Soft-Collinear Effective Theory for Collider Processes

Part I: Power-suppressed processes in SCET
Part II: SCET applications to BSM Physics

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Bianka Meçaj

Born 20 July 1993 in Albania

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Supervisor: Prof. Matthias Neubert

To my grandparents
and in memory of Prof. Artan Boriçi

Abstract

In this thesis we discuss details of soft-collinear effective theory in the context of application to power suppressed processes at particle colliders and to beyond the Standard Model physics. In the first part of this work we derive the first renormalized factorization theorem for a subleading power process in soft-collinear effective theory, with an application to the radiative decay $h \to \gamma\gamma$ mediated by a light quark loop. We derive this factorization formula using a "plus-type" subtraction scheme to regularize the endpoint divergences and two D-dimensional refactorization conditions for one of the operator matrix elements and a Wilson coefficient. We present the scale evolution of the soft-quark soft function that enters this formula as an important generic feature of soft-collinear effective theory at subleading power. We use the renormalization group evolution equations of the operator matrix elements and their Wilson coefficients to predict the large logarithms of order $\alpha\alpha_s^2 L^k$ for $k = 6, 5, 4, 3$ and $L = \ln(-M_h^2/m_b^2)$. This is the first three loop result for an observable at subleading power in soft-collinear effective theory. We discuss here the resummation of the large logarithms at leading order in renormalization group improved perturbation theory. In the second part of the thesis we discuss the decay rates of three exotic non-singlet particles using soft-collinear effective theory framework. We present the leading and subleading power operator basis for two scalar leptoquarks S_1 and S_3 and a vector U_1^μ and show the numerical resummation for their Wilson coefficients. We also present a matching for these coefficients for a particular UV model.

Keywords: Soft-collinear effective theory, factorization theorems, renormalization, resummation, endpoint divergences, soft-quark soft function, beyond Standard Model physics, leptoquarks.

List of abbreviations

BSM - Beyond Standard Model

EFT - Effective Field Theory

HQET - Heavy Quark Effective Theory

HSET - Heavy Scalar Effective theory

HVET - Heavy Vector Effective Theory

LCDA - Light-Cone Distribution Amplitude

LHC - Large Hadron Collider

LL - Leading Logarithm

LO - Leading Order

NLO - Next-to-Leading Order

NNLO - Next-to-Next-to-Leading Order

NNLL - Next-to-Next-to-Leading-Logarithm

NNLO - Next-to-Next-to-Leading Order

QCD - Quantum Chromodynamics

RGE - Renormalization Group Equation

RGIPT - Renormalization Group Improved Perturbation Theory

SCET - Soft-Collinear Effective Theory

SM - Standard Model

SMEFT - Standard Model Effective Theory

UV - Ultraviolet

Review of Publications

This thesis is based on work that has appeared in the publications and preprints [1–4] and done by the author during her PhD. We provide below an overview of the main results of these papers and the author's main contributions.

[1] Z. L. Liu, B. Mecaj, M. Neubert, X. Wang and S. Fleming, "*Renormalization and Scale Evolution of the Soft-Quark Soft Function*", JHEP **07** (2020) 104, [2005.03013].

In this work we have presented the renormalization and the scale evolution of the soft-quark soft function that appear in the $h \to \gamma\gamma$ decay process mediated by a b-quark loop. We have derived the one-loop renormalization factor for the soft function based on a conjecture that one of the amplitude terms (T_3) must be scale invariant and proved that this factor removes all the $1/\epsilon^n$ poles from the one-loop bare soft function. In addition we have derived the two-loop renormalization group equation and showed an exact solution in momentum space, Laplace space and in the so-called "diagonal scape". We have shown that the evolution of the soft function becomes local in the diagonal space.

In here all the authors in the paper have derived the one-loop renormalized expression for the soft function, shown in equations (17) and (18) in the paper. All the authors have shown that the expression in (30) gives a solution to the evolution equation. All the derivations presented in Section 4, 5, 6 have been derived by the author, Z. L. Liu, M. Neubert and X. Wang. The plots for the publication in these Section were made by M. Neubert. The author has cross checked the result in Figure (2), the plot on the left-hand side on Figure (3) and Figure (6) as well as Figure (5). Construction of the diagonal space in Section 7.1 and the derivation of equation (76) was done by the author, M. Neubert and X. Wang. The result in equation (92) was derived by Z. L. Liu, M. Neubert, X. Wang and the author. M. Neubert and the author have derived the results in Appendix C. The author has contributed to the text.

The Figures (3.3) and (5.1) in this thesis are taken from this publication. The rest of the Figures from this publication, shown in this thesis were remade by the author for the purpose of this dissertation work.

[2] Z. L. Liu, B. Mecaj, M. Neubert and X. Wang " *Factorization at subleading power and endpoint divergences in $h \to \gamma\gamma$ decay. Part II. Renormalization and scale evolution*", JHEP **01** (2021) 077, [2009.06779].

In this paper we derived the first renormalized factorization theorem for a power suppressed process in soft-collinear effective theory, applied to the decay process $h \to \gamma\gamma$ mediated by a b-quark loop. This factorization formula was derived using two D-

dimensional refactorization conditions proved to all orders in perturbation theory together with a "plus-type" subtraction scheme to remove the endpoint divergences. We proved that the extra contribution terms that arise from the fact that the regularization of the endpoints does not commute with the renormalization, can be all absorbed into a redefinition of one of the matching coefficients in the formula. We derived here the renormalization group evolution equations for all the quantities and used them to predict the large logarithms of order $\alpha_b \alpha_s^2 L^k$ for $k = 3, 4, 5, 6$ and $L = \ln(-M_h^2/m_b^2)$.

All the authors have derived the renormalized matrix elements and the renormalized matching coefficients in Section 4.3 and 4.4 and the corresponding renormalization group equations in Section 5. All the authors have derived the renormalization group equation for the hard coefficient $H_1(\mu)$ in (5.15) and the renormalized factorization theorem for $h \to \gamma\gamma$ in equation (4.1). The author has used slightly different intermediate steps from the ones presented in the paper in Sections 4.4 and 4.5, to arrive to this result. In this thesis we present in details these steps in Chapter 6 and in Appendix D. The method is the same as the one followed in the paper with some small differences in combining particular terms in intermediate steps of the derivation and using a partially different notation. All the authors have derived the final results for $H_1(\mu)$ in (4.48), (5.31) and (5.32). All the authors have derived the three-loop large logarithms in Sections 6.1, 6.2 and 6.3.

The Figures (3.2) and (4.2) are take from this publication.

[3] Z. L. Liu, B. Mecaj, M. Neubert and X. Wang, "*Factorization at Subleading Power, Sudakov Resummation and Endpoint Divergences in Soft-Collinear Effective Theory*", Phys. Rev. D **104** (2021) no.1, 014004, [2009.04456].

In this letter we have used the renormalized factorization formula for the $h \to \gamma\gamma$ to present the resummation of the Sudakov logarithms in the presence of endpoint divergences for this process. We have explicitly derived the resummation of Sudakov logarithms in the amplitude term T_3 at leading order in renormalization group improved perturbation theory. All the authors have derived the main result of the paper presented in equation (7). Its expansion in equation (9) was derived by Z. L. Liu.

[4] B. Mecaj and M. Neubert, "*Effective Field Theory for Leptoquarks*", [2012.02186].

In this work we have applied soft-collinear effective theory in the context of beyond Standard Model physics. We have presented there a detailed analysis of the decay rates of the three leptoquarks that appear the most in literature, the scalars S_1, S_3 and the vector U_1^μ at leading and subleading power in the expansion parameter $\lambda \ll 1$. Using the renormalization group evolution equations we have resummed the large logarithms at next-to-leading logarithmic order arising from the evolution of the Wilson coefficients. We have also performed a tree level matching for the Wilson coefficients for a specific UV model.

The author has performed all the calculations and made all the figures that appear in this work, under the supervision of M. Neubert. The text was written by the author. All the work presented there has been literally inserted in this thesis except for some minor modifications in the text.

Contents

Chapter 1

Introduction

The Standard Model of Particle Physics is one of the most well-established theories of Nature and it has been extremely successful at describing interactions of elementary particles. In the past fifty years it has been tested to unprecedented precision at particle laboratories around the world and it counts many important discoveries such as the electroweak gauge bosons [6, 7], top quark discovery [8, 9] and the Higgs boson [10, 11]. Despite its success there are many other observed phenomena that remain unexplained within the Standard Model such as neutrino masses, matter-antimatter asymmetry, the nature of dark matter etc. There has been a lot of effort to accommodate these and other physical processes within different theoretical frameworks that attempt to extend the current Standard Model with new particles and/or new symmetries. In literature they are referred to as beyond the Standard Model (BSM) physics.

In fact it is generally known that the Standard Model itself must be an *effective theory* of a more fundamental theory which would also account for these unexplained observations. While this fundamental theory has yet to be discovered the effective theory approach of studying elementary particle processes has proven to be a very useful tool to parametrize effects of possible new physics beyond the Standard Model. Indeed effective theories have been known for a long time in Physics. It is a property of physical laws that they are usually tied to a certain characteristic length and energy scale. Such a hierarchy of scales has historically allowed us to gain understanding for a lot of physical processes, when the underlying theory was not yet known. For instance this is precisely why the Fermi Theory could describe the beta decays of the nuclei long before the underlying gauge interactions were discovered. Similarly in quantum field theory (QFT) one can write effective Lagrangians with operators that contain quantum fields that describe only the fluctuations that are relevant for a certain energy regime. This method has been used also in the case of the Standard Model for certain processes of interest. By comparing such calculations with experimental data we can learn more about the scale of the new physics or impose constrains on certain BSM models.

In the modern days the experiments at particle colliders are always pushing to new frontiers of energy and luminosity to further improve their uncertainty bands. In precision measurements it is possible to capture possible discrepancies between the experimental values and the calculated Standard Model values of physical observables. In particular the Large Hadron Collider (LHC) has a dedicated program on precision measurements which include measurements in the flavour physics sector and Higgs boson production. In general precision measurements are important because they could teach us more about the nature of the gauge interaction in the Standard Model, the Higgs boson itself and show possible hints of beyond Standard Model physics. Inevitably such measurements require the same matching level of precision also on the theoretical calculations. Also in these cases effective theories become very powerful because of their *factorization theorems*. In these theorems physical processes are described via products of functions that depend only on a single scale. These factorization theorems then hold the basis for *resummation* of large logarithms of scale ratios that appear in fixed order calculations in perturbation theory, which can easily break the perturbative expansion. In the era of precision physics resummation of such logarithms to all orders in coupling constant is of crucial importance. Traditional quantum field theory methods start to become less reliable if one requires resummation of higher logarithmic terms. Effective theories offer a solution to these limitation by using renormalization group methods to resum higher powers of large logarithms.

On the other hand most of the processes at particle colliders nowadays are complex, multi-scale problems and often they include non-local interaction along highly boosted final states. Tradtional effective theories have their limitiations also in such interactions, which are properly described by means of non-local effective theories. The soft-collinear effective theory (SCET) [21–23] naturally addresses such scenarios. SCET was initially invented to describe B-meson decays, though in the past twenty years it has matured to be a very useful framework for a large variety of QCD processes in the so-called soft-collinear limit. Some of the most interesting collider processes are observed away from the hard scattering point, at the edges of the detector. The soft and collinear limit in SCET has the advantage of capturing such effects through its factorisation theorems. Moreover, recently SCET has been also used to describe on shell-decays of exotic particles [55]. This is an example of implementation of SCET to new physics searches.

For several collider processes factorization and resummation has been already achieved for leading power contributions using SCET technology. Extending these calculations for power corrections (in some small power counting $\lambda \ll 1$) is both a necessity to match the experimental precision for these observables and also a theoretical need for understanding of SCET as an EFT beyond the leading power. In particular renormalized versions of the factorization theorems are needed to achieve higher order resummations of large logarithms. At subleading power this turns out to be a highly non-trivial task mostly due to the presence of certain power suppressed interactions in SCET such as emissions of soft fermions, which are captured in factorization functions called soft functions.

In this thesis we address two important aspects of SCET. In the first part of the thesis, which is covered in Chapters 3 - 7 we focus on subleading power SCET with an application to the power suppressed process $h \to \gamma\gamma$ mediated by a light quark loop. The main goal of this part is to set up a theoretical framework for deriving renormalized factorization theorems at subleading power in SCET in the presence of endpoint divergences. We derive the first renormalized factorization formula for a power suppressed process in SCET and the framework we develop here can be applied to other phenomenologically important collider processes. We start in Chapter 2 with an overview of the most important theoretical foundations we need for our work, then we present an introduction on the bare factorization for $h \to \gamma\gamma$ in Chapter

3, we proceed with the renormalization in Chapter 4 and show the scale evolution of the soft function that emerges in this process in Chapter 5. In Chapter 6 we derive the renormalized factorization theorem for the decay $h \to \gamma\gamma$ mediated by a b-quark loop and finally in Chapter 7 we present a discussion on the resummation of the large logarithms on the amplitude at leading order in renormalization group improved perturbation theory.

In the second part of this thesis we use SCET in the context of beyond the Standard Model applications. We present a detailed analysis for leading and subleading power two body decays and leading contributions to three body decays of three leptoquarks. These are particles that have recently received a lot of attention in literature as possible solutions to several extensions of the Standard Model. In Chapter 9 and Chapter 10 we discuss two scalar leptoquarks S_1 and S_3 and in Chapter 11 we present the same analysis for a vector leptoquark U_1^μ. In Chapter 12 we discuss the resummation at next-to-leading logarithmic order for each of these particles and in Chapter 13 we present a matching of the effective theory into specific models for a leptoquark extension of the Standard Model. We finally conclude our main results of this thesis in Chapter 14. We collect some of the technical details of our derivations in several appendices.

All the Figures and plots presented here were made by the author if otherwise mentioned in the Section “Review of Publications”. The author has used Mathematica to calculate the derivations presented here and to make the plots. The Feynman diagrams were made with the JaxoDraw program.

Chapter 2

Theoretical Foundation

In this chapter we present a summary of the theoretical background that is needed as a foundation for the work we discuss in this thesis. We start with a general introduction to effective field theories. We then proceed with a short summary of basic elements of soft-collinear effective theory in Section 2.2. In Section 2.3 we present the heavy particle effective theory for scalar and vector particles. We will use elements of this effective theory in the second part of the thesis. In here we assume the reader is familiar with Quantum Field Theory and the Standard Model of Particle Physics. There already exists an extensive excellent literature in these topics. For some comprehensive introductory level readings in these topics we refer the reader to [12–16].

2.1 Effective Field Theories

In general there are two kinds of effective field theories, the bottom-up and the top-down constructions [24, 25]. In the top-down approach the full theory is known and one integrates out the high energy regime to construct an effective theory at some lower energy scale. The integrated out modes are contained in the Wilson coefficients that multiply the low energy operators. These operators are constructed out of low energy fields and they must obey the allowed symmetries. SCET is an effective theory built in the top-down context. The dynamics of the interactions between quantum fields in SCET is described by copies of the QCD Lagrangian written in terms of the SCET fields. In this case the full theory (QCD) is well known, though still SCET is very useful for calculations that are either complicated or not possible in the full theory. As we will see later, by construction the Lagrangians in SCET are non-local. We will have a more detailed look on basic elements of SCET in the following Section.

In the bottom-up concept the full UV completion of the theory at some higher scale M is not known. In this case one can still write effective operators that live at a lower scale. In practice this is useful to test properties of possible UV extensions of the Standard Model by a matching procedure of the operator matrix elements of the effective theory to operator matrix elements of possible candidates of the UV completion theory. A well-studied EFT of this bottom-up construction is the Standard Model effective theory (SMEFT) [18–20]. This is an extension of the Standard Model to higher dimension operators. In general the effective Lagrangian in d dimensions can be written as a product of the low energy effective operators $\mathcal{O}_i$ of mass dimension D_i and some coupling g_i

$$\mathcal{L}_{\text{eff}}(x) = \sum_i g_i \mathcal{O}_i(x) \,. \tag{2.1}$$

The low energy operators $\mathcal{O}_i$ are built out of Standard Model fields in the usual way of respecting all the allowed symmetries. By dimensional analysis the coupling g_i has mass dimension $d - D_i$. We can further write the g_i in terms of a dimensionless coupling $\mathcal{C}_i$ such that

$$g_i = \mathcal{C}_i M^{d-D_i} \,. \tag{2.2}$$

The dimensionless coefficients $\mathcal{C}_i$ are the Wilson coefficients in this case. Note that while in a top-down effective theory the Wilson coefficients can be estimated by direct calculations in the full theory, in a bottom-up effective theory, they need to be extracted from some experimental data. The effective Lagrangian with the higher dimension contributions can be written as

$$\mathcal{L}_{\text{eff}}(x) = \sum_i \frac{\mathcal{C}_i}{M^{D_i-d}} \mathcal{O}_i(x) \equiv \mathcal{L}_{\text{eff}}^{D=d}(x) + \frac{\mathcal{L}_{\text{eff}}^{D=d+1}(x)}{M} + \frac{\mathcal{L}_{\text{eff}}^{D=d+2}(x)}{M^2} + \ldots . \tag{2.3}$$

where the dots represent Lagrangians of dimension $D > d+2$. Contrary to SCET the effective Lagrangian in SMEFT contains only local operators. Moreover since they are constructed out of Standard Model fields they scale as powers of the electroweak scale v. It is obvious from (2.3) that higher dimension contributions are suppressed by higher powers of the large scale M and physical observables computed out of the operators $\mathcal{O}_i$ for higher orders will be suppressed by powers of v/M. One can choose to truncate this series based on a desired level of precision. Each of the higher power terms in this expression will contribute a correction of $\mathcal{O}(1/M)$ to the lower effective terms taken into account already. With this simple example we have introduced a very important concept in EFT, which is the power counting. In this case and in general in traditional EFTs power counting is done in canonical mass dimensions and the higher order dimension operators are always suppressed by higher powers of the canonical dimension. In fact any consistent EFT should have a well defined procedure of doing such a power counting. This should be consistent in the sense that physical observables computed using an operator of a certain dimension should only receive small corrections by higher order operators in the series in (2.3).

In more complicated processes sometimes different momentum components of particles have different scaling and the power counting in canonical dimensions is not relevant any more. In particular this is the case for SCET. The power counting in SCET is done in powers of a scaling parameter $\lambda \ll 1$ that defines the offshellness of the momenta. In the simplest case there is only one such parameter defined by the ratio of a small scale $m \sim v$ present in the problem with another larger scale M. The scale M is usually defined either by a heavier particle or the energy of some highly energetic process. In such cases the relevant fluctuations in the Lagrangian are then determined power by power in the expansion parameter λ. For the processes we will discuss here M is the mass (or of that order) of a heavy particle. In particular when discussing decays of heavy exotic particles, as we do in the second part of the thesis, M represents the scale of a whole physics sector that has been integrated out which in principle could contain other heavy particles with masses around the scale M. Note that for simplicity in notation, not to confuse the scale M with the mass of the heavy exotic particles we denote the higher scale by Λ there, where $\Lambda \sim M$.

2.1.1 Factorization and Renormalization Group Evolution

In the EFT the effective operators are constructed at a certain scale μ that is usually a characteristic scale for some observables of interest in the experiment. Any physical observable $\mathcal{M}(m, M)$ that depends on some low scale m and the UV scale M can now be computed by integrating in two steps such that [24]

$$\int_m^M d\phi \mathcal{M}(m, M) = \int_m^\mu d\phi \mathcal{M}(m, \mu) + \int_\mu^M d\phi \mathcal{M}(M, \mu)\,, \tag{2.4}$$

where ϕ here is the phase space of integration and it is process dependent. The values of $\mathcal{M}(m, \mu)$ and $\mathcal{M}(M, \mu)$ are obviously not the same but the μ dependence must cancel in their sum. Then the integrating out process is done above the scale μ such that the second term in (2.4) that depends on the high scale M accounts for the contribution to the Wilson coefficients $\mathcal{C}(M, \mu)$. The first term in (2.4) that depends on the low scale m is matched onto the effective theory and describes the contribution to the low energy operators $\mathcal{O}(m, \mu)$. The scale μ is called the factorization scale since this kind of scale separation is the starting point for physical observables such as $\mathcal{M}(m, M)$ to be expressed as products (often in the convolution sense) of functions that have a single scale dependence in factorization theorems. In this simple example the factorization of $\mathcal{M}(m, M)$ would look like

$$\mathcal{M}(m, M) = \sum_i \mathcal{C}_i(M, \mu) \times \mathcal{O}_i(m, \mu)\,, \tag{2.5}$$

where we are considering here the general case when the matching procedure yields more than one operator. The structure of the operators should not change depending on the choice of the scale μ as the symmetries and the low energy field content would remain the same, though their values and the ones of the Wilson coefficients would change.

Both the operators and the Wilson coefficients in (2.5) receive quantum corrections from loop effects that contribute logarithms of the scale ratios. For the Wilson coefficients these logarithms would be of the form $\alpha^n \ln^k(\mu^2/M^2)$ for $k = 1, 2...$ for any $\mathcal{O}(\alpha^n)$, where α is the concerning coupling constant. Similarly for the operators the logarithms would have the structure $\alpha^n \ln^k(\mu^2/m^2)$. In reality it is useful to know the values of the operators and the Wilson coefficients at scales different from μ. This has then the immediate consequence that the above logarithms can become significantly large and break the perturbative series expansion in the coupling constant. These large logarithms must be resummed systematically to all orders in the coupling constant. The renormalization group equation (RGE) addresses both the scale evolution of the operators and Wilson coefficients and the resummation of these large logarithms. These equations are a consequence of the fact that any physical observable must be independent on the choice of the scale μ. For the quantity $\mathcal{M}(m, M)$ we would have

$$\frac{dC_i(M,\mu)}{d\ln\mu}\mathcal{O}_i(m,\mu) + C_i(M,\mu)\frac{d\mathcal{O}_i(m,\mu)}{d\ln\mu} = 0. \tag{2.6}$$

The operators $\mathcal{O}_i$ span a linearly independent operator basis which means that for any operator $\mathcal{O}_i$ we have [24]

$$\frac{d\mathcal{O}_i(m,\mu)}{d\ln\mu} \equiv -\gamma_{ij}(\mu)\mathcal{O}_j(m,\mu)\,, \qquad \gamma_{ij}(\mu) = \frac{d\ln Z_{ij}(\mu)}{d\ln\mu}\,, \tag{2.7}$$

where $Z_{ij}(\mu)$ is the renormalization constant that removes the divergences from the operator $\mathcal{O}_i$ and γ_{ij} are entries of anomalous dimensions in the anomalous dimension matrix. The indices $\{i\,,j\}$ here indicate that during renormalization there might also be operator mixing among operators with the same power counting in the EFT. From the equation (2.6) it follows that the Wilson coefficients obey a similar RGE such that

$$\frac{d\mathcal{C}_j(M,\mu)}{d\ln\mu} = \mathcal{C}_i(M,\mu)\gamma_{ij}(\mu)\,. \tag{2.8}$$

Solving the above evolution equations resumms the large logarithms of the form $\alpha^n \ln^k(\mu^2/M^2)$ for $k = 1, 2....$ In this way effective theories offer a lot of leverage when it comes to precision calculations for processes at particle colliders. We emphasise here that resummation of the large logarithms through RG evolution is consistent only if the operators have a single scale dependence. For processes when there are more scales present in the problem, it is necessary to do further scale separation and the factorization theorems take more complicated structures. We will look at this case in more details with the application to the Higgs boson decay $h \to \gamma\gamma$.

2.2 Basic elements of SCET

The central idea of SCET lies in identifying the leading order momentum regions with respect to the parameter λ for any given process and assign those momentum regions to quantum fields. In particular the relevant momenta for the decays of a heavy particle are the collinear momenta carried by the energetic decay products, the hard momenta that has been integrated out and the soft momenta. The large energy flow in the final states defines the collinear directions $\vec{n}_i$. For each such directions we define two vectors $n_i^\mu = \{1, \vec{n}_i\}$ and $\bar{n}_i^\mu = \{1, -\vec{n}_i\}$ such that $n_i \cdot \bar{n}_i = 2$. The freedom to rescale these light-like reference vectors leads to the so-called reparametrization invariance in SCET [59], which is a remnant of Lorentz invariance. All the operators of a SCET Lagrangian must be reparametrization invariant.

The four momentum of a particle moving in the n_i collinear direction is written in terms of the reference vectors

$$p_i^\mu = p_i \cdot \bar{n}_i \frac{n_i^\mu}{2} + p_i \cdot n_i \frac{\bar{n}_i^\mu}{2} + p_{i\perp}^\mu\,, \tag{2.9}$$

The components of the collinear momenta scale as $(p_i \cdot n_i, p_i \cdot \bar{n}_i, p_{i\perp}) \sim M(\lambda^2, 1, \lambda)$. The soft momentum components are defined such that they all vanish in the limit where $\lambda \to 0$. The exact λ - scaling in the soft region depends on the specific process but for most cases they are either ultra-soft, where $k_{us} \sim M(\lambda^2, \lambda^2, \lambda^2)$ or soft, where $k_s \sim M(\lambda, \lambda, \lambda)$. The fields in SCET have a well defined scaling with respect to the power counting parameter λ as well. Acting with an arbitrary number of anti-collinear derivatives on a collinear field leaves its λ-scaling unchanged. Then a collinear field $\psi_{n_i}(x)$ can contain a series of such derivatives. To account for this effect operators built out of collinear fields are non-local along the light like direction. For instance

$$\psi_{n_i}(x + t\bar{n}_i) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\bar{n}_i \cdot \partial)^k \, \psi_{n_i}(x) \,, \tag{2.10}$$

where t is the displacement in the anti-collinear direction. In the Lagrangian, operators built out of such collinear fields always appear multiplied by Wilson coefficients that also depend on the t parameters, and these products are integrated over the variables t. In this way an arbitrary dependence on the large derivatives $\bar{n}_i \cdot \partial$ is allowed. This property of collinear fields makes SCET a non-local EFT and it is therefore necessary to introduce Wilson lines in each collinear direction n_i defined as [60, 61]

$$W^{(A)}_{n_i}(x) \; = P \, \exp\left[i \, g_A \, t^a_A \int_{-\infty}^{0} ds \, \bar{n}_i \cdot A^{\,a}_{\,n_i} \, (x + s\bar{n}_i) \right] \,, \tag{2.11}$$

where A_{n_i} is a collinear gauge field and the t^a is the corresponding group generator in the representation of the field where the Wilson line is acting upon. The g_A here is the gauge coupling constant corresponding to the gauge field A. For the gauge group $U(1)_Y$ the gauge field A^a is replaced by the gauge field B and the generators are the hypercharge Y of the field.

In SCET the collinear spinor fields in n_i direction are defined with the help of a projector operator $P_{n_i} = \frac{\not{n}_i \bar{\not{n}}_i}{4}$, such that this operator projects out only the large momentum component of the spinor and $P^2_{n_i} = P_{n_i}$. Then at leading power in λ a collinear Standard Model fermion scales as $\mathcal{O}(\lambda)$ and it is defined as

$$\Psi_{n_i}(x) = \frac{\not{n}_i \bar{\not{n}}_i}{4} \, W^{\dagger}_{n_i}(x) \, \psi(x) \,. \tag{2.12}$$

Here the Wilson line W_{n_i} without the superscript (A) is a product of Wilson lines $W^{(A)}_{n_i}$, one for each gauge group where the field $\psi(x)$ is transformed. The fermionic field $\Psi_{n_i}(x)$ obeys the constraint

$$\not{n}_i \Psi_{n_i}(x) = 0 \,. \tag{2.13}$$

A Standard Model collinear scalar field is also dressed in Wilson lines such that

$$\Phi_{n_i}(x) = W^{\dagger}_{n_i}(x) \, \phi(x) \,, \tag{2.14}$$

and it scales $\sim \lambda$. These collinear fields in SCET defined in terms of Wilson lines are referred to as collinear gauge invariant building blocks. The gauge invariant building block for a boson $\mathcal{A}^{\mu}_{n_i}$ is defined as the corresponding strength tensor sandwiched between two Wilson lines [22, 64], where we have

$$\mathcal{A}^{\mu}_{n_i}(x) = g_A \int_{-\infty}^{0} ds \, \bar{n}_{i,\nu} \left[W^{(A)\dagger}_{n_i} \, F^{\nu\mu}_{n_i} \, W^{(A)}_{n_i} \right] (x + s\bar{n}_i) \,. \tag{2.15}$$

For an Abelian gauge group such as $U(1)_Y$ the definition simplifies to

$$\mathcal{B}^{\mu}_{n_i}(x) = g_B \int_{-\infty}^{0} ds\, \bar{n}_{i\nu}\, B^{\nu\mu}_{n_i}(x + s\bar{n}_i)\,. \tag{2.16}$$

The different components of a vector field in SCET follow the same scaling as the corresponding momentum component. Then from the definition of the Wilson line in (2.11) a collinear field can emit any number of gauge bosons along its collinear direction suppressed only by powers of the coupling constant. In addition the gauge invariant building block for a gauge field obeys the following property

$$\bar{n}_i \cdot \mathcal{A}_{n_i} = 0\,. \tag{2.17}$$

The remaining components of a collinear gauge field with momentum $p \sim M(\lambda^2, 1, \lambda)$ will scale as the corresponding momentum component [65]

$$\mathcal{A}^{\mu}_{n_i,\perp} \sim \lambda, \qquad\qquad n_i \cdot \mathcal{A}_{n_i} \sim \lambda^2\,. \tag{2.18}$$

where $\mathcal{A}^{\mu}_{n_i,\perp}$ is defined

$$\mathcal{A}^{\mu}_{n_i,\perp} = \mathcal{A}^{\mu}_{n_i} - n_i \cdot \mathcal{A}_{n_i} \frac{\bar{n}^{\mu}_i}{2}\,. \tag{2.19}$$

The component $n_i \cdot \mathcal{A}_{n_i}$ is power suppressed and in fact it can always be eliminated using a field redefinition [66]. This implies that at leading power only the transverse component of a collinear gauge boson will be produced in a decay process.

In principle it is possible to introduce SCET fields to describe also Standard Model particles carrying soft momenta and these fields have well defined scaling with respect to λ, similarly to the collinear fields. Though the leading power SCET Lagrangian does not contain soft fermionic fields and the soft gauge bosons can be completely decoupled via field redefinitions and absorbed into soft Wilson lines. This results into the so-called exponentiation or eikonalization of soft emissions at leading power [22]. We introduce the emission of soft fermions in more details in Chapter 3.

2.3 Heavy Particle Effective Theory

Another useful effective theory that we also use in this thesis is heavy particle effective theory. This is a convenient framework to describe decays of a heavy particle with mass M into particles with a much smaller mass, similarly to the heavy quark effective theory (HQET) in QCD [61–63]. In this case we integrate out the heavy degrees of freedom around the large scale M. This restricts the interactions of these particles only through soft momentum transfer $k \sim M(\lambda^2, \lambda^2, \lambda^2)$ such that the heavy particle would still remain on-shell via these interactions. The four momentum of such a particle with mass M can be written as

$$p^{\mu} = Mv^{\mu} + k^{\mu}\,, \tag{2.20}$$

where $v^{\mu} = (1, 0, 0, 0)$ and k^{μ} is some residual momentum of the order of the electroweak scale in this case. In the on-shell limit the heavy scalar can be described by a quantum field

$S_v(x)$ such that it admits the field redefinition where $S(x) \to e^{-iM_S v\cdot x} S_v(x)$. Inserting this expression in the Lagrangian of a complex scalar field

$$\mathcal{L}_{\text{scalar}} = (D^\mu S)^\dagger D_\mu S - M_S^2 S^\dagger S\,, \tag{2.21}$$

we get

$$\mathcal{L}_{\text{scalar}} = 2M_S\, S_v^\dagger\ (iv\cdot DS_v) + (D^\mu S_v)^\dagger D_\mu S_v\,, \tag{2.22}$$

where $D_\mu = \partial_\mu - igG_\mu^a t^a - ig_2 W_\mu^a \tau^a - ig_1 Y B_\mu$. The covariant derivative will pick up only the soft component of the momentum p^μ since $S_v(x) \propto e^{-ix\cdot k} S_v(k)$. The second term in (2.22) is suppressed by $(1/M_S)$ relative to the first term. At leading power then the scalar Lagrangian becomes the effective Lagrangian $\mathcal{L}_{\text{HSET}}$ that describes the propagation of the heavy field S_v in heavy scalar effective theory (HSET) such that

$$\mathcal{L}_{\text{HSET}} = 2M_S \left[S_v^\dagger\ (iv\cdot DS_v) + \mathcal{O}\left(\frac{1}{M_S}\right)\right]. \tag{2.23}$$

The physical quantities described by $\mathcal{L}_{\text{HSET}}$ are mass independent at first order. This observation is similar to HQET and the accident symmetries that arise there [63]. This is due to the fact that the exchange of an ultra-soft gauge boson cannot probe the quantum numbers of the particle, instead one would need a hard momentum exchange. At leading power we can neglect the second term in (2.22).

In a similar fashion, a consistent description of decays of a heavy vector particle U^μ requires a heavy vector effective theory (HVET) that separates the leading power contribution from the vector field $U^\mu(x)$. We separate the transverse and longitudinal components of the vector U^μ using the projector operators defined in terms of the reference vector v^μ

$$T_\parallel^{\mu\nu} = v^\mu v^\nu\ , \qquad\qquad T_\perp^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu\,, \tag{2.24}$$

where v^μ is a reference vector in the direction of the four momentum of U^μ. The way we define v^μ here we have $v^2 = 1$. Then

$$U_\parallel^\mu = T_{\parallel\mu\nu} U^\nu\ , \qquad\qquad U_\perp^\mu = T_{\perp\mu\nu} U^\nu\,, \tag{2.25}$$

where $U_\parallel^\mu$ is the component parallel to the direction of four momentum of the heavy particle and $U_\perp^\mu$ is the component perpendicular to its four momentum. Looking at the two point correlation functions [65] for these two fields it is not difficult to show that the $U_\parallel^\mu$ is power suppressed compared to $U_\perp^\mu$. This means at leading power the vector U^μ is produced perpendicularly polarized and $v\cdot U = 0$. Its longitudinal component is then integrated out in the effective Lagrangian. We derive this Lagrangian starting from the most general gauge invariant Lagrangian for a massive vector field $U^\mu(x)$[1]

$$\mathcal{L} = (D^\mu U^\nu)^\dagger D_\mu U_\nu - (D^\nu U^\mu)^\dagger D_\mu U_\nu - M_U^2\, U^{\mu\dagger} U_\mu\,, \tag{2.26}$$

[1]The minus sing in front of the second term is necessary to get the equation $D_\mu U^\mu = 0$ in the massless limit.

with covariant derivative $D_\mu = \partial_\mu - igG^a_\mu t^a - ig_1 Y_U B_\mu$. We perform a field transformation on $U^\mu(x)$ such that: $U^\mu(x) \to e^{-i\,M_U v\cdot x} U^\mu_v(x)$, where $U^\mu_v(x)$ contains only the soft momentum fluctuations similarly to before. Then the Lagrangian that describes the interactions of the heavy field $U^\mu_v(x)$ reads

$$\mathcal{L} = 2M_U\, U_v^{\mu\dagger}\left(iv\cdot DU_{v\mu}\right) + (D_\mu U_{v\nu})^\dagger (D^\mu {U_v}^\nu) - (D^\nu {U_v}^\mu)^\dagger (D_\mu U_{v\nu}) + \mathcal{L}\left(v\cdot U\right)\,, \tag{2.27}$$

where the second and the third term are suppressed relatively to the first one by $1/M_U$. The $\mathcal{L}\left(v\cdot U\right)$ contains power suppressed terms that are integrated out. We then define the HVET Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{HVET}} &= 2M_U\left[{U^\mu_{v\perp}}^\dagger\left(iv\cdot DU_{v\perp\mu}\right) + \mathcal{O}\left(\frac{1}{M_U}\right)\right]\\ &\equiv 2M_U\left[T_{\perp\mu\nu}\Big[\,U_v^{\mu\dagger}\left(iv\cdot DU_v^\nu\right)\Big] + \mathcal{O}\left(\frac{1}{M_U}\right)\right],\end{aligned} \tag{2.28}$$

where $U^\mu_{v\perp}$ is the perpendicular component of the field U^μ_v. The leading term in the above Lagrangian is the same as the one in (2.22). In other words at leading power the renormalization of the heavy vector field $U^\mu_v(x)$ and its interaction vertices are similar to the renormalization for a heavy scalar.

Part I

Soft-collinear effective theory at subleading power

Chapter 3

Introduction to SCET at subleading power

SCET has been a very powerful tool for studing QCD factorization theorems. A lot of recent work in the community has been focused on understanding the structure of these factorization theorems and resummation beyond the leading power in the expansion parameter. The current state of the art for most of the power suppressed processes are factorization theorems expressed in terms of bare functions. Progress has been made in understanding power corrections for several collider processes such as B- meson decays [85–87], some interesting event shape observables for jets (thrust, N-jettiness) [88, 89], resummation for Drell-Yan and Higgs production in the limit of soft and collinear gluon emittions [90–95], resummations for power correction in the tail of transverse momentum spectra [96] and recently on exclusive Higgs decays [2, 3, 98]. For resummation of higher order large logarithms one needs to derive factorization theorems in terms of renormalized functions in order to be able to employ RG evolution equations to exponentiate these logarithms. On the other hand there are several technical challenges that show up when trying to extend these studies to renormalized factorization theorems. As we will see in this first part of the thesis, usually after renormalization the factorization for several terms in the amplitude breaks down.

Factorization theorems at subleading power in most cases are given as a sum over convolution functions at different scales that suffer from endpoint divergences [97, 98]. These endpoint divergences are caused by the fact that one integrates over an infinite range of rapidity. Their presence hints for a violation of the naiive scale separation as well as a breaking of the dimensional regularization scheme. In most cases dimensional regularization is not enough to cure such divergences and a suitable rapidity regulator is required. This turns out to be a subtle complication that makes it challenging deriving the factorization theorems in terms of renormalized functions because renormalization does not necessarily commute with regularization of such divergences.

Another generic feature of subleading power SCET is the emission of soft fermions. The treatment of soft-fermionic soft functions is rather complicated both to compute them as well as renormalize and solve their RG evolution. Moreover the endpoint divergences are closely related to these soft emissions. For an exclusive process such a function was recently computed at $\mathcal{O}(\alpha_s)$ in [98], in the context of the power suppressed radiative decay $h \to \gamma\gamma$ mediated by

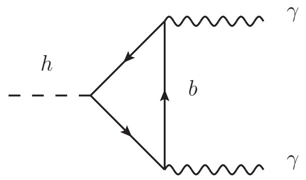

Figure 3.1: Feynman diagram for the radiative decay $h \to \gamma\gamma$ with a bottom quark loop.

a b- quark loop.

Better understanding SCET beyond leading power is an important step also from the theoretical perspective. As we have already mentioned a consistent EFT should provide a well-defined procedure to calculate observables without ambiguity at any order in power counting. In this first part of the thesis, we address the above complications of subleading power SCET with an application to the decay process $h \to \gamma\gamma$ mediated by a b-quark loop as shown in Fig.(3.1) This is a power suppressed process in SCET, which makes it simple enough to study since the operator basis contains fewer contributions and at the same time allows us to address very generic features of SCET at subleading power.

A detailed discussion for this process was already initiated in [98], where the SCET factorization at the bare level was derived. This factorization theorem describes the amplitude of $h \to \gamma\gamma$ decay in terms of bare SCET operators and the corresponding bare Wilson coefficients. In this work we extend this analysis to derive a factorization theorem for the $h \to \gamma\gamma$ process in terms of renormalized SCET operators and Wilson coefficients. Regarding the issue of the endpoint divergences one might hope that after renormalization such divergences might be cured by the scale evolution of the functions present in the convolution. We find that this is indeed not the case and the remaining endpoint divergences need to be removed. We regularize the endpoint divergences using a "plus-type" subtraction scheme together with some refactorization conditions for one of the Wilson coefficients and one of the operator matrix elements.

The renormalized factorization formula we derive is the first renormalized factorization theorem derived in SCET at subleading power and the approach we develop here can be applied to other power suppressed processes. For instance it can be extended for Higgs boson production at the LHC ($gg \to h$), subleading contributions to transverse momentum distributions for Drell-Yan and Higgs production, non leptonic B-meson decays etc.

In the next section we present a short summary of the bare factorization for $h \to \gamma\gamma$ from [98], which is the starting point for our derivations. Then in Chapter 4 we start by renormalizing the bare quantities that appear in the bare factorization theorem and we derive their RG equations. In particular we discuss in more details here the renormalization of the soft-quark soft function. We dedicate a long discussion to the scale evolution of the soft-quark soft function in Chapter 5. In Chapter 6 we derive the renormalized factorization formula for $h \to \gamma\gamma$ in details. Moreover we predict there the higher order logarithms for this decay process using the previously derived RG equations. Lastly in Chapter 7 we conclude this first part of the thesis with a discussion on the resummation of the large logarithms in the decay amplitude at leading order (LO) in RG-improved perturbation theory.

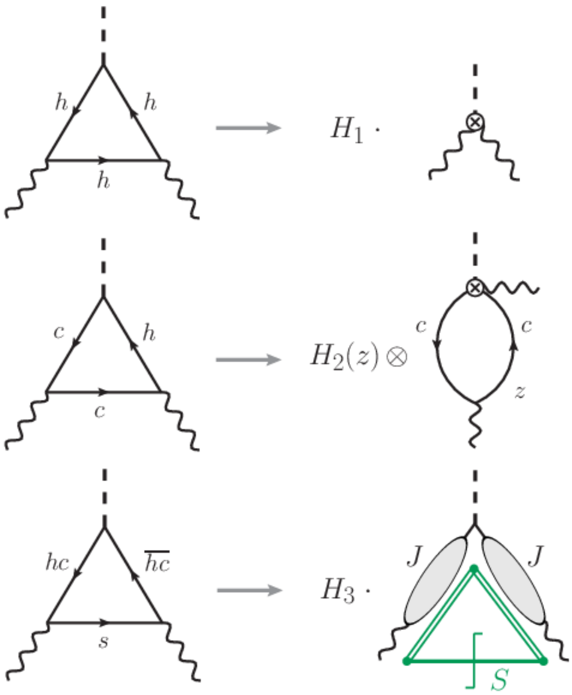

Figure 3.2: Matching of the leading regions from the full theory graph into the effective theory graphs. The quantities H_1, H_2 and H_3 are the hard matching coefficients and the diagrams on the right represent the operator matrix elements in SCET for each momentum region. The symbol $\otimes$ represents a convolution over the z variable.

3.1 $h \to \gamma\gamma$ decay in SCET

We start with a short summary of the bare factorization of $h \to \gamma\gamma$ mediated by a b-quark loop. We present here the most essential ingredients that we will also need for our analysis. For a more detailed discussion we refer the reader to the original work in [98]. The construction of the operator basis for this process was achieved from a two step matching of the full theory graph in Fig.(3.1), with on-shell collinear final state photons, into SCET operators. This matching is shown schematically in Fig.(3.2) The leading power momentum regions that can appear in the b-quark propagator are hard, (anti)collinear, soft and hard-collinear such that

$$
\begin{aligned}
\text{hard } (h) :& \ell^\mu \sim M_h(1,1,1) \\
n_1 \text{ -collinear } (c) :& \ell^\mu \sim M_h\left(\lambda^2, 1, \lambda\right) \\
n_2 \text{ -collinear } (\bar{c}) :& \ell^\mu \sim M_h\left(1, \lambda^2, \lambda\right) \\
\text{soft } (s) :& \ell^\mu \sim M_h(\lambda, \lambda, \lambda) \\
n_1 \text{ -hard-collinear } (hc) :& \ell^\mu \sim M_h\left(\lambda, 1, \lambda^{\frac{1}{2}}\right) \\
n_2 \text{ -hard-collinear } (\overline{hc}) :& \ell^\mu \sim M_h\left(1, \lambda, \lambda^{\frac{1}{2}}\right) ,
\end{aligned}
\tag{3.1}
$$

where $\lambda = m_b/M_h$ with m_b the b-quark mass and M_h the Higgs boson mass, and the hard-collinear scale is an intermediate scale set by $\sqrt{M_h m_b}$. In a first step the hard loop momenta are integrated out, leaving only hard-collinear modes off-shell and the full theory is matched into so-called SCET$_I$ theory (type one SCET). The result is a factorization of the decay amplitude, which in a simple form can be written as a sum of product terms of operator matrix elements $\langle O_i^{(0)}\rangle$ emerging from the matching procedure and the corresponding Wilson coefficients $H_i^{(0)}$

$$\mathcal{M}_b(h \to \gamma\gamma) = H_1^{(0)}\langle O_1^{(0)}\rangle + H_2^{(0)} \otimes \langle O_2^{(0)}\rangle + H_3^{(0)}\langle O_3^{(0)}\rangle\,, \tag{3.2}$$

where $\otimes$ represents convolution integrals and $\langle O_i^{(0)}\rangle \equiv \langle\gamma\gamma|O_i^{(0)}|h\rangle$. The bare operator $O_1^{(0)}$ is a SCET operator that describes the coupling of the Higgs boson to two collinear photons each moving in n_1 and $n_2 \equiv \bar{n}_1$ directions

$$O_1^{(0)} = \frac{m_{b,0}}{e_b^2} h \mathcal{A}_{n_1}^{\perp\mu}\mathcal{A}_{n_2,\mu}^{\perp}\,, \tag{3.3}$$

where e_b^2 is the electric charge of the b-quark, $m_{b,0}$ is the bare b-quark mass, h and $\mathcal{A}_{n_i}^{\perp\mu}$ are the Higgs field after spontaneous symmetry breaking and the perpendicular component of the gauge invariant photon field with respect to the (n_1, n_2) plane. The definition of the field $\mathcal{A}_{n_i}^{\perp\mu}$ is the same as the one given in (2.19) for a generic perpendicular gauge invariant gauge boson field in SCET. The operator $O_2^{(0)}$ contains the Higgs field, two collinear b-quark fields in n_1 direction and a collinear photon field in the other collinear direction n_2

$$O_2^{(0)}(z) = h\left[\bar{\chi}_{n_1}\gamma_\perp^\mu \frac{\bar{\not{n}}_1}{2}\delta\left(z\bar{n}_1\cdot k_1 + i\bar{n}_1\cdot\partial\right)\chi_{n_1}\right]\mathcal{A}_{n_2,\mu}^{\perp}\,. \tag{3.4}$$

In this case the two collinear b-quarks share the momentum of the other collinear photon $\mathcal{A}_{n_1,\mu}^{\perp}$ and the delta function in the operator ensures a momentum conservation along the collinear direction n_1. The field χ_{n_1} is the gauge invariant fermionic field defined in terms of collinear Wilson lines (defined in general in equation (2.11)), such that $\not{n}_1\chi_{n_1} = 0$. In the amplitude the operator matrix element $\langle O_2^{(0)}(z)\rangle$ is convoluted with the hard coefficient $H_2^{(0)}(z)$ and this convolution is divergent at both endpoints; $z = 0$ and $z = 1$. This is represented by the second term in (3.2). The last operator $O_3^{(0)}$ has a slightly more complicated structure due to the presence of a soft quark. It is defined as the time ordered product of the Higgs boson field with two subleading power SCET Lagrangian terms that describe the coupling of a soft fermion with a hard-collinear gauge boson and a hard-collinear fermion

$$O_3^{(0)} = T\left\{h\bar{\chi}_{n_1}\chi_{n_2}, i\int d^Dx\mathcal{L}_{q\xi_{n_1}}^{(1/2)}(x), i\int d^Dy\mathcal{L}_{\xi_2 q}^{(1/2)}(y)\right\} + \text{ h.c. }\,, \tag{3.5}$$

where

$$\begin{aligned}\mathcal{L}_{q\xi_{n_1}}^{(1/2)}(x) &= \bar{q}_s\left(x_-\right)\left[\mathcal{A}_{n_1}^{\perp}(x) + \not{\mathcal{G}}_{n_1}^{\perp}(x)\right]\chi_{n_1}(x)\,,\\ \mathcal{L}_{q\xi_{n_2}}^{(1/2)}(x) &= \bar{\chi}_{n_2}\left(y\right)\left[\mathcal{A}_{n_2}^{\perp}(y) + \not{\mathcal{G}}_{n_2}^{\perp}(y)\right]q_s(y_+)\,.\end{aligned} \tag{3.6}$$

In here $\mathcal{G}_{n_2}^{\perp}$ is the gauge invariant building block of the hard-collinear gluon field. The appearance of the $x_-^\mu = (\bar{n}_1\cdot x)\,\frac{n_1^\mu}{2}$ and $y_+^\mu = (\bar{n}_2\cdot y)\,\frac{n_2^\mu}{2}$ in this expression is due to the multipole expansion of the soft fields. Such an expasion ensures consitency of the terms with regard to power counting [82, 116].

At a second step also the hard-collinear modes are integrated out and a matching into SCET_{II} (type two) is performed. This results into a further factorization of the operator $O_3^{(0)}$ into a soft-quark soft function $S^{(0)}$ and two jet functions $J^{(0)}$. This result reads

$$\begin{aligned}\langle\gamma\gamma|O_3^{(0)}|h\rangle =& \frac{g_\perp^{\mu\nu}}{2}\int_0^\infty \frac{d\ell_+}{\ell}\int_0^\infty \frac{d\ell_-}{\ell_-} \\ &\times \left[J^{(0)}\left(M_h\ell_+\right)J^{(0)}\left(-M_h\ell_-\right)+J^{(0)}\left(-M_h\ell_+\right)J^{(0)}\left(M_h\ell_-\right)\right]S^{(0)}\left(\ell_+\ell_-\right)\,,\end{aligned} \tag{3.7}$$

where $\ell\pm$ ($\ell_+ = n_1\cdot\ell$ and $\ell_- = n_2\cdot\ell$) are the plus and minus components of the loop momentum ℓ. The above expression similarly to the convolution $H_2^{(0)}\otimes\langle O_2^{(0)}\rangle$ in (3.2) develops divergences at the endpoint region for $\ell_+\to\infty$ or $\ell_-\to\infty$ for fixed $\ell_+\ell_-$ (this product is fixed around the soft scale $\ell_+\ell_-\sim m_b$). All these divergences need to be regularized consistently and in a way that would still preserve the factorized structure of the amplitude.

3.1.1 Endpoint divergences and the refactorization conditions

In [98] it was shown that a rapidity regulator η imposed on the divergent integrals presented above regulates the infinities in both these cases. It was observed that in order for the remaining $1/\eta$ terms to cancel among different contributions, then the convolution containing the $O_2^{(0)}$ operator and the one with the jet and soft functions must behave similarly in the region where $z\to 0$ and $z\to 1$. This condition gave rise to two D-dimensional refactorization conditions to all orders in perturbation theory

$$\begin{aligned}[\![\bar{H}_2^{(0)}(z)]\!] =& -H_3^{(0)}J^{(0)}\left(zM_h^2\right)\,,\\ [\![\langle\gamma\gamma|O_2^{(0)}(z)|h\rangle]\!] =& -\frac{g_\perp^{\mu\nu}}{2}\int_0^\infty\frac{d\ell_+}{\ell_+}J^{(0)}\left(-M_h\ell_+\right)S^{(0)}\left(zM_h\ell_+\right)\,,\end{aligned} \tag{3.8}$$

where the function $[\![f(z)]\!]$ is defined from taking the limit $z\to 0$ of the function $f(z)$. It is also useful to mention that $z=\ell_-/M_h$. Here the function $[\![\bar{H}(z)^{(0)}]\!]$ is defined as

$$[\![H_2^{(0)}(z)]\!] = \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z}\,. \tag{3.9}$$

A similar definition that we will use in our calculations can be presented for the function $H_2^{(0)}(z)$ such that

$$H_2^{(0)}(z) = \frac{\bar{H}_2^{(0)}(z)}{z(1-z)}\,. \tag{3.10}$$

This redefinition is useful because the singularities in the function $\bar{H}_2^{(0)}(z)$ are logarithmic. There is no ambiguity in imposing the above refactorization conditions from the requirement that all the $1/\eta$ poles must cancel in the final result since the QCD amplitude for $h\to\gamma\gamma$ process is a finite quantity. Nevertheless the same refactorization conditions were later derived independently of this requirement based on first principles in SCET in [2].

These refactorization conditions were then used to rearrange different terms in the amplitude combined with an infinity-bin subtraction scheme that regularizes the endpoint divergences by means of subtractions. The final result for the amplitude was then found to take the form

$$\begin{aligned}\mathcal{M}_b &= \left(H_1^{(0)} + \Delta H_1^{(0)}\right) \langle O_1^{(0)} \rangle \\ &+ 2 \lim_{\delta\to 0} \int_\delta^{1-\delta} dz \left[H_2^{(0)}(z) \langle O_2^{(0)}(z) \rangle - \frac{[\![\bar{H}_2^{(0)}(z)]\!]}{z} [\![\langle O_2^{(0)}(z) \rangle]\!] - \frac{[\![\bar{H}_2^{(0)}(1-z)]\!]}{1-z} [\![\langle O_2^{(0)}(1-z) \rangle]\!] \right] \\ &+ g_\perp^{\mu\nu} H_3^{(0)} \lim_{\sigma\to -1} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} \int_0^{M_h} \frac{d\ell_-}{\ell_-} J^{(0)}\left(-M_h \ell_+\right) J^{(0)}\left(M_h \ell_-\right) S^{(0)}\left(\ell_+ \ell_-\right) \Big|_{\text{leading power}} ,\end{aligned} \tag{3.11}$$

where the limit $\delta \to 0$ is smooth now. The function $\Delta H_1^{(0)}$ in the first term in this expression and the cutoffs in the double convolution emerge naturally in the subtraction scheme employed here. The hard cutoffs in the last term break the homogeneous power counting due to the so-called collinear anomaly [17] and for consistency one should keep only the leading power contributions here.

Lastly we note that the Wilson coefficient $H_2^{(0)}(z)$ and the operator matrix element $\langle O_2^{(0)}(z) \rangle$ are symmetric under $z \to (1-z)$. We can use this symmetry property together with (3.10) to write

$$\int_0^1 dz H_2^{(0)}(z) \langle O_2^{(0)}(z) \rangle = 2 \int_0^1 \frac{dz}{z} \bar{H}^{(0)}(z) \langle O_2^{(0)}(z) \rangle . \tag{3.12}$$

With this relation the z-divergence is now concentrated in the region around $z = 0$. This relation also shortens the second line in (3.11) and sometimes it is more convenient for calculations. From now on we will refer to the first, second and the third terms in the formula (3.11) as T_1, T_2 and T_3 correspondingly. We present the one-loop expressions for the bare matrix elements and their Wilson coefficients together with the one-loop bare soft and jet function in Appendix A.

