

Barabási–Albert Model

While the Erdős–Rényi model assumes uniform edge probabilities, many real-world networks such as social, biological, or communication networks display *scale-free* degree distributions that follow a power law. The *Barabási–Albert (BA)* model captures this feature through the principle of *preferential attachment*.

Starting from a small connected seed network of m_0 vertices, new vertices are added one by one. Each new vertex connects to $m \leq m_0$ existing vertices, where the probability $\Pi(v_i)$ of connecting to vertex v_i is proportional to its current degree k_i :

$$\Pi(v_i) = \frac{k_i}{\sum_j k_j}.$$

This iterative process generates networks where a few vertices (“hubs”) acquire a large number of connections, while most vertices have relatively few. The resulting degree distribution asymptotically follows

$$P(k) \propto k^{-3},$$

a hallmark of scale-free behavior. These insights are crucial for understanding the emergence of influential nodes in complex networks.

Waxman Model

The *Waxman model* introduces spatial embedding into network generation, making it suitable for physical or communication networks constrained by distance. Vertices are placed randomly (typically uniformly) in a bounded Euclidean region, and edges between two vertices u and v are created with probability that decays exponentially with their distance $d(u, v)$:

$$\mathbb{P}[\{u, v\} \in E] = \beta e^{-d(u, v)/(\alpha L)},$$

where L is the maximum possible distance between any two vertices, and $\alpha, \beta \in (0, 1]$ are tunable parameters controlling the graph’s density and spatial decay rate.

For small α , connections are strongly local, producing highly clustered but sparsely connected networks. For larger α , long-range connections become more likely, shortening average path lengths and approximating small-world characteristics. The Waxman model therefore provides a bridge between purely random and spatially constrained network structures.

4 Results

In this chapter we present the main results of the thesis. The work can be divided into two broad themes: quantum repeaters and quantum networks. We begin with the results on quantum networks, which are covered in Papers I [52] and II [53], and then turn to the quantum repeater results discussed in Papers III [54] and IV [55].

4.1 Generation of Entanglement in Quantum Networks (Paper I)

Papers I and II present the results on quantum networks. In broad terms, we address the problem of efficiently distributing highly entangled GHZ states across networks using only LOCC together with an initial supply of Bell pairs. The aim is to optimize the process while taking into account Bell pair consumption, the number of gates required, and constraints imposed by the network connectivity. Paper II also includes an implementation on IBM quantum hardware together with noisy simulations.

Problem setting and figures of merit

Highly entangled GHZ states [58, 59], cluster states, and more general graph states serve as essential resources for distributed tasks in quantum communication, measurement based quantum computation, and sensing [109–115]. Much of the existing literature assumes that suitable multipartite entangled states have already been distributed across the network [116–119]. Generating such resource states across multiple nodes, however, is a fundamental and non-trivial challenge. Several architectures prepare these states locally at a central node and then teleport them to remote nodes using long-distance Bell pairs supplied by quantum repeaters [120–123]. This approach is costly in terms of Bell-pair consumption as well as the number of qubits required at a central node. Some protocols construct graph states that follow the underlying network topology [124], while others can generate GHZ states, but rely on solving computationally difficult (NP-hard) graph problems such as the minimal Steiner tree problem [125] to minimize Bell-pair usage. This work is also known as MMG Protocol [126]. In nearly all of these approaches, LOCC is treated as a free resource, and performance is evaluated primarily in terms of Bell-pair consumption, waiting time, and final-state fidelity.

In this work, we adopt a different viewpoint that is motivated by realistic, noisy, large-scale implementations. We model the quantum network as a connected simple graph $G = (V, E)$ with $|V| = N$ nodes, where each vertex represents a quantum node and each edge denotes the ability to establish a Bell pair between the corresponding pair of nodes as shown in figure 4.1(a).

Node $v \in V$ stores $\deg(v)$ qubits, one per incident edge, and an initial round of entanglement distribution produces a Bell pair on every used edge. The maximum possible Bell pairs are shown in figure 4.1(b). Our goal is to use only LOCC to convert this Bell-pair network into an N -party GHZ state, or more generally into a GHZ state on a specified subset of nodes, while explicitly accounting for all relevant resources. Besides the conventional figures of merit—Bell-pair consumption, waiting time, and output fidelity—we place particular emphasis on three additional, experimentally crucial quantities: (i) the classical

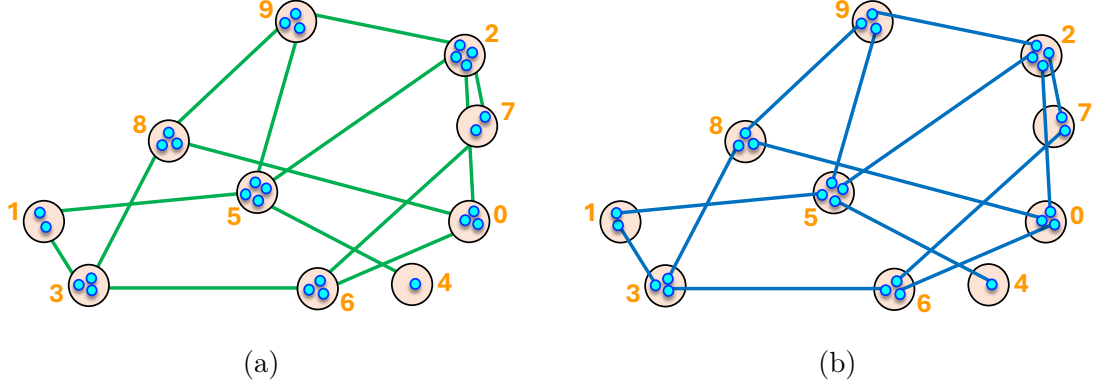


Figure 4.1: Example of a quantum network and the Bell pairs generated on its physical links. (a) Network topology where orange circles are nodes, green lines are physical connections, and blue circular discs are qubits at each node. (b) Established Bell pairs between connected nodes.

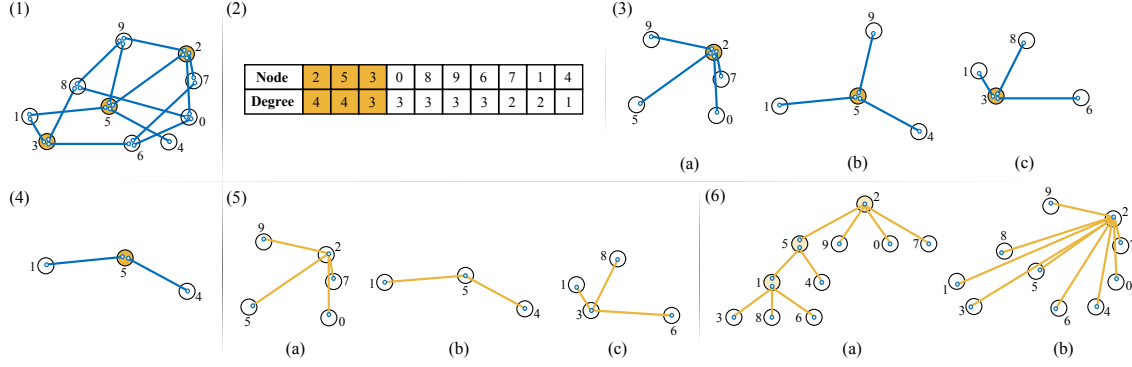
computational complexity of the orchestration algorithm (no NP-hardness), (ii) the total number of local two-qubit gates, which dominate the noise budget in current platforms, and (iii) the number and spatial placement of Bell-pair sources required in the underlying physical network. Since realistic devices suffer from gate errors and finite quantum-memory coherence times, we treat LOCC as a costly resource rather than a free one, and we design our protocol to be simultaneously efficient with respect to all of these figures of merit.

Protocol 1: GHZ creation on all nodes

The goal of Protocol 1 is to take as input a connected Bell-pair network on a graph $G = (V, E)$ and output a concrete sequence of local operations that produces a single GHZ state shared by all $N = |V|$ nodes. We achieve this in two stages. In the *classical* stage, we process the graph to decide where we create local GHZ states and how we later fuse them; the output of this stage is a list of subgraphs (a.k.a stars) arranged in a tree structure. In the *quantum* stage, we use this list to (i) locally convert each star of Bell pairs into a small GHZ state using Protocol 2 and (ii) fuse these small GHZ states along the tree using Protocol 3 until only one global GHZ state remains. An explicit example of all steps of Protocol 1 on a 10-node network is shown in Figure 4.2.

The classical part of Protocol 1 is easiest to understand if we think in terms of *stars* and follow the example in Figure 4.2. A star in our setting is a small subgraph built around a single node v : it contains v together with its neighbors, and only the edges that connect v to those neighbors, so that v plays the role of a local hub at which we will later create a small GHZ state (for example, see Figure 4.2(3)).

Starting from the full network in Figure 4.2(1), we first compute the degree of every node and sort the nodes in descending order of degree, which gives the table shown in Figure 4.2(2). We begin with the node at the top of the table: we take this node, form the star consisting of this node and all of its current neighbors as depicted in Figure 4.2(3a). We then add this star to a list SG. We go back to the degree table and take the next node, and repeat the same procedure as illustrated for the second and third stars in Figure 4.2(3b). We continue this process until all the nodes are present at least in one of the stars. The full collection of stars shown in Figure 4.2(3). Next, we delete duplicate nodes of the stars in SG and add the modified stars into a list MSG. We begin by selecting the largest star in SG as a root star S_0 , moving it from SG to MSG. Then, while SG is non-empty, we repeatedly pick a star from SG that shares at least one node with any star already in MSG. Before



adding such a star to MSG, we prune it by removing all but one of the nodes it has in common with MSG, together with the corresponding edges. This guarantees that, in MSG, any two stars intersect in at most a single node. Intuitively, MSG now encodes a fusion tree whose vertices are stars and whose edges correspond to shared nodes; in Figure 4.2, this step corresponds to going from the raw stars in panel (3) to the pruned configuration in panel (4). In this example, only the star in Figure 4.2(3)b as an extra unnecessary node 9 as node 9 is already included in Figure 4.2(3)a. At the end of the classical stage, MSG contains exactly the information we need: which local neighborhoods will host small GHZ states, how these neighborhoods are connected via shared nodes, and in which pattern we should perform fusion operations.

Protocol 2 GHZ state from Bell pairs in a star topology

Input: $k \in \{2, 3, \dots\}$ Bell pairs $|\Phi\rangle_{A_i B_i}$, $k+1$ nodes in a star topology. (The central node shares a Bell pair with each of the k outer nodes.)

Output: GHZ state $|\text{GHZ}_{k+1}\rangle$ shared by all $k+1$ nodes.

- 1: Apply the CNOT gates $\text{CNOT}_{A_1 A_2}$, $\text{CNOT}_{A_1 A_3}$, $\dots$, $\text{CNOT}_{A_1 A_k}$ on the qubits in the central node.
 - 2: Measure the qubits $A_2, A_3, \dots, A_k$ in the Z -basis $\{|0\rangle, |1\rangle\}$.
 - 3: Communicate the outcome of the measurement on qubit A_i to the qubit B_i , $i \in \{2, 3, \dots, k\}$. For every $i \in \{2, 3, \dots, k\}$: if the outcome communicated by A_i is 0, then nothing is done on qubit B_i ; if the outcome communicated by A_i is 1, then apply the Pauli- X gate to qubit B_i .
-

Protocol 3 Fusion of GHZ states in a tree topology**Input:** GHZ states in a tree configuration, with a root node identified.**Output:** GHZ state shared by all of the nodes.

- 1: Apply a CNOT gate at every node except for the root node and the leaf nodes. The source qubit is in a GHZ state with qubits in the next highest level of the tree, and the target qubits is in a GHZ with qubits in the next lowest level of the tree.
- 2: Measure the target qubits Pauli- Z basis $\{|0\rangle, |1\rangle\}$. The outcomes are communicated to all nodes.
- 3: For every leaf node of the tree, apply the Pauli- X correction $X^{z_1 \oplus z_2 \oplus \dots \oplus z_{\ell-1}}$, where $z_1, z_2, \dots, z_{\ell} \in \{0, 1\}$ are the measurement outcomes obtained at the nodes along the path going from the leaf node to the root node.

The quantum part of Protocol 1 then follows this precomputed structure. For every star in MSG, we locally apply Protocol 2 at its central node to convert the incident Bell pairs into a small GHZ state on the nodes of that star. After this first round, each star in MSG carries its own GHZ state, as depicted in Fig. 3(5). We then interpret MSG as a tree and apply Protocol 3 along this tree: at each internal node of the tree we perform a single CNOT and a Z -basis measurement to fuse the two GHZ states associated with the adjacent stars, followed by Pauli- X corrections on the leaves determined by the communicated outcomes. Repeating these fusion steps from the leaves toward the root gradually merges all local GHZ states into one global GHZ state shared by all N nodes, as shown in Fig. 3(6). In this way, Protocol 1 systematically turns the original Bell-pair network into a single multipartite GHZ state, while ensuring near-optimal use of Bell pairs, two-qubit gates, and Bell-pair sources.

Protocol 4: GHZ state only among a subset of nodes

For scenarios where only a chosen subset of desired nodes $(DN) \subseteq V$ should share a GHZ state, naively running Protocol 1 on the full graph and then measuring out unwanted nodes in the X -basis would create the desired GHZ state but is highly inefficient in gates and Bell pairs. Instead, we introduce Protocol 4, which first reduces G to a connected subgraph G_S that contains all desired nodes but only as few additional nodes as needed to ensure connectivity. Please note that it is impossible to create a GHZ state among a subset of nodes if these nodes are disconnected in the graph.

Protocol 4 proceeds by first forming the induced subgraph on DN and checking whether it is connected. If not, it picks a random node among DN and runs a breadth-first search (BFS) on the full graph, forming a BFS tree that passes through all desired nodes as shown in Figure 4.3b. Then it iteratively prunes leaf nodes from this BFS tree that are not in DN until all remaining leaves lie in DN , and finally takes the subgraph induced by the remaining BFS-tree vertices as G_S .

In this “subset case”, the overall protocol is: run Protocol 4 to get G_S , run Protocol 1 on G_S , and finally measure out any extra nodes in the GHZ with X -measurements to obtain a GHZ state on exactly the desired subset DN . This approach avoids solving a Steiner tree problem and yet empirically yields subgraphs that are close to Steiner trees in size and thus near-optimal in Bell-pair consumption.

Resource costs

Here we summarize the main resource-scaling results of our protocol; all derivations and proofs are given in the paper I of this thesis. On the classical side, the offline graph-processing needed for Protocol 1 has time complexity $O(N^2)$, dominated by constructing and sorting the degree table, generating all stars, and pruning and ordering them into

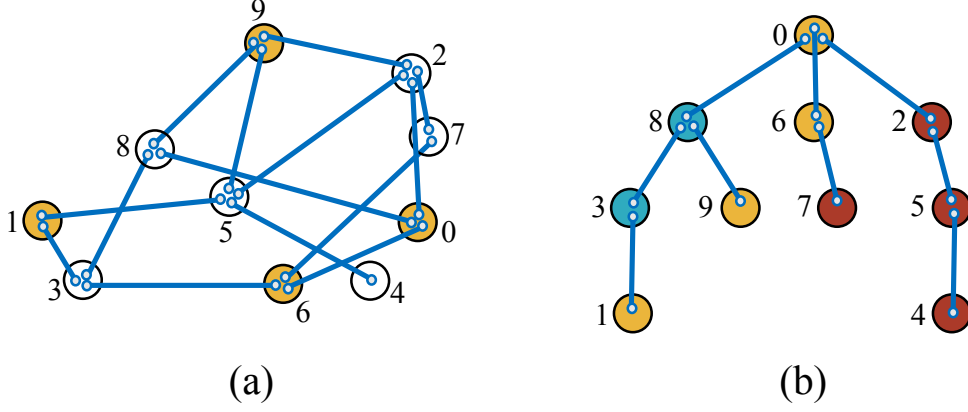


Figure 4.3: An example of using Protocol 4 to determine a connected subgraph containing a desired subset of nodes. (a) The original network, with the desired nodes highlighted in yellow, forming the set $DN = \{0, 1, 6, 9\}$. (b) The tree from root node 0 using BFS search. The nodes highlighted in red are not necessary and can be deleted, while the nodes highlighted in blue are necessary to have a connected subgraph. The resulting connected subgraph G consists of the desired yellow nodes and the blue nodes [52].

the fusion list MSG. Protocol 4, which finds and prunes a suitable connected subgraph for a given terminal set, also runs in $O(N^2)$ time in the worst case, so the total classical complexity of preparing a GHZ state on any subset of nodes using Protocols 4 and 1 together is $O(N^2)$, independently of the subset size. On the quantum side, the number of two-qubit gates required to create a GHZ state on all N nodes is exactly $N - 2$, and when we first restrict to a connected subgraph of size S , the cost is exactly $S - 2$ gates, so the gate count scales linearly in the number of participating nodes and is independent of the network topology. For Bell-pair consumption, Protocol 1 attains the information-theoretic lower bound in the complete case, using exactly $N - 1$ Bell pairs to entangle N nodes, while in the subset case the number of Bell pairs lies between $S - 1$ and $d_S(S - 1)$, where S and d_S are the size and diameter of the selected subgraph; numerically, this is very close to the performance of good Steiner-tree heuristics on a wide range of random network models. Finally, for Bell-pair sources, we prove that the optimal source-placement problem is equivalent to the classical minimum dominating set problem on the network graph [127, 128], and we use the star structure produced by our protocol to construct simple, efficiently computable upper bounds on the number of required sources that perform near-optimally in our simulations.

Performance on random network models

To assess performance in realistic regimes, we simulate our protocol and MMG on three ensembles of random networks: Erdős–Rényi $G(N, p)$ [105, 106], Barabási–Albert (scale-free) [97, 107, 129] with preferential attachment, and Waxman spatial graphs modeling fiber networks with distance-dependent edge probabilities. We use NetworkX [130] for graph generation and Mehlhorn’s Steiner-tree approximation as the MMG baseline, and perform Monte Carlo averaging over many instances for various N , densities, and subset fractions. For gate cost versus subgraph size S , the simulations confirm that their protocol scales linearly in S while MMG grows quadratically, in agreement with the analytic bounds as shown in Figure 4.4.

Regarding Bell-pair consumption in the subset case, the ratio $|V(G_S)|/|DN|$ for Protocol 4 is observed to be very close to that of the Mehlhorn’s Steiner tree algorithm in both ER

and BA models, with the difference shrinking as N grows, indicating asymptotic near-optimality as shown in Figure 4.5.

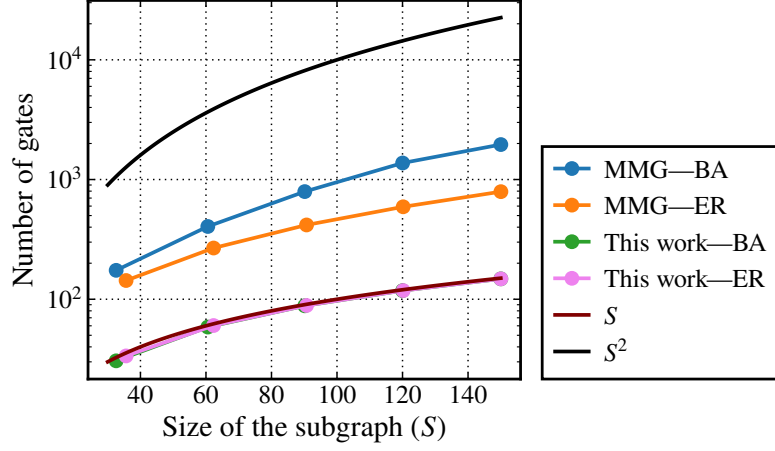


Figure 4.4: Number of gates versus subgraph size for Erdős–Rényi and Barabási–Albert networks with up to 500 nodes. We consider a fraction $f = 0.3$ of nodes to be entangled, and we set $p = 0.05$ [52].

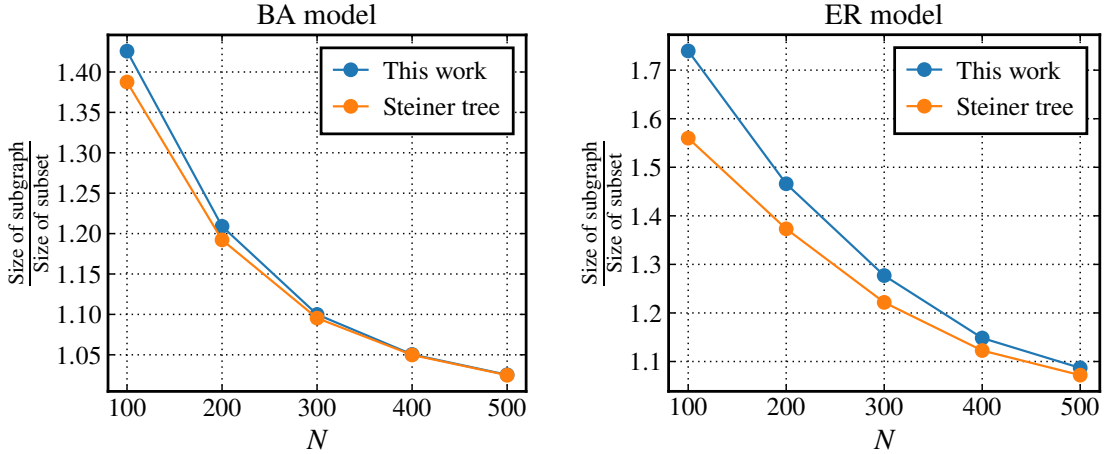


Figure 4.5: Ratio of the subgraph size to the number of nodes to be entangled, as a function of the total number of nodes, for both the Steiner tree algorithm [131] and our Protocol 4. We consider the Erdős–Rényi and Barabási–Albert models with $p = 0.05$ and a fraction $f = 0.1$ of nodes to be entangled [52].

Performance with Noise

We analyze how our protocols perform in the presence of noise affecting Bell pairs, locally created GHZ states, and quantum gates. In contrast to previous work that mainly considered Pauli noise models (e.g., depolarizing noise) [132], we allow for *arbitrary* quantum noise channels, including non-Pauli and non-unital noise. This makes our analysis applicable to a wider range of physical quantum hardware.

We first consider the fidelity of a GHZ state produced from n noisy Bell pairs using the Protocol 1 of Paper I [52]. Each Bell pair is described by a density operator $\rho_{A_i B_i}^{(i)}$ (where A, B represents the two qubits involved and i is the index). The resulting GHZ fidelity depends on the overlaps of these noisy Bell states with the Bell basis states $|\Phi^{0,0}\rangle$ and

$|\Phi^{1,0}\rangle$. In general, the final fidelity is a weighted sum over parity combinations of these overlaps. If all Bell pairs are affected by the same noise model, the expression simplifies to

$$F = \frac{1}{2}((\mu_0 - \mu_1)^n + (\mu_0 + \mu_1)^n),$$

where μ_0 and μ_1 denote the fidelities of the noisy Bell pair with respect to $|\Phi^{0,0}\rangle$ and $|\Phi^{1,0}\rangle$. This expression allows us to determine how much noise can be tolerated while still achieving a target GHZ fidelity F .

We then evaluate this condition for several common noise models:

- **Depolarizing noise.** For depolarizing noise with parameter p , the fidelity condition becomes

$$\left(1 - \frac{4p}{3}\right)^n + \left(1 - \frac{2p}{3}\right)^n \geq 2F.$$

This relation can be solved numerically to determine the maximum tolerable depolarizing noise.

- **Dephasing noise.** For dephasing noise with parameter q , the condition simplifies to

$$(1 - 2q)^n + 1 \geq 2F,$$

which leads to the bound

$$q \leq \frac{1}{2} \left(1 - (2F - 1)^{1/n}\right).$$

- **General Pauli noise.** For a Pauli noise channel with probabilities p_I, p_X, p_Y, p_Z , the fidelity depends only on p_I and p_Z , giving

$$\frac{1}{2}((p_I - p_Z)^n + (p_I + p_Z)^n) \geq F.$$

For example, in the bit-flip noise model ($p_X = p$, $p_I = 1 - p$), this reduces to

$$(1 - p)^n \geq F \quad \Rightarrow \quad p \leq 1 - F^{1/n}.$$

- **Amplitude damping noise.** For amplitude damping with parameter γ , the fidelity condition becomes

$$(1 - \gamma)^{n/2} + \left(1 - \frac{\gamma}{2}\right)^n \geq 2F.$$

- **Fidelity based on (T_1, T_2) noise.** For a realistic noise model characterized by relaxation and dephasing times T_1 and T_2 , the condition becomes

$$e^{-nt/T_2} + \frac{1}{2^n}(1 + e^{-t/T_1})^n \geq 2F.$$

This relation determines how long Bell pairs can be stored in memory while maintaining the desired fidelity.

We also analyze the protocol that merges m noisy GHZ states into a larger GHZ state. If the GHZ states used in the merging step are themselves created from noisy Bell pairs, the overall fidelity becomes

$$F = \frac{1}{2}((\mu_0 + \mu_1)^{nm} + (\mu_0 - \mu_1)^{nm}).$$

This shows that the final fidelity depends primarily on the *total number of entangling operations* performed during the protocol. In particular, the dominant factor governing noise accumulation is the total number of CNOT gates used to generate and merge the GHZ states.

Conclusion

Overall, we have presented a protocol for distributing multipartite entanglement via LOCC, in particular GHZ states, directly from Bell-pair networks that simultaneously addresses all relevant practical resources. Our construction combines a simple, offline classical orchestration with local star creation and GHZ fusion, leading to favorable performance across Bell-pair usage, local gate count, and the number and placement of Bell-pair sources, while remaining competitive with or outperforming existing Steiner-tree-based strategies on realistic network models. By relating Bell-pair and source costs to well-studied graph-theoretic problems and deriving general fidelity formulas for arbitrary local noise on both Bell pairs and gates, we provide a framework that connects high-level network design with concrete hardware parameters. We expect that extending these ideas to more general graph states, adaptive network operation, and application-specific architectures will further broaden the impact of our approach on future quantum network implementations.

4.2 Efficient GHZ State Generation on NISQ Hardware (Paper II)

This paper II [53] investigates how to generate large multipartite Greenberger–Horne–Zeilinger (GHZ) states on noisy intermediate-scale quantum (NISQ) devices with limited qubit connectivity, comparing a measurement-based “merging” protocol with a purely unitary “growing” protocol that both explicitly respect hardware topology.

Motivation and Problem Statement

Another important motivation for studying GHZ states is that the ability to efficiently generate high-fidelity GHZ states at scale has become a key benchmark of device performance for NISQ platforms and a stepping stone toward practical quantum networking [133]. Apart from the communication-based applications discussed in Paper I, quantum sensing is another application that requires GHZ states [134–136]. GHZ state preparation on current devices remains difficult because resource demands, such as circuit depth and the number of entangling operations, grow with system size, while noise, decoherence, and sparse or inhomogeneous connectivity restrict the attainable fidelity [137–143]. Existing methods either rely on simple linear-depth constructions, which lead to deep circuits [137–140], or on log- or constant-depth measurement-based approaches that assume idealized connectivity [137, 138, 141], making them costly or impractical on realistic architectures such as IBM’s heavy-hex chips or Google’s grid layouts [144, 145]. The preparation of GHZ states from graph state transformations was also studied [76, 142, 146] but the resulting GHZ state had only half the number of qubits as the original graph state. Recent experiments have demonstrated large-scale GHZ states by combining topology-aware scheduling with error suppression and mitigation [139, 140, 143], yet a systematic treatment of hardware connectivity within measurement-based protocols remains limited.

The present work addresses this gap by designing and benchmarking two connectivity-aware GHZ protocols on realistic device layouts and on random graphs that model distributed architectures, with circuit depth, two-qubit gates, measurements, and fidelity as central performance measures. The two protocols and their performance is discussed below.

Measurement-based Merging Protocol: Merging Method

This merging method is inspired by Protocols 1 and 2 of Paper I of this thesis [52], which generate smaller GHZ states at the highest degree nodes and merge them. The hardware setting introduces important differences because each node contains only one physical qubit, unlike quantum networks where nodes may hold multiple qubits, and because gate execution time, scheduling, and qubit reuse play a central role in reducing circuit depth and growing the GHZ state. Apart from selecting high-degree nodes for creating small GHZ states and merging them, the details of the method differ significantly from the network setting.

Step 1: Star selection The layout graph is partitioned into stars, each consisting of a center qubit and its neighbors, by iteratively selecting a qubit of highest degree and grouping it with its adjacent qubits, removing each selected star from the graph until all qubits are assigned. Alternative heuristics constrain star sizes by fixing a target degree derived from the average degree or by enforcing approximately equal star sizes to balance local preparation times. An example based on an IBM layout graph is shown in the Figure 4.6.

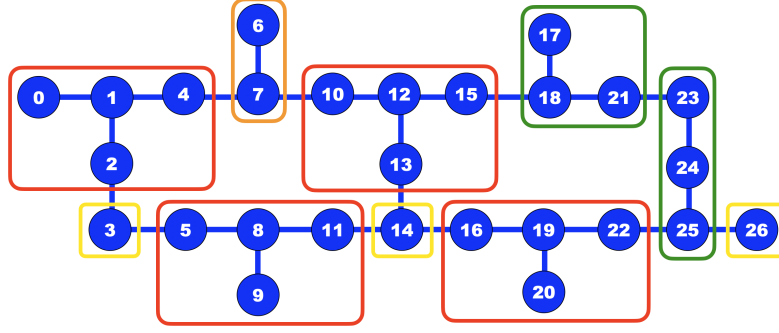


Figure 4.6: The figure illustrates star selection in the IBMQ device layout for the merging method. Coloured boxes indicate 4-, 3-, 2-, and 1-qubit stars (red, green, orange, and yellow, respectively), selected by iteratively choosing the highest-degree node. These stars, consisting of a central qubit and its neighboring leaves, form the basic units of the merging protocol.

Step 2: Local GHZ preparation On each star, a small GHZ state is created by applying a Hadamard gate to the center qubit followed by CX gates from the center to each leaf. A star of degree d therefore becomes a $(d+1)$ -qubit GHZ state. The number of CX gates, and hence the local contribution to depth, grows with the star degree, which makes star-size control directly relevant for depth optimization. Because different stars act on disjoint sets of qubits, their local GHZ preparation can be performed simultaneously, reducing total circuit depth.

Step 3: Recursive merging The small GHZ states are then merged pairwise whenever two stars are connected by an edge, using the one-CX-plus-measurement rule to expand the global GHZ state (Protocol 3 of Paper I). Each merge uses one measurement and may require a layer of feedforward single-qubit corrections. This process is repeated until a single GHZ state spans the desired set of qubits.

Step 4: Re-adding measured qubits Each merge measures and discards one qubit as explained in Step 3. That qubit is reintroduced by reinitializing it in $|0\rangle$ and applying a CX from a neighboring GHZ qubit, which adds one extra two-qubit gate per merge. This step ensures that all physical qubits originally considered are included in the final GHZ state, at the cost of a modest increase in the total number of CX gates.

The star-selection strategy induces a fundamental trade-off. Larger stars reduce the number of merges and measurements but require more CX gates per star and can increase local depth. Smaller or more balanced stars lower the local GHZ-preparation depth but increase the number of merges and mid-circuit measurements.

Unitary protocol - Growing

The growing protocol is fully unitary and avoids mid-circuit measurements and classical feedforward, instead expanding a GHZ state via BFS over the connectivity graph. It proceeds as follows.

First, a node of highest degree is chosen as an initial center, and a local GHZ state is prepared on this center and its neighbors using a Hadamard followed by CX gates from the center to its neighbors. Next, the neighbors of all qubits currently in the GHZ state are identified (the BFS frontier), and each such new neighbor is entangled by using one of its already-entangled neighbors as the control of a CX, ensuring that the overall state

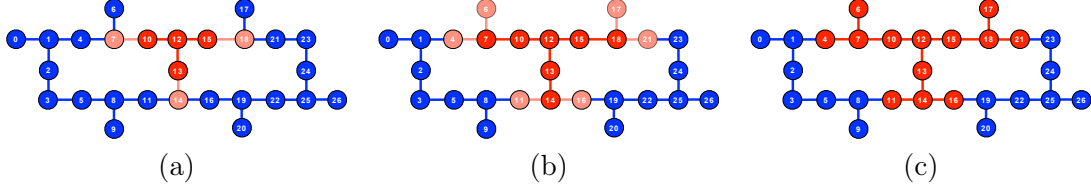


Figure 4.7: The first three steps of the GHZ state growing method on the IBM layout as shown in this figure. The red nodes correspond to qubits that are included in the current GHZ state and the light red nodes are to-be-included in the next step. Panel (a) shows the initial GHZ state generation, picking node 12 as the highest degree node and the first BFS step identifying the direct neighbors of the initial GHZ state. Panel (b) shows the GHZ state after adding the neighbors from the first BFS step via CX gates and the second BFS step, identifying the direct neighbors of the current GHZ state. Panel (c) shows the final GHZ state after adding the neighbors from the second BFS step, reaching a final size of 13 qubits [53].

remains a GHZ state over the expanded set. An example is visualized in Figure 4.7. This BFS expansion step is repeated layer by layer until the target GHZ-state size is reached, and by construction, all two-qubit gates follow existing edges in the layout graph.

Hardware experiments on IBM Eagle r3

To assess performance under realistic noise, both protocols are implemented on IBM’s 127-qubit Eagle r3 chip (IBM Brussels) for a range of GHZ-state sizes. Because full state tomography is infeasible for larger systems, we evaluate performance via the classical Hellinger fidelity between the ideal GHZ measurement distribution and the experimentally observed distribution [147, 148],

$$F_H(P, Q) = \left(\sum_i \sqrt{p_i q_i} \right)^2,$$

where p_i and q_i are the probabilities of P and Q respectively.

For each GHZ-state size and for each protocol, 100 random samples on distinct subsets of physical qubits are generated, each circuit is executed with 4096 shots, and both the average and best-case fidelities are recorded. On real hardware, the growing protocol consistently achieves higher average and maximum fidelities than the merging protocol (see Figure 4.8), a result attributed to avoiding mid-circuit measurements and using slightly fewer two-qubit gates, both of which reduce accumulated noise.

We also simulate both protocols using IBM’s calibrated noise model for the same device and time window, finding that simulated fidelities systematically overestimate experimental ones, indicating that current noise models do not fully capture hardware imperfections or temporal drifts. No error mitigation is applied in any of these results, so the reported fidelities represent raw device performance and could be improved with techniques such as zero-noise extrapolation or error-detecting circuits. [140, 149]

Further simulations

We perform further simulations on connectivity layouts inspired by IBM’s and Google’s chip architectures, as well as on Erdős–Rényi random graphs that represent distributed quantum computing networks. Rather than executing these circuits or performing full quantum state simulations, we analyze the circuits classically by extracting metrics such

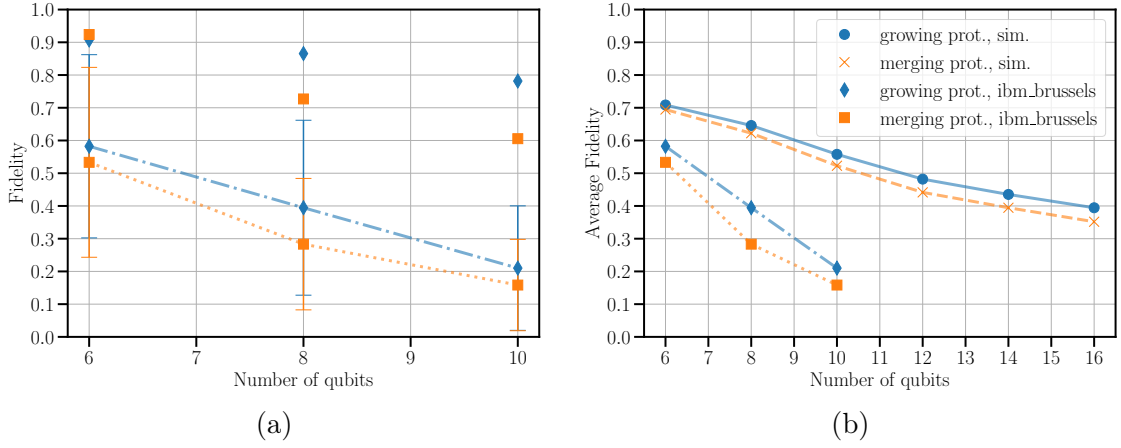


Figure 4.8: GHZ state fidelity over the number of qubits in the final GHZ state. Here we compare the *growing* protocol (blue circles, blue diamonds) with our *merging* protocol (orange crosses, orange squares) using the highest degree star size. For each GHZ-state size and each protocol, we created and executed 100 different circuits. Panel (a) shows the average fidelity values obtained from executing the circuits of the *growing* protocol (blue diamonds) and the *merging* protocol (orange squares) on quantum hardware `IBM_Brussels` and additionally the maximum fidelity achieved for each size is indicated as another marker. The standard deviation is shown as error bars on the average values. Panel (b) shows the average fidelity values obtained from noisy simulations using the full `IBM_Brussels` noise model (blue circles, orange crosses) and from running the same circuits on the quantum hardware `IBM_Brussels` (blue diamonds, orange squares). The legend of Panel (b) is also valid for Panel (a). For each GHZ-state size, we created and executed 100 different circuits. Each circuit as been executed with 4096 shots [53].

as circuit depth, two-qubit gate count, and the number of mid-circuit measurements from the gate schedules generated by our merging and growing protocols. This classical analysis approach enables us to efficiently evaluate protocol scalability and topology dependence while keeping computational effort low. We investigate different star-selection heuristics and graph parameters to understand the trade-offs between depth, measurement overhead, and gate usage across various device architectures and connectivity regimes.

Conclusions and future directions

We identify a clear trade-off between measurement-based and unitary GHZ-generation protocols when accounting for realistic device connectivity. On current NISQ superconducting hardware, the unitary growing protocol is favored, as it avoids noisy mid-circuit measurements and uses fewer two-qubit gates, resulting in higher fidelities despite greater circuit depth. Architecturally, the measurement-based merging protocol offers significantly reduced circuit depths and is expected to become advantageous on future devices with improved gate fidelities and fast, reliable mid-circuit measurements with feedforward. The discrepancy between noisy simulations and experimental outcomes highlights the need for more accurate noise modeling and calibration.

Future work includes extending these comparisons to other quantum platforms (photonic, trapped-ion, neutral-atom), employing reinforcement learning to optimize star-selection within the merging protocol, utilizing specialized large-scale simulators, and incorporating realistic constraints such as node or link failures and targeted GHZ-state subsets. Many of these challenges connect naturally to graph-optimization problems, inviting advances at the intersection of quantum networking, algorithmic graph theory, and machine learning-based control of entanglement distribution [28, 150–152].

4.3 Bosonic Quantum Error Correction for Quantum Repeaters (Paper III)

As we discussed in Chapter 2, quantum repeaters are one of the important tools to have long-distance quantum communication. The paper III [54] analyzes how bosonic quantum error correction (QEC) based on Binomial codes can enhance secret-key rates in second-generation quantum repeaters with lossy memories, and shows that realistic microwave-cavity plus transmon setups can implement all required primitives: encoded state preparation, syndrome extraction, recovery, and logical Bell-state measurements.

Problem setting and objective

Long-distance quantum communication relies on distributing entanglement across distant nodes, yet exponential fiber attenuation prevents entanglement from being shared over practical distances without additional infrastructure [20]. Quantum repeaters [24] address this limitation by dividing the channel into shorter segments and storing intermediate states in quantum memories, and among the possible architectures [29], second-generation repeaters are particularly attractive because they are compatible with near-term hardware while requiring relatively few stations. Their performance is limited by memory decoherence fundamentally leading to dephasing and loss errors [27, 153]. Although dephasing has been extensively analyzed in the literature [28, 154–158], loss errors have received considerably less attention even though many leading physical platforms are dominated by excitation loss [25, 159]. A natural approach to mitigating loss is to use bosonic quantum error correction, and prior work has examined repeaters based on GKP codes [31, 66, 160], while more recent results indicate that Binomial codes [63, 161, 162] offer important advantages over both GKP and Cat codes [64, 65, 163]. Motivated by these observations, we study a two-segment second-generation repeater in which the memories suffer both dephasing and loss, and where stored information is protected using a Binomial code, while the photons transmitted between stations remain unencoded. We analyze single shot and repeated recovery strategies and compute the resulting secret key rates. As a concrete implementation platform, we consider cQED systems that provide long-lived cavity photons and high-fidelity control [164–166], together with microwave to optical transduction that enables interfacing the memories with optical channels [167–169].

Model and Workflow

We analyze a minimal but nontrivial architecture: a two-segment repeater with four memory units labelled 1 to 4, where memories 2 and 3 form the central station and 1 and 4 are the remote end nodes as shown in figure 4.9.

Entanglement is first generated independently in segments (1, 2) and (3, 4) by creating entangled microwave–optical states at each node and projecting the optical photons via a Bell measurement in the middle of each segment, with success probability per attempt

$$p = p_{\text{link}} \eta_0, \quad \eta_0 = e^{-L_0/L_{\text{att}}},$$

where p_{link} describes intrinsic hardware efficiencies and η_0 accounts for fiber loss over distance L_0 . Each attempt consumes a time $T_0 = L_0/c$, dominated by the classical communication delay in the fiber of speed $c \simeq 2 \times 10^8$ m/s. This T_0 is one time step. Since the two segments succeed at random times, the number of attempts n_{12} and n_{34} for each segment are geometric random variables, and typically one segment succeeds earlier and must wait while its entangled state is stored in the memories and undergoes loss. The relevant waiting time is quantified by $n_{\text{diff}} = |n_{12} - n_{34}|$ which determines the duration over which one entangled pair experiences memory decoherence. It is during this waiting time the

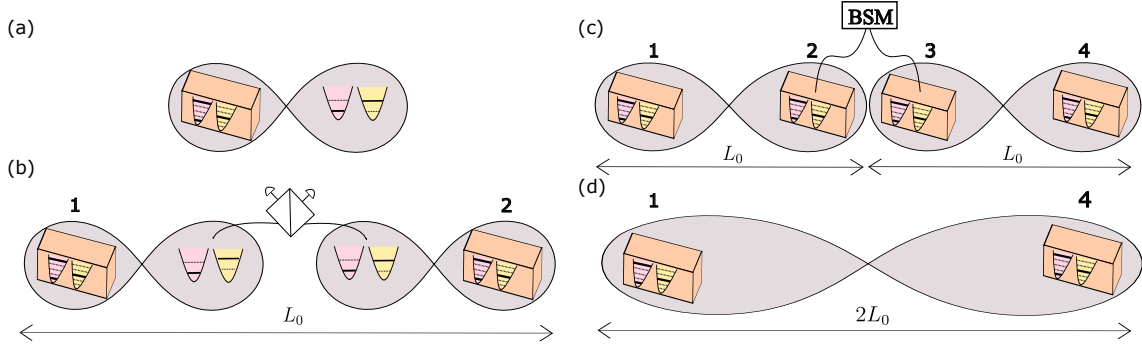


Figure 4.9: Two-segment, second-generation quantum repeater, with memory implemented as a microwave cavity (shown as an example). (a) Entanglement between a travelling photon and a microwave cavity occurs at all four memory units. (b) Entanglement within a single repeater segment of distance L_0 is established between two memory units via projection measurement (using beam splitters) on travelling photons, which are conducted independently in both repeater segments. (c) Entanglement swapping occurs locally between two memory units at the middle station of the two segments via a logical Bell state measurement (BSM). (d) The final entangled state is achieved between distant memory units 1 and 4. Throughout these steps, the superposition states of the photon are represented by yellow and pink photons, though only one entangled pair of photons is distributed per round of the protocol. Cuboids represent microwave cavities, and the encoded states of the photon are depicted by the thick lines in the harmonic oscillator [54].

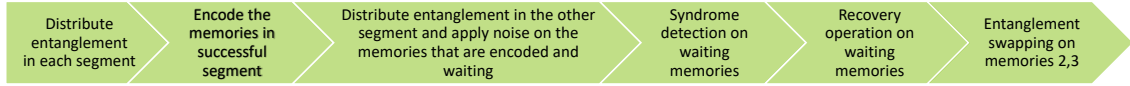


Figure 4.10: The workflow of the two-segment quantum repeater equipped with quantum error correction.

memories are encoded with Binomial code to combat loss errors. Once both segments are successful, a Bell-state measurement (BSM) on memories 2 and 3 performs entanglement swapping, thereby establishing an entangled pair between memories 1 and 4, conditioned on a successful BSM outcome. The entire sequence is given in the flow chart shown in figure 4.10

The figure of merit is the secret key rate as given in equation 2.22 and the raw rate for this case is given by [27]

$$R = \frac{\langle P_s \rangle}{\langle n_{\max} \rangle T_0},$$

where P_s is the success probability of the entanglement swapping, $n_{\max} = \max(n_{12}, n_{34})$, and $\langle \cdot \rangle$ denotes average.

The corresponding memory noise is modeled as an amplitude-damping (photon-loss) channel with Kraus operators

$$E_k = \sum_{n=k}^{\infty} \sqrt{\binom{n}{k}} (1-\gamma)^{n-k} \gamma^k |n-k\rangle \langle n|,$$

where γ is the probability of losing a single excitation in one time step T_0 . For an entangled state in one segment ρ_{12} , the decayed state is

$$\rho_{12}^{\text{dec}} = \sum_{k_1, k_2} E_{k_1,1} E_{k_2,2} \rho_{12} E_{k_1,1}^\dagger E_{k_2,2}^\dagger,$$

with mode indices indicating which memory is affected.

To protect against such loss, the memories are encoded using bosonic Binomial codes, whose logical codewords are finite superpositions of Fock states with binomial coefficients chosen to cancel leading-order loss errors. For the lowest-order Binomial code (LBC) that corrects up to a single excitation loss, the codewords are

$$|\bar{0}\rangle = \frac{|0\rangle + |4\rangle}{\sqrt{2}}, \quad |\bar{1}\rangle = |2\rangle,$$

which live in the $\{0, 2, 4\}$ -photon Fock subspace. For the higher-order Binomial code (HBC), which corrects up to two photon losses, the logical states are

$$|\bar{0}\rangle_h = \frac{|0\rangle + \sqrt{3}|6\rangle}{2}, \quad |\bar{1}\rangle_h = \frac{\sqrt{3}|3\rangle + |9\rangle}{2},$$

occupying Fock levels up to 9 photons.

Error correction proceeds via syndrome extraction followed by a recovery operation that depends on the detected loss number. For the LBC, if there were no losses, the recovery operation to be performed, up to linear order of γ , is

$$R_0 = |\bar{0}\rangle \langle \bar{0}^*| + |\bar{1}\rangle \langle \bar{1}^*|, \quad (4.1)$$

where $|\bar{0}^*\rangle = |0\rangle + (1 - \gamma)^2 |4\rangle / \sqrt{2(1 - 2\gamma)}$ and $|\bar{1}^*\rangle = (1 - \gamma) |2\rangle / \sqrt{(1 - 2\gamma)}$. Note that $R_0 = \text{id}$ when $\gamma = 0$. If a single loss was found, the recovery operation is

$$R_1 = |\bar{0}\rangle \langle 3| + |\bar{1}\rangle \langle 1|. \quad (4.2)$$

For the HBC, the expressions are given in the paper.

The final state after entanglement swapping is

$$\rho_{14} = \frac{\langle \psi^+ |_{23} \rho_{1234} | \psi^+ \rangle_{23}}{\text{Tr}[\langle \psi^+ |_{23} \rho_{1234} | \psi^+ \rangle_{23}]},$$

but an important subtlety is that the memories are encoded. Entanglement swapping therefore requires a *logical* Bell-state measurement (BSM), which is considerably more challenging than a standard, physical BSM. The paper shows two feasible approaches: a circuit-QED scheme using microwave cavities that achieves deterministic (100%) logical BSM, and an all-optical scheme based on linear optics that identifies half of the logical Bell states (50% efficiency). Both methods are described in detail in Sections III and IV and Appendix A.

Numerical results

For the numerical study, we consider a two-segment repeater with lossy memories subject to amplitude damping, and compare three cases: unencoded memories, the lowest-order Binomial code (LBC), and the higher-order Binomial code (HBC). The simulation parameters cover coherence times $\tau_{\text{coh}} \in \{0.01, 0.1, 1\}$ s, an attenuation length $L_{\text{att}} = 22$ km, and memory read-out efficiencies $\eta_m \in \{0.9, 0.95, 1.0\}$. Logical entanglement swapping is assumed ideal for the cQED-based architecture, and we additionally evaluate the case of a 50%-efficient logical BSM corresponding to the all-optical scheme.

During the waiting time between successful entanglement generation events, error correction can be performed either once (single-time error correction, SEC) or repeatedly (multiple-time error correction, MEC). In principle, applying error correction multiple times should maintain the state closer to the code space, but as the recoveries are approximate to first order, it is unclear if we benefit from MEC over SEC. Therefore both strategies are included in the numerical comparison.

The resulting SKRs as a function of the segment length L_0 are shown in Figure 4.11. Across all parameter regimes, Binomial encoding provides a clear advantage over unencoded memories. For high interface efficiency ($\eta_m = 0.95$), both LBC and HBC significantly enhance the SKR over a broad distance range, with the HBC giving the largest boost at long distances where loss during storage dominates. Even for poorer interfaces ($\eta_m = 0.9$), where unencoded memories fail to produce a positive key rate, both Binomial codes still achieve nonzero SKR, highlighting their robustness to photon loss. Notably, the HBC can outperform unencoded memories that possess an order of magnitude longer coherence time, thanks to its ability to correct two excitation losses.

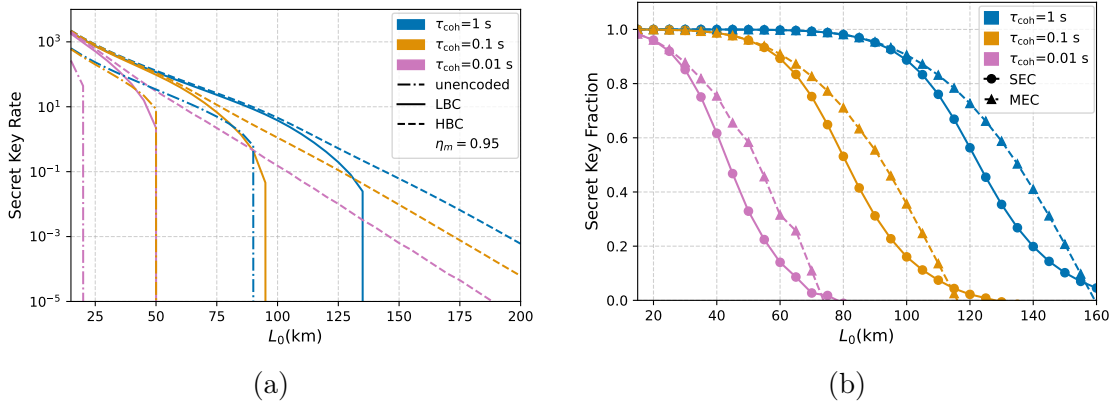


Figure 4.11: (a) Secret key rates for unencoded memories, the Lower Binomial Code (LBC), and the Higher Binomial Code (HBC) for different memory coherence times when SEC is used. The memory interface parameter is $\eta_m = 0.95$. (b) Secret key fraction for LBC at different memory coherence times for both SEC and MEC. The memory interface parameter is $\eta_m = 1.0$ [54].

