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Abstract

This thesis deals with the Nine Dragon Tree conjectures, which concern partitioning the edge set of a graph into as few forests as possible while ensuring that one of the forests is as “sparse” as possible. The well-known Nash-Williams Theorem states that the edge set of a loopless graph G can be partitioned into k forests if $\gamma(G) := \max_{H \subseteq G,\, v(H)>1} \frac{e(H)}{v(H)-1} \leq k$. Thus, $\lceil \gamma(G) \rceil$ is the minimum number of forests into which $E(G)$ can be partitioned. The Nine Dragon Tree Theorem generalizes this relationship: if $k, d \in \mathbb{N}_0$ and $\gamma(G) \leq k + \frac{d}{d+k+1}$, then G admits a decomposition into $k+1$ forests, where one of the forests F has maximum degree at most d. The Strong Nine Dragon Tree Conjecture even states that a decomposition can be found in which each tree of F contains at most d edges. While the Nine Dragon Tree Theorem has already been proven, its strong variant remains open in most cases. In this thesis, we prove it for $d \leq 2(k+1)$ and show an approximation for the remaining cases in which the sizes of the connected components are bounded by $\mathcal{O}(d^2/k)$. We also show that, as conjectured by Kim, Kostochka, West, Wu, and Zhu, there exists a weaker density condition that still implies the stated decompositions. Analogous conjectures can be formulated for pseudoforests, for which even the strong variant has already been proven. In this thesis, we show that an even stronger decomposition is possible in which one of the pseudoforests F is actually a forest, and its diameter can be further restricted. We prove that the diameter bounds we establish are best possible, in the case that F is also required to have its degree bounded above by any constant.

We further show that, as in the case of the Nine Dragon Tree conjectures for forests, a weaker density condition exists for the pseudoforest variants that also allows us to achieve the additional diameter bounds.

Abstract (german)

Diese Arbeit beschäftigt sich mit den Nine-Dragon-Tree-Vermutungen, bei denen es darum geht, die Kantenmenge eines Graphen in so wenige Wälder wie möglich zu partitionieren und zusätzlich einen der Wälder so “dünn” wie möglich zu machen. Der bekannte Nash-Williams-Satz besagt, dass die Kantenmenge eines schleifenfreien Graphen G in k Wälder zerlegt werden kann, wenn $\gamma(G) := \max_{H \subseteq G, v(H)>1} e(H)/(v(H)-1) \leq k$ gilt. $\lceil \gamma(G) \rceil$ ist somit die minimale Anzahl an Wäldern, in die $E(G)$ partitioniert werden kann. Der Nine-Dragon-Tree-Satz verallgemeinert diesen Zusammenhang: Wenn $k, d \in \mathbb{N}_0$ und $\gamma(G) \leq k + \frac{d}{d+k+1}$, dann besitzt G eine Zerlegung in $k+1$ Wälder und in einem der Wälder ist der Knotengrad höchstens d. Die starke Nine-Dragon-Tree-Vermutung besagt sogar, dass eine Zerlegung existiert, in der jeder Baum in einem der Wälder höchstens d Kanten besitzt. Während der Nine-Dragon-Tree-Satz bereits bewiesen ist, ist die starke Variante für die meisten Fälle noch unbewiesen. In dieser Arbeit wird sie für $d \leq 2(k+1)$ bewiesen und für die restlichen Fälle eine Approximation gezeigt, in der die Größe der Zusammenhangskomponenten auf $\mathcal{O}(d^2/k)$ beschränkt wird. Es wird außerdem gezeigt, dass wie von Kim, Kostochka, West, Wu und Zhu vermutet, eine schwächere Dichtebedingung existiert, die ebenfalls die genannten Zerlegungen impliziert.

Analog lassen sich Pseudowald-Vermutungen aufstellen, von denen sogar die starke Variante schon bewiesen wurde. In dieser Arbeit zeigen wir, dass eine stärkere Zerlegung möglich ist, in der der Pseudowald F mit kleinen Zusammenhangskomponenten ein Wald ist und der Durchmesser noch weiter beschränkt werden kann. Wir zeigen, dass diese Durchmesserschranken bestmöglich sind, sofern der Knotengrad in F durch eine beliebige Konstante beschränkt sein soll.

Wir zeigen außerdem, dass wie bei den Nine-Dragon-Tree-Sätzen für Wälder eine schwächere Dichtebedingung für die Pseudowaldvarianten existiert, mit der auch die zusätzlichen Durchmesserschranken erreicht werden können.

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Contents

Introduction

1

Decompositions of graphs are a heavily studied area, since if one can decompose a graph into only a few simple pieces, then one can in some sense deduce that the entire graph structure is simple. A decomposition can also be helpful in an algorithmic context. If a problem is (efficiently) solvable on every simple part of the decomposition, the problem, or at least an approximation of it, may also be (efficiently) solvable for the entire graph.

In this thesis, we are interested in decompositions that partition the edge set of a graph. More formally, $G_1, \dots, G_k$ is a *decomposition of a graph* G if for every $i \in \{1, \dots, k\}$, G_i is a graph with $V(G_i) = V(G)$, and $E(G_1), \dots, E(G_k)$ is a partition of $E(G)$. Unless stated otherwise, graphs in this thesis are undirected and may contain parallel edges and loops. Furthermore, we use the notation $v(G) := |V(G)|$ and $e(G) := |E(G)|$.

One of the simplest classes of graphs is the class of forests, which are just graphs without cycles. Thus, a natural question is: when can a graph be decomposed into k forests? This question was answered completely by a beautiful theorem of Nash-Williams from 1964.

Theorem 1.1 (Nash-Williams' Theorem [45]). *A graph G without loops decomposes into k forests if and only if for every subgraph $H \subseteq G$ with $v(H) > 1$, we have $\frac{e(H)}{v(H)-1} \leq k$.*

The term

$$\gamma(G) := \max_{\substack{H \subseteq G, \\ v(H) > 1}} \frac{e(H)}{v(H) - 1}$$

is called *the fractional arboricity of* G, and $\lceil \gamma(G) \rceil$ is the *arboricity of* G. By Nash-Williams' Theorem, this is the minimum number of forests into which G can be decomposed.

Note that if G has a decomposition into k forests, then it is clear that $e(H) \leq k(v(H) - 1)$ for every $H \subseteq G$, since $v(H) - 1$ is the maximum number of edges that can be contained in a forest with vertex set $V(H)$. However, the other direction of the proof of Nash-Williams' Theorem is non-trivial. We will provide a proof in Section 3.1. We note that for any $k \in \mathbb{N}_0$, a decomposition into k forests of a graph G, or a subgraph $H \subseteq G$ with $\frac{e(H)}{v(H)-1} > k$, can be computed in polynomial time; see [12], [18], [17].

Interestingly, if one asks for a decomposition of a graph into subgraphs that are all isomorphic to one specific graph, the problem becomes NP-complete very quickly — even if the specific graph is a forest.

Theorem 1.2 ([11]). *Deciding whether a graph has a decomposition into subgraphs isomorphic to a graph H is NP-complete if H contains a component with at least three edges.*

It seems that Nash-Williams' Theorem leaves room for improvement. Suppose we have two loopless graphs G_1, G_2 with $\gamma(G_1) = k + 0.001$ and $\gamma(G_2) = k + 0.9$. Then Nash-Williams' Theorem only guarantees that both graphs have a decomposition into $k+1$ forests. It would be desirable to have a theorem that takes the fractional part of the fractional arboricity into account, and thus yields a "better" decomposition for G_1 than for G_2. For a loopless graph G with $\gamma(G) = k + \epsilon$, where $0 < \epsilon < 1$ is small, we may be able to move as many edges as possible into the k forests such that the $(k+1)$-th forest F has sparse structure. Given that k forests can contain at most $k(v(G) - 1)$ edges, we cannot in general expect F to have fewer than $\epsilon(v(G) - 1)$ edges. However, the following theorem shows that the bound of $\epsilon(v(G) - 1)$ is achievable and is, of course, optimal.

Theorem 1.3 ([14]). *Let G be a loopless graph with $\gamma(G) \leq k + \epsilon$, where $0 \leq \epsilon < 1$. Then G decomposes into k forests and a forest with at most $\epsilon(v(G) - 1)$ edges, and the bound of $\epsilon(v(G) - 1)$ is best possible.*

Note that Theorem 1.3 can be generalized to matroids.

The Nine Dragon Tree Conjectures, which are the focus of this thesis, also aim to make the $(k+1)$-th forest F as sparse as possible, but they do so by enforcing structure within each component of F. for this purpose, the following three sparseness properties — each increasingly stronger — have been proposed.

Let $k, d \in \mathbb{N}_0$.

We call a graph $(k, d)^-$*-decomposable* if it has a decomposition into $k+1$ graphs, where k of them are forests and in the remaining graph, every vertex has degree at most d.

We call a graph (k, d)*-decomposable* if it has a decomposition into $k+1$ forests,

where in one of the forests, every vertex has degree at most d.
We call a graph $(k, d)^+$-*decomposable* if it has a decomposition into $k + 1$ forests, where in one of the forests, every component has at most d edges.
Montassier, Ossana de Mendez, Raspaud, and Zhu [41] proposed the following corresponding conjectures for these properties.

Theorem 1.4 (Weak Nine Dragon Tree Theorem, [27]). *Let $k, d \in \mathbb{N}_0$, and let G be a loopless graph with $\gamma(G) \leq k + \frac{d}{d+k+1}$. Then G is $(k, d)^-$-decomposable.*

Theorem 1.5 (Nine Dragon Tree Theorem, [27]). *Let $k, d \in \mathbb{N}_0$, and let G be a loopless graph with $\gamma(G) \leq k + \frac{d}{d+k+1}$. Then G is (k, d)-decomposable.*

Conjecture 1.6 (Strong Nine Dragon Tree Conjecture, [41]). *Let $k, d \in \mathbb{N}_0$, and let G be a loopless graph with $\gamma(G) \leq k + \frac{d}{d+k+1}$. Then G is $(k, d)^+$-decomposable.*

The interesting name of the conjectures originated as follows: *"On the top of the Longevity Mountain in Kaohsiung, Taiwan, there is a giant banyan tree, called the 'Nine Dragon Tree'. It resembles a number of dragons resting on the top of the mountain. Many graph theorists visited this tree and are amazed at the fact that this tree is far from being acyclic. We would like to associate the conjecture below with this beautiful tree, whose fractional arboricity remains a mystery."* [41]

Note that Conjecture 1.6 implies Theorem 1.5, which in turn implies Theorem 1.4. In [41], it was shown that the Weak Nine Dragon Tree Conjecture is already best possible.

Theorem 1.7 ([41]). *Let $k \in \mathbb{N}_{\geq 1}$, $d \in \mathbb{N}_0$. For every $\epsilon > 0$, there exist simple graphs G such that $\gamma(G) \leq k + \frac{d}{d+k+1} + \epsilon$, and yet G is not $(k, d)^-$-decomposable.*

Note that Theorem 1.7 also shows the optimality of Theorem 1.5 and Conjecture 1.6. We provide a brief history of the (partial) resolutions of Theorem 1.4, Theorem 1.5, and Conjecture 1.6. When the conjectures were proposed in 2011 in [41], the authors proved the Nine Dragon Theorem for the cases $(k, d) \in \{(1, 1), (1, 2)\}$.
Slightly before that, an approximation of the Weak Nine Dragon Tree Theorem with a degree bound of $d + 1$ was proven by Király and Lau [30]. Another result towards the $(k, d) = (1, 1)$ case of the Nine Dragon Tree Theorem was provided by Kaiser, Montassier, and Raspaud [28]. Kim, Kostochka, West, Wu and Zhu [29] proved the Weak Nine Dragon Tree Theorem for $d > k$, and the Nine Dragon Tree Theorem for $d = k + 1$, and the case where $k = 1$, $d \leq 6$. For the case $(k, d) = (1, 2)$, they also proved the Strong Nine Dragon Tree Conjecture.

Subsequently, Chen, Kim, Kostochka, West and Zhu [9] proved the Nine Dragon Tree Theorem for $k \leq 2$, except for the case where $(k, d) = (2, 1)$.
In [53], Yang proved the Strong Nine Dragon Tree Conjecture for $d = 1$. Note that in this case, the Nine Dragon Tree Theorem and the Strong Nine Dragon Tree Conjecture are equivalent.
Finally, Jiang and Yang [27] gave a complete proof of the Nine Dragon Tree Theorem. This also proved the Weak Nine Dragon Tree Theorem. The proof was submitted in 2015.
After that, for several years, the Strong Nine Dragon Tree Conjecture was only proven for the cases $d = 1$ and $(k, d) = (1, 2)$.
Blumenstock and Fischer [7] implicitly proved the Strong Nine Dragon Tree Conjecture when $k = d$ for simple graphs G in which the pseudoarboricity (we will define this term later in the introduction) is $\lceil \gamma(G) \rceil + 1$ (note that the pseudoarboricity of a simple graph G is either $\lceil \gamma(G) \rceil$ or $\lceil \gamma(G) \rceil + 1$, see [47]).
Moore [42] proved an approximation of the Strong Nine Dragon Tree Conjecture in the case $(k, d) = (1, 3)$.
In [38], the author and Moore proved the Strong Nine Dragon Tree Conjecture for $d \leq k + 1$ and provided an approximate version for $d \leq 2(k + 1)$. This result was part of the author's Master's thesis [35].
In this thesis, we prove the Strong Nine Dragon Tree Conjecture for $d \leq 2(k + 1)$.

Theorem 1.8. *The Strong Nine Dragon Tree Conjecture is true when* $d \leq 2(k + 1)$.

This result implies that every planar graph with girth at least 5 is $(1, 4)^+$-decomposable, see Theorem 5.2.3. In Section 5.4, we discuss an application of this result to so-called ϵ-thin spanning trees, showing that every planar graph with girth at least 5 contains a $\frac{4}{5}$-thin spanning tree, see Theorem 5.4.3. This result is best possible in the sense that there exist planar graphs with girth 4 that do not contain an ϵ-thin spanning tree for any fixed $0 \leq \epsilon < 1$.
To date, these are the only cases of the Strong Nine Dragon Tree Conjecture that have been resolved.
In this thesis, we also prove an approximate version of the Strong Nine Dragon Tree Conjecture for all $k, d \in \mathbb{N}_0$.

Theorem 1.9. *Let* $k, d \in \mathbb{N}_0$, *and let* G *be a loopless graph with* $\gamma(G) \leq k + \frac{d}{d+k+1}$. *Then* G *is* $(k,\ d + \lceil k \lfloor \frac{d-1}{k+1} \rfloor (\frac{d}{k+1} - \frac{1}{2} \lceil \frac{d}{k+1} \rceil) \rceil)^+$*-decomposable.*

We note that we have heard that Yang and Zhang have submitted a similar approximation result in 2024 around the time Moore and the author did. At the time of submission of this thesis, this result is not yet published.

In [29], a modification of the Nine Dragon Conjectures with a slightly stronger weaker criterion was proposed. To understand this criterion, we need to provide some context.
Note that a loopless graph G has $\gamma(G) \leq k+\frac{d}{d+k+1}$ if and only if for every non-empty subgraph $H \subseteq G$, we have

$$v(H)(k+1)(d+k) - e(H)(d+k+1) - k^2 - kd - k - d \geq 0.$$

One could ask how much can be added to the left-hand side such that G is still $(k,d)^-$-, (k,d)- or $(k,d)^+$-decomposable. As the term $kd+k+d$ appeared promising, they defined the following notion in [29].
G is (k,d)-*sparse* if G is loopless and for every non-empty subgraph $H \subseteq G$, we have

$$\beta(H) := v(H)(k+1)(d+k) - e(H)(d+k+1) - k^2 \geq 0.$$

In [29], the following analogue of the Weak Nine Dragon Tree Theorem was suggested.

Theorem 1.10. *(k,d)-sparse graphs are $(k,d)^-$-decomposable for any $k,d \in \mathbb{N}_0$.*

Note that not every (k,d)-sparse graph has a decomposition into $k+1$ forests: for $\varnothing \neq H \subseteq G$, the condition $\beta(H) \geq 0$ is equivalent to

$$\frac{e(H)}{v(H)-1} \leq k + \frac{1}{d+k+1}\left(d + \frac{kd+k+d}{v(G)-1}\right).$$

Thus, there may be a subgraph H with $v(H) \leq d+1$ whose density causes G to have $\gamma(G) > k+1$. For example, it was noted in [9] that the graph consisting of two vertices and $k+2$ edges connecting them does not have a decomposition into $k+1$ forests, but is (k,d)-sparse if $k(d-1) \geq 2$.
Hence, if we want to relate (k,d)-sparseness to (k,d)- or $(k,d)^+$-decomposability, then we must additionally require that the graph has a decomposition into $k+1$ forests. In [29], the following two statements were conjectured.

Theorem 1.11. *Let $k,d \in \mathbb{N}_0$, and let G be a (k,d)-sparse graph with $\gamma(G) \leq k+1$. Then G is (k,d)-decomposable.*

Conjecture 1.12 ([29]). *Let $k, d \in \mathbb{N}_0$, and let G be a (k,d)-sparse graph with $\gamma(G) \leq k+1$. Then G is $(k,d)^+$-decomposable.*

Of course Theorem 1.10 implies the Weak Nine Dragon Tree Theorem, Theorem 1.11 implies the Nine Dragon Tree Theorem, and Conjecture 1.12 implies the Strong Nine Dragon Tree Conjecture.

The results in [29] and [9] towards the (Weak/Strong) Nine Dragon Tree Conjectures mentioned earlier were achieved with the relaxed density criterion of (k,d)-sparseness. Consequently, Theorem 1.10 is proven for $d > k$, Theorem 1.11 is proven for $d = k+1$, the case $k = 1, d \leq 6$, and $k \leq 2$, except for the case $(k,d) = (2,1)$, and Conjecture 1.12 is proven when $(k,d) \in \{(1,1),(1,2)\}$.

In this thesis, we prove Theorem 1.10 and Theorem 1.11 by proving the following, slightly stronger theorem.

Theorem 1.13. *Let $k, d \in \mathbb{N}_0$, and let G be a (k,d)-sparse graph. Then G has a decomposition into k forests and a graph F, where in F, every vertex has degree at most d and in every component of F, the number of edges on cycles is at most d. Furthermore, if $\gamma(G) \leq k+1$, then G is (k,d)-decomposable.*

Moreover, our results towards the Strong Nine Dragon Tree Conjecture (Theorem 1.8 and Theorem 1.9) also cover (k,d)-sparse graphs with fractional arboricity at most $k+1$.

Theorem 1.14. *Conjecture 1.12 is true when $d \leq 2(k+1)$.*

Theorem 1.15. *Let $k, d \in \mathbb{N}_0$ and let G be a (k,d)-sparse graph with $\gamma(G) \leq k+1$. Then G is $(k,\ d + \lceil k \lfloor \frac{d-1}{k+1} \rfloor (\frac{d}{k+1} - \frac{1}{2} \lceil \frac{d}{k+1} \rceil) \rceil)^+$-decomposable.*

Note that Theorem 1.10, and thus Theorem 1.11 and Conjecture 1.12 are best possible in the following sense.

Theorem 1.16 ([29]). *Let $k \in \mathbb{N}_{\geq 1}, d \in \mathbb{N}_0$. There are simple graphs G such that $\gamma(G) \leq k+1$, $\beta(G) = -1$, $\beta(H) \geq 0$ for all $\varnothing \neq H \subsetneq G$, but G is not $(k,d)^-$-decomposable.*

Now, we turn to the pseudoforest analogues of the Nine Dragon Tree Conjectures. A *pseudoforest* is a graph in which every component contains at most one cycle. We call such a component a *pseudotree*. It is very interesting that everything we have discussed so far also works with pseudoforests. Note that in what follows, we allow graphs to contain loops. We begin with an analogue of Nash-Williams' Theorem.

Theorem 1.17 ([25], Hakimi's Theorem). *A graph G decomposes into k pseudoforests if and only if for every $\varnothing \neq H \subseteq G$, we have $\frac{e(H)}{v(H)} \leq k$.*

We call the term

$$\gamma'(G) := \max_{\varnothing \neq H \subseteq G} \frac{e(H)}{v(H)}$$

the *fractional pseudoarboricity of G*, and $\lceil \gamma'(G) \rceil$ is the *pseudoarboricity of G*. By Hakimi's Theorem, this is the minimum number of pseudoforests into which G can be decomposed. We will provide a proof of Hakimi's Theorem in Section 4.1.
We note that in the literature, the term *maximum average degree* $\mathrm{mad}(G) := 2 \cdot \gamma'(G)$ *of G* is often used instead of fractional pseudoarboricity.
We call a graph $(k,d)^-$*-pseudo-decomposable* if it has a decomposition into $k+1$ graphs, where k of them are pseudoforests, and in the remaining graph, every vertex has degree at most d.
We call a graph (k,d)*-pseudo-decomposable* if it has a decomposition into $k+1$ pseudoforests, where in one of the pseudoforests, every vertex has degree at most d.
We call a graph $(k,d)^+$*-pseudo-decomposable* if it has a decomposition into $k+1$ pseudoforests, where in one of the pseudoforests, every component has at most d edges.
We can now state the pseudoforest analogues of the Nine Dragon Tree Conjectures.

Theorem 1.18 (Weak Pseudoforest Nine Dragon Tree Theorem, [15]). *Let $k, d \in \mathbb{N}_0$, and let G be a graph with $\gamma'(G) \leq k + \frac{d}{d+k+1}$. Then G is $(k,d)^-$-pseudo-decomposable.*

Theorem 1.19 (Pseudoforest Nine Dragon Tree Theorem, [15]). *Let $k, d \in \mathbb{N}_0$, and let G be a graph with $\gamma'(G) \leq k + \frac{d}{d+k+1}$. Then G is (k,d)-pseudo-decomposable.*

Theorem 1.20 (Strong Pseudoforest Nine Dragon Tree Theorem, [23]). *Let $k, d \in \mathbb{N}_0$, and let G be a graph with $\gamma'(G) \leq k + \frac{d}{d+k+1}$. Then G is $(k,d)^+$-pseudo-decomposable.*

Obviously, Theorem 1.20 implies Theorem 1.19, which in turn implies Theorem 1.18.
Note that, as in the forest case, Theorem 1.18 is already best possible in the following sense.

Theorem 1.21 (cf. [15]). *For any $k \in \mathbb{N}_{\geq 1}$, $d \in \mathbb{N}_0$, and $\epsilon > 0$, there exists a simple graph G with $\gamma'(G) \leq k + \frac{d}{d+k+1} + \epsilon$, but G is not $(k, d)^-$-pseudo-decomposable.*

Not long after the Nine Dragon Tree Conjectures (for forests) were proposed, Fan, Li, Song, and Yang proved the Pseudoforest Nine Dragon Tree Theorem (and thus the Weak Pseudoforest Nine Dragon Tree Theorem) in [15].
Several years later, Grout and Moore [23] also proved the Strong Pseudoforest Nine Dragon Tree Theorem.
Thus, it appears that the pseudoforest case is much simpler than the forest case.
In fact, in this thesis, we strengthen the Strong Pseudoforest Nine Dragon Tree Theorem even further by adding diameter constraints.

Theorem 1.22. *Let $k, d \in \mathbb{N}_0$, $\ell := \left\lfloor \frac{d-1}{k+1} \right\rfloor$, and let G be a graph with $\gamma'(G) \leq k + \frac{d}{d+k+1}$. Then G decomposes into $k+1$ pseudoforests, where one of the pseudoforests F satisfies the following:*

- *F is acyclic,*
- *every component K of F has $e(K) \leq d$,*
- $\operatorname{diam}(K) \leq 2\ell + 2$, *and if* $d \equiv 1 \mod (k+1)$, *then* $\operatorname{diam}(K) \leq 2\ell + 1$,
- *for every component K of F with $e(K) \geq d - z(k-1) + 1$, we have* $\operatorname{diam}(K) \leq 2z$ *for any $z \in \mathbb{N}_0$.*

Note that Theorem 1.22 implies Theorem 1.20.
The following corollary clarifies which upper bound on the fractional pseudoarboricity is necessary in Theorem 1.22 if we are interested in a specific overall diameter bound.

Corollary 1.23. *Let $k, q \in \mathbb{N}_0$, let $\alpha \in \{0, 1\}$, and let G be a graph with $\gamma'(G) \leq k + \frac{q + \frac{\alpha}{k+1}}{q + 1 + \frac{\alpha}{k+1}}$. Then G decomposes into $k+1$ pseudoforests, where one of the pseudoforests is a forest of diameter at most $2q + \alpha$.*

Proof. The corollary follows from Theorem 1.22 by letting $d := q(k+1) + \alpha$. □

The following immediate corollary shows in which cases we can obtain a star forest.

Corollary 1.24. *Let $k \in \mathbb{N}_0$, and let G be a graph with $\gamma'(G) \leq k + \frac{1}{2}$. Then G decomposes into k pseudoforests and a star forest, where every star has at most $k+1$ edges.*

This corollary can be applied to graphs with odd maximum degree.
In general, a graph G with maximum degree at most $\Delta \in \mathbb{N}_0$ — for example the complete graph on $\Delta + 1$ vertices — satisfies $\frac{e(H)}{v(H)} \leq \frac{\Delta}{2}$ for every non-empty subgraph $H \subseteq G$, and thus G has a decomposition into $\lceil \frac{\Delta}{2} \rceil$ pseudoforests by Hakimi's Theorem. If Δ is odd, it is a bit unsatisfactory that Hakimi's Theorem does not provide a "better" decomposition than it does for $\Delta+1$. However, Corollary 1.24 yields an improvement:

Corollary 1.25. *Let $\Delta \in \mathbb{N}_{\geq 1}$ be odd, and let G be a graph in which every vertex has degree at most Δ. Then G has a decomposition into $\frac{\Delta-1}{2}$ pseudoforests and a star forest, where every star has at most $\frac{\Delta+1}{2}$ edges.*

In Section 5.1, we show in Theorem 5.1.6 that this corollary is best possible with respect to the diameter: there exist simple graphs for which the last pseudoforest cannot be forced to have diameter 1. We also discuss in Section 5.1 which decompositions into forests can be obtained from graphs with bounded maximum degree.

Now we turn to the question of how good the bounds in Theorem 1.22 are. The constructions proving the optimality of the Weak (Pseudoforest) Nine Dragon Tree Theorem have a decomposition in which one of the (pseudo-)forests is a star forest in which every star has either d or 0 edges, except for one star that has $d+1$ edges; see Theorem 4.2.5 and Theorem 3.2.7. This suggests that a strengthening of the Strong (Pseudoforest) Nine Dragon Tree Theorem exists, where the trees with at most d edges can always be transformed into stars with at most d edges. However, in general, this is not the case.
The following theorem shows that the overall diameter bound of pseudoforest F in Theorem 1.22 cannot be improved if F is required to have any bound on the degree. Thus, Theorem 1.22 constitutes an optimal diameter extension of the (Weak/Strong) Pseudoforest Nine Dragon Tree Theorem.

Theorem 1.26. *Let $k \in \mathbb{N}_{\geq 1}$, $\ell, D \in \mathbb{N}_0$, $\epsilon > 0$, and $\alpha \in \{0, 1\}$. There are simple graphs G with $\gamma(G) < k + \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon$ that do not have a decomposition into k pseudoforests and another graph F such that F has maximum degree at most D and the diameter of every component in F is less than $2\ell + 1 + \alpha$. Furthermore, G is planar if $k = 1$.*

Note that Theorem 1.26 uses $\gamma(G)$ ($> \gamma'(G)$) as a density bound. Thus, it also provides a lower bound for a potential diameter refinement of the (Weak/Strong) Nine

Dragon Tree Theorem (/Conjecture) (for forests). The same holds for the following Theorem, which shows the optimality of the additional diameter bound for large components in Theorem 1.22 as a refinement of the (Weak/Strong) Pseudoforest Nine Dragon Tree Theorem.

Theorem 1.27. *Let $k, D, d, z \in \mathbb{N}_0$, $\epsilon > 0$, $k \geq 2$, $z \leq \left\lfloor \frac{d-1}{k+1} \right\rfloor$. There are simple graphs G with $\gamma(G) < k + \frac{d}{d+k+1} + \epsilon$, such that, in any decomposition of G into k pseudoforests and another graph F with maximum degree at most D, F contains a component K with $e(K) \geq d - z(k-1) + 1$ and $\operatorname{diam}(K) \geq 2(z+1)$. Furthermore, G is planar if $k = 1$.*

Now, we know that we cannot obtain stars with at most d edges (or other pseudotrees of diameter at most 2) in the last (pseudo-)forest. But what if we do not care about the number of edges of the components? Unfortunately, we cannot answer this question for pseudoforest decompositions. But the following theorem shows that, in general, such a decomposition is not possible in the case of forest decompositions.

Theorem 1.28. *Let $k, \ell \in \mathbb{N}_{\geq 1}$, and let $\ell \equiv 0 \mod 4$. There is a loopless (multi-) graph G with $\gamma(G) = k + \frac{\ell}{\ell+1}$ that does not have a decomposition into $k+1$ forests in which one of the forests has diameter at most $\frac{\ell}{4}$.*

Next, we discuss the pseudoforest analogue of (k, d)-sparseness. To the best of our knowledge, this has not been examined before.
Note that a graph G has $\gamma'(G) \leq k$ if and only if for every subgraph $H \subseteq G$, we have

$$v(H)(k+1)(d+k) - e(H)(d+k+1) \geq 0.$$

We will show that the largest term we can add to the left-hand side such that the (Weak/Strong) Pseudoforest Nine Dragon Tree Theorems still apply to G is k, and thus we define

$$\beta'(H) := v(H)(k+1)(d+k) - e(H)(d+k+1) + k$$

for any graph H. Furthermore, we call a graph G (k, d)-*pseudo-sparse* if for every subgraph $H \subseteq G$, we have $\beta'(H) \geq 0$. In contrast to the forest analogue, (k, d)-pseudo-sparse graphs always have a decomposition into $k+1$ pseudoforests since every subgraph H of a (k, d)-pseudo-sparse graph satisfies $\frac{e(H)}{v(H)} \leq k + \frac{d + \frac{k}{v(H)}}{d+k+1} \leq$

$k+1$. Hence, we state the pseudoforest analogues of Theorem 1.10, Theorem 1.11, and Conjecture 1.12 as follows.

Theorem 1.29. *(k,d)-pseudo-sparse graphs are $(k,d)^-$-pseudo-decomposable for any $k,d \in \mathbb{N}_0$.*

Theorem 1.30. *(k,d)-pseudo-sparse graphs are (k,d)-pseudo-decomposable for any $k,d \in \mathbb{N}_0$.*

Theorem 1.31. *(k,d)-pseudo-sparse graphs are $(k,d)^+$-pseudo-decomposable for any $k,d \in \mathbb{N}_0$.*

We will show that these three theorems are best possible in the following sense.

Theorem 1.32. *Let $k,d \in \mathbb{N}_0$. There are simple graphs G with $\beta'(G) = -1$ and $\beta'(H) \geq 0$ for all $H \subsetneq G$, but G is not $(k,d)^-$-pseudo-decomposable.*

Furthermore, Theorem 1.22 also applies to (k,d)-pseudo-sparse graphs.

Theorem 1.33. *Let $k,d \in \mathbb{N}_0$, $\ell := \left\lfloor \frac{d-1}{k+1} \right\rfloor$, and let G be (k,d)-pseudo-sparse. Then G decomposes into $k+1$ pseudoforests where one of the pseudoforests F satisfies the following:*

- *F is acyclic,*
- *every component K of F has $e(K) \leq d$,*
- $\operatorname{diam}(K) \leq 2\ell + 2$*, and if* $d \equiv 1 \mod (k+1)$*, then* $\operatorname{diam}(K) \leq 2\ell + 1$*,*
- *for every component K of F with $e(K) \geq d - z(k-1) + 1$, we have $\operatorname{diam}(K) \leq 2z$ for any $z \in \mathbb{N}_0$.*

Note that Theorem 1.33 implies Theorem 1.31, which in turn implies Theorem 1.30, which in turn implies Theorem 1.29.

In the rest of the introduction, we explain the proof framework used to establish the four main theorems of the thesis, Theorems 1.13, 1.14, 1.15, and 1.33. This framework was first used in the proof of the Pseudoforest Nine Dragon Tree Theorem in [15], then for the Nine Dragon Tree Theorem when $d = 1$ in [53], then for the complete resolution of the Nine Dragon Tree Theorem in [27], and later for the proof of the Strong Pseudoforest Nine Dragon Tree Theorem in [23]. Although we use the framework for forest decompositions, pseudoforest decompositions, and decompositions into k forests and a graph that may contain multiple cycles in a component, we will, for simplicity, refer in the following only to decompositions into k forests $T_1, \ldots, T_k$ and another forest F, where F is the forest for which we aim to obtain certain bounds. We will say that the edges of $T_1, \ldots, T_k$ are blue and the edges of F are red. Furthermore, for simplicity, we will first outline how the theorems can be proven when the graph has fractional (pseudo-)arboricity at most $k + \frac{d}{d+k+1}$. At the end, we will explain how the results can be extended to (k, d)-(pseudo-)sparse graphs.

The proof begins by considering a vertex-minimal counterexample graph G and showing that we may assume $T_1, \ldots, T_k$ to be spanning trees (or pseudoforests with $v(G)$ edges each). This is shown by proving that, for general graphs, edges between subgraphs in which k spanning trees can be packed can be coloured blue.

For each of the theorems we will define a residue function for F. The smaller its value, the closer F is to satisfying the respective theorem. An example of a suitable residue function value of F for the Strong Nine Dragon Tree Conjecture would be the sum of the number of edges of the components of F that have more than d edges. However, note that we will define other residue functions.

We choose a decomposition that minimizes the residue function value. The red forest of the decomposition will contain a component R^* that does not satisfy the theorem to prove since G is assumed to be a counterexample.

We will choose a vertex of R^* and call it r. We orient all the blue edges of the spanning trees towards r (for pseudoforest decompositions, we do not need to choose a vertex r and just transform the k pseudoforests into a k-orientation).

After that, we build a tree of red components which is rooted at R^*. We obtain this tree by an exploration of red components starting at R^*: a new red component C is added to the tree as a child of an already explored component K of the tree if there is a blue arc from K to C. If we cannot extend the tree of components further, we stop the exploration and call the induced subgraph the exploration subgraph H. It has the nice property that every vertex (except for r) has exactly k blue outgoing arcs. This will let us deduce that $\frac{e_r(H)}{v(H)-1} \leq \frac{d}{d+k+1}$, where $e_r(H)$ denotes the number of red edges of H.

Now, our goal is to "push" red edges away from R^*. If we manage to do this such that R^* has small child components, then we may find an edge exchange that reduces the size of R^*, and thereby reduces the residue function value of the red forest, which is a contradiction.

We also want to have a measure to track our progress in pushing away the red edges from R^*. This measurement is a tuple of the red components of the tree, ordered by the order in which they are explored by the greedy exploration strategy that chooses the child with the fewest (red) edges first. We call such an order of components a legal order. We compare legal orders of different decompositions using lexicographic ordering (for an ordering $(R_1, \ldots, R_t)$ we will use $(e(R_1), \ldots, e(R_t))$ for the comparison). An example of such a legal order can be seen in Figure 1.1, where $(k, d) = (1, 2)$ and the goal is to reduce the number of red edges in each component to at most two. We refine our choice of the decomposition of G by selecting one that minimizes the residue function, and as a secondary criterion, has a minimal legal order.

It will be useful to categorize red components into small and non-small components, where a small (red) component K has $\frac{e(K)}{v(K)} < \frac{d}{d+k+1}$. In a sense, these components are bad, since we aim to derive a contradiction by showing that $\frac{e_r(H)}{v(H)} \geq \frac{d}{d+k+1}$.

We can ensure that a small child component has a non-small parent: suppose two small components are connected by a blue arc. This arc can be coloured red, and in exchange, a red edge closer to the root component R^* can be coloured blue. An example can be seen in Figure 1.2. This will not create a new component violating the theorem because the components are small, but it will produce a smaller legal order; therefore, this case cannot happen in our optimal decomposition. This method is called the special path augmentation since the two edges being exchanged are connected by a blue directed path.

In order to show that $\frac{e_r(H)}{v(H)} \geq \frac{d}{d+k+1}$, we aim to partition the red subgraph of H into units such that every unit contains non-small and potentially small components while still being non-small on average. E.g., for Theorems 1.13, 1.15 and 1.33, a unit is formed by taking a non-small component together with all of its small children. If a non-small component has at most k small children (at most one "generated" by every spanning tree), then the density of the unit is at least $\frac{d}{d+k+1}$, making this bound very desirable.

The second kind of exchange comes into play when this bound cannot be achieved resulting in a non-small component K having two small children C_1 and C_2 that are generated by blue arcs $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ belonging to the same spanning tree. It is always possible to exchange one of e_1, e_2 with a red edge e on the red path in K from x_1 to x_2. Suppose we can exchange e_2 with e like in Figure 1.3, where $e = vx_1$. If we are lucky, this exchange reduces the size of K and therefore also decreases the legal order. For pseudoforests, it is easy to guarantee such an exchange. Here, it takes relatively little effort to maintain a valid decomposition

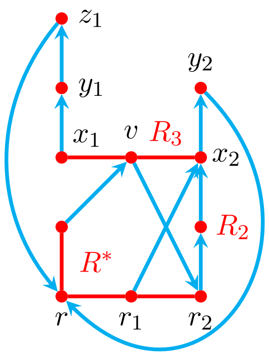

Fig. 1.1: An exploration subgraph with legal order $(R^*, R_2, R_3, \{y_2\}, \{y_1\}, \{z_1\})$ with component sizes $(3, 0, 2, 0, 0, 0)$.

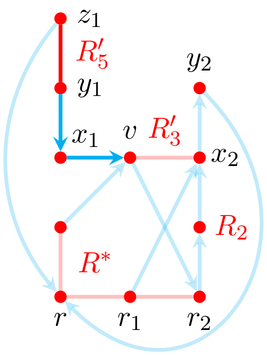

Fig. 1.2: The decomposition of Fig. 1.1 after performing a special path augmentation, colouring y_1z_1 red. A legal order is $(R^*, R_2, R'_3, \{y_2\})$ with component sizes $(3, 0, 1, 0)$. Edges that have not changed are drawn with reduced opacity.

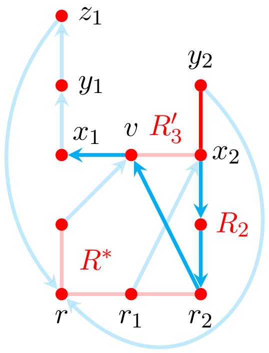

Fig. 1.3: The decomposition of Fig. 1.1 after (x_2, y_2) and vx_1 switched colours. A legal order is $(R^*, R'_3, R_2, \{x_1\}, \{y_1\}, \{z_1\})$ with component sizes $(3, 2, 0, 0, 0, 0)$. Edges that have not changed are drawn with reduced opacity.

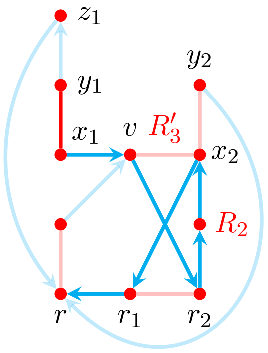

Fig. 1.4: The decomposition of Fig. 1.3 after performing a special path augmentation, colouring x_1y_1 red. This decomposition does not contain a red component with more than two edges anymore. Edges that have not changed are drawn with reduced opacity.

with k pseudoforests since these simply form a k-orientation. However, for the forest case, it is much more difficult to maintain k spanning trees, because we are not allowed to create blue cycles. Moreover, other blue arcs might need to be reoriented in order to maintain the orientation of the blue arcs towards r. Such reorientations can fundamentally change the legal order.

Note that in Figure 1.3, the exchange does not decrease the legal order because the blue path from v to x_2 is reoriented by the exchange and thus, R_2 cannot come after R^* in a legal order. In this case, further exchanges must be made. For example, by performing a special path augmentation colouring e_1 red and (r_1, r) blue, we can decrease the size of the root component, thereby improving the residue function value, see Figure 1.4.

Like in this example, we will always first carry out a number of exchanges of the second kind, where we exchange edges on the red path between "generating" vertices with generating edges. This can significantly worsen the legal order. But after that we can apply a special path augmentation where the special path goes back in the legal order at least to the vertex where the legal order was destructed (in our example the "destruction" occurred at vertex r_2 because the blue arc from r_2 to R_2 was reoriented). Therefore, this final augmentation yields a genuine improvement of the legal order. Using this technique, we will be able to bound the number of small children per non-small component.

The proof framework ends with showing that the augmentations are sufficient to ensure that a unit non-small on average. This is particularly easy in the proofs of Theorem 1.15 and 1.33, where a unit consists solely of a non-small component and its small children. For Theorem 1.14, units are formed in a somewhat more complicated way, and the exchanges involve not only blue arcs between parents and their small children, but also between parents and their parents (technically, these need not be the actual grandparents of the small children, but they must have a blue arc to the parents of the small children).

For Theorem 1.13, we proceed similarly to the proof of the Nine Dragon Tree Theorem of Jiang and Yang when performing the density calculation. The non-small components may be very large and have many small children, but only vertices of degree at least d can generate small children.

In order to guarantee the respective decompositions not only for graphs with fractional (pseudo-)arboricity bounded by $k + \frac{d}{d+k+1}$ but also for (k, d)-(pseudo-)sparse graphs, it suffices for the three forest-related theorems to show that the root component does not have small children. For the pseudoforest-related theorem, Theorem

1.33, we cannot enforce more structure around the root component than we can for other non-small components, but the “bad” properties of the root component violating Theorem 1.33 will imply enough density such that $\beta'(H) < 0$.

1.1 Collaboration and Publications

Theorem 1.11 was proven in [40] together with Benjamin Moore and contains a large part of the proof of Theorem 1.13 in Section 3.4.
Theorem 1.14 (as well as Theorem 5.4.3 as a corollary) was proven in [37] together with Benjamin Moore.
The proof of Theorem 1.9 (and thus, a major part of the proof of Theorem 1.15 in Section 3.6) was published in [36] together with Benjamin Moore.
The proof of Theorem 1.22 (and thus a major part of the proof of Theorem 1.33 in Section 4.3) was published in [39] together with Benjamin Moore and Evelyne Smith-Roberge. This paper also covers Theorem 1.26 and Theorem 1.27.
Apart from that, all results not explicitly designated otherwise are contributions of the author.

1.2 Structure of the Thesis

Chapter 2 introduces important notations that are used throughout the thesis.

In Chapter 3, we prove the aforementioned Nine Dragon Tree results for forests. Section 3.1 provides a proof of Nash-Williams’ Theorem. In Section 3.2, we review the optimality of the Nine Dragon Tree Conjectures by proving Theorem 1.7 and Theorem 1.16. In Sections 3.4, 3.6, and 3.7, we prove Theorems 1.13, 1.14, 1.15, respectively. Sections 3.3 and 3.5 provide notation and lemmas shared across the proofs of these theorems. Finally, Section 3.8 covers the proof of Theorem 1.28.

Chapter 4 covers the Nine Dragon Tree results for pseudoforests. Section 4.1 contains basic definitions and facts about pseudoforests and orientations, and provides a proof of Hakimi’s Theorem. Section 4.2 addresses the optimality of the Pseudoforest Nine Dragon Tree Theorems and proves Theorem 1.21 and Theorem

1.32. Section 4.3 proves Theorem 1.33 and Sections 4.4, 4.5 cover the proofs of the lower bound Theorems 1.26 and 1.27.

Chapter 5 presents a few applications of Nine Dragon Tree results on graphs with bounded degree, planar graphs with high girth, game chromatic number and game colouring numbers, and thin spanning trees.

Chapter 6 discusses open questions.

Notation and Preliminaries

2

In this chapter, we introduce the notation used throughout the thesis and recall a few basic results from graph theory.

2.1 Sets and Numbers

If S is a set and a is any object, let $S + a := S \cup \{a\}$ and $S - a := S \setminus \{a\}$.

We denote $\{0, 1, 2, \ldots\}$ by $\mathbb{N}_0$ and for $n \in \mathbb{N}_0$, we denote $\{n, n+1, \ldots\}$ by $\mathbb{N}_{\geq n}$.

Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}_{\geq 1}$. We use the notation $n \bmod k := n - k \left\lfloor \frac{n}{k} \right\rfloor$. Note that $n \bmod k \in \{0, \ldots, k-1\}$. For $n_1, n_2 \in \mathbb{Z}$, we write $n_1 \equiv n_2 \mod k$ if $n_1 \bmod k = n_2 \bmod k$.

2.2 Lexicographic Ordering

For $n \in \mathbb{N}_{\geq 1}$, let $X_n := \bigl\{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{N}_0\bigr\}$ for the following definition of *lexicographic ordering* of elements of $\bigcup_{n \in \mathbb{N}_{\geq 1}} X_n$.
Let $n, m \in \mathbb{N}_{\geq 1}$, $x = (x_1, \ldots, x_n) \in X_n$, and $y = (y_1, \ldots, y_m) \in X_m$. For $i \in \{n+1, \ldots, \max\{n, m\}\}$, let $x_i = 0$, and for $j \in \{m+1, \ldots, \max\{n, m\}\}$, let $y_i = 0$. We write $x \leq y$ if there is an $i \in \bigl\{1, \ldots, \max\{n, m\}\bigr\}$ such that for all $j \in \{1, \ldots, i\}$, we have $x_j = y_j$, and $x_i \leq y_i$. Furthermore, we write $x < y$ if $x \leq y$ and $x \neq y$. Note that this defines a total order for the elements of $\bigcup_{n \in \mathbb{N}_{\geq 1}} X_n$.

2.3 Graphs

A *mixed (multi-)graph* G is a tuple $(V(G), E(G))$, where $V(G)$ is a set of objects and $E(G)$ is a multiset in which every element is either a multiset $\{u, v\}$ with

$u, v \in V(G)$, or a tuple (u, v) with $u, v \in V(G)$.
If $v \in V(G)$, we call v a *vertex of* G. If $e \in E(G)$, we call e an *edge of* G. Furthermore, we write $e(G) := |E(G)|$ and $v(G) := |V(G)|$. If $u, v \in V(G)$, we also call the (multi-) set $\{u, v\}$ an undirected edge and write uv. We also call $e = (u, v)$ an *arc*, and call u the *tail of* e and v the *head of* e. Furthermore, we call vv and (v, v) a *loop*. If an edge is contained more than once in $E(G)$, we call it a *parallel edge (of* G*)*. If G does not contain a loop, we call it a *loopless graph*. If G is loopless and does not contain parallel edges, we call it a *simple graph*. If G does not contain any arcs, we call it an *undirected graph* or just *graph*. If G does not contain undirected edges, we call it a *directed graph* or a *digraph*.

We say that a graph G *is isomorphic to a graph* H if there is a bijection $\phi\colon V(G) \cup E(G) \longrightarrow V(H) \cup E(H)$ such that $\phi(V(G)) = V(G)$, for every undirected edge $e = \{u, v\} \in E(G)$, we have that $\phi(e) = \{\phi(u), \phi(v)\}$, and for every arc $f = (u, v) \in E(G)$, we have that $\phi(f) = (\phi(u), \phi(v))$.

Let G be a mixed graph, and let $v \in V(G)$. We say that $w \in V(G)$ is a *neighbour of* v *in* G if $vw \in E(G)$, $(v, w) \in E(G)$ or $(w, v) \in E(G)$. Let $U \subseteq V(G)$. We say that $w \in V(G)$ is a *neighbour of* U *in* G if w is a neighbour of a vertex $u \in U$.

Let G, H be mixed graphs. We say that H is a *subgraph of* G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We let $\varnothing$ denote the *empty graph* $(\varnothing,\ \varnothing)$. Let V' be a set of vertices and E' be a set of edges. We write

- $G \cup V' := \big(V(G) \cup V',\ E(G)\big)$,
- $G \setminus V' := \big(V(G) \cup V',\ \{xy \in E(G) \mid x, y \notin V'\}\big)$,
- $G \cup E' := \big(V(G),\ E(G) \cup E'\big)$.
- $G \setminus E' := \big(V(G),\ E(G) \setminus E'\big)$.

For a single vertex v or a single edge e, we write $G + v := G \cup \{v\}$, $G - v := G \setminus \{v\}$, $G + e := G \cup \{e\}$, and $G - e := G \setminus \{e\}$.
Let G_1, G_2 be mixed graphs. We let $G_1 \cap G_2 := \big(V(G_1) \cap V(G_2),\ E(G_1) \cap E(G_2)\big)$.
Let G be a mixed graph and let $V' \subseteq V(G)$. We call $G[V'] := \big(V',\ \{xy \in E(G) \mid x, y \in V'\}\big)$ the *subgraph of* G *induced by* V'. We also write $E(X) := E(G[X])$ and $e(X) := e(G[X])$ if it is clear from context which graph G is underlying.
Let G be a mixed graph and let V_1, V_2 be sets of vertices. Let $E_G(V_1, V_2)$ denote the set of edges and arcs of $E(G)$ that have one end-vertex in V_1 and the other one in

V_2. We will just write $E(V_1, V_2)$ when it is clear from context which graph G we refer to. We also write $e(V_1, V_2) := |E(V_1, V_2)|$.

Let $n \in \mathbb{N}_{\geq 1}$. The *complete graph on* n *vertices* is the simple graph G with $v(G) = n$ and $E(G) = \{xy \mid x, y \in V(G), x \neq y\}$.

Let G be a graph and let $X, Y \subseteq V(G)$ such that $X \dot{\cup} Y = V(G)$. G is *bipartite on* X *and* Y if $E(G) = E_G(X, Y)$. Furthermore, we call G *the complete bipartite graph on* X *and* Y if $E(G) = X \times Y$.

2.3.1 Paths and Cycles

For $n \in \mathbb{N}_{\geq 1}$, we call a graph isomorphic to $\big(\{v_1, \dots, v_n\}, \{(v_i, v_{i+1}) | i \in \{1, \dots, n-1\}\}\big)$, where $v_i \neq v_j$ if $i \neq j$, a *directed path* or *dipath (from* v_1 *to* v_n*)* and use the notation $[v_1, \dots, v_n]$ for it. We call the same graph, but with undirected edges, a *path* and use the same notation $[v_1, \dots, v_n]$ for it. It will be clear from context whether the path is undirected or directed.

If P is a (directed) path, we call $e(P)$ the *length of* P.

A subgraph P of a graph G is called a *Hamiltonian path (of* G*)* if P is a path and $V(P) = V(G)$.

Let G be a mixed graph and let G' be the (undirected) graph obtained from G by replacing every arc (u, v) by uv. We call G *connected* if for every $u, v \in V(G)$, there is a path $P \subseteq G'$ from u to v in G'. If G is not connected, we call it *disconnected*.

We call a graph G n*-edge-connected* if n is the minimum number such that there is an edge set $E' \subseteq E(G)$ with $|E'| = n$ such that $G \setminus E'$ is disconnected.

Let u, v be two vertices of a connected (undirected) graph G. Let $\mathrm{dist}_G(u, v)$ denote the minimum length of all paths from u to v. We call $\max_{u,v \in V(G)} \mathrm{dist}_G(u, v)$ the *diameter of* G and denote it by $\mathrm{diam}(G)$. If G is not connected and $\mathcal{C}$ is the set of components of G, then we call $\max_{C \in \mathcal{C}} \mathrm{diam}(C)$ the *diameter of* G.

A *cycle* is a loop, an undirected graph isomorphic to $[v_1, \dots, v_n] + v_n v_1$ or a directed graph isomorphic to $[v_1, \dots, v_n] + (v_n, v_1)$. We call a graph G *cyclic* if a subgraph of G is a cycle.

A cycle with exactly $n \in \mathbb{N}_{\geq 1}$ edges is called an n*-cycle*.

A subgraph C of a graph G is called a *Hamiltonian cycle (of* G*)* if C is a cycle and $V(C) = V(G)$. The *girth of a graph* G is the smallest number $g \in \mathbb{N}_{\geq 1}$ such that a g-cycle is a subgraph of G. If G is acyclic, the girth of G is ∞.

2.3.2 Degrees

Let G be a mixed graph and $v \in V(G)$. Note that, in the following, the cardinalities of multisets (not sets) are considered. If G is undirected, we let $\deg_G(v) := |\{uv \in E(G) | u \in V(G) - v\}| + 2 \cdot |\{vv \in E(G)\}|$ and call this number the ***degree of*** v ***in*** G. If G is directed, we denote the *outdegree of* v *in* G by $\deg_G^+(v) := |\{(v,u) \in E(G) \,|\, u \in V(G)\}|$, and we denote the *indegree of* v *in* G by $\deg_G^-(v) := |\{(u,v) \in E(G)\}|$. For $X \in V(G)$, we write $\deg_X(v) := \deg_{G[X]}(v)$, $\deg_X^+(v) := \deg_{G[X]}^+(v)$, and $\deg_X^-(v) := \deg_{G[X]}^-(v)$ if it is clear from context that G is the underlying graph.
For $\Delta \in \mathbb{N}_0$, a graph G is Δ*-regular* if every vertex $v \in V(G)$ has $\deg_G(v) = \Delta$.

2.3.3 Forests and Trees

A *forest* is a graph that is not cyclic. A connected forest is called a *tree*.
First, note that if T is a tree, then $e(T) = v(T) - 1$. Furthermore, for any two vertices u, v of a tree, there is a unique path from u to v. In a tree T with undirected edges and $u, v \in V(T)$, let $P^T(u,v)$ denote the unique path in T from u to v.
Let T be a tree and $r \in V(T)$. We may call T ***a tree with root*** r. In this context, let $u, v \in V(T)$ such that $u \in V(P^T(v,r))$. Then we call u *an ancestor of* v *in* T, and we call v *a descendant of* u *in* T. Note that v is a descendant of v and v is an ancestor of v.
We call v a *leaf of* T if v is its only descendant.
We call a vertex of T that is not a leaf of T an *inner vertex of* T.
We call $e(P^T(v,r))$ the *depth of* v *in* T and denote it by $\mathrm{depth}_T(v)$.
A *subtree* T' *of a rooted tree* T *with root* r' is the subgraph of T induced by r' and all of its descendants in T. For a tree T that has its edges directed towards a root vertex, and $x, y \in V(T)$ such that x is a descendant of y in T, we let $P^T(x,y)$ denote the unique directed path from x to y in T.

We call a graph isomorphic to $\big(\{c, \ell_1, \dots, \ell_n\}, \{c\ell_i \,|\, i \in \{1, \dots, n\}\}\big)$ a *star with centre* c. Note that a star is a tree.

2.3.4 Pseudoforests and Pseudotrees

A *pseudotree* is a connected graph having at most one cycle. Note that if G is a pseudotree, then $e(G) \in \{v(G) - 1, v(G)\}$.
A *pseudoforest* is a graph in which every component is a pseudotree.
For a detailed discussion of k-orientations ($k \in \mathbb{N}_0$), which are closely connected to pseudoforests, see Section 4.1.

2.3.5 Matchings

A *matching* M *of a graph* G is a subgraph $M \subseteq G$ such that every vertex $v \in V(M)$ has $\deg_M(v) \leq 1$.
We call M *perfect* if every vertex $v \in V(M)$ has $\deg_M(v) = 1$.

Theorem 2.3.1 (Tutte's Theorem (1947), [50]). *A graph G has a perfect matching if and only if for every $U \subseteq V(G)$, we have that $G \setminus U$ has at most $|U|$ components whose number of vertices is odd.*

2.3.6 Planar Graphs

A graph that can be drawn on the plane without any edges crossing each other is called a *planar graph*. We call such a drawing a *planar embedding of* G. For a formal definition of these terms, we refer to [10].

Theorem 2.3.2 (Euler's Formula). *Let G be a connected planar graph and let $f \in \mathbb{N}_{\geq 1}$ be the number of faces of a planar embedding of G. Then $v(G) - e(G) + f = 2$.*

For $g \in \mathbb{N}_{\geq 1}$, we let $\mathcal{P}_g$ denote the set of planar graphs with girth at least g.

3

Forest Variants of the Nine Dragon Tree Conjectures

3.1 A Proof of Nash-Williams' Theorem

In this section, we present an (implicit) proof from [27] of Nash-Williams' Theorem, which was already stated in the introduction.