3.2 Soft-quark soft function in SCET

For the $h \to \gamma\gamma$ process mediated by a b-quark loop, the soft-quark soft function is defined from the vacuum expectation value of the soft quark propagator dressed by two finite length soft Wilson lines in direction n and $\bar{n}$ that come together at the Higgs boson vertex [98]

$$-\frac{(4\pi)^{1-\epsilon}}{N_c} e^{\epsilon\gamma_E} \mu^{2\epsilon} \langle 0 \left| T \operatorname{Tr} S_{\bar{n}}\left(0, r_1 \bar{n}\right) q_s\left(r_1 \bar{n}\right) \bar{q}_s\left(r_2 n\right) S_n\left(r_2 n, 0\right) \right| 0 \rangle . \tag{3.13}$$

The prefactor in front is chosen for later convenience, where μ is some renormalization scale, $\epsilon = (4-D)/2$ is the dimensional regulator and $N_c = 3$ for $SU(3)$. The trace inside the time

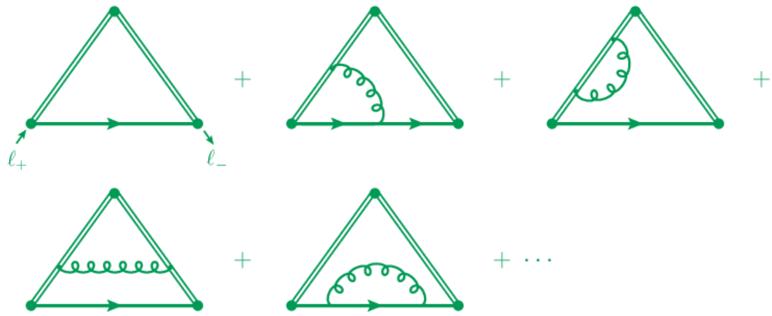

Figure 3.3: Feynman diagrams that contribute to the soft-quark soft function at one-loop.

ordered product runs over the color indices. The soft Wilson lines here are defined as

$$\begin{aligned} S_n\left(x_-,0\right) &= P\exp\left[ig_A\int_0^{\bar{n}\cdot x/2} dt\, n\cdot A_s\left(tn\right)\right]\,, \\ S_{\bar{n}}\left(0,y_+\right) &= P\exp\left[ig_A\int_{\bar{n}_2\cdot y/2}^{0} dt\, \bar{n}\cdot A_s\left(t\bar{n}\right)\right]\,, \end{aligned} \tag{3.14}$$

where in this case A_s can be both a gluon and a photon field. Due to the presence of the Wilson lines it is instructive to define the Fourier transform $\mathcal{S}(w)$ of the above definition such that the soft-quark soft function is now defined as

$$\begin{aligned} \frac{i}{2}\mathcal{S}\left(\ell_+\ell_-\right)P_n = &-\frac{(4\pi)^{1-\epsilon}}{N_c}e^{\epsilon\gamma_E}\int dr_1 e^{ir_1\ell_-}\int dr_2 e^{-ir_2\ell_+} \\ &\times \mu^{2\epsilon}\left\langle 0\left|T\,\mathrm{Tr}\,\bar{P}_{\bar{n}}S_{\bar{n}}\left(0,r_1\bar{n}\right)q_s\left(r_1\bar{n}\right)\bar{q}_s\left(r_2 n\right)S_n\left(r_2 n,0\right)P_n\right|0\right\rangle\,, \end{aligned} \tag{3.15}$$

where $P_n = \frac{\not{n}\not{\bar{n}}}{4}$ and $P_{\bar{n}} = \frac{\not{\bar{n}}\not{n}}{4}$ are the projector operators. At one-loop order in α_s variables ℓ_+ and ℓ_- are respectively the large and the small momentum components $\bar{n}\cdot\ell$ and $n\cdot\ell$ of the momentum ℓ^μ that flows in the soft quark propagator. This statement does not hold beyond leading order and this is easy to see if for instance a virtual gluon is emitted from the soft quark or its Wilson line. The soft function defined in equation (3.15) develops a discontinuity in the complex plane. In the ℓ_- plane for values of $\ell_+ > 0$ the discontinuities are above the real axis and for $\ell_+ > 0$ they are located below the real axis [98]. We can then define a new soft function as the discontinuity of the function $\mathcal{S}(w)$ such that

$$S(w) = \frac{1}{2\pi i}\left[\mathcal{S}(w+i0)-\mathcal{S}(w-i0)\right], \tag{3.16}$$

where $w = \ell_+\ell_-$. At one-loop in QCD this soft function was computed in [98] and given there in terms of generalized hypergeometric series. In Fig.(3.3) we show some of the Feynman diagrams that contribute at one-loop. After expanding that result around $\epsilon = 0$ the bare soft function $S^{(0)}(w)$ is expressed as a sum of two theta functions each multiplied by two bare functions that contain the $1/\epsilon^n$ poles

$$S^{(0)}(w) = m_{b,0}\mu^{2\epsilon}\left[S_a^{(0)}(w)\theta\left(w-m_{b,0}^2\right)+S_b^{(0)}(w)\theta\left(m_{b,0}^2-w\right)\right]\,, \tag{3.17}$$

where the functions $S_a^{(0)}(w)$ and $S_b^{(0)}(w)$ are

$$
\begin{aligned}
S_a^{(0)}(w) =& \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(w - m_{b,0}^2\right)^{-\epsilon} \left[1 + \epsilon \frac{C_F \alpha_{s,0}}{4\pi} 2e^{\epsilon\gamma_E} \frac{3-2\epsilon}{1-2\epsilon} \Gamma(\epsilon) \frac{\left(m_{b,0}^2\right)^{1-\epsilon}}{w - m_{b,0}^2}\right] \\
&+ \frac{C_F \alpha_{s,0}}{4\pi} \left[\left(-\frac{2}{\epsilon^2} + \frac{6}{\epsilon} + \frac{2}{\epsilon} \ln\left(1 - \frac{1}{\hat{w}_0}\right) + 12 - \frac{\pi^2}{3}\right) \left(w - m_{b,0}^2\right)^{-2\epsilon} \right. \\
&\left. -2\mathrm{Li}_2\left(\frac{1}{\hat{w}_0}\right) - 2\left(\ln \hat{w}_0 - 1\right) \ln\left(1 - \frac{1}{\hat{w}_0}\right) - 3\ln^2\left(1 - \frac{1}{\hat{w}_0}\right) + \mathcal{O}(\epsilon)\right] , \\
S_b^{(0)}(w) =& \frac{C_F \alpha_{s,0}}{4\pi} \left(m_{b,0}^2\right)^{-2\epsilon} \left[-\frac{4}{\epsilon} \ln\left(1 - \hat{w}_0\right) + 6\ln^2\left(1 - \hat{w}_0\right) + \mathcal{O}(\epsilon)\right] .
\end{aligned}
\tag{3.18}
$$

The terms that vanish for $\epsilon \to 0$ are collected in $\mathcal{O}(\epsilon)$. Here we have defined $\hat{w}_0 = w/m_{b,0}^2$, where here $m_{b,0}$ is the bare pole mass of the $b-$ quark and $\alpha_{s,0}$ is the bare strong coupling constant. At leading order the soft function has coverage only for $w > m_{b,0}^2$ and the expression has a relatively simpler form compared to the $\mathcal{O}(\alpha_{s,0})$ terms, where logarithmic and dilogarithmic functions appear.

Chapter 4

Renormalization

In this chapter we derive the one-loop renormalization of the quantities that appear in bare factorization theorem in (3.11). We start with the parameter renormalization in the following section. We then present a detailed discussion on the soft function renormalization in Section 4.2 and the one-loop renormalization of the bare matrix elements for operators $O_1^{(0)}$, $O_2^{(0)}(z)$, $[\![O_2(z)^{(0)}]\!]$ in Section 4.3. Then in Section 4.4 we discuss the one-loop renormalization of the Wilson coefficients $H_2^{(0)}$ and $H_3^{(0)}$. We postpone the discussion on the renormalization of the hard coefficient $H_1^{(0)}$ until Chapter 6, since it is closely related to the derivation the renormalized factorization formula for $h \to \gamma\gamma$. The renormalization of the jet function was achieved at two-loop order in [102] and we will only state that result here. From the renormalization conditions it is straightforward to derive the corresponding RGEs for the various functions. We collect these equations in Section 4.5.

4.1 Parameter renormalization

The parameters that appear in the bare amplitude include the b-quark mass, the b-quark Yukawa coupling, the $b-$quark electromagnetic coupling α_0 and the strong coupling $\alpha_{s,0}$. We renormalize all these parameters in the $\overline{\text{MS}}$ scheme, where we have

$$\begin{aligned} m_{b,0} &= Z_m m_b(\mu)\,, \qquad & y_{b,0} &= \mu^{\epsilon} Z_y y_b(\mu)\,, \\ \alpha_0 &= \mu^{2\epsilon} Z_\alpha \alpha\,, \qquad & \alpha_{s,0} &= \mu^{2\epsilon} Z_{\alpha_s} \alpha_s(\mu)\,. \end{aligned} \tag{4.1}$$

At one-loop order in QCD the above renormalization factors read

$$\begin{aligned} Z_\alpha &= 1\,, \\ Z_{\alpha_s} &= 1 - \beta_0 \frac{\alpha_s}{4\pi\epsilon} + \mathcal{O}\left(\alpha_s^2\right)\,, \\ Z_y = Z_m &= 1 - 3C_F \frac{\alpha_s}{4\pi\epsilon} + \mathcal{O}\left(\alpha_s^2\right)\,, \end{aligned} \tag{4.2}$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$ is the one-loop coefficient of the QCD β-function with $n_f = 5$ (five active flavours with the top quark integrated out) and $C_F = 4/3$ for the $SU(3)$ gauge group.

In certain steps of our calculations it will be useful to renormalize the b-quark mass in the pole scheme. The renormalized pole mass m_b is defined as the pole of the renormalized b-quark propagator such that [117]

$$m_b^2 = m_{b,0}^2 + \frac{C_F \alpha_{s,0}}{4\pi} 2e^{\epsilon\gamma_E} \frac{3-2\epsilon}{1-2\epsilon} \Gamma(\epsilon) \left(m_{b,0}^2\right)^{1-\epsilon} + \mathcal{O}(\alpha_s^2) \,. \tag{4.3}$$

The renormalized pole mass is related to the running mass $m_b(\mu)$ by the following relation [117]

$$\begin{aligned} m_b(\mu) = m_b \Bigg\{ & 1 + \frac{C_F \alpha_s}{4\pi} (3L_m - 4) \\ & + C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\left(\frac{9}{2} C_F - \frac{3}{2}\beta_0\right) L_m^2 + \left(-\frac{21}{2} C_F + \frac{185}{6} C_A - \frac{26}{3} T_F n_f\right) L_m + \dots \right] \\ & + \mathcal{O}\left(\alpha_s^3\right) \Bigg\} , \end{aligned} \tag{4.4}$$

where $L_m = \ln\left(m_b^2/\mu^2\right)$. The dots in the second line in the above expression denote scale independent terms that we will not need in our analysis. This quantity is known to very high orders in QCD [118].

4.2 Soft function renormalization

We start the soft function renormalization by renormalizing the bare b-quark mass and the coupling constant in (3.17) and (3.18). In here it is convenient to use the pole scheme in (4.3). In fact the mass renormalization in the leading term in $S_a^{(0)}(w)$ entirely removes the pole in the second term there, since for $\epsilon \to 0$ we can write

$$\left(\omega - m_{b,0}^2\right)^{-\epsilon} = \left(\omega - m_b^2\right)^{-\epsilon} \left(1 - \epsilon \frac{C_F \alpha_{s,0}}{4\pi} 2e^{\epsilon\gamma_E} \frac{3-2\epsilon}{1-2\epsilon} \Gamma(\epsilon) \frac{(m_b^2)^{1-\epsilon}}{\omega - m_b^2}\right) + \mathcal{O}(\alpha^2) \,. \tag{4.5}$$

Expanding the rest of the terms around ϵ and keeping up to $\mathcal{O}(\alpha_s)$ corrections we find

$$S^{(0)}(w) = m_b \left[S_a^{(0)}(w) \theta\left(w - m_b^2\right) + S_b^{(0)}(w) \theta\left(m_b^2 - w\right) \right] , \tag{4.6}$$

where we have absorbed a factor of $\mu^{2\epsilon}$ to the $S_a^{(0)}(w)$ and $S_b^{(0)}(w)$

$$\begin{aligned} S_a^{(0)}(w) = & \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{w - m_b^2}{\mu^2}\right)^{-\epsilon} \left[1 - \frac{C_F \alpha_s}{4\pi} \left(\frac{3}{\epsilon} - 3 \ln \frac{m_b^2}{\mu^2} + 4 + \mathcal{O}(\epsilon) \right) \right] \\ & + \frac{C_F \alpha_s}{4\pi} \left[\left(-\frac{2}{\epsilon^2} + \frac{6}{\epsilon} + \frac{2}{\epsilon} \ln\left(1 - \frac{1}{\hat{w}}\right) + 12 - \frac{\pi^2}{3} \right) \left(\frac{w - m_b^2}{\mu^2}\right)^{-2\epsilon} \right. \\ & \left. - 2\mathrm{Li}_2\left(\frac{1}{\hat{w}}\right) - 3\ln^2\left(1 - \frac{1}{\hat{w}}\right) + 2\left(1 - \ln\hat{w}\right) \ln\left(1 - \frac{1}{\hat{w}}\right) + \mathcal{O}(\epsilon) \right] , \\ S_b^{(0)}(w) = & \frac{C_F \alpha_s}{4\pi} \left(\frac{m_b^2}{\mu^2}\right)^{-2\epsilon} \left[-\frac{4}{\epsilon} \ln(1 - \hat{w}) + 6 \ln^2(1 - \hat{w}) + \mathcal{O}(\epsilon) \right] . \end{aligned} \tag{4.7}$$

Similarly to before here $\hat{w} = w/m_b^2$. From the above result it is evident that renormalizing only the bare parameters is not enough to remove all the $1/\epsilon^n$ poles. Though this should not come as a surprise due to the presence of the $1/\epsilon$ in $S_b^{(0)}(w)$, which only starts at $\mathcal{O}(\alpha_s)$. This observation indicates that the soft function does not renormalize trivially and the one-loop coefficient of the renormalization factor should contain some non-local part in w in order for the poles from $S_b^{(0)}$ to cancel entirely.

The renormalization condition for the soft-quark soft function is far from obvious and we will derive it based on a conjecture that the amplitude T_3 in its original form (without cutoffs) is scale invariant. In other words

$$\frac{d}{d\ln\mu}T_3 = 0\,, \tag{4.8}$$

where

$$T_3 = g_\perp^{\mu\nu} H_3^{(0)} \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} J^{(0)}\left(M_h\ell_-\right) J^{(0)}\left(-M_h\ell_+\right) S^{(0)}\left(\ell_+\ell_-\right)\,. \tag{4.9}$$

When computing explicitly the bare results for T_1, T_2 and T_3, it was observed in [98] (see equation (69) there) that T_3 is the only term that has only one $1/\epsilon$ pole remaining, while the other terms have also higher order $1/\epsilon$ poles[1]. This has been used as a hint to conjecture that this third term in the amplitude is scale invariant. At this point we note that as it stands in (4.9) this convolution is not well defined at the endpoint and suffers from divergences. We will neglect this for now for the purpose of deriving the Z_s renormalization factor and revisit it later. To derive the one-loop Z_s from this conjecture we also need the one-loop renormalization factor for the jet functions and the Wilson coefficient that appear in (4.9). These results are both known to two [101, 102] and three-loop [103] order respectively. The hard function is renormalized multiplicatively in momentum space and its renormalization factor contains only the cusp terms

$$H_3(\mu) = Z_{33}^{-1}(\mu) H_3^{(0)}\,, \tag{4.10}$$

where

$$Z_{33}^{-1}(\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\left(\ln\frac{-M_h^2}{\mu^2} - \frac{3}{2}\right)\right] + \mathcal{O}\left(\alpha_s^2\right)\,. \tag{4.11}$$

The jet function renormalization is less trivial because its renormalization factor contains non-local terms

$$J\left(\pm M_h\ell,\mu\right) = \int_0^\infty d\ell' Z_J\left(\pm M_h\ell, \pm M_h\ell';\mu\right) J^{(0)}\left(\pm M_h\ell'\right)\,, \tag{4.12}$$

where Z_J reads [102]

$$Z_J\left(\pm M_h\ell, \pm M_h\ell';\mu\right) = \left[1 + \frac{C_F\alpha_s}{4\pi}\left(-\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mp M_h\ell}{\mu^2}\right)\right]\delta\left(\ell-\ell'\right) + \frac{C_F\alpha_s}{2\pi\epsilon}\ell\,\Gamma\left(\ell,\ell'\right) + \mathcal{O}\left(\alpha_s^2\right)\,. \tag{4.13}$$

In here and below if otherwise mentioned we use $-M_h^2 \equiv -M_h^2 - i0$ and $-p^2 \equiv -p^2 - i0$. The function $\Gamma(\ell,\ell')$ is a symmetric plus distribution defined as

$$\Gamma\left(\omega,\omega'\right) = \left[\frac{\theta\left(\omega-\omega'\right)}{\omega\left(\omega-\omega'\right)} + \frac{\theta\left(\omega'-\omega\right)}{\omega'\left(\omega'-\omega\right)}\right]_+\,. \tag{4.14}$$

[1] All these poles cancel though in the sum $T_1 + T_2 + T_3$ as they should.

When $\Gamma(\omega,\omega')$ is integrated with a function $f(\omega')$ one should replace $f(\omega') \to f(\omega') - f(\omega)$ and it vanishes if it is integrated with a constant. Using the inverse of the equations (4.10) and (4.12) we can rewrite T_3 in terms of renormalized functions. Then combining it with the requirement of scale invariance we have

$$\begin{aligned} T_3 =& Z_{33}H_3(\mu)\int_0^\infty d\ell_- \int_0^\infty \frac{d\ell'_-}{\ell'_-} Z_J^{-1}\left(M_h\ell'_-, M_h\ell_-;\mu\right) J(M_h\ell'_-,\mu) \\ &\times \int_0^\infty d\ell_+ \int_0^\infty \frac{d\ell'_+}{\ell'_+} Z_J^{-1}\left(-M_h\ell'_+, -M_h\ell_+;\mu\right) J\left(-M_h\ell'_+,\mu\right) S^{(0)}\left(\ell_+\ell_-\right) \\ \equiv& H_3(\mu)\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+} J\left(M_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right) . \end{aligned} \tag{4.15}$$

At one-loop the inverse renormalization factor Z_J^{-1} is derived from Z_J by changing the sign in front of the terms multiplied by α_s. The same for Z_{33}. In this last expression the variables $\ell_\pm$ and $\ell'_\pm$ in Z_J^{-1} are interchanged because we have used the following symmetry relation

$$\frac{\ell'}{\ell} Z_J^{-1}(\ell,\ell';\mu) = Z_J^{-1}(\ell',\ell;\mu)\,. \tag{4.16}$$

It is then easy to see that the soft function should have the following renormalization condition

$$\begin{aligned} S\left(\ell_+\ell_-,\mu\right) = Z_{33}(\mu) &\int_0^\infty d\ell'_-\, Z_J^{-1}\left(M_h\ell_-, M_h\ell'_-;\mu\right) \\ \times &\int_0^\infty d\ell'_+ Z_J^{-1}\left(-M_h\ell_+, -M_h\ell'_+;\mu\right) S^{(0)}\left(\ell'_+\ell'_-\right). \end{aligned} \tag{4.17}$$

This expression can be further simplified as

$$S(w,\mu) = \int_0^\infty dw' Z_S\left(w,w';\mu\right) S^{(0)}\left(w'\right) , \tag{4.18}$$

where from the convolution of the known three Z-factors in (4.17) we derive the one-loop expression for the renormalization factor of the soft-quark soft function

$$Z_S\left(w,w';\mu\right) = \left[1 + \frac{C_F\alpha_s}{4\pi}\left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{w}{\mu^2} - \frac{3}{\epsilon}\right)\right]\delta\left(w-w'\right) - \frac{C_F\alpha_s}{\pi\epsilon} w\Gamma\left(w,w'\right) + \mathcal{O}\left(\alpha_s^2\right) . \tag{4.19}$$

The only difference between the non-local term in Z_S and Z_J is the factor of 2 in the denominator, which in fact will have some non-trivial consequences in the solution of the RG equation for the soft function. In this result $\ln(w/\mu^2)$ comes from the product of the log term in Z_{33} and the one in Z_J^{-1} proportional to the delta function. Here $\Gamma\left(w,w'\right)$ is the same plus distribution as in (4.14).

4.2.1 Renormalized soft function at one-loop

Even though the renormalization factor Z_S was derived based on a conjecture we find that it entirely removes all the remaining $1/\epsilon^n$ poles from the one-loop bare soft function in (4.7).

The most non-trivial part in this derivation is the integration of the plus distribution with the function $S_a^{(0)}(w)$ in (4.7). When $\Gamma(w,w')$ is integrated with the function $S_a^{(0)}(w)$, it yields a term proportional to $\theta(m_b^2 - w)$. This is then combined with results from the convolution of $S_b^{(0)}(w)$ with Z_S to fully cancel the divergences in the renormalized $S_b(w)$. As we anticipated before the non-local term in Z_S has already an important effect at leading order and its presence is crucial in cancelling the remaining poles. After several steps of calculations we find that the result for the renormalized soft-quark soft function reads

$$S(w,\mu) = m_b\Big[S_a(w,\mu)\theta\left(w - m_b^2\right) + S_b(w,\mu)\theta\left(m_b^2 - w\right)\Big], \tag{4.20}$$

with

$$\begin{aligned} S_a(w,\mu) =& 1 + \frac{C_F\alpha_s}{4\pi}\left[-L_w^2 - 6L_w + 3L_m + 8 - \frac{\pi^2}{2} + 2\text{Li}_2\left(\frac{1}{\hat{w}}\right)\right.\\ &\left. -4\ln\left(1-\frac{1}{\hat{w}}\right)\left(L_m + 1 + \ln\left(1-\frac{1}{\hat{w}}\right) + \frac{3}{2}\ln\hat{w}\right)\right], \\ S_b(w,\mu) =& \frac{C_F\alpha_s}{\pi}\ln(1-\hat{w})\Big[L_m + \ln(1-\hat{w})\Big]. \end{aligned} \tag{4.21}$$

We have defined here $L_m = \ln(m_b^2/\mu^2)$ and $L_w = \ln(w/\mu^2)$. In Fig.(4.1) we plot the renormalized soft function in units of m_b as a function of the dimensionless variable $\hat{w} = w/m_b^2$ at tree level and at one-loop. We use $m_b = 4.8$ GeV for the pole mass and we fix the renormalization scale to $\mu = m_b$. In this graph we notice that the discontinuity around $\hat{w} = 1$ is smoothed out after renormalization. From the expressions in (4.21) it is worth mentioning that the

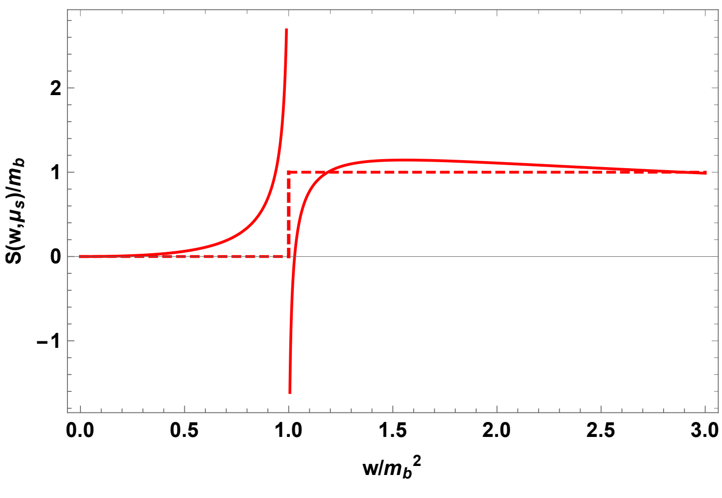

Figure 4.1: Renormalized soft function $S(w,\mu_s)/m_b$ at tree level (dashed line) and one-loop (solid line), for $\mu_s = m_b$.

soft function contains both double and single logarithms and it is the source of leading large logarithms in the amplitude T_3. It is therefore important to consistently resum these contributions. Lastly we want to point out that the soft function renormalization factor Z_S that we have derived based on a conjecure has been recently confirmed in an independent calculation in [99].

4.3 Operator renormalization

We start with the bare operator $O_1^{(0)}$ defined in (3.3). This operator is renormalized multiplicatively such that

$$O_1(\mu) = Z_{11} O_1^{(0)} \,. \tag{4.22}$$

Its renormalization is fully defined by the b-quark mass renormalization to all orders in QCD since no higher order corrections can emerge from the photon final states. In other words $Z_{11} = Z_m^{-1}$, where Z_m is known to five-loop in QCD [118]. At lowest order we have

$$Z_{11} = Z_m^{-1} = 1 + \frac{3}{\epsilon}\frac{C_F \alpha_s}{4\pi} + \mathcal{O}\left(\alpha_s^2\right) \,. \tag{4.23}$$

In the case of $O_2^{(0)}$ defined in (3.4) there is a mixing under renormalization with the operator $O_1^{(0)}$. The renormalization condition in this case contains a diagonal part with renormalization factor $Z_{22}(z, z')$ that has a non-local dependence on the variable z and a non-diagonal part $Z_{21}(z)$ local in z

$$O_2(z, \mu) = \int_0^1 dz' Z_{22}\left(z, z'\right) O_2^{(0)}\left(z'\right) + Z_{21}(z) O_1^{(0)} \,. \tag{4.24}$$

The $Z_{22}(z, z')$ is derived from the operator $O_2^{(0)}$, by isolating its UV divergent part. The Feynman diagrams that contribute at one-loop in QCD are shown in Fig.(4.2). This result should be combined with the wave-function renormalization and finally we have [2]

$$\begin{aligned} Z_{22}\left(z, z'\right) = {} & \left(1 + \frac{C_F \alpha_s}{4\pi\epsilon}\right) \delta\left(z - z'\right) \\ & - \frac{C_F \alpha_s}{2\pi\epsilon} \frac{1}{z'\left(1 - z'\right)} \left[z\left(1 - z'\right) \frac{\theta\left(z' - z\right)}{z' - z} + z'(1 - z) \frac{\theta\left(z - z'\right)}{z - z'} \right]_+ + \mathcal{O}\left(\alpha_s^2\right) \,, \end{aligned} \tag{4.25}$$

where $z, z' \in [0, 1]$ and $[...]_+$ in the second line is a plus distribution that acts on functions on the right the usual way. The origin of the non-local terms in (4.25) results from emissions of gauge bosons from the collinear Wilson lines that are then absorbed by the b-quark propagators. This expression is in fact the well-known "Brodsky-Lepage" kernel that enters in the renormalization of the leading-twist light cone distribution amplitude in B-mesons [120, 121]. We can integrate this result with the expression of the one-loop bare operator $O_2^{(0)}$ presented in Appendix A (renormalizing also the bare parameters α_0, $\alpha_{s,0}$ and $m_{b,0}$) and require that the renaming poles must be cancelled by the off diagonal term $Z_{21}(z) O_1^{(0)}$ in order for the operator $O_2(z, \mu)$ to be free of $1/\epsilon$ poles. The bare operator $O_1^{(0)}$ is given by the bare b-quark mass. In the minimal

Figure 4.2: Feynman diagrams that contribute to the one-loop expression of the diagonal renormalization factor Z_{22}.

subtraction scheme we find that

$$\begin{aligned} Z_{21}(z) =& \frac{N_c\alpha_b}{2\pi}\left\{-\frac{1}{\epsilon}+\frac{C_F\alpha_s}{4\pi}\left[\frac{\ln z+\ln(1-z)}{\epsilon^2}\right.\right. \\ &\left.\left.-\frac{1}{\epsilon}\left(\frac{\ln^2 z+\ln^2(1-z)}{2}-2\ln z\ln(1-z)+\frac{11}{2}-\frac{\pi^2}{3}\right)\right]+\mathcal{O}\left(\alpha_s^2\right)\right\}. \end{aligned} \tag{4.26}$$

It is now easy to derive the renormalization for the matrix element $[\![\langle O_2^{(0)}(z)\rangle]\!]$ from the renormalization of the operator $O_2^{(0)}$ by taking $z\to 0$ in $Z_{22}(z,z')$ and $Z_{21}(z)$. In this case the upper boundary on the integral runs to infinity. This is due to the fact that now z and z' are treated as infinitesimally small variables. Then we have

$$[\![\langle O_2(z,\mu)\rangle]\!]=\int_0^\infty dz'[\![Z_{22}\left(z,z'\right)]\!][\![\langle O_2^{(0)}\left(z'\right)\rangle]\!]+[\![Z_{21}(z)]\!]\langle O_1^{(0)}\rangle\,, \tag{4.27}$$

where

$$\begin{aligned} [\![Z_{22}\left(z,z'\right)]\!] &= \left[1-\frac{C_F\alpha_s}{4\pi\epsilon}(2\ln z+3)\right]\delta\left(z-z'\right)-\frac{C_F\alpha_s}{2\pi\epsilon}z\left[\frac{\theta\left(z'-z\right)}{z'\left(z'-z\right)}+\frac{\theta\left(z-z'\right)}{z\left(z-z'\right)}\right]_+ +\mathcal{O}\left(\alpha_s^2\right)\,, \\ [\![Z_{21}\left(z\right)]\!] =& \frac{N_c\alpha_b}{2\pi}\left\{-\frac{1}{\epsilon}+\frac{C_F\alpha_s}{4\pi}\left[\frac{\ln z}{\epsilon^2}-\frac{1}{\epsilon}\left(\frac{\ln^2 z}{2}+\frac{11}{2}-\frac{\pi^2}{3}\right)\right]+\mathcal{O}\left(\alpha_s^2\right)\right\}. \end{aligned} \tag{4.28}$$

We have checked that the renormalization factors in (4.25), (4.26) and(4.28) remove all the $1/\epsilon^n$ poles at one loop.

The renormalization factor of the operator $O_3^{(0)}$ follows from the renormalization of the current $h\bar{\chi}_{n_1}\chi_{n_2}$ in (3.5). The time ordered product in this definition contains also two subleading power Lagrangians but as it is known the SCET Lagrangian is non-renormalizable at all orders [82]. For this operator there is no mixing due to the presence of cutoffs in the convolution T_3 in (3.11). As mentioned before these cutoffs break the homogeneous power counting, preventing in this way mixing of O_3 with other operators at the same power. The renormalization of the current has been calculated in [122] and at leading order in α_s it reads

$$Z_{33}=1+\frac{C_F\alpha_s}{4\pi}\left[-\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\left(L_h-\frac{3}{2}\right)\right]+\mathcal{O}\left(\alpha_s^2\right)\,. \tag{4.29}$$

with $L_h = \ln(-M_h^2/\mu^2)$. Since the operator $O_3^{(0)}$ can be written as a convolution of two jet functions and the soft function as shown in (3.7) its renormalization factor Z_{33} is also related to the renormalization factor of the soft and jet function by

$$Z_S(w,w') = \frac{w}{w'} Z_{33} \int_0^\infty \frac{dx}{x} Z_J^{-1}\left(\frac{M_h w'}{x\ell_+}, \frac{M_h w}{\ell_+}\right) Z_J^{-1}\left(-xM_h\ell_+, -M_h\ell_+\right) . \tag{4.30}$$

where at the end the ℓ_+ dependence cancels out in the above expression. Equation (4.30) is just another way of expressing the Z_S, Z_J and Z_{33} relation to each other to what we have already seen in the soft quark renormalization in (4.17).

4.3.1 Renormalized matrix elements

At all orders the renormalized matrix element of the operator $O_1(\mu)$ is given by the renormalized b-quark mass

$$\langle O_1(\mu)\rangle = m_b(\mu) g_\perp^{\mu\nu} . \tag{4.31}$$

To derive the one-loop renormalized matrix element $\langle O_2(z,\mu)\rangle$ we apply equation (4.24) with the one-loop bare operators in this equation and the corresponding renormalization factors from the previous section. We find

$$\begin{aligned}\langle O_2(z,\mu)\rangle =& \frac{N_c\alpha_b}{2\pi} m_b(\mu) g_\perp^{\mu\nu} \Bigg\{ -L_m + \frac{C_F\alpha_s}{4\pi}\Bigg[L_m^2\Big(\ln z + \ln(1-z) + 3\Big) \\ & - L_m\left(\ln^2 z + \ln^2(1-z) - 4\ln z\ln(1-z) + 11 - \frac{2\pi^2}{3}\right) \\ & + F(z) + F(1-z)\Bigg] + \mathcal{O}\left(\alpha_s^2\right) \Bigg\} ,\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}F(z) =& \frac{\ln^3 z}{6} + z\ln^2 z - \ln^2 z\ln(1-z) - \ln z\ln(1-z) - \frac{1+3z}{2}\ln z \\ & - (4\ln z + 2z)\,\mathrm{Li}_2(z) + 6\,\mathrm{Li}_3(z) + \frac{11}{2} - 4\zeta_3 .\end{aligned} \tag{4.33}$$

We note here that in order for the $1/\epsilon^n$ poles to cancel properly one needs to keep $\mathcal{O}(\epsilon)$ terms in the bare one-loop expressions when they are multiplied with the factorization factor $Z_{21}(z)$. This is because already at leading order this expression has a $1/\epsilon$ pole as shown in (4.26). If we have a look at the structure of the logarithms in (4.32) we see that for a large hierarchy between the renormalization scale μ and m_b the operator O_2 will develop large logarithms already at $\mathcal{O}(\alpha_s^0)$. The renormalized expression for $[\![\langle O_2^{(0)}(z)\rangle]\!]$ can be readily derived from taking the $z \to 0$ limit in the above result. We then have

$$
\begin{aligned}
[\![\langle O_2^{(0)}(z)\rangle]\!] =& \frac{N_c\alpha_b}{2\pi} m_b(\mu) g_\perp^{\mu\nu} \Bigg\{ -L_m + \frac{C_F\alpha_s}{4\pi}\left[L_m^2(\ln z + 3) - L_m\left(\ln^2 z + 11 - \frac{2\pi^2}{3}\right)\right. \\
& \left. + \frac{\ln^3 z}{6} - \frac{\ln z}{2} + 11 - \frac{\pi^2}{3} - 2\zeta_3\right] + \mathcal{O}\left(\alpha_s^2\right)\Bigg\} .
\end{aligned}
\tag{4.34}
$$

The same result can be obtained by directly applying the renormalization condition for $[\![\langle O_2^{(0)}(z)\rangle]\!]$ in (4.27) together with the renormalization factors in (4.28).

To derive the renormalized matrix elements $\langle O_3(z,\mu)\rangle$ we need the renormalized one-loop expressions for the soft function and the jet functions. We have already derived the renormalized soft function in Section 4.2.1. If we factor out the running b-quark mass in the expression derived there one gets a relatively simpler result for the soft function

$$
S(w,\mu) = -\frac{N_c\alpha_b}{\pi} m_b(\mu)\Big[S_a(w,\mu)\theta\left(w - m_b^2\right) + S_b(w,\mu)\theta\left(m_b^2 - w\right)\Big] , \tag{4.35}
$$

where

$$
\begin{aligned}
S_a(w,\mu) =& 1 + \frac{C_F\alpha_s}{4\pi}\left[-L_w^2 - 6L_w + 12 - \frac{\pi^2}{2} + 2\mathrm{Li}_2\left(\frac{1}{\hat{w}}\right)\right. \\
& \left. -4\ln\left(1-\frac{1}{\hat{w}}\right)\left(L_m + 1 + \ln\left(1-\frac{1}{\hat{w}}\right) + \frac{3}{2}\ln\hat{w}\right)\right] + \mathcal{O}\left(\alpha_s^2\right) , \\
S_b(w,\mu) =& \frac{C_F\alpha_s}{\pi}\ln(1-\hat{w})\Big[L_m + \ln(1-\hat{w})\Big] + \mathcal{O}\left(\alpha_s^2\right) .
\end{aligned}
\tag{4.36}
$$

Note that here we have adopted a prefactor of $(N_c\alpha_b/\pi)$ in front of the soft function that does not appear in our original discussion on the soft function in Section 3.2. This factor multiplies each of the three amplitudes T_1, T_2 and T_3 and one can choose to absorb it, though for completeness we recover it here for the soft function. The one-loop expression for the renormalized jet function reads [101, 102]

$$
J\left(p^2,\mu\right) = 1 + \frac{C_F\alpha_s}{4\pi}\left[\ln^2\left(\frac{-p^2 - i0}{\mu^2}\right) - 1 - \frac{\pi^2}{6}\right] + \mathcal{O}\left(\alpha_s^2\right) . \tag{4.37}
$$

We can then combine these results in the following double convolution from the amplitude T_3

$$
\lim_{\sigma\to -1}\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+} J\left(M_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right) , \tag{4.38}
$$

where the limit $\sigma \to -1$ should be taken after integration. We obtain the following expression for the renormalized matrix element $\langle O_3(\mu)\rangle$

$$
\begin{aligned}
\langle O_3(\mu)\rangle =& -\frac{N_c\alpha_b}{\pi} m_b(\mu) g_\perp^{\mu\nu}\Bigg\{\frac{L^2}{2} + \frac{C_F\alpha_s}{4\pi}\left[\frac{5}{12}L^4 - L^3 + \left(5 - \frac{5\pi^2}{12}\right)L^2 + \left(\frac{2\pi^2}{3} + 8\zeta_3\right)L\right. \\
& \left. -4\zeta_3 - \frac{\pi^4}{9} + \frac{1}{2}L_m^2 L^2 + L_m\left(L^3 - 3L^2 - 8\zeta_3\right)\right] + \mathcal{O}\left(\alpha_s^2\right)\Bigg\} ,
\end{aligned}
\tag{4.39}
$$

where $L = \ln(-M_h^2/m_b^2)$. These are large rapidity logarithms that appear due to the double integration over ℓ_+ and ℓ_- ($\ell_+\ell_- \sim m_b^2$) in (4.38) in the presence of cutoffs. There are up to two powers of such logarithms generated at each loop order, which need to be resummed in the final result. This resummation should be derived from RG evolution of single scale dependence functions in order for it to be consistent. This treatment requires derivation and solution of RG evolution equations for both the soft and jet functions and the Wilson coefficient $H_3(\mu)$.

4.4 Renormalized Wilson coefficients

In the operator basis we are working for the $h \to \gamma\gamma$ decay, the renormalization matrix for the Wilson coefficients is the inverse of the renormalization matrix for the corresponding operators such that [2]

$$\boldsymbol{Z}^{-1} = \begin{pmatrix} Z_{11}^{-1} & 0 & 0 & 0 \\ Z_{21}^{-1} & Z_{22}^{-1} & 0 & 0 \\ [\![Z_{21}^{-1}]\!] & 0 & [\![Z_{22}^{-1}]\!] & 0 \\ 0 & 0 & 0 & Z_{33}^{-1} \end{pmatrix} . \tag{4.40}$$

There is one subtlety here related to the renormalization of the hard coefficient H_1. As we have indicated before this term in the renormalized amplitude requires a more careful consideration and we discuss it in more details in Chapter 6. For the remaining Wilson coefficients we have

$$\begin{aligned} H_2(z,\mu) &= \int_0^1 dz' H_2^{(0)}(z')\, Z_{22}^{-1}(z',z)\,, \\ \frac{[\![\bar{H}_2(z,\mu)]\!]}{z} &= \int_0^\infty dz' \frac{[\![\bar{H}_2^{(0)}(z')]\!]}{z'} [\![Z_{22}^{-1}(z',z)]\!], \\ H_3(\mu) &= H_3^{(0)} Z_{33}^{-1}\,. \end{aligned} \tag{4.41}$$

The renormalization factors $Z_{22}^{-1}(z,z')$, $[\![Z_{22}^{-1}(z,z')]\!]$ and Z_{33}^{-1} can be derived from the renormalization factors $Z_{22}(z,z')$, $[\![Z_{22}(z,z')]\!]$ and Z_{33} simply by changing the sign in front of the terms proportional to α_s. This is not the case for the off-diagonal terms $Z_{21}^{-1}(z)$ and $[\![Z_{21}^{-1}(z)]\!]$. They should be carefully derived from the renormalization conditions of the operator O_2 and this result reads

$$\begin{aligned} Z_{21}^{-1}(z) &= -\int_0^1 dz' Z_{22}^{-1}(z,z')\, Z_{21}(z')\, Z_{11}^{-1}, \\ [\![Z_{21}^{-1}(z)]\!] &= -\int_0^\infty dz' [\![Z_{22}^{-1}(z,z')]\!][\![Z_{21}(z')]\!] Z_{11}^{-1}. \end{aligned} \tag{4.42}$$

From these results we can now derive the expressions for the renormalized Wilson coefficients at one-loop

$$
\begin{aligned}
H_2(z,\mu) &= \frac{y_b(\mu)}{\sqrt{2}}\frac{1}{z(1-z)}\left\{1+\frac{C_F\alpha_s}{4\pi}\left[2L_h\Big(\ln z+\ln(1-z)\Big)+\ln^2 z+\ln^2(1-z)-3\right]+\mathcal{O}\left(\alpha_s^2\right)\right\}, \\
[\![\bar{H}_2(z,\mu)]\!] &= \frac{y_b(\mu)}{\sqrt{2}}\left[1+\frac{C_F\alpha_s}{4\pi}\Big(2L_h\ln z+\ln^2 z-3\Big)+\mathcal{O}\left(\alpha_s^2\right)\right], \\
H_3(\mu) &= \frac{y_b(\mu)}{\sqrt{2}}\left[-1+\frac{C_F\alpha_s}{4\pi}\left(L_h^2+2-\frac{\pi^2}{6}\right)+\mathcal{O}\left(\alpha_s^2\right)\right].
\end{aligned}
\tag{4.43}
$$

To derive these expressions we have used the bare one-loop results for the bare Wilson coefficients we show in Appendix A.

4.5 Renormalization group evolution

For the matrix elements $\langle O_1(\mu)\rangle$, $\langle O_2(z,\mu)\rangle$, $[\![\langle O_2(z,\mu)\rangle]\!]$ and the Wilson coefficients $H_2(z,\mu)$, $H_3(\mu)$ the evolution equations can be derived from the corresponding renormalization conditions we have derived above. The respective anomalous dimensions can be derived from the renormalization constants by the usual relation

$$
\gamma_{ij} = \left(2\alpha_b\frac{\partial}{\partial\alpha_b}+2\alpha_s\frac{\partial}{\partial\alpha_s}\right)Z_{ij}^{[1]}\,, \tag{4.44}
$$

where $Z_{ij}^{[1]}$ denotes the single pole $1/\epsilon$ coefficient in the anomalous dimension Z_{ij}. For the diagonal terms in the anomalous dimension matrix we find

$$
\begin{aligned}
\gamma_{11} &= \frac{3C_F\alpha_s}{2\pi}+\mathcal{O}\left(\alpha_s^2\right), \\
\gamma_{22}\left(z,z'\right) &= -\frac{C_F\alpha_s}{\pi}\left\{\left[\ln z+\ln(1-z)+\frac{3}{2}\right]\delta\left(z-z'\right)\right. \\
&\qquad \left. +z(1-z)\left[\frac{1}{z'(1-z)}\frac{\theta\left(z'-z\right)}{z'-z}+\frac{1}{z\left(1-z'\right)}\frac{\theta\left(z-z'\right)}{z-z'}\right]_+\right\}+\mathcal{O}\left(\alpha_s^2\right), \\
[\![\gamma_{22}\left(z,z'\right)]\!] &= -\frac{C_F\alpha_s}{\pi}\left\{\left(\ln z+\frac{3}{2}\right)\delta\left(z-z'\right)+z\left[\frac{\theta\left(z'-z\right)}{z'\left(z'-z\right)}+\frac{\theta\left(z-z'\right)}{z\left(z-z'\right)}\right]_+\right\}+\mathcal{O}\left(\alpha_s^2\right), \\
\gamma_{33} &= \frac{C_F\alpha_s}{\pi}\left(L_h-\frac{3}{2}\right)+\mathcal{O}\left(\alpha_s^2\right).
\end{aligned}
\tag{4.45}
$$

The off-diagonal contributions are simpler and do not exhibit a non-local behaviour

$$\begin{aligned} \gamma_{21}(z) &= -\frac{N_c \alpha_b}{\pi}\left\{1+\frac{C_F \alpha_s}{4\pi}\left[\ln^2 z+\ln^2(1-z)-4\ln z\ln(1-z)+11-\frac{2\pi^2}{3}\right]+\mathcal{O}\left(\alpha_s^2\right)\right\}, \\ [\![\gamma_{21}(z)]\!] &= -\frac{N_c \alpha_b}{\pi}\left\{1+\frac{C_F \alpha_s}{4\pi}\left(\ln^2 z+11-\frac{2\pi^2}{3}\right)+\mathcal{O}\left(\alpha_s^2\right)\right\}. \end{aligned} \tag{4.46}$$

The anomalous dimension γ_{33} is given to all orders by [70]

$$\gamma_{33}=\Gamma_{\text{cusp}}\left(\alpha_s\right)\ln\frac{-M_h^2}{\mu^2}+2\gamma_q\left(\alpha_s\right), \tag{4.47}$$

where Γ_{cusp} is the light-like cusp anomalous dimension [104] in the fundamental representation of $SU(N_c)$ and γ_q is the quark field anomalous dimension given in light cone gauge [122]. The relation between the renormalization factors of the Wilson coefficient H_3, jet function and the soft function in (4.30) implies the following relation among various anomalous dimensions

$$\gamma_{33}=\gamma_J\left(\frac{M_h w}{\ell_+},x\frac{M_h w}{\ell_+}\right)+\gamma_J\left(-M_h\ell_+,-xM_h\ell_+\right)+\gamma_S(w,w/x), \tag{4.48}$$

where γ_J is given in (4.51) and γ_S is the soft quark anomalous dimension we derive explicitly in (4.56).

4.5.1 Renormalization group equations

The operator matrix elements obey the following RG evolution equations:

$$\begin{aligned} \frac{d}{d\ln\mu}\left\langle O_1(\mu)\right\rangle &= -\gamma_{11}\left\langle O_1(\mu)\right\rangle, \\ \frac{d}{d\ln\mu}\left\langle O_2(z,\mu)\right\rangle &= -\int_0^1 dz'\gamma_{22}\left(z,z'\right)\left\langle O_2\left(z',\mu\right)\right\rangle-\gamma_{21}(z)\left\langle O_1(\mu)\right\rangle, \\ \frac{d}{d\ln\mu}[\![\langle O_2(z,\mu)\rangle]\!] &= -\int_0^\infty dz'[\![\gamma_{22}\left(z,z'\right)]\!][\![\langle O_2\left(z',\mu\right)\rangle]\!]-[\![\gamma_{21}(z)]\!]\left\langle O_1(\mu)\right\rangle, \end{aligned} \tag{4.49}$$

where $\gamma_{11}=-\gamma_m$ to all orders. The two-loop RG evolution for the jet function in the double convolution in (4.9) have been recently derived in [102] using results from the two-loop anomalous dimension of the light cone distribution amplitude (LCDA) of the B-meson

$$\frac{d}{d\ln\mu}J\left(p^2,\mu\right)=-\int_0^\infty dx\gamma_J\left(p^2,xp^2;\mu\right)J\left(xp^2,\mu\right), \tag{4.50}$$

where the two-loop anomalous dimension is

$$\begin{aligned} \gamma_J\left(p^2,xp^2;\mu\right) &= \left[\Gamma_{\text{cusp}}\left(\alpha_s\right)\ln\frac{-p^2}{\mu^2}-\gamma'\left(\alpha_s\right)\right]\delta(1-x)+\Gamma_{\text{cusp}}\left(\alpha_s\right)\Gamma(1,x) \\ &\quad +C_F\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\theta(1-x)}{1-x}h(x). \end{aligned} \tag{4.51}$$

Here $\Gamma(1,x)$ is a plus distribution defined in (4.14). The local term has a simpler structure given to all orders by the cusp anomalous dimension and an anomalous dimension $\gamma'(\alpha_s)$. The non-local term $\Gamma(1,x)$ is multiplied by $\Gamma_{\rm cusp}$ and it is also related to the logarithmic terms. The function $h(x)$ can be written in a simple form as

$$h(x) = \ln x \left[\beta_0 + 2C_F\left(\ln x - \frac{1+x}{x}\ln(1-x) - \frac{3}{2}\right)\right] . \tag{4.52}$$

It is now straightforward to write the RGEs for the hard coefficients $H_2(z,\mu)$, $[\![H_2(z,\mu)]\!]$ and $H_3(\mu)$

$$\begin{aligned} \frac{d}{d\ln\mu}H_2(z,\mu) &= \int_0^1 dz' H_2\left(z',\mu\right)\gamma_{22}\left(z',z\right) , \\ \frac{d}{d\ln\mu}[\![\bar{H}_2(z,\mu)]\!] &= \int_0^\infty dz' [\![\bar{H}_2\left(z',\mu\right)]\!] \frac{z}{z'} [\![\gamma_{22}\left(z',z\right)]\!] , \\ \frac{d}{d\ln\mu}H_3(\mu) &= \gamma_{33}H_3(\mu) . \end{aligned} \tag{4.53}$$

4.5.2 Two-loop RG equation for the soft function

The scale dependence of the renormalized soft-quark soft function is governed by the following RG evolution equation

$$\frac{d}{d\ln\mu}S(w,\mu) = -\int_0^\infty dw' \gamma_S\left(w,w';\mu\right) S\left(w',\mu\right) . \tag{4.54}$$

From the one-loop expression of Z_S in (4.19) we can read its one-loop anomalous dimension

$$\gamma_S\left(w,w';\mu\right) = -\frac{C_F\alpha_s}{4\pi}\left[\left(4\ln\frac{w}{\mu^2}+6\right)\delta\left(w-w'\right) + 8w\Gamma\left(w,w'\right)\right] + \mathcal{O}\left(\alpha_s^2\right) . \tag{4.55}$$

At one-loop order we have proven that this anomalous dimension together with the renormalized soft function presented in (4.21) satisfy the evolution equation in (4.54). For academic reasons it is though beneficial to derive and solve the RG evolution for the soft-quark soft function at two-loop order, namely the evolution with the two-loop anomalous dimension. We will actually see that discussing the above RGE beyond $\mathcal{O}(\alpha_s)$ has some interesting and non-trivial characteristics. To derive the two-loop anomalous dimension for the soft function without any ambiguity it requires the explicit two-loop expression for the renormalization factor Z_S, that cancels all the $1/\epsilon^n$ poles from the two-loop bare soft function as well. Extending the bare soft-quark soft function calculation to $\mathcal{O}(\alpha_s^2)$ is a highly non-trivial task and it is beyond the scope of this work, though for pedagogical reasons in here we can derive the explicit $\mathcal{O}(\alpha_s^2)$ terms for the anomalous dimension γ_S. This is feasible since the RG evolution for the jet function and the hard function that appear in T_3 are both known at two-loop order. In fact the anomalous dimension for the hard function is given to all orders by (4.47). Combining the evolution equations for $H_3(\mu)$ and jet function in (4.53) and (4.50) and their corresponding anomalous dimensions, along with the conjecture that T_3 is RG invariant we derive the two-loop anomalous dimension for the soft function

$$\begin{aligned}\gamma_S\left(w, w'; \mu\right) = & -\left[\Gamma_{\text{cusp}}\left(\alpha_s\right) \ln \frac{w}{\mu^2} - \gamma_s\left(\alpha_s\right)\right] \delta\left(w - w'\right) - 2\Gamma_{\text{cusp}}\left(\alpha_s\right) w\, \Gamma\left(w, w'\right) \\ & - 2C_F\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{w\, \theta\left(w' - w\right)}{w'\left(w' - w\right)} h\left(\frac{w}{w'}\right) + \mathcal{O}\left(\alpha_s^3\right) ,\end{aligned} \tag{4.56}$$

where we define

$$\gamma_s\left(\alpha_s\right) = 2\gamma_q\left(\alpha_s\right) + 2\gamma'\left(\alpha_s\right) . \tag{4.57}$$

The plus distribution $\Gamma(w, w')$ is defined the same as before

$$\Gamma\left(w, w'\right) = \left[\frac{\theta\left(w - w'\right)}{w\left(w - w'\right)} + \frac{\theta\left(w' - w\right)}{w'\left(w' - w\right)}\right]_+ . \tag{4.58}$$

We see that at $\mathcal{O}(\alpha_s^2)$, γ_S is non-local in w and it contains the function $h(x)$ defined in (4.52) that originates from the jet function anomalous dimensions. We collect in Appendix B the two-loop expressions for the various anomalous dimensions γ', γ_q and γ_s.

Chapter 5

Scale evolution of the soft function

In this Chapter we discuss in details the scale evolution of the soft-quark soft function that enters the factorization of the $h \to \gamma\gamma$ decay, mediated by a b-quark loop. The renormalized soft function in (4.21) is the source of the leading large double logarithms in the amplitude T_3 and it is important that these large logarithms are resummed consistently. Providing a solution to the RG evolution equation in (4.54) naturally tackles this problem. In addition this solution is theoretically interesting for SCET since soft functions originating from soft fermion emissions are generic features at subleading power SCET. In here we present an exact solution to the soft function RGE in momentum space, in Laplace space and in the so-called "diagonal" space, where we find the evolution becomes local. By the end of this Chapter in Section 5.6 we present a short discussion on the RG invariance of the double convolution T_3.

5.1 Exact solution to the RG equation

As we have seen before the soft-quark soft function obeys an integro-differential equation

$$\frac{d}{d\ln\mu} S(w,\mu) = -\int_0^\infty dw' \gamma_S\,(w,w';\mu)\, S\,(w',\mu)\ . \tag{5.1}$$

The main complications in finding an exact solution to this RGE arise from the presence of the non-local terms in the anomalous dimension and in fact also from the presence of a factor of 2 in front of the plus distribution in (4.56). The treatment of the local terms is straightforward, while for the non-local part the easiest way would be to find a function of w that is an eigenfunction of the evolution kernel. If we look closer at the RG evolution and keep in mind that w is a dimension-full variable, all the integrals over w must preserve this dimensionality. For instance from the second term from (4.56) we would have

$$2\Gamma_{\text{cusp}}(\alpha)\int_0^\infty dw'\, w\left[\frac{\theta(w-w')}{w(w-w')}+\frac{\theta(w'-w)}{w'(w'-w)}\right]_+ S(w',\mu)\,, \tag{5.2}$$

which means that on dimensional grounds the solution of the above will have powers of w coming from the function $S(w',\mu)$ only. The same is true for the two-loop non local term.