The comparison between SEC and MEC, shown in Figure 4.11(b), reveals that MEC offers clear benefits at large distances and long coherence times, while SEC can be slightly favorable at short distances where additional cycles are not necessary. In all scenarios, the overall SKR decreases with increasing L_0 due to the reduced entanglement-generation probability and longer waiting times, even when the secret fraction remains high.

Finally, we also extract thresholds for achieving a positive SKR for a minimal memory-readout efficiency. For instance, at $L_0 = 10$ km and $\tau_{\text{coh}} = 1$ s, the LBC requires $\eta_m > 0.78$ to obtain a nonzero key rate, while the HBC relaxes this requirement to $\eta_m > 0.73$.

Physical Implementation: circuit-QED

We propose the physical realization of the quantum repeater system based on circuit quantum electrodynamics (cQED), employing microwave cavities coupled to superconducting transmon qubits [164, 165]. Each repeater station hosts two high-quality microwave cavities, which serve as bosonic quantum memories capable of encoding the logical qubits using Binomial codes. Every cavity is locally coupled to a transmon qubit, an anharmonic oscillator with nonlinear energy levels, which enables universal control over the microwave photons through external drives and tunable detuning. The interaction between each

transmon and its associated cavity is described by a beam-splitter-type Hamiltonian that facilitates coherent excitation exchange when the transmon frequency is brought into resonance with the cavity mode via dynamic tuning of the detuning parameter. Entangled microwave-transmon states can be generated with high fidelity (greater than 97%) [170], limited primarily by cavity photon loss and transmon decoherence times.

Formally, the cavity mode Hamiltonian is

$$H_a^{(i)} = \hbar\omega_{a,i}a_i^\dagger a_i,$$

where $\omega_{a,i}$ is the cavity frequency and $a_i^\dagger, a_i$ are the photon creation and annihilation operators. The transmon qubit Hamiltonian is approximated by an anharmonic oscillator,

$$\frac{H_q^{(i)}}{\hbar} = \omega_{q,i}q_i^\dagger q_i - \frac{\alpha_i}{2}q_i^\dagger q_i(q_i^\dagger q_i - 1),$$

where $\omega_{q,i}$ is the transmon frequency and α_i denotes the anharmonicity, with $q_i^\dagger, q_i$ the transmon raising and lowering operators. The total driven Hamiltonian for transmon i includes external control,

$$\frac{H_{q,d}^{(i)}(t)}{\hbar} = \Delta_i(t)q_i^\dagger q_i + \epsilon_i(t)q_i^\dagger + \epsilon_i^*(t)q_i,$$

where $\Delta_i(t)$ is the time-dependent detuning and $\epsilon_i(t)$ an applied drive pulse.

When the transmon and cavity are tuned near resonance, their interaction is modeled as a beam-splitter Hamiltonian,

$$H_{aq}^{(i)} = \hbar g_{aq,i}(q_i^\dagger a_i + q_i a_i^\dagger),$$

with coupling strength $g_{aq,i}$ controlling excitation exchange for state preparation and control. For joint logical Bell-state measurements, a third transmon is dispersively coupled to both cavities:

$$H_{\text{disp}} = \hbar\chi(t)q_3^\dagger q_3 \left(a_1^\dagger a_1 + a_2^\dagger a_2 \right),$$

where $\chi(t)$ is a tunable dispersive interaction strength controlled by the third transmon's detuning. This term enables photon-number-dependent phase shifts on the transmon state, which, combined with appropriate microwave pulses, realize parity measurements and distinguish encoded Bell states deterministically. We assume that the entangled transmon states are converted into flying optical photons through an ideal microwave-to-optical quantum transducer, facilitating long-distance entanglement distribution over optical fiber.

Conclusions

We demonstrate that Binomial bosonic QEC codes are a powerful tool for mitigating memory loss in near-term second-generation quantum repeaters, with both lower- and higher-order codes significantly outperforming unencoded schemes for realistic loss and coherence-time parameters. A concrete cQED-based architecture is outlined that, in principle, realizes high-fidelity encoded state preparation, parity-based syndrome detection, approximate recovery, and deterministic logical Bell-state measurements, assuming continued progress in microwave-to-optical transduction. The analysis also indicates that multiple rounds of loss correction (MEC) can be beneficial under idealized assumptions, motivating future work that includes operational errors from QEC cycles to determine optimal correction frequencies. Extensions to multi-segment repeater chains, all-optical implementations with improved Bell-measurement efficiencies beyond the 50% linear-optics bound, and integration with other bosonic code families are identified as promising directions for approaching ultimate SKR limits in realistic quantum repeaters and networks.

4.4 Quantum Repeater Performance with Memory Buffer

Another important aspect of second-generation quantum repeaters is the use of quantum memories, which are a critical resource for enabling efficient long-distance quantum communication. Since these memories are costly and limited, their optimal utilization plays a key role in reducing the resource overhead of quantum repeater network. In this work, we investigate how different strategies for distributing quantum memories across a repeater network impact entanglement distribution efficiency. By exploring various allocation schemes, we identify memory configurations that enhance performance and provide insights into optimal resource management. The results of this work are currently being prepared for publication [55].

Problem and model

We consider a second-generation quantum repeater chain of total length L , divided into N elementary segments of length $L_0 = L/N$, with one optical channel per segment and probabilistic entanglement swapping at the intermediate stations. The central problem addressed in this work is the following: given a fixed total number M of quantum memories (memory buffers) distributed over the N segments, and in general $M > 2N$ (where $M = 2N$ is a quantum repeater with no additional memories), how should these memories be allocated across the half stations so that the raw entanglement distribution rate is maximized (equivalently, the average waiting time (AWT) is minimized)? An example configuration for $N = 4$ and $M = 16$ with some possible entanglement status is illustrated in Figure 4.12 (this configuration is not the optimal distribution of memories).

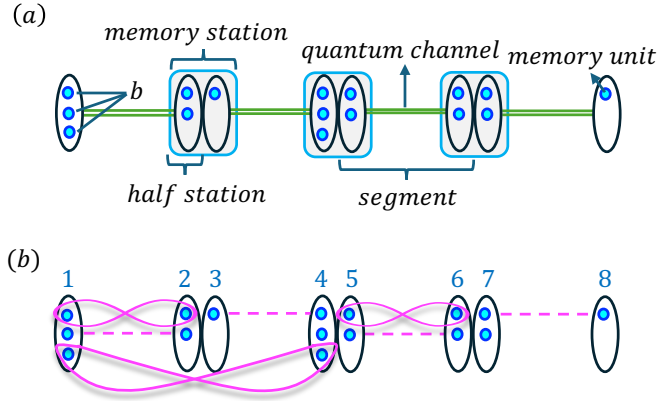


Figure 4.12: (a) Example physical layout of a quantum repeater setup with memory buffers, where b denotes the number of memories allocated to each half-station. (b) Example entanglement status for (a), with pink infinity symbols indicating entangled memories and pink dotted lines showing ongoing entanglement distribution.

The model assumes that each segment has a single source attempting to generate one elementary link per time step, with a distance-dependent success probability $p = \exp(-L_0/L_{\text{att}})$. Entanglement swapping at the intermediate stations succeeds with a constant, distance-independent probability $a < 1$, which captures local hardware inefficiencies but is otherwise treated as deterministic in the sense that it does not fluctuate with segment length or network size. In our model, the entanglement swapping protocol is fixed to the doubling scheme. Memory decoherence is neglected, so stored links remain available until they are consumed in a swapping operation. The resulting dynamics depend only on the stochasticity of link generation and swapping, and on how the finite memory budget

m is distributed among stations.

This setting is distinct from standard multiplexing schemes. In multiplexing, each elementary segment is equipped with multiple parallel channels and multiple memories, so that many link-generation attempts can be made simultaneously in different modes. In contrast, the present work considers only a single physical channel per segment, but allows multiple memories per half station; the additional memories serve purely as a *memory buffer* along that single channel, enabling continued use of the channel after one link has already been established on the segment. This “buffered single-channel” architecture leads to a different optimization problem than multiplexing: the goal is to find, for fixed n and m , the memory layout that best exploits the probabilistic swapping process.

Both analytic and numerical methods are employed. Exact analytic expressions for the average waiting time can be derived only for small systems (in particular, short chains and low memory numbers), because the state space of the underlying Markov process grows exponentially with the number of segments and memories. These analytic results are complemented with large-scale Monte Carlo simulations, which allow exploration of more complex memory distributions and segmentations, including the regimes where a direct analytic treatment is computationally prohibitive. In particular a few cases are discussed in the next subsections.

Case 1: Varying segment length with fixed total distance

A central contribution of the paper is the systematic study of how the segmentation of a fixed total distance L interacts with a fixed total memory budget B . Here, both the number of segments N (and thus the segment length $L_0 = L/N$) and the number of memories per half station b are varied, while keeping L and B fixed. This isolates the trade-off between (i) shorter segments with higher link generation probability p but more stations and fewer memories per station, and (ii) longer segments with lower p , fewer stations, and more memories per station. The example configurations are shown in Figure 4.13.

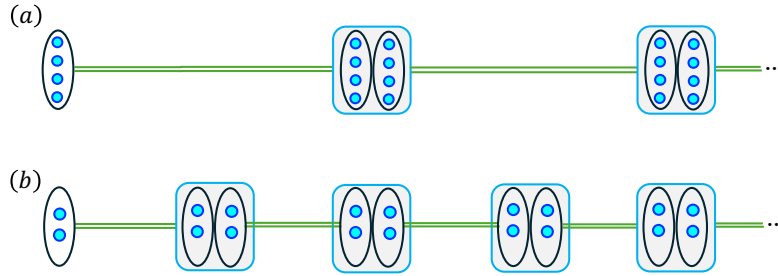


Figure 4.13: Different buffer distributions and different partitions of the fixed length. The segments of the repeater chain in (a) are twice as long as those in (b). Therefore, the link generation probability differs by a power of two. Also, due to the shorter segment length in (b), we need more repeater stations. Therefore, we can have less memory units per station. The tradeoff between having better link generation probabilities p and having more memory units per station is plotted in fig. 4.14.

We study the case where total distance $L = 400$ km, total memories: $B = 64$ and four choices for $N : \{4, 8, 16, 32\}$. The corresponding configuration details are summarized in Table 4.1. For each configuration, the memories are distributed homogeneously, so that every half station has $b = B/(2N)$ buffers.

The results are shown in Figure 4.14. We observe a trade-off between the number of segments and the swapping probability a . For values of a close to one, shorter segments with more nodes generally perform better due to higher link generation probabilities and the

N	L_0	p	b
4	100km	0.01	8
8	50km	0.10	4
16	25km	0.32	2
32	12.5km	0.57	1

Table 4.1: Possible repeater configurations for total length $L = 400\text{km}$. It gets divided into N segments of length L_0 given in the table. The link generation probability $p = e^{-\frac{L_0}{22\text{km}}}$ depends on the length L_0 . The number of memories in each half station is given by b (note that there are two half stations per segment).

need for fewer memories per station. Conversely, for smaller a , fewer but longer segments become favorable as they allow allocating more memories per station to compensate for the increased swapping failures. As a result, the performance curves for different segmentations cross each other, indicating no universally optimal choice. A commonly used segment length of 100 km is only competitive within a narrow range of intermediate swapping probabilities. Overall, the optimal segmentation depends subtly on both the channel loss characteristics and swapping efficiency.

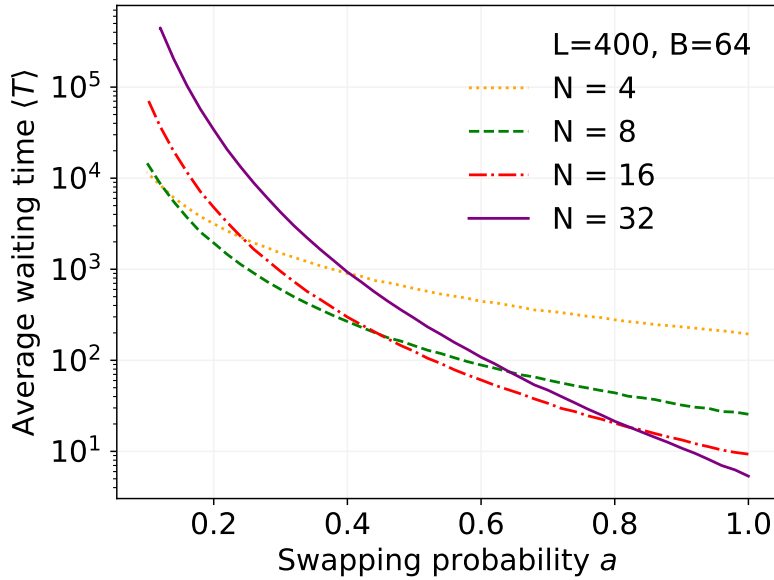


Figure 4.14: Comparison of different repeater configurations from the Table 4.1. L is the total distance in kilometers and B is the total number of memories used. The AWT $T(p, a)$ is plotted against the swapping probability a .

Case 2: Unequal segment lengths

We extend our model to include unequal segment lengths to better represent realistic quantum repeaters, which often rely on previously installed fiber infrastructure with non-uniform segment sizes. In particular, we consider a total distance of $L = 400\text{km}$ divided into eight segments, where some segments are designated “short” and others “long”. The elementary link probabilities are adjusted according to their specific lengths. With a fixed total number of memories (e.g., 32 for $L = 400\text{km}$), three allocation strategies are compared: (1) equal memories on all half stations, (2) more memories on short segments, and (3) more memories on long segments. All these configurations are shown in Figure 4.15.

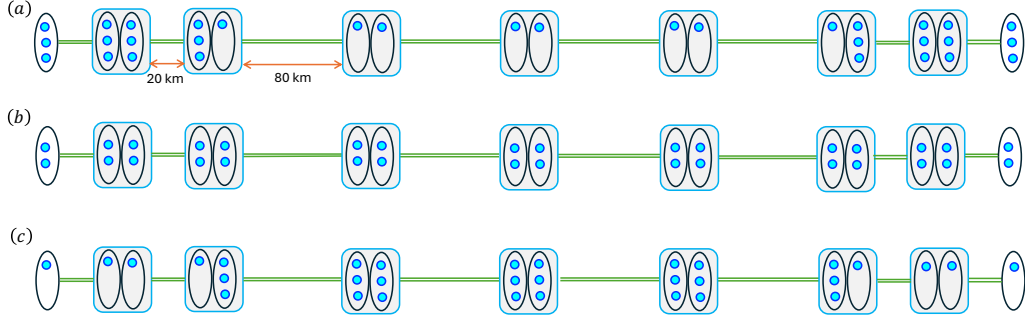


Figure 4.15: Physical structure of quantum repeater with long and short segment lengths with three possible memory configurations.

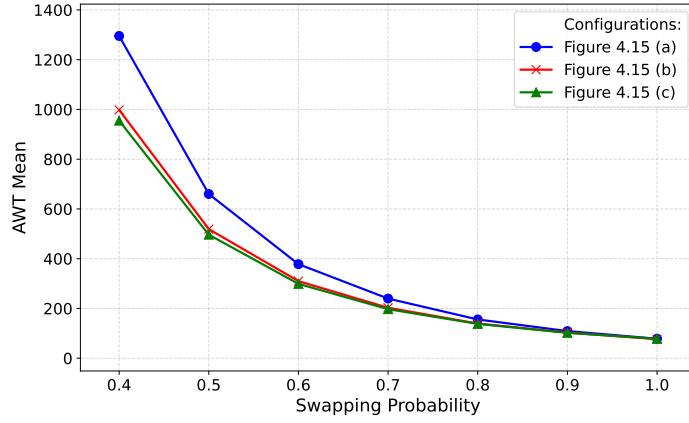


Figure 4.16: Average waiting time (AWT) versus swapping probability for different memory distributions. The setup consists of 8 segments over a total distance of 400 km. The shorter segment distance is 20 km and the longer segment distance is 80 km.

The numerical results are shown in Figure 4.16. It is interesting that unlike in the equal-length case (where homogeneous distributions are typically near-optimal [see, paper IV]), for unequal lengths it can be beneficial to allocate extra buffers to the longer segments. Specifically, for an example with four segments of lengths 20 km and 80 km, the best AWT at moderate and low swapping probabilities is obtained when the long segments carry more memories (e.g., three memories per half station on long segments), slightly outperforming both the homogeneous configuration and the allocation biased toward shorter segments.

This trend has a clear physical interpretation. Longer segments have smaller generation probabilities and thus take more time to generate successful elementary links; additional buffers on these segments allow them to accumulate multiple stored links, which can be consumed across repeated swapping attempts without idling the channel. In contrast, short segments already generate links more frequently, so increasing their memory capacity yields limited extra benefit, since they are less often memory-limited. We have also tested the unequal configurations with only four segments and a total distance of 200 km. and across various tested patterns of short and long segments (including highly asymmetric splits such as 10/90 or 1/99 km), we observe that the homogeneous distribution remains optimal.

Conclusions

Overall, the study shows that memory buffers and segment lengths must be co-designed to minimize the AWT in realistic quantum repeaters. For equal segment lengths and realistic parameters, homogeneous memory distributions are usually close to optimal, and only a modest number of memories per half station is required to approach the infinite-buffer limit. When the total distance and memory budget are fixed, there is a nontrivial optimal segmentation that depends on the swapping probability a , with more, shorter segments favored at high a and fewer, longer segments favored at low a . For networks with unequal segment lengths, allocating additional buffers to the longest, most lossy segments is generally more beneficial than favoring shorter links, highlighting the importance of segment-dependent memory planning in realistic quantum repeater architectures.

As future research directions, it would be interesting to consider swap-asap configurations, which attempt entanglement swapping immediately once two adjacent links are established, potentially reducing waiting times. Additionally, incorporating various memory decoherence models and cut-off strategies would provide a more realistic assessment of repeater performance and help optimize buffer allocation under practical constraints.

5 Conclusion and Outlook

In this thesis we investigate bipartite and multipartite entanglement generation across distant nodes for quantum communication and distributed quantum computing across two complementary layers: abstract quantum networks and concrete quantum repeater architectures. The work on quantum networks introduced LOCC-only classical-quantum hybrid protocols for generating GHZ states directly from Bell-pair networks with efficient classical overhead, near-optimal Bell-pair and source usage, and a number of two-qubit gates that scales linearly in the number of participating nodes, supported by simulations on Erdős–Rényi, Barabási–Albert and Waxman network models as well as experiments on IBM quantum hardware.

At the level of quantum repeater architectures, we perform a rate analysis of second-generation quantum repeaters incorporating quantum error correction based on bosonic Binomial codes. This analysis quantifies the achievable performance of such repeaters while accounting for realistic operational constraints. In this context, we also provide insights into aspects of the physical implementation, including state engineering and logical operations in microwave cavities coupled to transmon qubits. In addition, we investigate all-optical entanglement swapping as a complementary photonic approach. Finally, we optimize memory-buffer allocation strategies in repeater chains, demonstrating improvements in raw rates and uncovering non-trivial trade-offs between segment length, entanglement-swapping success probability, and buffer size.

Taken together, these results show that entanglement distribution protocols, error correction, and resource allocation can be treated as a unified network-design problem, yielding architectures that remain efficient under realistic noise, connectivity, and hardware constraints. The methods developed here connect high-level graph-theoretic formulations and complexity-theoretic insights with concrete physical models of memories, gates, and channels, providing a versatile toolbox that is directly applicable to both communication-oriented quantum networks and modular quantum computing platforms.

A natural direction for future research in quantum networks is to extend the GHZ-distribution techniques presented here to more general graph and cluster states and to benchmark these protocols on heterogeneous hardware platforms and larger-scale devices. This includes integrating more accurate, time-dependent noise models, advanced error-mitigation strategies, and learning-based approaches for star selection, fusion ordering, routing, and scheduling in dynamic quantum networks with failing nodes or links. A rigorous rate analysis for quantum networks remains a longstanding open problem.

For quantum repeaters, promising avenues include studying bosonic quantum error correction with more than two segments, imperfect syndrome extraction, and realistic microwave-to-optical transduction efficiencies. Extending the rate analysis of memory buffers to account for memory decay is also an important open problem.

Progress along these directions will further strengthen the connection between network-level protocols, repeater architectures, and physical hardware, bringing scalable quantum communication and distributed quantum computing closer to practical realization.

6 Bibliography

- [1] R. P. Feynman, *Feynman Lectures On Computation*. CRC Press, 1st ed., 2001.
- [2] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. S. L. Brandao, D. A. Buell, B. Burkett, Y. Chen, Z. Chen, B. Chiaro, R. Collins, W. Courtney, A. Dunsworth, E. Farhi, B. Foxen, A. Fowler, C. Gidney, M. Giustina, R. Graff, K. Guerin, S. Habegger, M. P. Harrigan, M. J. Hartmann, A. Ho, M. Hoffmann, T. Huang, T. S. Humble, S. V. Isakov, E. Jeffrey, Z. Jiang, D. Kafri, K. Kechedzhi, J. Kelly, P. V. Klimov, S. Knysh, A. Korotkov, F. Kostritsa, D. Landhuis, M. Lindmark, E. Lucero, D. Lyakh, S. Mandrà, J. R. McClean, M. McEwen, A. Megrant, X. Mi, K. Michielsen, M. Mohseni, J. Mutus, O. Naaman, M. Neeley, C. Neill, M. Y. Niu, E. Ostby, A. Petukhov, J. C. Platt, C. Quintana, E. G. Rieffel, P. Roushan, N. C. Rubin, D. Sank, K. J. Satzinger, V. Smelyanskiy, K. J. Sung, M. D. Trevithick, A. Vainsencher, B. Villalonga, T. White, Z. J. Yao, P. Yeh, A. Zalcman, H. Neven, and J. M. Martinis, “Quantum supremacy using a programmable superconducting processor,” *Nature*, **574**, pp. 505–510, Oct 2019.
- [3] P. Jurcevic, A. Javadi-Abhari, L. S. Bishop, I. Lauer, D. F. Bogorin, M. Brink, L. Capelluto, O. Günlük, T. Itoko, N. Kanazawa, A. Kandala, G. A. Keefe, K. Kruslich, W. Landers, E. P. Lewandowski, D. T. McClure, G. Nannicini, A. Narasgond, H. M. Nayfeh, E. Pritchett, M. B. Rothwell, S. Srinivasan, N. Sundaresan, C. Wang, K. X. Wei, C. J. Wood, J.-B. Yau, E. J. Zhang, O. E. Dial, J. M. Chow, and J. M. Gambetta, “Demonstration of quantum volume 64 on a superconducting quantum computing system,” *Quantum Science and Technology*, **6**, p. 025020, mar 2021.
- [4] J. M. Arrazola, V. Bergholm, K. Brádler, T. R. Bromley, M. J. Collins, I. Dhand, A. Fumagalli, T. Gerrits, A. Goussev, L. G. Helt, J. Hundal, T. Isacsson, R. B. Israel, J. Izaac, S. Jahangiri, R. Janik, N. Killoran, S. P. Kumar, J. Lavoie, A. E. Lita, D. H. Mahler, M. Menotti, B. Morrison, S. W. Nam, L. Neuhaus, H. Y. Qi, N. Quesada, A. Repeatingon, K. K. Sabapathy, M. Schuld, D. Su, J. Swinerton, A. Száva, K. Tan, P. Tan, V. D. Vaidya, Z. Vernon, Z. Zabaneh, and Y. Zhang, “Quantum circuits with many photons on a programmable nanophotonic chip,” *Nature*, **591**, pp. 54–60, Mar 2021.
- [5] J.-S. Chen, E. Nielsen, M. Ebert, V. Inlek, K. Wright, V. Chaplin, A. Maksymov, E. Páez, A. Poudel, P. Maunz, and J. Gamble, “Benchmarking a trapped-ion quantum computer with 30 qubits,” *Quantum*, **8**, p. 1516, 2024.
- [6] L. Abdurakhimov, J. Adam, H. Ahmad, O. Ahonen, M. Algaba, *et al.*, “Technology and performance benchmarks of iqm’s 20-qubit quantum computer,” *arXiv preprint arXiv:2408.12433*, 2024.
- [7] Qiskit Community, “Qiskit: An open-source framework for quantum computing,” 2017. The official way to cite Qiskit is using the included BibTeX file from the repository.
- [8] V. Bergholm, J. Izaac, M. Schuld, C. Gogolin, C. Blank, K. McKiernan, and N. Killoran, “Pennylane: Automatic differentiation of hybrid quantum-classical computations,” *arXiv preprint arXiv:1811.04968*, 2018.

- [9] Cirq Developers, “Cirq,” 2024.
- [10] L. K. Grover, “A fast quantum mechanical algorithm for database search,” in *Proceedings of the twenty-eighth annual ACM symposium on Theory of Computing*, pp. 212–219, ACM, 1996.
- [11] P. W. Shor, “Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer,” *SIAM Journal on Computing*, **26**, no. 5, pp. 1484–1509, 1997.
- [12] R. L. Rivest, A. Shamir, and L. Adleman, “A method for obtaining digital signatures and public-key cryptosystems,” *Communications of the ACM*, **21**, pp. 120–126, Feb 1978.
- [13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010.
- [14] C. H. Bennett and G. Brassard, “Quantum cryptography: Public key distribution and coin tossing,” *Theoretical Computer Science*, **560**, p. 7–11, Dec 2014.
- [15] A. K. Ekert, “Quantum cryptography based on bell’s theorem,” *Phys. Rev. Lett.*, **67**, pp. 661–663, Aug 1991.
- [16] D. J. Bernstein and T. Lange, “Post-quantum cryptography,” *Nature*, **549**, pp. 188–194, Sep 2017.
- [17] H. Nejatollahi, N. Dutt, S. Ray, F. Regazzoni, I. Banerjee, and R. Cammarota, “Post-quantum lattice-based cryptography implementations: A survey,” *ACM Computing Surveys*, 2019.
- [18] A. Meyer, “Post-quantum cryptography: An analysis of code-based and lattice-based cryptosystems,” *arXiv preprint arXiv:2505.08791*, 2025.
- [19] T. Miya, Y. Terunuma, T. Hosaka, and T. Miyashita, “Ultimate low-loss single-mode fibre at 1.55 μm ,” *Electronics Letters*, **15**, no. 4, pp. 106 – –108, 1979.
- [20] S. Pirandola, R. Laurenza, C. Ottaviani, and L. Banchi, “Fundamental limits of repeaterless quantum communications,” *Nature Communications*, **8**, p. 15043, Apr 2017.
- [21] S. Ecker, B. Liu, J. Handsteiner, M. Fink, D. Rauch, F. Steinlechner, T. Scheidl, A. Zeilinger, and R. Ursin, “Strategies for achieving high key rates in satellite-based qkd,” *npj Quantum Information*, **7**, Jan. 2021.
- [22] J. Yin, Y. Cao, Y.-H. Li, S.-K. Liao, L. Zhang, J.-G. Ren, W.-Q. Cai, W.-Y. Liu, B. Li, H. Dai, G.-B. Li, Q.-M. Lu, Y.-H. Gong, Y. Xu, S.-L. Li, F.-Z. Li, Y.-Y. Yin, Z.-Q. Jiang, M. Li, J.-J. Jia, G. Ren, D. He, Y.-L. Zhou, X.-X. Zhang, N. Wang, X. Chang, Z.-C. Zhu, N.-L. Liu, Y.-A. Chen, C.-Y. Lu, R. Shu, C.-Z. Peng, J.-Y. Wang, and J.-W. Pan, “Satellite-based entanglement distribution over 1200 kilometers,” *Science*, **356**, no. 6343, pp. 1140–1144, 2017.
- [23] S.-K. Liao, W.-Q. Cai, W.-Y. Liu, L. Zhang, Y. Li, J.-G. Ren, and J. Y. et al., “Satellite-to-ground quantum key distribution,” *Nature*, **549**, no. 7670, p. 43–47, 2017.
- [24] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, “Quantum Repeaters: The Role of Imperfect Local Operations in Quantum Communication,” *Phys. Rev. Lett.*, **81**, pp. 5932–5935, Dec 1998.

- [25] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, “Long-distance quantum communication with atomic ensembles and linear optics,” *Nature*, **414**, pp. 413–418, Nov 2001.
- [26] E. Shchukin, F. Schmidt, and P. van Loock, “Waiting time in quantum repeaters with probabilistic entanglement swapping,” *Phys. Rev. A*, **100**, p. 032322, Sep 2019.
- [27] Y. Wu, J. Liu, and C. Simon, “Near-term performance of quantum repeaters with imperfect ensemble-based quantum memories,” *Phys. Rev. A*, **101**, p. 042301, Apr 2020.
- [28] S. Haldar, P. J. Barge, S. Khatri, and H. Lee, “Fast and reliable entanglement distribution with quantum repeaters: Principles for improving protocols using reinforcement learning,” *Phys. Rev. Appl.*, **21**, p. 024041, 2024.
- [29] S. Muralidharan, L. Li, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang, “Optimal architectures for long distance quantum communication,” *Scientific Reports*, **6**, p. 20463, Feb 2016.
- [30] K. Fukui, R. N. Alexander, and P. van Loock, “All-optical long-distance quantum communication with gottesman-kitaev-preskill qubits,” *Phys. Rev. Res.*, **3**, p. 033118, Aug 2021.
- [31] S. Häussler and P. van Loock, “Quantum repeaters based on stationary gottesman-kitaev-preskill qubits,” *Phys. Rev. A*, **111**, p. 062611, Jun 2025.
- [32] F. Rozpędek, K. Noh, Q. Xu, S. Guha, and L. Jiang, “Quantum repeaters based on concatenated bosonic and discrete-variable quantum codes,” *npj Quantum Information*, **7**, June 2021.
- [33] F. Rozpędek, K. P. Seshadreesan, P. Polakos, L. Jiang, and S. Guha, “All-photonic gottesman-kitaev-preskill–qubit repeater using analog-information-assisted multiplexed entanglement ranking,” *Phys. Rev. Res.*, **5**, p. 043056, Oct 2023.
- [34] W. J. Munro, A. M. Stephens, S. J. Devitt, K. A. Harrison, and K. Nemoto, “Quantum communication without the necessity of quantum memories,” *Nature Photonics*, **6**, pp. 777–781, Nov 2012.
- [35] S. Muralidharan, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang, “Ultrafast and Fault-Tolerant Quantum Communication across Long Distances,” *Phys. Rev. Lett.*, **112**, p. 250501, Jun 2014.
- [36] F. Ewert, M. Bergmann, and P. van Loock, “Ultrafast Long-Distance Quantum Communication with Static Linear Optics,” *Phys. Rev. Lett.*, **117**, p. 210501, Nov 2016.
- [37] P. Li, J. Dias, W. J. Munro, P. van Loock, K. Nemoto, and N. Lo Piparo, “Performance of rotation-symmetric bosonic codes in a quantum repeater network,” *Advanced Quantum Technologies*, **7**, May 2024.
- [38] P. W. Shor, “Scheme for reducing decoherence in quantum computer memory,” *Physical Review A*, **52**, no. 4, p. R2493, 1995.
- [39] A. M. Steane, “Error correcting codes in quantum theory,” *Physical Review Letters*, **77**, no. 5, p. 793, 1996.
- [40] A. R. Calderbank and P. W. Shor, “Good quantum error-correcting codes exist,” *Physical Review A*, **54**, no. 2, p. 1098, 1996.
- [41] D. Gottesman, *Stabilizer codes and quantum error correction*. PhD thesis, California Institute of Technology, 1997.

- [42] E. Knill and R. Laflamme, “Theory of quantum error-correcting codes,” *Physical Review A*, **55**, no. 2, p. 900, 1997.
- [43] H. J. Kimble, “The quantum internet,” *Nature*, **453**, p. 1023–1030, June 2008.
- [44] W. Kozłowski and S. Wehner, “Towards large-scale quantum networks,” in *Proceedings of the Sixth Annual ACM International Conference on Nanoscale Computing and Communication*, NANOCOM ’19, (New York, NY, USA), Association for Computing Machinery, 2019.
- [45] A. M. Childs, “Secure assisted quantum computation,” *Quantum Info. Comput.*, **5**, p. 456–466, sep 2005.
- [46] J. F. Fitzsimons, “Private quantum computation: an introduction to blind quantum computing and related protocols,” *npj Quantum Information*, **3**, p. 23, Jun 2017.
- [47] M. Caleffi, M. Amoretti, D. Ferrari, D. Cuomo, J. Illiano, A. Manzalini, and A. S. Cacciapuoti, “Distributed quantum computing: A survey,” *Computer Networks*, **254**, p. 110672, 2024.
- [48] D. Barral, C. Moro, *et al.*, “Review of distributed quantum computing: From single processors to networks,” *Physics Reports*, 2025.
- [49] D. Main, J. P. Gaebler, M. Heinrich, C. Hempel, and et al., “Distributed quantum computing across an optical network link,” *Nature*, **606**, pp. 287–291, 2025.
- [50] R. Jozsa, D. S. Abrams, J. P. Dowling, and C. P. Williams, “Quantum Clock Synchronization Based on Shared Prior Entanglement,” *Phys. Rev. Lett.*, **85**, pp. 2010–2013, Aug 2000.
- [51] P. Kómár, E. M. Kessler, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, “A quantum network of clocks,” *Nature Physics*, **10**, pp. 582–587, Aug 2014.
- [52] S. S. Chelluri, S. Khatri, and P. van Loock, “A resource- and computationally-efficient protocol for multipartite entanglement distribution in bell-pair networks.” arXiv: 2412.04252, 2024.
- [53] S. S. Chelluri, S. Schuster, Sumeet, and R. Roma, “Shallow-depth ghz state generation on nisc devices.” arXiv:2507.19145, 2025.
- [54] S. S. Chelluri, S. Sharma, F. Schmidt, S. V. Kusminskiy, and P. van Loock, “Bosonic quantum error correction with microwave cavities for quantum repeaters.” arXiv:2503.21569, 2025.
- [55] L. Vandr , E. shchukin, S. S. Chelluri, and P. van Loock, “Effects of memory buffer distribution in quantum repeaters.” paper in preparation.
- [56] A. Peres, “Separability criterion for density matrices,” *Phys. Rev. Lett.*, **77**, pp. 1413–1415, Aug 1996.
- [57] M. Horodecki, P. Horodecki, and R. Horodecki, “Separability of mixed states: Necessary and sufficient conditions,” *Phys. Lett. A*, **223**, pp. 1–8, Nov 1996.
- [58] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Going Beyond Bell’s Theorem*, pp. 69–72. Dordrecht: Springer Netherlands, 1989.

- [59] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, “Experimental entanglement of three photons and quantum nonlocality,” *Nature*, **403**, no. 6769, pp. 515–519, 2000.
- [60] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, “Elementary gates for quantum computation,” *Physical Review A*, **52**, no. 5, pp. 3457–3467, 1995.
- [61] C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, “Demonstration of a fundamental quantum logic gate,” *Physical Review Letters*, **75**, no. 25, pp. 4714–4717, 1995.
- [62] D. Gottesman, “The Heisenberg representation of quantum computers,” 1998.
- [63] M. H. Michael, M. Silveri, R. T. Brierley, V. V. Albert, J. Salmilehto, L. Jiang, and S. M. Girvin, “New Class of Quantum Error-Correcting Codes for a Bosonic Mode,” *Phys. Rev. X*, **6**, p. 031006, Jul 2016.
- [64] Z. Leghtas, G. Kirchmair, B. Vlastakis, R. J. Schoelkopf, M. H. Devoret, and M. Mirrahimi, “Hardware-efficient autonomous quantum memory protection,” *Phys. Rev. Lett.*, **111**, p. 120501, 2013.
- [65] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, “Dynamically protected cat-qubits: a new paradigm for universal quantum computation,” *New Journal of Physics*, **16**, no. 4, p. 045014, 2014.
- [66] D. Gottesman, A. Kitaev, and J. Preskill, “Encoding a qubit in an oscillator,” *Phys. Rev. A*, **64**, p. 012310, Jun 2001.
- [67] M. Schlosshauer, “Quantum decoherence,” *Physics Reports*, **831**, pp. 1–57, 2019. Quantum decoherence.
- [68] C. Erven, X. Ma, R. Laflamme, and G. Weihs, “Entangled quantum key distribution with a biased basis choice,” *New Journal of Physics*, **11**, p. 045025, Apr. 2009.
- [69] D. Bruß, “Optimal eavesdropping in quantum cryptography with six states,” *Phys. Rev. Lett.*, **81**, pp. 3018–3021, Oct 1998.
- [70] D. Orsucci, P. Kleinpaß, J. Meister, I. De Marco, S. Häusler, T. Strang, N. Walenta, and F. Moll, “Assessment of practical satellite quantum key distribution architectures for current and near-future missions,” *International Journal of Satellite Communications and Networking*, **43**, p. 164–192, Feb. 2025.
- [71] N. M. Lawrence, A. M. Davis, and M. J. Smith, “Vulnerability of satellite quantum key distribution to disruption from ground-based lasers,” *Sensors*, **21**, no. 23, p. 7904, 2021.
- [72] J. Calsamiglia and N. Lütkenhaus, “Maximum efficiency of a linear-optical bell-state analyzer,” *Applied Physics B*, **72**, pp. 67–71, Jan 2001.
- [73] J.-W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, “Experimental Entanglement Swapping: Entangling Photons That Never Interacted,” *Phys. Rev. Lett.*, **80**, pp. 3891–3894, May 1998.
- [74] P. van Loock, T. D. Ladd, K. Sanaka, F. Yamaguchi, K. Nemoto, W. J. Munro, and Y. Yamamoto, “Hybrid Quantum Repeater Using Bright Coherent Light,” *Phys. Rev. Lett.*, **96**, p. 240501, Jun 2006.

- [75] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. V. den Nest, and H. J. Briegel, “Entanglement in graph states and its applications,” 2006.
- [76] A. Dahlberg, J. Helsen, and S. Wehner, “How to transform graph states using single-qubit operations: computational complexity and algorithms,” *Quantum Science and Technology*, **5**, p. 045016, sep 2020.
- [77] N. Claudet, “Local equivalences of graph states,” 2025.
- [78] S. Wehner, D. Elkouss, and R. Hanson, “Quantum internet: A vision for the road ahead,” *Science*, **362**, no. 6412, p. eaam9288, 2018.
- [79] M. Caleffi, A. S. Cacciapuoti, and G. Bianchi, “Quantum internet: from communication to distributed computing!,” in *Proceedings of the 5th ACM International Conference on Nanoscale Computing and Communication*, NANOCOM ’18, (New York, NY, USA), Association for Computing Machinery, 2018.
- [80] R. Van Meter, *Quantum Networking*. Hoboken: Wiley, 2014.
- [81] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, “Everything you always wanted to know about locc (but were afraid to ask),” *Communications in Mathematical Physics*, **328**, p. 303–326, Mar. 2014.
- [82] E. Knill, R. Laflamme, and G. J. Milburn, “A scheme for efficient quantum computation with linear optics,” *Nature*, **409**, p. 46–52, Jan. 2001.
- [83] J. L. O’Brien, “Optical quantum computing,” *Science*, **318**, no. 5856, pp. 1567–1570, 2007.
- [84] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, “Linear optical quantum computing with photonic qubits,” *Rev. Mod. Phys.*, **79**, pp. 135–174, Jan 2007.
- [85] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, “Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation,” *Physical Review A*, **69**, no. 6, p. 062320, 2004.
- [86] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, “Quantum-information processing with circuit quantum electrodynamics,” *Physical Review A*, **75**, no. 3, p. 032329, 2007.
- [87] S. Haroche and J.-M. Raimond, *Exploring the Quantum: Atoms, Cavities, and Photons*. Oxford: Oxford University Press, 2006.
- [88] J. Q. You and F. Nori, “Atomic physics and quantum optics using superconducting circuits,” *Nature*, **474**, no. 7353, pp. 589–597, 2011.
- [89] D. N. Klyshko, “Photons and nonlinear optics,” Apr. 2018.
- [90] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, “New high-intensity source of polarization-entangled photon pairs,” *Phys. Rev. Lett.*, **75**, pp. 4337–4341, Dec 1995.
- [91] C. Gerry and P. Knight, *Introductory Quantum Optics*. Cambridge University Press, Oct. 2004.
- [92] P. van Loock, W. Alt, C. Becher, O. Benson, H. Boche, C. Deppe, J. Eschner, S. Höfling, D. Meschede, P. Michler, F. Schmidt, and H. Weinfurter, “Extending quantum links: Modules for fiber- and memory-based quantum repeaters,” *Advanced Quantum Technologies*, **3**, Oct. 2020.

- [93] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, “Circuit quantum electrodynamics,” *Rev. Mod. Phys.*, **93**, p. 025005, May 2021.
- [94] B. W. Shore and P. L. Knight, “The jaynes-cummings model,” *Journal of Modern Optics*, **40**, no. 7, pp. 1195–1238, 1993.
- [95] J. Larson and T. Mavrogordatos, *The Jaynes–Cummings Model and its Descendants (Second Edition)*. 2053-2563, IOP Publishing, 2024.
- [96] M. Newman, *Networks*. Oxford University Press, 07 2018.
- [97] A.-L. Barabási, *Network Science*. Cambridge University Press, 2016.
- [98] U. M. Adrian Bondy, *Graph theory*. Graduate texts in mathematics 244, Springer, 3 ed., 2008.
- [99] B. Bollobás, *Modern Graph Theory*. Graduate Texts in Mathematics 184, Springer-Verlag New York, 1 ed., 1998.
- [100] T. Nishizeki, *Planar Graph Problems*, pp. 53–68. Springer Vienna, 1990.
- [101] S. Arora and B. Barak, *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.
- [102] M. Sipser, *Introduction to the Theory of Computation*. Cengage Learning, 3rd ed., 2012.
- [103] C. M. Papadimitriou, *Computational Complexity*. Addison-Wesley, 1994.
- [104] R. M. Karp, “Reducibility among combinatorial problems,” in *Complexity of Computer Computations* (R. E. Miller and J. W. Thatcher, eds.), (New York), pp. 85–103, Plenum Press, 1972.
- [105] P. Erdős and A. Rényi, “On random graphs. I,” *Publicationes Mathematicae Debrecen*, **6**, pp. 290–297, 11 1959.
- [106] P. Erdős and A. Rényi, “On the evolution of random graphs,” *Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei*, **5**, pp. 17–61, 1960.
- [107] A.-L. Barabási and R. Albert, “Emergence of Scaling in Random Networks,” *Science*, **286**, no. 5439, pp. 509–512, 1999.
- [108] B. M. Waxman, “Routing of multipoint connections,” *IEEE Journal on Selected Areas in Communications*, **6**, no. 9, pp. 1617–1622, 1988.
- [109] R. Raussendorf and H. J. Briegel, “A one-way quantum computer,” *Phys. Rev. Lett.*, **86**, pp. 5188–5191, May 2001.
- [110] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, “Measurement-based quantum computation,” *Nature Physics*, **5**, p. 19–26, Jan. 2009.
- [111] R. Jozsa, “An introduction to measurement based quantum computation,” *arXiv:quant-ph/0508124*, 2005.
- [112] F. Grasselli, H. Kampermann, and D. Bruß, “Conference key agreement with single-photon interference,” *New Journal of Physics*, **21**, p. 123002, dec 2019.
- [113] G. Murta, F. Grasselli, H. Kampermann, and D. Bruß, “Quantum Conference Key Agreement: A Review,” *Advanced Quantum Technologies*, **3**, no. 11, p. 2000025, 2020.

- [114] G. Tóth, “Multipartite entanglement and high-precision metrology,” *Physical Review A*, **85**, p. 022322, Feb 2012.
- [115] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, “Fisher information and multiparticle entanglement,” *Physical Review A*, **85**, p. 022321, Feb 2012.
- [116] A. Pirker, J. Wallnöfer, and W. Dür, “Modular architectures for quantum networks,” *New Journal of Physics*, **20**, p. 053054, May 2018.
- [117] A. Pirker and W. Dür, “A quantum network stack and protocols for reliable entanglement-based networks,” *New Journal of Physics*, **21**, p. 033003, March 2019.
- [118] F. Hahn, A. Pappa, and J. Eisert, “Quantum network routing and local complementation,” *npj Quantum Information*, **5**, p. 76, 2019.
- [119] J. Freund, A. Pirker, and W. Dür, “Flexible quantum data bus for quantum networks,” *Physical Review Research*, **6**, p. 033267, Sep 2024.
- [120] M. Cuquet and J. Calsamiglia, “Growth of graph states in quantum networks,” *Physical Review A*, **86**, p. 042304, October 2012.
- [121] M. Epping, H. Kampermann, and D. Bruß, “Large-scale quantum networks based on graphs,” *New Journal of Physics*, **18**, no. 5, p. 053036, 2016.
- [122] A. Fischer and D. Towsley, “Distributing Graph States Across Quantum Networks,” in *2021 IEEE International Conference on Quantum Computing and Engineering (QCE)*, pp. 324–333, 2021.
- [123] G. Avis, F. Rozpędek, and S. Wehner, “Analysis of multipartite entanglement distribution using a central quantum-network node,” *Physical Review A*, **107**, p. 012609, Jan 2023.
- [124] M. Cuquet and J. Calsamiglia, “Growth of graph states in quantum networks,” *Phys. Rev. A*, **86**, p. 042304, Oct 2012.
- [125] F. K. Hwang, D. S. Richards, and P. Winter, *The Steiner Tree Problem*, vol. 53 of *Annals of Discrete Mathematics*. North-Holland, 1992.
- [126] C. Meignant, D. Markham, and F. Grosshans, “Distributing graph states over arbitrary quantum networks,” *Physical Review A*, **100**, p. 052333, November 2019.
- [127] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Series of Books in the Mathematical Sciences, New York: W. H. Freeman and Company, 1st ed., 1979.
- [128] V. V. Vazirani, *Approximation Algorithms*. Springer Science & Business Media, 2001.
- [129] B. Waclaw and I. M. Sokolov, “Finite-size effects in Barabási-Albert growing networks,” *Physical Review E*, **75**, p. 056114, May 2007.
- [130] A. A. Hagberg, D. A. Schult, and P. J. Swart, “Exploring Network Structure, Dynamics, and Function using NetworkX,” in *Proceedings of the 7th Python in Science Conference* (G. Varoquaux, T. Vaught, and J. Millman, eds.), (Pasadena, CA USA), pp. 11–15, 2008. <https://networkx.org/>.
- [131] K. Mehlhorn, “A faster approximation algorithm for the Steiner problem in graphs,” *Information Processing Letters*, **27**, no. 3, pp. 125–128, 1988.

- [132] M. Cuquet and J. Calsamiglia, “Growth of graph states in quantum networks,” *Phys. Rev. A*, **86**, p. 042304, Oct 2012.
- [133] J. Preskill, “Quantum Computing in the NISQ era and beyond,” *Quantum*, **2**, p. 79, Aug. 2018.
- [134] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, “Fisher information and multiparticle entanglement,” *Phys. Rev. A*, **85**, p. 022321, Feb 2012.
- [135] P. L. and S. A., “Quantum theory of phase estimation,” *Proceedings of the International School of Physics "Enrico Fermi"*, **188**, p. 691–741, 2014.
- [136] T. Kielinski, P. O. Schmidt, and K. Hammerer, “Ghz protocols enhance frequency metrology despite spontaneous decay,” *Science Advances*, **10**, Oct. 2024.
- [137] D. Cruz, R. Fournier, F. Gremion, A. Jeannerot, K. Komagata, T. Tosić, J. Thiesbrummel, C. L. Chan, N. Macris, M. Dupertuis, and C. Javerzac-Galy, “Efficient quantum algorithms for ghz and w states, and implementation on the ibm quantum computer,” *Advanced Quantum Technologies*, **2**, Apr. 2019.
- [138] E. Bäumer, V. Tripathi, D. S. Wang, P. Rall, E. H. Chen, S. Majumder, A. Seif, and Z. K. Mineev, “Efficient long-range entanglement using dynamic circuits,” *PRX Quantum*, **5**, p. 030339, Aug 2024.
- [139] H. Liao, G. S. Hartnett, A. Kakkar, A. Tan, M. Hush, P. S. Mundada, M. J. Biercuk, and Y. Baum, “Achieving computational gains with quantum error-correction primitives: Generation of long-range entanglement enhanced by error detection,” *PRX Quantum*, **6**, p. 020331, May 2025.
- [140] K.-C. Chen, “Short-depth circuits and error mitigation for large-scale ghz-state preparation, and benchmarking on ibm’s 127-qubit system,” in *2023 IEEE International Conference on Quantum Computing and Engineering (QCE)*, vol. 02, pp. 207–210, 2023.
- [141] S. A. Moses, C. H. Baldwin, M. S. Allman, R. Ancona, L. Ascarrunz, C. Barnes, J. Bartolotta, B. Bjork, P. Blanchard, M. Bohn, J. G. Bohnet, N. C. Brown, N. Q. Burdick, W. C. Burton, S. L. Campbell, and . others, “A race-track trapped-ion quantum processor,” *Phys. Rev. X*, **13**, p. 041052, Dec 2023.
- [142] J. de Jong, F. Hahn, N. Tcholtchev, M. Hauswirth, and A. Pappa, “Extracting ghz states from linear cluster states,” *Physical Review Research*, **6**, p. 013330, Mar. 2024.
- [143] Z. Bao, S. Xu, Z. Song, K. Wang, L. Xiang, Z. Zhu, J. Chen, F. Jin, X. Zhu, Y. Gao, Y. Wu, C. Zhang, N. Wang, Y. Zou, Z. Tan, A. Zhang, Z. Cui, F. Shen, J. Zhong, T. Li, J. Deng, X. Zhang, H. Dong, P. Zhang, Y.-R. Liu, L. Zhao, J. Hao, H. Li, Z. Wang, C. Song, Q. Guo, B. Huang, and H. Wang, “Creating and controlling global greenberger-horne-zeilinger entanglement on quantum processors,” *Nature Communications*, **15**, Oct. 2024.
- [144] P. Ball, “First quantum computer to pack 100 qubits enters crowded race,” *Nature*, **599**, Nov 2021.
- [145] R. Acharya, D. A. Abanin, L. Aghababaie-Beni, I. Aleiner, T. I. Andersen, M. Ansmann, F. Arute, K. Arya, A. Asfaw, N. Astrakhantsev, J. Atalaya, R. Babbush, D. Bacon, B. Ballard, J. C. Bardin, others, G. Q. AI, and Collaborators, “Quantum error correction below the surface code threshold,” *Nature*, **638**, pp. 920–926, Feb 2025.