Theorem 1.1 (Nash-Williams' Theorem [45]). *A graph G without loops decomposes into k forests if and only if for every subgraph $H \subseteq G$ with $v(H) > 1$, we have $\frac{e(H)}{v(H)-1} \leq k$.*

Proving this theorem is actually only a byproduct of the proof. We are interested in this proof technique because it enables us to show that vertex-minimal counterexamples to Theorems 1.13, 1.14, 1.15 have a decomposition into $k+1$ forests, where k of them are spanning trees. This was also done in [27] for the Nine Dragon Tree Theorem.

For $k \in \mathbb{N}_0$, a *maximal k-colouring of a loopless graph G* is a decomposition $(F_1, \dots, F_k, A)$ of G such that $F_1, \dots, F_k$ are forests and there is no $e \in E(A)$ such that there is a decomposition of $(F_1 \cup \dots \cup F_k) + e$ into k forests.

In the following let $(F_1, \dots, F_k, A)$ be a maximal k-colouring of a loopless graph G. A *component tree* of a maximal k-colouring of an undirected graph G is a rooted tree where each vertex v is associated with a vertex set $X_v \subseteq V(G)$ and with an index $t(v) \in \{1, \dots, k\}$ such that:

- The root r has $X_r = V(G)$.
- If a vertex v has children, then $F_{t(v)}[X_v]$ is disconnected and for every component C of $F_{t(v)}[X_v]$, v has exactly one child associated with C and these are the only children of v.

Furthermore, an edge $e = u_1u_2 \in E(G)$ is *between two children of* $v \in V(T)$ if v has two children associated with vertex sets X_1, X_2 such that $u_1 \in X_1$ and $u_2 \in X_2$.
The next lemma implicitly shows how we can obtain a maximal k-colouring of a graph.

Lemma 3.1.1. *Let T be a component tree T of a maximal k-colouring $(F_1, \dots, F_k, A)$, then no edge of $E(A)$ is between two children of a vertex of T.*

Proof. Suppose the lemma is not true and that there is an edge e between two children of a component tree vertex v with depth h. We prove the lemma by showing that there is another component tree T' of another maximal k-colouring of G having an edge between two children of a vertex with depth $h' < h$. Using this argument several times we obtain a contradiction since the depth is lower-bounded by 0.
Let $i := t(v)$ and note that $F_i + e$ contains a cycle C or otherwise $(F_1, \dots, F_k, A)$ would not be a maximal k-colouring. There is an ancestor $u \neq v$ of v such that $V(C)$ is contained in X_u. If we choose u with maximum depth, there is an edge $e' \in E(C)$ between two children of u. Note that $F_i' := (F_i - e') + e$ is a forest, and since the vertex sets of the components of F_i' are the same as in F_i, we have that $(F_1, \dots, F_{i-1}, F_i', F_{i+1}, \dots, F_k, (A - e) + e')$ is a maximal k-colouring of G, which completes the proof. □

A component tree T of a maximal k-colouring is *maximal* if for the associated vertex set X_v of any leaf v, $F_i[X_v]$ is connected for every $i \in \{1, \dots, k\}$ (and thus, we cannot add children to retrieve another component tree).

Corollary 3.1.2. *Let $\mathcal{T} = (F_1, \dots, F_k, A)$ be a maximal k-colouring of a graph G, let T be a component tree of $\mathcal{T}$, and let $e \in E(A)$. Then there is a leaf vertex ℓ of T such that $e \in E(G[X_\ell])$.*

Proof. Let $u_1u_2 := e$ and let $\ell_1, \dots, \ell_n$ be the leaves of T. Note that $\dot{\bigcup}_{j=1}^{n} X_{\ell_j} = V(G)$. Thus, there is a leaf ℓ such that $u_1 \in X_\ell$. If ℓ is the root of T, then $u_2 \in V(G) = X_\ell$. If ℓ is the root of T, then by Lemma 3.1.1, we have that $u_2 \in X_\ell$. □

Lemma 3.1.3. *If a loopless graph G has a maximal k-colouring $\mathcal{T} = (F_1, \dots, F_k, A)$ such that $A \neq \varnothing$, then $\gamma(G) > k$.*

Proof. Let T be a maximal component tree of $\mathcal{T}$ and let $e \in E(A)$. Let $\ell_1, \dots, \ell_n$ be the leaves of T. By Corollary 3.1.2, there is a leaf vertex ℓ of T such that $e \in E(G[X_\ell])$. As G does not contain loops, e has two distinct end-vertices, and thus, $|X_\ell| \geq 2$.

Since T is maximal, we have that $F_i[X_{\ell_j}]$ is a spanning tree for every $i \in \{1, \dots, k\}$ and every $j \in \{1, \dots, n\}$. Thus, $e(G[X_\ell]) \geq \sum_{i=1}^{k} e(F_i[X_\ell]) + |\{e\}| = k(|X_\ell| - 1) + 1$. $\square$

We are now able to prove Nash-Williams' Theorem.

Proof of Theorem 1.1. If G does not have a decomposition into k forests, then it has a maximal k-colouring such that $A \neq \varnothing$, and by Lemma 3.1.3 we have $\gamma(G) > k$. If $\gamma(G) > k$, then it has a subgraph H with $e(H) > k(v(H) - 1)$. Since an acyclic subgraph of H has at most $v(H) - 1$ edges, there is no decomposition of G into k forests. $\square$

3.2 Optimality of the Nine Dragon Tree Conjectures

In this section, we discuss the optimality of the Nine Dragon Tree Conjectures. Recall Theorem 1.7 and Theorem 1.16 from the introduction.

Theorem 1.7 ([41]). *Let $k \in \mathbb{N}_{\geq 1}$, $d \in \mathbb{N}_0$. For every $\epsilon > 0$, there exist simple graphs G such that $\gamma(G) \leq k + \frac{d}{d+k+1} + \epsilon$, and yet G is not $(k, d)^-$-decomposable.*

Theorem 1.16 ([29]). *Let $k \in \mathbb{N}_{\geq 1}, d \in \mathbb{N}_0$. There are simple graphs G such that $\gamma(G) \leq k + 1$, $\beta(G) = -1$, $\beta(H) \geq 0$ for all $\varnothing \neq H \subsetneq G$, but G is not $(k, d)^-$-decomposable.*

We will present the proof of Theorem 1.16 from [29] with more rigorous reasoning, and with the same construction, we will reprove Theorem 1.7.
Note that all results will only discuss the optimality of the weak variants of the conjectures, i.e. Theorem 1.4 and Theorem 1.10. The optimality for the stronger versions will immediately follow.
When the Nine Dragon Tree Conjectures were proposed, the optimality was justified by the following theorem.

Theorem 3.2.1 ([41]). *Let $k, d \in \mathbb{N}_{\geq 1}$. There are simple, arbitrarily large graphs G and edges $e^* \in E(G)$ such that $\gamma(G - e^*) = k + \frac{d}{d+k+1}$, however, there are vertices $v \in V(G)$ such that G does not have a decomposition into k forests and a graph F, where in F, every vertex has degree at most d and v is isolated. Furthermore, v and e are incident.*

In fact, demanding one vertex to be isolated in the degree-/component size-constrained graph is not a restriction.

Theorem 3.2.2 ([41]). *Let $k, d \in \mathbb{N}_0$, let G be a loopless graph with $\gamma(G) \leq k + \frac{d}{d+k+1}$, and let $v \in G$. If the (Weak/Strong) Nine Dragon Tree Conjecture is true, then there is a decomposition satisfying the respective conjecture, and furthermore, in the degree-/component size-constrained graph of the decomposition, v is isolated.*

Proof. Let $\mathcal{C}$ be one of the three mentioned conjectures and let $\mathcal{C}$ be true.
Let G' be the graph obtained by taking $d+1$ copies $G_1, \dots, G_{d+1}$ of G and identifying the copies of v. Let $H \subseteq G'$ and for $i \in \{1, \dots, d+1\}$, let $H_i = H \cap G_i$. Note that $\frac{e(H_i)}{v(H_i)-1} \leq k + \frac{d}{d+k+1}$ for every $i \in \{1, \dots, d+1\}$. Furthermore, let $\alpha = 1$ if $v \in V(H)$, and $\alpha = 0$ otherwise. Then

$$\frac{e(H)}{v(H)-1} = \frac{\sum\limits_{i=1}^{d+1} e(H_i)}{v(H_1) + \sum\limits_{i=2}^{d+1} \big(v(H_i) - \alpha\big) - 1} \leq \frac{\sum\limits_{i=1}^{d+1} e(H_i)}{\sum\limits_{i=1}^{d+1} \big(v(H_i) - 1\big)} \leq k + \frac{d}{d+k+1}.$$

Thus, by $\mathcal{C}$, G' has a decomposition into k forests and a graph F, where in F, every vertex has degree at most d. Thus, in F, v is adjacent to vertices of at most d copies of G, which completes the proof. □

Using the technique of the proof of Theorem 3.2.2, a more direct lower bound to the Weak Nine Dragon Tree Conjecture can be proven, that is also mentioned in [38].

Theorem 3.2.3. *Let $k, d \in \mathbb{N}_{\geq 1}$. There are simple graphs G and edge sets $S \subseteq E(G)$ with $|S| = d+1$ such that $\gamma(G \setminus S) = k + \frac{d}{d+k+1}$, however, G is not $(k,d)^-$-decomposable.*

Proof. For $k, d \in \mathbb{N}_{\geq 1}$, let G' be a graph, $v \in V(G)$ and $e^* \in E(G)$ as described in Theorem 3.2.1. Let G' be the graph obtained by taking $d+1$ copies of G and identifying the copies of v. Let S be the set of copies of e^*. Analogously to the proof of Theorem 3.2.2, it can be proven that $\gamma(G \setminus S) = k + \frac{d}{d+k+1}$.
Furthermore, if there was a decomposition of G' into k forests and a graph F, where in F, every vertex has degree at most d, then this also holds for v and thus, in at least one of the $d+1$ copies, it has degree 0. But this is a contradiction to Theorem 3.2.1. □

The authors of [41] also stated Theorem 1.7 as a corollary of Theorem 3.2.1 without providing a proof. Note that the statement of Theorem 3.2.3 does not directly imply Theorem 1.7: even if G is very large, its densest subgraph could be of small size and contain one of the edges of S. In this case, $\gamma(G)$ approaching $k + \frac{d}{d+k+1}$ arbitrarily closely is not guaranteed.

We will prove Theorem 1.7 properly using the construction that was used in [29] to prove Theorem 1.16.

Recall that

$$\beta(H) := v(H)(k+1)(d+k) - e(H)(d+k+1) - k^2$$

for a non-empty graph H.

Before we turn to the actual proof, we consider the edge cases $k = 0$ and $d = 0$. For the case $k = 0$, we need the following technical lemma.

Lemma 3.2.4. *Let G be a loopless graph and $k \in \mathbb{N}_0$. Then G is $(k,0)$-sparse if and only if $\gamma(G) \leq k$.*

Proof. It is clear that G is $(k,0)$-sparse if $\gamma(G) \leq k$.

Thus, let G be $(k,0)$-sparse and hence, $\frac{e(H)-\frac{k}{k+1}}{v(H)-1} \leq k$ for all $H \subseteq G$ with $v(H) \geq 2$. Suppose that there is a subgraph H with $\frac{e(H)}{v(H)-1} > k$. Then we have that

$$\frac{e(H) - \frac{k}{k+1}}{v(H)-1} \geq \frac{k(v(H)-1) + 1 - \frac{k}{k+1}}{v(H)-1} > k,$$

which is a contradiction. □

Lemma 3.2.5. *Theorem 1.7 and Theorem 1.16 are true for $d = 0$.*

Proof. Let $k \in \mathbb{N}_{\geq 1}$ and let G be a simple graph with $v(G) \geq \frac{\epsilon+1}{\epsilon}$ that is decomposable into $k-1$ Hamiltonian paths $P_1, \dots, P_{k-1}$ and a Hamiltonian Cycle C. It is clear that such a graph can be constructed. We have that $e(G) = k(v(G)-1)+1$ and thus, $\beta(G) = -1$. By Nash-Williams' Theorem, G is not decomposable into k forests and thus, it is not $(k,0)^-$-decomposable. Note that G is the only subgraph that is not decomposable into k forests, since for every $V' \subsetneq V(G)$, each of $P_1[V'], \dots, P_{k-1}[V'], C[V']$ is acyclic. Thus, $\gamma(G) = \frac{e(G)}{v(G)-1} \leq k + \epsilon$ and by Lemma 3.2.4, we have that $\beta(H) \geq 0$ for every $\varnothing \neq H \subsetneq G$. □

Lemma 3.2.6. *Theorem 1.16 is true for $k = 0$.*

Proof. Let $d \in \mathbb{N}_0$. Let G be a star with $d+1$ edges. Then $\beta(G) = (d+2)d - (d+1)^2 = -1$, $\beta(H) \geq 0$ for all $\varnothing \neq H \subsetneq G$ and G is not $(0,d)^-$-decomposable since its centre has degree $d+1$. □

Note that Theorem 1.7 does not hold for $k = 0$ since there is no graph G with $0 < \gamma(G) < 1$ since the graph with two vertices connected by an edge has fractional arboricity 1.

Now, we turn to the actual proof from [29] and define a graph G that satisfies Theorem 1.7 and Theorem 1.16.
Because of Lemma 3.2.5 and Lemma 3.2.6, we let $k, d \in \mathbb{N}_{\geq 1}$. Furthermore, we let $t \in \mathbb{N}_{\geq 1}$ with $t \geq k+1$, and $s := t(d+k) - k + 1$. Additionally, for the proof of Theorem 1.7, we let $\epsilon > 0$ and $t \geq \frac{1+\frac{dk}{d+k+1}+k\epsilon}{\epsilon(d+k+1)}$. We define a bipartite graph G on the set of vertices $X = \{x_0, \dots, x_{s-1}\}$ and $Y = \{y_0, \dots, y_{t-1}\}$. The edge set of G is formed by the edges $x_i y_i, \dots, x_{i+k} y_{i+k}$ for every $i \in \{0, \dots, s-1\}$, where the indices of the vertices of Y are always understood to be taken modulo t, including the rest of the section. G is depicted in Figure 3.2.1 for fixed values of k, d, and t.
Note that $v(G) = s + t$, $e(G) = s(k+1)$, and thus

$$\begin{aligned}
\frac{e(G)}{v(G)-1} &= \frac{s(k+1)}{s+t-1} \\
&= \frac{k\left(t(d+k)-k+1\right)+t(d+k)-k+1}{t(d+k+1)-k} \\
&= k + \frac{td+1}{t(d+k+1)-k} \\
&= k + \frac{d+\frac{1}{t}}{d+k+1-\frac{k}{t}}.
\end{aligned}$$

In order to get a better understanding of the structure of G, we show that there is a decomposition of G that is almost $(k,d)^+$-decomposable.

Theorem 3.2.7. *G has a decomposition into k forests and a star forest only consisting of stars with exactly 0 or d edges, except for one star which has $d+1$ edges.*

Proof. We describe the decomposition G by colouring its edges with colours $\{0, \dots, k\}$. k, which we will also call "red", will be the colour of the forest with the star components. We describe the colouring by considering the incident edges of each vertex of X. For this purpose we partition X into $R = \{x_0, \dots, x_{dt-1}\}$, $B = x_{dt}, \dots, x_{s-2}$ and $Z = \{x_{s-1}\}$. Note that the colouring is depicted in Figure 3.2.2.

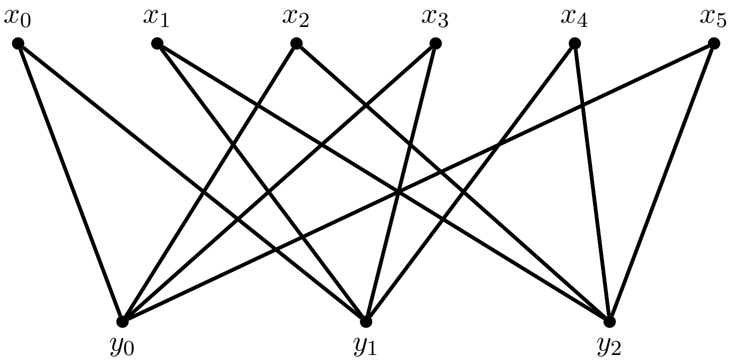

Fig. 3.2.1: G when $k = 1$, $d = 1$, $t = 3$.

First, we consider R: for $i \in \{0, \ldots, dt - 1\}$ let $x_i y_i$ be coloured red and let the other k incident edges to x_i have all the colours of $\{0, \ldots, k - 1\}$.

For B, note that the index of every vertex of B can be written as $\alpha_{b,i} := dt + b(t-1) + i$ where $b \in \{0, \ldots, k-1\}$ and $i \in \{0, \ldots, t-2\}$. Let $x_{\alpha_{b,i}} y_{\alpha_{b,i}}$ and $x_{\alpha_{b,i}} y_{\alpha_{b,i}+1}$ be coloured with colour b and let the $k - 1$ remaining edges incident to $x_{\alpha_{b,i}}$ be coloured with all the colours of $\{0, \ldots, b - 1, b + 1, \ldots, k - 1\}$.

Finally, we consider $Z = \{x_{s-1}\} =: \{z\}$. Let $z' := y_{s-1}$. We colour zz' red and we colour all remaining edges incident to z with all the colours of $\{0, \ldots, k - 1\}$. This completes the description of the colouring, which we will call f in the following.

Let F be the red subgraph of G induced by f. Note that in F every vertex of Y is the centre of a star with exactly d edges except for z', which is the centre of a star with $d + 1$ edges. The leaves of the stars are exactly the vertices of $R \cup Z$. Furthermore, the vertices of B are isolated in F.

In order to prove the theorem, it remains to show that for every $b \in \{0, \ldots, k - 1\}$ the graph B_b induced by colour b of f is acyclic.

Suppose that there was a cycle C in B_b. Note that $V_b := \{x_{\alpha_{b,0}}, \ldots, x_{\alpha_{b,t-2}}\} \cup Y$ is the set of vertices that have degree at least 2 in B_b. Thus, $C \subseteq G[V_b]$. However, $G[V_b]$ is the path $[y_{\alpha_{b,0}}, x_{\alpha_{b,0}}, y_{\alpha_{b,1}}, x_{\alpha_{b,1}}, \ldots, y_{\alpha_{b,t-2}}, x_{\alpha_{b,t-2}}, y_{\alpha_{b,t-2}+1}]$, which is a contradiction. Thus, B_b is acyclic. □

In the rest of the section, we will prove Theorem 1.7 and Theorem 1.16. Before we examine the density of G, we prove that the desired decomposition is not possible.

Lemma 3.2.8. *G is not $(k, d)^-$-decomposable.*

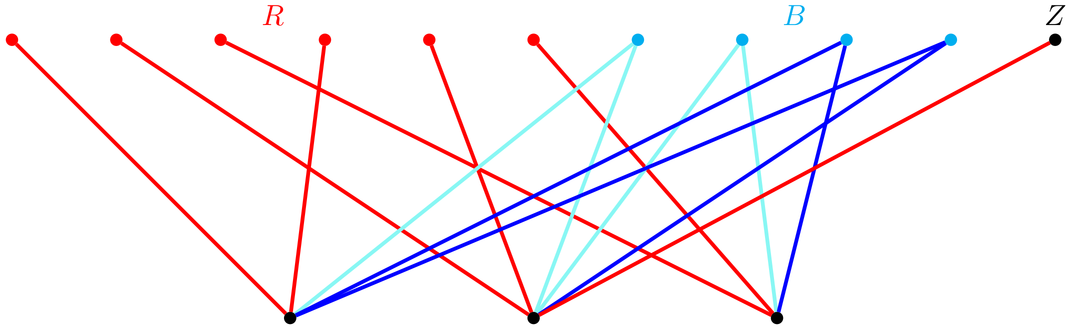

Fig. 3.2.2: The colouring of G described in the proof of Theorem 3.2.7 in the case where $k = 2$, $d = 2$, $t = 3$. For simplicity's sake, only a few edges that are crucial for the proof are depicted.

Proof. Note that since every vertex of X has exactly $k+1$ neighbours and G is bipartite, we have that $e(G) = (k+1)s = (k+1)t(d+k) - k^2 + 1$. Removing the edges of a graph from G, where every vertex has degree at most d results in a graph whose number of edges is at least

$$e(G) - dt = kt(d+k) + tk - k^2 + 1 = k(s+t-1) + 1 = k(v(G)-1) + 1.$$

Thus, the resulting graph cannot be partitioned into k forests. □

Next, we quickly calculate $\beta(G)$.

Lemma 3.2.9. $\beta(G) = -1$.

Proof. We have that

$$\begin{aligned}\beta(G) &= (s+t)(k+1)(d+k) - s(k+1)(d+k+1) - k^2\\ &= (k+1)\big((d+k)t - s\big) - k^2\\ &= (k+1)(k-1) - k^2\\ &= -1.\end{aligned}$$

□

Let H^* be a non-empty subgraph of G minimizing β. Note that by the definition of β, we have that $H^* = G[V(H^*)]$.

For the proof of Theorem 1.7, we define a similar function for subgraphs $H \subseteq G$ with $V(H) > 1$.

$$\beta_t(H) := (v(H) - 1)\,(k+1)\left(d + k - \frac{k-1}{t}\right) - e(H)\left(d + k - \frac{k}{t} + 1\right).$$

Note that $\beta_t(H) \leq 0$ if and only if $\frac{e(H)}{v(H)-1} \geq k + \frac{d+\frac{1}{t}}{d+k+1-\frac{k}{t}}$. In particular, $\beta_t(G) = 0$.

Similarly to the definition of H^*, we let H_t^* be a subgraph of G with at least two vertices minimizing β_t. We also have that $H_t^* = G[V(H_t^*)]$.
In the next four lemmas, we show simple cases in which we can add or remove a vertex and its adjacent edges to increase the density.

Lemma 3.2.10. *Every vertex of H^* has degree at least $k+1$ in H^*.*

Proof. Suppose to the contrary that there is a vertex $v \in V(H^*)$ that has degree at most k in H^*. If $V(H^*) = \{v\}$, then $\beta(H^*) = k + kd + d > -1$, which is a contradiction to Lemma 3.2.9. Thus, $V(H^*) > 1$. Let $H := H^* - v$. Then

$$\beta(H^*) - \beta(H) \geq (k+1)(d+k) - (d+k+1)k = d,$$

which is a contradiction to the minimality of H^*. ☐

Lemma 3.2.11. *Every vertex of H_t^* has degree at least $k+1$ in H_t^*.*

Proof. Suppose to the contrary that there is a vertex $v \in V(H_t^*)$ that has degree at most k in H_t^*. By the definition of β_t, H_t^* contains at least two vertices.
If $v(H_t^*) = 2$, then $e(H_t^*) \leq 1$ since G is simple, and thus we would have

$$\begin{aligned}\beta_t(H_t^*) &= (k+1)\left(d + k - \frac{k-1}{t}\right) - \left(d + k - \frac{k}{t} + 1\right)\\ &= kd - 1 + k^2 \cdot \frac{t-1}{t} + \frac{k+1}{t}\\ &> 0,\end{aligned}$$

which is a contradiction to the minimality of H_t^* since $\beta_t(G) = 0$.
Thus, $v(H_t^*) \geq 3$. Let $H := H_t^* - v$. Then

$$\begin{aligned}\beta_t(H_t^*) - \beta_t(H) &\geq (k+1)\left(d + k - \frac{k-1}{t}\right) - k\left(d + k - \frac{k}{t} + 1\right) \\ &= \frac{k}{t} + d + k - \frac{k-1}{t} - k \\ &= d - \frac{1}{t},\end{aligned}$$

which is a contradiction to the minimality of H_t^*. □

Lemma 3.2.12. *If $k+1$ neighbours of $v \in V(G)$ are in $V(H^*)$, then v is also contained in $V(H^*)$.*

Proof. Suppose to the contrary that v is not in $V(H^*)$, and let $H := H^* + v$. Then

$$\beta(H) - \beta(H^*) \leq (k+1)(d+k) - (k+1)(d+k+1) = -k-1,$$

which is a contradiction to the minimality of H^*. □

Lemma 3.2.13. *If $k+1$ neighbours of $v \in V(G)$ are in $V(H_t^*)$, then v is also contained in $V(H_t^*)$.*

Proof. Suppose to the contrary that v is not in $V(H_t^*)$, and let $H := H_t^* + v$. Then

$$\begin{aligned}\beta_t(H) - \beta_t(H_t^*) &\leq (k+1)\left(d + k - \frac{k-1}{t}\right) - (k+1)\left(d + k - \frac{k}{t} + 1\right) \\ &= (k+1)\left(\frac{1}{t} - 1\right),\end{aligned}$$

which is a contradiction to the minimality of H_t^*. □

For $i \in \{0, \ldots, t-1\}$, let X_i be the set of vertices of X whose neighbours in G are exactly $y_i, \ldots, y_{i+k}$. The indices of $X_\bullet$ are always meant to be taken modulo t. Note that $X_0, \ldots, X_{t-1}$ is a partition of X. With the help of the previous four lemmas, we are now in position to prove that G is its densest subgraph. After that, proving Theorem 1.7 and Theorem 1.16 will be easy.

Lemma 3.2.14. *$H^* = G$ and $H_t^* = G$.*

Proof. Let $H \in \{H^*, H_t^*\}$. Note that if $Y \subseteq V(H)$, then $H = G$ by Lemmas 3.2.12, 3.2.13. Thus, suppose that there is a vertex of Y that is not in $V(H)$. Note that by the definition of β and β_t, we have $V(H) \neq \varnothing$. Thus, $Y \cap V(H) \neq \varnothing$ (if $x \in X \cap V(H)$, then the $k+1$ neighbours of x in Y are also in $V(H)$ by Lemmas 3.2.10, 3.2.11). Thus, $\varnothing \neq Y \cap V(H) \subsetneq Y$ and furthermore, there are integers l, r such that $0 \leq \ell \leq r \leq 2t-1$ and $y_{\ell-1}, y_{r+1} \in V(H)$ and for every $i \in \{\ell, \dots, r\}$, we have that $y_i \notin V(H)$. Note that this situation is depicted in Figure 3.2.3. By Lemmas 3.2.10, 3.2.11, we have that $X_{\ell-k} \cap V(H), \dots, X_r \cap V(H) = \varnothing$. Thus, the only neighbours of $y_{\ell-1}$ in H are in $X_{\ell-k-1}$, and again by Lemma 3.2.10, 3.2.11, at least $k+1$ vertices of $X_{\ell-k-1}$ are in $V(H)$. Thus, $y_{\ell-k-1}, \dots, y_{\ell-1} \in V(H)$. Analogously, we have that $y_{r+1}, \dots, y_{r+1+k} \in V(H)$.

We obtain $H' \subseteq G$ from H by adding vertices $y_\ell, \dots, y_r$ and the vertices of $X_{\ell-k}, \dots, X_r$, and adding all edges that are incident to $X_{\ell-k}, \dots, X_r$ in G.

Note that we add $y := r - \ell + 1$ vertices from Y. To determine the number of vertices we add from X, we need to analyze our previous partition of X a little bit more.

Note that $s \bmod t = t - k + 1 =: m$. Furthermore, $|X_i| = \lceil \frac{s}{t} \rceil$ if $i \in \{0, \dots, m-1\}$ and $|X_i| = \lfloor \frac{s}{t} \rfloor$ if $i \in \{m, \dots, t-1\}$. Thus, at most $k-1$ of the sets of vertices $X_0, \dots, X_{t-1}$ have exactly $\lfloor \frac{s}{t} \rfloor$ vertices. Furthermore, note that $\lceil \frac{s}{t} \rceil = d + k - \lfloor \frac{k-1}{t} \rfloor = d + k$ since $t > k$. Thus, when obtaining H' from H, the number of vertices added from X is

$$\begin{aligned} x &:= \sum_{i=\ell-k}^{r} |X_i| \\ &\geq (r - (\ell - k) + 1) \left\lceil \frac{s}{t} \right\rceil - (k-1) \\ &= (y+k)(d+k) - k + 1. \end{aligned}$$

Furthermore, the number of added edges is $(k+1)x$.

For $H = H^*$, we have that

$$\begin{aligned} \beta(H') - \beta(H^*) &= (k+1)(d+k)(x+y) - (d+k+1)(k+1)x \\ &= (k+1)\big((d+k)y - x\big) \\ &\leq (k+1)\big(-(d+k)k + k - 1\big) \\ &= -(k+1)\big((d+k-1)k + 1\big) \\ &< 0, \end{aligned}$$

which is a contradiction to the minimality of H^*.
For $H = H_t^*$, we have that

$$\begin{aligned}
\beta_t(H') - \beta_t(H_t^*) &= (x+y)(k+1)\left(d+k-\frac{k-1}{t}\right) - x(k+1)\left(d+k-\frac{k}{t}+1\right)\\
&= (k+1)\left(-x\cdot\frac{t-1}{t} + y\left(d+k+\frac{1}{t}(1-k)\right)\right)\\
&\leq (k+1)\left(-\frac{t-1}{t}\big((y+k)(d+k)-k+1\big) + \frac{y}{t}\big(t(d+k)+1-k\big)\right)\\
&= (k+1)\left(\frac{y}{t}(d+k+1-k) - \frac{t-1}{t}\big(k(d+k)-k+1\big)\right)\\
&\overset{y\,\leq\, t-1}{\leq} \frac{(t-1)(k+1)}{t}\big(d+1-(kd+k^2-k+1)\big)\\
&< 0,
\end{aligned}$$

which is a contradiction to the minimality of H_t^*.

□

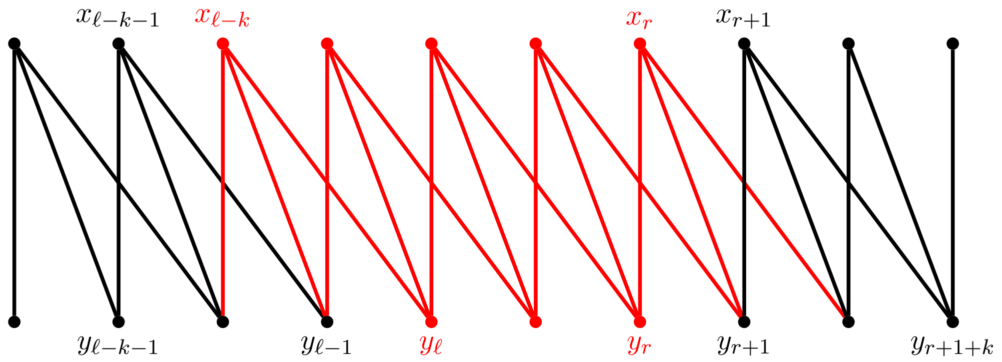

Fig. 3.2.3: An example of ℓ and r in the proof of Lemma 3.2.14. The black vertices and edges are from H. The red vertices and edges are the ones we add to obtain H'.

Now, we just have to combine the previous lemmas to prove Theorem 1.7 and Theorem 1.16.

Proof of Theorem 1.7. G is not $(k,d)^-$-decomposable by Lemma 3.2.8.
By Lemma 3.2.14, we have that G is the only subgraph of G with at least two vertices minimizing β_t, and $\beta_t(G) = 0$. Thus,

$$\gamma(G) = \frac{e(G)}{v(G)-1} = k + \frac{d+\frac{1}{t}}{d+k+1-\frac{k}{t}} \leq k + \frac{d}{d+k+1} + \epsilon$$

since $t \geq \frac{1+\frac{dk}{d+k+1}+k\epsilon}{\epsilon(d+k+1)}$. $\square$

Proof of Theorem 1.16. By Lemma 3.2.9, we have that $\beta(G) = -1$. By Lemma 3.2.14, G minimizes β and we have that there is no other (non-empty) subgraph of G that also minimizes β. Thus, $\beta(H) \geq 0$ for all $\varnothing \neq H \subsetneq G$.
Furthermore, by Theorem 3.2.7, we have that there is a decomposition of G into $k+1$ forests and thus, $\gamma(G) \leq k+1$. Finally, G is not $(k,d)^-$-decomposable by Lemma 3.2.8. $\square$

3.3 Notation and Preliminaries

In the following sections, we prove Theorems 1.13, 1.14, 1.15. Since the framework for these proofs is the same, we use this section to define terms and prove lemmas used in all of the three proofs.
We begin by showing that the theorems hold in the trivial cases $k = 0$ or $d = 0$ by proving this for the two strongest variations that imply all the others.

Theorem 3.3.1. *Conjecture 1.12 and Theorem 1.13 are true if* $k = 0$ *or* $d = 0$.

Proof. Let G be a loopless graph. First, let G be $(0,d)$-sparse where $d \in \mathbb{N}_0$. Thus, we have that $\frac{e(H)}{v(H)} \leq \frac{d}{d+1}$ for all $\varnothing \neq H \subseteq G$. This implies that G is acyclic and every component has at most d edges, which proves Conjecture 1.12 and Theorem 1.13 for $k = 0$.
Now, let G be $(k,0)$-sparse where $k \in \mathbb{N}_0$. By Lemma 3.2.5, we have that $\gamma(G) \leq k+1$, and thus, the theorem follows by Nash-Williams' Theorem. $\square$

Note that the Strong Nine Dragon Tree Conjecture (Conjecture 1.6) is even more trivial for $k = 0$ or $d = 0$: if $d = 0$, it is equivalent to Nash-Williams' Theorem. In the case where $k = 0$, the condition $\gamma(G) \leq \frac{d}{d+1}$ holds only if $e(G) = 0$ since the graph with two vertices connected by an edge is a spanning tree and thus has fractional arboricity 1.
Because of Theorem 3.3.1, we will assume $k, d \in \mathbb{N}_{\geq 1}$ in the following sections.

Lemma 3.3.2 ([27]). *A vertex-minimal counterexample graph* G *of Theorem 1.13, 1.14 or 1.15 has a decomposition into* $k+1$ *forests, where* k *of them are spanning trees.*

Proof. Let G be a vertex-minimal counterexample graph of Theorem X, where $X \in \{1.13, 1.14, 1.15\}$. Let $\mathcal{T} = (T_1, \ldots, T_k, F)$ be a maximal k-colouring of G. Let T be a maximal component tree of $\mathcal{T}$ and suppose that the lemma is not true. Thus, the root of T is not a leaf. Let $l_1, \ldots, l_n$ be the leaves of T. By Corollary 3.1.2, we have that $E(F) \subseteq \bigcup_{i=1}^{n} E(G[X_{l_i}])$. Since G is a vertex-minimal counterexample to Theorem X, there are decompositions $\mathcal{T}_1, \ldots, \mathcal{T}_n$ for $G[X_{l_1}], \ldots, G[X_{l_n}]$, respectively, each of which satisfies Theorem X.

We define the decomposition $(T_1', \ldots, T_k', F')$ of G such that $E(F')$ is the union of the edges of the $(k+1)$-th subgraphs of $\mathcal{T}_1, \ldots, \mathcal{T}_n$ and for $i \in \{1, \ldots, k\}$, let $E(T_i')$ be the union of the edges of the i-th forests of $\mathcal{T}_1, \ldots, \mathcal{T}_n$ and additionally contain all edges of T_i that are not contained in $E(G[X_{l_1}]) \cup \ldots \cup E(G[X_{l_n}])$. It is easy to verify that $(T_1', \ldots, T_k', F')$ also satisfies Theorem X, which is a contradiction. $\square$

Note that G does not contain loops, as Theorems 1.13, 1.14, 1.15 also do not allow loops.

For each of the Theorems 1.13, 1.14, 1.15, we will define a *residue function* ρ that maps the $(k+1)$-th "constrained" subgraph of a decomposition to a value indicating how close the subgraph is to satisfying the respective theorem. The smaller the value, the closer the constrained subgraph is to satisfying the theorem. For example, in the case of proving Theorem 1.14, a forest with exactly one component K having more than d edges will have a smaller residue function value than a forest with two components K_1, K_2 with $e(K) \leq e(K_1), e(K_2)$.

Among all decompositions of G into k spanning trees and a graph $\hat{F}$, we choose one in which $\hat{F}$ minimizes ρ. We call this minimal value ρ^*. For each of the Theorems 1.13, 1.14, 1.15, we will carefully choose a vertex $r \in V(G)$ that lies in a "bad" component of $\hat{F}$, i.e., a component with too many edges, or containing a vertex with too high degree, or having too many edges in cycles. r will later be a root vertex for different kinds of trees that we will construct from G.

For each Theorem 1.13, 1.14, 1.15, we will also define what a *root component* is. This subgraph of G will always be the component of the constrained subgraph containing r. It will also always be a bad component.

We only want to consider decompositions of G in which the constrained subgraph contains a root component. Thus, we define $\mathcal{F}$ to be the set of decompositions $(T_1, \ldots, T_k, F)$ of G such that $T_1, \ldots, T_k$ are directed spanning trees of G where the arcs of $T_1, \ldots, T_k$ are directed towards r and F is an undirected graph containing a root component. Furthermore, for Theorems 1.14 and 1.15, we require F to be a forest (and we require $\hat{F}$ to be a forest as well). We let $\mathcal{F}^* \subseteq \mathcal{F}$ be the set of decompositions $(T_1, \ldots, T_k, F) \in \mathcal{F}$ such that $\rho(F) = \rho^*$.

Let $\mathcal{T} = (T_1, \ldots, T_k, F) \in \mathcal{F}$. We say that the arcs of $T_1, \ldots, T_k$ are *blue*, and the (undirected) edges of F are *red*. We define $E(\mathcal{T}) := E(T_1) \cup \cdots \cup E(T_k) \cup E(F)$. Furthermore, we let $\mathcal{R}(\mathcal{T}) := F$, and for any $b \in \{1, \ldots, k\}$, we let $\mathcal{B}_b(\mathcal{T}) := T_b$. For any induced subgraph H of $(V(G), E(\mathcal{T}))$, we call the connected components of $\mathcal{R}(\mathcal{T})[V(H)]$ *red components of* H and let $E_r(H)$ denote the set of red edges of H, as well as $e_r(H) := |E_r(H)|$. Furthermore, a *blue (directed) path in* $\mathcal{T}$ is a directed path in which all edges are from $\bigcup_{b\in\{1,\ldots,k\}} E(\mathcal{B}_b(\mathcal{T}))$.

The *exploration subgraph* $H_{\mathcal{T}}$ of $\mathcal{T} \in \mathcal{F}$ is the subgraph of the mixed graph $(V(G), E(\mathcal{T}))$ that is induced by the vertex set consisting of all vertices v for which there is a sequence of vertices $r = x_1, \ldots, x_l = v$ such that for all $i \in \{1, \ldots, l-1\}$ we have that $(x_i, x_{i+1}) \in \bigcup_{b\in\{1,\ldots,k\}} E(\mathcal{B}_b(\mathcal{T}))$ or $x_i x_{i+1} \in E(\mathcal{R}(\mathcal{T}))$.

We now want to examine the density of the red subgraph of an exploration subgraph. For a graph H, let

$$\hat{\beta}(H) := d \cdot v(H) - (d + k + 1)e(H).$$

Lemma 3.3.3. *Let $H := (V(H_{\mathcal{T}}),\ E_r(H_{\mathcal{T}}))$. Then $\hat{\beta}(H) \geq -k(d+1)$.*

Proof. Note that every vertex of $V(H) - r$ has exactly k outgoing blue arcs, and the heads of these arcs are again in H by its definition. r, on the other hand, does not have any outgoing blue arcs. Thus, $e(H) = k(v(H) - 1) + e_r(H)$. We have that

$$\begin{aligned}
0 &\leq \beta(H) \\
&= (d+k)(k+1)v(H) - e(H)(d+k+1) - k^2 \\
&= \big(k(d+k+1) + d\big)v(H) - e_r(H)(d+k+1) - k\big(v(H) - 1\big)(d+k+1) - k^2 \\
&= d \cdot v(H) - e_r(H)(d+k+1) + k(d+k+1) - k^2 \\
&= \hat{\beta}(H) + k(d+1).
\end{aligned}$$

□

Now we turn our focus to the notion of legal orders, which are orderings of the red components of an exploration subgraph that loosely tell us in what order we should augment the decomposition.

Let $\mathcal{T} \in \mathcal{F}$, and let $\sigma = (R_1, \ldots, R_t)$, $t \in \mathbb{N}_{\geq 1}$, be a sequence of all red components in $H_{\mathcal{T}}$. We say σ is a *legal order* for $\mathcal{T}$ if R_1 is a root component, and furthermore, for each $j \in \{2,\dots,t\}$, there is an $i_j < j$ such that there is a blue arc (x_j, y_j) with $x_j \in V(R_{i_j})$ and $y_j \in V(R_j)$.

Note that we are generally not interested in the number of components in the legal order. Thus, when defining a legal order $\sigma = (R_1,\dots,R_t)$, it is understood that t is a new variable that is integral and positive.

We will compare legal orders for possibly different decompositions of $\mathcal{F}$ using lexicographic ordering, as defined in Section 2.2: let $\mathcal{T}, \mathcal{T}' \in \mathcal{F}$ and suppose that $\sigma = (R_1,\dots,R_t)$ and $\sigma' = (R'_1,\dots,R'_{t'})$ are legal orders for $\mathcal{T}$ and $\mathcal{T}'$, respectively. We write $\sigma \leq \sigma'$ if $(e(R_1),\dots,e(R_t)) \leq (e(R'_1),\dots,e(R'_{t'}))$.

For any $\{(x_j, y_j) \mid j \in \{2,\dots,t\}\}$ chosen as described above in the definition of legal orders, we obtain an *auxiliary tree of* $\mathcal{T}$ *and* σ from $\mathcal{R}(\mathcal{T})$ by removing a minimal number of edges until the graph is acyclic, and then adding the arcs $\{(x_j, y_j) \mid j \in \{2,\dots,t\}\}$. Note that an auxiliary tree is, in fact, a tree.

Let $\mathrm{Aux}(\mathcal{T},\sigma)$ denote the set of auxiliary trees of $\mathcal{T}$ and σ. We always consider auxiliary trees to be rooted at r. Note that in the blue spanning trees of decompositions of $\mathcal{F}$, the arcs are directed towards r while in an auxiliary tree, blue arcs are directed away from the root r. Let $W \in \mathrm{Aux}(\mathcal{T},\sigma)$ and $j \in \{2,\dots,t\}$. Let $\eta_j(W)$ and $\eta_{R_j}(W)$ both denote the blue arc of W whose head is in R_j. Furthermore, we denote this head vertex by $w_j(W)$ or $w_{R_j}(W)$.

Suppose $\sigma = (R_1,\dots,R_t)$ is a legal order for $\mathcal{T} \in \mathcal{F}$. If $v \in V(R_j)$, we write $i_\sigma(v) := j$. For $U \subseteq V(H_{\mathcal{T}})$, we define $i_\sigma(U) := \min_{u \in U} i_\sigma(v)$ (in particular, $i_\sigma(\varnothing) = \infty$) and for any subgraph $H \subseteq H_{\mathcal{T}}$ we let $i_\sigma(H) := i_\sigma(V(H))$. Moreover, for $\mathcal{T}, \mathcal{T}' \in \mathcal{F}$, let $\Delta(\mathcal{T},\mathcal{T}')$ denote the set of vertices v for which there is a blue arc $(v,u) \in E(\mathcal{B}_b(\mathcal{T}))$ for some $b \in \{1,\dots,k\}$, but this arc is not contained in $E(\mathcal{B}_b(\mathcal{T}'))$.

With this, we are in position to fully define our counterexample decomposition. As already outlined, G is a (loopless) vertex-minimal counterexample graph to Theorem 1.13, 1.14 or 1.15. Furthermore, we pick a legal order $\sigma^* = (R^*_1,\dots,R^*_{t^*})$ for a decomposition $\mathcal{T}^* \in \mathcal{F}^*$ such that there is no legal order σ with $\sigma < \sigma^*$ for any $\mathcal{T} \in \mathcal{F}^*$. We also let $R^* := R^*_1$ and fix an auxiliary tree $W^* \in \mathrm{Aux}(\mathcal{T}^*,\sigma^*)$. We fix these definitions for the rest of the chapter.

Let $j \in \{2,\dots,t^*\}$, $(x,y) = \eta_j(W^*)$, and let K_x and K_y be the components in $\mathcal{R}(\mathcal{T}^*)$ of x and y, respectively. We say that K_x is the parent of K_y. On the other hand, we call K_y a child of K_x (that is generated by (x,y)). Note that by this definition, every red component of $H_{\mathcal{T}^*}$, except for R^*, has a unique parent component.

Furthermore, note that the definition of children and parents also makes sense for decompositions and legal orders other than $\mathcal{T}^*$ and σ^*. However, we will not need these terms for other contexts and therefore, they always refer to $\mathcal{T}^*$ and σ^*.

For the rest of the chapter, let $\ell := \lfloor \frac{d-1}{k+1} \rfloor = \lceil \frac{d}{k+1} \rceil - 1$. We call a graph K *small* if K is a tree and $e(K) \leq \ell$. Note that $e(K) \leq \ell$ if and only if $e(K) < \frac{d}{k+1}$, which yields the following observation.

Observation 3.3.4. *A connected graph K is small if and only if $\hat{\beta}(K) > 0$.*

Let $\mathcal{K}$ denote the set of non-small red components of $H_{\mathcal{T}^*}$.

3.3.1 Special Paths

We will now review the special path augmentation from [27], which has already been discussed in the introduction.

Let $\sigma = (R_1, \dots, R_t)$ be a legal order for $\mathcal{T} \in \mathcal{F}$ and let $W \in \mathrm{Aux}(\mathcal{T}, \sigma)$. We call a blue directed path $P = [v_0, v_1, \dots, v_l] \subseteq \bigcup_{b \in \{1,\dots,k\}} \mathcal{B}_b(\mathcal{T})$ that is contained in $H_{\mathcal{T}}$ *special with respect to $\mathcal{T}, \sigma, W$ and* (v_{l-1}, v_l) if v_{l-1} and v_l are in different components of $\mathcal{R}(\mathcal{T})$, $i_\sigma(v_l) > i_\sigma(v_0)$, and furthermore v_0 needs to be an ancestor of v_{l-1} in W if both vertices are in the same component of $\mathcal{R}(\mathcal{T})$.

For two special paths $P = [v_0, v_1, \dots, v_l]$ and $P' = [v'_0, v'_1, \dots, v'_{l'}]$ with respect to $\mathcal{T}, \sigma, W$ and (v_{l-1}, v_l), we write $P \leq P'$ if $i_\sigma(v_0) < i_\sigma(v'_0)$, or if $i_\sigma(v_0) = i_\sigma(v'_0)$ and in W, v_0 is an ancestor (with respect to the root r) of v'_0.

We call a special path P with respect to $\mathcal{T}, \sigma, W$ and (x, y) *minimal* if there is no special path $P' \neq P$ with respect to $\mathcal{T}, \sigma, W$ and (x, y) such that $P' \leq P$.

Note that for every special path P' with respect to $\mathcal{T}, \sigma, W$ and (x, y), there exists a minimal special path P with respect to $\mathcal{T}, \sigma, W$ and (x, y) such that $P \leq P'$. Furthermore, note that if we have a minimal special path $P = [v_0, v_1, \dots, v_l]$ with respect to $\mathcal{T}, \sigma, W$ and (v_{l-1}, v_l), we have $v_0 \neq r$ because r has no outgoing blue arc by construction. Therefore, v_0 has a parent vertex in W, which we denote by v_{-1}. Note that the edge $v_{-1}v_0$ is red because of the minimality of P and since all blue arcs in W are directed away from r in the auxiliary tree.

The following lemma describes which modifications to the decomposition can be made if a minimal special path exists and how they change the legal order. An illustration of these modifications can be seen in Figure 3.3.1.

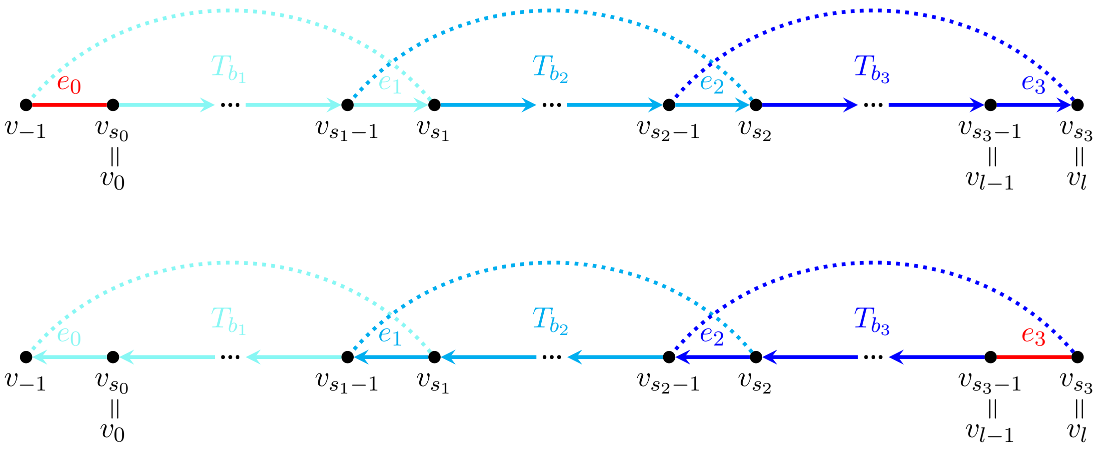

Fig. 3.3.1: An example of the augmentation of a minimal special path with three segments in the proof of Lemma 3.3.5.

Lemma 3.3.5 (cf. Lemma 2.4 from [27]). *Let* $\sigma = (R_1, \dots, R_t)$ *be a legal order for* $\mathcal{T} = (T_1, \dots, T_k, F) \in \mathcal{F}$ *and let* $W \in \operatorname{Aux}(\mathcal{T}, \sigma)$*. Furthermore, let* $P = [v_0, v_1, \dots, v_l]$ *be a minimal special path with respect to* $\mathcal{T}, \sigma, W$ *and* (v_{l-1}, v_l)*. Let* $i_0 := i_\sigma(v_0)$*. Then there is a decomposition* $\mathcal{T}' = (T'_1, \dots, T'_k, F')$ *of* G *such that* $T'_1, \dots, T'_k$ *are spanning trees rooted at* r *whose arcs are directed to the respective parent vertex and we have that*

(1) $F' = \big(F + v_{l-1}v_l\big) - v_{-1}v_0$.

(2) $i_{\sigma^*}(\Delta(\mathcal{T}, \mathcal{T}')) = i_0$.

(3) If F' *contains a root component, then* $\mathcal{T}' \in \mathcal{F}$ *and there exists a legal order* $\sigma' = (R'_1, \dots, R'_{t'})$ *for* $\mathcal{T}'$ *with* $R'_j = R_j$ *for all* $j < i_0$ *and* $e(R'_{i_0}) < e(R_{i_0})$*, where* R'_{i_0} *is the component of* v_{-1} *in* F'*. Thus,* $\sigma' < \sigma$*. Furthermore, there is an auxiliary tree* $W' \in \operatorname{Aux}(\mathcal{T}', \sigma')$ *such that* $\eta_j(W') = \eta_j(W)$ *for all* $j \in \{2, \dots, i_0\}$*.*

Proof. We define the *segments* of a special path to be the maximal subpaths that are part of one of the k spanning trees. Let P consist of the segments $P_1, \dots, P_n$ where $P_j = [v_{s_{(j-1)}}, \dots, v_{s_j}] \subseteq T_{b_j}$. Furthermore, let the *label of* P be $(s_0, \dots, s_{n-1})$. We choose P such that it is a minimal special path with respect to $\mathcal{T}, \sigma, W$ and (v_{l-1}, v_l) starting at v_0 and minimizing the number of segments and subject to that, minimizing the label with respect to lexicographic ordering.

Claim 3.3.5.1. *For every $j \in \{1, \dots, n\}$, $v_{s_{(j-1)}}$ is the first vertex on P which has a directed path in T_{b_j} to $v_{(s_j-1)}$.*

Proof. Suppose to the contrary that there is a vertex v_i with $i < s_{(j-1)}$ that has a path $\bar{P}$ in T_{b_j} to $v_{(s_j-1)}$. By replacing the subpath from $v_{s_{(j-1)}}$ to $v_{(s_j-1)}$ in P by $\bar{P}$, we obtain a minimal special path P' with respect to $\mathcal{T}, \sigma, W$ and (v_{l-1}, v_l) starting at v_0. If $i < s_{(j-2)}$, we have that P' has a smaller number of segments than P, which is a contradiction. If $i \geq s_{(j-2)}$, we have that the label of P' starts with $(s_0, \dots, s_{(j-2)}, v_i, s_j, \dots, s_n)$, which is lexicographically smaller than the label of P, which is again a contradiction. *(End of proof of the claim)* ■

Claim 3.3.5.2. *The unique path in T_{b_j} (in an undirected sense) between $v_{s_{(j-1)}-1}$ and $v_{s_{(j-1)}}$ contains P_j.*

Proof. Suppose to the contrary that there is a path (in an undirected sense) from $v_{s_{(j-1)}-1}$ to a vertex v of $V(P) - v_{(s_j-1)}$ not containing $v_{(s_j-1)}$. As every vertex of $V(P) - v_{(s_j-1)}$ has an outgoing arc in P_j and no vertex has two outgoing arcs in T_{b_j}, we have that there must be a directed path from $v_{s_{(j-1)}-1}$ to v, which is a contradiction to Claim 3.3.5.1. *(End of proof of the claim)* ■

For $j \in \{0, \dots, n\}$, let $e_j = v_{(s_j-1)}v_{s_j}$. Note that $e_0 = v_{-1}v_0$ and $e_n = v_{l-1}v_l$. We modify the decomposition $(T_1, \dots, T_k, F)$ in the following way: we let $F' := (F + e_0) - e_n$ and modify $T_1, \dots, T_k$ in the following way to finally obtain $T'_1, \dots, T'_k$ (first, we will describe the edge exchanges ignoring orientations and after that we explain how the added edges have to be oriented): for $j \in \{1, \dots, n\}$ add e_{j-1} and remove e_j from T_{b_j}. Furthermore, orient every edge that is in P when ignoring orientations such that there is a directed blue path $[v_{l-1}, \dots, v_{-1}]$ in the resulting decomposition. By Claim 3.3.5.2 this procedure results in $T'_1, \dots, T'_k$ being spanning trees since in every exchange, we remove an edge from the cycle which was created by the edge that was added. Furthermore, as every vertex of P (still) has exactly one outgoing arc in every tree $T'_1, \dots, T'_k$, we have that all arcs in these trees are directed towards the root r. Note that (1) holds for F'. Since P is a minimal special path, we have that $i_{\sigma^*}(i_0) \leq i_{\sigma^*}(v_i)$ for every $i \in \{0, \dots, l\}$. Thus, (2) holds. Since v_{-1} is the parent of v_0 in W, we have that v_{-1} is in the component of $w_{i_0}(W)$ in $R_{i_0} - v_{-1}v_0$. Thus, σ' and W' exist as described in (3), and (3) follows. □

3.3.2 Simple Edge Exchanges Between Forests

In the rest of the section, we formalize the obvious condition in which we can exchange a blue arc with a red edge. In certain cases, we also need to fix the orientation of other blue arcs if we perform such an exchange.

Definition 3.3.6. Let $\mathcal{T} \in \mathcal{F}$, let $e \in E(\mathcal{B}_b(\mathcal{T}))$ for some $b \in \{1, \dots, k\}$ and $e' \in E(\mathcal{R}(\mathcal{T}))$. If $(\mathcal{B}_b(\mathcal{T}) - e) + e'$ is a spanning tree and $(F - e') + e$ does not contain a cycle containing e (ignoring orientations), we say that e' can be exchanged with e, and say that $e \leftrightarrow e'$ holds in $\mathcal{T}$.

The next lemma is obvious and we omit the proof, but it usefully characterizes when $e \leftrightarrow e'$. However, we refer the reader to Figure 3.3.2 for an illustration.

Lemma and Definition 3.3.7. *Let $u \in V(G) - r$, u' be the parent vertex of u in $\mathcal{B}_b(\mathcal{T})$ for some $b \in \{1, \dots, k\}$ and $e = v_1v_2 \in E(\mathcal{R}(\mathcal{T}))$ such that $(\mathcal{R}(\mathcal{T}) + uu') - e$ does not contain a cycle containing uu'.*
Then, the following are equivalent:

(1) $(u, u') \leftrightarrow e$ holds in $\mathcal{T}$.

(2) The edge (u, u') lies in the unique cycle (ignoring orientations) of $\mathcal{B}_b(\mathcal{T}) + e$.

(3) Up to relabelling v_1 as v_2, v_1 is a descendant of u in $\mathcal{B}_b(\mathcal{T})$ and v_2 is not.

