The influence of correlations on random energy models

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Abstract

This thesis explores how correlations influence the extremes of *variable-speed branching Brownian motion (VSBBM)* and the limiting free energy of the *continuous random energy model (CREM)*. These models are closely related and serve as toy models for mean field spin glass models such as the *Sherrington-Kirkpatrick (SK) model*.

In Chapter 1, we give a detailed introduction to VSBBM and the CREM. Also, we summarise the results of Chapter 2 and 3 and place them in the broader research context.

In Chapter 2, we study the extremes of a VSBBM where the time-dependent *speed functions*, which describe the time-inhomogeneous variance, converge to the identity function from below. We show that the log-correction for the order of the maximum depends only on the rate of convergence of the speed function near 0 and 1 and exhibits a smooth interpolation between the correction in the i.i.d. case, $\frac{1}{2\sqrt{2}}\ln t$, and that of standard branching Brownian motion (BBM), $\frac{3}{2\sqrt{2}}\ln t$. We prove that the limiting law of the maximum and the extremal process essentially coincide with those of standard BBM, using a first and second moment method which relies on the localisation of extremal particles.

In Chapter 3, we study the free energy of the CREM with the so-called *Hamilton-Jacobi approach*. This approach compares the limiting free energy of mean field spin glass models to the so-called *viscosity solution* of a *Hamilton-Jacobi equation* (HJE). For some of these models, this viscosity solution exactly matches the limiting free energy. In other cases such as the bipartite SK model, particularly when the so-called *nonlinearity* of the HJE is nonconvex, it is only proven that the viscosity solution provides a bound to the limiting free energy.

In the case of the CREM with a convex speed function, we establish a HJE on an infinite-dimensional space and prove that a unique viscosity solution exists, which is characterised by a variational formula. Also, we give an explicit description of the so-called *initial condition* of that HJE, which allows to simplify the variational formula of the viscosity solution. The limiting free energy of the CREM with a convex speed function is equal to the aforementioned viscosity solution.

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1 Introduction

In this thesis, we investigate *variable speed branching Brownian motion* (VSBBM) and the *continuous random energy model* (CREM). These models from statistical mechanics serve as toy models for *spin glasses*.

This introduction is structured as follows: In Section 1.1, we introduce important terms from statistical mechanics and the fundamentals of mathematical models of magnetism. Furthermore, we give an overview of the phenomenon of spin glass magnetism. Section 1.2 deals with Derrida's random energy models including the CREM and their extreme values. We introduce VSBBM in Section 1.3, explain its relation to the CREM and present important results regarding the extreme values of VSBBM. These results form the basis of the original contributions of this thesis regarding VSBBM in Chapter 2. Section 1.3.5 contains an overview of these original contributions. In Section 1.4, we summarise the central results of Chapter 3 and explain the Hamilton-Jacobi approach. With this approach, we study the free energy of the CREM in Chapter 3.

1.1 Introduction to spin glass magnetism and statistical mechanics

For a ferromagnetic element such as iron, nickel or cobalt, the magnetic field is stronger, the more spins of its atoms are aligned. The phenomenon of *spin glass magnetism* emerges e.g. in alloys of copper with manganese, iron or cobalt. Here, the spins of the atoms show no pattern in their alignment. This difference of spin glass magnets and ferromagnets is illustrated in Figure 1.1. A similar difference occurs in the crystalline structure of glass and quartz crystals (see Figure 1.2). The term "spin glass" is derived from this observation.

Before we take a deeper dive into certain (toy) models of spin glass magnetism, we give a general introduction to the terminology of statistical mechanics, following the structure

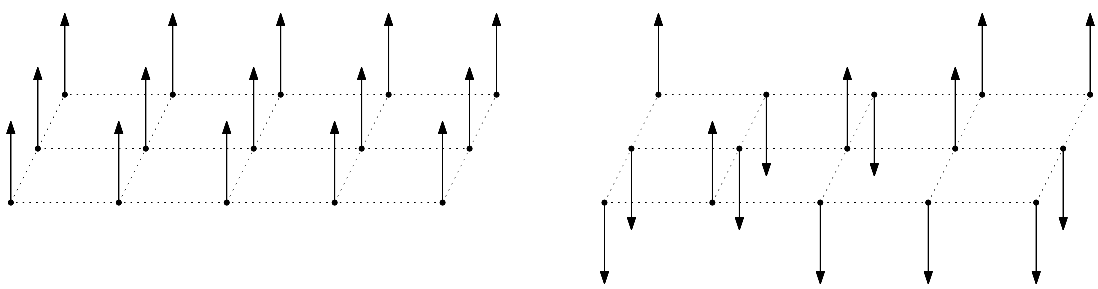

Figure 1.1: In ferromagnets, ordered spin configurations (left) are energetically favoured. In spin glasses, disordered spin configurations (right) are energetically favoured.

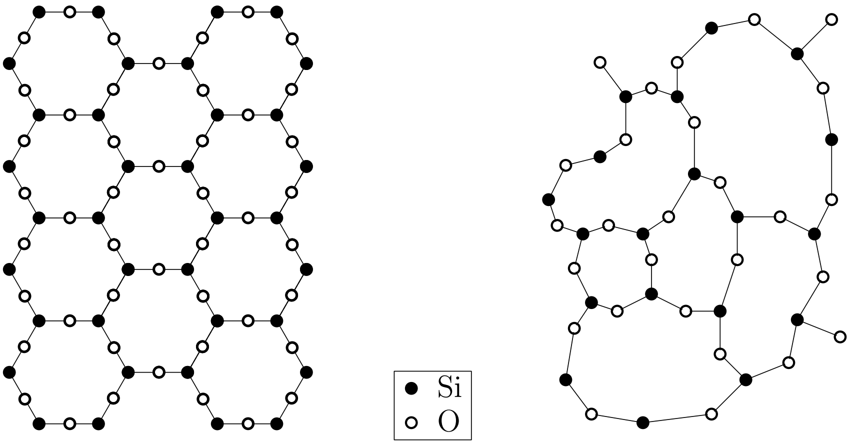

Figure 1.2: Quartz has an ordered crystalline structure while that of glass is disordered.