- [146] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. V. den Nest, and H. J. Briegel, “Entanglement in graph states and its applications,” 2006.
- [147] Qiskit Developers, “Hellinger fidelity,” 2025. Accessed: 2025-07-10.
- [148] E. Hellinger, “Neue begründung der theorie quadratischer formen von unendlichvielen veränderlichen.,” *Journal für die reine und angewandte Mathematik*, **1909**, pp. 210–271, July 1909.
- [149] K. F. Koenig, F. Reinecke, W. Hahn, and T. Wellens, “Inverted-circuit zero-noise extrapolation for quantum-gate error mitigation,” *Phys. Rev. A*, **110**, p. 042625, Oct 2024.
- [150] S. D. Reiß and P. van Loock, “Deep reinforcement learning for key distribution based on quantum repeaters,” *Phys. Rev. A*, **108**, p. 012406, Jul 2023.
- [151] F. Hahn, J. de Jong, and A. Pappa, “Anonymous quantum conference key agreement,” *PRX Quantum*, **1**, p. 020325, Dec 2020.
- [152] V. Kasirajan, T. Battelle, and B. Wold, “Empowering large scale quantum circuit development: Effective simulation of sycamore circuits,” 2024.
- [153] R. Laurenza, N. Walk, J. Eisert, and S. Pirandola, “Rate limits in quantum networks with lossy repeaters,” *arXiv preprint*, 2021. arXiv:2110.10168.
- [154] L. Kamin, E. Shchukin, F. Schmidt, and P. van Loock, “Exact rate analysis for quantum repeaters with imperfect memories and entanglement swapping as soon as possible,” *Phys. Rev. Res.*, **5**, p. 023086, May 2023.
- [155] M. Razavi, M. Piani, and N. Lütkenhaus, “Quantum repeaters with imperfect memories: Cost and scalability,” *Phys. Rev. A*, **80**, p. 032301, Sep 2009.
- [156] A. Romanova and P. van Loock, “Memory-corrected quantum repeaters with adaptive syndrome identification,” *Phys. Rev. Res.*, **7**, p. 013117, Jan 2025.
- [157] A. Reiserer, N. Kalb, M. S. Blok, K. J. M. van Bemmelen, T. H. Taminiau, R. Hanson, D. J. Twitchen, and M. Markham, “Robust quantum-network memory using decoherence-protected subspaces of nuclear spins,” *Phys. Rev. X*, **6**, p. 021040, Jun 2016.
- [158] N. Kalb, P. C. Humphreys, J. J. Slim, and R. Hanson, “Dephasing mechanisms of diamond-based nuclear-spin memories for quantum networks,” *Phys. Rev. A*, **97**, p. 062330, Jun 2018.
- [159] N. Sangouard, R. Dubessy, and C. Simon, “Quantum repeaters with imperfect memories: Cost and scalability,” *Physical Review A*, **80**, p. 032301, 2009.
- [160] K. Fukui, R. N. Alexander, and P. van Loock, “All-optical long-distance quantum communication with Gottesman-Kitaev-Preskill qubits,” *Phys. Rev. Res.*, **3**, p. 033118, Aug 2021.
- [161] P. Laha and P. van Loock, “Arbitrary high-fidelity binomial codes from multiphoton spin-boson interactions,” *Phys. Rev. Res.*, **8**, p. 013237, Mar 2026.
- [162] P. Li, J. Dias, W. Munro, P. van Loock, K. Nemoto, and N. Lo Piparo, “Towards More Feasible Quantum Networks Based on Cavity-QED and Continuous-variable Codes,” in *APS March Meeting Abstracts*, vol. 2024 of *APS Meeting Abstracts*, p. J00.287, Mar. 2024.

- [163] R. W. Heeres, P. Reinhold, N. Ofek, L. Frunzio, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, “Implementing a universal gate set on a logical qubit encoded in an oscillator,” *Nature Communications*, **8**, p. 94, Jul 2017.
- [164] X. Gu, A. F. Kockum, A. Miranowicz, Y. xi Liu, and F. Nori, “Microwave photonics with superconducting quantum circuits,” *Physics Reports*, **718-719**, pp. 1–102, 2017. Microwave photonics with superconducting quantum circuits.
- [165] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, “Circuit quantum electrodynamics,” *Rev. Mod. Phys.*, **93**, p. 025005, May 2021.
- [166] S. Kuhr, S. Gleyzes, C. Guerlin, J. Bernu, U. B. Hoff, S. Deléglise, S. Osnaghi, M. Brune, J.-M. Raimond, S. Haroche, E. Jacques, P. Bosland, and B. Visentin, “Ultrahigh finesse Fabry-Pérot superconducting resonator,” *Applied Physics Letters*, **90**, p. 164101, 04 2007.
- [167] N. Lauk, N. Sinclair, S. Barzanjeh, J. P. Covey, M. Saffman, M. Spiropulu, and C. Simon, “Perspectives on quantum transduction,” *Quantum Science and Technology*, **5**, p. 020501, mar 2020.
- [168] R. W. Andrews, R. W. Peterson, T. P. Purdy, K. Cicak, R. W. Simmonds, C. A. Regal, and K. W. Lehnert, “Bidirectional and efficient conversion between microwave and optical light,” *Nature Physics*, **10**, pp. 321–326, Apr 2014.
- [169] R. Sahu, L. Qiu, W. Hease, G. Arnold, Y. Minoguchi, P. Rabl, and J. M. Fink, “Entangling microwaves with light,” *Science*, **380**, no. 6646, pp. 718–721, 2023.
- [170] S. Sharma, V. S. V. Bittencourt, and S. V. Kusminskiy, “Protocol for generating an arbitrary quantum state of the magnetization in cavity magnonics,” *Journal of Physics: Materials*, **5**, p. 034006, jul 2022.

7 Publications

Contribution to publications

Removed for privacy reasons.

Paper I

A resource- and computationally-efficient protocol for multipartite entanglement distribution in Bell-pair networks

S. Siddardha Chelluri, Sumeet Khatri and Peter van Loock

arXiv:2412.04252, in review at Nature Physics Journal, Quantum Information
(npj-QI)

A resource- and computationally-efficient protocol for multipartite entanglement distribution in Bell-pair networks

S. Siddardha Chelluri,^{1,*} Sumeet Khatri,^{2,3,4,†} and Peter van Loock^{1,‡}

¹*Institute of Physics, Johannes-Gutenberg University of Mainz, Staudingerweg 7, 55128 Mainz, Germany*

²*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

³*Department of Computer Science, Virginia Tech, Blacksburg, VA 24061, USA*

⁴*Virginia Tech Center for Quantum Information Science and Engineering, Blacksburg, VA 24061, USA*

(Dated: July 29, 2025)

Multipartite entangled states, particularly Greenberger–Horne–Zeilinger (GHZ) and other graph states, are important resources in multiparty quantum network protocols and measurement-based quantum computing. We consider the problem of generating such states from networks of bipartite entangled (Bell) pairs. We adopt the perspective that, in practice, unlike the traditional information-theoretic setting, local operations and classical communications are not free. Consequently, protocols should not only be efficient with respect to the number of consumed Bell pairs, as typically considered, but also efficient with respect to the number of (local) gates, number of Bell-pair sources, and computational complexity. In this work, we present a protocol for producing GHZ states in arbitrary Bell-pair networks that is efficient with respect to all of these figures of merit. We prove that our protocol: (1) requires $O(N)$ gates in a network with N nodes, independent of the topology of the network; (2) has time complexity $O(N^2)$, thereby avoiding finding a Steiner tree or solving any other computationally hard problem; and (3) maintains nearly the optimal number of consumed Bell pairs. We prove that the minimal Bell-pair source cost is equivalent to the graph-theoretic dominating set problem, and via numerical simulations on real-world network models, we demonstrate that our protocol is nearly optimal with respect to Bell-pair source cost. Numerically, our protocol also outperforms existing protocols based on (approximate) Steiner trees with respect to both number of gates and Bell-pair sources. Finally, we provide a detailed analysis of the impact of noisy Bell pairs and gates on the fidelity of the distributed GHZ states.

I. INTRODUCTION

Recent years have seen a rapid rise in the development and realization of small-scale quantum devices, such as small-scale quantum computers and quantum sensors [1–4]. Individually, these small-scale devices are not particularly powerful nor useful. However, by *networking* these small-scale devices, it may become possible to execute large-scale applications in a distributed fashion [5–7], and thus make strides towards the goal of quantum advantage.

Entanglement is the resource that enables distributed quantum information processing tasks, such as quantum communication, computation, and sensing, over long distances. Both bipartite and multipartite entanglement can be used to achieve these tasks. Indeed, bipartite entanglement, along with local operations and classical communication (LOCC) between spatially separated nodes, enables the fundamental protocol of teleportation [8, 9], and multipartite entanglement enables various distributed and measurement-based quantum computing strategies [10–19], and can enhance quantum sensing [20–24].

The creation of bipartite entanglement (e.g., Bell pairs) between nearest neighbors in a network has been realized experimentally in recent years [25–35], and will arguably form the backbone of large-scale quantum networks. Therefore, when considering the distribution of multipartite entanglement in quantum networks, it is meaningful to consider protocols in which the aim is to transform a network of Bell pairs, via LOCC,

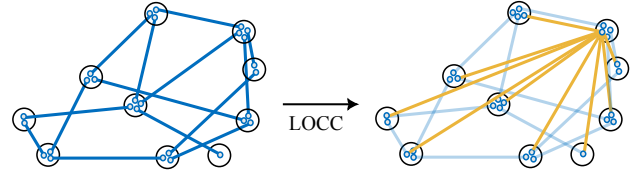


FIG. 1. (Left) A network of Bell pairs, consisting of nodes (black circles) containing qubits (blue circles). Every edge connecting two qubits represents a Bell pair. Our goal is to transform the Bell-pair network, via local operations and classical communication, into a graph state. (Right) A star graph state (which is local-unitary-equivalent to a GHZ state), indicated in yellow, shared by all nodes in the network.

into a multipartite entangled state shared by a given set of nodes in the network; see Fig. 1. Some prior works assume that these multipartite entangled states are already distributed in advance in the network [36–39], without consideration of how this might be done in the first place. Other works consider first creating the multipartite states locally, and then distributing them over quantum channels and/or via teleportation [40–43] using the network of nearest-neighbor Bell pairs. The latter strategy requires quantum repeater protocols [44–48] for creating Bell pairs between one node and all of the others, which is in general costly in terms of the number of Bell pairs and local qubits.

In this work, we consider LOCC-based entanglement distribution of multipartite entangled states directly from a network of Bell pairs, as illustrated in Fig. 1. Previously, Ref. [40] has provided a protocol in this setting that creates a graph state shared among all nodes in the network, such that the topology of the corresponding graph mimics that of the Bell-pair network. They have shown that their protocol can outperform the

* schellur@uni-mainz.de

† skhatri@vt.edu

‡ loock@uni-mainz.de

repeater-based protocol that first creates bipartite entanglement between one node and all of the others and then teleports the locally-created graph state. Similarly, Ref. [49] provides a protocol for distributing an arbitrary graph state using a network of Bell pairs, even among a chosen subset of nodes, but this protocol involves (in the most general case) solving the NP-hard Steiner tree problem [50, 51] as an initial step. The protocol in Ref. [42] involves generating the graph state locally at a node and then distributing the state to the other nodes via appropriately selected paths using the Bell-pair network. We refer to App. A for a more detailed summary of prior work.

The main contribution of our work is an alternative protocol for distributing graph states, specifically Greenberger–Horne–Zeilinger (GHZ) states [52], in arbitrary network topologies directly from the Bell pairs that connect the nearest-neighbor nodes of the network. GHZ states are useful, for instance, for conference key agreement [53], and are necessary for achieving optimal precision in certain quantum metrological tasks [20, 21]. Unlike Ref. [49], our protocol is not based on solving the Steiner tree problem, nor any other computationally hard problem. Also, unlike Ref. [42], our protocol does not require first creating Bell pairs between one node in the network and all of the others. Nevertheless, we can provably achieve either comparable or better performance compared to Refs. [42, 49].

The usual figures of merit for assessing the performance of entanglement distribution protocols include the number of consumed Bell pairs, waiting time, and fidelity. In this work, we consider also the computational efficiency, the total number of (local) gates, and the number of Bell-pair sources. These additional figures of merit are motivated by the fact that gates are currently prone to errors, which means that applying more gates comes at the cost of reduced fidelity. The amount of classical communication, particularly between distant nodes for the purpose of communicating measurement outcomes, is also constrained in practice due to limitations on the coherence times of quantum memories. From these facts, we can conclude that LOCC operations are not free in practice, unlike the usual information-theoretic setting of entanglement distribution in which LOCC operations are considered free. (We note that LOCC operations are considered free in Refs. [42, 49].) Furthermore, when we envision the development of large-scale quantum networks, the question of how many Bell-pair sources are needed in the network naturally arises, and it is meaningful to consider keeping the number of Bell-pair sources to a minimum. With all of these points in mind, we are motivated to seek protocols that are efficient with respect to the following figures of merit: (1) computational efficiency; (2) number of gates applied; and (3) number of Bell-pair sources.

Our protocol, which we present in Sec. II, is efficient with respect to all three of the above figures of merit. It is based on successively creating small GHZ states, starting from the highest-degree nodes, and then merging them to obtain larger GHZ states that eventually span either the entire network or some given subset of nodes. We present our main results in Sec. III. In Sec. IV, we assess the performance of our protocol on models of real networks, which exhibit the small-world property, along with high density and central hubs. We consider the

Erdős–Rényi, Barabási–Albert, and Waxman models [54–61], which capture these realistic network features.

In Sec. III A, we present our first result, which is that our protocol has time complexity $O(N^2)$. Then, in Sec. III B, we prove that our protocol requires $O(N)$ gates to produce a GHZ state shared by all nodes in the network. On the other hand, we prove that the protocol of Ref. [49] requires at most $O(N^2)$ gates, and in Sec. IV we provide numerical evidence that our protocol indeed improves the gate count compared to Ref. [49]. It is also remarkable that the gate count for our protocol depends *only* on the number N of nodes, and not on any other property of the network, e.g., topology and density of nodes.

We also prove in Sec. III C that our protocol consumes the same optimal number of Bell pairs as in Ref. [49] when entangling all nodes in the network, and in Sec. IV we provide numerical evidence that our protocol is nearly optimal with respect to consumed Bell pairs when entangling a subset of nodes, i.e., the number of consumed Bell pairs in our protocol is close to that of the Steiner tree approach of Ref. [49]. As such, our protocol represents a heuristic algorithm for the Steiner tree problem.

Regarding the number of Bell-pair sources, our protocol has the notable feature that it automatically prescribes the location of the Bell-pair sources. Indeed, by creating small-scale GHZ states at the highest-degree nodes within the network, these sources only need to be located at the highest-degree nodes, rather than at every individual node or at the midpoint of every edge of the network. It is then natural to ask whether this placement of Bell-pair sources is optimal. We address this question in Sec. III D, where we prove that minimizing the number of Bell-pair sources is equivalent to the *minimum dominating set problem* from graph theory [62], which is an NP-hard problem in general [63]. This result represents an interesting connection to graph theory, analogous to the well-known connection between the minimal Bell-pair cost and the Steiner tree problem. With this result, we are able to show numerically in Sec. IV that our protocol is nearly optimal with respect to the number of Bell-pair sources. In other words, our protocol provides a heuristic algorithm for both the Steiner tree and minimum dominating set problems. Furthermore, by using an approximate algorithm for the minimum dominating set problem [62], we can evaluate the Bell-pair source counts of algorithms based on the Steiner tree, and we find in Sec. IV that our protocol outperforms these existing algorithms for both the Erdős–Rényi and Barabási–Albert network models.

Finally, in Sec. V, we consider our protocol in the presence of noise, both noisy gates and noisy Bell pairs, and we derive exact analytical expressions for the fidelity of the output state with respect to the desired GHZ state. These analytical expressions hold for arbitrary noise models for both the Bell pairs and the gates.

II. OUR PROTOCOL

We model a network with a graph. The vertices of the graph correspond to the nodes in the network. Every node has a number of qubits equal to the degree of the corresponding vertex.

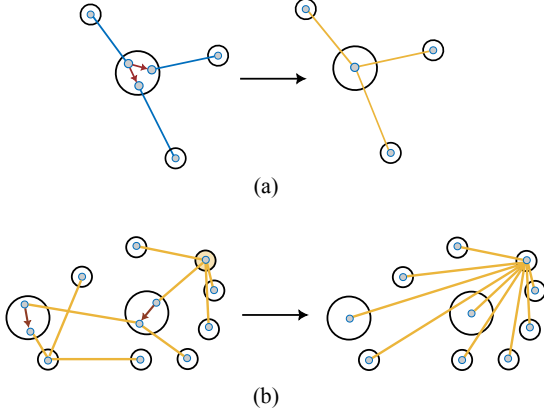


FIG. 2. Fusion protocols [36, 64]. (a) Fusion of Bell pairs connected to a central node to create a GHZ state (Protocol 2). Arrows indicate CNOT gates, pointing from source qubits to target qubits. (b) Fusion of GHZ states in a tree topology (Protocol 3), with the root node highlighted. In both protocols, after the CNOT gates, the target qubits are measured in the Pauli-Z basis.

The edges in the graph dictate how these qubits within the nodes are connected to the qubits in other nodes via quantum channels, which are used to generate Bell pairs $|\Phi\rangle := \frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle)$ between the qubits in different nodes. In particular, if the graph has a vertex with degree k , then its corresponding network has a node with k qubits, and each one of these qubits can be connected via a Bell pair to a qubit in the neighboring node; see Fig. 1 for an example. Starting from the network of Bell pairs, we consider the task of distributing a GHZ state among either all nodes of the network (the *complete case*) or a given subset of nodes in the network (the *subset case*). Every qubit of the GHZ state should belong to one of the desired nodes.

The N -party GHZ state [52] is defined as $|\text{GHZ}_N\rangle := \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$. Graphically, GHZ states can be depicted using a star graph; see Fig. 2. Notably, because the GHZ state is permutation invariant, the center of the star can be placed at any of the qubits, a fact that we make use of extensively in our protocol. (This fact is in contrast to the star graph state, which has a distinguished central qubit, and to change the central qubit requires local operations; see App. B.) Throughout the rest of this work, we use the term “star” to refer to both a GHZ state and its graphical depiction.

In Protocol 1, we present our protocol for creating a GHZ state in an arbitrary network of Bell pairs, and in Fig. 3 we provide an example of how it works. The protocol makes use of two fusion protocols, illustrated in Fig. 2: Protocol 2, which creates local, small-scale GHZ states centered around nodes with the highest degree; and Protocol 3, which merges small-scale GHZ states in a tree topology to achieve the desired GHZ state among all of the desired nodes. We note that Protocol 2 is a well-known method for creating a GHZ state from Bell pairs in a star topology, and Protocol 3 is based on a well-known method for merging GHZ states; see, e.g., Refs. [36, 64].

In the subset case, we can first generate a GHZ state for the entire graph using our Protocol 1 and then use X -measurements to remove the qubits outside of the target subset, resulting in

a GHZ state shared among the desired nodes. However, this method consumes significantly more gates and Bell pairs than necessary. Instead, we first apply Protocol 4, which constructs a connected subgraph that includes the desired nodes, and then using this subgraph as the input to Protocol 1 we generate the GHZ state within this subgraph. Finally, we X -measure out any qubits in the subgraph that are not part of the target subset. (If the set of desired nodes already forms a connected subgraph, then Protocol 4 is not required.) We provide an example of how Protocol 4 works in Fig. 4. Note that this initial step of constructing a connected subgraph is in contrast to the protocol in Ref. [49]—which we refer to as the “MMG protocol” from now on—in which the subset case involves first identifying a Steiner tree (a process known to be NP-hard [65]) and then executing the star expansion protocol of the same work to obtain the GHZ state among the selected nodes.

Protocol 1 GHZ creation in a Bell-pair network

Input: Network of Bell pairs, represented by a graph G with N nodes.

Output: GHZ state shared among all N nodes.

- 1: Create an empty list, SG.
 - 2: Create a table listing the degree of each node in G , sorted in descending order. If multiple nodes have the same degree, then order them randomly.
 - 3: Select the top-most entry of the table. Save the star subgraph, consisting of the corresponding node and its neighboring nodes, as an element in SG.
 - 4: Delete the top most entry of the table, and remove the corresponding node and its associated edges from G .
 - 5: Repeat steps 3 and 4 until all nodes in G are included in at least one star.
 - 6: If any edges remain in G , add them along with their two incident nodes to SG.
 - 7: Select the largest star in SG, label it S_0 , and remove it from SG. Create a new list called MSG and add S_0 to it.
 - 8: Find another star in SG that shares at least one node in common with any star in MSG and call it S_1 .
 - 9: Check if all nodes of S_1 are already in any stars of MSG. If so, remove S_1 from SG and return to the previous step.
 - 10: If the central node of S_1 is in common with any star in MSG, remove all common nodes (and the associated edges) from S_1 except its central node. If not, remove all common nodes (and the associated edges) except one randomly-chosen common node. Remove S_1 from SG and add the modified S_1 to MSG.
 - 11: Repeat steps 8–10, naming stars as S_2, S_3 , etc., until SG is empty.
 - 12: For every star in MSG, apply Protocol 2 to create a GHZ state among the nodes in the star.
 - 13: Using Protocol 3, merge all of the small GHZ states into one.
-

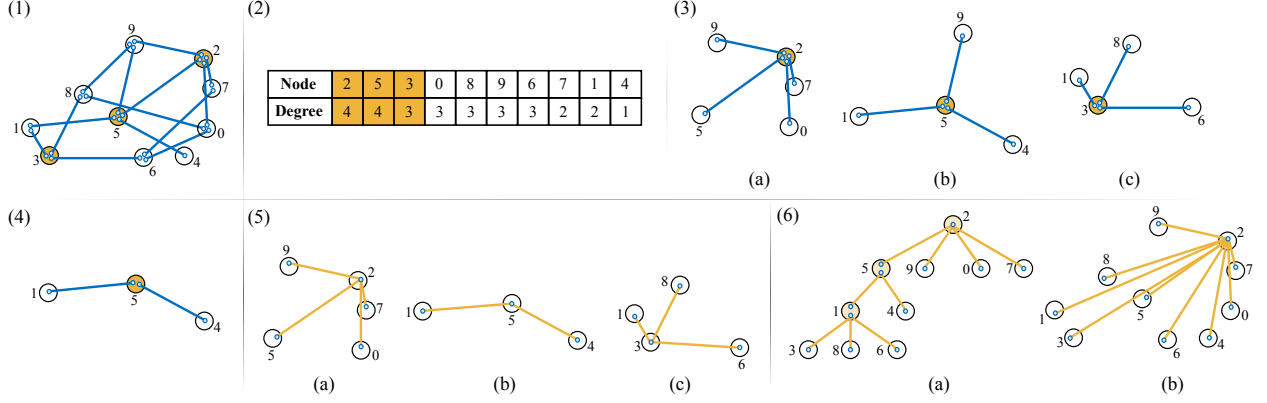


FIG. 3. Example execution of Protocol 1 for distributing a GHZ state among all network nodes. (1) The initial network of Bell pairs. (2) Table of nodes and their degrees in descending order, as per Step 2. (3) The selection of star subgraphs according to Steps 3–6, with (a), (b), and (c) representing the consecutive stars obtained from the first three entries of the table. At this point, all nodes are included in at least one star. (4) Duplicate nodes and edges are removed, according to Steps 7–11, where stars (3a) and (3c) are unaltered, but the edge between nodes 5 and 9 in star (3b) is deleted. (5) The locally created GHZ states as in Step 12. (6a) GHZ states from (5) arranged as a tree, with the center of (5c) shifted to node 1. (6b) The final distributed GHZ state after Step 13.

Protocol 2 GHZ state from Bell pairs in a star topology

Input: $k \in \{2, 3, \dots\}$ Bell pairs $|\Phi\rangle_{A_i B_i}$, $k + 1$ nodes in a star topology. (The central node shares a Bell pair with each of the k outer nodes.)

Output: GHZ state $|\text{GHZ}_{k+1}\rangle$ shared by all $k + 1$ nodes.

- 1: Apply the CNOT gates $\text{CNOT}_{A_1 A_2}, \text{CNOT}_{A_1 A_3}, \dots, \text{CNOT}_{A_1 A_k}$ on the qubits in the central node.
- 2: Measure the qubits $A_2, A_3, \dots, A_k$ in the Z-basis $\{|0\rangle, |1\rangle\}$.
- 3: Communicate the outcome of the measurement on qubit A_i to the qubit B_i , $i \in \{2, 3, \dots, k\}$. For every $i \in \{2, 3, \dots, k\}$: if the outcome communicated by A_i is 0, then nothing is done on qubit B_i ; if the outcome communicated by A_i is 1, then apply the Pauli-X gate to qubit B_i .

Protocol 3 Fusion of GHZ states in a tree topology

Input: GHZ states in a tree configuration, with a root node identified.

Output: GHZ state shared by all of the nodes.

- 1: Apply a CNOT gate at every node except for the root node and the leaf nodes. The source qubit is in a GHZ state with qubits in the next highest level of the tree, and the target qubits is in a GHZ with qubits in the next lowest level of the tree.
- 2: Measure the target qubits Pauli-Z basis $\{|0\rangle, |1\rangle\}$. The outcomes are communicated to all nodes.
- 3: For every leaf node of the tree, apply the Pauli-X correction $X^{z_1 \oplus z_2 \oplus \dots \oplus z_{\ell-1}}$, where $z_1, z_2, \dots, z_{\ell} \in \{0, 1\}$ are the measurement outcomes obtained at the nodes along the path going from the leaf node to the root node.

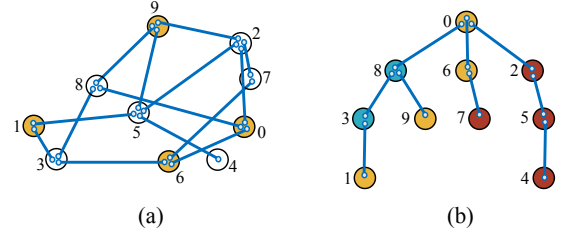


FIG. 4. An example of using Protocol 4 to determine a connected subgraph containing a desired subset of nodes. (a) The original network, with the desired nodes highlighted in yellow, forming the set $\text{DN} = \{0, 1, 6, 9\}$. (b) The nodes highlighted in red are not necessary and can be deleted, while the nodes highlighted in blue are necessary to have a connected subgraph. The resulting subgraph G consists of the desired yellow nodes and the blue nodes.

Protocol 4 Connected subgraph generation

Input: Network of Bell pairs, represented by a graph G with N nodes, and a list of desired nodes, DN .

Output: A connected subgraph of G containing all of the nodes in DN .

- 1: Generate the subgraph induced by DN , call it G' . This subgraph may not be connected; if it is connected, output G' .
- 2: Randomly choose a node from DN to serve as the root node. Using a breadth-first search (BFS) algorithm [66], create a tree-like structure starting from the root node by visiting its first-level (i.e., nearest) neighbors, then the second-nearest neighbors, and so on, without repeating any nodes (see Fig. 4).
- 3: Iteratively delete leaf nodes (red nodes in Fig. 4) from the BFS tree unless they are part of the set of desired nodes (DN). Continue this process until all remaining leaf nodes belong to DN , and no further removals are possible.
- 4: Construct a subgraph using nodes from BFS tree (yellow and blue nodes from Fig. 4) and corresponding edges from G .

A. Remarks

1. The graph G in Protocol 1 can be arbitrary, as long as it is connected, i.e., if there exists a path connecting every pairs of vertices in G . Note that because we are in the LOCC setting, an LOCC protocol can create a GHZ among all nodes of G only if G is connected.
2. In our protocols, the locations of the star creations are not unique, in part because the highest-degree nodes (selected in Step 2 of the protocol) need not be unique and could be chosen randomly.
3. Protocol 1 requires a number of Bell pairs equal to the total sum of Bell pairs across all the stars in the list MSG from Step 12. The fusion protocols do not require any additional Bell pairs. It is also sufficient to use a number of Bell pair sources that is equal to the number of elements in the list MSG. We consider the number of Bell-pair sources in more detail in Sec. III D.
4. In any given network, Protocols 1 and 4 need not be executed in real time, they can be executed offline. The resulting information about where the local GHZ states should be created, and where the fusions of these local GHZ states should take place, can then be conveyed to all nodes in the network, such that the desired GHZ state (in either the complete case or the subset case) can be created.
5. Protocol 1 will produce a tree. If we are in the complete case, then this is a minimum spanning tree for G . In the subset case, it will be a minimum spanning tree corresponding to the subgraph containing the desired nodes, as generated by Protocol 4. However, we emphasize that while our algorithm produces a tree, much like prior algorithms, ours is not just any tree — it features a specific structure aimed at minimizing the number of Bell pair sources required for creation of the GHZ state. We elaborate on this point throughout Sec. III.
6. If the given network has one node with a degree equal to $N - 1$ —that is, a node connected to all other nodes—then the protocol finishes in one step.

B. Classical communication cost

Protocol 2 requires $k - 1$ bits of classical communication, where k is the number of nodes in the star graph. In Protocol 3, the number of bits is determined as follows. We first note that all correction operations in Protocol 3 are done by the leaf nodes only — the intermediate internal nodes (i.e., non-leaf and non-root nodes) do not have to apply correction operations. Every internal node performs a single fusion operation, consisting of the CNOT gate followed by a Pauli-Z measurement, and the measurement outcome is communicated downward in the tree to its corresponding leaf nodes. Every leaf node, v_ℓ , therefore receives L_{v_ℓ} bits, where L_{v_ℓ} is the number of internal nodes along the path from v_ℓ to the root node. This means that the total classical communication cost of Protocol 3 is $\sum_\ell L_{v_\ell}$, where we sum over all leaf nodes. In total, the classical

communication cost of Protocol 1 is the total number of bits needed to create the local GHZ states from the set MSG using Protocol 2 combined with the total number of bits needed in Protocol 3 to fuse the local GHZ states into the overall GHZ state shared by the desired nodes.

C. Creation of arbitrary graph states

While our primary focus is on creating GHZ states, it is possible to use our protocol to create other graph states. A first strategy is one based on Ref. [49]. Specifically, from a GHZ state shared by all of the nodes, we can create the complete graph (all-to-all connected graph) via local complementation. Then, if a desired graph state can be created from the complete graph by deleting vertices and their corresponding edges, implemented via Z-basis measurements, or via other local operations on the complete graph state, then such states can be generated by first using Protocol 1 to create a GHZ state in the network and then performing the appropriate local operations. This strategy can be enhanced with additional qubits, as described in Ref. [49], in order to create the so-called edge-decorated complete graph using GHZ states in order to generate an arbitrary graph state. Alternatively, instead of proceeding via the GHZ state, we can modify Protocol 1 by replacing Protocol 2 therein with the Bipartite B protocol of Ref. [40] (also described in App. C), and then perform the appropriate fusion operations similar to those in Protocol 3. In this case, determining the appropriate fusion operations in order to achieve the desired graph state is non-trivial, depending in general on both the desired graph state and the network topology, and thus we leave this as an interesting direction for future work.

III. RESULTS

We now evaluate our protocol with respect to time complexity (Sec. III A), number of gates (Sec. III B), number of Bell pairs (Sec. III C), and number of Bell pair sources (Sec. III D). We assume that all gates have equal and unit efficiencies, and that the Bell pairs are distributed without losses and with unit fidelity. We present numerical results in Sec. IV. We provide a detailed analysis of our protocol in the presence of noise in Sec. V.

A. Time complexity

We now present results on the time complexity of our protocols in the worst-case scenario, focusing solely on the classical steps and neglecting the complexity associated with implementing quantum operations.

Theorem 1. The time complexity of Protocol 1 is $O(N^2)$, where N is the total number of nodes in the graph.

Proof. Protocol 1 consists of three main components:

1. Picking stars, which contains two elements: (i) Computing the degree distribution table, which takes $O(N + E)$ time steps [66]; (ii) Sorting the degree distribution in decreasing order takes $O(N \log N)$ time steps, using standard comparison-based sorting algorithms such as mergesort or heapsort [66].
2. Modifying stars, which takes $O(E + K)$ time steps, where K is the total number of stars, because we visit all of the nodes for each star to remove extra common nodes. The number of nodes visited in each star corresponds to its degree plus one. Consequently, the total number of nodes visited across all stars is $E + K$.
3. Fusions, which take $O(N \cdot K)$ time steps, because N nodes are visited for each of the $K - 1$ fusion operations.

Putting everything together, the total time complexity is $O(N + E + N \log N + E + K + N \cdot K)$. Since a graph can have at most $E = O(N^2)$ edges (as in the case of a complete graph), and because $K = O(N)$, we conclude that the total time complexity is $O(E) = O(N^2)$. ■

Theorem 2. The time complexity of Protocol 4 is $O(N + E) = O(N^2)$, where N is the total number of nodes in the graph.

Proof. Protocol 4 consists of two main components: (1) BFS traversal, which has a complexity of $O(N + E)$ [66], because it visits all N nodes and E edges; (2) leaf node removal, with a complexity of $O(N)$, as it involves going through every node in the BFS tree one time. Combining these, the total time complexity is $O(N + E + N) = O(N + E)$. In a general graph, the number E of edges is at most $O(N^2)$ (as in the case of a complete graph), so the overall time complexity can be simplified to $O(N^2)$. ■

To create a GHZ state on a subset of nodes, Protocol 4 should be executed first, followed by Protocol 1. Since each protocol is applied exactly once, regardless of the size of initial network or subset, the overall complexity is $O(N^2)$. It is worth pointing out that while polynomial-time approximation algorithms for Steiner trees exist (see, e.g., Refs. [67–69]), the algorithms in Refs. [67, 69] have time complexity $O(S' \cdot N^2)$, where S' is the number of desired nodes, and the widely-implemented algorithm of Ref. [68] has time complexity $O(E + N \log N)$. Our Protocol 4 has a faster time complexity of $O(E + N)$ compared to Refs. [67–69]. At the same time, let us remark that if we are only in the complete case, then we need Protocol 1 only, which produces a minimum spanning tree, and its time complexity of $O(N^2)$ is comparable to known algorithms for minimum spanning trees; see, e.g., Ref. [70]. However, while the minimum spanning tree would be enough if we were concerned only with the number of Bell pairs, as we detail in Sec. III D below, the advantage of Protocol 1 becomes clear when considering the number of Bell-pair sources as a figure of merit in addition to the number of Bell pairs.

B. Gate cost

To determine the gate cost of our protocol, we start by counting the number of gates needed in Protocols 2 and 6.

1. Protocol 2 requires $n - 1 = N - 2$ gates, where n is the number of Bell pairs input to the protocol and $N = n + 1$ is the number of nodes involved in the protocol.
2. Protocol 6 requires one gate, independent of the sizes of the GHZ states being merged.

Note that we do not consider the Pauli correction gates in the gate count because in near-term hardware, experiments show that two-qubit gates are typically one or two orders of magnitude noisier than single-qubit gates [71].

Theorem 3 (Number of gates: complete case). The gate cost of Protocol 1 in the complete case is $N - 2$ gates, where N is the total number of nodes in the network.

Proof. Let us assume there are a total of N nodes and these are distributed in k GHZ states of different sizes. The number of nodes in each GHZ state is given by $n_1, n_2, n_3, \dots, n_k$. The number of gates required to create these GHZ states is given by (according to point 1 above) $(n_1 - 2), (n_2 - 2), (n_3 - 2), \dots, (n_k - 2)$. Adding up all of these gives us $n_1 + n_2 + n_3 + \dots + n_k - 2k$. As there are k GHZ states, there will be $k - 1$ nodes that are repeated one time. Therefore, the total number of nodes in all of the small GHZ states should be $N + k - 1$, i.e., $n_1 + n_2 + n_3 + \dots + n_k = N + k - 1$, which means that the gate cost is $N + k - 1 - 2k = N - k - 1$. Now, let us add the gates for the merging of the GHZ states (explained in point 2 above). As there are k GHZ states, there will be $k - 1$ merges in total, and each merge requires one gate (see point 2 above), which means that the total number of gates required is $N - k - 1 + k - 1 = N - 2$. ■

Theorem 4 (Number of gates: subset case). In the subset case, the gate cost of our protocol is $S - 2$ gates where S is the total number of nodes in the subgraph (the subgraph obtained using Protocol 4).

Proof. The proof is analogous to the proof of Theorem 3. ■

It is notable here that the number of gates depends *only* on the number of nodes in the subgraph—no specific properties of the structure/topology/connectivity of the graph matters, except for the fact that it should be connected. In particular, this means that if we take any fraction $f \in (0, 1]$ of the nodes N of our network to comprise our subset, then the size S of the subgraph containing these nodes satisfies $S \geq Nf$, which means that the number of gates is bounded from below as $S - 2 \geq Nf - 2$.

Comparison to Ref. [49]. The gate cost of the star expansion protocol of Ref. [49] is as follows. If k is the degree of the node on which the star expansion protocol is performed, then:

- If the central node is to be kept, then the number of gates is $g_1(k) := \binom{k}{2} + 2k - 1 = \frac{1}{2}k^2 + \frac{3}{2}k - 1$.
- If the central node is to be discarded, then the number of gates is $g_2(k) := \binom{k}{2} + k = \frac{1}{2}k^2 + \frac{1}{2}k$.

Using this, we can establish the following fact about the number of gates required to create a GHZ state according to the protocol in Ref. [49].

Proposition 5 (Upper bound on gate cost of the MMG protocol [49]). For any graph with N nodes, the number of gates required to create a GHZ state among all N nodes, according to the MMG protocol, scales as $O(N^2)$.

Proof. The MMG protocol of Ref. [49] proceeds by starting with a minimum spanning tree for the graph. Then, an arbitrary leaf node of the tree is chosen, and the star expansion protocol is applied successively on non-leaf neighbors of the chosen leaf node until all of the nodes are included in the GHZ state. If the number of non-leaf neighbors is L , then the gate cost is $\sum_{i=1}^L g_1(k_i)$. We can bound this from above by summing over all nodes, i.e.,

$$\sum_{i=1}^L g_1(k_i) \leq \sum_{i=1}^N g_1(k_i) \quad (1)$$

$$= \frac{1}{2} \sum_{i=1}^N k_i^2 + \frac{3}{2} \sum_{i=1}^N k_i - N \quad (2)$$

$$\leq \frac{1}{2} \left(\sum_{i=1}^N k_i \right)^2 + \frac{3}{2} \cdot 2(N-1) - N \quad (3)$$

$$= \frac{1}{2} (2(N-1))^2 + 2N - 3 \quad (4)$$

$$= 2N^2 - 2N - 1 \quad (5)$$

$$= O(N^2), \quad (6)$$

where we have made use of the fact that $\sum_{i=1}^N k_i = 2(N-1)$, i.e., the sum of the degrees is equal to twice the number of edges, and the number of edges in the minimum spanning tree is necessarily $N-1$. ■

Let us now also observe that we can significantly reduce the gate cost of the star expansion Protocol by noticing that the end result of the protocol is a GHZ state, either with or without the central node. If the central node is to be excluded, then by the Bipartite B protocol (see App. C), the gate cost is $k-1$. If the central node is to be included, then the gate cost is also $k-1$, using the protocol in App. D. This represents a quadratic reduction in gate cost, from $O(k^2)$ (as established in Ref. [49]) to $O(k)$.

Comparison to Ref. [42]. The lower bound on gate count in [42] is reached when a single node is connected to all other nodes in the graph, resulting in $N-2$ gates—consistent with our protocol’s gate count. However, as we prove below, the upper bound in [42] scales as $O(N^2)$. Therefore, our protocol generally achieves a lower gate count, except in the specific scenario where one node connects to all others, in which case our protocol matches the gate count in [42].

Proposition 6 (Upper bound on gate cost in Ref. [42]). For a graph with N nodes, the upper bound on the number of gates required to create a GHZ state across all N nodes, as specified by the protocol in Ref. [42], scales as $O(N^2)$.

Proof. Consider a chain topology for the network, with nodes labeled from 1 to N from left to right. Suppose that the root node is node 1. In this setup, the protocol of Ref. [42] requires “connection transfers” from each node to node 1, with each transfer utilizing one gate. The farthest node, node N , requires $N-2$ connection transfers to reach node 1. The next farthest node, node $N-1$, requires $N-3$ connection transfers, and so on. Consequently, the total number of gates used is given by $(N-2) + (N-3) + (N-4) + \dots + (N-N) = \sum_{j=2}^N (N-j) = \frac{N(N-3)}{2} + 1 = O(N^2)$. ■

C. Bell-pair cost

Theorem 7 (Number of Bell pairs used: complete case). For the complete case of our protocol, the number of Bell pairs used is equal to $N-1$. This matches the lower bound of $N-1$, which is the minimum number of Bell pairs needed to create an arbitrary graph state.

Proof. The proof consists of two parts. First, we establish that the lower bound is $N-1$; second, we show that our protocol achieves this bound.

To begin, the network must be connected, as it is otherwise impossible to generate entanglement among all nodes in the LOCC setting. The minimum number of edges required to connect a graph of N nodes is $N-1$, which implies that at least $N-1$ Bell pairs are needed. This gives us the lower bound of $N-1$ Bell pairs.

To minimize the number of Bell pairs, a single common node between any two stars is necessary and sufficient for merging. If more than one common node is present, additional Bell pairs are consumed. Therefore, in step 10 of Protocol 1, we modify the stars to ensure that only one common node connects with the rest. This ensures that we require only $N-1$ Bell pairs to create the GHZ state. ■

Let us now examine the number of Bell pairs in the subset case by introducing the concept of the graph diameter. The diameter of an unweighted graph is defined as the length of the longest shortest path between any two nodes. In simpler terms, it is the greatest distance, measured by the number of edges, between any pair of nodes. Here, the shortest path refers to the path connecting two nodes with the fewest edges.

Theorem 8 (Number of Bell pairs used: subset case). Let S be the size of the subgraph containing the desired subset of nodes, as generated from Protocol 4. The number of Bell pairs needed to generate a GHZ state is bounded from below by $S-1$ and bounded from above by $d_S(S-1)$, where d_S is the diameter of the connected subgraph containing the desired subset of nodes, as generated from Protocol 4.

Proof. For the lower bound, the minimum number of Bell pairs used is determined analogous to the complete case, resulting in $S-1$.

For the upper bound, the maximum number of Bell pairs will be used when all the nodes in the subset are maximally distant from each other. By definition, the diameter gives the

maximum distance in terms of the number of Bell pairs. Since there are S maximally distant nodes, the maximum number of Bell pairs used is $d_S(S - 1)$. ■

Comparison to Refs. [42, 49]. In the complete case, for the MMG protocol [49], the number of Bell pairs required is $N - 1$, as it corresponds to the size of the minimum spanning tree. This is the same number of Bell pairs used in our protocol (Theorem 7). However, the approach in Ref. [42] requires additional Bell pairs to connect the desired node to every other node, unlike our protocol and the MMG protocol. In the subset case, the number of Bell pairs needed in the MMG protocol is equal to the size of the tree, which will generally exceed $N_S - 1$, where N_S is the number of nodes in the subset. This is because the Steiner tree will typically include more nodes than just the terminal nodes. In Ref. [42], the same logic applies as in the complete case.

D. Bell-pair source cost

It is commonly assumed that a Bell pair source must be placed at the midpoint of each utilized edge. Ideally, this allows all Bell pairs to be distributed simultaneously in a single time step. However, sources such as quantum dots can generate Bell pairs at very high rates (GHz range) [72]. This opens up an alternative approach to designing the placement of Bell pair sources. Instead of placing a source on every edge, one can place sources at selected nodes, and then have each source generate and distribute Bell pairs to its neighbors. While this introduces a small overhead due to sequential pair generation, it does not significantly affect timing thanks to the high repetition rates. Furthermore, and importantly, the number of edges in a network scales quadratically with the number of nodes, while the number of nodes grows linearly. Therefore, placing sources at the nodes rather than at edges can offer a significant advantage in terms of reducing the total number of sources required. With careful placement, a small number of node-based sources can efficiently cover the entire network. Now the key question becomes: given a network topology, what is the minimal number of Bell pair sources required? As we now show, this optimization problem is equivalent to the well-known minimum dominating set problem.

Theorem 9. The problem of determining the minimum number of Bell pair sources required to cover all nodes in a quantum network is equivalent to the minimum dominating set problem on the network graph.

Proof. Let the quantum network be represented by an undirected graph $G = (V, E)$, where we recall that each node $v \in V$ is a quantum node (e.g., a repeater or memory station), and each edge $\{u, v\} \in E$ represents a quantum channel capable of distributing a Bell pair between adjacent nodes.

A Bell pair source placed at a node $s \in V$ can generate a Bell pair locally and distribute one qubit to itself and the other to any one of its neighbors via the quantum channel. Thus, a node $v \in V$ is said to be *covered* if it either hosts a source or is adjacent to a node that hosts a source.

We define a set $D \subseteq V$ of nodes that host Bell pair sources. Then, the set of covered nodes is given by

$$\bigcup_{v \in D} (\{v\} \cup N(v)), \quad (7)$$

where $N(v)$ denotes the set of neighbors of node v . The objective is to find the smallest such set D such that this union equals V , i.e.,

$$\bigcup_{v \in D} (\{v\} \cup N(v)) = V. \quad (8)$$

This is precisely the definition of a *dominating set* in graph theory: a set $D \subseteq V$ such that every node $u \in V$ is either in D or adjacent to a node in D . Therefore, finding the minimum number of Bell pair sources required to cover the network is equivalent to finding a minimum dominating set in G . Specifically, the *domination number* $\gamma(G) := \min \{|D| : D \text{ is a dominating set of } G\}$ is a lower bound on the minimum number of Bell-pair sources in the network. ■

Since the minimum dominating set problem is known to be NP-hard [63], the Bell pair source placement problem inherits this computational complexity. However, a simple structural observation provides a useful upper bound on the number of sources required. In any connected graph, each leaf node (a node of degree one) is connected to exactly one neighbor, which must be an internal node (i.e., node with degree greater than one). We begin by placing a source at every internal node. By placing sources on all internal nodes, we ensure that every leaf node is adjacent to a source. Since a source can distribute Bell pairs to all its neighbors, each leaf node can receive a Bell pair from its connected internal node. Therefore, placing a source at a leaf node is unnecessary. Thus, placing sources only at all internal nodes suffices to cover the graph, giving a simple upper bound on the minimum number of Bell pair sources required. We emphasize that this is generally a better upper bound compared to placing the sources at the midpoints of the edges, based on the reasoning presented at the beginning of this subsection.

IV. PERFORMANCE ON RANDOM NETWORKS

To evaluate our protocol, i.e., calculate the gate and Bell pairs costs, we implemented our algorithms and performed Monte Carlo simulations on various types of random networks [73, 74]. We made use of the Python package NetworkX [75], and for finding Steiner trees, we applied Mehlhorn's approximate algorithm [68].

A. Random network models

Let us first briefly describe the three random network models that we consider.

Erdős–Rényi model. The Erdős–Rényi (ER) random network ensemble, commonly referred to as the “ $G(N, p)$ model”, is defined as follows [54, 55]:

- $N \in \{1, 2, \dots\}$: The number of nodes in the graph.
- For every pair of (distinct) nodes, add an edge between them with probability $p \in [0, 1]$.

In general, graphs in the ER ensemble will not be connected, particularly for small values of p . However, in order to ensure that all nodes of the network are included in the final GHZ state, we require the graph to be connected. To address this, we introduce a slight modification to ensure the graph remains connected. The modification ensures connectivity by initially adding a random edge for each node, guaranteeing at least one connection, and then adding additional edges based on the probability p .

Barabási–Albert model. The Barabási–Albert (BA) model [59] generates scale-free networks, which are characterized by a power-law degree distribution. In such scale-free networks, a few nodes (hubs) have many connections, while most nodes have only a few. As such, it represents a realistic model of the internet. The construction of a random graph in the BA model is done as follows. We let $c \in \{1, 2, \dots\}$, and we suppose that the total number N of nodes satisfies $N \geq c + 1$. Then:

1. Start with the complete graph (i.e., all-to-all connected graph) of $c + 1$ nodes.
2. Until the total number N of nodes is reached, add a new node with c edges that links the new node to c different nodes already present in the network. The probability that a new node connects to an existing node i with degree k_i is given by the so-called “preferential attachment” model, in which nodes of higher degree are more likely to receive a new edge. Specifically, the probability is $k_i / \sum_j k_j$, where the sum in the denominator is over all existing nodes j in the network.

Observe that the parameter c is the minimum possible degree of a node. In other words, every node in every graph generated according to this model has degree at least c . For the sake of comparison with the ER model, we consider values of c given by some fraction of N , i.e., $c = \lceil Np \rceil$, where $p \in [0, 1]$.

Waxman model. In order to analyse our protocol on more realistic networks, namely, photonic networks based on fiber-optic links, we consider the Waxman model [56]. The Waxman model incorporates aspects of spatial and geographical proximity by using random geometric graphs [76] with a particular distance-dependent probability distribution for the edges. Graphs in this model are generated as follows:

- Nodes are distributed uniformly at random over a circle of diameter D .
- The probability of creating an edge between any two nodes u and v depends on their Euclidean distance $d(u, v)$. Specifically, the probability $P(u, v)$ of an edge between two nodes u and v is given by $P(u, v) =$

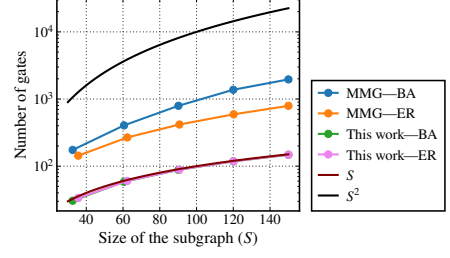


FIG. 5. Number of gates versus subgraph size for Erdős–Rényi and Barabási–Albert networks with up to 500 nodes. We consider a fraction $f = 0.3$ of nodes to be entangled, and we set $p = 0.05$.