Furthermore, if these conditions are met, then after exchanging (u, u') and e between $\mathcal{B}_b(\mathcal{T})$ and $\mathcal{R}(\mathcal{T})$, orienting e towards v_2, removing the orientation of e and reorienting the path $P^{\mathcal{B}_b(\mathcal{T})}(v_1, u)$, the resulting decomposition is again in $\mathcal{F}$, and we say we obtain the resulting decomposition from $\mathcal{T}$ by performing $(u, u') \leftrightarrow e$.

The next lemma shows a case in which we can exchange edges.

Lemma 3.3.8. *Let $b \in \{1, \dots, k\}$. Let K be a component of $\mathcal{R}(\mathcal{T})$. Let $u, v \in V(K)$ such that v is not a descendant of u in $\mathcal{B}_b(\mathcal{T}^*)$. Let $[v_1, \dots, v_n]$ be a path from u to v in K. Let $(u, u') \in \mathcal{B}_b(\mathcal{T}^*)$.*
Then there is a $j \in \{1, \dots, n-1\}$ such that $(u, u') \leftrightarrow v_jv_{j+1}$, where v_j is a descendant of u in $\mathcal{B}_b(\mathcal{T}^*)$.

Proof. Let $j \in \{1, \dots, n\}$ be minimal such that for all $j' \in \{1, \dots, j\}$, $v_{j'}$ is a descendant of u in $\mathcal{B}_b(\mathcal{T}^*)$. Note that j exists and $j < n$ since u is a descendant of u in $\mathcal{B}_b(\mathcal{T}^*)$ and v is not.

☐

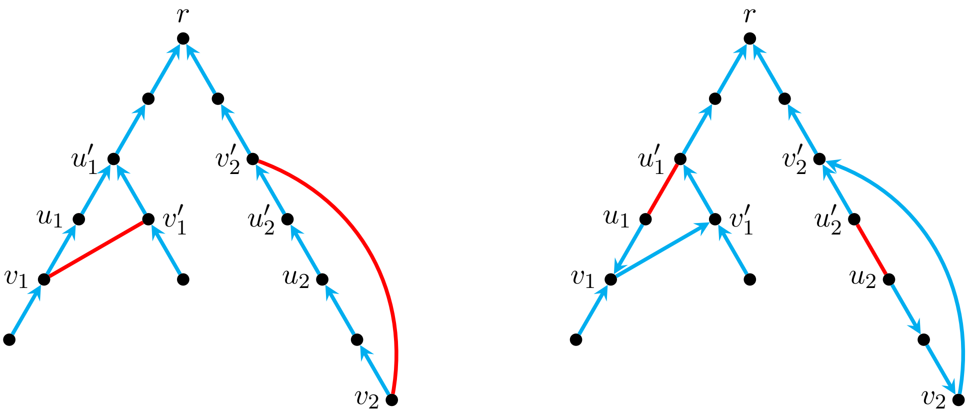

Fig. 3.3.2: An example where we have $(u_1, u_1') \leftrightarrow v_1 v_1'$ and $(u_2, u_2') \leftrightarrow v_2 v_2'$.

3.4 The Nine Dragon Tree Theorem for (k, d)-Sparse Graphs

In this section, we prove Theorem 1.13, building on Section 3.3.

Theorem 1.13. *Let $k, d \in \mathbb{N}_0$, and let G be a (k, d)-sparse graph. Then G has a decomposition into k forests and a graph F, where in F, every vertex has degree at most d and in every component of F, the number of edges on cycles is at most d. Furthermore, if $\gamma(G) \leq k + 1$, then G is (k, d)-decomposable.*

We start by defining a few objects for this section which were introduced in Section 3.3 in a more abstract sense.
We begin by defining the residue function ρ for this section.
For a graph $F \subseteq G$, let $\tau(F) := (\tau_{e(G)}(F), \dots, \tau_1(F))$, where $\tau_i(F)$ is the number of components K of F whose number of edges contained in a cycle of K is exactly i.
For a graph $F \subseteq G$, let $\delta(F) := (\delta_{e(G)}(F), \dots, \delta_{d+1}(F))$, where $\delta_i(F)$ is the number of vertices v of F with $\deg_F(v) = i$.
Let $\rho(F) := (\tau(F), \delta(F))$. We compare the values of τ and δ using lexicographic

ordering, and we also use lexicographic ordering to compare values of $\rho(F)$.
With the help of ρ, we obtain the graph $\hat{F}$, as described in Section 3.3. Next, we discuss how r is picked out of $V(G)$ and how a root component is defined in this section.
We call a graph H *very cyclic* if it has more than d edges contained in a cycle of H. If $\hat{F}$ has a very cyclic component R, then we pick r from one of the cycles of R. In this case, a root component is a cyclic graph containing r.
Otherwise, if $\hat{F}$ does not have a cyclic component, we choose r with maximum degree in $\hat{F}$. Note that in this case we have $\deg_{\hat{F}}(r) > d$, and furthermore, in every decomposition $\mathcal{T} \in \mathcal{F}^*$, $\mathcal{R}(\mathcal{T})$ does not contain a very cyclic component.

3.4.1 Structure of the Exploration Subgraph

In this subsection, we will use the augmentation techniques presented in Section 3.3 to show some properties of the exploration subgraph. We will use these properties in the next subsection to show the contradiction that the exploration subgraph of our counterexample graph is too dense.
We start by showing the consequences of the definition of r, ρ and root components in this section are for the special paths lemma (Lemma 3.3.5).

Corollary 3.4.1 (cf. Lemma 2.4 from [27]). *Let $\sigma = (R_1, \dots, R_t)$ be a legal order for $\mathcal{T} \in \mathcal{F}^*$ and let $W \in \mathrm{Aux}(\mathcal{T}, \sigma)$. Furthermore, let $P = [v_0, v_1, \dots, v_l]$ be a minimal special path with respect to $\mathcal{T}, \sigma, W$ and (v_{l-1}, v_l) such that $\rho(\mathcal{R}(\mathcal{T}) + v_{l-1}v_l) = \rho^*$. Let $i_0 := i_\sigma(v_0)$.*
Then there is a decomposition $\mathcal{T}'$ of G such that

(1′) $\mathcal{T}' \in \mathcal{F}^*$.

(2′) $\mathcal{R}(\mathcal{T}') = \big(\mathcal{R}(\mathcal{T}) + v_{l-1}v_l\big) - v_{-1}v_0$.

(3′) *$v_{-1}v_0$ is not in a cycle of $\mathcal{R}(\mathcal{T})$.*

(4′) *There exists a legal order $\sigma' = (R'_1, \dots, R'_{t'})$ for $\mathcal{T}'$ with $R'_j = R_j$ for all $j < i_0$ and $e(R'_{i_0}) < e(R_{i_0})$, where R'_{i_0} is the component of v_{-1} in $\mathcal{R}(\mathcal{T}')$. Thus, $\sigma' < \sigma$. Furthermore, there is an auxiliary tree $W' \in \mathrm{Aux}(\mathcal{T}', \sigma')$ such that $\eta_j(W') = \eta_j(W)$ for all $j \in \{2, \dots, i_0\}$.*

Proof. From Lemma 3.3.5, we obtain the decomposition $\mathcal{T}' = (T_1', \dots, T_k', F')$, and show that the corollary holds for $\mathcal{T}'$.

Note that (2′) trivially holds. Furthermore, the component of r in F' is a root component, or otherwise we had that $v_{-1}v_0 \in R_1$, and thus $\rho(F') < \rho^*$, which is a contradiction.

Thus, (3) from Lemma 3.3.5 holds for $\mathcal{T}'$, which implies (1′) and (4′). Finally, (3′) holds since otherwise, we had $\tau(\mathcal{R}(\mathcal{T}')) < \tau(\mathcal{R}(\mathcal{T}^*))$. □

Lemma 3.4.2 (Corollary 2.5 from [27]). *Let K be a red component of $H_{\mathcal{T}^*}$, and let C be a small child of K generated by $(x, x') \in \mathcal{B}_b(\mathcal{T}^*)$. Then $\deg_{\mathcal{R}(\mathcal{T}^*)}(x) \geq d$.*

Proof. Suppose to the contrary that $\deg_{\mathcal{R}(\mathcal{T}^*)}(x) < d$. Then $\rho(\mathcal{R}(\mathcal{T}^*) + xx') = \rho^*$ since $\deg_{\mathcal{R}(\mathcal{T}^*)}(x') \leq e(C) \leq \ell < d$. Let P be a minimal special path with respect to $\mathcal{T}^*, \sigma^*, W^*$ and (x, x'). By Corollary 3.4.1, we obtain a contradiction to the minimality of σ^*. □

Corollary 3.4.3. *Let K be a small red component of $H_{\mathcal{T}^*}$. Then K does not have small children.*

Let $b \in \{1, \dots, k\}$, and let $x, x' \in V(H_{\mathcal{T}^*})$. We call x *a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex (with respect to x')* if there is a red small child component in $H_{\mathcal{T}^*}$ generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$.

Next, we will show that in a cycle of K, there can be only one $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex.

Lemma 3.4.4. *Let K be a red component of $H_{\mathcal{T}^*}$, let $L \subseteq K$ be a cycle, and $b \in \{1, \dots, k\}$. Then, at most one vertex of L is a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex.*

Proof. Suppose that $x \in V(L)$ is a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex with respect to x', and $y \in V(L) - x$ is a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex with respect to y'. Without loss of generality, let y not be a descendant of x in $\mathcal{B}_b(\mathcal{T}^*)$. By Lemma 3.3.8, there is an edge uv on a path from x to y in K such that $(x, x') \leftrightarrow uv$. We obtain $\mathcal{T}$ from $\mathcal{T}^*$ by performing $(x, x') \leftrightarrow uv$. Note that $\tau(\mathcal{R}(\mathcal{T})) < \tau(\mathcal{R}(\mathcal{T}^*))$ since uv is contained in a cycle of K and the small child generated by (x, x') does not contain a cycle. But this is a contradiction to the minimality of $\mathcal{T}^*$. □

In general, we are not able to prevent that there are two $\mathcal{B}_b(\mathcal{T}^*)$-generating vertices in K. However, it will be enough to show that they cannot be neighbours in K.

Lemma 3.4.5 (Lemma 2.6 from [27]). *Let K be a red component of $H_{\mathcal{T}^*}$, let $xy \in E(K)$, and let $b \in \{1,\dots,k\}$. Then, x and y cannot both be $\mathcal{B}_b(\mathcal{T}^*)$-generating vertices.*

Proof. Suppose to the contrary that x is a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex with respect to x', and y is a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex with respect to y'.
Without loss of generality, let y not be a descendant of x in $\mathcal{B}_b(\mathcal{T}^*)$. We obtain $\mathcal{T}$ from $\mathcal{T}^*$ by performing $(x, x') \leftrightarrow xy$. Note that the exchange only affects these two edges, and no further blue arcs are reoriented. We have that xy is not contained in a cycle by Lemma 3.4.4. Furthermore, note that $\deg_{\mathcal{R}(\mathcal{T})}(x) = \deg_{\mathcal{R}(\mathcal{T}^*)}(x)$, $\deg_{\mathcal{R}(\mathcal{T})}(x') \leq d$ and $\deg_{\mathcal{R}(\mathcal{T})}(y) = \deg_{\mathcal{R}(\mathcal{T}^*)}(y) - 1$. Thus, $\rho(\mathcal{R}(\mathcal{T})) = \rho^*$. Note that $x, y \neq r$ since r does not have outgoing blue arcs in $\mathcal{T}^*$. Thus, the component of r in $\mathcal{R}(\mathcal{T})$ is (still) a root component and $\mathcal{T} \in \mathcal{F}^*$. Let K_x and K_y be the components in $\mathcal{R}(\mathcal{T})$ of x and y, respectively. Furthermore, let $C_{x'}$ be the component of x' in $\mathcal{R}(\mathcal{T}^*)$.

Case 1: $K = R^*$ and $r \in V(K_y)$, or $K \neq R^*$ and $w_K(W^*) \in V(K_y)$:
Then there is a legal order $\sigma = (R_1, \dots, R_t)$ for $\mathcal{T}$ with $R_i = R_i^*$ for all $i \in \{1, \dots, i_{\sigma^*}(K) - 1\}$ and $R_{i_{\sigma^*}(K)} = K_y$. Thus, $\sigma < \sigma^*$, which is a contradiction.

Case 2: $K = R^*$ and $r \in V(K_x) \cap V(K)$, or $K \neq R^*$ and $w_K(W^*) \in V(K_x) \cap V(K)$:
Then there is a legal order $\sigma = (R_1, \dots, R_t)$ for $\mathcal{T}$ such that $R_i = R_i^*$ for all $i \in \{1, \dots, i_{\sigma^*}(K) - 1\}$, $R_{i_{\sigma^*}(K)} = K_x$ and $R_{i_{\sigma^*}(K)+1} = K_y$. Furthermore, there is $W \in \operatorname{Aux}(\mathcal{T}, \sigma)$ such that $\eta_j(W) = \eta_j(W^*)$ for all $j \in \{2, \dots, i_{\sigma^*}(K)\}$. Let $P = [v_0, \dots, v_l]$ be a minimal special path with respect to $\mathcal{T}, \sigma, W$ and (y, y'). Note that $P \leq [x, y, y']$ and thus, $i_\sigma(v_0) \leq i_\sigma(x)$. Let $i_0 := i_{\sigma^*}(v_0) = i_\sigma(v_0)$.
Note that $\deg_{\mathcal{R}(\mathcal{T}')}(y) = \deg_{\mathcal{R}(\mathcal{T}^*)}(y)$ and $\deg_{\mathcal{R}(\mathcal{T}')}(y') = \deg_{\mathcal{R}(\mathcal{T}^*)}(y') + 1 \leq d$. Thus, $\rho(\mathcal{R}(\mathcal{T}) + yy') = \rho^*$. We apply Corollary 3.4.1 to $\mathcal{T}, \sigma, W, P$ and obtain a legal order $\sigma' = (R'_1, \dots, R'_{t'})$ for a decomposition $\mathcal{T}' \in \mathcal{F}^*$ such that $R'_i = R_i = R_i^*$ for all $i \in \{1, \dots, i_0 - 1\}$ and $e(R'_{i_0}) < e(R_{i_0})$. Furthermore, we obtain $W' \in \operatorname{Aux}(\mathcal{T}', \sigma')$ such that $\eta_j(W') = \eta_j(W)$ for all $j \in \{2, \dots, i_0\}$, and $v_0 v_{-1}$ is not contained in a cycle of K.
If $i_0 < i_\sigma(K_x)$, we have that $\sigma' < \sigma^*$, which is a contradiction.
Thus, $i_0 = i_\sigma(K_x) = i_{\sigma^*}(K)$. By Lemma 3.3.5, we have that R'_{i_0} is the component of v_{-1} and $w_{K_x}(W) = w_K(W^*)$ in $K_x - v_{-1}v_0$, and thus R'_{i_0} does not contain a vertex of $V(C_{x'}) + x$. Hence, $e(R'_{i_0}) < e(K_x \cap K) < e(K) = e(R^*_{i_0})$. Thus, again we have $\sigma' < \sigma^*$, which is a contradiction. ☐

Finally, we show two augmentations for root components. They are necessary so that Theorem 1.13 holds for (k,d)-sparse graphs (with $\gamma(G) \leq k+1$) and not just for graphs with $\gamma(G) \leq k + \frac{d}{d+k+1}$.

Lemma 3.4.6. *Let* $\deg_{R^*}(r) > d$*,* $b \in \{1,\dots,k\}$*, and let* $x \in V(R^*)$ *be a* $\mathcal{B}_b(\mathcal{T}^*)$*-generating vertex. Then* $x \neq r$*, and* $xr \notin E(R^*)$*.*

Proof. Let $x \in V(R^*)$ be a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex with respect to x'.
We have that $x \neq r$ since r has no outgoing edges in $\mathcal{B}_b(\mathcal{T}^*)$.
Now suppose that $xr \in E(R^*)$. We obtain $\mathcal{T}$ from $\mathcal{T}^*$ by performing $(x,x') \leftrightarrow xr$.
We have that $\tau(\mathcal{R}(\mathcal{T})) \leq \tau(\mathcal{R}(\mathcal{T}^*))$ since the red component of x' does not contain a cycle.
Furthermore, we have that $\deg_{\mathcal{R}(\mathcal{T})}(x') = \deg_{\mathcal{R}(\mathcal{T}^*)}(x')+1 \leq d$, $\deg_{\mathcal{R}(\mathcal{T})}(x) = \deg_{\mathcal{R}(\mathcal{T}^*)}(x)$ and $\deg_{\mathcal{R}(\mathcal{T})}(r) = \deg_{\mathcal{R}(\mathcal{T}^*)}(r) - 1$. But this implies that $\rho(\mathcal{R}(\mathcal{T})) < \rho(\mathcal{R}(\mathcal{T}^*))$, a contradiction. □

Lemma 3.4.7. *Let* r *be contained in a cycle of* R^**. Then there is no* $\mathcal{B}_b(\mathcal{T}^*)$*-generating vertex in a cycle of* R^**.*

Proof. Suppose to the contrary that there is a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex x with respect to x' such that x is contained in a cycle of R^*.
Note that $x \neq r$ since r does not have outgoing blue arcs in $\mathcal{T}^*$.
Let $[v_1,\dots,v_n]$ be a path from x to r in R^*. By Lemma 3.3.8, we have that there is a $j \in \{1,\dots,n-1\}$ such that $(x,x') \leftrightarrow v_j v_{j+1}$.
We obtain $\mathcal{T}$ from $\mathcal{T}^*$ by performing $(x,x') \leftrightarrow v_j v_{j+1}$. Note that $v_j v_{j+1}$ is either contained in a cycle or x and r are in different components of $\mathcal{R}(\mathcal{T})$. In both cases we have that $\tau(\mathcal{R}(\mathcal{T})) < \tau(\mathcal{R}(\mathcal{T}^*))$, which is a contradiction to the minimality of $\rho(\mathcal{R}(\mathcal{T}^*))$. □

3.4.2 The Density Calculation

In this section we show that our exploration graph is not (k,d)-sparse by contradicting Lemma 3.3.3. For $x \in V(H_{\mathcal{T}^*})$ and its component K in $\mathcal{R}(\mathcal{T}^*)$, let $\mathcal{C}(x)$ be the set of small children of K that are generated by an arc whose tail is x. Furthermore, for any red subgraph $H \subseteq H_{\mathcal{T}^*}$, let $\mathcal{C}(H) := \mathcal{C}(V(H)) := \bigcup_{x \in V(H)} \mathcal{C}(x)$ and

$$\operatorname{Ext}(H) := \Big(V(H) \cup \bigcup_{C \in \mathcal{C}(H)} V(C),\ E(H) \cup \bigcup_{C \in \mathcal{C}(H)} E(C)\Big).$$

Now, observe the following partitioning of the red subgraph of the exploration subgraph, which is justified by Lemma 3.4.3 and the fact that R^* is not small.

Observation 3.4.8. $\big(V(H_{\mathcal{T}^*}),\ E_r(H_{\mathcal{T}^*})\big) = \dot{\bigcup}_{K\in\mathcal{K}} \operatorname{Ext}(K)$.

We aim to show that $\hat{\beta}(K) \leq 0$ for all $K \in \mathcal{K}$. We will use the same counting argument as in [27], but we extend it so that it can deal with cyclic components. Furthermore, we will show that $\hat{\beta}(R^*) < -k(d+1)$ in order to find a contradiction to Lemma 3.3.3.

We will do this by induction on the size of K. For a subgraph $L \subseteq K$ and a vertex $x \in V(L)$, we only want $\mathcal{C}(x)$ to be part of $\operatorname{Ext}(K)$ if all neighbours of x in K are in $V(L)$. Thus, for $L \subseteq \mathcal{R}(\mathcal{T}^*)$ and $x \in V(L)$, we define

$$\mathcal{C}_L(x) = \begin{cases} \mathcal{C}(x) & \text{if } \deg_L(x) = \deg_{\mathcal{R}(\mathcal{T}^*)}(x), \\ \varnothing & \text{otherwise.} \end{cases}$$

As before, for $L' \subseteq L$, we let $\mathcal{C}_L(L') := \mathcal{C}_L(V(L')) := \bigcup_{x\in V(L')} \mathcal{C}_L(x)$ and

$$\operatorname{Ext}_L(L') := \Big(V(L') \cup \bigcup_{C\in\mathcal{C}_L(L')} V(C),\ E(L') \cup \bigcup_{C\in\mathcal{C}_L(L')} E(C)\Big).$$

Note that for a red component K of $H_{\mathcal{T}^*}$, we have $\mathcal{C} = \mathcal{C}_K$ and $\operatorname{Ext} = \operatorname{Ext}_K$.

For better structure, we derive a tree $\bar{L}$ from L.

In order to construct $\bar{L}$, we define an equivalence relation on $V(L)$, where x and y are equivalent if and only if $x = y$ or if there is a cycle in L containing x and y. We denote the equivalence class of $x \in V(L)$ by $\bar{x}$.

We obtain $\bar{L}$ by removing the edges between vertices of the same equivalence class and then identifying the vertices of each equivalence class. Note that $\bar{L}$ is a tree (if $\bar{L}$ had a cycle, the vertices on the cycle would be in the same equivalence class and we would have contracted the cycle).

Note that $\mathcal{C}_L(\bar{v}) \leq k$ for every $\bar{v} \in V(\bar{L})$ by Lemma 3.4.4 and since every vertex has at most k outgoing blue arcs in $\mathcal{T}^*$.

Let $X_{\bar{L}}$ be the set of vertices $\bar{v}$ of $\bar{L}$ for which $\mathcal{C}_L(\bar{v}) \neq \varnothing$.

Before we come to the main induction lemma, we need the following lemma and corollary as preparation.

Lemma 3.4.9. *Let H be a connected graph. Then $\hat{\beta}(H) \leq d$.*

Proof.

$$\hat{\beta}(H) \leq d \cdot \big(e(H)+1\big) - (d+k+1)e(H) \leq d.$$

□

The following corollary follows directly from Lemma 3.4.9 and Lemma 3.4.2.

Corollary 3.4.10. *Let $L \subseteq K$ be connected such that for every $v \in V(L)$, we have $\deg_L(v) < d$. Then $\hat{\beta}(\mathrm{Ext}_L(L)) \leq d$.*

Lemma 3.4.11 (cf. Claim 2.8 from [27]). *Let L be a connected subgraph of a red component of $H_{\mathcal{T}^*}$ containing a vertex $v \in V(L)$ with $\deg_L(v) \geq d$.*

$$\hat{\beta}(\mathrm{Ext}_L(L)) \leq 0.$$

Proof. We prove the lemma by induction on $|X_{\bar{L}}|$.
As the base case, suppose that $|X_{\bar{L}}| \leq 1$. Note that $e(L) \geq d$ since L contains a vertex of degree at least d. Furthermore, $|\mathcal{C}_L(L)| \leq k$. We have that

$$\begin{aligned}
\hat{\beta}(\mathrm{Ext}_L(L)) &= d\Big(v(L) + \sum_{C \in \mathcal{C}_L(L)} v(C)\Big) - (d+k+1)\Big(e(L) + \sum_{C \in \mathcal{C}_L(L)} e(C)\Big) \\
&\leq d\Big(e(L) + 1 + \sum_{C \in \mathcal{C}_L(L)} \big(e(C)+1\big)\Big) - (d+k+1)\Big(e(L) + \sum_{C \in \mathcal{C}_L(L)} e(C)\Big) \\
&\leq d\big(1 + |\mathcal{C}_L(L)|\big) - (k+1)e(L) \\
&\leq d(1+k) - (k+1)d \\
&= 0.
\end{aligned}$$

Next, suppose that $|X_{\bar{L}}| \geq 2$.
We consider $\bar{L}$ to be rooted at any vertex $\bar{s}$ of $X_{\bar{L}}$.
Let $\bar{x} \in X_{\bar{L}}$ such that there is no $\bar{x}' \in X_{\bar{L}}$ with $\bar{x}' \neq \bar{x}$ that is a descendant of $\bar{x}$ in $\bar{L}$.
Let $\bar{y}$ be the parent of $\bar{x}$ in $\bar{L}$. Without loss of generality, we let x and y be the vertices such that $xy \in E(L)$.

Case 1: $\mathcal{C}_L(\bar{x}) \neq \mathcal{C}_L(x)$**:**
Let $\bar{M}_x$ be the subtree of $\bar{L}$ with root $\bar{x}$. Let $\bar{M}$ be the subtree of $\bar{L}$ obtained by taking the component of $\bar{y}$ in $\bar{L} - \bar{x}$ and adding vertex $\bar{x}$ and edge $\bar{x}\bar{y}$.
Note that $E(\bar{L}) = E(\bar{M}) \mathbin{\dot{\cup}} E(\bar{M}_x)$ and $V(\bar{M}) \cap V(\bar{M}_x) = \{\bar{x}\}$.
Let $M \subseteq L$ be the subgraph corresponding to $\bar{M}$, and $M_x \subseteq L$ be the subgraph

corresponding to $\bar{M}_x$ such that $E(L) = E(M) \dot\cup E(M_x)$ and $V(M) \cap V(M_x) = \{x\}$.
By the choice of $\bar{x}$, we have that $\mathcal{C}_L(M_x) = \mathcal{C}_L(\bar{x})$. Since $\mathcal{C}_L(\bar{x}) \neq \mathcal{C}_L(x)$, we have that there is a vertex $x' \in \bar{x}$ with $x' \neq x$ and $\deg_{M_x}(x') \geq d$ by Lemma 3.4.2. Thus, $e(M_x) \geq d$, and $v(M_x) \leq e(M_x)$ since $M_x[\bar{x}]$ contains a cycle.
Furthermore, $\bar{s} \in V(\bar{M})$ and for every vertex $s' \in \bar{s}$, we have that $\deg_M(s') = \deg_L(s')$, and thus we can apply the induction hypothesis to M. We have that

$$\begin{aligned}
\hat{\beta}(\mathrm{Ext}_L(L)) &= \hat{\beta}(\mathrm{Ext}_M(M)) + d\Big(v(M_x) - 1 + \sum_{C \in \mathcal{C}_L(M_x)} v(C)\Big) - (d+k+1)\Big(e(M_x) + \sum_{C \in \mathcal{C}_L(M_x)} e(C)\Big) \\
&\leq d\Big(e(M_x) - 1 + \sum_{C \in \mathcal{C}_L(\bar{x})} (e(C)+1)\Big) - (d+k+1)\Big(e(M_x) + \sum_{C \in \mathcal{C}_L(\bar{x})} e(C)\Big) \\
&\leq d\big(-1 + |\mathcal{C}_L(\bar{x})|\big) - (k+1)e(M_x) \\
&\leq d(-1+k) - (k+1)d \\
&= -2d.
\end{aligned}$$

Case 2: $\mathcal{C}_L(\bar{x}) = \mathcal{C}_L(x)$**:**
We let $\bar{N} := \bar{L} \setminus V(\bar{M}_x)$ and $\bar{N}_x := \big(V(\bar{M}_x) + \bar{y},\ E(\bar{M}_x) + \bar{x}\bar{y}\big)$. Note that $E(\bar{L}) = E(\bar{N}) \dot\cup E(\bar{N}_x)$ and $V(\bar{N}) \cap V(\bar{N}_x) = \{\bar{y}\}$.
Let $N \subseteq L$ be the subgraph corresponding to $\bar{N}$, and $N_x \subseteq L$ be the subgraph corresponding to $\bar{N}_x$ such that $E(L) = E(N) \dot\cup E(N_x)$ and $V(N) \cap V(N_x) = \{y\}$.
Note that $\hat{\beta}(\mathrm{Ext}_N(N)) \leq d$ by the induction hypothesis (if N contains a vertex v with $\deg_N(v) \geq d$), or by Corollary 3.4.10 (if N does not contain such a vertex).
Since $\varnothing \neq \mathcal{C}_L(\bar{x}) = \mathcal{C}_L(x)$ and $\deg_L(x) = \deg_{N_x}(x)$, we have that $\deg_{N_x}(x) \geq d$ by Lemma 3.4.2, and thus, $e(N_x) \geq d$.
By the choice of $\bar{x}$, we have that $\mathcal{C}_L(N_x) = \mathcal{C}_L(\bar{x} + y) = \mathcal{C}_L(\{x, y\})$. Note that we have $\mathcal{C}_L(\{x, y\}) \leq k$ by Lemma 3.4.5.

$$\begin{aligned}
\hat{\beta}(\mathrm{Ext}_L(L)) &= \hat{\beta}(\mathrm{Ext}_N(N)) + d\Big(v(N_x) - 1 + \sum_{C \in \mathcal{C}_L(N_x)} v(C)\Big) - (d+k+1)\Big(e(N_x) + \sum_{C \in \mathcal{C}_L(N_x)} e(C)\Big) \\
&\leq d + d\Big(e(N_x) + \sum_{C \in \mathcal{C}_L(\{x,y\})} (e(C)+1)\Big) - (d+k+1)\Big(e(N_x) + \sum_{C \in \mathcal{C}_L(\{x,y\})} e(C)\Big) \\
&\leq d\Big(1 + |\mathcal{C}_L(\{x, y\})|\Big) - (k+1)e(N_x) \\
&\leq d(1+k) - (k+1)d \\
&= 0.
\end{aligned}$$

□

In the following two lemmas, we show that $\mathrm{Ext}(R^*)$ is even denser by exploiting Lemma 3.4.6 and Lemma 3.4.7.

Lemma 3.4.12. *If* $\deg_{R^*}(r) > d$*, then* $\hat{\beta}(\mathrm{Ext}(R^*)) < -k(d+1)$.

Proof. Let N_r be the set of neighbours of r in R^*, and let $R := R^*[N_r + r]$. Furthermore, let $R_1, \ldots, R_m$ be the components of $R^* - r$. Note that $E(R), E(R_1), \ldots, E(R_m)$ is a partition of $E(R^*)$, and furthermore, $v(R^*) = v(R) + \sum_{i=1}^{m} \big(v(R_i) - 1\big)$. By Lemma 3.4.6, we have that $\mathrm{Ext}_{R^*}(R) = \mathrm{Ext}_R(R)$, and for every $i \in \{1, \ldots, m\}$, we have that $\mathrm{Ext}_{R^*}(R_i) = \mathrm{Ext}_{R_i}(R_i)$.
Thus,

$$\begin{aligned}
\hat{\beta}(\mathrm{Ext}(R^*)) &= d \cdot v(\mathrm{Ext}_{R^*}(R^*)) - (d+k+1)e(\mathrm{Ext}_{R^*}(R^*)) \\
&= d\Big(v(R) + \sum_{i=1}^{m}\big(v(\mathrm{Ext}_{R_i}(R_i)) - 1\big)\Big) - (d+k+1)\Big(e(R) + \sum_{i=1}^{m} e(\mathrm{Ext}_{R_i}(R_i))\Big) \\
&\overset{\substack{\text{Cor. 3.4.10,}\\ \text{Lemma 3.4.11}}}{\leq} d \cdot \big(e(R) + 1\big) - (d+k+1) \cdot e(R) \\
&\leq d - (k+1)e(R) \\
&\leq d - (k+1)(d+1) \\
&= -k(d+1) - 1.
\end{aligned}$$

□

Lemma 3.4.13. $\hat{\beta}(\mathrm{Ext}(R^*)) < -k(d+1)$.

Proof. By Lemma 3.4.12, we have that R^* is very cyclic, and r is contained in a cycle of R^*.
Again, we construct a partitioning of $E(R^*)$. Let $\bar{s}_1, \ldots, \bar{s}_n$ be all the vertices $\bar{s}$ of $\overline{R^*}$ with $|\bar{s}| > 1$. For $i \in \{1, \ldots, n\}$, let $S_i := R^*[\bar{s}_i]$.
Let $R_1, \ldots, R_m$ be all maximal connected subgraphs of R^* containing at least one edge such that no edge is contained in a cycle of R^*.
Note that $E(S_1), \ldots, E(S_n)$, $E(R_1), \ldots, E(R_m)$ is a partition of $E(R^*)$, and we have that

$$v(R^*) = \sum_{i=1}^{n}(v(S_i) - 1) + \sum_{i=1}^{m}(v(R_i) - 1).$$

By Lemma 3.4.7, we have that $\mathrm{Ext}_{R^*}(S_i) = S_i$ for every $i \in \{1, \ldots, n\}$.
Let $i \in \{1, \ldots, m\}$ and $v \in V(R_i)$ with $\deg_{R^*}(v) \neq \deg_{R_i}(v)$. Then, by the construction of R_i, v is contained in a cycle of R^* and thus, is not a $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex for any $b \in \{1, \ldots, k\}$ by Lemma 3.4.7. Hence, $\mathrm{Ext}_{R^*}(R_i) = \mathrm{Ext}_{R_i}(R_i)$.
Now, we are in position to prove the lemma:

$$\begin{aligned}
\hat{\beta}(\mathrm{Ext}(R^*)) &= d \cdot v(\mathrm{Ext}_{R^*}(R^*)) - (d+k+1)e(\mathrm{Ext}_{R^*}(R^*)) \\
&= d \cdot \left(\sum_{i=1}^{n} \big(v(S_i) - 1\big) + \sum_{i=1}^{m} \big(v(\mathrm{Ext}_{R_i}(R_i)) - 1\big) \right) \\
&\quad - (d+k+1) \cdot \left(\sum_{i=1}^{n} e(S_i) + \sum_{i=1}^{m} e(\mathrm{Ext}_{R_i}(R_i)) \right) \\
&\overset{\substack{\text{Cor. 3.4.10,} \\ \text{Lemma 3.4.11}}}{\leq} d \cdot \left(\sum_{i=1}^{n} \big(e(S_i) - 1\big) \right) - (d+k+1) \cdot \left(\sum_{i=1}^{n} e(S_i) \right) \\
&= -dn - (k+1) \sum_{i=1}^{n} e(S_i) \\
&\leq -d - (k+1)(d+1) \\
&= -k(d+1) - 1 - 2d.
\end{aligned}$$

□

We are now in position to prove Theorem 1.13 by showing that the counterexample graph G is not (k,d)-sparse.

Proof of Theorem 1.13. Let $K \in \mathcal{K}$ such that every vertex $v \in V(K)$ has $\deg_K(v) < d$. Then by Lemma 3.4.2, there is no $\mathcal{B}_b(\mathcal{T}^*)$-generating vertex in K for any $b \in \{1, \ldots, k\}$. Thus, $\hat{\beta}(\mathrm{Ext}(K)) = \hat{\beta}(K) \leq 0$ by Observation 3.3.4. Hence, by Lemma 3.4.11, for every $K \in \mathcal{K}$, we have that $\hat{\beta}(\mathrm{Ext}(K)) \leq 0$.
Recall that by Lemma 3.4.13, we have that $\hat{\beta}(\mathrm{Ext}(R^*)) < -k(d+1)$.
Let $H := (V(H_{\mathcal{T}^*}),\ E_r(H_{\mathcal{T}^*}))$. By Observation 3.4.8, we conclude that

$$\hat{\beta}(\mathrm{Ext}(H)) = \hat{\beta}(\mathrm{Ext}(R^*)) + \sum_{K \in \mathcal{K} - R^*} \hat{\beta}(\mathrm{Ext}(K)) < -k(d+1),$$

which is a contradiction to Observation 3.3.3. □

We note that Theorem 1.13 is not true if we allow loops. For $k, d \in \mathbb{N}_{\geq 1}$, the graph G with one vertex and $\lfloor \frac{d}{2} \rfloor + 1$ loops is (k, d)-sparse, but its only vertex has degree greater than d.

3.5 Notation and Preliminaries for the Strong Nine Dragon Tree Conjecture Results

In this section, we do a few preparations for the following two sections, in which we prove Theorem 1.14 and Theorem 1.15.
Let $d' := d + \lceil k \lfloor \frac{d-1}{k+1} \rfloor (\frac{d}{k+1} - \frac{1}{2} \lceil \frac{d}{k+1} \rceil) \rceil = d + \lceil k\ell(\frac{d}{k+1} - \frac{1}{2}(\ell + 1)) \rceil$. Furthermore, let $D := d$ in the context of proving Theorem 1.14, and let $D := d'$ in the context of proving Theorem 1.15.

The following fact will be useful throughout the rest of the chapter:

Observation 3.5.1. $2\ell + 1 \leq d \leq d'$.

Proof. It is clear that $2\ell + 1 \leq d$ holds. For the second inequality, note that if $\ell = 0$, we have $d = d'$. If $\ell \geq 1$, we have that $d > k + 1$ and thus, $\frac{1}{2} \lceil \frac{d}{k+1} \rceil < \frac{d}{k+1}$, which yields $d < d'$. ☐

For Theorem 1.14 and Theorem 1.15, we define the residue function value $\rho(F)$ of a forest F as $(\rho_{v(G)-1}(F), \rho_{v(G)-2}(F), \ldots, \rho_{D+1}(F))$, where $\rho_i(F)$ is the number of components of F having i edges. We compare the function values using lexicographic ordering.
As described at the beginning of Section 3.3, we choose a decomposition of G into k spanning trees and a forest $\hat{F}$ minimizing ρ. Since G is a counterexample graph, $\hat{F}$ contains a component with more than D edges. We pick such a component, call it R^*, and define that $R \subseteq G$ is a root component if and only if $R = R^*$. Thus, this definition of R^* is compliant with the definition $R^* := R_1^*$ from Section 3.3. The next lemma shows how we pick r from $V(R^*)$. The idea is that r lies at the centre of R^*.

Lemma 3.5.2. *The component R^* contains a vertex r such that for every edge $e \in E(R^*)$ that is incident to r we have that the component of r in $R^* - e$ has at least $\ell + 1$ edges.*

Proof. For every $v \in V(R^*)$ let $\alpha(v) \in \mathbb{N}_0$ be the maximum number such that for every edge $e \in E(R^*)$ that is incident to v we have that the component of v in $R^* - e$ has at least $\alpha(v)$ edges.
Let $r' \in V(R^*)$ such that $\alpha(r')$ is maximized over all vertices. Suppose that the lemma is false, and thus $\alpha(r') \leq \ell$. We will show that a neighbour of r' has strictly larger α. Let $r''r' \in E(R^*)$ such that the component of r' in $R^* - r''r'$ has exactly $\alpha(r')$ edges. We show that $\alpha(r'') > \alpha(r')$. First, note that the component of r'' in $R^* - r''r'$ has at least

$$e(R^*) - (\alpha(r') + 1) \geq d' + 1 - (\ell + 1) \geq \ell + 1 > \alpha(r')$$

edges by Observation 3.5.1. Now, let $x \in V(R^*)$ such that $r''x \in E(R^*)$ and $x \neq r'$. Then the component of r'' in $R^* - r''x$ has at least $\alpha(r') + 1$ edges, contradicting our choice of r'. □

With this definition of r, we can prevent R^* from having any small children, as the following lemma shows.

Lemma 3.5.3. *The component R^* does not have small children.*

Proof. Suppose to the contrary that R^* has a small child C generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$ for some $b \in \{1, \ldots, k\}$. By Lemma 3.3.8, there is an edge uv on the path in R^* from x to r such that $(x, x') \leftrightarrow uv$. We obtain $\mathcal{T}$ from $\mathcal{T}^*$ by performing $(x, x') \leftrightarrow uv$. The component K_r of r in $\mathcal{R}(\mathcal{T})$ contains at least $\ell + 1$ edges by the definition of r in Lemma 3.5.2. For the component K_x of x in $\mathcal{R}(\mathcal{T})$, we have

$$e(K_x) \leq e(R^*) - |\{uv\}| - e(K_r) + |\{xx'\}| + e(C) < e(R^*).$$

Since K_r is a proper subgraph of R^*, we have that $\rho(\mathcal{R}(\mathcal{T})) < \rho^*$, which is a contradiction. □

Now, we turn to a few applications of Lemma 3.3.5 involving special paths.

Lemma 3.5.4. *Let $\sigma = (R_1, \ldots, R_t)$ be a legal order for $\mathcal{T} \in \mathcal{F}$, and $W \in \mathrm{Aux}(\mathcal{T}, \sigma)$. If there is a special path $P = [v_0, \ldots, v_l]$ with respect to $\mathcal{T}, \sigma, W$ and (v_{l-1}, v_l) such that $e(R_i) \leq e(R_i^*)$ for all $i \in \{1, \ldots, i_\sigma(v_0)\}$ and $\eta_i(W) = \eta_i(W^*)$ for all $i \in \{2, \ldots, i_\sigma(v_0)\}$, then $\rho(\mathcal{R}(\mathcal{T}) + v_{l-1}v_l) > \rho^*$.*

Proof. Suppose to the contrary that such a path P exists, and furthermore, $\rho(\mathcal{R}(\mathcal{T}) + v_{l-1}v_l) = \rho^*$.

First, suppose that $i_\sigma(v_0) = 1$. Then for the partition $\mathcal{T}' = (T_1', \dots, T_k', F')$ we obtain by applying Lemma 3.3.5 to $\mathcal{T}, \sigma, W, P$, we have $\rho(F') < \rho(\mathcal{R}(\mathcal{T})) \leq \rho^*$, which is a contradiction.
Thus, $i_\sigma(v_0) > 1$ and thus, R^* is a component of $\mathcal{R}(\mathcal{T})$. Hence, we can apply (3) from Lemma 3.3.5 and obtain a legal order smaller than σ^*, which is a contradiction. □

The next lemma is the counterpart to Lemma 3.4.2.

Lemma 3.5.5. *Let K be a red component of $H_{\mathcal{T}^*}$, and let C be a child of K that is generated by (x, y). Then $e(K) + e(C) \geq D$.*

Proof. Suppose to the contrary that $e(K) + e(C) < D$. Since $i_{\sigma^*}(y) > i_{\sigma^*}(x)$, we have that $[x, y]$ is a special path with respect to $\mathcal{T}^*, \sigma^*, W^*$ and (x, y). Furthermore, $\rho(\mathcal{R}(\mathcal{T}^*) + xy) = \rho^*$, but this contradicts Lemma 3.5.4. □

Lemma 3.5.6. *Let C be a small red component of $H_{\mathcal{T}^*}$. Then C does not have small children, and thus every small red component of $H_{\mathcal{T}^*}$ has a parent component that is not small.*

Proof. Let C be a small red component of $H_{\mathcal{T}^*}$. If C does not have a parent component, then $C = R^*$, which is a contradiction since $e(R^*) \geq D + 1 > \ell$.
Thus, let K be the parent of C. By Lemma 3.5.5, we have that $e(K) \geq D - e(C) \geq D - \ell \geq \ell + 1$ by Observation 3.5.1, and hence K is not small. □

We now outline how we will show that the counterexample graph is not (k, d)-sparse. Let $\mathcal{C}$ be the set of small red components of $H_{\mathcal{T}^*}$. For both Theorem 1.15 and Theorem 1.14, we will characterize the structure of $H_{\mathcal{T}^*}$ and σ^*, and in the end, we will be able to assign small components to non-small components such that the high density of the non-small component compensates for the low density of the small components. We will find a function f that is forbidden by the following lemma. This will yield the desired contradiction to prove the theorems.

Lemma 3.5.7. *There is no function $f : \mathcal{C} \longrightarrow \mathcal{K} - R^*$ such that for all $K \in \mathcal{K} - R^*$, we have*

$$\hat{\beta}(K) + \sum_{C \in f^{-1}(K)} \hat{\beta}(C) \leq 0.$$

Proof. Suppose that such a function f exists, and let $H := (V(H_{\mathcal{T}^*}),\ E_r(H_{\mathcal{T}^*}))$. Then

$$\begin{aligned}\hat{\beta}(H) &= \hat{\beta}(R^*) + \sum_{K \in \mathcal{K} - R^*} \Big(\hat{\beta}(K) + \sum_{C \in f^{-1}(K)} \hat{\beta}(C) \Big) \\ &\leq \hat{\beta}(R^*) \\ &= d\,(e(R^*) + 1) - (d + k + 1)e(R^*) \\ &= d - (k+1)e(R^*) \\ &\leq d - (k+1)(d+1) \\ &= -k(d+1) - 1,\end{aligned}$$

which is a contradiction to Lemma 3.3.3. ☐

3.6 An Approximate Version of the Strong Nine Dragon Tree Conjecture

In this section, we prove Theorem 1.15, building on Section 3.3 and Section 3.5.

Theorem 1.15. *Let $k, d \in \mathbb{N}_0$ and let G be a (k, d)-sparse graph with $\gamma(G) \leq k + 1$. Then G is $(k,\ d + \lceil k \lfloor \frac{d-1}{k+1} \rfloor (\frac{d}{k+1} - \frac{1}{2} \lceil \frac{d}{k+1} \rceil) \rceil)^+$-decomposable.*

3.6.1 Non-small Components Do Not Have Many Small Children

In this subsection, we fix $b \in \{1, \ldots, k\}$ and a non-small red component $K \neq R^*$ of $H_{\mathcal{T}^*}$. Our goal is to show that K has at most $\ell + 1$ small children generated by $\mathcal{B}_b(\mathcal{T}^*)$. In fact, we will show a slightly stronger statement, which is Lemma 3.6.8. Let us give some intuition for the approach before delving into the technical details. We will assume that K has too many small children. First, we perform edge exchanges as defined in Definition 3.3.6. These exchanges will cut off a part of K and link it with a small child. After we have done this a sufficient number of times, every part will be linked to at most one small child, which will guarantee us that no part will have more than $e(K)$ edges. If the exchanges have not messed up the legal order before K, then we will have arranged that the component of $w_K(W^*)$ has less than $e(K)$ edges, which immediately gives us a smaller legal order than σ^*. If the legal order is jumbled by the exchanges, we will have arranged that there is a blue arc to a small child of K, which we have not exchanged and which is the end of a minimal special path whose start is so far back in the legal order where it is still equal to σ^*. If we augment this minimal special path we will retrieve a smaller legal order than σ^* as we previously did. We will later call the second to last vertex of this path (which is in K and has an arc to a small child) a special vertex. More specifically, each intermediate decomposition will have a such a special vertex, even if we are not aiming to perform a special path augmentation, yet. However, it might change after an exchange operation and pass its “special” properties to another vertex.

Another challenge lies in not accidentally reorienting generating edges while performing edge exchanges. Since a blue directed path ending at a “generating vertex” (i.e. a tail of a generating edge) is reoriented by the exchange operation introduced in Lemma 3.3.7, we aim to make an edge exchange with a generating vertex that does not have another generating vertex as a descendant. In Lemma 3.6.6, this vertex will be called x or it will be the special vertex v_*. However, for technical reasons, we might have to take another vertex (which will be called y) for the exchange, which also has the nice property of not reorienting generating edges. When we are forced to perform an edge exchange with v_*, we will arrange that after the exchange there is a blue path from v_* to another generating vertex, which will then meet all the criteria for being a special vertex in the resulting decomposition. When it comes to not reorienting generating edges, we will actually make an exception: during the phase of edge exchange operations, our goal is to decrease the size of the largest component that stems from K. In the worst case we can only split one edge from it (i.e. the edge that is exchanged with a generating edge). Thus, if we split off a generating vertex that is not v_* from the largest component together with at least another vertex, then we can allow ourselves to not consider this vertex for future operations, since more progress than expected has already been made, and thus, we may also reorient it. This is covered by (5) in the following “Definition of Valid Intermediate States”. In this part of the subsection, we describe which properties intermediate decompositions before the minimal special path augmentation should have. Of course, these properties also hold for $\mathcal{T}^*$. In the second part of the subsection, we then fully describe and prove the augmentation steps we just encountered.

Definition of Valid Intermediate States

Let $\ell' \in \{0,\dots,\ell\}$. By $X_{\ell'}(K)$ we denote the set of vertices x such that there is a small child of K having at most ℓ' edges that is generated by $(x,x') \in E(\mathcal{B}_b(\mathcal{T}^*))$. If not described otherwise, x' denotes the parent of $x \in X_{\ell'}(K)$ in $\mathcal{B}_b(\mathcal{T}^*)$.
Let $\mathcal{C}(K)$ denote the set of small children of K. For $\ell' \in \{0,\dots,\ell\}$ let $\mathcal{C}_{\ell'}(K)$ denote the number of small children of K having at most ℓ' edges.

The following definition will help us describing which vertices may be reoriented by performing edge exchanges. This might not be clear straightaway, but we will give an intuition afterwards.

Definition 3.6.1. Let $\mathcal{T} \in \mathcal{F}$ and $U \subseteq V(K)$. Furthermore, for $v \in V(G)$ let $\Delta(v)$ denote the set of descendants of v in $\mathcal{B}_b(\mathcal{T})$. For every $u \in U$, let $T_u(U,\mathcal{T})$ be the subtree of $\mathcal{B}_b(\mathcal{T})$ with root u and vertex set $\Delta(u) \setminus \left(\bigcup_{u' \in \left(U \cap \Delta(u)\right)-u} \Delta(u')\right)$. Let $\mathcal{I}_{\ell'}(K) := \min_{x \in X_{\ell'}(K)} i_{\sigma^*}(T_x(X_{\ell'}(K),\mathcal{T}^*))$.

Note that in Definition 3.6.1, we have that $V(T_u(U,\mathcal{T}))$ and $V(T_{u'}(U,\mathcal{T}))$ are disjoint for any distinct vertices $u, u' \in U$. Later, we will use this definition with $U \subseteq X_{\ell'}(K)$ and our setup will guarantee that when performing $(u,u') \leftrightarrow e$, where $u \in U$, then only blue arcs of $T_u(U,\mathcal{T})$ will be reoriented and in particular, no other "child-generating" arcs will be reoriented whose tails are in U.
Let $\ell' \in \{0,\dots,\ell\}$, $\mathcal{T} \in \mathcal{F}$ and $a \in X_{\ell'}(K)$. In the following paragraphs, we enumerate nine conditions $(1),\dots,(9)$ which all need to hold such that we may call $(\mathcal{T},a)$ a *valid intermediate state for K and ℓ'*. For better readability we will sometimes use the notation $\mathcal{T}_a := (\mathcal{T},a)$.

Let $\mathcal{L}(\mathcal{T})$ be the set of components of $\mathcal{R}(\mathcal{T})[V(K)]$, and for every $L \in \mathcal{L}(\mathcal{T})$ let $\bar{L}$ be the component of $\mathcal{R}(\mathcal{T})$ containing L.

(1) All components of $\mathcal{R}(\mathcal{T}^*)$ that are not from $\mathcal{C}_{\ell'}(K) \cup \{K\}$ are also components of $\mathcal{R}(\mathcal{T})$, $\bigcup_{C \in \mathcal{C}_{\ell'}(K)} E(C) \subseteq E(\mathcal{R}(\mathcal{T}))$ and the remaining edges of $\mathcal{R}(\mathcal{T})$ form a subset of $E(K) \cup \bigcup_{x \in X_{\ell'}(K)} \{xx'\}$.

(2a) If $\mathcal{I}_{\ell'}(K) < i_{\sigma^*}(K)$, then there is a legal order $(R_1,\dots,R_t)$ for $\mathcal{T}$ such that $R_i = R_i^*$ for all $i \in \{0,\dots,\mathcal{I}_{\ell'}(K)\}$, and there is a $W \in \mathrm{Aux}(\mathcal{T},\sigma)$ such that $\eta_j(W) = \eta_j(W^*)$ for all $j \in \{2,\dots,\mathcal{I}_{\ell'}(K)\}$.

(2b) If $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$, then there is a legal order $(R_1, \dots, R_t)$ for $\mathcal{T}$ such that $R_i = R_i^*$ for all $i \in \{1, \dots, i_{\sigma^*}(K) - 1\}$, and there is a $W \in \mathrm{Aux}(\mathcal{T}, \sigma)$ such that $\eta_j(W) = \eta_j(W^*)$ for all $j \in \{2, \dots, i_{\sigma^*}(K)\}$.

(3) $\min_{x \in X_{\ell'}(K)} i_{\sigma^*}(T_x(X_{\ell'}(K), \mathcal{T})) \geq \mathcal{I}_{\ell'}(K)$.

(4) For all $L \in \mathcal{L}(\mathcal{T})$, $\bar{L}$ contains at most one edge of $\bigcup_{x \in X_{\ell'}(K)} \{xx'\}$.

Note that at most one component of $\mathcal{L}(\mathcal{T})$ has at least $\max\{d', e(K)\} - \ell'$ edges (or otherwise $e(K) \geq 2(\max\{d', e(K)\} - \ell') + 1 > \max\{d', e(K)\}$ by Observation 3.5.1). We denote this component by $K'(\mathcal{T})$ if it exists and otherwise we let $K'(\mathcal{T})$ be the subgraph of K not containing any vertices, denoted by $\varnothing$.

(5) For all $x \in X_{\ell'}(K) \cap V(K'(\mathcal{T}))$, we have that $(x', x) \notin E(\mathcal{B}_b(\mathcal{T}))$ (and thus, $(x, x') \in E(\mathcal{B}_b(\mathcal{T}))$ or $xx' \in E(\mathcal{R}(\mathcal{T}))$).

In the following, we define special vertices. They are linked to special paths in the following way: our augmentations will finally resolve in a valid intermediate state, which might have a special vertex v_*. If so, we will do a special path augmentation in the proof of Lemma 3.6.7, where the minimal special path ends with (v_*, v_*'). By the definition of special vertices, this minimal special path will start relatively close to R^* with respect to σ^*. As all our other changes on the decomposition will be later in the legal order, the special path augmentation will provide a smaller legal order than σ^*, which will be the desired contradiction.
Let $S(\mathcal{T})$ be the set of vertices $s \in X_{\ell'}(K) \cap V(K'(\mathcal{T}))$ with $(s, s') \in E(\mathcal{B}_b(\mathcal{T}))$. Furthermore, let $w(K) := w_K(W^*)$.

Definition 3.6.2. Let $v_* \in X_{\ell'}(K)$ and define $\bar{S}_{v_*} := S(\mathcal{T}) + v_*$ (only for the scope of this definition).

- If $\mathcal{I}_{\ell'}(K) < i_{\sigma^*}(K)$, we call v_* *special for* $\mathcal{T}$ if $(v_*, v_*') \in E(\mathcal{B}_b(\mathcal{T}))$ and $i_{\sigma^*}(T_{v_*}(\bar{S}_{v_*}, \mathcal{T})) = \mathcal{I}_{\ell'}(K)$.
- If $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$, we call v_* *special for* $\mathcal{T}$ if $(v_*, v_*') \in E(\mathcal{B}_b(\mathcal{T}))$ and $w(K) \in V(T_{v_*}(\bar{S}_{v_*}, \mathcal{T}))$.

(6a) If there is a special vertex for $\mathcal{T}$, then a is a special vertex. In this case let $v_* := a$.

(6b) If there is no special vertex for $\mathcal{T}$, then $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$ and $a = w(K)$.

(7) Let $L \in \mathcal{L}(\mathcal{T})$ contain a. Then $\bar{L}$ does not contain an edge of $\bigcup_{x \in X_{\ell'}(K)} \{xx'\}$ and thus, $\bar{L} = L$.

Note that some of the notations defined in the following paragraphs are illustrated in Figure 3.6.1. If a is a special vertex for $\mathcal{T}$, let $\bar{S}(\mathcal{T}_a) := S(\mathcal{T}) + v_*$ and $\mathring{S}(\mathcal{T}_a) := S(\mathcal{T}) - v_*$.
If there is no special vertex for $\mathcal{T}$, let $\bar{S}(\mathcal{T}_a) := \mathring{S}(\mathcal{T}_a) := S(\mathcal{T})$.
If there is a vertex $t \in X_{\ell'}(K) \cap V(K'(\mathcal{T}))$ such that $tt' \in E(\mathcal{R}(\mathcal{T}))$, then we say that t *extends* $K'(\mathcal{T})$ *in* $\mathcal{T}$. Note that if (4) and (7) hold for $\mathcal{T}_a$, then there can be at most one such vertex and this vertex is not a. If there is a vertex extending $K'(\mathcal{T})$ in $\mathcal{T}$, we let $u(\mathcal{T}_a)$ be this vertex, and otherwise we let $u(\mathcal{T}_a) := a$. Note that if (4) and (5) hold for $\mathcal{T}_a$, $u(\mathcal{T}_a)$ always exists and is unique.

(8) If $K'(\mathcal{T}) \neq \varnothing$, then $u(\mathcal{T}_a) \in V(K'(\mathcal{T}))$.

(9) If $u(\mathcal{T}_a)$ extends $K'(\mathcal{T})$ in $\mathcal{T}$, then $P^{\mathcal{B}_b(\mathcal{T})}(u(\mathcal{T}_a), r)$ either does not contain a vertex of $\bar{S}(\mathcal{T}_a)$ or the first vertex on this path which is in $\bar{S}(\mathcal{T}_a)$ is v_*.