In other words for a dimensionless ratio $(w'/w)^a$ integrated with the non-local parts of the anomalous dimension, the result will simply be a function of a by dimensional analysis. It follows that any power function is an eigenfunction of the symmetric kernel

$$\Gamma(w, w') = \left[\frac{\theta(w - w')}{w(w - w')} + \frac{\theta(w' - w)}{w'(w' - w)}\right]_+ . \tag{5.3}$$

In literature (5.3) is known as the "Lange-Neubert" kernel and this important observation was first noticed in [108] and used to solve the RG evolution for the B-meson LCDA. The result from the integration of the non-local terms with the ratio (w'/w) yields

$$\int_0^\infty dw'\, w\Gamma(w, w') \left(\frac{w'}{w}\right)^a = -\left[H(a) + H(-a)\right] \equiv \mathcal{F}(a)\,, \tag{5.4}$$

where $H(a)$ is the harmonic number and it can be written in terms of the digamma function ψ and the Euler gamma

$$H(a) = \psi(1 + a) + \gamma_E\,. \tag{5.5}$$

Similarly for the two-loop non-local term

$$\frac{1}{\beta_0}\int_0^\infty dw'\, \frac{w\,\theta(w' - w)}{w'(w' - w)} h\left(\frac{w}{w'}\right)\left(\frac{w'}{w}\right)^a \equiv \mathcal{H}(-a)\,, \tag{5.6}$$

where [1]

$$\begin{aligned}\mathcal{H}(a) =& \frac{1}{\beta_0}\int_0^1 \frac{dx}{1-x} h(x) x^a \\ =& \left(\frac{3C_F}{\beta_0} - 1\right)\psi'(1 + a) + \frac{2C_F}{\beta_0}\left[\frac{\psi'(1 + a)}{a} - \left(\frac{1}{a^2} + 2\psi'(1 + a)\right) H(a)\right] .\end{aligned} \tag{5.7}$$

With this observation it then follows that the solution of the RG evolution equation for the soft function is given by

$$\begin{aligned}&\left(\frac{w}{\mu_s^2}\right)^{\eta - a_\Gamma(\mu_s,\mu)} \exp\left[2\,\mathcal{S}(\mu_s, \mu) + a_{\gamma_s}(\mu_s, \mu)\right] \exp\left[2\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} \mathcal{F}(\eta - a_\Gamma(\mu_s, \mu_\alpha))\right] \\ &\times \exp\left[\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\left[2C_F\left(\frac{\alpha}{2\pi}\right)^2 \beta_0 \mathcal{H}(a_\Gamma(\mu_s, \mu_\alpha) - \eta) + \mathcal{O}(\alpha^3)\right]\right] ,\end{aligned} \tag{5.8}$$

where $\alpha_s(\mu_\alpha) \equiv \alpha$, $\beta(\alpha_s) = d\alpha_s/d\ln\mu$ is the QCD $\beta-$ function and (w/μ_s^2) is the initial condition at the soft scale μ_s. The fixing of the initial condition is not unique and we have conveniently chosen it to have a power law, though for consistency the scale μ_s should be fixed such that the renormalized soft function must be free of large logarithms at μ_s. In here we have defined the Sudakov factor $\mathcal{S}(\mu_s, \mu)$ and the function $a_\Gamma(\mu_s, \mu)$ as

$$\begin{aligned}\mathcal{S}(\mu_s, \mu) &= -\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\mu_s)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}\,, \\ a_\Gamma(\mu_s, \mu) &= -\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\,.\end{aligned} \tag{5.9}$$

The $a_{\gamma_s}(\mu_s, \mu)$ is defined similarly to $a_\Gamma(\mu_s, \mu)$ with Γ_{cusp} substituted by γ_s. Notice that $a_\Gamma(\mu_s, \mu) = -a_\Gamma(\mu, \mu_s)$ though in general $\mathcal{S}(\mu_s, \mu) \neq -\mathcal{S}(\mu, \mu_s)$. Both $\mathcal{S}(\mu_s, \mu)$ and $a_\Gamma(\mu_s, \mu)$ take negative values if the running scale μ is higher than the soft scale μ_s and $\mathcal{S}(\mu_s, \mu_s) = a_\Gamma(\mu_s, \mu_s) = 0$. We present in Appendix B the explicit expressions for these functions at next-to leading order (NLO) in RG-improved perturbation theory. It is not difficult to check that (5.8) satisfies the RG evolution in (5.1) with the two-loop anomalous dimension in (4.56). The shift in the exponent of (w/μ_s^2) and in the argument of $\mathcal{F}$ and $\mathcal{H}$ by $a_\Gamma(\mu_s, \mu)$ and $a_\Gamma(\mu_s, \mu_\alpha)$ is necessary for (5.8) to provide a solution to the RG equation. It is indeed in the nature of the anomalous dimensions that they changes the scaling dimension of the correlation functions due to the running effects. To derive a closed form solution we perform the integral in the second exponent in (5.8) by changing variables to $a_\Gamma(\mu_s, \mu)$ and keeping only leading terms in α_s. We find

$$\exp\left[2\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} \mathcal{F}\left(\eta - a_\Gamma(\mu_s, \mu_\alpha)\right)\right] = e^{4a_\Gamma \gamma_E} \frac{\Gamma^2(1-\eta+a_\Gamma)\Gamma^2(1+\eta)}{\Gamma^2(1+\eta-a_\Gamma)\Gamma^2(1-\eta)}, \tag{5.10}$$

where $a_\Gamma = a_\Gamma(\mu_s, \mu)$. The additional factor of 2 in the non-local term in the anomalous dimension has now manifested as powers of the gamma functions. It is then instructive to define the evolution function $U_S(w; \mu, \mu_s)$

$$U_S\left(w; \mu, \mu_s\right) = \left(\frac{w e^{-4\gamma_E}}{\mu_s^2}\right)^{-a_\Gamma(\mu_s, \mu)} \exp\left[2\mathcal{S}\left(\mu_s, \mu\right) + a_{\gamma_s}\left(\mu_s, \mu\right)\right], \tag{5.11}$$

where $U_S\left(w; \mu_s, \mu_s\right) = 1$. Then the exact closed form solution for the RG equation for the soft function can be written

$$\begin{aligned} &U_S\left(w; \mu, \mu_s\right) \left(\frac{w}{\mu_s^2}\right)^\eta \frac{\Gamma^2\left(1-\eta+a_\Gamma\left(\mu_s, \mu\right)\right)\Gamma^2(1+\eta)}{\Gamma^2\left(1+\eta-a_\Gamma\left(\mu_s, \mu\right)\right)\Gamma^2(1-\eta)} \\ &\times \exp\left[-C_F \int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} \frac{d\alpha}{\pi} \Big[\mathcal{H}\left(a_\Gamma\left(\mu_s, \mu_\alpha\right) - \eta\right) + \mathcal{O}(\alpha)\Big]\right]. \end{aligned} \tag{5.12}$$

To arrive to this expression we have kept only the leading order terms in the second line after writing $\beta\left(\alpha_s\right) = -\beta_0 \alpha_s^2/(2\pi) + \mathcal{O}(\alpha_s^3)$.

As we argued in the beginning of this section we need to cast this solution in terms of powers of (w/μ_s^2). To achieve this we need to use an integral transform for the function $S(w, \mu_s)$ and then find the scale evolution of that transform. A suitable choice in this case is to use the Laplace transform with respect to $\ln(w/m_b^2)$ defined as

$$\tilde{S}(\eta, \mu) = \int_0^\infty \frac{dw}{w} S(w, \mu) \left(\frac{w}{m_b^2}\right)^{-\eta}, \tag{5.13}$$

where the integral converges for $0 < \eta < 1$. Its inverse transform is then given by

$$S\left(w, \mu_s\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\eta \tilde{S}\left(\eta, \mu_s\right) \left(\frac{w}{m_b^2}\right)^\eta, \tag{5.14}$$

for $0 < c < 1$. The above expression has the convenient power law dependence and $(w/m_b^2)^\eta$ is an eigenfunction of the evolution kernel. We can then use the solution of the RG equation to express the evolution of the soft function to a new scale μ

$$\begin{aligned} S(w,\mu) =& U_S(w;\mu,\mu_s)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta \tilde{S}(\eta,\mu_s)\left(\frac{w}{m_b^2}\right)^\eta \frac{\Gamma^2(1-\eta+a_\Gamma)\Gamma^2(1+\eta)}{\Gamma^2(1+\eta-a_\Gamma)\Gamma^2(1-\eta)} \\ &\times \exp\left[-C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}\frac{d\alpha}{\pi}\mathcal{H}\left(a_\Gamma(\mu_s,\mu_\alpha)-\eta\right)+\mathcal{O}(\alpha_s^2)\right]. \end{aligned} \tag{5.15}$$

This expression relates the soft function in momentum space $S(w,\mu)$ at the scale μ to its Fourier transform $\tilde{S}(\eta,\mu_s)$ at the scale μ_s. To relate the function $S(w,\mu)$ to the soft function $S(w,\mu_s)$ in momentum space we can apply the integral transform in (5.13). Lastly together with the definition (5.7) we can write

$$\begin{aligned} S(w,\mu) =& U_S(w;\mu,\mu_s)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta\int_0^\infty \frac{dw'}{w'}S(w',\mu_s)\left(\frac{w'}{w}\right)^{-\eta} \frac{\Gamma^2(1-\eta+a_\Gamma)\Gamma^2(1+\eta)}{\Gamma^2(1+\eta-a_\Gamma)\Gamma^2(1-\eta)} \\ &\times\left[1-\frac{C_F}{\beta_0\pi}\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}d\alpha\int_0^1\frac{dx}{1-x}h(x)x^{a_\Gamma(\mu_s,\mu_\alpha)-\eta}+\mathcal{O}\left(\alpha_s^2\right)\right], \end{aligned} \tag{5.16}$$

where $0 < c < 1+\min(0,a_\Gamma(\mu_s,\mu))$. The $\eta-$ dependent integrand gives a non-vanishing result if the poles reside on opposite sides of integration contour. The poles are located at $\eta = n + a_\Gamma(\mu_s,\mu)$ and $\eta = -n$ for $n \in \mathbb{N}$ and this condition implies that $a_\Gamma(\mu_s,\mu) > -1$. This is in practice always satisfied by running the soft function from a large scale μ down to the soft scale μ_s for $n_f = 5$. With this condition, in principle it is possible to integrate over η, though the presence of the squared gamma functions makes it challenging to write the result in an analytic closed form. The contour integration gives a result in terms of hypergeometric function (see appendix C in [1]) though such a result is impractical, especially for later numerical evaluations. Instead we present the η integration in terms of the so called Meijer G-functions [109] where we find

$$\begin{aligned} S(w,\mu) = U_S(w;\mu,\mu_s)\int_0^\infty\frac{dw'}{w'}S(w',\mu_s)\Bigg[&G_{4,4}^{2,2}\left(\begin{matrix}-a,-a,1-a,1-a\\1,1,0,0\end{matrix}\,\middle|\,\frac{w'}{w}\right) \\ &-\frac{C_F}{\beta_0\pi}\int_0^1\frac{dx}{1-x}h(x)\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}d\alpha\,x^{a_\Gamma(\mu_s,\mu_\alpha)}G_{4,4}^{2,2}\left(\begin{matrix}-a,-a,1-a,1-a\\1,1,0,0\end{matrix}\,\middle|\,\frac{xw'}{w}\right)+\mathcal{O}(\alpha_s^2)\Bigg], \end{aligned} \tag{5.17}$$

where for simplicity we write $a = a_\Gamma(\mu_s,\mu)$ and $h(x)$ comes from the definition in (5.7). In (5.17) "a" must be smaller than zero for the solution to be valid, which again holds if the evolution scale is above the soft scale ($\mu > \mu_s$). The Meijer G-function has the following useful property that can be used to simplify numerical calculations

$$
\begin{aligned}
G^{2,2}_{4,4}\left(\begin{matrix} -a,-a,1-a,1-a \\ 1,1,0,0 \end{matrix}\,\middle|\,\frac{1}{z}\right) &= z^a G^{2,2}_{4,4}\left(\begin{matrix} -a,-a,1-a,1-a \\ 1,1,0,0 \end{matrix}\,\middle|\,z\right) \\
&= G^{2,2}_{4,4}\left(\begin{matrix} 0,0,1,1 \\ 1+a,1+a,a,a \end{matrix}\,\middle|\,z\right) .
\end{aligned} \tag{5.18}
$$

For $a < 0$ the Meijer G-function has an integrable singularity when its last argument approaches 1 and in this limit its asymptotic approximation is [1]

$$
\lim_{z\to 1} G^{2,2}_{4,4}\left(\begin{matrix} -a,-a,1-a,1-a \\ 1,1,0,0 \end{matrix}\,\middle|\,z\right) = -\frac{\sin 2\pi a}{\pi}\frac{\Gamma(1+4a)}{|1-z|^{1+4a}} + \mathcal{O}(1) . \tag{5.19}
$$

It is useful to further simplify the solution in (5.17) by substituting the explicit expression for $a_\Gamma(\mu_s,\mu)$ at leading order which is

$$
a_\Gamma\left(\mu_s,\mu_\alpha\right) = \frac{2C_F}{\beta_0}\ln\frac{\alpha}{\alpha_s\left(\mu_s\right)} + \mathcal{O}\left(\alpha_s\right) , \tag{5.20}
$$

with $\beta_0 = 23/3$ for α_s running with five flavours. Then the final result for the exact evolution of the soft-quark soft function from a low scale μ_s to a different scale μ reads

$$
\begin{aligned}
S(w,\mu) = U_S\left(w;\mu,\mu_s\right)\Bigg[&\int_0^\infty \frac{dx}{x} S\left(w/x,\mu_s\right) G^{2,2}_{4,4}\left(\begin{matrix} -a,-a,1-a,1-a \\ 1,1,0,0 \end{matrix}\,\middle|\,\frac{1}{x}\right) \\
&- m_b\frac{C_F\alpha_s\left(\mu_s\right)}{\pi}\int_0^1 \frac{dx}{1-x}\frac{h(x)}{\beta_0}\frac{r^{1+\frac{2C_F}{\beta_0}\ln x}-1}{1+\frac{2C_F}{\beta_0}\ln x} G^{2,2}_{4,4}\left(\begin{matrix} -a,-a,1-a,1-a \\ 0,1,0,0 \end{matrix}\,\middle|\,\frac{xm_b^2}{w}\right) + \mathcal{O}(\alpha_s^2)\Bigg] ,
\end{aligned} \tag{5.21}
$$

where we have defined $r = \alpha_s(\mu)/\alpha_s(\mu_s)$. This result presents a solution of the RG equation at NLO in RG-improved perturbation theory, which corresponds to providing a resummation of the large logarithms at next-to-next-to-leading logarithmic (NNLL) order. In other words the solution (5.21) resums large logarithms of the form $\alpha_s^n L^k$, with $2n-3 \le k \le 2n$ [110]. To do that one should use the one-loop expression for the soft function $S(w,\mu_s)$ at the matching scale, the two-loop expression for the anomalous dimension γ_s and the three-loop expression for the cusp anomalous dimension. This expressions can be found in Appendix B. For a leading order result one needs to consider only the tree level soft function, the one-loop and two-loop expressions for γ_s and Γ_{cusp} respectively. This corresponds to a solution at LO in RG-improved where logarithms with $2n-1 \le k \le 2n$ are resummed. In this case the integrations are much simpler and we find

$$
S_{\text{LO}}(w,\mu) = m_b U_S\left(w;\mu,\mu_s\right) G^{2,2}_{4,4}\left(\begin{matrix} -a,-a,1-a,1-a \\ 0,1,0,0 \end{matrix}\,\middle|\,\frac{m_b^2}{w}\right) , \tag{5.22}
$$

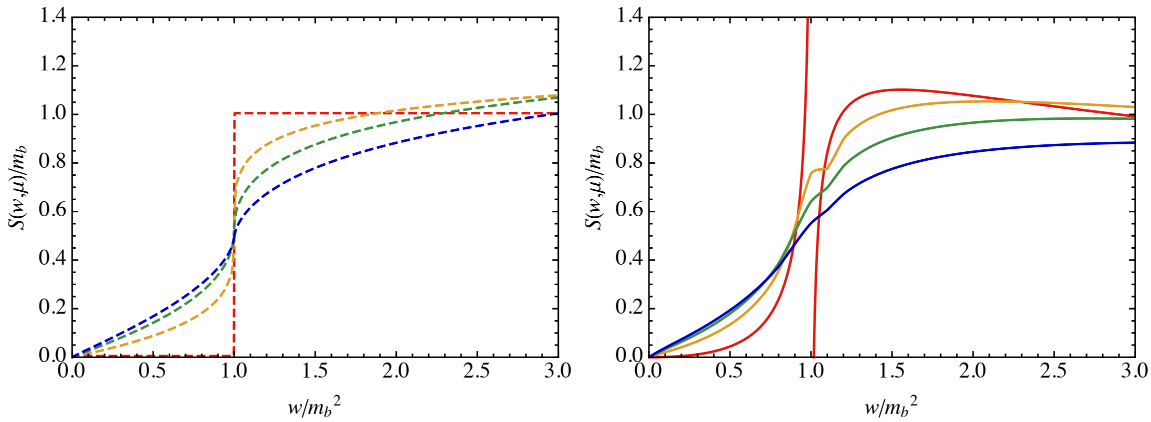

Figure 5.1: Renormalized soft function at LO (left-dashed line) and NLO (right-solid line) in rg-improved perturbation theory at $\mu = mb_b$ (red), μ = 10 GeV (orange), μ = 20 GeV (green), $\mu = 40GeV$ (blue). The initial scale $\mu_s = m_b$ in all these cases.

with $a = a_\Gamma(\mu_s, \mu)$. The Meijer G-function in (5.22) is obtained by integrating over the first Meijer G-function in (5.21) with the theta function from the leading order soft function. Beyond the leading order the numerical integration with the Meijer G-function becomes more difficult. We show in Fig.(5.1) the renormalized soft function $S(w, \mu)/m_b$ at different scales μ_s at LO and NLO in RG-improved perturbation theory. In both cases the discontinuity around b-quark mass is smoothed out by scale evolution. In these graphs it is also obvious that the soft function vanishes at the origin, which is due to the presence of the Meijer G-function.

5.2 Asymptotic behaviour and dynamical scale setting

In order for the soft function solution we have just derived, to be consistent it should be free of large logarithms at the initial scale μ_s. This means that the ratios that appear in the logarithms $\ln(w/\mu^2)$ and $\ln(m_b^2/\mu^2)$ in the fixed order solution in (4.21) must be of $\mathcal{O}(1)$ at $\mu = \mu_s$, namely $\mu_s \sim \sqrt{w} \sim m_b$. In practice it is useful to be able to extend the solution for values of $w \gg m_b^2$. Since there is no fixed scale constrain in the expression of the anomalous dimension for the soft function, because it does not depend on the m_b^2 we can formally generalize our solution also for larger values of w. Then from the expression of the renormalized soft function in (4.21), for $w > m_b^2$ in the limit $w/m_b^2 \to \infty$ we find

$$\begin{aligned} S_\infty(w,\mu) =& m_b\left[1+\frac{C_F\alpha_s}{4\pi}\left(-L_w^2-6L_w+3L_m+8-\frac{\pi^2}{2}\right)\right] \\ =& m_b(\mu)\left[1+\frac{C_F\alpha_s}{4\pi}\left(-L_w^2-6L_w+12-\frac{\pi^2}{2}\right)\right] \\ \equiv& m_b(\mu)\mathcal{S}_\infty(L_w,\mu)\,, \end{aligned} \tag{5.23}$$

where we have used equation (4.4) to substitute the pole mass with the running mass of the b-quark. This is necessary to avoid the pole in L_m for $m_b \to 0$ in the expression of m_b. Then in the fixed order calculation in (5.23) there are only terms of $L_w = \ln(w/\mu^2)$ that would generate large logarithms. The scale μ and w should be at the same order now and in principle do not need to be constrained by the b-quark mass. Similarly to before, since the anomalous dimension remains the same, even in this limit we have

$$\frac{d}{d\ln\mu} S_\infty(w,\mu) = -\int_0^\infty dw' \gamma_S(w,w',\mu) S_\infty(w',\mu)\,. \tag{5.24}$$

We could go though the same steps as for the previous soft function $S(w,\mu)$ but now the function S_∞ given at the initial scale has a much simpler form and it only depends on w via $\ln w/\mu^2$, which is indeed the same structure as in the case of the jet function in [102]. Therefore in a similar fashion we write the solution at the initial scale μ_s as a power law $\mathcal{S}_\infty(\partial_\eta,\mu_s)\left(\frac{w}{\mu_s}\right)^\eta$ and set $\eta = 0$. Then using also (5.12), the exact solution to the RG equation in (5.24) reads

$$\begin{aligned} S_\infty(w,\mu) =& m_b(\mu_s) U_S(w;\mu,\mu_s) \mathcal{S}_\infty(\partial_\eta,\mu_s) \left(\frac{w}{\mu_s}\right)^\eta \frac{\Gamma^2\left(1-\eta+a_\Gamma(\mu_s,\mu)\right)\Gamma^2(1+\eta)}{\Gamma^2\left(1+\eta-a_\Gamma(\mu_s,\mu)\right)\Gamma^2(1-\eta)} \\ &\times \exp\left[1 - C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} \frac{d\alpha}{\pi} \mathcal{H}\left(a_\Gamma(\mu_s,\mu_\alpha)-\eta\right) + \mathcal{O}(\alpha_s^2)\right]\Bigg|_{\eta=0}\,. \end{aligned} \tag{5.25}$$

This evolution resumms the large logarithms $\ln w/\mu^2$ at NLO in RG-improved perturbation theory. The explicit expression for the function $\mathcal{H}$ is shown in (5.7).

It is obvious that in the limit $w \gg m_b^2$ it is inconsistent to fix the scale μ_s to m_b as we did for the solution in (5.21). In this case the soft scale must be fixed *dynamically* such that $\mu_s^2 = r_s w$, where $r_s = \mathcal{O}(1)$. Then for certain values of w the scale μ can also be smaller than μ_s here. Note that this was not allowed in the previous solution in (5.21), where it is required to fix the running scale above the soft scale μ_s. Moreover that solution is consistent up to values of $w \sim 3m_b^2$ and breaks down for $w > 3m_b^2$ (see [1]). For arbitrary large values of w one should use the asymptotic solution in (5.25) with dynamical scale setting.

5.3 Soft function in Laplace space

It is also practical to derive the scale evolution of the Laplace transform $\tilde{S}(\eta,\mu)$. In fact the convolution between the soft and the jet functions takes a much simpler form in Laplace space. This is due to the fact that after RG evolution the jet function can be written in terms of a derivative operator acting on a power function $(p^2/\mu_j^2)^a$ [102, 113]. The solution of the RGE in this case takes a much simpler form. Starting from (5.15) and using the definition in (5.13) we get

$$\begin{aligned} \tilde{S}(\eta,\mu) =& U_S\left(m_b^2;\mu,\mu_s\right) \frac{\Gamma^2\left(1+\eta+a_\Gamma\left(\mu_s,\mu\right)\right)\Gamma^2(1-\eta)}{\Gamma^2\left(1-\eta-a_\Gamma\left(\mu_s,\mu\right)\right)\Gamma^2(1+\eta)} \\ &\times \exp\left[-C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} \frac{d\alpha}{\pi}\mathcal{H}\left(-\eta+a_\Gamma\left(\mu,\mu_\alpha\right)\right) + \mathcal{O}\left(\alpha_s^2\right)\right] \tilde{S}\left(\eta+a_\Gamma\left(\mu_s,\mu\right),\mu_s\right)\,. \end{aligned} \tag{5.26}$$

This is a much simpler solution that involves no integration over the soft function. Since the Laplace transform exists for $0 < \eta < 1$, due to the argument shift in the soft function here it follows that $-a_\Gamma(\mu_s, \mu) = |a_\Gamma(\mu_s, \mu)| < \eta < 1$. The RG evolution in Laplace space can be derived from the original RG evolution in (4.54) using the definition of the Laplace transform of $\tilde{S}(w, \mu)$ at the scale μ. It can then be shown that the expression in (5.26) is a solution of the following RG evolution equation

$$\left(\frac{d}{d\ln\mu} + \Gamma_{\text{cusp}}(\alpha_s)\frac{\partial}{\partial\eta}\right)\tilde{S}(\eta,\mu)$$
$$= \left[\Gamma_{\text{cusp}}(\alpha_s)\left(\ln\frac{m_b^2}{\mu^2} + 2\mathcal{F}(-\eta)\right) - \gamma_s(\alpha_s) + 2C_F\beta_0\left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{H}(-\eta) + \mathcal{O}\left(\alpha_s^3\right)\right]\tilde{S}(\eta,\mu). \tag{5.27}$$

The function $\mathcal{F}$ and $\mathcal{H}$ are the same as in (5.4) and (5.7).

From the momentum space solution in (4.21) we can also derive the fixed order soft function at the matching scale μ_s in Laplace space

$$\begin{aligned}\tilde{S}(\eta,\mu_s) =& \frac{m_b}{\eta}\Bigg\{1 + \frac{C_F\alpha_s(\mu_s)}{4\pi}\Bigg[-\frac{2}{\eta^2} - \frac{2}{\eta}(L_m+3) - L_m^2 - 3L_m + 8 - \frac{3\pi^2}{2}\\ &+ 4L_m\left[H(\eta) + H(-\eta)\right] + \frac{4(1+\eta)H(\eta)}{\eta}\\ &-4\left[H^2(\eta) + H^2(-\eta)\right] - 2\psi'(1+\eta) + 4\psi'(1-\eta)\Bigg]\Bigg\},\end{aligned} \tag{5.28}$$

where $H(a)$ is the harmonic-number function, ψ' is the first derivative of the digamma function and $L_m = \ln(m_b^2/\mu_s^2)$. This last result together with the derivation in (5.26) can be used to find the evolution of the soft function in Laplace space from the soft scale μ_s to another scale μ.

5.4 Soft function in diagonal space

We have seen that many of the complications in the evolution of the soft function result from the fact that it obeys a non-local RG evolution. Already in Laplace space we found that the solution of the RGE simplifies compared to momentum space solution. It is thus natural to pose the question if we can find another functional transform of the soft function, such that the evolution of the new function becomes local to all orders in perturbation theory. This would be similar to the approach found in literature for the B-meson where the evolution of the LCDA becomes local at one-loop in the so-called *dual space* [111]. In our case the soft function would evolve locally in the dual space only up to $\mathcal{O}(\alpha_s)$ because starting from $\mathcal{O}(\alpha_s^2)$

the anomalous dimension γ_S still contains a non-local term [1] such that

$$\begin{aligned}\gamma_S^{\text{dual}}\left(w,w';\mu\right) = &-\left[\Gamma_{\text{cusp}}\left(\alpha_s\right)\ln\frac{we^{-4\gamma_E}}{\mu^2}-\gamma_s\left(\alpha_s\right)\right]\delta\left(w-w'\right)\\ &-2C_F\left(\frac{\alpha_s}{2\pi}\right)^2\frac{w\theta\left(w'-w\right)}{w'\left(w'-w\right)}h\left(\frac{w}{w'}\right)+\mathcal{O}\left(\alpha_s^3\right)\,.\end{aligned}\tag{5.29}$$

Nevertheless we find that it is possible to diagonalize the anomalous dimension beyond $\mathcal{O}(\alpha_s)$ by building what we call the “diagonal space”. As we will show below in this space the evolution of the soft-quark soft function is local to all orders in coupling constant.

5.4.1 Construction of the diagonal space

We start with two simple observations. First from the definition of the function a_Γ in (5.9) for $\mu_s < \mu < \mu_\alpha$ we have $a_\Gamma\left(\mu_s,\mu\right)+a_\Gamma\left(\mu,\mu_\alpha\right)=a_\Gamma\left(\mu_s,\mu_\alpha\right)$. Second in the Laplace space solution in (5.26) we can split the integration boundaries in the higher order term and still have two well defined integrals in the exponent such that

$$\begin{aligned}&\exp\left[-C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta+a_\Gamma(\mu,\mu_\alpha))\right]\\ &=\exp\left[-C_F\left(\int_{\alpha_s(\mu_s)}^{\alpha_s(\rho)}\frac{d\alpha}{\pi}+\int_{\alpha_s(\rho)}^{\alpha_s(\mu)}\frac{d\alpha}{\pi}\right)\mathcal{H}(-\eta+a_\Gamma(\mu,\mu_\alpha)\right]\,,\end{aligned}\tag{5.30}$$

where the scale ρ can be any intermediate scale between μ_s and μ. From the above split of integration boundaries and looking at the expression in (5.26) it is useful to define the function

$$\tilde{s}(w,\mu,\rho,\eta)=\frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\exp\left[-C_F\int_{\alpha_s(\mu)}^{\alpha_s(\rho)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta+a_\Gamma(\mu,\mu_\alpha))+\mathcal{O}(\alpha_s^2)\right]\tilde{S}(\eta,\mu)\,.\tag{5.31}$$

We now use the solution in (5.26) to express the function $\tilde{S}(\eta,\mu)$ here at the initial scale such that

$$\begin{aligned}\tilde{s}(w,\mu,\rho,\eta)=U_S(m_b^2;\mu,\mu_s)&\frac{\Gamma^2(1+\eta+a_\Gamma)}{\Gamma^2(1-\eta-a_\Gamma)}\\ \times\exp&\left[-C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\rho)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta-a_\Gamma(\mu_s,\mu)+a_\Gamma(\mu_s,\mu_\alpha))+\mathcal{O}(\alpha_s^2)\right]\tilde{S}(\eta+a_\Gamma,\mu_s)\,,\end{aligned}\tag{5.32}$$

where $a_\Gamma=a_\Gamma(\mu_s,\mu)$ and we have used $a_\Gamma(\mu,\mu_\alpha)=a_\Gamma(\mu_s,\mu_\alpha)-a_\Gamma(\mu_s,\mu)$. It is possible to define the Laplace transform of the function $\tilde{s}(w,\mu,\rho,\eta)$ in the diagonal space similarly to what we have done before

$$s(w,\mu,\rho)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\eta\,\tilde{s}(w,\mu,\rho,\eta)\left(\frac{w}{m_b^2}\right)^\eta\,,\tag{5.33}$$

where $-a_\Gamma(\mu_s,\mu) < c < 1 + \min(0, a_\Gamma(\mu,\rho))$. Then using this last definition and performing a shift in η by $a_\Gamma(\mu_s,\mu)$ in (5.32) we find the following expression for the function $s(w,\mu,\rho)$ in diagonal space

$$\begin{aligned} s(w,\mu,\rho) =& U_S(w;\mu,\mu_s)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta \frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\left(\frac{w}{m_b^2}\right)^\eta \\ &\times \exp\left[-C_F\int_{\alpha_s(\mu_s)}^{\alpha_s(\rho)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta + a_\Gamma(\mu_s,\mu_\alpha)) + \mathcal{O}(\alpha_s^2)\right]\tilde{S}(\eta,\mu_s) \\ =& U_S(w;\mu,\mu_s)s(w,\mu_s,\rho)\,. \end{aligned} \tag{5.34}$$

Clearly the soft function $s(w,\mu,\rho)$ obeys a local RG evolution equation in the diagonal space.

Note that we have explicitly included here only the two-loop non local terms in the anomalous dimension, but this procedure can be easily extended for higher order correction to the exponent in (5.31). The diagonal space function at the scale μ also depends on the intermediate scale ρ. The scale ρ does not appear in physical quantities and it is a remnant of the fact that we can perform the integral split in (5.30) at any intermediate scale. This function obeys a well defined evolution equation with respect to ρ. At leading order we find

$$\frac{d}{d\ln\rho}s(w,\mu,\rho) = \left[2C_F\left(\frac{\alpha_s(\rho)}{2\pi}\right)^2\int_0^1\frac{dx}{1-x}h(x)x^{a_\Gamma(\mu,\rho)} + \mathcal{O}\left(\alpha_s^3\right)\right]s(w/x,\mu,\rho)\,. \tag{5.35}$$

There are no additional large logarithms generated by this scale splitting. All the logarithms are resummed in the exponent in the above expression.

5.4.2 Transformation between momentum and diagonal space

The expression in (5.34) relates the soft function in diagonal space with the Laplace transform of the soft function in the original space. Instead it is convenient to find a *transfer function* that relates the function $s(w,\mu,\rho)$ to the soft function in momentum space $S(w,\mu)$. Combining (5.34) with the definition in (5.13) it is not difficult to find this transformation. We have

$$\begin{aligned} s(w,\mu,\rho) =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta\frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\int_0^\infty\frac{dx}{x}S(wx,\mu)x^{-\eta} \\ &\times\exp\left[-C_F\int_{\alpha_s(\mu)}^{\alpha_s(\rho)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta - a_\Gamma(\mu_\alpha,\mu)) + \mathcal{O}(\alpha_s^2)\right] \\ \equiv& \int_0^\infty\frac{dx}{\sqrt{x}}F_S(x,\mu,\rho)S(xw,\mu)\,, \end{aligned} \tag{5.36}$$

where $x = w'/w$ and we define $F_S(x,\mu,\rho)$ such that

$$\sqrt{x}F_S(x,\mu,\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta\,\frac{\Gamma^2(1+\eta)}{\Gamma^2(1-\eta)}\,x^{-\eta}\exp\left[-C_F\int_{\alpha_s(\mu)}^{\alpha_s(\rho)}\frac{d\alpha}{\pi}\mathcal{H}(-\eta + a_\Gamma(\mu,\mu_\alpha)) + \mathcal{O}(\alpha_s^2)\right]. \tag{5.37}$$

The function F_S is called the transfer function of the soft function from momentum space to the diagonal space. In analogy we can also derive the inverse of such a transformation from the relation

$$S(w,\mu) = \int_0^\infty \frac{dx}{\sqrt{x}} F_S^{\rm inv}(x,\mu,\rho) s(w/x,\mu,\rho) \,. \tag{5.38}$$

Now we use the inverse Laplace transform $S(w,\mu)$ to derive

$$\sqrt{x} F_S^{\rm inv}(x,\mu,\rho) = \frac{1}{2\pi i}\int_{c-\infty}^{c+i\infty} d\eta \frac{\Gamma^2(1-\eta)}{\Gamma^2(1+\eta)} x^\eta \exp\left[C_F \int_{\alpha_s(\mu)}^{\alpha_s(\rho)} \frac{d\alpha}{\pi} \mathcal{H}\left(-\eta + a_\Gamma\left(\mu,\mu_\alpha\right)\right) + \mathcal{O}\left(\alpha_s^2\right)\right] . \tag{5.39}$$

In principle one can then derive an explicit expression for $F_S(x)$ for a given $\mathcal{H}$ at a given order in α_s. At next-to leading order both F_S and $F_S^{\rm inv}$ can be expanded as [1]

$$F_S(x,\mu,\rho) = F_{S,0}(x) + \frac{C_F \alpha_s(\mu)}{\pi} F_{S,1}(x,r) + \mathcal{O}\left(\alpha_s^2\right); \quad r = \frac{\alpha_s(\rho)}{\alpha_s(\mu)} \,, \tag{5.40}$$

where

$$\begin{aligned} F_{S,0}(x) =& F_{S,0}^{\rm inv}(x) = \frac{1}{\sqrt{x}} G_{0,4}^{2,0}\left({}_{1,1,0,0} \mid x\right) \,, \\ F_{S,1}(x,r) =& -\frac{1}{\sqrt{x}} \int_0^1 \frac{dy}{1-y} \frac{h(y)}{\beta_0} \frac{r^{1+\frac{2C_F}{\beta_0}\ln y} - 1}{1 + \frac{2C_F}{\beta_0}\ln y} G_{0,4}^{2,0}\left({}_{1,1,0,0} \mid xy\right) \,, \\ F_{S,1}^{\rm inv}(x,r) =& \frac{1}{\sqrt{x}} \int_0^1 \frac{dy}{1-y} \frac{h(y)}{\beta_0} \frac{r^{1+\frac{2C_F}{\beta_0}\ln y} - 1}{1 + \frac{2C_F}{\beta_0}\ln y} G_{0,4}^{2,0}\left({}_{1,1,0,0} \Big| \frac{x}{y}\right) . \end{aligned} \tag{5.41}$$

These transfer functions enjoy the useful property

$$\int_0^\infty dx F_S(xa,\mu,\rho) F_S^{\rm inv}(xb,\mu,\rho) = \delta(a-b) \,. \tag{5.42}$$

Moreover powers of the variable x have a simple transformation form in the diagonal space [1]

$$\int_0^\infty \frac{dx}{\sqrt{x}} F_S(x,\mu,\rho) x^b = \frac{\Gamma^2(1+b)}{\Gamma^2(1-b)} \exp\left[-C_F \int_{\alpha_s(\mu)}^{\alpha_s(\rho)} \frac{d\alpha}{\pi} \mathcal{H}\left(-b + a_\Gamma\left(\mu,\mu_\alpha\right)\right) + \mathcal{O}\left(\alpha_s^2\right)\right] . \tag{5.43}$$

This last relation is useful to transform convolutions of both soft and jet function in the diagonal apace.

5.4.3 Leading order soft function in diagonal space

From the results we have derived so far we present here the diagonal space soft function at LO in RG-improved perturbation theory. To arrive to this result we use the transformation in (5.36) with the lowest order F_S and the leading order renormalized soft function in (4.20) at its natural scale μ_s. We can then use the RGE in (5.34) to find the evolution at another

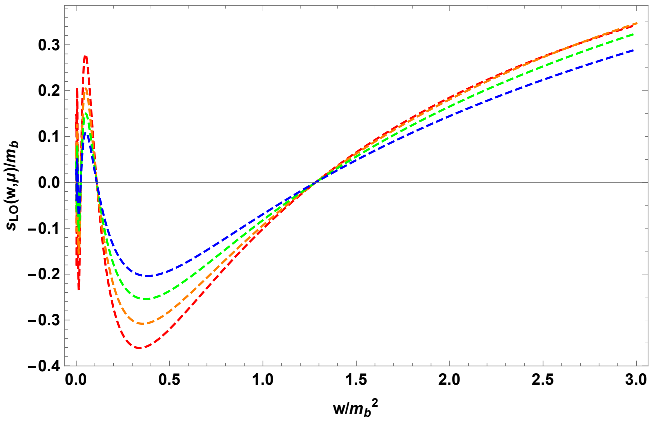

Figure 5.2: Renormalized soft function in the diagonal space at LO in RG-improved perturbation theory. The different lines refer to different evolution scales, $\mu = m_b$ (red), $\mu = 10$ GeV (orange), $\mu = 20$ GeV(green), $\mu = 40$ GeV (blue), with $m_b = 4.8$ GeV. The matching scale $\mu_s = m_b$ in all cases.

scale $\mu \neq \mu_s$. The momentum space soft function is given by a theta function and the integration over the Meijer G-function for the leading order term in the transfer function F_S is straightforward. We obtain

$$\begin{aligned} s_{\rm LO}(w,\mu) =& m_b U_S\left(w;\mu,\mu_s\right) \int_0^\infty \frac{dx}{\sqrt{x}} F_{S,0}(x)\theta\left(xw - m_b^2\right) \\ =& m_b U_S\left(w;\mu,\mu_s\right) G^{2,0}_{0,4}\left(\left. \begin{matrix} \\ 0,1,0,0 \end{matrix} \right| \frac{m_b^2}{w} \right) . \end{aligned} \tag{5.44}$$

The structure is similar to the momentum space result in (5.22) with a new Meijer G-function. In Fig.(5.2) we plot this result for different values of μ, where we fix the soft scale to the b-quark pole mass $m_b = 4.8$ GeV. For values of w much smaller than m_b the plots are oscillatory, while for $w \gg m_b^2$ the behaviour is very similar compared to the momentum space plots in Fig.(5.1). We note that beyond the leading order these oscillations become even more pronounced and therefore it becomes difficult to use the diagonal space results for practical numerical calculations. This happens because of the presence of the transfer function, which also rapidly become more complicated at higher orders.

5.5 Convolution T_3 in the diagonal space

It is instructive for our discussion to derive the double convolution T_3 in the diagonal space. For that we also need to find the transformation of the jet function in the diagonal space. The Wilson coefficient H_3 transforms trivially from momentum space to diagonal space since it obeys a local RGE.

The discussion of the jet function in diagonal space follows the same steps as the ones for the soft function. In complete analogy and using the solution of the RG evolution equation for the jet function from [102] we can derive the corresponding jet function in diagonal space. Again we find that the jet function obeys a local RG evolution in the diagonal space

$$j(p^2,\mu,\rho) = V_j(p^2;\mu,\mu_j)j(p^2,\mu_j,\rho)\,, \tag{5.45}$$

where μ_j is the initial scale for the jet function and the evolution function $V_j(p^2;\mu,\mu_j)$ is defined as

$$V_j(\mu,\mu_j) = \exp\Big[-2\mathcal{S}(\mu_j,\mu) - a_{\gamma'}(\mu_j,\mu) - 2\gamma_E a_\Gamma(\mu_j,\mu)\Big]\,. \tag{5.46}$$

The function $\mathcal{S}(\mu_j,\mu)$ is the same as in (5.9). We find that the jet function in diagonal space is related to the jet function in momentum space by the following relation

$$j(p^2,\mu,\rho) = \int_0^\infty \frac{dx}{\sqrt{x}} F_J(x,\mu,\rho)(xp^2,\mu)\,, \tag{5.47}$$

with

$$\sqrt{x}F_J(x,\mu,\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta x^{-\eta}\frac{\Gamma(1+\eta)}{\Gamma(1-\eta)}\exp\left[C_F\int_{\alpha_s(\mu)}^{\alpha_s(\rho)}\frac{d\alpha}{2\pi}\mathcal{H}(-\eta + a_\Gamma(\mu,\mu_\alpha,) + \mathcal{O}(\alpha_s^2)\right]\,. \tag{5.48}$$

Moreover it can be shown that the jet function obeys the following differential equation

$$\frac{d}{d\ln\rho}j\left(p^2,\mu,\rho\right) = \left[-C_F\left(\frac{\alpha_s(\rho)}{2\pi}\right)^2\int_0^1\frac{dx}{1-x}h(x)x^{a_\Gamma(\mu,\rho)} + \mathcal{O}\left(\alpha_s^3\right)\right]j\left(xp^2,\mu,\rho\right)\,. \tag{5.49}$$

The transformation from momentum space to diagonal space is given by

$$J\left(p^2,\mu\right) = \int_0^\infty \frac{dx}{\sqrt{x}}F_J^{\rm inv}(x,\mu,\rho)j\left(p^2/x,\mu,\rho\right)\,, \tag{5.50}$$

where the inverse transfer function for the jet function reads

$$\sqrt{x}F_J^{\rm inv}(x,\mu,\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\eta x^{-\eta}\frac{\Gamma(1+\eta)}{\Gamma(1-\eta)}\exp\left[-C_F\int_{\alpha_s(\mu)}^{\alpha_s(\rho)}\frac{d\alpha}{2\pi}\mathcal{H}(-\eta + a_\Gamma(\mu,\mu_\alpha))\right]\,. \tag{5.51}$$

At leading order in $\mathcal{O}(\alpha_s)$ we find

$$F_{J,0}(x) = F_{J,0}^{\rm inv}(x) = \mathrm{J}_1(2\sqrt{x})\,. \tag{5.52}$$

This expression seems different from the leading order transfer function for the soft function in (5.41) though in fact they behave very similarly.

In Fig.(5.3) we plot the leading terms for $F_{S,0}$ and $F_{J,0}$. They both oscillate for large values of x. In the case of the jet function the oscillation appears more frequent. The transfer functions for the jet and soft function are related by the following relation

$$F_S(x,\mu,\rho) = \int_0^\infty \frac{dz}{z}F_J^{\rm inv}(z,\mu,\rho)F_J^{\rm inv}\left(\frac{x}{z},\mu,\rho\right)\,. \tag{5.53}$$

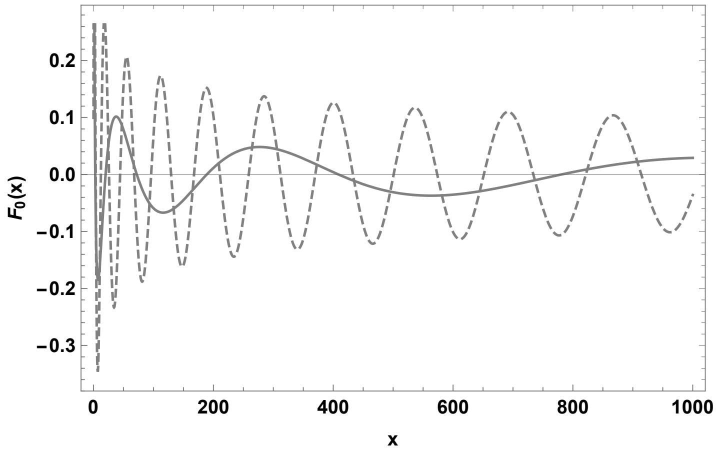

Figure 5.3: Leading order transfer functions for the soft function (solid) and jet function (dashed).

This above expression can be derived by the definitions in (5.40) and (5.48) together with the following identity presented in [1]

$$\int_0^\infty \frac{dx}{x} x^{\eta'-\eta} = \int_{-\infty}^\infty dy e^{ity} = 2\pi\delta(t) = 2\pi i \delta\left(\eta' - \eta\right), \tag{5.54}$$

Using all the above ingredients remarkably we find that the convolution T_3 factorizes to its original form also in the diagonal space such that

$$T_3 = H_3(\mu) \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} j\left(M_h\ell_-, \mu, \rho\right) j\left(-M_h\ell_+, \mu, \rho\right) s\left(\ell_+\ell_-, \mu, \rho\right) . \tag{5.55}$$

We note that the ρ dependence in T_3 cancels out between soft and jet function if they are evaluated for the same intermediate scale ρ. This can be seen by observing their evolutions with respect to ρ in (5.35) and (5.49).

5.6 RG invariance of the convolution T_3

There are two important things concerning the amplitude T_3 that call for further explanations. First it is our initial assumption that T_3 is RG invariant, on which the whole derivation of Z_S is Section 4.2 was based on, and second this RG invariance should be satisfied while at the same time T_3 is free of endpoint divergences. While we have already proved that the renormalization factor Z_S factor successfully removes all the divergences from the one-loop bare soft function, it would be useful to find a regularization scheme for T_3 that is explicitly compatible with its RG invariance. Using known rapidity regulators in momentum space breaks the scale invariance of T_3 and the origin for this seems to be the non-local nature of the anomalous dimensions γ_S and γ_J. We show this explicitly for two different regulators in Appendix C. In other words if we want to derive a formula for T_3 that is RG invariant and free of endpoint divergences first we need to diagonalize the anomalous dimensions that enter in the evolution in T_3. We have already shown in Sections 5.4 and 5.5 that this is exactly what happens to the anomalous dimensions for the jet and the soft function in the diagonal space. The evolution of all the component functions of the T_3 is local in the diagonal space and this means that T_3 is scale invariant point by point in this space. In other words for each point in the (ℓ_+, ℓ_-) plane T_3 is a scale invariant quantity.

For the purpose of showing that we can regularize T_3 in a way compatible with its scale invariance we present here the results for the analytic regulator-δ written in the diagonal space. From (5.55) in this case we can write

$$\begin{aligned} T_3^{\text{analytic}} =& H_3(\mu) \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} j\left(M_h \ell_-, \mu\right) j\left(-M_h \ell_+, \mu\right) s\left(\ell_+ \ell_-, \mu\right) \\ & \times \left(\frac{M_h\left(\ell_+ - \ell_-\right) - i0}{\nu^2}\right)^{-2\delta} . \end{aligned} \tag{5.56}$$

It is sufficient to show the explicit result at LO in RG improved perturbation theory, since the behaviour regarding the divergences should not be significantly modified by higher order corrections. At leading order the resummed results for the hard, jet and soft function read

$$\begin{aligned} H_{3,\text{LO}}(\mu) =& \frac{N_c \alpha_b}{\pi} \frac{y_b\left(\mu_h\right)}{\sqrt{2}} \exp\left[2\mathcal{S}\left(\mu_h, \mu\right) - 2a_{\gamma_q}\left(\mu_h, \mu\right)\right] \left(\frac{-M_h^2}{\mu_h^2}\right)^{-a_\Gamma(\mu_h, \mu)} , \\ j_{\text{LO}}\left(p^2, \mu\right) =& \exp\left[-2\mathcal{S}\left(\mu_j, \mu\right) - a_{\gamma'}\left(\mu_j, \mu\right)\right] \left(\frac{-p^2 e^{-2\gamma_E}}{\mu_j^2}\right)^{a_\Gamma(\mu_j, \mu)} , \\ s_{\text{LO}}(w, \mu) =& m_b \exp\left[2\mathcal{S}\left(\mu_s, \mu\right) + a_{\gamma_s}\left(\mu_s, \mu\right)\right] \left(\frac{w e^{-4\gamma_E}}{\mu_s^2}\right)^{-a_\Gamma(\mu_s, \mu)} \int_0^\infty \frac{dx}{\sqrt{x}} F_{S,0}(x) \theta\left(xw - m_b^2\right) . \end{aligned} \tag{5.57}$$

The soft and jet function above are given in the diagonal space while for the hard function we have written the known expression in momentum space. This should not be confusing, since the hard function evolves locally also in momentum space and the transformation from one space of the other is trivial. The hard matching scale μ_h is chosen such that the fixed ordered result is free of large logarithms. A natural choice is $\mu_h^2 \approx M_h^2$ or $\mu_h^2 \approx -M_h^2$. Then the expression in (5.56) takes the form

$$\begin{aligned} T_{3,\text{LO}}^{\text{analytic}} =& \frac{N_c \alpha_b}{\pi} \frac{y_b\left(\mu_h\right)}{\sqrt{2}} m_b \exp\left[2\mathcal{S}\left(\mu_s, \mu_j\right) + 2\mathcal{S}\left(\mu_h, \mu_j\right) + a_{\gamma_s}\left(\mu_s, \mu_j\right) - 2a_{\gamma_q}\left(\mu_h, \mu_j\right)\right] \\ & \times \left(\frac{-M_h^2}{\mu_h^2}\right)^{-a_\Gamma(\mu_h, \mu_j)} \left(\frac{m_b^2}{\mu_s^2}\right)^{-a_\Gamma(\mu_s, \mu_j)} \frac{\Gamma^2(\delta)}{2\Gamma(2\delta)} \left(\frac{-M_h^2 m_b^2}{\nu^4}\right)^{-\delta} \\ & \times e^{4\gamma_E a_\Gamma(\mu_s, \mu_j)} \int_0^\infty \frac{dw}{w} \left(\frac{w}{m_b^2}\right)^{-a_\Gamma(\mu_s, \mu_j) - \delta} \int_0^\infty \frac{dx}{\sqrt{x}} F_{S,0}(x) \theta\left(xw - m_b^2\right) . \end{aligned} \tag{5.58}$$

To arrive to this result we have used the following identities to relate various RG functions to each other

$$
\begin{aligned}
a_\Gamma(\nu_1,\mu) - a_\Gamma(\nu_2,\mu) &= a_\Gamma(\nu_1,\nu_2)\,, \\
\mathcal{S}(\nu_1,\mu) - \mathcal{S}(\nu_2,\mu) &= S(\nu_1,\nu_2) - a_\Gamma(\nu_2,\mu)\ln\frac{\nu_1}{\nu_2}\,.
\end{aligned} \tag{5.59}
$$

We can integrate over w in (5.58) by a change of variables $w \to z = xw/m_b^2$. Then this integration gives

$$
\int_1^\infty \frac{dz}{z} z^{-a-\delta} \int_0^\infty \frac{dx}{\sqrt{x}} F_{S,0}(x) x^{a+\delta} = \frac{\Gamma(a+\delta)\Gamma(1+a+\delta)}{\Gamma^2(1-a-\delta)}\,. \tag{5.60}
$$

where $a = a_\Gamma(\mu_s,\mu_j)$. Here $a+\delta$ should be positive for this result to converge. Eventually we want to take the limit $\delta \to 0$, which implies $a > 0$ and $\mu_j < \mu_s$. This is not physical since the soft scale is the lowest scale, but we can formally solve it with this condition and then analytically continue for values of a such that $\mu_s < \mu_j$. Finally expanding in δ we find

$$
\begin{aligned}
T_{3,\mathrm{LO}}^{\mathrm{analytic}} =& \frac{N_c\alpha_b}{\pi}\frac{y_b(\mu_h)}{\sqrt{2}} m_b \exp\Big[2\mathcal{S}(\mu_s,\mu_j) + 2\mathcal{S}(\mu_h,\mu_j) + a_{\gamma_s}(\mu_s,\mu_j) - 2a_{\gamma_q}(\mu_h,\mu_j)\Big] \\
&\times \left(\frac{-M_h^2}{\mu_h^2}\right)^{-a_\Gamma(\mu_h,\mu_j)} \left(\frac{m_b^2}{\mu_s^2}\right)^{-a_\Gamma(\mu_s,\mu_j)} e^{4\gamma_E a}\frac{\Gamma(a)\Gamma(1+a)}{\Gamma^2(1-a)} \\
&\times \left[\frac{1}{\delta} + \ln\frac{\nu^4}{-M_h^2 m_b^2} + 2\psi(1+a) + 2\psi(1-a) - \frac{1}{a}\right]\,.
\end{aligned} \tag{5.61}
$$

where still $a = a_\Gamma(\mu_s,\mu_j)$. The remaining $1/\delta$ pole and the logarithm on the auxiliary scale ν must cancel among other contributions in the total decay amplitude for $h \to \gamma\gamma$.

With this discussion we have shown that the double convolution T_3 in the diagonal space can be cast into an expression that is both RG invariant and free of endpoint divergences. This result explains why our initial conjecture lead to consist results on the renormalization of the soft-quark soft function.

Chapter 6

Renormalized factorization formula for $h \to \gamma\gamma$

With the analysis we have presented so far on the renormalization of the functions present in the bare amplitude for the $h \to \gamma\gamma$ decay mediated by a b-quark loop, we have already set the foundation to derive a renormalized version of the factorized amplitude. In this chapter we want to derive a factorization formula that will have the following structure

$$\begin{aligned}
\mathcal{M}_b &= H_1(\mu)\,\langle O_1(\mu)\rangle \\
&\quad + 2\int_0^1 dz\left[H_2(z,\mu)\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right] \\
&\quad + g_\perp^{\mu\nu}\lim_{\sigma\to -1} H_3(\mu)\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+} J\left(M_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right)\Bigg|_{\text{leading power}},
\end{aligned} \tag{6.1}$$

where $\bar{z} = 1 - z$. This formula has the same structure as the bare factorization theorem in (3.11), though it is not at all obvious that (6.1) follows directly from (3.11) by simply substituting the operator matrix elements and the Wilson coefficients by their renormalized quantities. As we have already seen most of these quantities renormalize non-locally and for the convolution terms T_2 and T_3 simply replacing the operator matrix elements and the Wilson coefficients with their renormalized versions breaks the factorized structure of the amplitude. The reason for this is that the boundaries of the convolution integrals do not always commute with the renormalization of various functions. This is in particular pronounced in the double convolution T_3, where the cutoffs clearly do not commute with the renormalization of the soft and jet function in (4.18) and (4.12).

On the other hand we must ensure that the endpoint divergences are cured also after renormalization. Because of this, imposing renormalization on all the terms in T_2 and T_3 and at the same time requiring that the amplitude respects the well defined (free of endpoint divergences) factorized structure as shown in (6.1) is highly non-trivial. For both cases there will be some remaining extra terms that need to be treated carefully. We find that all these "extra" terms depend only on the hard scale M_h and it is therefore natural to absorb them into a redefinition of the Wilson coefficient $H_1(\mu)$. From here we then derive the renormalization of this Wilson coefficient. Most of the calculations in this Chapter are quite technical and we collect some of the gory details of the derivations in Appendix D. We start in Section 6.1 with a detailed derivation of the renormalized factorization formula for $h \to \gamma\gamma$. In Section 6.2 we derive the RG equation for the hard coefficient $H_1(\mu)$ with which we complete the set of RG equations for all the functions in the decay amplitude $h \to \gamma\gamma$. In Section 6.3 we show the final results for the renormalized amplitudes T_1, T_2 and T_3. Using the RG evolution equations for the Wilson coefficients and the operator matrix elements we derive in Section 6.4 the large logarithms at $\mathcal{O}(\alpha_s^2)$, which is the first three-loop result for an observable in SCET.

