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A C_2-equivariant Gabber presentation lemma

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ABSTRACT

We establish a version of Gabber's presentation lemma in the setting of varieties with an action by the finite group of order 2.

1. Introduction

In [4, §3], O. Gabber established a certain technical "presentation lemma" which he used to prove exactness of so-called Gersten complexes. This innocuous looking statement has since become a cornerstone of motivic homotopy theory over fields [11]. Much energy has been expended into generalizing it—it is now known also over finite fields [6] and, with an additional "shift", also over much more general bases [13,3].

In this short note we establish a variant of Gabber's lemma in the setting of varieties with an action by the group C_2 of order two. From this we deduce a modified form of Gersten injectivity.[1]

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[1] It is well-known that Gersten injectivity in its usual form does not hold equivariantly, see e.g. [8, Remark 6.3].

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Using these results, we extend some of the standard properties of motivic homotopy theory over a field to the C_2-equivariant setting (stable connectivity, homotopy t-structure). We only give some of the most immediate applications, leaving many extensions to future work. This is because I view the results as somewhat "experimental", "a first foray" into equivariant Gabber lemmas. Similar results should be provable for many other groups, and the good consequences for motivic homotopy theory should then follow for all these groups.

I leave it to someone with more geometric clout than myself to tackle these issues.

Notation. We denote by $\mathbb{A}^\sigma$ the affine line with C_2-action given by the sign representation, and also by $\sigma \in C_2$ the generator. Given a scheme X and a point $x \in X$, we write $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$ for the associated local scheme, and $X_x^h = \mathrm{Spec}(\mathcal{O}_{X,x}^h)$ for the henselization. Given a scheme X and subscheme $Z \subset X$, by an étale neighborhood of Z (in X) we mean an étale morphism $X' \to X$ such that $X' \times_X Z \to Z$ is an isomorphism.[2]

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2. Presentation lemma

Our main result is the following. It is a generalization in both statement and proof of [4, Lemma 3.1, Remark 3.7].

Theorem 2.1. *Let k be an infinite field of characteristic $\neq 2$. Let X be a smooth k-scheme with an action by C_2. Let $Z \subset X$ be a closed invariant subscheme of positive codimension and $z \in Z$ be a point (not necessarily closed). Then there exist:*

- *A C_2-equivariant étale neighborhood $X' \to X$ of $C_2 z$.*
- *A smooth C_2-scheme W.*
- *A C_2-equivariant étale neighborhood $X' \to \mathbb{A}^1_W$ of $Z_{X'}$. Here $\mathbb{A}^1$ carries either the trivial or the sign action.*

Moreover the following hold:

(1) *The composite $Z_{X'} \to \mathbb{A}^1_W \to W$ is finite.*
(2) *If the action on $\mathbb{A}^1_W$ is the sign action, then $X' \to W$ admits a section.*

[2] We apply this definition in particular in the case where $Z = \{x\}$ is a point. Thus for us an étale neighborhood of x in X is $X' \to X$ such that the fiber of X' above x contains a unique point x', and $k(x') = k(x)$.

Remark 2.2. The existence of the lift of W in the case of a non-trivial action on $\mathbb{A}^1$ is important in the applications. Luckily this also naturally arises in the proof.

Remark 2.3. To illustrate the challenges in proving Theorem 2.1, I propose to discuss some special cases.

(1) Suppose $Z = X^{C_2}$ is smooth of dimension n, codimension 1 in X. Since $Z_{X'} \to W$ factors through W^{C_2} and is finite, we find for dimension reasons that $W = W^{C_2}$ has trivial action. Since the action on X is non-trivial, we see that $\mathbb{A}^1$ must carry the sign action in this case. Now $Z_{X'} \to \mathbb{A}^1_W$ factors through $(\mathbb{A}^1_W)^{C_2} = W$, and so we conclude that $Z_{X'} \to W$ is a clopen immersion.
Gabber's proof of his lemma starts by constructing a map $X \to \mathbb{A}^n$ with finite restriction to Z, and then uses the remaining coordinate to convert the finite map into a closed immersion. This cannot work in our case (unless Z is isomorphic to an open subset of $\mathbb{A}^n$). Our method is to instead pull back the map to $\mathbb{A}^n$ along $W \to \mathbb{A}^n$; this will yield an étale neighborhood of z since $z \in W$.
(2) Now suppose in contrast that $z \notin X^{C_2}$. If we attempt to mimic the same strategy, we must find a smooth closed subscheme $W \subset X$ of codimension 1 containing z. This is not possible in general unless k is perfect (because z may be a point of codimension 1 with closure which is not generically smooth). We must thus use a somewhat different argument, and construct a map to $\mathbb{A}^1_W$ where $\mathbb{A}^1$ has the trivial action.