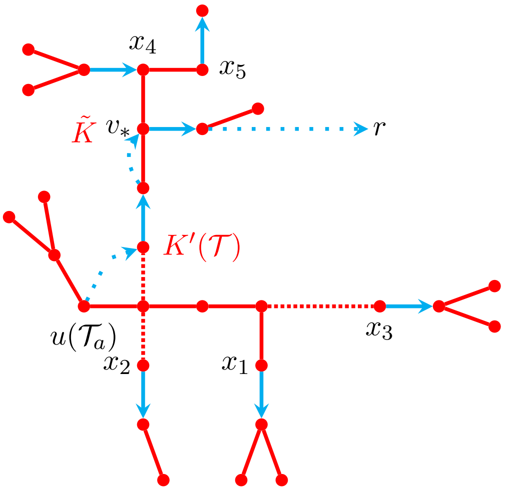

Fig. 3.6.1: A part of a valid intermediate state $\mathcal{T}_a$ where only the changes to components involving K are shown, such that we have the following properties:

- $V(K) = V(K'(\mathcal{T})) \ \dot{\cup}\ V(\tilde{K})$.
- $X_2(K) = \{x_1, \ldots, x_5, u(\mathcal{T}_a), v_*\}$.
- $S(\mathcal{T}_a) = \{x_1, x_2, x_3\} = \mathring{S}(\mathcal{T}_a)$.
- $\bar{S}(\mathcal{T}_a) = \{x_1, x_2, x_3, v_*\}$.
- $u(\mathcal{T}_a)$ extends $K'(\mathcal{T})$ in $\mathcal{T}$.

If conditions $(1), \ldots, (9)$ are met, we call $\mathcal{T}_a$ a *valid intermediate state for K and ℓ'* and denote the set of all valid intermediate states for K and ℓ' by $\mathcal{F}_{K,\ell'}$. It is routine to check that $\mathcal{F}_{K,\ell'}$ is not empty: we state this as an observation.

Observation 3.6.3. *If $\mathcal{I}_{\ell'}(K) < i_{\sigma^*}(K)$, let $a^* \in X_{\ell'}(K)$ be a vertex minimizing $i_{\sigma^*}(T_x(X_{\ell'}(K), \mathcal{T}^*))$. If $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$, let a^* be a special vertex for $\mathcal{T}^*$ if one exists, and otherwise let $a^* = w(K)$.*
Then $(\mathcal{T}^*, a^*) \in \mathcal{F}_{K,\ell'}$ and thus, $\mathcal{F}_{K,\ell'} \neq \varnothing$.

In the rest of the subsection, if we say (i) holds for $(\mathcal{T}, a)$, (i) always refers to the respective item in the definition of valid intermediate states. Note that for $i \in \{1, \ldots, 5\}$, we might just say that (i) holds for $\mathcal{T}$ since in these cases, (i) does not depend on a.

Augmenting Valid Intermediate States

In Lemma 3.6.6, we will show that we can perform an edge exchange that gives us a "better" valid intermediate state. Before we turn to this lemma, we show that after any obvious edge exchange, (1), (2a), (2b) and (3) will still hold in the new decomposition.

Lemma 3.6.4. *Let $\ell' \in \{0, \ldots, \ell\}$, $(\mathcal{T}, a) \in \mathcal{F}_{K,\ell'}$ and $x \in \bar{S}(\mathcal{T}_a)$. Furthermore, let v be a descendant of x in $\mathcal{B}_b(\mathcal{T})$ and v' not be a descendant of x such that $vv' \in E(K) \cap E(\mathcal{R}(\mathcal{T}))$. Then (1), (2a), (2b) and (3) hold for the decomposition $\mathcal{T}'$ that can be obtained from $\mathcal{T}$ by performing $(x, x') \leftrightarrow vv'$.*

Proof. It is clear that (1) holds for $\mathcal{T}'$. Now, we prove that (2a) and (2b) hold for $\mathcal{T}'$. First, observe that since (3) holds for $\mathcal{T}_a$, we have $i_{\sigma^*}(P^{\mathcal{B}_b(\mathcal{T})}(v, x)) \geq \mathcal{I}_{\ell'}(K)$. First, let $\mathcal{I}_{\ell'}(K) < i_{\sigma^*}(K)$. Since $i_{\sigma^*}(K \cup \bigcup_{C \in \mathcal{C}_{\ell'}(K)} C) > \mathcal{I}_{\ell'}(K)$ and $i_{\sigma^*}(P^{\mathcal{B}_b(\mathcal{T})}(v, x)) \geq \mathcal{I}_{\ell'}(K)$, there is a legal order $(R'_1, \ldots, R'_{t'})$ for $\mathcal{T}'$ such that $R'_i = R_i = R^*_i$ for all $i \in \{0, \ldots, \mathcal{I}_{\ell'}(K)\}$ and hence, (2a) holds for $\mathcal{T}'$.
If $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$, then $i_{\sigma^*}(K \cup \bigcup_{C \in \mathcal{C}_{\ell'}(K)} C) \geq i_{\sigma^*}(K)$ and $i_{\sigma^*}(P^{\mathcal{B}_b(\mathcal{T})}(v, x)) \geq i_{\sigma^*}(K)$, and thus, there is a legal order $(R'_1, \ldots, R'_{t'})$ for $\mathcal{T}'$ such that $R'_i = R_i = R^*_i$ for all $i \in \{0, \ldots, i_{\sigma^*}(K) - 1\}$, and there is a $W' \in \mathrm{Aux}(\mathcal{T}', \sigma')$ such that $\eta_j(W') = \eta_j(W) = \eta_j(W^*)$ for all $j \in \{2, \ldots, i_{\sigma^*}(K)\}$.
Hence, (2b) holds for $\mathcal{T}'$.
Finally, suppose (3) does not hold for $\mathcal{T}'$. Thus, there is a vertex z with $i_{\sigma^*}(z) < \mathcal{I}_{\ell'}(K)$

that has a path in $\mathcal{B}_b(\mathcal{T}')$ to some vertex of $X_{\ell'}(K)$, but it does not have such a path in $\mathcal{B}_b(\mathcal{T})$. Let (u, u') be the first arc on this path that is not in $\mathcal{B}_b(\mathcal{T}')$. Then $u \in V(P^{\mathcal{B}_b(\mathcal{T})}(v, x))$ and thus, z has a path to x in $\mathcal{B}_b(\mathcal{T})$, which is a contradiction. $\square$

Lemma 3.6.5. *Let $\ell' \in \{0, \ldots, \ell\}$ and $\mathcal{T}_a \in \mathcal{F}_{K,\ell'}$. Then $P^{\mathcal{B}_b(\mathcal{T})}(u(\mathcal{T}_a), r)$ either does not contain a vertex of $\bar{S}(\mathcal{T}_a)$ or a is special and the first vertex on this path which is in $\bar{S}(\mathcal{T}_a)$ is a.*

Proof. The lemma trivially holds if $u(\mathcal{T}_a) = a$ is special. If $u(\mathcal{T}_a)$ extends $K'(\mathcal{T})$ in $\mathcal{T}$, then the lemma holds since (9) holds for $\mathcal{T}$. If $u(\mathcal{T}_a) = a = w(K)$ is not special, then $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$ and there is no special vertex for $\mathcal{T}$ by (6a). Thus, $w(K)$ is not a descendant of any vertex of $\bar{S}(\mathcal{T}_a)$. $\square$

After these preparations, we turn to the main augmentation lemma.

Lemma 3.6.6. *Let $\ell' \in \{0, \ldots, \ell\}$ and $\mathcal{T}_a \in \mathcal{F}_{K,\ell'}$ such that $\mathring{S}(\mathcal{T}_a) \neq \varnothing$ and $K'(\mathcal{T}) \neq \varnothing$. Then there is a tuple $(\mathcal{T}', a') \in \mathcal{F}_{K,\ell'}$ such that $K'(\mathcal{T}') \subsetneq K'(\mathcal{T})$.*

Proof. If a is special for $\mathcal{T}$, then let $a =: v_*$. Consider the tree Q with vertex set $\bar{S}(\mathcal{T}_a) + r$ where two vertices v_1, v_2 are linked by an arc (v_1, v_2) whenever v_1 is a descendant of v_2 in $\mathcal{B}_b(\mathcal{T})$. Observe that Q has at least two vertices which are not a since $\mathring{S}(\mathcal{T}_a) \neq \varnothing$. Let Q be rooted at r such that every vertex has a directed path to r. If a is special for $\mathcal{T}$ and the only leaf in Q, then let x be the parent of a (in this case Q is a path from a to r). Otherwise, let x be a leaf of Q. Note that $x \notin \{u(\mathcal{T}_a), r\}$ and furthermore, if a is special we have $x \neq v_*$. Since (8) holds for $\mathcal{T}_a$, we have that $u(\mathcal{T}_a) \in K'(\mathcal{T})$, and as $x \in \mathring{S}(\mathcal{T}_a)$, it follows that both x and $u(\mathcal{T}_a)$ are in $K'(\mathcal{T})$.
Let $P^{K'(\mathcal{T})}(u(\mathcal{T}_a), x) = [v_1, \ldots, v_n]$. For $i \in \{2, \ldots, n\}$, let L_i and L_i' be the components of $K'(\mathcal{T}) - v_{i-1}v_i$ containing v_i and v_{i-1}, respectively. Note that the notations defined in the following are illustrated on the left side of Figure 3.6.2. Let i be minimal with the property that $P^{\mathcal{B}_b(\mathcal{T})}(v_i, r)$ visits a vertex of $V(L_i) \cap \mathring{S}(\mathcal{T}_a)$ before it reaches any vertex of $\{v_*, r\}$. Note that i exists since $x \in V(L_n) \cap \mathring{S}(\mathcal{T}_a)$, and we have $v_i \neq u(\mathcal{T}_a)$ by Lemma 3.6.5 and thus, $i \geq 2$. Let $v := v_i$ and $v' := v_{i-1}$, and note that $P^{\mathcal{B}_b(\mathcal{T})}(v', r)$ either does not contain a vertex of $\bar{S}(\mathcal{T}_a)$, or the first vertex on this path which is in $\bar{S}(\mathcal{T}_a)$ is v_* (if $i = 2$ and thus, $v' = u(\mathcal{T}_a)$, this is true by Lemma 3.6.5). Let $L := L_i$ and $L' := L_i'$. We split into cases depending on how many edges L' has.

Case 1: $e(L') \leq \max\{d', e(K)\} - \ell' - 1$**:**
Let y be the first vertex on $P^{\mathcal{B}_b(\mathcal{T})}(v, r)$ that is in $V(L) \cap \mathring{S}(\mathcal{T}_a)$.

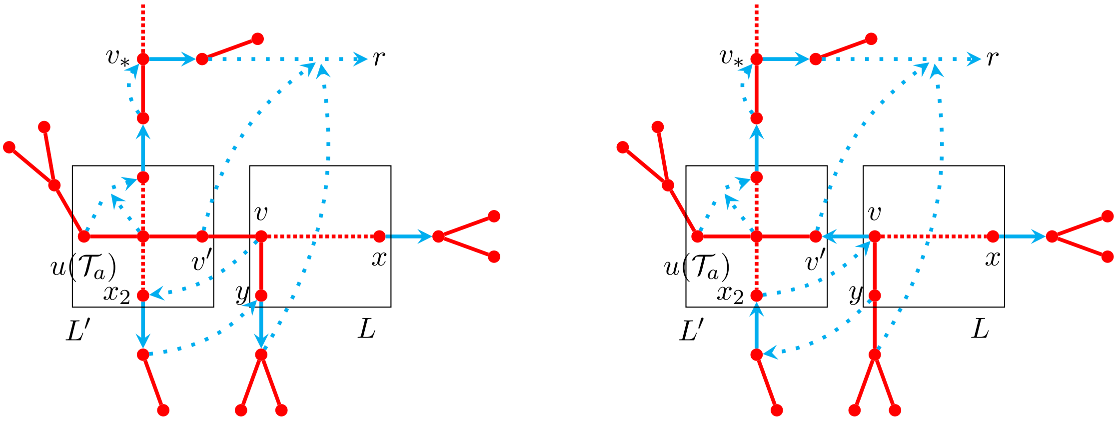

Fig. 3.6.2: $\mathcal{T}_a$ (left side) and $\mathcal{T}_a'$ (right side) in Subcase 1.1 in the proof of Lemma 3.6.6.

Subcase 1.1: v' is not a descendant of y in $\mathcal{B}_b(\mathcal{T})$:

Note that v' is a neighbour of v in the red path from $u(\mathcal{T}_a)$ to v, while y' shall denote the parent of y in $\mathcal{B}_b(\mathcal{T}^*)$. We obtain $\mathcal{T}'$ from $\mathcal{T}$ by performing $(y, y') \leftrightarrow vv'$ (which reorients $P^{\mathcal{B}_b(\mathcal{T})}(v, y)$), and show that $\mathcal{T}_a' := (\mathcal{T}', a) \in \mathcal{F}_{K,\ell'}$, and that $\mathcal{T}_a'$ satisfies the lemma. Note that $\mathcal{T}_a'$ is depicted on the right side of Figure 3.6.2.

Note that (1), (2a), (2b) and (3) hold for $\mathcal{T}'$ by Lemma 3.6.4. Hence, we focus on the remaining conditions.

To see that (4) holds for $\mathcal{T}'$, first observe that since (4) holds for $\mathcal{T}$, we only need to check the condition on the components L and L' affected by the exchange. Observe if there is a vertex $z \in X_{\ell'}(K) \cap V(K'(\mathcal{T}))$ such that $zz' \in E(\mathcal{R}(\mathcal{T}))$, then $z = u(\mathcal{T}_a)$ since (4) holds for $\mathcal{T}$. Since $u(\mathcal{T}_a)$ is in L' and y is in L, we have that (4) also holds for $\mathcal{T}'$.

Note that either $K'(\mathcal{T}') = \varnothing$ or $K'(\mathcal{T}') = L$. As (5) holds for $\mathcal{T}$ and since y is the only vertex of $X_{\ell'}(K) \cap V(L)$ contained in $P^{\mathcal{B}_b(\mathcal{T})}(v, y)$, we have that (5) also holds for $\mathcal{T}'$. Thus, $\mathring{S}(\mathcal{T}_a') = (\mathring{S}(\mathcal{T}_a) \cap V(L)) - y$ if $K'(\mathcal{T}) \neq \varnothing$.

Next, we want to show (6a) and suppose that $a = v_*$ is special for $\mathcal{T}$. Since $P^{\mathcal{B}_b(\mathcal{T})}(v, y) \subseteq T_y(\bar{S}(\mathcal{T}_a), \mathcal{T})$ and the vertex set of the supergraph is disjoint from $V(T_{v_*}(\bar{S}(\mathcal{T}_a), \mathcal{T}))$, we have that $T_{v_*}(\bar{S}(\mathcal{T}_a), \mathcal{T}) \subseteq T_{v_*}(\bar{S}(\mathcal{T}_a'), \mathcal{T}')$. Thus, v_* is also special for $\mathcal{T}'$ and (6a) holds for $\mathcal{T}_a'$.

Now we consider (6b). Suppose that $a = w(K)$ is not special for $\mathcal{T}$ and thus, there is no special vertex for $\mathcal{T}$ and $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$. Then $P^{\mathcal{B}_b(\mathcal{T})}(w(K), r)$ does not contain a vertex of $X_{\ell'}(K)$ and in particular, it does not contain x. As all tails of the arcs of $E(\mathcal{B}_b(\mathcal{T}')) \setminus E(\mathcal{B}_b(\mathcal{T}))$ are descendants of x, we have that $P^{\mathcal{B}_b(\mathcal{T})}(w(K), r) \subseteq \mathcal{B}_b(\mathcal{T}')$. Thus, there is no special vertex for $\mathcal{T}'$. Hence, (6b) holds for $\mathcal{T}_a'$.

We show that $a \notin V(L)$ and thus, (7) holds for $\mathcal{T}'_a$: if $u(\mathcal{T}_a)$ extends $K'(\mathcal{T})$ in $\mathcal{T}$, then a is not contained in $K'(\mathcal{T})$ since (7) holds for $\mathcal{T}$. If, on the other hand, we have $u(\mathcal{T}_a) = a$, then $a \in V(L')$. Thus, (7) holds for $\mathcal{T}'_a$.

Note that if $\varnothing \neq K'(\mathcal{T}') = L$, then $u(\mathcal{T}'_a) = y$ extends L in $\mathcal{T}'$ and thus, (8) holds for $\mathcal{T}'_a$. Furthermore, note that $P^{\mathcal{B}_b(\mathcal{T}')}(y, r)$ contains (v, v'). By the definition of v and v', we have that (9) holds for $\mathcal{T}'_a$.

Thus, $\mathcal{T}'_a \in \mathcal{F}_{K,\ell'}$ and furthermore, since $K'(\mathcal{T}') \in \{L, \varnothing\}$, we have that $K'(\mathcal{T}') \subseteq L \subseteq K'(\mathcal{T}) - vv'$, which completes the claim in this case.

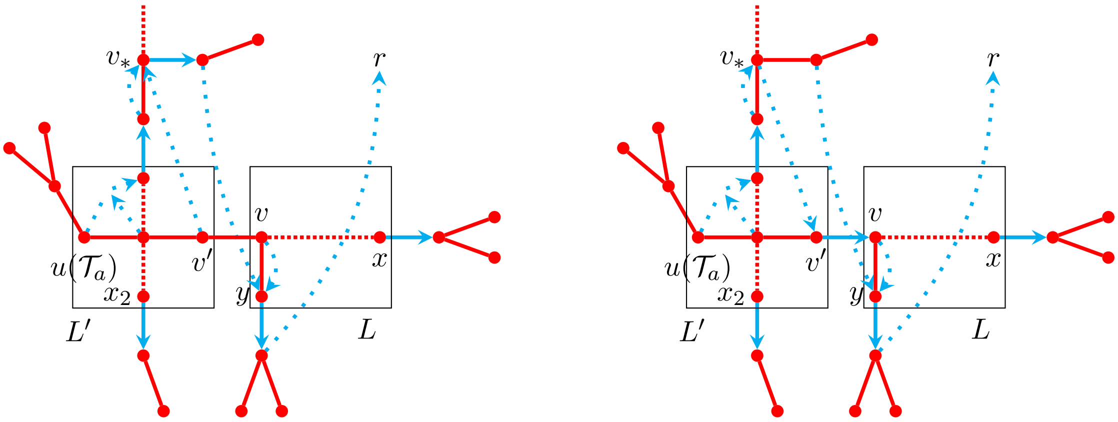

Fig. 3.6.3: $\mathcal{T}_a$ (left side) and $\mathcal{T}'_y$ (right side) in Subcase 1.2 in the proof of Lemma 3.6.6

Subcase 1.2: v' is a descendant of y in $\mathcal{B}_b(\mathcal{T})$:

This case is depicted in Figure 3.6.3. In this case, it follows that $a = v_*$ is special for $\mathcal{T}$ and $P^{\mathcal{B}_b(\mathcal{T})}(v', y)$ contains v_*. Note that v is not a descendant of v_* in $\mathcal{B}_b(\mathcal{T})$ or otherwise $\mathcal{B}_b(\mathcal{T})$ contains a cycle. Thus, we may obtain $\mathcal{T}'$ from $\mathcal{T}$ by performing $(v_*, v'_*) \leftrightarrow v'v$ (which reorients $P^{\mathcal{B}_b(\mathcal{T})}(v', v_*)$) and show that $(\mathcal{T}', y)$ is a valid intermediate state for K and ℓ' and satisfies the lemma. Note that (1), (2a), (2b) and (3) hold for $\mathcal{T}'$ by Lemma 3.6.4.

Now we prove (4) holds. As (7) holds for $\mathcal{T}_a$, we have that the component of v_* in $\mathcal{R}(\mathcal{T}')$ contains exactly one edge of $\bigcup_{x \in X_{\ell'}(K)} \{xx'\}$, which is $v_*v'_*$, and thus, (4) holds for $\mathcal{T}'$.

Note that if $K'(\mathcal{T}') \neq \varnothing$, then $K'(\mathcal{T}') = L$. As (5) holds for $\mathcal{T}$ and since v_* is the only vertex of $X_{\ell'}(K) \cap V(L)$ contained in $P^{\mathcal{B}_b(\mathcal{T})}(v', v_*)$, we have that (5) also holds for $\mathcal{T}'$.

We have that $V(T_{v_*}(\bar{S}(\mathcal{T}_a), \mathcal{T})) \subseteq V(T_y(\bar{S}(\mathcal{T}'_y), \mathcal{T}'))$ since in $\mathcal{B}_b(\mathcal{T}')$ there is a path from v_* to y going over (v', v) and this path does not contain any vertices of $\mathring{S}(\mathcal{T}'_y) - y$. Thus, y is a special vertex for $\mathcal{T}'$ and (6a) holds for $\mathcal{T}'_y$. As there is a special vertex

for $\mathcal{T}'$, (6b) trivially holds for $\mathcal{T}'_y$.
Note that if $\overline{K'(\mathcal{T})}$ contains an edge of $\bigcup_{x \in X_{\ell'}(K)} \{xx'\}$, then $u(\mathcal{T}_a)$, which is in L', extends $K'(\mathcal{T})$ in $\mathcal{T}$. Since $y \in V(L)$, we have that (7) holds for $\mathcal{T}'_y$.
Using the same argument, we have that there is no vertex extending L in $\mathcal{T}'$, and thus, $u(\mathcal{T}'_y) = y \in V(L)$. Hence, (8) and (9) hold for $\mathcal{T}'_y$.
Thus, $\mathcal{T}'_y \in \mathcal{F}_{K,\ell'}$ and furthermore, since $K'(\mathcal{T}') \in \{L, \varnothing\}$, we have that $K'(\mathcal{T}') \subseteq L \subseteq K'(\mathcal{T}) - vv'$.

Case 2: $e(L') \geq \max\{d', e(K)\} - \ell'$**:**
Recall that $P^{K'(\mathcal{T})}(u(\mathcal{T}_a), x) = [v_1, \ldots, v_n]$. Let $j \in \{1, \ldots, n\}$ be maximal such that v_j is not a descendant of x in $\mathcal{B}_b(\mathcal{T})$ or $P^{\mathcal{B}_b(\mathcal{T})}(v_j, r)$ visits v_* and after that it visits x. Note that j exists and $i - 1 \leq j < n$. Let $\bar{v} := v_{j+1}$ and $\bar{v}' := v_j$. Furthermore, let M and M' be the components in $K'(\mathcal{T}) - \bar{v}\bar{v}'$ of x and $u(\mathcal{T}_a)$, respectively. Note that $e(M') \geq e(L')$. We note that the edge exchange operations in the following two subcases are similar to the ones in the Subcases 1.1 and 1.2.

Subcase 2.1: $\bar{v}'$ **is not a descendant of** x **in** $\mathcal{B}_b(\mathcal{T})$**:**
We obtain $\mathcal{T}'$ from $\mathcal{T}$ by performing $(x, x') \leftrightarrow \bar{v}\bar{v}'$ and show that $(\mathcal{T}', a)$ is a valid intermediate state for K and ℓ' and satisfies the lemma.
Note that (1), (2a), (2b) and (3) hold for $\mathcal{T}'$ by Lemma 3.6.4.
Since x is not in the component of $u(\mathcal{T}_a)$ in $\mathcal{R}(\mathcal{T}')$, we have that (4) holds for $\mathcal{T}'$.
Note that $K'(\mathcal{T}') = M'$. As (5) holds for $\mathcal{T}$ and since by the definition of x and $\bar{v}$ we have that x is the only vertex of $X_{\ell'}(K) \cap V(M)$ contained in $P^{\mathcal{B}_b(\mathcal{T})}(\bar{v}, x)$, we have that (5) also hold for $\mathcal{T}'$.
Analogously to Subcase 1.1 it can be proven that (6a) and (6b) hold for $\mathcal{T}'_a$.
We show that $a \notin V(M)$ and thus, (7) holds for $\mathcal{T}'_a$: if $u(\mathcal{T}_a)$ extends M in $\mathcal{T}'$, then a is not contained in $K'(\mathcal{T})$ since (7) holds for $\mathcal{T}_a$. If, on the other hand, we have that $u(\mathcal{T}_a) = a$, then $a \in V(M')$. Thus, (7) holds for $\mathcal{T}'_a$.
Observe that (8) holds for $\mathcal{T}'_a$ since $u(\mathcal{T}'_a) = u(\mathcal{T}_a) \in V(M')$.
Finally, suppose that (9) does not hold for $\mathcal{T}'_a$ and let (z, z') be the first arc on $P^{\mathcal{B}_b(\mathcal{T})}(u(\mathcal{T}_a), r)$ that is not on $P^{\mathcal{B}_b(\mathcal{T}')}(u(\mathcal{T}_a), r)$. Then (z, z') is in $P^{\mathcal{B}_b(\mathcal{T})}(\bar{v}, x)$. As (9) holds for $\mathcal{T}_a$, we have that $a = v_*$ is special for $\mathcal{T}$ and $v_* \in V(P^{\mathcal{B}_b(\mathcal{T})}(u(\mathcal{T}_a), x))$. Since $v_* \notin V(P^{\mathcal{B}_b(\mathcal{T})}(\bar{v}, x))$, we have that v_* is in $P^{\mathcal{B}_b(\mathcal{T})}(u(\mathcal{T}_a), z) \subseteq \mathcal{B}_b(\mathcal{T}')$ and thus, (9) holds for $\mathcal{T}'_a$.
Thus, $\mathcal{T}'_a \in \mathcal{F}_{K,\ell'}$ and furthermore, $K'(\mathcal{T}') = M' \subseteq K'(\mathcal{T}) - \bar{v}\bar{v}'$.

Subcase 2.2: $\bar{v}'$ **is a descendant of** x **in** $\mathcal{B}_b(\mathcal{T})$**:**
Then $a = v_*$ is special for $\mathcal{T}$ and $P^{\mathcal{B}_b(\mathcal{T})}(\bar{v}', x)$ contains v_*. Note that $\bar{v}$ is not a

descendant of v_* in $\mathcal{B}_b(\mathcal{T})$ or otherwise $\mathcal{B}_b(\mathcal{T})$ contains a cycle. Thus, we may obtain $\mathcal{T}'$ from $\mathcal{T}$ by performing $(v_*, v'_*) \leftrightarrow v'v$ and show that $(\mathcal{T}', x)$ is a valid intermediate state for K and ℓ' and satisfies the lemma.

We have that (1), (2a), (2b) and (3) hold for $\mathcal{T}'$ by Lemma 3.6.4.

As (7) holds for $\mathcal{T}_a$, we have that the component of v_* in $\mathcal{R}(\mathcal{T}')$ contains exactly one edge of $\bigcup_{x \in X_{\ell'}(K)} \{xx'\}$, and thus, (4) holds for $\mathcal{T}'$.

Note that $K'(\mathcal{T}') = M'$. Since v_* is a descendant of x in $\mathcal{B}_b(\mathcal{T})$, we have by the definition of x that there are no vertices of $\mathring{S}(\mathcal{T}_a)$ on $P^{\mathcal{B}_b(\mathcal{T})}(\bar{v}', v_*)$, the path that is reoriented in $\mathcal{B}_b(\mathcal{T}')$. Thus, since (5) holds for $\mathcal{T}$, we have that (5) also holds for $\mathcal{T}'$.

We have that $V(T_{v_*}(\bar{S}(\mathcal{T}_a), \mathcal{T})) \subseteq V(T_x(\bar{S}(\mathcal{T}'_x), \mathcal{T}'))$ since in $\mathcal{B}_b(\mathcal{T}')$ there is a path from v_* to x going over $(\bar{v}', \bar{v})$ and this path does not contain any vertices of $\mathring{S}(\mathcal{T}'_x) - x$. Thus, x is a special vertex for $\mathcal{T}'$ and (6a) holds for $\mathcal{T}'_x$ and furthermore, (6b) trivially holds for $\mathcal{T}'_x$.

Note that if $\overline{K'(\mathcal{T})}$ contains an edge of $\bigcup_{x \in X_{\ell'}(K)} \{xx'\}$, then $u(\mathcal{T}_a)$, which is in M', extends $K'(\mathcal{T})$ in $\mathcal{T}$. Since $x \in V(M)$, we have that (7) holds for $\mathcal{T}'_x$.

Using the same argument, we have that there is no vertex extending M in $\mathcal{T}'$, and thus, $u(\mathcal{T}'_x) = x \in V(M)$. Hence, (8) and (9) hold for $\mathcal{T}'_x$.

Thus, $(\mathcal{T}', x) \in \mathcal{F}_{K,\ell'}$ and furthermore, $K'(\mathcal{T}') = M' \subseteq K'(\mathcal{T}) - \bar{v}\bar{v}'$. $\square$

In the following, let $\alpha_K := \max\{d' - e(K), 0\}$.

Lemma 3.6.7. *Let* $\ell' \in \{0, \dots, \ell\}$. *If* $|\mathcal{C}_{\ell'}(K)| \geq \ell' - \alpha_K + 2$, *then there is a tuple* $(\mathcal{T}, a) \in \mathcal{F}_{K,\ell'}$ *such that* $K'(\mathcal{T}) = \varnothing$.

Proof. Recall that by Observation 3.6.3 we have that $\mathcal{F}_{K,\ell'} \neq \varnothing$. Suppose the lemma is not true and let $(\mathcal{T}, a) \in \mathcal{F}_{K,\ell'}$ such that $e(K'(\mathcal{T}))$ is minimal. By Lemma 3.6.6, we have that $\mathring{S}(\mathcal{T}_a) = \varnothing$. As (4) and (5) hold for $\mathcal{T}$, we have that $X_{\ell'}(K) \cap V(K'(\mathcal{T})) \subseteq \{a, t\}$ where t is a vertex such that $tt' \in E(\mathcal{R}(\mathcal{T}))$. Since (7) holds for $\mathcal{T}_a$, we have that $|X_{\ell'}(K) \cap V(K'(\mathcal{T}))| \leq 1$. Thus, $|X_{\ell'}(K) \setminus V(K'(\mathcal{T}))| \geq \ell' - \alpha_K + 1$ and $e(K'(\mathcal{T})) \leq e(K) - (\ell' - \alpha_K + 1) \leq \max\{d', e(K)\} - \ell'$, which is a contradiction to the definition of $K'(\mathcal{T})$. $\square$

Note that the lower bound of $|\mathcal{C}_{\ell'}(K)|$ cannot easily be lowered. In the "worst case", the size of K' is only decreased by one edge in every exchange operation. An example for this can be seen in Figure 3.6.4.

Finally, we show that the decomposition $\mathcal{T}$ from Lemma 3.6.7 either already contradicts the minimality of σ^*, or there is a special path with respect to $\mathcal{T}, \sigma, W$ that we can use to retrieve a decomposition having a smaller legal order than σ^*.

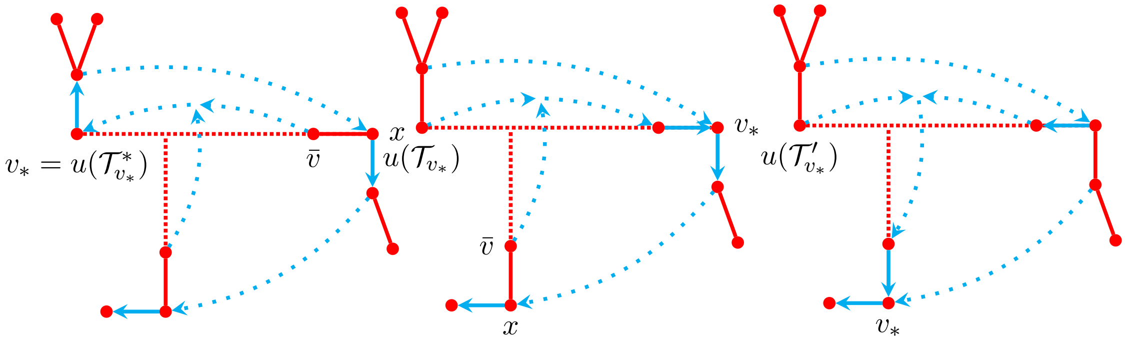

Fig. 3.6.4: An example where $\ell' = 2, \alpha_K = 0, |\mathcal{C}_2(K)| = 3$ showing the tightness of Lemma 3.6.7 when applying the augmentations of Lemma 3.6.6 starting with $\mathcal{T}^*$ on the left and progressing to the right with every operation suggested by the proof of Lemma 3.6.6. Note that every vertex on the dotted red paths is supposed to have a path to $u(\mathcal{T}^*_{v_*})$. After the two augmentations we have that $K'(\mathcal{T}') \neq \varnothing$ has $e(K) - 2$ edges.

Lemma 3.6.8. *For every $b \in \{1, \dots, k\}$ and $\ell' \in \{0, \dots, \ell\}$, the number of children of K generated by $\mathcal{B}_b(\mathcal{T}^*)$ each having at most ℓ' edges is at most $\ell' - \alpha_K + 1$.*

Proof. Let $\ell' \in \{0, \dots, \ell\}$ and suppose that K has $\ell' - \alpha_K + 2$ children, each with at most ℓ' edges. Thus, by Lemma 3.5.5 we have $e(K) \geq d' - \ell'$ and $\alpha_K \leq \ell'$. By Lemma 3.6.7 there is a tuple $(\mathcal{T}, a) \in \mathcal{F}_{K,\ell'}$ such that for every $L \in \mathcal{L}(\mathcal{T})$ we have $e(L) \leq \max\{d', e(K)\} - \ell' - 1 < e(K)$. If a is special for $\mathcal{T}$, let $v_* := a$. Let $R' := \mathcal{R}(\mathcal{T}) + v_* v'_*$ if a is special, and $R' := \mathcal{R}(\mathcal{T})$ otherwise.

Claim. $\rho(R') \leq \rho^*$

Proof. Let A be the set of components of R' that contain a vertex of K. Note that every component $\bar{L} \in A$ has $e(\bar{L}) \leq \max\{d', e(K)\}$ including the component of v_* (if a is special for $\mathcal{T}$) since (7) holds for $\mathcal{T}_a$. If $e(\bar{L}) \leq \max\{d', e(K) - 1\}$ for every $\bar{L} \in A$, we have that $\rho(R') \leq \rho^*$. Thus, suppose that there is an $\bar{L} \in A$ such that $e(\bar{L}) = e(K) > d'$. Then $|E(\bar{L}) \cap E(K)| \geq e(K) - \ell' - 1$ and thus, $|(E(K) \cap E(R')) \setminus E(\bar{L})| \leq \ell'$ since at least one edge of $E(K)$ is coloured blue in $\mathcal{T}$. Thus, for every $\bar{L}' \in A \setminus \{\bar{L}\}$ we have that $e(\bar{L}') \leq \ell' + (\ell' + 1) \leq d'$ by Observation 3.5.1 and the claim follows. *(End of proof of the claim)* ■

Suppose that $a = w(K)$ is not special for $\mathcal{T}$. Note that since (7) holds for $\mathcal{T}_a$, the component of $w(K)$ in $\mathcal{R}(\mathcal{T})$ is contained in $\mathcal{L}(\mathcal{T})$ and thus has less than $e(K)$ edges. Since (2b) holds for $\mathcal{T}$, there is a smaller legal order than σ^* for $\mathcal{T}$, which is a contradiction.

Thus, a is special for $\mathcal{T}$. Let σ and W be as mentioned in (2a) and (2b). Note that there is a minimal special path $P := [v_0, \ldots, v_l]$ with respect to $\mathcal{T}, \sigma, W$ and (v_*, v'_*) such that $i_{\sigma^*}(v_0) \leq i_{\sigma^*}(T_{v_*}(\bar{S}(\mathcal{T}_a), \mathcal{T})) = \mathcal{I}_{\ell'}(K)$. If $\mathcal{I}_{\ell'}(K) = i_{\sigma^*}(K)$, then $P^{\mathcal{B}_b(\mathcal{T})}(w(K), v_*)$ preceded by $\eta_{i_{\sigma^*}(K)}(W^*)$ is a special path and thus, we even have $i_{\sigma^*}(v_0) \leq \mathcal{I}_{\ell'}(K) - 1$ in this case. But this contradicts Corollary 3.5.4. $\square$

3.6.2 Density Calculations

In Lemma 3.6.8, we managed to bound the number and size of small children to such an extent that the function mapping small children to their parents contradicts Lemma 3.5.7. This contradiction will prove Theorem 1.15.
Before we can prove the theorem, we need one technical lemma.

Lemma 3.6.9. *For any $K \in \mathcal{K} - R^*$, we have that*

$$\sum_{C \in \mathcal{C}(K)} (d - (k+1)e(C)) \leq k \sum_{i=\alpha_K}^{\ell} (d - (k+1)i)$$

Proof. Let $\mathcal{G}$ be the set of functions $g\colon \{0,\dots,\ell\} \longrightarrow \{0,\dots,\ell\}$ with $g(i) = 0$ if $i < \alpha_K$, and $\sum_{j=\alpha_K}^{i} g(j) \leq (i - \alpha_K + 1)k$ for all $i \in \{0,\dots,\ell\}$.
Note that the function $\ell' \mapsto |\mathcal{C}_{\ell'}(K)|$ is in $\mathcal{G}$ by Lemma 3.6.8. Thus, in order to prove the lemma, it suffices to show that for every $g \in \mathcal{G}$, we have

$$\mathcal{A}(g) := \sum_{i=\alpha_K}^{\ell} g(i)\,(d - (k+1)i) \leq k \sum_{i=\alpha_K}^{\ell} (d - (k+1)i)\,.$$

Note that the inequality holds if $g(i) \leq k$ for every $i \in \{0,\dots,\ell\}$ since $d-(k+1)\ell > 0$. Let $g^* \in \mathcal{G}$ maximize $\mathcal{A}$ and suppose that g^* does not satisfy the inequality. Let $j_1 \in \{0,\dots,\ell\}$ be minimal such that $g^*(j_1) > k$. Then there is a $j_2 \in \{\alpha_K,\dots,j_1-1\}$ with $g^*(j_2) < k$ since $g^* \in \mathcal{G}$.
Let $g\colon \{0,\dots,\ell\} \longrightarrow \{0,\dots,\ell\}$ be the function such that $g'(j_1) = g^*(j_1) - 1$, $g'(j_2) = g^*(j_2) + 1$, and $g'(i) = g^*(i)$ for all $i \in \{0,\dots,\ell\} \setminus \{j_1, j_2\}$. Note that $g' \in \mathcal{G}$, and furthermore,

$$\begin{aligned}
\mathcal{A}(g') &= \mathcal{A}(g^*) - \big(g^*(j_1) - g'(j_1)\big)\big(d-(k+1)j_1\big) - \big(g^*(j_2) - g'(j_2)\big)\big(d-(k+1)j_2\big)\\
&= \mathcal{A}(g^*) - \big(d-(k+1)j_1\big) + \big(d-(k+1)j_2\big)\\
&= \mathcal{A}(g^*) + (k+1)(j_1 - j_2)\\
&> \mathcal{A}(g^*),
\end{aligned}$$

which is a contradiction to the minimality of $\mathcal{A}(g^*)$. □

Now, we are able to prove Theorem 1.15.

Proof of Theorem 1.15. Let $f\colon \mathcal{C} \longrightarrow \mathcal{K} - R^*$ be the function that maps small red components of $H_{\mathcal{T}^*}$ to their respective parent component. We show that f contradicts Lemma 3.5.7, which proves Theorem 1.15.
Let $K \in \mathcal{K} - R^*$. Note that $\hat{\beta}(K) \leq 0$ by Observation 3.3.4. Now, suppose that K has a small child. Then $\alpha_K \leq \ell$ by Lemma 3.5.5. We have that

$$\begin{aligned}
&\hat{\beta}(K) + \sum_{C \in \mathcal{C}(K)} \hat{\beta}(C) \\
&= d\Big(e(K) + 1 + \sum_{C \in \mathcal{C}(K)} \big(e(C) + 1\big)\Big) - (d + k + 1)\Big(e(K) + \sum_{C \in \mathcal{C}(K)} e(C)\Big) \\
&= d - (k+1)e(K) + \sum_{C \in \mathcal{C}(K)} \big(d - (k+1)e(C)\big) \\
&\overset{\text{Lemma 3.6.9}}{\leq} d - (k+1)(d' - \alpha_K) + k \sum_{i=\alpha_K}^{\ell} (d - (k+1)i) \\
&\leq d - (k+1)\Big(d + k\ell\Big(\frac{d}{k+1} - \frac{\ell+1}{2}\Big) - \alpha_K\Big) \\
&\qquad + k(\ell - \alpha_K + 1)d - k(k+1)\Big(\frac{\ell(\ell+1) - \alpha_K(\alpha_K - 1)}{2}\Big) \\
&= (k+1)\alpha_K - k\alpha_K d + k(k+1)\frac{\alpha_K(\alpha_K - 1)}{2} \\
&= \alpha_K k\Big(1 + \frac{(k+1)(\alpha_K - 1)}{2} - d\Big) + \alpha_K \\
&\leq \alpha_K k\Big(1 + \frac{(k+1)(\ell - 1)}{2} - d\Big) + \alpha_K \\
&\leq \alpha_K k\Big(1 + \frac{d - 1 - (k+1)}{2} - d\Big) + \alpha_K \\
&= -\alpha_K k\frac{d+k}{2} + \alpha_K \\
&\leq 0.
\end{aligned}$$

But this contradicts Lemma 3.5.7. ☐

3.7 The Strong Nine Dragon Tree Conjecture for $d \leq 2(k+1)$

In this section, we prove Theorem 1.14, showing that (k, d)-sparse graphs with $\gamma(G) \leq k+1$ are (k, d)-decomposable for $d \leq 2(k+1)$, building on Section 3.3 and Section 3.5.

Recall that the Strong Nine Dragon Tree Conjecture for $d \leq k+1$ was proven in [38] by the author and Moore. Furthermore, Theorem 1.15 reproves this case. Thus, for the rest of the section, we let $k+1 < d \leq 2(k+1)$ (and we still let $k \geq 1$ due to 3.3.1). In particular, $d \geq 3$. Thus, we have $\ell = 1$, so that small components either have zero or one edge. The techniques we use for this case build on the same ideas from [38] and from the previous section: we bound the number of small children as in Lemma 3.6.8. For $d \leq k+1$, where $\ell = 0$, this bound suffices to prove the Strong Nine Dragon Tree Conjecture. But this is not the case if $d > k+1$, where $\ell = 1$. For example, a non-small component K with $e(K) = d$ can have k children with zero edges and k children with one edge. Then, $\hat{\beta}(K) + \sum_{C \in \mathcal{C}(K)} \hat{\beta}(C) = k\big(d-(k+1)\big) > 0$, which prevents us from contradicting Lemma 3.5.7 using the function f that maps small children to their parents, as in Section 3.6.

We will not be able to bound the number and sizes of small children further. However, we will introduce new augmentations that are less local, and introduce a new counting argument to eventually find a function f that contradicts Lemma 3.5.7.

The idea is the following. Suppose K is “bad”, meaning it has two children generated by $\mathcal{B}_b(\mathcal{T}^*)$, one with zero edges generated by (x, x'), and one with one edge. Then we can force a certain structure on K, which allows us to isolate x from K, link x to another component K', and then use (x, x') in a “usual” edge exchange with a red edge from K'. In this case, we will call K an interesting neighbour of K', and K' can “use” K for similar augmentations as if K was a child of K' with one edge.

Finally, the idea of the counting argument is that the child of K with one edge is not assigned to K, but to K' (and it might be reassigned to another component K'' if K' is an interesting neighbour of K'' and so on).

Now, let us start with the formal definition of interesting neighbours. Note that the terms and the situation described in the following is depicted on the left side of Figure 3.7.1.

Let K be a red component of $H_{\mathcal{T}^*}$ and let $x \in V(K)$. If x has degree 1 in K, then let n_x be its only neighbour in K. If $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$ and $x' \neq r$, then let x'' denote the parent of x' in $\mathcal{B}_b(\mathcal{T}^*)$.

Let K be a red component of $H_{\mathcal{T}^*}$ and $x \in V(K)$. Let $b \in \{1, \ldots, k\}$ such that $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$, where $x' \neq r$ has exactly one incident red edge $x'n_{x'}$ within its red component $L \neq K$ in $H_{\mathcal{T}^*}$. Furthermore, let the red component in $H_{\mathcal{T}^*}$ of x'' not contain an edge and be a child of L. Moreover, suppose there is a directed path from $n_{x'}$ to x in $\mathcal{B}_b(\mathcal{T}^*)$. Then we say L *is an interesting neighbour of* K *generated by* (x, x') *(or generated by* $\mathcal{B}_b(\mathcal{T}^*)$*)*. Furthermore, we call a red component a *relevant neighbour of* K if it is a small child of K or if it is an interesting neighbour of K.

Fig. 3.7.1: An interesting neighbour generated by (x, x') in $\mathcal{T}^*$ and in $\mathcal{T}_x$.

Observation 3.7.1. *Let* C *be an interesting neighbour of* K *generated by* (x, x'). *Then* $e(C) \geq d$ *by Corollary 3.5.5 (as* x'' *has no incident red edges) and thus, a relevant neighbour cannot be both interesting and small.*

We introduce more notation that will be useful when performing exchanges. Let C_x be a relevant neighbour of K generated by (x, x').

If C_x is interesting, we have that $n_{x'}$ is a descendant of x in $\mathcal{B}_b(\mathcal{T}^*)$ and x' is not. In this case, let $\mathcal{T}_x$ denote the decomposition we obtain from $\mathcal{T}^*$ by performing $(x, x') \leftrightarrow n_{x'}x'$. Note that $\mathcal{T}_x$ is depicted on the right side of Figure 3.7.1.

If C_x is small, then we let $\mathcal{T}_x := \mathcal{T}^*$.

Furthermore, we let $(\bar{x}, \bar{x}') := (x, x')$ if C_x is small, and $(\bar{x}, \bar{x}') := (x', x'')$ if C_x is interesting. Note that $(\bar{x}, \bar{x}')$ always generates a small child.

Moreover, we let $c_x = 0$ if $e(C_x) = 0$, and $c_x = 1$ otherwise. If C_x is interesting, then let $C'_x := C_x - x'$.

The next lemma is a generalization of Lemma 3.5.3, and is the first showcase on how we can treat interesting neighbours in the same way as small children with one edge.

Lemma 3.7.2. *The component* R^* *does not have relevant neighbours.*

Proof. Note that the decompositions of this proof are depicted in Figure 3.7.2. Suppose to the contrary that R^* has a relevant neighbour C_x generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$ for some $b \in \{1, \dots, k\}$. Let $P = [x_1, \dots, x_n]$ be the path from x to r in $\mathcal{R}(\mathcal{T}_x)$. Let $i \in \{1, \dots, n\}$ such that x_i is the first vertex on P that is not a descendant of $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$. This vertex exists and $i > 1$ since x is a descendant of $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$ and r is not. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow x_{i-1}x_i$. The component K_r of r in $\mathcal{R}(\mathcal{T}')$ contains at least two edges by the way we chose r in Lemma 3.5.2. For the component K_x of x in $\mathcal{R}(\mathcal{T}')$ we have

$$e(K_x) \leq e(R^*) - |\{x_{i-1}x_i\}| - e(K_r) + |\{xx'\}| + c_x < e(R^*).$$

Since K_r is a proper subgraph of R^*, we obtain a contradiction to the minimality of ρ^*. ☐

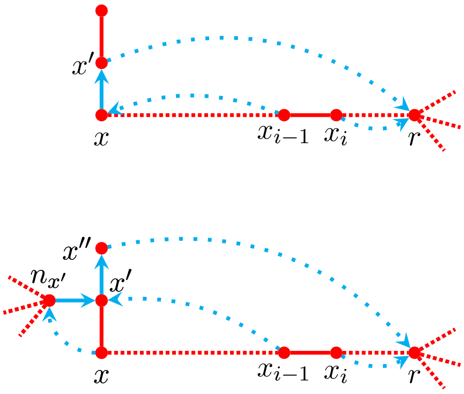

Fig. 3.7.2: The decomposition $\mathcal{T}_x$ and $\mathcal{T}'$ in the proof of Lemma 3.7.2: in the first row, we have the case where R^* has a small child and in the second row, R^* has an interesting neighbour.

Lemma 3.5.4 shows in which cases we can improve the legal order by a special path augmentation. We will very often perform such an augmentation after performing other edge exchanges. The next lemma simply states other conditions under which we can apply Lemma 3.5.4. These conditions may look a bit artificial at first glance, however, they will be very easy to check and will save us a lot of time.

Recall that for $\mathcal{T}, \mathcal{T}' \in \mathcal{F}$, $\Delta(\mathcal{T}, \mathcal{T}')$ is the set of vertices v for which there is a blue arc $(v, u) \in E(\mathcal{B}_b(\mathcal{T}))$ for some $b \in \{1, \dots, k\}$, but this arc is not contained in $E(\mathcal{B}_b(\mathcal{T}'))$.

Lemma 3.7.3. *Let $\bar{\mathcal{T}} \in \mathcal{F}$ be a decomposition containing a blue arc (a, a'). Let A be the set of vertices which have a blue directed path to a in $\bar{\mathcal{T}}$ and let B be the set* *of all the vertices of* $\Delta(\mathcal{T}^*, \bar{\mathcal{T}})$ *which are not in* A*. Let* $i_A := i_{\sigma^*}(A)$ *and let* L *be the component of* $w_{i_A}(W^*)$ *in* $\mathcal{R}(\bar{\mathcal{T}})$*. We have that* $\rho(\mathcal{R}(\bar{\mathcal{T}}) + aa') > \rho^*$ *if the following conditions are met:*

a) $i_{\sigma^*}(a') > i_A$*.*

b) $i_{\sigma^*}(B) \geq i_A$*.*

c) If $L \neq R^*_{i_A}$ *and* $e(L) = e(R^*_{i_A})$*, then* L *contains a vertex of* A*.*

d) If the component of a vertex $v \in V(R^*_{i_A}) \cap V(L)$ *in* $\mathcal{R}(\bar{\mathcal{T}})$ *contains more than* $e(R^*_{i_A})$ *edges, then* $v \in A$*.*

Proof. Suppose to the contrary that $\rho(\mathcal{R}(\bar{\mathcal{T}}) + aa') = \rho^*$ (recall that by definition of ρ^*, $\rho(\mathcal{R}(\bar{\mathcal{T}}) + aa') < \rho^*$ never occurs).
By b), there is a legal order $\bar{\sigma} = (\bar{R}_1, \ldots, \bar{R}_{\bar{t}})$ for $\bar{\mathcal{T}}$ with $\bar{R}_i = R^*_i$ for all $i < i_A$ and $\bar{R}_{i_A} = L$, and furthermore, there is a $\bar{W} \in \operatorname{Aux}(\bar{\mathcal{T}}, \bar{\sigma})$ with $\eta_j(\bar{W}) = \eta_j(W^*)$ for all $j \in \{2, \ldots, i_A\}$.
Furthermore, we have that $w_{i_A}(W^*) \notin A$, since otherwise the tail t of $\eta_{i_A}(W^*)$, which has $i_{\sigma^*}(t) = i_A - 1$, would also be in A. Hence, $e(L) \leq e(R^*_{i_A})$ by d). In fact, $e(L) = e(R^*_{i_A})$ as otherwise we obtain $\bar{\sigma} < \sigma^*$, contradicting our choice of σ^*.
Note that by the definition of i_A, there is a vertex of A in $V(R^*_{i_A})$. Considering c), we have that L always contains a vertex of A.
Thus, we can complete $\bar{\sigma}$ after L by traversing a blue path in $\bar{\mathcal{T}}$ from a vertex of $V(A) \cap V(L)$ to a' going over a, and adding every red component to $\bar{\sigma}$ that is not yet in $\bar{\sigma}$. We obtain $i_{\bar{\sigma}}(a') > i_A$ by a). Thus, there is a minimal special path starting at L with respect to $\bar{\mathcal{T}}$, $\bar{\sigma}$, $\bar{W}$ and (a, a'), which is a contradiction to Lemma 3.5.4. □

We can now generalize Lemma 3.5.5 by considering the case where C is an interesting neighbour of K.

Lemma 3.7.4. *Let* K *be a red component of* $H_{\mathcal{T}^*}$ *and* C_x *a relevant neighbour of* K*. Then* $e(K) \geq d - c_x$ *and in particular,* K *is not small.*

Proof. By Lemma 3.5.5, we know that if the lemma is not true, then $e(K) \leq d - 2$ and C_x is an interesting neighbour of K. Let C_x be generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$ for some $b \in \{1, \ldots, k\}$. We want to find a contradiction to Lemma 3.7.3 and choose $\bar{\mathcal{T}} := \mathcal{T}_x$ and $(a, a') := (x', x'')$. Note that $B = \varnothing$ since all vertices in $\Delta(\mathcal{T}^*, \bar{\mathcal{T}})$ belong to the path $P^{\mathcal{B}_b(\mathcal{T}^*)}(n_{x'}, x)$, but this path is reoriented in $\mathcal{B}_b(\mathcal{T}_x)$ and $\mathcal{B}_b(\mathcal{T}_x)$ also contains the arc $(n_{x'}, x')$, and thus, all of these vertices have a blue path to x' in

$\mathcal{B}_b(\mathcal{T}_x)$, and so cannot be in B. Thus, b) holds and it is also clear that a) holds. Let K' be the component of x in $\mathcal{R}(\mathcal{T}_x) + x'x''$ and note that C'_x is also a component of this forest. We have $e(C'_x) = e(C_x) - 1$ and $e(K') = e(K) + |\{xx', x'x''\}| \leq d$. Thus, d) holds and $\rho(\mathcal{R}(\mathcal{T}_x) + x'x'') = \rho^*$. As $x' \in A$, we also have that c) holds contradicting Lemma 3.7.3. ☐

We are looking for exchanges improving the legal order if a red component has two relevant neighbours generated by edges $(x, x'), (y, y')$ of the same tree. The next lemma shows that we can find an edge e on the red path from x to y that can be exchanged with either $(\bar{x}, \bar{x}')$ or $(\bar{y}, \bar{y}')$. It also distinguishes between two cases depending on properties of the red path between x and y. These cases can be seen in Figure 3.7.3.

Lemma 3.7.5. *Let $b \in \{1, \dots, k\}$, let $K \neq R^*$ be a red component of $H_{\mathcal{T}^*}$ and let x and y be vertices of K. Furthermore, let C_x be a relevant neighbour of K generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$, and let y not be a descendant of $\bar{x}$. Let $y' \neq x'$ be the parent vertex of y in $\mathcal{B}_b(\mathcal{T}^*)$. Furthermore, let $[x_1, \dots, x_n]$ be the path from x to y in $\mathcal{R}(\mathcal{T}^*)$. Then one of the following two cases applies:*

- *There is an $i \in \{1, \dots, n-1\}$ such that x_i is a descendant of x in $\mathcal{B}_b(\mathcal{T}_x)$ and x_{i+1} is not. Thus, $(\bar{x}, \bar{x}') \leftrightarrow x_i x_{i+1}$ holds in $\mathcal{T}_x$. Furthermore, x_{i+1} is a descendant of y in $\mathcal{B}_b(\mathcal{T}_x)$.*
  In this case we say that $x \xrightarrow{(2,b)} y$ holds with edge (x_i, x_{i+1}).
- *It does not hold $x \xrightarrow{(2,b)} y$. There is an $i \in \{1, \dots, n-2\}$ such that x_i is a descendant of x in $\mathcal{B}_b(\mathcal{T}_x)$ and x_{i+1} is not and i is maximal with this property. Thus, $(\bar{x}, \bar{x}') \leftrightarrow x_i x_{i+1}$ holds in $\mathcal{T}_x$. Furthermore, there is a $j \in \{i+2, \dots, n\}$ such that x_j is a descendant of y in $\mathcal{B}_b(\mathcal{T}_x)$ and x_{j-1} is not and j is minimal with this property. Thus, we have $(y, y') \leftrightarrow x_j x_{j-1}$ in $\mathcal{T}_x$. In this case we say that $x \xrightarrow{(1,b)} y$ holds with edges $(x_i, x_{i+1}), (x_{j-1}, x_j)$.*

Proof. Suppose it does not hold $x \xrightarrow{(2,b)} y$. Note that y also does not have a path to $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$: if it had one, then it would have a path in $\mathcal{B}_b(\mathcal{T}^*)$ to a vertex of $\Delta(\mathcal{T}^*, \mathcal{T}_x) = P^{\mathcal{B}_b(\mathcal{T}^*)}(n_{x'}, x)$ and thus, would be a descendant of x in $\mathcal{B}_b(\mathcal{T}^*)$, a contradiction. Thus, there is an $i \in \{1, \dots, n-2\}$ as described in $\xrightarrow{(1,b)}$ since x is a descendant of $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$ and y is not. To see that $i \neq n-1$, if it were, we would have $x \xrightarrow{(2,b)} y$ with edge (x_i, x_{i+1}). Now, let $j \geq i+2$ be minimal such that x_j is a descendant of y in

$\mathcal{B}_b(\mathcal{T}^*)$. Observe that j exists since y is a descendant of y in $\mathcal{B}_b(\mathcal{T}^*)$. If x_{j-1} was a descendant of y, then $j = i + 2$ and thus, we have $x \xrightarrow{(2,b)} y$, a contradiction. $\square$

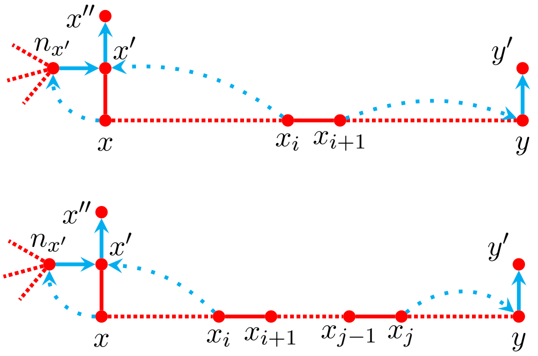

Fig. 3.7.3: We have $x \xrightarrow{(2,b)} y$ in $\mathcal{T}_x$ above and $x \xrightarrow{(1,b)} y$ in $\mathcal{T}_x$ below.