$\beta \exp[-d(u, v)/L_0]$, where $d(u, v)$ is the distance between the nodes, L_0 is the attenuation length, and $\beta \in (0, 1]$ is a parameter that controls the density of the graphs.

As with the ER model, this model does not guarantee a connected graph. To address this, we generate data by varying the diameter D and focusing exclusively on creating a GHZ state among the nodes in the largest connected component of the graph.

B. Results and discussion

For our Monte Carlo simulations, we considered network sizes ranging from 100 to 500 nodes for each model, with 500 samples used for each network size.

In Fig. 5, we plot the gate cost as a function of the size S of the subgraph. We can see that the gate cost of our protocol agrees with the theoretical bounds discussed in Sec. III, specifically Theorem 4. Similarly, the gate cost of the MMG protocol [49] agrees with the upper bound in Proposition 5.

In addition to understanding how the gate cost scales with the number of nodes (as presented in Fig. 5), we are interested in understanding how the gate cost depends on graphical properties of the networks. In Fig. 6, we plot the gate cost as a function of the parameter p characterizing the ER and BA network models. In both models, the parameter p gives an indication of the density of the network, with larger values of p indicating higher density. Intuitively, as the density increases, we might expect the gate cost to decrease; however, as we can see in Fig. 6 (left), for the MMG protocol the gate cost actually *increases* with p . In contrast, our protocol maintains a nearly constant number of gates because the size of the subgraph does not change beyond a certain value of p , as we can see in Fig. 6 (right). In particular, beyond that value of p , the connected subgraph obtained from Protocol 4 contains only those nodes that are to be entangled, with no additional nodes.

In Fig. 7, we consider the Waxman model, which incorporates the distance between nodes to better reflect realistic networks. We set β to unity, and we set the attenuation length between any two nodes to be $L_0 = 22$ km. Because the Waxman model does not always produce a connected graph, we focus on the largest connected component when analyzing this model. In

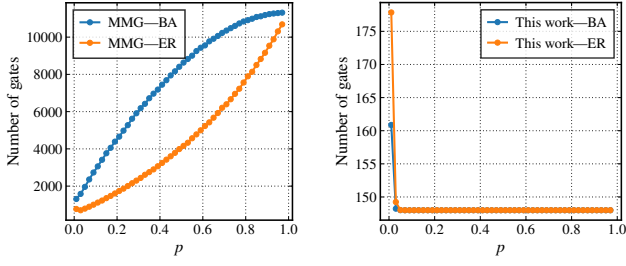


FIG. 6. Gate cost versus p for Erdős-Rényi and Barabási-Albert networks with 500 nodes and a fraction $f = 0.3$ of nodes to be entangled.

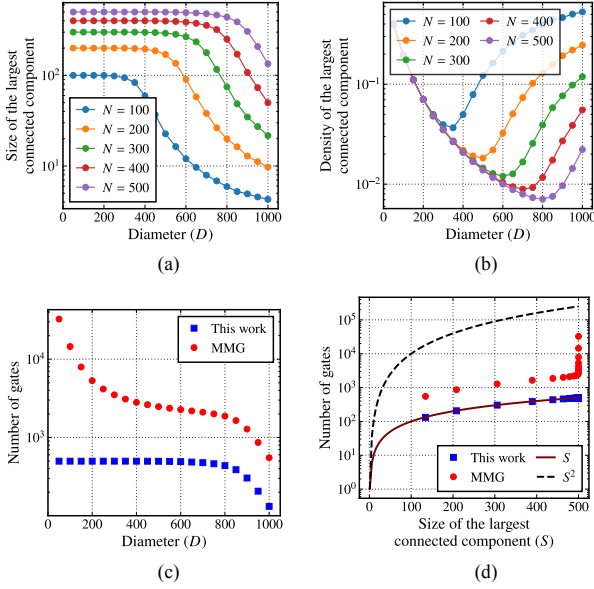


FIG. 7. Results for the Waxman model with $\beta = 1$ and $L_0 = 22$ km. (a) and (b): size and density of the largest connected component as a function of diameter. (c): Gate cost as a function of diameter for networks with 500 nodes. (d): Gate cost as a function of the size of the largest connected component.

this model, the largest connected component undergoes a phase transition with respect to diameter; see Fig. 7(a). This phase transition is also reflected in the change of density of the largest connected component, as shown in Fig. 7(b). In Fig. 7(c), for networks with 500 nodes, we can also see the effect of the phase transition on the gate cost of our protocol. Finally, in Fig. 7(d), we show how the gate cost varies with the size of the largest connected component. As with Fig. 5, the results are in agreement with the analytical results of Sec. III B. We observe multiple values for the gate cost when $S = 500$, which reflects the dependence of the gate cost of the MMG protocol [49] on the density of the graph.

Let us now consider the number of consumed Bell pairs. In particular, it is of interest to know, particularly in the subset case, how close to optimal is the number of Bell pairs consumed using Protocol 4 compared to using the Steiner tree. In Fig. 8, we plot the ratio of the size of the subgraph to the size of

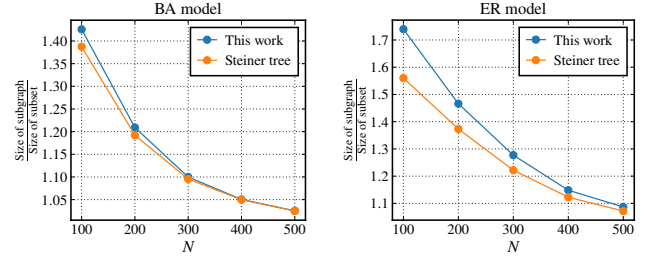


FIG. 8. Ratio of the subgraph size to the number of nodes to be entangled, as a function of the total number of nodes, for both the Steiner tree algorithm [68] and our Protocol 4. We consider the Erdős-Rényi and Barabási-Albert models with $p = 0.05$ and a fraction $f = 0.1$ of nodes to be entangled.

the subset (i.e., the number of nodes to be entangled). This ratio tells us how many additional nodes are needed in the subgraph generated by our Protocol 4 compared to the Steiner tree subgraph (generated using the protocol of Ref. [68]). We can see that for both the BA and ER models, the ratio of our subgraph is close to that of the Steiner tree, implying the number of consumed Bell pairs is also close. In particular, the ratios appear to be getting closer as the total number N of nodes increases, indicating that our protocol may be optimal asymptotically.

Finally, we assess the number of Bell-pair sources required by our protocol; see Fig. 9. From Sec. III D, we know that the size of a minimum dominating set (i.e., the domination number) provides a lower bound on the number of Bell-pair sources, while the number of internal nodes provides an upper bound. In Fig. 9, we compare these bounds for the trees generated by our Protocol 1 and those produced by a Minimum Spanning Tree (MST) of the same network. We also include the number of stars, that is, the number of elements in MSG, from Protocol 1, since MSG captures the list of stars that form the entire network (see Fig. 3, panel (5)), and one source at the center of each star is sufficient. Importantly, computing the number of elements in MSG requires no additional effort; it is a direct byproduct of executing our Protocol 1, unlike computing a minimum dominating set or identifying internal nodes. Since finding a Minimum Weight Dominating Set (MWDS) is NP-hard, we employ the approximation algorithm from [62] in our simulations. In our setting, all nodes are assigned equal weight (set to 1), so the MWDS problem reduces to the classic Minimum Dominating Set (MDS) problem. Notably, we find that the lowest number of Bell-pair sources is achieved only when the MWDS algorithm is applied to the graph generated by Protocol 1; applying MWDS directly to the MST does not yield better results (see the red and purple curves in Fig. 9). Finally, we observe that Barabási-Albert networks require fewer Bell-pair sources than Erdős-Rényi networks. Interestingly, as the network size increases in the Barabási-Albert model, the gap between our protocol and the MST approach also widens.

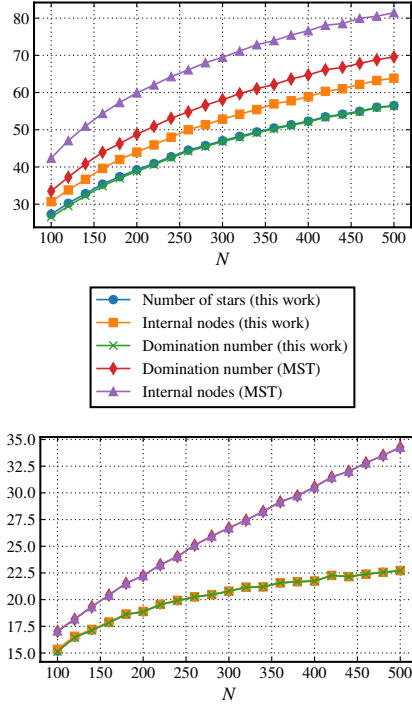


FIG. 9. The number of Bell pair sources needed as a function of the number of nodes (N) for the Erdős-Rényi model (top) and Barabási-Albert model (bottom) with $p = 0.05$. We consider the complete case (GHZ among all the nodes) using Protocol 1 of this work, and we compare it to counting the internal nodes and the (approximate) domination number of the tree produced by both Protocol 1 and the minimum spanning tree (MST).

V. PERFORMANCE WITH NOISE

How does our protocol perform in the presence of noise? In this section, we consider the case that both the Bell pairs, the locally-created GHZ states, and the local gates are noisy. We present formulas for the fidelity of the GHZ state after Protocol 2 and Protocol 6.

Theorem 10 (Fidelity after Protocol 2 with noisy Bell pairs). If in Protocol 2 the Bell pairs are noisy and given by arbitrary two-qubit density operators $\rho_{A_1 B_1}^{(1)}, \rho_{A_2 B_2}^{(2)}, \dots, \rho_{A_n B_n}^{(n)}$, then the fidelity of the state after Protocol 2 with respect to the target GHZ state is given by

$$\sum_{\vec{z} \in \{0,1\}^{n-1}} \mu_{z'}^{(1)} \mu_{z_1}^{(2)} \mu_{z_2}^{(3)} \dots \mu_{z_{n-1}}^{(n)}, \quad (9)$$

where for every $\vec{z} = (z_1, z_2, \dots, z_{n-1}) \in \{0, 1\}^{n-1}$, we have let $z' = z_1 \oplus z_2 \oplus \dots \oplus z_{n-1}$, and $\mu_z^{(i)} = \langle \Phi^{z,0} | \rho^{(i)} | \Phi^{z,0} \rangle$ for $z \in \{0, 1\}$. We have also made use of the Bell state vectors

$$|\Phi^{z,x}\rangle := (Z^z X^x \otimes \mathbb{1})|\Phi\rangle, \quad (10)$$

for $z, x \in \{0, 1\}$.

Proof. See App. D 1. ■

We emphasize that the expression in (9) holds for noisy Bell pairs with an arbitrary noise model for each Bell pair. Also, the expression in (9) involves not only the fidelity of the states, given by $\langle \Phi^{0,0} | \rho_{A_i B_i}^{(i)} | \Phi^{0,0} \rangle$, but also the overlap of the states with respect to $|\Phi^{1,0}\rangle = \frac{1}{\sqrt{2}}(|0, 0\rangle - |1, 1\rangle)$. Also, while a noise analysis was performed in Ref. [40], the model considered therein was depolarizing noise only. Our results hold for arbitrary noise models, which allows for a more precise and meaningful analysis for specific qubit architectures.

To demonstrate the usefulness of Theorem 10, suppose that the Bell pairs are subjected to the same noise, meaning that $\mu_z = \langle \Phi^{z,0} | \rho_{A_i B_i}^{(i)} | \Phi^{z,0} \rangle$ for all $i \in \{1, 2, \dots, n\}$, $z \in \{0, 1\}$. Then, it is straightforward to show that the fidelity in (9) is equal to $\frac{1}{2}((\mu_0 - \mu_1)^n + (\mu_0 + \mu_1)^n)$. Now, given a desired fidelity F for the final GHZ state and a number n of noisy Bell pairs impacted by the same type of noise, how much noise can be tolerated while achieving a fidelity of at least F after Protocol 2? Using Theorem 10, we can simply evaluate the condition $\frac{1}{2}((\mu_0 - \mu_1)^n + (\mu_0 + \mu_1)^n) \geq F$ for any given noise model. Letting the noisy Bell pair be given by $\rho_{AB} = (\text{id} \otimes \mathcal{N})(\Phi_{AB}^{0,0})$, where $\mathcal{N}$ is the quantum channel describing the noise, we obtain the following results for different choices of $\mathcal{N}$.

- *Depolarizing noise:* $\mathcal{N}(\cdot) = (1-p)(\cdot) + \frac{p}{3}(X(\cdot)X + Y(\cdot)Y + Z(\cdot)Z)$, where $p \in [0, 1]$ is the depolarizing parameter and X, Y, Z are the single-qubit Pauli operators. The corresponding noisy Bell pair is given by $\rho_{AB} = (1-p)\Phi_{AB}^{0,0} + \frac{p}{3}(\Phi_{AB}^{1,0} + \Phi_{AB}^{0,1} + \Phi_{AB}^{1,1})$, where $\Phi_{AB}^{z,x} = |\Phi^{z,x}\rangle\langle\Phi^{z,x}|$. We have that $\mu_0 = 1-p$ and $\mu_1 = \frac{p}{3}$, so our condition is

$$\left(1 - \frac{4p}{3}\right)^n + \left(1 - \frac{2p}{3}\right)^n \geq 2F, \quad (11)$$

which can be solved numerically to get an upper bound on p ; see Fig. 10.

- *Dephasing noise:* $\mathcal{N}(\cdot) = (1-q)(\cdot) + qZ(\cdot)Z$, where $q \in [0, 1]$ is the dephasing parameter. The corresponding noisy Bell pair is given by $\rho_{AB} = (1-q)\Phi_{AB}^{0,0} + q\Phi_{AB}^{1,0}$. In this case, $\mu_0 = 1-q$ and $\mu_1 = q$, so our condition is

$$(1-2q)^n + 1 \geq 2F \quad (12)$$

$$\Rightarrow q \leq \frac{1}{2}(1 - (2F-1)^{1/n}). \quad (13)$$

- *Amplitude damping noise:* $\mathcal{N}(\cdot) = K_1(\cdot)K_1^\dagger + K_2(\cdot)K_2^\dagger$, where $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$, $K_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$, and $\gamma \in [0, 1]$ is the damping parameter. The corresponding noisy Bell pair is given by $\rho_{AB} = \frac{\gamma}{2}|1, 0\rangle\langle 1, 0| + (1 - \frac{\gamma}{2})|v\rangle\langle v|$, where $|v\rangle = \frac{1}{\sqrt{2-\gamma}}(|0, 0\rangle + \sqrt{1-\gamma}|1, 1\rangle)$. In this case, we have $\mu_0 = \frac{1}{4}(1 + \sqrt{1-\gamma})^2$ and $\mu_1 = \frac{1}{4}(1 - \sqrt{1-\gamma})^2$, which leads to the condition

$$(1-\gamma)^{n/2} + \left(1 - \frac{\gamma}{2}\right)^n \geq 2F, \quad (14)$$

which can also be solved numerically to get an upper bound on γ ; see Fig. 10.

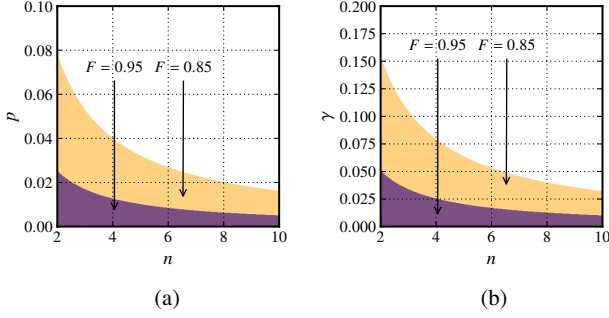


FIG. 10. Noise tolerance of Protocol 2 when the output fidelity of the protocol should be at least $F = 0.85, 0.95$, as a function of the number n of Bell pairs. Shaded regions indicate the values of n and the noise parameter for which the desired fidelity can be achieved. (a) Depolarizing noise, with the tolerance regions given by the condition in (11). (b) Amplitude-damping noise, with the tolerance regions given by the condition in (14).

Theorem 11 (Fidelity after Protocol 3). Consider Protocol 3 with $m \in \{2, 3, \dots\}$ noisy GHZ states (of arbitrary size). Then, the fidelity of the state after the protocol with respect to the target GHZ state is given by

$$\sum_{\vec{z} \in \{0,1\}^{m-1}} \lambda_{z_1}^{(1)} \lambda_{z_2}^{(2)} \dots \lambda_{z_{m-1}}^{(m-1)} \lambda_{z'}^{(m)}, \quad (15)$$

where $z' = z_1 \oplus z_2 \oplus \dots \oplus z_{m-1}$, $\lambda_z^{(i)} = \langle \text{GHZ}^{(z, \vec{0})} | \rho^{(i)} | \text{GHZ}^{(z, \vec{0})} \rangle$ is the fidelity of the i^{th} state with respect to $|\text{GHZ}^{(z, \vec{0})}\rangle$, $i \in \{1, 2, \dots, m\}$, and

$$|\text{GHZ}_n^{(z, \vec{x})}\rangle := (Z^z \otimes X^{\vec{x}}) |\text{GHZ}_n\rangle, \quad (16)$$

with $z \in \{0, 1\}$ and $\vec{x} \in \{0, 1\}^{n-1}$, is the n -qubit GHZ basis.

Proof. See App. E 4. ■

We note that the fidelity after Protocol 3 depends only on the number of input GHZ states, and not on the star topology of the network. Also, similarly to the fidelity after Protocol 2, the fidelity after Protocol 3 depends only on the diagonal elements of the noisy GHZ states with respect to the GHZ basis. In particular, only the fidelities of the noisy GHZ states with respect to $|\text{GHZ}_n^{(0, \vec{0})}\rangle \equiv |\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ and $|\text{GHZ}_n^{(1, \vec{0})}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} - |1\rangle^{\otimes n})$ are relevant.

Consider fusing $m \in \{2, 3, \dots\}$ noisy GHZ states according to Protocol 3, where each of the noisy GHZ states has been created using Protocol 2. Assuming (for simplicity and illustrative purposes) that all of the (non-leaf) nodes in the tree have the same number $n \in \{2, 3, \dots\}$ of children nodes, and that all of the Bell pairs have the same noise acted upon them, we know from Theorem 10 that the fidelity with respect to $|\text{GHZ}^{(0, \vec{0})}\rangle$ is given by

$$\lambda_0^{(i)} \equiv \lambda_0 = \frac{1}{2}((\mu_0 + \mu_1)^n + (\mu_0 - \mu_1)^n), \quad (17)$$

for all $i \in \{1, 2, \dots, m\}$. By reasoning similar to the proof of Theorem 10, it is possible to show that the fidelity of the state after Protocol 2 with respect to $|\text{GHZ}^{(1, \vec{0})}\rangle$ is

$$\lambda_1^{(i)} \equiv \lambda_1 = \frac{1}{2}((\mu_0 + \mu_1)^n - (\mu_0 - \mu_1)^n), \quad (18)$$

for all $i \in \{1, 2, \dots, m\}$. Therefore, by Theorem 11, the final fidelity with respect to the GHZ state is

$$\sum_{\vec{z} \in \{0,1\}^{m-1}} \lambda_{z_1} \lambda_{z_2} \dots \lambda_{z_{m-1}} \lambda_{z'} \\ = \sum_{\vec{z} \in \{0,1\}^{m-1}} \lambda_1^{|\vec{z}|} \lambda_0^{m-1-|\vec{z}|} \lambda_{z'} \quad (19)$$

$$= \frac{1}{2}((\lambda_0 + \lambda_1)^m + (\lambda_0 - \lambda_1)^m) \quad (20)$$

$$= \frac{1}{2}((\mu_0 + \mu_1)^{nm} + (\mu_0 - \mu_1)^{nm}). \quad (21)$$

This tells us that, for every noise model, the final fidelity depends essentially on the number of CNOT gates performed among all fusion operations, which is nm .

A. Noise in the gates

Now, in addition to noise on the input states of Protocols 2 and 3, we want to consider noise in the local CNOT gates. Recall that the two-qubit CNOT gate is $\text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$, where X is the Pauli- X gate. Let us denote the sequence of CNOT gates in Protocol 2 as

$$\text{CNOT}_n \equiv \text{CNOT}_{A_1 A_n} \text{CNOT}_{A_1 A_{n-1}} \dots \text{CNOT}_{A_1 A_3} \text{CNOT}_{A_1 A_2}, \quad (22)$$

where $n \in \{2, 3, \dots\}$ is the number of qubits. It is straightforward to show that

$$\text{CNOT}_n = |0\rangle\langle 0| \otimes \mathbb{1}^{\otimes(n-1)} + |1\rangle\langle 1| \otimes X^{\otimes(n-1)}. \quad (23)$$

Now, as our model of the noisy CNOT_n gate, let us assume that an arbitrary n -qubit quantum channel $\mathcal{M}$ is applied immediately after the CNOT_n gate, which means that the noisy CNOT_n gate is the quantum channel $(\cdot) \mapsto \mathcal{M}(\text{CNOT}_n(\cdot)\text{CNOT}_n)$.

In order to apply Theorems 10 and 11 while taking gate noise into account, we can attempt to absorb the gate noise into the input states. This would mean finding a channel $\mathcal{M}'$ such that

$$\mathcal{M}(\text{CNOT}_n(\cdot)\text{CNOT}_n) = \text{CNOT}_n \mathcal{M}'(\cdot) \text{CNOT}_n. \quad (24)$$

Because CNOT_n is a unitary, we immediately see that

$$\mathcal{M}'(\cdot) = \text{CNOT}_n \mathcal{M}(\text{CNOT}_n(\cdot)\text{CNOT}_n) \text{CNOT}_n. \quad (25)$$

Letting $\mathcal{M}(\cdot) = \sum_{j=1}^r K_j(\cdot) K_j^\dagger$ be a Kraus representation of $\mathcal{M}$, it follows that

$$\mathcal{M}'(\cdot) = \sum_{j=1}^r \text{CNOT}_n K_j \text{CNOT}_n(\cdot) \text{CNOT}_n K_j^\dagger \text{CNOT}_n \quad (26)$$

$$= \sum_{j=1}^r K'_j(\cdot) K_j'^\dagger, \quad (27)$$

where $K'_j = \text{CNOT}_n K_j \text{CNOT}_n$.

If $\mathcal{M}$ is a Pauli channel, then it can be written as $\mathcal{M}(\cdot) = \sum_{\{1, X, Y, Z\}^{\otimes n}} p_P P(\cdot) P$, where the quantities p_P , $P \in \{1, X, Y, Z\}^{\otimes n}$, form a probability distribution. This implies that the transformed Kraus operators are given by $P' = \text{CNOT}_n P \text{CNOT}_n$. Now, because CNOT_n is a Clifford gate, it holds that $P' \in \{1, X, Y, Z\}^{\otimes n}$ is some other n -qubit Pauli operator. In other words, when the noise is any Pauli channel, the CNOT_n gate can be brought outside of the action of the noise, and can therefore be absorbed into the input states of Protocol 2, while still retaining the Pauli-covariant structure of the noise. This means that Theorem 10 holds even when the CNOT_n gate is impacted by arbitrary Pauli noise.

It is straightforward to determine the form of the transformed Pauli operator P' . Let $P \equiv i^{\vec{x} \cdot \vec{z}} X^{\vec{x}} Z^{\vec{z}}$, where $\vec{x}, \vec{z} \in \{0, 1\}^n$ and $X^{\vec{x}} \equiv X^{x_1} \otimes X^{x_2} \otimes \dots \otimes X^{x_n}$, $Z^{\vec{z}} \equiv Z^{z_1} \otimes Z^{z_2} \otimes \dots \otimes Z^{z_n}$. Then, it holds that $P' = \text{CNOT}_n(i^{\vec{x} \cdot \vec{z}} X^{\vec{x}} Z^{\vec{z}}) \text{CNOT}_n = i^{\vec{x}' \cdot \vec{z}'} X^{\vec{x}'} Z^{\vec{z}'}$, where $\vec{x}' = (x_1, x_1 \oplus x_2, x_1 \oplus x_3, \dots, x_1 \oplus x_n)$ and $\vec{z}' = (z_1 \oplus z_2 \oplus \dots \oplus z_n, z_2, \dots, z_n)$. As an example, let us consider $n = 2$, and assume a tensor-product Pauli noise channel, $\mathcal{M} = \mathcal{P}^{\otimes 2}$, where $\mathcal{P}(\cdot) = \sum_{x,z \in \{0,1\}} p_{x,z} X^x Z^z (\cdot) Z^z X^x$, with the coeffi-

cients $p_{x,z}$ forming an arbitrary probability distribution. This means that $\mathcal{M}(\cdot) = \sum_{x_1, z_1, x_2, z_2 \in \{0,1\}} p_{x_1, z_1} p_{x_2, z_2} (X^{x_1} Z^{z_1} \otimes X^{x_2} Z^{z_2})(\cdot) (Z^{z_1} X^{x_1} \otimes Z^{z_2} X^{x_2})$. Then, the transformed channel is $\mathcal{M}'(\cdot) = \sum_{x_1, z_1, x_2, z_2 \in \{0,1\}} p_{x_1, z_1 \oplus z_2} p_{x_1 \oplus x_2, z_2} (X^{x_1} Z^{z_1} \otimes X^{x_2} Z^{z_2})(\cdot) (Z^{z_1} X^{x_1} \otimes Z^{z_2} X^{x_2})$. Observe that this channel is not necessarily a tensor-product channel, due to the correlations in the transformed probability distributions. As such, this example illustrates the important point that transforming an uncorrelated noise channel will generally result in a correlated noise channel. In particular, after absorbing the noise into the input states via this transformation, the initial product state of noisy Bell pairs may not retain the product state form, i.e., the initial input state to Protocol 2 may not be a product state, as assumed in Theorem 10. However, Theorem 10 holds even for arbitrarily correlated input states, and we provide the precise statement of this general case in App. D 1.

For non-Pauli noise, the form of $\mathcal{M}'$ depends heavily on the specific form of $\mathcal{M}$. Taking $\mathcal{M} = \mathcal{A}_\gamma^{\otimes 2}$ as an example, where $\mathcal{A}_\gamma$ is the amplitude-damping channel defined above, it is straightforward to determine the transformed Kraus operators. Let $K'_{i,j} = \text{CNOT}_n(K_i \otimes K_j) \text{CNOT}_n$ be the transformed Kraus operators, for $i, j \in \{0, 1\}$. Then,

$$K'_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad K'_{1,2} = \begin{pmatrix} 0 & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma(1-\gamma)} & 0 \end{pmatrix}, \quad (28)$$

$$K'_{2,1} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{\gamma} \\ 0 & 0 & \sqrt{\gamma(1-\gamma)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K'_{2,2} = \begin{pmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (29)$$

As in the case of Pauli channels, the transformed channel is not a tensor-product channel.

Analogous reasoning can be applied to deal with gate noise in Protocol 3, such that Theorem 11 can be applied by absorbing the gate noise into the input states.

VI. SUMMARY AND OUTLOOK

In this work, we consider the distribution of GHZ states in arbitrary Bell-pair networks. While this task is not new, we adopt a novel perspective on evaluating protocols to achieve this task by considering the gate cost and the Bell-pair source cost of the protocol, in addition to the number of Bell pairs used. Notably, the gate count imposes a more restricted form of local operations and classical communications (LOCC). Unlike the traditional LOCC framework, in which all local operations are considered free, our approach emphasizes the constraints associated with gate usage. Furthermore, to minimize the number of consumed Bell pairs when distributing a GHZ state among a subset of nodes in a network, a minimum Steiner

tree must generally be identified; however, the Steiner tree problem is NP-hard [65]. It is therefore interesting to ask whether other, polynomial-time strategies can be employed to distribute a GHZ state, while still maintaining near-optimal Bell pair consumption, gate usage, source cost. To the best of our knowledge, the number of Bell-pair sources has not been considered in previous works.

We present protocols for GHZ entanglement distribution that use a number of gates that is linear in the size of the subgraph corresponding to the desired nodes to be entangled. Notably, this gate cost is independent of the structure and topology of the network. We also propose an alternative protocol that identifies connected subgraphs using the breadth-first search (BFS) algorithm. While this approach does not yield the optimal Bell pair count, it avoids the need to find Steiner trees, it runs in polynomial time, and we provide numerical evidence that it achieves near-optimal Bell pair consumption. Furthermore, by relaxing optimality in Bell-pair consumption for the sake of computational efficiency, our protocol has better complexity compared to polynomial-time Steiner tree approximation algorithms [67–69].

Unlike prior work, which focuses primarily on minimizing the number of Bell pairs and therefore involves solving the Steiner tree problem (either exactly or with an approximate algorithm), our objective has been stronger: our goal was to minimize not only the number of Bell pairs, but also the number of Bell pair sources. What we have shown is that doing this necessitates a different perspective. Specifically, we have shown that we must find not only an approximation of a minimum Steiner tree, in order to achieve a minimal number of Bell pairs, but also an approximation of the minimum dominating set size. In other words, we have shown that focusing on Steiner trees alone will not achieve the minimum number of Bell-pair sources, and this is where our algorithm has its advantage, because it can simultaneously achieve near-optimal Bell-pair consumption and near-optimal Bell-pair source count.

To evaluate our protocols, we performed Monte Carlo simulations on Erdős–Rényi, Barabási–Albert, and Waxman random network models, which represent increasingly realistic network models. For these models, our protocol shows enhanced performance compared to the MMG protocol in dense networks, achieving a nearly quadratic advantage compared to the MMG protocol by minimizing the number of gates required while keeping the number of Bell pairs and sources of Bell pairs as low as possible. Additionally, we analyzed our protocol in the context of noisy Bell pairs and noisy gates, and we provided explicit analytic expressions for the fidelity of the states after our protocols for arbitrary noise models.

Future directions. While our current focus has been on creating GHZ states, exploring the distribution of other graph states, such as cluster states, would be relevant for distributed and measurement-based quantum computing. Additionally, analyzing in detail the robustness of our approach against link and node failures is of particular interest, especially in light of classical and quantum research on network resilience, as demonstrated by Refs. [77–80]. Intuitively, by creating local, small-scale GHZ states independently, our protocol is less susceptible to the failure of any single GHZ state creation, which

might result from the failure of nearest-neighbor entanglement generation in networks of lossy optical fiber channels. In cases of failure, the other GHZ states can be stored in memory, potentially reducing overall waiting times in the presence of link and node failures. Studying the waiting time of our protocol would generalize studies on the waiting time of linear repeater chains for bipartite entanglement, e.g., Refs. [81, 82], to the multipartite setting.

Currently, our protocol requires full, global knowledge of the network. Therefore, developing variants of our protocol that rely primarily on local knowledge and require only a few rounds of classical communication to establish global awareness, leveraging the ideas of, e.g., Ref. [83], would be beneficial.

Beyond theoretical studies, implementing our algorithm on various experimental platforms is also of great interest. Another intriguing approach is to add an optimization layer to our protocol enhance the way the stars are selected and merged, particularly in the context of lossy Bell pairs and lossy memory stations. We anticipate that this can be accomplished using reinforcement learning, by expressing our problem as a Markov decision process (MDP) and leveraging the tools from Refs. [82, 84–88]. Moreover, employing further tools such as (probability) generating functions, the protocols could be optimized in the presence of noise [89, 90].

ACKNOWLEDGMENTS

We thank Frederik Hahn, Ernst Althaus, Luzie Marianczuk, Eleanor Reiffel, Kurt Mehlhorn, Lina Vandr , Kenneth Goode-nough and Annirudha Sen for useful discussions. We acknowledge funding from the BMBF in Germany (QR.X, PhotonQ, QuKuK, QuaPhySI, HYBRID), the EU’s Horizon Research and Innovation Actions (CLUSTEC), and from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)–Project-ID 429529648 – TRR 306 QuCoLiMa (“Quantum Cooperativity of Light and Matter”).

-
- [1] S. Pirandola, B. R. Bardhan, T. Gehring, C. Weedbrook, and S. Lloyd, Advances in photonic quantum sensing, *Nature Photonics* **12**, 724 (2018), 1811.01969.
 - [2] M. Tse *et al.*, Quantum-Enhanced Advanced LIGO Detectors in the Era of Gravitational-Wave Astronomy, *Physical Review Letters* **123**, 231107 (2019).
 - [3] J. Gambetta, IBM’s roadmap for scaling quantum technology, *IBM Research Blog* (2020).
 - [4] W. Luo, L. Cao, Y. Shi, L. Wan, H. Zhang, S. Li, G. Chen, Y. Li, S. Li, Y. Wang, S. Sun, M. F. Karim, H. Cai, L. C. Kwek, and A. Q. Liu, Recent progress in quantum photonic chips for quantum communication and internet, *Light: Science & Applications* **12**, 175 (2023).
 - [5] A. S. Cacciapuoti, M. Caleffi, F. Tafuri, F. S. Cataliotti, S. Gherardini, and G. Bianchi, Quantum Internet: Networking Challenges in Distributed Quantum Computing, *IEEE Network* **34**, 137 (2020), 1810.08421.
 - [6] D. Awschalom, K. K. Berggren, H. Bernien, S. Bhave, L. D. Carr, P. Davids, S. E. Economou, D. Englund, A. Faraon, M. Fejer, S. Guha, M. V. Gustafsson, E. Hu, L. Jiang, J. Kim, B. Korzh, P. Kumar, P. G. Kwiat, M. Lon ar, M. D. Lukin, D. A. Miller, C. Monroe, S. W. Nam, P. Narang, J. S. Orcutt, M. G. Raymer, A. H. Safavi-Naeini, M. Spiropulu, K. Srinivasan, S. Sun, J. Vu kovi , E. Waks, R. Walsworth, A. M. Weiner, and Z. Zhang, Development of Quantum Interconnects (QuICs) for Next-Generation Information Technologies, *PRX Quantum* **2**, 017002 (2021).
 - [7] T. S. Humble, A. McCaskey, D. I. Lyakh, M. Gowrishankar, A. Frisch, and T. Monz, Quantum Computers for High-Performance Computing, *IEEE Micro* **41**, 15 (2021).
 - [8] C. H. Bennett, G. Brassard, C. Cr peau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Physical Review Letters* **70**, 1895 (1993).
 - [9] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of noisy entanglement

- and faithful teleportation via noisy channels, *Physical Review Letters* **76**, 722 (1996).
- [10] D. Gottesman and I. L. Chuang, Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations, *Nature* **402**, 390 (1999), quant-ph/9908010.
 - [11] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Optimal local implementation of nonlocal quantum gates, *Physical Review A* **62**, 052317 (2000), quant-ph/0005101.
 - [12] R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, *Physical Review Letters* **86**, 5188 (2001).
 - [13] D. W. Leung, Two-qubit Projective Measurements are Universal for Quantum Computation, *arXiv:quant-ph/0111122* (2001).
 - [14] M. A. Nielsen, Quantum computation by measurement and quantum memory, *Physics Letters A* **308**, 96 (2003), quant-ph/0108020.
 - [15] D. W. Leung, Quantum Computation by Measurements, *International Journal of Quantum Information* **02**, 33 (2004), quant-ph/0310189.
 - [16] R. Jozsa, An introduction to measurement based quantum computation, *arXiv:quant-ph/0508124* (2005).
 - [17] V. Danos, E. D'Hondt, E. Kashefi, and P. Panangaden, Distributed Measurement-based Quantum Computation, *Electronic Notes in Theoretical Computer Science* **170**, 73 (2007), Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005), quant-ph/0506070.
 - [18] C. Piveteau and D. Sutter, Circuit Knitting With Classical Communication, *IEEE Transactions on Information Theory* **70**, 2734 (2024), 2205.00016.
 - [19] A. Broadbent, J. Fitzsimons, and E. Kashefi, Universal Blind Quantum Computation, in *2009 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009)* (2009) pp. 517–526, 0807.4154.
 - [20] G. Tóth, Multipartite entanglement and high-precision metrology, *Physical Review A* **85**, 022322 (2012), 1006.4368.
 - [21] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Fisher information and multiparticle entanglement, *Physical Review A* **85**, 022321 (2012), 1006.4366.
 - [22] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Distributed quantum sensing using continuous-variable multipartite entanglement, *Physical Review A* **97**, 032329 (2018).
 - [23] Y. Xia, Q. Zhuang, W. Clark, and Z. Zhang, Repeater-enhanced distributed quantum sensing based on continuous-variable multipartite entanglement, *Physical Review A* **99**, 012328 (2019).
 - [24] X. Guo, C. R. Breum, J. Borregaard, S. Izumi, M. V. Larsen, T. Gehring, M. Christandl, J. S. Neergaard-Nielsen, and U. L. Andersen, Distributed quantum sensing in a continuous-variable entangled network, *Nature Physics* **16**, 281 (2020), 1905.09408.
 - [25] A. Delteil, Z. Sun, W.-b. Gao, E. Togan, S. Faelst, and A. Imamoglu, Generation of heralded entanglement between distant hole spins, *Nature Physics* **12**, 218 (2016).
 - [26] R. Stockill, M. J. Stanley, L. Huthmacher, E. Clarke, M. Hugues, A. J. Miller, C. Matthiesen, C. Le Gall, and M. Atatüre, Phase-tuned entangled state generation between distant spin qubits, *Physical Review Letters* **119**, 010503 (2017).
 - [27] P. C. Humphreys, N. Kalb, J. P. J. Morits, R. N. Schouten, R. F. L. Vermeulen, D. J. Twitchen, M. Markham, and R. Hanson, Deterministic delivery of remote entanglement on a quantum network, *Nature* **558**, 268 (2018).
 - [28] M. Pompili, S. L. N. Hermans, S. Baier, H. K. C. Beukers, P. C. Humphreys, R. N. Schouten, R. F. L. Vermeulen, M. J. Tiggeleman, L. dos Santos Martins, B. Dirkse, S. Wehner, and R. Hanson, Realization of a multi-node quantum network of remote solid-state qubits, *Science* **372**, 259 (2021).
 - [29] S. L. N. Hermans, M. Pompili, H. K. C. Beukers, S. Baier, J. Borregaard, and R. Hanson, Qubit teleportation between non-neighbouring nodes in a quantum network, *Nature* **605**, 663 (2022).
 - [30] M. Pompili, C. Delle Donne, I. te Raa, B. van der Vecht, M. Skrzypczyk, G. Ferreira, L. de Kluijver, A. J. Stolk, S. L. N. Hermans, P. Pawełczak, W. Kozłowski, R. Hanson, and S. Wehner, Experimental demonstration of entanglement delivery using a quantum network stack, *npj Quantum Information* **8**, 121 (2022).
 - [31] A. J. Stolk, K. L. van der Enden, M.-C. Slater, I. te Raa-Derckx, P. Botma, J. van Rantwijk, J. J. B. Biemond, R. A. J. Hagen, R. W. Herfst, W. D. Koek, A. J. H. Meskers, R. Vollmer, E. J. van Zwet, M. Markham, A. M. Edmonds, J. F. Geus, F. Elsen, B. Jungbluth, C. Haefner, C. Tresp, J. Stuhler, S. Ritter, and R. Hanson, Metropolitan-scale heralded entanglement of solid-state qubits, *Science Advances* **10**, eadp6442 (2024), 2404.03723.
 - [32] C. M. Knaut, A. Suleymanzade, Y.-C. Wei, D. R. Assumpcao, P.-J. Stas, Y. Q. Huan, B. Machielse, E. N. Knall, M. Sutula, G. Baranes, N. Sinclair, C. De-Eknankul, D. S. Levonian, M. K. Bhaskar, H. Park, M. Lončar, and M. D. Lukin, Entanglement of nanophotonic quantum memory nodes in a telecom network, *Nature* **629**, 573 (2024), 2310.01316.
 - [33] S. Kucera, C. Haen, E. Arenskötter, T. Bauer, J. Meiers, M. Schäfer, R. Boland, M. Yahyapour, M. Lessing, R. Holzwarth, C. Becher, and J. Eschner, Demonstration of quantum network protocols over a 14-km urban fiber link, *npj Quantum Information* **10** (2024), 2404.04958.
 - [34] Y. Zhou, P. Malik, F. Fertig, M. Bock, T. Bauer, T. van Leent, W. Zhang, C. Becher, and H. Weinfurter, Long-Lived Quantum Memory Enabling Atom-Photon Entanglement over 101 km of Telecom Fiber, *PRX Quantum* **5**, 020307 (2024), 2308.08892.
 - [35] L. Hartung, M. Seubert, S. Welte, E. Distant, and G. Rempe, A quantum-network register assembled with optical tweezers in an optical cavity, *Science* **385**, 179 (2024), 2407.09109.
 - [36] A. Pirker, J. Wallnöfer, and W. Dür, Modular architectures for quantum networks, *New Journal of Physics* **20**, 053054 (2018).
 - [37] A. Pirker and W. Dür, A quantum network stack and protocols for reliable entanglement-based networks, *New Journal of Physics* **21**, 033003 (2019).
 - [38] F. Hahn, A. Pappa, and J. Eisert, Quantum network routing and local complementation, *npj Quantum Information* **5**, 76 (2019).
 - [39] J. Freund, A. Pirker, and W. Dür, Flexible quantum data bus for quantum networks, *Physical Review Research* **6**, 033267 (2024), 2404.06578.
 - [40] M. Cuquet and J. Calsamiglia, Growth of graph states in quantum networks, *Physical Review A* **86**, 042304 (2012), 1208.0710.
 - [41] M. Epping, H. Kampermann, and D. Bruß, Large-scale quantum networks based on graphs, *New Journal of Physics* **18**, 053036 (2016).
 - [42] A. Fischer and D. Towsley, Distributing Graph States Across Quantum Networks, in *2021 IEEE International Conference on Quantum Computing and Engineering (QCE)* (2021) pp. 324–333, 2009.10888.
 - [43] G. Avis, F. Rozpędek, and S. Wehner, Analysis of multipartite entanglement distribution using a central quantum-network node, *Physical Review A* **107**, 012609 (2023), 2203.05517.
 - [44] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Quantum repeaters: The role of imperfect local operations in quantum communication, *Physical Review Letters* **81**, 5932 (1998).
 - [45] W. Dür, H.-J. Briegel, J. I. Cirac, and P. Zoller, Quantum repeaters based on entanglement purification, *Physical Review*

- A **59**, 169 (1999).
- [46] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Quantum repeaters based on atomic ensembles and linear optics, *Reviews of Modern Physics* **83**, 33 (2011).
 - [47] K. Azuma, S. Bäuml, T. Coopmans, D. Elkouss, and B. Li, Tools for quantum network design, *AVS Quantum Science* **3**, 014101 (2021).
 - [48] K. Azuma, S. E. Economou, D. Elkouss, P. Hilaire, L. Jiang, H.-K. Lo, and I. Tzitrin, Quantum repeaters: From quantum networks to the quantum internet, *Reviews of Modern Physics* **95**, 045006 (2023), 2212.10820.
 - [49] C. Meignant, D. Markham, and F. Grosshans, Distributing graph states over arbitrary quantum networks, *Physical Review A* **100**, 052333 (2019).
 - [50] F. K. Hwang, D. S. Richards, and P. Winter, *The Steiner Tree Problem*, Annals of Discrete Mathematics, Vol. 53 (North-Holland, 1992).
 - [51] M. Brazil and M. Zachariasen, Steiner Trees in Graphs and Hypergraphs, in *Optimal Interconnection Trees in the Plane: Theory, Algorithms and Applications* (Springer International Publishing, 2015) pp. 301–317.
 - [52] D. M. Greenberger, M. A. Horne, and A. Zeilinger, Going Beyond Bell’s Theorem, in *Bell’s Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Springer Netherlands, Dordrecht, 1989) pp. 69–72.
 - [53] G. Murta, F. Grasselli, H. Kampermann, and D. Bruß, Quantum Conference Key Agreement: A Review, *Advanced Quantum Technologies* **3**, 2000025 (2020), 2003.10186.
 - [54] P. L. Erdős and A. Rényi, On random graphs. I., *Publicationes Mathematicae Debrecen* **6**, 290 (1959).
 - [55] P. L. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **5**, 17 (1960).
 - [56] B. Waxman, Routing of multipoint connections, *IEEE Journal on Selected Areas in Communications* **6**, 1617 (1988).
 - [57] D. J. Watts and S. H. Strogatz, Collective dynamics of ‘small-world’ networks, *Nature* **393**, 440 (1998).
 - [58] R. Albert, H. Jeong, and A.-L. Barabási, Diameter of the World-Wide Web, *Nature* **401**, 130 (1999), cond-mat/9907038.
 - [59] A.-L. Barabási and R. Albert, Emergence of Scaling in Random Networks, *Science* **286**, 509 (1999), cond-mat/9910332.
 - [60] A.-L. Barabási, *Network Science* (Cambridge University Press, 2016).
 - [61] M. Newman, *Networks* (Oxford University Press, 2018).
 - [62] V. V. Vazirani, *Approximation Algorithms* (Springer Science & Business Media, 2001).
 - [63] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, 1st ed., Series of Books in the Mathematical Sciences (W. H. Freeman and Company, New York, 1979).
 - [64] C. Kruszynska, A. Miyake, H. J. Briegel, and W. Dür, Entanglement purification protocols for all graph states, *Physical Review A* **74**, 052316 (2006), quant-ph/0606090.
 - [65] M. R. Garey, R. L. Graham, and D. S. Johnson, The Complexity of Computing Steiner Minimal Trees, *SIAM Journal on Applied Mathematics* **32**, 835 (1977).
 - [66] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 3rd ed. (MIT Press, Cambridge, MA, 2009).
 - [67] L. Kou, G. Markowsky, and L. Berman, A fast algorithm for steiner trees, *Acta Informatica* **15**, 141 (1981).
 - [68] K. Mehlhorn, A faster approximation algorithm for the Steiner problem in graphs, *Information Processing Letters* **27**, 125 (1988).
 - [69] G. Robins and A. Zelikovsky, Tighter Bounds for Graph Steiner Tree Approximation, *SIAM Journal on Discrete Mathematics* **19**, 122 (2005).
 - [70] M. L. Fredman and D. E. Willard, Trans-dichotomous algorithms for minimum spanning trees and shortest paths, *Journal of Computer and System Sciences* **48**, 533 (1994).
 - [71] S. A. Moses, C. H. Baldwin, M. S. Allman, R. Ancona, L. Ascarunz, C. Barnes, J. Bartolotta, B. Bjork, P. Blanchard, M. Bohn, J. G. Bohnet, N. C. Brown, N. Q. Burdick, W. C. Burton, S. L. Campbell, J. P. Campora, C. Carron, J. Chambers, J. W. Chan, Y. H. Chen, A. Chernoguzov, E. Chertkov, J. Colina, J. P. Curtis, R. Daniel, M. DeCross, D. Deen, C. Delaney, J. M. Dreiling, C. T. Ertsgaard, J. Esposito, B. Estey, M. Fabrikant, C. Figgatt, C. Foltz, M. Foss-Feig, D. Francois, J. P. Gaebler, T. M. Gatterman, C. N. Gilbreth, J. Giles, E. Glynn, A. Hall, A. M. Hankin, A. Hansen, D. Hayes, B. Higashi, I. M. Hoffman, B. Horning, J. J. Hout, R. Jacobs, J. Johansen, L. Jones, J. Karcz, T. Klein, P. Lauria, P. Lee, D. Liefer, S. T. Lu, D. Lucchetti, C. Lytle, A. Malm, M. Matheny, B. Mathewson, K. Mayer, D. B. Miller, M. Mills, B. Neyenhuis, L. Nugent, S. Olson, J. Parks, G. N. Price, Z. Price, M. Pugh, A. Ransford, A. P. Reed, C. Roman, M. Rowe, C. Ryan-Anderson, S. Sanders, J. Sedlacek, P. Shevchuk, P. Siegfried, T. Skripka, B. Spaun, R. T. Sprenkle, R. P. Stutz, M. Swallows, R. I. Tobey, A. Tran, T. Tran, E. Vogt, C. Volin, J. Walker, A. M. Zolot, and J. M. Pino, A Race-Track Trapped-Ion Quantum Processor, *Physical Review X* **13**, 041052 (2023).
 - [72] P. van Loock, W. Alt, C. Becher, O. Benson, H. Boche, C. Deppe, J. Eschner, S. Höfling, D. Meschede, P. Michler, F. Schmidt, and H. Weinfurter, Extending Quantum Links: Modules for Fiber- and Memory-Based Quantum Repeaters, *Advanced Quantum Technologies* **3** (2020).
 - [73] B. Bollobás, *Random Graphs*, 2nd ed., Cambridge Studies in Advanced Mathematics (Cambridge University Press, 2001).
 - [74] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Random graphs with arbitrary degree distributions and their applications, *Physical Review E* **64**, 026118 (2001).
 - [75] A. A. Hagberg, D. A. Schult, and P. J. Swart, Exploring Network Structure, Dynamics, and Function using NetworkX, in *Proceedings of the 7th Python in Science Conference*, edited by G. Varoquaux, T. Vaught, and J. Millman (Pasadena, CA USA, 2008) pp. 11–15, <https://networkx.org/>.
 - [76] M. Penrose, *Random geometric graphs* (Oxford University Press, 2003).
 - [77] R. Albert, H. Jeong, and A.-L. Barabási, Error and attack tolerance of complex networks, *Nature* **406**, 378 (2000), cond-mat/0008064.
 - [78] S. Das, S. Khatri, and J. P. Dowling, Robust quantum network architectures and topologies for entanglement distribution, *Physical Review A* **97**, 012335 (2018).
 - [79] B. C. Coutinho, W. J. Munro, K. Nemoto, and Y. Omar, Robustness of noisy quantum networks, *Communications Physics* **5**, 105 (2022), 2103.03266.
 - [80] A. Sadhu, M. A. Somayajula, K. Horodecki, and S. Das, Practical limitations on robustness and scalability of quantum Internet, *arXiv:2308.12739* (2023).
 - [81] S. Khatri, C. T. Matyas, A. U. Siddiqui, and J. P. Dowling, Practical figures of merit and thresholds for entanglement distribution in quantum networks, *Physical Review Research* **1**, 023032 (2019).
 - [82] E. Shchukin, F. Schmidt, and P. van Loock, Waiting time in quantum repeaters with probabilistic entanglement swapping, *Physical Review A* **100**, 032322 (2019), 1710.06214.
 - [83] S. Haldar, P. J. Barge, X. Cheng, K.-C. Chang, B. T. Kirby,

- S. Khatri, C. W. Wong, and H. Lee, Reducing classical communication costs in multiplexed quantum repeaters using hardware-aware quasi-local policies, [arXiv:2401.13168](#) (2024).
- [84] S. Khatri, Policies for elementary links in a quantum network, *Quantum* **5**, 537 (2021).
- [85] S. Khatri, On the design and analysis of near-term quantum network protocols using Markov decision processes, *AVS Quantum Science* **4**, 030501 (2022), 2207.03403.
- [86] S. Haldar, P. J. Barge, S. Khatri, and H. Lee, Fast and reliable entanglement distribution with quantum repeaters: Principles for improving protocols using reinforcement learning, *Physical Review Applied* **21**, 024041 (2024), 2303.00777.
- [87] S. D. Reiß and P. van Loock, Deep reinforcement learning for key distribution based on quantum repeaters, *Physical Review A* **108**, 012406 (2023), 2207.09930.
- [88] E. Shchukin and P. van Loock, Optimal Entanglement Swapping in Quantum Repeater, *Physical Review Letters* **128**, 150502 (2022), 2109.00793.
- [89] K. Goodenough, T. Coopmans, and D. Towsley, On noise in swap ASAP repeater chains: exact analytics, distributions and tight approximations, [arXiv:2404.07146](#) (2024).
- [90] L. Kamin, E. Shchukin, F. Schmidt, and P. van Loock, Exact rate analysis for quantum repeaters with imperfect memories and entanglement swapping as soon as possible, *Physical Review Research* **5**, 023086 (2023), 2203.10318.
- [91] J. Walln  fer, M. Zwerger, C. Muschik, N. Sangouard, and W. D  r, Two-dimensional quantum repeaters, *Physical Review A* **94**, 052307 (2016).
- [92] V. V. Kuzmin, D. V. Vasilyev, N. Sangouard, W. D  r, and C. A. Muschik, Scalable repeater architectures for multi-party states, *npj Quantum Information* **5**, 115 (2019).
- [93] S. Brito, A. Canabarro, R. Chaves, and D. Cavalcanti, Statistical Properties of the Quantum Internet, *Physical Review Letters* **124**, 210501 (2020).
- [94] S. de Bone, R. Ouyang, K. Goodenough, and D. Elkouss, Protocols for Creating and Distilling Multipartite GHZ States With Bell Pairs, *IEEE Transactions on Quantum Engineering* **1**, 1 (2020), 2010.12259.
- [95] L. Bugalho, B. C. Coutinho, F. A. Monteiro, and Y. Omar, Distributing Multipartite Entanglement over Noisy Quantum Networks, *Quantum* **7**, 920 (2023).
- [96] W. Roga, R. Ikuta, T. Horikiri, and M. Takeoka, Efficient Dicke-state distribution in a network of lossy channels, *Physical Review A* **108**, 012612 (2023), 2211.15138.
- [97] H. Shimizu, W. Roga, D. Elkouss, and M. Takeoka, Simple loss-tolerant protocol for GHZ-state distribution in a quantum network, [arXiv:2404.19458](#) (2024).
- [98] A. Sen, K. Goodenough, and D. Towsley, Multipartite Entanglement in Quantum Networks using Subgraph Complementations, [arXiv:2308.13700](#) (2023).
- [99] H. J. Briegel and R. Raussendorf, Persistent entanglement in arrays of interacting particles, *Physical Review Letters* **86**, 910 (2001).
- [100] H. J. Briegel, Cluster States, in *Compendium of Quantum Physics*, edited by D. Greenberger, K. Hentschel, and F. Weinert (Springer, Berlin, Heidelberg, 2009) pp. 96–105.
- [101] Y.-A. Chen, X. Liu, C. Zhu, L. Zhang, J. Liu, and X. Wang, Quantum Entanglement Allocation through a Central Hub, [arXiv:2409.08173](#) (2024).