We now proceed with the proof.

Lemma 2.4. *Let X be an affine finite type k-scheme, k an infinite field. Let $S \subset X$ be a finite set of closed points. There exists a function $f : X \to \mathbb{A}^1$ inducing a universal injection $S \hookrightarrow \mathbb{A}^1$.*

More generally, suppose given $T \subset X$ another finite set of closed points, disjoint from S. Let $X \hookrightarrow \mathbb{A}^N$ be an embedding. Suppose that after geometric base change, T consists of n points. Among the set of polynomial functions on X of degree $\le n+1$ (with respect to the embedding $X \hookrightarrow \mathbb{A}^N$) vanishing on T, those which induce a universal injection $S \hookrightarrow \mathbb{A}^1 \setminus 0$ are dense.

Proof. The first claim follows from the second one by choosing an embedding $X \hookrightarrow \mathbb{A}^N$ and setting $T = \emptyset$ (and using that open dense subsets of affine spaces have points over k).

The desired condition (injectivity after geometric base change) is open. To prove it can be satisfied, we may thus assume that k is algebraically closed. We must find a polynomial P of degree $\le n+1$ vanishing on T satisfying that: for every $s \in S$ we have $P(s) \neq 0$, and for $s' \neq s \in S$ we have $P(s) \neq P(s')$. Each of these conditions is open, so we may satisfy them separately. By temporarily adding s' to T, it will suffice to show: there exists a polynomial P of degree $\leq |T|$ vanishing at T but not at s. Taking a product over terms for each $t \in T$, it suffices to find a linear polynomial vanishing at some specific $t \in T$, but not at s. Since $t \neq s$ this is clearly possible. □

Lemma 2.5. *Let X be an affine finite type k-scheme with a C_2-action, k an infinite field of characteristic $\neq 2$. Let $S \subset X$ be a finite set of closed points.*

(1) *For $x \in X^{C_2} \setminus S$ there exists an equivariant map $f : X \to \mathbb{A}^1$ (where $\mathbb{A}^1$ has the trivial action) with $f(x) \notin f(S)$.*
(2) *If $S \subset X \setminus X^{C_2}$ then there exists an equivariant map $f : X \to \mathbb{A}^\sigma$ (where $\mathbb{A}^\sigma$ denotes $\mathbb{A}^1$ with the sign action) with $f(s) \neq 0$ for $s \in S$.*
(3) *Suppose $S \subset X \setminus X^{C_2}$ and $T \subset X$ is another finite set of closed points, disjoint from $S \cup \sigma S$. There exists $f : X \to \mathbb{A}^\sigma$ vanishing on T and non-vanishing on S.*

Proof. (1) Apply Lemma 2.4 to X/C_2 and $S/C_2 \cup \{x\}$.

(2) Apply Lemma 2.4 to X and $C_2 S$ to obtain a (non-equivariant) function $f_0 : X \to \mathbb{A}^1$. We claim that $f = f_0 - f_0\sigma$ has the desired property, where $\sigma \in C_2$ denotes the generator. By construction f is an equivariant map to $\mathbb{A}^\sigma$. To check that $f(s) \neq 0$ for $s \in S$ we may base change to the algebraic closure. Now by assumption $s \neq \sigma s$, and hence $f(s) = f_0(s) - f_0(\sigma s)$ is non-vanishing by construction of f_0.

(3) Choose a (non-equivariant) embedding $X \hookrightarrow \mathbb{A}^N$ and consider the space of (non-equivariant) polynomials P of degree $\leq n$ such that P vanishes on T. Here n is the sum of the geometric numbers of points in S and T. The subspace of polynomials such that $P - \sigma P$ does not vanish on S is open; it will suffice to show it is non-empty. This we may do after geometric base change. Now $S \cup \sigma S = S' \amalg \sigma S'$. Set $T' = T \cup \sigma S'$. Lemma 2.4 yields P vanishing on T' and non-vanishing on S'. This has the desired property. □

Proof of Theorem 2.1. First reductions. If the action of C_2 on X is trivial (in some étale neighborhood of z), then we may appeal to the original Gabber lemma. Hence we may assume that the action of C_2 is non-trivial. Moreover, since the problem is Nisnevich local around z, we may assume that X is affine [7, Lemma 2.20].

We shall treat two cases, each in several steps.

Case $z \in X^{C_2}$. Pick a closed specialization $\bar{z} \in X^{C_2}$ of z. We can decompose the tangent space $T_{\bar{z}}X$ as a sum of trivial and sign representations, and then write $T_z X \simeq k(\bar{z})^n \oplus k(\bar{z})^{(m+1)\sigma}$. If the action of C_2 is trivial on $T_{\bar{z}}X$, then it is trivial on an open neighborhood[3] of $\bar{z}$, which we have excluded. Thus $m \geq 0$.