The next lemma is an observation for the case in which the reoriented blue path of an exchange is trivial.

Lemma 3.7.6. *Let K be a red component of $H_{\mathcal{T}^*}$ and C_x a relevant neighbour of K generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$ for some $b \in \{1, \ldots, k\}$. Furthermore, let $xu \in E(K)$ be a red edge, and let K_x be the component of $K - xu$ containing x.*
If $e(K_x) = 0$ and there is no path in $\mathcal{B}_b(\mathcal{T}_x)$ from u to $\bar{x}$ (and thus, no path from u to x), then $d = 3$, $e(K) = 2$ and $c_x = 1$.

Proof. Suppose that $e(K_x) = 0$ and there is no path in $\mathcal{B}_b(\mathcal{T}_x)$ from u to $\bar{x}$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow xu$. Let K'_x and K'_u be the components in $\mathcal{R}(\mathcal{T}')$ of x and u, respectively. Note that if C_x is small, we have $\mathcal{B}_b(\mathcal{T}') = (\mathcal{B}_b(\mathcal{T}^*) - (x, x')) + (x, u)$ and otherwise $\mathcal{B}_b(\mathcal{T}') = \mathcal{B}_b(\mathcal{T}^*) \setminus \{(x, x'), (x', x'')\} \cup \{(x, u), (x', n_{x'})\}$ since $P^{\mathcal{B}_b(\mathcal{T}^*)}(n_{x'}, x)$ is reoriented when obtaining $\mathcal{T}_x$ from $\mathcal{T}^*$, but the resulting path is again reoriented when obtaining $\mathcal{T}'$ from $\mathcal{T}_x$. Hence, there is a legal order $\sigma' = (R'_1, \ldots, R'_{t'})$ for $\mathcal{T}'$ with $R'_j = R^*_j$ for all $j < i := \min\{i_{\sigma^*}(K), i_{\sigma^*}(C_x)\}$ and $(R^*_i, R'_i) \in \{(K, K'_x), (K, K'_u), (C_x, C'_x), (C_x, K'_x)\}$ (where $R'_i = C'_x$ or $R^*_i = C_x$ can only be possible if C_x is interesting). Note that $e(K'_u) < e(K)$ and if C_x is interesting, we have $e(C'_x) < e(C_x)$ and $e(C_x) \geq d$ by Observation 3.7.1. Since $e(K'_x) = 1 + c_x < d$, we have $\rho(\mathcal{R}(\mathcal{T}')) = \rho^*$. Thus, it must be $R^*_i = K$, $R'_i = K'_x$ and $e(K'_x) \geq e(K)$ or otherwise we have $\sigma' < \sigma^*$, which is a contradiction. Thus, $e(K) \leq 2$ and it cannot be $e(K) = 1$ because of Lemma 3.7.4. Hence, $e(K) = 2$, $c_x = 1$ and furthermore, $d = 3$ by Lemma 3.7.4. $\square$

3.7.1 Bounding Relevant Neighbours - The Case $x \xrightarrow{(2,b)} y$

In this subsection, we want to characterize the case where K has two relevant neighbours generated by $(x,x'),(y,y')$ and $x \xrightarrow{(2,b)} y$ holds. Before that, we consider the following lemma, which describes how the components around K will look like in $\bar{\mathcal{T}}$ when we are applying Lemma 3.7.3 in the upcoming sections.

Lemma 3.7.7. *Let K be a red component of $H_{\mathcal{T}^*}$ and let C_x, C_y be two distinct relevant neighbours of K that are generated by the edges (x,x') and (y,y') of the same tree $\mathcal{B}_b(\mathcal{T}^*)$, respectively, where $b \in \{1,\dots,k\}$. Let e be an edge on the path from x to y in $\mathcal{R}(\mathcal{T}^*)$. Let F' be a forest that can be obtained from $(\mathcal{R}(\mathcal{T}^*)-e) \cup \{xx', yy'\}$ by adding $z'z''$ and removing $z'n_{z'}$ for every $z \in \{x,y\}$ such that C_z is interesting. Let K_z be the component in $K-e$ of z for any $z \in \{x,y\}$. If $e(K_x) \geq c_y$ and $e(K_y) \geq c_x$, then $\rho(F') = \rho^*$, and the components of x and y in F' each have at most $e(K)$ edges.*

Proof. Let K'_z be the component of z in F' for $z \in \{x,y\}$. If C_z is interesting, then F' contains the component C'_z with $e(C'_z) = e(C_z) - 1$. Further, we have that $e(K'_x) = e(K) - e(K_y) - |\{e\}| + |\{xx'\}| + c_x \leq e(K)$ and similarly $e(K'_y) \leq e(K) - e(K_x) - |\{e\}| + |\{yy'\}| + c_y \leq e(K)$.
Furthermore, $e(K) + |\{xx', yy'\}| + c_x + c_y = e(K'_x) + e(K'_y) + |\{e\}|$. Thus, if one of the components K'_x, K'_y has $e(K)$ edges, then the other one has $1 + c_x + c_y \leq d$ edges. Hence, $\rho(F') = \rho^*$. ☐

Lemma 3.7.8. *Let K be a red component of $H_{\mathcal{T}^*}$ and let C_x, C_y be two distinct relevant neighbours of K generated by $(x,x'),(y,y') \in E(\mathcal{B}_b(\mathcal{T}^*))$, respectively, where $b \in \{1,\dots,k\}$. Furthermore, suppose we have $x \xrightarrow{(2,b)} y$ with edge (u,v). Then $e(K) \geq d$, $c_x = 1$ and the component of y in $K-uv$ has no edges.*

Proof. Let K_x and K_y be the components of $K-uv$ containing x and y, respectively. Suppose towards a contradiction that either $e(K) < d$, $c_x = 0$ or the component of y in $K-uv$ has an edge. Note that if one of the latter two conditions is true, we have $e(K_y) \geq c_x$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow uv$. We have that $\Delta(\mathcal{T}^*, \mathcal{T}') = V(P^{\mathcal{B}_b(\mathcal{T}^*)}(u,\bar{x}))$. Note that all these vertices have a directed blue path towards y in $\mathcal{B}_b(\mathcal{T}')$ through (u,v). Thus, we can obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(y,y') \leftrightarrow n_{y'}y'$ if C_y is interesting. If C_y is small, let $\mathcal{T}'' := \mathcal{T}'$. Note that $\mathcal{T}''$ is depicted in Figure 3.7.4. Let K'_x and K'_y be the components in $\mathcal{R}(\mathcal{T}'') + (\bar{y}, \bar{y}')$ of x and y, respectively. We want to obtain a contradiction to Lemma 3.7.3 and

choose $\bar{\mathcal{T}} := \mathcal{T}''$, $(a, a') := (\bar{y}, \bar{y}')$. Using the notation from Lemma 3.7.3, we have $\Delta(\mathcal{T}^*, \bar{\mathcal{T}}) \subseteq A$ and thus, $B = \varnothing$. Recalling that $i_{\sigma^*}(B) = \infty$ if $B = \varnothing$, we have that b) holds and it is also clear that a) holds. Further, c) holds since $x, y \in A$. Therefore, it suffices to show that d) holds and $\rho(\mathcal{R}(\mathcal{T}'') + aa') = \rho^*$. We split into cases:
Case 1: $e(K) \geq d$:
In this case, $e(K_y) \geq c_x$. By Lemma 3.7.6, we have that $e(K_x) \geq 1$, and by Lemma 3.7.7, we have that $\rho(\mathcal{R}(\mathcal{T}'') + aa') = \rho^*$ and $e(K'_x), e(K'_y) \leq e(K)$. Furthermore, $\bar{x}, \bar{y} \in A$ and hence, d) holds, a contradiction.
Case 2: $e(K) < d$:
In this case we have $e(K) = d - 1$ and $c_x = 1$ by Lemma 3.7.4. For $z \in \{x, y\}$ we have that $e(K'_z) \leq e(K) - |\{e\}| + 1 + c_z \leq d$ and thus, $\rho(\mathcal{R}(\mathcal{T}'') + aa') = \rho^*$. We split into subcases:
Subcase 2.1: $R^*_{i_A} \neq K$: In this case, we immediately obtain that d) holds.
Subcase 2.2: $R^*_{i_A} = K$ **and** $w_{i_A}(W^*) \in V(K_y)$:
In this case, there is a legal order $\sigma' = (R'_1, \dots, R'_{t'})$ for $\mathcal{T}'$ such that $R'_i = R^*_i$ for all $i < i_A$ and $R'_{i_A} = K_y$. As $e(K_y) < e(K)$, we have $\sigma' < \sigma^*$. Furthermore, $\rho(\mathcal{R}(\mathcal{T}')) = \rho^*$, a contradiction.
Subcase 2.3: $R^*_{i_A} = K$ **and** $w_{i_A}(W^*) \in V(K_x)$:
In this case, there is a legal order $\sigma'' = (R''_1, \dots, R''_{t''})$ for $\mathcal{T}''$ such that $R''_i = R^*_i$ for all $i < i_A$, $R''_{i_A} = K'_x$.
There also is a $W'' \in \mathrm{Aux}(\mathcal{T}'', \sigma'')$ with $\eta_j(W'') = \eta_j(W^*)$ for all $j \in \{2, \dots, i_A\}$.
Furthermore, if C_y is interesting, we can choose $i_{\sigma^*}(K'_y - y'') > i_A$, and otherwise we have that C_y is small and we can choose $i_A < i_{\sigma^*}(K'_y), i_{\sigma^*}(C_y)$.
By the definition of A, we have $w_{i_A}(W^*) \notin A$ and thus, $x \neq w_{i_A}(W^*) (= w_{i_A}(W''))$. Let $P = [v_0, \dots, v_l]$ be a minimal special path with respect to $\mathcal{T}''$, σ'', W'' and $(\bar{y}, \bar{y}')$ that is smaller or equal to the special path from x to $\bar{y}'$ in $\mathcal{B}_b(\mathcal{T}'')$. Note that v_0 lies on the red path from x to $w_{i_A}(W'')$ in $K'_x \subseteq W''$. Thus, the component of v_{-1} in $K'_x - x_0x_{-1}$ has at most $e(K'_x) - |\{\bar{x}\bar{x}'\}| - c_x \leq d - 2 < e(K)$ edges, which implies that the legal order we obtain from (3) in Lemma 3.3.5 is smaller than σ^*, a contradiction (we can apply (3) by the same reasoning as in the proof of Lemma 3.5.4). □

3.7.2 Bounding Relevant Neighbours - The Case $x \xrightarrow{(1,b)} y$

In this subsection, let K again be a red component of $H_{\mathcal{T}^*}$ and let C_x, C_y be two distinct relevant neighbours of K generated by $(x, x'), (y, y') \in E(\mathcal{B}_b(\mathcal{T}^*))$, respec-

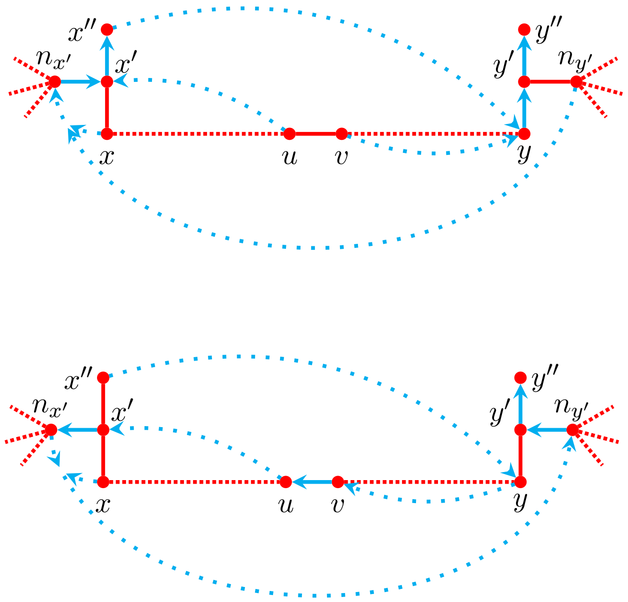

Fig. 3.7.4: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Lemma 3.7.8 with C_x and C_y being interesting. Note that $P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y)$ might not visit x' and x'' and there might not be a path from x'' to y in $\mathcal{B}_b(\mathcal{T}'')$.

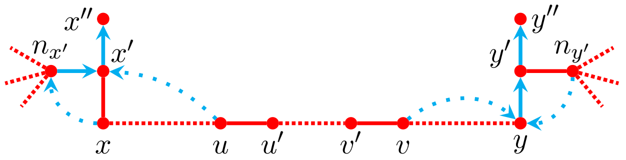

Fig. 3.7.5: The component K and its neighbours in $\mathcal{T}_x$ in the case where C_x and C_y are interesting.

tively, where $b \in \{1, \dots, k\}$. Furthermore, let $x \xrightarrow{(1,b)} y$ with edges (u, u'), (v', v). See Figure 3.7.5 as an illustration.
Let K_x be the component of $K - uu'$ containing x, and let K_y be the component of $K - vv'$ containing y.

For $v \in V(G)$ let D_v^b denote the set of descendants of v in $\mathcal{B}_b(\mathcal{T}_x)$. Furthermore, for $v_1, v_2 \in V(G)$ let $D_{v_2,v_1}^b = D_{v_2}^b \setminus D_{v_1}^b$. Let $\bar{Y}$ be the set of vertices having a blue directed path P to $\bar{y}$ in $\mathcal{B}_b(\mathcal{T}_x)$ such that $V(P) \cap D_{\bar{x}}^b = \varnothing$.
Note that for the following two definitions b is fixed to our choice from the beginning of the subsection. Let $\bar{X}_{v_1}$ be the set of vertices having a blue directed path P

to $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$ such that $V(P) \cap D^b_{v1,\bar{x}} = \varnothing$.

As an illustration of these definitions, note that for the four dotted blue paths belonging to $\mathcal{B}_b(\mathcal{T}_x)$ in Figure 3.7.5, we have $V(P^{\mathcal{B}_b(\mathcal{T}_x)}(u,\bar{x})), V(P^{\mathcal{B}_b(\mathcal{T}_x)}(x,n_{x'})) \subseteq D^b_{\bar{x}} \subseteq \bar{X}_{\bar{y}}$ and $V(P^{\mathcal{B}_b(\mathcal{T}_x)}(v,y)), P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'},y)) \subseteq D^b_{\bar{y},\bar{x}} \subseteq \bar{Y}$.

We will end the subsection with a similar characterization of K and its relevant neighbours like in Lemma 3.7.8. An intermediate objective will be Claim 3.7.11 showing that C_y is interesting and that $P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'},y')$ goes over $(\bar{x},\bar{x}')$.

Claim 3.7.9. *If $i_{\sigma^*}(\bar{X}_{\bar{y}}) = i_{\sigma^*}(K)$, then $w_{i_{\sigma^*}(K)}(W^*) \in V(K_x)$.*

Proof. Suppose to the contrary that $i_{\sigma^*}(\bar{X}_{\bar{y}}) = i_{\sigma^*}(K)$ and $w_{i_{\sigma^*}(K)}(W^*) \in V(K) \setminus V(K_x)$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x},\bar{x}') \leftrightarrow uu'$. Note that $\Delta(\mathcal{T}^*,\mathcal{T}') \subseteq D^b_{\bar{x}}$ and thus, $\Delta(\mathcal{T}^*,\mathcal{T}') \subseteq \bar{X}_{\bar{y}}$. Let K'_x and K'_y be the components of x and y in $\mathcal{R}(\mathcal{T}')$, respectively. We have $e(K'_x) \leq e(K) + 1 + c_x - |\{uu',vv'\}| \leq e(K)$ edges and $e(K'_y) \leq e(K) - |\{uu'\}| < e(K)$. Furthermore, $e(K'_x) + e(K'_y) + |\{uu'\}| = e(K) + |\{xx'\}| + c_x$. Thus, if $e(K'_x) = e(K)$, then $e(K'_y) = c_x < d$ and thus, $\rho(\mathcal{R}(\mathcal{T}')) = \rho^*$. There also exists a legal order $\sigma' = (R'_1,\dots,R'_{t'})$ with $R'_i = R^*_i$ for all $i < i_{\sigma^*}(K)$ and $R'_{i_{\sigma^*}(K)} = K'_y$. Thus, $\sigma' < \sigma^*$, which is a contradiction. ☐

Claim 3.7.10. *We have $i_{\sigma^*}(\bar{Y}) \geq i_{\sigma^*}(\bar{X}_{\bar{y}})$ if either C_y is small, or C_y is interesting and $\bar{x} \notin V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'},y'))$. Further, if C_y is interesting and $\bar{x} \notin V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'},y'))$, then there is no path from u' to y' in $\mathcal{B}_b(\mathcal{T}_x)$.*

Proof. Throughout, if C_y is interesting, we assume that $\bar{x} \notin V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'},y'))$. Suppose to the contrary that

(1) $i_{\sigma^*}(\bar{Y}) < i_{\sigma^*}(\bar{X}_{\bar{y}})$ or

(2) C_y is interesting and there is a path from u' to y' in $\mathcal{B}_b(\mathcal{T}_x)$.

We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x},\bar{x}') \leftrightarrow uu'$. If C_y is interesting, then the path from $n_{y'}$ to y in $\mathcal{B}_b(\mathcal{T}_x)$ does not visit a vertex of $\bar{X}_{\bar{y}}$ by our assumptions. As $\Delta(\mathcal{T}_x,\mathcal{T}') \subseteq D^b_{\bar{x}}$, there is (still) a path from $n_{y'}$ to y in $\mathcal{B}_b(\mathcal{T}')$ and as $(y,y') \in E(\mathcal{B}_b(\mathcal{T}'))$, there is no path from y' to y in this tree. We obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(y,y') \leftrightarrow y'n_{y'}$ if C_y is interesting. If C_y is small, we let $\mathcal{T}'' := \mathcal{T}'$. For an illustration of $\mathcal{T}''$ see Figure 3.7.6. We want to find a contradiction to Lemma 3.7.3 and choose $\bar{\mathcal{T}} := \mathcal{T}''$, $(a,a') := (\bar{y},\bar{y}')$. In Case (2), we have that $\bar{Y} \cup \bar{X}_{\bar{y}} \subseteq A$ and $B = \varnothing$ since every vertex of $\bar{X}_{\bar{y}}$ has a path to u' in $\mathcal{B}_b(\mathcal{T}'')$. In Case (1), we have $\bar{Y} \subseteq A$ and $B \subseteq D^b_{\bar{x}} \subseteq \bar{X}_{\bar{y}}$. Thus, in both cases we have that b) holds and it is also clear that a) holds. Note that in Case (1), we have that $K \neq R^*_{i_A}$ and $K'_x \neq L$. We have $y \in A$ and in Case (2) we also have $x \in A$. Thus, c) holds and in Case (1) we also have that d) holds since $\bar{y} \in A$. It remains to show that we have $\rho(\mathcal{R}(\mathcal{T}'') + \bar{y}\bar{y}') = \rho^*$ and we also have to show that d) holds in Case (2).

First, we suppose that $e(K_x) \geq 1$. Since the component of y in $K - uu'$ also contains an edge vv', we have by Lemma 3.7.7 that $\rho(\mathcal{R}(\mathcal{T}'') + \bar{y}\bar{y}') = \rho^*$ and the components K'_x and K'_y of x and y in $\mathcal{R}(\mathcal{T}'') + \bar{y}\bar{y}'$, respectively, have $e(K'_x), e(K'_y) \leq e(K)$. As $\bar{x}, \bar{y} \in A$ we have that d) holds in Case (2), which is a contradiction to Lemma 3.7.3.

Now, let $e(K_x) = 0$. By Lemma 3.7.6 we have that $d = 3$, $e(K) = 2$, $c_x = 1$ and thus, $E(K) = \{uu', vv'\}$. We have that $e(K'_x) = 2$, $e(K'_y) = 3$ and thus, $\rho(\mathcal{R}(\mathcal{T}'') + \bar{y}\bar{y}') = \rho^*$. In Case (2) we have $x, \bar{x}, y, \bar{y} \in A$ and thus, d) holds. ☐

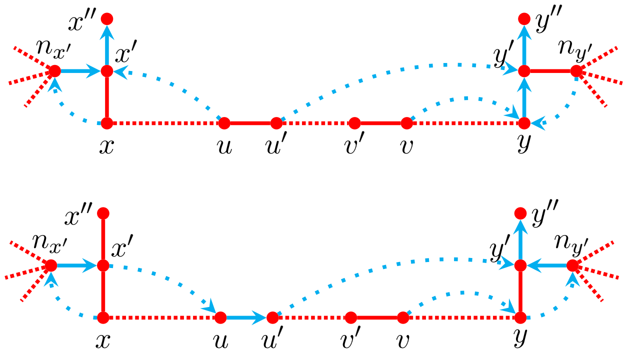

Fig. 3.7.6: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Claim 3.7.10 in the case where C_x and C_y are interesting and $P^{\mathcal{B}_b(\mathcal{T}_x)}(u', y')$ exists.

Claim 3.7.11. *We have that* C_y *is interesting and* $\bar{x} \in V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y'))$.

Proof. Suppose that either C_y is small or that C_y is interesting and $\bar{x} \notin V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y'))$. By Claim 3.7.10 we have that $i_{\sigma^*}(\bar{Y}) \geq i_{\sigma^*}(\bar{X}_{\bar{y}})$ and if C_y is interesting, then there is no path from u' to y' in $\mathcal{B}_b(\mathcal{T}_x)$.

If C_y is small, we let $\mathcal{T}' := \mathcal{T}_x$ and otherwise we obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(y, y') \leftrightarrow n_{y'}y'$. We choose vertex v_i on the path $[v_1, \ldots, v_m]$ from u' to v in $\mathcal{R}(\mathcal{T}')$ such that i is minimal and there is a path in $\mathcal{B}_b(\mathcal{T}')$ from v_i to $\bar{y}$. This vertex exists and $i > 1$ considering that v has a path to y in $\mathcal{B}_b(\mathcal{T}')$, and u' does not by our assumptions. We obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(\bar{y}, \bar{y}') \leftrightarrow v_i v_{i-1}$. The decomposition $\mathcal{T}''$ is depicted in Figure 3.7.7. We will now find a contradiction to Lemma 3.7.3 by letting $\bar{\mathcal{T}} := \mathcal{T}''$ and $(a, a') = (\bar{x}, \bar{x}')$. As a consequence we have that $\bar{X}_{\bar{y}} \subseteq A$, $B \subseteq D^b_{\bar{y},\bar{x}} \subseteq \bar{Y}$ and thus, b) holds. It is also clear that a) holds. Let K'_x and K'_y be the components of x and y, respectively, in $\mathcal{R}(\mathcal{T}'') + \bar{x}\bar{x}'$. Note that by Claim 3.7.9 we have that $L \neq K'_y$.

First, suppose that $|E(K'_y) \cap E(K)| \geq 1$. Since the component of x in $K - v_{i-1}v_i$ contains uu', we have by Lemma 3.7.7 that $\rho(\mathcal{R}(\mathcal{T}'') + \bar{x}\bar{x}') = \rho^*$ and $e(K'_x), e(K'_y) \leq e(K)$. As $\bar{x} \in A$ and $L \neq K'_y$, we have that c) and d) hold, which is a contradiction.
Now, suppose that $|E(K'_y) \cap E(K)| = 0$. By Lemma 3.7.6 we have that $d = 3$, $c_y = 1$, $e(K) = 2$ and thus, $E(K) = \{uu', v_{i-1}v_i\}$ and $c_x = 1$. Thus, $e(K'_x) = 3$, $e(K'_y) = 2$ and $\rho(\mathcal{R}(\mathcal{T}'') + \bar{x}\bar{x}') = \rho^*$. Note that c) holds since $\bar{x} \in A$. For the same reason d) holds if $R^*_{i_A} \neq K$.
Thus, suppose that $R^*_{i_A} = K$. By Claim 3.7.9 we have that $w_{i_A}(W^*) = x$, which is a contradiction since $x \in A$. ☐

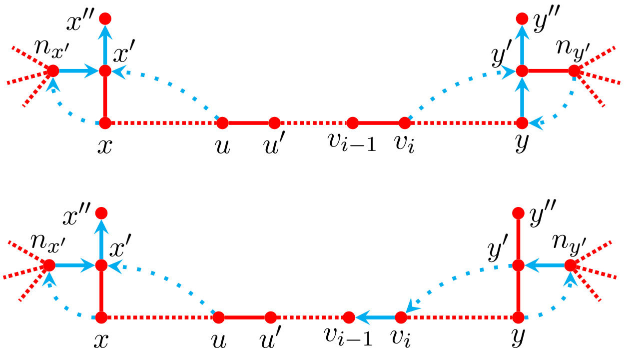

Fig. 3.7.7: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Claim 3.7.11 in the case where C_x and C_y are interesting and $\bar{x} \notin V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y'))$.

Let Y be the set of vertices having a blue directed path to y in $\mathcal{B}_b(\mathcal{T}_x)$ such that $V(P) \cap D^b_{\bar{x}} = \varnothing$.
Let $\mathring{Y}'$ be the set of vertices having a blue directed path to y' in $\mathcal{B}_b(\mathcal{T}_x)$ such that $V(P) \cap D^b_y = \varnothing$.
In the proofs of the remaining claims of this subsection, we will implicitly make use of Claim 3.7.11.

Claim 3.7.12. *We have that $i_{\sigma^*}(Y) < i_{\sigma^*}(\bar{X}_{y'} \cup \mathring{Y}')$ and additionally, we have that there is no path from v' to y' in $\mathcal{B}_b(\mathcal{T}_x)$ or we have that $e(K_y) = 0$, $e(K) \geq d$ and $c_x = 1$.*

Proof. Suppose towards a contradiction that $i_{\sigma^*}(Y) \geq i_{\sigma^*}(\bar{X}_{y'} \cup \mathring{Y}')$ or there is a path from v' to y' in $\mathcal{B}_b(\mathcal{T}_x)$. If this path exists, we also suppose that $e(K_y) \geq c_x$ or $e(K) = d - 1$. Note that if this path does not exist, then we either have $e(K_y) \geq 1$ or $e(K) = d - 1$ by Lemma 3.7.6. Thus, we suppose in any case that $e(K_y) \geq c_x$ or $e(K) = d - 1$ holds. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(y, y') \leftrightarrow vv'$. Note that $\Delta(\mathcal{T}_x, \mathcal{T}') \subseteq D^b_{y,\bar{x}}$ and thus, $P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, \bar{x}')$ and $P^{\mathcal{B}_b(\mathcal{T}_x)}(y', r)$ also exist in $\mathcal{B}_b(\mathcal{T}')$. Thus, we may obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(\bar{x}, \bar{x}') \leftrightarrow y'n_{y'}$. The decomposition $\mathcal{T}''$ is depicted in Figure 3.7.8. If $P^{\mathcal{B}_b(\mathcal{T}_x)}(v', y')$ exists, then it also exists in $\mathcal{B}_b(\mathcal{T}'')$ since $\Delta(\mathcal{T}_x, \mathcal{T}'') \subseteq D^b_{\bar{x}} \cup D^b_{y,\bar{x}}$. We again want to obtain a contradiction to Lemma 3.7.3 by choosing $\bar{\mathcal{T}} := \mathcal{T}''$ and $(a, a') := (y', y'')$. It is clear that a) holds. If $P^{\mathcal{B}_b(\mathcal{T}_x)}(v', y')$ exists, then $\Delta(\mathcal{T}^*, \mathcal{T}'') \subseteq A$ and thus, $B = \varnothing$. If the path does not exist, then $\bar{X}_{y'} \cup \mathring{Y}' \subseteq A$ and $B \subseteq D^b_{y,\bar{x}} \subseteq Y$. In both cases we have that b) holds. Furthermore, c) holds since $x, y' \in A$. Let K'_x and K'_y be the components in $\mathcal{R}(\mathcal{T}'') + y'y''$ of x and y, respectively. For d), first note that $\bar{x}, y' \in A$. If $e(K_y) \geq c_x$, then d) holds by Lemma 3.7.7 and we also have that $\rho(\mathcal{R}(\mathcal{T}'') + aa') = \rho^*$, which is a contradiction.

Thus, suppose that $e(K_y) = 0$, $e(K) = d - 1$ and thus, $c_x = 1$. Then we have that $e(K'_y) = |\{yy', y'y''\}| = 2 < d$, $e(K'_x) = e(K) + |\{xx'\}| + c_x - |\{vv'\}| = d$ and thus, $\rho(\mathcal{R}(\mathcal{T}'') + y'y'') = \rho^*$. If $R^*_{i_A} \neq K$, we have that d) holds and obtain a contradiction.

Thus, suppose that $R^*_{i_A} = K$. By Claim 3.7.9, we have $w_{i_A}(W^*) \in V(K_x)$. Further, there is a legal order $\sigma'' = (R''_1, \ldots, R''_{t''})$ for $\mathcal{T}''$ such that $R''_i = R^*_i$ for $i < i_A$ and $R''_{i_A} = K'_x$ and $i_{\sigma''}(y') > i_A$. Furthermore, there is a $W'' \in \mathrm{Aux}(\mathcal{T}'', \sigma'')$ with $\eta_j(W'') = \eta_j(W^*)$ for all $j \in \{2, \ldots, i_A\}$.

We obtain a smaller legal order than σ^* similar to Subcase 2.3 in the proof of Lemma 3.7.8:

let $P = [v_0, \ldots, v_l]$ be a minimal special path with respect to $\mathcal{T}''$, σ'', W'' and (y', y'') such that P is smaller or equal to the special path $P^{\mathcal{B}_b(\mathcal{T}'')}(x, y'')$. We have that $v_0 \in V(K_x)$. Applying (3) from Lemma 3.3.5, we receive a decomposition $\mathcal{T}''' \in \mathcal{F}^*$ having a legal order $\sigma''' = (R'''_1, \ldots, R'''_{t'''})$ with $R'''_i = R''_i = R^*_i$ for all $i < i_A$ and R'''_{i_A} is the component of v_{-1} having at most $e(K'_x) - c_x - |\{xx', v_0v_{-1}\}| = d - 3 < e(K)$ edges and obtain $\sigma''' < \sigma^*$, which is a contradiction. □

Claim 3.7.13. *There is a path from u' to y' in $\mathcal{B}_b(\mathcal{T}_x)$.*

Proof. Suppose to the contrary that there is no path from u' to y' in $\mathcal{B}_b(\mathcal{T}_x)$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow uu'$. Note that $\Delta(\mathcal{T}_x, \mathcal{T}') \subseteq D^b_{\bar{x}}$. In $\mathcal{B}_b(\mathcal{T}')$ there is a path from every vertex of $D^b_{\bar{x}}$ to u', which only visits vertices of $D^b_{\bar{x}} + u'$ and in particular, it does not go over y'. Since $u' \notin \Delta(\mathcal{T}_x, \mathcal{T}')$, the path from u' to r in $\mathcal{B}_b(\mathcal{T}_x)$ also exists in $\mathcal{B}_b(\mathcal{T}')$. We conclude that there is no path in $\mathcal{B}_b(\mathcal{T}')$ from $n_{y'}$ to y' and thus, we have $(y', y'') \leftrightarrow y'n_{y'}$. We obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing this exchange. Note that $n_{y'}y'$ and $y'y''$ are the only edges affected by this exchange and thus, $\Delta(\mathcal{T}^*, \mathcal{T}'') \subseteq D^b_{\bar{x}} + y'$. The decomposition $\mathcal{T}''$ is depicted in Figure 3.7.9. We will again find a contradiction to Lemma 3.7.3 and choose $\bar{\mathcal{T}} := \mathcal{T}''$, $(a, a') := (y, y')$, which implies $Y \subseteq A$ and $B \subseteq \bar{X}_{y'} \cup \mathring{Y}'$. Thus, b) holds by Claim 3.7.12 and it is

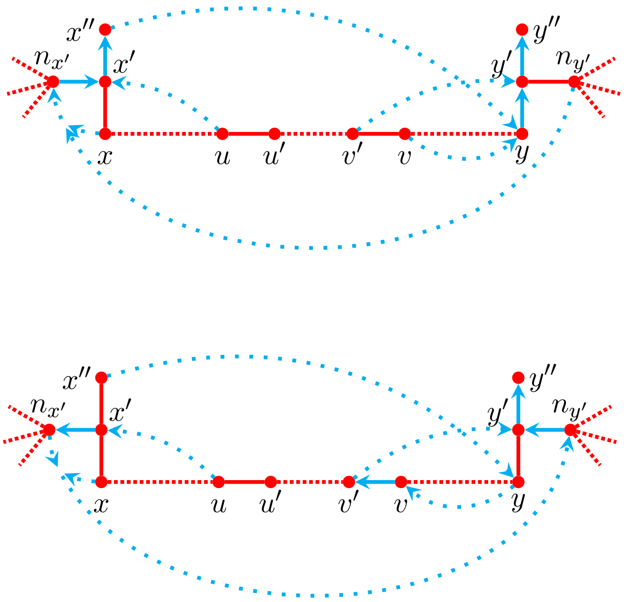

Fig. 3.7.8: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Claim 3.7.12 in the case where C_x is interesting and $P^{\mathcal{B}_b(\mathcal{T}_x)}(v', y')$ exists.

also clear that a) holds. Let K'_x, K'_y be the components in $\mathcal{R}(\mathcal{T}'') + yy'$ of x and y, respectively. Since $i_{\sigma^*}(Y) < i_{\sigma^*}(\bar{X}_{y'} \cup \mathring{Y}')$, we have that $K, C_y \neq R^*_{i_A}$, $L \neq K'_x$ and L is not the component of y in $\mathcal{R}(\mathcal{T}'')$. Thus, c) and d) hold. Finally, we have $\rho(\mathcal{R}(\mathcal{T}'') + yy') = \rho^*$ if $e(K_x) \geq 1$ by Lemma 3.7.7. If $e(K_x) = 0$, then $d = 3$, $e(K) = 2$ and $c_x = 1$ by Lemma 3.7.6. Thus, $e(K'_x) = 2$, $e(K'_y) = 3$ and $\rho(\mathcal{R}(\mathcal{T}'') + yy') = \rho^*$. □

Claim 3.7.14. *There is a path from v' to y' in $\mathcal{B}_b(\mathcal{T}_x)$.*

Proof. Suppose to the contrary that there is no path from v' to y' in $\mathcal{B}_b(\mathcal{T}_x)$. Let P be the red path in $\mathcal{T}_x$ from u' to v'. Note that by the definition of $\xrightarrow{(1,b)}$ no vertex of P has a path to y in $\mathcal{B}_b(\mathcal{T}_x)$. Let w' be the first vertex on P from which there is not a path in $\mathcal{B}_b(\mathcal{T}_x)$ to y'. Such a vertex exists and $w' \neq u'$ since u' has a path to y' by Claim 3.7.13 and v' has not. Let w be the vertex in P before w' in P having a path to y'. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow y'n_{y'}$. Note that $\Delta(\mathcal{T}_x, \mathcal{T}') \subseteq D^b_{\bar{x}}$ and thus, the paths in $\mathcal{B}_b(\mathcal{T}')$ from the vertices of P to r are the same as in $\mathcal{B}_b(\mathcal{T}_x)$. Hence, w also has a path to y' in $\mathcal{B}_b(\mathcal{T}')$ and w' does not. We obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(y', y'') \leftrightarrow ww'$. Note that $\mathcal{T}''$ is depicted in Figure 3.7.10. We want to obtain a contradiction to Lemma 3.7.3 by choosing $\bar{\mathcal{T}} := \mathcal{T}''$ and $(a, a') := (y, y')$. Note that

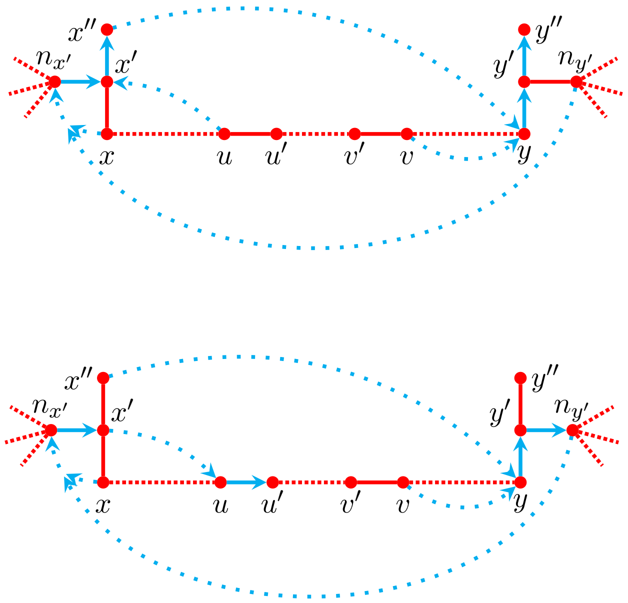

Fig. 3.7.9: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Claim 3.7.13 in the case where C_x is interesting.

$Y \subseteq A$ and $B \subseteq D^b_{\bar{x}} + y' \subseteq \bar{X}_{y'} \cup \mathring{Y}'$. Thus, b) holds and it is also clear that a) holds. Since $i_{\sigma^*}(\bar{x}') > i_{\sigma^*}(\bar{x}), i_{\sigma^*}(C'_y) \geq i_{\sigma^*}(\bar{X}_{y'}) > i_{\sigma^*}(Y)$, we have that $C_x, C_y, K \neq R^*_{i_A}$ and thus, $R^*_{i_A} = L$. Thus, c) and d) hold. Finally, since the component in $K - ww'$ of x contains uu' and the other component contains vv' we have $\rho(\mathcal{R}(\mathcal{T}'') + yy') = \rho^*$ by Lemma 3.7.7. ☐

We summarize the results of the Claims 3.7.11, 3.7.12, 3.7.14 in the following lemma:

Lemma 3.7.15. *Let K be a red component of $H_{\mathcal{T}^*}$ and let C_x, C_y be two distinct relevant neighbours of K generated by $(x, x'), (y, y') \in E(\mathcal{B}_b(\mathcal{T}^*))$, respectively, where $b \in \{1, \ldots, k\}$. Furthermore, let $x \xrightarrow{(1,b)} y$ with edges (u, u'), (v', v) and let K_y be the component of y in $K - vv'$.*

Then $e(K) \geq d$, C_y is interesting, $\bar{x} \in V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y'))$, $e(K_y) = 0$, $c_x = 1$ and there is a path from v' to y' in $\mathcal{B}_b(\mathcal{T}_x)$.

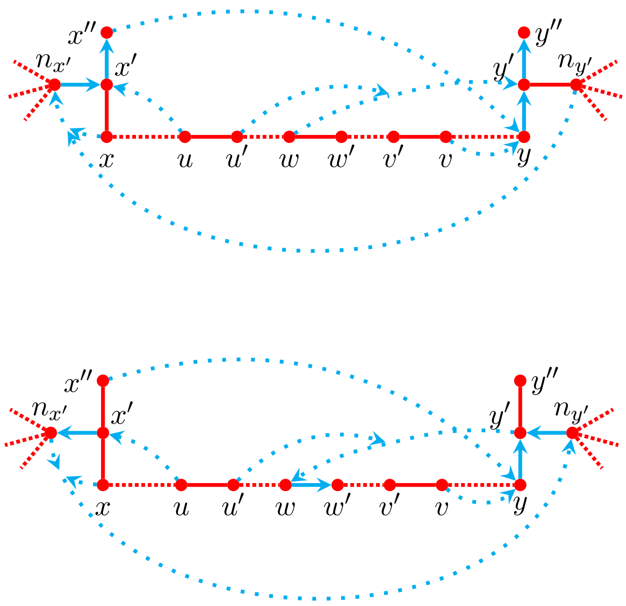

Fig. 3.7.10: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Claim 3.7.14 in the case where C_x is interesting.

3.7.3 Bounding the Number of Relevant Neighbours

We summarize the results of the last two subsections by describing the structure of K if it has two relevant neighbours generated by the same tree:

Lemma 3.7.16. *Let K be a red component of $H_{\mathcal{T}^*}$ and let C_x, C_y be two distinct relevant neighbours of K generated by $(x, x'), (y, y') \in E(\mathcal{B}_b(\mathcal{T}^*))$, respectively, where $b \in \{1, \ldots, k\}$. Without loss of generality, let y not be a descendant of $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}^*)$. Then $e(K) \geq d$, $c_x = 1$ and one of the following two cases applies:*

(1) We have $x \xrightarrow{(1,b)} y$ with edges $(u, u'), (v', v)$, C_y is interesting, $\bar{x} \in V(P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y'))$, there is a path from v' to y' in $\mathcal{B}_b(\mathcal{T}_x)$, the component of y in $K - vv'$ has no edges and x is a descendant of y in $\mathcal{B}_b(\mathcal{T}^*)$.

(2) We have $x \xrightarrow{(2,b)} y$ with edge (u, v), the component of y in $K - uv$ has no edges and x is a descendant of $\bar{y}$ in $\mathcal{B}_b(\mathcal{T}^*)$.

Proof. The lemma follows directly from Lemmas 3.7.5, 3.7.8 and 3.7.15 except for the fact that in Case (2) there is a path from x to $\bar{y}$ in $\mathcal{B}_b(\mathcal{T}^*)$. Thus, suppose that

(2) holds and no such path exists. Then u, which is the only neighbour of y, is not a descendant of y in $\mathcal{B}_b(\mathcal{T}^*)$ since it has a path to $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}^*)$. By Lemma 3.7.6 we have that $e(K) = 2 < 3 = d$, which contradicts $e(K) \geq d$. $\square$

The following lemma bounds the number of relevant neighbours of K. This also bounds the number of small children of K, which will help us to show that the density around K is high.

Lemma 3.7.17. *Let K be a red component of $H_{\mathcal{T}^*}$. Then K has at most two relevant neighbours generated by $\mathcal{B}_b(\mathcal{T}^*)$ for any $b \in \{1, \dots, k\}$.*

Proof. Suppose to the contrary that there are three relevant neighbours of K generated by (x, x'), (y, y'), $(z, z') \in E(\mathcal{B}_b(\mathcal{T}^*))$, respectively. By Lemma 3.7.16, we may assume that there is a path in $\mathcal{B}_b(\mathcal{T}^*)$ from x over $\bar{y}$ to $\bar{z}$ and that y and z only have one neighbour in K, n_y and n_z, respectively.

- If we have $x \xrightarrow{(1,b)} y$, then C_y is interesting and we obtain $\mathcal{T}'_x$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow n_{y'}y'$. This is possible because $P^{\mathcal{B}_b(\mathcal{T}_x)}(n_{y'}, y')$ goes over $(\bar{x}, \bar{x}')$. Note that $P^{\mathcal{B}_b(\mathcal{T}_x)}(n_y, y')$ (still) exists in $\mathcal{B}_b(\mathcal{T}'_x)$ since all vertices of $\Delta(\mathcal{T}_x, \mathcal{T}'_x)$ have a path to y in $\mathcal{B}_b(\mathcal{T}_x)$ and n_y does not. Thus, we may obtain $\mathcal{T}_1$ from $\mathcal{T}'_x$ by performing $(y, y') \leftrightarrow n_y y$.
  If we have $x \xrightarrow{(2,b)} y$, we obtain $\mathcal{T}'_x$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow y n_y$. Note that if C_y is interesting, there (still) is a path from $n_{y'}$ to y in $\mathcal{B}_b(\mathcal{T}'_x)$ since every vertex of $\Delta(\mathcal{T}_x, \mathcal{T}'_x)$ has a path to y in $\mathcal{B}_b(\mathcal{T}'_x)$ over (n_y, y). Thus, we may obtain $\mathcal{T}_1$ from $\mathcal{T}'_x$ by performing $(y, y') \leftrightarrow n_{y'}y'$. If C_y is small, we let $\mathcal{T}_1 := \mathcal{T}'_x$.

- We perform the exchanges of the previous bullet point but for y and z instead of x and y. For this, note that every vertex of $\Delta(\mathcal{T}_x, \mathcal{T}_1)$ has a path to $\bar{y}$ in $\mathcal{B}_b(\mathcal{T}_x)$, as well as in $\mathcal{B}_b(\mathcal{T}_1)$. Thus, the set of all descendants of $\bar{y}$ in $\mathcal{B}_b(\mathcal{T}_1)$ is the same as in $\mathcal{B}_b(\mathcal{T}_x)$. The same holds for z since it is an ancestor of y in $\mathcal{B}_b(\mathcal{T}_x)$, as well as in $\mathcal{B}_b(\mathcal{T}_1)$. Thus, the following exchanges are performable with the same reasoning as in the previous bullet point.
  If we have $y \xrightarrow{(1,b)} z$, we obtain $\mathcal{T}_2$ from $\mathcal{T}_1$ by first performing $(\bar{y}, \bar{y}') \leftrightarrow n_{z'}z'$ and then performing $(z, z') \leftrightarrow n_z z$.
  If we have $y \xrightarrow{(2,b)} z$, we obtain $\mathcal{T}'_1$ from $\mathcal{T}_1$ by performing $(\bar{y}, \bar{y}') \leftrightarrow z n_z$. If C_z is small, we let $\mathcal{T}_2 := \mathcal{T}'_1$, otherwise we obtain $\mathcal{T}_2$ from $\mathcal{T}'_1$ by performing $(z, z') \leftrightarrow n_{z'}z'$.

Note that $\mathcal{T}_2$ is depicted in Figure 3.7.11. We want to obtain a contradiction to Lemma 3.7.3 and choose $\bar{T} := \mathcal{T}_2$ and $(a, a') := (\bar{z}, \bar{z}')$. Note that we have $\Delta(\mathcal{T}^*, \mathcal{T}_2) \subseteq A$ and thus, $B = \varnothing$. It is clear that a) and b) hold. Let K'_x, K'_y, K'_z be the components in $\mathcal{R}(\mathcal{T}_2) + \bar{z}\bar{z}'$ of x, y, z, respectively. We have that $e(K'_y) = |\{yy'\}| + c_y < d \leq e(K)$, $e(K'_z) = |\{zz'\}| + c_z < d \leq e(K)$ and $e(K'_x) = e(K) - |\{n_y y, n_z z\}| + |\{xx'\}| + c_x = e(K)$. Thus, $\rho(\mathcal{R}(\mathcal{T}_2) + \bar{z}\bar{z}') = \rho^*$. Furthermore, c) holds since $x, y, z \in A$ and finally, d) holds since for all $v \in \{x, y, z\}$ we have that $v' \in A$ if C_v is not small. □

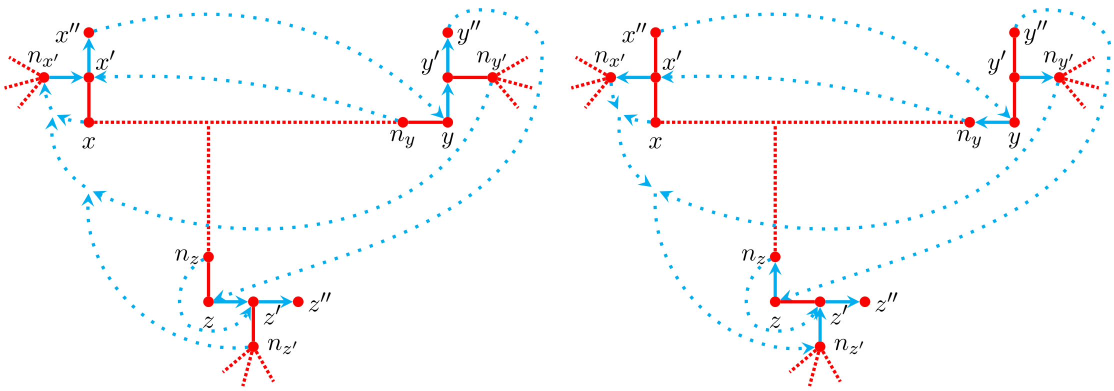

Fig. 3.7.11: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}_2$ in the proof of Lemma 3.7.17 in the case where $x \xrightarrow{(2,b)} y$, $y \xrightarrow{(1,b)} z$ and C_x, C_y are interesting.

3.7.4 Bad Components Are Interesting Neighbours

Note that in the case of $x \xrightarrow{(2,b)} y$, it could be that (x, x') generates a child with exactly one edge, and (y, y') generates a child without any edges. If several blue spanning trees generate two such children for K and if f assigned these children to K, then we might have $\hat{\beta}(K) + \sum_{C \in f^{-1}(K)} \hat{\beta}(C) > 0$. Thus, we need another definition of f in order to contradict Lemma 3.5.7 (and by this, prove Theorem 1.14).

In this subsection, we want to show that such a “bad” component is an interesting neighbour of some other component L, and we will let f assign the child with one edge to L. If L also has a child with zero edges, we want to call it bad as well and, using the same technique as for K, assign the child of K with one edge to another component instead, until it is received by a component that does not have many small children. First, we formalize in which case we call a component bad.

Definition 3.7.18. Let K be a red component of $H_{\mathcal{T}^*}$ having two relevant neighbours C_x and C_y generated by $(x, x'), (y, y') \in E(\mathcal{B}_b(\mathcal{T}^*))$, respectively. Furthermore, we let $e(C_y) = 0$ and thus, we have $x \xrightarrow{(2,b)} y$ by Lemma 3.7.16. Then we say that K *is* b*-bad (due to* x *and* y*).*

For the rest of this subsection, let K be b-bad due to x and y as in Definition 3.7.18. Thus, y has exactly one neighbour n_y in K. In this subsection, we aim to show that the last arc (z, y) of the path from n_y to y in $\mathcal{B}_b(\mathcal{T}^*)$ has its tail z outside of K. This would imply that K is an interesting neighbour of the red component of z. For the rest of the subsection, we suppose to the contrary that $z \in V(K)$.

Claim 3.7.19. *We have that* $x \xrightarrow{(1,b)} z$*.*

Proof. Note that there is a path from $\bar{x}$ to z in $\mathcal{B}_b(\mathcal{T}^*)$ (and thus also in $\mathcal{B}_b(\mathcal{T}_x)$) and hence, z is not a descendant of $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$. Thus, suppose to the contrary that $x \xrightarrow{(2,b)} z$ holds with edge (u, v). We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow uv$. Note that every vertex of $\Delta(\mathcal{T}^*, \mathcal{T}')$ has a path to z in $\mathcal{B}_b(\mathcal{T}^*)$ and in $\mathcal{B}_b(\mathcal{T}')$. Thus, there (still) is a path from n_y to z in $\mathcal{B}_b(\mathcal{T}')$ and $(z, y) \in E(\mathcal{B}_b(\mathcal{T}'))$. We obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(z, y) \leftrightarrow n_y y$. Note that this does not create a red cycle since y (still) has degree one in $\mathcal{R}(\mathcal{T}'')$. Note that $\mathcal{T}''$ is depicted in Figure 3.7.12. We want to obtain a contradiction to Lemma 3.7.3 by choosing $\bar{\mathcal{T}} := \mathcal{T}''$ and $(a, a') := (y, y')$, which gives us $\Delta(\mathcal{T}^*, \mathcal{T}'') \subseteq A$ and $B = \varnothing$. It is clear that a) and b) hold. For c) note that $\bar{x}, y \in A$. Let K'_x be the component of x and K'_y be the component of y and z in $\mathcal{R}(\mathcal{T}'') + yy'$. We have that $e(K'_x) \leq e(K) - |\{uv, n_y y\}| + |\{xx'\}| + c_x = e(K)$ and $e(K'_y) \leq e(K) - |\{uv, n_y y\}| + |\{zy, yy'\}| = e(K)$. Furthermore, we have that

$$e(K) + |\{zy, yy', xx'\}| + c_x = e(K'_x) + e(K'_y) + |\{uv, n_y y\}|.$$

Thus, if one of the components K'_x, K'_y has $e(K)$ edges, then the other one has only $2 < d$ edges. Thus, $\rho(\mathcal{R}(\mathcal{T}'') + yy') = \rho^*$. Furthermore, d) holds since $x' \in A$, which leads to a contradiction to Lemma 3.7.3. $\square$

For the rest of the subsection, suppose that $x \xrightarrow{(1,b)} z$ holds with edges (u, u') and (v', v) due to Claim 3.7.19. Let K_z be the component of z in $K - v'v$.

Claim 3.7.20. $e(K_z) \geq 1$.

Proof. Suppose towards a contradiction that $e(K_z) = 0$ and thus, $z = v$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(z, y) \leftrightarrow zv'$. Note that the exchange only affects these

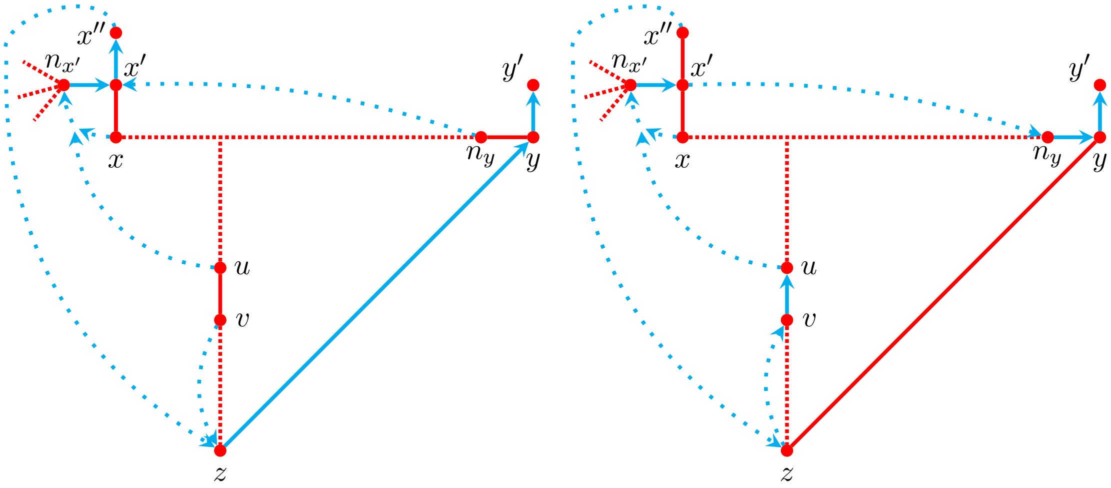

Fig. 3.7.12: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Claim 3.7.19 in the case where C_x is interesting.

two edges and $\mathcal{T}'$ does not contain a red cycle because z (still) has degree 1 in $\mathcal{R}(\mathcal{T}')$. Thus, if we replace K by $K - zv' + zv$ in σ^* we obtain a legal order for $\mathcal{T}'$ that is lexicographically equal to σ^*. Thus, Lemma 3.7.8 also applies to $\mathcal{T}'$ and σ', which is a contradiction since the component of y in $\mathcal{R}(\mathcal{T}') - n_y y$ contains yz. ☐

We maintain the definitions of $D^b_{\bar{x}}$ and $D^b_{y,\bar{x}}$ for $\mathcal{T}_x$ of Subsection 3.7.2. Furthermore, let Z be the set of vertices having a blue directed path P to z in $\mathcal{B}_b(\mathcal{T}_x)$ such that $V(P) \cap D^b_{\bar{x}} = \varnothing$ and let $\bar{X}_z$ still be the set of vertices having a blue directed path P to $\bar{x}$ in $\mathcal{B}_b(\mathcal{T}_x)$ such that $V(P) \cap D^b_{z,\bar{x}} = \varnothing$.