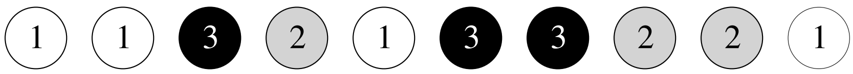

Figure 1.3: An example of a microscopic state configuration $x \in \{1,2,3\}^{10}$. Here, the macroscopic state configuration is $N_1(x) = 4, N_2(x) = N_3(x) = 3$.

of [58, Section 1.1]. Consider a system of $N \in \mathbb{N}$ *particles*. Each particle is assigned a *state* $k \in \{1,\dots,K\}$, where $K \in \mathbb{N}$ can be dependent on N. One often studies the model for large N. We call $x \in \{1,\dots,K\}^N$ a *microscopic state configuration*. Each state $k \in K$ is assigned an *energy level* $e_k \geq 0$. Let $N_k(x) := \sum_{n=1}^{N} \mathbb{1}_{x_n=k}$ denote the number of particles of a microscopic state configuration x in state k. The *total energy* of x is $\sum_{k=1}^{K} N_k(x)e_k$. For $\bar{e} \geq 0$ with $\min_{1\leq k\leq K} e_k \leq \bar{e} \leq \max_{1\leq k\leq K} e_k$, we denote by

$$M_{\bar{e}} := \left\{x \in \{1,\dots,K\}^N \colon\ \sum_{k=1}^{K} N_k(x)e_k = N\bar{e}\right\} \tag{1.1.1}$$

the space of all microscopic state configurations with total energy $N\bar{e}$. From now on, we sample microscopic state configurations uniformly from $M_{\bar{e}}$. In this context, we call $\bar{e}$ the *average energy*.

Zooming out to a more macroscopic level, we investigate $(N_k(x))_{k\in K}$, which we call a *macroscopic state configuration*. In particular, the following question is of interest:

Which macroscopic state configuration is the most likely?

Since the microscopic state configurations are sampled uniformly from $M_{\bar{e}}$, we can answer the previous question by finding weights $p = (p_k)_{k=1}^{K} \in [0,1]^K$ with $\sum_{k=1}^{K} p_k = 1$ and $\sum_{k=1}^{K} p_k e_k = \bar{e}$ such that

$$\#\{x \in M_{\bar{e}} \colon N_k(x) = Np_k\ \forall\, k = 1,\dots,K\} \tag{1.1.2}$$

is maximal. This quantity denotes the number of microscopic state configurations with total energy $\bar{e}$ which assume the macroscopic state configuration $(Np_1, \ldots, Np_K)$. Here, we neglect whether $Np_k, k = 1, \ldots, K$ is an integer or not. We have

$$\begin{aligned}\#\{x \in M_{\bar{e}} \colon N_k(x) = Np_k \; \forall\, k = 1, \ldots, K\} &= \#\left\{x \in \{1, \ldots, K\}^N : N_k(x) = Np_k \; \forall\, k = 1, \ldots, K\right\} \\ &= \binom{N}{Np_1}\binom{N - Np_1}{Np_2} \cdots \binom{N - N\sum_{k=1}^{K-1} p_k}{Np_K} \\ &= \frac{N!}{(Np_1)!(Np_2)! \cdots (Np_K)!}. \qquad (1.1.3)\end{aligned}$$

Then, Stirling's formula, $N! = \exp(N \ln N - N + O(\ln N))$, implies that

$$\#\{x \in M_{\bar{e}} \colon N_k(x) = Np_k \; \forall\, k = 1, \ldots, K\} = \exp\left(-N \sum_{k=1}^{K} p_k \ln p_k + O(\ln N)\right). \qquad (1.1.4)$$

We call

$$S(p) := -\sum_{k=1}^{K} p_k \ln p_k \qquad (1.1.5)$$

the *entropy* of p. If $S(p) > S(p')$, there will be exponentially more macroscopic state configurations corresponding to the weights p than to the weights p'. Thus, we will most likely observe the macroscopic state configurations

$$(N_1(x), \ldots N_K(x)) \approx \left(p_1^* N, \ldots, p_K^* N\right), \qquad (1.1.6)$$

where p^* is a solution of

$$\begin{aligned}\max \; S(p) \; \text{ w.r.t. } p \in [0,1]^K&, \\ \sum_{k=1}^{K} p_k = 1&, \\ \sum_{k=1}^{K} p_k e_k = \bar{e}&. \qquad (1.1.7)\end{aligned}$$

To solve (1.1.7) for p^*, we use the method of Lagrange multipliers with Lagrange function Λ given by

$$\Lambda(p, \alpha, \beta) := S(p) - \alpha\left(\sum_{k=1}^{K} p_k - 1\right) - \beta\left(\sum_{k=1}^{K} p_k e_k - \bar{e}\right), \qquad p \in [0,1]^K, \alpha, \beta \in \mathbb{R}. \qquad (1.1.8)$$

The first-order conditions are

$$
\begin{aligned}
0 &\overset{!}{=} \tfrac{\partial}{\partial_{p_k}}\Lambda(p,\alpha,\beta) = -1 - \ln p_k - \alpha - \beta e_k, \qquad k = 1,\dots,K,\\
0 &\overset{!}{=} \tfrac{\partial}{\partial_{\alpha}}\Lambda(p,\alpha,\beta) = \sum_{k=1}^{K} p_k - 1,\\
0 &\overset{!}{=} \sum_{k=1}^{K} p_k e_k - \bar{e}.
\end{aligned} \tag{1.1.9}
$$