6.1 Derivation of the renormalized factorization formula

6.1.1 Renormalized T_3

We start by investigating the amplitude T_3 written in terms of renormalized functions, where we have

$$T_3^{\text{ren}} = g_\perp^{\mu\nu} \lim_{\sigma \to -1} H_3(\mu) \int_0^{M_h} \frac{\mathrm{d}\ell_-}{\ell_-} \int_0^{\sigma M_H} \frac{\mathrm{d}\ell_+}{\ell_+} J\left(M_h \ell_-, \mu\right) J\left(-M_h \ell_+, \mu\right) S\left(\ell_+ \ell_-, \mu\right) . \tag{6.2}$$

We can use the inverse of the renormalization conditions for the jet and the soft function in (4.18) and (4.12) to rewrite this expression in terms of bare functions and the corresponding renormalization factors. Then with a rearrangement of the integration boundaries we find that the result in (6.2) is equivalent to the sum of T_3 written in terms of bare quantities as it appears in (3.11) and a remaining *mismatch* term we call δT_3 such that

$$T_3^{\text{ren}} = T_3 + \delta T_3 \,, \tag{6.3}$$

where

$$\begin{aligned} \delta T_3 = & g_\perp^{\mu\nu} \lim_{\sigma \to -1} H_3(\mu) \int_0^\infty dw' \left[\int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} \int_0^\infty d\ell'_- \int_0^\infty d\ell'_+ \right. \\ & \left. - \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \int_0^{M_h} d\ell'_- \int_0^{\sigma M_h} d\ell'_+ \right] S^{(0)}(w') \\ & \times J^{(0)}\left(M_h \ell'_-\right) J^{(0)}\left(-M_h \ell'_+\right) Z_S\left(\ell_- \ell_+, w'\right) Z_J\left(\ell_-, \ell'_-\right) Z_J\left(\ell_+, \ell'_+\right) . \end{aligned} \tag{6.4}$$

This is a mismatch term in the sense that it results from the integration boundaries being mismatched before and after renormalization. Moreover the integration boundaries between the (ℓ_+, ℓ_-) and (ℓ'_+, ℓ'_-) are interchanged in δT_3. We present the intermediate steps of this derivation in Appendix D, where we have used the already mentioned symmetry properties of the Z_J and Z_S and the relation (4.30). Lastly we note that in general this term can depend both on the hard scale M_h and the low scale of $\mathcal{O}(m_b)$ since the momenta are all integrated from zero to the higher scale.

6.1.2 Renormalized T_2

In a similar fashion we start here by writing the amplitude T_2 in terms of the renormalized functions and name it $T_2^{\rm ren}$. We have

$$T_2^{\rm ren} = 2\int_0^1 dz \left[H_2(z,\mu)\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right], \tag{6.5}$$

where the renormalization conditions are given in Chapter 4. We can use the corresponding renormalization conditions for the operator matrix elements $\langle O_2(z,\mu)\rangle$ and $[\![\langle O_2(z,\mu)\rangle]\!]$ in (4.24) and (4.27) and their Wilson coefficients to write the above expression in terms of bare functions. After a few steps of calculation (see Appendix D) we find that the renormalized convolution in (6.5) can be written as

$$\begin{aligned} T_2^{\rm ren} &= T_2 + \delta T_2 \\ &+ 2\int_0^1 dz \left[H_2(z,\mu)\, Z_{21}(z) - \frac{[\![\,\bar{H}_2(z,\mu)\,]\!]}{z}[\![\,Z_{21}(z)\,]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![Z_{21}(\bar{z})]\!]\right]\langle O_1^{(0)}\rangle\,, \end{aligned} \tag{6.6}$$

where T_2 is the bare T_2 convolution in the bare factorization theorem in (3.11). We have defined here another mismatch term δT_2 which can be written in a compact form as

$$\delta T_2 = -4\left(\int_0^1 \frac{dz}{z}\int_0^\infty dz' - \int_0^1 dz'\int_0^\infty \frac{dz}{z}\right)[\![\,\bar{H}_2(z,\mu)\,]\!]\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\,\langle O_2^{(0)}(z')\rangle\,]\!]\,. \tag{6.7}$$

Notice that the convolution $H_2(z,\mu)\otimes\langle O_2(z,\mu)\rangle$ does not contribute to the mismatch term. This is because for such terms the renormalization conditions contain integrals with boundaries $(0,1)$ that commute with the convolution itself. We also recall that both the Wilson coefficient $[\![\bar{H}_2(z,\mu)]\!]$ and the operator matrix element $[\![\langle O_2^{(0)}(z)\rangle]\!]$ are symmetric under $z\to(1-z)$.

Lastly the expression in the second line in (6.6) is a mixing term that results from the mixing nature of the renormalization of $\langle O_2^{(0)}\rangle$ and $[\![\langle O_2^{(0)}\rangle]\!]$. We then define a function δH_{21} such that

$$\delta H_{21} = 2\int_0^1 dz \left[H_2(z,\mu)\, Z_{21}(z) - \frac{[\![\,\bar{H}_2(z,\mu)\,]\!]}{z}[\![\,Z_{21}(z)\,]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![Z_{21}(\bar{z})]\!]\right]. \tag{6.8}$$

This function is free of endpoint divergences. The subtracted terms serve as regulators for the integration just like in the amplitude. The expression in (6.8) still contains $1/\epsilon$ poles though. This should be clear since $H_2^{(0)}(z)$ and $[\![\bar{H}_2^{(0)}(z)]\!]$ are renormalized by $Z_{22}(z,z')$ and $[\![Z_{22}(z,z')]\!]$. In fact at leading order in α_s we find the following result for δH_{21}

$$\delta H_{21} = \frac{y_b(\mu_h)}{\sqrt{2}}\frac{N_c\alpha_b}{\pi}\frac{C_F\alpha_s}{4\pi}\left[-\frac{\pi^2}{3\epsilon^2} + \frac{4\zeta_2 L_h - 2\zeta_3}{\epsilon} - \frac{\pi^2}{3}L_h^2 + 4\zeta_3 L_h + 4\zeta_2 + \zeta_4\right], \tag{6.9}$$

where $L_h = \ln(-M_h^2/\mu^2)$ as before.

The mixing contribution in (6.8) can be absorbed into a redefinition of the Wilson coefficient H_1, since it only depends on the hard scale M_h. Moreover it multiplies the bare operator matrix element $\langle O_1^{(0)} \rangle$ in (6.6). This is not clear in the case of the mismatch terms $\delta T_2 + \delta T_3$ from (6.4) and (6.7), which can also depend on the low scale. At a next step we rewrite both δT_2 and δT_3 in such ways that it will be obvious that the low scale dependence cancels in their sum $\delta T_2 + \delta T_3$.

6.1.3 Cancellation of the low scale dependence

We start by rewriting the mismatch term δT_2 in (6.7) using the refactorization conditions shown in (3.8). The refactorization of the Wilson coefficient $[\![\bar{H}_2^{(0)}(z)]\!]$ holds also for its renormalized version, since this is a multiplicative and local identity such that

$$[\![\, \bar{H}_2(z,\, \mu) \,]\!] = -H_3(\mu)\, J\,(z\, M_h^2\,, \mu)\,, \tag{6.10}$$

where $z = \ell_-/M_h$. In fact from this refactorization condition we can derive the useful relation of the following renormalization constants

$$[\![Z_{22}(\ell_-, \ell'_-) \,]\!] = \int_0^\infty \mathrm{d}\ell_+ \ell'_+ Z_J\left(\ell'_+, \ell_+\right) Z_S\left(\ell_-\ell_+, \ell'_-\ell'_+\right) \,. \tag{6.11}$$

Two other expressions we will use here are the following symmetry relations

$$Z_J(\ell_-, \ell'_-)\, \frac{\ell'_-}{\ell_-} = Z_J(\ell'_-, \ell_-)\,, \qquad\qquad Z_S(\ell_+\ell_-, w)\, \frac{\omega}{\ell_+\ell_-} = Z_S(w, \ell_+\ell_-)\,, \tag{6.12}$$

together with the relation between the jet, soft function and the $H_3(\mu)$ renormalization factors

$$Z_{33}^{-1} \int_0^\infty \mathrm{d}\ell_- \int_0^\infty \mathrm{d}\ell_+ Z_J\left(\ell'_-, \ell_-\right) Z_J\left(\ell'_+, \ell_+\right) Z_S\left(\ell_-\ell_+, w'\right) = \delta\left(w' - \ell'_-\ell'_+\right) \,. \tag{6.13}$$

Then after several steps of manipulating the δT_2 expression from (6.7) we find that

$$\begin{aligned}
\delta T_2 = -\, & \Bigg[\Bigg(\int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \int_0^\infty d\ell'_- \int_0^\infty d\ell'_+ - \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \int_0^{M_h} d\ell'_- \int_0^\infty d\ell'_+ \Bigg) \\
& + \Bigg(\int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} \int_0^\infty d\ell'_- \int_0^\infty d\ell'_+ - \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \int_0^\infty \mathrm{d}\ell'_- \int_0^{\sigma M_h} d\ell'_+ \Bigg)\Bigg] \\
& \times H_3(\mu) \int_0^\infty dw' S(w') Z_J(\ell_-, \ell'_-) Z_J(\ell_+, \ell'_+) Z_S(\ell_+\ell_-, w') J^{(0)}(M_h\ell'_-) J^{(0)}(-M_h\ell'_+) S^{(0)}(w')\,.
\end{aligned} \tag{6.14}$$

We have collected details of this derivation in Appendix D.2. In (6.14) we have separated the integrals in such a way that will be convenient in a later step to combine this result with the mismatch term in (6.4). The integrand in both (6.14) and (6.4) expression is the same though in the form that they are written it is not clear if the sum $\delta T_2 + \delta T_3$ depends only on the hard scale M_h.

It is convenient to further work out the different integration boundaries. In this sum the terms that are proportional to the integral of $\ell'_\pm$ integrated to infinity read

$$\begin{aligned}
&\int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} - \int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} - \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} \\
&= - \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} - \int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} \\
&= \int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} - \int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} - \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} \\
&\quad + \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} - \int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} + \int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} \\
&= \int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} - \int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} .
\end{aligned} \tag{6.15}$$

The second term gives vanishing contributions because it is an overall scaleless integral since also ℓ'_+ and ℓ'_- are integrated in $(0,\infty)$ in these terms. Exactly the same procedure of combining different integration boundaries can be applied for the remaining terms where the boundaries on $\ell_\pm$ are now $(0,\infty)$ and the prime momenta are integrated from $(0, M_h)$. Then finally combining all these integrals together we find that the sum $\delta T_2 + \delta T_3$ reads

$$\begin{aligned}
H_3(\mu) \int_0^{\infty} dw' \Bigg(&\int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} \int_0^{\infty} d\ell'_- \int_0^{\infty} d\ell'_+ - \int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} \int_{M_h}^{\infty} d\ell'_- \int_{\sigma M_h}^{\infty} d\ell'_+ \Bigg) \\
&\times Z_J(\ell_-, \ell'_-) Z_J(\ell_+, \ell'_+) Z_S(\ell_+\ell_-, w') J^{(0)}(M_h \ell'_-) J^{(0)}(-M_h \ell'_+) S^{(0)}(w') .
\end{aligned} \tag{6.16}$$

In the first term here the integration over ℓ'_+ and ℓ'_- can be readily done using the renormalization conditions for the jet and soft function. In the second term in (6.16) we can integrate the ℓ_+ and ℓ_- using the first relation in (6.12) for jet function renormalization factors together with the identity in (6.13) and then write this term as a convolution of bare functions such that

$$\begin{aligned}
H_3(\mu) &\int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} \int_{M_h}^{\infty} d\ell'_- \int_{\sigma M_h}^{\infty} d\ell'_+ \int_0^{\infty} dw' Z_J(\ell_-, \ell'_-) Z_J(\ell_+, \ell'_+) Z_S(\ell_+\ell_-, w') \\
&\qquad \times J^{(0)}(M_h \ell'_-) J^{(0)}(-M_h \ell'_+) S^{(0)}(w') \qquad (6.17) \\
= H_3^{(0)} &\int_{M_h}^{\infty} \frac{d\ell'_-}{\ell'_-} \int_{\sigma M_h}^{\infty} \frac{d\ell'_+}{\ell'_+} J^{(0)}(M_h \ell'_-) J^{(0)}(-M_h \ell'_+) S^{(0)}(\ell'_+ \ell'_-) .
\end{aligned}$$

In this last result we can relabel the (ℓ'_+, ℓ'_-) momenta to (ℓ_+, ℓ_-) and combine it together with the first term in (6.16). Finally the total contribution from the mismatch terms as a result of renormalization in the convolutions T_2 and T_3 reads

$$\begin{aligned} \delta T_2 + \delta T_3 = & H_3(\mu) \lim_{\sigma\to -1} \int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} J\left(M_h \ell_-, \mu\right) J(-M_h\ell_+, \mu) S(\ell_+\ell_-, \mu) \\ & - H_3^{(0)} \lim_{\sigma\to -1} \int_{M_h}^{\infty} \frac{d\ell_-}{\ell_-} \int_{\sigma M_h}^{\infty} \frac{d\ell_+}{\ell_+} J^{(0)}(M_h\ell_-) J^{(0)}(-M_h\ell_+) S^{(0)}(\ell_+\ell_-)\,. \end{aligned} \tag{6.18}$$

In this last expression we can see that all the integration boundaries are above the Higgs boson mass. This means that all these integrals will only generate terms that depend on the hard scale, more precisely they will be logarithms of the form $\ln(-M_h^2/\mu^2)$ to all orders.

At this point, since we know the one-loop results from all the quantities involved in (6.18) we can compute it explicitly at $\mathcal{O}(\alpha_s)$. In addition the integration boundaries in the hard region allow us to replace the soft function in these expressions with its asymptotic approximation $S_\infty^{(0)}$ which at leading order is

$$S_\infty^{(0)}(w) = -m_{b,0} \frac{N_c \alpha_{b,0}}{\pi} \frac{e^{e_E}}{\Gamma(1-\epsilon)} w^{-\epsilon}\,. \tag{6.19}$$

We then obtain

$$\delta T_2 + \delta T_3 = m_b \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \frac{\alpha_S C_F}{4\pi} \left(-\frac{8\zeta_3}{\epsilon} + 8\zeta_3 L_h \right) + \mathcal{O}(\alpha_s^2)\,, \tag{6.20}$$

where $L_h = \ln(-M_h^2/\mu^2)$.

From the whole analysis presented above we see that as a result of the renormalization of the bare factorized theorem using a plus-type subtraction to regularize the endpoint divergences, there are additional terms than depend only on the hard scale. It is thus natural to absorb these terms into a redefinition of the Wilson coefficient H_1 in the amplitude T_1. This is an all order analysis and even though we have computed the integrals explicitly up to $\mathcal{O}(\alpha_s)$ there should be no ambiguity here that the framework holds to all order.

6.1.4 Renormalized Wilson coefficient H_1

The physical decay amplitude $\mathcal{M}_b(h \to \gamma\gamma)$ expressed in terms of bare quantities in (3.11) should be the same as the amplitude expressed in terms of renormalized quantities in (6.1). By imposing this requirement we can now derive the renormalized Wilson coefficient $H_1(\mu)$ from the renormalized factorization theorem. We find that

$$H_1(\mu) = \left(H_1^{(0)} + \Delta H_1^{(0)} - \frac{\delta T_2 + \delta T_3}{\langle O_1^{(0)} \rangle} - \delta H_{21} \right) Z_{11}^{-1}\,. \tag{6.21}$$

The expressions for the Wilson coefficient $H_1^{(0)}$ and the infinity-bin subtraction term $\Delta H_1^{(0)}$ are given in the Appendix (A). The expressions for the sum $\delta T_2 + \delta T_3$ and δH_{21} are the results in (6.18) and (6.8) we just derived in the previous Section.

In the third term in (6.21) the b-quark mass dependence in $\delta T_2 + \delta T_3$, that comes from the presence of the soft function cancels out with the b-quark mass in $\langle O_1^{(0)} \rangle$. For an explicit calculation at one-loop we obtain the result

$$H_1(\mu) = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \left\{ -2 + \frac{C_F \alpha_s}{4\pi} \left[-\frac{\pi^2}{3} L_h^2 + (12 + 8\zeta_3)\, L_h - 36 - \frac{2\pi^2}{3} - \frac{11\pi^4}{45} \right] + \mathcal{O}\left(\alpha_s^2\right) \right\} , \tag{6.22}$$

where it is obvious that depends only on the hard scale M_h. Also the fact that in this expression all the $1/\epsilon^n$ have cancelled out is a very non-trivial cross check of our analysis.

6.2 RG evolution of the Wilson coefficient H_1

In this Section we discuss the derivation of the RG evolution equation for the Wilson coefficient $H_1(\mu)$. We begin from the fact that the total decay amplitude is independent on the renormalization scale μ. This means that we can write

$$\frac{d}{d\ln\mu} \mathcal{M}_b = \frac{d\,T_1^{\text{ren}}}{d\ln\mu} + \frac{d\,T_2^{\text{ren}}}{d\ln\mu} + \frac{d\,T_3^{\text{ren}}}{d\ln\mu} \equiv 0 \,, \tag{6.23}$$

where we have defined $T_1^{\text{ren}} = H_1(\mu) O_1(\mu)$ here. We have already derived all the RGEs for the various function in (6.23) in Chapter 4. Combing those results with the corresponding expressions for the renormalized T_2^{ren} and T_3^{ren} in (6.6) and (6.4) it is not difficult to obtain

$$\frac{d}{d\ln\mu} H_1(\mu) = \gamma_{11} H_1(\mu) - \frac{1}{\langle O_1(\mu) \rangle} \frac{d}{d\ln\mu} \left(\delta T_2 + \delta T_3\right) - Z_{11}^{-1} \frac{d}{d\ln\mu} \delta H_{21} \,. \tag{6.24}$$

The first term here results from the RGE of the operator matrix element $\langle O_1(\mu) \rangle$ in (4.49). The explicit structure of the other two terms is less obvious.

We start with the differentiation on the second term using the result presented in (6.18). Only the first line there, which depends on renormalized functions will give a non-vanishing contribution. We find

$$\begin{aligned} \frac{d}{d\ln\mu} (\delta T_2 + \delta T_3) = H_3(\mu) \int_0^\infty & dx K(x,\mu) \\ \times \Bigg(\int_{M_h/x}^{M_h} \frac{d\ell_-}{\ell_-} & \int_{\sigma M_h}^\infty \frac{d\ell_+}{\ell_+} S_\infty\left(\ell_-\ell_+, \mu\right) J\left(x M_h \ell_-, \mu\right) J\left(-M_h \ell_+, \mu\right) \\ + \int_{M_h}^\infty \frac{d\ell_-}{\ell} & \int_{\sigma M_h/x}^{\sigma M_h} \frac{d\ell_+}{\ell_+} S_\infty\left(\ell_-\ell_+, \mu\right) J\left(M_h \ell_-, \mu\right) J\left(-x M_h \ell_+, \mu\right) \Bigg) . \end{aligned} \tag{6.25}$$

The function $K(x,\mu)$ was computed at $\mathcal{O}(\alpha_s)$ in [1] under the discussion of the scale invariance of the amplitude T_3 and it reads

$$K(x,\mu) = \Gamma_{\text{cusp}}\ (\alpha_s)\, \Gamma(1,x) + C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\theta(1-x)}{1-x} h(x) + \mathcal{O}\left(\alpha_s^3\right) \,. \tag{6.26}$$

All the local terms of the anomalous dimensions have cancelled in the expression in (6.25) and $K(x,\mu)$ contains only the combined non-local contributions from the RG equations.

The last term in (6.24) calls for a more careful discussion since the mixing hard function δH_{21} results from a term that mixes the operators O_2 and O_1 under renormalization (see eq.(6.6)). If we differentiate the mixing term $\delta H_{21}\langle O_1^{(0)}\rangle$ we find that this contribution can be written as

$$\begin{aligned}
\left(\frac{d}{d\ln\mu}\delta H_{21}\right) Z_{11}^{-1}O_1(\mu) =& -2\int_0^1 dz H_2(z,\mu)\gamma_{21}(z)\langle O_1(\mu)\rangle \\
& -2\int_0^1 dz\left[-\frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\gamma_{21}(z)]\!]-\frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![\gamma_{21}(\bar{z})]\!]\right]\langle O_1(\mu)\rangle \\
& +4\left(\int_0^1 dz'\int_0^\infty \frac{dz}{z}-\int_0^1 dz\int_0^\infty\frac{dz'}{z'}\right)[\![\bar{H}_2\left(z',\mu\right]\!][\![\gamma_{22}\left(z',z\right)]\!] \\
& \times[\![Z_{21}(z)]\!]Z_{11}^{-1}O_1(\mu)\,.
\end{aligned} \tag{6.27}$$

We have defined here the mixing anomalous dimensions $\gamma_{21}(z)$ as

$$\gamma_{21}(z) = -\left(\int_0^1 dz'\gamma_{22}(z,z')Z_{21}(z')+\frac{d}{d\ln\mu}Z_{21}(z)\right)Z_{11}^{-1}\,. \tag{6.28}$$

In the same way $[\![\gamma_{21}(z)]\!]$ in defined by integration to infinity and replacing the corresponding quantities with their $[\![\,..]\!]$ versions. In fact the anomalous dimension in (6.28) can be derived also from combining together the RG evolution for $\langle O_2(z,\mu)\rangle$ and $[\![\langle O_2(z,\mu)\rangle]\!]$ in (4.49) and the corresponding renormalization conditions.

This first and the second line in (6.27) are a result of the mixing of the operator $O_2(z,\mu)$ with the operator $O_1(\mu)$. The expression in the last two lines is a left over term that originates from the fact that the upper integration boundary for the RG evolution of the operator matrix elements $[\![\langle O_2(z,\mu)\rangle]\!]$ is different from 1. This means that an exchange of integration sequence for the variables z and z' is not possible for such terms. Contrary this kind of exchange is possible for the terms with the $H_2(z,\mu)$ and $\langle O_2(z,\mu)\rangle$. This subtle detail turns out to be the reason for the presence of the left over term in this case.

We can rewrite the last lines in (6.27) using the refactorization condition for $[\![\bar{H}_2(z,\mu)]\!]$ and it becomes

$$4H_3(\mu)\left(\int_0^1 dz\int_0^\infty\frac{dz'}{z'}-\int_0^1\frac{dz'}{z'}\int_0^\infty dz\right)J\left(M_h^2z',\mu\right)[\![\gamma_{22}\left(z',z\right)]\!][\![Z_{21}(z)]\!]Z_{11}^{-1}O_1(\mu)\,. \tag{6.29}$$

This result is proportional to $H_3(\mu)$ similarly to the contribution from the mixing terms in (6.25). In fact both these terms are non-homogeneous, in the sense that do not follow a the usual structure of an RG evolution equation, in this case of $H_1(\mu)$. It is useful to define the part of the expression in (6.29) that contributes to the evolution in (6.24) such that

$$\delta H_{21}' = 4H_3(\mu)\left(\int_0^1 dz\int_0^\infty\frac{dz'}{z'}-\int_0^1\frac{dz'}{z'}\int_0^\infty dz\right)J\left(M_h^2z',\mu\right)[\![\gamma_{22}\left(z',z\right)]\!][\![Z_{21}(z)]\!]Z_{11}^{-1}\,. \tag{6.30}$$

We combine both of the contributions proportional to the Wilson coefficient $H_3(\mu)$ in a quantity we call D_{cut} such that

$$-\left(\frac{1}{\langle O_1(\mu)\rangle}\frac{d}{d\ln\mu}\left(\delta T_2+\delta T_3\right)+\delta H'_{21}\right)\equiv D_{\text{cut}}(\alpha_s)\,. \tag{6.31}$$

We follow here the same notation as in [2], where the subscript "cut" is used to indicate that these left over terms are a result of the cutoffs that emerged as rapidity regulators in our regularization scheme. We note here that this quantity should not be seen as a mixing term of the operator matrix elements $\langle O_1(\mu)\rangle$ and $\langle O_3(\mu)\rangle$ because of the presence of the $H_3(\mu)$. The non-homogeneous nature of these contributions is also an indicator of this fact.

One would need to compute these terms explicitly order by order for the full result of the RG evolution of $H_1(\mu)$. We compute them here at $\mathcal{O}(\alpha_s^2)$ using the already derived one-loop results of the various component functions in (6.25) and (6.30). We find

$$-\left(\frac{1}{\langle O_1(\mu)\rangle}\frac{d}{d\ln\mu}\left(\delta T_2+\delta T_3\right)+\delta H'_{21}\right)=-\frac{N_c\alpha_b}{\pi}\frac{y_b(\mu)}{\sqrt{2}}\left[\frac{C_F\alpha_s}{4\pi}16\zeta_3+\left(\frac{\alpha_s}{4\pi}\right)^2 d_{\text{cut},2}\right]\,, \tag{6.32}$$

where

$$\begin{aligned}
d_{\text{cut},2}=&C_F^2\left[-48\zeta_3L_h^2+\left(-48\zeta_3+\frac{8\pi^4}{15}\right)L_h+136\zeta_3+\frac{4\pi^4}{5}-32\zeta_5+\frac{16\pi^2}{3}\zeta_3\right]\\
&+C_FC_A\left(-\frac{176}{3}\zeta_3L_h+\frac{1072}{9}\zeta_3-\frac{44\pi^4}{45}-\frac{16\pi^2}{3}\zeta_3\right)\\
&+C_FT_Fn_f\left(\frac{64}{3}\zeta_3L_h-\frac{320}{9}\zeta_3+\frac{16\pi^4}{45}\right)\,.
\end{aligned} \tag{6.33}$$

Lastly combining (6.24), (6.27) and (6.31) we can write the final result for the RG evolution equation of the Wilson coefficient $H_1(\mu)$

$$\begin{aligned}
\frac{d}{d\ln\mu}H_1(\mu)=&\,D_{\text{cut}}(\alpha_s)+\gamma_{11}H_1(\mu)\\
&+2\int_0^1dz\left[H_2(z,\mu)\gamma_{21}(z)-\frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\gamma_{21}(z)]\!]-\frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![\gamma_{21}(\bar{z})]\!]\right]\,.
\end{aligned} \tag{6.34}$$

This is an inhomogeneous RGE and it is challenging to find an all order solution due to the presence of the inhomogeneous term $D_{\text{cut}}(\alpha_s)$. A way to avoid this issue is to fix the scale for the decay process $h\to\gamma\gamma$ to M_h. Then the renormalized function $H_1(\mu)$ (and all the other Wilson coefficients) would be free of large logarithms. In this case the resummation of the remaining large logarithms in the process would need to be taken care of by the RG evolution of the operator matrix elements.

6.3 Contribution to the decay amplitude

As a cross check we can see that all the terms in the renormalized amplitude produce the separate T_i terms such that when they are summed up the total decay amplitude is reproduced.

We have

$$
\begin{aligned}
T_1 =& \mathcal{M}_0 \left\{ -2 + \frac{C_F \alpha_s}{4\pi} \left[-\frac{\pi^2}{3} L_h^2 + (12 + 8\zeta_3) L_h - 36 - \frac{2\pi^2}{3} - \frac{11\pi^4}{45} \right] \right\}, \\
T_2 =& \mathcal{M}_0 \frac{C_F \alpha_s}{4\pi} \left[\frac{2\pi^2}{3} L_h L_m - \frac{\pi^2}{3} L_m^2 + \frac{2\pi^2}{3} + 8\zeta_3 + \frac{7\pi^4}{45} \right] \\
T_3 =& \mathcal{M}_0 \left\{ \frac{L^2}{2} + \frac{C_F \alpha_s}{4\pi} \left[\frac{5L^4}{12} + (L_m - 1) L^3 + \left(4 - \frac{\pi^2}{3} + \frac{L_m^2}{2} - \frac{L_h^2}{2} - 3L_m \right) L^2 \right. \right. \\
& \left. \left. + \left(\frac{2\pi^2}{3} + 8\zeta_3 \right) L - 8\zeta_3 L_m - 4\zeta_3 - \frac{\pi^4}{9} \right] \right\},
\end{aligned}
\tag{6.35}
$$

where

$$
\mathcal{M}_0 = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} m_b(\mu) \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2). \tag{6.36}
$$

These results produce the known total decay amplitude for the $h \to \gamma\gamma$ (leading order in α_s and $\mathcal{O}(m_b^2/M_h^2)$) [2]

$$
\begin{aligned}
\mathcal{M}_b = \mathcal{M}_0 \Bigg\{ & \left(\frac{L^2}{2} - 2 \right) + \frac{C_F \alpha_s(\mu)}{4\pi} \left[-\frac{L^4}{12} - L^3 + \left(4 - \frac{2\pi^2}{3} \right) L^2 \right. \\
& \left. + \left(12 + \frac{2\pi^2}{3} + 16\zeta_3 \right) L - 36 + 4\zeta_3 - \frac{\pi^4}{5} - \left(3L^2 - 12 \right) \ln \frac{m_b^2}{\mu^2} \right] \Bigg\},
\end{aligned}
\tag{6.37}
$$

where $L = \ln(-M_h^2/m_b^2)$ is the physical logarithm.

6.4 Large logarithms in the three-loop decay amplitude

In Section 4.5 we have derived the RG evolution equations for the operators and the hard coefficients at one-loop. Using the one-loop results for the renormalized corresponding expressions at their natural scale we derive here higher order logarithms at three-loop for the b-quark induced amplitude of $h \to \gamma\gamma$. In this way we are extending the $\mathcal{O}(\alpha_s)$ results we have found in Chapter 4 for the renormalized operator matrix elements and their Wilson coefficients.

6.4.1 Higher order logarithms in the matrix elements

For the matrix elements of the operator $O_1(\mu)$ in (4.31) the structure of the large logarithms remains the same at higher orders in QCD since at all orders it is given by the renormalized b-quark mass.

For the operator matrix elements $\langle O_2(z,\mu)\rangle$, $[\![\langle O_2(z,\mu)\rangle]\!]$ and $\langle O_3(z,\mu)\rangle$ the problem is less trivial. Solving the corresponding RG equations iteratively by using the one-loop results for the anomalous dimensions and the renormalized matrix elements we can explicitly calculate coefficients of logarithmic terms of $\mathcal{O}\left(\alpha_b \alpha_s^2 L_m^3\right)$. We start then with the operator matrix elements $\langle O_2(z,\mu)\rangle$, where generically we can write the three-loop result as

$$\langle O_2(z,\mu)\rangle = \frac{N_c \alpha_b}{2\pi} m_b(\mu) g_\perp^{\mu\nu} \left\{ -L_m + \mathcal{O}\left(\alpha_s\right) - C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left[L_m^3 f_3(z) + \mathcal{O}\left(L_m^2\right)\right] \right\} . \tag{6.38}$$

From the evolution from the second RGE in (4.49) we find

$$\begin{aligned} f_3(z) =& C_F \left(\frac{2}{3}\ln^2 z + 4\ln z + 3\right) + \frac{\beta_0}{3}\left(\ln z + \frac{3}{2}\right) \\ &+ C_F \left(\frac{2}{3}\ln^2(1-z) + 4\ln(1-z) + 3\right) + \frac{\beta_0}{3}\left(\ln(1-z) + \frac{3}{2}\right) . \end{aligned} \tag{6.39}$$

This coefficient is symmetric for z and $(1-z)$ which is exactly what we would expect for the operator O_2. Since the matrix element $\langle O_2(z,\mu)\rangle$ starts at $\mathcal{O}(\alpha_s^0)$, to derive the coefficeints of L_m^2 requires the two loop expression of the γ_{22}. In principle this result can be extracted from literature and then used to derive the $\mathcal{O}(L_m^2)$ term as well. The result for the matrix element $[\![\langle O_2(z,\mu)\rangle]\!]$ can be derived from the above expression by taking the appropriate $z \to 0$ limits. Similarly we can follow the same steps to find the three-loop large logarithms for the jet function, though in this case we can directly use the two-loop RG equation for the jet function from [102].

Next we look at the higher order logarithms for the soft function. The procedure is the same, though the derivations in this case are more complicated due to the presence of polylogarithms in the one-loop expression for the soft function in (4.21). To derive our result we have used several identities of polylogarithmic functions [26]. We can parametrize the final result here as

$$\begin{aligned} S_a(w,\mu) =& 1 + \mathcal{O}\left(\alpha_s\right) + C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg[\frac{C_F}{2} L_w^4 + \left(6C_F + \frac{\beta_0}{3}\right) L_w^3 + r_2 L_w^2 + r_1 L_w + \mathcal{O}\left(L_w^0\right) \\ &+ s_{3a}(\hat{w}) L_m^3 + s_{2a}(\hat{w}) L_m^2 + s_{1a}(\hat{w}) L_m + \mathcal{O}\left(L_m^0\right) \Bigg] , \\ S_b(w,\mu) =& \mathcal{O}\left(\alpha_s\right) + C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left[s_{3b}(\hat{w}) L_m^3 + s_{2b}(\hat{w}) L_m^2 + s_{1b}(\hat{w}) L_m + \mathcal{O}\left(L_m^0\right) \right] . \end{aligned} \tag{6.40}$$

After some rather cumbersome calculations we find

$$\begin{aligned} r_2 &= \left(6 + \frac{\pi^2}{2}\right) C_F + \left(\frac{32}{9} + \frac{\pi^2}{3}\right) C_A - \frac{16}{9} T_F n_f , \\ r_1 &= \left(-75 + 3\pi^2\right) C_F + \left(-\frac{1297}{27} + \frac{11\pi^2}{9} - 2\zeta_3\right) C_A + \left(\frac{428}{27} - \frac{4\pi^2}{9}\right) T_F n_f . \end{aligned} \tag{6.41}$$

The coefficients $s_{ia}(\hat{w})$ and $s_{ib}(\hat{w})$ in (6.40) vanish for $\hat{w} \to \infty$ and $\hat{w} \to 0$ respectively, where

$\hat{w} = w/m_b^2$ and $L_w = \ln(w/m_b^2)$. For the coefficients s_{ia} we obtain the following expressions

$$
\begin{aligned}
s_{3a}(\hat{w}) =& 4C_F \ln\left(1-\frac{1}{\hat{w}}\right) , \\
s_{2a}(\hat{w}) =& C_F \left[28\ln\left(1-\frac{1}{\hat{w}}\right) + 20\ln^2\left(1-\frac{1}{\hat{w}}\right) + 18\ln\hat{w}\ln\left(1-\frac{1}{\hat{w}}\right) + 10\,\mathrm{Li}_2\left(\frac{1}{\hat{w}}\right)\right] \\
&+ 2\beta_0 \ln\left(1-\frac{1}{\hat{w}}\right) , \\
s_{1a}(\hat{w}) =& C_F \Bigg[\left(-24+\frac{22\pi^2}{3}\right)\ln\left(1-\frac{1}{\hat{w}}\right) + 48\ln^2\left(1-\frac{1}{\hat{w}}\right) + 32\ln^3\left(1-\frac{1}{\hat{w}}\right) \\
&+ 60\ln\hat{w}\ln\left(1-\frac{1}{\hat{w}}\right) + 72\ln\hat{w}\ln^2\left(1-\frac{1}{\hat{w}}\right) + 24\ln^2\hat{w}\ln\left(1-\frac{1}{\hat{w}}\right) \\
&+ \left[12+12\ln\hat{w} - 8\ln\left(1-\frac{1}{\hat{w}}\right)\right]\mathrm{Li}_2\left(\frac{1}{\hat{w}}\right) - 48\mathrm{Li}_3\left(1-\frac{1}{\hat{w}}\right) + 48\zeta_3 \Bigg] \\
&+ C_A\left(-\frac{16}{3}+\frac{4\pi^2}{3}\right)\ln\left(1-\frac{1}{\hat{w}}\right) \\
&+ \beta_0\left[-\frac{8}{3}\ln\left(1-\frac{1}{\hat{w}}\right) + 4\ln^2\left(1-\frac{1}{\hat{w}}\right) + 6\ln\hat{w}\ln\left(1-\frac{1}{\hat{w}}\right) - 2\,\mathrm{Li}_2\left(\frac{1}{\hat{w}}\right)\right] .
\end{aligned}
\tag{6.42}
$$

To derive these expressions we have used the two-loop coefficients of the anomalous dimensions $\Gamma_{\rm cusp}$, γ_s and γ_m shown in Appendix B. For the coefficient functions $s_{ib}(\hat{w})$ we find

$$
\begin{aligned}
s_{3b}(\hat{w}) =& -4C_F\ln(1-\hat{w}) , \\
s_{2b}(\hat{w}) =& -C_F\Big[24\ln(1-\hat{w}) + 20\ln(1-\hat{w})^2 - 4\ln\hat{w}\ln(1-\hat{w}) + 4\,\mathrm{Li}_2(\hat{w})\Big] - 2\beta_0\ln(1-\hat{w}) , \\
s_{1b}(\hat{w}) =& C_F\Bigg[\left(48-\frac{22\pi^2}{3}\right)\ln(1-\hat{w}) - 32\ln^2(1-\hat{w}) - 32\ln^3(1-\hat{w}) \\
&- 12\ln\hat{w}\ln(1-\hat{w}) + 24\ln\hat{w}\ln^2(1-\hat{w}) + 8\ln^2\hat{w}\ln(1-\hat{w}) \\
&+(-4+8\ln\hat{w}+8\ln(1-\hat{w}))\,\mathrm{Li}_2(\hat{w}) - 8\mathrm{Li}_3(\hat{w}) + 48\mathrm{Li}_3(1-\hat{w}) - 48\zeta_3\Bigg] \\
&+ C_A\left(\frac{16}{3}-\frac{4\pi^2}{3}\right)\ln(1-\hat{w}) \\
&+ \beta_0\left[\frac{20}{3}\ln(1-\hat{w}) - 4\ln^2(1-\hat{w}) + 4\ln\hat{w}\ln(1-\hat{w}) + 4\mathrm{Li}_2(\hat{w})\right] .
\end{aligned}
\tag{6.43}
$$

The prediction of terms of $\mathcal{O}(\alpha_s^2 L^0_{w,m})$ would require an explicit NNLO calculation.

6.4.2 Higher order logarithms in the matching coefficients

We now use the RG equation for the Wilson coefficients shown in (4.53) to find the corresponding higher order large logarithms. For the matching coefficient H_2 the calculation is straightforward and we have

$$H_2(z,\mu) = \frac{y_b(\mu)}{\sqrt{2}} \frac{1}{z(1-z)} \left\{ 1 + \mathcal{O}(\alpha_s) + C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left[L_h^2 f_2(z) + \mathcal{O}(L_h) \right] \right\} , \tag{6.44}$$

where

$$f_2(z) = 2C_F \left(\ln^2 z - \beta_0 \ln z + \ln^2(1-z) - \beta_0 \ln(1-z) \right) . \tag{6.45}$$

The matching coefficient $[\![H_2(z,\mu)]\!]$ can be derived by taking the limit $z \to 0$ in the above result. For terms $\mathcal{O}(\alpha_s^2 L_h)$ also here we would need the two-loop coefficient of the γ_{22}.

In complete analogy for $H_3(z,\mu)$ we have

$$H_3(\mu) = -\frac{y_b(\mu)}{\sqrt{2}} \left[1 + \mathcal{O}(\alpha_s) + C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{C_F}{2} L_h^4 + \frac{\beta_0}{3} L_h^3 + c_2 L_h^2 + c_1 L_h + \mathcal{O}\left(L_h^0\right) \right) \right] , \tag{6.46}$$

where

$$\begin{aligned} c_2 &= \left(2 - \frac{\pi^2}{6}\right) C_F + \left(-\frac{67}{9} + \frac{\pi^2}{3}\right) C_A + \frac{20}{9} T_F n_f \,, \\ c_1 &= \left(-2\pi^2 + 24\zeta_3\right) C_F + \left(\frac{242}{27} + \frac{11\pi^2}{9} - 26\zeta_3\right) C_A + \left(-\frac{112}{27} - \frac{4\pi^2}{9}\right) T_F n_f . \end{aligned} \tag{6.47}$$

Lastly we can use the evolution equation we derived in (6.34) to predict the large logarithms for the Wilson coefficient H_1. Using also the result from (6.44) for H_2 it is not difficult to perform the integration in (6.34). We get

$$H_1(\mu) = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} \left[-2 + \mathcal{O}(\alpha_s) + C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left(k_3 L_h^3 + \mathcal{O}\left(L_h^2\right) \right) \right] , \tag{6.48}$$

where

$$k_3 = \left(\frac{2\pi^2}{3} - \frac{16}{3}\zeta_3 \right) C_F + \frac{2\pi^2}{9}\beta_0 . \tag{6.49}$$

The unknown term $\mathcal{O}(\alpha_s^2 L_h^2)$ here would require the $\mathcal{O}(\alpha_s^2 L_h)$ result from the Wilson coefficient $H_2(z,\mu)$.

6.4.3 Three-loop large logarithms in the decay amplitude

We can now combine the results on the higher order logarithmic structure of the Wilson coefficients and the operator matrix elements to predict the three-loop structure of the large logarithms in the total decay amplitude $\mathcal{M}_b(h \to \gamma\gamma)$. We write the results as

$$
\begin{aligned}
\mathcal{M}_b =& \mathcal{M}_0(\mu) \Bigg\{ \left(\frac{L^2}{2} - 2\right) + \frac{C_F \alpha_s(\mu)}{4\pi} \left[-\frac{L^4}{12} - L^3 + \left(4 - \frac{2\pi^2}{3}\right) L^2 \right. \\
&+ \left. \left(12 + \frac{2\pi^2}{3} + 16\zeta_3\right) L - 36 + 4\zeta_3 - \frac{\pi^4}{5} - \left(3L^2 - 12\right) L_m \right] \\
&+ C_F \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} + \frac{\beta_0}{20}\right) L^5 + d_4 L^4 + d_3 L^3 \right. \\
&+ L_m \left[\left(\frac{C_F}{2} + \frac{\beta_0}{12}\right) L^4 + (6C_F + \beta_0) L^3 + d_{1,2} L^2 \right] \\
&\left. + L_m^2 L^2 \left(9C_F + \frac{3}{2}\beta_0\right) + \dots \right] + \mathcal{O}\left(\frac{m_b^2}{M_h^2}\right) \Bigg\} ,
\end{aligned}
\tag{6.50}
$$

where the prefactor in front contains the renormalized b-quark Yukawa $y_b(\mu)$ coupling and the running b-quark mass $m_b(\mu)$

$$
\mathcal{M}_0(\mu) = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu)}{\sqrt{2}} m_b(\mu) \varepsilon_\perp^* (k_1) \cdot \varepsilon_\perp^* (k_2) . \tag{6.51}
$$

For the various coefficients in front of the logarithmic terms we find

$$
\begin{aligned}
d_4 &= \left(\frac{5}{6} + \frac{\pi^2}{18}\right) C_F + \left(\frac{8}{27} + \frac{\pi^2}{36}\right) C_A - \frac{4}{27} T_F n_f , \\
d_3 &= \left(-\frac{17}{2} + \frac{7\pi^2}{9} + \frac{20}{3}\zeta_3\right) C_F + \left(-\frac{199}{18} + \frac{44\pi^2}{27} - 4\zeta_3\right) C_A + \left(\frac{22}{9} - \frac{16\pi^2}{27}\right) T_F n_f , \\
d_{1,2} &= \left(-\frac{51}{2} + 4\pi^2\right) C_F + \left(-\frac{185}{6} + \frac{22\pi^2}{9}\right) C_A + \left(\frac{26}{3} - \frac{8\pi^2}{9}\right) T_F n_f .
\end{aligned}
\tag{6.52}
$$

The dots in (6.50) represent three-loop contributions with less than three powers of logarithms L or L_m. Note that the amplitude in (6.50) is scale independent because the overall μ dependence is cancelled among $y_b(\mu)$, $m_b(\mu)$, $\alpha_s(\mu)$ and $L_m = \ln(m_b^2/\mu^2)$.

To get an estimate of the large logarithmic effects we can have a look at the numerical values of the coefficients. For simplicity here we fix $\mu = m_b$ to eliminate the terms multiplied by L_m. Then we find

$$
C_F \left[\frac{C_F}{90} L^6 + \left(\frac{C_F}{10} + \frac{\beta_0}{20}\right) L^5 + d_4 L^4 + d_3 L^3 \right] \approx C_F \left[0.015 L^6 + 0.517 L^5 + 2.232 L^4 + 1.736 L^3\right] . \tag{6.53}
$$

This simple computation shows that the coefficient of other logarithms, beyond the large double logarithms, can still be significant and in fact here the coefficient of L^3 is larger by a factor of 10^2 compared to the coefficient of L^6. We see in this example that higher order logarithms are not necessarily negligible and need to be resummed correctly.

Chapter 7

Resummation

In this Chapter we discuss aspects of the resummation of the large logarithms in the b-quark induced decay amplitude for $h \to \gamma\gamma$. The leading and the subleading logarithms at some scale μ are contained in the double convolution T_3. We have already seen in fact that due to the collinear anomaly there is a logarithmic enhancement in this term. Moreover the soft-quark soft function resides in this last term in the amplitude and it contributes double logarithms starting from $\mathcal{O}(\alpha_s)$. The logarithmic structure in T_3 is of the $\mathcal{O}(\alpha_b \alpha_s^n L^{2n+2})$ for $L = \ln(-M_h^2/m_b^2)$. This is a different structure compared to T_1 and T_2, where they have logarithms of $\mathcal{O}(\alpha_b \alpha_s^n L^{n+1})$. We showed in the previous Chapter that the tower of large logarithms can be significantly large and reliable results for the decay $h \to \gamma\gamma$ requires exponentiation of these large logarithms.

We have already derived in Chapter 4 the one-loop RG evolution equations for all the operator matrix elements and their Wilson coefficients. With these results we can perform the resummation of large logarithms at leading order in RG-improved perturbation theory for the terms T_1 and T_3. For the amplitude T_2 with our results we can derive an almost leading order RG-improved result. For a complete calculation in this case we would need also the two-loop expressions for the anomalous dimension $\gamma_{22}(z, z')$, which can in principle be extracted from literature. For the other anomalous dimensions we need their one-loop QCD results, for the cusp anomalous dimensions and the β-function the two-loop expressions are needed. The renormalized operator matrix elements and their Wilson coefficients are needed at lowest order and calculated at their natural scale free of large logarithms.

Resummation at leading order in RG-improved perturbation theory corresponds to keeping up to $\mathcal{O}(\alpha_s^0)$ terms in the solutions of the RG evolution equations. In here we fix the scale for the evaluation of the amplitude to the hard scale $\mu_h = -M_h^2 - i0$. This is convenient since it is the natural scale for all the Wilson coefficients and we then only need to run the operator matrix elements. In addition this is in particular convenient for the hard coefficient $H_1(\mu)$ which obeys the rather complicated RGE in (6.34). This is difficult to solve due to the presence of the non-homogeneous term D_{cut}. As a low running scale it is natural to choose μ_s around the b-quark mass m_b, since all the operator matrix elements are free of large logarithms as m_b. Though we note here that for the case of T_3 the soft scale μ_s should be treated dynamically. We will see in Section 7.4 why this is the case there.

We start in the next Section with the resummation in T_1, then discuss carefully the scale evolution and the cancellation of the endpoint divergences in the amplitude T_2. We show in Section 7.2 that after resummation in our subtraction scheme the amplitude T_2 is well defined and the divergences at the endpoint cancel. In Section 7.3 we present a semi-numerical solution for the resummed T_2 and in Section 7.4 we present the result for the leading order T_3 in RG-improved perturbation theory.

7.1 Resummation in T_1

The resummation for the matrix elements of $O_1(\mu)$ is the most straightforward one. It obeys the local RG evolution presented in equation (4.49), which at all orders has the simple solution

$$\langle O_1(\mu_h)\rangle = \langle O_1(\mu_s)\rangle \exp\left[-\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu_h)} \mathrm{d}\alpha \frac{\gamma_{11}(\alpha)}{\beta(\alpha)}\right] . \tag{7.1}$$

At leading order, relevant for our discussion we have

$$\langle O_1(\mu_h)\rangle = \langle O_1(\mu_s)\rangle \left(\frac{\alpha_s(\mu_h)}{\alpha_s(\mu_s)}\right)^{3C_F/\beta_0} . \tag{7.2}$$

Here $\langle O_1(\mu_s)\rangle$ is the renormalized operator matrix element evaluated at the low scale μ_s at leading order in α_s in this case.

7.2 Endpoint divergences in T_2

The resummation of the amplitude T_2 is less obvious not only because of the non-local RG evolution of the matrix elements $\langle O_2(z,\mu)\rangle$ and $[\![\langle O_2(z,\mu)\rangle]\!]$ in (4.49), but also due to the presence of the subtraction terms in T_2. We recall that these terms are necessary to fully cancel the endpoint divergences in the amplitude. It is important to make sure that these divergences are cancelled also after the scale evolution in T_2. We show this explicitly in this Section.

7.2.1 RG evolution of $\langle O_2(z,\mu)\rangle$ in Gegenbauer space

To prove that the endpoint divergences are cancelled after RG evolution we start with the observation that the anomalous dimension $\gamma_{22}(z,z')$ in (4.45) is related to the well-known "Brodsky-Lepage" kernel for the B-meson LCDA. It is known that at leading order Gegenbauer polynomials $C_m^{3/2}(2z-1)$ are eigenfunctions of this kernel such that [119, 120]

$$\int_0^1 dz C_m^{(3/2)}(2z-1)\gamma_{22}(z,z') = \tilde{\gamma}_{22}(m)\, C_m^{(3/2)}(2z'-1) , \tag{7.3}$$

where now the function $\tilde{\gamma}_{22}(m)$ is local. The new anomalous dimension $\tilde{\gamma}_{22}(m)$ depends also on the running scale $\alpha_s(\mu)$ just like $\gamma_{22}(z,z')$ but for simplicity in notation we suppress it here. The Gegenbauer polynomials enjoy the following orthogonal and completeness relations

$$
\begin{aligned}
&6N(m)\int_0^1 dzz(1-z)C_m^{3/2}(2z-1)C_n^{3/2}(2z-1)=\delta_{mn}\,,\\
&6z(1-z)\sum_{m=0}^{\infty}N(m)C_m^{3/2}(2z-1)C_m^{3/2}\left(2z'-1\right)=\delta\left(z-z'\right)\,,
\end{aligned}
\tag{7.4}
$$

where $N(m) = 2(2m+3)/\left(3(m+1)(m+2)\right)$. For $m = 2k$ with $k \in \mathbb{N}$, the polynomials $C_m^{(3/2)}(2z-1)$ are symmetric for $z \to 1-z$. We recall that this is also a property of $\langle O_2(z,\mu)\rangle$.

The realtion (7.3) becomes very useful to solve the non-local RGE of $\langle O_2(z,\mu)\rangle$ if we can write this matrix elements in Gegenbauer space. In fact it is possible to express the operator matrix elements $\langle O_2(z,\mu)\rangle$ in terms of Gegenbauer polynomials such that [120, 121]

$$
\langle O_2(z,\mu)\rangle = \frac{N_c\alpha_b}{2\pi}6z(1-z)\sum_{m=0}^{\infty}b_{2m}(\mu)C_{2m}^{(3/2)}(2z-1)\,. \tag{7.5}
$$

The functions $b_m(\mu)$ are called Gegenbauer moments. At $\mathcal{O}(\alpha_s)$ we can find their expressions using the known result for the operator matrix element $\langle O_2(z,\mu)\rangle$ at $\mathcal{O}(\alpha_s)$ together with the relations in (7.4).

After this transformation the scale dependence is contained in the Gegenbauer moments. We then derive the corresponding RGE for these coefficients from the RGE of $\langle O_2(z,\mu)\rangle$ in (4.49), using also the property in (7.3). We find that they obey a multiplicative RG evolution

$$
\frac{d\,b_m(\mu)}{d\ln\mu} = -\tilde{\gamma}_{22}\left(m\right)b_m(\mu) - \tilde{\gamma}_{21}\left(m\right)\left\langle O_1(\mu)\right\rangle\,, \tag{7.6}
$$

for $m = 2k$ where $k \in \mathbb{N}$ as before. The functions $\tilde{\gamma}_{22}$ and $\tilde{\gamma}_{21}$ are the new anomalous dimensions in the Gegenbauer space. The analytic expression for $\tilde{\gamma}_{22}(m)$ is relatively simple, where at leading order we obtain

$$
\tilde{\gamma}_{22}(m) = \frac{C_F\alpha_s}{2\pi}\left(4H_{m+1}-3\right)+\mathcal{O}\left(\alpha_s^2\right)\,. \tag{7.7}
$$

For the non-diagonal term in (7.6) we have

$$
\begin{aligned}
\tilde{\gamma}_{21}\left(m\right) = -\frac{4(2m+3)}{3(m+1)(m+2)}\Bigg\{1+\frac{C_F\alpha_s}{4\pi}\Bigg[&4H_{m+1}^2-2\psi^{(1)}\left(\frac{m}{2}+1\right)+2\psi^{(1)}\left(\frac{m+3}{2}\right)\\
&+11-\frac{2\pi^2}{3}\Bigg]+\mathcal{O}(\alpha_s^2)\Bigg\}\,,
\end{aligned}
\tag{7.8}
$$

where $\psi^{(1)}(x)$ is the first derivative of the digamma function $\psi(x)$.

The scale evolution takes a much simpler structure now and (7.6) can be solved analytically. We can start by solving the homogeneous equation

$$
\frac{d\,b_m(\mu)}{d\ln\mu} = -\tilde{\gamma}_{22}\left(m\right)b_m(\mu)\,, \tag{7.9}
$$

and then substitute it to (7.6) to find a solution to the complete inhomogeneous RGE. For some initial condition $b_m(\mu_s)$ we find

$$b_m(\mu) = e^{a_{\tilde{\gamma}_{22}}(\mu_s,\mu)}\Big[b_m(\mu_s) + 2N(m)\langle O_1(\mu_s)\rangle a_{\tilde{\gamma}_{21}}(\mu_s,\mu)\, e^{a_{\gamma_{11}}(\mu_s,\mu_\alpha)-a_{\tilde{\gamma}_{22}}(\mu_s,\mu_\alpha)}\Big]\,, \tag{7.10}$$

where $a_\gamma(\mu_s,\mu)$ is defined as

$$a_\gamma\left(\mu_s,\mu\right) = -\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)} d\alpha \frac{\gamma(\alpha)}{\beta(\alpha)}\,. \tag{7.11}$$

The renormalized $\langle O_2(z,\mu)\rangle$ in (4.32) at leading order is proportional to L_m and as a result also $b_m(\mu_s)$ will be proportional to L_m. Therefore for $\mu_s = m_b$ the initial condition in the above solution vanishes. In fact it is not difficult to see from (7.5) that

$$b_m(\mu) = -g_\perp^{\mu\nu}\frac{2}{3m}m_b\left(\frac{\alpha_s(\mu)}{\alpha_s(m_b)}\right)^{-\frac{\gamma_{m,0}}{2\beta_0}} L_m + \mathcal{O}(\alpha_s)\,, \tag{7.12}$$

where $L_m = \ln(m_b^2/\mu^2)$.

For our purpose of discussing the endpoint divergences in T_2, we only need to consider the asymptotic behaviour of the solution in (7.10). This corresponds to taking the limit $m \to \infty$ there. Using the one-loop expressions for the various anomalous dimensions we find that in the asymptotic limit the solution in (7.10) reads

$$b_m(\mu) = \frac{4\pi}{\alpha_s(\mu_s)}\frac{N(m)}{\beta_0 + 2C_F\left(2H_{m+1}-3\right)} r^{\frac{3C_F}{\beta_0}-1}\langle O_1(\mu_s)\rangle + \dots\,, \tag{7.13}$$

where $r = \alpha_s(\mu)/\alpha_s(\mu_s)$ and the dots represent finite terms of $\mathcal{O}(1)$. We note here thought that for the complete resummation at leading order in RG-improved perturbation theory we would also need to consider these $\mathcal{O}(1)$ contributions. For this discussion we can safely neglect these terms as they cannot change the functional behaviour of the amplitude terms to affect the cancellation of the divergences.