Step 1: We claim that there exist C_2-equivariant maps $f_1, \ldots, f_n : X \to \mathbb{A}^1$ and $g_1, \ldots, g_m : X \to \mathbb{A}^\sigma$ such that if $(f, g) : X \to \mathbb{A}^{n+m\sigma}$ denotes the induced map, then $(f,g)^{-1}((f,g)(\bar{z})) \cap Z$ has dimension 0. We first find $f_1, \ldots, f_n$ such that $\dim f^{-1}(f(\bar{z})) \cap$

[3] Let $X = Spec(A)$ and $\bar{z}$ correspond to m. Write $A_\pm$ for the eigenspaces of the action. Then $A_+ \to A$ is finite. If the action on m/m^2 is trivial, then $(m_m)_-/((m_m)_+(m_m)_-) = 0$, and hence $(m_m)_- = 0$ by Nakayama's lemma applied to the finite $(A_m)_+$-module $(m_m)_-$. It follows that $A_m = (A_m)_+$.

$Z \le \dim Z - n$ and $\dim f^{-1}(f(\bar{z})) \cap Z^{C_2} = 0$, where $f : X \to \mathbb{A}^n$ is the induced map. Note that $\dim Z^{C_2} \le \dim X^{C_2} = n$. Working inductively,[4] it will be enough to find f_1 such that $\dim f_1^{-1}(f_1(\bar{z})) \cap Z < \dim Z$ and $\dim f_1^{-1}(f_1(\bar{z})) \cap Z^{C_2} < \dim Z^{C_2}$. Choosing one point in each component of positive dimension of Z and Z^{C_2} we can use Lemma 2.5(1) to find f_1 such that $f_1^{-1}(f_1(\bar{z}))$ does not completely contain any component of positive dimension of Z or Z^{C_2}, which implies what we need. Next note that $\dim Z \le \dim X - 1 = n + m$, so $\dim f^{-1}(f(\bar{z})) \cap Z \le m$. Each component of $f^{-1}(f(\bar{z})) \cap Z$ of positive dimension contains a point outside Z^{C_2} (since $\dim Z(f_1, \ldots, f_n) \cap Z^{C_2} = 0$). Hence we can use Lemma 2.5(2) to find g_1 with $\dim(f, g_1)^{-1}((f, g_1)(\bar{z})) \cap Z \le m - 1$. Iterating this procedure we obtain what we need.

Step 2: Let $p_1, \ldots, p_N \in \mathcal{O}_X(X)$ generate $\mathcal{O}_X(X)$. By symmetrizing and anti-symmetrizing, we can assume that $\sigma p_i = \pm p_i$, where $\sigma \in C_2$ denotes the generator (recall that k has characteristic not 2). We further add the functions from step (1) to this set. This way we obtain an equivariant closed immersion $X \hookrightarrow \mathbb{A}^{N+M\sigma}$.

Step 3: We claim that a general equivariant linear projection $\varphi : \mathbb{A}^{N+M\sigma} \to \mathbb{A}^{n+m\sigma} \times \mathbb{A}^{\sigma}$ has the following properties:

(a) The composite $Z \to \mathbb{A}^{N+M\sigma} \to \mathbb{A}^{n+m\sigma} \times \mathbb{A}^{\sigma} \to \mathbb{A}^{n+m\sigma}$ is quasi-finite at $\bar{z}$.
(b) The composite $X \to \mathbb{A}^{N+M\sigma} \to \mathbb{A}^{n+m\sigma} \times \mathbb{A}^{\sigma}$ is étale at $\bar{z}$.

Indeed both conditions are open,[5] so we need only check that the sets are non-empty. For (a), this holds because among the coordinates are the functions $f_1, \ldots, f_n, g_1, \ldots, g_m$ of step (1). For (b), we may first base change to an algebraic closure of k. Doing so splits $\bar{z}$ into finitely many points; it will suffice to show that being étale at each one is an open non-empty condition. Thus we may assume that $\bar{z}$ is a rational point. Now the claim follows using [5, 17.11.1], because the target is isomorphic to $T_z X$ and the linear inclusion $T_z X \to k^{N+M\sigma}$ equivariantly splits (k having characteristic $\neq 2$).

Step 4: In this step we will produce étale maps $X \leftarrow X_4 \to W_z^h \times \mathbb{A}^{\sigma}$ such that the preimage of Z is finite over W_z^h and maps via a closed immersion to $W_z^h \times \mathbb{A}^{\sigma}$.