Claim 3.7.21. $i_{\sigma^*}(\bar{X}_z) \leq i_{\sigma^*}(Z)$.

Proof. Suppose to the contrary that $i_{\sigma^*}(\bar{X}_z) > i_{\sigma^*}(Z)$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow uu'$. Note that $\mathcal{T}'$ is depicted in Figure 3.7.13. We want to obtain a contradiction to Lemma 3.7.3 by choosing $\bar{\mathcal{T}} := \mathcal{T}'$ and $(a, a') := (y, y')$. This implies $Z \subseteq A$ and $B \subseteq D^b_{\bar{x}} \subseteq \bar{X}_z$. This implies b) and it is also clear that a) holds. Let K'_x and K'_z be the components of x and z in $\mathcal{R}(\mathcal{T}') + yy'$. Note that it is not clear whether y is contained in K'_x or K'_z. We have that $e(K'_x) \leq e(K) + c_x + |\{xx', yy'\}| - |\{uu', vv'\}| - e(K_z) \leq e(K)$ by Claim 3.7.20 and $e(K'_z) \leq e(K) - |\{uu'\}| + |\{yy'\}| \leq e(K)$. Furthermore,

$$e(K) + |\{xx', yy'\}| + c_x = e(K'_x) + e(K'_z) + |\{uu'\}|.$$

Thus, if one of the components K'_x, K'_z contains $e(K)$ edges, then the other one contains $2 < d$ edges. Thus, $\rho(\mathcal{R}(\mathcal{T}') + yy') = \rho^*$. Furthermore, we have that $L \neq K'_x$ since $i_{\sigma^*}(K'_x) \geq i_{\sigma^*}(\bar{X}_z) > i_{\sigma^*}(Z) \geq i_A$. Thus, d) holds and c) holds as well considering that $z \in A$. $\square$

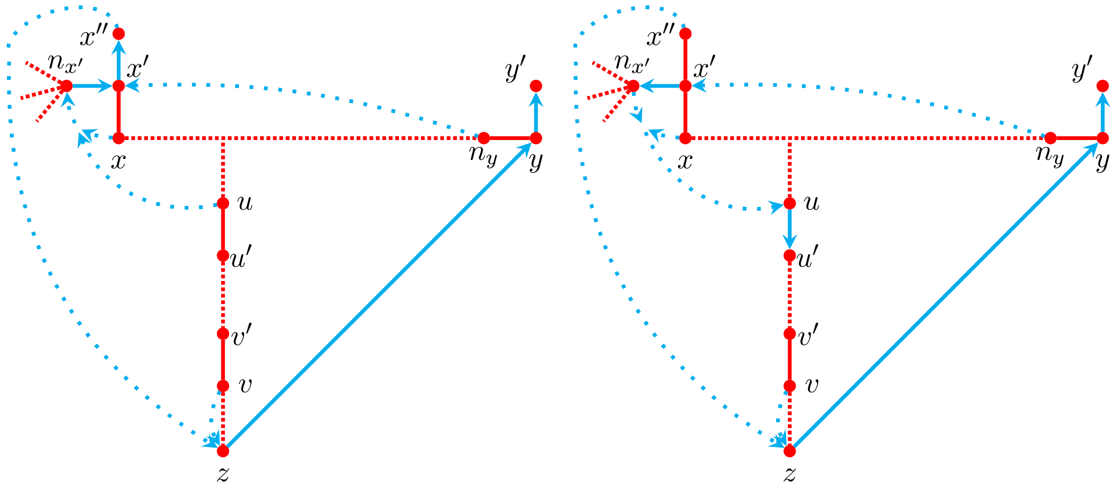

Fig. 3.7.13: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}'$ in the proof of Claim 3.7.21 in the case where C_x is interesting.

We now state the main result of this subsection, where we reintroduce notation for clarity:

Lemma 3.7.22. *Let K be a b-bad component due to x and y and let z be the vertex before y on the path from n_y to y in $\mathcal{B}_b(\mathcal{T}^*)$. Then $z \notin V(K)$.*

Proof. Suppose that $z \in V(K)$. Then by Claim 3.7.19, 3.7.20 and 3.7.21 we have that $x \xrightarrow{(1,b)} z$ with edges (u, u') and (v', v), $e(K_z) \geq 1$ and $i_{\sigma^*}(\bar{X}_z) \leq i_{\sigma^*}(Z)$. We obtain $\mathcal{T}'$ from $\mathcal{T}_x$ by performing $(\bar{x}, \bar{x}') \leftrightarrow n_y y$. Note that $\Delta(\mathcal{T}_x, \mathcal{T}') \subseteq \bar{X}_z$ and thus, we may obtain $\mathcal{T}''$ from $\mathcal{T}'$ by performing $(z, y) \leftrightarrow vv'$. The decomposition $\mathcal{T}''$ is depicted in Figure 3.7.14. We want to obtain a contradiction to Lemma 3.7.3 and choose $\bar{\mathcal{T}} := \mathcal{T}''$ and $(a, a') := (y, y')$. This implies $\bar{X}_z \subseteq A$ and $B \subseteq D^b_{z,\bar{x}} \subseteq Z$. Thus, b) holds and it is also clear that a) holds. c) holds as well since $x, y \in A$. Let K'_x be the component of x and K'_y be the components of y and z in $\mathcal{R}(\mathcal{T}'') + yy'$. We have that $e(K'_x) = e(K) - |\{n_y y, vv'\}| - e(K_z) + |\{xx'\}| + c_x < e(K)$ and $e(K'_y) \leq e(K) - |\{vv', n_y y, uu'\}| + |\{yy', zy\}| < e(K)$. Thus, $\rho(\mathcal{R}(\mathcal{T}') + yy') = \rho^*$. Finally, we have that d) holds since $x' \in A$ and we obtain a contradiction. $\square$

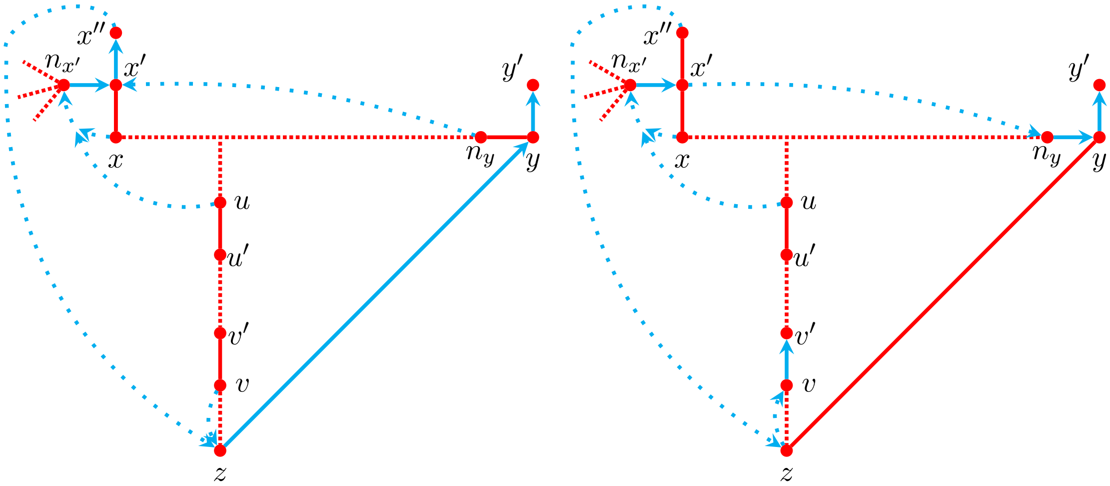

Fig. 3.7.14: The component K and its neighbours in $\mathcal{T}_x$ and $\mathcal{T}''$ in the proof of Lemma 3.7.22 in the case where C_x is interesting.

3.7.5 Density Calculations

In this final subsection we will finally define a function f that contradicts Lemma 3.5.7, which proves Theorem 1.14. First, we will show to which component of $\mathcal{K} - R^*$ a child of a b-bad component generated by $\mathcal{B}_b(\mathcal{T}^*)$ containing exactly one edge will be assigned to. This is K_n in the following definition, when the sequence is a sink sequence.

Definition 3.7.23. Let $n \geq 1$ and $b \in \{1, \dots, k\}$. Let $K_1, \dots, K_n$ be red components of $H_{\mathcal{T}^*}$ and $(x_i, x_i') \in E(\mathcal{B}_b(\mathcal{T}^*))$ for $i \in \{1, \dots, n\}$. We call $(K_1, x_1), \dots, (K_n, x_n)$ a *partial sink sequence for* b if the following conditions are met:

- K_1 is b-bad due to x_1 and x_2' such that the relevant neighbour C_{x_1} of K_1 that is generated by $(x_i, x_i') \in E(\mathcal{B}_b(\mathcal{T}^*))$ has $e(C_{x_1}) = 1$.
- For $i \in \{2, \dots, n\}$ we have that K_{i-1} is an interesting neighbour of K_i generated by $(x_i, x_i') \in E(\mathcal{B}_b(\mathcal{T}^*))$.

Furthermore, we call $(K_1, x_1), \dots, (K_n, x_n)$ a *sink sequence for* b if it is a partial sink sequence for b and additionally, K_n is not b-bad.

Let us make some critical observations about partial sink sequences. First, note that in a sink sequence we have that $K_1 \neq K_n$ and thus, $n \geq 2$.

Observation 3.7.24. *In a partial sink sequence* $(K_1, x_1), \ldots, (K_n, x_n)$ *for any* $i \in \{1, \ldots, n-1\}$*, the component* K_i *is* b*-bad due to* x_i *and* x'_{i+1}*.*

Observation 3.7.25. *In a partial sink sequence* $(K_1, x_1), \ldots, (K_n, x_n)$*, the components* $K_1, \ldots, K_n$ *are pairwise distinct.*

Proof. Let $i \in \{1, \ldots, n-1\}$. Note that since K_i is b-bad due to x_i and x'_{i+1} by Observation 3.7.24, there is a path in $\mathcal{B}_b(\mathcal{T}^*)$ from $n_{x'_{i+1}}$ to x'_{i+1} visiting $\bar{x}_i$ and then x_{i+1}. Thus, x_i is a proper descendant of x_{i+1} in $\mathcal{B}_b(\mathcal{T}^*)$ and the claim follows. □

Observation 3.7.26. *If* $(K_1, x_1), \ldots, (K_i, x_i)$ *is a partial sink sequence for* $b \in \{1, \ldots, k\}$ *which is not a sink sequence, then there exists a pair* (K_{i+1}, x_{i+1}) *such that* $(K_1, x_1), \ldots, (K_i, x_i), (K_{i+1}, x_{i+1})$ *is a partial sink sequence.*

Proof. Let $(K_1, x_1), \ldots, (K_i, x_i)$ be a partial sink sequence that is not a sink sequence. Thus, K_i is b-bad due to x_i and another vertex y. By Lemma 3.7.22, the vertex z on the path from n_y to y in $\mathcal{B}_b(\mathcal{T}^*)$ which is adjacent to y is not in K_i. Let K_{i+1} be the component containing z. Then K_i is an interesting neighbour of K_{i+1} and thus, $(K_1, x_1), \ldots, (K_i, x_i), (K_{i+1}, z)$ is a partial sink sequence. □

Corollary 3.7.27. *Every partial sink sequence extends to a sink sequence.*

Lemma 3.7.28. *Let* $b \in \{1, \ldots, k\}$ *and let* L *be a red component of* $H_{\mathcal{T}^*}$ *that is not* b*-bad having* $l \in \{0, 1, 2\}$ *interesting neighbours generated by* $\mathcal{B}_b(\mathcal{T}^*)$*. Then* L *is contained in at most* l *sink sequences for* b *(and always is the end of such a sequence).*

Proof. Since L is not b-bad, it follows that L can only be the end of any sink sequence for b. To see that L belongs in at most l sink sequences for b, it suffices to show that for every vertex $x \in V(L)$ such that L has an interesting neighbour generated by an arc $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$, there is at most one sink sequence for b ending with the tuple (L, x). Thus, suppose that there are two sink sequences $(K_1, x_1), \ldots, (K_{n-1}, x_{n-1}), (K_n, x_n)$, and $(K'_1, y_1), \ldots, (K'_{m-1}, y_{m-1}), (K'_m, y_m)$, where $(K_n, x_n) = (L, x) = (K'_m, y_m)$, and suppose that

$$(K_{n-i}, x_{n-i}) = (K'_{m-i}, y_{m-i}), \ldots, (K_n, x_n) = (K'_m, y_m)$$

for some $i \in \{0, \ldots, \min\{n-2, m-2\}\}$ and let x'_j and y'_j be the parents of x_j and y_j in $\mathcal{B}_b(\mathcal{T}^*)$, respectively.

Claim. $(K_{n-(i+1)}, x_{n-(i+1)}) = (K'_{m-(i+1)}, y_{m-(i+1)})$

Proof. Clearly, $x'_{n-i} = y'_{m-i}$ and thus, $K_{n-(i+1)} = K'_{m-(i+1)}$. Note that $K_{n-(i+1)}$ is b-bad due to $x_{n-(i+1)}$ and x'_{n-i} and $K'_{m-(i+1)}$ is b-bad due to $y_{m-(i+1)}$ and y'_{m-i} by Observation 3.7.24. By Lemma 3.7.17, $K_{n-(i+1)}$ does not have more relevant neighbours generated by $\mathcal{B}_b(\mathcal{T}^*)$ than the two that are generated by $(x_{n-(i+1)}, x'_{n-(i+1)})$ and $(x'_{n-i}, x''_{n-i}) = (y'_{m-i}, y''_{m-i})$. Thus, $(x_{n-(i+1)}, x'_{n-(i+1)}) = (y_{m-(i+1)}, y'_{m-(i+1)})$.

(End of proof of the claim) ■

From the claim it follows (without loss of generality) that $(K_1, x_1) = (K'_{m-(n-1)}, y_{m-(n-1)})$, $\ldots, (K_n, x_n) = (K'_m, y_m)$. To prove the lemma it only remains to show that $n = m$ and thus, the sink sequences are equal. This follows directly from the fact that x'_1 is contained in a red component having exactly one edge and $x'_1 = y'_{m-(n-1)}$. Thus, the component $K'_{m-(n-1)}$ containing $y_{m-(n-1)}$ can only be in a first tuple of a sink sequence. □

Putting these together we obtain the following:

Lemma 3.7.29. *Let $b \in \{1, \ldots, k\}$. Let K be a red component of $H_{\mathcal{T}^*}$ that is b-bad due to x and y and let C_x be the relevant neighbour of K generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$. Then K is contained in exactly one sink sequence for b and if $e(C_x) = 1$, then K is the beginning of this sink sequence.*
Let L be a red component of $H_{\mathcal{T}^*}$ that is not b-bad having $l \in \{0, 1, 2\}$ interesting neighbours generated by $\mathcal{B}_b(\mathcal{T}^*)$. Then L is contained in at most l sink sequences for b (and always is the end of such a sequence).

Now we are in position to define f.

Lemma 3.7.30. *There is a function $f\colon \mathcal{C} \longrightarrow \mathcal{K} - R^*$ such that for every $K \in \mathcal{K} - R^*$ we have:*

- *if $e(K) < d - 1$, then $f^{-1}(K) = \varnothing$.*
- *if $e(K) = d - 1$, then $|f^{-1}(K)| \leq k$ and each of the components of $f^{-1}(K)$ has exactly one edge.*
- *if $e(K) \geq d$, then there are integers $q_0, q_1 \geq 0$ with $q_0 + q_1 \leq k$ such that $f^{-1}(K)$ contains exactly q_0 components having zero edges and exactly $2q_1$ components having one edge.*

Proof. Let C be a small component. By Lemma 3.7.4 we have that C has some parent $K \in \mathcal{K} - R^*$ having at least $d - e(C)$ edges. By Lemma 3.7.2, we have that $K \neq R^*$. Let C be generated by $(x, x') \in E(\mathcal{B}_b(\mathcal{T}^*))$. We assign C to K except if

$e(C) = 1$ and K has another small child C' generated by $\mathcal{B}_b(\mathcal{T}^*)$ with $e(C') = 0$ and hence, K is b-bad: In this case, by Lemma 3.7.29, there exists a sink sequence $(K_1, x_1), \dots, (K_n, x_n)$ for b where $(K_1, x_1) = (K, x)$ and we assign C to K_n. Note that by Lemma 3.7.4 we have $e(K_n) \geq d - 1$.

With this definition of f, note that by Lemma 3.7.29 the number of small children with zero edges assigned to some $K \in \mathcal{K} - R^*$ is at most the number of small children of K not having an edge and thus bounded by k by Lemma 3.7.16. Further, the number of small children with one edge assigned to some $K \in \mathcal{K}$ is at most the number of relevant neighbours of K containing at least one edge, which is at most $2k$ by Lemma 3.7.17 (and thus, we obtain the desired integers q_0 and q_1). The lemma follows by Lemma 3.7.4, 3.7.16 and 3.7.17. □

We are now able to prove Theorem 1.14.

Proof of Theorem 1.14. Recall that Theorem 1.14 is true for $d \leq k+1$ and for $k = 0$ by Theorem 1.15 and Theorem 3.3.1, and thus, we let $k \geq 1$ and $k+1 < d \leq 2(k+1)$. Let f be defined as in Lemma 3.7.30. We will show that f contradicts Lemma 3.5.7. Let $K \in \mathcal{K} - R^*$, and let $\alpha := \max\{d - e(K), 0\}$. For $\alpha = 0$, let q_0 and q_1 be defined as in Lemma 3.7.30. For $\alpha = 1$, let $q_0 = 0$ and $q_1 \in \{0, \dots, k\}$. For $\alpha \geq 2$, let $q_0 = q_1 = 0$. Then the number of components of $|f^{-1}(K)|$ with one edge is at most $(2-\alpha)q_1$, and the number of components of $|f^{-1}(K)|$ having no edge is at most q_0. Furthermore, recall that $q_0 + q_1 \leq k$ if $\alpha = 0$ by Lemma 3.7.30. Thus,

$$\begin{aligned}
\hat{\beta}(K) + \sum_{C \in f^{-1}(K)} \hat{\beta}(C) &= d\Big(e(K) + 1 + \sum_{C \in f^{-1}(K)} (e(C) + 1)\Big) - (d + k + 1)\Big(e(K) + \sum_{C \in f^{-1}(K)} e(C)\Big) \\
&= d - (k+1)e(K) + \sum_{C \in f^{-1}(K)} \big(d - (k+1)e(C)\big) \\
&\leq d - (k+1)(d - \alpha) + q_0 d + (2-\alpha)q_1\big(d - (k+1)\big) \\
&= d\big(q_0 + (2-\alpha)q_1 - k\big) + \alpha(k+1) - (2-\alpha)(k+1)q_1.
\end{aligned}$$

We obtain a contradiction to Lemma 3.5.7 if this term is at most 0. This is the case if $\alpha \geq 2$. If $\alpha = 0$, we have

$$\hat{\beta}(K) + \sum_{C \in f^{-1}(K)} \hat{\beta}(C) \leq dq_1 - 2(k+1)q_1 \leq 0.$$

Finally, if $\alpha = 1$, then

$$\begin{aligned}
\hat{\beta}(K) + \sum_{C \in f^{-1}(K)} \hat{\beta}(C) &\leq d(q_1 - k) + k + 1 - (k+1)q_1 \\
&= q_1\big(d - (k+1)\big) - kd + k + 1 \\
&\leq k\big(d - (k+1)\big) - kd + k + 1 \\
&= -(k-1)(k+1) \\
&\leq 0.
\end{aligned}$$

□

3.8 A Lower Bound for a Diameter-Constrained Version of the Nine Dragon Tree Theorem

In this section, we will prove Theorem 1.28, which provides a lower bound to the diameter that could be achieved in a diameter version of the Nine Dragon Tree Conjectures, without containing any degree of component size constraints.

Theorem 1.28. *Let $k, \ell \in \mathbb{N}_{\geq 1}$, and let $\ell \equiv 0 \mod 4$. There is a loopless (multi-) graph G with $\gamma(G) = k + \frac{\ell}{\ell+1}$ that does not have a decomposition into $k+1$ forests in which one of the forests has diameter at most $\frac{\ell}{4}$.*

For the rest of the section, let $k, \ell \in \mathbb{N}_{\geq 1}$ and let $\ell \equiv 0 \mod 4$. We now define G for Theorem 1.28. Note that in Figure 3.8.1, G is depicted for $k = 2$ and $\ell = 16$. Let $s = \frac{\ell}{2}$ and let $V(G) = \{a_0, \ldots, a_s, b_0, \ldots, b_s\}$. For $i \in \{0, \ldots, s-1\}$, $E(G)$ contains $a_i a_{i+1}$ and $b_i b_{i+1}$. For $i \in \{0, \ldots, s\}$, $E(G)$ contains k parallel edges $a_i b_i$. For $i \in \{0, \ldots, s-1\}$, $E(G)$ contains k parallel edges $b_i a_{i+1}$. Note that G has $2(s+1)$ vertices and $k(2s+1) + 2s$ edges.

Lemma 3.8.1. *In every decomposition of G into $k+1$ forests, every forest has diameter at least $\frac{\ell}{4} + 1$.*

Proof. Suppose that there is decomposition into $k+1$ forests $T_1, \ldots, T_k, F$ where $\operatorname{diam}(F) \leq \frac{\ell}{4} = \frac{s}{2}$. We show that F has exactly two components. First, note that F is not spanning or otherwise there would be a path from a_0 to a_s in F, which has length at least s. As $v(G) = 2(s+1)$, each of $T_1, \ldots, T_k$ has at most $2s+1$ edges. Since

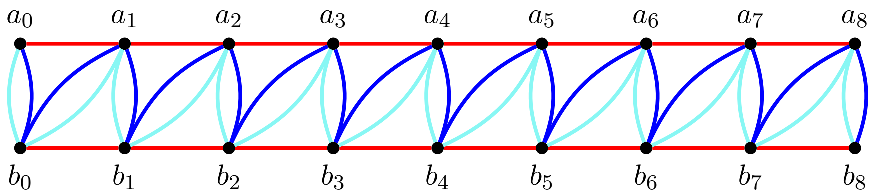

Fig. 3.8.1: G for $k = 2, \ell = 16$, where the edges are coloured such that there are two blue Hamiltonian paths and a red forest consisting of two paths of (non-optimal) length $s = 8$.

$e(G) = k(2s + 1) + 2s$, F must contain at least $2s$ edges. Thus, F has exactly two components. Let C_1 be the component of F containing a_0 and C_2 be the component of F containing b_s. We have that $C_1 \neq C_2$ or otherwise $\operatorname{diam}(F) \geq s + 1$.
Next, note that $c_1 := a_{\frac{s}{2}}$ is contained in C_1 and $c_2 := b_{\frac{s}{2}}$ is contained in C_2 or otherwise $\operatorname{diam}(F) \geq \frac{s}{2}+1$. There is only one path in C_1 from a_0 to c_1 that has length at most $\frac{s}{2}$, which is $[a_0, \dots, a_{\frac{s}{2}}]$ and thus, this path is contained in C_1. Analogously, $[b_{\frac{s}{2}}, \dots, b_s]$ is contained in C_2.
Now, consider $H := G[\{c_1, a_{\frac{s}{2}+1}, c_2\}]$. None of the (parallel) edges of this subgraph belong to F since otherwise $\operatorname{diam}(F) \geq \frac{s}{2} + 1$. But $\frac{e(H)}{v(H)-1} = \frac{2k+1}{2} > k$ and thus, not all edges of H can be contained in $T_1, \dots, T_k$, a contradiction. ☐

Note that Figure 3.8.2 shows that G has a decomposition into $k + 1$ forests, where one of the forests consists of two components, each of which has diameter $\frac{\ell}{4} + 1$.

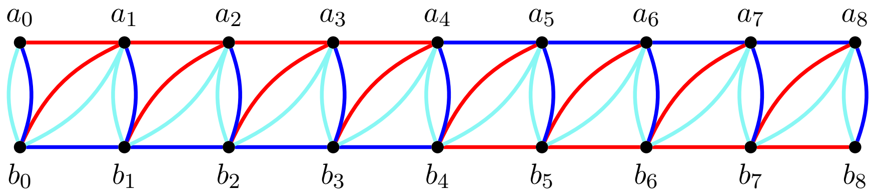

Fig. 3.8.2: A colouring of G minimizing the diameter in the red forest.

In order to prove Theorem 1.28, it remains to show that $\gamma(G) = k + \frac{\ell}{\ell+1} (= k + \frac{2s}{2s+1})$.
Note that $\frac{e(G)}{v(G)-1} = k + \frac{2s}{2s+1}$. We will show that G is its densest subgraph.
Let *a triangle of* G be the induced subgraph Δ of G such that there is an $i \in \{0, \dots, s - 1\}$ such that $V(\Delta)$ is either $\{a_i, a_{i+1}, b_i\}$ or $\{b_i, a_{i+1}, b_{i+1}\}$. Note that every edge of G is contained in a triangle.

From the subgraphs of G with at least two vertices, let H minimize $f(H) := \big(k(\ell + 1) + \ell\big)v(H) - (\ell + 1)e(H)$.

Lemma 3.8.2. *If two vertices of a triangle are in $V(H)$, then the third one is also in $V(H)$.*

Proof. Suppose not. Then adding the third one to H results in a subgraph H' having at least $k+1$ more edges than H and thus,

$$f(H) - f(H') = -\big(k(\ell+1)+\ell\big) + (\ell+1)(k+1) = 1,$$

which is a contradiction. □

Lemma 3.8.3. $H = G$.

Proof. Note that $f(G) = k(\ell+1)+\ell$. Since $v(H) \geq 2$, we have that $E(H) \neq \varnothing$, or otherwise $f(H) \geq 2\,(k(\ell+1)+\ell)$. Let $e \in E(H)$ and let Δ be a triangle of G containing e. Then the third vertex of Δ that is not incident with e is also in $V(H)$ by Lemma 3.8.2. Thus, $\Delta \subseteq H$.
We call two triangles of G adjacent if they have two common vertices. Note that these two vertices are always linked by k (parallel) edges in G. Note that the corresponding graph of this adjacency is a path of all triangles of G. Since the adjacent triangles of Δ in this path have two vertices in $V(H)$, they are also contained in H by Lemma 3.8.2. Using this argument inductively, we have that $H = G$. □

Now, we just have to combine the previous lemmas to prove Theorem 1.28.

Proof of Theorem 1.28. Since G minimizes f by Lemma 3.8.3 and $f(G) = k(\ell+1)+\ell$, we have that $\gamma(G) = k + \frac{\ell}{\ell+1}$. The theorem follows by Lemma 3.8.1. □

A Diameter Refinement of the Strong Pseudoforest Nine Dragon Tree Theorem

4

4.1 Pseudoforests and Orientations

In this section, we define pseudoforests and orientations, and show their close relationship. At the end of the section, we will give a proof of Hakimi's Theorem (Theorem 1.17).

For $k \in \mathbb{N}_0$, a *k-orientation* is a digraph G in which every vertex has at most k outgoing arcs ($\deg_G^+(v) \leq k$ for every $v \in V(G)$).

We say that a digraph D is *an orientation of a graph* G if we can obtain D from G by replacing every edge of G by an arc with the same end-vertices.

If D is an orientation of a graph G and D is a k-orientation, then we say that D is *a k-orientation of* G or G *has a k-orientation* D.

The following lemma and corollary are well-known and describe the strong connection between orientations and pseudoforest decompositions.

Lemma 4.1.1. *A graph G is a pseudoforest if and only if it has a 1-orientation.*

Proof. Let G be a pseudoforest and let K be a component of G. We orient the edges of K in the following way: if K contains a cycle C, we orient the edges of C such that they form a directed cycle. All other edges of K are directed towards C. If K does not contain a cycle, then it is a spanning tree and we pick any vertex of K and orient the edges towards it. In both cases we obtain a 1-orientation of K.

Now, let G be a graph having a 1-orientation, and let K be a component of G. Since every vertex of G has at most one outgoing edge, K has at most $v(K)$ edges. Thus, K is a pseudotree (if K had more than one cycle, then by removing one edge of a cycle the resulting graph is still connected and cyclic, which is a contradiction to having at most $v(K) - 1$ edges). □

Corollary 4.1.2. *An undirected graph G has a decomposition into k pseudoforests if and only if G has a k-orientation.*

We will make use of Corollary 4.1.2 throughout the chapter. Next, we will give a standard proof of Hakimi's Theorem using path reversal that arose in the 2000s, see [6], [51], [4].

Let D be a digraph. We now define a measure of how close D is to being a k-orientation. Let $\phi_i(D)$ denote the number of vertices $v \in V(D)$ with $\deg_D^+(v) = i$. Furthermore, let $\Phi_k(D) := (\phi_{e(D)}(D), \ldots, \phi_{k+1}(D))$. Note that D is a k-orientation if and only if $\Phi_k(D) = (0, \ldots, 0)$. We compare the values of Φ_k using lexicographic ordering (see Section 2.2).

Let a *k-optimal orientation of an undirected graph* G be an orientation D of G minimizing $\Phi_k(D)$.

Lemma 4.1.3 ([6], [51], [4]). *Let D be a k-optimal orientation of an undirected graph. Then there is no dipath in D from a vertex with outdegree at least $k + 1$ to a vertex with outdegree at most $k - 1$.*

Proof. Suppose there is such a dipath P from v to u. By reorienting the edges of P, the outdegrees of the inner vertices of P do not change, the outdegree of v decreases by one and the outdegree of u increases by one. But this is a contradiction to D being a k-optimal orientation. □

Using this path reversal operation, we can prove Hakimi's Theorem.

Proof of Theorem 1.17 **([6], [51], [4]).** Let $k \in \mathbb{N}_0$. We show that a graph has a k-orientation if and only if for every subgraph $H \subseteq G$, we have $e(H) \leq k \cdot v(H)$. Theorem 1.17 will then follow by Corollary 4.1.2.

First, let G be a graph that has a k-orientation D. Then every vertex has at most k outgoing arcs in D and thus, $e(G) \leq k \cdot v(H)$.

Now, let G not have a k-orientation and let D be a maximal k-orientation of G. Then there is a vertex $u \in V(G)$ with $\deg_D^+(u) \geq k+1$. Let U be the set of vertices to which there is a dipath from u. By Lemma 4.1.3, every vertex of U has at least k outgoing arcs and the head of every of these outgoing arcs is also in U. Furthermore, note that $u \in U$. Thus, we have $e(G[U]) > k \cdot |U|$. □

4.2 Optimality of the Pseudoforest Nine Dragon Tree Theorems

In this section, we prove that the Pseudoforest Nine Dragon Tree Theorems are best possible. I.e., we will prove Theorem 1.21 and Theorem 1.32 at the end of the section.

Theorem 1.21 (cf. [15]). *For any $k \in \mathbb{N}_{\geq 1}$, $d \in \mathbb{N}_0$, and $\epsilon > 0$, there exists a simple graph G with $\gamma'(G) \leq k + \frac{d}{d+k+1} + \epsilon$, but G is not $(k,d)^-$-pseudo-decomposable.*

Theorem 1.32. *Let $k, d \in \mathbb{N}_0$. There are simple graphs G with $\beta'(G) = -1$ and $\beta'(H) \geq 0$ for all $H \subsetneq G$, but G is not $(k,d)^-$-pseudo-decomposable.*

In [15], where they proved the Pseudoforest Nine Dragon Tree Theorem, they also proved the following theorem showing that the weak version is already best possible.

Theorem 4.2.1 ([15]). *For any $k, d \in \mathbb{N}_{\geq 1}$, there are arbitrarily large, simple graphs G such that there is an edge $e^* \in E(G)$ with $\gamma'(G - e^*) = k + \frac{d}{d+k+1}$, but G is not $(k,d)^-$-pseudo-decomposable.*

Their construction is a pseudoforest version of the construction in [41] proving Theorem 3.2.1. Note that the statement of Theorem 4.2.1 does not imply Theorem 1.21: even if the graph G in Theorem 4.2.1 is very large, its densest subgraph could be small and contain e^*. In this case, $\gamma'(G)$ approaching $k + \frac{d}{d+k+1}$ arbitrarily closely is not guaranteed.
Note that the graph construction we use to prove Theorem 1.21 and Theorem 1.21 is a pseudoforest version of the construction described in [29], which they used to prove Theorem 1.16. We presented this (forest) construction in Section 3.2.
Recall that

$$\beta'(H) = v(H)(k+1)(d+k) - e(H)(d+k+1) + k$$

for a graph H.

Before we turn to the actual proof, we consider the edge cases $k = 0$ and $d = 0$. For the case $k = 0$, we need the following technical lemma.

Lemma 4.2.2. *Let G be a graph and $k \in \mathbb{N}_0$. Then G is $(k,0)$-pseudo-sparse if and only if $\gamma'(G) \leq k$.*

Proof. It is clear that G is $(k,0)$-pseudo-sparse if $\gamma'(G) \leq k$.
Thus, let G be $(k,0)$-pseudo-sparse and hence, $k \cdot v(H) \geq e(H) - \frac{k}{k+1}$ for all $H \subseteq G$. Suppose that there is a subgraph H with $\frac{e(H)}{v(H)} > k$. Then we have that

$$\frac{e(H) - \frac{k}{k+1}}{v(H)} \geq \frac{k \cdot v(H) + 1 - \frac{k}{k+1}}{v(H)} > k,$$

which is a contradiction. □

Lemma 4.2.3. *Theorem 1.21 and Theorem 1.32 are true for $d = 0$.*

Proof. Let $k \in \mathbb{N}_{\geq 1}$ and let G be a simple graph with $v(G) \geq \frac{1}{\epsilon}$ that is decomposable into k Hamiltonian cycles $C_1, \dots, C_k$ and one additional edge e. It is clear that such a graph can be constructed. We have that $e(G) = k \cdot v(G) + 1$ and thus, $\beta'(G) = -1$. By Hakimi's Theorem, G is not decomposable into k pseudoforests and thus, it is not $(k,0)^-$-pseudo-decomposable. Note that $C_k + e$ has exactly two cycles and every vertex is contained in at least one cycle. Thus, for every $V' \subsetneq V(G)$, each of $C_1[V'], \dots, C_{k-1}[V'], C_k[V'] + e$ contains at most one cycle. Hence, G is the only subgraph that is not decomposable into k pseudoforests. Thus, $\gamma'(G) = \frac{e(G)}{v(G)} \leq k + \epsilon$ and by Lemma 4.2.2, we have that $\beta'(H) \geq 0$ for every $H \subsetneq G$. □

Lemma 4.2.4. *Theorem 1.32 is true for $k = 0$.*

Proof. Let $d \in \mathbb{N}_0$. Let G be a star with $d+1$ edges. Then $\beta'(G) = (d+2)d - (d+1)^2 = -1$, $\beta'(H) \geq 0$ for all $H \subsetneq G$ and G is not $(0,d)^-$-pseudo-decomposable since its centre has degree $d+1$. □

Lastly, note that Theorem 1.21 is not true when $k = 0$: let G be a graph with $0 < \gamma'(G) < 1$. Then G is a forest. Let c be the number of edges of the largest component of G. Then $\gamma'(G) = \frac{c}{c+1}$. Thus, it is easy to find an $\epsilon(d)$ such that there is no graph G with $\frac{d}{d+1} < \gamma'(G) \leq \frac{d}{d+1} + \epsilon(d)$.

Now, we turn to the actual proof and define a graph G that will satisfy Theorem 1.21 and Theorem 1.32.
Because of Lemma 4.2.3 and Lemma 4.2.4, we let $k, d \in \mathbb{N}_{\geq 1}$. Furthermore, we let $t \in \mathbb{N}_{\geq 1}$ with $t \geq k+1$ and $s := t(d+k) + 1$. Additionally, for the proof of Theorem 1.21, we let $\epsilon > 0$ and $t \geq \frac{k+1-\epsilon(d+k+1)}{\epsilon(d+k+1)^2}$. We define a bipartite graph G on the set

of vertices $X = \{x_0, \dots, x_{s-1}\}$ and $Y = \{y_0, \dots, y_{t-1}\}$. The edge set of G is formed by the edges $x_i y_i, \dots, x_{i+k} y_{i+k}$ for every $i \in \{0, \dots, s-1\}$, where the indices of the vertices of Y are always understood to be taken modulo t, including the following. G is depicted in Figure 4.2.1 for fixed values of k, d, and t.
Note that

$$\frac{e(G)}{v(G)} = \frac{(k+1)s}{s+t} = k + \frac{d + \frac{1}{t}}{d + k + 1 + \frac{1}{t}}.$$

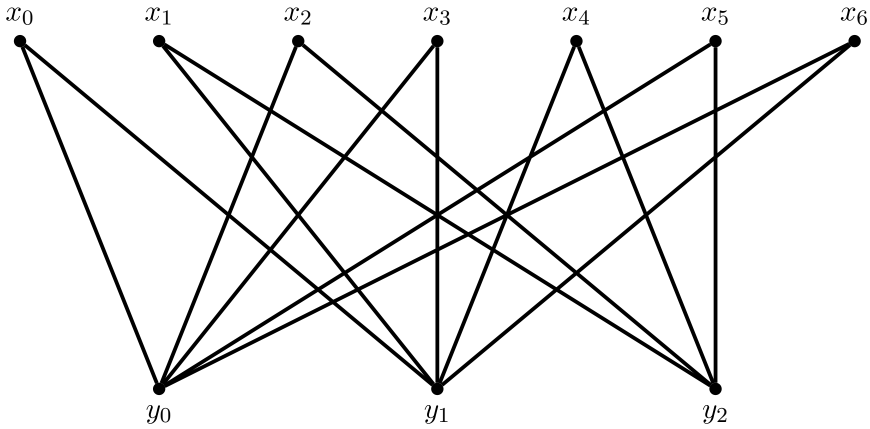

Fig. 4.2.1: G when $k = 1$, $d = 1$, $t = 3$.

In order to get a better understanding of the structure of G, we show that there is a decomposition of G that is almost $(k, d)^+$-pseudo-decomposable.

Theorem 4.2.5. *G has a decomposition into k pseudoforests and a star forest only consisting of stars with exactly 0 or d edges, except for one star which has $d+1$ edges.*

Proof. We describe the decomposition G by colouring its edges with colours $\{0, \dots, k\}$. k, which we will also call “red”, will be the colour of the forest with the star components. We describe the colouring by considering the incident edges of each vertex of X. For this purpose we partition X into $R = \{x_0, \dots, x_{dt-1}\}$, $B = x_{dt}, \dots, x_{s-2}$ and $Z = \{x_{s-1}\}$. Note that the colouring is depicted in Figure 4.2.2.
First, we consider R: for $i \in \{0, \dots, dt-1\}$, let $x_i y_i$ be coloured red and let the other k incident edges to x_i have all the colours of $\{0, \dots, k-1\}$.
For B, note that the index of every vertex of B can be written as $\alpha_{b,i} := dt + bt + i$ where $b \in \{0, \dots, k-1\}$ and $i \in \{0, \dots, t-1\}$. Let $x_{\alpha_{b,i}} y_{\alpha_{b,i}}$ and $x_{\alpha_{b,i}} y_{\alpha_{b,i}+1}$ be coloured

with colour b and let the $k-1$ remaining edges incident to $x_{\alpha_{b,i}}$ be coloured with all the colours of $\{0, \dots, b-1, b+1, \dots, k-1\}$.

Finally, we consider $Z = \{x_{s-1}\} =: \{z\}$. Let $z' := y_{s-1}$. We colour zz' red and we colour all remaining edges incident to z with all the colours of $\{0, \dots, k-1\}$. This completes the description of the colouring, which we will call f in the following.

Let F be the red subgraph of G induced by f. Note that in F, every vertex of Y is the centre of a star with exactly d edges except for z', which is the centre of a star having $d+1$ edges. The leaves of these stars are exactly the vertices of $R \cup Z$. Furthermore, the vertices of B are isolated in F.

In order to prove the theorem, we will show that for every $b \in \{0, \dots, k-1\}$, the graph B_b induced by colour b of f has exactly one cycle.

Note that $V_b := \{x_{\alpha_{b,0}}, \dots, x_{\alpha_{b,t-1}}\} \cup Y$ is the set of vertices that have degree at least 2 in B_b. Thus, all cycles of B_b are contained in $G[V_b]$. Now, observe that $G[V_b]$ is one cycle

$$[y_{\alpha_{b,0}}, x_{\alpha_{b,0}}, y_{\alpha_{b,1}}, x_{\alpha_{b,1}}, \dots, y_{\alpha_{b,t-1}}, x_{\alpha_{b,t-1}}, y_{\alpha_{b,t-1}+1}]$$

(note that $\alpha_{b,t-1} + 1 \equiv \alpha_{b,0} \mod t$), which completes the proof. □

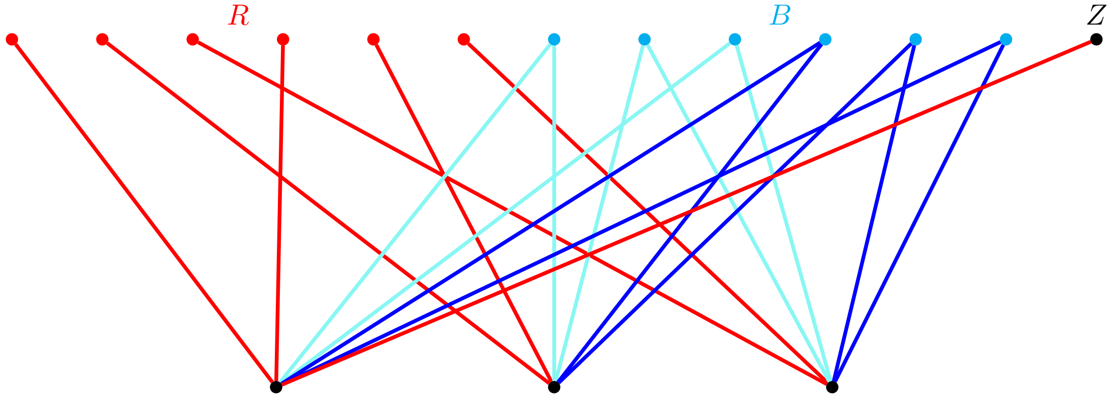

Fig. 4.2.2: The colouring of G described in the proof of Theorem 4.2.5 in the case where $k = 2$, $d = 2$, $t = 3$. For simplicity's sake, only a few edges that are crucial for the proof are depicted.

Before we examine the density of G, we prove that the desired decomposition is not possible.

Lemma 4.2.6. *G is not $(k, d)^{-}$-pseudo-decomposable.*

Proof. Removing the edges of a graph from G where every vertex has degree at most d results in a graph whose number of edges is at least

$$\begin{aligned} e(G) - dt &= (k+1)t(d+k) + k + 1 - dt \\ &= kt(d+k) + tk + k + 1 \\ &= k(s+t) + 1 \\ &= k \cdot v(G) + 1. \end{aligned}$$

Thus, the resulting graph cannot be partitioned into k pseudoforests. □

Next, we quickly calculate $\beta'(G)$.

Lemma 4.2.7. $\beta'(G) = -1$.

Proof. We have that

$$\begin{aligned} \beta'(G) &= (s+t)(k+1)(d+k) - s(k+1)(d+k+1) + k \\ &= (k+1)\big((d+k)t - s\big) + k \\ &= -1. \end{aligned}$$

□

Let H^* be a subgraph of G minimizing β'. Note that by the definition of β'_t, we have that $H^* = G[V(H^*)]$.

For the proof of Theorem 1.21, we define a similar function for $\varnothing \neq H \subseteq G$.

$$\beta'_t(H) := \big(kv(H) - e(H)\big)\left(d + k + 1 + \frac{1}{t}\right) + v(H)\left(d + \frac{1}{t}\right).$$

Note that $\beta'_t(H) \leq 0$ if and only if $\frac{e(H)}{v(H)} \geq k + \frac{d+\frac{1}{t}}{d+k+1+\frac{1}{t}}$. In particular, $\beta'_t(G) = 0$.

Similarly to the definition of H^*, we let H^*_t be a subgraph of G minimizing β'_t. We also have that $H^*_t = G[V(H^*_t)]$. In the next four lemmas, we show simple cases in which we can add or remove a vertex and its adjacent edges to increase the density.

Lemma 4.2.8. *Every vertex of H^* has degree at least $k+1$ in H^*.*

Proof. Suppose to the contrary that there is a vertex $v \in V(H^*)$ that has degree at most k in H^*. If $V(H^*) = \{v\}$, then $\beta'(H^*) = (k+1)(d+k) + k > 0$, which is a contradiction since $\beta'(G) = -1$. Thus, $v(H^*) > 1$. Let $H := H^* - v$. Then

$$\beta'(H^*) - \beta'(H) \geq (k+1)(d+k) - k(d+k+1) = d,$$

which is a contradiction to the minimality of H^*. □

Lemma 4.2.9. *Every vertex of H_t^* has degree at least $k+1$ in H_t^*.*

Proof. Suppose to the contrary that there is a vertex $v \in V(H_t^*)$ that has degree at most k in H_t^*. If $V(H_t^*) = \{v\}$, then $\beta_t'(H_t^*) = (k+1)(d+k) + \frac{2k}{t} > 0$, which is a contradiction since $\beta_t'(G) = 0$. Thus, $V(H_t^*) > 1$. Let $H := H_t^* - v$. Then

$$\beta_t'(H) \leq \beta_t'(H_t^*) - (k \cdot 1 - k)\left(d + k + 1 + \frac{1}{t}\right) - 1 \cdot \left(d + \frac{1}{t}\right) = \beta_t'(H_t^*) - d - \frac{1}{t},$$

which is a contradiction to the minimality of H_t^*. □

Lemma 4.2.10. *If $k+1$ neighbours of $v \in V(G)$ are in $V(H^*)$, then v is also contained in $V(H^*)$.*

Proof. Suppose to the contrary that v is not in $V(H^*)$ and let $H = H^* + v$. Then

$$\beta'(H) - \beta'(H^*) \leq (k+1)(d+k) - (k+1)(d+k+1) = -k-1,$$

which is a contradiction to the minimality of H^*. □

Lemma 4.2.11. *If* $k+1$ *neighbours of* $v \in V(G)$ *are in* $V(H_t^*)$*, then* v *is also contained in* $V(H_t^*)$*.*

Proof. Suppose to the contrary that v is not in $V(H_t^*)$ and let $H = H_t^* + v$. Then

$$\beta_t'(H) - \beta_t'(H_t^*) \leq -1 \cdot \left(d + k + 1 + \frac{1}{t}\right) + 1 \cdot \left(d + \frac{1}{t}\right) = -k - 1,$$

which is a contradiction to the minimality of H_t^*. □

For $i \in \{0, \dots, t-1\}$, let X_i be the set of vertices of X whose neighbours in G are exactly $y_i, \dots, y_{i+k}$. The indices of $X_\bullet$ are always meant to be taken modulo t. Note that $X_0, \dots, X_{t-1}$ is a partition of X. Furthermore, $|X_0| = \lceil \frac{s}{t} \rceil = d + k + 1$, and for $i \in \{1, \dots, t-1\}$, we have $|X_i| = \lfloor \frac{s}{t} \rfloor = d + k$.
With the help of the previous four lemmas, we are now in position to prove that G is its densest subgraph. After that, proving Theorem 1.21 and Theorem 1.32 will be easy.

Lemma 4.2.12. $H^* = G$ *and* $H_t^* = G$*.*

Proof. Let $H \in \{H^*, H_t^*\}$. Note that if $Y \subseteq V(H)$, then $H = G$ by Lemmas 4.2.10, 4.2.11. Thus, suppose that there is a vertex of Y that is not in $V(H)$. Note that $H \neq \varnothing$ by the definition of β_t' and since $\beta'(\varnothing) = k > \beta'(G)$. Thus, $Y \cap V(H) \neq \varnothing$ (if $x \in X \cap V(H)$, then the $k+1$ neighbours of x in Y are also in $V(H)$ by Lemmas 4.2.8, 4.2.9). Thus, $\varnothing \neq Y \cap V(H) \subsetneq Y$ and furthermore, there are integers l, r such that $0 \leq \ell \leq r \leq 2t - 1$ and $y_{\ell-1}, y_{r+1} \in V(H)$ and for every $i \in \{\ell, \dots, r\}$, we have that $y_i \notin V(H)$. Note that this situation is depicted in Figure 4.2.3. By Lemmas 4.2.8, 4.2.9, we have that $X_{\ell-k} \cap V(H), \dots, X_r \cap V(H) = \varnothing$. Thus, the only neighbours of $y_{\ell-1}$ in H are in $X_{\ell-k-1}$, and again by Lemmas 4.2.8, 4.2.9, at least $k+1$ vertices of $X_{\ell-k-1}$ are in $V(H)$. Thus, $y_{\ell-k-1}, \dots, y_{\ell-1} \in V(H)$. Analogously, we have that $y_{r+1}, \dots, y_{r+1+k} \in V(H)$.
We obtain $H' \subseteq G$ from H by adding vertices $y_\ell, \dots, y_r$ and the vertices of $X_{\ell-k}, \dots, X_r$, and adding all edges that are incident to $X_{\ell-k}, \dots, X_r$ in G. Let $\alpha := 1$ if $X_0 \in \{X_{\ell-k}, \dots, X_r\}$ and $\alpha := 0$ otherwise. Note that we add $y := r - \ell + 1$ vertices from Y, and the number of vertices added from X is

$$x := \sum_{i=\ell-k}^{r} |X_i| = (r - \ell + k + 1)(d + k) + \alpha = (y + k)(d + k) + \alpha.$$

Furthermore, the number of added edges is $(k+1)x$.
For $H = H_t^*$, we have that

$$\begin{aligned}
\beta_t'(H') - \beta_t'(H_t^*) &= \big(k(x+y) - x(k+1)\big)\left(d+k+1+\frac{1}{t}\right) + (x+y)\left(d+\frac{1}{t}\right) \\
&= (ky-x)\left(d+k+1+\frac{1}{t}\right) + (x+y)\left(d+\frac{1}{t}\right) \\
&= -(k+1)x + y\left(k(d+k+1)+d+\frac{k+1}{t}\right) \\
&\leq -(k+1)(y+k)(d+k) + y(d+k)(k+1) + y\cdot\frac{k+1}{t} \\
&= (k+1)\left(\frac{y}{t} - k(d+k)\right) \\
&< 0
\end{aligned}$$

since $\frac{y}{t} < 1$. However, this is a contradiction to the minimality of H^*.
For $H = H^*$, we have that

$$\begin{aligned}
\beta'(H') - \beta'(H^*) &= (x+y)(k+1)(d+k) - x(k+1)(d+k+1) \\
&= (k+1)\big(-x + y(d+k)\big) \\
&\leq (k+1)\big(-(y+k)(d+k) + y(d+k)\big) \\
&= -(k+1)k(d+k),
\end{aligned}$$

which is a contradiction to the minimality of H^*. □

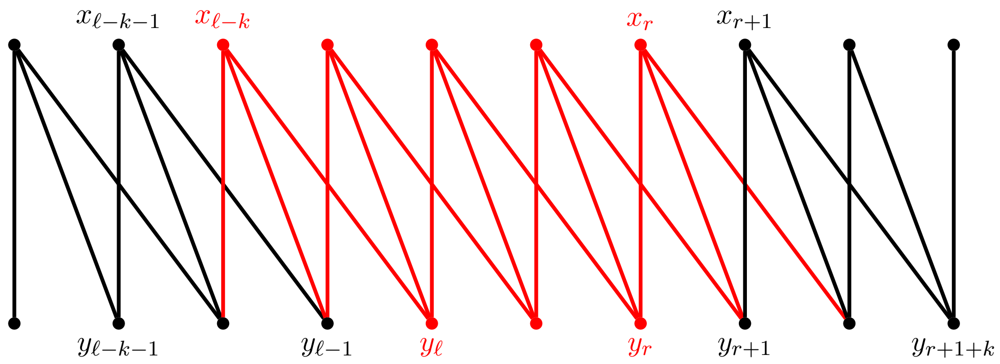

Fig. 4.2.3: An example of ℓ and r in the proof of Lemma 4.2.12. The black vertices and edges are from H. The red vertices are the ones we add to obtain H'.

Now, we only have to combine the previous lemmas to prove Theorem 1.21 and Theorem 1.32.

Proof of Theorem 1.21. G is not $(k,d)^-$-pseudo-decomposable by Lemma 4.2.6. Furthermore, we have that $\beta'_t(G) = 0$ and $\beta'_t(H) > 0$ for every $\varnothing \neq H \subsetneq G$ by Lemma 4.2.12. Thus,

$$\gamma'(G) = \frac{e(G)}{v(G)} = k + \frac{d + \frac{1}{t}}{d + k + 1 + \frac{1}{t}} \leq k + \frac{d}{d+k+1} + \epsilon.$$

$\square$

Proof of Theorem 1.32. Again, G is not $(k,d)^-$-pseudo-decomposable by Lemma 4.2.6.

Furthermore, we have that $\beta'(G) = -1$ by Lemma 4.2.7 and $\beta'(H) \geq 0$ for every $H \subsetneq G$ by Lemma 4.2.12. $\square$

4.3 Proof of the Upper Diameter Bounds

In this section, we prove Theorem 1.33.

Theorem 1.33. *Let $k, d \in \mathbb{N}_0$, $\ell := \left\lfloor \frac{d-1}{k+1} \right\rfloor$, and let G be (k,d)-pseudo-sparse. Then G decomposes into $k+1$ pseudoforests where one of the pseudoforests F satisfies the following:*

- *F is acyclic,*
- *every component K of F has $e(K) \leq d$,*
- $\operatorname{diam}(K) \leq 2\ell + 2$, *and if* $d \equiv 1 \mod (k+1)$, *then* $\operatorname{diam}(K) \leq 2\ell + 1$,
- *for every component K of F with $e(K) \geq d - z(k-1) + 1$, we have $\operatorname{diam}(K) \leq 2z$ for any $z \in \mathbb{N}_0$.*

The proof technique is very similar to the proof of the Theorems 1.13, 1.14, 1.15 in Chapter 3 and was also applied to pseudoforests in the proofs of the Pseudoforest Nine Dragon Tree Theorem in [15] and the Strong Pseudoforest Nine Dragon Tree Theorem in [23].

First, let us get the trivial cases out of the way.

Theorem 4.3.1. *Theorem 1.33 is true if* $k = 0$ *or* $d = 0$.

Proof. Let G be a graph. First, let G be $(0, d)$-pseudo-sparse where $d \in \mathbb{N}_0$. Thus, we have that $d \cdot v(G) \geq (d + 1)e(G)$ for all $H \subseteq G$. This implies that G is acyclic and every component has at most d edges. It is easy to see that the diameter constraints of Theorem 1.33 also hold.
Now, let G be $(k, 0)$-pseudo-sparse where $k \in \mathbb{N}_0$. By Lemma 4.2.2, we have that $\gamma'(G) \leq k$ and thus, in this case, the theorem follows by Hakimi's Theorem (Theorem 1.17). ☐

Due to Theorem 4.3.1, we let $k, d \in \mathbb{N}_{\geq 1}$ for the rest of the section.

4.3.1 Picking the Counterexample

We start by picking a counterexample graph and we will also define certain structure within this graph in this subsection. Most of the terms we define are also contained in Section 3.3. First, we show that when picking the counterexample graph, it suffices to deal with the case where k of the pseudoforests are maximal, or, in the context of orientations, the counterexample graph has an orientation where every vertex has at least k outgoing arcs.

Lemma 4.3.2 (Lemma 2.1 from [15]). *If* G *is a vertex-minimal counterexample to Theorem 1.33, then there exists an orientation of* G *such that for all* $v \in V(G)$*, we have* $k \leq \deg^+_G(v) \leq k + 1$.