From this we obtain the solution

$$
\begin{aligned}
p_k^* &= \tfrac{1}{Z_{\beta,K}} \mathrm{e}^{-\beta e_k}, \qquad k = 1,\dots,K,\\
Z_{\beta,K} &:= \sum_{k=1}^{K} \mathrm{e}^{-\beta e_k},
\end{aligned} \tag{1.1.10}
$$

where we choose β such that $\frac{1}{Z_{\beta,K}} \sum_{k=1}^{K} \mathrm{e}^{-\beta e_k} e_k = \bar{e}$. We call β the *inverse temperature*. The normalisation term $Z_{\beta,K}$ is also referred to as the *partition function* at inverse temperature β. We call p^* the *Gibbs weights* or *Gibbs measure* at inverse temperature β with energy levels $e_1,\dots,e_K$. We now have an answer to the question posed above:

The most likely macroscopic state configuration is the one where the weights p in (1.1.2) *are the Gibbs weights p^* from* (1.1.10).

Furthermore,

$$
S(p^*) = \sum_{k=1}^{K} p_k^* \left(\beta e_k + \ln Z_{\beta,K}\right) = \beta\left(\bar{e} + \tfrac{1}{\beta}\ln Z_{\beta,K}\right). \tag{1.1.11}
$$

This means that the entropy can be written as the product of the inverse temperature β with the sum of the average energy $\bar{e}$ and the term $f(\beta) := \frac{1}{\beta}\ln Z_{\beta,K}$. The function f is called *free energy*[1]. We have

$$
f(\beta) \geq \tfrac{1}{\beta}\ln\left(\exp\left(-\beta \min_{k=1,\dots,K} e_k\right)\right) = -\min_{k=1,\dots,K} e_k, \tag{1.1.12}
$$

and

$$
f(\beta) \leq \tfrac{1}{\beta}\ln\left(K\exp\left(-\beta \min_{k=1,\dots,K} e_k\right)\right) = -\min_{k=1,\dots,K} e_k + \frac{\ln K}{\beta}. \tag{1.1.13}
$$

Thus, the free energy gives a good indication of the order of the lowest energy level for large inverse temperatures.

[1] In [58, Section 1.1], the authors argue that the term *free entropy* seems to be more appropriate in the setting of (1.1.11). However, the term *free energy* is widely used.

1.1.1 Models on the Ising spin space

From now on, we consider models of ferromagnets or spin glasses on the space $\{-1,1\}^N$, the *Ising spin space*. This means that we have N atoms with spins pointing up $(+1)$ or down (-1). In these models, the parameter $\beta \geq 0$ describes the inverse temperature. Each possible spin configuration $\sigma \in \{-1,1\}^N$ is assigned an energy level $H_N(\sigma)$ called *Hamiltonian*. The probability to observe the spin configuration $\sigma \in \{-1,1\}^N$ at inverse temperature β is given by the *Gibbs weight*

$$\mu_{\beta,N}(\sigma) := \frac{\mathrm{e}^{-\beta H_N(\sigma)}}{Z_{\beta,N}}, \tag{1.1.14}$$

where

$$Z_{\beta,N} := \sum_{\sigma\in\{-1,1\}^N} \mathrm{e}^{-\beta H_N(\sigma)} \tag{1.1.15}$$

denotes the *partition function*. The (normalised) free energy is denoted by

$$F_N(\beta) := -\tfrac{1}{N} \ln Z_{\beta,N} \tag{1.1.16}$$

and the limiting free energy by

$$F(\beta) := \lim_{N\uparrow\infty} F_N(\beta). \tag{1.1.17}$$

A simple example of a model of a ferromagnet is the *Curie-Weiss* model. Its Hamiltonian

$$H_N^{\mathrm{CW}}(\sigma) := -\frac{1}{N}\sum_{i,j=1}^{N} \sigma_i\sigma_j, \qquad \sigma \in \{-1,1\}^N. \tag{1.1.18}$$

assigns lower energy to configurations of spins where many pairs of spins are aligned. This means that the higher the inverse temperature β, the more concentration on ordered configurations occurs in the Gibbs weights

$$\mu_{\beta,N}^{\mathrm{CW}}(\sigma) = \frac{\mathrm{e}^{-\beta H_N^{\mathrm{CW}}(\sigma)}}{Z_{\beta,N}^{\mathrm{CW}}}, \qquad \sigma \in \{-1,1\}^N. \tag{1.1.19}$$

For $\beta = 0$, the weights of $\mu_{0,N}^{\mathrm{CW}}$ simplify to

$$\mu_{0,N}^{\mathrm{CW}}(\sigma) = \frac{1}{2^N}, \qquad \sigma \in \{-1,1\}^N, \tag{1.1.20}$$

so $\mu_{0,N}^{\mathrm{CW}}$ is the uniform distribution on $\{-1,1\}^N$. If $\beta \uparrow \infty$, $\mu_{\beta,N}^{\mathrm{CW}}$ concentrates on the spin configuration with the least disorder and thus the lowest energy, namely $\sigma = (1,\dots,1)$ and $\sigma = (-1,\dots,-1)$.