Choose a projection as in step (3) (this is possible since k is infinite). We obtain a map $\varphi : X \to \mathbb{A}^{n+m\sigma} \times \mathbb{A}^{\sigma}$ which is étale at $\bar{z}$ and such that $Z \to \mathbb{A}^{n+m\sigma}$ is quasi-finite at $\bar{z}$. Since the étale and quasi-finite loci are open [14, Tags 02GI and 01TI], shrinking X we may assume φ is étale and quasi-finite. Let $W \subset X$ denote the vanishing locus of $g_{m+1} : X \to \mathbb{A}^{\sigma}$. This is a closed subscheme étale over $\mathbb{A}^{n+m\sigma}$, whence smooth. Moreover $X^{C_2} \subset W$, so $z \in W$. Consider the pullback squares

[4] Observe that $(f, g)^{-1}((f, g)(\bar{z})) \subset \cap_i f_i^{-1}(f_i(\bar{z})) \cap \cap_j g_j^{-1}(g_j(\bar{z}))$.

[5] Note that spaces of equivariant maps are fixed points on spaces of all maps, so it suffices to show the openness ignoring equivariance. For étaleness see e.g. [9, §3.2.2]. For the quasifiniteness, apply openness of the quasi-finite locus [14, Tag 01TI] to the universal projection $Z \times P \to \mathbb{A}^{n+m\sigma} \times P$.

$$\begin{array}{ccccc} X_1 & \longrightarrow & W \times \mathbb{A}^\sigma & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{A}^{n+m\sigma} \times \mathbb{A}^\sigma & \longrightarrow & \mathbb{A}^{n+m\sigma}. \end{array}$$

We have $W \hookrightarrow X$ and so $W \times_{\mathbb{A}^{n+m\sigma}} W \hookrightarrow X_1$. The diagonal $W \to W \times_{\mathbb{A}^{n+m\sigma}} W$ is open (both schemes being étale over $\mathbb{A}^{n+m\sigma}$) and closed ($W \to \mathbb{A}^{n+m\sigma}$ being separated); hence $W \times_{\mathbb{A}^{n+m\sigma}} W = W \amalg W'$. Set $X_1' = X \setminus W'$ and consider the pullback squares

$$\begin{array}{ccccccc} W_z^h & \longrightarrow & X_2 & \longrightarrow & W_z^h \times \mathbb{A}^\sigma & \longrightarrow & W_z^h \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & X_1' & \longrightarrow & W \times \mathbb{A}^\sigma & \longrightarrow & W. \end{array}$$

Since W_z^h lifts to X_2 as indicated, z lifts to X_2. Because $Z \to \mathbb{A}^{n+m\sigma}$ is quasi-finite at z so is $Z_{X_2} \to W_z^h$. We can write [14, Tag 04GJ]

$$Z_{X_2} = A_1 \amalg \cdots \amalg A_n \amalg B$$

with A_i local and finite over W_z^h and $B \to W_1$ not quasi-finite at any point over z. It follows that $z \in A_i$ for some i, say $i = 1$. Set

$$X_3 = X_2 \setminus (\bigcup_{i>1} A_i \cup B).$$

Then $Z_{X_3} \to W_z^h$ is finite and the composite $Z_{X_3} \to X_3 \to W_z^h \times \mathbb{A}^\sigma$ is unramified and finite. Since $Z_{X_3} (\simeq A_1)$ has fiber over z consisting only of z, the fiber of $Z_{X_3} \to W_z^h \times \mathbb{A}^\sigma$ over z is a closed immersion (use [14, Tag 04XV]), and hence so is $Z_{X_3} \to W_z^h \times \mathbb{A}^\sigma$ by Nakayama's lemma.

Step 5: In this step we perform some standard cleanup: we transform our map $X_3 \to W_z^h \times \mathbb{A}^\sigma$ into an étale neighborhood of the preimage of Z, and then we descend from W_z^h to some finite type scheme.