7.2.2 Cancellation of endpoint divergences

In order to study the endpoint behaviour in the amplitude T_2 we need to find the asymptotic limit for the convolutions$H_2(z,\mu)\otimes\langle O_2(z,\mu)\rangle$ and the subtraction terms of the form $[\![\bar{H}_2(z,\mu)]\!]\otimes[\![\langle O_2(z,\mu)\rangle]\!]$ that appear in the factorization theorem in (6.1). Eventually we fix $\mu = \mu_h$ so it is enough to just transform the fixed order results for the Wilson coefficients in Gegenbauer space. We can define the function $h_m(\mu)$, for $m = 2k$ such that

$$h_m(\mu) = \int_0^1 dz H_2(z,\mu) z(1-z) C_m^{(3/2)}(2z-1) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-4L_hH_{m+1} + 4H_{m+1}^2 - 3\right]\,, \tag{7.14}$$

where $L_h = \ln(-M_h^2/\mu^2)$. Here we will use only the leading term in (7.14) which is trivial. Then at lowest order and for $m \to \infty$ combing the result in (7.14) with the solution in (7.13) we find that the convolution of the operator matrix element $\langle O_2(z,\mu)\rangle$ with its Wilson coefficient in Gegenbauer space takes the following form

$$\int_0^1 dz H_2(z,\mu)\langle O_2(z,\mu)\rangle\,|_{\rm div} = \frac{3N_c\alpha_b}{\pi}\sum_{m=0}^{\infty}\frac{4\pi}{\alpha_s(\mu_s)}\frac{N(2m)}{\beta_0+2C_F\left(2H_{2m+1}-3\right)}r^{\frac{3C_F}{\beta_0}-1}\langle O_1(\mu_s)\rangle\ . \tag{7.15}$$

The sum in this expression is divergent and this is a manifestation of the remaining endpoint divergences after resummation. We are already familiar with the fact that the scale evolution does not necessarily remove these divergences.

In the case of the subtraction term it is not convenient to solve the RGE in (4.49) in Gegenbauer space because the anomalous dimension $[\![\gamma_{22}(z,z')]\!]$ in (4.45) is not related to the "Brodsky-Lepage" kernel. Instead we make use of another property of $[\![\gamma_{22}(z,z')]\!]$. We observe that $[\![\gamma_{22}(z,z')]\!]$ has a similar non-local structure to $\gamma_J(z,z')$. This implies we should be able to transform the operator matrix element $[\![\langle O_2(z,\mu)\rangle]\!]$ in the diagonal space with the same transformation given in (5.47) and derive a local RGE for $[\![\langle O_2(z,\mu)\rangle]\!]$. In principle this derivation is very similar to the same calculation for the jet function in Chapter 5. We show different steps of this derivation in Appendix E.

We find that the diagonal space solution $[\![\langle \tilde{O}_2(z,\mu)\rangle]\!]$ has a very similar structure to the solution in (7.10) (see equation (E.4)), which can then be readily transformed to a momentum space solution in complete analogy to the transformation (5.50). Then we find that the subtraction term at the lowest order yields the following result in momentum space

$$\begin{aligned}
&-\int_0^1 dz\left[\frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\langle O_2(z,\mu)\rangle]\!]+\frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right]\\
&=-\frac{4\pi}{\alpha_s(\mu_s)}\frac{1}{\beta_0-2C_F\left(\partial_\eta+3\right)}\frac{\Gamma(1-\eta)}{\Gamma(1+\eta)}r^{\frac{3C_F}{\beta_0}-1}e^{-2\gamma_E}\int_0^1 dz\,\left(z^{\eta-1}+\bar{z}^{\eta-1}\right)\Big|_{\eta=0}\langle O_1(\mu_s)\rangle+\dots\,.
\end{aligned} \tag{7.16}$$

The other neglected terms are proportional to $z^{\eta-1-a_\Gamma}$ and a_Γ serves as a regulator for those cases. The above integrand can be written in terms of Gegenbauer polynomials by using the orthogonal relation in (7.4). We then have

$$z^{-1+\eta}+\bar{z}^{-1+\eta}=\sum_{m=0}^{\infty}6N(2m)\frac{(2m+1)(2m+2)\Gamma(1+\eta)\Gamma(2m+1-\eta)}{\Gamma(1-\eta)\Gamma(2m+3+\eta)}C_{2m}^{3/2}(2z-1)\,, \tag{7.17}$$

where for any m positive integer, the integral over z here gives 1.

The Wilson coefficient $[\![\bar{H}_2(z,\mu)]\!]$ is a constant at lowest order like in (7.14). Then the divergent part from the subtraction term in T_2 reads

$$\begin{aligned}
&\int_0^1 dz\left[\frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\langle O_2(z,\mu)\rangle]\!]+\frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}\;[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right]\Bigg|_{\rm div}\\
&=-\frac{3N_c\alpha_b}{\pi}\sum_{m=0}^{\infty}\frac{4\pi}{\alpha_s(\mu_s)}\;\frac{N(2m)}{\beta_0+2C_F\left(2H_{2m+1}-3-\mathcal{O}(1/m^2)\right)}r^{\frac{3C_F}{\beta_0}-1}\langle O_1(\mu_s)\rangle\,,
\end{aligned} \tag{7.18}$$

where the suppressed term of $\mathcal{O}(1/m^2)$ is $1/\left((2m+1)(2m+2)\right)$. Comparing this last result with the result in (7.15) it is clear that the subtraction term does cancel the endpoint divergences also after resummation. This is another consistency check of our method of regulating the divergences by means of a plus-type subtraction scheme.

7.3 Resummation in T_2

The analytical approach we have just presented is useful to check explicitly that our approach of dealing with the endpoint divergences in the decay process $h \to \gamma\gamma$ is consistent also with the scale evolution. For practical purposes it is very inconvenient to use this approach to resum the large logarithms in T_2 because the calculations become quite complicated for higher order terms. A better way is to solve the RGEs for the matrix elements of $O_2(z,\mu)$ and $[\![O_2(z,\mu)]\!]$ with a semi-numerical method. In the following Section we explain in details the derivation steps for this approach. The reader who is not interested in the technical details can skip this part and jump to the numerical result we present in paragraph 7.3.3 of this Section.

7.3.1 Semi-numerical resummation

We start with solving the RG evolution of $[\![\langle O_2(z,\mu)\rangle]\!]$ at leading order in RG-improved perturbation theory. Formally the solution of its RG equation can be written as

$$[\![\langle O_2(z,\mu_h)\rangle]\!] = [\![\langle O_2(z,\mu_s)\rangle]\!] - \int_{\mu_s}^{\mu_h} d\ln\mu \left[\int_0^\infty dz' [\![\gamma_{22}(z,z')]\!][\![\langle O_2(z',\mu)\rangle]\!] + [\![\gamma_{21}(z)]\!]\,\langle O_1(\mu)\rangle\right], \tag{7.19}$$

where we run from the soft scale μ_s to the hard scale μ_h. We can integrate numerically over μ by slicing the scale interval into infinitesimal intervals $\Delta\mu_i \equiv \mu_{i+1} - \mu_i$ for $i \in \{0,1,2,..,n\}$, $n \in \mathbb{N}$. At leading order the initial condition $[\![\langle O_2(z,\mu_s)\rangle]\!] = 0$, for $\mu_s = m_b$, which follows immediately from (4.34).

The first non-vanishing contribution in the numerical iteration of (7.19) comes from the off-diagonal term with the anomalous dimension $[\![\gamma_{21}(z)]\!]$. The scale integration for this term can be performed in a closed form for any integration interval $\Delta\mu_i$. After changing variables from $\ln\mu$ to α for any value of $i \in \mathbb{N}$ we find

$$\begin{aligned}\int_{\alpha_s(\mu_i)}^{\alpha_s(\mu_{i+1})} \frac{\mathrm{d}\alpha}{\beta(\alpha)} [\![\gamma_{21}(z,\mu_\alpha)]\!] \langle O_1(\mu_\alpha)\rangle = \frac{N_c\alpha_b}{2\pi} m_b(\mu_h) &\left[\frac{4\pi}{\alpha_s(\mu_h)} \frac{1}{3C_F-\beta_0}\left(r_{i+1}^{1-3C_F/\beta_0} - r_i^{1-3C_F/\beta_0}\right)\right. \\ &\left.+\frac{1}{3}\left([\![\gamma_{21}^{(1)}(z)]\!] - \frac{\beta_1}{C_F\beta_0}\right)\left(r_{i+1}^{-3C_F/\beta_0} - r_i^{-3C_F/\beta_0}\right) + \mathcal{O}(\alpha_s)\right],\end{aligned} \tag{7.20}$$

where $r_i = \alpha_s(\mu_h)/\alpha_s(\mu_i)$ and we have used the leading order solution in (7.2) for the evolution of the operator matrix element $\langle O_1(\mu_\alpha)\rangle$. Here $[\![\gamma_{21}^{(1)}(z)]\!]$ is the one-loop coefficient of $[\![\gamma_{21}(z)]\!]$ in (4.46) proportional to $C_F\alpha_s/(4\pi)$. The complete leading order RG-improved solution requires also the two-loop expressions for the anomalous dimension $[\![\gamma_{22}(z,z')]\!]$, though for the purpose of numerical resummation we believe these contributions are small.

For $i \geq 1$ the iteration includes integration over the variable z as well and this brings the main complication in the solution due to the presence of the non-local term in $[\![\gamma_{21}(z)]\!]$, which is in fact nothing but the "Lange-Neubert" kernel in (5.3). The dependence on z changes at any step over scale integration and this means that for any interval $\Delta\mu_i$ for $i \geq 1$ the integrand that is integrated with the plus distribution is different. In principle this would break the numerical iteration. In our case we observe that the z-dependence of this integrand is always a logarithmic function which has the advantage that it can be cast into a power function using the derivative operator. This is very useful because we have already seen that any power functions is an eigenfunction of the "Lange-Neubert" kernel. This property of the kernel makes it possible to use a simple iterative procedure to integrate slice by slice over μ and perform the integration over z-analytically for any value of μ. The remaining contribution from the local terms ($z = 1$) is trivial at any step. The final result we find for the resummed operator matrix elements $[\![\langle O_2(z,\mu)\rangle]\!]$ at leading order in RG-improved perturbation theory is a logarithmic function in z such that

$$\begin{aligned}[\![\langle O_2(z,\mu_h)\rangle]\!] \approx \frac{N_c\alpha_b}{\pi} m_b(\mu_h)\Big[&(3.461 - 1.622i) - (0.293 - 0.241i)\ln z \\ &+ (0.073 - 0.043i)\ln^2 z + (0.006 - 0.006i)\ln^3 z + \ldots\Big]\,.\end{aligned} \tag{7.21}$$

Higher logarithmic terms are suppressed in the above expression. The coefficient of $\ln^4 z$ is suppressed by a factors of 10^3 and higher logarithms are suppressed by 10^6 or larger factors.

7.3.2 T_2 at LO in RG-improved perturbation theory

We now discuss the resummation in the convolution T_2 defined as

$$T_2 = 2\int_0^1 dz \left[H_2\,(z,\mu)\langle O_2(z,\mu\,)\rangle - \frac{[\![\,\bar{H}_2(z,\mu)\,]\!]}{z}\,[\![\langle O_2\,(z,\mu)\rangle]\!] - \frac{[\![\,\bar{H}_2(\bar{z},\mu)\,]\!]}{\bar{z}}[\![\langle O_2\,(\bar{z},\mu)\rangle]\!]\right]. \tag{7.22}$$

The operator matrix elements that appear above should be substituted by their solutions after evolution to the hard scale μ_h and for the Wilson coefficients we use the lowest order renormalized expressions at the hard scale μ_h from (4.43). We will see that at leading order we only need the resummation of the operator matrix elements $[\![\langle O_2(z,\mu)\rangle]\!]$ we have just derived and the difference $\langle O_2(z,\mu)\rangle - [\![\langle O_2(z,\mu)\rangle]\!]$. It is not necessary to separately solve the RGE for the $\langle O_2(z,\mu)\rangle$. Moreover the non-local part of $\gamma_{22}(z,z')$ has a vanishing contribution at leading order.

To see this notice in (4.43) that $H_2(z,\mu_h)$ contains a factor of $z(1-z)$ in the denominator. At leading order we only need to consider $\mathcal{O}(1)$ terms in $H_2(z,\mu_h)$ since the anomalous dimension $\gamma_{22}(z,z')$ in (4.45) starts already at $\mathcal{O}(\alpha_s)$. Then the z-dependence on the Wilson coefficient here drops out. We can interchange the integration over z and z' (they have the same boundaries in this case) such that the contribution from the non-local term would be

$$\frac{y_b(\mu_h)}{\sqrt{2}}\int_0^1 dz'\int_0^1 dz\left[\frac{1}{z'(1-z)}\frac{\theta\,(z'-z)}{z'-z} + \frac{1}{z\,(1-z')}\frac{\theta\,(z-z')}{z-z'}\right]_+ \langle O_2(z',\mu_\alpha)\rangle\,. \tag{7.23}$$

This integral is clearly zero from the definition of the plus-distribution. Note that such an interchange is not possible for the subtraction term because the integration over z' runs up to infinity in that case. Then the only non-local contribution comes from the evolution term with the anomalous dimension $[\![\gamma_{22}(z,z')]\!]$, where we have

$$T_2^{\text{non-local}} = \frac{y_b(\mu_h)}{\sqrt{2}} \frac{8C_F}{\beta_0} \sum_{i\geq 0} \ln \frac{\alpha_s(\mu_{i+1})}{\alpha_s(\mu_i)} \int_0^1 dz \int_0^\infty dz' \left[\frac{\theta(z-z')}{z(z-z')} + \frac{\theta(z'-z)}{z'(z'-z)} \right]_+ [\![\langle O_2(z',\mu_i)\rangle]\!], \tag{7.24}$$

where the sum here is a result of the scale integration in the slice $\Delta\mu_i$. This expression can be written in a convenient way by using

$$\begin{aligned} &\int_0^\infty dz \int_0^\infty dz' \theta(1-z) \left[\frac{\theta(z-z')}{z(z-z')} + \frac{\theta(z'-z)}{z'(z'-z)} \right]_+ [\![\langle O_2(z',\mu_i)\rangle]\!] \\ &= \left[\int_0^1 \frac{\mathrm{d}z}{z} \ln(1-z) - \int_1^\infty \frac{\mathrm{d}z}{z} \ln\left(1-\frac{1}{z}\right) \right] [\![\langle O_2(z,\mu_i)\rangle]\!] . \end{aligned} \tag{7.25}$$

The diagonal terms in the convolution (7.22) are proportional to $\langle O_2(z,\mu)\rangle$ and $[\![\langle O_2(z,\mu)\rangle]\!]$, where the evolution of $\langle O_2(z,\mu)\rangle$ receives contribution only from the local part of the anomalous dimensions now. After a few steps of calculations we find the following result

$$\begin{aligned} T_2^{\text{diag}} = -\frac{y_b(\mu_h)}{\sqrt{2}} \frac{8C_F}{\beta_0} \sum_{i\geq 0} \ln \frac{\alpha_s(\mu_{i+1})}{\alpha_s(\mu_i)} \Bigg[&\int_1^\infty \frac{dz}{z} \ln\left(1-\frac{1}{z}\right) [\![\langle O_2(z,\mu_i)\rangle]\!] \\ &+ \int_0^1 \frac{dz}{z} \left(\ln z + \ln(1-z) + \frac{3}{2} \right) \left(\langle O_2(z,\mu_i)\rangle - [\![\langle O_2(z,\mu_i)\rangle]\!] \right) \Bigg] . \end{aligned} \tag{7.26}$$

We can also perform a change of variables in the last integral here to change the boundaries from $(1,\infty)$ to $(0,1)$.

From the result in (7.26) we can now explicitly see that we need to define the evolution of a new operator matrix element $\Delta\langle O_2(z,\mu)\rangle = \langle O_2(z,\mu_i)\rangle - [\![\langle O_2(z,\mu_i)\rangle]\!]$. This evolution can be solved iteratively in complete analogy with what we have presented for the matrix element $[\![O_2(z,\mu)]\!]$ in Section 7.3.1. In this case we obtain

$$\begin{aligned} \Delta O_2(z,\mu_{i+1}) = &\Delta O_2(z,\mu_i) - \int_{\alpha_z(\mu_i)}^{\alpha_\pi(\mu_{i+1})} \frac{\mathrm{d}\alpha}{\beta(\alpha)} \Delta\gamma_{21}(z,\alpha) \langle O_1(\mu_\alpha)\rangle - \frac{2C_F}{\beta_0} \ln \frac{\alpha_s(\mu_{i+1})}{\alpha_s(\mu_i)} \\ &\times \left[\left(\ln z + \ln(1-z) + \frac{3}{2} \right) \Delta O_2(z,\mu_i) + \ln(1-z) [\![\langle O_2(1/z,\mu_i)\rangle]\!] \right] , \end{aligned} \tag{7.27}$$

where $\Delta\gamma_{21}(z,\alpha) = \gamma_{21}(z,\alpha) - [\![\gamma_{21}(z,\alpha)]\!]$. Their one-loop explicit expressions are shown in (4.45). The initial condition for ΔO_2 vanishes. The integral in the first line above can be computed in a closed form and it yields

$$\int_{\alpha_z(\mu_i)}^{\alpha_\pi(\mu_{i+1})} \frac{\mathrm{d}\alpha}{\beta(\alpha)} \Delta\gamma_{21}(z,\alpha) \langle O_1(\mu_\alpha)\rangle = \frac{N_c\alpha_b}{6\pi} m_b(\mu_h) \left(r_{i+1}^{-3C_F/\beta_0} - r_i^{-3C_F/\beta_0} \right) . \tag{7.28}$$

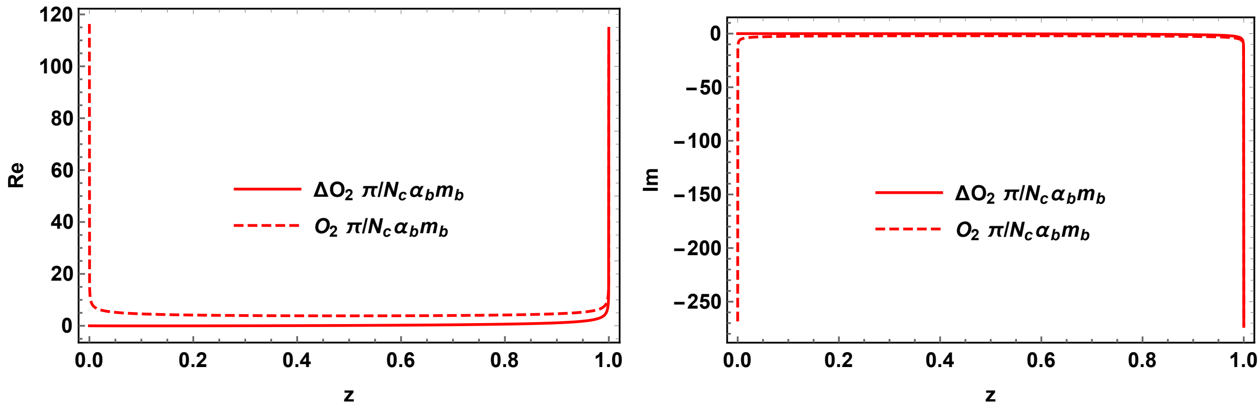

Figure 7.1: Dependence of the resummed functions $\Delta O_2(z,\mu_h)$ (solid line) and the operator matrix elements of $O_2(z,\mu_h)$ (dashed line) in units of $\pi/N_c\alpha_b m_b$. On the left side it is plotted the real part of the functions and the right plot shows the imaginary part.

We combine this result together with numerical results for the $[\![O_2(z,\mu)]\!]$ we derived in (7.21) in the numerical iteration and we find the following result for the evolution of $\Delta O_2(z,\mu)$

$$\begin{aligned}\Delta O_2(z,\mu_h) \approx \frac{N_c\alpha_b}{\pi} m_b(\mu_h)\Big[&(-0.003-2.04i)\ln(1-z)+(0.002-0.001i)\ln^2(1-z)\\ &+(0.009-0.005i)\ln(1-z)\ln z+\ldots\Big],\end{aligned} \tag{7.29}$$

where the dots represent suppressed higher logarithmic terms.

From the above result it is possible to extract numerically also the operator matrix elements $\langle O_2(z,\mu_h)\rangle$ after evolution. In Fig.(7.1) we show the z-dependence of the resummed function $\Delta O_2(z,\mu_h)$ and the operator matrix elements of $O_2(z,\mu_h)$ at the hard scale μ_h. On the left we present the dependence of the the real part of the function on the variable z and on the right the imaginary part dependence on z. It is clear from the plots that the $\Delta O_2(z,\mu_h)$ vanishes for $z\to 0$ which is an indicator that the endpoint divergences cancel numerically as well. The plot for the operator O_2 is symmetric for z and $1-z$ as we would expect in this case.

Lastly we look at the contribution from the mixing terms in the off-diagonal T_2 due to the evolution of $\langle O_2(z,\mu)\rangle$ and $[\![\langle O_2(z,\mu)\rangle]\!]$. This part of the scale evolution is much simpler and can be all computed in a closed form. In this case we obtain

$$\begin{aligned}T_2^{\text{off}-\text{diag}} &= -4\int_0^1 dz\int_{\alpha_s(m_b)}^{\alpha_s(\mu_h)}\frac{d\alpha}{\beta(\alpha)}\Big[\bar{H}_2(z,\mu_h)\,\gamma_{21}(z;\alpha)-[\![\bar{H}_2(z,\mu_h)]\!][\![\gamma_{21}(z;\alpha)]\!]\Big]\langle O_1(\mu_\alpha)\rangle\\ &=\frac{y_b(\mu_h)}{\sqrt{2}}\frac{N_c\alpha_b}{\pi}m_b(\mu_h)\,4\zeta_3\frac{\beta_0+r^{-3C_F/\beta_0}(3C_F-\beta_0-3C_F r)}{3(3C_F-\beta_0)}\,.\end{aligned} \tag{7.30}$$

Here we have used the relation (3.12) to write the result in a more compact form. The Wilson coefficient $\bar{H}_2(z,\mu)$ is defined as in (3.10).

With the discussion in this section we have completed the technical part of the calculations related to the resummation of T_2 at (almost) leading order in RG-improved perturbation theory. We recall that almost here stands for the limitation that we have not included terms proportional to the two-loop coefficients of the γ_{22} and the corresponding quantities in brackets. The method of computation would be the same though.

7.3.3 Numerical results for T_2 at LO in RG-improved PT

We combine now all the derivation in the two previous Sections to present the numerical evaluation of the resummed amplitude T_2. For the diagonal term we substitute in (7.26) the numerical results in (7.21) and (7.29) and perform the numerical integration over the scale μ by slicing the interval into infinitesimal intervals as before. We find the following contribution at leading order

$$T_2^{\text{diag}} = \frac{N_c \alpha_b}{\pi} \frac{y_b(\mu_h)}{\sqrt{2}} m_b(\mu_h)(2.31 - 2.22i). \tag{7.31}$$

The numerical evaluation from the off-diagonal term in (7.30) is straightforward. We find

$$T_2^{\text{off-diag}} = \frac{y_b(\mu_h)}{\sqrt{2}} \frac{N_c \alpha_b}{\pi} m_b(\mu_h)(-0.17 + 0.13i). \tag{7.32}$$

Then the final numerical result for the resummed T_2 at leading order in RG-improved perturbation theory is given by adding together the results in (7.31) and (7.32)

$$T_2^{\text{LO}} = \frac{y_b(\mu_h)}{\sqrt{2}} \frac{N_c \alpha_b}{\pi} m_b(\mu_h)(2.14 - 2.09i). \tag{7.33}$$

7.4 Resummation in T_3

We now discuss the resummation of T_3 at leading order in RG-improved perturbation theory. We recall that this double convolution is defined as

$$T_3 = \lim_{\sigma \to -1} H_3(\mu) \int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J(M_h \ell_-, \mu) J(-M_h \ell_+, \mu) S(\ell_+ \ell_-, \mu) \Big|_{\text{leading power}} \tag{7.34}$$

This part of the amplitude has the property that at the scale μ_h it contains all the leading and next-to-leading logarithmic contributions of the total amplitude. On the other hand the presence of the hard cutoffs in this double convolution breaks the homogeneous power counting. For consistency one should keep only the leading power contributions to the formula in (7.34).

In Chapters 4 and 5 we have already discussed in details the renormalization and the RG evolution of the functions present in T_3. We need to substitute in (7.34) their solutions at leading order in RG-improved perturbation theory. For the soft function this is shown in (5.22). Here we convert that result for the running mass $m_b(\mu)$, which is straightforward from the relation (4.4). For the jet function this solution is [102]

$$J_{\text{LO}}(p^2, \mu) = \exp\left[-2\mathcal{S}(\mu_j, \mu) - a_{\gamma'}(\mu_j, \mu)\right] \frac{\Gamma(1 - a_\Gamma(\mu_j, \mu))}{\Gamma(1 + a_\Gamma(\mu_j, \mu))} \left(\frac{-p^2 e^{-2\gamma_E}}{\mu_j^2}\right)^{a_\Gamma(\mu_j, \mu)}. \tag{7.35}$$

where μ_j is the jet function natural scale. The functions $\mathcal{S}(\mu_j,\mu)$ and $a_\gamma(\mu_j,\mu)$ were defined in (5.9). For the Wilson coefficient $H_3(\mu)$ we substitute the leading term for the fixed order result in (4.43) at the hard scale μ_h. Finally we find the following expression

$$
\begin{aligned}
T_3^{\mathrm{LO}} &= \frac{\alpha}{3\pi}\frac{y_b\left(\mu_h\right)}{\sqrt{2}} m_b\left(\mu_s\right) \int_0^{M_h} \frac{d\ell}{\ell_-} \int_0^{M_h} \frac{d\ell_+}{\ell_+} e^{2\mathcal{S}(\mu_s,\mu_h)-2\mathcal{S}(\mu_-,\mu_h)-2\mathcal{S}(\mu_+,\mu_h)} \\
&\times \left(\frac{-M_h\ell_-}{\mu_-^2}\right)^{a_\Gamma^-} \left(\frac{-M_h\ell_+}{\mu_+^2}\right)^{a_\Gamma^+} \left(\frac{-\ell_+\ell_-}{\mu_s^2}\right)^{-a_\Gamma^s} \left(\frac{\alpha_s\left(\mu_s\right)}{\alpha_s\left(\mu_h\right)}\right)^{-\frac{\gamma_{s,0}}{2\beta_0}} \\
&\times e^{-2\gamma_E a_\Gamma^+} \frac{\Gamma\left(1-a_\Gamma^+\right)}{\Gamma\left(1+a_\Gamma^+\right)} e^{-2\gamma_E a_\Gamma^-} \frac{\Gamma\left(1-a_\Gamma^-\right)}{\Gamma\left(1+a_\Gamma^-\right)} e^{4\gamma_E a_\Gamma^s} \left(\begin{matrix} -a_\Gamma^s\,, -a_\Gamma^s\,, 1-a_\Gamma^s\,, 1-a_\Gamma^s \\ 0, \quad 1, \quad 0, \quad 0 \end{matrix} \middle| \frac{m_b^2}{-\ell_+\ell_-}\right),
\end{aligned} \tag{7.36}
$$

where the scale $\mu_\pm$ is the jet scale associated to the jet function that depends on momentum $\ell_\pm$ and $\gamma_{s,0} = -6C_F$ is the one-loop anomalous dimension from the soft function in (4.57). The functions in the exponent are the Sudakov exponent $\mathcal{S}$ and a_Γ. Here we need only their expressions up to $\mathcal{O}(1)$ which read [103]

$$
\begin{aligned}
\mathcal{S}\left(\mu_i,\mu_h\right) &= \frac{\Gamma_0}{4\beta_0^2}\left[\frac{4\pi}{\alpha_s\left(\mu_i\right)}\left(1-\frac{1}{r}-\ln r\right)\right. \\
&\left.+\left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)\left(1-r+\ln r\right)+\frac{\beta_1}{2\beta_0}\ln^2 r\right], \\
a_\Gamma\left(\mu_i,\mu_h\right) &= \frac{\Gamma_0}{2\beta_0}\ln\frac{\alpha_s\left(\mu_h\right)}{\alpha_s\left(\mu_i\right)} \equiv a_\Gamma^i\,.
\end{aligned} \tag{7.37}
$$

where $r = \alpha_s(\mu_h)/\alpha_s(\mu_i)$. The one and two-loop coefficients of the cusp anomalous dimension Γ_0 and Γ_0 are presented in Appendix B.

The scale ratios in the second line in (7.36) come from the initial conditions of the jet functions and the soft function. All these ratios must be small in order for this solution to be consistent. It is non-trivial to simultaneously fix the scales $\mu_\pm$ and μ_s such that $(-M_h\ell_\pm/\mu_\pm^2)$ and $(\ell_+\ell_-/\mu_s^2)$ remain small since they depend on the integration variables $\ell_\pm$. In this case one should introduce dynamical scale setting. It is also important to be careful here that the soft function $S(\ell_+\ell_-,\mu)$ should not develop large logarithms of the form $\ln^n(\ell_+\ell_-/m_b^2)$ in higher orders. This is ensured by the fact that the soft funtion anomalous dimension does not depend on $\ell_+\ell_-$, as we have seen already in Chapter 5. In addition notice that from the presence of the Meijer G-function in (7.36) it follows that the region where $\ell_+\ell_- \ll m_b^2$ would give power suppressed contributions in the resummed T_3. This is because the Meijer G-function vanishes for $m_b^2/\ell_+\ell_- \to \infty$.

The result we have derived in (7.36) represents the first resummation of Sudakov logarithms in RG-improved perturbation theory for a power suppressed quantity. In principle it is possible to extend this result numerically beyond leading order in RG-improved perturbation theory. Though there are complications related to the fact that one should carefully fix the soft and jet functions scales through dynamical scale setting and analytically continue the integrations for $\sigma \to -1$. We leave this discussion to future work.

Part II

Effective Field Theory for Leptoquarks

Chapter 8

Introduction

Leptoquarks are hypothetical particles that couple both to leptons and quarks and appear in several extensions of the Standard Model [27–30]. They are color triplets and were initially predicted in the Pati-Salam model [31] and other unified theories [32–34]. Leptoquark vertices can violate the lepton univerality in the Standard Model and introduce generation changing interactions. In resent years the observation of the B-meson anomalies in $R_{(D^*)}$ and $R_{(K)}$ measurements [35, 36] have raised an interest in both vector (spin 1) and scalar (spin 0) leptoquarks. Indeed leptoquarks have become prominent candidates responsible for these observed deviations from the Standard Model. Other authors have used leptoquarks as a possible solution to the long standing problem of the $(g-2)_\mu$ anomaly [37–40]. In several of the theoretical models that try to fit the anomalies the predicted leptoquark particles are a vector U_1^μ and two scalars S_1 and S_3, where S_1 is a singlet under $SU(2)_L$ and S_3 transfors as a triplet in $SU(2)_L$ [39, 41–52].

Current searches for leptoquark pair production at the LHC at $\sqrt{s} = 13$ TeV have set a lower mass limit at round 1.7 TeV [53, 54]. Considering also the future upgrade of the LHC, if leptoquarks exist, in principle it would be possible to have one of these particles produced on-shell. Then the next natural step would be to study their properties in an effective field theory approach. In this sense SCET offers a consistent framework to describe the decay rates of these particles. As we have already seen it, SCET is a non-local EFT that quantitatively describes the decays of heavy particles into light and energetic ones. This approach of using SCET to analyse the decays of beyond the Standard Model particles was intially introduced in [55] as the SCET-BSM framework for the decays of a heavy singlet and was later applied to a model with vector-like quark mediators in [56].

In this second part of the thesis we use the SCET formalism to build the effective Lagrangians that describe the decays of the leptoquarks S_1, S_3 and U_1^μ. We construct the operator basis for two and three body final states at leading and subleading order in the power counting parameter λ. The parameter λ is of the order $v/\Lambda \ll 1$, where Λ is some large scale that characterizes the whole high energy sector where the leptoquarks live and v is the electroweak scale. In addition we use renormalization group techniques to resum the QCD and electroweak large logarithms in the Wilson coefficients of the leading power operators. Here we assume that a leptoquark can couple to different families of leptons and quarks at the same time, which is different from the original assumption on leptoquark couplings in the

Buchmüller-Rückl-Wyler model [30].

The final states consist of collinear Standard Model particles and, to make the discussion more interesting we also allow for the existance of a light right handed neutrino in the particle spectrum. This is a singlet under the Standard Model gauge group that we denote by $\nu_R(1,1,0)$ and enters in several models with neutrino mass generation [57, 58]. The Standard Model collinear particles are described by the collinear gauge invariant building blocks we have introduced in Chapter 2 and the right handed neutrino is represented by a collinear field $\nu_R(x)$. Contrary to the other fields the right handed neutrino is not dressed in Wilson lines since it transforms trivially under the gauge group. Lastly, for a charged heavy particle such as a leptoquark, a complete consistent description requires them to be treated within a heavy particle effective theory framework that we have also discussed in Chapter 2. We note that in this work we are only interested in analysing leading and subleading power two jet final states and leading power three jet final states. Operators with soft fields are further suppressed by higher powers of λ.

This part of the thesis has the following structure; we start with the construction of the operator basis for the leptoquark S_1 in Chapter 9, then follow in Chapter 10 and 11 with the leptoquarks S_3 and U_1^μ. In all these cases we present the $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^3)$ operators for leading and subleading two jet final states and leading power $\mathcal{O}(\lambda^3)$ three jet final states. Then in Chapter 12 we show the running of the Wilson coefficients of the operators and sum their large logarithms using renormalization group equations. Lastly in Chapter 13 we present a tree level matching for certain new physics models.

Chapter 9

SCET formalism for the scalar leptoquark $S_1(3, 1, -\frac{1}{3})$

The scalar leptoquark S_1 is a color triplet, an $SU(2)_L$ singlet and has hypercharge $Y = -1/3$. It couples to Standard Model particles similarly to a right handed down type quark. This particular leptoquark has been studied as a viable solution both to the flavour anomalies and the $(g-2)_\mu$ anomaly. Note that its quantum numbers allow the S_1 to couple in operators that would induce proton decays though we neglect those operators here. In literature they are usually suppressed assuming the realization of certain symmetries such as Peccei-Quinn symetry or other discrete symmetries [39, 41, 67].

9.1 Leading power two jet operators for S_1

We start with the SCET Lagrangian that describes the decays of the S_1 at leading order in λ. The decay products are all collinear gauge invariant building blocks. The leptoquark should be thought of as a heavy field that interacts with the other particles only through soft momentum interactions and it is described within the heavy scalar effective theory by the heavy field $S_{1v}(x)$ the same way presented in Section 2.3. At lowest power in λ the symmetries allow for S_1 to couple to two collinear fermions moving in opposite directions n_1 and n_2. The leading power operators are of $\mathcal{O}(\lambda^2)$. We use the subscript n_i to denote a collinear field in SCET moving in the n_i direction here. Then the SCET Lagrangian at $\mathcal{O}(\lambda^2)$ reads

$$\begin{aligned}\mathcal{L}_{S_1}^{(\lambda^2)} =& C^{ij}_{S_1^* u_R^c \ell_R}\, \bar{u}^{\,c,i}_{R,n_1}\, \ell^j_{R,n_2}\, S^*_{1v} + C^{ij}_{S_1^* Q_L^c L_L}\, \bar{Q}^{\,c,i}_{L,n_1}\, i\sigma_2\, L^j_{L,n_2}\, S^*_{1v} \\ &+ C^{ij}_{S_1^* d_R^c \nu_R}\, \bar{d}^{\,c,i}_{R,n_1}\, \nu^j_{R,n_2}\, S^*_{1v} + (n_1 \leftrightarrow n_2) + \text{ h.c.}\,,\end{aligned} \tag{9.1}$$

where $C^{ij}_{S_1 f_1 f_2}$ are the Wilson coefficients of the corresponding operators. We label the operators and their Wilson coefficient by their field content. The fields Q_{L,n_i} and L_{L,n_1} are the collinear quark and lepton doublets while u_{R,n_i} and d_{R,n_i}, ℓ_{R,n_i} and ν_{R,n_i} stand for the up and down type collinear quarks, right handed collinear lepton and right handed collinear neutrino respectively. The indices i, j where $\{i, j\} \in \{1, 2, 3\}$ labell the fermion families. As mentioned before we are considering here the most general case where the leptoquark can decay into a quark and a lepton of different generations. This is the case which gives rise to flavour changing neutral currents (FCNC) in the model [68]. The fields that carry a superscript c are the charge conjugate field defined as $\Psi^c = C\bar{\Psi}^T$ with C being the charge conjugate operator here. As a result all the operators in (9.1) violate fermion number conservation by $\Delta F = 2$. The above Lagrangian contains all the non-vanishing operators at $\mathcal{O}(\lambda^2)$ that are Standard Model gauge invariant, Lorentz invariant and reparametrization invariant in SCET. For simplicity we keep the coordinate and scale dependence of the fields and the Wilson coefficients implicit but according to equation (2.10) the above operator products should be understood as products of non-local fields. Considering for instance the first term in (9.1) we would have

$$C^{ij}_{S_1^* u_R^c \ell_R}\, \bar{u}^{c,i}_{R,n_1}\, \ell^j_{R,n_2} S^*_{1v} \equiv \int ds\, dt\, \bar{C}^{ij}_{S_1^* u_R^c \ell_R}\,(\Lambda, s, t, \mu)\, \bar{u}^{c,i}_{R,n_1}\,(x + s\bar{n}_1)\, \ell^j_{R,n_2}(x + t\bar{n}_2)\, S^*_{1v}(x)\,, \quad (9.2)$$

where Λ represents the large scale that has been integrated out and μ is the factorization scale of the operators. Inserting the exponential form of the series in (2.10) and apply it on the Fourier transform of the fields we end up with the following expression

$$\int ds\, dt\, \bar{C}^{ij}_{S_1^* u_R^c \ell_R}(\Lambda, s, t, \mu)\, e^{it\bar{n}_1\cdot\mathcal{P}_1}\, e^{is\bar{n}_2\cdot\mathcal{P}_2}\, \bar{u}^{c,i}_{R,n_1}(x)\, \ell^j_{R,n_2}(x)\, S^*_{1v}(x), \quad (9.3)$$

where $\mathcal{P}_i$ is a momentum label. This is a generalization to the four momentum p_i carried by the field with the index i where now $\mathcal{P}_i$ denotes the total momentum carried by all the fields with the index i. The momentum label $\bar{n}_i \cdot \mathcal{P}_i$ picks up the total large momentum component in the direction n_i. The Wilson coefficients appearing in the Lagrangian in (9.1) are defined as the Fourier transform of the Wilson coefficients $\bar{C}(\Lambda, s, t, \mu)$ such that

$$C \,\equiv\, C(\Lambda, \bar{n}_1 \cdot \mathcal{P}_1, \bar{n}_2 \cdot \mathcal{P}_2, \mu) \,=\, \int ds\, dt\, \bar{C}(\Lambda, s, t, \mu)\, e^{is\bar{n}_1\cdot\mathcal{P}_1}\, e^{it\bar{n}_2\cdot\mathcal{P}_2}\,. \quad (9.4)$$

Using arguments of Lorentz and reparametrization invariance it follows that at leading order the dependence of $C\,(\Lambda, \bar{n}_1 \cdot \mathcal{P}_1, \bar{n}_2 \cdot \mathcal{P}_2, \mu)$ on the momenta can only be proportional to the operator $\mathcal{P}^2$, where $\mathcal{P}$ is the operator carrying the total momentum of all the final states. The eigenvalue of this operator is the mass M of the leptoquark in this case [55]. From now on it is implied that all the Wilson coefficients of the two jet operators are defined as in (9.4). Because of reparametrization invariance the $\bar{n}_i \cdot \mathcal{P}$ scalar product can only depend on the leptoquark mass M where we have

$$C \equiv C(\Lambda, \bar{n}_1 \cdot \mathcal{P}_1, \bar{n}_2 \cdot \mathcal{P}_2, \mu) = C(\Lambda, M, \mu)\,. \quad (9.5)$$

The Lagrangian in (9.1) contains only dimension four operators and therefore the Wilson coefficients are dimensionless. We write the Lagrangian in a compact form such that

$$\begin{aligned}\mathcal{L}^{(\lambda^2)}_{S_1} &= C^{ij}_{S_1^* u_R^c \ell_R}(\Lambda, M_{S_1}, \mu)\, \mathcal{O}^{ij}_{S_1^* u_R^c \ell_R}(\mu) \,+\, C^{ij}_{S_1^* Q_L^c L_L}(\Lambda, M_{S_1}, \mu)\, \mathcal{O}^{ij}_{S_1^* Q_L^c L_L}(\mu) \\ &\quad +\, C^{ij}_{S_1^* d_R^c \nu_R}(\Lambda, M_{S_1}, \mu)\, \mathcal{O}^{ij}_{S_1^* d_R^c \nu_R}(\mu) \,+\, \text{h.c.}\,,\end{aligned} \quad (9.6)$$

where M_{S_1} is the mass of S_1 and we have defined the operator basis

$$\begin{aligned}
\mathcal{O}^{ij}_{S_1^* u_R^c \ell_R} &= \bar{u}^{c,i}_{R,n_1}\, \ell^j_{R,n_2} S^*_{1v} + (n_1 \leftrightarrow n_2)\,, \\
\mathcal{O}^{ij}_{S_1^* Q_L^c L_L} &= \bar{Q}^{c,i}_{L,n_1}\, i\,\sigma_2\, L^j_{L,n_2}\, S^*_{1v} + (n_1 \leftrightarrow n_2)\,, \\
\mathcal{O}^{ij}_{S_1^* d_R^c \nu_R} &= \bar{d}^{c,i}_{R,n_1}\, \nu^j_{R,n_2}\, S^*_{1v} + (n_1 \leftrightarrow n_2)\,.
\end{aligned} \tag{9.7}$$

The first two operators in the above equation define a two jet final state while the decay into a ν_R is a mono jet signature plus missing energy. From this operator basis it is straightforward to calculate the tree level decay rates of the leptoquark S_1. Then the Standard Model fields and the Wilson coefficients are transformed from the weak basis to the mass basis. We use the notation "C" for the Wilson coefficients in the mass basis and collect the components C^{ij} of these Wilson coefficients in the matrix $\mathbf{C}$. The new Wilson coefficients transform with the transformation matrices of the various fields in the corresponding operator. The two body decays at $\mathcal{O}(\lambda^2)$ are fixed by kinematics and in the limit of massless final states the total decay rates for the singlet S_1 are

$$\begin{aligned}
\Gamma(S_1 \to \bar{u}^{c,i}_R\, \ell^j_R) &= \frac{M_{S_1}}{16\pi}\, |\, \mathrm{C}^{ij}_{S_1^* u_R^c \ell_R}\, |^2\,, \\
\Gamma(S_1 \to \bar{u}^{c,i}_L \ell^j_L) &= \frac{M_{S_1}}{16\pi}\, |\, \mathrm{C}^{ij}_{S_1^* Q_L^c L_L}\, |^2\,, \\
\Gamma(S_1 \to \bar{d}^{c,i}_L\, \nu^j_L) &= \frac{M_{S_1}}{16\pi}\, |\, \mathrm{C}^{ij}_{S_1^* Q_L^c L_L}\, |^2\,, \\
\Gamma(S_1 \to \bar{d}^{c,i}_R\, \nu^j_R) &= \frac{M_{S_1}}{16\pi}\, |\, \mathrm{C}^{ij}_{S_1^* d_R^c \nu_R}\, |^2\,.
\end{aligned} \tag{9.8}$$

For different final states the decay rates differ only by the Wilson coefficient of the operator. The decays into left handed leptons result from the second term in (9.6) and the decay rates are identical. Here and below for all the decay rates we present we use $\ell = e, \mu, \tau$.

9.2 Subleading power two jet operators for S_1

It is of interest to further explore the BSM-SCET Lagrangian beyond $\mathcal{O}(\lambda^2)$. At $\mathcal{O}(\lambda^3)$ the leptoquark S_1 can decay into two and three jet final states. In both cases there can be three collinear fields in one Lagrangian operator. Though for the two jet final states at $\mathcal{O}(\lambda^3)$ there are two collinear particles in n_i direction that share the total jet momentum $\mathcal{P}_i$ such that one of the particles will carry momentum $u\mathcal{P}_i$, with $0 < u < 1$ and the other one momentum $(1-u)\mathcal{P}_i$. Since u can have any value between 0 and 1 one has to integrate over this parameter in the Lagrangian. Applying the same arguments of gauge invariance, Lorentz invariance and reparametrization invariance we find that the two body decay Lagrangian for the leptoquark S_1 at $\mathcal{O}(\lambda^3)$ is

$$
\begin{aligned}
\mathcal{L}_{S_1}^{(\lambda^3)}\bigg|_{2\text{ jet}} &= \frac{1}{\Lambda}\bigg[C^{(0)^{ij}}_{S_1^* L_L \Phi d_R}(\Lambda, M_{S_1}, \mu)\, \mathcal{O}^{(0)^{ij}}_{S_1^* L_L \Phi d_R}(\mu) \\
&\qquad + C^{(0)^{ij}}_{S_1 Q_L \Phi \nu_R}(\Lambda, M_{S_1}, \mu)\, \mathcal{O}^{(0)^{ij}}_{S_1 Q_L \Phi \nu_R}(\mu) \bigg] \\
&+ \frac{1}{\Lambda}\bigg[\sum_{k=1,2} \int_0^1 du \bigg(C^{(k)^{ij}}_{S_1^* L_L \Phi d_R}(\Lambda, M_{S_1}, \mu, u)\, \mathcal{O}^{(k)^{ij}}_{S_1^* L_L \Phi d_R}(\mu, u) \\
&\qquad + C^{(k)^{ij}}_{S_1 Q_L \Phi \nu_R}(\Lambda, M_{S_1}, \mu, u)\, \mathcal{O}^{(k)^{ij}}_{S_1 Q_L \Phi \nu_R}(\mu, u) \\
&\qquad + C^{(k)^{ij}}_{S_1 d_R B \nu_R}(\Lambda, M_{S_1}, \mu, u)\, \mathcal{O}^{(k)^{ij}}_{S_1 d_R B \nu_R}(\mu, u) \bigg) + \text{h.c.} \bigg] .
\end{aligned}
\tag{9.9}
$$

We label the operators by their field content and B is the $U(1)_Y$ gauge boson. To distinguish the two jet operators at $\mathcal{O}(\lambda^3)$ we use the superscript (k) for $k = 1, 2$ which denotes the collinear direction in which the third field with momentum $u\mathcal{P}_i$ is emitted. The operators labelled by (0) contain a zero momentum field $\Phi^{(0)}$ such that

$$
\begin{aligned}
\mathcal{O}^{(0)^{ij}}_{S_1^* L_L \Phi d_R} &= \bar{L}^i_{L,n_1}\, \tilde{\Phi}^{(0)}\, d^j_{R,n_2}\, S^*_{1v} + (n_1 \leftrightarrow n_2)\,, \\
\mathcal{O}^{(0)^{ij}}_{S_1 Q_L \Phi\, \nu_R} &= \bar{Q}^i_{L,n_1}\, \Phi^{(0)} \nu^j_{R,n_2}\, S_{1v} + (n_1 \leftrightarrow n_2)\,,
\end{aligned}
\tag{9.10}
$$

where $\tilde{\Phi}^{(0)} = i\sigma_2 \Phi^{(0)^*}$. The zero momentum field $\Phi^{(0)}$ has the gauge quantum numbers of the Higgs doublet but it does not transform under gauge transformations in SCET. After electroweak symmetry breaking it can be rotated to

$$
\Phi^{(0)} = \frac{1}{\sqrt{2}}\,(0, v)^T . \tag{9.11}
$$

These operators will give a non-vanishing contribution to the two body decay rates of S_1 at $\mathcal{O}(\lambda^3)$. The second equation in (9.10) describes a mono-jet signature in the detector plus missing energy from the ν_R. All the fields in the remaining operators in the Lagrangian (9.9) carry momentum different from zero.

The Wilson coefficients for the two jet Lagrangian at $\mathcal{O}(\lambda^3)$ depend on the parameter u if a particle with non-zero momentum is emitted within the same jet. The superscript (u) on the field implies the presence of a δ- function which is there to ensure that the large momentum component of the second particle in the i^{th} jet is fixed by $u\,(\bar{n}_i \cdot \mathcal{P}_i)$. For an explicit derivation see [55]. For example for the following operator from (9.9) we have

$$
\begin{aligned}
\mathcal{O}^{(1)ij}_{S_1^* L_L \Phi d_R}(u) &= \bar{L}^i_{L,n_1}(x)\, \tilde{\Phi}^{(u)}_{n_1}(x)\, d^j_{R,n_2}(x)\, S^*_{1v}(x) \\
&\equiv \bar{L}^i_{L,n_1}(x)\, \delta(u - \frac{\bar{n}_1 \cdot \mathcal{P}_\Phi}{\bar{n}_1 \cdot \mathcal{P}_1})\, \tilde{\Phi}_{n_1}(x)\, d^j_{R,n_2}(x)\, S^*_{1v}(x) .
\end{aligned}
\tag{9.12}
$$

The other three operators with the scalar $\Phi(x)$ are

$$\begin{aligned}
\mathcal{O}^{(2)^{ij}}_{S_1^* L_L \Phi d_R}(u) &= \bar{L}^i_{L,n_1}\, \tilde{\Phi}^{(u)}_{n_2}\, d^j_{R,n_2}\, S^*_{1v} + (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(1)^{ij}}_{S_1 Q_L \Phi \nu_R}(u) &= \bar{Q}^i_{L,n_1}\, \Phi^{(u)}_{n_1}\, \nu^j_{R,n_2}\, S_{1v} + (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)^{ij}}_{S_1 Q_L \Phi \nu_R}(u) &= \bar{Q}^i_{L,n_1}\, \Phi^{(u)}_{n_2}\, \nu^j_{R,n_2}\, S_{1v} + (n_1 \leftrightarrow n_2)\,.
\end{aligned} \tag{9.13}$$

Note that two fermionic fields cannot be emitted in the same n_i direction since that would give a vanishing contribution due to $n^2 = \bar{n}^2 = 0$. The last line in (9.9) contains the same chirality operators built out of a down-type quark, a right handed neutrino and the $U(1)_Y$ gauge boson. To maintain the subleading order power counting we can only include the perpendicular component of the gauge invariant building block $\mathcal{B}^{\perp}_{\mu} \sim \mathcal{O}(\lambda)$ of the gauge field. We then find

$$\begin{aligned}
\mathcal{O}^{(1)^{ij}}_{S_1 d_R B \nu_R}(u) &= \bar{d}^i_{R,n_1}\, \not{\mathcal{B}}^{\perp,(u)}_{n_1}\, \nu^j_{R,n_2}\, S_{1v} + (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)^{ij}}_{S_1 d_R B \nu_R}(u) &= \bar{d}^i_{R,n_1}\, \not{\mathcal{B}}^{\perp,(u)}_{n_2}\, \nu^j_{R,n_2}\, S_{1v} + (n_1 \leftrightarrow n_2)\,,
\end{aligned} \tag{9.14}$$

where the $\not{\mathcal{B}}^{\perp}_n = \gamma^{\mu}_{\perp}\, \mathcal{B}^{\perp,\mu}_n$ and the perpendicular component of the $\mathcal{B}_n$ is defined as

$$\mathcal{B}^{\perp,\mu}_n = \mathcal{B}^{\mu}_n - n\cdot\mathcal{B}\,\frac{\bar{n}^{\mu}}{2}\,. \tag{9.15}$$

The gamma matrix $\gamma^{\mu}_{\perp}$ is defined by

$$\gamma^{\mu}_{\perp} = \gamma^{\mu} - \frac{\not{n}_1}{n_1\cdot n_2}\, n_2^{\mu} - \frac{\not{n}_2}{n_1\cdot n_2}\, n_1^{\mu}\,. \tag{9.16}$$

There are no charge conjugate fields arising at $\mathcal{O}(\lambda^3)$ and therefore all the operators conserve the fermion number. Moreover this implies no mixing between the leading and subleading operators for the S_1. Since now all the operators are of canonical dimension five we multiply each term by $1/\Lambda$ so that the Wilson coefficients are dimensionless. It is instructive to do so because effectively the Wilson coefficients play the role of coupling constants.

From the operator basis at $\mathcal{O}(\lambda^3)$ it is not difficult to derive the power suppressed two body decays of S_1, where we find

$$\begin{aligned}
\Gamma(S_1 \to \bar{\nu}^i_L d^j_R) &= \frac{v^2}{2}\,\frac{M_{S_1}}{16\pi}\,\frac{|\,\mathrm{C}^{(0)^{ij}}_{S_1^* L_L \Phi d_R}\,|^2}{\Lambda^2}\,,\\
\Gamma(S_1 \to \bar{\nu}^i_R\, d^j_L) &= \frac{v^2}{2}\,\frac{M_{S_1}}{16\pi}\,\frac{|\,\mathrm{C}^{(0)^{ij}}_{S_1 Q_L \Phi \nu_R}\,|^2}{\Lambda^2}\,.
\end{aligned} \tag{9.17}$$

They are both suppressed by a factor of v^2/Λ^2 compared to the two body decay rates from $\mathcal{O}(\lambda^2)$ operators and result from the operator basis in (9.10). Even though the final states differ only by an interchange in chirality, in general the two decay rates in (9.17) can be different. Such examples of decay rates can be very interesting to study in the context of new physics models with different couplings for left and right handed particles because the suppression or the enhancement of one of these decays can be used as a discriminator for the model.