APPENDICES

A. Relation to prior work	17
B. Overview of graph states	18
C. Graph state distribution in the star topology	19
D. Analysis of Protocol 2	19
1. Proof of Theorem 10	20
E. Protocols for fusing GHZ states	21
1. Fusion of two GHZ states (Protocol 6)	22
2. Fusion of GHZ states in a star topology (Protocol 7)	23
3. Fusion of GHZ states in a linear topology (Protocol 8)	25
4. Fusion of GHZ states in a tree topology (Protocol 3)	28

Appendix A: Relation to prior work

Prior work on the distribution of multipartite entanglement in genuine network settings, going beyond repeater chains, includes Refs. [36, 37, 40, 42, 43, 49, 80, 91–98]. Below, we highlight and compare our work to some of the most relevant of these prior works.

- Reference [40] considers the distribution of arbitrary graph states, in which the graph of the corresponding graph state matches the topology of the underlying network of Bell pairs.
- Reference [42] involves local preparation of the desired graph state, followed by teleportation of the state to the desired parties using the Bell pair network, which can be arbitrary. The basis of their protocol is the so-called “Bipartite A” protocol

of Ref. [40]. Essentially, their protocol is the Bipartite A protocol supplemented by an analysis of completion time based on particular algorithms for finding paths in the Bell-pair network for the purpose of teleporting the locally-created graph state.

- Reference [43] considers the distribution of GHZ states in a star Bell-pair network topology, in which the central “factory” node locally produces the GHZ states and then uses the Bell pairs to teleport it to the outer parties. This is also essentially the Bipartite A protocol of Ref. [40], with noisy Bell pairs and an analysis of the rate at which the GHZ states can be distributed.
- Reference [49] uses the Steiner tree, along with a sub-routine called “star expansion”, to distribution a GHZ state between a given set of nodes in the network.
- Reference [95] also makes use of the Steiner tree problem, and includes an analysis with noisy Bell pairs.
- Similarly to Ref. [43], Ref. [98] requires a star Bell-pair topology. Here, an arbitrary graph state can be produced using a “subgraph complementation system” associated with the graph of the desired graph state. Reference [97] also considers a star topology only.

Appendix B: Overview of graph states

A graph state [12, 99, 100] is a multi-qubit quantum state defined using graphs. Consider a graph $G = (V, E)$, which consists of a set V of vertices and a set E of edges. For the purposes of this example, G is an undirected graph, and E is a set of two-element subsets of V . The graph state $|G\rangle$ is an n -qubit quantum state $|G\rangle_{A_1 \dots A_n}$, with $n = |V|$, that is defined as

$$|G\rangle_{A_1 \dots A_n} := \frac{1}{\sqrt{2^n}} \sum_{\vec{x} \in \{0,1\}^n} (-1)^{\frac{1}{2} \vec{x}^T M_G \vec{x}} |\vec{a}\rangle, \quad (\text{B1})$$

where M_G is the adjacency matrix of G , which is defined as

$$(M_G)_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{B2})$$

and $\vec{x}$ is the column vector $(x_1, x_2, \dots, x_n)^T$. It is straightforward to show that

$$|G\rangle_{A_1 \dots A_n} = \text{CZ}(G)(|+\rangle_{A_1} \otimes \dots \otimes |+\rangle_{A_n}), \quad (\text{B3})$$

where $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and

$$\text{CZ}(G) := \bigotimes_{\{v_i, v_j\} \in E} \text{CZ}_{A_i A_j}, \quad (\text{B4})$$

with $\text{CZ}_{A_i A_j} := |0\rangle\langle 0|_{A_i} \otimes \mathbb{1}_{A_j} + |1\rangle\langle 1|_{A_i} \otimes Z_{A_j}$ being the controlled-Z gate.

Consider a star graph of n nodes, labeled $\{1, 2, \dots, n\}$, such that the center of the star is node $k \in \{1, 2, \dots, n\}$. We denote this graph by $S_{n,k}$. It is straightforward to show that the graph state corresponding to this graph, defined equivalently according to (B1) or (B3), is

$$|S_{n,k}\rangle = (H^{\otimes(k-1)} \otimes \mathbb{1} \otimes H^{\otimes(n-k)})|\text{GHZ}_n\rangle, \quad (\text{B5})$$

where $H = |+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard operator and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. This means that the star graph state is equivalent to the GHZ state up to local Hadamard gates. In particular, observe in (B5) that the center of the star graph (namely, the k -th qubit) is the only qubit that does not have a Hadamard gate applied to it. This implies that to change the center of the star graph state involves only *two* Hadamard gates: one Hadamard gate applied to the current central qubit, and the other Hadamard gate applied to the new central qubit. Note that we apply only two gates to change the center regardless of the size n of the star. We also note that for $n = 2$, we have that the two-party GHZ state is equal to the Bell state, i.e., $|\text{GHZ}_2\rangle = \frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle) = |\Phi\rangle$. It follows that $|S_{2,1}\rangle = (\mathbb{1} \otimes H)|\Phi\rangle = (H \otimes \mathbb{1})|\Phi\rangle = |S_{2,2}\rangle$. Finally, performing an X -basis measurement on any subset of the qubits in a GHZ state produces a GHZ state among the remaining qubits, up to a Pauli-Z correction. More precisely, if we measure the first $m < n$ of the qubits of the state $|\text{GHZ}_n\rangle$ in the X -basis, then we obtain

$$(\langle z_1|H \otimes \langle z_2|H \otimes \dots \otimes \langle z_m|H \otimes \mathbb{1})|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2^m}} (Z^{z_1 \oplus z_2 \oplus \dots \oplus z_m} \otimes \mathbb{1})|\text{GHZ}_{n-m}\rangle, \quad (\text{B6})$$

for all $z_1, z_2, \dots, z_m \in \{0, 1\}$.

Appendix C: Graph state distribution in the star topology

Consider the scenario in which n nodes share Bell states with a central node. The task is for the central node to distribute the graph state $|G\rangle$ to the n outer nodes. One possible procedure is for the central node to locally prepare the graph state and then to teleport the individual qubits using the Bell states—this is the “Bipartite A” protocol of Ref. [40]. However, it is possible to perform a slightly simpler procedure that does not require the additional qubits needed to prepare the graph state locally. In fact, the following deterministic procedure produces the required graph state $|G\rangle$ shared by the nodes $A_1, \dots, A_n$. It is known as the “Bipartite B” protocol in Ref. [40, Sec. III.B] and is presented in Protocol 5.

Protocol 5 Graph state from Bell pairs in a star topology (Bipartite B protocol of Ref. [40])

Input: $n \in \{1, 2, \dots\}$ Bell pairs $|\Phi\rangle$, $N = n + 1$ nodes in a star topology (The central node shares a Bell pair with each of the n outer nodes.); graph G with $N - 1 = n$ nodes.

Output: Graph state $|G\rangle$ shared by the n outer nodes.

- 1: The central node applies $CZ(G)$ to its qubits.
 - 2: Measure each central qubit in the Pauli- X basis. The outcome is an n -bit string $\vec{x} = (x_1, \dots, x_n)$. The central node communicates outcome x_i to the i -th outer node.
 - 3: At the i -th outer node: if $x_i = 0$, then no gate is applied, and if $x_i = 1$, then the Pauli- Z gate is applied.
-

We note that Protocol 5 uses $|E|$ two-qubit gates, where $|E|$ is the number of edges in the graph G . (We exclude the single-qubit Pauli-gate corrections in the gate count.) It has also been recently shown in Ref. [101] that the Bipartite B protocol is optimal in terms of the communication cost of distributing a graph state in a star Bell-pair topology.

Remark 1. Note that at the end of this protocol, the central node is not included in the final graph state. We can modify this protocol slightly by adding an additional Bell pair that is contained entirely within the central node, and then doing the protocol just as above. With this modification, the final graph state includes the central node. ◀

In Ref. [85, Sec. III.B.3], it is shown explicitly that this protocol achieves the desired graph state $|G\rangle$ between the outer nodes. It is also shown in Ref. [85] that if, instead of perfect Bell pairs, the central node is connected to the outer nodes via imperfect Bell pairs (and if the gates are perfect), then the fidelity of the state at the end of the protocol has fidelity with respect to the desired graph state according to the following formula.

Proposition 12 (Fidelity after the noisy Bipartite B protocol [85]). Consider a network with a star topology, consisting of n outer nodes. Suppose that the central node shares imperfect Bell pairs with each of the outer nodes, and that these imperfect Bell pairs are represented by arbitrary two-qubit density operators $\rho_1, \rho_2, \dots, \rho_n$. Let ρ_{final} the state resulting from applying the Bipartite B protocol. The fidelity of ρ_{final} with respect to a given, desired graph state $|G\rangle$ is equal to the following expression:

$$\langle G | \rho_{\text{final}} | G \rangle = \sum_{\vec{x} \in \{0,1\}^n} \langle \Phi^{z_1, x_1} | \rho_1 | \Phi^{z_1, x_1} \rangle \langle \Phi^{z_2, x_2} | \rho_2 | \Phi^{z_2, x_2} \rangle \dots \langle \Phi^{z_n, x_n} | \rho_n | \Phi^{z_n, x_n} \rangle, \quad (\text{C1})$$

where the column vector $\vec{z} = (z_1, \dots, z_n)^T$ is given by $\vec{z} = A(G)\vec{x}$, with $A(G)$ being the adjacency matrix of the graph G . Also, we have used the definition

$$|\Phi^{z,x}\rangle := (Z^z X^x \otimes \mathbb{1})|\Phi\rangle, \quad |\Phi\rangle := \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle). \quad (\text{C2})$$

Appendix D: Analysis of Protocol 2

As mentioned in Remark 1, Protocol 5 creates a final graph state that excludes the central node. In particular, when creating a star graph state using Protocol 5, the central node is not part of the final state. (Recall from (B5) that the star graph state is local-unitary equivalent to the GHZ state.) Here, we analyze Protocol 2, which, without any additional Bell pairs, creates a GHZ state that includes the central node.

We start with $n \in \{2, 3, \dots\}$ Bell pairs $|\Phi\rangle_{A_1 B_1}, |\Phi\rangle_{A_2 B_2}, \dots, |\Phi\rangle_{A_n B_n}$, where $n \in \{2, 3, \dots\}$. The following protocol deterministically transforms these Bell pairs into the GHZ state $|\text{GHZ}_{n+1}\rangle_{A_1 B_1 \dots B_n}$. Let us prove that Protocol 2 performs as claimed. We use the abbreviation $A_{1:n} \equiv A_1 A_2 \dots A_n$, similarly for $B_{1:n}$. The initial state is

$$|\Phi\rangle_{A_1 B_1} \otimes |\Phi\rangle_{A_2 B_2} \otimes \dots \otimes |\Phi\rangle_{A_n B_n} = \frac{1}{\sqrt{2^n}} \sum_{\vec{\alpha} \in \{0,1\}^n} |\vec{\alpha}\rangle_{A_{1:n}} |\vec{\alpha}\rangle_{B_{1:n}} \quad (\text{D1})$$

$$= \frac{1}{\sqrt{2^n}} \sum_{\vec{\beta} \in \{0,1\}^{n-1}} \left(|0\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} |0\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} + |1\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} |1\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} \right). \quad (\text{D2})$$

Then, after applying the CNOT gates, we obtain

$$\begin{aligned} \frac{1}{\sqrt{2^n}} \sum_{\vec{\beta} \in \{0,1\}^{n-1}} & \left(|0\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} |0\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} + |1\rangle_{A_1} |\vec{\beta} \oplus \vec{1}\rangle_{A_{2:n}} |1\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} \right) \\ &= \frac{1}{\sqrt{2^n}} \sum_{\vec{\beta} \in \{0,1\}^{n-1}} \left(|0\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} |0\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} + |1\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} |1\rangle_{B_1} |\vec{\beta} \oplus \vec{1}\rangle_{B_{2:n}} \right). \end{aligned} \quad (D3)$$

From this, we can see that upon measurement of the qubits $A_2, A_3, \dots, A_n$, we obtain every outcome string $\vec{\beta} \in \{0, 1\}^{n-1}$ with probability $\frac{1}{2^{n-1}}$, and conditioned on this outcome the state of the remaining qubits is

$$\frac{1}{\sqrt{2}} \left(|0\rangle_{A_1} |0\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} + |1\rangle_{A_1} |1\rangle_{B_1} |\vec{\beta} \oplus \vec{1}\rangle_{B_{2:n}} \right) = X_{B_2}^{\beta_1} \otimes X_{B_3}^{\beta_2} \otimes \dots \otimes X_{B_n}^{\beta_{n-1}} |\text{GHZ}_{n+1}\rangle_{A_1 B_{1:n}}, \quad (D4)$$

where $\beta_i \in \{0, 1\}$ corresponds to the measurement outcome for the qubit A_i , $i \in \{2, 3, \dots, n\}$. With the convention that $X^0 \equiv \mathbb{1}$, we see that by applying the Pauli- X gate to every qubit B_i whose corresponding measured qubit A_i had outcome 1, we obtain the desired GHZ state on the qubits $A_1, B_1, B_2, \dots, B_n$.

This protocol has the following representation as an LOCC quantum channel:

$$\mathcal{L}(\rho_{A_{1:n} B_{1:n}}) = \sum_{\vec{\beta} \in \{0,1\}^{n-1}} \left(\langle \vec{\beta} |_{A_{2:n}} \text{CNOT}_n^\dagger \otimes X_{B_{2:n}}^{\vec{\beta}} \right) (\rho_{A_{1:n} B_{1:n}}) \left(\text{CNOT}_n |\vec{\beta}\rangle_{A_{2:n}} \otimes X_{B_{2:n}}^{\vec{\beta}} \right), \quad (D5)$$

where we have made use of the abbreviations

$$\text{CNOT}_n \equiv \text{CNOT}_{A_1 A_n} \text{CNOT}_{A_1 A_{n-1}} \dots \text{CNOT}_{A_1 A_3} \text{CNOT}_{A_1 A_2} \quad (D6)$$

$$|\vec{\beta}\rangle_{A_{2:n}} \equiv |\beta_1\rangle_{A_2} \otimes |\beta_2\rangle_{A_3} \otimes \dots \otimes |\beta_{n-1}\rangle_{A_n}, \quad (D7)$$

$$X_{B_{2:n}}^{\vec{\beta}} \equiv X_{B_2}^{\beta_1} \otimes X_{B_3}^{\beta_2} \otimes \dots \otimes X_{B_n}^{\beta_{n-1}}. \quad (D8)$$

for all $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_{n-1}) \in \{0, 1\}^{n-1}$.

Remark 2. Protocol 2 can be modified in the case that the inputs are not the Bell state vectors $|\Phi\rangle$ but instead the state vectors $|S_{2,1}\rangle = (\mathbb{1} \otimes H)|\Phi\rangle$. In this case, the steps of the protocol are exactly as before, except that the correction operations change: instead of applying the Pauli- X to the qubits $B_2, B_3, \dots, B_n$ according to the measurement outcomes, we apply the Pauli- Z gate to those qubits. The result of the protocol is then the star graph state $|S_{n+1,1}\rangle_{A_1 B_{1:n}}$, with the center at qubit A_1 . ◀

1. Proof of Theorem 10

Theorem 13 (Restatement of Theorem 10). If in Protocol 2 the Bell pairs are noisy and given by arbitrary two-qubit density operators $\rho_{A_1 B_1}^1, \rho_{A_2 B_2}^2, \dots, \rho_{A_n B_n}^n$, then the fidelity of the state after Protocol 2 with respect to the target GHZ state is given by

$$\begin{aligned} \langle \text{GHZ}_{n+1} |_{A_1 B_{1:n}} \mathcal{L} \left(\rho_{A_1 B_1}^1 \otimes \rho_{A_2 B_2}^2 \otimes \dots \otimes \rho_{A_n B_n}^n \right) | \text{GHZ}_{n+1} \rangle_{A_1 B_{1:n}} \\ = \sum_{\vec{z} \in \{0,1\}^{n-1}} \langle \Phi^{z',0} |_{A_1 B_1} \rho_{A_1 B_1}^1 | \Phi^{z',0} \rangle \langle \Phi^{z_1,0} |_{A_2 B_2} \rho_{A_2 B_2}^2 | \Phi^{z_1,0} \rangle \langle \Phi^{z_2,0} |_{A_2 B_2} \rho_{A_2 B_2}^2 | \Phi^{z_2,0} \rangle \dots \langle \Phi^{z_{n-1},0} |_{A_n B_n} \rho_{A_n B_n}^n | \Phi^{z_{n-1},0} \rangle, \end{aligned} \quad (D9)$$

where $z' = z_1 \oplus z_2 \oplus \dots \oplus z_{n-1}$, and we recall the definition of the Bell states $|\Phi^{z_i,0}\rangle$ in (C2).

Proof. We start by noting that

$$|v(\vec{\beta})\rangle := \left(\text{CNOT}_n |\vec{\beta}\rangle_{A_{2:n}} \otimes X_{B_{2:n}}^{\vec{\beta}} \right) |\text{GHZ}_{n+1}\rangle_{A_1 B_{1:n}} \quad (D10)$$

$$= \frac{1}{\sqrt{2}} \left(\text{CNOT}_n |0\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} \otimes |0\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} + \text{CNOT}_n |1\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} \otimes |1\rangle_{B_1} |\vec{\beta} \oplus \vec{1}\rangle_{B_{2:n}} \right) \quad (D11)$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle_{A_1} |\vec{\beta}\rangle_{A_{2:n}} \otimes |0\rangle_{B_1} |\vec{\beta}\rangle_{B_{2:n}} + |1\rangle_{A_1} |\vec{\beta} \oplus \vec{1}\rangle_{A_{2:n}} \otimes |1\rangle_{B_1} |\vec{\beta} \oplus \vec{1}\rangle_{B_{2:n}} \right) \quad (D12)$$

$$= \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |x\rangle_{A_1} |x\rangle_{B_1} \bigotimes_{i=1}^{n-1} |\beta_i \oplus x\rangle_{A_{i+1}} |\beta_i \oplus x\rangle_{B_{i+1}}. \quad (D13)$$

Now, we use the fact that

$$|x\rangle_A |x\rangle_B = \frac{1}{\sqrt{2}} \sum_{z \in \{0,1\}} (-1)^{xz} |\Phi^{z,0}\rangle, \quad (\text{D14})$$

for all $x \in \{0,1\}$. With this, we obtain

$$|v(\vec{\beta})\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}} \sum_{z_0, z_1, \dots, z_{n-1} \in \{0,1\}} (-1)^{xz_0} |\Phi^{z_0,0}\rangle_{A_1 B_1} \bigotimes_{i=1}^{n-1} \sum_{z_i \in \{0,1\}} (-1)^{(\beta_i \oplus x)z_i} |\Phi^{z_i,0}\rangle_{A_{i+1} B_{i+1}} \quad (\text{D15})$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n}} \sum_{z_0, z_1, \dots, z_{n-1} \in \{0,1\}} \sum_{x \in \{0,1\}} (-1)^{x(z_0 + z_1 + \dots + z_{n-1})} (-1)^{\beta_1 z_1 + \beta_2 z_2 + \dots + \beta_{n-1} z_{n-1}} |\Phi^{z_0,0}\rangle_{A_1 B_1} \bigotimes_{i=1}^{n-1} |\Phi^{z_i,0}\rangle_{A_{i+1} B_{i+1}} \quad (\text{D16})$$

$$= \frac{1}{\sqrt{2^{n-1}}} \sum_{\vec{z} \in \{0,1\}^{n-1}} (-1)^{\vec{\beta} \cdot \vec{z}} |\Phi^{z',0}\rangle_{A_1 B_1} \bigotimes_{i=1}^{n-1} |\Phi^{z_i,0}\rangle_{A_{i+1} B_{i+1}}, \quad (\text{D17})$$

where for the final equality we used the fact that

$$\sum_{x \in \{0,1\}} (-1)^{x(z_0 + z_1 + \dots + z_{n-1})} = 2\delta_{z_0 + z_1 + \dots + z_{n-1}, 0} = 2\delta_{z_0, z_1 \oplus z_2 \oplus \dots \oplus z_{n-1}}, \quad (\text{D18})$$

and we let $z' \equiv z_1 \oplus z_2 \oplus \dots \oplus z_{n-1}$. Therefore, for any arbitrary state $\rho_{A_{1:n} B_{1:n}}$, we have

$$\begin{aligned} & \langle \text{GHZ}_{n+1} |_{A_1 B_{1:n}} \mathcal{L}(\rho_{A_{1:n} B_{1:n}}) | \text{GHZ}_{n+1} \rangle_{A_1 B_{1:n}} \\ &= \sum_{\vec{\beta} \in \{0,1\}^{n-1}} \langle v(\vec{\beta}) | \rho_{A_{1:n} B_{1:n}} | v(\vec{\beta}) \rangle \end{aligned} \quad (\text{D19})$$

$$= \frac{1}{2^{n-1}} \sum_{\vec{\beta}, \vec{z}, \vec{y} \in \{0,1\}^{n-1}} (-1)^{\vec{\beta} \cdot \vec{z}} (-1)^{\vec{\beta} \cdot \vec{y}} \left(\langle \Phi^{z',0} |_{A_1 B_1} \bigotimes_{i=1}^{n-1} \langle \Phi^{z_i,0} |_{A_{i+1} B_{i+1}} \right) (\rho_{A_{1:n} B_{1:n}}) \left(| \Phi^{y',0} \rangle_{A_1 B_1} \bigotimes_{i=1}^{n-1} | \Phi^{y_i,0} \rangle_{A_{i+1} B_{i+1}} \right) \quad (\text{D20})$$

$$= \sum_{\vec{z} \in \{0,1\}^{n-1}} \left(\langle \Phi^{z',0} |_{A_1 B_1} \bigotimes_{i=1}^{n-1} \langle \Phi^{z_i,0} |_{A_{i+1} B_{i+1}} \right) (\rho_{A_{1:n} B_{1:n}}) \left(| \Phi^{z',0} \rangle_{A_1 B_1} \bigotimes_{i=1}^{n-1} | \Phi^{z_i,0} \rangle_{A_{i+1} B_{i+1}} \right), \quad (\text{D21})$$

where to obtain the final equality we used the fact that

$$\sum_{\vec{\beta} \in \{0,1\}^{n-1}} (-1)^{\vec{\beta} \cdot \vec{z}} (-1)^{\vec{\beta} \cdot \vec{y}} = \sum_{\vec{\beta} \in \{0,1\}^{n-1}} (-1)^{\vec{\beta} \cdot (\vec{z} \oplus \vec{y})} = 2^{n-1} \delta_{\vec{z}, \vec{y}}, \quad (\text{D22})$$

for all $\vec{z}, \vec{y} \in \{0,1\}^{n-1}$. As (D21) holds for all states $\rho_{A_{1:n} B_{1:n}}$, it holds also for the tensor-product state in the statement of the proposition. The proof is therefore complete. ■

Appendix E: Protocols for fusing GHZ states

In this section, we present a protocols for fusing GHZ states in various configurations. All of these protocols involve performing the CNOT gate on two qubits, one from each GHZ state, at a common node; see Fig. 11. We also provide expressions for the fidelity with respect to the desired GHZ state of the state resulting from the protocols when the input GHZ states are noisy.

Protocol 6 Fusion of two GHZ states

Input: Two GHZ states, $|\text{GHZ}_{n+1}\rangle_{A_1 B_{1:n}}$ and $|\text{GHZ}_{m+1}\rangle_{A_2 C_{1:m}}$, $n, m \in \{1, 2, \dots\}$, such that qubits A_1 and A_2 are located at the same node.

Output: GHZ state $|\text{GHZ}_{n+m+1}\rangle_{A_1 B_{1:n} C_{1:m}}$.

- 1: Perform the gate $\text{CNOT}_{A_1 A_2}$ between the qubits A_1 and A_2 .
- 2: Measure the qubit A_2 in the Z-basis (i.e., the $\{|0\rangle, |1\rangle\}$ basis).
- 3: Communicate the outcome $x \in \{0, 1\}$ of the measurement to the nodes $C_1, C_2, \dots, C_m$. If the outcome is $x = 0$, then $C_1, C_2, \dots, C_m$ do nothing; if the outcome is $x = 1$, then $C_1, C_2, \dots, C_m$ apply the Pauli-X gate to their qubits.

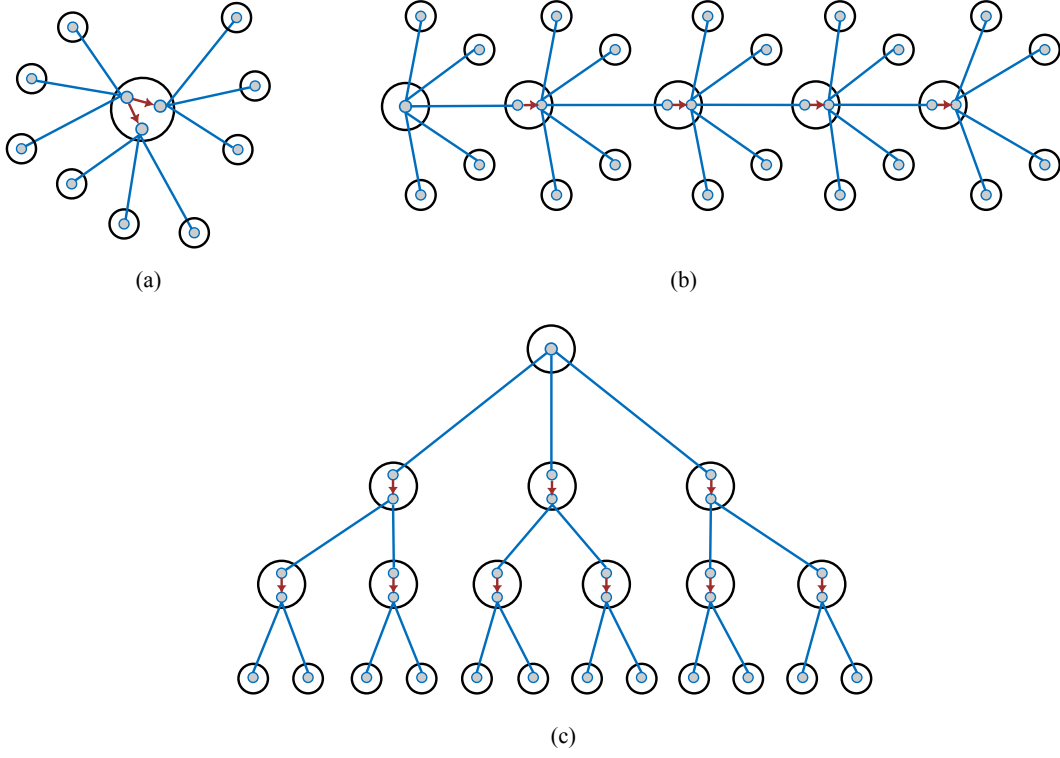


FIG. 11. Fusion of GHZ states in various configurations. The arrows indicate CNOT gates, pointing from the source qubit to the target qubit. The target qubit is then measured in the Pauli-Z basis. (a) Three GHZ states in a star configuration (Protocol 7). (b) Five GHZ states in a linear chain (Protocol 8). (c) 10 GHZ states in a tree configuration (Protocol 3).

1. Fusion of two GHZ states (Protocol 6)

Consider the following two GHZ states:

$$|\text{GHZ}_{n+1}\rangle_{A_1 B_{1:n}} = \frac{1}{\sqrt{2}} \left(|0\rangle_{A_1} \otimes |0\rangle_{B_{1:n}}^{\otimes n} + |1\rangle_{A_1} \otimes |1\rangle_{B_{1:n}}^{\otimes n} \right), \quad (\text{E1})$$

$$|\text{GHZ}_{m+1}\rangle_{A_2 C_{1:m}} = \frac{1}{\sqrt{2}} \left(|0\rangle_{A_2} \otimes |0\rangle_{C_{1:m}}^{\otimes m} + |1\rangle_{A_2} \otimes |1\rangle_{C_{1:m}}^{\otimes m} \right), \quad (\text{E2})$$

such that the qubits A_1 and A_2 are located at the same node. The protocol for fusing these two states into a larger GHZ state is provided in Protocol 6.

Let us now prove that after executing Protocol 6 we deterministically have the GHZ state $|\text{GHZ}_{n+m+1}\rangle_{A_1 B_{1:n} C_{1:m}}$ shared by $A_1, B_1, \dots, B_n, C_1, \dots, C_m$. First,

$$\begin{aligned} |\text{GHZ}_{n+1}\rangle \otimes |\text{GHZ}_{m+1}\rangle &= \frac{1}{2} \left(|0\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \right. \\ &\quad + |0\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} \\ &\quad + |1\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ &\quad \left. + |1\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} \right). \end{aligned} \quad (\text{E3})$$

Then, after the CNOT between A_1 and A_2 , the state becomes

$$\begin{aligned} & \frac{1}{2} (|0\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ & + |0\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} \\ & + |1\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ & + |1\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m}). \end{aligned} \quad (\text{E4})$$

From this, we can see immediately that, when A_2 measures its qubit in the $\{|0\rangle, |1\rangle\}$ basis, then the outcome “0” occurs with probability $\frac{1}{2}$, and the state conditioned on this outcome is

$$\frac{1}{\sqrt{2}} (|0\rangle_{A_1} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} + |1\rangle_{A_1} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m}) = |\text{GHZ}_{n+m+1}\rangle. \quad (\text{E5})$$

On the other hand, if the outcome “1” occurs, also with probability $\frac{1}{2}$, then the state conditioned on this outcome is

$$\frac{1}{\sqrt{2}} (|0\rangle_{A_1} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} + |1\rangle_{A_1} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m}). \quad (\text{E6})$$

Thus, if $C_1, C_2, \dots, C_m$ apply the Pauli X gate to their qubits, then we obtain the required GHZ state $|\text{GHZ}_{n+m+1}\rangle$. This completes the proof.

Remark 3. Let us note that in Protocol 6, we fused the two GHZ states using the first two qubits, A_1 and A_2 , of the two GHZ states. It is possible, however, to fuse two GHZ states using any two qubits, one from each state, as long as both qubits belong to the same node. ◀

2. Fusion of GHZ states in a star topology (Protocol 7)

Protocol 6 generalizes to fusing multiple (more than two) GHZ states in a star topology; see Fig. 11(a). It is analogous to Protocol 2. Consider $N \in \{2, 3, \dots\}$ GHZ states, $|\text{GHZ}_{n_1+1}\rangle_{A_1 B_{1:n_1}^1}, |\text{GHZ}_{n_2+1}\rangle_{A_1 B_{1:n_2}^2}, \dots, |\text{GHZ}_{n_N+1}\rangle_{A_N B_{1:n_N}^N}$, such that the qubits $A_i, i \in \{1, 2, \dots, N\}$, are located at the same node. The protocol for fusing all of these GHZ states into the one large GHZ state $|\text{GHZ}_{1+N_{\text{tot}}}\rangle_{A_1 B_{1:n_1}^1 \dots B_{1:n_N}^N}$, with $N_{\text{tot}} = n_1 + n_2 + \dots + n_N$, is as follows.

Protocol 7 Fusion of GHZ states in a star topology

Input: $N \in \{2, 3, \dots\}$ GHZ states, $|\text{GHZ}_{n_1+1}\rangle_{A_1 B_{1:n_1}^1}, |\text{GHZ}_{n_2+1}\rangle_{A_2 B_{1:n_2}^2}, \dots, |\text{GHZ}_{n_N+1}\rangle_{A_N B_{1:n_N}^N}$, such that the qubits $A_i, i \in \{1, 2, \dots, N\}$, are located at the same node.

Output: GHZ state $|\text{GHZ}_{1+N_{\text{tot}}}\rangle_{A_1 B_{1:n_1}^1 \dots B_{1:n_N}^N}$, with $N_{\text{tot}} = n_1 + n_2 + \dots + n_N$.

- 1: Apply the gates $\text{CNOT}_{A_1 A_2}, \text{CNOT}_{A_1 A_3}, \dots, \text{CNOT}_{A_1 A_N}$.
 - 2: Measure each of the qubits $A_2, A_3, \dots, A_N$ in the basis $\{|0\rangle, |1\rangle\}$.
 - 3: The outcome of measurement on A_2 is communicated to $B_{1:n_2}^2$, the outcome of measurement on A_3 is communicated to $B_{1:n_3}^3$, etc.
 - 4: For every set $B_{1:n_i}^i$ of qubits, $i \in \{2, 3, \dots, N\}$: if the outcome communicated by A_i is 0, then do nothing; if the outcome communicated by A_i is 1, then apply the Pauli- X gate to all qubits $B_{1:n_i}^i$.
-

The number of gates needed in Protocol 7 is the number of CNOT gates in the first step, which is $N - 1$.

We can verify that Protocol 7 works as claimed. The initial state is

$$\begin{aligned} \bigotimes_{j=1}^N |\text{GHZ}_{n_j+1}\rangle_{A_j B_{1:n_j}^j} &= \frac{1}{\sqrt{2^N}} \left(\sum_{x_1 \in \{0,1\}} |x_1\rangle_{A_1} \otimes |x_1\rangle_{B_{1:n_1}^1}^{\otimes n_1} \right) \otimes \left(\sum_{x_2 \in \{0,1\}} |x_2\rangle_{A_2} \otimes |x_2\rangle_{B_{1:n_2}^2}^{\otimes n_2} \right) \otimes \dots \\ &\quad \otimes \left(\sum_{x_N \in \{0,1\}} |x_N\rangle_{A_N} \otimes |x_N\rangle_{B_{1:n_N}^N}^{\otimes n_N} \right). \end{aligned} \quad (\text{E7})$$

After the CNOT gates $\text{CNOT}_{A_1 A_2}, \text{CNOT}_{A_1 A_3}, \dots, \text{CNOT}_{A_1 A_N}$, we obtain the state

$$\frac{1}{\sqrt{2^N}} \sum_{x_1, x_2, \dots, x_N \in \{0,1\}} |x_1\rangle_{A_1} \otimes |x_1\rangle_{B_{1:n_1}^{\otimes n_1}} \otimes |x_1 \oplus x_2\rangle_{A_2} \otimes |x_2\rangle_{B_{1:n_2}^{\otimes n_2}} \otimes \dots \otimes |x_1 \oplus x_N\rangle_{A_N} \otimes |x_N\rangle_{B_{1:n_N}^{\otimes n_N}}. \quad (\text{E8})$$

Then, for every collection $(y_1, y_2, \dots, y_{N-1}) \in \{0,1\}^{N-1}$ of measurement outcomes corresponding to the Z-basis measurement of the qubits $A_2, A_3, \dots, A_N$, we obtain the state

$$\frac{1}{\sqrt{2}} \sum_{x_1, x_2, \dots, x_N \in \{0,1\}^N} |x_1\rangle_{A_1} \otimes |x_1\rangle_{B_{1:n_1}^{\otimes n_1}} \otimes \langle y_1 | x_1 \oplus x_2 \rangle |x_2\rangle_{B_{1:n_2}^{\otimes n_2}} \otimes \dots \otimes \langle y_{N-1} | x_1 \oplus x_N \rangle |x_N\rangle_{B_{1:n_N}^{\otimes n_N}} \quad (\text{E9})$$

$$= \sum_{x_1 \in \{0,1\}} |x_1\rangle_{A_1} \otimes |x_1\rangle_{B_{1:n_1}^{\otimes n_1}} \otimes |x_1 \oplus y_1\rangle_{B_{1:n_2}^{\otimes n_2}} \otimes \dots \otimes |x_1 \oplus y_{N-1}\rangle_{B_{1:n_N}^{\otimes n_N}} \quad (\text{E10})$$

$$= \left(\mathbb{1}_{A_1 B_{1:n_1}^1} \otimes (X^{y_1})_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes (X^{y_{N-1}})_{B_{1:n_N}^N}^{\otimes n_N} \right) |\text{GHZ}_{1+N_{\text{tot}}}\rangle_{A_1 B_{1:n_1} \dots B_{1:n_N}}, \quad (\text{E11})$$

as required.

We can write down the action of Protocol 7 as an LOCC channel as follows:

$$\begin{aligned} \mathcal{L}(\rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \dots A_N B_{1:n_N}^N}) &= \sum_{\vec{y} \in \{0,1\}^N} \left(\langle \vec{y} |_{A_{2:N}} \text{CNOT}_N \otimes (X^{y_1})_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes (X^{y_{N-1}})_{B_{1:n_N}^N}^{\otimes n_N} \right) \rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \dots A_N B_{1:n_N}^N \\ &\quad \times \left(\text{CNOT}_N | \vec{y} \rangle_{A_{2:N}} \otimes (X^{y_1})_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes (X^{y_{N-1}})_{B_{1:n_N}^N}^{\otimes n_N} \right), \quad (\text{E12}) \end{aligned}$$

where $\rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \dots A_N B_{1:n_N}^N}$ is an arbitrary state, and we recall the definition of CNOT_N from (22).

Proposition 14 (Fidelity after Protocol 7). Consider states $\rho_{A_1 B_{1:n_1}^1}^1, \rho_{A_2 B_{1:n_2}^2}^2, \dots, \rho_{A_N B_{1:n_N}^N}^N$, representing noisy input states to Protocol 7. With these input states, the fidelity of the state produced by Protocol 7 with respect to the final, ideal GHZ state is

$$\begin{aligned} \langle \text{GHZ}_{1+N_{\text{tot}}} | \mathcal{L}(\rho_{A_1 B_{1:n_1}^1}^1 \otimes \rho_{A_2 B_{1:n_2}^2}^2 \otimes \dots \otimes \rho_{A_N B_{1:n_N}^N}^N) | \text{GHZ}_{1+N_{\text{tot}}} \rangle \\ = \sum_{\vec{z} \in \{0,1\}^{N-1}} \langle \text{GHZ}_{n_1+1}^{(z', \vec{0})} | \rho_{A_1 B_{1:n_1}^1}^1 | \text{GHZ}_{n_1+1}^{(z', \vec{0})} \rangle \langle \text{GHZ}_{n_2+1}^{(z_1, \vec{0})} | \rho_{A_2 B_{1:n_2}^2}^2 | \text{GHZ}_{n_2+1}^{(z_1, \vec{0})} \rangle \dots \langle \text{GHZ}_{n_N+1}^{(z_{N-1}, \vec{0})} | \rho_{A_N B_{1:n_N}^N}^N | \text{GHZ}_{n_N+1}^{(z_{N-1}, \vec{0})} \rangle, \quad (\text{E13}) \end{aligned}$$

$z' = z_1 \oplus z_2 \oplus \dots \oplus z_{N-1}$, and we have introduced the following n -qubit GHZ basis, for $n \in \{2, 3, \dots\}$:

$$|\text{GHZ}_n^{(z, \vec{x})}\rangle := (Z^z \otimes X^{\vec{x}}) |\text{GHZ}_n\rangle, \quad (\text{E14})$$

with $z \in \{0,1\}$ and $\vec{x} \in \{0,1\}^{n-1}$.

Proof. We start by defining

$$|w(\vec{y})\rangle := \left(\text{CNOT}_N | \vec{y} \rangle_{A_{2:N}} \otimes (X^{y_1})_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes (X^{y_{N-1}})_{B_{1:n_N}^N}^{\otimes n_N} \right) |\text{GHZ}_{1+N_{\text{tot}}}\rangle_{A_1 B_{1:n_1}^1 B_{1:n_2}^2 \dots B_{1:n_N}^N} \quad (\text{E15})$$

$$= \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} \text{CNOT}_N |x\rangle_{A_1} | \vec{y} \rangle_{A_{2:N}} \otimes |x\rangle_{B_{1:n_1}^1}^{\otimes n_1} \otimes (X^{y_1})_{B_{1:n_2}^2}^{\otimes n_2} |x\rangle_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes (X^{y_{N-1}})_{B_{1:n_N}^N}^{\otimes n_N} |x\rangle_{B_{1:n_N}^N}^{\otimes n_N} \quad (\text{E16})$$

$$= \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |x\rangle_{A_1} |y_1 \oplus x\rangle_{A_2} \dots |y_{N-1} \oplus x\rangle_{A_N} \otimes |x\rangle_{B_{1:n_1}^1}^{\otimes n_1} \otimes |y_1 \oplus x\rangle_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes |y_{N-1} \oplus x\rangle_{B_{1:n_N}^N}^{\otimes n_N} \quad (\text{E17})$$

$$= \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |x\rangle_{A_1} |x\rangle_{B_{1:n_1}^1}^{\otimes n_1} \otimes |y_1 \oplus x\rangle_{A_2} |y_1 \oplus x\rangle_{B_{1:n_2}^2}^{\otimes n_2} \otimes \dots \otimes |y_{N-1} \oplus x\rangle_{A_N} |y_{N-1} \oplus x\rangle_{B_{1:n_N}^N}^{\otimes n_N}. \quad (\text{E18})$$

Now, it is straightforward to show that

$$|x\rangle^{\otimes n} = \frac{1}{\sqrt{2}} \sum_{z \in \{0,1\}} (-1)^{zx} |\text{GHZ}_n^{(z, \vec{0})}\rangle, \quad (\text{E19})$$

for $x \in \{0, 1\}$. With this, we find that

$$|w(\vec{y})\rangle = \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} \left(\frac{1}{\sqrt{2}} \sum_{z_1 \in \{0,1\}} (-1)^{z_1 x} |\text{GHZ}_{n_1+1}^{(z_1, \vec{0})}\rangle_{A_1 B_{1:n_1}^1} \right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{z_2 \in \{0,1\}} (-1)^{z_2 (y_1 \oplus x)} |\text{GHZ}_{n_2+1}^{(z_2, \vec{0})}\rangle_{A_2 B_{1:n_2}^2} \right) \otimes \cdots$$

$$\cdots \otimes \left(\frac{1}{\sqrt{2}} \sum_{z_N \in \{0,1\}} (-1)^{z_N (y_{N-1} \oplus x)} |\text{GHZ}_{n_N+1}^{(z_N, \vec{0})}\rangle_{A_N B_{1:n_N}^N} \right) \quad (\text{E20})$$

$$= \frac{1}{\sqrt{2^{N+1}}} \sum_{z_1, z_2, \dots, z_N \in \{0,1\}} (-1)^{z_2 y_1 + z_3 y_2 + \cdots + z_N y_{N-1}} \underbrace{\left(\sum_{x \in \{0,1\}} (-1)^{x(z_1 \oplus z_2 \oplus \cdots \oplus z_N)} \right)}_{2\delta_{z_1, z_2 \oplus \cdots \oplus z_N}}$$

$$\times |\text{GHZ}_{n_1+1}^{(z_1, \vec{0})}\rangle_{A_1 B_{1:n_1}^1} \otimes |\text{GHZ}_{n_2+1}^{(z_2, \vec{0})}\rangle_{A_2 B_{1:n_2}^2} \otimes \cdots \otimes |\text{GHZ}_{n_N+1}^{(z_N, \vec{0})}\rangle_{A_N B_{1:n_N}^N} \quad (\text{E21})$$

$$= \frac{1}{\sqrt{2^{N-1}}} \sum_{z_2, z_3, \dots, z_N \in \{0,1\}} (-1)^{z_2 y_1 + z_3 y_2 + \cdots + z_N y_{N-1}} |\text{GHZ}_{n_1+1}^{(z', \vec{0})}\rangle_{A_1 B_{1:n_1}^1} \otimes |\text{GHZ}_{n_2+1}^{(z_2, \vec{0})}\rangle_{A_2 B_{1:n_2}^2} \otimes \cdots \otimes |\text{GHZ}_{n_N+1}^{(z_N, \vec{0})}\rangle_{A_N B_{1:n_N}^N}, \quad (\text{E22})$$

where in the final line we let $z' = z_2 \oplus z_3 \oplus \cdots \oplus z_N$. Therefore, the fidelity of the state in (E12) is given by

$$\langle \text{GHZ}_{1+N_{\text{tot}}} | \mathcal{L}(\rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \cdots A_N B_{1:n_N}^N}) | \text{GHZ}_{1+N_{\text{tot}}} \rangle$$

$$= \sum_{\vec{y} \in \{0,1\}^{N-1}} \langle w(\vec{y}) | \rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \cdots A_N B_{1:n_N}^N | w(\vec{y}) \rangle \quad (\text{E23})$$

$$= \frac{1}{2^{N-1}} \sum_{\vec{y}, \vec{z}, \vec{w} \in \{0,1\}^{N-1}} (-1)^{\vec{z} \cdot \vec{y}} (-1)^{\vec{w} \cdot \vec{y}} \left(\langle \text{GHZ}_{n_1+1}^{(z', \vec{0})} |_{A_1 B_{1:n_1}^1} \otimes \langle \text{GHZ}_{n_2+1}^{(z_1, \vec{0})} |_{A_2 B_{1:n_2}^2} \otimes \cdots \otimes \langle \text{GHZ}_{n_N+1}^{(z_N, \vec{0})} |_{A_N B_{1:n_N}^N} \right)$$

$$\times \rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \cdots A_N B_{1:n_N}^N} \left(|\text{GHZ}_{n_1+1}^{(w', \vec{0})}\rangle_{A_1 B_{1:n_1}^1} \otimes |\text{GHZ}_{n_2+1}^{(w_1, \vec{0})}\rangle_{A_2 B_{1:n_2}^2} \otimes \cdots \otimes |\text{GHZ}_{n_N+1}^{(w_N, \vec{0})}\rangle_{A_N B_{1:n_N}^N} \right) \quad (\text{E24})$$

$$= \sum_{\vec{z} \in \{0,1\}^{N-1}} \left(\langle \text{GHZ}_{n_1+1}^{(z', \vec{0})} |_{A_1 B_{1:n_1}^1} \otimes \langle \text{GHZ}_{n_2+1}^{(z_1, \vec{0})} |_{A_2 B_{1:n_2}^2} \otimes \cdots \otimes \langle \text{GHZ}_{n_N+1}^{(z_N, \vec{0})} |_{A_N B_{1:n_N}^N} \right)$$

$$\times \rho_{A_1 B_{1:n_1}^1 A_2 B_{1:n_2}^2 \cdots A_N B_{1:n_N}^N} \left(|\text{GHZ}_{n_1+1}^{(z', \vec{0})}\rangle_{A_1 B_{1:n_1}^1} \otimes |\text{GHZ}_{n_2+1}^{(z_1, \vec{0})}\rangle_{A_2 B_{1:n_2}^2} \otimes \cdots \otimes |\text{GHZ}_{n_N+1}^{(z_N, \vec{0})}\rangle_{A_N B_{1:n_N}^N} \right), \quad (\text{E25})$$

as required. ■

3. Fusion of GHZ states in a linear topology (Protocol 8)

Fusing GHZ states in a linear topology (see Fig. 11(b)) proceeds similarly, as we outline in Protocol 8.

Protocol 8 Fusion of GHZ states in a linear topology

Input: $N \in \{2, 3, \dots\}$ GHZ states, $|\text{GHZ}_{n_1+1}\rangle_{A_{1:n_1}^1 R_1}$, $|\text{GHZ}_{n_2+2}\rangle_{R_2 A_{1:n_2}^2 R_3}$, $\dots$, $|\text{GHZ}_{n_{N-1}+2}\rangle_{R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3}}$, $|\text{GHZ}_{n_N+1}\rangle_{R_{2N-2} A_{1:n_N}^N}$, such that the qubits R_i and R_{i+1} , $i \in \{1, 3, 5, \dots, 2N-1\}$, are located at the same node; see Fig. 11(b).