Note that for $i > 1$, $W_z^h \cap A_i \subset W_z^h \subset X_2$ is closed but does not contain z, so must be empty. Similarly for $W_z^h \cap B$. It follows that $W_z^h \subset X_3$. The map $Z_{X_3} \to \varphi^{-1}(\varphi(Z_{X_3}))$ is open (both schemes being étale over $\varphi(Z_{X_3}) \simeq Z_{X_3}$) and closed (both schemes being closed in X_3). Hence $\varphi^{-1}(\varphi(Z_{X_3})) \simeq Z_{X_3} \amalg Z'$. As before $z \notin Z'$ and so $Z' \cap W_z^h = \emptyset$. Thus $W_z^h \subset X_4 := X_3 \setminus Z'$. By construction, $X_4 \to \mathbb{A}^\sigma \times W_z^h$ is an étale neighborhood of Z_{X_4}. The composite $W_z^h \to X_4 \to \mathbb{A}^\sigma \times W_z^h$ consists of finitely presented W_z^h-schemes. Writing W_z^h as the cofiltered limit of étale neighborhoods of z in W, we conclude by continuity that there exists such an étale neighborhood $\tilde{W} \to W$ together with a model $\tilde{W} \to \tilde{X}_4 \to \mathbb{A}^\sigma \times \tilde{W}$ of the previous composite, still satisfying all the same properties (in particular $\tilde{X}_4 \to \mathbb{A}^\sigma \times \tilde{W}$ is an étale neighborhood of $Z_{\tilde{X}_4}$, and $\tilde{X}_4$ is an open subscheme of $X_1' \times_W \tilde{W}$). Since $\tilde{W} \to W$ and $X_1' \to X$ are étale neighborhoods of z, so is $\tilde{X}_4 \to X$.

This concludes the case where $z \in X^{C_2}$.

Case $z \notin X^{C_2}$**.** Replacing X by $X \setminus X^{C_2}$, we may assume that the action of C_2 is free.

Trivial subcase. Suppose there is a smooth map $X \to B$, where B is a smooth 0-dimensional k-scheme with a free action by C_2. In this case everything happening equivariantly over B is equivalent to things happening non-equivariantly over B/C_2, and so the result follows by appeal to the usual Gabber lemma.

Reduction to z **set-theoretically fixed.** Suppose that $\sigma z \neq z$. Then the map $C_2 \times X \to X$ is an equivariant étale neighborhood of z (since $C_2 z \simeq C_2 \times z$). We may thus replace X by $C_2 \times X$, and consequently assume that X has a smooth map to C_2. This is handled by the trivial subcase. Consequently from now on we may assume that z is set-theoretically fixed.

Reduction to $\dim X > 1$**.** Existence of $z \in Z$ precludes $\dim X = 0$. Hence we may assume $\dim X = 1$, so $z \in X$ is a closed point. Since C_2 acts non-trivially on $k(z)$, the separable degree of $k(z)/k$ is positive (this follows from [14, Tag 09HK]). Let $k_s \subset k(z)$ denote the maximal separable subextension, and set $B = Spec(k_s)$. (This has a canonical C_2-action.) Since $B \to B \times B$ is a clopen immersion, $X_B \to X$ contains an equivariant étale neighborhood of z. If the action of C_2 on B is non-trivial, we are done by the trivial subcase. Otherwise we can repeat the construction, with the base field replaced by k_s. Note that this construction has decreased the degree of $k(z)/k$, whence the process must terminate.

Step 0: To summarize, we have X with a free action by C_2, $z \in Z \subset X$ with $C_2 z = \{z\}$, $\dim X = n + 1$, $n > 0$. Let $\bar{z}$ be a closed specialization of z.

Step 1: We claim that there is a map $\varphi : X \to \mathbb{A}^{n\sigma}$ which is smooth at $\bar{z}$, its restriction to Z is quasi-finite at $\bar{z}$, and has $\varphi(\bar{z}) \neq 0$. After choosing an open embedding of X into a C_2-representation and considering linear projections, all three conditions are open. Hence we need only prove that they can be satisfied. By adding further generators to the embedding, we in fact just need to construct functions $X \to \mathbb{A}^{n\sigma}$ satisfying the smoothness/quasifiniteness/non-vanishing condition.

For the non-vanishing, we can use Lemma 2.5(2).

For the quasi-finiteness, it will be enough to prove that given $Z' \subset X$ closed of dimension $\leq d$ ($d \geq 1$) with $\bar{z} \in Z'$, there exists $f : X \to \mathbb{A}^{\sigma}$ vanishing at $\bar{z}$ with $Z' \cap Z(f)$ of dimension $< d$. In every component of Z' of dimension d we can find a closed point not conjugate to $\bar{z}$; call the resulting set S. We thus need $f(\bar{z}) = 0$ and $f(S) \not\ni 0$. This we find by Lemma 2.5(3).