9.3 Leading power three body decays for S_1

It is possible to have the same field content as in (9.9) with each collinear field emitted in one separate collinear direction n_i. These operators describe the decays of the leptoquark S_1 into three jet final states though the phase space would be much smaller and the decay rates further suppressed. Then the S_1 three jet Lagrangian at $\mathcal{O}(\lambda^3)$ reads

$$\begin{aligned}\mathcal{L}_{S_{1v}}^{(\lambda^3)}\Big|_{3\text{ jet}} = \frac{1}{\Lambda}\Bigg[& C^{ij}_{S_1^* L_L \Phi d_R}(\Lambda, M_{S_1}, \{m^2_{k\ell}\},\mu)\, \mathcal{O}^{ij}_{S_1^* L_L \Phi d_R}(\mu) \\ & + C^{ij}_{S_1 Q_L \Phi \nu_R}(\Lambda, M_{S_1}, \{m^2_{k\ell}\},\mu)\, \mathcal{O}^{ij}_{S_1 Q_L \Phi \nu_R}(\mu) \\ & + C^{ij}_{S_1 d_R B \nu_R}(\Lambda, M_{S_1}, \{m^2_{k\ell}\},\mu)\, \mathcal{O}^{ij}_{S_1 d_R B \nu_R}(\mu) \,+\, \text{h.c.}\Bigg].\end{aligned} \tag{9.18}$$

The Wilson coefficients in this case depend also on the invariant mass $m^2_{k\ell}$ for any (k,ℓ) pair of final state particles, where $k \neq \ell \in \{1,2,3\}$. The operators basis is the following

$$\begin{aligned}\mathcal{O}^{ij}_{S_1^* L_L \Phi d_R} &= \bar{L}^i_{L,n_1}\, \tilde{\Phi}_{n_3} d^j_{R,n_2}\, S^*_{1v} \,+\, (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{ij}_{S_1 Q_L \Phi \nu_R} &= \bar{Q}^i_{L,n_1}\, \Phi_{n_3} \nu^j_{R,n_2}\, S_{1v} \,+\, (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{ij}_{S_1 d_R B \nu_R} &= \bar{d}^i_{R,n_1}\, \not{\mathcal{B}}^{\perp}_{n_3}\, \nu^j_{R,n_2}\, S_{1v} \,+\, (n_1 \leftrightarrow n_2)\,,\end{aligned} \tag{9.19}$$

where n_1, n_2, n_3 are the three collinear directions each defining a jet signature in the experiment. As shown in (2.14) the collinear scalar field Φ_{n_i} is defined from the Higgs quantum field multiplied by a collinear Wilson line. After spontaneous symmetry breaking the collinear scalar can be written as

$$\Phi_{n_i}(0) = \frac{1}{\sqrt{2}} W^\dagger_{n_i}(0) \begin{pmatrix} 0 \\ v + h_{n_i}(0) \end{pmatrix}. \tag{9.20}$$

The Wilson line is now expressed in terms of mass eigenstates of collinear electroweak gauge bosons $W^\pm_{n_i}$, Z_{n_i} and the photon A_{n_i}

$$W_{n_i}(0) = P\exp\left[\frac{ig}{2}\int_{-\infty}^0 ds \begin{pmatrix} \frac{c_w^2 - s_w^2}{c_w}\bar{n}_i\cdot Z_{n_i} + 2 s_w \bar{n}_i \cdot A_{n_i} & \sqrt{2}\bar{n}_i\cdot W^+_{n_i} \\ \sqrt{2}\bar{n}_i\cdot W^-_{n_i} & -\frac{1}{c_w}\bar{n}_i\cdot Z_{n_i}\end{pmatrix}(s\bar{n}_i)\right], \tag{9.21}$$

where the $c_w \equiv \cos\theta_W$ and $s_w \equiv \sin\theta_W$ are respectively the cosine and sine of the weak mixing angle. This Wilson line will give rise to additional three body decays for all the leptoquark operators where the field Φ_{n_i} is present. From the three jet Lagrangian in (9.18) we can compute the squared matrix elements for the various operators. For decays into two fermions and a Higgs boson we find that

$$\begin{aligned}
\mid \mathcal{M}(S_1 \to \bar{\nu}_L^i d_R^j h) \mid^2 &= \frac{\mid \mathrm{C}^{ij}_{S_1^* L_L \Phi d_R} \mid^2}{4\Lambda^2} (n_1 \cdot n_2)\,(\bar{n}_1 \cdot p_1)\,(\bar{n}_2 \cdot p_2) \\
&\approx \frac{|\mathrm{C}^{ij}_{S_1 L_L \Phi d_R}|^2}{2\Lambda^2}\, m^2_{\nu d}\,, \\
\mid \mathcal{M}^2(S_1 \to \bar{\nu}_R^i d_L^i h) \mid^2 &= \frac{\mid \mathrm{C}^{ij}_{S_1 Q_L \Phi \nu_R} \mid^2}{4\Lambda^2} (n_1 \cdot n_2)\,(\bar{n}_1 \cdot p_1)\,(\bar{n}_2 \cdot p_2) \\
&\approx \frac{\mid \mathrm{C}^{ij}_{S_1 Q_L \Phi \nu_R} \mid^2}{2\Lambda^2}\, m^2_{\nu d}\,,
\end{aligned} \tag{9.22}$$

where $m^2_{\nu d} = (p_\nu + p_d)^2$ and for simplicity in notation we keep the field chirality labels implicit in the m^2_{ij}. We have used the following approximation here

$$\begin{aligned}
m^2_{12} &= \frac{1}{2}(n_1 \cdot n_2)(\bar{n}_1 \cdot p_1)(\bar{n}_2 \cdot p_2) + \mathcal{O}(\lambda^2) \\
&\approx \frac{1}{2}(n_1 \cdot n_2)(\bar{n}_1 \cdot p_1)(\bar{n}_2 \cdot p_2)\,.
\end{aligned} \tag{9.23}$$

Then the differential decay rates from the above contributions are

$$\begin{aligned}
\frac{d^2\Gamma(S_1 \to \bar{\nu}_L^i d_R^j h)}{dm^2_{dh}\, dm^2_{\nu d}} &= \frac{1}{512\pi^3} \frac{\mid \mathrm{C}^{ij}_{S_1^* L_L \Phi d_R} \mid^2}{\Lambda^2} \frac{m^2_{\nu d}}{M^3_{S_1}}\,, \\
\frac{d^2\Gamma(S_1 \to \bar{\nu}_R^i d_L^j h)}{dm^2_{dh}\, dm^2_{\nu d}} &= \frac{1}{512\pi^3} \frac{\mid \mathrm{C}^{ij}_{S_1 Q_L \Phi \nu_R} \mid^2}{\Lambda^2} \frac{m^2_{\nu d}}{M^3_{S_1}}\,.
\end{aligned} \tag{9.24}$$

For the operator with the Wilson line in (9.21) it follows that the leptoquark can also decay into a $W^\pm$ or a Z boson. The decay rates are similar in both cases and they yield

$$\begin{aligned}
\frac{d^2\Gamma(S_1 \to \bar{\nu}_L^i d_R^j Z)}{dm^2_{\nu d}\, dm^2_{dZ}} &= \frac{1}{512\pi^3} \frac{\mid \mathrm{C}^{ij}_{S_1^* L_L \Phi d_R} \mid^2}{\Lambda^2} \frac{m^2_{\nu d}}{M^3_{S_1}}\,, \\
\frac{d^2\Gamma(S_1 \to \bar{\ell}_L^i d_R^j W^-)}{dm^2_{\ell d}\, dm^2_{dW}} &= \frac{1}{256\pi^3} \frac{\mid \mathrm{C}^{ij}_{S_1^* L_L \Phi d_R} \mid^2}{\Lambda^2} \frac{m^2_{\ell d}}{M^3_{S_1}}\,, \\
\frac{d^2\Gamma(S_1 \to \bar{\nu}_R^i d_L^j Z)}{dm^2_{\nu d}\, dm^2_{dZ}} &= \frac{1}{512\pi^3} \frac{\mid \mathrm{C}^{ij}_{S_1 Q_L \Phi \nu_R} \mid^2}{\Lambda^2} \frac{m^2_{\nu d}}{M^3_{S_1}}\,, \\
\frac{d^2\Gamma(S_1 \to \bar{\nu}_R^i u_L^j W^-)}{dm^2_{\nu u}\, dm^2_{uW}} &= \frac{1}{256\pi^3} \frac{\mid \mathrm{C}^{ij}_{S_1 Q_L \Phi \nu_R} \mid^2}{\Lambda^2} \frac{m^2_{\nu u}}{M^3_{S_1}}\,.
\end{aligned} \tag{9.25}$$

Similarly we compute the differential decay rates from the same chirality operator in (9.19). In this case we have to write the electroweak gauge boson B in terms of the fields $Z(x)$ and

$A(x)$ of the Z boson and the photon γ respectively. This is because Z and γ are the observable physical final states. Using the relation $B(x) = c_w A(x) - s_w Z(x)$ [14] we find these results

$$\begin{aligned}
\frac{d^2\Gamma(S_1 \to \bar{\nu}^i_R d^j_R \gamma)}{dm^2_{\nu d}\, dm^2_{d\gamma}} &= \frac{\alpha}{32\pi^2} \frac{|\mathrm{C}^{ij}_{S_1 d_R B \nu_R}|^2}{\Lambda^2} \frac{m^2_{\nu d}}{M^3_{S_1}} \frac{(m^2_{d\gamma})^2 + (m^2_{\nu\gamma})^2}{(M^2_{S_1} - m^2_{d\nu})^2}\,, \\
\frac{d^2\Gamma(S_1 \to \bar{\nu}^i_R d^j_R Z)}{dm^2_{\nu d}\, dm^2_{dZ}} &= \frac{\alpha}{32\pi^2} \frac{s^2_w}{c^2_w} \frac{|\mathrm{C}^{ij}_{S_1 d_R B \nu_R}|^2}{\Lambda^2} \frac{m^2_{\nu d}}{M^3_{S_1}} \frac{(m^2_{dZ})^2 + (m^2_{\nu Z})^2}{(M^2_{S_1} - m^2_{\nu d})^2}\,,
\end{aligned} \tag{9.26}$$

where the gauge coupling dependence in the decay rates follows from the Feynman rule of the SCET field $\mathcal{B}^\mu_\perp$ that is $g'\epsilon^\star_\perp(p)$, where p is the momentum carried by the field B. In here α is the electromagnetic coupling constant. In this computation we consider all the gauge bosons massless, which is a reasonable approximation given that the masses of the leptoquarks are of $\mathcal{O}$(TeV). We sum over the two perpendicular polarization vectors of the gauge bosons such that

$$\sum_{i=1}^{2} \epsilon^\mu_\perp(p_3)\, \epsilon^{\star\nu}_\perp(p_3) = -g^{\mu\nu}_\perp, \tag{9.27}$$

where

$$g^{\mu\nu}_\perp = g^{\mu\nu} - \frac{n^\mu_3\, \bar{n}^\nu_3}{n_3 \cdot \bar{n}_3} - \frac{\bar{n}^\mu_3\, n^\nu_3}{n_3 \cdot \bar{n}_3}\,. \tag{9.28}$$

Notice that in all the above results, the cases with a neutrino in the final sate are still a three jet final state in the SCET context, though experimentally the neutrinos are not seen as a jet like the other electrically charged particles. With these last results we have completed the discussion on the decays of the scalar leptoquark S_1. We follow the same formalism we have introduced here to drive similar results for the other two leptoquarks in the next following Chapters.

Chapter 10

SCET formalism for the scalar leptoquark $S_3(3, 3, -\frac{1}{3})$

There are several possible extensions of the Standard Model that try to interpret the observed anomalies in B-physics systems. Most of these theoretical models that use scalar leptoquarks as a viable explanation, contain both the singlet S_1 and another scalar leptoquark S_3. The leptoquark $S_3(3, 3, -1/3)$ transforms as a triplet under $SU(2)_L$ with hypercharge $-1/3$ [44, 51]. Such models seem to give promising solution both to the $R_{(D^*)}$ and to the decay process $b \to s\mu^+\mu^-$. It is therefore of interest to apply our framework to the triplet S_3 and find its tree level decay rates. We present in the following section the S_3 effective Lagrangian in SCET at $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^3)$ for two and three body final states.

10.1 Leading power two jet operators for S_3

We start by constructing the leading power Lagrangian which we refer to as $\mathcal{L}_{S_3}^{(\lambda^2)}$. Since S_3 is an $SU(2)$ triplet it should be understood as $S_3 \equiv t^a S_3^a$, for $a = 1, 2, 3$ and t^a generators of the $SU(2)$. As a result gauge invariance constrains the operator basis a lot more in this case. Indeed we find only one operator that describes the decays of the S_3 into two energetic particles going into the collinear directions n_1 and n_2. This is a dimension four operator built out of a quark and a lepton doublet and the triplet S_3. The doublets couple to S_3 similarly to an $SU(2)$ gauge field in the Standard Model. Then the leading power Lagrangian reads

$$\mathcal{L}_{S_3}^{(\lambda^2)} = C_{S_3^* Q_L^c L_L}^{ij}(\Lambda, M_{S_3}, \mu)\, \mathcal{O}_{S_3^* Q_L^c L_L}^{ij} + \text{h.c.}, \tag{10.1}$$

with

$$\mathcal{O}_{S_3^* Q_L^c L_L}^{ij} = \bar{Q}^{i\,c,a}_{L,n_1}\, \epsilon^{ab}\, S_{3v}^{*bd}\, L_{L,n_2}^{j,d} + (n_1 \leftrightarrow n_2)\,, \tag{10.2}$$

where M_{S_3} is the mass of the leptoquark S_3 and and i, j are flavor indices and the a, b, c are $SU(2)$ indices. The heavy particle S_3 is treated within the HSEFT as described before where $S_{3v}(x)$ contains only the soft momentum fluctuations. The Wilson coefficients are defined in the same way as in equation (9.4) and they are dimensionless. Similarly to the $\mathcal{L}_{S_1}^{(\lambda^2)}$ the Lagrangian in (10.1) violates fermion number conservation. The leading order two body decay rates of the leptoquark S_3 are governed by the matrix elements of the Lagrangian in (10.1), which allows for a decay into a left handed quark and a left handed lepton. The leptoquark triple S_3 contains three particles of different electric charge. We can then write the field S_{3v} in terms of eigenstates of the charge operator such that

$$S_{3v}^{2/3} = \frac{S_3^1 - iS_{3v}^2}{\sqrt{2}}\,, \qquad S_{3v}^{-4/3} = \frac{S_{3v}^1 + iS_{3v}^2}{\sqrt{2}}\,, \qquad S_{3v}^{-1/3} = S_{3v}^3\,, \tag{10.3}$$

where the superscript denotes the electric charge of the corresponding particle. Then the various two body decay rate at $\mathcal{O}(\lambda^2)$ for each particle in the triplet evaluate to

$$\begin{aligned}
\Gamma(S_3^{2/3} \to \bar{u}_L^{c,i}\, \nu_L^j) &= \frac{M_{S_3}}{32\pi}\, |\mathrm{C}^{ij}_{S_3^* Q_L^c L_L}|^2\,, \\
\Gamma(S_3^{-4/3} \to \bar{d}_L^{c,i}\, e_L^j) &= \frac{M_{S_3}}{32\pi}\, |\mathrm{C}^{ij}_{S_3^* Q_L^c L_L}|^2\,, \\
\Gamma(S_3^{-1/3} \to \bar{d}_L^{c,i} \nu_L^j) &= \frac{M_{S_3}}{64\pi}\, |\mathrm{C}^{ij}_{S_3^* Q_L^c L_L}|^2\,, \\
\Gamma(S_3^{-1/3} \to \bar{u}_L^{c,i} \ell_L^j) &= \frac{M_{S_3}}{64\pi}\, |\mathrm{C}^{ij}_{S_3^* Q_L^c L_L}|^2\,.
\end{aligned} \tag{10.4}$$

The decay rates of the third component particle the $S_3^{-1/3}$ are suppressed by a factor of two compared to the decay of the other two particles in the triplet. Moreover the leptoquark with charge $-4/3$ is the only one that does not contain the mono-jet+missing energy signature in this case.

10.2 Subleading power two jet operators for S_3

We now look at the subleading power two jet operator basis for the leptoquark S_3. At $\mathcal{O}(\lambda^3)$ the symmetries allow for a larger number of operators and we find six of them with mixed chirality and two others of the same chirality fields such that

$$\begin{aligned}
\mathcal{L}_{S_3}^{(\lambda^3)}\Big|_{2\text{ jet}} &= \frac{1}{\Lambda}\Bigg[C^{(0)^{ij}}_{S_3 Q_L \Phi \nu_R}(\Lambda, M_{S_3}, \mu)\, \mathcal{O}^{(0)^{ij}}_{S_3 Q_L \Phi \nu_R} \\
&\qquad + C^{(0)^{ij}}_{S_3 d_R \Phi L_L}(\Lambda, M_{S_3}, \mu)\, \mathcal{O}^{(0)^{ij}}_{S_3 d_R \Phi L_L}(\mu) \Bigg] \\
&+ \frac{1}{\Lambda}\Bigg[\sum_{k=1,2} \int_0^1 du \Bigg(C^{(k)^{ij}}_{S_3 Q_L \Phi \nu_R}(\Lambda, M_{S_3}, \mu, u)\, \mathcal{O}^{(k)^{ij}}_{S_3 Q_L \Phi \nu_R}(\mu, u) \\
&\qquad + C^{(k)^{ij}}_{S_3 d_R \Phi L_L}(\Lambda, M_{S_3}, \mu, u)\, \mathcal{O}^{(k)^{ij}}_{S_3 d_R \Phi L_L}(\mu, u) \\
&\qquad + C^{(k)^{ij}}_{S_3 d_R W \nu_R}(\Lambda, M_{S_3}, \mu, u)\, \mathcal{O}^{(k)^{ij}}_{S_3 d_R W \nu_R}(\mu, u) \Bigg) \Bigg] + \text{h.c.}\,.
\end{aligned} \tag{10.5}$$

We name the operators and the Wilson coefficients based on their field content as before and where W here is the $SU(2)$ gauge boson. The operators in the first line in (10.5) contain the zero momentum field $\Phi^{(0)}(x)$ and $\tilde{\Phi}^{(0)}(x)$ with the following explicit structure

$$\begin{aligned} \mathcal{O}^{(0)ij}_{S_3 Q_L \Phi \nu_R} &= \bar{Q}^i_{L,n_1}\, S_{3v}\, \Phi^{(0)}\, \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{(0)ij}_{S_3 d_R \Phi L_L} &= \bar{d}^i_{R,n_1}\, \tilde{\Phi}^{\dagger(0)}\, S_{3v}\, L^j_{L,n_2} + (n_1 \leftrightarrow n_2)\,, \end{aligned} \tag{10.6}$$

which are all mono-jet detector signatures. The rest of the mixed chirality operators in (10.5) read as follows

$$\begin{aligned} \mathcal{O}^{(1)ij}_{S_3 Q_L \Phi \nu_R}(u) &= \bar{Q}^i_{L,n_1}\, S_{3v}\, \Phi^{(u)}_{n_1}\, \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{(2)ij}_{S_3 Q_L \Phi \nu_R}(u) &= \bar{Q}^i_{L,n_1}\, S_{3v}\, \Phi^{(u)}_{n_2}\, \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{(1)ij}_{S_3 d_R \Phi L_L}(u) &= \bar{d}^i_{R,n_1}\, \tilde{\Phi}^{\dagger(u)}_{n_1}\, S_{3v}\, L^j_{L,n_2} + (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{(2)ij}_{S_3 d_R \Phi L_L}(u) &= \bar{d}^i_{R,n_1}\, \tilde{\Phi}^{\dagger(u)}_{n_2}\, S_{3v}\, L^j_{L,n_2} + (n_1 \leftrightarrow n_2)\,. \end{aligned} \tag{10.7}$$

The triplet S_3 very similarly to S_1 will decay into right handed neutrinos, left handed quarks and a Higgs boson and into left handed leptons together with right handed down-type quarks. Lastly the same chirality operators in equation (10.5) contain the perpendicular component of a $SU(2)_L$ gauge boson $W^{\perp}_{\mu}$, a down type quark and a right handed neutrino

$$\begin{aligned} \mathcal{O}^{(1)ij}_{S_3 d_R W \nu_R}(u) &= \bar{d}^i_{R,n_1}\, \mathrm{Tr}\left[S_{3v}\, \not{\mathcal{W}}^{\perp,(u)}_{n_1} \right] \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,, \\ \mathcal{O}^{(2)ij}_{S_3 d_R W \nu_R}(u) &= \bar{d}^i_{R,n_1}\, \mathrm{Tr}\left[S_{3v}\, \not{\mathcal{W}}^{\perp,(u)}_{n_2} \right] \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,, \end{aligned} \tag{10.8}$$

where the trace here runs over $SU(2)$ indices and the perpendicular component of the gauge invariant building block $\mathcal{W}^{\mu}$ is defined as usually

$$\mathcal{W}^{\perp,\mu}_{n} = \mathcal{W}^{\mu}_{n} - n \cdot \mathcal{W}\, \frac{\bar{n}^{\mu}}{2}\,. \tag{10.9}$$

The subleading operators with the zero momentum field $\Phi^{(0)}$ in (10.6) contribute to the power suppressed two body decays for each of the three leptoquarks of the triplet. We find the following decay rates for each case

$$\begin{aligned} \Gamma(S_3^{2/3} \to \bar{\nu}^i_R u^j_L) &= \frac{v^2}{4}\, \frac{M_{S_3}}{16\pi}\, \frac{|\mathrm{C}^{(0)ij}_{S_3 Q_L \Phi \nu_R}|^2}{\Lambda^2}\,, \\ \Gamma(S_3^{-1/3} \to \bar{\nu}^i_R d^j_L) &= \frac{v^2}{8}\, \frac{M_{S_3}}{16\pi}\, \frac{|\mathrm{C}^{(0)ij}_{S_3 Q_L \Phi \nu_R}|^2}{\Lambda^2}\,, \\ \Gamma(S_3^{-1/3} \to \bar{\nu}^i_L d^j_R) &= \frac{v^2}{8}\, \frac{M_{S_3}}{16\pi}\, \frac{|\mathrm{C}^{(0)ij}_{S_3 d_R \Phi L_L}|^2}{\Lambda^2}\,, \\ \Gamma(S_3^{2/3} \to \bar{\ell}^i_L d^j_R) &= \frac{v^2}{4}\, \frac{M_{S_3}}{16\pi}\, \frac{|\mathrm{C}^{(0)ij}_{S_3 d_R \Phi L_L}|^2}{\Lambda^2}\,. \end{aligned} \tag{10.10}$$

10.3 Leading power three body decays for S_3

The leading power three jet operators start from $\mathcal{O}(\lambda^3)$ and from the field content the allowed interactions are very similar to the subleading power two jet operators we presented in the previous Section. In this case though each collinear particle is emitted in a separate collinear directions. Then the three jet Lagrangian at $\mathcal{O}(\lambda^3)$ for the scalar S_3 is

$$\begin{aligned}
\mathcal{L}^{(\lambda^3)}_{S_3}\bigg|_{3\text{ jet}} = \frac{1}{\Lambda}\bigg[& C^{ij}_{S_3 d_R \Phi L_L}(\Lambda, M_{S_3}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{S_3 d_R \Phi L_L}(\mu) \\
& + C^{ij}_{S_3 Q_L \Phi \nu_R}(\Lambda, M_{S_3}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{S_3 Q_L \Phi \nu_R}(\mu) \\
& + C^{ij}_{S_3 d_R W \nu_R}(\Lambda, M_{S_3}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{S_3 d_R W \nu_R}(\mu) + \text{h.c.}\bigg].
\end{aligned} \tag{10.11}$$

where the operators read

$$\begin{aligned}
\mathcal{O}^{ij}_{S_3 d_R \Phi L_L} &= \bar{d}^i_{R,n_1}\, \tilde{\Phi}^\dagger_{n_3}\, S_{3v}\, L^j_{L,n_2} + (n_1 \leftrightarrow n_2)\,, \\
\mathcal{O}^{ij}_{S_3 Q_L \Phi \nu_R} &= \bar{Q}^i_{L,n_1}\, S_{3v}\, \Phi_{n_3}\, \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,, \\
\mathcal{O}^{ij}_{S_3 d_R W \nu_R} &= \bar{d}^i_{R,n_1}\, \mathrm{Tr}\left[S_{3v}\, \mathcal{W}\!\!\!/^{\perp}_{n_3}\right] \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,.
\end{aligned} \tag{10.12}$$

The field content of the above operators is the same as in the corresponding operators $\mathcal{O}^{(k)ij}$ in (10.7) and (10.8) though experimentally they would have very different angular distributions. The first two operators in (10.12) describe S_3 decays into two different chirality fermions and a Higgs or a gauge boson. The gauge boson final states from these operators result from the Wilson line of the scalar Φ in (9.21) and they are emitted in the third n_3 collinear direction according to the above operator basis. We present below the results for the differential decay rates for each of the three charged particles $S_3^{2/3}$, $S_3^{-4/3}$ and $S_3^{-1/3}$. For the Higgs boson in the final sate we find

$$\begin{aligned}
\frac{d^2\,\Gamma(\,S_3^{2/3} \to \bar{\ell}^i_L\, d^j_R\, h\,)}{dm^2_{\ell d}\, dm^2_{dh}} &= \frac{1}{2048\pi^3}\, \frac{|\mathrm{C}^{ij}_{S_3 d_R \Phi L_L}|^2}{\Lambda^2}\, \frac{m^2_{\ell d}}{M^3_{S_3}}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-1/3} \to \bar{\nu}^i_L\, d^j_R\, h\,)}{dm^2_{\nu d}\, dm^2_{dh}} &= \frac{1}{4096\pi^3}\, \frac{|\mathrm{C}^{ij}_{S_3 d_R \Phi L_L}|^2}{\Lambda^2}\, \frac{m^2_{\nu d}}{M^3_{S_3}}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-1/3} \to \bar{\nu}^i_R\, d^j_L\, h)}{dm^2_{\nu d}\, dm^2_{dh}} &= \frac{1}{4096\pi^3}\, \frac{|\mathrm{C}^{ij}_{S_3 Q_L \Phi \nu_R}|^2}{\Lambda^2}\, \frac{m^2_{\nu d}}{M^3_{S_3}}\,, \\
\frac{d^2\,\Gamma(\,S_3^{2/3} \to \bar{\nu}^i_R\, u^j_L\, h)}{dm^2_{\nu u}\, dm^2_{uh}} &= \frac{1}{2048\pi^3}\, \frac{|\mathrm{C}^{ij}_{S_3 Q_L \Phi \nu_R}|^2}{\Lambda^2}\, \frac{m^2_{\nu u}}{M^3_{S_3}}\,.
\end{aligned} \tag{10.13}$$

For the remaining rates with a gauge bosons in the final state we obtain the following results

$$\begin{aligned}
\frac{d^2\,\Gamma(S_3^{-1/3} \to \bar{\nu}_R^i u_L^j W^-)}{dm_{\nu u}^2\, dm_{uW}^2} &= \frac{1}{1024\pi^3} \frac{|\mathrm{C}_{S_3 Q_L \Phi \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu u}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{2/3} \to \bar{\nu}_R^i\, u_L^j\, Z)}{dm_{\nu u}^2\, dm_{uZ}^2} &= \frac{1}{1024\pi^3} \frac{|\mathrm{C}_{S_3 Q_L \Phi \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu u}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-4/3} \to \bar{\nu}_R^i\, d_L^j W^-)}{dm_{\nu d}^2\, dm_{dW}^2} &= \frac{1}{2048\pi^3} \frac{|\mathrm{C}_{S_3 Q_L \Phi \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-1/3} \to \bar{\nu}_R^i\, d_L^j Z)}{dm_{\nu d}^2\, dm_{dZ}^2} &= \frac{1}{8192\pi^3} \frac{|\mathrm{C}_{S_3 Q_L \Phi \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-1/3} \to \bar{\nu}_L^i\, d_R^j Z)}{dm_{\nu d}^2\, dm_{dZ}^2} &= \frac{1}{8192\pi^3} \frac{|\mathrm{C}_{S_3 d_R \Phi L_L}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-4/3} \to \bar{\nu}_L^i\, d_R^j W^-)}{dm_{\nu d}^2\, dm_{dW}^2} &= \frac{1}{2048\pi^3} \frac{|\mathrm{C}_{S_3 d_R \Phi L_L}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{-1/3} \to \bar{\ell}_L^i\, d_R^j W^-)}{dm_{\ell d}^2\, dm_{dW}^2} &= \frac{1}{1024\pi^3} \frac{|\mathrm{C}_{S_3 d_R \Phi L_L}^{ij}|^2}{\Lambda^2} \frac{m_{\ell d}^2}{M_{S_3}^3}\,, \\
\frac{d^2\,\Gamma(\,S_3^{2/3} \to \bar{\ell}_L^i\, d_R^j Z)}{dm_{\ell d}^2\, dm_{dZ}^2} &= \frac{1}{1024\pi^3} \frac{|\mathrm{C}_{S_3 d_R \Phi L_L}^{ij}|^2}{\Lambda^2} \frac{m_{\ell d}^2}{M_{S_3}^3}\,.
\end{aligned} \tag{10.14}$$

The structure is very similar for the same initial state leptoquark depending on the final states. For the same initial state particle, the decays in two fermion and a Higgs or two fermions and a gauge boson differ only by the Wilson coefficient of the corresponding operator. The same chirality operator with a right handed neutrino calculate to a less trivial result. In this case we find

$$\begin{aligned}
\frac{d^2\Gamma(S_3^{2/3} \to \bar{\nu}_R^i d_R^j W^+)}{dm_{\nu d}^2\, dm_{dW}^2} &= \frac{1}{s_w^2} \frac{\alpha}{128\pi^2} \frac{|\mathrm{C}_{S_3 d_R W \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3} \frac{(m_{\nu d}^2)^2 + (m_{dW}^2)^2}{(M_{S_3}^2 - m_{\nu d}^2)^2}\,, \\
\frac{d^2\Gamma(S_3^{-4/3} \to \bar{\nu}_R^i d_R^j W^-)}{dm_{\nu d}^2\, dm_{dW}^2} &= \frac{1}{s_w^2} \frac{\alpha}{128\pi^2} \frac{|\mathrm{C}_{S_3 d_R W \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3} \frac{(m_{\nu W}^2)^2 + (m_{dW}^2)^2}{(M_{S_3}^2 - m_{\nu d}^2)^2}\,, \\
\frac{d^2\Gamma(S_3^{-1/3} \to \bar{\nu}_R^i d_R^j Z)}{dm_{\nu d}^2\, dm_{dZ}^2} &= \frac{c_w^2}{s_w^2} \frac{\alpha}{128\pi^2} \frac{|\mathrm{C}_{S_3 d_R W \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}^3} \frac{(m_{\nu Z}^2)^2 + (m_{dZ}^2)^2}{(M_{S_3}^2 - m_{\nu d}^2)^2}\,, \\
\frac{d^2\Gamma(S_3^{-1/3} \to \bar{\nu}_R^i d_R^j \gamma)}{dm_{\nu d}^2\, dm_{d\gamma}^2} &= \frac{\alpha}{128\pi^2} \frac{|\mathrm{C}_{S_3 d_R W \nu_R}^{ij}|^2}{\Lambda^2} \frac{m_{\nu d}^2}{M_{S_3}} \frac{(m_{\nu\gamma}^2)^2 + (m_{d\gamma}^2)^2}{(M_{S_3}^2 - m_{\nu d}^2)^2}\,.
\end{aligned} \tag{10.15}$$

Chapter 11

SCET formalism for the vector leptoquark $U_1^\mu(3,1,\frac{2}{3})$

The vector U_1^μ is another interesting example from the family of leptoquarks that has been introduced as a solution to departures from the Standard Model in the flavour sector [44, 46]. It is a color triplet, $SU(2)$ singlet and has hypercharge 2/3. In the following section we analyse its decays in leading and subleading order in power counting λ. All interactions of the field U_1^μ here are described in the soft limit by the HVEFT presented earlier in Section 2.3.

11.1 Leading power two jet operators for U_1^μ

The operator basis is built following the same reasoning as for the other two leptoquarks, where we construct all the possible particle combinations that preserve gauge variance, Lorentz invariance, reparametrization invariance and do not vanish under chirality transformations. Also in this case we find non-vanishing operators with the right handed neutrino. At leading order in SCET we find that the Lagrangian for the leptoquark U_1^μ contains three operators such that

$$\begin{aligned}\mathcal{L}_{U_1}^{(\lambda^2)} = &\, C^{ij}_{U_1 Q_L L_L}(\Lambda, M_{U_1}, \mu)\, \mathcal{O}^{ij}_{U_1 Q_L L_L}(\mu) + C^{ij}_{U_1 d_R \ell_R}(\Lambda, M_{U_1}, \mu)\, \mathcal{O}^{ij}_{U_1 d_R \ell_R}(\mu) \\ &+ C^{ij}_{U_1 u_R \nu_R}(\Lambda, M_{U_1}, \mu)\, \mathcal{O}^{ij}_{U_1 u_R \nu_R}(\mu) + \text{h.c.}\,,\end{aligned} \tag{11.1}$$

with the following dimension four operators

$$\begin{aligned}\mathcal{O}^{ij}_{U_1 Q_L L_L} &= \bar{Q}^i_{L,n_1}\, \not{U}_{1v\perp}\, L^j_{L,n_2} + (n_1 \leftrightarrow n_2)\,,\\ \mathcal{O}^{ij}_{U_1 d_R \ell_R} &= \bar{d}^i_{R,n_1}\, \not{U}_{1v\perp}\, \ell^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,,\\ \mathcal{O}^{ij}_{U_1 u_R \nu_R} &= \bar{u}^i_{R,n_1}\, \not{U}_{1v\perp}\, \nu^j_{R,n_2} + (n_1 \leftrightarrow n_2)\,,\end{aligned} \tag{11.2}$$

where $\not{U}_{1v\perp} = \gamma_\perp^\mu \cdot U_{1v\perp\mu}$ with $\gamma_\perp^\mu$ defined in (9.16). Then at leading power the vector leptoquark will decay into two fermions of the same chirality with the following unpolarized decay rates

$$\begin{aligned}
\Gamma(U_1 \to \bar{\nu}_L^i u_L^j) &= \frac{M_{U_1}}{24\pi} \mid \mathrm{C}^{ij}_{U_1 Q_L L_L} \mid^2 , \\
\Gamma(U_1 \to \bar{\ell}_L^i d_L^j) &= \frac{M_{U_1}}{24\pi} \mid \mathrm{C}^{ij}_{U_1 Q_L L_L} \mid^2 , \\
\Gamma(U_1 \to \bar{\ell}_R^i d_R^j) &= \frac{M_{U_1}}{24\pi} \mid \mathrm{C}^{ij}_{U_1 d_R \ell_R} \mid^2 , \\
\Gamma(U_1 \to \bar{\nu}_R^i u_R^j) &= \frac{M_{U_1}}{24\pi} \mid \mathrm{C}^{ij}_{U_1 u_R \nu_R} \mid^2 .
\end{aligned} \tag{11.3}$$

In the same way as before at leading power the two body decay rates for the vector U_1^μ differ only from the Wilson coefficients.

11.2 Subleading power two jet operators for U_1^μ

At $\mathcal{O}(\lambda^3)$ we find a much richer basis of operators for the leptoquark U_1^μ. Now both the longitudinal and perpendicular components of the heavy field appear in the operators and there are various decays into all Standard Model particles and the right handed neutrino ν_R. To describe the subleading power decays for the vector U_1^μ, it is useful to define the reparametrization invariant quantity Π^μ such that

$$\Pi^\mu = \frac{(v\cdot\bar{n})n^\mu - (v\cdot n)\bar{n}^\mu}{2} , \tag{11.4}$$

where $\Pi^\mu \to -\Pi^\mu$ under hermitian conjugate and it is odd for $n \leftrightarrow \bar{n}$ [69]. The vector v is the reference vector $v^\mu = (1,0,0,0)$ and $v \cdot U_{1v} = 0$. There is a handful of interactions in this case, and for convenience we write the total Lagrangian as a sum of two Lagrangian

$$\mathcal{L}_{U_1}^{(\lambda^3)}\Big|_{2\text{ jet}} = \mathcal{L}_{U_1}^{(\lambda^3),A} + \mathcal{L}_{U_1}^{(\lambda^3),\Phi} . \tag{11.5}$$

The first Lagrangian $\mathcal{L}_{U_1}^{(\lambda^3),A}$ contains operators with two fermions and a gauge field A described by the gauge invariant building block $\mathcal{A}_\mu$ (at leading power only $\mathcal{A}_\mu^\perp$ appears), which can be either a $U(1)_Y$, $SU(2)_L$ or $SU(3)$ gauge boson. The second term in (11.5) is built out of operators with two fermions and a scalar field $\Phi_{n_i}^{(u)}$ or $\Phi^{(0)}$. The gauge boson Lagrangian reads

$$\begin{aligned}
\mathcal{L}_{U_1}^{(\lambda^3),A} =& \frac{1}{\Lambda}\sum_{k=1,2}\int_0^1 du \Bigg[\sum_{A=G,W,B} C^{(k)\,ij}_{U_1 Q_L A L_L}(\Lambda, M_{U_1},\mu,u)\,\mathcal{O}^{(k)\,ij}_{U_1 Q_L A L_L}(\mu,u) \\
&+ \sum_{A=G,B}\Bigg(C^{(k)\,ij}_{U_1 u_R A \nu_R}(\Lambda, M_{U_1},\mu,u)\,\mathcal{O}^{(k)\,ij}_{U_1 u_R A \nu_R}(\mu,u) \\
&+ C^{(k)\,ij}_{U_1 d_R A \ell_R}(\Lambda, M_{U_1},\mu,u)\,\mathcal{O}^{(k)\,ij}_{U_1 d_R A \ell_R}(\mu,u)\Bigg)\Bigg] + \text{h.c.}\,.
\end{aligned} \tag{11.6}$$

The operators have the following field content

$$
\begin{aligned}
\mathcal{O}^{(1)ij}_{U_1 Q_L A L_L}(u) &= \bar{Q}^i_{L,n_1}\, \mathcal{A}^{\perp,(u)}_{n_1}\, \Pi\cdot U_{1v}\, L^j_{L,n_2} \,-(n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1 Q_L A L_L}(u) &= \bar{Q}^i_{L,n_1}\, \mathcal{A}^{\perp,(u)}_{n_2}\, \Pi\cdot U_{1v}\, L^j_{L,n_2} \,-(n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(1)ij}_{U_1 u_R A \nu_R}(u) &= \bar{u}^i_{R,n_1}\, \mathcal{A}^{\perp,(u)}_{n_1}\, \Pi\cdot U_{1v}\, \nu^j_{R,n_2} \,-(n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1 u_R A \nu_R}(u) &= \bar{u}^i_{R,n_1} \mathcal{A}^{\perp,(u)}_{n_2}\, \Pi\cdot U_{1v}\, \nu^j_{R,n_2} \,-(n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(1)ij}_{U_1 d_R A \ell_R}(u) &= \bar{d}^i_{R,n_1} \mathcal{A}^{\perp,(u)}_{n_1}\, \Pi\cdot U_{1v}\, \ell^j_{R,n_2} \,-(n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1 d_R A \ell_R}(u) &= \bar{d}^i_{R,n_1}\, \mathcal{A}^{\perp,(u)}_{n_2}\, \Pi\cdot U_{1v}\, \ell^j_{R,n_2} \,-(n_1 \leftrightarrow n_2)\,.
\end{aligned}
\tag{11.7}
$$

These operators account for all the possible non-vanishing gauge invariant interactions between a leptoquark U_1^μ and a collinear building block of a Standard Model gauge boson at $\mathcal{O}(\lambda^3)$, with two jet final states. For each operator the allowed gauge boson should be easily understood by the gauge symmetry. We show them for each case in the sum in the first and second line in the expression in (11.6).

The second Lagrangian in (11.5) is built out of operators with two fermions and the scalar field $\Phi^{(u)}_{n_i}$ or the zero momentum scalar $\Phi^{(0)}$ and it contains even more term such that

$$
\begin{aligned}
\mathcal{L}^{(\lambda^3),\Phi}_{U_1} &= \frac{1}{\Lambda}\Bigg[C^{(0)\,ij}_{U_1 Q_L \Phi \ell_R}(\Lambda, M_{U_1},\mu)\, \mathcal{O}^{(0)\,ij}_{U_1 Q_L \Phi \ell_R}(\mu)\\
&\qquad + C^{(0)\,ij}_{U_1 u_R \Phi L_L}(\Lambda, M_{U_1},\mu)\, \mathcal{O}^{(0)\,ij}_{U_1 u_R \Phi L_L}(\mu)\\
&\qquad + C^{(0)\,ij}_{U_1 Q_L \Phi \nu_R}(\Lambda, M_{U_1},\mu)\, \mathcal{O}^{(0)\,ij}_{U_1 Q_L \Phi \nu_R}(\mu)\\
&\qquad + C^{(0)\,ij}_{U_1 u_R \Phi L^c_L}(\Lambda, M_{U_1},\mu)\, \mathcal{O}^{(0)\,ij}_{U_1 u_R \Phi L^c_L}(\mu)\\
&\qquad + C^{(0)\,ij}_{U_1 Q_L \Phi \nu^c_R}(\Lambda, M_{U_1},\mu)\, \mathcal{O}^{(0)\,ij}_{U_1 Q_L \Phi \nu^c_R}(\mu)\Bigg]\\
&+ \frac{1}{\Lambda}\Bigg[\sum_{k=1,2}\int_0^1 du \Bigg(C^{(k)\,ij}_{U_1 Q_L \Phi \ell_R}(\Lambda, M_{U_1},\mu,u)\, \mathcal{O}^{(k)\,ij}_{U_1 Q_L \Phi \ell_R}(\mu,u)\\
&\qquad + C^{(k)\,ij}_{U_1 u_R \Phi L_L}(\Lambda, M_{U_1},\mu,u)\, \mathcal{O}^{(k)\,ij}_{U_1 u_R \Phi L_L}(\mu,u)\\
&\qquad + C^{(k)\,ij}_{U_1 Q_L \Phi \nu_R}(\Lambda, M_{U_1},\mu,u)\, \mathcal{O}^{(k)\,ij}_{U_1 Q_L \Phi \nu_R}(\mu,u)\\
&\qquad + C^{(k)ij}_{U_1 u_R \Phi L^c_L}(\Lambda, M_{U_1},\mu,u)\, \mathcal{O}^{(k)\,ij}_{U_1 u_R \Phi L^c_L}(\mu,u)\\
&\qquad + C^{(k)\,ij}_{U_1 Q_L \Phi \nu^c_R}(\Lambda, M_{U_1},\mu,u)\, \mathcal{O}^{(k)\,ij}_{U_1 Q_L \Phi \nu^c_R}(\mu,u)\Bigg)\Bigg]\,.
\end{aligned}
\tag{11.8}
$$

In this case the operators also include the perpendicular component of the U_1^μ field and charge conjugate fermions as well. The operators with the zero momentum scalar $\Phi^{(0)}$ read

$$\begin{aligned}
\mathcal{O}^{(0)ij}_{U_1Q_L\Phi\ell_R} &= \bar{Q}^i_{L,n_1}\Phi^{(0)}\,\Pi\cdot U_{1v}\,\ell^j_{R,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(0)ij}_{U_1u_R\Phi L_L} &= \bar{u}^i_{R,n_1}\tilde{\Phi}^{\dagger(0)}\,\Pi\cdot U_{1v}\,L^j_{L,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(0)ij}_{U_1Q_L\Phi\nu_R} &= \bar{Q}^i_{L,n_1}\tilde{\Phi}^{(0)}\,\Pi\cdot U_{1v}\,\nu^j_{R,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(0)ij}_{U_1u_R\Phi L^c_L} &= \bar{u}^i_{R,n_1}\not{U}^\perp_{1v}\,\tilde{\Phi}^{(0)}\,L^{c\,j}_{L,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(0)\,ij}_{U_1Q_L\Phi\nu^c_R} &= \bar{Q}^i_{L,n_1}\not{U}^\perp_{1v}\,\tilde{\Phi}^{(0)}\,\nu^{c\,j}_{R,n_2} -(n_1\leftrightarrow n_2)\,.
\end{aligned}\tag{11.9}$$

In the last two operators the presence of the charge conjugate right handed neutrino ensures that the product of the chirality projectors give a non-vanishing result. The remaining operators in (11.8) are quite similar to the operators in (11.9) in the kind of interactions they induce, though they contain the scalar $\Phi^{(u)}_{n_i}$ now. For an operator with the same particle content the scalar $\Phi^{(u)}_{n_i}$ with momentum $u\mathcal{P}_i$ can be emitted in either of the two collinear directions. We find

$$\begin{aligned}
\mathcal{O}^{(1)ij}_{U_1Q_L\Phi\ell_R}(u) &= \bar{Q}^i_{L,n_1}\Pi\cdot U_{1v}\;\Phi^{(u)}_{n_1}\,\ell^j_{R,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1Q_L\Phi\ell_R}(u) &= \bar{Q}^i_{L,n_1}\Pi\cdot U_{1v}\,\Phi^{(u)}_{n_2}\,\ell^j_{R,n_2} \;-(n_1\leftrightarrow n_2)\,.\\
\mathcal{O}^{(1)ij}_{U_1u_R\Phi L_L}(u) &= \bar{u}^i_{R,n_1}\tilde{\Phi}^{\dagger(u)}_{n_1}\,\Pi\cdot U_{1v}\,L^j_{L,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1u_R\Phi L_L}(u) &= \bar{u}^i_{R,n_1}\tilde{\Phi}^{\dagger(u)}_{n_2}\,\Pi\cdot U_{1v}\,L^j_{L,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(1)ij}_{U_1Q_L\Phi\nu_R}(u) &= \bar{Q}^i_{L,n_1}\Pi\cdot U_{1v}\,\tilde{\Phi}^{(u)}_{n_1}\,\nu^j_{R,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1Q_L\Phi\nu_R}(u) &= \bar{Q}^i_{L,n_1}\,\Pi\cdot U_{1v}\,\tilde{\Phi}^{(u)}_{n_2}\,\nu^j_{R,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(1)ij}_{U_1u_R\Phi L^c_L}(u) &= \bar{u}^i_{R,n_1}\tilde{\Phi}^{(u)}_{n_1}\,\not{U}^\perp_{1v}\,L^{cj}_{L,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)ij}_{U_1u_R\Phi L^c_L}(u) &= \bar{u}^i_{R,n_1}\tilde{\Phi}^{(u)}_{n_2}\,\not{U}^\perp_{1v}\,L^{cj}_{L,n_2} \;-(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(1)\,ij}_{U_1Q_L\Phi\nu^c_R}(u) &= \bar{Q}^i_{L,n_1}\,\not{U}^\perp_{1v}\tilde{\Phi}^{(u)}_{n_1}\,\nu^{c\,j}_{R,n_2} -(n_1\leftrightarrow n_2)\,,\\
\mathcal{O}^{(2)\,ij}_{U_1Q_L\Phi\nu^c_R}(u) &= \bar{Q}^i_{L,n_1}\,\not{U}^\perp_{1v}\tilde{\Phi}^{(u)}_{n_2}\,\nu^{c\,j}_{R,n_2} -(n_1\leftrightarrow n_2)\,.
\end{aligned}\tag{11.10}$$

Similarly to the other two leptoquarks, the operators with the zero momentum field $\Phi^{(0)}$ in (11.9) contribute to more power suppressed two body decays. We find the following results in this case

$$\begin{aligned}
\Gamma(U_1 \to \bar{\ell}^i_R d^j_L) &= \frac{v^2}{96\pi}\frac{M_{U_1}}{\Lambda^2} \mid \mathrm{C}^{(0)\,ij}_{U_1 Q_L \Phi \ell_R} \mid^2 , \\
\Gamma(U_1 \to \bar{\nu}^i_L u^j_R) &= \frac{v^2}{96\pi}\frac{M_{U_1}}{\Lambda^2} \mid \mathrm{C}^{(0)\,ij}_{U_1 u_R \Phi L_L} \mid^2 , \\
\Gamma(U_1 \to \bar{\nu}^i_R u^j_L) &= \frac{v^2}{96\pi}\frac{M_{U_1}}{\Lambda^2} \mid \mathrm{C}^{(0)\,ij}_{U_1 Q_L \Phi \nu_R} \mid^2 , \\
\Gamma(U_1 \to \bar{\nu}^{c\,i}_L u^j_R) &= \frac{v^2}{16\pi}\frac{M_{U_1}}{\Lambda^2} \mid \mathrm{C}^{(0)\,ij}_{U_1 u_R \Phi L^c_L} \mid^2 , \\
\Gamma(U_1 \to \bar{\nu}^{c\,i}_R u^j_L) &= \frac{v^2}{16\pi}\frac{M_{U_1}}{\Lambda^2} \mid \mathrm{C}^{(0)\,ij}_{U_1 Q_L \Phi \nu^c_R} \mid^2 .
\end{aligned} \tag{11.11}$$

Notice that the contributions from the $\not{U}^\perp_{1v}$ have a different prefactor compared to decays of its longitudinal components.

11.3 Leading power three body decays for U_1^μ

In this section we present an analysis of the leading power three body decays for the vector leptoquark U_1^μ. We find several operators both for the longitudinal $\Pi\cdot U_1$ and the perpendicular component $U_1^{\mu\perp}$. The operator basis is constructed based on requirements of gauge invariance, reparametrization and Lorentz invariance. The Lagrangian can be written in a compact form such as

$$\begin{aligned}
\mathcal{L}^{(\lambda^3)}_{U_1}\bigg|_{3\text{ jet}} = \frac{1}{\Lambda}\Bigg[& C^{ij}_{U_1 Q_L \ell_R}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 Q_L \ell_R}(\mu) \\
& + C^{ij}_{U_1 u_R \Phi L_L}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 u_R \Phi L_L}(\mu) \\
& + C^{ij}_{U_1 Q_L \Phi \nu_R}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 Q_L \Phi \nu_R}(\mu) \\
& + \sum_{A=G,W,B} C^{ij}_{U_1 Q_L A L_L}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 Q_L A L_L}(\mu) \\
& + \sum_{A=G,B} \bigg(C^{ij}_{U_1 u_R A \nu_R}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 u_R A \nu_R}(\mu) \\
& \qquad + C^{ij}_{U_1 d_R A \ell_R}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 d_R A \ell_R}(\mu) \bigg) \\
& + C^{ij}_{U_1 u_R \Phi L^c_L}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 u_R \Phi L^c_L}(\mu) \\
& + C^{ij}_{U_1 Q_L \Phi \nu^c_R}(\Lambda, M_{U_1}, \{m^2_{k\ell}\}, \mu)\, \mathcal{O}^{ij}_{U_1 Q_L \Phi \nu^c_R}(\mu) \Bigg] + \text{h.c.} ,
\end{aligned} \tag{11.12}$$

where $m^2_{k\ell}$ is the invariant mass of the particle pair (k,ℓ) and all the Wilson coefficients are dimensionless as usual. The operator basis reads

$$
\begin{aligned}
\mathcal{O}^{ij}_{U_1 Q_L \Phi \ell_R} &= \bar{Q}^i_{L,n_1}\,\Phi_{n_3}\,\Pi\cdot U_{1v}\,\ell^j_{R,n_2} - (n_1 \leftrightarrow n_2)\\
\mathcal{O}^{ij}_{U_1 u_R \Phi L_L} &= \bar{u}^i_{R,n_1}\,\tilde{\Phi}^\dagger_{n_3}\,\Pi\cdot U_{1v}\,L^j_{L,n_2} - (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{ij}_{U_1 Q_L \Phi \nu_R} &= \bar{Q}^i_{L,n_1}\,\tilde{\Phi}_{n_3}\,\Pi\cdot U_{1v}\,\nu^j_{R,n_2} - (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{ij}_{U_1 Q_L A L_L} &= \bar{Q}^i_{L,n_1}\,\not{\mathcal{A}}^\perp_{n_3}\,\Pi\cdot U_{1v}\,L^j_{L,n_2} - (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{ij}_{U_1 u_R A \nu_R} &= \bar{u}^i_{R,n_1}\,\not{\mathcal{A}}^\perp_{n_3}\,\Pi\cdot U_{1v}\,\nu^j_{R,n_2} - (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{ij}_{U_1 d_R A \ell_R} &= \bar{d}^i_{R,n_1}\,\not{\mathcal{A}}^\perp_{n_3}\,\Pi\cdot U_{1v}\ell^j_{R,n_2} - (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{ij}_{U_1 u_R \Phi L^c_L} &= \bar{u}^i_{R,n_1}\,\tilde{\Phi}_{n_3}\,\not{U}^\perp_{1v}\,L^{cj}_{L,n_2} - (n_1 \leftrightarrow n_2)\,,\\
\mathcal{O}^{ij}_{U_1 Q_L \Phi \nu^c_R} &= \bar{Q}^i_{L,n_1}\,\not{U}^\perp_{1v}\tilde{\Phi}_{n_3}\,\nu^{cj}_{R,n_2} - (n_1 \leftrightarrow n_2)\,.
\end{aligned}
\tag{11.13}
$$

In all the above operators the third particle emitted in the third collinear direction n_3 is a gauge boson or a Higgs boson. We use the same notation as before, where $\mathcal{A}^\perp_\mu$ can denote either a gluon, an $SU(2)$ or $U(1)$ gauge boson as presented by the sum over $A = G, W$ and B in the Lagrangian in (11.13). From these operators we can now derive the leading power three body decays of the vector U_1^μ. We start with contributions from operators with the longitudinal component $\Pi\cdot U_{1v}$ and the scalar field Φ. The final states in this case are two fermions and a Higgs boson and two fermions plus a Z or $W^\pm$. The latter are contributions from the Wilson line shown in (9.21). From the first operator in (11.13) we have

$$
\begin{aligned}
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}^i_R d^j_L\, h\right)}{dm^2_{\ell d}\, dm^2_{dh}} &= \frac{1}{1536\pi^3}\frac{|\,\mathrm{C}^{ij}_{U_1 Q_L \Phi \ell_R}\,|^2}{\Lambda^2}\frac{m^2_{\ell d}}{M^3_{U_1}}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}^i_R u^j_L W^-\right)}{dm^2_{\ell u}\, dm^2_{uW}} &= \frac{1}{768\pi^3}\frac{|\,\mathrm{C}^{ij}_{U_1 Q_L \Phi \ell_R}\,|^2}{\Lambda^2}\frac{m^2_{\ell u}}{M^3_{U_1}}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}^i_R d^j_L Z\right)}{dm^2_{\ell d}\, dm^2_{dZ}} &= \frac{1}{1536\pi^3}\frac{|\,\mathrm{C}^{ij}_{U_1 Q_L \Phi \ell_R}\,|^2}{\Lambda^2}\frac{m^2_{\ell d}}{M^3_{U_1}}\,.
\end{aligned}
\tag{11.14}
$$

For Higgs boson and Z final state the numerical coefficient in front is the same. The coefficient for $W^\pm$ final state is smaller by a factor of two, because these fields are multiplied by $\sqrt{2}$ in 9.21. The other two remaining operators of the same kind, with the scalar Φ yield a very similar structure with the corresponding Wilson coefficients changed such that

$$\begin{aligned}
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_L^i u_R^j\, h\right)}{dm_{\nu u}^2\, dm_{uh}^2} &= \frac{1}{1536\pi^3}\frac{\mid \mathrm{C}_{U_1 u_R \Phi L_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}_L^i u_R^j W^-\right)}{dm_{\ell u}^2\, dm_{uW}^2} &= \frac{1}{768\pi^3}\frac{\mid \mathrm{C}_{U_1 u_R \Phi L_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\ell u}^2}{M_{U_1}^3}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_L^i u_R^j Z\right)}{dm_{\nu u}^2\, dm_{uZ}^2} &= \frac{1}{1536\pi^3}\frac{\mid \mathrm{C}_{U_1 u_R \Phi L_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^i u_L^j h\right)}{dm_{\nu u}^2\, dm_{uh}^2} &= \frac{1}{1536\pi^3}\frac{\mid \mathrm{C}_{U_1 Q_L \Phi \nu_R}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^i u_L^j Z\right)}{dm_{\nu u}^2\, dm_{uZ}^2} &= \frac{1}{1536\pi^3}\frac{\mid \mathrm{C}_{U_1 Q_L \Phi \nu_R}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\,,\\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^i d_L^j W^+\right)}{dm_{\nu d}^2\, dm_{dW}^2} &= \frac{1}{768\pi^3}\frac{\mid \mathrm{C}_{U_1 Q_L \Phi \nu_R}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu d}^2}{M_{U_1}^3}\,.
\end{aligned} \tag{11.15}$$