Of major interest in models of ferromagnets or spin glasses is whether a *phase transition* occurs. A phase transition is a drastic change in magnetism depending on the temperature, which is also observed both for ferromagnets and spin glasses: Above a threshold temperature called *Curie temperature*, a ferromagnetic material enters a disordered phase and does not exhibit a magnetic field itself. Below the Curie temperature, the material enters the *ferromagnetic phase*, where the alignment of the spins produces a magnetic field. Spin glasses behave similarly above the so-called *transition temperature*, also known as *freezing temperature*. Below this threshold, the material enters the *spin glass phase*: The spin configuration seems frozen in a disordered state. But on a larger time scale, a spin glass exhibits *metastable* behaviour. This means that it alternates to a different spin configuration from time to time, with periods of stability in between. Typically, the freezing temperature of a spin glass is below −200°C and therefore significantly lower than the Curie temperature of a ferromagnet such as iron (770°C) or cobalt (1130°C).

Models of such magnets exhibit a phase transition at the *critical inverse temperature* $\beta_c > 0$, in which the limiting free energy is not differentiable. We refer to Section 1.2.5 for a further discussion of this notion based on the example of the random energy model, which is introduced in the following section.

1.2 Random energy models on the Ising spin space

To take into account the disordered structure of spin configurations in spin glasses, many models of spin glasses use random Hamiltonians. One of the simplest examples of such a model is the *random energy model* (REM). For $N \in \mathbb{N}$, the Hamiltonian $(H_N^{\mathrm{REM}}(\sigma))_{\sigma\in\{-1,1\}^N}$ of the REM is defined as a family of i.i.d. Gaussian random variables with mean 0 and variance N. Derrida [53, 54] introduced the REM as a toy model for the *Sherrington-Kirkpatrick* (SK) model. Giorgio Parisi was awarded the Nobel Prize in Physics 2021 particularly for his contributions towards a description of the free energy of the SK model [102, 103] by the *Parisi formula*. The REM reduces the complex structure of the SK model but nevertheless exhibits significant phenomena of spin glass magnetism such as a phase transition, as we will see in Section 1.2.5.

1.2.1 The extreme values of the REM

We observe that for large β, the Gibbs weights $\mu_{\beta,N}^{\mathrm{REM}}(\sigma) = \frac{1}{Z_{\beta,N}^{\mathrm{REM}}} \mathrm{e}^{-\beta H_N^{\mathrm{REM}}(\sigma)}$ concentrate on the spin configurations $\sigma \in \{-1,1\}^N$ for which $H_N^{\mathrm{REM}}(\sigma)$ is minimal. Thus, to describe how much energy in the REM is assigned to the most favoured spin configuration, we aim to study the asymptotics of the extreme values of $(H_N^{\mathrm{REM}}(\sigma))_{\sigma\in\{-1,1\}^N}$ as $N \uparrow \infty$. To keep notation simple in spin glass models with a Gaussian Hamiltonian, it is more common to determine $\max_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma) \overset{\mathrm{d}}{=} -\min_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma)$. This is justified by the symmetry of the Gaussian distribution.

To compute the asymptotics of $\max_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma)$ as $N \uparrow \infty$, we aim to find $(m_N^{\mathrm{REM}})_{N\in\mathbb{N}}$ so that the function

$$y \mapsto \lim_{N\uparrow\infty} \mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma) \leq m_N^{\mathrm{REM}} + y \right) \tag{1.2.1}$$

exists for $y \in \mathbb{R}$ and is a nontrivial distribution function. This is where the following elementary Gaussian tail asymptotics are a helpful tool, as they are in many other contexts.

Lemma 1.2.1 [See e.g. 63, Chapter VII, Lemma 2]**.** *For $X \sim \mathcal{N}(0,\sigma^2)$, $\sigma > 0$ and $u > 0$,*

$$\mathbb{P}(X > u) = \frac{\sigma}{\sqrt{2\pi}u} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \left(1 + O\left(\frac{\sigma^2}{u^2}\right)\right), \tag{1.2.2}$$

as $\frac{\sigma}{u} \to 0$. Furthermore, dropping the error term gives an upper bound for $\mathbb{P}(X > u)$.

Since the $(H_N^{\mathrm{REM}}(\sigma))_{\sigma\in\{-1,1\}^N}$ are i.i.d.,

$$\mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma) \leq m_N^{\mathrm{REM}} + y \right) = \mathbb{P}\left(H_N^{\mathrm{REM}}(\tilde{\sigma}) \leq m_N^{\mathrm{REM}} + y \right)^{2^N}, \tag{1.2.3}$$

for any $\tilde{\sigma} \in \{-1,1\}^N$. We rewrite

$$\begin{aligned}\mathbb{P}\left(H_N^{\mathrm{REM}}(\tilde{\sigma}) \le m_N^{\mathrm{REM}} + y\right)^{2^N} &= \left(1 - \mathbb{P}\left(H_N^{\mathrm{REM}}(\tilde{\sigma}) > m_N^{\mathrm{REM}} + y\right)\right)^{2^N} \\ &= \left(1 - \frac{2^N \mathbb{P}\left(H_N^{\mathrm{REM}}(\tilde{\sigma}) > m_N^{\mathrm{REM}} + y\right)}{2^N}\right)^{2^N}. \qquad (1.2.4)\end{aligned}$$

This means that we need to find $(m_N^{\mathrm{REM}})_{N\in\mathbb{N}}$ so that $2^N\mathbb{P}\left(H_N^{\mathrm{REM}}(\tilde{\sigma}) > m_N^{\mathrm{REM}} + y\right)$ has a positive limit c as $N \uparrow \infty$. In this case, the right-hand side of (1.2.4) converges to e^{-c} as $N \uparrow \infty$.