Proof. Let D be a k-optimal orientation of G. Note that since there exists a $(k + 1)$-orientation of G by Theorem 1.17, we have that $\phi_i(D) = 0$ for all $i \in \{k+2, \ldots, e(G)\}$, and thus, D is a $(k + 1)$-orientation.
Now, suppose the lemma is not true. Then there is a vertex w' with $\deg^+_D(w') \leq k-1$. Let W be the set of vertices that have a dipath to w' in D. By Lemma 4.1.3, every vertex w of W has $\deg^+_D(w) \leq k$. Let $N(W)$ be the set of neighbours of W in G that are not in W. By Lemma the definition of W, no vertex of $N(W)$ has an outgoing edge to W in D. Thus, $D_W := D[W \cup N(W)] \setminus E(D[N(W)])$ is a k-orientation.
Let $U := V(G) \setminus W$. Note that $\varnothing \neq U \subsetneq V(G)$. Thus, there is a pseudoforest decomposition of $G[U]$ satisfying Theorem 1.33. Note that $E[D] = E(D_W) \dot\cup E(D[U])$. By canonically adding the arcs of the k-orientation D_W to the k non-constrained

pseudoforests, we obtain a pseudoforest decomposition of G satisfying Theorem 1.33, a contradiction. □

Motivated by Lemma 4.3.2, we let G be a vertex-minimal counterexample graph to Theorem 1.33 for the rest of the section.

We call $f = (D, F)$ a *red-blue colouring* if D is a digraph in which every vertex has exactly k outgoing arcs, F is a pseudoforest, and the multisets $E(G)$ and $\{\{u, v\} \mid (u, v) \in E(D)\} \cup E(F)$ are equal. We call the arcs of D *blue*, and the edges of F *red*. Furthermore, we write $\mathcal{B}(f) := D$ and $\mathcal{R}(f)$.

We let $\mathcal{F}$ denote the set of all red-blue colourings.

Note that $\mathcal{F}$ is non-empty by Lemma 4.3.2. Furthermore, by Corollary 4.1.2, we can extract a decomposition into $k+1$ pseudoforests from a red-blue colouring, where k pseudoforests are coloured blue and one is coloured red. If $k \geq 2$, we have multiple options to form the blue pseudoforests. The red pseudoforest will be the one in which we wish to eliminate cycles and bound the size and diameter of each component. As a note for the reader, as we modify a red-blue colouring in the following pages, it will be easier to focus on maintaining a blue k-orientation rather than considering the k individual blue pseudoforests. Furthermore, note that we call components of the red pseudoforest *red components*.

We now categorize the red components that contradict Theorem 1.33. Recall that $\ell := \left\lfloor \frac{d-1}{k+1} \right\rfloor$.

Definition 4.3.3. A red component K of a pseudoforest F is *bad* if K satisfies any of the following:

(1) contains a cycle, or

(2) has more than d edges, or

(3) has $\operatorname{diam}(K) > 2\ell + 2$, or

(4) has $\operatorname{diam}(K) = 2\ell + 2$ when $d \equiv 1 \mod (k+1)$, or

(5) has $\operatorname{diam}(K) > 2z$ and $e(K) \geq d - z(k-1) + 1$ for a $z \in \mathbb{N}_{\geq 1}$ with $z \leq \ell$.

We say K is *(i)-bad* if (i) is true for K and (j) is not true for K for all $j < i$.

Note that for (5)-bad components, we chose the range $1 \leq z \leq \ell$ since if $z = 0$ then K is (2)-bad, and if $z > \ell$ then K is (3)-bad.

We are now able to define a measure for red-blue colourings that indicates how close such a colouring is to satisfying Theorem 1.33. Namely: we say that an (i)-bad component is always worse than a (j)-bad component if $i < j$; that an (i)-bad component K is worse than an (i)-bad component K' if $e(K) > e(K')$. We will formalize this now.

Given a red-blue colouring f, let $\Delta^{(i)}(f) := \big(\Delta^{(i)}_{e(G)}(f), \Delta^{(i)}_{e(G)-1}(f), \dots, \Delta^{(i)}_0(f)\big)$ where $\Delta^{(i)}_j(f)$ is the number of red components of f that are (i)-bad and have exactly j edges. Let $\Delta'(f) = \big(\Delta^{(1)}(f), \dots, \Delta^{(5)}(f)\big)$. We will compare values of Δ' using lexicographic ordering (see Section 2.2).

As G is a counterexample, in every red-blue colouring f, there is at least one bad component in $\mathcal{R}(f)$. Our goal will be to augment the colouring in order to reduce $\Delta'(f)$. We will not be able to do that in a single step necessarily, so instead we will focus on one bad component and a specific subgraph that stems from this component where we will perform possibly many augmentations to eventually make the bad component less bad. To that end, we define the following subgraph.

Let $f \in \mathcal{F}$, and let R be a bad component of $\mathcal{R}(f)$. We define the *exploration subgraph* $H_{f,R}$ *of* f *rooted at* R in the following manner: let $S \subseteq V(G)$ where $v \in S$ if and only if there exists a path $P = [v_1, \dots, v_m]$ such that $v_m = v$, $v_1 \in V(R)$, and for each $i \in \{1, \dots, m-1\}$ either $(v_i, v_{i+1}) \in E(\mathcal{B}(f))$ or $v_i v_{i+1} \in E(\mathcal{R}(f))$. Let $H_{f,R}$ be the (mixed) subgraph of $\big(V(G),\ E(\mathcal{B}(f)) \cup E(\mathcal{R}(f))\big)$ induced by S.

We call a component of $\mathcal{R}(f)$ that is contained in $H_{f,R}$ a *red component of* $H_{f,R}$.

We want to augment red-blue colourings, and we want a measure of progress for the case where we cannot improve Δ'. For this purpose, we introduce a notion of a “legal order" of the red components of an exploration subgraph. This order describes how close red components are to the root component. When rating a legal order, we will consider it good if it has very small components close to the root component, because in this case we might find an augmentation involving the root component and an adjacent small component, which makes the root component “less bad” or even not bad at all. We formalize this in the following manner.

We call the sequence $(R_1, \dots, R_t)$ a *legal order of* $H_{f,R}$ if the sequence contains exactly all red components of $H_{f,R}$, R_1 is the root component R, and for each $j \in \{2, \dots, t\}$, there exists an integer i with $1 \leq i < j$ such that there is a blue arc (u, v) in $H_{f,R}$ with $u \in V(R_i)$ and $v \in V(R_j)$.

We will compare legal orders using lexicographic ordering, as defined in Section 2.2: let $H_{f,R}$, $H_{f',R'}$ be exploration subgraphs and let $\sigma = (R_1, \dots, R_t)$ and $\sigma' =$

$(R'_1, \dots, R'_{t'})$ be legal orders of $H_{f,R}$ and $H_{f',R'}$, respectively. We say that σ *is smaller than* σ' if $(e(R_1), \dots, e(R_t)) \leq (e(R'_1), \dots, e(R'_{t'}))$.

We are now able to define the exploration subgraph and legal order we will focus on in the proof. According to our measurements, they will be best possible. Thus, if we are able to improve them, this will be a contradiction. At the end, we will show that the exploration subgraph is not (k, d)-pseudo-sparse, which will prove Theorem 1.33.

Let $\sigma(f, R)$ denote the lexicographically smallest tuple $(e(R_1), \dots, e(R_t))$ over all legal orders $(R_1, \dots, R_t)$ of $H_{f,R}$. We call a legal order $(R'_1, \dots, R'_{t'})$ with $\sigma(f, R) = (e(R'_1), \dots, e(R'_{t'}))$ a *smallest legal order for* f, R.

For our vertex-minimal counterexample G, we choose a colouring $f^* \in \mathcal{F}$ and a bad component R^* of $F^* := \mathcal{R}(f)$ such that $\Delta(f^*, R^*) = \big(\Delta'(f^*), \sigma(f^*, R^*)\big)$ is lexicographically smallest.

From here on out, we will fix f^*, R^*, F^* picked in the manner described above. Furthermore, we let $H^* := H_{f^*,R^*}$ and let $(R^*_1, \dots, R^*_t)$ be a smallest legal order for f^*, R^*.

Next, we define a few terms that help us talking about the structure of $(R^*_1, \dots, R^*_t)$. Note that these terms can be defined for any legal order, however, we will only need them for our optimal legal order.

We say that R^*_i is a *parent of* R^*_j if $i < j$ and there is a blue arc (v_i, v_j) in H^* where $v_i \in V(R^*_i)$ and $v_j \in V(R^*_j)$. In this case, we also say that R^*_j is a *child of* R^*_i *generated by* (v_i, v_j).

Note that a red component may have many parents, unlike in Section 3.3. We say a red component R^*_i is an *ancestor of* R^*_j if there exists a sequence of red components $R^*_{i_1}, \dots, R^*_{i_m}$ such that $R^*_{i_1} = R^*_i$, $R^*_{j_m} = R^*_j$, and $R^*_{i_q}$ is a parent of $R^*_{i_{q+1}}$ for all $q \in \{1, \dots, m-1\}$.

We say a vertex $v \in V(R^*_j)$ *witnesses the legal order* if there is a blue arc (u, v) and integer $1 \leq i < j$ such that $u \in V(R^*_i)$.

Observe that there may be many vertices that witness the legal order. More importantly, for every red component except the root, there exists a vertex that witnesses the legal order.

In order to augment our decomposition, we will only use the following simple procedure taken from [23].

Definition 4.3.4. Let K be a red component of H^* and let C be an acyclic child of K generated by (x, y). Suppose that $e = xv$ is a red edge in K incident to x. To *exchange* e *and* (x, y) is to perform the following procedure: first, change the colour of (x, y) to red and remove its orientation. Second, change the colour of e to blue and orient it to (x, v).

Observation 4.3.5 ([23]). *Suppose we exchange the edge* $e = xv$ *and* (x, y)*. Then the resulting red-blue colouring is in* $\mathcal{F}$*.*

We will implicitly use Observation 4.3.5 throughout the rest of the section.

Next, we want to analyze the density of H^*. For a graph H, an important density measure will be

$$\hat{\beta}(H) := d \cdot v(H) - (d + k + 1)e(H).$$

We call a graph K *small* if K is a tree and $e(K) \leq \ell$. Note that $e(K) \leq \ell$ if and only if $e(K) < \frac{d}{k+1}$, which yields the following observation.

Observation 4.3.6. *A connected graph* K *is small if and only if* $\hat{\beta}(K) > 0$*.*

A small component cannot be bad; we prove this in Corollary 4.3.12.

The next lemma shows how we will obtain the contradiction that G is not (k, d)-pseudo-sparse. For this, we define a partitioning of the red components of F^*, and show that every part H has $\hat{\beta}(H) \leq 0$. The part H' including R^* will even have $\hat{\beta}(H') < -k$. This will be enough to show that H^* is too dense.

The way we form the partitioning is simple: we assign every small child component to its parent component. Together they will form a part. Furthermore, we will later show that small components cannot have small children. Thus, a part will consist of a non-small component and its small children (and no further descendants). Let us formalize this.

Let $\mathcal{K}$ denote the set of red components of H^* that are not small. Furthermore, let $\mathcal{C}(K)$ be the set of small children of a red component K. Let

$$\operatorname{Ext}(K) := \Big(V(K) \cup \bigcup_{C \in \mathcal{C}(K)} V(C),\ E(K) \cup \bigcup_{C \in \mathcal{C}(K)} E(C)\Big).$$

Lemma 4.3.7. *Assume that in* H^**,*

(1) small components do not have small children,

(2) we have $\hat{\beta}(\operatorname{Ext}(K)) < -k$ *for every bad component* K *in* H^**, and*

(3) we have $\hat{\beta}(\operatorname{Ext}(K)) \leq 0$ *for every non-bad red component* K *of* H^**.*

Then we obtain the contradiction $\beta'(H^*) < 0$, which proves Theorem 1.33.

Proof. Note that by the definition of f^* and H^*, the number of blue arcs in H^* is $k \cdot v(H^*)$ since every vertex has exactly k outgoing blue arcs. Let $e_r(H^*)$ denote the number of red edges in H^*. We have that

$$\begin{aligned}
\beta'(H^*) &= v(H^*)(k+1)(d+k) - e(H^*)(d+k+1) + k \\
&= v(H^*)\big(k(d+k+1) + d\big) - \big(k \cdot v(H^*) + e_r(H^*)\big)(d+k+1) + k \\
&= v(H^*) \cdot d - e_r(H^*)(d+k+1) + k \\
&\overset{(1)}{=} \sum_{K \in \mathcal{K}} \hat{\beta}(\mathrm{Ext}(K)) + k.
\end{aligned}$$

By (2) and (3), the lemma follows. □

The setup for our proof is now done. It remains to show that the three conditions of Lemma 4.3.7 hold for our optimally chosen red-blue colouring f^* and legal order $(R_1^*, \dots, R_t^*)$.
In the following subsections, we will show the second condition for all types of bad components. In the final Subsection 4.3.5, we will also show the first and third condition.

4.3.2 Density of $\mathrm{Ext}(K)$ if K is (1)-bad

In this subsection, we show that if we have a component which is (1)-bad, then it has no small children, and thus contributes to showing that the average degree of our exploration subgraph is too high.

Observation 4.3.8 ([23]). *If C is an acyclic child of K generated by (x, y), then x does not lie in a cycle of F^*.*

Proof. Suppose towards a contradiction that x lies in a cycle of F^*. Let e be an edge incident to x which lies in the cycle coloured red. Now exchange e and (x, y). As (x, y) was an arc between two distinct red components and e was in the cycle coloured red, after performing the exchange, we reduce the number of cycles in F^* by one and do not affect other cyclic components. Thus the exchange results in a red-blue colouring with a smaller $\Delta^{(1)}$, which is a contradiction. □

Lemma 4.3.9 ([23]). *If K is a cyclic red component of H^*, then there is no blue arc (x, y) from K to an acyclic red component C.*

Proof. Suppose towards a contradiction that there is such an arc. By Observation 4.3.8 we know that x does not lie on a cycle of F^*. There is a unique red path from x to the cycle of K. On this path, let w be the neighbour of x. Exchange xw and (x, y). In the resulting red pseudoforest, x and y are in an acyclic component and w is in a cyclic component containing fewer edges than K. Again, this results in a red-blue colouring with smaller $\Delta^{(1)}$, a contradiction. □

We are now equipped to prove the main result of this subsection.

Lemma 4.3.10. *If a red component K of H^* is (1)-bad, then $\hat{\beta}(\mathrm{Ext}(K)) < -k$.*

Proof. If K is cyclic, then $v(K) = e(K) \geq 1$, and K does not have any small children by Lemma 4.3.9. Thus,

$$\hat{\beta}(\mathrm{Ext}(K)) = d \cdot v(K) - (d + k + 1)e(K) = -(k + 1)e(K) < -k.$$

□

4.3.3 Density of $\mathrm{Ext}(K)$ if K is (3)-bad

In this subsection, we build towards bounding the density of $\mathrm{Ext}(K)$ when K is (3)-bad. Recall that a red component is (3)-bad if it is acyclic, has at most d edges, and has diameter at least $2\ell + 3$. First, we aim to show that small components are not bad. To that end, we prove the following.

Lemma 4.3.11. *We have that $d > \ell(k + 1)$; and if $z \in \{0, \ldots, \ell\}$, we have moreover that $d - z(k - 1) + 1 > 2\ell + 1$.*

Proof. We have

$$\ell(k + 1) = \left\lfloor \frac{d - 1}{k + 1} \right\rfloor (k + 1) \leq d - 1.$$

The second part of the lemma follows immediately since $k \geq 1$. □

This leads us to the useful corollary below.

Corollary 4.3.12. *If K is an acyclic red component with $e(K) \leq 2\ell + 1$, then K is not bad. In particular, small red components are not bad.*

The following technical lemma will be important for several future manipulations we will perform upon F^*.

Lemma 4.3.13. *If K is a red acyclic component of H^* and there is a colouring $f \in \mathcal{F}$ with a red pseudoforest F whose set of components can be obtained from the set of components of F^* by:*

- *removing K,*
- *possibly adding acyclic components $K_1, ..., K_q$ with $q \in \mathbb{N}_0, e(K_i) < e(K)$ and $\operatorname{diam}(K_i) \leq \operatorname{diam}(K)$ for every $1 \leq i \leq q$, and*
- *possibly adding or removing non-bad components,*

then $K, K_1, ..., K_q$ are not bad and thus $\Delta'(f) \leq \Delta'(f^*)$.

Proof. Obviously, the addition and removal of non-bad components does not change Δ'.
First, suppose that K is bad. Since K is acyclic, it follows that K is (b)-bad for some $b \in \{2, \ldots, 5\}$. Then $\Delta^{(b)}_{e(K)}$ decreases and $\Delta^{(b)}_i$ remains the same for all $i > e(K)$ when manipulating F^* as described above. Thus, Δ' decreases, which is a contradiction. Next, suppose that K_i is (b)-bad, where $i \in \{1, ..., q\}$ and $b \in \{2, ..., 5\}$. But then K is also (b)-bad and we again obtain a contradiction as Δ' decreases when performing the operations described in the lemma. □

In the next two lemmas we will show that the large diameter of (3)-bad components prevents them from having small children in our optimal colouring f^*.

Observation 4.3.14. *For any vertex v of a tree T, there is a path of length at least $\left\lceil \frac{\operatorname{diam}(T)}{2} \right\rceil$ starting at v. If v does not lie on a longest path of T, then this bound can be increased by one.*

Proof. Let P be a path that attains the diameter of T. If $v \in V(P)$, then the result is immediate. Otherwise, $v \notin V(P)$ and let $P' = [v, \ldots, v']$ be the shortest path from v to P. Concatenating P' and the longest path in P starting at v' gives the result. □

Lemma 4.3.15. *If there is a small child C of a red acyclic component K of H^* generated by (x, y), then every red path starting at x has length at most $e(C) + 1$, and the diameter of K is at most $2e(C) + 2$.*

Proof. Suppose towards a contradiction that there is a path inside K of length at least $e(C)+2$ starting at x. Let P be this path, and let e be the edge on this path that is incident with x. Exchange e and (x,y). This does not create any cyclic component. Let K' and K'' be the resulting new components, where $x \in V(K')$. We have $e(K'') < e(K)$ and $\mathrm{diam}(K'') \leq \mathrm{diam}(K)$, since K'' is a subgraph of $K - e$. Furthermore, we have that $e(K') \leq e(K) - e(P) + e(C) + 1 < e(K)$ and it is also easy to see that $\mathrm{diam}(K') \leq \mathrm{diam}(K)$. By Lemma 4.3.13 we get that K, K' and K'' are not bad and thus Δ' has not increased.
Finally, we can construct a smaller legal order by taking $(R_1^*, \dots, R_t^*)$ up until K, and then replacing K with one of K' and K'' containing a vertex that witnesses this legal order, and completing the order arbitrarily. By this contradiction, it follows that every path of K starting at x has length at most $e(C)+1$. If $\mathrm{diam}(K) \geq 2e(C)+3$, then by Observation 4.3.14 there would be a path of length $e(C)+2$ starting at x, which is a contradiction. ☐

For red acyclic components of large diameter, Lemma 4.3.15 gives the following.

Corollary 4.3.16. *Let K be an acyclic red component of H^*. If $\mathrm{diam}(K) > 2\ell + 2$, then K does not have any small children.*

We are now equipped to prove the main result of this subsection.

Lemma 4.3.17. *If a red component K of H^* is (3)-bad, then $\hat{\beta}(\mathrm{Ext}(K)) < -k$.*

Proof. If K is (3)-bad, then by Corollary 4.3.16 it does not have any small children. Thus,

$$\begin{aligned}
\hat{\beta}(\mathrm{Ext}(K)) &= d \cdot \big(e(K)+1\big) - (d+k+1)e(K) \\
&\leq d - (k+1)(2\ell+3) \\
&< d - (k+1)\left(2 \cdot \frac{d}{k+1} + 1\right) \\
&= -d-k-1
\end{aligned}$$

☐

4.3.4 Density of $\mathrm{Ext}(K)$ if K is (2)-, (4)- or (5)-bad

We will not be able to get rid of all the small children of non-small red components that are are not (1)- or (3)-bad like we did in the previous subsections. However, we will bound the number of small child components in the following lemma. A similar lemma can be found in [23], but in our case we have to carry out a more careful analysis to ensure that we do not create new bad components when manipulating f^*.

Lemma 4.3.18. *If K is an acyclic red component of H^*, then K has at most k small children.*

Proof. Suppose that K has $k+1$ small children. As each vertex has only k outgoing blue edges, there exist two distinct vertices $x_1, x_2 \in V(K)$ such that there are two distinct small children C_1 and C_2 generated by blue arcs (x_1, y_1) and (x_2, y_2), respectively. Let e_1 and e_2 be the edges on the red path between x_1 and x_2 that are incident to x_1 and x_2, respectively. Note that e_1 and e_2 are not necessarily distinct. For $i \in \{1, 2\}$, let K_i be the component of $K - e_i$ containing x_i and let $K_i' = (V(K_i) \cup V(C_i),\ E(K_i) \cup E(C_i) + x_i y_i)$. Let $L_i = K \setminus V(K_j)$, where $j = 3 - i$. Furthermore, we define *doing exchange i* to mean exchanging e_i and (x_i, y_i). In the course of the proof, we will always do only one of the exchanges, not both. Note that after doing exchange i, the red pseudoforest of the resulting colouring contains the components K_i' and L_j with $e(L_j) < e(K)$. Thus, we call K_1', L_2, C_2 and K_2', L_1, C_1 *new components*. The described components are depicted in Figure 4.3.1. Without loss of generality let L_2 contain a vertex that witnesses the legal order (if K is not the root).

Claim 4.3.18.1. *We have $e(K_i') < e(K)$ for each $i \in \{1, 2\}$.*

Proof. Let $i \in \{1, 2\}$, $j = 3 - i$ and suppose towards a contradiction that we have $e(K_i') \geq e(K)$. Then we have

$$\begin{aligned} e(K_j) &\leq e(K) - 1 - e(K_i) \\ &= e(K) - 1 - (e(K_i') - e(C_i) - 1) \\ &\leq \ell, \qquad\qquad \text{since } C_i \text{ is small.} \end{aligned}$$

First, suppose that $j = 2$ and $e(K) \leq 2\ell + 1$. It follows that $e(K_1) \leq \ell$ since $e(K_2) \leq \ell$ and $e(K) \geq e(K_1) + e(K_2) + 1$. After doing exchange 1, we have $e(K_1') \leq 2\ell + 1$

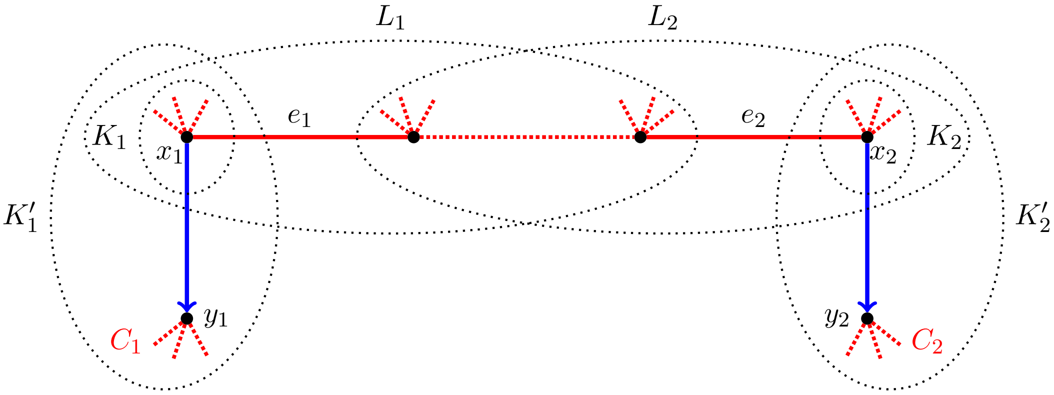

Fig. 4.3.1: A diagram showing the various components in Lemma 4.3.18. The red dotted lines indicate possibly undrawn vertices and edges.

and $e(L_2) < e(K)$. Thus, none of the new components are bad by Corollary 4.3.12 and also K is not bad. We see that after the exchange, we obtain a smaller legal order by taking the same legal order up to K but replacing K with L_2 and then completing the order arbitrarily.

Thus, this case cannot happen and it must be that $j = 1$ or $e(K) > 2\ell+1$. In this case we perform exchange j instead of exchange i, then $e(K'_j) \leq 2\ell + 1$, since $e(K_j) \leq \ell$. Note that K'_j is not bad by Corollary 4.3.12, and so by Lemma 4.3.13 we get that neither K nor any of the new components are bad and in particular, $K \neq R^*$.

We can again obtain a smaller legal order by taking the same legal order up to K but replacing K with K'_2 if $j = 2$ and $e(K) \leq 2\ell + 1$, or with L_2 otherwise, and then completing the order arbitrarily. *(End of proof of the claim)* ■

For each $i \in \{1, 2\}$, let $j = 3 - i$ and let r_i be the number of edges in a longest path in K_i starting at x_i. Note that $r_i \leq e(C_j)$, as otherwise there would be a path of size $e(C_j) + 1 + |\{e_j\}|$ starting at x_j, which is a contradiction to Lemma 4.3.15. Similarly, $r_i \leq e(C_i) + 1$.

Claim 4.3.18.2. $\mathrm{diam}(K'_1), \mathrm{diam}(K'_2) \leq 2\ell + 1$.

Proof. Suppose $\mathrm{diam}(K'_i) \geq 2\ell + 2$. As $e(C_i) \leq \ell$ and thus $r_i \leq \ell$, a longest path in K'_i that contains x_i has at most $2\ell + 1$ edges and thus, x_i does not lie on a longest path of K'_i. But by Observation 4.3.14, this gives a path of size $\ell + 2$ starting at x_i, which is a contradiction since every path in K'_i starting at x_i is fully contained in either K_i or $C_i + x_i y_i$. *(End of proof of the claim)* ■

Let us look at the consequences of the two claims regarding whether or not K is bad: we defined K to be acyclic, thus it is not (1)-bad and neither are any new components.

Using Claim 4.3.18.1, we know that K is not (2)-bad or otherwise $\Delta^{(2)}$ would decrease when doing exchange 1 or 2. Thus, none of the new components are (2)-bad. Analogously, using Claim 4.3.18.2 we could decrease Δ' if K was (3)-bad or (4)-bad and also none of the new components are (3)-bad or (4)-bad.

If K was (5)-bad, it is again clear by Claim 4.3.18.1 that we could decrease Δ'.

If there is an $i \in \{1, 2\}$ such that K'_i is not (5)-bad (and thus not bad), then this either decreases Δ' or we find a smaller legal order after doing exchange i by taking the same legal order up to K but replacing K with the component K'_2, if $i = 2$, or L_2, if $i = 1$ and then completing the order arbitrarily.

Thus, for the rest of the proof we assume that both K'_1 and K'_2 are (5)-bad and we aim to show that K is also (5)-bad. This proves the lemma, as in this case we can do either exchange 1 or 2 and get a smaller Δ', by Claim 4.3.18.1, a contradiction.

If a longest path P of K'_i does not contain x_i, then either $P \subseteq C_i$ and thus $\operatorname{diam}(K'_i) \leq \ell$, or $P \subseteq K_i$, thus $\operatorname{diam}(K'_i) = \operatorname{diam}(K_i)$ and by Observation 4.3.14 we obtain $\operatorname{diam}(K'_i) \leq 2r_i$. As $r_i \leq e(C_i) + 1$, it follows that $\operatorname{diam}(K'_i) \leq r_i + e(C_i) + 1$. On the other hand if there exists a longest path of K'_i containing x_i, we also have $\operatorname{diam}(K'_i) \leq r_i + e(C_i) + 1$. Thus, in any case we have that $\operatorname{diam}(K'_i) \leq r_i + e(C_i) + 1$.

As both K'_1 and K'_2 are (5)-bad, there are natural numbers z_1 and z_2 such that for each $i \in \{1, 2\}$, we have $\operatorname{diam}(K'_i) \geq 2z_i + 1$ and $e(K'_i) \geq d - z_i(k-1) + 1$. Thus

$$\begin{aligned} e(K'_i) &\geq d - \left\lfloor \frac{\operatorname{diam}(K'_i) - 1}{2} \right\rfloor (k-1) + 1 \\ &\geq d - \left\lfloor \frac{r_i + e(C_i)}{2} \right\rfloor (k-1) + 1. \end{aligned}$$

We know that $\operatorname{diam}(K) \geq r_1 + r_2 + 1$. Thus, it suffices to prove $e(K) \geq d - \left\lfloor \frac{r_1+r_2}{2} \right\rfloor (k-1) + 1$ in order to prove that K is (5)-bad, completing the proof of the lemma as explained above.

Observe

$$\begin{aligned} e(K) &\geq e(K_1) + e(K_2) + 1 \\ &= \sum_{i=1}^{2} \big(e(K_i') - e(C_i) - 1\big) + 1 \\ &\geq \sum_{i=1}^{2} \left(d - \left\lfloor \frac{r_i + e(C_i)}{2} \right\rfloor (k-1) - e(C_i)\right) + 1. \end{aligned}$$

From here, let $\alpha_i = 1$ if $r_i \not\equiv e(C_i) \mod 2$ and 0 otherwise. Let $\delta = 1$ if $e(C_1) \not\equiv e(C_2) \mod 2$, and 0 otherwise. Note that $\left\lfloor \frac{r_i+e(C_i)}{2} \right\rfloor = \frac{r_i+e(C_i)-\alpha_i}{2}$. Then

$$\begin{aligned} e(K) &\geq 2d - \left(\frac{e(C_1) + r_1 - \alpha_1}{2} + \frac{e(C_2) + r_2 - \alpha_2}{2}\right)(k-1) - (e(C_1) + e(C_2)) + 1 \\ &= d - \frac{r_1 + r_2 - (\alpha_1 + \alpha_2)}{2}(k-1) + 1 + d - \frac{k+1}{2}(e(C_1) + e(C_2)) \\ &\geq d - \frac{r_1 + r_2 - (\alpha_1 + \alpha_2)}{2}(k-1) + 1 + d - \frac{k+1}{2}(2\ell - \delta) \text{ since } e(C_i) \leq \ell \text{ for } i \in \{1,2\} \\ &> d - \frac{r_1 + r_2 - (\alpha_1 + \alpha_2 + \delta)}{2}(k-1) + 1. \end{aligned}$$

If $1 \in \{\alpha_1, \alpha_2, \delta\}$, then $\left\lfloor \frac{r_1+r_2}{2} \right\rfloor \geq \frac{r_1+r_2-(\alpha_1+\alpha_2+\delta)}{2}$. If $1 \notin \{\alpha_1, \alpha_2, \delta\}$, then by definition of α_1, α_2, and δ we have that $r_1 + r_2$ is even, and so again $\left\lfloor \frac{r_1+r_2}{2} \right\rfloor \geq \frac{r_1+r_2-(\alpha_1+\alpha_2+\delta)}{2}$. Thus in either case K is (5)-bad since

$$e(K) \geq d - \left\lfloor \frac{r_1 + r_2}{2} \right\rfloor (k-1) + 1.$$

□

Lemma 4.3.18 provides a first bound for $\hat{\beta}(K)$.

Lemma 4.3.19. *Let $K \in \mathcal{K}$ be acyclic and let $c \in \mathbb{N}_0$ such that $e(C) \geq c$ for all $C \in \mathcal{C}(K)$ if K has a small child, and $c := 0$ otherwise. Then*

$$\hat{\beta}(\mathrm{Ext}(K)) \leq k \cdot \big(d - (k+1)c\big) + d - (k+1)e(K).$$

Proof.

$$\begin{aligned}\hat{\beta}(\mathrm{Ext}(K)) &= d \cdot v(\mathrm{Ext}(K)) - (d+k+1) \cdot e(\mathrm{Ext}(K)) \\ &= d \cdot \big(e(\mathrm{Ext}(K)) + 1 + |\mathcal{C}(K)|\big) - (d+k+1) \cdot e(\mathrm{Ext}(K)) \\ &\leq d \cdot \big(1 + |\mathcal{C}(K)|\big) - (k+1)\big(e(K) + |\mathcal{C}(K)| \cdot c\big) \\ &= |\mathcal{C}(K)| \cdot \big(d - (k+1)c\big) + d - (k+1)e(K).\end{aligned}$$

The lemma follows since $|\mathcal{C}(K)| \leq k$ by Lemma 4.3.18, and since $d - (k+1)c > 0$ because $c \leq \ell$ and by Lemma 4.3.11. ☐

Lemma 4.3.20. *If a red component K of H^* is (2)-bad, then $\hat{\beta}(\mathrm{Ext}(K)) < -k$.*

Proof. Suppose that K is (2)-bad. By Lemma 4.3.19, we have that

$$\hat{\beta}(\mathrm{Ext}(K)) \leq k \cdot d + d - (k+1)(d+1) = -k - 1.$$

☐

In order to show the same for (4)-bad components, we only need the following simple corollary from Lemma 4.3.15:

Corollary 4.3.21. *Let K be an acyclic red component of H^*. If $\mathrm{diam}(K) \geq 2\ell + 1$, then for all small children C of K, we have $e(C) = \ell$.*

Lemma 4.3.22. *If a red component K of H^* is (4)-bad, then $\hat{\beta}(\mathrm{Ext}(K)) < -k$.*

Proof. Let K be (4)-bad. Since $d \equiv 1 \mod (k+1)$, we have that $(k+1)\ell = d - 1$. By Lemma 4.3.19, we have that

$$\begin{aligned}\hat{\beta}(\mathrm{Ext}(K)) &\leq k\big(d - (k+1)\ell\big) + d - (k+1)(2\ell+2) \\ &= k + d - 2(d-1) - 2(k+1) \\ &= -k - d.\end{aligned}$$

☐

Lemma 4.3.23. *If a red component K of H^* is (5)-bad, then $\hat{\beta}(\mathrm{Ext}(K)) < -k$.*

Proof. Suppose K is (5)-bad. Then we have that $e(K) \geq d - z(k-1) + 1$ and $\operatorname{diam}(K) > 2z$ for a $z \in \mathbb{N}_{\geq 1}$ with $z \leq \ell$. Thus, $e(C) \geq z$ for every small child C of K by Lemma 4.3.15. By Lemma 4.3.19, we have that

$$\hat{\beta}(\mathrm{Ext}(K)) \leq k\bigl(d - (k+1)z\bigr) + d - (k+1)\bigl(d - (k-1)z + 1\bigr) = -(k+1)z - (k+1).$$

□

4.3.5 Density of $\mathrm{Ext}(K)$ if K is not bad

In this subsection we will show that the density of K_C for non-bad components is large if K is not small. If K is small, we will show that K has no small children. Before this we start with a technical lemma:

Lemma 4.3.24. *Let K be a red acyclic component of H^* and C be a small child of K generated by (x, y). Then $K' := \bigl(V(K) \cup V(C),\ E(K) \cup E(C) + xy\bigr)$ has diameter at most $2e(C) + 2$.*

Proof. If K' contains a path of size $2e(C) + 3$, then by Lemma 4.3.15 this path must contain xy and therefore contain x. But again by Lemma 4.3.15, there are only paths of size at most $e(C) + 1$ starting at x in K and $E(C) + xy$ also has at most $e(C) + 1$ edges. □

Lemma 4.3.25 ([23]). *Let K be a red component of H^* that is not bad. Let C be a small child of K generated by (x, y). Furthermore, suppose that K is small or $d \not\equiv 1 \mod (k+1)$ or $e(C) < \ell$. Then $e(K) \geq d - e(C)k$.*

Proof. Assume to the contrary that $e(K) + e(C) + 1 < d - e(C)(k-1) + 1$. Let $L := (V(K) \cup V(C), E(K) \cup E(C) + xy)$.

Claim 4.3.25.1. *L is not bad.*

Proof. Note that L is acyclic and thus not (1)-bad. By our assumption we have $e(L) < d - e(C)(k-1) + 1$ and thus it is not (2)-bad. Further, L is not (5)-bad. To see this, suppose there exists $z \in \{1, \ldots, \ell\}$ such that $e(K) \geq d - z(k-1) + 1$. By the assumption, $z > e(C)$. By Lemma 4.3.24, it follows that the diameter of L is at most $2e(C) + 2$, and thus the diameter of L is at most $2z$ implying that L is not (5)-bad. Similarly, by Lemma 4.3.24 we obtain that $\operatorname{diam}(L) \leq 2e(C) + 2$

and thus it is not (3)-bad. Finally, we have that L is not (4)-bad, since if K is small we have $e(L) \leq 2\ell + 1$ and if $e(C) < \ell$, then by Lemma 4.3.24 we even have $\operatorname{diam}(L) \leq 2\ell$. *(End of proof of the claim)* ■

Since K is not bad, it is not the root component R^* and thus there is a vertex $w \in V(K)$ witnessing the legal order.

Case 1: $w \neq x$.
Let e be the red edge incident to x in K such that e lies on the path from x to w in K. Then exchange e and (x, y) and let K' and K'' be the new red components containing x and w, respectively. As K' and K'' are subgraphs of L and L is not bad by Claim 4.3.25.1, it follows that K' and K'' are not bad. Furthermore, find a smaller legal order by taking the same legal order up to K but replacing K with its proper subgraph K'' and completing the order arbitrarily.

Case 2: $w = x$.
We refer the reader to Figure 4.3.2 for an illustration. As K is not the root component, K has an ancestor.

As x witnesses the legal order, there is a parent component S_1 of K that has a blue arc to x. If S_1 does not have any edges and thus only consists of a single vertex x_1, then x_1 also witnesses the legal order. In this manner we can find an ancestor of K that contains an edge, since the root component R^* contains at least one edge. Let $S_1, \dots, S_n$ be a sequence of red components such that K is a child of S_1, $e(S_n) \geq 1$ and for $i \in \{1, \dots, n-1\}$, S_i is a child of S_{i+1} and $e(S_i) = 0$. There is a blue directed path $P = x_n, \dots, x_1, x, y$ with $x_i \in V(S_i)$ for all $i \in \{1, \dots n\}$. Let e be a red edge incident to x_j. Now do the following. Colour (x, y) red, remove its orientation and reverse the direction of all remaining arcs in P. Colour e blue, and orient e away from x_j. The resulting coloured mixed graph is in $\mathcal{F}$, which contains the red and non-bad component L. By Lemma 4.3.13 we can conclude that neither S_j nor the components of $S_n - e$ are bad. Thus, none of the components that have been manipulated by the exchange are bad and in particular, S_n is not the root.
Finally, we can find a smaller legal order in the colouring of this orientation, as we simply take the same legal order up to S_n, replace S_n with one of the two components of $S_n - e$ and complete the remaining order arbitrarily. □

Corollary 4.3.26 ([23]). *If K is a small red component of H^*, then K does not have any small red children. Furthermore, every small red component of H^* has a parent component which is not small.*

Fig. 4.3.2: An illustration of Case 2 in Lemma 4.3.25, where $k = 1$, $d = 3$, $x = w$ and $n = 1$.

Proof. Suppose K is small and has a small child C. Then $e(K) \leq \ell$ by the definition of small, and by Lemma 4.3.25, we have $e(K) \geq d - e(C)k$. Moreover, since C is small, $e(K) \geq d - \ell k$. Thus $d - \ell k \leq \ell$, or $d \leq \ell(k+1)$. This contradicts the definition of ℓ.
For the second part of the corollary remember that the root component is bad and therefore not small due to Corollary 4.3.12. □

Lemma 4.3.27. *If a red component $K \in \mathcal{K}$ of H^* is not bad, then $\hat{\beta}(\mathrm{Ext}(K)) \leq 0$.*

Proof. First, suppose that K does not have small children. Then $\hat{\beta}(\mathrm{Ext}(K)) = \hat{\beta}(K) \leq 0$ by Observation 4.3.6.
Next, suppose that $d \equiv 1 \mod (k+1)$ (and thus $(k+1)\ell = d - 1$) and all small children of K have ℓ edges. By Lemma 4.3.19, we have that

$$\begin{aligned}\hat{\beta}(\mathrm{Ext}(K)) &\leq k \cdot \big(d - (k+1)\ell\big) + d - (k+1)(\ell+1)\\ &= k + d - (d-1) - (k+1)\\ &= 0.\end{aligned}$$

Finally, suppose that $d \not\equiv 1 \mod (k+1)$ or that K has a small child with less than ℓ edges. Thus, we can apply Lemma 4.3.25 and obtain $e(K) \geq d - k \cdot e(C)$, where C is a small child of K with the smallest number of edges. By Lemma 4.3.19, we have that

$$\hat{\beta}(\mathrm{Ext}(K)) \geq k(d - (k+1)e(C)) + d - (k+1)(d - k \cdot e(C)) = 0.$$

□

Proof of Theorem 1.33. For $k = 0$ or $d = 0$, the theorem follows by Theorem 4.3.1. Otherwise, the theorem follows by Lemma 4.3.7: Condition (1) follows from Corollary 4.3.26, Condition (2) follows from Lemma 4.3.10, 4.3.20, 4.3.17, 4.3.22, 4.3.23, and Condition (3) follows from Lemma 4.3.27. □

4.4 Lower Bound of the Overall Diameter

In this section, we prove Theorem 1.26, which establishes the optimality of the overall diameter bound of $2\ell + 2$ (or $2\ell + 1$, respectively) stated in Theorem 1.22.

Theorem 1.26. *Let $k \in \mathbb{N}_{\geq 1}$, $\ell, D \in \mathbb{N}_0$, $\epsilon > 0$, and $\alpha \in \{0, 1\}$. There are simple graphs G with $\gamma(G) < k + \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon$ that do not have a decomposition into k pseudoforests and another graph F such that F has maximum degree at most D and the diameter of every component in F is less than $2\ell + 1 + \alpha$. Furthermore, G is planar if $k = 1$.*

We will construct a graph G parameterized by δ and p that satisfies Theorem 1.26 for fixed k, D, ϵ, α. We do this in phases. First we define a specific tree with a particular orientation and edge-colouring with colours $\{1, \dots, k + 1\}$.
Let T be a tree with root r_T and odd depth $\delta \geq \frac{2(\ell+1)}{\epsilon} - 1$ formed from a 1-coloured dipath of length δ starting at r_T by adding another $k - 1$ outgoing arcs (to new vertices) with colours $2, \dots, k$ to each even-depth vertex of this path.
From T, we construct a tree C in the following manner. For each $t \in V(T)$, we add a path P_t which is disjoint from T except for at t, and P_t has length l_t where

$$l_t = \begin{cases} 2\ell + 1 + \alpha & \text{if } t = r_T, \\ \ell + \alpha & \text{if } t \neq r_T \text{ and } \mathrm{depth}_T(t) \text{ is even,} \\ \ell & \text{if } \mathrm{depth}_T(t) \text{ is odd.} \end{cases}$$

If $t \neq r_T$, then t is an endpoint of P_t; and r_T is in the middle of P_{r_T} (i.e. there are two edge-disjoint subpaths of P_{r_T} starting at r_T of length at least ℓ). Further, for all $t \in V(T)$, we colour all edges of P_t with $k + 1$, and we orient the edges towards t. We call C a *colourful tree* and consider it to be rooted at $r_C := r_T$ as well.
Finally, we obtain our desired graph G by taking p pairwise disjoint copies of C, where $p \in \mathbb{N}_{\geq 1}$ such that $p \geq kD + k^2 + 1$, a set of new vertices $S := \{s_1, \dots, s_k\}$ and

for every colour $i \in \{1, \dots k\}$ and every vertex x belonging to a colourful tree, we add an arc (x, s_i) if x does not have an outgoing arc coloured i. For each copy of C, we let T_C denote that copy of T contained in C.

We will refer to the colours $\{1, \dots, k\}$ as *blue*, and the colour $k+1$ as *red*. Note that unlike for the upper bound, where it is useful to think of all of the different blue colours as the same, for this construction each blue monochromatic component should be thought of as having a specific blue colour i. We call the established colouring and orientation of G the *example colouring*. A colourful tree having the colours of the example colouring is depicted in Figure 4.4.1. Note the example colouring is a colouring where with the tools developed in Section 4.3, we would not be able to reduce the diameter further, and the red graph contains a path of length $2\ell + 1 + \alpha$. The example colouring will be especially useful in Subsection 4.4.2 as it allows us to concisely refer to different structures within G.

Our first point of order is to show that G is planar for $k = 1$.

Lemma 4.4.1. *G is planar if $k = 1$.*

Proof. The union of all colourful trees of G is outerplanar since it is a forest, see, for example, [2]. The lemma follows by placing s_1 in the outer face of the forest. □

Next, we will show that in any decomposition of G into k pseudoforests and a red graph, the red graph has large diameter. We do this in Subsection 4.4.1. In Subsection 4.4.2, we lower-bound the fractional arboricity of G, which completes the proof of Theorem 1.26.

4.4.1 Bounding the Diameter

The goal of this subsection is to prove the following theorem for the graph G we constructed earlier.

Theorem 4.4.2. *In any decomposition of G into k blue pseudoforests and one red graph where every vertex has at most D incident red edges, there is a red component that has diameter at least $2\ell + 1 + \alpha$.*

We assume there is a colouring f of the edges of G that contradicts Theorem 4.4.2. In the following lemma and corollary, we will easily force colours on many edges of at least one colourful tree. The result is depicted in Figure 4.4.2. The rest of the subsection will show that we cannot find colours for the remaining black

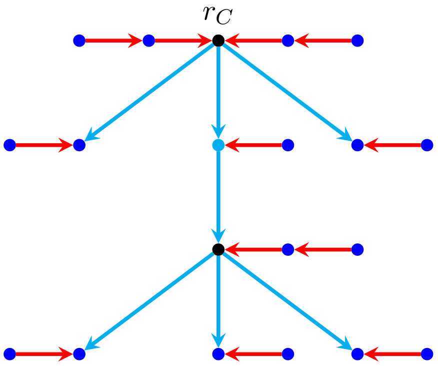

Fig. 4.4.1: The example colouring of a colourful tree if $k = 3, \ell = 1, \alpha = 1, \delta = 3$. Blue [light blue] vertices have an edge to S in every colour of $\{1, \dots, k\}$ [of $\{2, \dots k\}$].

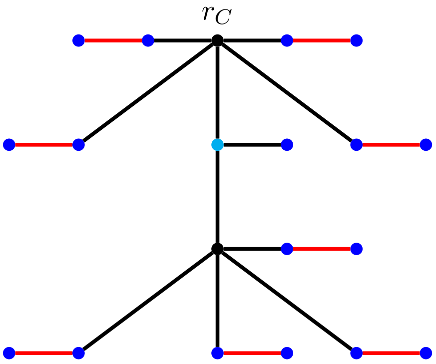

Fig. 4.4.2: C after enforcing the edge-colouring of Corollary 4.4.4. Blue [light blue] vertices have k [$k-1$] blue edges to S.

edges in the figure without creating red paths which are too long or creating blue components with too many cycles.

In what follows, an *S-C-S-path* is a path or a cycle with endpoints in S and whose inner vertices are all from one colourful tree C.

Lemma 4.4.3. *There is a colourful tree C in G such that every edge of $E(C, S)$ is coloured blue in f. Furthermore, there is no monochromatic S-C-S-path.*

Proof. There are at most kD red edges incident to S. For any colour $b \in \{1, \dots, k\}$, there can be at most k S-C-S paths having colour b in f as otherwise we would have monochromatic components containing two cycles. As $p > kD + k^2$, there is at least one colourful tree C that satisfies the lemma. □

For the rest of the subsection, we let C be a colourful tree satisfying Lemma 4.4.3. The following corollary follows easily from Lemma 4.4.3.

Corollary 4.4.4. *In f, every vertex of C that is not an inner vertex of T_C has a b-coloured edge to S for every $b \in \{1, \dots, k\}$, and every vertex of T_C with odd depth has $k-1$ incident edges to S in pairwise distinct colours of $\{1, \dots, k\}$. Every edge of $E(C)$ that is not incident with an inner vertex of T_C is coloured red in f. Furthermore, every red component containing at least one vertex in C is acyclic in f.*

For $i \in \{\ell, \ell+1, 2\ell+1+\alpha\}$, let V_i be the set of vertices of $V(T_C)$ contained in red components with exactly i edges in the example colouring.

Given a colour $b \in \{1, \dots, k\}$, we say a vertex $v \in V(C)$ has a *b-coloured S-path* if there is a path of colour b in f that goes from v to a vertex of S and for any inner vertex w of this path, $w \in V(C)$ and $\mathrm{depth}_C(w) \geq \mathrm{depth}_C(v)$. Further, we say a vertex t *is the end of a low i-path* if t is an endpoint of a red path P in f with $e(P) \geq i$ and $\mathrm{depth}_C(v) \geq \mathrm{depth}_C(t)$ for every $v \in V(P)$.

Lemma 4.4.5. *In f, for each $i \in \{\ell, \ell+1\}$, every $t \in V_i$ is the end of a low i-path and has a b-coloured S-path for every $b \in \{1, \dots, k\}$.*

Proof. The lemma is clear for any leaf of T_C due to Corollary 4.4.4.
Next, suppose that $t \in V_{\ell+\alpha}$ has even depth, is not the root of T_C and that the lemma is true for all of its children $c_1, \dots, c_k$ in T_C. Let u be the vertex such that there is a red arc (u, t) in the example colouring. Note that u is also a child of t in C. By Corollary 4.4.4, u is the end of a low $(\ell+\alpha-1)$-path in f and it also has a b-coloured S-path (of length 1) for any $b \in \{1, \dots, k\}$.
No two edges of $tc_1, \dots, tc_k, tu$ have the same colour $j \in \{1, \dots, k\}$ in f or there would be a monochromatic S-C-S-path, contradicting the choice of C. Furthermore, no two of these edges are red or f would have a red path with at least $(\ell+\alpha-1)+2+\ell$ edges, a contradiction. Thus, the $k+1$ edges have pairwise distinct colours, from which the lemma follows.
Now, let $t \in V_\ell$ have odd depth, not be a leaf of T_C and again suppose that the lemma is true for the only child t' of t in T_C. Let u be the other child of t in C, which is the neighbour of t in P_t. Note that t' has a low $(\ell+\alpha)$-path, and by Corollary 4.4.4 u has a low $(\ell-1)$-path. Since t has $k-1$ S-paths of length 1, at most one of the edges to the children of t can be blue. If both edges were red, then there would be a red path of length $(\ell+\alpha)+2+(\ell-1)$, which is contrary to our assumptions. The lemma follows. ☐

The following corollary shows that r_C is contained in a red component with too high diameter, contradicting our initial assumption and thus completing the proof of Theorem 4.4.2.

Corollary 4.4.6. *The diameter of the red component of r_C in f is at least $2\ell+1+\alpha$.*

Proof. By Lemma 4.4.5 we know that the k children of r_C in T_C are the ends of low ℓ-paths and that they each have a b-coloured S-path for all $1 \leq b \leq k$. Furthermore, by Corollary 4.4.4 the two neighbours of r_C in P_{r_C} are also ends of low ℓ- and $(\ell+\alpha-1)$-paths, respectively, and they have b-coloured S-paths of length 1 for every $1 \leq b \leq k$.

We again conclude that there cannot be two edges from r_C to one of its $k+2$ children in C that have the same colour b with $1 \leq b \leq k$ as otherwise we have a monochromatic S-C-S-path, contradicting Lemma 4.4.3. Thus, two of the $k+2$ edges from r_T to its children are red and thus — together with the low ℓ- (and perhaps $(\ell+\alpha-1)$-)paths described above — form a red path with at least $(\ell+\alpha-1)+\ell+2 = 2\ell+1+\alpha$ edges. □

4.4.2 Bounding the Fractional Arboricity

The last point of order is to upper-bound the fractional arboricity of G. As mentioned in the introduction, this will also give us a bound on its fractional pseudoarboricity. We aim to show that the entire graph is the densest subgraph of G. We have chosen δ large enough such that the red components of a colourful tree in the example colouring that do not contain a root of a colourful tree, which have a density of roughly $\frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha}$, compensate for the largest red component (which contains the root). We will prove the following theorem in this subsection.

Theorem 4.4.7. *The fractional arboricity of G is less than $k + \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon$.*

We assume towards a contradiction that there is a subgraph $H \subseteq G$ with $v(H) \geq 2$ and $\frac{e(H)}{v(H)-1} \geq k + \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon$ for the rest of the subsection. Let $\mathcal{X}$ be the set of all subsets $X \subseteq V(G)$ with $S \subsetneq X$ and

$$\frac{e(X)}{|X \setminus S|} > k + \mu,$$

where

$$\mu = \frac{\ell(k+1)+\alpha+\frac{2(\ell+1)}{\delta+1}}{(\ell+1)(k+1)+\alpha+\frac{2(\ell+1)}{\delta+1}}$$

Lemma 4.4.8. $\frac{e(G)}{|V(G)\setminus S|} = k + \mu$ *and thus* $V(G) \notin \mathcal{X}$.

Proof. In the example colouring every colourful tree C has $\frac{\delta+1}{2}$ red components P_t containing ℓ edges (where $\operatorname{depth}_{T_C}(t)$ is odd) and C has $\frac{\delta-1}{2}$ red components P_t containing $\ell+\alpha$ edges (where $\operatorname{depth}_{T_C}(t)$ is even and non-zero). Furthermore, note that every vertex of $V(G) \setminus S$ has exactly k blue outgoing arcs in the example colouring and there are no blue outgoing arcs from S. It follows that

$$\frac{e(G)}{|V(G) \setminus S|} = k + \frac{p\big(2\ell + \alpha + 1 + \frac{\delta-1}{2}(\ell+\alpha) + k\frac{\delta+1}{2}\ell\big)}{p\big(2\ell + \alpha + 2 + \frac{\delta-1}{2}(\ell+\alpha+1) + k\frac{\delta+1}{2}(\ell+1)\big)} = k + \mu.$$

□

Lemma 4.4.9.

$$\frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} < \mu < \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon.$$

Proof. The first inequality is easy to see. For the second inequality we use the lower bound of $\delta \geq \frac{2(\ell+1)}{\epsilon} - 1$ and the fact that for any numbers $a \geq 0, b \geq 1$, and $c > 0$, it holds that $\frac{a+c}{b+c} < \frac{a}{b} + c$.

□

Corollary 4.4.10. *If Theorem 4.4.7 is false, then $\mathcal{X}$ is not empty.*

Proof. Let H be an induced subgraph of G with $v(H) \geq 2$ and $\frac{e(H)}{v(H)-1} \geq k + \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon$. Note that $S \subseteq V(H)$, as otherwise H decomposes into at most $k-1$ star forests, where the centres lie in S, and one forest whose edges are from colourful trees, contradicting Theorem 1.1 since $\frac{e(H)}{v(H)-1} > k$ by assumption. We have

$$k + \mu < k + \frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha} + \epsilon \leq \frac{e(H)}{v(H)-1} \leq \frac{e(H)}{|V(H) \setminus S|}.$$

□

Our goal is to show that $V(G) \in \mathcal{X}$, which contradicts Lemma 4.4.8. For this we will show that if there is an $X \in \mathcal{X}$, then we can manipulate X in multiple steps such that after every step the new set of vertices is still in $\mathcal{X}$, and the resulting set is $V(G)$. The technical lemma that we will use implicitly in the remaining lemmas is the following, which follows from the definition of $\mathcal{X}$:

Lemma 4.4.11. *Let $X \in \mathcal{X}$ and $Z \subseteq V(G) \setminus S$.*
If $Z \subseteq X$ and

$$\frac{e(Z) + e(X \setminus Z, Z)}{|Z|} \leq k + \mu,$$

then $X \setminus Z \in \mathcal{X}$ *and in particular,* $X \setminus Z \neq S$*. If* $X \cap Z = \varnothing$ *and*

$$\frac{e(Z) + e(X, Z)}{|Z|} \geq k + \mu,$$

then $X \cup Z \in \mathcal{X}$*.*

Let C be a colourful tree and $t \in V(T_C)$ for the rest of the subsection.