Next we show the three body decays from the same chirality operators in (11.13) that contain the field $\Pi \cdot U_{1v}$ and a perpendicular gauge boson. Depending on the specific operator the final state gauge boson can be either a gluon, a $W^\pm$, Z or a photon. When a gluon $A = G$ is emitted in the n_3 direction, the averaged squared matrix element in this case evaluates to

$$\mid \mathcal{M}(U_1 \to \bar{f}_{n_1} f_{n2}\, g_{n_3}) \mid^2 = 2C_F \frac{g_s^2}{3} m_{12}^2 \frac{\mid \mathrm{C} \mid^2}{\Lambda^2}\frac{(m_{13}^2)^2 + (m_{23}^2)^2}{\left(M_{U_1}^2 - m_{12}^2\right)^2}\,, \tag{11.16}$$

where denotes g is the gluon final state in here and below. For the case of a $W^\pm$, Z and γ the coupling constant and the coefficient get slightly modified because of the symmetry breaking and the amplitude will also depend on the weak mixing angle. The rest of the formula in (11.16) that captures the kinematics remains the same. For simplicity we omit the subscript n_i in the decay rates. The decay rates from the operator with two left handed fermion doublets in (11.13) are

$$
\begin{aligned}
\frac{d^2\Gamma\left(U_1\to\bar{\nu}_L^i u_L^j g\right)}{dm_{\nu u}^2\,dm_{ug}^2} &= \frac{\alpha_3 C_F}{96\pi^2}\frac{\mid \mathrm{C}_{U_1Q_LGL_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\frac{(m_{\nu g}^2)^2+(m_{ug}^2)^2}{\left(M_{U_1}^2-m_{\nu u}^2\right)^2}\,,\\
\frac{d^2\Gamma\left(U_1\to\bar{\nu}_L^i u_L^j Z\right)}{dm_{\nu u}^2\,dm_{uZ}^2} &= \frac{\alpha}{384\pi^2}\frac{\mid \tilde{\mathrm{C}}_{U_1Q_LL_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\frac{(m_{\nu Z}^2)^2+(m_{uZ}^2)^2}{\left(M_{U_1}^2-m_{\nu u}^2\right)^2}\,,\\
\frac{d^2\Gamma\left(U_1\to\bar{\nu}_L^i u_L^j \gamma\right)}{dm_{\nu u}^2\,dm_{uZ}^2} &= \frac{\alpha}{384\pi^2}\frac{\mid \mathrm{C}_{U_1Q_LL_L}^{\prime ij}\mid^2}{\Lambda^2}\frac{m_{\nu u}^2}{M_{U_1}^3}\frac{(m_{\nu\gamma}^2)^2+(m_{u\gamma}^2)^2}{\left(M_{U_1}^2-m_{\nu u}^2\right)^2}\,,\\
\frac{d^2\Gamma\left(U_1\to\bar{\ell}_L^i u_L^j W^-\right)}{dm_{\ell u}^2\,dm_{uW}^2} &= \frac{1}{s_w^2}\frac{\alpha}{192\pi^2}\frac{\mid \mathrm{C}_{U_1Q_LWL_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\ell u}^2}{M_{U_1}^3}\frac{(m_{\ell W}^2)^2+(m_{uW}^2)^2}{\left(M_{U_1}^2-m_{\ell u}^2\right)^2}\,,\\
\frac{d^2\Gamma\left(U_1\to\bar{\ell}_L^i d_L^j Z\right)}{dm_{\ell d}^2\,dm_{dZ}^2} &= \frac{\alpha}{384\pi^2}\frac{\mid \tilde{\mathrm{C}}_{U_1Q_LL_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\ell d}^2}{M_{U_1}^3}\frac{(m_{\ell Z}^2)^2+(m_{dZ}^2)^2}{\left(M_{U_1}^2-m_{\ell d}^2\right)^2}\,,\\
\frac{d^2\Gamma\left(U_1\to\bar{\ell}_L^i d_L^j \gamma\right)}{dm_{\ell d}^2\,dm_{d\gamma}^2} &= \frac{\alpha}{384\pi^2}\frac{\mid \mathrm{C}_{U_1Q_LL_L}^{\prime\prime ij}\mid^2}{\Lambda^2}\frac{m_{\ell d}^2}{M_{U_1}^3}\frac{(m_{\ell\gamma}^2)^2+(m_{d\gamma}^2)^2}{\left(M_{U_1}^2-m_{\ell d}^2\right)^2}\,,\\
\frac{d^2\Gamma\left(U_1\to\bar{\nu}_L^i d_L^j W^+\right)}{dm_{\nu d}^2\,dm_{dW}^2} &= \frac{1}{s_w^2}\frac{\alpha}{192\pi^2}\frac{\mid \mathrm{C}_{U_1Q_LWL_L}^{ij}\mid^2}{\Lambda^2}\frac{m_{\nu d}^2}{M_{U_1}^3}\frac{(m_{\nu W}^2)^2+(m_{dW}^2)^2}{\left(M_{U_1}^2-m_{\nu d}^2\right)^2}\,.
\end{aligned}
\tag{11.17}
$$

For a photon or a Z boson final state the rates depend on a combination of different Wilson coefficients. In these cases we have defined the following composite Wilson coefficients:

$$
\begin{aligned}
\tilde{\mathrm{C}}_{U_1Q_LL_L}^{ij} &= \frac{c_w}{s_w}\mathrm{C}_{U_1Q_LWL_L}^{ij} + 2\,\frac{s_w}{c_w}\mathrm{C}_{U_1Q_LBL_L}^{ij}\,,\\
\mathrm{C}_{U_1Q_LL_L}^{\prime ij} &= \mathrm{C}_{U_1Q_LWL_L}^{ij} + 2\,\mathrm{C}_{U_1Q_LBL_L}^{ij}\,,\\
\mathrm{C}_{U_1Q_LL_L}^{\prime\prime ij} &= -\mathrm{C}_{U_1Q_LWL_L}^{ij} + 2\,\mathrm{C}_{U_1Q_LBL_L}^{ij}\,.
\end{aligned}
\tag{11.18}
$$

The decay rates from the remaining operators with a perpendicular gauge boson in (11.13) are

$$\begin{aligned}
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^i u_R^j g\right)}{dm_{\nu u}^2\, dm_{ug}^2} &= \frac{\alpha_3 C_F}{96\pi^2} \frac{|\ \mathrm{C}_{U_1 u_R G \nu_R}^{ij}\ |^2}{\Lambda^2} \frac{m_{\nu u}^2}{M_{U_1}^3} \frac{(m_{\nu g}^2)^2 + (m_{ug}^2)^2}{\left(M_{U_1}^2 - m_{\nu u}^2\right)^2}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^i u_R^j Z\right)}{dm_{\nu u}^2\, dm_{uZ}^2} &= \frac{s_w^2}{c_w^2} \frac{\alpha}{96\pi^2} \frac{|\ \mathrm{C}_{U_1 u_R B \nu_R}^{ij}\ |^2}{\Lambda^2} \frac{m_{\nu u}^2}{M_{U_1}^3} \frac{(m_{\nu Z}^2)^2 + (m_{uZ}^2)^2}{\left(M_{U_1}^2 - m_{\nu u}^2\right)^2}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^i u_R^j \gamma\right)}{dm_{\nu u}^2\, dm_{u\gamma}^2} &= \frac{\alpha}{96\pi^2} \frac{|\ \mathrm{C}_{U_1 u_R B \nu_R}^{ij}\ |^2}{\Lambda^2} \frac{m_{\nu u}^2}{M_{U_1}^3} \frac{(m_{\nu \gamma}^2)^2 + (m_{u\gamma}^2)^2}{\left(M_{U_1}^2 - m_{\nu u}^2\right)^2}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}_R^i d_R^j g\right)}{dm_{\ell d}^2\, dm_{dg}^2} &= \frac{\alpha_3 C_F}{96\pi^2} \frac{|\ \mathrm{C}_{U_1 d_R G \ell_R}^{ij}\ |^2}{\Lambda^2} \frac{m_{\ell d}^2}{M_{U_1}^3} \frac{(m_{\ell g}^2)^2 + (m_{dg}^2)^2}{\left(M_{U_1}^2 - m_{\ell d}^2\right)^2}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}_R^i d_R^j Z\right)}{dm_{\ell d}^2\, dm_{dZ}^2} &= \frac{s_w^2}{c_w^2} \frac{\alpha}{96\pi^2} \frac{|\ \mathrm{C}_{U_1 d_R B \ell_R}^{ij}\ |^2}{\Lambda^2} \frac{m_{\ell d}^2}{M_{U_1}^3} \frac{(m_{\ell Z}^2)^2 + (m_{dZ}^2)^2}{\left(M_{U_1}^2 - m_{\ell d}^2\right)^2}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}_R^i d_R^j \gamma\right)}{dm_{\ell d}^2\, dm_{d\gamma}^2} &= \frac{\alpha}{96\pi^2} \frac{|\ \mathrm{C}_{U_1 d_R B \ell_R}^{ij}\ |^2}{\Lambda^2} \frac{m_{\ell d}^2}{M_{U_1}^3} \frac{(m_{\ell \gamma}^2)^2 + (m_{d\gamma}^2)^2}{\left(M_{U_1}^2 - m_{\ell d}^2\right)^2}\,.
\end{aligned} \tag{11.19}$$

The last two operators in (11.13) give a slightly different structure for the decay rate formula due to the presence of the $U_1^{\mu,\perp}$. The squared matrix elements in this case contain separate products of $n_1 \cdot n_2 = (k_1 \cdot k_2)/(E_1 E_2)$ which can be rewritten in terms of invariant masses using [55]

$$E_1 = \frac{M_{U_1}^2 - m_{23}^2}{2M_{U_1}}\,, \qquad E_2 = \frac{M_{U_1}^2 - m_{13}^2}{2M_{U_1}}\,. \tag{11.20}$$

From these relations it follows that the kinematic dependence in the amplitude for these decays is

$$\frac{M_{U_1}^2 m_{12}^2 + \left(m_{23}^2 - M_{U_1}^2\right)\left(m_{13}^2 - M_{U_1}^2\right)}{M_{U_1}^2}\,. \tag{11.21}$$

Then the differential three body decays of the operator with the charge conjugate lepton are

$$\begin{aligned}
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_L^{ic} u_R^j h\right)}{dm_{\nu u}^2\, dm_{uh}^2} &= \frac{1}{1536\pi^3} \frac{|\ \mathrm{C}_{U_1 u_R \Phi L_L^c}^{ij}\ |^2}{\Lambda^2} \frac{M_{U_1}^2 m_{\nu u}^2 + \left(m_{uh}^2 - M_{U_1}^2\right)\left(m_{\nu h}^2 - M_{U_1}^2\right)}{M_{U_1}^5}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_L^{ic} u_R^j Z\right)}{dm_{\nu u}^2\, dm_{uh}^2} &= \frac{1}{1536\pi^3} \frac{|\ \mathrm{C}_{U_1 u_R \Phi L_L^c}^{ij}\ |^2}{\Lambda^2} \frac{M_{U_1}^2 m_{\nu u}^2 + \left(m_{uZ}^2 - M_{U_1}^2\right)\left(m_{\nu Z}^2 - M_{U_1}^2\right)}{M_{U_1}^5}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\ell}_L^{ic} u_R^j W^+\right)}{dm_{\ell u}^2\, dm_{uh}^2} &= \frac{1}{768\pi^3} \frac{|\ \mathrm{C}_{U_1 u_R \Phi L_L^c}^{ij}\ |^2}{\Lambda^2} \frac{M_{U_1}^2 m_{\ell u}^2 + \left(m_{uW}^2 - M_{U_1}^2\right)\left(m_{\ell W}^2 - M_{U_1}^2\right)}{M_{U_1}^5}\,.
\end{aligned} \tag{11.22}$$

Similarly for the remaining $U_1^{\mu,\perp}$ decays into the charge conjugate right handed neutrino we find

$$\begin{aligned}
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^{i\,c} u_L^j h\right)}{dm_{\nu u}^2\, dm_{uh}^2} &= \frac{1}{1536\pi^3} \frac{\mid \mathrm{C}^{ij}_{U_1 Q_L \Phi \nu_R^c} \mid^2}{\Lambda^2} \frac{M_{U_1}^2 m_{\nu u}^2 + \left(m_{uh}^2 - M_{U_1}^2\right)\left(m_{\nu h}^2 - M_{U_1}^2\right)}{M_{U_1}^5}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^{i\,c} u_L^j Z\right)}{dm_{\nu u}^2\, dm_{uh}^2} &= \frac{1}{1536\pi^3} \frac{\mid \mathrm{C}^{ij}_{U_1 Q_L \Phi \nu_R^c} \mid^2}{\Lambda^2} \frac{M_{U_1}^2 m_{\nu u}^2 + \left(m_{uZ}^2 - M_{U_1}^2\right)\left(m_{\nu Z}^2 - M_{U_1}^2\right)}{M_{U_1}^5}\,, \\
\frac{d^2\Gamma\left(U_1 \to \bar{\nu}_R^{i\,c} d_L^j W^+\right)}{dm_{\ell u}^2\, dm_{uh}^2} &= \frac{1}{768\pi^3} \frac{\mid \mathrm{C}^{ij}_{U_1 Q_L \Phi \nu_R^c} \mid^2}{\Lambda^2} \frac{M_{U_1}^2 m_{\ell u}^2 + \left(m_{uW}^2 - M_{U_1}^2\right)\left(m_{\ell W}^2 - M_{U_1}^2\right)}{M_{U_1}^5}\,.
\end{aligned} \tag{11.23}$$

All the three body decays of the perpendicular $U_1^{\mu,\perp}$ are highly suppressed by powers of the leptoquark mass.

In this Chapter and before we have discussed in great details all the leading and subleading power two body decays and the leading three body decays for the the three leptoquarks S_1, S_3 and U_1^μ. In these formulas one can directly substitute the corresponding values of the various quantities in these decay rates and evaluate their numerical results modulus their Wilson coefficients. These coefficients are model dependent and one would have to evaluate them within a certain new physics model by a well defined matching procedure. We discuss this more in Chapter 13.

Chapter 12

Running of the Wilson coefficients

In the process of scale separation in EFT the Wilson coefficients are the functions that capture the hard scale dependence. They correspond to diagrams with vertices from the full theory that have been integrated out. Inherently they will depend on logarithms of ratios of the hard scale Λ and the factorization scale μ. We are already familiar with the fact that a reliable result on the decay rates of the heavy particles considered within an EFT approach requires resummation of these large logarithms in the Wilson coefficients. Also in the SCET-BSM framework it is possible to achieve this using renormalization group techniques. Wilson coefficients obey well defined RGEs which are derived from renormalization of their corresponding operators. At one loop the type of diagrams that contribute to the renormalization of the $\mathcal{O}(\lambda^2)$ operators are shown in Fig.(12.1). In most general case the RG evolution equation is a matrix equation such that

$$\frac{d\,\boldsymbol{C}(\mu)}{d\ln\mu} = \mathbf{\Gamma}(\mu) \otimes \boldsymbol{C}(\mu)\,, \tag{12.1}$$

where $\mathbf{\Gamma}$ is the anomalous dimension matrix in generation space and $\boldsymbol{C}$ is the matrix of Wilson coefficients in generation space. The symbol $\otimes$ takes into account that the ordering of $\mathbf{\Gamma}$ and $\boldsymbol{C}$ matters. The solution of (12.1) can then be formally written as

$$\boldsymbol{C}(\mu) \;=\; \boldsymbol{C}(\Lambda)\mathbf{P}\exp\left[\int_\Lambda^\mu d\ln\mu\,\mathbf{\Gamma}(\mu)\right], \tag{12.2}$$

and it systematically resums the large logarithms of type $\ln(\Lambda^2/\mu^2)$.

The anomalous dimension $\mathbf{\Gamma}$ of a SCET operator with three external lines, one heavy particle and two collinear particles in n_1 and n_2 directions depends on the cusps anomalous dimensions $\gamma^{(r)}_{\text{cusp}}(\alpha_r)$, on the single-particle anomalous dimensions $\boldsymbol{\gamma}^i$ and on the leptoquark anomalous dimension γ^{LQ} such that [55, 70, 71]

$$\begin{aligned}\mathbf{\Gamma}\left(\{p,p_1,p_2\},M,\mu\right) = \frac{1}{2}\sum_r \left(C^{(r)} - C^{(r)}_1 - C^{(r)}_2\right)\gamma^{(r)}_{\text{cusp}}(\alpha_r)\left(\ln\frac{\mu^2}{M^2} + i\pi\right) \\ - \sum_r C^{(r)}\,\gamma^{(r)}_{\text{cusp}}(\alpha_r)\,\ln\frac{\mu}{M} + \sum_{i=1,2}\boldsymbol{\gamma}^i + \gamma^{LQ}\,,\end{aligned} \tag{12.3}$$

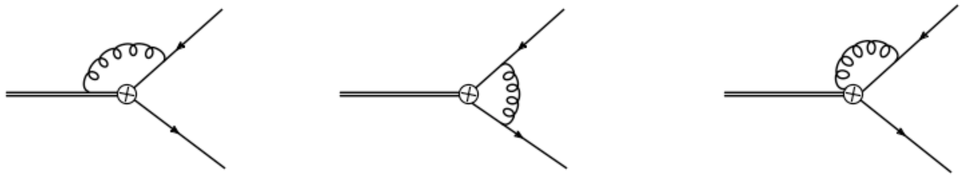

Figure 12.1: Soft and collinear gluon emissions for the one-loop renormalization of operators at $\mathcal{O}(\lambda^2)$. The double line indicates a heavy leptoquark. The first diagram corresponds to soft gauge boson emissions, the second diagram describes final state interactions, the last diagram accounts for the type of diagrams where gauge bosons are emitted from the collinear Wilson lines. In the first two diagrams gauge bosons have soft momentum scaling, in the third diagram they have collinear scaling.

where M is the mass of the leptoquark and $C^{(r)}$, $C_1^{(r)}$, $C_2^{(r)}$ are the Casimir operators of the leptoquark, the n_1 and n_2 collinear particles respectively for the gauge group (r) where these particles transform. For a non-Abelian group $SU(N)$ the Casimir operator is $C_i = (N^2-1)/2N$ for the fundamental representation and $C_i = N$ for the adjoint representation. For the Abelian group $U(1)_Y$ we have $C_i = Y_i^2$, where Y_i is the hypercharge of the particle. The $\gamma^{(r)}_{\text{cusp}}(\alpha_r)$ are functions of the coupling constant arising from light like Wilson loops [72, 104]. Up to NNLO they depend only on α_r for each symmetry group $G^{(r)}$ [74–76]. For SM particles, $\boldsymbol{\gamma^i}$ are matrices in generation space that contain Standard Model Yukawa matrices. They multiply the corresponding Wilson coefficient either from the left or from the right as described below. Here we present the resummation of the Wilson coefficients in the mass basis for the leading power two jet operators for the leptoquarks S_1, S_3 and U_1^μ. We show in the Appendix F the anomalous dimensions for the three jet operators at $\mathcal{O}(\lambda^3)$ and a very similar procedure can be straightforwardly extended for the Wilson coefficients of these operators. We work at leading order in RG-improved perturbation theory, which is equivalent to resumming the large logarithms at next-to-leading logarithmic (NLL) order. This requires the two loop expressions for $\gamma^{(r)}_{\text{cusp}}(\alpha_r)$, one loop expression for $\boldsymbol{\gamma}^i$ and one loop γ^{LQ}. This estimate would give a prediction of the running effects to the tree level matching coefficients for various decay rates. In the Appendix B we collect the explicit expressions for the cusp anomalous dimensions and the beta functions at two loop.

12.1 Resummation effects on the singlet S_1

From the formula (12.3) we derive the evolution of the Wilson coefficients of the $\mathcal{O}(\lambda^2)$ operators for S_1 shown in (9.7). They are governed by the following anomalous dimensions in generation space

$$\begin{aligned}
\boldsymbol{\Gamma}_{S_1^* u_R^c \ell_R} &= \left(-\frac{4}{3}\gamma_{\rm cusp}^{(3)} - \frac{1}{9}\gamma_{\rm cusp}^{(1)}\right)\ln\frac{\mu}{M_{S_1}} - \frac{2}{3}\gamma_{\rm cusp}^{(1)}\left(\ln\frac{\mu^2}{M_{S_1}^2} + i\pi\right) + \gamma^{S_1} \\
&\quad + \left(\gamma^{\ell_R}, .\right) + \left(.\,, \boldsymbol{\gamma}^{u_R}\right)\,, \\
\boldsymbol{\Gamma}_{S_1^* Q_L^c L_L} &= \left(-\frac{4}{3}\gamma_{\rm cusp}^{(3)} - \frac{1}{9}\gamma_{\rm cusp}^{(1)}\right)\ln\frac{\mu}{M_{S_1}} - \left(\frac{3}{4}\gamma_{\rm cusp}^{(2)} + \frac{1}{12}\gamma_{\rm cusp}^{(1)}\right)\left(\ln\frac{\mu^2}{M_{S_1}^2} + i\pi\right) \\
&\quad + \gamma^{S_1} + \left(\boldsymbol{\gamma}^{L_L}, .\right) + \left(.\,, \boldsymbol{\gamma}^{Q_L}\right)\,, \\
\boldsymbol{\Gamma}_{S_1 d_R \nu_R} &= \left(-\frac{4}{3}\gamma_{\rm cusp}^{(3)} - \frac{1}{9}\gamma_{\rm cusp}^{(1)}\right)\ln\frac{\mu}{M_{S_1}} + \gamma^{S_1} + \left(.\,, \boldsymbol{\gamma}^{d_R}\right)\,,
\end{aligned} \tag{12.4}$$

where we use the notations $(.\,,\boldsymbol{\gamma})$ and $(\boldsymbol{\gamma}\,,.)$ for the single particle anomalous dimensions to indicate a multiplication with the Wilson coefficient from the left and from the right respectively, such that

$$\begin{aligned}
(.\,,\boldsymbol{\gamma})\;\boldsymbol{C} &\equiv \boldsymbol{C}\,\boldsymbol{\gamma}\,, \\
(\boldsymbol{\gamma}\,,.)\;\boldsymbol{C} &\equiv \boldsymbol{\gamma}\,\boldsymbol{C}\,.
\end{aligned} \tag{12.5}$$

If $\boldsymbol{\gamma}$ is the anomalous dimension of an antiparticle the multiplication with the Wilson coefficient is from the left and for a particle it becomes a multiplication from the right. The various single field anomalous dimensions in (12.4) are [55]

$$\begin{aligned}
\boldsymbol{\gamma}^{\ell_R} &= -\frac{\alpha_1}{4\pi} + \frac{1}{16\pi^2}\mathbf{Y}_\ell^\dagger\mathbf{Y}_\ell\,, \\
\boldsymbol{\gamma}^{L_L} &= -\frac{9\alpha_2}{16\pi} - \frac{\alpha_1}{16\pi} + \frac{1}{32\pi^2}\mathbf{Y}_\ell\mathbf{Y}_\ell^\dagger \\
\boldsymbol{\gamma}^{u_R} &= -\frac{\alpha_3}{\pi} - \frac{\alpha_1}{9\pi} + \frac{1}{16\pi^2}\mathbf{Y}_u^\dagger\mathbf{Y}_u\,, \\
\boldsymbol{\gamma}^{d_R} &= -\frac{\alpha_3}{\pi} - \frac{\alpha_1}{36\pi} + \frac{1}{16\pi^2}\mathbf{Y}_u^\dagger\mathbf{Y}_u\,, \\
\boldsymbol{\gamma}^{Q_L} &= -\frac{\alpha_3}{\pi} - \frac{9\alpha_2}{16\pi} - \frac{\alpha_1}{144\pi} + \frac{1}{32\pi^2}\left(\mathbf{Y}_u\mathbf{Y}_u^\dagger + \mathbf{Y}_d\mathbf{Y}_d^\dagger\right)\,, \\
\gamma^{S_1} &= -\frac{2\,\alpha_3}{3\pi} - \frac{\alpha_1}{18\pi}\,,
\end{aligned} \tag{12.6}$$

where $\mathbf{Y}_\ell$ is the Yukawa matrix for the lepton ℓ, $\mathbf{Y}_u$ and $\mathbf{Y}_d$ are the Yukawa matrices for up and down-type quarks. In practice we transform the Wilson coefficient in mass basis since this is the relevant basis for physical quantities such as decay rates. In the mass basis the Yukawa matrices in (12.6) become diagonal expect for the case of $\boldsymbol{\gamma}^{Q_L}$. In this case one needs to distinguish between the up-type quark and the down-type quark in the doublet [69]

$$\begin{aligned}
\boldsymbol{\gamma}^{u_L} &= -\frac{\alpha_3}{\pi} - \frac{9\alpha_2}{16\pi} - \frac{\alpha_1}{144\pi} + \frac{1}{32\pi^2}\Big[(\mathrm{diag}\left(y_u^2, y_c^2, y_t^2\right) + \boldsymbol{V}\mathrm{diag}\left(y_d^2, y_s^2, y_b^s\right)\boldsymbol{V}^\dagger\Big]\,, \\
\boldsymbol{\gamma}^{d_L} &= -\frac{\alpha_3}{\pi} - \frac{9\alpha_2}{16\pi} - \frac{\alpha_1}{144\pi} + \frac{1}{32\pi^2}\Big[\boldsymbol{V}^\dagger(\mathrm{diag}\left(y_u^2, y_c^2, y_t^2\right)\boldsymbol{V} + \mathrm{diag}\left(y_d^2, y_s^2, y_b^s\right)\Big]\,,
\end{aligned} \tag{12.7}$$

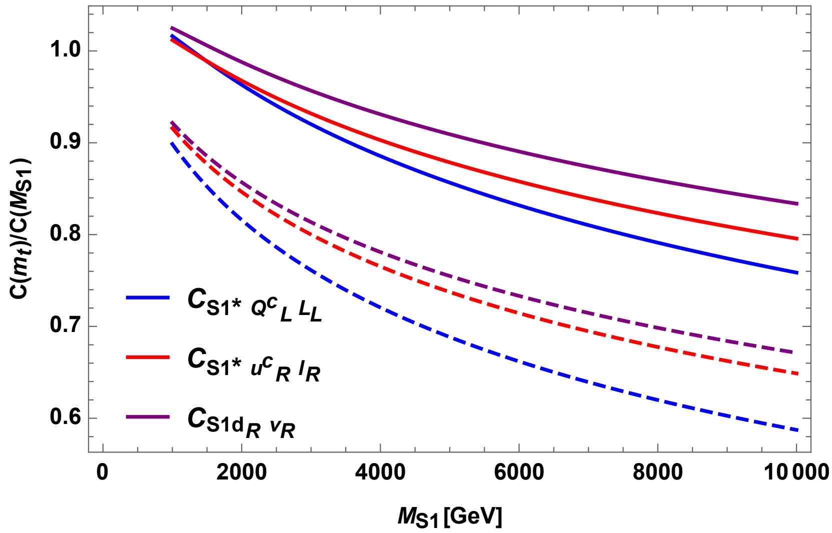

Figure 12.2: Resummation effects on Wilson coefficients of the $\mathcal{O}(\lambda^2)$ operators for S_1 as a function of M_{S_1} with top quark final state jets in all cases. The running is performed from the leptoquark scale to the top mass. The solid lines show the whole contribution and the dashed lines show the resummation only for the double logarithms.

where y_q is the Yukawa coupling of the quark q and $\boldsymbol{V} = \boldsymbol{U}_u^\dagger \boldsymbol{U}_d$ is the CKM matrix. For the numerical estimates we present here we take into account only the top quark Yukawa coupling. All the other quark Yukawas, including in $\boldsymbol{\gamma}^{u_R}$ have tiny effects in the resummation. We also neglect the Yukawa coupling of the leptons in $\boldsymbol{\gamma}^{\ell_R}$ and in $\boldsymbol{\gamma}^{L_L}$. This means in practice for the running of the Wilson coefficients in the mass basis the only relevant term including Yukawa couplings in the expressions in (12.6) is $\operatorname{diag}(y_u^2, y_c^2, y_t^2)$ as in the first line in (12.7) which becomes $\operatorname{diag}(0, 0, y_t^2)$, in this approximation. The evolution of the Yukawa coupling of the top quark is given by [77]

$$\frac{d}{d\ln\mu}\, y_t(\mu) = \frac{9\, y_t^3}{32\pi^2} - y_t\left(\frac{17\,\alpha_1}{48\pi} + \frac{9\,\alpha_2}{16\pi} + \frac{2\,\alpha_3}{\pi}\right). \tag{12.8}$$

In the case of leptoquarks the full gauge group running must be included for consistent numerical estimates at leading order RG improved PT.

We now present numerical results for the resummation of the Wilson coefficients from the leptoquark scale to a lower scale for the $\mathcal{O}(\lambda^2)$ operators for the S_1 shown in (9.7). For these operators the largest effects comes from $\bar{t}\ell$ final states. Therefore we fix the low scale to the top quark mass and we consider $M_{S_1} = 3$ TeV. We numerically integrate the evolution function $\mathcal{U}(M_{S_1}, \mu)$ such that the Wilson coefficients at different scales are related by

$$\mathrm{C}^{t\ell}(m_t) = \mathrm{C}^{t\ell}(M_{S_1})\,\mathcal{U}(M_{S_1}, m_t)\,, \tag{12.9}$$

where ℓ here stands for a lepton either left handed or right handed. We find the following

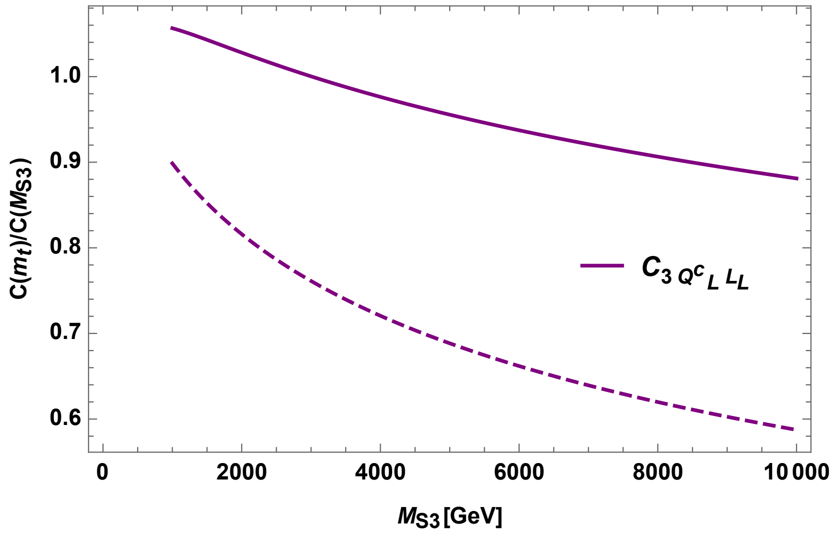

Figure 12.3: Variation of the resummation effects on the $\mathrm{C}_{S_3 Q_L^c L_L}$ with the mass of S_3, for left handed top quark and left handed lepton final states with initial scale around the leptoquark mass.The solid line shows the whole contribution and the dashed line represents only the resummation of the double logarithms.

results

$$\begin{aligned} \mathrm{C}^{t\ell}_{S_1^* u_R^c \ell_R}(m_t) &\approx 0.93\, e^{0.02i}\, \mathrm{C}^{t\ell}_{S_1^* u_R^c \ell_R}(M_{S_1})\,, \\ \mathrm{C}^{t\ell}_{S_1^* Q_L^c L_L}(m_t) &\approx 0.92\, e^{0.07i}\, \mathrm{C}^{t\ell}_{S_1^* Q_L^c L_L}(M_{S_1})\,. \end{aligned} \tag{12.10}$$

For the operator $\mathcal{O}^{ij}_{S_1 d_R \nu_R}$ the running is practically independent on the specific final state lepton flavour. In this case we find

$$\mathrm{C}^{ij}_{S_1 d_R \nu_R}(m_t) \approx 0.96\, \mathrm{C}^{ij}_{S_1 d_R \nu_R}(M_{S_1})\,. \tag{12.11}$$

In Fig.(12.2) we show how the resummation effects vary for different values of M_{S_1}. We compare between the full running effects and the contribution only from terms multiplied by $\gamma^{(r)}_{\text{cusp}}$ for the full gauge group. In the latter case the single logarithmic terms coming from single-particle anomalous dimensions are neglected. In practice this is often the case, where one resumms only the double logarithms which exponentiate. Though in Fig.(12.2) it can be seen that this difference is at least of $\mathcal{O}(10\%)$. In fact this is a merit of the effective theory that RG methods allow for a consistent resummation of the large logarithms, both single and double logarithms.

12.2 Resummation effects on the triplet S_3

As presented in Section 10.1 at leading order the triplet leptoquark S_3 decays only into a left handed quark and left handed lepton. In this case the Wilson coefficient $\mathrm{C}^{ij}_{S_3 Q_L^c L_L}$ obeys an

RG equation with anomalous dimension:

$$\begin{aligned}\mathbf{\Gamma}_{S_3^* Q_L^c L_L} &= \left(-\frac{4}{3}\gamma_{\text{cusp}}^{(3)} - 2\gamma_{\text{cusp}}^{(2)} - \frac{1}{9}\gamma_{\text{cusp}}^{(1)}\right) \ln\frac{\mu}{M_{S_3}} \\ &+ \left(\frac{1}{4}\gamma_{\text{cusp}}^{(2)} - \frac{1}{12}\gamma_{\text{cusp}}^{(1)}\right)\left(\ln\frac{\mu^2}{M_{S_3}^2} + i\pi\right) + \gamma^{S_3} + \left(\boldsymbol{\gamma}^L, .\right) + \left(., \boldsymbol{\gamma}^Q\right) ,\end{aligned} \tag{12.12}$$

where $\boldsymbol{\gamma}^{L_L}$, $\boldsymbol{\gamma}^{Q_L}$ are shown in (12.6) and the field anomalous dimension γ^{S_3} reads

$$\gamma^{S_3} = -\frac{2\,\alpha_3}{3\pi} - \frac{\alpha_2}{\pi} - \frac{\alpha_1}{18\pi} . \tag{12.13}$$

The QCD running for double logarithmic terms has not changed and it is indeed the same for the three leptoquarks since there are no QCD interactions between final sates at $\mathcal{O}(\lambda^2)$. For a 3 TeV leptoquark the effects are tiny in this case. They become more seizable for $M_{S_3} \geq 4$ TeV. For instance for $M_{S_3} = 4.5$ TeV we find

$$\mathrm{C}^{t\ell}_{S_3 Q_L^c L_L}(m_t) \;\approx\; 0.97\, e^{-0.02i}\, \mathrm{C}^{t\ell}_{S_3 Q_L^c L_L}(M_{S_3}) . \tag{12.14}$$

In Fig.(12.3) we show how these effects change significantly when single logarithmic terms are not included. For all the mass range the difference accounts for a change of $\mathcal{O}(20\%)$ in the Wilson coefficients.

12.3 Resummation effects on the vector U_1^μ

In a similar fashion we derive the anomalous dimensions of the leading order two jet operators for U_1^μ shown in (11.2). We find that

$$\begin{aligned}\mathbf{\Gamma}_{U_1 Q_L L_L} &= \left(-\frac{4}{3}\gamma_{\text{cusp}}^{(3)} - \frac{4}{9}\gamma_{\text{cusp}}^{(1)}\right) \ln\frac{\mu}{M_{U_1}} \\ &+ \left(-\frac{3}{4}\gamma_{\text{cusp}}^{(2)} + \frac{1}{12}\gamma_{\text{cusp}}^{(1)}\right)\left(\ln\frac{\mu^2}{M_{U_1}^2} + i\pi\right) + \gamma^{U_1} + \left(., \boldsymbol{\gamma}^Q\right) + \left(\boldsymbol{\gamma}^L, .\right) , \\ \mathbf{\Gamma}_{U_1 d_R \ell_R} &= \left(-\frac{4}{3}\gamma_{\text{cusp}}^{(3)} - \frac{4}{9}\gamma_{\text{cusp}}^{(1)}\right) \ln\frac{\mu}{M_{U_1}} - \frac{1}{3}\gamma_{\text{cusp}}^{(1)}\left(\ln\frac{\mu^2}{M_{U_1}^2} + i\pi\right) + \gamma^{U_1} \\ &+ \left(., \boldsymbol{\gamma}^{d_R}\right) + \left(\boldsymbol{\gamma}^{\ell_R}, .\right) , \\ \mathbf{\Gamma}_{U_1 u_R \nu_R} &= \left(-\frac{4}{3}\gamma_{\text{cusp}}^{(3)} - \frac{4}{9}\gamma_{\text{cusp}}^{(1)}\right) \ln\frac{\mu}{M_{U_1}} + \gamma^{U_1} + \left(., \boldsymbol{\gamma}^{u_R}\right) ,\end{aligned} \tag{12.15}$$

where the anomalous dimension of the leptoquark U_1^μ reads

$$\gamma^{U_1} = -\frac{2\,\alpha_3}{3\pi} - \frac{2\,\alpha_1}{9\pi} . \tag{12.16}$$

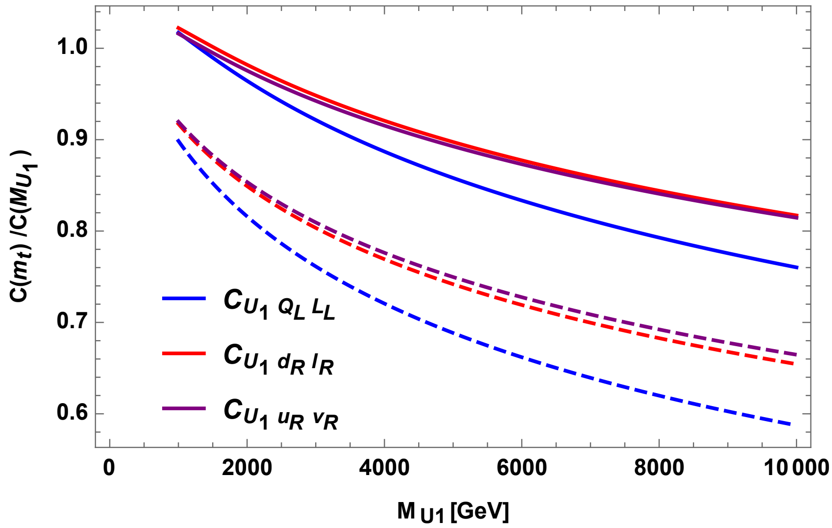

Figure 12.4: Resummation effects on the Wilson coefficients of $\mathcal{O}(\lambda^2)$ operators for U_1^μ as a function of M_{U_1}. The results are for top quark and lepton final state for $\mathrm{C}_{U_1 Q_L L_L}$ and $\mathrm{C}_{U_1 d_R \ell_R}$ and top quark and right handed neutrino for $\mathrm{C}_{U_1 u_R \nu_R}$. In both cases the initial scale is set to the M_{U_1}. The solid lines show the full effects and the dashed lines take into account only the double logs.

The results for top quark final states are shown in Fig.(12.4) both for the complete resummation and for the separate double log contribution. For $M_{U_1} = 3$ TeV we find the following numerical results

$$\begin{aligned}
\mathrm{C}^{t\ell}_{U_1 Q_L L_L}(m_t) &\approx 0.92\, e^{0.06i}\, \mathrm{C}^{t\ell}_{U_1 Q_L L_L}(M_{U_1})\,, \\
\mathrm{C}^{t\ell}_{U_1 d_R \ell_R}(m_t) &\approx 0.95\, e^{0.01i}\, \mathrm{C}^{t\ell}_{U_1 d_R \ell_R}(M_{U_1})\,, \\
\mathrm{C}^{t\nu}_{U_1 u_R \nu_R}(m_t) &\approx 0.94\, \mathrm{C}^{t\nu}_{U_1 u_R \nu_R}(M_{U_1})\,.
\end{aligned} \tag{12.17}$$

Also in this case there is a significant difference of about 20% in neglecting the single log resummation.

12.4 Example of an analytic solution of the RGE

At one-loop contribution it is possible to derive an analytic solution of the RG evolution equations of the Wilson coefficients for the full $SU(3) \times SU(2)_L \times U(1)_Y$ interactions. Beyond one loop this is challenging because cusp anomalous dimensions and beta functions start to mix with each other. Here we show an example of the exact solution for the evolution of the Wilson coefficient for a NLL resummation. We consider the Wilson coefficient of the operator $\mathcal{O}^{ij}_{S_1^* u_R^c \ell_R}$ in (9.7) neglecting the $SU(2)_L$, $U(1)_Y$ and Yukawa running. We define the anomalous dimension $\Gamma^{\mathrm{QCD}}_{S_1^* u_R^c \ell_R}$ such that

$$\Gamma^{\mathrm{QCD}}_{S_1^* u_R^c \ell_R} = -\frac{4}{3}\gamma^{(3)}_{\mathrm{cusp}} \ln\frac{\mu}{M_{S_1}} + \gamma^{S_1}_{\mathrm{QCD}} + \gamma^{u_R}_{\mathrm{QCD}}\,, \tag{12.18}$$

where $\gamma^{u_R}_{\text{QCD}} = -\alpha_3/\pi$ and $\gamma^{S_1}_{\text{QCD}} = -2\alpha_3/3\pi$. Then it can be shown that the following expression is a solution to the RGE with anomalous dimension $\Gamma^{\text{QCD}}_{S_1^* u_R^c \ell_R}$ [110, 113]

$$C^{ij}_{S_1^* u_R^c \ell_R}(\Lambda, M_{S_1}, \mu_2) = \left(\frac{\mu_1}{M_{S_1}}\right)^{-a_{\gamma_3}(\mu_1,\mu_2)} \exp\left[\mathcal{S}(\mu_1,\mu_2) - a_\gamma(\mu_1,\mu_2)\right] \times C^{ij}_{S_1^* u_R^c \ell_R}(\Lambda, M_{S_1}, \mu_1)\,, \tag{12.19}$$

where the Sudakov exponent $\mathcal{S}(\mu_1,\mu_2)$ is given by

$$\mathcal{S}(\mu_1,\mu_2) = -\frac{4}{3}\int_{\alpha_3(\mu_1)}^{\alpha_3(\mu_2)} d\alpha\, \frac{\gamma^{(3)}_{\text{cusp}}(\alpha)}{\beta(\alpha)} \int_{\alpha_3(\mu_1)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}\,. \tag{12.20}$$

We have defined $\gamma = \gamma^{S_1}_{\text{QCD}} + \gamma^{u_R}_{\text{QCD}}$ and $\gamma_3 = -\frac{4}{3}\gamma^{(3)}_{\text{cusp}}$. The quantity a_{γ_i} is defined as

$$a_{\gamma_i}(\mu_1,\mu_2) = -\int_{\alpha_3(\mu_1)}^{\alpha_3(\mu_2)} d\alpha\, \frac{\gamma_i(\alpha)}{\beta(\alpha)}\,. \tag{12.21}$$

In the above solution μ_1 should be of the order of M_{S_1} so that the initial condition is free of large logarithms. Using the one loop expressions of the one-particle anomalous dimensions, two loop cusps and two loop QCD β-function the scale evolution of the Wilson coefficient $C^{ij}_{S_1^* u_R^c \ell_R}$ is then [110]

$$C^{ij}_{S_1^* u_R^c \ell_R}(\Lambda, M_{S_1}, \mu_2) = C^{ij}_{S_1^* u_R^c \ell_R}(\Lambda, M_{S_1}, \mu_1)\left(\frac{\mu_1}{M_{S_1}}\right)^{-a_{\gamma_3}} \times \exp\left\{\frac{4\pi}{\alpha_3(\mu_1)}\left(1 - \frac{1}{r} - \ln r\right) + \left(\frac{251}{21} - \pi^2\right)(1 - r + \ln r) + \frac{13}{7}\ln^2 r - \frac{10}{21\pi}\ln r\right\}, \tag{12.22}$$

where $r = \alpha_3(\mu_2)/\alpha_3(\mu_1)$ and at LO in RG-improved perturbation theory $a_{\gamma_3} = (56/9\pi)\ln r$.

Chapter 13

Matching for tree level Wilson coefficients

In this section we look at certain UV models for each of the leptoquarks we have considered and perform a tree level matching of the matrix elements into the corresponding SCET Lagrangians. We match the operators that describe the two jet final states at leading and subleading power in the parameter λ. We start with the renormalizable Lagrangian that describes the S_1 interactions in a speficic leptoquark extension of the Standard Model. We follow a similar notation as in [49] and include an additional term for a right handed neutrino such that

$$\mathcal{L}_{S_1} = g_{1L}^{ij}\, \bar{Q}_L^{c,i} i\sigma_2 L_L^j S_1^\star + g_1^{ij}\, \bar{u}_R^{c,i} \ell_R^j S_1^\star + g_{1\nu}^{ij}\, \bar{d}_R^{c,i} \nu_R^j S_1^\star + \text{h.c.}\,, \tag{13.1}$$

where g^{ij} is the coupling strength between a quark of generation i and a lepton of the generation j in the full theory. Tree level matching of the matrix elements of the above Lagrangian into our effective Lagrangian in (9.1) yields the following Wilson coefficients

$$\begin{aligned} C^{ij}_{S_1^* u_R^c \ell_R} &= g^{ij}_{1R}\,, \\ C^{ij}_{S_1^* Q_L^c L_L} &= g^{ij}_{1L}\,, \\ C^{ij}_{S_1^* d_R^c \nu_R} &= g^{ij}_{1\nu}\,. \end{aligned} \tag{13.2}$$

Matching of the power suppressed SCET Lagrangian (9.9) into the (13.1) gives vanishing Wilson coefficients for all the subleading operators for two body decays of the S_1

$$\begin{aligned} C^{(0)^{ij}}_{S_1^* L_L \Phi d_R} &= C^{(0)^{ij}}_{S_1 Q_L \Phi \nu_R} = C^{(1)^{ij}}_{S_1^* L_L \Phi d_R} = C^{(2)^{ij}}_{S_1^* L_L \Phi d_R} = 0\,, \\ C^{(1)^{ij}}_{S_1 Q_L \Phi \nu_R} &= C^{(2)^{ij}}_{S_1 Q_L \Phi \nu_R} = C^{(1)^{ij}}_{S_1 d_R B \nu_R} = C^{(2)^{ij}}_{S_1 d_R B \nu_R} = 0\,. \end{aligned} \tag{13.3}$$

This is a remnant of the fact that at leading power the S_1 couples only to the charge conjugate of the quark field, while the Standard Model Higgs boson and the gauge bosons do not have conjugate particle vertices. In such a case it is not possible to get a hard propagator, which would have been integrated out in the effective theory.

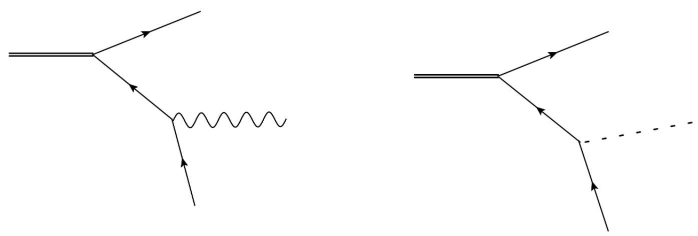

Figure 13.1: Feynman diagrams for the matching of the Wilson coefficients for the subleading power two jet operators. The diagram on the left contributes to operators with two fermions and a gauge boson and the one on the right contributes to operators with two fermions and the Higgs scalar.

The full theory Lagrangian that describes the S_3 interactions in four dimensions is given by the following

$$\mathcal{L}_{S_3} = g_{3L}^{ij}\,\bar{Q}^{i\,c,a}_{L}\,\epsilon^{ab}\,S_3^{*bd}\,L_L^{j,d} + \text{h.c.}\,. \tag{13.4}$$

Then it is straightforward to derive that at tree level the Wilson coefficient from the SCET Lagrangian in (10.1) reads

$$C^{ij}_{S_3^* Q_L^c L_L} = g^{ij}_{3L}\,. \tag{13.5}$$

Also in this case the Wilson coefficients at subleading power are equal to zero because of the charge conjugate fields present in (13.4).

Lastly we look at a UV Lagrangian that describes the interactions of the vector leptoquark U_1^μ. Since this leptoquark is a Lorentz vector there are more subtle issues that arise regarding the UV completed model. The cases that are usually considered in literature are either a gauge model where U_1^μ arises from the breaking of a gauge symmetry into the Standard Model gauge group [31, 80] or strongly interacting models [81]. The most general Lagrangian describing the interaction of the leptoquark U_1^μ reads

$$\begin{aligned}\mathcal{L}_{U_1} = {} & \frac{g_U}{\sqrt{2}}\left(\beta_L^{ij}\,\bar{Q}^i_L\gamma_\mu L^j_L U_1^\mu + \beta_R^{ij}\,\bar{d}^i_R\gamma_\mu \ell^j_R U_1^\mu\right) + g_U^\nu\,\beta_\nu^{ij}\,\bar{u}^i_R\gamma_\mu\nu^j_R U_1^\mu \\ & - i g_s(1-\kappa_U)U^\dagger_{1\mu}T^a U_{1\nu}G^{a\mu\nu} - i g_Y\frac{2}{3}(1-\tilde{\kappa}_U)U^\dagger_{1\mu}U_{1\nu}B^{\mu\nu} + \text{h.c.}\,.\end{aligned} \tag{13.6}$$

For simplicity here we consider the minimal coupling scenario for the leptoquark which corresponds to $\kappa_U = \tilde{\kappa}_U = 1$ [81]. Then the last two terms in (13.6) do not contribute to the matrix elements of the SCET operators for a tree level matching in our case.

To compute the Wilson coefficients here we require that in the collinear limit for the same initial and final states the matrix elements of the Lagrangian in equation (11.1) and in (13.6) give the same result. At $\mathcal{O}(\lambda^2)$ we obtain

$$
\begin{aligned}
C^{ij}_{U_1 Q_L L_L} &= \frac{g_U}{\sqrt{2}} \beta^{ij}_L \,, \\
C^{(k)ij}_{U_1 d_R \ell_R} &= \frac{g_U}{\sqrt{2}} \beta^{ij}_R \,, \\
C^{ij}_{U_1 u_R \nu_R} &= \frac{g^{\nu}_U}{\sqrt{2}} \beta^{ij}_{\nu} \,.
\end{aligned}
\tag{13.7}
$$

We find several non-trivial matching results for the Wilson coefficients at $\mathcal{O}(\lambda^3)$ as well. We show the type of Feynman diagrams that contribute in these cases in Fig.(13.1). For the operator basis in (11.7) we find that the corresponding non-vanishing Wilson coefficients are

$$
\begin{aligned}
C^{(k)ij}_{U_1 Q_L A L_L} &= \frac{g_U}{\sqrt{2}} \, \beta^{ij}_L \, \frac{2\Lambda}{M_{U_1}} \,, \\
C^{(2)ij}_{U_1 u_R A \nu_R} &= g^{\nu}_U \, \beta^{ij}_{\nu} \, \frac{2\Lambda}{M_{U_1}} \,, \\
C^{(k)ij}_{U_1 d_R A \ell_R} &= \frac{g_U}{\sqrt{2}} \, \beta^{ij}_R \, \frac{2\Lambda}{M_{U_1}} \,,
\end{aligned}
\tag{13.8}
$$

where $k = 1, 2$. The Wilson coefficient $C^{(1)ij}_{U_1 u_R A \nu_R}$ vanishes because the gauge boson in this case cannot be emitted in the same jet with the u-quark. This is because in the full theory there is no ν_R and gauge boson vertex. Also the various gauge couplings cancel in all cases because the Feynman rule for gauge bosons in SCET contains also the corresponding coupling constant. We note that there are several allowed possibilities for the gauge field "A" for each operator in (11.7). We have discussed this in details in Chapter 11, though for the purpose of matching the Wilson coefficients the discussion would be the same for each case, since the allowed gauge boson vertices in the EFT are the same as the ones allowed in the Standard Model for all these operators.

Next we match the SCET Feynman diagrams that generate from the operator basis in (11.10) into the fully theory graphs with the same final states. In this case there is always the scalar $\Phi^{(u)}$ present and the full theory diagrams are represented by the graph on the right in

Fig.(13.1). We obtain these results for the Wilson coefficients

$$
\begin{aligned}
C^{(1)ij}_{U_1 Q_L \Phi \ell_R} &= -i\frac{g_U}{\sqrt{2}}\,\beta^{ij}_L\, y_{\ell_j}\frac{2\Lambda}{M_{U_1}}\,, \\
C^{(2)ij}_{U_1 Q_L \Phi \ell_R} &= -i\frac{g_U}{\sqrt{2}}\,\beta^{ij}_R\, y_{d_i}\frac{2\Lambda}{M_{U_1}}\,, \\
C^{(1)ij}_{U_1 u_R \Phi L_L} &= 0 \\
C^{(2)ij}_{U_1 u_R \Phi L_L} &= -i\frac{g_U}{\sqrt{2}}\,\beta^{ij}_L\, y_{u_i}\frac{2\Lambda}{M_{U_1}}\,, \\
C^{(1)ij}_{U_1 Q_L \Phi \nu_R} &= 0 \\
C^{(2)ij}_{U_1 Q_L \Phi \nu_R} &= -i\frac{g_U}{\sqrt{2}}\,\beta^{ij}_\nu\, y_{u_i}\frac{2\Lambda}{M_{U_1}}\,,
\end{aligned}
\tag{13.9}
$$

where y_{d_i} is the Yukawa coupling of the down-type quark in generation i and y_{ℓ_j} is the Yukawa coupling of the lepton ℓ of generation j. The Wilson coefficients $C^{(1)ij}_{U_1 u_R \Phi L_L}$ and $C^{(1)ij}_{U_1 Q_L \Phi \nu_R}$ in (13.9) vanish as a result of the right handed neutrino not coupling to the Higgs field in the full theory. The remaining Wilson coefficients associated to the operators with a charge conjugate field in (11.10) are zero, as well as all the Wilson coefficients from the operator list in (11.9).

Chapter 14

Conclusions

In this work we have addressed several important aspects of soft-collinear effective theory from both the theoretical and application point of view. In the first part of the thesis we have presented a detailed analysis of the radiative decay process $h \to \gamma\gamma$ mediated by a b-quark loop, which is a power suppressed process in SCET. One of the main results of this part was the derivation of the renormalized factorization formula for this process in the presence of endpoint divergences. This is the first formula of its kind derived for an observable at subleading power in SCET. To arrive to this derivation we have studied other very important features of the effective theory at subleading power such as the soft-quark soft function and the regularization of divergences at the endpoints of convolution integrals in the factorization theorem. We have regularized the endpoint divergences by using a "plus-type" subtraction scheme.

We have presented in Chapter 4 the one-loop renormalization of all the quantities in the factorization formula for $h \to \gamma\gamma$. We have derived in details the renormalized one-loop soft-quark soft function based on a conjecture that one of the amplitude terms where the soft function enters, the T_3 is scale invariant. We proved that the renormalization factor derived with this assumption successfully removes all the $1/\epsilon^n$ poles of the one-loop bare soft function. From the same conjecture and using also already derived two-loop results for the jet function's and one of the Wilson coefficients' anomalous dimensions we extended also the derivation of the anomalous dimension for the soft function to two-loop order.

We have then derived in Chapter 5 an exact solution to the RG evolution equation of the soft function in momentum space, in Laplace space and in the "diagonal" space. In momentum space the analytic solution gives a result in terms of the so-called Meijer G-function. We showed that the soft function obeys a local RGE in the diagonal space and exists a well-defined transformation through some transfer functions from momentum space to diagonal space and vice-versa. Due to the local nature of the evolutions in diagonal space we could show explicitly that here the amplitude T_3 is free of endpoint divergences in a way that is compatible with the RG invariance.

We have used our results on renormalization to derive the renormalized version of the factorization theorem in Chapter 6. This was a highly non-trivial derivation due to the fact that renormalization for various terms does not commute with the regularization of the endpoint divergences. We regulate these divergences by means of a "plus-type" subtraction scheme, though there are extra terms that emerge after renormalization. At first it seemed that these terms would break the factorization. With several not so obvious rearrangements of terms together with two important "refactorization conditions" for one of the operator matrix elements and the Wilson coefficients we proved that all the extra terms could be absorbed into a redefinition of one of the matching coefficients and finally have a well-defined factorized form for the renormalized amplitude. We have combined our results to predict the large logarithms in the three-loop decay amplitude for $h \to \gamma\gamma$.

We concluded our analysis for $h \to \gamma\gamma$ with a discussion on the resummation of large logarithms at leading-order RG-improved perturbation theory in Chapter 7. We presented there a closed form solution for the resummation in the amplitude terms T_1 and T_3 and a semi-numerical result for the convolution T_2. Consistently resumming the large logarithms to all orders in α_s is a requirement for precision calculations.

The results we have derived here such as the evolution of the soft-quark soft function, the renormalization and factorization in the presence of endpoint divergences and resummation beyond the leading logarithms for a power suppressed process are very important steps towards deepening the understand of SCET as an effective theory beyond leading order in power counting. Moreover the framework we develop is general and can be applied to several other important processes at particle colliders such as Higgs boson production, Drell-Yan processes, B-meson exclusive decays etc.