For $N \in \mathbb{N}$, let

$$m_N^{\mathrm{REM}} := \sqrt{2\ln 2}\, N - \frac{1}{2\sqrt{2\ln 2}} \ln N - \frac{\ln(4\pi \ln 2)}{2\sqrt{2\ln 2}}. \qquad (1.2.5)$$

Then, for N large enough such that $m_N^{\mathrm{REM}}+y$ is positive, we have by Lemma 1.2.1,

$$\begin{aligned}&2^N\mathbb{P}\left(H_N^{\mathrm{REM}}(\tilde{\sigma}) > m_N^{\mathrm{REM}} + y\right) \\ &= 2^N \frac{\sqrt{N}}{m_N^{\mathrm{REM}} + y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2N}\left(m_N^{\mathrm{REM}} + y\right)^2\right)(1+o(1)) \\ &= 2^N \frac{\sqrt{N}}{\sqrt{2\ln 2}\, N + o(N)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2N}\left(\sqrt{2\ln 2}\, N - \frac{1}{2\sqrt{2\ln 2}} \ln N - \frac{\ln(4\pi\ln 2)}{2\sqrt{2\ln 2}} + y\right)^2\right)(1+o(1)) \\ &= 2^N \frac{1}{\sqrt{2\ln 2}\sqrt{N}} \frac{1}{\sqrt{2\pi}} 2^{-N}\sqrt{N}\sqrt{4\pi\ln 2}\, \mathrm{e}^{-\sqrt{2\ln 2}\, y}(1+o(1)) \\ &= \mathrm{e}^{-\sqrt{2\ln 2}\, y}(1+o(1)). \qquad (1.2.6)\end{aligned}$$

We have obtained in (1.2.3)–(1.2.6) that

$$\lim_{N\uparrow\infty} \mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma) \le m_N^{\mathrm{REM}} + y\right) = \mathrm{e}^{-\mathrm{e}^{-\sqrt{2\ln 2}\, y}}, \qquad (1.2.7)$$

which is the distribution function of a *Gumbel distribution*.

Another quantity of interest is the *extremal process* of the REM, which is defined as

$$\mathcal{E}_N^{\mathrm{REM}} := \sum_{\sigma\in\{-1,1\}^N} \delta_{H_N^{\mathrm{REM}}(\sigma) - m_N^{\mathrm{REM}}}. \qquad (1.2.8)$$

The *limiting extremal process* is denoted by $\mathcal{E}^{\mathrm{REM}} := \lim_{N\uparrow\infty} \mathcal{E}_N^{\mathrm{REM}}$, where the convergence is in law. The process $\mathcal{E}^{\mathrm{REM}}$ gives a description of the asymptotic distribution not only of the maximum of the Hamiltonian but also of spin configurations close to the maximum. These configurations are called *extremal*. It can be shown (see e.g. [24, Proposition 8.6]) that the limiting extremal process $\mathcal{E}^{\mathrm{REM}}$ of the REM is a Poisson point process with intensity $\sqrt{2\ln 2}\, \mathrm{e}^{-\sqrt{2\ln 2}\, y}\mathrm{d}y$.

1.2.2 The generalised REM and the continuous REM

In the REM, the random energy levels of all possible spin configurations are independent. However, in spin glasses, the spins of atoms exhibit a mixture of positive and negative correlations (ferromagnetic and antiferromagnetic behaviour), which lead to the disordered structure of the energetically favoured spin configurations.

We introduce a generalisation of the REM regarding the correlations of the Hamiltonian. We consider $\{-1,1\}^N$ to be the set of leaves of an N-level binary tree, see Figure 1.4 for an illustration of the labelling of the vertices.

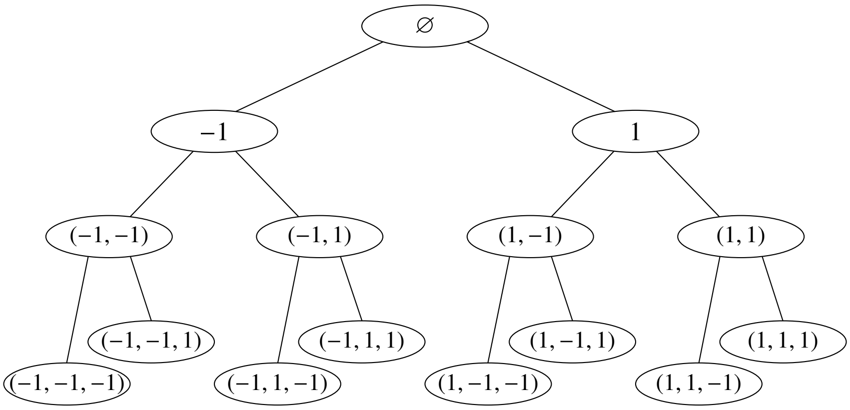

Figure 1.4: The 3-level binary tree with leaves $\{-1,1\}^3$

The *ancestor* of $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1,1\}^N$ at level i, $i = 1, \ldots, N$, is denoted by $\sigma|_i = (\sigma_1, \ldots, \sigma_i)$ and $\sigma|_0$ denotes the root $\varnothing$. The *overlap* of $\sigma, \tilde{\sigma} \in \{-1,1\}^N$ is defined as

$$\sigma \wedge \tilde{\sigma} := \max\{i = 0, \ldots, N \colon \sigma|_i = \tilde{\sigma}|_i\}. \tag{1.2.9}$$

This means that the overlap is the level of the most recent common ancestor, see Figure 1.5.