Output: GHZ state $|\text{GHZ}_{N_{\text{tot}}}\rangle$, with $N_{\text{tot}} = n_1 + n_2 + \cdots + n_N + N - 1$, shared by $A_{1:n_1}^1, R_1, A_{1:n_2}^2, R_3, \dots, A_{1:n_{N-1}}^{N-1}, R_{2N-3}, A_{1:n_N}^N$.

- 1: Apply the CNOT gates $\text{CNOT}_{R_i R_{i+1}}$, $i \in \{1, 3, 5, \dots, 2N-3\}$.
 - 2: Measure the target qubits $R_2, R_4, R_6, \dots, R_{2N-2}$ in the Z-basis $\{|0\rangle, |1\rangle\}$. The outcomes $z_1, z_2, \dots, z_{N-1} \in \{0, 1\}$ are communicated to all nodes.
 - 3: The qubits associated with the second GHZ state apply the Pauli-X correction X^{z_1} ; the qubits associated with the third GHZ state apply the Pauli-X correction $X^{z_1 \oplus z_2}$; etc. In general, the qubits associated with the k -th node, $k \in \{2, 3, \dots, N\}$, apply the Pauli-X correction $X^{z_1 \oplus z_2 \oplus \cdots \oplus z_{k-1}}$.
-

Let us verify that Protocol 8 performs as expected. The initial state is

$$\begin{aligned} & |\text{GHZ}_{n_1+1}\rangle_{A_{1:n_1}^1 R_1} \otimes |\text{GHZ}_{n_2+2}\rangle_{R_2 A_{1:n_2}^2 R_3} \otimes |\text{GHZ}_{n_3+2}\rangle_{R_4 A_{1:n_3}^3 R_5} \cdots \otimes |\text{GHZ}_{n_{N-1}+2}\rangle_{R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3}} \otimes |\text{GHZ}_{n_N+1}\rangle_{R_{2N-2} A_{1:n_N}^N} \\ &= \frac{1}{\sqrt{2^N}} \sum_{x_1, x_2, x_3, \dots, x_{N-1}, x_N \in \{0,1\}} |x_1\rangle_{A_{1:n_1}^1}^{\otimes n_1} |x_1\rangle_{R_1} \otimes |x_2\rangle_{R_2} |x_2\rangle_{A_{1:n_2}^2}^{\otimes n_2} |x_2\rangle_{R_3} \otimes |x_3\rangle_{R_4} |x_3\rangle_{A_{1:n_3}^3}^{\otimes n_3} |x_3\rangle_{R_5} \otimes \cdots \\ &\quad \cdots \otimes |x_{N-1}\rangle_{R_{2N-4}} |x_{N-1}\rangle_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} |x_{N-1}\rangle_{R_{2N-3}} \otimes |x_N\rangle_{R_{2N-2}} |x_N\rangle_{A_{1:n_N}^N}^{\otimes n_N}. \end{aligned} \quad (\text{E26})$$

After the CNOT gates, we obtain

$$\begin{aligned} & \frac{1}{\sqrt{2^N}} \sum_{x_1, x_2, x_3, \dots, x_{N-1}, x_N \in \{0,1\}} |x_1\rangle_{A_{1:n_1}^1}^{\otimes n_1} |x_1\rangle_{R_1} \otimes |x_1 \oplus x_2\rangle_{R_2} |x_2\rangle_{A_{1:n_2}^2}^{\otimes n_2} |x_2\rangle_{R_3} \otimes |x_2 \oplus x_3\rangle_{R_4} |x_3\rangle_{A_{1:n_3}^3}^{\otimes n_3} |x_3\rangle_{R_5} \otimes \cdots \\ &\quad \cdots \otimes |x_{N-2} \oplus x_{N-1}\rangle_{R_{2N-4}} |x_{N-1}\rangle_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} |x_{N-1}\rangle_{R_{2N-3}} \otimes |x_{N-1} \oplus x_N\rangle_{R_{2N-2}} |x_N\rangle_{A_{1:n_N}^N}^{\otimes n_N}. \end{aligned} \quad (\text{E27})$$

Then, upon measuring the target qubits $R_2, R_4, \dots$ in the $\{|0\rangle, |1\rangle\}$ basis, to every outcome string $(z_1, z_2, \dots, z_{N-1}) \in \{0, 1\}^{N-1}$ we have the following associated (unnormalized) post-measurement state:

$$\begin{aligned} & \frac{1}{\sqrt{2^N}} \sum_{x_1, x_2, x_3, \dots, x_{N-1}, x_N \in \{0,1\}} |x_1\rangle_{A_{1:n_1}^1}^{\otimes n_1} |x_1\rangle_{R_1} \otimes \langle z_1 | x_1 \oplus x_2 \rangle_{R_2} |x_2\rangle_{A_{1:n_2}^2}^{\otimes n_2} |x_2\rangle_{R_3} \otimes \langle z_2 | x_2 \oplus x_3 \rangle_{R_4} |x_3\rangle_{A_{1:n_3}^3}^{\otimes n_3} |x_3\rangle_{R_5} \otimes \cdots \\ &\quad \cdots \otimes \langle z_{N-2} | x_{N-2} \oplus x_{N-1} \rangle_{R_{2N-4}} |x_{N-1}\rangle_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} |x_{N-1}\rangle_{R_{2N-3}} \otimes \langle z_{N-1} | x_{N-1} \oplus x_N \rangle_{R_{2N-2}} |x_N\rangle_{A_{1:n_N}^N}^{\otimes n_N}, \end{aligned} \quad (\text{E28})$$

which we can simplify to

$$\begin{aligned} & \frac{1}{\sqrt{2^N}} \sum_{x_1 \in \{0,1\}} |x_1\rangle_{A_{1:n_1}^1}^{\otimes n_1} |x_1\rangle_{R_1} \otimes |z_1 \oplus x_1\rangle_{A_{1:n_2}^2}^{\otimes n_2} |z_1 \oplus x_1\rangle_{R_3} \otimes |z_1 \oplus z_2 \oplus x_1\rangle_{A_{1:n_3}^3}^{\otimes n_3} |z_1 \oplus z_2 \oplus x_1\rangle_{R_5} \otimes \cdots \\ &\quad \cdots \otimes |z_1 \oplus z_2 \oplus \cdots \oplus z_{N-2} \oplus x_1\rangle_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} |z_1 \oplus z_2 \oplus \cdots \oplus z_{N-2} \oplus x_1\rangle_{R_{2N-3}} \otimes |z_1 \oplus z_2 \oplus \cdots \oplus z_{N-1} \oplus x_1\rangle_{A_{1:n_N}^N}^{\otimes n_N} \\ &= \left((X^{z_1})_{A_{1:n_2}^2 R_3}^{\otimes (n_2+1)} \otimes (X^{z_1 \oplus z_2})_{A_{1:n_3}^3 R_5}^{\otimes (n_3+1)} \otimes \cdots \otimes (X^{z_1 \oplus z_2 \oplus \cdots \oplus z_{N-2}})_{A_{1:n_{N-1}}^{N-1} R_{2N-3}}^{\otimes (n_{N-1}+1)} \otimes (X^{z_1 \oplus z_2 \oplus \cdots \oplus z_{N-1}})_{A_{1:n_N}^N}^{\otimes n_N} \right) |\text{GHZ}_{N_{\text{tot}}}\rangle. \end{aligned} \quad (\text{E29})$$

We thus see that, as expected, the resulting state is the required GHZ state up to the appropriate Pauli-X corrections.

As an LOCC channel, we can write the action of Protocol 8 as follows:

$$\begin{aligned} & \mathcal{L} \left(\rho_{A_{1:n_1}^1 R_1 R_2 A_{1:n_2}^2 R_3 \cdots R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3} R_{2N-2} A_{1:n_N}^N} \right) \\ &= \sum_{\vec{z} \in \{0,1\}^{N-1}} \left((X^{y_1})_{A_{1:n_2}^2 R_3}^{\otimes (n_2+1)} \otimes (X^{y_2})_{A_{1:n_3}^3 R_5}^{\otimes (n_3+1)} \otimes \cdots \otimes (X^{y_{N-2}})_{A_{1:n_{N-1}}^{N-1} R_{2N-3}}^{\otimes (n_{N-1}+1)} \otimes (X^{y_{N-1}})_{A_{1:n_N}^N}^{\otimes n_N} \right) \\ &\quad \times \left(\langle \vec{z} | R_2 R_4 R_6 \cdots R_{2N-2} \text{CNOT}'_N \right) \rho_{A_{1:n_1}^1 R_1 R_2 A_{1:n_2}^2 R_3 \cdots R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3} R_{2N-2} A_{1:n_N}^N} \left(\text{CNOT}'_N | \vec{z} \rangle_{R_2 R_4 R_6 \cdots R_{2N-2}} \right) \\ &\quad \times \left((X^{y_1})_{A_{1:n_2}^2 R_3}^{\otimes (n_2+1)} \otimes (X^{y_2})_{A_{1:n_3}^3 R_5}^{\otimes (n_3+1)} \otimes \cdots \otimes (X^{y_{N-2}})_{A_{1:n_{N-1}}^{N-1} R_{2N-3}}^{\otimes (n_{N-1}+1)} \otimes (X^{y_{N-1}})_{A_{1:n_N}^N}^{\otimes n_N} \right), \end{aligned} \quad (\text{E30})$$

where

$$y_k := z_1 \oplus z_2 \oplus \cdots \oplus z_k, \quad k \in \{1, 2, \dots, N-1\}, \quad (\text{E31})$$

$$\text{CNOT}'_N := \text{CNOT}_{R_1 R_2} \text{CNOT}_{R_3 R_4} \cdots \text{CNOT}_{R_{2N-3} R_{2N-2}}. \quad (\text{E32})$$

Proposition 15 (Fidelity after Protocol 8). Consider states $\rho_{A_{1:n_1}^1 R_1}^1, \rho_{R_2 A_{1:n_2}^2 R_3}^2, \dots, \rho_{R_{2N-2} A_{1:n_N}^N}^N$, representing noisy input states to Protocol 8. With these input states, the fidelity of the state produced by Protocol 8 with respect to the final, ideal GHZ state is

$$\begin{aligned} & \langle \text{GHZ}_{N_{\text{tot}}} | \mathcal{L} \left(\rho_{A_{1:n_1}^1 R_1}^1 \otimes \rho_{R_2 A_{1:n_2}^2 R_3}^2 \otimes \cdots \otimes \rho_{R_{2N-2} A_{1:n_N}^N}^N \right) \\ &= \sum_{y_1, y_2, \dots, y_{N-1} \in \{0,1\}} \langle \text{GHZ}_{n_1+1}^{(y_1, \vec{0})} | \rho_{A_{1:n_1}^1 R_1}^1 | \text{GHZ}_{n_1+1}^{(y_1, \vec{0})} \rangle \langle \text{GHZ}_{n_2+2}^{(y_2, \vec{0})} | \rho_{R_2 A_{1:n_2}^2 R_3}^2 | \text{GHZ}_{n_2+2}^{(y_2, \vec{0})} \rangle \cdots \\ &\quad \cdots \langle \text{GHZ}_{n_N+1}^{(y', \vec{0})} | \rho_{R_{2N-2} A_{1:n_N}^N}^N | \text{GHZ}_{n_N+1}^{(y', \vec{0})} \rangle, \end{aligned} \quad (\text{E33})$$

where $y' = y_1 \oplus y_2 \oplus \dots \oplus y_{N-1}$.

Proof. We start by defining the vector

$$|u(\vec{z})\rangle := (\text{CNOT}'_N |\vec{z}\rangle_{R_2 R_4 R_6 \dots R_{2N-2}}) \left((X^{y_1})_{A_{1:n_2}^2 R_3}^{\otimes(n_2+1)} \otimes (X^{y_2})_{A_{1:n_3}^3 R_5}^{\otimes(n_3+1)} \otimes \dots \right. \\ \left. \dots \otimes (X^{y_{N-2}})_{A_{1:n_{N-1}}^{N-1} R_{2N-3}}^{\otimes(n_{N-1}+1)} \otimes (X^{y_{N-1}})_{A_{1:n_N}^N}^{\otimes n_N} \right) |\text{GHZ}_{N_{\text{tot}}}\rangle_{A_{1:n_1}^1 R_1 A_{1:n_2}^2 R_3 \dots A_{1:n_{N-1}}^{N-1} R_{2N-3} A_{1:n_N}^N}, \quad (\text{E34})$$

which can be simplified as follows:

$$|u(\vec{z})\rangle = \frac{1}{\sqrt{2}} \sum_{\vec{x} \in \{0,1\}} |x\rangle_{A_{1:n_1}^1}^{\otimes n_1} \text{CNOT}_{R_1 R_2} |x, z_1\rangle_{R_1 R_2} \otimes (X^{y_1} |x\rangle)_{A_{1:n_2}^2}^{\otimes n_2} \text{CNOT}_{R_3 R_4} X_{R_3}^{y_1} |x, z_2\rangle_{R_3 R_4} \otimes (X^{y_2} |x\rangle)_{A_{1:n_3}^3}^{\otimes n_3} \text{CNOT}_{R_5 R_6} X_{R_5}^{y_2} |x, z_3\rangle_{R_5 R_6} \otimes \dots \\ \dots \otimes (X^{y_{N-2}} |x\rangle)_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} \text{CNOT}_{R_{2N-3} R_{2N-2}} X_{R_{2N-3}}^{y_{N-2}} |x, z_{N-1}\rangle_{R_{2N-3} R_{2N-2}} (X^{y_{N-1}} |x\rangle)_{A_{1:n_N}^N}^{\otimes n_N} \quad (\text{E35})$$

$$= \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |x\rangle_{A_{1:n_1}^1}^{\otimes n_1} |x\rangle_{R_1} \otimes |x \oplus z_1\rangle_{R_2} |x \oplus y_1\rangle_{A_{1:n_2}^2}^{\otimes n_2} |x \oplus y_1\rangle_{R_3} \otimes |x \oplus y_1 \oplus z_2\rangle_{R_4} |x \oplus y_2\rangle_{A_{1:n_3}^3}^{\otimes n_3} |x \oplus y_2\rangle_{R_5} \otimes |x \oplus y_2 \oplus z_3\rangle_{R_6} \otimes \dots \\ \dots \otimes |x \oplus y_{N-2}\rangle_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} |x \oplus y_{N-2}\rangle_{R_{2N-3}} \otimes |x \oplus y_{N-2} \oplus z_{N-1}\rangle_{R_{2N-2}} |x \oplus y_{N-1}\rangle_{A_{1:n_N}^N}^{\otimes n_N} \quad (\text{E36})$$

$$= \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |x\rangle_{A_{1:n_1}^1}^{\otimes n_1} |x\rangle_{R_1} \otimes |x \oplus z_1\rangle_{R_2} |x \oplus z_1\rangle_{A_{1:n_2}^2}^{\otimes n_2} |x \oplus z_1\rangle_{R_3} \otimes |x \oplus z_1 \oplus z_2\rangle_{R_4} |x \oplus z_1 \oplus z_2\rangle_{A_{1:n_3}^3}^{\otimes n_3} |x \oplus z_1 \oplus z_2\rangle_{R_5} \\ \otimes |x \oplus z_1 \oplus z_2 \oplus z_3\rangle_{R_6} \otimes \dots \otimes |x \oplus z_1 \oplus \dots \oplus z_{N-2}\rangle_{A_{1:n_{N-1}}^{N-1}}^{\otimes n_{N-1}} |x \oplus z_1 \oplus \dots \oplus z_{N-2}\rangle_{R_{2N-3}} \\ \otimes |x \oplus z_1 \oplus \dots \oplus z_{N-2} \oplus z_{N-1}\rangle_{R_{2N-2}} |x \oplus z_1 \oplus \dots \oplus z_{N-1}\rangle_{A_{1:n_N}^N}^{\otimes n_N}, \quad (\text{E37})$$

where we have used (E31) and (E32). Let us now use (E19) to make further simplifications, as follows:

$$|u(\vec{z})\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^N}} \sum_{x \in \{0,1\}} \sum_{y_1, y_2, \dots, y_N \in \{0,1\}} (-1)^{x y_1} |\text{GHZ}_{n_1+1}^{(y_1, \vec{0})}\rangle_{A_{1:n_1}^1 R_1} \otimes (-1)^{(x \oplus z_1) y_2} |\text{GHZ}_{n_2+2}^{(y_2, \vec{0})}\rangle_{R_2 A_{1:n_2}^2 R_3} \\ \otimes (-1)^{(x \oplus z_1 \oplus z_2) y_3} |\text{GHZ}_{n_3+2}^{(y_3, \vec{0})}\rangle_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \otimes (-1)^{y_{N-1} (x \oplus z_1 \oplus \dots \oplus z_{N-2})} |\text{GHZ}_{n_{N-1}+2}^{(y_{N-1}, \vec{0})}\rangle_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \quad (\text{E38})$$

$$\otimes (-1)^{y_N (x \oplus z_1 \oplus \dots \oplus z_{N-1})} |\text{GHZ}_{n_N+1}^{(y_N, \vec{0})}\rangle_{R_{2N-2} A_{1:n_N}^N} \quad (\text{E39})$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^N}} \sum_{y_1, y_2, \dots, y_N \in \{0,1\}} \left(\sum_{x \in \{0,1\}} (-1)^{x(y_1 + y_2 + \dots + y_N)} \right) (-1)^{z_1(y_2 + \dots + y_N)} (-1)^{z_2(y_3 + \dots + y_N)} \dots (-1)^{z_{N-2}(y_{N-1} + y_N)} (-1)^{z_{N-1} y_N} \\ \times |\text{GHZ}_{n_1+1}^{(y_1, \vec{0})}\rangle_{A_{1:n_1}^1 R_1} \otimes |\text{GHZ}_{n_2+2}^{(y_2, \vec{0})}\rangle_{R_2 A_{1:n_2}^2 R_3} \otimes |\text{GHZ}_{n_3+2}^{(y_3, \vec{0})}\rangle_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \\ \dots \otimes |\text{GHZ}_{n_{N-1}+2}^{(y_{N-1}, \vec{0})}\rangle_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \otimes |\text{GHZ}_{n_N+1}^{(y_N, \vec{0})}\rangle_{R_{2N-2} A_{1:n_N}^N} \quad (\text{E40})$$

$$= \frac{1}{\sqrt{2^{N-1}}} \sum_{y_1, y_2, \dots, y_{N-1} \in \{0,1\}} (-1)^{z_1 y_1} (-1)^{z_2(y_1 + y_2)} \dots (-1)^{z_{N-2}(y_1 + y_2 + \dots + y_{N-2})} (-1)^{z_{N-1}(y_1 + y_2 + \dots + y_{N-1})} \\ \times |\text{GHZ}_{n_1+1}^{(y_1, \vec{0})}\rangle_{A_{1:n_1}^1 R_1} \otimes |\text{GHZ}_{n_2+2}^{(y_2, \vec{0})}\rangle_{R_2 A_{1:n_2}^2 R_3} \otimes |\text{GHZ}_{n_3+2}^{(y_3, \vec{0})}\rangle_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \\ \dots \otimes |\text{GHZ}_{n_{N-1}+2}^{(y_{N-1}, \vec{0})}\rangle_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \otimes |\text{GHZ}_{n_N+1}^{(y', \vec{0})}\rangle_{R_{2N-2} A_{1:n_N}^N}, \quad (\text{E41})$$

where $y' = y_1 + y_2 + \dots + y_{N-1}$. Therefore, the fidelity of the state in (E30) is

$$\sum_{z_1, z_2, \dots, z_{N-1} \in \{0,1\}} \langle u(\vec{z}) | \rho_{A_{1:n_1}^1 R_1 R_2 A_{1:n_2}^2 R_3 \dots R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3} R_{2N-2} A_{1:n_N}^N | u(\vec{z}) \rangle \quad (\text{E42})$$

$$= \frac{1}{2^{N-1}} \sum_{\substack{z_1, z_2, \dots, z_{N-1} \in \{0,1\} \\ y_1, y_2, \dots, y_{N-1} \in \{0,1\} \\ x_1, x_2, \dots, x_{N-1} \in \{0,1\}}} (-1)^{z_1(x_1+y_1)} (-1)^{z_2(y_1+y_2+x_1+x_2)} \dots (-1)^{z_{N-2}(y_1+\dots+y_{N-2}+x_1+\dots+x_{N-2})} (-1)^{z_{N-1}(y_1+\dots+y_{N-1}+x_1+\dots+x_{N-1})} \\ \times \left(\langle \text{GHZ}_{n_1+1}^{(y_1, \vec{0})} |_{A_{1:n_1}^1 R_1} \otimes \langle \text{GHZ}_{n_2+2}^{(y_2, \vec{0})} |_{R_2 A_{1:n_2}^2 R_3} \otimes \langle \text{GHZ}_{n_3+2}^{(y_3, \vec{0})} |_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \right. \\ \dots \otimes \langle \text{GHZ}_{n_{N-1}+2}^{(y_{N-1}, \vec{0})} |_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \otimes \langle \text{GHZ}_{n_N+1}^{(y', \vec{0})} |_{R_{2N-2} A_{1:n_N}^N} \left. \right) \rho_{A_{1:n_1}^1 R_1 R_2 A_{1:n_2}^2 R_3 \dots R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3} R_{2N-2} A_{1:n_N}^N} \\ \times \left(| \text{GHZ}_{n_1+1}^{(x_1, \vec{0})} \rangle_{A_{1:n_1}^1 R_1} \otimes | \text{GHZ}_{n_2+2}^{(x_2, \vec{0})} \rangle_{R_2 A_{1:n_2}^2 R_3} \otimes | \text{GHZ}_{n_3+2}^{(x_3, \vec{0})} \rangle_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \right. \\ \dots \otimes | \text{GHZ}_{n_{N-1}+2}^{(x_{N-1}, \vec{0})} \rangle_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \otimes | \text{GHZ}_{n_N+1}^{(x', \vec{0})} \rangle_{R_{2N-2} A_{1:n_N}^N} \left. \right) \quad (\text{E43})$$

$$= \sum_{y_1, y_2, \dots, y_{N-1} \in \{0,1\}} \left(\langle \text{GHZ}_{n_1+1}^{(y_1, \vec{0})} |_{A_{1:n_1}^1 R_1} \otimes \langle \text{GHZ}_{n_2+2}^{(y_2, \vec{0})} |_{R_2 A_{1:n_2}^2 R_3} \otimes \langle \text{GHZ}_{n_3+2}^{(y_3, \vec{0})} |_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \right. \\ \dots \otimes \langle \text{GHZ}_{n_{N-1}+2}^{(y_{N-1}, \vec{0})} |_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \otimes \langle \text{GHZ}_{n_N+1}^{(y', \vec{0})} |_{R_{2N-2} A_{1:n_N}^N} \left. \right) \rho_{A_{1:n_1}^1 R_1 R_2 A_{1:n_2}^2 R_3 \dots R_{2N-4} A_{1:n_{N-1}}^{N-1} R_{2N-3} R_{2N-2} A_{1:n_N}^N} \\ \times \left(| \text{GHZ}_{n_1+1}^{(y_1, \vec{0})} \rangle_{A_{1:n_1}^1 R_1} \otimes | \text{GHZ}_{n_2+2}^{(y_2, \vec{0})} \rangle_{R_2 A_{1:n_2}^2 R_3} \otimes | \text{GHZ}_{n_3+2}^{(y_3, \vec{0})} \rangle_{R_4 A_{1:n_3}^3 R_5} \otimes \dots \right. \\ \dots \otimes | \text{GHZ}_{n_{N-1}+2}^{(y_{N-1}, \vec{0})} \rangle_{R_{2N-4} A_{1:n_{N-1}}^N R_{2N-3}} \otimes | \text{GHZ}_{n_N+1}^{(y', \vec{0})} \rangle_{R_{2N-2} A_{1:n_N}^N} \left. \right), \quad (\text{E44})$$

as required. ■

4. Fusion of GHZ states in a tree topology (Protocol 3)

Protocol 8 can be used almost identically in the case that the GHZ states are in a tree topology, as shown in Fig. 11. (Observe that, in fact, the previous examples of a star topology and a linear topology are both special cases of tree graphs.) In this case (see Protocol 3), we start by taking the graph and arranging it explicitly as a tree, as in Fig. 11. (Note that this can be done by taking an arbitrary node as the root.) Then, we perform CNOT gates “downward” through the tree, followed by measuring the target qubits in the Pauli-Z basis. Finally, the measurement outcomes are announced globally. All of the nodes apply Pauli-X corrections based on their level in the tree, with the root node being level 0.

When the input GHZ states in Protocol 3 are noisy, the output state after executing Protocol 3 has fidelity to the ideal GHZ state given by an expression analogous to the one in (E33).

Paper II

Shallow-depth GHZ state generation on NISQ devices

S. Siddardha Chelluri, Stephan Schuster, Sumeet and Riccardo Roma

arXiv:2507.19145, in review at Quantum Science and Technology.

Shallow-depth GHZ state generation on NISQ devices

S. Siddardha Chelluri,^{1,*} Stephan Schuster,^{2,*} Sumeet,² and Riccardo Roma^{3,4}

¹*Institute of Physics, Johannes-Gutenberg University of Mainz, Staudingerweg 7, 55128 Mainz, Germany*

²*Department of Physics, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), 91058 Erlangen, Germany*

³*Theoretical Physics, Saarland University, 66123 Saarbrücken, Germany*

⁴*Institute for Quantum Computing Analytics (PGI-12),
Forschungszentrum Jülich, 52425 Jülich, Germany*

(Dated: July 28, 2025)

In this work, we focus on GHZ state generation under the practical constraint of limited qubit connectivity, a hallmark of current NISQ hardware. We study the GHZ state preparation across different connectivity graphs inspired by IBM and Google chip architectures, as well as random graphs that reflect distributed quantum systems. Our approach is a measurement-based protocol designed to utilize qubit connectivity constraints for the generation of GHZ states on NISQ devices. We benchmark this against a tailored version of state-of-the-art unitary-based protocols, also incorporating physical connectivity limitations. To evaluate the performance of the protocols under realistic conditions, we conducted implementations on the IBM Eagle r3 chip. Additionally, to explore near-term scalability, we performed simulations across a range of graph sizes and connectivity configurations, assessing performance based on circuit depth, the number of two-qubit gates, and measurement overhead. We observe a trade-off between the two protocols across different figures of merit. For current state-of-the-art NISQ architectures, the unitary-based protocol is more suitable, as it avoids mid-circuit measurements and classical feedforward. However, the measurement-based protocol is expected to become more advantageous in the future with more error-resilient quantum devices, owing to its reduced circuit depth and consequently shorter execution times. In both settings, our proposed method provides an efficient means of leveraging the topology of qubit connections available on a given device.

I. INTRODUCTION

Efficient generation and control of multipartite entanglement is a fundamental requirement for implementing a wide range of quantum information protocols. Among multipartite entangled states, Greenberger-Horne-Zeilinger (GHZ) states are particularly notable for exhibiting genuinely nonlocal correlations [1]. These states are an essential resource for quantum internet [2, 3], including quantum key distribution (QKD) [4, 5], as well as for quantum sensing [6–8] and distributed quantum computing [9–11]. With the rapid development of quantum hardware across several platforms, the ability to efficiently generate high-fidelity GHZ states at scale is a key milestone for near-term, noisy intermediate-scale quantum (NISQ) devices. However, in the NISQ era, generating GHZ states remains a fundamental challenge, as resource requirements often scale poorly with system size, leading to deeper circuits, more entangling operations, and higher sensitivity to noise, gate errors, and decoherence [12].

Numerous methods have been proposed for generating GHZ states on digital quantum computing hardware [13–19]. Linear depth constructions are straightforward but result in deep circuits that are especially vulnerable to noise in systems with limited qubit connectivity [13–16].

In contrast, logarithmic-depth circuits and measurement-based protocols can mitigate circuit depth, but they typically assume idealized qubit connectivity resulting in additional operations that are costly to implement on current quantum hardware [13, 14, 17]. Bäumer et al. [14] compared a constant-depth measurement-based method to a linear-depth unitary method for GHZ state generation on a 27-qubit IBM quantum hardware. Both methods considered a linear qubit connectivity to generate the GHZ states.

The unitary method outperformed the measurement-based approach, which was partly attributed to inefficient implementation of classical feedforward on this IBM device. Moses et al. [17] tested a log-depth unitary and a constant-depth measurement-based method on ion-trap devices, achieving high fidelities (> 0.5) for up to 32 qubits, and identified the measurement-based method as promising for devices with limited qubit connectivity. Liao et al. [15] proposed a unitary, BFS-based approach, which considers the qubit connectivity layout of the 156-qubit IBM hardware, achieving multipartite entanglement in GHZ states of up to 75 qubits using error suppression and sparse parity checks. Bao et al. [19] enhanced unitary GHZ state generation by leveraging device connectivity and discrete time crystal techniques, demonstrating multipartite entanglement in GHZ states of up to 60 qubits on a superconducting qubit device. Chen et al. [16] employed error mitigation techniques to reduce the circuit depth required for GHZ state preparation. GHZ states can also be generated via graph state transformations [20, 21]. For instance, a GHZ state was derived from a linear cluster state in [18], though the re-

* S.S.C. and S.S. contributed equally to this work. Correspondence should be addressed to S.S.C. (schellur@uni-mainz.de) and S.S. (stephan.schuster@fau.de).

sulting GHZ state had only half the number of qubits as the original cluster state.

While all these works so far achieved impressive results in generating large GHZ states using various error suppression and error correction techniques, the available qubit connectivity on the devices has not been rigorously considered, especially in measurement-based approaches. Our work addresses this gap by exploring two connectivity-aware protocols for GHZ state generation: a measurement-based protocol (i.e. *merging* protocol) and a unitary-based protocol (i.e. *growing* protocol).

The *merging* protocol was initially developed for random graph architectures without accounting for implementation on circuit-based hardware [22]. We adapt this protocol for local hardware layouts by considering practical constraints such as gate scheduling and average degree of graphs. It constructs small GHZ states on star-like subgraphs and recursively fuses them using mid-circuit measurements, enabling significant parallelization and shallower circuit depth through strategic subgraph selection.

In contrast, the *growing* protocol, introduced in [15] is a purely unitary approach that initiates from a central qubit and expands in a breadth-first manner by entangling neighboring qubits, always respecting hardware topology. This method eliminates measurement overhead and improves robustness against noise, though often at the cost of deeper circuits. We begin by implementing both protocols on real quantum hardware (IBM Eagle r3 chip) to evaluate the fidelity of the generated GHZ states. However, due to the high error rates of current NISQ devices, we also perform extensive simulations to explore performance in near-future scenarios. These simulations cover a range of hardware-relevant topologies, including IBM’s 127-qubit Eagle architecture [23], rectangular lattices inspired by Google’s Willow layout [24], and Erdős–Rényi random graphs [25, 26], representative of distributed quantum devices. We assess performance using multiple figures of merit, such as circuit depth, the number of two-qubit gates, measurements, and the final GHZ state fidelity. The choice between unitary-based and measurement-based protocols presents a trade-off that depends on the specific performance metrics and hardware capabilities. Unitary-based approaches are currently more compatible with near-term quantum devices, as they avoid mid-circuit measurements and classical feedforward, which remain error-prone on today’s NISQ hardware. In contrast, measurement-based protocols, characterized by shallower circuit depth, are likely to become more effective as quantum technology advances, particularly in the context of fault-tolerant or more error-resilient architectures where measurements and feedforward operations are more reliable and efficient.

The paper is organized as follows. In Section II, we discuss the two protocols, the *merging* protocol in Section II A, and the *growing* protocol in Section II B. We further discuss the benchmarking of the two protocols in Section III based on different figures of merit. Specifi-

cally, the performance evaluations on hardware are discussed in Section III A. Further simulations on near-future hardware and random qubit layouts are provided in Sections III B and III C, respectively. Finally, Section IV summarizes the main results and outlines potential directions for future work.

II. METHODS

In this section, we discuss two protocols: the measurement-based merging protocol and the unitary-based growing protocol. We demonstrate both protocols on the IBM Eagle r3 chip layout; however, they are applicable to any architecture with arbitrary qubit connectivity.

A. Merging Protocol

The main idea of this protocol is the “merging” of multiple small GHZ states via measurements to create a large GHZ state efficiently. The protocol consists of the following four main steps.

Step 1: star selection

This step involves identifying groups of nodes, referred to as *stars*, within the layout graph. First, we select the node with the highest degree in the graph layout, along with its neighboring nodes. This first *star* is then removed from the layout, and the procedure is repeated: the next highest-degree node (among the remaining nodes) is selected along with its neighbors, forming the next star. This continues iteratively until all nodes have been grouped to a star. An illustrative example for star selection on the layout of an IBM device is shown in Figure 1, where the selected stars are highlighted by colored boxes. The subgraphs in these boxes resemble stars.

Step 2: construction of small GHZ states

Each selected star is used to construct a GHZ state through a sequence of controlled-X (CX) gates. For the central node i with its neighboring nodes $\mathcal{N}(i)$, the corresponding GHZ state is given by

$$|\text{GHZ}\rangle_{i,\mathcal{N}(i)} = \prod_{j \in \mathcal{N}(i)} CX_{i,j} H_i |0\rangle_{i,\mathcal{N}(i)}, \quad (1)$$

where H_i denotes the Hadamard gate applied to the central qubit i , and $CX_{i,j}$ represents the CX gate with control qubit i and target qubit j .

Step 3: merging of GHZ states

After constructing the small GHZ states, they are recursively merged to form a single large GHZ state. The protocol for merging two GHZ states involves applying a CX gate followed by a single-qubit measurement.

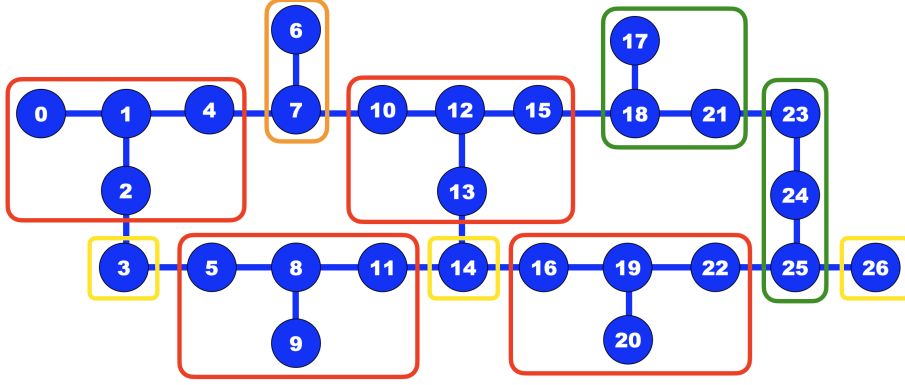


FIG. 1. The figure illustrates star selection in the IBMQ device layout for the merging method. Coloured boxes indicate 4-, 3-, 2-, and 1-qubit stars (red, green, orange, and yellow, respectively), selected by iteratively choosing the highest-degree node. These stars, consisting of a central qubit and its neighboring leaves, form the basic units of the merging protocol.

For example, consider two GHZ states, $|\text{GHZ}_{n+1}\rangle_{A_1 B_{1:n}}$ and $|\text{GHZ}_{m+1}\rangle_{C_1 D_{1:m}}$, of sizes $n+1$ and $m+1$, respectively. These states involve the qubits A_1 and $B_{1:n} = \{B_1, B_2, \dots, B_n\}$, and C_1 and $D_{1:m} = \{D_1, D_2, \dots, D_m\}$. If qubits A_1 and C_1 share an edge, the two GHZ states can be merged into a new GHZ state of size $n+m+1$ by applying a controlled-NOT gate CX_{A_1, C_1} , followed by measuring qubit C_1 in the X -basis. Depending on the measurement outcome $x \in \{0, 1\}$, a corrective X gate may need to be applied to the qubits $D_{1:m}$ to obtain the desired GHZ state exactly. A more detailed discussion of this merging procedure is provided in Appendix A.

Step 4: re-adding/re-using the measured qubits

During each merge, a qubit is measured (see Step 3). Afterwards, this qubit is re-integrated into the GHZ state by initializing it to $|0\rangle$ and applying a CX gate between the measured qubit and one of its neighboring qubits already in the GHZ state.

Note that in step 1, it is also possible to select stars based on alternative criteria rather than choosing nodes with the highest degree. We explore this option extensively in Appendix B, where stars are chosen such that the degree of each star is the same or nearly the same (throughout the article, degree of star/star degree refers to the degree of the star's center node), resulting in similar star sizes.

The motivation behind this approach is to reduce the total time required for constructing all stars in step 2. Since the CX gates involved in this step are applied sequentially and their number depends on the degree of the star, the total time to prepare the state scales with the star size. The preparation time of a star is considered in our benchmark in the circuit depth. A shorter preparation time is indicated by a smaller circuit depth. By choosing similar star sizes, we aim to reduce the circuit depth of the star preparation. However, this may come at the cost of requiring more two-qubit gates and measurements overall, due to an increasing number of stars and

thus an increasing number of merges. To implement this, we consider two possibilities: First, we define a scaling factor as the ratio of the chosen star degree to the average degree of the layout graph. A scaling factor of exactly 1 indicates that the selected star degree matches the average degree. If the factor is less than 1, the stars degrees are smaller than the average degree; if it is greater than 1, the stars degrees are larger than the average degree. Secondly, we also set the star size explicitly to fixed values. Note that it is not always possible to select all stars with exactly the same size, as the initial layout graph is fixed. In such cases, stars are chosen with the closest possible size to the target value.

B. Growing Protocol

In addition to the *merging* approach, we present here a purely unitary method for efficiently generating a GHZ state, tailored to a given qubit connectivity layout. The growing method is explained in the stepwise fashion as follows:

Step 1: initial GHZ state generation

We begin by selecting the node i with the highest degree and construct an initial GHZ state involving this node and its immediate neighbors $\mathcal{N}(i)$ (cf. Figure 2 (a)):

$$|\text{GHZ}\rangle_{i, \mathcal{N}(i)} = \prod_{j \in \mathcal{N}(i)} CX_{i,j} H_i |0\rangle_{i, \mathcal{N}(i)}, \quad (2)$$

where H_i is the Hadamard gate acting on the qubit corresponding to the highest degree node i in the layout graph and $CX_{i,j}$ is the controlled- X gate with control qubit i and target qubit j .

Step 2: expanding the GHZ state via BFS

Next, we expand the state by identifying the neighbors of the qubits already in the GHZ state and include them

via CX gates (cf. Figure 2 (b) and (c)). Therefore, the qubits already in the GHZ state have to be the control qubits of the CX gates and the neighboring qubits have to be the target qubits, such that the resulting state is again a GHZ state. This step is repeated until the desired state size is reached.

While this method avoids any merging operations and relies solely on unitary gates, the BFS naturally incorporates the qubit connectivity of the underlying hardware. Although we anticipate that this approach may yield deeper circuits than our *merging* protocol, it is expected to outperform other purely unitary techniques, such as linear constructions [14] or logarithmic-depth methods that ignore qubit connectivity [17]-in terms of overall circuit depth.

Because this approach begins with a small GHZ state centered on the highest-degree node and incrementally grows it through BFS, we refer to it as GHZ state *growing* for the rest of the article. We will compare the proposed *merging* protocol to this unitary method in terms of the resulting circuit depth, measurement overhead, number of two qubit gates and for the IBM layout also in terms of the resulting GHZ state fidelity.

III. BENCHMARK RESULTS

To benchmark the two methods presented in Sections II A and II B, we implemented them on IBM's 127-qubit Eagle chip topology [23], using fidelity as the primary figure of merit. The results were compared with noisy simulations of the same hardware, as discussed in Section III A. While running experiments on real quantum hardware is both interesting and valuable, the current devices are still subject to significant noise and errors. As a result, such experiments alone are insufficient for estimating the feasibility of these protocols on near-future devices. To address this, we conducted additional simulations across several graph layouts: IBM's 127-qubit Eagle topology [23], rectangular grids inspired by Google's Willow architecture [24] (both detailed in Section III B), and randomly generated graphs representing distributed quantum computing scenarios (discussed in Section III C). For these simulations, we evaluated several figures of merit, including circuit depth, the number of two-qubit gates, and the number of measurements across all layouts. The circuit depth is measured before hardware-specific compilation and is thus independent of any backend. Single-qubit gates alone are not explicitly discussed, as their error rates are typically an order of magnitude lower than those of two-qubit gates. Nevertheless, they are included in our circuit-depth calculations. All simulations were implemented in Python using the `NetworkX` [27] and `Qiskit` [28] libraries. Each configuration was executed 100 times to gather statistical data.

A. Fidelity analysis for IBM Eagle chip

Quantum state tomography can fully characterize any quantum state. However, it quickly becomes infeasible as the number of measurements grows exponentially with the number of qubits. To benchmark the performance of our protocols, we therefore rely on the Hellinger fidelity [29, 30] as our evaluation metric. It quantifies the similarity between two (discrete) probability distributions P and Q (e.g., ideal and measured outcomes) as:

$$F_H(P, Q) = \left[1 - \left(\frac{1}{\sqrt{2}} \left\| \sqrt{P} - \sqrt{Q} \right\|_2 \right)^2 \right]^2 \quad (3)$$

$$= \left(\sum_i \sqrt{p_i q_i} \right)^2$$

where p_i and q_i are the probabilities of P and Q respectively, and $\|\cdot\|_2$ denotes the Euclidean norm.

We use the Hellinger fidelity (Equation 3) to evaluate the performance of our two protocols. To this end, we implemented GHZ state generation circuits on the 127-qubit Eagle-chip device (`IBM_Brussels`). For each GHZ state size and each protocol, we generated and executed 100 random circuits on `IBM_Brussels`, each using a different set of physical qubits, and computed the average fidelity across these runs. These averages are presented in Figure 3 (a), along with the highest fidelity observed among the 100 instances, illustrating the best-case performance. The standard deviations, shown as error bars on the average values, indicate substantial variation in fidelity for a given state size. This spread is largely due to heterogeneous two-qubit gate and measurement error rates across the device. Notably, some instances yield high-fidelity GHZ states despite overall variability. To complement the hardware experiments, we performed noisy simulations using the full noise model of the same device. The simulated fidelities are shown in Figure 3 (b). While the simulations rely on a noise model calibrated from the hardware itself, and simulations and experiments were conducted within the same calibration window, we observe a consistent discrepancy between simulated and experimental results.

Furthermore, we find that the *growing* protocol outperforms the *merging* protocol in terms of fidelity. This is expected, as the *growing* protocol avoids mid-circuit measurements and also requires slightly fewer two-qubit gates, both of which contribute significantly to noise accumulation. It should be noted that no error mitigation techniques were employed to enhance the fidelity in our results.

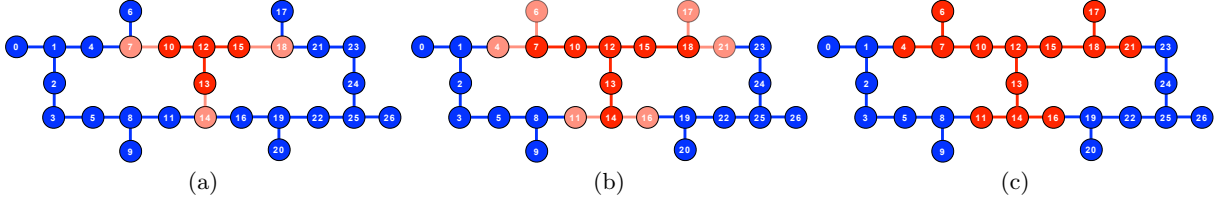


FIG. 2. The first three steps of the GHZ state growing method on the IBM layout as shown in this figure. The red nodes correspond to qubits that are included in the current GHZ state and the light red nodes are to-be-included in the next step. Panel (a) shows the initial GHZ state generation according to Equation (2), picking node 12 as the highest degree node and the first BFS step identifying the direct neighbors of the initial GHZ state. Panel (b) shows the GHZ state after adding the neighbors from the first BFS step via CX gates and the second BFS step, identifying the direct neighbors of the current GHZ state. Panel (c) shows the final GHZ state after adding the neighbors from the second BFS step, reaching a final size of 13 qubits.

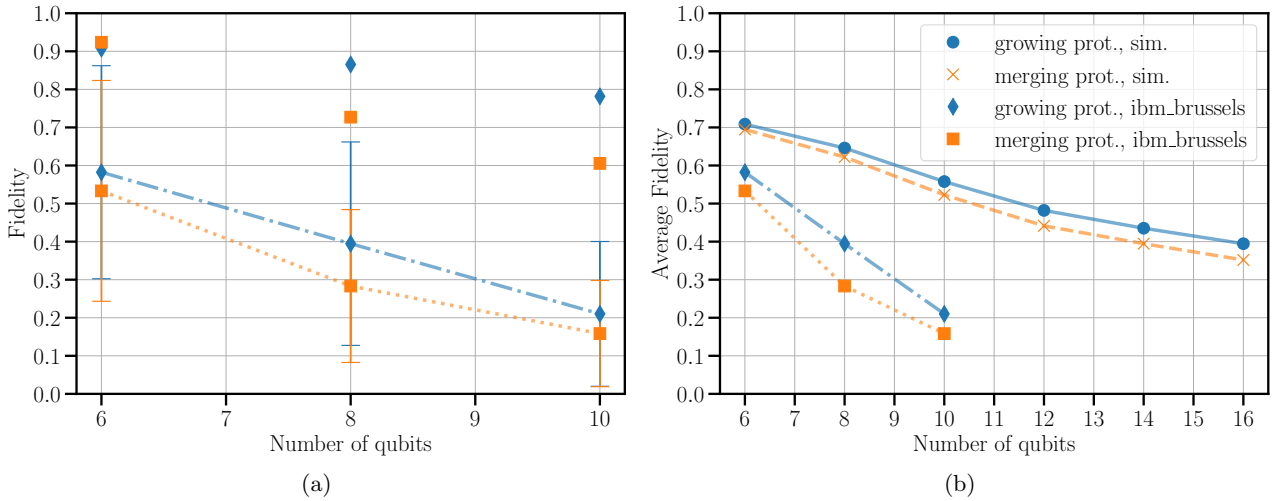


FIG. 3. GHZ state fidelity (cf. Equation (3)) over the number of qubits in the final GHZ state. Here we compare the *growing* protocol (blue circles, blue diamonds) with our *merging* protocol (orange crosses, orange squares) using the highest degree star size. For each GHZ state size and each protocol, we created and executed 100 different circuits. Panel (a) shows the average fidelity values obtained from executing the circuits of the *growing* protocol (blue diamonds) and the *merging* protocol (orange squares) on quantum hardware `IBM_Brussels` and additionally the maximum fidelity achieved for each size is indicated as another marker. The standard deviation is shown as error bars on the average values. Panel (b) shows the average fidelity values obtained from noisy simulations using the full `IBM_Brussels` noise model (blue circles, orange crosses) and from running the same circuits on the quantum hardware `IBM_Brussels` (blue diamonds, orange squares). The legend of Panel (b) is also valid for Panel (a). For each GHZ state size, we created and executed 100 different circuits. Each circuit as been executed with 4096 shots.

B. Performance on quantum hardware layouts

In this section, we examine the results for IBM’s 127-qubit Eagle layout [23] and Google’s Willow [24] layout shown in Figures 4 and 5 respectively. We compare the *merging* protocol with the *growing* protocol on grounds of averaged circuit depth, number of measurements and two-qubit gates.

Interestingly, we observe that the *merging* protocol performs best when selecting stars based on the highest degree. Consequently, only the highest-degree *merging* protocol is compared with the *growing* protocol, while the comparison of different star selection strategies within the *merging* protocol is discussed in Appendix B.

As shown in Figures 4 and 5, the *merging* protocol results in shallower circuits compared to *growing*, with the improvement becoming more pronounced as the GHZ state size increases. However, this advantage comes with a trade-off: unlike *growing*, *merging* requires measurements and leads to a higher number of two-qubit gates due to qubit reuse.