In the second part of this thesis we have presented an application of SCET for on-shell decays of heavy exotic particles charged under the Standard Model gauge group. The leptoquarks we have studied are considered main candidates for solving several observed deviations from the Standard Model in the flavour sector. We have discussed in details the decay rates of two scalar leptoquarks S_1, S_3 and a vector leptoquark U_1^μ. We have shown that a consistent analysis of this problem requires treating the leptoquarks as heavy degrees of freedom that interact with their lighter decay products described by SCET operators. We have shown that at leading order in the effective theory the leptoquarks decay into two Standard Model particles and in the case of U_1^μ and S_1 a right handed neutrino is also allowed as a final state. In addition we have presented the subleading power two jet operators and the leading power Lagrangians for three jet final states at $\mathcal{O}(\lambda^3)$.

We have computed all the leading and subleading power two body decay rates together with the differential decay rates for leading three body decays for the S_1, S_3 and U_1^μ. We have used RG equations of the SCET operators at leading order to resum the large logarithms in their Wilson coefficients at next-to-leading logarithmic order. We have given numerical estimates of these effects on the decay rates for some of the decays with most phenomenological interest. We have found that for the two jet operators, for all the three leptoquarks, there is a significant effect coming from the single logarithmic terms in the running of the Wilson coefficients. The decay rates would change by as much as about 20% if the single logarithmic terms are not properly resummed. We have observed that the leading power two body decays of the scalar leptoquark S_1 receive the largest correction from resummation.

Lastly we have done a matching procedure of our effective Lagrangians for the leptoquark S_1, S_3 and U_1^μ into three corresponding extensions of the Standard Model with these heavy particles and show the relations between the Wilson coefficients in our effective theory and various coupling constants in these models. On application grounds this work provides an estimation of the effects of resummation on the main decay rates of the singlet leptoquark S_1, the triplet S_3 and the vector leptoquark U_1^μ.

Appendix A

Bare matrix elements and Wilson coefficients

In here we present the one-loop expressions for the bare matrix elements and their Wilson coefficients that appear in the bare amplitude for the $h \to \gamma\gamma$ decay. These results are taken from Appendix B in reference [2].

A.1 Bare matrix elements

The bare operator $O_1^{(0)}$ is given by (omitting the polarization vectors of the photon)

$$\langle\gamma\gamma|O_1^{(0)}|h\rangle = m_{b,0}g_\perp^{\mu\nu} \,. \tag{A.1}$$

There are no QCD corrections to its matrix elements since the operator $O_1^{(0)}$ has no coloured particles in its definition. The bare matrix elements for the operator $O_2^{(0)}$ are

$$\langle\gamma\gamma|O_2^{(0)}(z)|h\rangle = \frac{N_c\alpha_{b,0}}{2\pi}m_{b,0}g_\perp^{\mu\nu}\left[e^{\epsilon\gamma_E}\Gamma(\epsilon)\left(m_{b,0}^2\right)^{-\epsilon} + \frac{C_F\alpha_{s,0}}{4\pi}\left(m_{b,0}^2\right)^{-2\epsilon}\left[K(z)+K(1-z)\right]\right] , \tag{A.2}$$

where

$$\begin{aligned} K(z) =& \frac{1}{\epsilon^2}\left(\ln z + \frac{3}{2}\right) + \frac{1}{\epsilon}\left(\frac{\ln^2 z}{2} - \ln z\ln(1-z) - \frac{1}{4} - \frac{\pi^2}{6}\right) \\ &+ 6\mathrm{Li}_3(z) + (1-2z-2\ln z)\,\mathrm{Li}_2(z) + \frac{\ln^3 z}{6} + [z+\ln(1-z)]\ln^2 z \\ &+ \left(2\mathrm{Li}_2(1-z) - \frac{1}{2}\ln(1-z) - \frac{1+3z}{2} - \frac{\pi^2}{6}\right)\ln z + \frac{3}{2} + \frac{\pi^2}{6} - 4\zeta_3 + \mathcal{O}(\epsilon) \,. \end{aligned} \tag{A.3}$$

The $\alpha_{b,0}$ and $\alpha_{s,0}$ are the bare electromagnetic and strong couplings. The functions $K(z)$ and $K(1-z)$ contain terms singular for $z \to 0$ and $z \to 1$ in the above expression. From this result it is straightforward to obtain the matrix elements for the operator $[\![O_2^{(0)}(z)]\!]$ simply by taking the $z \to 0$ limit. We have

$$\begin{aligned}[\![K(z)+K(1-z)]\!] =& \frac{e^{2\epsilon\gamma_E}}{1-2\epsilon}\Big[2\left(2-3\epsilon+2\epsilon^2\right)\Gamma^2(\epsilon)+2(1-\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(-\epsilon) \\ &+z^\epsilon\left(2-4\epsilon-\epsilon^2\right)\frac{\Gamma(2\epsilon)\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\Big]\,.\end{aligned} \tag{A.4}$$

The bare jet function that enters in the bare operator $O_3^{(0)}$ reads

$$J^{(0)}\left(p^2\right)=1+\frac{C_F\alpha_{s,0}}{4\pi}\left(-p^2-i0\right)^{-\epsilon}e^{\epsilon\gamma_E}\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\left(2-4\epsilon-\epsilon^2\right)\,. \tag{A.5}$$

In section (3.2) we have already presented the bare soft function expanded around $\epsilon=0$. For completeness we give here the closed form result in ϵ as originally derived in [98]:

$$S^{(0)}(w)=-\frac{N_c\alpha_{b,0}}{\pi}m_{b,0}\left[S_a^{(0)}(w)\theta\left(w-m_{b,0}^2\right)+S_b^{(0)}(w)\theta\left(m_{b,0}^2-w\right)\right]\,, \tag{A.6}$$

where

$$\begin{aligned}S_a^{(0)}(w) =& \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(w-m_{b,0}^2\right)^{-\epsilon} \\ &+\frac{C_F\alpha_{s,0}}{4\pi}\Big[\Big[C_1(\epsilon)+\frac{2}{\epsilon}\ln(1-r)\Big]\left(w-m_{b,0}^2\right)^{-2\epsilon}+C_2(\epsilon)\left(m_{b,0}^2\right)^{1-\epsilon}\left(w-m_{b,0}^2\right)^{-1-\epsilon} \\ &-2\mathrm{Li}_2(r)+2\ln r\ln(1-r)-3\ln^2(1-r)+2\ln(1-r)+\ldots\Big] \\ S_b^{(0)}(w) =& \frac{C_F\alpha_{s,0}}{4\pi}\left(m_{b,0}^2\right)^{-2\epsilon}\left[-\frac{4}{\epsilon}\ln(1-\hat{w})+6\ln^2(1-\hat{w})+\ldots\right]\end{aligned} \tag{A.7}$$

Here $C_1(\epsilon)$ and $C_2(\epsilon)$ are:

$$\begin{aligned}C_1(\epsilon) =& \frac{2e^{2\epsilon\gamma_E}}{\Gamma(1-2\epsilon)}\left[\frac{(1+\epsilon)\Gamma(-\epsilon)^2}{\Gamma(2-2\epsilon)}+2\Gamma(\epsilon)\Gamma(-\epsilon)\right]\,, \\ C_2(\epsilon) =& -2e^{2\epsilon\gamma_E}\frac{3-2\epsilon}{1-2\epsilon}\frac{\Gamma(\epsilon)}{\Gamma(-\epsilon)}\,.\end{aligned} \tag{A.8}$$

The dimensionless ratios r and $\hat{w}$ are defined as $r=m_{b,0}^2/w$ and $\hat{w}=w/m_{b,0}^2$. The dots in (A.7) refer to terms that vanish for $r\to 0$ and $\hat{w}\to 0$.

A.2 Bare Wilson coefficients

The bare Wilson coefficients derived in [98] reads

$$\begin{aligned}H_1^{(0)} =& \frac{y_{b,0}}{\sqrt{2}} \frac{N_c \alpha_{b,0}}{\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} (1-3\epsilon) \frac{2\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(3-2\epsilon)} \\ &\times \left\{1 - \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{\Gamma(2-3\epsilon)} \right. \\ &\times \left[\frac{2(1-\epsilon)\left(3-12\epsilon+9\epsilon^2-2\epsilon^3\right)}{1-3\epsilon} + \frac{8}{1-2\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(2-\epsilon)\Gamma(2-3\epsilon)}{\Gamma(1+2\epsilon)\Gamma^3(1-2\epsilon)} \right. \\ &\left.\left. - \frac{4\left(3-18\epsilon+28\epsilon^2-10\epsilon^3-4\epsilon^4\right)}{1-3\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(1+\epsilon)\Gamma(2-2\epsilon)}\right]\right\} .\end{aligned} \tag{A.9}$$

The $H_2^{(0)}$ and $H_3^{(0)}$ are

$$\begin{aligned}H_2^{(0)}(z) =& \frac{y_{b,0}}{\sqrt{2}} \left\{\frac{1}{z} + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \right. \\ &\left. \times \left[\frac{2-4\epsilon-\epsilon^2}{z^{1+\epsilon}} - \frac{2(1-\epsilon)^2}{z} - 2\left(1-2\epsilon-\epsilon^2\right) \frac{1-z^{-\epsilon}}{1-z}\right]\right\} + (z \to 1-z), \end{aligned} \tag{A.10}$$

$$H_3^{(0)} = \frac{y_{b,0}}{\sqrt{2}} \left[-1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} e^{\epsilon\gamma_E} 2(1-\epsilon)^2 \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\right] .$$

We can easily take the $z \to 0$ limit for $H_2^{(0)}$ and find that $[\![H_2^{(0)}(z)]\!]$

$$[\![H_2^{(0)}(z)]\!] = \frac{y_{b,0}}{\sqrt{2}} \left\{1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2\right)^{-\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \left[\left(2-4\epsilon-\epsilon^2\right) z^{-\epsilon} - 2(1-\epsilon)^2\right]\right\} . \tag{A.11}$$

The infinity-bin subtraction term $\Delta H_1^{(0)}$ is given by

$$\begin{aligned}\Delta H_1^{(0)} =& -\frac{y_{b,0}}{\sqrt{2}} \frac{N_c \alpha_{b,0}}{\pi} \left(-M_h^2 - i0\right)^{-\epsilon} \frac{e^{\epsilon\gamma_E}}{\epsilon^2 \Gamma(1-\epsilon)} \left\{1 + \frac{C_F \alpha_{s,0}}{4\pi} \left(-M_h^2 - i0\right)^{-\epsilon} \right. \\ &\left. \times \frac{e^{\epsilon\gamma_E} \Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left[\left(1-2\epsilon+3\epsilon^2\right)\Gamma(\epsilon) + \frac{1+\epsilon}{2} \frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\right]\right\} .\end{aligned} \tag{A.12}$$

Appendix B

Anomalous dimensions and RG functions

We collect in here the one and two loop expressions for the cusp anomalous dimension, for the β-function and other various anomalous dimensions that appear in our work together with the $\mathcal{O}(\alpha_s)$ results for the RG functions $a_\Gamma(\mu_i, \mu)$ and $\mathcal{S}(\mu_i, \mu)$. We start with the β-function where for various gauge groups we have

$$\beta(\alpha_r) \;=\; -2\alpha_r \left[\beta_0^{(r)}\,\frac{\alpha_r}{4\pi} \;+\; \beta_1^{(r)}\left(\frac{\alpha_r}{4\pi}\right)^2 + \ldots\right], \tag{B.1}$$

for $r \in \{SU(3), SU(2)_L, U(1)_Y\}$. The two-loop coefficients read:

$$\begin{aligned}\beta_0^{(r)} &= \frac{11}{3}C_A^{(r)} - \frac{4}{3}T_F^{(r)} n_f \\ \beta_1^{(r)} &= \frac{34}{3}{C_A^{(r)}}^2 - \frac{20}{3}C_A^{(r)}T_F^{(r)} n_f - 4C_F^{(r)}T_F^{(r)} n_f\,,\end{aligned} \tag{B.2}$$

where the Casimir operators for each gauge symmetry for the fundamental and adjoint representations are $C_F^{(3)} = \frac{4}{3}$, $C_F^{(2)} = \frac{3}{4}$, $C_F^{(1)} = 1$, $C_A^{(3)} = 3$, $C_A^{(2)} = 2$, $C_A^{(1)} = 0$. $T_F = \frac{1}{2}$ for the non abelian groups and for $U(1)_Y$ it is the sum $\sum_i Y_i^2$ over the active flavours of fermions. In particular the two-loop coefficients of the QCD β-function for $n_f = 5$ read

$$\begin{aligned}\beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_f = \frac{23}{3}, \\ \beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F n_f - 4C_F T_F n_f \approx 38.6667 \\ \beta_2 &= \frac{2857}{54}C_A^3 + \left(2C_F^2 - \frac{205}{9}C_F C_A - \frac{1415}{27}C_A^2\right)T_F n_f + \left(\frac{44}{9}C_F + \frac{158}{27}C_A\right)T_F^2 n_f^2 \\ &\approx 180.907.\end{aligned} \tag{B.3}$$

where for simplicity in notation we have removed the group label (3) in this case.

For the cusp anomalous dimension Γ_{cusp} we present here only the QCD contributions which are relevant for our calculations for the $h \to \gamma\gamma$. Again in this case since the top quark cannot appear in the loop we fix $n_f = 5$. We then have:

$$\Gamma_{\text{cusp}}\ (\alpha_s) = \Gamma_0 \frac{\alpha_s}{4\pi} + \Gamma_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \Gamma_2 \left(\frac{\alpha_s}{4\pi}\right)^3 + \dots, \tag{B.4}$$

with

$$\begin{aligned}
\Gamma_0 =& 4C_F = \frac{16}{3}, \\
\Gamma_1 =& 4C_F \left[C_A \left(\frac{67}{9} - \frac{\pi^2}{3}\right) - \frac{20}{9} T_F n_f\right] \approx 36.8436 \\
\Gamma_2 =& 4C_F \left[C_A^2 \left(\frac{245}{6} - \frac{134\pi^2}{27} + \frac{11\pi^4}{45} + \frac{22}{3}\zeta_3\right) + C_A T_F n_f \left(-\frac{418}{27} + \frac{40\pi^2}{27} - \frac{56}{3}\zeta_3\right) \right. \\
& \left. + C_F T_F n_f \left(-\frac{55}{3} + 16\zeta_3\right) - \frac{16}{27} T_F^2 n_f^2\right] \approx 239.208
\end{aligned} \tag{B.5}$$

The one-loop QCD coefficients for the γ_s, γ_q and γ' anomalous dimensions are respectively:

$$\begin{aligned}
\gamma_{s,0} &= -6C_F = -8, \\
\gamma_{s,1} &= C_F^2 \left(-3 + 4\pi^2 - 48\zeta_3\right) + C_F C_A \left(\frac{655}{27} - \frac{55\pi^2}{9} - 4\zeta_3\right) + C_F T_F n_f \left(-\frac{188}{27} + \frac{20\pi^2}{9}\right) \\
&\approx -151.280.
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\gamma_{q,0} &= -3C_F, \\
\gamma_{q,1} &= C_F^2 \left(-\tfrac{3}{2} + 2\pi^2 - 24\zeta_3\right) + C_F C_A \left(-\tfrac{961}{54} - \tfrac{11\pi^2}{6} + 26\zeta_3\right) + C_F T_F n_f \left(\tfrac{130}{27} + \tfrac{2\pi^2}{3}\right),
\end{aligned} \tag{B.7}$$

$$\gamma'_0 = 0, \quad \gamma'_1 = C_F \left[C_A \left(\frac{808}{27} - \frac{11\pi^2}{9} - 28\zeta_3\right) - T_F n_f \left(\frac{224}{27} - \frac{4\pi^2}{9}\right)\right]. \tag{B.8}$$

For the mass anomalous dimension we have

$$\gamma_{m,0} = -6C_F, \quad \gamma_{m,1} = -3C_F^2 - \frac{97}{3} C_F C_A + \frac{20}{3} C_F T_F n_f\,. \tag{B.9}$$

Lastly we present the $\gamma^{(r)}_{\text{cusp}}$ from the second part of the thesis, for different groups read [55]. In here we need to use all the six possible flavours, since the heavy exotic particles can decay into any of the six flavours. We then have:

$$\begin{aligned}
\gamma^{(1)}_{\text{cusp}} &= \frac{\alpha_1}{\pi} - \frac{17}{6}\left(\frac{\alpha_1}{\pi}\right)^2 + .. \\
\gamma^{(2)}_{\text{cusp}} &= \frac{\alpha_2}{\pi} + \left(2 - \frac{\pi^2}{6}\right)\left(\frac{\alpha_2}{\pi}\right)^2 + .. \\
\gamma^{(3)}_{\text{cusp}} &= \frac{\alpha_3}{\pi} + \left(\frac{47}{12} - \frac{\pi^2}{4}\right)\left(\frac{\alpha_3}{\pi}\right)^2 + ..
\end{aligned} \tag{B.10}$$

where $\alpha_3 \equiv \alpha_s$ here. We use in (B.10) the same notation we follow in the main text for these quantities.

The RG function $a_\Gamma(\mu_i, \mu)$ and the Sudakov factor $\mathcal{S}(\mu_i, \mu)$ for the QCD contribution at NLO in RG-improved perturbation theory are given by [103]:

$$
\begin{aligned}
a_\Gamma\left(\mu_i, \mu\right) =& \frac{\Gamma_0}{2\beta_0}\left[\ln\frac{\alpha_s(\mu)}{\alpha_s\left(\mu_s\right)} + \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0}\right)\frac{\alpha_s(\mu) - \alpha_s\left(\mu_s\right)}{4\pi} + \mathcal{O}\left(\alpha_s^2\right)\right], \\
\mathcal{S}\left(\mu_i, \mu\right) =& \frac{\Gamma_0}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s\left(\mu_s\right)}\left(1 - \frac{1}{r} - \ln r\right) + \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0}\right)\left(1 - r + \ln r\right) + \frac{\beta_1}{2\beta_0}\ln^2 r\right. \\
& + \frac{\alpha_s\left(\mu_s\right)}{4\pi}\left[\left(\frac{\beta_1\Gamma_1}{\beta_0\Gamma_0} - \frac{\beta_2}{\beta_0}\right)\left(1 - r + r\ln r\right) + \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0}\right)\left(1 - r\right)\ln r\right. \\
& \left.\left. - \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} - \frac{\beta_1\Gamma_1}{\beta_0\Gamma_0} + \frac{\Gamma_2}{\Gamma_0}\right)\frac{(1-r)^2}{2}\right] + \mathcal{O}\left(\alpha_s^2\right)\right\}.
\end{aligned}
\tag{B.11}
$$

where $r = \alpha_s(\mu)/\alpha_s(\mu_i)$. This results can be readily used for other gauge groups by substituting the corresponding coefficients and the coupling constants accordingly.

Appendix C

T_3 regularization in momentum space

The convolution amplitude T_3 remains ill-defined at the endpoints also after resummation. We can see this explicitly even at leading order. For simplicity consider the leading order jet functions in RG-improved perturbation theory which at the jet scale μ_j reads

$$J_{\mathrm{LO}}\left(p^2,\mu\right) = \exp\left[-2S\left(\mu_j,\mu\right) - a_{\gamma'}\left(\mu_j,\mu\right)\right] \frac{\Gamma\left(1-a_\Gamma\left(\mu_j,\mu\right)\right)}{\Gamma\left(1+a_\Gamma\left(\mu_j,\mu\right)\right)} \left(\frac{-p^2 e^{-2\gamma_E}}{\mu_j^2}\right)^{a_\Gamma(\mu_j,\mu)} \tag{C.1}$$

and insert it in the above expression. We have

$$\begin{aligned} T_{3,\mathrm{LO}} =& H_3(\mu) \exp\left[-4S\left(\mu_j,\mu\right) - 2a_{\gamma'}\left(\mu_j,\mu\right)\right] \frac{\Gamma^2\left(1-a_\Gamma\left(\mu_j,\mu\right)\right)}{\Gamma^2\left(1+a_\Gamma\left(\mu_j,\mu\right)\right)} \\ &\times \int_0^\infty \frac{dw}{w} S(w,\mu) \left(\frac{-M_h^2 w e^{-4\gamma_E}}{\mu_j^4}\right)^{a_\Gamma(\mu_j,\mu)} \int_0^\infty \frac{d\ell_-}{\ell_-} . \end{aligned} \tag{C.2}$$

In this expression it is clear that the divergence in w, for $w \to \infty$ has been regulated by the RG evolution, where $a_\gamma(\mu_j,\mu)$ acts as a regulator, while the divergence in ℓ_- is left unregulated after the resummation. Note that for $w \to 0$ there is no singularity because of the presence of the Meijer G-function in the soft function in (5.22). The presence of this remaining divergence requires a rapidity regularization on the convolution.

C.1 RG invariance and rapidity regularization

There are two main schemes one can consider to regularize the rapidity divergences in the convolution T_3. The first one is by using an analytic rapidity regulator δ such that

$$\int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \to \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \left(\frac{M_h\left(\ell_+ - \ell_-\right) - i0}{\nu^2}\right)^{-2\delta} , \tag{C.3}$$

where ν is the associated scale. This regulator has already been used in [98] when deriving the bare factorization theorem for the $h \to \gamma\gamma$ process. Another option is to impose hard cut offs on the integrals

$$\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+} \to \lim_{\sigma\to -1}\int_0^{M_h} \frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+}\,, \tag{C.4}$$

where the parameter σ here is considered positive and the limit $\sigma \to (-1 - i0)$ should be taken only after the integration by analytic continuation. It should be obvious by now that are interested in a regulator that is compatible with the RG invariance of the T_3. In fact in this case we find that both these regulator schemes break T_3 scale invariance already at $\mathcal{O}(\alpha_s)$. Using the RG equations for all the component functions together with the two loop jet anomalous dimension in (4.51) we find

$$\begin{aligned}\frac{dT_3^{\text{analytic}}}{d\ln\mu} =& H_3(\mu)\int_0^\infty dx K(x,\mu)\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+} S\left(\ell_+\ell_-,\mu\right)\\ &\times\left\{\left[\left(\frac{M_h\left(\ell_+ - x\ell_-\right)}{\nu^2}\right)^{-2\delta} - \left(\frac{M_h\left(\ell_+ - \ell_-\right)}{\nu^2}\right)^{-2\delta}\right] J\left(xM_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right)\right.\\ &+\left.\left[\left(\frac{M_h\left(x\ell_+ - \ell_-\right)}{\nu^2}\right)^{-2\delta} - \left(\frac{M_h\left(\ell_+ - \ell_-\right)}{\nu^2}\right)^{-2\delta}\right] J\left(M_h\ell_-,\mu\right) J\left(-xM_h\ell_+,\mu\right)\right\}\,,\end{aligned} \tag{C.5}$$

where $K(x,\mu)$ captures the non-local part of the anomalous dimension:

$$K(x,\mu) = \Gamma_{cusp}(\alpha_s)\Gamma(1,x) + C_F\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\theta(1-x)}{1-x} h(x) + \mathcal{O}(\alpha_s)\,. \tag{C.6}$$

All the local terms have cancelled out and the violation of scale invariance comes from the fact that the non-local contributions do not vanish, more precisely they do not vanish at $x = 1$ as seen in (C.6). This fact causes that the regularization of rapidity divergences to break the RG invariance. The integrations over $\ell_\pm$ is not scale invariant any more. A similar interpretation holds for the cut off regulator

$$\begin{aligned}\frac{dT_3^{\text{cutoff}}}{d\ln\mu} =& \lim_{\sigma\to-1} H_3(\mu)\int_0^\infty dx K(x,\mu)\\ &\times\left[\int_{M_h}^{M_h/x} \frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} J\left(xM_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right)\right.\\ &+\left.\int_0^{M_h} \frac{d\ell_-}{\ell_-}\int_{\sigma M_h}^{\sigma M_h/x} \frac{d\ell_+}{\ell_+} J\left(M_h\ell_-,\mu\right) J\left(-xM_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right)\right]\,,\end{aligned} \tag{C.7}$$

with $K(x,\mu)$ the same as in (C.6). We can use the leading order contributions for the renormalized hard, soft and jet function and derive the leading order expressions in both these cases. At leading order the the function is a constant

$$J\left(p^2,\mu\right) = 1 + \mathcal{O}\left(\alpha_s\right) \tag{C.8}$$

The hard and the soft function read:

$$\begin{aligned} H_3(\mu) &= \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} + \mathcal{O}(\alpha_s) \ , \\ S(w,\mu) &= m_b \theta\left(w - m_b^2\right) + \mathcal{O}(\alpha_s) \, . \end{aligned} \tag{C.9}$$

For the analytic regulator we find:

$$\frac{dT_3^{\text{analytic}}}{d\ln\mu} = \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} m_b \frac{C_F \alpha_s}{4\pi} \left(\frac{-M_h^2 m_b^2}{\nu^4}\right)^{-\delta} \frac{8\Gamma^2(\delta)}{\Gamma(1+2\delta)} [-H(\delta) - H(-\delta)] + \mathcal{O}\left(\alpha_s^2\right) \ . \tag{C.10}$$

And similarly for the cut off regulator

$$\frac{dT_3^{\text{cutoff}}}{d\ln\mu} = \frac{N_c \alpha_b}{\pi} \frac{y_b}{\sqrt{2}} m_b \frac{C_F \alpha_s}{4\pi} 16\zeta_3 + \mathcal{O}\left(\alpha_s^2\right) \tag{C.11}$$

The above expressions give the same result if we take the limit $\delta \to 0$ in (C.10). While this is a consistency check that both these schemes independently can regularize the rapidity divergences in the convolution T_3, our goal is to show that one can regularize the T_3 in a scale invariant way. Since we noticed that the origin of the violation of the scale invariance resides in the non-local term $K(x,\mu)$ it is intuitive to think that if one can diagonalize these non-local terms the scale invariance should be restored.

Appendix D

Detailed calculations from the renormalized amplitude

D.1 T_3 mismatch term

In here we derive the mismatch term δT_3 from the amplitude T_3 written in terms of renormalized functions. We have (for simplicity we omit the $\sigma \to -1$ in all the following derivations)

$$
\begin{aligned}
T_3^{\mathrm{ren}} &= g_\perp^{\mu\nu} H_3(\mu) \int_0^{M_h} \frac{\mathrm{d}\ell_-}{\ell_-} \int_0^{\sigma M_H} \frac{\mathrm{d}\ell_+}{\ell_+} J\left(M_h \ell_-, \mu\right) J\left(-M_h \ell_+, \mu\right) S\left(\ell_+ \ell_-, \mu\right) \\
&= g_\perp^{\mu\nu} H_3(\mu) \int_0^\infty dw' \Bigg[\Bigg(\int_0^{M_h} \frac{d\ell_-}{\ell_-} \int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+} \int_0^\infty d\ell'_- \int_0^\infty d\ell'_+ \\
&\qquad - \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \int_0^{M_h} d\ell'_- \int_0^{\sigma M_h} d\ell'_+ \Bigg) + \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \int_0^{M_h} d\ell'_- \int_0^{\sigma M_h} d\ell'_+ \Bigg] \\
&\qquad \times Z_J\left(\ell_-, \ell'_-\right) Z_J\left(\ell_+, \ell'_+\right) Z_S\left(\ell_- \ell_+, w'\right) J^{(0)}\left(M_h \ell'_-\right) J^{(0)}\left(-M_h \ell'_+\right) S^{(0)}(w')\,.
\end{aligned}
\tag{D.1}
$$

For the last term it is easy to show that it is in fact the T_3 amplitude written in terms of bare functions

$$\begin{aligned}
&H_3(\mu)\int_0^\infty dw'\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+}\int_0^{M_h} d\ell'_-\int_0^{\sigma M_h} d\ell'_+ Z_J\left(\ell_-,\ell'_-\right) Z_J\left(\ell_+,\ell'_+\right) Z_S\left(\ell_+\ell_-,w,'\right)\\
&\qquad\qquad\qquad\qquad\qquad \times J^{(0)}\left(M_h\ell'_-\right) J\left(-M_h\ell'_+\right) S^{(0)}(w')\\
&\quad = Z_{33}^{-1} H_3^{(0)}\int_0^\infty dw'\int_0^\infty d\ell_-\int_0^\infty d\ell_+\int_0^{M_h} \frac{d\ell'_-}{\ell'_-}\int_0^{\sigma M_h} \frac{d\ell'_+}{\ell'_+} Z_J\left(\ell'_-,\ell_-\right) Z_J\left(\ell'_+,\ell_+\right)\\
&\qquad\qquad\qquad \times Z_S\left(\ell_-\ell_+,w'\right) J^{(0)}\left(M_h\ell'_-\right) J\left(-M_h\ell'_+\right) S^{(0)}(w')\\
&\quad = H_3^{(0)}\int_0^{M_h} \frac{d\ell'_-}{\ell'_-}\int_0^{\sigma M_h} \frac{d\ell'_+}{\ell'_+} J^{(0)}\left(M_h\ell'_-\right) J^{(0)}\left(-M_h\ell'_+\right) S^{(0)}(\ell'_+\ell'_-)\,,
\end{aligned} \tag{D.2}$$

where we have used the identity

$$\int_0^\infty \mathrm{d}\ell_-\int_0^\infty \mathrm{d}\ell_+ Z_S\left(w,\ell_-\ell_+\right) Z_J\left(\ell_-,\ell'_-\right) Z_J\left(\ell_+,\ell'_+\right) = Z_{33}\delta\left(w-\ell'_-\ell'_+\right)\,, \tag{D.3}$$

together with the property of the soft function renormalization factor that $Z_S(w,w') = w/w' Z_s(w^{'},w)$. The remaining terms in (D.1) can be interpreted as a mismatch between the amplitude T_3 written in terms of bare quantities and the expression written in terms of renormalized quantities. We call this mismatch contribution δT_3

$$\begin{aligned}
\delta T_3 &= g_\perp^{\mu\nu} H_3(\mu)\int_0^{M_h} \frac{\mathrm{d}\ell_-}{\ell_-}\int_0^{\sigma M_H} \frac{\mathrm{d}\ell_+}{\ell_+} J\left(M_h\ell_-,\mu\right) J\left(-M_h\ell_+,\mu\right) S\left(\ell_+\ell_-,\mu\right)\\
&= g_\perp^{\mu\nu} H_3(\mu)\int_0^\infty dw'\Bigg[\int_0^{M_h} \frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h} \frac{d\ell_+}{\ell_+}\int_0^\infty d\ell'_-\int_0^\infty d\ell'_+\\
&\qquad\qquad\qquad\qquad - \int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+}\int_0^{M_h} d\ell'_-\int_0^{\sigma M_h} d\ell'_+\Bigg]\\
&\qquad \times Z_J\left(\ell_-,\ell'_-\right) Z_J\left(\ell_+,\ell'_+\right) Z_S\left(\ell_-\ell_+,w'\right) J^{(0)}\left(M_h\ell'_-\right) J^{(0)}\left(-M_h\ell'_+\right) S^{(0)}(w')\,.
\end{aligned} \tag{D.4}$$

D.2 T_2 extra contributions

We can write the amplitude T_2 in terms of renormalized functions such that

$$\begin{aligned}
T_2^{\text{ren}} &= 2\int_0^1 dz\left[H_2(z,\mu)\langle O_2(z,\mu)\rangle - \frac{[\![\bar{H}_2(z,\mu)]\!]}{z}[\![\langle O_2(z,\mu)\rangle]\!] - \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{\bar{z}}[\![\langle O_2(\bar{z},\mu)\rangle]\!]\right] \\
&= 2\int_0^1 dz\, H_2\left(z,\mu\right)\left(\int_0^1 dz'\, Z_{22}(z,z')\,\langle O_2^{(0)}(z')\rangle \;+\; Z_{21}(z)\,\langle O_1^{(0)}\rangle\right) \\
&- 2\int_0^1 dz\,\frac{[\![\,\bar{H}_2(z,\mu)\,]\!]}{z}\left(\int_0^\infty dz'\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\langle O_2^{(0)}\,(z')\rangle]\!] + [\![\,Z_{21}(z)\,]\!]\langle O_1^{(0)}\rangle\right) \\
&- 2\int_0^1 dz\,\frac{[\![\,\bar{H}_2(\bar{z},\mu)\,]\!]}{\bar{z}}\left(\int_0^\infty dz'\,[\![\,Z_{22}(\bar{z},\bar{z}')\,]\!]\,[\![\langle O_2^{(0)}\,(\bar{z}')\rangle]\!] + [\![\,Z_{21}(\bar{z})\,]\!]\langle O_1^{(0)}\rangle\right)\,.
\end{aligned} \tag{D.5}$$

From the renormalization condition for the hard coefficient $H_2(z,\mu)$ in (4.41) it is straightforward to see that the first term in the second line in the above expression is part of the bare amplitude T_2, such that

$$2\int_0^1 dz\, H_2(z,\mu)\int_0^1 dz' Z_{22}(z,z')\langle O_2^{(0)}(z')\rangle = 2\int_0^1 dz\, H_2^{(0)}(z)\langle O_2^{(0)}(z)\rangle\,, \tag{D.6}$$

where we have renamed the variable from z' to z in the last step. The terms proportional to the operator matrix elements $\langle O_1^{(0)}\rangle$ are mixing contribution.

For the remaining terms in (D.5) proportional to the operator matrix element $[\![\langle O_2^{(0)}\,(z')\rangle]\!]$ we have

$$\begin{aligned}
&-2\int_0^1 dz\int_0^\infty dz'\left[\frac{[\![\,\bar{H}_2(z,\mu)\,]\!]}{z}\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\langle O_2^{(0)}\,(z')\rangle]\!] + \frac{[\![\,\bar{H}_2(\bar{z},\mu)\,]\!]}{\bar{z}}\,[\![\,Z_{22}(\bar{z},\bar{z}')\,]\!]\,[\![\langle O_2^{(0)}\,(\bar{z}')\rangle]\!]\right] \\
&= -\,4\int_0^1 dz\int_0^\infty dz'\frac{[\![\,\bar{H}_2(z,\mu)\,]\!]}{z}\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\langle O_2^{(0)}\,(z')\rangle]\!]\,,
\end{aligned} \tag{D.7}$$

since both integrands are symmetric for $z \longleftrightarrow 1-z$. Then we can write

$$\begin{aligned}
&-4\left(\int_0^1 \frac{dz}{z}\int_0^\infty dz' \;-\; \int_0^1 dz'\int_0^\infty \frac{dz}{z} \;+\; \int_0^1 dz'\int_0^\infty \frac{dz}{z}\right)[\![\,\bar{H}_2(z,\mu)\,]\!]\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\,\langle O_2^{(0)}\,(z')\rangle\,]\!] \\
&= -\,4\left(\int_0^1 \frac{dz}{z}\int_0^\infty dz' \;-\; \int_0^1 dz'\int_0^\infty \frac{dz}{z}\right)[\![\,\bar{H}_2(z,\mu)\,]\!]\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\,\langle O_2^{(0)}\,(z')\rangle\,]\!] \\
&\quad -\,4\int_0^1 \frac{dz'}{z'}\,[\![\,\bar{H}_2^{(0)}(z')\,]\!][\![\,\langle O_2^{(0)}\,(z')\rangle\,]\!] \\
&\equiv \delta T_2 - 4\int_0^1 \frac{dz'}{z'}\,[\![\,\bar{H}_2^{(0)}(z')\,]\!][\![\,\langle O_2^{(0)}\,(z')\rangle\,]\!]\,.
\end{aligned} \tag{D.8}$$

The second term here is part of the bare amplitude T_2, from the identity (3.12) and we have defined a mismatch term δT_2 similarly to the case of the T_3 amplitude such that

$$\delta T_2 = -4\left(\int_0^1 \frac{dz}{z}\int_0^\infty dz' - \int_0^1 dz'\int_0^\infty \frac{dz}{z}\right)[\![\,\bar{H}_2(z,\mu)\,]\!]\,[\![\,Z_{22}(z,z')\,]\!]\,[\![\,\langle O_2^{(0)}(z')\rangle\,]\!]\,. \quad \text{(D.9)}$$

Using the refactorization conditions in (3.8) together with (6.11) this result can be further written as

$$\begin{aligned}
\delta T_2 =& -2H_3(\mu)\Bigg(\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^\infty d\ell_+\int_0^\infty d\ell'_-\int_0^\infty d\ell'_+\int_0^\infty d\ell''_- \\
&\qquad - \int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^\infty d\ell_+\int_0^{M_h} d\ell'_-\int_0^\infty d\ell'_+\int_0^\infty d\ell''_-\Bigg) \\
&\quad\times Z_J(\ell_-,\ell''_-)Z_J(\ell'_+,\ell_+)Z_S(\ell_+\ell_-,\ell'_+\ell'_-)J^{(0)}(M_h\ell''_-)J^{(0)}(-M_h\ell'_+)S^{(0)}(\ell'_+\ell'_-) \\
=& -2H_3(\mu)\Bigg(\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^\infty d\ell'_-\int_0^\infty d\ell'_+\int_0^\infty dw' \\
&\qquad - \int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^{M_h} d\ell'_-\int_0^\infty d\ell'_+\int_0^\infty dw'\Bigg) \\
&\quad\times Z_J(\ell_-,\ell'_-)Z_J(\ell_+,\ell'_+)Z_S(\ell_+\ell_-w')J^{(0)}(M_h\ell'_-)J^{(0)}(-M_h\ell'_+)S^{(0)}(w')\,.
\end{aligned} \quad \text{(D.10)}$$

In an intermediate step we have used the identity

$$Z_{33}^{-1}\int_0^\infty \mathrm{d}\ell_-\int_0^\infty \mathrm{d}\ell_+ Z_J\left(\ell'_-,\ell_-\right)Z_J\left(\ell'_+,\ell_+\right)Z_S\left(\ell_-\ell_+,w'\right) = \delta\left(w'-\ell'_-\ell'_+\right)\,. \quad \text{(D.11)}$$

Notice that the refactorization for $[\![\bar{H}_2(z)]\!]$ holds for both the bare and factorized expression for this Wilson coefficient. In order to combine this result with δT_3 we can rewrite it as

$$\begin{aligned}
\delta T_2 =& \Bigg[-H_3(\mu)\Bigg(\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^\infty d\ell'_-\int_0^\infty d\ell'_+ - \int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^{M_h} d\ell'_-\int_0^\infty d\ell'_+\Bigg) \\
&- H_3(\mu)\Bigg(\int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+}\int_0^\infty d\ell'_-\int_0^\infty d\ell'_+ - \int_0^\infty\frac{d\ell_-}{\ell_-}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^\infty \mathrm{d}\ell'_-\int_0^{\sigma M_h} d\ell'_+\Bigg)\Bigg] \\
&\times\int_0^\infty dw' S(w')Z_J(\ell_-,\ell'_-)Z_J(\ell_+,\ell'_+)Z_S(\ell_+\ell_-,w')J^{(0)}(M_h\ell'_-)J^{(0)}(-M_h\ell'_+)S^{(0)}(w')\,.
\end{aligned}$$

(D.12)

Appendix E

RG evolution of operator matrix elements $[\![\langle O_2(z,\mu)\rangle]\!]$ in diagonal space

We can transform the operator $[\![O_2(z,\mu)]\!]$ in the diagonal space with the same transformation given in (5.47) where at leading order the transfer function is the Bessel function $\mathrm{J}_1(2\sqrt{x})$ such that

$$\begin{aligned}
[\![\tilde{O}_2(z,\mu)]\!] &= \int_0^\infty \frac{dx}{\sqrt{x}} \mathrm{J}_1(2\sqrt{x})[\![O_2(xz,\mu)]\!], \\
[\![O_2(z,\mu)]\!] &= \int_0^\infty \frac{dx}{\sqrt{x}} \mathrm{J}_1(2\sqrt{x})[\![\tilde{O}_2(z/x,\mu)]\!].
\end{aligned} \tag{E.1}$$

Then using the initial RG equation for $[\![\langle O_2(z,\mu)\rangle]\!]$ shown in (4.49) we find that

$$\frac{d\,[\![\langle \tilde{O}_2(z)\rangle]\!]}{d\ln\mu} = -[\![\tilde{\gamma}_{22}\,(z)]\!][\![\tilde{O}_2(z,\mu)]\!] - \frac{N_c\alpha_b}{\pi}[\![\tilde{\gamma}_{21}\,(z)]\!]\langle O_1(\mu)\rangle\,. \tag{E.2}$$

The anomalous dimensions in the diagonal space now become

$$\begin{aligned}
[\![\tilde{\gamma}_{22}(z,\mu)]\!] &= -\frac{C_F\alpha_s}{4\pi}(4\ln\tilde{z}+6) + \mathcal{O}\left(\alpha_s^2\right)\,, \\
[\![\tilde{\gamma}_{21}(z,\mu)]\!] &= -\frac{C_F\alpha_s}{\pi}\left[1+\frac{C_F\alpha_s}{4\pi}\left(\ln^2\tilde{z}+11-\frac{2\pi^2}{3}\right)+\mathcal{O}\left(\alpha_s^2\right)\right]\,,
\end{aligned} \tag{E.3}$$

where $\tilde{z} \equiv ze^{-2\gamma_E}$. These anomalous dimensions are diagonalized and the evolution in the diagonal space is local. An exact solution to the RGE can be found in complete analogy to (7.6), where we obtain

$$[\![\tilde{O}_2(z,\mu)]\!] = e^{a_{[\![\tilde{\gamma}_{22}]\!]}(\mu_s,\mu)}\Big[[\![\tilde{O}_2(z,\mu_s)]\!] + \frac{N_c\alpha_b}{\pi}\langle O_1(\mu_s)\rangle a_{[\![\tilde{\gamma}_{21}]\!]}(\mu_s,\mu)\, e^{a_{\gamma_{11}}(\mu_s,\mu_\alpha)-a_{[\![\tilde{\gamma}_{22}]\!]}(\mu_s,\mu_\alpha)}\Big], \tag{E.4}$$

where $a_{[\![\gamma]\!]}$ is defined as in (7.11) with anomalous dimension $[\![\gamma]\!]$. Similarly to before for $\mu_s = m_b$ the initial condition in (E.4) vanishes. For the explicit lowest order explicit expression we need the one-loop expressions for the anomalous dimensions in (E.3). We then find

$$[\![\tilde{O}_2(z,\mu)]\!] = \frac{N_c\alpha_b}{\alpha_s(\mu_s)}\frac{2}{\beta_0-2C_F\left(\partial_\eta+3\right)}\left[r^{\frac{3C_F}{\beta_0}-1}\tilde{z}^\eta - r^{-\frac{3C_F}{\beta_0}}\tilde{z}^{\eta-a_\Gamma}\right]_{\eta=0}\langle O_1(\mu_s)\rangle. \tag{E.5}$$

We can use now the transformation in (5.50) to find the momentum space solution. We obtain the following result for the leading order contribution

$$
\begin{aligned}
[\![\langle O_2(z,\mu)\rangle]\!] &= \int_0^\infty \frac{dx}{\sqrt{x}} \mathrm{J}_1(2\sqrt{x})[\![\langle \tilde{O}_2(z/x,\mu)\rangle]\!] \\
&= \frac{N_c \alpha_b}{\alpha_s(\mu_s)} \frac{2}{\beta_0 - 2C_F\,(\partial_\eta + 3)} \frac{\Gamma(1-\eta)}{\Gamma(1+\eta)} \left[r^{\frac{3C_F}{\beta_0}-1} \tilde{z}^{\eta} - r^{-\frac{3C_F}{\beta_0}} \tilde{z}^{\eta - a_\Gamma} \right]_{\eta=0} \langle O_1(\mu_s)\rangle .
\end{aligned}
\tag{E.6}
$$

For the total subtraction term we use the solution (E.6) to derive the convolution with the hard coefficient $[\![\bar{H}_2(z,\mu)]\!]$. Note that the subtraction term in (6.1) has also a symmetric contribution for $\bar{z} = 1 - z$. For the divergent part we find

$$
\begin{aligned}
&-\int_0^1 dz \left[\frac{[\![\bar{H}_2(z,\mu)]\!]}{z} [\![\langle O_2(z,\mu)\rangle]\!] + \frac{[\![\bar{H}_2(\bar{z},\mu)]\!]}{1-z} [\![\langle O_2(\bar{z},\mu)\rangle]\!] \right] \\
&= -\frac{4\pi}{\alpha_s(\mu_s)} \frac{1}{\beta_0 - 2C_F\,(\partial_\eta + 3)} \frac{\Gamma(1-\eta)}{\Gamma(1+\eta)} r^{\frac{3C_F}{\beta_0}-1} e^{-2\gamma_E} \int_0^1 dz\, \left(z^{\eta-1} + \bar{z}^{\eta-1} \right) \Big|_{\eta=0} \langle O_1(\mu_s)\rangle + \ldots
\end{aligned}
\tag{E.7}
$$

where the dots are finite terms.

Appendix F

Anomalous dimensions for three jet operators at $\mathcal{O}(\lambda^3)$

For completeness we include here the anomalous dimensions of the three jet operators at $\mathcal{O}(\lambda^3)$ though for practical purposes they contribute only suppressed effects. Below we write the anomalous dimensions for the operator basis in (9.19) that describe three jet final states for the leptoquark S_1

$$
\begin{aligned}
\boldsymbol{\Gamma}_{S_1^\star L_L \Phi d_R} &= -\frac{4}{3}\gamma_{\text{cusp}}^{(3)} \ln\frac{\mu}{2v\cdot p_d} - \frac{1}{9}\gamma_{\text{cusp}}^{(1)} \ln\frac{\mu}{2v\cdot p_d} + \frac{1}{6}\gamma_{\text{cusp}}^{(1)} \ln\frac{\mu}{2v\cdot p_L} \\
&\quad - \frac{1}{6}\gamma_{\text{cusp}}^{(1)} \ln\frac{\mu}{2v\cdot p_H} - \frac{3}{4}\gamma_{\text{cusp}}^{(2)} \left(\ln\frac{\mu^2}{m_{L\Phi}^2} + i\pi\right) - \frac{1}{6}\gamma_{\text{cusp}}^{(1)} \left(\ln\frac{\mu^2}{m_{dL}^2} + i\pi\right) \\
&\quad + \frac{1}{6}\gamma_{\text{cusp}}^{(1)} \left(\ln\frac{\mu^2}{m_{d\Phi}^2} + i\pi\right) + \gamma^{S_1} + \boldsymbol{\gamma}^{L_L} + \boldsymbol{\gamma}^{d_R} + \gamma^{\Phi} \\
\boldsymbol{\Gamma}_{S_1 Q_L \Phi \nu_R} &= -\frac{4}{3}\gamma_{\text{cusp}}^{(3)} \ln\frac{\mu}{2v\cdot p_Q} + \frac{1}{18}\gamma_{\text{cusp}}^{(1)} \ln\frac{\mu}{2v\cdot p_Q} - \frac{1}{6}\gamma_{\text{cusp}}^{(1)} \ln\frac{\mu}{2v\cdot p_\Phi} \\
&\quad - \frac{1}{12}\gamma_{\text{cusp}}^{(1)} \left(\ln\frac{\mu^2}{m_{QL}^2} + i\pi\right) - \frac{3}{4}\gamma_{\text{cusp}}^{(2)} \left(\ln\frac{\mu^2}{m_{Q\Phi}^2} + i\pi\right) + \gamma^{S_1} \\
&\quad + \boldsymbol{\gamma}^{Q_L} + \gamma^{\Phi} \\
\boldsymbol{\Gamma}_{S_1 d_R B \nu_R} &= -\frac{4}{3}\gamma_{\text{cusp}}^{(3)} \ln\frac{\mu}{2v\cdot p_d} - \frac{1}{9}\gamma_{\text{cusp}}^{(1)} \ln\frac{\mu}{2v\cdot p_d} + \gamma^{S_1} + \boldsymbol{\gamma}^{d_R} + \gamma^{B}
\end{aligned}
\tag{F.1}
$$

The quantity $m_{k\ell}^2$ is the invariant mass for the particle pair (k,ℓ) and p_i is the four momentum of particle i. The three jet anomalous dimensions for the scalar S_3 as listed in (10.12) are

$$\begin{aligned}
\boldsymbol{\Gamma}_{S_3 Q_L \Phi \nu_R} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{1}{2}\gamma^{(2)}_{\text{cusp}} + \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_Q} \\
&+ \left(-\frac{1}{2}\gamma^{(2)}_{\text{cusp}} - \frac{1}{6}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_\Phi} \\
&+ \left(\frac{1}{4}\gamma^{(2)}_{\text{cusp}} - \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{Q\Phi}} + i\pi\right) + \gamma^{S_3} + \boldsymbol{\gamma}^{Q_L} + \gamma^{\Phi} \\
\boldsymbol{\Gamma}_{S_3 d_R \Phi L_L} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_d} + \left(-\frac{1}{2}\gamma^{(2)}_{cusp} + \frac{1}{6}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_\Phi} \\
&+ \left(\frac{1}{4}\gamma^{(2)}_{\text{cusp}} + \frac{1}{4}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{\Phi L}} + i\pi\right) + \gamma^{S_3} + \boldsymbol{\gamma}^{d_R} + \gamma^{\Phi} \\
\boldsymbol{\Gamma}_{S_3 d_R W \nu_R} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_d} - 2\gamma^{(2)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_W} \\
&+ \gamma^{S_3} + \boldsymbol{\gamma}^{d_R} + \gamma^{W}
\end{aligned} \tag{F.2}$$

Leading order three jet operators for the leptoquark U_1^μ are presented in equation (11.13) and the corresponding anomalous dimensions are the following

$$\begin{aligned}
\mathbf{\Gamma}_{U_1 Q_L \Phi \ell_R} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right) \ln\frac{\mu}{2v\cdot p_Q} - \frac{2}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_\ell} \\
&+ \frac{1}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_\Phi} + \left(-\frac{3}{4}\gamma^{(2)}_{\text{cusp}} - \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{Q\Phi}} + i\pi\right) \\
&+ \frac{1}{6}\gamma^{(1)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{Q\ell}} + i\pi\right) + \gamma^{U_1} + \boldsymbol{\gamma}^{Q_L} + \gamma^{\Phi} + \boldsymbol{\gamma}^{\ell_R} \\
\mathbf{\Gamma}_{U_1 u_R \Phi L_L} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} + \frac{4}{9}\gamma^{(1)}_{\text{cusp}}\right) \ln\frac{\mu}{2v\cdot p_u} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_L} \\
&+ \frac{1}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_\Phi} + \left(-\frac{3}{4}\gamma^{(2)}_{\text{cusp}} - \frac{1}{4}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{L\Phi}} + i\pi\right) \\
&+ \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{uL}} + i\pi\right) + \gamma^{U_1} + \boldsymbol{\gamma}^{u_R} + \gamma^{\Phi} + \boldsymbol{\gamma}^{L_L} \\
\mathbf{\Gamma}_{U_1 Q_L \Phi \nu_R} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right) \ln\frac{\mu}{2v\cdot p_Q} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_\Phi} \\
&+ \left(-\frac{3}{4}\gamma^{(2)}_{\text{cusp}} + \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{Q\Phi}} + i\pi\right) \\
&+ \gamma^{U_1} + \boldsymbol{\gamma}^{Q_L} + \gamma^{\Phi} \\
\mathbf{\Gamma}_{U_1 Q_L A L_L}\Big|_{A=G} &= \left(\frac{1}{6}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right) \ln\frac{\mu}{2v\cdot p_Q} - \frac{3}{2}\gamma^{(3)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_A} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_L} \\
&+ \left(-\frac{3}{4}\gamma^{(2)}_{\text{cusp}} + \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{QL}} + i\pi\right) - \frac{3}{2}\gamma^{(3)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{QA}} + i\pi\right) \\
&+ \gamma^{U_1} + \boldsymbol{\gamma}^{Q_L} + \gamma^{L_L} + \boldsymbol{\gamma}^{A} \\
\mathbf{\Gamma}_{U_1 Q_L A L_L}\Big|_{A=W} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right) \ln\frac{\mu}{2v\cdot p_Q} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}} \ln\frac{\mu}{2v\cdot p_L} \\
&+ \left(\frac{1}{4}\gamma^{(2)}_{\text{cusp}} + \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{QL}} + i\pi\right) - \gamma^{(2)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{QA}} + i\pi\right) \\
&- \gamma^{(2)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{LA}} + i\pi\right) + \gamma^{U_1} + \boldsymbol{\gamma}^{Q_L} + \gamma^{L_L} + \boldsymbol{\gamma}^{A}
\end{aligned} \tag{F.3}$$

$$
\begin{aligned}
\mathbf{\Gamma}_{U_1 Q_L A L_L}\bigg|_{A=B} &= \left(\frac{1}{6}\gamma^{(3)}_{\text{cusp}} - \frac{1}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_Q} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_L} \\
&+ \left(-\frac{3}{4}\gamma^{(2)}_{\text{cusp}} + \frac{1}{12}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{QL}} + i\pi\right) \\
&+ \gamma^{U_1} + \boldsymbol{\gamma}^{Q_L} + \gamma^{L_L} + \boldsymbol{\gamma}^{A} \\
\mathbf{\Gamma}_{U_1 u_R A \nu_R}\bigg|_{A=G} &= \left(\frac{1}{6}\gamma^{(3)}_{\text{cusp}} - \frac{4}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_u} - \frac{3}{2}\gamma^{(3)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_A} \\
&- \frac{3}{2}\gamma^{(3)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{uA}} + i\pi\right) + \gamma^{U_1} + \boldsymbol{\gamma}^{u_R} + \boldsymbol{\gamma}^{A} \\
\mathbf{\Gamma}_{U_1 u_R A \nu_R}\bigg|_{A=B} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{4}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_u} + \gamma^{U_1} + \boldsymbol{\gamma}^{u_R} + \boldsymbol{\gamma}^{A} \\
\mathbf{\Gamma}_{U_1 d_R A \ell_R}\bigg|_{A=G} &= \left(\frac{1}{6}\gamma^{(3)}_{\text{cusp}} + \frac{2}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_d} - \frac{2}{3}\gamma^{(1)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_\ell} \\
&- \frac{3}{2}\gamma^{(3)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_A} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{d\ell}} + i\pi\right) \\
&- \frac{3}{2}\gamma^{(3)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{dA}} + i\pi\right) + \gamma^{U_1} + \boldsymbol{\gamma}^{d_R} + \boldsymbol{\gamma}^{A} + \boldsymbol{\gamma}^{\ell_R} \\
\mathbf{\Gamma}_{U_1 d_R A \ell_R}\bigg|_{A=B} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} + \frac{2}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_d} - \frac{2}{3}\gamma^{(1)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_\ell} \\
&- \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{d\ell}} + i\pi\right) - \frac{3}{2}\gamma^{(3)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{dA}} + i\pi\right) \\
&+ \gamma^{U_1} + \boldsymbol{\gamma}^{d_R} + \boldsymbol{\gamma}^{A} + \boldsymbol{\gamma}^{\ell_R} \\
\mathbf{\Gamma}_{U_1 u_R \Phi L_L^c} &= \left(-\frac{4}{3}\gamma^{(3)}_{\text{cusp}} - \frac{4}{9}\gamma^{(1)}_{\text{cusp}}\right)\ln\frac{\mu}{2v\cdot p_u} + \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_L} \\
&- \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\ln\frac{\mu}{2v\cdot p_\Phi} - \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{uL}} + i\pi\right) \\
&+ \frac{1}{3}\gamma^{(1)}_{\text{cusp}}\left(\ln\frac{\mu^2}{m^2_{u\Phi}} + i\pi\right) + \left(-\frac{3}{4}\gamma^{(2)}_{\text{cusp}} - \frac{1}{4}\gamma^{(1)}_{\text{cusp}}\right)\left(\ln\frac{\mu^2}{m^2_{L\Phi}} + i\pi\right) \\
&+ \gamma^{U_1} + \boldsymbol{\gamma}^{u_R} + \boldsymbol{\gamma}^{\Phi} + \boldsymbol{\gamma}^{L_L}
\end{aligned}
\tag{F.4}
$$

Notice that for all the above anomalous dimensions, in the RG equation the ordering of multiplication matters. The same reasoning described in details in Chapter 12 applies in here as well.

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List of Figures

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