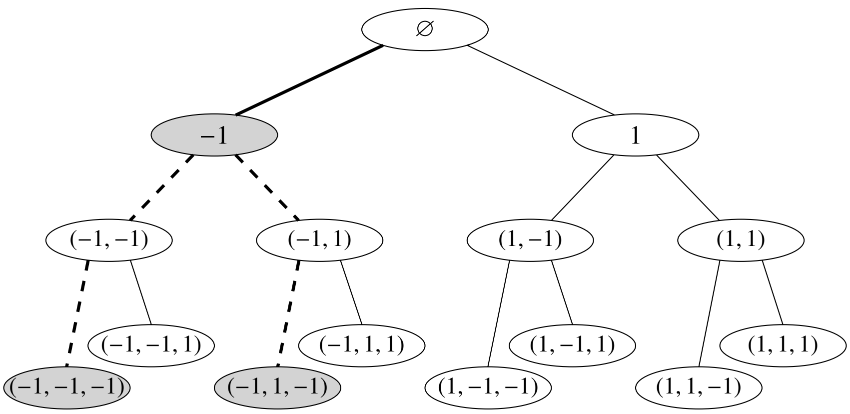

Figure 1.5: The most recent common ancestor of $(-1,-1,-1)$ and $(-1,1,-1)$ is the vertex -1. The level of this vertex is 1, so $(-1,-1,-1) \wedge (-1,1,-1) = 1$.

Each vertex v of the binary tree is assigned an i.i.d. copy z_v of a standard Gaussian random variable. The *branching random walk* (BRW) on the binary tree of depth N with standard Gaussian increments is defined as

$$H_N^{\mathrm{BRW}}(\sigma) := \sum_{i=1}^{N} z_{\sigma|_i}, \qquad \sigma \in \{-1,1\}^N . \tag{1.2.10}$$

We see in (1.2.10) that for $H_N^{\mathrm{BRW}}(\sigma)$, we sum up the Gaussian random variables $z_{\sigma|_i}$ assigned to all ancestors $\sigma|_i, i = 1, \dots, N$, of σ except the root. For $\sigma, \tilde{\sigma} \in \{-1,1\}^N$, since z_v and $z_{\tilde{v}}$ are independent for $v \neq \tilde{v}$,

$$\begin{aligned} \mathbb{E}\left[H_N^{\mathrm{BRW}}(\sigma)\, H_N^{\mathrm{BRW}}(\tilde{\sigma})\right] &= \mathbb{E}\left[\left(\textstyle\sum_{i=1}^N z_{\sigma|_i}\right)\left(\textstyle\sum_{j=1}^N z_{\tilde{\sigma}|_j}\right)\right] = \sum_{i=1}^N \sum_{j=1}^N \mathbb{E}\left[z_{\sigma|_i} z_{\tilde{\sigma}|_j}\right] \\ &= \sum_{i=1}^N \sum_{j=1}^N \mathbb{1}_{\sigma|_i = \tilde{\sigma}|_j} = \sum_{i=1}^N \mathbb{1}_{\sigma|_i = \tilde{\sigma}|_i} = \sigma \wedge \tilde{\sigma}. \end{aligned} \tag{1.2.11}$$

Thus, $(H_N^{\mathrm{BRW}}(\sigma))_{\sigma\in\{-1,1\}^N}$ can be formally defined as the centred Gaussian process with covariance

$$\mathbb{E}\left[H_N^{\mathrm{BRW}}(\sigma)\, H_N^{\mathrm{BRW}}(\tilde{\sigma})\right] = \sigma \wedge \tilde{\sigma}, \qquad \sigma, \tilde{\sigma} \in \{-1,1\}^N . \tag{1.2.12}$$

In this view, we interpret N as a point of time and $(H_N^{\mathrm{BRW}}(\sigma))_{\sigma\in\{-1,1\}^N}$ as the spatial position of 2^N *particles*. Alternatively, this BRW can be constructed in the following way: It starts with

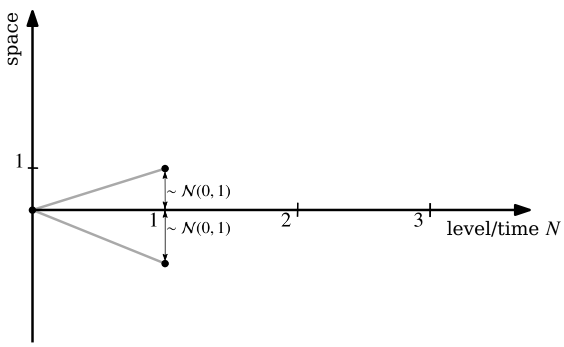

Figure 1.6: At time $N = 0$, there is one particle at height 0. At time $N = 1$, the particle splits into two particles, whose height change at time 1 is i.i.d. standard Gaussian.

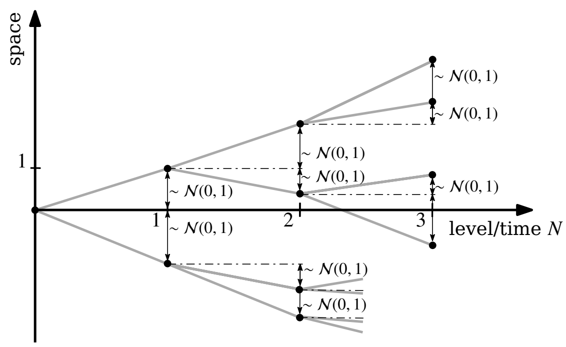

Figure 1.7: At time $N = 2$, each of the particles splits into two again and each child gains an i.i.d. standard Gaussian height. The procedure at time $N = 2$ is repeated at each integer time.

one particle at time 0. At time 1, it branches into two particles, which change their position independently according to a standard Gaussian distribution, see Figure 1.6. At time 2, each particle branches into two particles with i.i.d. standard Gaussian height changes again. This is repeated at each integer point of time, see Figure 1.7. In the light of this construction, we use the terms “spin configurations” and “particles” alternating from now on. The random energy levels described by the Hamiltonian are in this view understood as the height of particles.

Branching random walks, in particular those with Gaussian increments, have been investigated e.g. in [34, 71]. We refer to [2, 88] for an analysis of the extremes of the BRW with Gaussian increments.