C. Performance on random graphs

Random graph layouts naturally arise in modular quantum computing architectures, where small-scale quantum processors are interconnected to form a larger

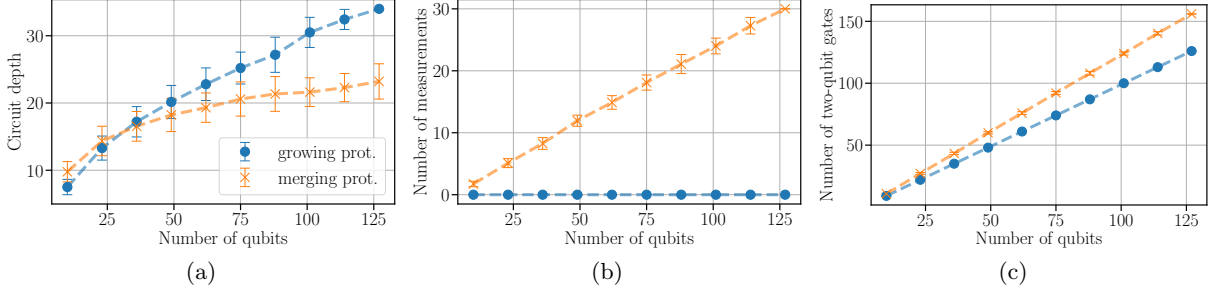


FIG. 4. Random subgraph sampling from IBM's 127-qubit Eagle layout [23]. Here we compare the *growing* protocol (blue circles) with our *merging* protocol (orange crosses) using the highest degree star size. Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and number of two-qubit gates in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

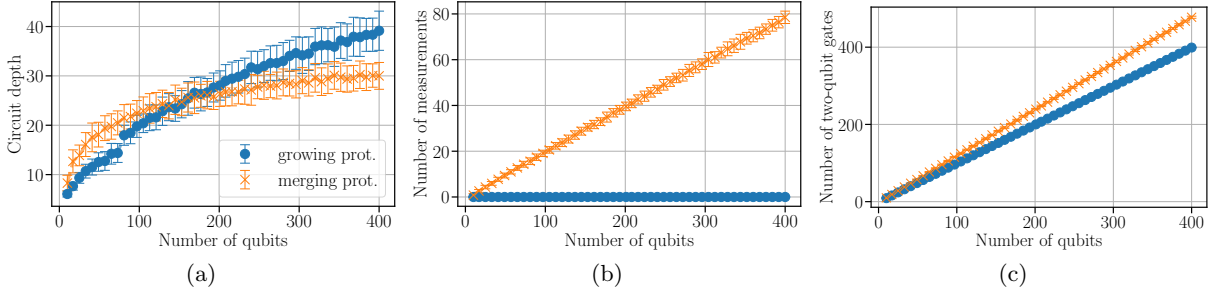


FIG. 5. Random subgraph sampling from rectangular grid layout inspired by Google's Willow chip [24]. Here we compare the *growing* protocol (blue circles) with our *merging* protocol (orange crosses) using the highest degree star size. Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and number of two-qubit gates in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

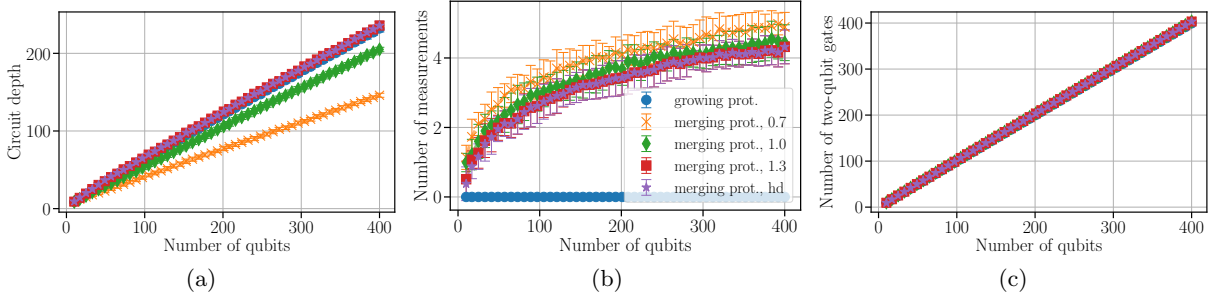


FIG. 6. Sampling from Erdős-Rényi random graphs [25, 26]. Here we compare the *growing* protocol with our *merging* protocol using a varying target size for the star selection. The curves represent *growing* protocol (blue circles) as well as the *merging* protocol with a star selection based on scaling factors 0.7 (orange crosses), 1.0 (green diamonds), 1.3 (red squares) and the highest degree (violet stars). As explained in Section II A, the scaling factor relates the target star size for every star selection during *merging* with the average star size in the input graph. A larger scaling factor thus corresponds to a larger star size (cf. Panels (c), (f) and (i) of Figure 13 in Appendix B). For the graph generation in the Erdős-Rényi model, we used $p = 0.5$. Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and number of two-qubit gates for *merging* in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 Erdős-Rényi random graphs of this size. The error bars show the standard deviation obtained from averaging over those samples.

system. Examples include distributed quantum computing based on ion-trap architectures or optical fiber networks linking independent quantum nodes [10, 11]. In such settings, the physical connectivity between modules

can be modeled as a graph structure. To model such scenarios, we simulated random graphs based on the Erdős-Rényi model [25, 26], denoted as $G(N, p)$, where N is the number of nodes and p is a fixed probability that any

given pair of nodes is connected by an edge, independently of all other pairs. The Erdős-Rényi model does not guarantee connectedness for arbitrary parameters. However, entanglement generation using LOCC (Local Operations and Classical Communication) requires the overall graph to be connected. Therefore, we adopted a modified version of the Erdős-Rényi model introduced in [22], where connectivity is explicitly enforced. In this version, each node is initially connected to at least one other node to ensure connectivity, and additional edges are then added according to the standard $G(N, p)$ process.

We studied the performance of the *growing* and *merging* protocols for creating GHZ states. We determined the circuit depth, number of measurements and the number of two-qubit gates for each generated circuit. For every GHZ state size we created 100 circuits based on different random graphs. The averaged circuit depth, number of measurements and number of two-qubit gates are shown in Figure 6 for the different GHZ state sizes. Unlike the results for the IBM or rectangular layout, we observe that selecting the stars with the highest degree is no longer the global best strategy. Therefore, we also consider another selection criteria which allows us to fix the target star size during the selection (see last paragraph of Section II A). We adjust this target size in relation to the average star size in the input graph via a scaling factor, where larger scaling factors correspond to larger target sizes (cf. Panels (c), (f) and (i) in Figure 13 in Appendix B).

We find that smaller star sizes lead to reduced circuit depth, while larger star sizes require fewer measurements during the *merging* process. This reveals an inherent trade-off, where the optimal strategy must be chosen on a case-by-case basis. Additionally, in Appendix B, we examine Erdős-Rényi graphs with varying values of p . Where we observe an overall increasing number of measurements with decreasing p values. Due to the re-adding of qubits after merging (cf. step 4 in Section II A), we get an additional two-qubit gate for each merge, i.e., for each measurement. Since the number of measurements is very small compared to the minimal required number of two-qubit gates to generate the GHZ state of large sizes, we see all five curves overlapping for the number of two-qubit gates in Panel (c) of Figure 6.

IV. SUMMARY AND OUTLOOK

As quantum computers continue to improve, generating large-scale entangled states has become an important benchmarking tool. In this work, we explore the feasibility of creating GHZ states on quantum hardware using two distinct approaches: *merging* and *growing*. The *merging* protocol involves creating small GHZ states and then “merging” them into a larger GHZ state via measurements. In contrast, the *growing* protocol is fully unitary and does not involve any measurements. It begins

by entangling a small group of neighboring qubits into a GHZ state and then iteratively expands the entanglement to include their neighbors, continuing this process until the desired size is achieved. To compare the two protocols and analyze their respective strengths and limitations, we first implemented both methods on state-of-the-art IBM quantum hardware. Given the limitations of current NISQ devices, we then conducted simulations across various graph topologies inspired by IBM, Google, and distributed quantum computing architectures to gain deeper insights under prospective near-term conditions. We evaluated the performance of the *merging* and *growing* protocols using several figures of merit. Although the *growing* protocol performs better on today’s hardware, our results indicate that the *merging* protocol has the potential to surpass it as two-qubit gate fidelities and readout capabilities continue to improve. We also observe a significant discrepancy between the fidelities of GHZ states obtained from noisy simulations and those measured on actual hardware. The fidelities presented were computed in the absence of error mitigation, and are therefore expected to improve upon the application of appropriate mitigation strategies [15, 31].

We have identified several promising directions for future work. While our hardware experiments were conducted exclusively on IBM quantum devices, it would be valuable to extend this study to other architectures such as photonic quantum computers, trapped-ion, and neutral-atom platforms. Since the *merging* protocol can be formulated as a Markov decision process, it is, in principle, possible to apply reinforcement learning to discover more sophisticated strategies for star selection beyond simple heuristics like highest or average degree [32, 33]. Another key area is the development of more accurate methods for estimating the noise models of different hardware systems. Additionally, using specialized SDKs, such as Quantum Rings, could enable more efficient and tailored simulations [34]. Further optimization of the protocols under realistic conditions, including node or edge failures and creating entangled states only among pre-selected nodes also presents an important avenue for research [35]. We conjecture that many of these layout specific challenges ultimately reduce to optimization problems on graphs.

V. ACKNOWLEDGMENTS

We thank Peter van Loock, Joachim von Zanthier, Kai Phillip Schmidt, Frank Wilhelm-Mauch, Christian Melzer, and Sumeet Khatri for useful discussions. Additionally, we thank Karl Jansen for valuable support during the completion of this work. We thank Lucas Marti and Pradip Laha for proof reading the manuscript. We received funds by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Project-ID 429529648—TRR 306 QuCoLiMa (“Quantum Cooperativity of Light and Matter”) and support by its inte-

grated Research Training Group (iRTG). Sumeet thank the support by the Munich Quantum Valley, which is

supported by the Bavarian state government with funds from the Hightech Agenda Bayern Plus.

-
- [1] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Experimental entanglement of three photons and quantum nonlocality, *Nature* **403**, 515 (2000).
 - [2] H. J. Kimble, The quantum internet, *Nature* **453**, 1023–1030 (2008).
 - [3] S. Wehner, D. Elkouss, and R. Hanson, Quantum internet: A vision for the road ahead, *Science* **362**, eaam9288 (2018).
 - [4] C. H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, *Theoretical Computer Science* **560**, 7 (2014), theoretical Aspects of Quantum Cryptography – celebrating 30 years of BB84.
 - [5] A. K. Ekert, Quantum cryptography based on bell’s theorem, *Phys. Rev. Lett.* **67**, 661 (1991).
 - [6] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Fisher information and multiparticle entanglement, *Physical Review A* **85**, 022321 (2012).
 - [7] P. L. and S. A., Quantum theory of phase estimation, *Proceedings of the International School of Physics “Enrico Fermi”*; **188**, 691–741 (2014).
 - [8] T. Kielinski, P. O. Schmidt, and K. Hammerer, Ghz protocols enhance frequency metrology despite spontaneous decay, *Science Advances* **10**, 10.1126/sciadv.adr1439 (2024).
 - [9] A. S. Cacciapuoti, M. Caleffi, F. Tafuri, F. S. Cataliotti, S. Gherardini, and G. Bianchi, Quantum Internet: Networking Challenges in Distributed Quantum Computing, *IEEE Network* **34**, 137 (2020), 1810.08421.
 - [10] M. Caleffi, M. Amoretti, D. Ferrari, J. Illiano, A. Manzalini, and A. S. Cacciapuoti, Distributed quantum computing: A survey, *Computer Networks* **254**, 110672 (2024).
 - [11] D. Main, P. Drmota, D. P. Nadlinger, E. M. Ainley, A. Agrawal, B. C. Nichol, R. Srinivas, G. Araneda, and D. M. Lucas, Distributed quantum computing across an optical network link, *Nature* **638**, 383–388 (2025).
 - [12] J. Preskill, Quantum Computing in the NISQ era and beyond, *Quantum* **2**, 79 (2018).
 - [13] D. Cruz, R. Fournier, F. Gremion, A. Jeannerot, K. Komagata, T. Tosić, J. Thiesbrummel, C. L. Chan, N. Macris, M. Dupertuis, and C. Javerzac-Galy, Efficient quantum algorithms for ghz and w states, and implementation on the ibm quantum computer, *Advanced Quantum Technologies* **2**, 10.1002/qute.201900015 (2019).
 - [14] E. Bäumer, V. Tripathi, D. S. Wang, P. Rall, E. H. Chen, S. Majumder, A. Seif, and Z. K. Mineev, Efficient long-range entanglement using dynamic circuits, *PRX Quantum* **5**, 030339 (2024).
 - [15] H. Liao, G. S. Hartnett, A. Kakkar, A. Tan, M. Hush, P. S. Mundada, M. J. Biercuk, and Y. Baum, Achieving computational gains with quantum error-correction primitives: Generation of long-range entanglement enhanced by error detection, *PRX Quantum* **6**, 020331 (2025).
 - [16] K.-C. Chen, Short-depth circuits and error mitigation for large-scale ghz-state preparation, and benchmarking on ibm’s 127-qubit system, in *2023 IEEE International Conference on Quantum Computing and Engineering (QCE)*, Vol. 02 (2023) pp. 207–210.
 - [17] S. A. Moses, C. H. Baldwin, M. S. Allman, R. Ancona, L. Ascarrunz, C. Barnes, J. Bartolotta, B. Bjork, P. Blanchard, M. Bohn, J. G. Bohnet, N. C. Brown, N. Q. Burdick, W. C. Burton, S. L. Campbell, *et al.*, A race-track trapped-ion quantum processor, *Phys. Rev. X* **13**, 041052 (2023).
 - [18] J. de Jong, F. Hahn, N. Tcholtchev, M. Hauswirth, and A. Pappa, Extracting ghz states from linear cluster states, *Physical Review Research* **6**, 013330 (2024).
 - [19] Z. Bao, S. Xu, Z. Song, K. Wang, L. Xiang, Z. Zhu, J. Chen, F. Jin, X. Zhu, Y. Gao, Y. Wu, C. Zhang, N. Wang, Y. Zou, Z. Tan, A. Zhang, Z. Cui, F. Shen, J. Zhong, T. Li, J. Deng, X. Zhang, H. Dong, P. Zhang, Y.-R. Liu, L. Zhao, J. Hao, H. Li, Z. Wang, C. Song, Q. Guo, B. Huang, and H. Wang, Creating and controlling global greenberger-horne-zeilinger entanglement on quantum processors, *Nature Communications* **15**, 10.1038/s41467-024-53140-5 (2024).
 - [20] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. V. den Nest, and H. J. Briegel, Entanglement in graph states and its applications (2006), arXiv:quant-ph/0602096 [quant-ph].
 - [21] A. Dahlberg, J. Helsen, and S. Wehner, How to transform graph states using single-qubit operations: computational complexity and algorithms, *Quantum Science and Technology* **5**, 045016 (2020).
 - [22] S. S. Chelluri, S. Khatri, and P. van Loock, A resource- and computationally-efficient protocol for multipartite entanglement distribution in bell-pair networks (2025), arXiv:2412.04252 [quant-ph].
 - [23] P. Ball, First quantum computer to pack 100 qubits enters crowded race, *Nature* **599** (2021).
 - [24] R. Acharya, D. A. Abanin, L. Aghababaie-Beni, I. Aleiner, T. I. Andersen, M. Ansmann, F. Arute, K. Arya, A. Asfaw, N. Astrakhantsev, J. Atalaya, R. Babbush, D. Bacon, B. Ballard, J. C. Bardin, others, G. Q. AI, and Collaborators, Quantum error correction below the surface code threshold, *Nature* **638**, 920 (2025).
 - [25] P. L. Erdős and A. Rényi, On random graphs. I., *Publicationes Mathematicae Debrecen* **6**, 290 (1959).
 - [26] P. L. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **5**, 17 (1960).
 - [27] A. A. Hagberg, D. A. Schult, and P. J. Swart, Exploring Network Structure, Dynamics, and Function using NetworkX, in *Proceedings of the 7th Python in Science Conference*, edited by G. Varoquaux, T. Vaught, and J. Millman (Pasadena, CA USA, 2008) pp. 11–15, <https://networkx.org/>.
 - [28] A. Javadi-Abhari, M. Treinish, K. Krsulich, C. J. Wood, J. Lishman, J. Gacon, S. Martiel, P. D. Nation,

- L. S. Bishop, A. W. Cross, B. R. Johnson, and J. M. Gambetta, Quantum computing with Qiskit (2024), arXiv:2405.08810 [quant-ph].
- [29] Qiskit Developers, Hellinger fidelity (2025), accessed: 2025-07-10.
- [30] E. Hellinger, Neue begründung der theorie quadratischer formen von unendlichvielen veränderlichen., Journal für die reine und angewandte Mathematik **1909**, 210 (1909).
- [31] K. F. Koenig, F. Reinecke, W. Hahn, and T. Wellens, Inverted-circuit zero-noise extrapolation for quantum-gate error mitigation, Phys. Rev. A **110**, 042625 (2024).
- [32] S. D. Reiß and P. van Loock, Deep reinforcement learning for key distribution based on quantum repeaters, Phys. Rev. A **108**, 012406 (2023).
- [33] S. Haldar, P. J. Barge, S. Khatri, and H. Lee, Fast and reliable entanglement distribution with quantum repeaters: Principles for improving protocols using reinforcement learning, Phys. Rev. Appl. **21**, 024041 (2024).
- [34] V. Kasirajan, T. Battelle, and B. Wold, Empowering large scale quantum circuit development: Effective simulation of sycamore circuits (2024), arXiv:2411.12131 [quant-ph].
- [35] F. Hahn, J. de Jong, and A. Pappa, Anonymous quantum conference key agreement, PRX Quantum **1**, 020325 (2020).

Appendix A: Protocol for merging GHZ states

In this section, we present a protocol for merging two GHZ states. Consider the following two GHZ states:

$$|\text{GHZ}_{n+1}\rangle_{A_1 B_{1:n}} = \frac{1}{\sqrt{2}}(|0\rangle_{A_1} \otimes |0\rangle_{B_{1:n}}^{\otimes n} + |1\rangle_{A_1} \otimes |1\rangle_{B_{1:n}}^{\otimes n}), \quad (\text{A1})$$

$$|\text{GHZ}_{m+1}\rangle_{A_2 C_{1:m}} = \frac{1}{\sqrt{2}}(|0\rangle_{A_2} \otimes |0\rangle_{C_{1:m}}^{\otimes m} + |1\rangle_{A_2} \otimes |1\rangle_{C_{1:m}}^{\otimes m}), \quad (\text{A2})$$

where A_1 and A_2 are the centers of the two GHZ states, and we have used the abbreviation $B_{1:n} \equiv B_1 B_2 \cdots B_n$ (similarly for $C_{1:m}$). The protocol for merging these two states into a larger GHZ state is as follows:

1. Perform the gate $CX_{A_1 A_2}$ between the qubits A_1 and A_2 .
2. Measure the qubit A_2 in the Z -basis (i.e., the $\{|0\rangle, |1\rangle\}$ basis).
3. Communicate the outcome $x \in \{0, 1\}$ of the measurement to the nodes $C_1, C_2, \dots, C_m$.
4. (a) If the outcome is $x = 0$, then $C_1, C_2, \dots, C_m$ do nothing.
(b) If the outcome is $x = 1$, then $C_1, C_2, \dots, C_m$ apply the Pauli- X gate to their qubits.

Let us now prove that after this protocol we deterministically have the GHZ state $|\text{GHZ}_{n+m+1}\rangle_{A_1 B_{1:n} C_{1:m}}$ shared by $A_1, B_1, \dots, B_n, C_1, \dots, C_m$. First,

$$\begin{aligned} |\text{GHZ}_{n+1}\rangle \otimes |\text{GHZ}_{m+1}\rangle &= \frac{1}{2}(|0\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ &\quad + |0\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} \\ &\quad + |1\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ &\quad + |1\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m}). \end{aligned} \quad (\text{A3})$$

Then, after the CX between A_1 and A_2 , the state becomes

$$\begin{aligned} &\frac{1}{2}(|0\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ &\quad + |0\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} \\ &\quad + |1\rangle_{A_1} \otimes |1\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} \\ &\quad + |1\rangle_{A_1} \otimes |0\rangle_{A_2} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m}). \end{aligned} \quad (\text{A4})$$

From this, we can see immediately that, when A_2 measures its qubit in the $\{|0\rangle, |1\rangle\}$ basis, then the outcome “0” occurs with probability $\frac{1}{2}$, and the state conditioned on this outcome is

$$\frac{1}{\sqrt{2}}(|0\rangle_{A_1} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m} + |1\rangle_{A_1} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m}) = |\text{GHZ}_{n+m+1}\rangle. \quad (\text{A5})$$

On the other hand, if the outcome “1” occurs, also with probability $\frac{1}{2}$, then the state conditioned on this outcome is

$$\frac{1}{\sqrt{2}}(|0\rangle_{A_1} \otimes |0\rangle_{B_1^n}^{\otimes n} \otimes |1\rangle_{C_1^m}^{\otimes m} + |1\rangle_{A_1} \otimes |1\rangle_{B_1^n}^{\otimes n} \otimes |0\rangle_{C_1^m}^{\otimes m}). \quad (\text{A6})$$

Thus, if $C_1, C_2, \dots, C_m$ apply the Pauli X gate to their qubits, then we obtain the required GHZ state $|\text{GHZ}_{n+m+1}\rangle$. This completes the proof.

Appendix B: Evaluating different star selection criteria in the *merging* protocol

As already noted in Section II A, it is in principle possible to pick stars, in step 1 of the *merging* protocol, based on other criteria than the highest node degree. In this section, we investigated this possibility by picking stars based on target node degrees of the star center above, below and equal to the average degree of the input graph. Additionally, we also pick stars for the IBM and rectangular layouts based on absolute values of the target star size. Setting a target node degree is handled by the scaling factor, which relates the target degree to the average degree (cf. Section II A). In the following, we will evaluate the different criteria of picking stars for the considered layout topologies (IBM, rectangular and random), to find the best criteria for each layout topology.

For the considered IBM 127-qubit Eagle layout, the best ways to pick stars in the *merging* protocol is based on the highest degree. This criteria results in the smallest number of measurements, as the number of merges is smallest in that case. For the averaged circuit depth, we cannot determine a significant difference between the tested criteria, although we observe a tendency towards smaller circuit depth for larger star sizes. In Figure 7, we varied the absolute size of the selected stars and find that the largest size has the smallest number of measurements. A similar observation can be made for picking the stars based on a target node degree (cf. Figure 8). We find that the highest scaling factor results in the smallest number of measurements. From Figure 7 and Figure 8 we can conclude that, the larger the size of the selected stars is, the less measurements we have to do while still producing the smallest circuit depth. This shows that the circuit depth reduction we expected from a star generation with equal star sizes is insignificant compared to the increase in the circuit depth due to the higher number of merges. In order to always pick the largest stars, we have to pick stars with the highest degree. We compared the highest tested absolute size and the highest scaling factor to the highest degree criteria in Figure 9. Here we don't see any significant differences in the number of measurements, number of two-qubit gates or circuit depth. This can be explained by looking at the IBM layout graph (cf. Figure 1). The highest degree that occurs in the graph is three, corresponding to a target star size of four. This is already the target size for the largest tested size in Figure 7 and for the largest scaling factor in Figure 8. Hence, there is no difference anymore in the star selection compared to directly picking the nodes with the highest degree.

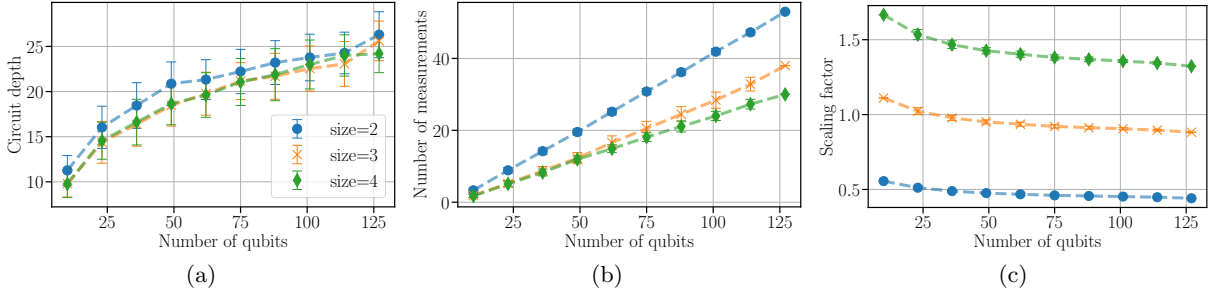


FIG. 7. Random subgraph sampling from IBM’s 127-qubit Eagle layout [23]. Here we compare our *merging* protocol with varying size for the star selection. The curves represent a star selection based on absolute target sizes of 2 (blue circles), 3 (orange crosses) and 4 nodes (green diamonds). Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and scaling factor in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

For the rectangular layout, we again find that selecting stars based on the highest degree is the best way, yielding the smallest number of measurements without any significant drawbacks in the averaged circuit depth. Thus, we identify the increasing number of merges as the most dominant factor, if we select stars based on similar star sizes, without any improvements in the circuit depth. We see no difference between the largest absolute star size, the largest scaling factor and the highest degree. Additionally, in this case there is also no difference to the average degree and

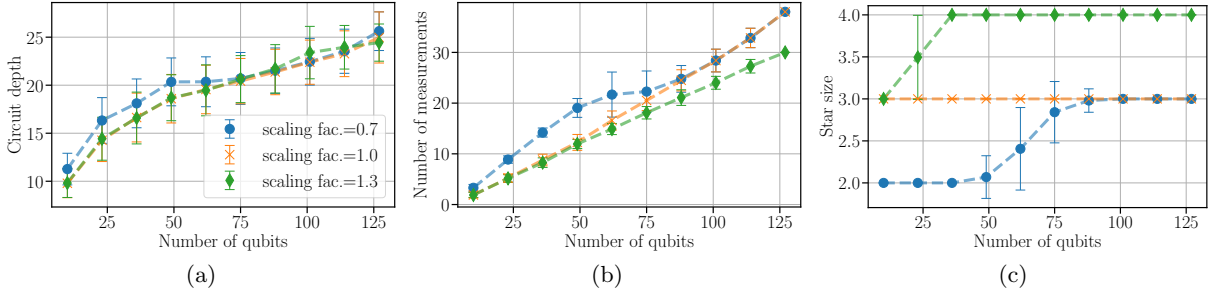


FIG. 8. Random subgraph sampling from IBM's 127-qubit Eagle layout [23]. Here we compare our *merging* protocol with varying size for the star selection. The curves represent a star selection based on scaling factors of 0.7 (blue circles), 1.0 (orange crosses) and 1.3 (green diamonds). Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and star size in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

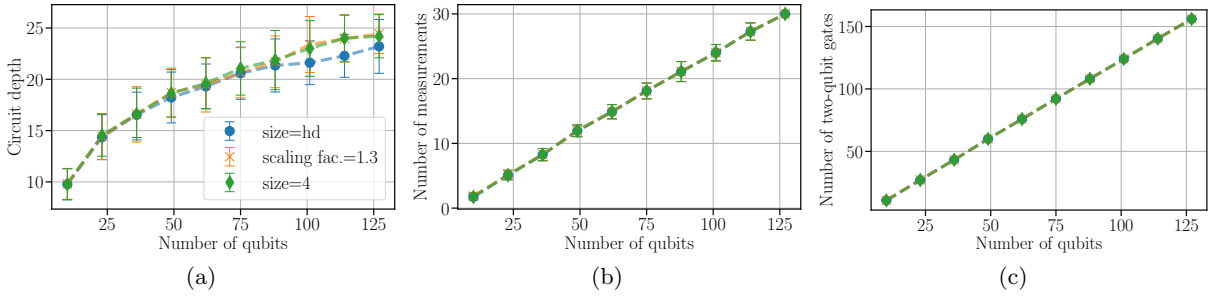


FIG. 9. Random subgraph sampling from IBM's 127-qubit Eagle layout [23]. Here we compare our *merging* protocol with varying size for the star selection. The curves represent a star selection based on the highest degree (blue circles), a scaling factor of 1.3 (orange crosses) and an absolute target size of 4 nodes (green diamonds). Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and number of two-qubit gates in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

the second largest absolute size (cf. Figures 10 and 11). Similar to the IBM layout, the reason for this behavior is that the highest node degree in a rectangular grid graph is four which corresponds to a target star size of five. The two largest absolute values in Figure 10 as well as all target node degrees equal or above the average degree in Figure 11 already result in a target size of five for the selected stars. Thus, we see no differences in the curves of Figure 12.

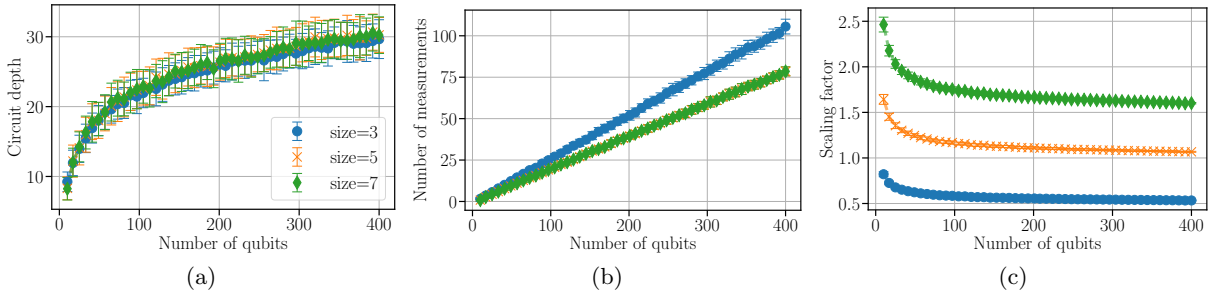


FIG. 10. Random subgraph sampling from rectangular grid layout inspired by Google's Willow chip [24]. Here we compare our *merging* protocol with varying size for the star selection. The curves represent a star selection based on absolute target sizes of 3 (blue circles), 5 (orange crosses) and 7 nodes (green diamonds). Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and scaling factor in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

Contrary to the IBM and rectangular layout, we cannot determine the best criteria to select stars in any random

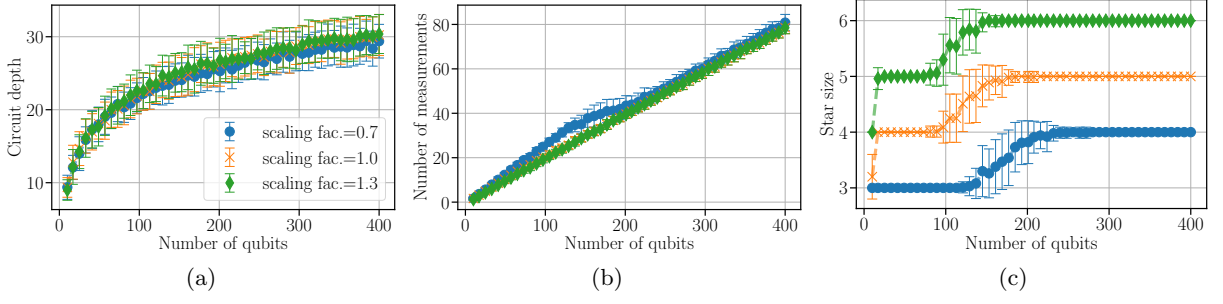


FIG. 11. Random subgraph sampling from rectangular grid layout inspired by Google’s Willow chip [24]. Here we compare our *merging* protocol with varying size for the star selection. The curves represent a star selection based on scaling factors of 0.7 (blue circles), 1.0 (orange crosses) and 1.3 (green diamonds). Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and star size in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

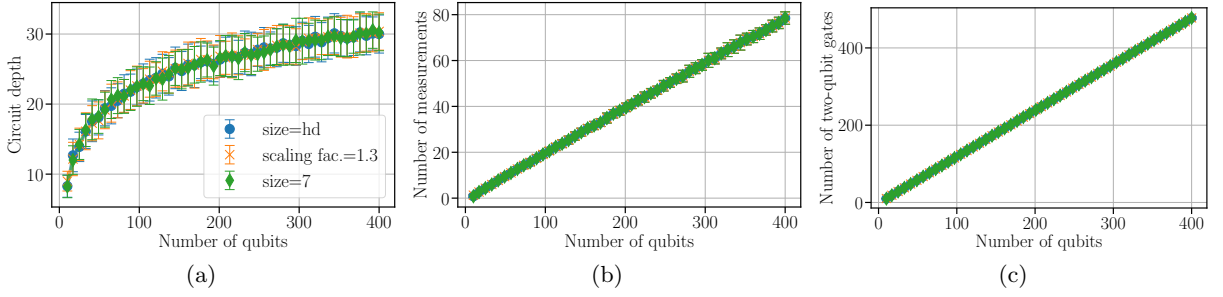


FIG. 12. Random subgraph sampling from rectangular grid layout inspired by Google’s Willow chip [24]. Here we compare our *merging* protocol with varying size for the star selection. The curves represent a star selection based on the highest degree (blue circles), a scaling factor of 1.3 (orange crosses) and an absolute target size of 7 nodes (green diamonds). Panels (a), (b) and (c) show the averaged circuit depth, number of measurements and number of two-qubit gates in dependence of the number of qubits in the final GHZ state. For each GHZ state size, we generated 100 subgraph samples of this size randomly from the initial layout graph. The error bars show the standard deviation obtained from averaging over the samples.

layout graph (cf. Figure 13). On the one hand, a small scaling factor results in a small circuit depth. Choosing a scaling factor of 1.3, i.e., a target degree above the average degree or directly using the highest degree criteria yields a larger average circuit depth than the *growing* protocol. On the other hand, larger scaling factors result in larger stars and thus in less merging processes, i.e., less measurements. Additionally, the absolute values of the circuits depth increase for increasing p values of the random graph and the number of measurements decrease. This can be explained by noting that the p value determines the average degree in the random graph. For increasing average degree the star sizes get larger for a fixed scaling factor. We thus conclude that the best criteria has to be chosen for each individual case separately, as a tradeoff between small circuit depth and resulting number of measurements. Note however that it is always possible in a random layout graph to artificially reduce the p value by leaving out existing edges. This will decrease the circuit depth and increase the required number of measurements and can be used to find the ideal compromise.

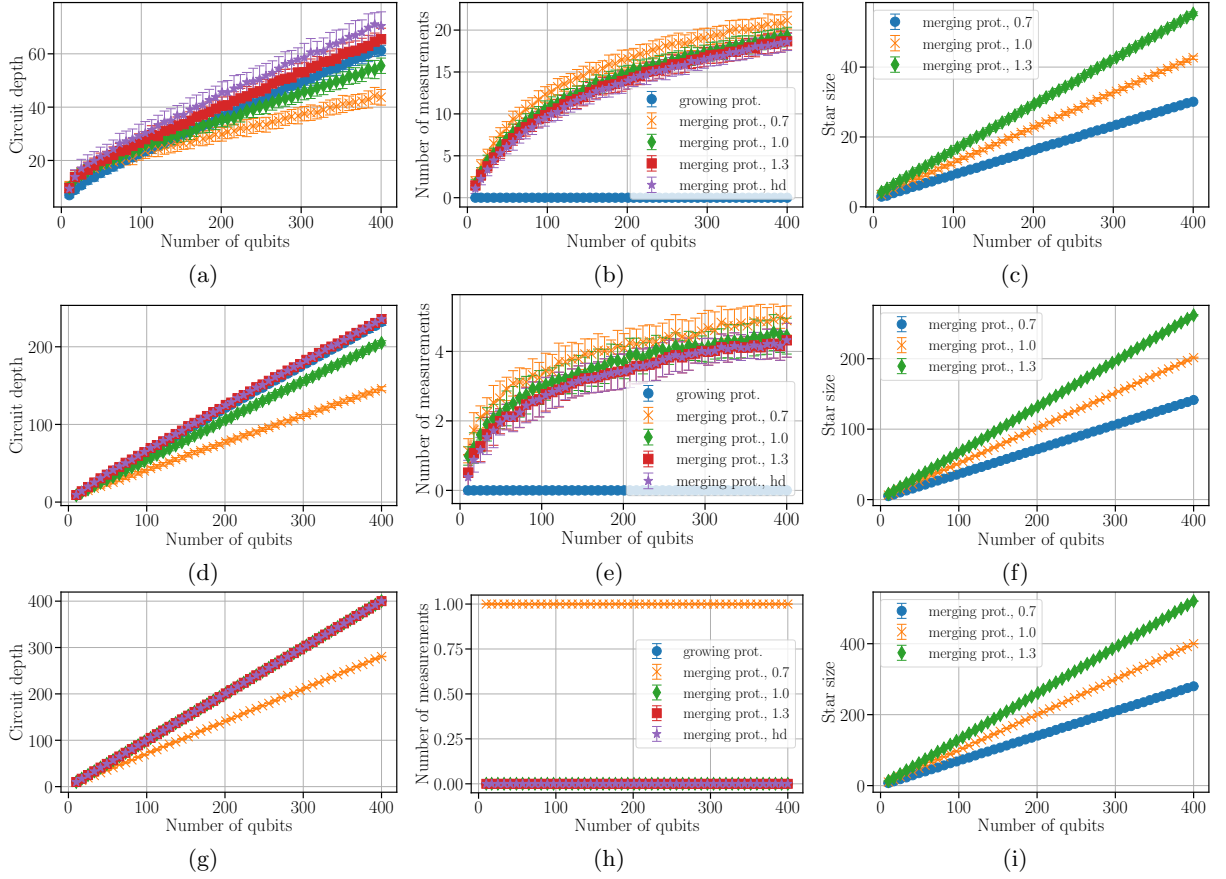


FIG. 13. Sampling from Erdős-Rényi random graphs [25, 26]. Here we compare the *growing* protocol with our *merging* protocol using a varying target size for the star selection. The curves represent *growing* protocol (blue circles) as well as the *merging* protocol with a star selection based on scaling factors 0.7 (orange crosses), 1.0 (green diamonds), 1.3 (red squares) and the highest degree (violet stars). As explained in Section II A, the scaling factor relates the target star size for every star selection during *merging* with the average star size in the input graph. A larger scaling factor thus corresponds to a larger star size (cf. Panels (c), (f) and (i)). The legends of panels (a) and (b), (d) and (e), (g) and (h) coincide and are thus only shown in panels (b), (e) and (h). The rows 1, 2, 3 (top to bottom) show the comparison for the graph generation in the Erdős-Rényi model with $p = 0.1$, $p = 0.5$ and $p = 1.0$ respectively. The columns 1, 2, 3 (from left to right) show the averaged circuit depth, number of measurements and star size for *merging* in dependence of the number of qubits in the final GHZ state, respectively. For each GHZ state size, we generated 100 Erdős-Rényi random graphs of this size. The error bars show the standard deviation obtained from averaging over those samples.

Paper III

Bosonic quantum error correction with microwave cavities for quantum repeaters

S. Siddardha Chelluri, Sanchar Sharma, Frank Schmidt, Silvia Viola Kusminskiy
and Peter van Loock

arXiv:2503.21569, in review at Phys. Rev. A.

Bosonic quantum error correction with microwave cavities for quantum repeaters

S. Siddardha Chelluri,^{1,*} Sanchar Sharma,² Frank Schmidt,¹ Silvia Viola Kusminskiy,^{3,4} and Peter van Loock^{1,†}

¹*Institute of Physics, Johannes-Gutenberg University of Mainz, Staudingerweg 7, 55128 Mainz, Germany*

²*Laboratoire de Physique de l'École Normale Supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université de Paris, F-75005 Paris, France*

³*Institute for Theoretical Solid State Physics, RWTH Aachen University, 52074 Aachen, Germany*

⁴*Max Planck Institute for the Science of Light, Staudtstraße 2, 91058 Erlangen, Germany*

(Dated: November 17, 2025)

Long-distance quantum communication necessitates the use of quantum repeaters, which typically include highly coherent quantum memories. We provide a theoretical analysis of the secret key rates for a quantum repeater system incorporating bosonic error correction and memory components. Specifically, we focus on the application of Binomial codes for two repeater segments. Using these codes, our investigation aims to suppress memory loss errors that commonly affect bosonic systems such as atomic gases, excitations of spin ensembles, and microwave cavities, in contrast to dephasing errors in single-spin memories. We further discuss a physical implementation of such a quantum repeater comprising a microwave cavity and a superconducting transmon, capable of state engineering with high fidelities ($> 97\%$) and logical Bell state measurements for successful entanglement swapping. As an alternative approach, we also discuss a realization in the all-optical domain.

I. INTRODUCTION

The distribution of entanglement over long distances and between multiple parties, forming quantum networks, enables numerous applications such as the quantum Internet [1], uncompromised secure communication through the establishment of private keys [2, 3], distributed quantum computing [4] and blind quantum computing [5]. However, the fiber channel transmittivity η decreases exponentially with distance between parties, making long-distance quantum communication infeasible. This limitation can be addressed by the use of quantum repeaters [6] in which a channel is divided into multiple, say n , segments. With the addition of quantum memories, this division effectively increases the channel attenuation length by a factor of n , thus enhancing the transmittivity to $\eta' = \eta^{1/n}$.

Quantum repeaters are typically classified into three generations based on their physical implementations [7]: the first relies on memories and entanglement distillation, the second on memories and error correction, and the third exclusively on error correction. As memories are crucial for the first two generations, various experimental platforms were explored as implementations, such as atomic ensembles [8], color centers in diamonds [9, 10], and quantum dots [11]. Due to decoherence in the memories, building quantum repeaters remains a significant challenge. Depending on the platform, either dephasing or loss dominates the decoherence. In the latter case, since the primary imperfection in the entire repeater system is excitation loss — both in the optical channel and the memory node — it is crucial to investigate the impact of memory loss [12]. While dephasing

in memories has been extensively studied [13–19], the loss of excitations in memories, particularly in the context of repeaters, has received relatively less attention, although prominent quantum repeater proposals would rely on memory systems subject to this type of error [8, 20]. To address loss errors, bosonic quantum error correction (QEC) can be applied, with examples including Binomial codes [21], Cat codes [22, 23], and Gottesman-Kitaev-Preskill (GKP) codes [24]. Clearly, bosonic QEC is then also applicable to those repeater systems based upon bosonic memories [25]. Bosonic qubit systems have also been applied in other areas, such as quantum simulation [26].

The performance of a repeater system is typically evaluated in terms of the rate at which it can transmit secret bits of information per second [27]. For third-generation repeaters, this figure of merit is relatively straightforward to calculate and has also been determined for repeaters using the Binomial code [28]. Although third-generation repeaters are promising, they remain a fairly distant prospect due to the challenges with the complex optical state engineering and the need for closely spaced repeater stations [7]. In contrast, second-generation repeaters are much closer to practical hardware implementation and require significantly fewer stations to cover the same distance. However, calculating their transmission rates is far more challenging and becomes exponentially difficult, as the number of possible states in a repeater system grows exponentially with the number of repeater stations [27].

In this work, we present a comprehensive analysis of a two-segment, second-generation repeater with loss errors where the information is encoded as a Binomial code in the memory. In these repeaters, photons remain unencoded during transmission, allowing for repeater stations to be spaced further apart than in typical third-generation systems. Two error correction schemes are

* schellur@uni-mainz.de

† loock@uni-mainz.de

studied: one where error correction is applied once, and another where it is applied multiple times. Our figure of merit will be the secret key rate of quantum key distribution, which we numerically calculate for our repeater protocols.

A promising implementation of this quantum repeater is provided by a circuit or cavity quantum electrodynamics (cQED) setups [29, 30], consisting of a microwave circuit or a cavity coupled to a superconducting transmon. cQED setup are interesting because of the long lifetimes of the cavity photons [31], required for hosting bosonic codes, and a nearly universal control of the microwaves [32], due to the highly non-linear nature of the accompanying transmon. We assume an ideal microwave-to-optical transducer, that can convert information from a cQED system to optical photons [33–35]. Note that a similar approach has been explored for a cavity using atom-light interactions [36], focusing on Cat codes in a third-generation quantum repeater, and in [37] for the engineering of Binomial code states. Here, we concentrate on transmon-enabled operations that have been used experimentally to demonstrate Binomial [38], Cat [39] and GKP [40] codes. Among the rotation-invariant bosonic codes, the orthogonality of the codewords in the Binomial code is beneficial [28]. Another advantage is that unlike other bosonic codes such as Cat or GKP codes, Binomial codes can be prepared without requiring an additional photon-number-related parameter (such as a phase-space displacement for Cat or a squeezing parameter for GKP), since they are truncated. Therefore, in this work, we focus on the Binomial code.

The structure of this article is as follows: In Sec. II, we concentrate on implementing QEC with Binomial codes for a second-generation quantum repeater protocol, and calculate the corresponding secret key rates. Section III introduces a deterministic algorithm for designing Binomial codeword states, syndrome detection, and logical Bell measurements for entanglement swapping in a microwave cavity system. Additionally, in Sec. IV, we propose a protocol for logical-level entanglement swapping, achieving 50% efficiency with linear optics. All numerical simulations and results are presented in Sec. V.

II. QUANTUM REPEATERS WITH QEC FOR LOSSY MEMORIES

A quantum repeater involves dividing the channel between the communicating parties into small segments, distributing entanglement in those segments, and then using entanglement swapping to distribute the entanglement across all segments. For the rest of the paper, we consider a two-segment repeater, as illustrated in Fig. 1, with a repeater node positioned in between (i.e., memory units 2 and 3 are located close to each other). The objective is to reliably distribute Bell pairs between memory units 1 and 4 (Fig. 1). As physical memory units, we consider microwave cavities which are manipulated

by superconducting transmons (not shown in the figure). Each segment requires two cavities at each end holding half of an entangled pair. Thus, there are two such cavities at each memory station, as shown in Fig. 1(c). In the first step, entanglement is independently distributed in the two segments, namely 1-2 and 3-4 (Fig. 1 (a),(b)). Since entanglement distribution is inherently probabilistic, the success probability for a single attempt is given by

$$p = p_{\text{link}} \eta_0, \quad (1)$$

where p_{link} represents the intrinsic success probability of local hardware effects such as coupling efficiency, while the dominant factor $\eta_0 = e^{-L_0/L_{\text{att}}}$ accounts for photon loss in the optical fiber channel. For example, at a distance of 100 km and an attenuation length $L_{\text{att}} = 22$ km, the typical success probability is on the order of 0.01. Consequently, multiple rounds of transmission attempts are required to successfully establish entanglement across each segment. The time taken for each attempt is primarily given by the classical communication time $T_0 = L_0/c$, where L_0 is the repeater segment length, and c is the speed of light in optical fiber. Once a segment succeeds in generating entanglement, the corresponding qubits are stored in their respective memories. In case of failure, the memories are reinitialized and another attempt is made. This process continues until both segments achieve success.

The second step is entanglement swapping between memory units 2 and 3 (Fig. 1(c)). Entanglement swapping requires a joint Bell state measurement of one half from each of two separately entangled pairs (2 from 1-2 pair and 3 from 3-4 pair), resulting in entanglement between the remaining two halves in the respective pairs (1 and 4, Fig. 1(d)) [41, 42].

Since the success of entanglement generation in each segment is independent of the other segment, we define two geometrically distributed random variables: n_{12} for segment 1-2 and n_{34} for segment 3-4. These random variables represent the number of attempts required to distribute entanglement in each segment. As the generation of entanglement is probabilistic, let us assume that segment 1-2 establishes entanglement earlier than segment 3-4. As a consequence, memory unit 2 will be waiting to swap. In general, with either memory unit 2 or 3 waiting, the waiting time is defined as $n_{\text{diff}} = |n_{12} - n_{34}|$ which follows a probability distribution p_{diff} . During this waiting period, either memory units 1 and 2 or 3 and 4 experience dissipation, leading to an exponential decrease in the swapping probability [43].

To mitigate memory loss, we encode the memory qubit using a Binomial code (see Eqs. (7), (8)) during the waiting period, thereby effectively enhancing the coherence time of the waiting memory units. The key performance metric for evaluating a quantum repeater is the (asymptotic) secret key rate, defined as

$$S = r \times R. \quad (2)$$

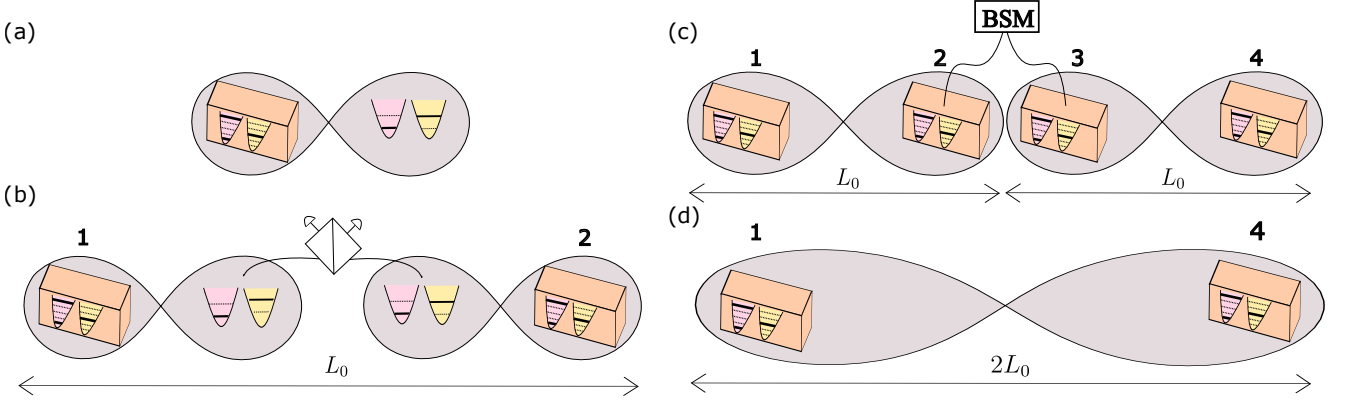


FIG. 1. Two-segment, second-generation quantum repeater, with memory implemented as a microwave cavity (shown as an example). (a) Entanglement between a travelling photon and a microwave cavity occurs at all four memory units. (b) Entanglement within a single repeater segment of distance L_0 is established between two memory units via projection measurement (using beam splitters) on travelling photons, which are conducted independently in both repeater segments. (c) Entanglement swapping occurs locally between two memory units at the middle station of the two segments via a logical Bell state measurement (BSM). (d) The final entangled state is achieved between distant memory units 1 and 4. Throughout these steps, the superposition states of the photon are represented by yellow and pink photons, though only one entangled pair of photons is distributed per round of the protocol. Cuboids represent microwave cavities, and the encoded states of the photon are depicted by the thick lines in the harmonic oscillator.

Here, R is the raw rate, i.e., the total number of bits transmitted per second. The fraction of the secret bits to the total bits transmitted is called secret key fraction and, considering the BB84 protocol, is given by

$$r = 1 - h(\langle e_x \rangle) - h(\langle e_z \rangle), \quad (3)$$

where the average is calculated by weighing it with the probability distribution function p_{diff} on condition that swapping is successful. Here $h(p)$ is the binary entropy function

$$h(p) = -p \log p - (1 - p) \log(1 - p), \quad (4)$$

and the quantum bit error rates (QBERs) e_x and e_z are defined as (assuming the desired state to be shared is $|\psi^+\rangle$)

$$e_z = \langle 00 | \rho_{14} | 00 \rangle + \langle 11 | \rho_{14} | 11 \rangle, \quad (5)$$

and

$$e_x = \langle +- | \rho_{14} | +- \rangle + \langle -+ | \rho_{14} | -+ \rangle. \quad (6)$$

where ρ_{14} is the density matrix of the state between memory units 1 and 4 (Fig. 1). To this end, we compare S between cases with and without encoding to assess the impact of error correction on the memories.

A. Binomial code

The Binomial code [21] is one of the bosonic codes to combat loss errors. The codewords of the lowest-order

Binomial code (LBC), in the Fock basis, are

$$|\bar{0}\rangle = \frac{|0\rangle + |4\rangle}{\sqrt{2}}, \quad |\bar{1}\rangle = |2\rangle. \quad (7)$$

This corrects single-excitation loss in the bosonic memories. The codewords of the next, higher-order Binomial code (HBC) are

$$|\bar{0}\rangle_h = \frac{|0\rangle + \sqrt{3}|6\rangle}{2}, \quad |\bar{1}\rangle_h = \frac{\sqrt{3}|3\rangle + |9\rangle}{2}. \quad (8)$$

This corrects for up to two excitation losses. We show the calculation of S explicitly for the lowest order. We then numerically simulate both codes for a two-segment repeater and compare their secret key rates S .

B. Secret key rate of a repeater with LBC

In this section, we discuss the calculation of the secret key rate when the memory units are encoded with LBC.

First, we generate one of the Bell states between the memory units 1-2 and 3-4. For any two memory units $\{i, j\}$, it is given by

$$|\psi^+\rangle_{ij} = \frac{|0\rangle_i |1\rangle_j + |1\rangle_i |0\rangle_j}{\sqrt{2}}. \quad (9)$$

The encoded Bell states are defined as $|\bar{\psi}^+\rangle$ by a similar definition as above with the replacements $|0, 1\rangle \rightarrow |\bar{0}, \bar{1}\rangle$. We also define the corresponding density matrix $\bar{\rho}_{ij} = |\bar{\psi}^+\rangle_{ij} \langle \bar{\psi}^+|_{ij}$.