Lemma 4.4.12. *If* $X \in \mathcal{X}$ *and* $t \in X$*, then* $X \cup V(P_t) \in \mathcal{X}$*.*

Proof. If we add all vertices of $V(P_t) \setminus X$ to X, then for each vertex $v \in V(P_t) \setminus X$ in the induced subgraph we add at least k edges from v to S and at least one other edge, namely the red outgoing arc of v in the example colouring. Thus,

$$e(V(P_t) \setminus X) + e(V(P_t) \setminus X, X) \geq (k+1)\,|V(P_t) \setminus X|.$$

□

Lemma 4.4.13. *If* $X \in \mathcal{X}$ *and* $t \notin X$*, then* $X \setminus V(P_t) \in \mathcal{X}$*.*

Proof. Let $P' = X \cap V(P_t) \neq \varnothing$. When subtracting P' from X, the induced subgraph will lose $k|P'|$ edges of $E(P', S)$ and another $e(P') \leq |P'| - 1$ edges. If $t \neq r_C$ or $G[P']$ has exactly one component, then we have

$$\frac{e(P') + e(P', S)}{|P'|} = \frac{k|P'| + e(P')}{|P'|} \leq k + \frac{|P'| - 1}{|P'|} \leq k + \frac{\ell}{\ell + 1} < k + \mu.$$

If $t = r_C$ and $G[P']$ has two components, we have that $e(P') \leq |P'| - 2$ and $|P'| \leq 2\ell + 1 + \alpha$. In this case we have

$$\frac{e(P') + e(P', S)}{|P'|} = \frac{k|P'| + e(P')}{|P'|} \leq k + \frac{|P'| - 2}{|P'|} \leq k + \frac{\ell + (\ell + \alpha - 1)}{\ell + 1 + (\ell + \alpha)} \leq k + \frac{\ell}{\ell + 1} < k + \mu.$$

□

Lemma 4.4.14. *If* $X \in \mathcal{X}$*,* $t \in X \cap V_\ell$ *and a neighbour of* t *in* T_C *is not in* X*, then* $X \setminus V(P_t) \in \mathcal{X}$*.*

Proof. Let $X' = X \cup V(P_t)$. We have $X' \in \mathcal{X}$ due to Lemma 4.4.12. When removing the $\ell + 1$ vertices of P_t from $G[X']$, we remove $k\ell$ edges of $E(V(P_t) - t, S)$, at most $k + 1$ edges that were incident to t and all $\ell - 1$ edges of $E(P_t - t)$. We have

$$\frac{k\ell + (k+1) + (\ell - 1)}{\ell + 1} = \frac{k(\ell+1) + \ell}{\ell + 1} = k + \frac{\ell}{\ell + 1} < k + \mu$$

and thus $X \setminus V(P_t) = X' \setminus V(P_t) \in \mathcal{X}$. □

By repeated application of the last three lemmas we get the following corollary:

Corollary 4.4.15. *If $\mathcal{X} \neq \varnothing$, then there is an $X \in \mathcal{X}$ such that for every $t \in V(T_C) \cap X$ we have $V(P_t) \subseteq X$ and if additionally $t \in V_\ell$, then $\deg_X(t) = k + 2$.*

We also want to prove the degree property of Corollary 4.4.15 for every $t \in V(T_C)$. We will tackle this in the following two lemmas.

Lemma 4.4.16. *Let $X \in \mathcal{X}$ such that for every $t \in V(T_C) \cap X$ we have $V(P_t) \subseteq X$ and if additionally $t \in V_\ell$, then $\deg_X(t) = k + 2$. Furthermore, suppose there is a vertex $r' \in V(T_C)$ such that all vertices of the tree T' containing the vertices $t \in V(T_C)$ with $\mathrm{depth}_{T_C}(t) \geq \mathrm{depth}_{T_C}(r')$ are also in X, but the parent of r' is not in X. Let $T'' = T' \cup \bigcup_{t \in V(T')} P_t$. Then $X \setminus V(T'') \in \mathcal{X}$.*

Proof. We have that $r' \in V_{\ell+1}$, as r' is the root of T' and thus has at most $k + 1$ neighbours in X, and all vertices in $V_\ell \cap X$ have $k + 2$ neighbours in X. It follows that $\alpha = 1$ since $V_{\ell+1} \neq \varnothing$. Let T'' contain x vertices of $V_{\ell+1}$ and thus kx vertices of V_ℓ. We have

$$\frac{e(T'') + e(X \setminus V(T''), V(T''))}{v(T'')} = k + \frac{x(\ell+1) + kx\ell}{x(\ell+2) + kx(\ell+1)} = k + \frac{\ell(k+1) + \alpha}{(\ell+1)(k+1) + \alpha} < k + \mu.$$

Thus, $X \setminus V(T'') \in \mathcal{X}$. □

Lemma 4.4.17. *If $\mathcal{X} \neq \varnothing$, then there is an $X \in \mathcal{X}$ where every $t \in V_\ell \cap X$ has $\deg_X(t) = k + 2$ and for every $t \in X \cap (V_{\ell+1} \cup V_{2\ell+1+\alpha})$ the child of t in T_C is also in X.*

Proof. Note that by Corollary 4.4.15, there exists an $X \in \mathcal{X}$ where every $t \in V_\ell \cap X$ has $\deg_X(t) = k + 2$. From all such X, we choose one set X where the number of vertices $t \in X \cap (V_{\ell+1} \cup V_{2\ell+1+\alpha})$, where the child of t in T_C is not in X, is minimal. Suppose towards a contradiction that this minimum value is greater than zero and let t be such a vertex with maximal depth and its child being $t' \notin X$.

Let T' be the subtree of C with root t'. By Lemma 4.4.16 we can choose X such that no vertex of T' is in X. Our desired contradiction will be $X \cup V(T') \cup \bigcup_{t \in V(T')} V(P_t) \in \mathcal{X}$, since this set contains t'. Let $x := |V_{\ell+1} \cap V(T')|$ and thus $|V_\ell \cap V(T')| = kx + |\{t'\}|$. Since $T' \neq T_C$, we have $x < \frac{\delta-1}{2}$ and thus

$$k + \frac{(\ell + |\{tt'\}|) + x(\ell + \alpha) + kx\ell}{(\ell + 1) + x(\ell + \alpha + 1) + kx(\ell + 1)} = k + \frac{\ell(k+1) + \alpha + \frac{\ell+1}{x}}{(\ell+1)(k+1) + \alpha + \frac{\ell+1}{x}} > k + \mu.$$

□

Corollary 4.4.18. *$\mathcal{X}$ is empty. Thus, Theorem 4.4.7 is true.*

Proof. Let $X \in \mathcal{X}$ such that it satisfies Lemma 4.4.17. We can choose X such that its induced subgraph does not have any subtrees T' with root r' as described in Lemma 4.4.16. But then we have either $X = S$ or $X = V(G)$, both of which give a contradiction: the former to the definition of $\mathcal{X}$, and the latter to Lemma 4.4.8. □

4.5 Lower Bound of the Diameter of Large Components

In this section, we will prove Theorem 1.27.

Theorem 1.27. *Let $k, D, d, z \in \mathbb{N}_0$, $\epsilon > 0$, $k \geq 2$, $z \leq \left\lfloor \frac{d-1}{k+1} \right\rfloor$. There are simple graphs G with $\gamma(G) < k + \frac{d}{d+k+1} + \epsilon$, such that, in any decomposition of G into k pseudoforests and another graph F with maximum degree at most D, F contains a component K with $e(K) \geq d - z(k-1) + 1$ and $\operatorname{diam}(K) \geq 2(z+1)$. Furthermore, G is planar if $k = 1$.*

We prove the theorem using a construction very similar to the one given in the previous section. The structure of this section is also identical to the last section and for some proofs we will just refer to the previous section. First, we give the construction. We will again describe the *example colouring of G*:
For every colourful tree C the tree T_C has odd depth $\delta > \frac{2(z+1)}{\epsilon} - 1$, and is comprised of a blue dipath of colour 1 and length $\delta - 1$ starting at the root r_C of T_C and C, where additionally, every vertex with even depth has $k-1$ other outgoing arcs with colours $2, \ldots, k$ in T_C. These $k-1$ children are leaves in T_C. We obtain C from T_C by adding the fewest number of new vertices and red edges such that no two

vertices of $V(T_C)$ are in the same red component and for every vertex $t \in V(T_C)$ we have:

- If $\mathrm{depth}_{T_C}(t)$ is odd, then there is an induced red path of length z ending in t.
- If $t = r_C$, then t is in a red acyclic component K with $e(K) = d - z(k-1) + 1$ and $\mathrm{diam}(K) = 2(z+1)$, t is exactly in the middle of a red path $Q_t \subseteq K$ of length $2(z+1)$ and $K - t$ consists of components with at most z edges.
- If $\mathrm{depth}_{T_C}(t)$ is even and $t \neq r_C$, then t is in a red acyclic component K having $d - zk$ edges and containing a red path Q_t of length $z+1$ ending in t, and $K - t$ consists of components with at most z edges.

Furthermore, all edges of a red component in C of a vertex $t \in V(T_C)$ are oriented towards t in the example colouring. Note that the paths Q_t and the diameter of K in the second and third case are well-defined by Lemma 4.3.11.

For vertices $t \in V(T_C)$, we again denote by P_t the red component containing t in the example colouring. Note that when creating C, there are many possible configurations for the red component P_t if $\mathrm{depth}_{T_C}(t)$ is even.
Finally, we obtain G by taking $p \geq kD + k^2 + 1$ disjoint copies of C and adding a set $S = \{s_1, \ldots, s_k\}$ of new vertices and for every colour $i \in \{1, \ldots, k\}$ and every vertex x of a colourful tree we add an edge xs_i — oriented towards s_i in the example colouring — if x does not already have an outgoing arc coloured i.
Analogously to Lemma 4.4.1, we have that G is planar for $k = 1$.

4.5.1 Bounding the Diameter

In this subsection, we prove the theorem below.

Theorem 4.5.1. *In any decomposition of G into k blue pseudoforests and one red graph where every vertex has at most D incident red edges, there is a red component K with $e(K) \geq d - z(k-1) + 1$ that has diameter at least $2(z+1)$.*

For the rest of the subsection, we assume that there is a colouring f of G contradicting Theorem 4.5.1. The following lemma and corollary can be proven analogously to Lemma 4.4.3 and Corollary 4.4.4.

Lemma 4.5.2. *There is a colourful tree C such that every edge of $E(C, S)$ is coloured blue in f. Furthermore, there is no monochromatic S-C-S-path.*

For the rest of the subsection, we let C be a colourful tree satisfying Lemma 4.5.2.

Corollary 4.5.3. *In f, every vertex of C that is not an inner vertex of T_C has a b-coloured edge to S for every $b \in \{1, \dots, k\}$, and every vertex of T_C with odd depth has $k-1$ incident edges to S in pairwise distinct colours of $\{1, \dots, k\}$. Every edge of $E(C)$ that is not incident to an inner vertex of T_C is coloured red in f. Furthermore, every red component containing at least one vertex of C is acyclic in f.*

For each vertex $t \in V(T_C)$, let K_t be the subgraph induced by the vertices v in the same red component in f as t and with $\mathrm{depth}_C(v) \geq \mathrm{depth}_C(t)$. Note that K_t is connected and all of its edges are coloured red in f. The following lemma bounds $e(K_t)$, and will be used to bound $e(K_{r_C})$ later. Recall that by Corollary 4.5.3, every edge not incident to an inner vertex of T_C is red. The main idea below is that if an edge from t to $V(C) \setminus V(T_C)$ is coloured blue in f (thus not matching the example colouring), then we will argue a corresponding edge from t to one of its children t' in T_C is coloured red in f; which, together with showing t' is the end of a low z-path, will be enough to bound $e(K_t)$. Note that the diameter bounds proceed similarly to the previous section, and the new argument is in bounding the number of edges.

Lemma 4.5.4. *If $t \in V(T_C)$ with $t \neq r_C$, then t has a b-coloured S-path for any $b \in \{1, \dots, k\}$ and t is the end of a low z-path P. If $\mathrm{depth}_{T_C}(t)$ is even, then $e(P) \geq z+1$ and $e(K_t) \geq d - zk$.*

Proof. The lemma is clear for any leaf of T_C due to Corollary 4.5.3. Now, let t have even depth and suppose that the lemma is true for the children of t in T_C. All of the children of t in C have a b-coloured S-path for any $b \in \{1, \dots, k\}$. Thus, at most k edges from t to the children of t in C can be blue or otherwise there is a monochromatic S-C-S-path, contradicting Lemma 4.5.2. At least $k+1$ children of t in C are ends of low z-paths (k of them are children of t in T_C and the other one is on Q_t; recall by Corollary 4.5.3 the edges of Q_t not incident with t are red). Let us call the set of edges to these $k+1$ children E_t. Since at least one of the edges of E_t is red, t is the end of a low $(z+1)$-path in f.
Recall that each of the components of $P_t - t$ has at most z edges, the components of the $k+1$ children of t that are not in P_t have at least z edges, and since at most k edges to the children of t in C are blue (like in the example colouring), we have $e(K_t) \geq e(P_t) = d - zk$.

If fewer than k edges of E_t were blue in f, then we had $\operatorname{diam}(K_t) \geq 2(z+1)$ and $e(K_t) \geq e(P_t) + (z+1) \geq d - z(k-1) + 1$, which is a contradiction. Thus, t has a b-coloured S-path for each $b \in \{1, \dots, k\}$.
Now, let t have odd depth and not be a leaf and again suppose that the lemma is true for the only child t' of t in T_C. Let u be the other child of t in C, which is the neighbour of t in P_t. Note that t' has a low $(z+1)$-path and u has a low $(z-1)$-path. Since t has $k-1$ S-paths of length 1, at most one of the edges to the children of t can be blue. If both edges were red, then we would have $e(K_t) \geq (d - zk) + (z-1) + 2 = d - z(k-1) + 1$ and $\operatorname{diam}(K_t) \geq (z+1) + 2 + (z-1) = 2(z+1)$, which is contrary to our assumptions. The lemma follows. □

Lemma 4.5.5. *We have* $e(K_{r_C}) \geq d - z(k-1) + 1$ *and* $\operatorname{diam}(K_{r_C}) \geq 2(z+1)$*, which is contradictory to our assumptions.*

Proof. The proof is analogous to the case in the proof of Lemma 4.5.4 where t has even depth, but this time $|E_t| = k+2$, which forces the statement of the lemma. □

4.5.2 Bounding the Fractional Arboricity

Theorem 4.5.6. *The fractional arboricity of* G *is less than* $k + \frac{d}{d+k+1} + \epsilon$*.*

We assume that there is a subgraph H with $\frac{e(H)}{v(H)-1} \geq k + \frac{d}{d+k+1} + \epsilon$ for the rest of the subsection. Let $\mathcal{X}$ be the set of all $X \subseteq V(G)$ having $S \subsetneq X$ and

$$\frac{e(X)}{|X \setminus S|} > k + \mu,$$

where

$$\mu = \frac{d + \frac{2(z+1)}{\delta+1}}{d + k + 1 + \frac{2(z+1)}{\delta+1}}.$$

Lemma 4.5.7. $\frac{e(G)}{|V(G) \setminus S|} = k + \mu$ *and thus* $V(G) \notin \mathcal{X}$*.*

Proof. In any colourful tree C, there are $\frac{\delta-1}{2}$ red components P_t in the example colouring, where $\operatorname{depth}_{T_C}(t)$ is even and non-zero, and there are $k\frac{\delta+1}{2}$ such components for which $\operatorname{depth}_{T_C}(t)$ is odd. Moreover, every vertex in a colourful tree has exactly one outgoing i-edge for each $i \in \{1, 2, \ldots, k\}$ in the example colouring. We have

$$\begin{aligned}\frac{e(G)}{|V(G) \setminus S|} &= k + \frac{p\big(d - z(k-1) + 1 + \frac{\delta-1}{2}(d - zk) + k\frac{\delta+1}{2}z\big)}{p\big(d - z(k-1) + 2 + \frac{\delta-1}{2}(d - zk + 1) + k\frac{\delta+1}{2}(z+1)\big)} \\ &= k + \frac{\frac{\delta+1}{2}d + (z+1)}{\frac{\delta+1}{2}(d+k+1) + (z+1)} \\ &= k + \mu.\end{aligned}$$

□

Immediately from the definitions it follows that:

Observation 4.5.8.

$$\frac{d}{d+k+1} < \mu < \frac{d}{d+k+1} + \epsilon$$

Corollary 4.5.9. *If Theorem 4.5.6 is false, then $\mathcal{X}$ is not empty.*

Proof. Analogously to Corollary 4.4.10. □

Let C be a colourful tree and $t \in V(T_C)$ for the rest of the subsection and let $\ell := \left\lfloor \frac{d-1}{k+1} \right\rfloor$. Note that since $z \leq \ell$, we have $\frac{z}{z+1} \leq \frac{d}{d+k+1}$.

Lemma 4.5.10. *If $X \in \mathcal{X}$ and $t \in X$, then $X \cup V(P_t) \in \mathcal{X}$.*

Proof. Analogously to Lemma 4.4.12. □

Lemma 4.5.11. *If $X \in \mathcal{X}$ and $t \notin X$, then $X \setminus V(P_t) \in \mathcal{X}$.*

Proof. Let $P' = X \cap V(P_t) \neq \varnothing$. When subtracting P' from X the induced subgraph will lose $k|P'|$ edges of $E(P', S)$ and another $e(P')$ edges. Let c be the number of components of $G[P']$. By construction every component of $G[P']$ has at most z edges. We have

$$\frac{k|P'| + e(P')}{|P'|} \leq k + \frac{cz}{c(z+1)} < k + \mu.$$

□

Let V_i, where $i \in \{z, d - zk, d - z(k-1) + 1\}$, be the set of vertices $t \in V(T_C)$ such that $e(P_t) = i$. Note that these three sets are pairwise disjoint since $z < d - zk$.

Lemma 4.5.12. *If $X \in \mathcal{X}$, $t \in X \cap V_z$ and at most one of the two neighbours of t in T_C are in X, then $X \setminus V(P_t) \in \mathcal{X}$.*

Proof. Analogously to Lemma 4.4.14 using $z \leq \ell$. ☐

By repeated application of the last three lemmas we get the following corollary:

Corollary 4.5.13. *If $\mathcal{X} \neq \varnothing$, then there is an $X \in \mathcal{X}$ such that for every $t \in V(T_C) \cap X$ we have $V(P_t) \subseteq X$ and if additionally $t \in V_z$, then $\deg_X(t) = k + 2$.*

Like in Section 3 we will prove this corollary for all the vertices of T_C now.

Lemma 4.5.14. *Let $X \in \mathcal{X}$ such that for every $t \in V(T_C) \cap X$ we have $V(P_t) \subseteq X$ and if additionally $t \in V_z$, then $\deg_X(t) = k + 2$. Furthermore, suppose there is a vertex $r' \in V(T_C)$ such that all vertices of the tree T' containing the vertices $t \in V(T_C)$ with $\operatorname{depth}_{T_C}(t) \geq \operatorname{depth}_{T_C}(r')$ are also in X, but the parent of r' is not in X. Let $T'' = T' \cup \bigcup_{t \in V(T')} P_t$. Then $X \setminus V(T'') \in \mathcal{X}$.*

Proof. It must be that $r' \in V_{d-zk}$ because of the degree property of X. Let T' contain x vertices of V_{d-zk} and thus kx vertices of V_z. We have

$$\begin{aligned}\frac{e(T'') + e(X \setminus V(T''), V(T''))}{v(T'')} &= k + \frac{x(d - zk) + xkz}{x(d + 1 - zk) + xk(z + 1)} \\ &= k + \frac{d}{d + k + 1} \\ &< k + \mu.\end{aligned}$$

Thus, $X \setminus V(T'') \in \mathcal{X}$. ☐

Lemma 4.5.15. *If $\mathcal{X} \neq \varnothing$, then there is an $X \in \mathcal{X}$ where every $t \in V_z \cap X$ has $\deg_X(t) = k + 2$ and for every $t \in X \cap (V_{d-zk} \cup V_{d-z(k-1)+1})$, the child of t in T_C is also in X.*

Proof. We choose X, t' and T' like we did in the proof of Lemma 4.4.17. Let $x := |V_{d-zk} \cap V(T')|$ and thus $|V_z \cap V(T')| = kx + |\{t'\}|$. Since $T' \neq T_C$, we have $x < \frac{\delta - 1}{2}$ and thus

$$k + \frac{(z + |\{tt'\}|) + x(d - zk) + kxz}{z + 1 + x(d - zk + 1) + kx(z + 1)} = k + \frac{xd + (z + 1)}{x(d + k + 1) + (z + 1)} > \mu.$$

☐

Corollary 4.5.16. *$\mathcal{X}$ is empty. Thus, Theorem 4.5.6 is true.*

Applications

5

5.1 (Pseudo-)Forest Decompositions of Graphs with Bounded Degree

In this section, we discuss some (pseudo-)forest decompositions in Nine Dragon Tree fashion for graphs with bounded degree.
We begin by determining the fractional (pseudo-)arboricity of degree-bounded graphs. While it is easy to see that a graph G with maximum degree at most $\Delta \in \mathbb{N}_0$ has $\gamma'(G) \leq \frac{\Delta}{2}$, a short calculation is required to determine the maximum of $\gamma(G)$.

Lemma 5.1.1. *Let G be a simple graph with maximum degree at most Δ. Then $\gamma(G) \leq \frac{\Delta+1}{2}$.*

Proof. Let $H \subseteq G$ with $v(H) > 1$. We have that $e(H) \leq \frac{v(H)}{2} \cdot \min\{\Delta, v(H) - 1\}$ since G is simple. If $v(H) - 1 < \Delta$, then

$$\frac{e(H)}{v(H) - 1} \leq \frac{v(H) \cdot (v(H) - 1)}{2(v(H) - 1)} = \frac{v(H)}{2} < \frac{\Delta + 1}{2}.$$

Otherwise, if $\Delta \leq v(H) - 1$, then

$$\frac{e(H)}{v(H) - 1} \leq \frac{\Delta}{2} \cdot \frac{v(H)}{v(H) - 1} \leq \frac{\Delta}{2} \cdot \frac{\Delta + 1}{\Delta} = \frac{\Delta + 1}{2}.$$

□

Note that if we allow parallel edges, then Lemma 5.1.1 does not hold: the graph G on two vertices with Δ edges between them has $\gamma(G) = \Delta$.
The following observation can be used to bound the degree in one of the (pseudo-)forests in a decomposition. Although we could not find this statement anywhere in the literature, we assume that it is common knowledge.

Theorem 5.1.2. *Let G be a simple graph G with maximum degree at most $\Delta \in \mathbb{N}_0$. Then G is $(\lceil \frac{\Delta-1}{2} \rceil, \lfloor \frac{\Delta+1}{2} \rfloor)$-decomposable.*

Proof. Let $k := \lceil \frac{\Delta-1}{2} \rceil$. Suppose the theorem is not true and let G be a counterexample graph. By Lemma 5.1.1 and Nash-Williams' Theorem, there is a decomposition into forests $(T_1, \dots, T_k, F)$ of G. We choose such a decomposition that minimizes the maximum degree d^* of F and, subject to this, minimizes the number of vertices of degree d^* in F.
Since G is a counterexample, we have that $d^* > \lfloor \frac{\Delta+1}{2} \rfloor$. Let $v \in V(G)$ with $\deg_F(v) = d^*$. Then the number of edges incident with v that are not in F is at most $\Delta - d^* \leq k - 1$. Thus, there is a forest $T \in \{T_1, \dots, T_k\}$ in which v is isolated. Moving one edge from F that is incident with v to T does not create a cycle in T and decreases the number of vertices with maximum degree in F, a contradiction to our choice of $(T_1, \dots, T_k, F)$. □

Now, let us apply the (Strong) Nine Dragon Tree Conjecture to a simple graph G with maximum degree at most Δ. Note that if Δ is odd, this is not possible with $k = \lceil \frac{\Delta-1}{2} \rceil$ since the fractional arboricity may be integral (i.e. in the case where G is the complete graph on $\Delta + 1$ vertices). However, if Δ is even, we can apply the (Strong) Nine Dragon Tree Conjecture for $d = k + 1$.

Corollary 5.1.3 (from Theorem 1.8). *Let $\Delta \in \mathbb{N}_0$ be even and let G be a simple graph in which every vertex has degree at most Δ. Then G is $\left(\frac{\Delta}{2}, \frac{\Delta}{2} + 1\right)^+$-decomposable.*

Interestingly, the trivial degree bound from Theorem 5.1.2 is smaller than the bound provided by the Nine Dragon Tree Theorem.

Now, we turn to pseudoforest decompositions, where similar observations can be made.

Theorem 5.1.4. *Let G be a graph with maximum degree $\Delta \in \mathbb{N}_0$. Then G is $(\lceil \frac{\Delta}{2} \rceil - 1, \lfloor \frac{\Delta}{2} \rfloor + 1)$-pseudo-decomposable.*

Proof. Analogously to the proof of Theorem 5.1.2. □

In the pseudoforest case, we cannot apply the pseudoforest analogues of the Nine Dragon Tree Conjectures if Δ is even. For the case where Δ is odd, Corollary 1.24 from the introduction applies our diameter refinement of the Strong Pseudoforest Nine Dragon Tree (Theorem 1.22).

In contrast to the forest case, the degree bound in the last forest from the Pseudoforest Nine Dragon Tree Theorem is the same as in Theorem 5.1.4 for odd Δ.

In the rest of the section, we show that the diameter bound in Corollary 1.24 is best possible, that is, we cannot replace "star forest" by "pseudoforest of diameter at most 1". Note that a component of such a pseudoforest is either a single edge or a triangle.

The graph G we construct to prove this will be triangle-free and will not have a perfect matching. Subtracting a maximum matching from G will result in a graph that has too many edges to be decomposable into $\frac{\Delta-1}{2}$ pseudoforests.

Lemma 5.1.5. *For every odd $\Delta \in \mathbb{N}_{\geq 3}$, there exists a simple, triangle-free, Δ-regular graph G_Δ that does not have a perfect matching.*

Proof. We use a construction inspired by [1]. Note that G_Δ is depicted in Figure 5.1.1. The graph G_Δ that we construct consists of a cut-vertex c that is connected to Δ copies of a connected graph H. We define H now. $V(H)$ is partitioned into $A = \{a_0, \dots, a_{\Delta-2}\}$, $B = \{b_0, \dots, b_{2\Delta-3}\}$, $C = \{c_0, \dots, c_{\Delta-2}\}$ and $\{v\}$, where v is the vertex of H that is connected to c.

As $v(H) = (\Delta-1) + (2\Delta-2) + (\Delta-1) + 1 = 4(\Delta-1) + 1$ is odd, we have that $G_\Delta - c$ has Δ components whose number of vertices is odd. Thus, by Tutte's Theorem (Theorem 2.3.1), G_Δ does not have a perfect matching.

Let $B_1 = \{b_0, \dots, b_{\Delta-2}\}$ and $B_2 = \{b_{\Delta-1}, \dots, b_{2\Delta-3}\}$ (and thus, $B = B_1 \cup B_2$). We now describe the edge set of H, which has the partition

$$E(H) = E_H(\{v\}, A) \cup E_H(A, B) \cup E_H(B, C) \cup E(H[B_1]),$$

where v has an edge to every vertex of A, and

$$E(H[B_1]) = \{b_i b_{i+1} \mid i \in \{0, 2, 4, \dots, \Delta-3\}\}.$$

Because of this structure, if H had a triangle, its vertex set would be $\{x, b_i, b_{i+1}\}$, where $x \in A \cup C$ and $i \in \{0, 2, 4, \dots, \Delta-3\}$. We will avoid such a subgraph when defining $E_H(A, B)$ and $E_H(B, C)$. We let

$$E_H(A, B) = \{a_i b_j \mid i \in \{0, \dots, \Delta-2\}, j \in \{0, \dots, 2\Delta-3\}, i \equiv j \mod 2\}.$$

Note that every vertex of A has one edge to v and $\Delta - 1$ edges to B, and every vertex of B has $\frac{\Delta-1}{2}$ edges to A. Furthermore, every vertex of B_1 has one edge within B_1 and its remaining $\frac{\Delta-1}{2}$ incident edges go to C:

$$E_H(B_1, C) = \big\{b_i c_j \mid i, j \in \{0, \dots, \Delta - 2\}, i \equiv j \mod 2\big\}.$$

Thus, every vertex of C has $\frac{\Delta-1}{2}$ edges to B_1. Now, it only remains to add $\frac{\Delta+1}{2}$ incident edges for every vertex of $B_2 \cup C$ in order for G_Δ to be Δ-regular:

$$\begin{aligned} E(C, B_2) = \big\{c_i b_{\Delta-1+j} \mid i \in \{0, \dots, \Delta - 2\},& \\ j \in \{i \bmod (\Delta - 1), \dots, \big(i + \frac{\Delta - 1}{2}\big) \bmod (\Delta - 1)\}\big\}&. \end{aligned}$$

□

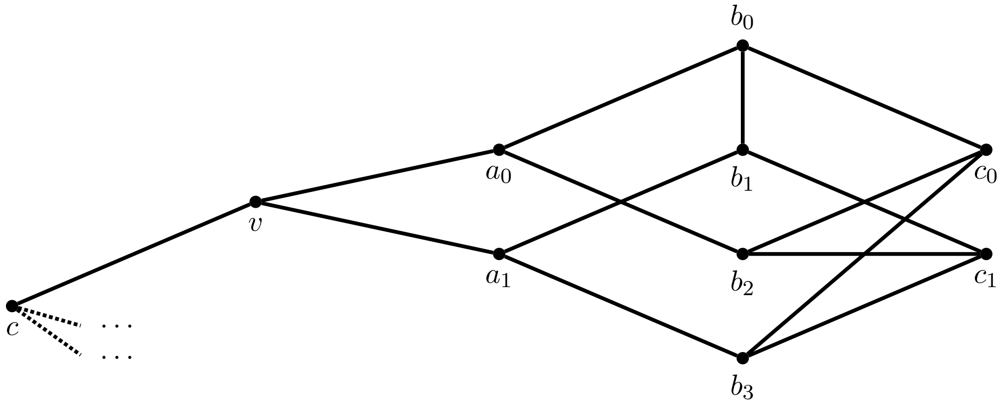

Fig. 5.1.1: G_Δ from the proof of Lemma 5.1.5 for $\Delta = 3$. Only one copy of subgraph H is depicted.

Note that for $\Delta = 3$, Lemma 5.1.5 relates to Petersen's Theorem [46], one of the oldest graph theory results, which states that every cubic, bridgeless graph contains a perfect matching. Lemma 5.1.5 shows that “bridgeless” is a necessary condition. We are now able to prove the optimality of the diameter bound of Corollary 1.24.

Theorem 5.1.6. *For every odd $\Delta \in \mathbb{N}_{\geq 3}$, there is a simple, Δ-regular graph which does not have a decomposition into $\frac{\Delta-1}{2}$ pseudoforests and another pseudoforest of diameter 1.*

Proof. Let G_Δ be a graph as described in Lemma 5.1.5 and let $k := \frac{\Delta-1}{2}$. Since $e(G_\Delta) = \frac{\Delta \cdot v(G_\Delta)}{2}$ is integral and Δ is odd, we have that $v(G_\Delta)$ is even. Since G_Δ

does not have a perfect matching, we have that a matching of G_Δ contains at most $\frac{v(G_\Delta)}{2} - 1$ edges.

Let $(P_1, \dots, P_k, F)$ be a decomposition of G_Δ, where $P_1, \dots, P_k$ are pseudoforests. Then

$$e(F) = e(G_\Delta) - \sum_{i=1}^{k} e(P_i) \geq \frac{\Delta}{2} \cdot v(G_\Delta) - k \cdot v(G_\Delta) = \frac{v(G_\Delta)}{2}.$$

Thus, F is not a matching. Since G_Δ does not contain triangles, F has diameter at least 2, which completes the proof. □

5.2 Planar Graphs With High Girth

In this section, we discuss the implications of the Nine Dragon Tree Conjectures for planar graphs with high girth.

Let $\mathcal{P}_g$ denote the set of planar graphs with girth at least $g \in \mathbb{N}_{\geq 1}$. Note that $\mathcal{P}_3$ is the set of simple planar graphs.

There is a very useful connection between the fractional arboricity and the girth of planar graphs, as shown by the following well-known corollary of Euler's Formula.

Theorem 5.2.1. *In every graph $G \in \mathcal{P}_g$ with $g \in \mathbb{N}_{\geq 3}$, every subgraph $H \subseteq G$ with $v(H) > 1$ has $\frac{e(H)}{v(H)-1} < \frac{g}{g-2}$.*

Proof. Without loss of generality, let $H \subseteq G$ be connected.

If H is acyclic, then $\frac{e(H)}{v(H)-1} \leq 1 < \frac{g}{g-2}$.

Thus, let H be cyclic and consider a planar embedding of H with $f \in \mathbb{N}_{\geq 1}$ faces. Every face borders at least g edges and every edge touches one or two faces. Thus, $g \cdot f \leq 2e(H)$, and by Euler's Formula (Theorem 2.3.2), we have that $2 = v(H) - e(H) + f \leq v(H) - e(H) + \frac{2}{g}e(H)$, which is equivalent to

$$\frac{e(H)}{v(H) - 1} + \frac{g}{g-2} \cdot \frac{1}{v(H) - 1} \leq \frac{g}{g-2}.$$

□

Corollary 5.2.2. *Every simple planar graph has a decomposition into three forests. Every graph of $\mathcal{P}_4$ has a decomposition into two forests.*

It is easy to see that the bounds in Corollary 5.2.2 are best possible: a 4-cycle is not a forest, and by removing an edge from the complete graph on 5 vertices, we obtain a simple planar graph G with $\frac{e(G)}{v(G)-1} = \frac{9}{4} > 2$.

Next, we discuss some results in Nine Dragon Tree fashion for planar graphs with different girths. Gonçalves [22] showed that every simple planar graph is $(2,4)$-decomposable. In [5], it is shown that there are simple planar graphs that are not $(2,3)^-$-decomposable. Surprisingly, Knauer and Ueckerdt [33] showed that 4-edge-connected simple planar graphs are $(2,2)$-decomposable.
However, there is no $d \in \mathbb{N}_0$ such that every simple planar graph is $(2,d)^+$-decomposable, as Merker and Postle noted in [34]:
Since a triangulation G satisfies $e(G) = 3v(G)-6$, every forest in a decomposition of G into three forests has at most four components. As triangulations of arbitrarily large diameter exist, these few components cannot cover all the vertices if the diameter of each component is bounded by a constant.
Analogously, quadrangulations (having $e(G) = 2v(G) - 4$) of arbitrarily large diameter exist, and thus, there is no $d \in \mathbb{N}_0$ such that every graph of $\mathcal{P}_4$ is $(1,d)^+$-decomposable.
Interestingly, no result concerning the $(1,d)$- or $(1,d)^-$-decomposability of $\mathcal{P}_4$ is known to the author, see Question 6.11. However, higher girth values appear to be understood much better, and the optimal bounds follow from the (Strong) Nine Dragon Tree Theorem:
By the resolution of the (Strong) Nine Dragon Tree Conjecture for $(k,d) = (1,1)$ in [41], every graph of $\mathcal{P}_8$ is $(1,1)^+$-decomposable. This was independently proven by Wang and Zhang [52] without using fractional arboricity arguments. In [41], it was also shown that there are planar graphs with girth 7 that are not $(1,1)^-$-decomposable.
By the resolution of the Strong Nine Dragon Tree Conjecture for $(k,d) = (1,2)$ in [29], every graph of $\mathcal{P}_6$ is $(1,2)^+$-decomposable. In [41], it was shown that there are planar graphs with girth 5 that are not $(1,2)$-decomposable. Their argument also shows that the graphs they constructed are not $(1,2)^-$-decomposable either. Thus, there are planar graphs with girth 5 that are not $(1,2)^-$-decomposable. This was apparently also shown by Kleitman [32] (unpublished).

It was shown by He, Hou, Lih, Shao, Wang, and Zhu [26] that every graph of $\mathcal{P}_5$ is $(1,4)^-$-decomposable and by the resolution of the Nine Dragon Tree Theorem for $(k,d) = (1,4)$, it follows that they are also $(1,4)$-decomposable. The following result emerges from our resolution of the Strong Nine Dragon Tree Conjecture for $d \leq 2(k+1)$.

Theorem 5.2.3. *Every planar graph with girth at least* 5 *is* $(1,4)^+$*-decomposable.*

Proof. By Theorem 5.2.1, we have that every graph of $\mathcal{P}_5$ has fractional arboricity at most $\frac{5}{3}$. The theorem follows by applying Theorem 1.8 for $(k,d)=(1,4)$. □

It remains open whether every graph of $\mathcal{P}_5$ is $(1,3)^-$-, $(1,3)$- or $(1,3)^+$-decomposable, see Question 6.12.

5.3 The Game Chromatic Number and the Game Colouring Number

In this section, we briefly consider the application of the Weak Nine Dragon Tree Theorem that originally motivated the conjectures and in particular, has highlighted the importance of finding $(1,d)^-$-decompositions with small d for planar graphs with large girth. This application concerns the Game Chromatic Number and the Game Colouring Number.

We start by introducing the *graph colouring game*. Two players, Alice and Bob, with Alice making the first move, alternately colour an uncoloured vertex of a graph G using a colour from a set of colours X such that no two adjacent vertices of G receive the same colour. The game ends when either every vertex has been coloured — in which case Alice wins — or there is an uncoloured vertex that cannot be coloured without violating the colouring rule, in which case Bob wins. The game colouring number $\chi_g(G)$ of a graph G is the smallest $|X|$ such that Alice has a winning strategy.

To establish upper bounds for $\chi_g(G)$, the game colouring number was first formally introduced by Zhu [54] in 1999 using the following game that was later referred to as the *marking game*.

Two players, Alice and Bob with Alice making the first move, alternately mark an unmarked vertex of a graph G. The game ends when every vertex is marked. For a marking order L of all vertices of G and a vertex $v \in V(G)$, let $b_L(v)$ denote the number of neighbours of v that have been marked before v. Furthermore, let $s(L) := 1 + \max_{v \in V(G)} b_L(v)$ be the *score* of the game. Alice's goal is to minimize $s(L)$

while Bob aims to maximize $s(L)$.

The *game colouring number* $\mathrm{col}_g(G)$ *of* G is the smallest s^* such that Alice has a strategy that guarantees the score of the game to be at most s^*. The next trivial theorem shows the usefulness of the game colouring number.

Theorem 5.3.1 ([54]). $\chi_g(G) \leq \mathrm{col}_g(G)$.

Proof. Let S be a strategy of the marking game guaranteeing a score of at most $\mathrm{col}_g(G)$.

When Alice plays the graph colouring game with $\mathrm{col}_g(G)$ colours, she can choose the next vertex to colour according to S. When doing this, at most $\mathrm{col}_g(G) - 1$ neighbours of the chosen vertex have already been coloured (or otherwise S would cause a score of at least $\mathrm{col}_g(G) + 1$ in the marking game), and thus, the vertex can be validly coloured. □

The next observation shows how graph decompositions can be useful for the marking game.

Theorem 5.3.2 ([54], [24]). *Let* (F, T) *be a decomposition of* G *such that* T *has maximum degree at most* $d \in \mathbb{N}_0$. *Then* $\mathrm{col}_g(G) \leq \mathrm{col}_g(F) + d$.

Proof. Follows directly when Alice applies an optimal strategy for F on G. □

Faigle, Kern, Kiersted and Trotter [13] showed that every forest F has $\chi_g(F), \mathrm{col}_g(F) \leq 4$, and Bodlaender [8] showed that there are trees T with $\chi_g(T) \geq 4$. This implies the following corollary.

Corollary 5.3.3. *If a graph* G *is* $(1, d)^-$*-decomposable, then* $\chi_g(G), \mathrm{col}_g(G) \leq 4 + d$.

For a set of graphs $\mathcal{G}$ let $\chi_g(\mathcal{G}) := \max_{G \in \mathcal{G}} \chi_g(G)$ and $\mathrm{col}_g(\mathcal{G}) := \max_{G \in \mathcal{G}} \mathrm{col}_g(G)$.

If we combine the corollary with the decompositions of planar graphs with higher girth mentioned in Section 5.2, we obtain the following results.

Corollary 5.3.4 ([26]). $\chi_g(\mathcal{P}_5), \mathrm{col}_g(\mathcal{P}_5) \leq 8$.

Corollary 5.3.5 ([41]). $\chi_g(\mathcal{P}_6), \mathrm{col}_g(\mathcal{P}_6) \leq 6$.

Corollary 5.3.6 ([41], [52]). $\chi_g(\mathcal{P}_8), \mathrm{col}_g(\mathcal{P}_8) \leq 5$.

Note that Sekiguchi [48] showed that $\mathrm{col}_g(\mathcal{P}_5) \geq 6$, and furthermore $7 \leq \mathrm{col}_g(\mathcal{P}_4) \leq 13$.
More recently, Nakprasit and Nakprasit [44] showed that $\mathrm{col}_g(\mathcal{P}_6) \geq 5$ and $\mathrm{col}_g(\mathcal{P}_7) \leq 5$. Thus $\mathrm{col}_g(\mathcal{P}_7), \mathrm{col}_g(\mathcal{P}_8) = 5$.

5.4 ϵ-Thin Spanning Trees in Highly Edge-Connected Planar Graphs

This section addresses a conjecture of Goddyn on ϵ-thin spanning trees in highly edge-connected graphs. Since, in the planar case, the dual graphs have high girth, we can apply the implications of the Strong Nine Dragon Tree Conjecture discussed in the previous section.
A spanning tree T is ϵ-*thin* with $0 \leq \epsilon < 1$, if for every edge-cut B of G, at most $\epsilon|B|$ edges in B are contained in T.

Conjecture 5.4.1 (Goddyn 2004, [21]). *There exists a function f such that, for $\epsilon > 0$, every $f(\epsilon)$-edge-connected graph has an ϵ-thin spanning tree.*

This conjecture is still open. However, for some graph classes, it has already been solved, see for example [20], [3]. For a recent and detailed overview of the progress towards this conjecture and its various applications, we refer to [31].
The application of the Strong Nine Dragon Tree Conjecture that we consider now deals with the question how small the edge-connectivity $f(\epsilon)$ can be in Conjecture 5.4.1 such that there still is a corresponding ϵ. Thomassen showed that it has to be at least 5:

Theorem 5.4.2 (Thomassen, cf. [34]). *There is no ϵ such that every 4-edge-connected planar graph contains an ϵ-thin spanning tree.*

Merker and Postle [34] showed that every 6-edge-connected planar graph has two edge-disjoint $\frac{18}{19}$-thin spanning trees. $\frac{18}{19}$ was then reduced to $\frac{14}{15}$ by Mousavi Haji [43].
Now, the question is whether there is an ϵ such that every 5-edge-connected planar graph has an ϵ-thin spanning tree. Merker and Postle's proof implies this if the Strong Nine Dragon Tree Conjecture is true for $(k, d) = (1, 4)$, which is verified by Theorem 1.8.

Theorem 5.4.3. *Every 5-edge-connected planar graph has an $\frac{4}{5}$-thin spanning trees.*

As the proof is quite short and elegant, we will reproduce it and prove Theorem 5.4.3 at the end of the section. The proof uses cut-cycle-duality, which is well-known: every minimal edge-cut of a planar graph G corresponds to a cycle in its dual graph G^* using the canonical bijection $\varphi : E(G) \to E(G^*)$. For a rigorous definition and proof of this concept, we refer to [10]. This implies that every edge-cut of G corresponds to an edge-disjoint union of cycles. The following theorem follows immediately.

Theorem 5.4.4. *Let $g \in \mathbb{N}_{\geq 3}$ and let G be a g-edge-connected planar graph. Then $G^* \in \mathcal{P}_g$.*

Thus, if G is a 6-edge-connected planar graph, then $G^* \in \mathcal{P}_6$, and hence G^* is $(1,2)^+$-decomposable (see Section 5.2). Such a decomposition can be transformed into two forests with bounded diameter, as the following theorem shows.

Theorem 5.4.5 ([43]). *A graph that has a decomposition into a forest and a star forest, also has a decomposition into two forests, where the diameter in both forests is at most 14.*

The final step of Merker, Postle and Mousavi Haji to show the existence of two edge-disjoint thin spanning trees is the following theorem.

Theorem 5.4.6 ([34]). *If a planar graph G^* has a decomposition into two forests, where the diameter of both of the forests is bounded by $d \in \mathbb{N}_{\geq 1}$, then G contains two edge-disjoint $\frac{d}{d+1}$-thin spanning trees.*

Now, let us turn to 5-edge-connected planar graphs G. As the dual graph G^* is in $\mathcal{P}_5$, we have by Theorem 5.2.3 that G^* has a decomposition into a forest and another forest that has diameter at most 4. Unfortunately, we do not know an analogous result to Theorem 5.4.5 for such decompositions. But still, there is an obvious simplification of the proof of Theorem 5.4.6 that yields at least one thin spanning tree for us.

Theorem 5.4.7 ([34]). *If a planar graph G^* has a decomposition into two forests, where the diameter of one of the forests is bounded by $d \in \mathbb{N}_{\geq 1}$, then G contains a $\frac{d}{d+1}$-thin spanning tree.*

Proof. Let (T^*, F^*) be a decomposition of G^* into two forests such that $\operatorname{diam}(F^*) \leq d$. We obtain $F \subseteq G$ from F^* using the canonical bijection φ^{-1}.
Now, consider a non-empty edge-cut $S \subseteq E(G)$. Note that $S^* := \varphi(S)$ decomposes into edge-disjoint cycles by cut-cycle-duality of planar graphs. As T^* is acyclic, S^* contains at least one edge of F^*, and thus S contains at least one edge of F. Thus, F is connected.
Furthermore, each cycle C^* of S^* has at most $\frac{d}{d+1}e(C^*)$ edges belonging to F^* since every path of F^* has at most d edges. Hence, at most $\frac{d}{d+1}|S|$ edges belong to F. Finally, we can obtain a $\frac{d}{d+1}$-thin spanning tree from F by removing edges on cycles until it is acyclic. □

We note that in the proof of Theorem 5.4.7, by making the same observations for T^* as for F^*, Theorem 5.4.6 follows immediately. We end the section by summarizing the proof of Theorem 5.4.3.

Proof of Theorem 5.4.3. Let G be a 5-edge-connected planar graph. By Theorem 5.4.4, we have $G^* \in \mathcal{P}_5$. By Theorem 5.2.3, G^* is $(1,4)^+$-decomposable. The theorem follows by Theorem 5.4.7. □

Note that, analogously, the results of [34] imply that every 6-edge-connected planar graph has a $\frac{2}{3}$-thin spanning tree.

Open Questions

6

This final chapter presents some open questions that arise from the previous chapters, as well as some other questions and conjectures from the literature related to the Nine Dragon Tree Conjectures.

We begin with the most obvious one: solving the Strong Nine Dragon Tree Conjecture (Conjecture 1.6, Conjecture 1.12). In Section 3.6, we established an approximate version of the conjecture and in Section 3.7, we showed how the approximate version can be made optimal for $d \leq 2(k+1)$ through the use of augmentations with interesting neighbours. This approach might be generalizable. A reasonable generalization of the notion of an interesting neighbour C generated by (x, x') could be that the component of x'' is either a small child (potentially with more than 0 edges) or it can also be an interesting neighbour (of C). If one manages to apply the augmentations of the approximate version to interesting neighbours instead of just small children, this could suffice to prove the Strong Nine Dragon Tree Conjecture. However, this approach has a lot of pitfalls since many of the blue directed paths that are redirected by the augmentations can be part of other important paths, e.g. a generating arc of a small child or an interesting neighbour. In the proofs of this thesis, we were able to handle this problem when $d \leq 2(k+1)$ (and with the restricted definition of interesting neighbours) and for the approximation result, but with the presence of “stacked” interesting neighbours, this seems to become significantly more complicated.

In light of Theorem 1.13, our strengthened version of the Weak Nine Dragon Tree Conjecture with the weaker density condition, it would be interesting to know how much the number of edges contained in a cycle within a single red component can be reduced. If the Strong Nine Dragon Tree Conjecture is solved, one could attempt to determine this number under the additional requirement that every red component must have at most d edges.

Another open topic is the complexity of computing decompositions in Nine Dragon Tree fashion. The approach of all recent Nine Dragon Tree results involving legal orders seems to be non-polynomial, as there is an exponential number of legal orders. It could be interesting to analyze the runtime of the first Nine Dragon Tree results in [41], [29] and [9], which used other methods. Furthermore, note that the author showed in [35] that the underlying algorithm of Gonçalves' proof in [22], which shows that simple planar graphs are $(2,4)$-decomposable, runs in time $\mathcal{O}(v(G)\log v(G))$.

In Theorem 1.26 and Theorem 1.27, we proved the optimality of the diameter constraints in Theorem 1.22. However, we could not prove the optimality if the graph is allowed to be (k,d)-(pseudo-)sparse (or, for Theorem 1.26, if the graph satisfies an analogue sparseness condition depending on k and ℓ). This is due to the following reason: in the graph constructions of the proofs, a colourful tree coloured with the example colouring has a red root component that, together with its small red child components, has high density. The other red components of even depth in the colourful tree, together with their small children, have exactly the density of $\frac{\ell(k+1)+\alpha}{(\ell+1)(k+1)+\alpha}$, or $\frac{d}{d+k+1}$, respectively. Since the colourful trees have large depth, the "allowed" fractional (pseudo-)arboricity density is exceeded just by ϵ. Having many copies of the colourful tree does not affect the (pseudo-)arboricity density since the ratio of root to non-root components remains the same. But this ratio argument does not hold for the density measures β' and β: adding a red root component (together with k outgoing blue arcs for every vertex) decreases β and β' while adding the rest of the colourful trees, which have exactly the allowed density, does not change the value of β or β'. Since we need to have many colourful trees in order to guarantee the existence of a colourful tree that does not have a red edge to S, $\beta(G)$ and $\beta'(G)$ are much smaller than -1.

Another obvious question is the following:

Question 6.1. *For any $k, d \in \mathbb{N}_0$, what is the smallest number $f(k,d) \in [0,1)$ such that every (loopless) graph G with $\gamma(G) \leq k + f(k,d)$ [or $\gamma'(G) \leq k + f(k,d)$] has a decomposition into k [pseudo-]forests and another graph (or [pseudo-]forest) of diameter at most $d \in \mathbb{N}_0$?*

Note that Theorem 1.9 and Theorem 1.22 provide probably non-optimal functions for both the forest and the pseudoforest case. Furthermore, Theorem 1.28 provides an upper bound to $f(k,d)$ in the forest case which is likely not optimal. It would also be nice for this result to be achieved using simple graphs. In the pseudoforest case, no non-trivial upper bound of $f(k,d)$ is known to the author. In particular, the graphs constructed in the proof of Theorem 1.28 do not yield such a bound for pseudoforests: it is easy to find a decomposition of the graph into k forests and a pseudoforest in which every component is a 3-cycle. Thus, this pseudoforest has diameter 1.

We also want to look at other interesting diameter-related problems mentioned in [34]. The following conjecture is a generalization of Theorem 5.4.5.

Conjecture 6.2 ([34]). *For any $d \in \mathbb{N}_0$, there is a number $f(d)$ such that every graph that has a decomposition into a forest and a forest of diameter at most d also has a decomposition into two forests, each of which has diameter at most $f(d)$.*

Merker and Postle also stated the following conjecture, which follows directly from Conjecture 6.2 and the Strong Nine Dragon Tree Conjecture (but also from Theorem 1.9).

Conjecture 6.3 ([34]). *For any $k, d \in \mathbb{N}_0$, there is a number $f(k, d)$ such that every loopless graph G with $\gamma(G) \leq k + f(k, d)$ has a decomposition into $k + 1$ forests, each of which has diameter at most d.*

Of course, the same conjecture can be made for pseudoforest decompositions.

Another conjecture in Nine Dragon Tree fashion concerns digraphs. A *branching* G is a 1-orientation that is acyclic in the undirected sense. We let $\Delta^+(G) := \max_{v \in V(G)} \deg^+_G(v)$ and $\Delta^-(G) := \max_{v \in V(G)} \deg^-_G(v)$. Frank proved the following digraph analogue to Nash-Williams' Theorem.

Theorem 6.4 ([16]). *A loopless digraph has a decomposition into k branchings if and only if $\gamma(G) \leq k$ and $\Delta^+(G) \leq k$.*

Gao and Yang conjectured the following digraph analogue of the Nine Dragon Tree Theorem.

Conjecture 6.5 ([19]). *Let $k, d \in \mathbb{N}_{\geq 1}$ and let G be a loopless digraph with $\gamma(G) \leq k + \frac{d-k}{d+1}$ and $\Delta^+(G) \leq k + 1$. Then G has a decomposition into $k + 1$ branchings, where one branching B has $\Delta^-(G) \leq d$.*

Gao and Yang proved the conjecture for $d \leq k$. Furthermore, they proved the pseudo-branching analogue of the conjecture (where a *pseudo-branching* is simply a 1-orientation).

Theorem 6.6 ([19]). *Let $k, d \in \mathbb{N}_{\geq 1}$ and let G be a loopless digraph with $\gamma'(G) \leq k + \frac{d-k}{d+1}$ and $\Delta^+(G) \leq k+1$. Then G has a decomposition into $k+1$ pseudo-branchings, where one pseudo-branching B has $\Delta^-(G) \leq d$.*

Gao and Yang also showed that the fractional (pseudo-)arboricity bounds in Conjecture 6.5 and Theorem 6.6 cannot be decreased.

Let us review another open Nine Dragon Tree problem. Note that arboricity can viewed as a colouring problem: it is the minimum number of colours with which the edges of a graph can be coloured such that there is no monochromatic cycle. When assigning colour lists of size k to every edge, then this number does not changed, as showed by Seymour.

Theorem 6.7 ([49]). *Let G be a loopless graph with $\gamma(G) \leq k$, $k \in \mathbb{N}_0$, and for every $e \in E(G)$, let $P(e) \subseteq \mathbb{N}_0$ with $|P(e)| \geq k$. Then every edge $e \in E(G)$ can be assigned a colour from $P(e)$ such that the resulting edge colouring contains no monochromatic cycle.*

Ronen Wdowinski and Penny Haxell asked if there also is a list version of the Nine Dragon Theorem(s) (cf. [42]).

Conjecture 6.8. *Let $d, k \in \mathbb{N}_0$, let G be a loopless graph with $\gamma(G) \leq k + \frac{d}{d+k+1}$, and for every $e \in E(G)$, let $P(e) \subseteq \mathbb{N}_0$ with $|P(e)| \geq k+1$. Then every edge $e \in E(G)$ can be assigned a colour from $P(e)$ such that the resulting edge colouring contains no monochromatic cycle and the forest induced by one of the colours has maximum degree at most d.*

The same conjecture can of course be stated for pseudoforests, and it is very believable that even the strong version holds.

Conjecture 6.9. *Let $d, k \in \mathbb{N}_0$, let G be a loopless graph with $\gamma'(G) \leq k + \frac{d}{d+k+1}$, and for every $e \in E(G)$, let $P(e) \subseteq \mathbb{N}_0$ with $|P(e)| \geq k+1$. Then every edge $e \in E(G)$ can be assigned a colour from $P(e)$ such that the subgraph induced by each colour is a pseudoforest, and in the pseudoforest induced by one of the colours, every component has at most d edges.*

Next, let us move on to the open questions that arose in Chapter 5. We begin with the class of degree-bounded graphs. In light of Theorem 5.1.2, Corollary 5.1.3, Theorem 5.1.4, and Corollary 1.24, the following question arises.

Question 6.10. *Let* $\Delta \in \mathbb{N}_0$, $k := \lceil \frac{\Delta-1}{2} \rceil$, $k' := \lceil \frac{\Delta}{2} \rceil - 1$. *What is the smallest number* d *such that every (simple) graph with maximum degree at most* Δ *is* $(k,d)^-$*-,* (k,d)*-,* $(k,d)^+$*-decomposable [*$(k',d)^-$*-,* (k',d)*-,* $(k',d)^+$*-pseudo-decomposable]?*

Next, we consider some questions concerning planar graphs that arose in Section 5.2. Recall that $\mathcal{P}_g$ denotes the set of planar graphs with girth at least $g \in \mathbb{N}_{\geq 1}$.

Question 6.11. *Is there a* $d \in \mathbb{N}_{\geq 1}$ *such that every graph in* $\mathcal{P}_4$ *is* $(1,d)^-$*- or* $(1,d)$*-decomposable?*

Question 6.12. *Is every graph in* $\mathcal{P}_5$ $(1,3)^-$*-,* $(1,3)$*-, or* $(1,3)^+$*-decomposable?*

Since the graphs in $\mathcal{P}_8$ are $(1,1)^+$-decomposable, but the graphs in $\mathcal{P}_7$ are not, Moore asked whether the same holds for $(1,1)^+$-pseudo-decomposability.

Question 6.13 ([42]). *Is every graph of* $\mathcal{P}_7$ $(1,1)^+$*-pseudo-decomposable?*

Question 6.14 ([42]). *Is every graph of* $\mathcal{P}_5$ $(1,2)^-$*-,* $(1,2)$*-, or* $(1,2)^+$*-pseudo-decomposable?*

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Colophon

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