We now study the *continuous random energy model* (CREM), a generalisation of the BRW with Gaussian increments, which was introduced by Bovier and Kurkova [32] in 2004. Let $A\colon [0,1] \to [0,1]$ with $A(0) = 0$ and $A(1) = 1$ be a non-decreasing and right-continuous function. Such a function is called *speed function* (or *covariance function*). Let $(H_N^A(\sigma))_{\sigma\in\{-1,1\}^N}$ be the centred Gaussian process with covariance

$$\mathbb{E}\left[H_N^A(\sigma)\, H_N^A(\tilde\sigma)\right] = NA\left(\tfrac{\sigma\wedge\tilde\sigma}{N}\right), \qquad \sigma, \tilde\sigma \in \{-1,1\}^N . \tag{1.2.13}$$

The model using this random Hamiltonian is called *CREM* with speed function A. For the sake of brevity, we also call the Hamiltonian $(H_N^A(\sigma))_{\sigma\in\{-1,1\}^N}$ the "CREM with speed function A".

We consider two examples:

1. $A_1\colon x \mapsto \mathbb{1}_{x=1}(x)$.
   Then, $\mathbb{E}\left[H_N^{A_1}(\sigma)\, H_N^{A_1}(\tilde\sigma)\right] = N\mathbb{1}_{\sigma\wedge\tilde\sigma=N}(\sigma,\tilde\sigma) = N\mathbb{1}_{\sigma=\tilde\sigma}(\sigma,\tilde\sigma)$. Thus, $(H_N^{A_1}(\sigma))_{\sigma\in\{-1,1\}^N}$ is a family of i.i.d. Gaussians with mean 0 and variance N, i.e. the Hamiltonian of the REM.

2. $A_2\colon x \mapsto x$, which is also called *identity function*.
   Then, $\mathbb{E}\left[H_N^{A_2}(\sigma)\, H_N^{A_2}(\tilde\sigma)\right] = \sigma \wedge \tilde\sigma$. Thus, $(H_N^{A_2}(\sigma))_{\sigma\in\{-1,1\}^N}$ is the BRW on the binary tree with standard Gaussian increments.

These examples show that the CREM is a class of models containing the REM as well as the BRW with standard Gaussian increments. If A is a piecewise constant speed function, the model is also called *generalised random energy model* (GREM). The GREM was introduced by Derrida [55] in 1985.

1.2.3 The extreme values of the GREM and CREM

We are interested in a description of the extreme values, not only for the CREM but also for any of the upcoming models of Section 1.3. This leads to a better understanding of the underlying process, in itself and in the context of spin glass models. We concentrate on the following questions:

1. What is the leading order of the maximum of the Hamiltonian H_N^A as $N \uparrow \infty$? What about subleading orders? Is there a sequence $(m_N^A)_{N\in\mathbb{N}}$ so that $\max_{\sigma\in\{-1,1\}^N} H_N^A(\sigma) - m_N^A$ is tight?

2. What is the best strategy for a particle before time N to reach the height m_N^A at time N?

3. What is the limit of the extremal process $\sum_{\sigma\in\{-1,1\}^N} \delta_{H_N^A(\sigma)-m_N^A}$?

We also call m_N^A *extremal height* from now on. This section contains an overview of results which answer the previous questions for the GREM and CREM.

Bovier and Kurkova have shown in [31] for the GREM and [32] for the CREM, both with speed function A, that the leading order $m_N^{A,\mathrm{lead}}$ of m_N^A satisfies

$$\boxed{m_N^{A,\mathrm{lead}} = \begin{cases} \sqrt{2\ln 2}\,N, & \text{if } A(x) \le x \text{ for all } x \in [0,1], \\ \sqrt{2\ln 2}\,N \int_0^1 \sqrt{\bar{A}'(x)}\,\mathrm{d}x, & \text{if } \bar{A}(x) > x \text{ for all } x \in (0,1), \end{cases}} \tag{1.2.14}$$

where we denote by $\bar{A}$ the concave hull of A. Note that, if $A(x) \le x$ for all $x \in [0,1]$, then $\bar{A}(x) = x$ for all $x \in [0,1]$, so $\int_0^1 \sqrt{\bar{A}'(x)}\,\mathrm{d}x = 1$. This means that for any speed function A,

$$m_N^{A,\mathrm{lead}} = \sqrt{2\ln 2}\,N \int_0^1 \sqrt{\bar{A}'(x)}\,\mathrm{d}x. \tag{1.2.15}$$

In the following Subsection 1.2.4, we compute $m_N^{A,\mathrm{lead}}$ for the two-level GREM. These computations also provide an illustration of the typical behaviour before time N of an extremal particle at time N, which answers Question 2 for the two-level GREM.

For the GREM with speed function A so that $A(x) \le x$, it holds by [31, Theorem 1.1] that $m_N^A = m_N^{\mathrm{REM}}$ even in subleading orders. Recall the full description of m_N^{REM} in (1.2.5). For a detailed computation of the subleading orders for the two-level GREM with $A(x) \le x$ for all $x \in [0,1]$ and $A(x_1) = x_1$ in one point $x_1 \in (0,1)$, we refer to Section 2.2.2 of Kistler's lecture notes in [68]. In the case of the BRW with Gaussian increments, i.e. the CREM with speed function $A_2 \colon x \mapsto x$, the subleading orders of $m_N^{A_2}$ are

$$-\frac{3}{2\sqrt{\ln 2}} \ln N + O(1), \tag{1.2.16}$$

see [1, Theorem 3]. For the CREM with speed function A in case "$\bar{A}(x) > x$ for all $x \in (0,1)$", the subleading orders depend on $\bar{A}$, as Bovier and Kurkova have shown in [31, Theorem 1.5] for the GREM case. In this work, they also provide results on the extremal process $\sum_{\sigma\in\{-1,1\}^N} \delta_{H_N^A(\sigma)-m_N^A}$ if A is the speed function of a GREM. We omit details in the description of their results to keep the necessary notation minimal.