Without loss of generality, we assume that the Bell state was successfully generated in memory units 1-2 first. While waiting for memory units 3-4, memory units 1-2 undergo amplitude damping given by the set of Kraus operators [44], E_k for all $k \in \mathbb{N}$,

$$E_k = \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k} \gamma^k} |n-k\rangle \langle n|, \quad (10)$$

where n is the number of excitations, k is the number of excitations lost, and γ is the probability of losing an excitation stored in the memory unit.

The state of the memory units 1 and 2 after being subject to the amplitude damping channel is

$$\bar{\rho}_{12}^{dec} = \sum_{k_1, k_2=0}^4 E_{k_1,1} E_{k_2,2} \bar{\rho}_{12} E_{k_1,1}^\dagger E_{k_2,2}^\dagger, \quad (11)$$

where $E_{k,j}$ denotes k losses in mode j .

To mitigate the effect of dissipation, error correction on the encoded qubits should be performed before entanglement swapping. Syndrome detection can be done by a parity measurement as discussed in Sec. III. The recovery operation to be performed depends on the syndrome outcome. If there were no losses, the recovery operation to be performed, up to linear order of γ , is

$$\mathcal{R}_0 = |\bar{0}\rangle \langle \bar{0}^*| + |\bar{1}\rangle \langle \bar{1}^*|, \quad (12)$$

where $|\bar{0}^*\rangle = |0\rangle + (1-\gamma)^2 |4\rangle / \sqrt{2(1-2\gamma)}$ and $|\bar{1}^*\rangle = (1-\gamma) |2\rangle / \sqrt{(1-2\gamma)}$. Note that $\mathcal{R}_0 = \text{id}$ when $\gamma = 0$. If a single loss was found, the recovery operation is

$$\mathcal{R}_1 = |\bar{0}\rangle \langle 3| + |\bar{1}\rangle \langle 1|. \quad (13)$$

In $\mathcal{R}_0$, the lack of complete orthogonality between error words and codewords is the cause of the significant disparity in expressions between $\mathcal{R}_0$ and $\mathcal{R}_1$.

The next step is to perform entanglement swapping, which is achieved by projecting the joint state of memory units 2 and 3 onto the Bell basis. For example,

$$\bar{\rho}_{14} = \frac{{}_{23}\langle \bar{\psi}^+ | \bar{\rho}_{1234} | \bar{\psi}^+ \rangle_{23}}{\text{Tr} [{}_{23}\langle \bar{\psi}^+ | \bar{\rho}_{1234} | \bar{\psi}^+ \rangle_{23}]}, \quad (14)$$

with similar expressions for the other Bell states. Here, $\bar{\rho}_{1234}$ is given by $\mathcal{R}^{\otimes 2}(\bar{\rho}_{12}^{dec}) \otimes \bar{\rho}_{34}$ and $\mathcal{R}$ is $\mathcal{R}_0$ or $\mathcal{R}_1$ depending on the error syndrome information.

The raw rate for the case of two segments with memory decay is given by [43]

$$R = \frac{\langle P_s \rangle}{\langle n_{max} \rangle T_0}, \quad (15)$$

where n_{max} is the maximum of n_{12} and n_{34} and where P_s is the swapping probability corresponding to the sum

of the traces of the conditional states of the Bell measurement (the denominator in Eq. (14)) for all four Bell measurement results. The average of P_s has to be taken, because the extent to which the state $\bar{\rho}_{1234}$ is subject to memory loss depends on the random memory storage (waiting) time n_{diff} . However, we will make a distinction between the cases where the loss error correction is only performed once at the end after storage and where it is done repeatedly for multiple times at a frequency given by the elementary time unit T_0 . Both cases are treated numerically by our simulation.

The figure of merit S is calculated from R and r (defined in Eqs. (15) and (3), respectively). The secret key rates for different regimes of γ are compared for the cases of encoded and unencoded memories in Sec. V.

III. QUANTUM REPEATER WITH MICROWAVE CAVITIES

In this section, we discuss a physical implementation of the quantum repeater, introduced in Sec. II, using cQED setups. As discussed in Fig. 1, there are two microwave (MW) cavities, say A_1 and A_2 , placed at every local station, each of which are loaded with a transmon qubit $Q_{1,2}$. The first step is the generation of an entangled MW-qubit state, using local operations as discussed below. Subsequently, the state of the transmon qubit is transferred to a photon which is sent to the middle station as shown in Fig. 1. Remote entanglement swapping is performed via a joint Bell-state measurement (BSM) on A_1 and A_2 in the Binomial-encoded basis. We discuss below a scheme for BSM using a third transmon, Q_3 , coupled to both MW cavities.

A. Hamiltonian

Here, we discuss the relevant Hamiltonians for our system, whose implementations are well-known in the literature [30]. The Hamiltonian for each cavity is that of a bosonic mode $H_i^a = \hbar \omega_{a,i} a_i^\dagger a_i$ for $i \in \{1, 2\}$, where $\omega_{a,i}$ and a_i are, respectively, the resonance frequency and the annihilation operator of A_i . For simplicity, we consider the resonant case $\omega_{a,1} = \omega_{a,2} = \omega_a$.

Each of the transmons are anharmonic oscillators, modelled by the Hamiltonian

$$\frac{H_i^{qs}}{\hbar} = \omega_{q,i} q_i^\dagger q_i - \frac{\alpha_i}{2} q_i^\dagger q_i (q_i^\dagger q_i - 1), \quad (16)$$

for $i \in \{1, 2, 3\}$ where $\omega_{q,i}$, α_i , and q_i are the resonance frequency, anharmonicity, and the annihilation operator of Q_i . Furthermore, each transmon can be externally controlled via the dynamic Hamiltonian

$$\frac{H_i^{qd}(t)}{\hbar} = \Delta_i(t) q_i^\dagger q_i + \varepsilon_i(t) q_i^\dagger + \varepsilon_i^*(t) q_i, \quad (17)$$

where $\Delta_i(t)$ is the externally controllable detuning and $\varepsilon_i(t)$ is the excitation amplitude corresponding to Q_i . We assume that the cavities and transmons are so far detuned that the coupling is negligible for no external detuning, i.e. $\Delta_i = 0$. Changing the detuning Δ_i can bring Q_i in resonance with one or both of the MW cavities, allowing for entangling operations.

The individual MW-qubit couplings are in the form of a beam-splitter Hamiltonian $H_i^{aq} = \hbar g_{aq,i} (q_i^\dagger a_i + q_i a_i^\dagger)$ for $i \in \{1, 2\}$ with coupling coefficients $g_{aq,i}$. The third transmon used for BSM is coupled to both MW cavities with an effective dispersive coupling χ , modelled with

$$H^{\text{disp}} = \hbar \chi(t) q_3^\dagger q_3 (a_1^\dagger a_1 + a_2^\dagger a_2). \quad (18)$$

The parameter χ can be tuned via the detuning Δ_3 [29, 30].

B. Entanglement generation

We want to create an entangled microwave and optical photon pair, as discussed in Fig. 1. In the first step, we create an entangled state in each MW-transmon subsystem. In this simulation, we can focus on only one MW-transmon pair ignoring the other, so we suppress the index, e.g. $a_i \rightarrow a$ and $q_i \rightarrow q$. Dissipation is added in standard Lindblad forms [45] of cavity loss, transmon loss, and transmon dephasing with corresponding Lindblad operators being a , q , and $q^\dagger q$ respectively. We consider nominal values for a cavity with $\omega_a = 2\pi \times 6$ GHz, and lifetime of 50 μs . We consider a transmon with $\omega_q = 2\pi \times 5$ GHz, $\alpha = 2\pi \times 300$ MHz, lifetime $T_1 = 10$ μs and total dephasing time $T_2^* = 5$ μs . We assume a coupling rate of $g = 2\pi \times 25$ MHz, which is sufficiently small to ensure that there is negligible MW-transmon coupling in equilibrium as $g \ll \omega_a - \omega_q$. The coupling is then induced by an input of $\Delta(t)$ that brings the MW and transmon in resonance. Complementarily, excitations in the MW-transmon subsystem are generated by exciting the transmon with the pulse $\varepsilon(t)$, as written above.

We adapt the code given in Ref. [46], to find $\Delta(t)$ and $\varepsilon(t)$ that leads to the entangled state

$$|\psi\rangle = |\bar{0}\rangle |e\rangle + |\bar{1}\rangle |g\rangle, \quad (19)$$

where the MW state is encoded. Our simulations show that this state can be generated with $> 97\%$ fidelity. The fidelity is primarily limited by cavity dissipation, and thus is significantly higher for cavities with higher lifetimes. The resonance frequencies have only a weak effect on the fidelity.

The state of the transmon can be swapped into a travelling photon via a microwave-to-optical converter, creating an entangled state between microwaves and photons [33–35]. We consider ideal transduction between microwave and optical photons. The feasibility and implications of this assumption are discussed in the next section.

C. Joint Bell-state measurement

Here, we discuss how to perform a joint BSM on the two MW cavities, A_1 and A_2 , using the transmon Q_3 coupled to both of them as discussed above. In the following, we ignore the first two transmons as they are expected to stay in the ground state.

A BSM results in a 2-bit output, where each of the four possibilities correspond to the four joint MW Bell states, namely

$$\begin{aligned} |\Phi^\pm\rangle &= \frac{|\bar{0}\bar{0}\rangle \pm |\bar{1}\bar{1}\rangle}{\sqrt{2}} \\ |\Psi^\pm\rangle &= \frac{|\bar{0}\bar{1}\rangle \pm |\bar{1}\bar{0}\rangle}{\sqrt{2}} \end{aligned} \quad (20)$$

The state $|\bar{i}\bar{j}\rangle$ refers to the first cavity being in the i th logical state and the second cavity in the j th state. If an error has occurred on any of the cavities, their states will instead be $|\bar{0}^*\rangle$ or $|\bar{1}^*\rangle$.

First, a parity check [47] is performed on each of the cavities to find if an error has occurred. If no error has occurred, then a general state in the logical subspace is

$$|\psi\rangle_{a_1 a_2 q_3} = \sum_{ij} \alpha_{ij} |\bar{i}\bar{j}\rangle |g\rangle, \quad (21)$$

where $i, j \in \{0, 1\}$.

To separate the Ψ Bell states from the Φ Bell states, we employ the following steps. First, we apply a Hadamard gate to the qubit Q_3 by choosing $\varepsilon_3(t) = -i\varepsilon(t)$ (in the rotating frame) satisfying

$$\int dt \varepsilon(t) = \frac{\pi}{4}, \quad (22)$$

to create an initial qubit state $\propto |g\rangle + |e\rangle$. Next, the dispersive coupling is turned on by reducing the detuning Δ_3 such that $\chi(t) = \chi_0$ for a time period of $t = \pi/(2\chi_0)$, and zero outside the time window. The effect of this coupling is to apply the entangling unitary on the joint system,

$$U = \exp \left[i \frac{\pi}{2} \left(a_1^\dagger a_1 + a_2^\dagger a_2 \right) |e\rangle \langle e| \right]. \quad (23)$$

This unitary applies a phase on the qubit, depending on the total number of microwave photons in both cavities. The phase is periodic with photon number 4. After the two operations, the joint state of the two cavities and Q_3 becomes

$$\sum_{ij} \alpha_{ij} |\bar{i}\bar{j}\rangle \frac{|g\rangle + (-1)^{i+j} |e\rangle}{\sqrt{2}}. \quad (24)$$

Third, we apply another Hadamard gate on the transmon to get the state

$$|g\rangle (\alpha_{00} |\bar{0}\bar{0}\rangle + \alpha_{11} |\bar{1}\bar{1}\rangle) + |e\rangle (\alpha_{01} |\bar{0}\bar{1}\rangle + \alpha_{10} |\bar{1}\bar{0}\rangle).$$

Now, we can measure the qubit to separate the Ψ -states from the Φ -states. The two measurement outcomes can be written in the Bell basis as

$$|g\rangle (A_+ |\bar{\Phi}^+\rangle + A_- |\bar{\Phi}^-\rangle), |e\rangle (B_+ |\bar{\Psi}^+\rangle + B_- |\bar{\Psi}^-\rangle)$$

for coefficients $A_{\pm}$ and $B_{\pm}$ that can be found in terms of α -coefficients.

On the binomial code, universal gate operations were demonstrated [38]. So, we can apply a Hadamard gate on the encoded qubits in both microwave cavities to get the two possibilities depending on the outcome of the above measurement,

$$|g\rangle (A_+ |\bar{\Phi}^+\rangle + A_- |\bar{\Psi}^+\rangle), |e\rangle (B_+ |\bar{\Phi}^-\rangle - B_- |\bar{\Psi}^-\rangle).$$

We can again apply the above three steps to separate $\Psi^{\pm}$ from $\Phi^{\pm}$, finishing the Bell state measurement.

If an error has occurred on the first cavity, Eq. (24) will be modified as

$$\sum_{ij} \alpha_{ij} |\bar{i}^* \bar{j}\rangle \frac{|g\rangle - i(-1)^{i+j} |e\rangle}{\sqrt{2}}. \quad (25)$$

Then, we can apply a Hadamard gate preceded by a phase gate to cancel off the factor $-i$. Note that both the gates can be applied in one step by choosing $\varepsilon_3(t) = -\varepsilon(t)$ with the condition in Eq. (22). Then, the rest of the process follows in the same way.

Similarly, if an error occurred on both cavities, Eq. (24) will be modified as

$$\sum_{ij} \alpha_{ij} |\bar{i}^* \bar{j}^*\rangle \frac{|g\rangle - (-1)^{i+j} |e\rangle}{\sqrt{2}}, \quad (26)$$

and the same procedure as in the no-error case can be applied.

D. Experimental considerations

The coupling of a superconducting transmon and a microwave cavity has been implemented in numerous experiments over decades [29, 30]. Binomial codes and their universal control has been demonstrated in microwave cavities [38]. For our simulations of the generation of entanglement between microwave and transmon systems, we use the Hamiltonian given in Eq. (18) to describe the coupling between two microwave cavities mediated by a transmon. The coupling parameters are taken from experimentally demonstrated values [48]. In such a system, care must be taken to minimize cross-talk between different transmons. To achieve this, the transmons are typically kept off-resonant with the cavities, effectively switching the couplings off. The couplings are activated only within specific time windows, and only one coupling is switched on at any given moment. This approach effectively suppresses cross-talk between transmons.

We have assumed an ideal microwave-to-optical converter in our analysis. While such converters are not yet

realizable, there has been a significant progress in achieving partial entanglement between microwaves and optics [33–35]. Since such a conversion is being coveted for several applications in quantum computing, we expect such converters to be available in the future.

If the microwave-to-optical conversion is not ideal and can instead be modeled as a probabilistic process—where, for instance, the conversion succeeds with a probability of 99% and fails with 1%—its effect can be directly incorporated into the secret key rate (SKR) calculation. In this case, the conversion efficiency modifies only the entanglement generation probability within a repeater segment (see Eq. 1). Specifically, the intrinsic link success probability p_{link} in Eq. 1 can be replaced by an effective probability

$$p'_{\text{link}} = p_{\text{cov}} p_{\text{link}},$$

where p_{cov} denotes the microwave-to-optical photon conversion probability during the state preparation stage. The entanglement swapping stage remains unaffected since it does not involve frequency conversion. Therefore, no additional modifications are required in our current theoretical model for evaluating the SKR. However, if the microwave-to-optical conversion cannot be described as such a simple probabilistic success-or-failure process, integrating its effect into the SKR analysis becomes significantly more complex.

IV. QUANTUM REPEATER WITH QEC IN THE OPTICAL DOMAIN

Another feasible experimental platform for quantum repeaters is optics. In an all-optical approach, we could bypass the need for a microwave-to-optical conversion. However, in this case, we need to store phase-sensitive photonic states in an optical cavity system. To a certain extent, this has been experimentally demonstrated already, at least for phase-sensitive states up to one photonic excitation [49–51]. Another benefit of this all-optical approach would be that the local state processing at each repeater node can be done much faster than with light-matter interactions. The hardest element in an all-optical domain is the state preparation. There are a few experimental demonstrations of generating higher Fock states [52, 53] which are crucial to generate Binomial codeword states. Concerning the logical Bell measurements, we show that entanglement swapping can be achieved with linear optics at a 50% success rate using photon-number-resolving detectors (PNRDs), provided that a recovery operation is performed prior to the swapping process. The corresponding derivation is presented in App. A, including a figure that shows that the overall behavior of the SKR remains the same.

V. SIMULATION OF SECRET KEY RATES

Here, we discuss the simulation and numerical evaluation of secret key rates for both cases, unencoded and encoded (LBC and HBC). In essence, we need to calculate Eq. (2). However, there are two possibilities in the calculation process: implementing error correction only once, referred to as single-time error correction (SEC), or implementing error correction multiple times at regular intervals during the waiting time, referred to as multiple-time error corrections (MEC). The motivation for MEC is to keep the memory states intact for longer while they are waiting. However, a typical problem is that the recovery operators are only approximate to first order, making it unclear whether repeated application would actually be beneficial. To address this, we implemented MEC under the assumption of no additional overheads, in order to determine whether it is feasible even in the idealized case. A detailed theoretical analysis of MEC is provided in App. C. The procedure to simulate the secret key rates S with SEC is described in Sec. II. Since it is SEC, $\mathcal{R}$ is applied only once. To numerically calculate the averages in Eq. (15), a cutoff time on the memories (apart from the exponential decay) was implemented. The swapping probability P_s is given by expressions such as the denominator of Eq. (14). For the SEC case, the cutoff was selected based on the time step corresponding to $\gamma = 0.5$. In the case of MEC, $\gamma = 0.1$ with 100 as an upper limit for the maximum number of times the recovery can be performed is considered. The parameters $L_{att} = 22$ km, $c = 2 \times 10^8$ m s⁻¹ were used throughout. The secret key rates in both cases are calculated under the assumption of 100% efficiency in the BSM during swapping. Three different coherence times are considered for the simulations, including realistic values from state-of-the-art experiments [31] and possible future values. All simulations were performed in Mathematica.

In Fig. 2, we show the secret key rate as a function of the segment length, L_0 (range 10-200 km), assuming a highly efficient memory interface, $\eta_m = 0.95$. We observe a clear improvement in the secret key rates of the encoded schemes in comparison with the unencoded one for all three different coherence times considered. For a more imperfect memory interface, $\eta_m = 0.9$, the results are shown in Fig. 3. They exhibit a similar trend but the unencoded case gives no positive secret key rates. Interestingly, memories with error correction outperform the unencoded memories even with coherence times one order of magnitude higher.

In Fig. 4, we show the secret key fraction (r) as a function of the segment length L_0 for LBC showing the difference between the cases of SEC and MEC. As expected, allowing for multiple error corrections increases r for long distances. For sufficiently short distances, a single recovery step is more appropriate. With increasing coherence times the distance at which SEC and MEC start to deviate is also increasing. Even though the value of r is almost unity for the blue curves ($\tau_{coh} = 1$ s) at

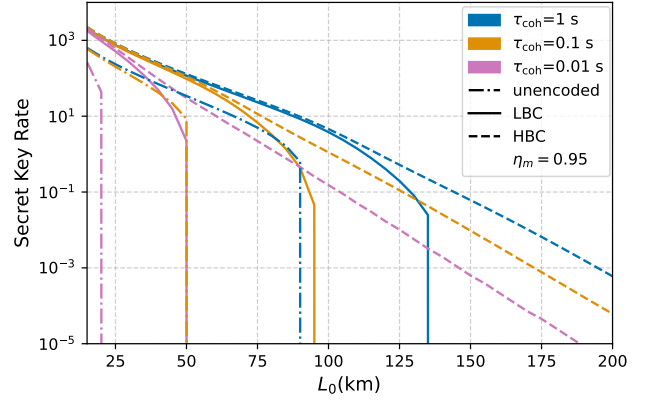


FIG. 2. Comparison of secret key rates with different memory coherence times for unencoded, Lower Binomial Code (LBC) and Higher Binomial code (HBC) when SEC is performed. The memory interface parameter $\eta_m = 0.95$.

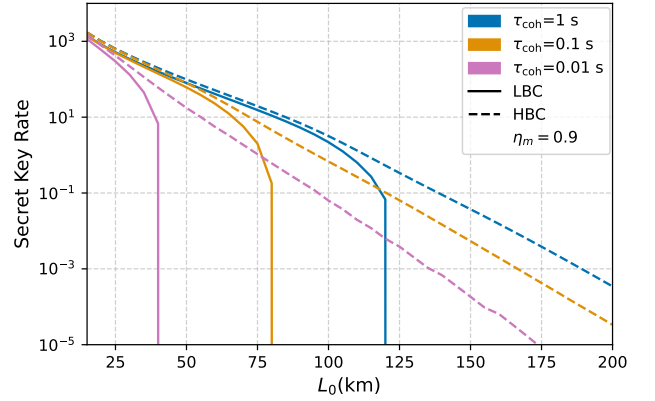


FIG. 3. Comparison of secret key rates with different memory coherence times for Lower Binomial Code (LBC) and Higher Binomial code (HBC) when SEC is performed. The memory interface parameter $\eta_m = 0.9$.

100 km for both SEC and MEC, from Fig. 2 it is clear that the secret key rates decrease by two orders of magnitude from the initial value due to the decrease in the raw rate (R).

We also find that in SEC, for a minimum segment length of 10 km and a memory coherence time (τ_{coh}) of 1 s, LBC yields positive secret key rates only when $\eta_m > 0.78$, whereas HBC requires $\eta_m > 0.73$. Our approach involves the full evolution of the density matrix, and performing a similar rate analysis for higher-order Binomial codes is challenging due to the exponential growth of terms in the density matrix. However, we expect a similar trend for higher orders, with a trade-off between the benefits of higher-order encoding and the coherence time of the memories.

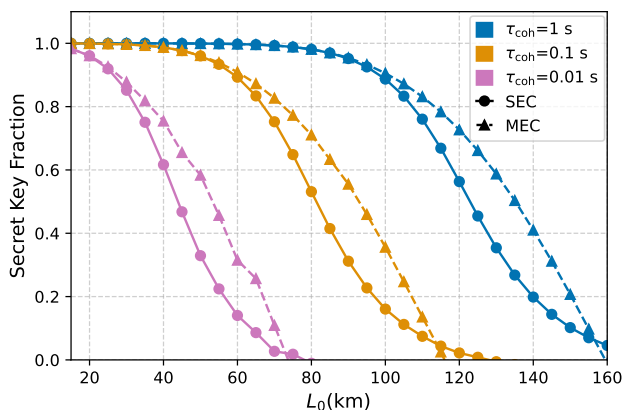


FIG. 4. Comparison of the secret key fraction of LBC with different coherence times for the cases of SEC and MEC. The memory interface parameter $\eta_m = 1.0$.

VI. CONCLUSION AND OUTLOOK

We have studied the near-term performance of a second-generation quantum repeater protocol under realistic conditions, in particular, focusing on a Binomial quantum error correction code with loss as an error model for the quantum memories at the repeater stations. In addition and more specifically, we have proposed a promising implementation using microwave cavities that achieves all four necessary key steps: state engineering, syndrome measurement, error correction, and entanglement swapping. Our simulations of a two-segment quantum repeater allowed us to examine memory loss errors and the influence of the Binomial encoding. We found that both schemes with the Lower Binomial Code (LBC) and with the Higher Binomial Code (HBC) outperform the unencoded schemes for the same coherence times of the quantum memories. The HBC, even with lower coherence times, surpassed the performance of the unencoded protocols with higher coherence times. Moreover, we observed that multiple-error correction (MEC) provides a better secret key fraction (SKF) than single-error correction (SEC). We considered two different memory interface efficiencies, η_m , and found that the SKR trends are similar for the encoded and the unencoded cases. Our methods and results offer valuable guidance for optimizing repeater protocols in upcoming experiments. Our analysis assumed ideal microwave-to-optical conversion. While such converters are not yet realized, partial implementations exist and there is also more research in this. Any non-idealities can be incorporated into the entanglement generation probability without fundamentally altering our model.

As a next step, it is interesting to analyze multi-segment repeater systems and networks. Since MEC turned out to be beneficial in the ideal case, it would be valuable to extend the analysis by including overhead losses, as there should exist an optimal number of re-

covery operations. This is because repeated application of error correction can itself introduce additional errors. Exploring Binomial-code error correction in the context of third-generation quantum repeaters, beyond existing schemes, is a possible, promising direction as well. In this context, it would be useful to investigate state engineering in the optical domain and consider improving the 50% bound for logical Bell measurements for entanglement swapping. A rigorous rate analysis in the context of η_m would also be interesting to pursue. In summary, our findings demonstrate the promise of Binomial-code quantum error correction in advancing quantum repeater protocols and suggest important directions for future work.

ACKNOWLEDGMENTS

S.C. thanks Simon Reiß and Evgeny Shchukin for useful discussions. We acknowledge support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Project-ID 429529648—TRR 306 QuCoLiMa (“Quantum Cooperativity of Light and Matter”). S.C., F.S., and P.v.L. acknowledge funding from the Bundesministerium für Bildung und Forschung (BMBF) under the projects QR.X/QR.N, PhotonQ, and QuKuK. S.S. and S.V.K. acknowledge funding from the Bundesministerium für Bildung und Forschung (BMBF) under the project QECHQS (Grant No. 16KIS1590K).

Appendix A: Logical linear-optics Bell measurements

In this appendix, we show that a 50% success rate in entanglement swapping/Bell-state measurement (BSM) using linear optics can be achieved for lower-order Binomial codes when there is no loss on the codewords. This means, in our repeater scheme, we have to assume that a recovery operation is performed before the entanglement swapping takes place such that all states are mapped back onto the original codespace.

The BSM requires to distinguish the logical two-qubit states given in Eq. (20). Now first notice that $|\bar{\Phi}^\pm\rangle$ only contain Fock states of 0, 4, and 8 photons, whereas $|\bar{\Psi}^\pm\rangle$ only have terms with 2 and 6 photons. We will apply a beam splitter that preserves the total photon number in each term and so accepting only 2 or 6 photons at the detectors will unambiguously identify the two Bell states $|\bar{\Psi}^\pm\rangle$. In order to discriminate among these two states, we have to examine the possible number patterns of the two-mode state at the output of the beam splitter operation, BS. A 50/50 beam splitter transforms $|\bar{\Psi}^\pm\rangle$

as

$$\begin{aligned}
|\bar{\Psi}^\pm\rangle &= \frac{|\bar{0}\bar{1}\rangle \pm |\bar{1}\bar{0}\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \left(\frac{|0\rangle + |4\rangle}{\sqrt{2}} |2\rangle \pm |2\rangle \frac{|0\rangle + |4\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{2} (|02\rangle \pm |20\rangle + |42\rangle \pm |24\rangle) \\
&\rightarrow \begin{cases} \frac{1}{2} [|20\rangle + |02\rangle + \hat{B}\hat{S}(|42\rangle + |24\rangle)] \\ \frac{1}{2} [\sqrt{2}|11\rangle + \hat{B}\hat{S}(|42\rangle - |24\rangle)] \end{cases} \quad (\text{A1})
\end{aligned}$$

Here we have used $\hat{B}\hat{S}|20\rangle = \frac{1}{2}(|20\rangle + |02\rangle) + \frac{1}{\sqrt{2}}|11\rangle$ and $\hat{B}\hat{S}|02\rangle = \frac{1}{2}(|20\rangle + |02\rangle) - \frac{1}{\sqrt{2}}|11\rangle$ which means that $\frac{1}{\sqrt{2}}(|20\rangle + |02\rangle)$ remains invariant under the beam splitter transformation, while $\frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$ becomes $|11\rangle$ (like an inverse Hong-Ou-Mandel effect). Thus, click patterns ‘11’ and ‘20/02’ unambiguously identify the states $|\bar{\Psi}^- \rangle$ and $|\bar{\Psi}^+ \rangle$, respectively. It then remains to be shown that also the patterns that originate from $\hat{B}\hat{S}(|42\rangle \pm |24\rangle)$ are in one-to-one correspondence to the two different Bell states. This can easily be inferred from the following beam splitter transformations:

$$\begin{aligned}
\hat{B}\hat{S}|24\rangle &= \frac{1}{8} \left[\sqrt{15}|06\rangle + \sqrt{10}|15\rangle \right. \\
&\quad \left. - |24\rangle - 2\sqrt{3}|33\rangle - |42\rangle \right. \\
&\quad \left. + \sqrt{10}|51\rangle + \sqrt{15}|60\rangle \right], \\
\hat{B}\hat{S}|42\rangle &= \frac{1}{8} \left[\sqrt{15}|06\rangle - \sqrt{10}|15\rangle \right. \\
&\quad \left. - |24\rangle + 2\sqrt{3}|33\rangle - |42\rangle \right. \\
&\quad \left. - \sqrt{10}|51\rangle + \sqrt{15}|60\rangle \right]. \quad (\text{A2})
\end{aligned}$$

From the above equations, it is clear that the states $\hat{B}\hat{S}(|42\rangle \pm |24\rangle)$ produce the unique click patterns ‘06’, ‘24’, ‘42’, ‘60’ and ‘15’, ‘33’, ‘51’, respectively. As a consequence, this proves that half of the Bell states can always be unambiguously identified, thus leading to a 50% BSM efficiency.

The secret key rate (SKR) for the all-optical entanglement swapping scheme is shown in Figure 5, while Figure 6 compares the performance of the all-optical and microwave-cavity-based entanglement swapping approaches.

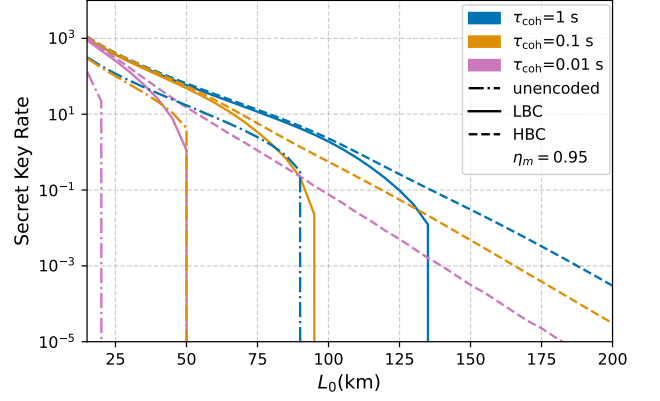


FIG. 5. Comparison of secret key rates with different memory coherence times for unencoded, Lower Binomial Code (LBC) and Higher Binomial code (HBC) when SEC is performed and entanglement swapping is performed using linear optics with 50% efficiency. The memory interface parameter $\eta_m = 0.95$

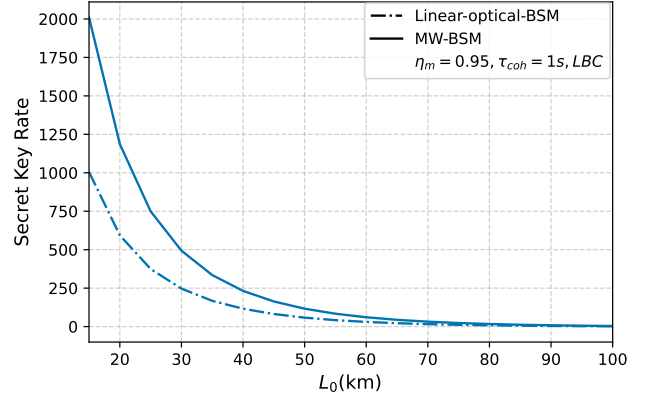


FIG. 6. The comparison of SKR for all-optical and microwave cavity approach for lower order binomial code (LBC).

Appendix B: Recovery operators

The no-loss recovery operator for the LBC in the linear loss order, as given in the main text by Eq. (12), can be equivalently written as (using the notation from Ref. [21])

$$\begin{aligned}
\mathcal{R}_0 &= (\sin(\log(1 - \gamma))) (|E_0^0\rangle \langle \bar{0}| - |\bar{0}\rangle \langle E_0^0|) \\
&\quad + (\cos(\log(1 - \gamma))) (|\bar{0}\rangle \langle \bar{0}| + |E_0^0\rangle \langle E_0^0|) \\
&\quad + |\bar{1}\rangle \langle \bar{1}|, \quad (\text{B1})
\end{aligned}$$

where $|E_0^0\rangle = (|0\rangle - |4\rangle)/\sqrt{2}$. Note that $\mathcal{R}_0 = \text{id}$ when $\gamma = 0$.

The no-loss recovery operator for the HBC is given by

[21]

$$\mathcal{R}_0 = \sum_{\sigma} \sqrt{1 - \frac{|\langle \hat{B}_0 \rangle|^2}{\beta_0}} (|W_{\sigma}\rangle \langle B_{\sigma}^0| - |B_{\sigma}^0\rangle \langle W_{\sigma}|) + \frac{\langle \hat{B}_0 \rangle}{\sqrt{\beta_0}} (|W_{\sigma}\rangle \langle W_{\sigma}| + |B_{\sigma}^0\rangle \langle B_{\sigma}^0|), \quad (\text{B2})$$

where $|W_{\sigma}\rangle$ is the codeword,

$$\beta_0 = \langle W_{\sigma} | \hat{B}_0^{\dagger} \hat{B}_0 | W_{\sigma} \rangle, \quad (\text{B3})$$

$\hat{B}_0 = 1 - \log(1 - \gamma)\hat{n}/2 + (\log(1 - \gamma))^2\hat{n}^2/8$, and

$$|B_{\sigma}^0\rangle = \frac{(1 - |W_{\sigma}\rangle \langle W_{\sigma}|) \hat{B}_0 |W_{\sigma}\rangle}{\sqrt{\beta_0 - |\langle \hat{B}_0 \rangle|^2}}. \quad (\text{B4})$$

The one-loss recovery operator for the HBC is given by

$$\mathcal{R}_1 = |\bar{0}\rangle \langle 5| - |5\rangle \langle \bar{0}| + \frac{1}{\sqrt{2}} |\bar{1}\rangle (\langle 2| + \langle 8|) - \frac{1}{\sqrt{2}} (|2\rangle + |8\rangle) \langle \bar{1}|. \quad (\text{B5})$$

The two-loss recovery operator for the HBC is given by

$$\mathcal{R}_2 = |\bar{0}\rangle \langle 4| - |4\rangle \langle \bar{0}| + |\bar{1}\rangle \left(\frac{1}{\sqrt{5}} \langle 1| + \frac{2}{\sqrt{5}} \langle 7| \right) - \left(\frac{1}{\sqrt{5}} |1\rangle + \frac{2}{\sqrt{5}} |7\rangle \right) \langle \bar{1}|. \quad (\text{B6})$$

Appendix C: Multiple rounds of error correction

Here, we consider a scheme with multiple rounds of quantum error correction where we perform one round thereof after a fixed number of time steps t_k . Thus, we calculate the concatenation of the loss channel after l time states followed by the recovery operation, which we will refer to as C^l , for $l \in \{1, \dots, t_k\}$. As an example let us consider that one quantum memory needs to wait for 100 time steps until entanglement is also distributed in the other segment and we choose t_k to be 30. The resulting final noise channel is then given by $C^{10}(C^{30}(C^{30}(\cdot)))$. Here we can see that we have to concatenate many channels. We do this concatenation by mapping the channel onto a vector of coefficients and the vector of coefficients after the concatenation can be obtained by multiplying the old vector of coefficients with a matrix. The mapping works as follows: For simplicity, we consider a qubit channel, i.e., we will project C^l onto a map that acts within the logical qubit space and a map that maps from the logical qubit space outside the codespace. Thus, we neglect parts mapping from outside the code space onto the codespace. Each single-qubit channel $\mathcal{C}$ can be written in Kraus-operator representation $\sum_{j=1}^4 \sqrt{p_j} \hat{U}_j \cdot \hat{U}_j^{\dagger} \sqrt{p_j}$, where p_j are probabilities and $\hat{U}_j$ are unitary operators. Pauli operators form an orthogonal basis with respect to the Frobenius inner product $\langle a, \hat{B} \rangle = \text{Tr}(\hat{a}^{\dagger} \hat{B})$. Thus, we can write $\hat{U}_j = \alpha_{j0}1 + \alpha_{j1}\sigma_1 + \alpha_{j2}\sigma_2 + \alpha_{j3}\sigma_3$. We can simplify the next expressions by defining $\sigma_0=1$ and $\mathcal{N} = \sum_{j=0}^3 \sum_{k,l} p_j \alpha_{jk} \sigma_k \cdot \sigma_l \alpha_{jl}^* = \sum_{k,l} p_{kl} \sigma_k \cdot \sigma_l$, where $p_{kl} = \sum_{j=0}^3 p_j \alpha_{jk} \alpha_{jl}^*$. The coefficients p_{kl} then contain the complete information of the channel. When considering the concatenation of two channels one can find that the new channel is simply given by the product of a matrix defined by the outer channel and the vector of coefficients of the inner channel. Thus, the multiple concatenations of the same channel can be represented by a matrix product.

-
- [1] H. J. Kimble, The quantum internet, *Nature* **453**, 1023–1030 (2008).
 - [2] G. B. C. H. Bennett, Quantum cryptography: Public key distribution and coin tossing, *IEEE* **175**, 8 (1984).
 - [3] A. K. Ekert, Quantum cryptography based on bell's theorem, *Phys. Rev. Lett.* **67**, 661 (1991).
 - [4] M. Caleffi, M. Amoretti, D. Ferrari, J. Illiano, A. Manzalini, and A. S. Cacciapuoti, Distributed quantum computing: A survey, *Computer Networks* **254**, 110672 (2024).
 - [5] J. F. Fitzsimons, Private quantum computation: an introduction to blind quantum computing and related protocols, *npj Quantum Information* **3**, 10.1038/s41534-017-0025-3 (2017).
 - [6] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Quantum repeaters: The role of imperfect local operations in quantum communication, *Phys. Rev. Lett.* **81**, 5932 (1998).
 - [7] S. Muralidharan, L. Li, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang, Optimal architectures for long distance quantum communication, *Scientific Reports* **6** (2016).
 - [8] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Long-distance quantum communication with atomic ensembles and linear optics, *Nature* **414**, 413 (2001).
 - [9] M. Pompili, S. L. N. Hermans, S. Baier, H. K. C. Beukers, P. C. Humphreys, R. N. Schouten, R. F. L. Vermeulen, M. J. Tiggelman, L. dos Santos Martins, B. Dirkse, S. Wehner, and R. Hanson, Realization of a multinode

- quantum network of remote solid-state qubits, *Science* **372**, 259 (2021).
- [10] Y. Jing and M. Razavi, Quantum repeaters with encoding on nitrogen-vacancy-center platforms, *Phys. Rev. Appl.* **18**, 024041 (2022).
 - [11] P. van Loock, W. Alt, C. Becher, O. Benson, H. Boche, C. Deppe, J. Eschner, S. Höfling, D. Meschede, P. Michler, F. Schmidt, and H. Weinfurter, Extending quantum links: Modules for fiber- and memory-based quantum repeaters, *Advanced Quantum Technologies* **3**, 1900141 (2020).
 - [12] R. Laurenza, N. Walk, J. Eisert, and S. Pirandola, Rate limits in quantum networks with lossy repeaters, *Phys. Rev. Res.* **4**, 023158 (2022).
 - [13] L. Kamin, E. Shchukin, F. Schmidt, and P. van Loock, Exact rate analysis for quantum repeaters with imperfect memories and entanglement swapping as soon as possible, *Phys. Rev. Res.* **5**, 023086 (2023).
 - [14] M. Razavi, M. Piani, and N. Lütkenhaus, Quantum repeaters with imperfect memories: Cost and scalability, *Phys. Rev. A* **80**, 032301 (2009).
 - [15] S. Santra, L. Jiang, and V. S. Malinovsky, Quantum repeater architecture with hierarchically optimized memory buffer times, *Quantum Science and Technology* **4**, 025010 (2019).
 - [16] S. D. Reiß and P. van Loock, Deep reinforcement learning for key distribution based on quantum repeaters, *Phys. Rev. A* **108**, 012406 (2023).
 - [17] A. Romanova and P. van Loock, Memory-corrected quantum repeaters with adaptive syndrome identification, *Phys. Rev. Res.* **7**, 013117 (2025).
 - [18] S. Haldar, P. J. Barge, S. Khatri, and H. Lee, Fast and reliable entanglement distribution with quantum repeaters: Principles for improving protocols using reinforcement learning, *Phys. Rev. Appl.* **21**, 024041 (2024).
 - [19] K. Goodenough, T. Coopmans, and D. Towsley, On noise in swap asap repeater chains: exact analytics, distributions and tight approximations (2024), arXiv:2404.07146 [quant-ph].
 - [20] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Quantum repeaters based on atomic ensembles and linear optics, *Rev. Mod. Phys.* **83**, 33 (2011).
 - [21] M. H. Michael, M. Silveri, R. T. Brierley, V. V. Albert, J. Salmilehto, L. Jiang, and S. M. Girvin, New class of quantum error-correcting codes for a bosonic mode, *Phys. Rev. X* **6**, 031006 (2016).
 - [22] Z. Leghtas, G. Kirchmair, B. Vlastakis, R. J. Schoelkopf, M. H. Devoret, and M. Mirrahimi, Hardware-efficient autonomous quantum memory protection, *Phys. Rev. Lett.* **111**, 120501 (2013).
 - [23] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, Dynamically protected cat-qubits: a new paradigm for universal quantum computation, *New Journal of Physics* **16**, 045014 (2014).
 - [24] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, *Phys. Rev. A* **64**, 012310 (2001).
 - [25] S. Häussler and P. van Loock, Quantum repeaters based on stationary gottesman-kitaev-preskill qubits (2024), arXiv:2406.07158 [quant-ph].
 - [26] Y.-H. Chen, Z.-C. Shi, F. Nori, and Y. Xia, Error-tolerant amplification and simulation of the ultrastrong-coupling quantum rabi model, *Phys. Rev. Lett.* **133**, 033603 (2024).
 - [27] E. Shchukin, F. Schmidt, and P. van Loock, Waiting time in quantum repeaters with probabilistic entanglement swapping, *Phys. Rev. A* **100**, 032322 (2019).
 - [28] P. Li, J. Dias, W. J. Munro, P. van Loock, K. Nemoto, and N. Lo Piparo, Performance of rotation-symmetric bosonic codes in a quantum repeater network, *Advanced Quantum Technologies* **7**, 10.1002/qute.202300252 (2024).
 - [29] X. Gu, A. F. Kockum, A. Miranowicz, Y. xi Liu, and F. Nori, Microwave photonics with superconducting quantum circuits, *Physics Reports* **718-719**, 1 (2017), microwave photonics with superconducting quantum circuits.
 - [30] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, Circuit quantum electrodynamics, *Rev. Mod. Phys.* **93**, 025005 (2021).
 - [31] S. Kuhr, S. Gleyzes, C. Guerlin, J. Bernu, U. B. Hoff, S. Deléglise, S. Osnaghi, M. Brune, J.-M. Raimond, S. Haroche, E. Jacques, P. Bosland, and B. Visentin, Ultra-high finesse Fabry-Pérot superconducting resonator, *Applied Physics Letters* **90**, 164101 (2007).
 - [32] A. Eickbusch, V. Sivak, A. Z. Ding, S. S. Elder, S. R. Jha, J. Venkatraman, B. Royer, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Fast universal control of an oscillator with weak dispersive coupling to a qubit, *Nature Physics* **18**, 1464 (2022).
 - [33] N. Lauk, N. Sinclair, S. Barzanjeh, J. P. Covey, M. Saffman, M. Spiropulu, and C. Simon, Perspectives on quantum transduction, *Quantum Science and Technology* **5**, 020501 (2020).
 - [34] R. W. Andrews, R. W. Peterson, T. P. Purdy, K. Cicak, R. W. Simmonds, C. A. Regal, and K. W. Lehnert, Bidirectional and efficient conversion between microwave and optical light, *Nature Physics* **10**, 321 (2014).
 - [35] R. Sahu, L. Qiu, W. Hease, G. Arnold, Y. Minoguchi, P. Rabl, and J. M. Fink, Entangling microwaves with light, *Science* **380**, 718 (2023).
 - [36] P. Li and P. van Loock, Memoryless quantum repeaters based on cavity-qed and coherent states, *Advanced Quantum Technologies* **6**, 10.1002/qute.202200151 (2023).
 - [37] P. Laha and P. van Loock, Arbitrary high-fidelity binomial codes from multiphoton spin-boson interactions (2025), arXiv:2507.08585 [quant-ph].
 - [38] L. Hu, Y. Ma, W. Cai, X. Mu, Y. Xu, W. Wang, Y. Wu, H. Wang, Y. P. Song, C.-L. Zou, S. M. Girvin, L.-M. Duan, and L. Sun, Quantum error correction and universal gate set operation on a binomial bosonic logical qubit, *Nature Physics* **15**, 503 (2019).
 - [39] R. W. Heeres, P. Reinhold, N. Ofek, L. Frunzio, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, Implementing a universal gate set on a logical qubit encoded in an oscillator, *Nature Communications* **8**, 94 (2017).
 - [40] V. V. Sivak, A. Eickbusch, B. Royer, S. Singh, I. Tsioutsios, S. Ganjam, A. Miano, B. L. Brock, A. Z. Ding, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Real-time quantum error correction beyond break-even, *Nature* **616**, 50 (2023).
 - [41] J.-W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, Experimental entanglement swapping: Entangling photons that never interacted, *Phys. Rev. Lett.* **80**, 3891 (1998).
 - [42] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, “event-ready-detectors” bell experiment via entanglement swapping, *Phys. Rev. Lett.* **71**, 4287 (1993).

- [43] Y. Wu, J. Liu, and C. Simon, Near-term performance of quantum repeaters with imperfect ensemble-based quantum memories, *Phys. Rev. A* **101**, 042301 (2020).
- [44] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2011).
- [45] H. P. Breuer and F. Petruccione, *The theory of open quantum systems* (Oxford University Press, 2002).
- [46] S. Sharma, V. S. V. Bittencourt, and S. V. Kusminskiy, Protocol for generating an arbitrary quantum state of the magnetization in cavity magnonics, *Journal of Physics: Materials* **5**, 034006 (2022).
- [47] L. Sun, A. Petrenko, Z. Leghtas, B. Vlastakis, G. Kirchmair, K. M. Sliwa, A. Narla, M. Hatridge, S. Shankar, J. Blumoff, L. Frunzio, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Tracking photon jumps with repeated quantum non-demolition parity measurements, *Nature* **511**, 444 (2014).
- [48] S. Rosenblum, Y. Y. Gao, P. Reinhold, C. Wang, C. J. Axline, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, A cnot gate between multiphoton qubits encoded in two cavities, *Nature Communications* **9**, 652 (2018).
- [49] F. Okamoto, M. Endo, M. Matsuyama, Y. Ishizuka, Y. Liu, R. Sakakibara, Y. Hashimoto, J.-i. Yoshikawa, P. van Loock, and A. Furusawa, Phase locking between two all-optical quantum memories, *Phys. Rev. Lett.* **125**, 260508 (2020).
- [50] Y. Hashimoto, T. Toyama, J.-i. Yoshikawa, K. Makino, F. Okamoto, R. Sakakibara, S. Takeda, P. van Loock, and A. Furusawa, All-optical storage of phase-sensitive quantum states of light, *Phys. Rev. Lett.* **123**, 113603 (2019).
- [51] J.-i. Yoshikawa, K. Makino, S. Kurata, P. van Loock, and A. Furusawa, Creation, storage, and on-demand release of optical quantum states with a negative wigner function, *Phys. Rev. X* **3**, 041028 (2013).
- [52] Z. Aqua, M. S. Kim, and B. Dayan, Generation of optical fock and w states with single-atom-based bright quantum scissors, *Photon. Res.* **7**, A45 (2019).
- [53] N. Rivera, J. Sloan, Y. Salamin, J. D. Joannopoulos, and M. Soljačić, Creating large fock states and massively squeezed states in optics using systems with nonlinear bound states in the continuum, *Proceedings of the National Academy of Sciences* **120**, e2219208120 (2023).
- [54] Z. Leghtas, G. Kirchmair, B. Vlastakis, M. H. Devoret, R. J. Schoelkopf, and M. Mirrahimi, Deterministic protocol for mapping a qubit to coherent state superpositions in a cavity, *Phys. Rev. A* **87**, 042315 (2013).
- [55] K. Fukui, R. N. Alexander, and P. van Loock, All-optical long-distance quantum communication with gottesman-kitaev-preskill qubits, *Phys. Rev. Res.* **3**, 033118 (2021).
- [56] F. Rozpędek, K. Noh, Q. Xu, S. Guha, and L. Jiang, Quantum repeaters based on concatenated bosonic and discrete-variable quantum codes, *npj Quantum Information* **7**, 102 (2021).
- [57] F. Rozpędek, K. P. Seshadreesan, P. Polakos, L. Jiang, and S. Guha, All-photonic gottesman-kitaev-preskill-qubit repeater using analog-information-assisted multiplexed entanglement ranking, *Phys. Rev. Res.* **5**, 043056 (2023).
- [58] F. Schmidt, D. Miller, and P. van Loock, Error-corrected quantum repeaters with gottesman-kitaev-preskill qubits, *Phys. Rev. A* **109**, 042427 (2024).
- [59] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Quantum repeaters based on atomic ensembles and linear optics, *Rev. Mod. Phys.* **83**, 33 (2011).
- [60] M. Reitz, C. Sommer, and C. Genes, Cooperative quantum phenomena in light-matter platforms, *PRX Quantum* **3**, 010201 (2022).
- [61] F. Wulschner, J. Goetz, F. R. Koessel, E. Hoffmann, A. Baust, P. Eder, M. Fischer, M. Haeberlein, M. J. Schwarz, M. Pernpeintner, E. Xie, L. Zhong, C. W. Zolnitsch, B. Peropadre, J.-J. Garcia Ripoll, E. Solano, K. G. Fedorov, E. P. Menzel, F. Deppe, A. Marx, and R. Gross, Tunable coupling of transmission-line microwave resonators mediated by an rf squid, *EPJ Quantum Technology* **3**, 10 (2016).
- [62] S. Pirandola, R. Laurenza, C. Ottaviani, and L. Banchi, Fundamental limits of repeaterless quantum communications, *Nature Communications* **8**, 10.1038/ncomms15043 (2017).
- [63] Ángel Rivas and S. F. Huelga, *Open Quantum Systems: An Introduction* (Springer Berlin, Heidelberg, 2012).