For the smoothness, first choose finitely many generators of $m_{\bar{z}}$, and write V for the affine space spanned by them. We claim that if $f_1, \dots, f_n$ are general elements of V, then $\varphi = (f_1 - \sigma f_1, \dots, f_n - \sigma f_n)$ has the desired property. We may verify this after base change to the algebraic closure. Then $\bar{z}$ splits into finitely many free C_2 orbits $C_2 z_1, \dots, C_2 z_r$. It will suffice to show that a general such φ smooth at $C_2 z_1$. The condition being open, we need only exhibit one specific set of maps $f_1, \dots, f_n$ satisfying the smoothness condition at $C_2 z_1$. The map $V \to \Omega_{z_1} X \oplus \Omega_{\sigma z_1} X$ is surjective (the right hand side being a quotient of $(m_{\bar{z}}/m_{\bar{z}}^2) \otimes_k \bar{k}$). Let $e_1, \dots, e_{n+1}$ be a basis of $\Omega_{z_1} X$. Pick f_i lifting $(e_i, 0)$ to get the desired result (note that the action of σ interchanges $(e_i, 0)$ and $(0, e_i)$) using [5, 17.11.1].

Step 2: In this step we will produce étale maps $X \leftarrow X_4 \to W \times \mathbb{A}^1$ such that the preimage of Z is finite over the henselian local scheme W and maps via a closed immersion to $W \times \mathbb{A}^\sigma$.

Shrinking X, we may thus assume given $\varphi : X \to \mathbb{A}^{n\sigma}$ which is smooth, and quasi-finite on Z, with $w := \varphi(z) \neq 0$. Note that w is set-theoretically fixed but has non-trivial action. Let $W = (\mathbb{A}^{n\sigma})^h_w$ and $X_1 = X \times_{\mathbb{A}^{n\sigma}} W$. Note that z lifts to X_1. Also $Z_{X_1} \to W$ is quasi-finite and hence as before we can obtain X_2 by removing finitely many clopen subsets of Z_{X_1} in such a way that $z \in X_2$ and $Z_{X_2} \to W$ is finite; in fact Z_{X_2} is local with closed point z and so z is the only point in Z_{X_1} above w (use incomparability in finite extensions [14, Tag 00GT]). We claim that there is a map $X \to \mathbb{A}^1_W$ (trivial action on $\mathbb{A}^1$) such that the special fiber $X_w \to \mathbb{A}^1_w$ is quasi-finite at z, étale at z and geometrically injective at z. Assuming this, the map is flat at z by miracle flatness [14, Tag 00R4], hence étale at z [14, Tag 02GU]. Let X_3 be obtained by shrinking X_2 such that the map becomes étale. Since Z_{X_2} is local, we must have $Z_{X_2} \subset X_3$ and so $Z_{X_3} = Z_{X_2} \to W$ is still finite. It follows as in Step 4 before (i.e. using Nakayama's lemma and [14, Tag 04XV]) that $Z_{X_3} \to \mathbb{A}^1_W$ is a closed immersion.

Proof of claim: Suppose $f_w : X_w \to \mathbb{A}^1_w$ has the desired properties. Let $f : X \to \mathbb{A}^1_W$ be any (non-equivariant) lift of f_w (this exists because f corresponds to an element of $\mathcal{O}(X)$ with specified image in $\mathcal{O}(X_w)$, and $X_w \to X$ is a closed immersion of affine schemes). The antisymmetrization of f will have the desired properties. We thus reduce to the case $W = w$. We are thus given X smooth of dimension 1 over w with a specified closed point z and must construct a map to $\mathbb{A}^1_w$ which is quasi-finite, étale and geometrically injective at z. Since w has a free action, by descent we can instead consider the non-equivariant situation over w/C_2. Thus given a smooth affine curve X over a field v with a closed point z, we must construct a map $X \to \mathbb{A}^1$ which is quasi-finite, étale and geometrically injective at z. Choosing an embedding $X \hookrightarrow \mathbb{A}^N_v$ and considering linear projections to $\mathbb{A}^1$, all three conditions become open. Being quasi-finite just requires being non-constant (at z), which can clearly be satisfied. The universal injectivity can be satisfied by Lemma 2.4. Generical étaleness is also well-known using arguments as in smoothness for step 1; see e.g. [9, §3.2.2].

Step 3: As in step 5 from case 1, we remove $\varphi^{-1}(\varphi(Z_{X_3})) \setminus Z_{X_3}$ from X_3 in order to obtain an étale neighborhood $X_4 \to \mathbb{A}^1_W$ of $Z_{X_4} = Z_{X_3}$. These are finitely presented schemes over W, which itself is the cofiltered limit of étale neighborhoods of $w \in \mathbb{A}^{n\sigma}$. It follows that there exists an étale neighborhood $\tilde{W} \to \mathbb{A}^{n\sigma}$ of w together with a model $\tilde{X}_4 \to \mathbb{A}^1_{\tilde{W}}$ having the same properties as before. Since $\tilde{W} \to \mathbb{A}^{n\sigma}$ is an étale neighborhood of W, $\tilde{X}_4 \to X$ is an étale neighborhood of z.

This concludes the proof. $\square$

3. Applications

Throughout let k be an infinite field of characteristic not 2. Denote by

$$\mathcal{SH}^{C_2,S^1}(k) \subset \mathrm{Fun}((\mathrm{Sm}_k^{C_2})^{\mathrm{op}}, \mathcal{SH})$$

the category of $\mathbb{A}^1$-invariant Nisnevich sheaves of spectra.[6]

Lemma 3.1. *Let $E \in \mathcal{SH}^{C_2,S^1}(k)$, $X \in \mathrm{Sm}_k^{C_2}$ and $x \in X$. Write S for the generic orbit of $X^h_{C_2x}$ and S' for the generic points of $(X^h_{C_2x})^{C_2}$ (of which there are 0 or 1). The map*

$$\pi_0 E(X^h_{C_2x}) \to \pi_0 E(S) \oplus \pi_0 E(S')$$

is injective.

Proof. We prove the result by induction on $\dim X$. The case of dimension 0 is tautological. Set $Y = X^h_{C_2x}$. Let $a \in \pi_0 E(Y)$ with $a|_S = 0$ and $a|_{S'} = 0$. Since the action on X^{C_2} is trivial, $\pi_0 E(Y^{C_2}) \to \pi_0 E(S')$ is injective (by non-equivariant Gersten injectivity; see e.g. [2, Corollary 6.2.4]); hence $a|_{Y^{C_2}} = 0$. By continuity we find an étale neighborhood $X_1 \to X$ of C_2x, a proper closed subscheme $Z \subset X_1$ and a class $b \in \pi_0 E(X_1)$ such that $b|_Y = a$, $b|_{X_1 \setminus Z} = 0$ and $b|_{X_1^{C_2}} = 0$. Applying Theorem 2.1 we obtain a further étale neighborhood $X_2 \to X_1$ of C_2x, a smooth C_2-scheme W and an étale neighborhood $X_2 \to \mathbb{A}^1_W$ of Z_{X_2}. Moreover if the action on $\mathbb{A}^1$ is non-trivial, we obtain a lift of W into X_2. We pull back the situation to the henselization of W in the image of C_2x; hence we may assume W henselian in the image of C_2x. Set $Z_2 = Z_{X_2}$ and $F = \pi_0 E$. From this we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
& F(W) & \\
F(X_2 \setminus Z_2) & F(X_2) \arrow[l] \arrow[u, "?"] & F(X_2/X_2 \setminus Z_2) \arrow[l] \\
F(\mathbb{A}^1_W \setminus Z_2) \arrow[u] & F(\mathbb{A}^1_W) \arrow[l] \arrow[u] & F(\mathbb{A}^1_W/\mathbb{A}^1_W \setminus Z_2). \arrow[l] \arrow[u, "\simeq"]
\end{tikzcd}$$

The map labeled ? only exists if the action on $\mathbb{A}^1$ is non-trivial. From the diagram we conclude that there is a class $c \in F(\mathbb{A}^1_W)$ with $c|_{X_2} = b|_{X_2}$ (so $c|_Y = a$) and $c|_{\mathbb{A}^1_W \setminus Z_2} = 0$. It will suffice to show that $c = 0$.

First consider the case where the action on $\mathbb{A}^1$ is trivial. In this case the map $\mathbb{A}^1_W \setminus Z_2 \to W$ has a section. (To see this, note that an invariant function on W defines

[6] That is, presheaves of spectra $F : (\mathrm{Sm}_k^{C_2})^{\mathrm{op}} \to \mathcal{SH}$ such that the canonical map $F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence for every $X \in \mathrm{Sm}_k^{C_2}$, $F(\emptyset)$ is contractible, and F transforms distinguished Nisnevich squares into pullback squares of spectra.

a section missing Z_2 if and only if this is true in the special fibers (Z_2 being finite over W). Since invariant functions can be lifted along closed immersions, to prove the claim we may replace w by its special orbit (the image of C_2x). This special orbit has either trivial or free action, and both cases are easily dealt with.) This implies that $F(W) \simeq F(\mathbb{A}^1_W) \to F(\mathbb{A}^1_W \setminus Z_2)$ is injective and hence $c = 0$, as needed.

Now we consider the case where the action on $\mathbb{A}^1$ is non-trivial; in particular we are given a lift of W into X_2 and the map ? exists. By assumption $b|_{X_2^{C_2}} = 0$ and hence also $c|_{W^{C_2}} = 0$. If the action of C_2 on W is trivial then $c = 0$ (since $F(\mathbb{A}^1_W) \simeq F(W)$). Otherwise the action of C_2 on the generic orbit $C_2\eta \in W$ must be non-trivial, so free. Since $Z_2 \to W$ is finite, the map $\mathbb{A}^1_{C_2\eta} \setminus Z_2 \to C_2\eta$ has a section. (Since $C_2\eta$ has free action, this problem corresponds to one over $(C_2\eta)/\eta$, and even though $\mathbb{A}^1$ has the sign action, the quotient corresponds to a line bundle on $(C_2\eta)/\eta$ which must be trivial.) This implies that $c|_{C_2\eta} = 0$, and hence $c = 0$ by induction. □

We arrive at the main point. Write $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_k^{C_2}, \mathcal{SH})$ for the category of spectral Nisnevich sheaves on $\mathrm{Sm}_k^{C_2}$. Recall that, as the stabilization of an ∞-topos, this has a standard t-structure with heart the sheaves of abelian groups on $\mathrm{Sm}_k^{C_2}$ [10, Proposition 1.3.2.1]. In fact for $E \in \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_k^{C_2}, \mathcal{SH})$ set $\underline{\pi}_i E = a_{\mathrm{Nis}} \pi_i E$. Then the non-negative/non-positive spectral presheaves are detected by the evident vanishing of homotopy sheaves $\underline{\pi}_i$, and the functor $\underline{\pi}_0$ is an equivalence when restricted to the heart. We call a Nisnevich sheaf of abelian groups on $\mathrm{Sm}_k^{C_2}$ *strictly* $\mathbb{A}^1$*-invariant* if all of its cohomology is $\mathbb{A}^1$-invariant; equivalently, the associated Eilenberg–MacLane spectrum is $\mathbb{A}^1$-invariant.

Proposition 3.2. *Let k be an infinite field of characteristic $\neq 2$.*

(1) *Let $E \in \mathcal{SH}^{C_2, S^1}(k)$. Then $\underline{\pi}_i E = 0$ if and only if, for every essentially smooth C_2-scheme O over k of dimension* 0 *(i.e. a finite disjoint union of spectra of finitely generated separable field extensions of k, with some action by C_2), we have $\underline{\pi}_i E(O) =$* 0.
(2) *Let $E \in \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_k^{C_2}, \mathcal{SH})_{\geq 0}$. Then $L_{\mathrm{mot}} E \in \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_k^{C_2}, \mathcal{SH})_{\geq 0}$.*
(3) *The standard t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_k^{C_2}, \mathcal{SH})_{\geq 0}$ restricts to the subcategory $\mathcal{SH}^{C_2, S^1}(k)$. The heart is the category of strictly $\mathbb{A}^1$-invariant sheaves on $\mathrm{Sm}_k^{C_2}$.*

Proof. (1) Necessity is clear, and sufficiency is immediate from Lemma 3.1.

(2) We must prove that $(L_{\mathrm{mot}} E)(O)$ a connective spectrum for every O as in (1). Since $(L_{\mathrm{mot}} E)(X)$ is obtained[7] as the singular construction $|E(X \times \mathbb{A}^\bullet)|$, for this it suffices to show that $E(O \times \mathbb{A}^n)$ is $(-n)$-connective. This holds since $O \times \mathbb{A}^n$ has Nisnevich cohomological dimension $\leq n$ [1, Proposition A.4.4].

[7] This singular construction is $\mathbb{A}^1$-equivalent to E and $\mathbb{A}^1$-invariant [12, §2 Corollaries 3.5 and 3.8], and it is a Nisnevich sheaf since the topology is defined by a cd structure; hence it is the motivic localization.

(3) Let $E \in \mathcal{SH}^{C_2,S^1}(k)$ and write $\tau_{\geq 0}^{\mathrm{Nis}} E \in \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_k^{C_2}, \mathcal{SH})_{\geq 0}$ for the connective cover. The canonical map $\tau_{\geq 0}^{\mathrm{Nis}} E \to E$ factors through $L_{\mathrm{mot}} \tau_{\geq 0}^{\mathrm{Nis}} E$, E being $\mathbb{A}^1$-invariant. On the other hand, $L_{\mathrm{mot}} \tau_{\geq 0}^{\mathrm{Nis}} E$ being connective, the map $L_{\mathrm{mot}} \tau_{\geq 0}^{\mathrm{Nis}} E \to E$ factors through $\tau_{\geq 0}^{\mathrm{Nis}} E$. We have now exhibited $\tau_{\geq 0}^{\mathrm{Nis}} E$ as a retract of the motivically local object $L_{\mathrm{mot}} \tau_{\geq 0}^{\mathrm{Nis}} E \to E$, whence $\tau_{\geq 0}^{\mathrm{Nis}} E \in \mathcal{SH}^{C_2,S^1}(k)$. It follows that the standard t-structure restricts as claimed. The identification of the heart follows by definition. □

Data availability

No data was used for the research described in the article.

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