1. If $A(x) \le x$ for all $x \in [0,1]$, then the extremal process converges in distribution to a Poisson point process with intensity $K\sqrt{2\ln 2}\,\mathrm{e}^{-\sqrt{2\ln 2}\,x}\mathrm{d}x$, where $K = 1$ if $A(x) < x$. If there exists $x \in (0,1)$ with $A(x) = x$, then $0 < K < 1$. An explicit formula for K is given in [31, Theorem 1.1][2].

[2]When checking the reference, note that Bovier and Kurkova use a slightly different parametrisation in the constant terms of $m_N^A = m_N^{\mathrm{REM}}$ which leads to different constant factors in the intensity of the limiting extremal process.

2. In the case $\bar{A}(x) > x$ for all $x \in (0,1)$, Bovier and Kurkova [31, Theorem 1.5] describe the limiting process in terms of a concatenation of m Poisson point processes, where $m \in \mathbb{N}$ is the number of points $x \in [0,1)$ for which $\bar{A}(x) = A(x)$. A concatenation of point processes is constructed the following way: Sample the first process, add to each atom the atoms of an independent copy of the second process, and so forth.

This means that if $A(x) < x$ for all $x \in (0,1)$, the extremal process and the first order of the maximum is the same as for the REM. In the case "$\bar{A}(x) > x$ for all $x \in (0,1)$", the results of [31] yield the following intuitive description of extremal particles: Particles which are extremal at time N are likely to also be extremal at any time xN where $x \in (0,1)$ satisfies $\bar{A}(x) = A(x)$. We refer to Section 10.1.1. in [25] for a more detailed discussion of these results.

1.2.4 Example: The leading order of the maximum of a two-level GREM

In this subsection, we study the extremes of the two-level GREM, i.e. of a CREM with speed function

$$\begin{aligned} A_3\colon [0,1] &\to [0,1], \\ x &\mapsto a_1 \mathbb{1}_{[x_1,1)}(x) + \mathbb{1}_{x=1}(x), \end{aligned} \tag{1.2.17}$$

where $x_1, a_1 \in (0,1)$. We refer to Figures 1.8 and 1.9 for illustrations of A_3 and its concave hull.

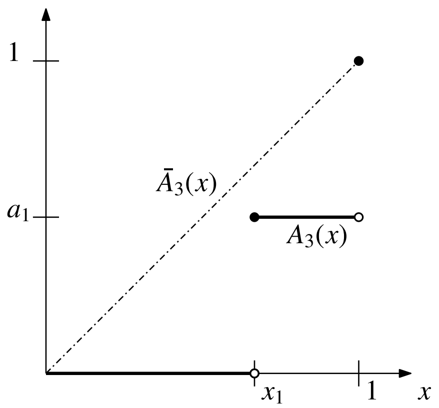

Figure 1.8: If $a_1 < x_1$, then $A_3(x) < x$ for all $x \in (0,1)$. In this case, $\bar{A}_3$ coincides with the identity function (dashdotted line).

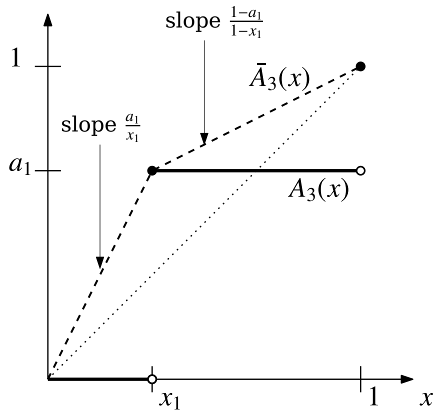

Figure 1.9: If $a_1 > x_1$, then $\bar{A}_3$ (dashed line) lies above the identity function (dotted line).

The GREM $\left(H_N^{A_3}(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ with speed function A_3 is a centred Gaussian process with covariances

$$\mathbb{E}\left[H_N^{A_3}(\sigma)\,H_N^{A_3}(\tilde\sigma)\right] = NA_3\left(\tfrac{\sigma\wedge\tilde\sigma}{N}\right) = \begin{cases} 0, & \sigma\wedge\tilde\sigma\in[0,x_1N),\\ a_1N, & \sigma\wedge\tilde\sigma\in[x_1N,N),\\ N, & \sigma=\tilde\sigma, \end{cases} \tag{1.2.18}$$

for $\sigma,\tilde\sigma\in\{-1,1\}^N$.

We prove in this section that the leading order of the maximum, $m_N^{A_3,\mathrm{lead}}$, satisfies

$$m_N^{A_3,\mathrm{lead}} = \begin{cases} \sqrt{2\ln 2}\,N, & \text{if } a_1\le x_1,\\ \sqrt{2\ln 2}\,N\left(\sqrt{x_1a_1}+\sqrt{(1-x_1)(1-a_1)}\right), & \text{if } \bar a_1 > x_1. \end{cases} \tag{1.2.19}$$

The case $a_1 < x_1$ translates to $A_3(x)\le x$ for all $x\in(0,1)$, see Figure 1.8. We will further distinguish between the case $A_3(x)<x$ for all $x\in(0,1)$, which is also called *weak correlation regime* in general, and the case $a_1 = A_3(x_1) = x_1$.

If $a_1 > x_1$, we have $\bar A_3(x) > x$ for all $x\in(0,1)$, see Figure 1.9. We also call this case the *concave regime*. In this regime,

$$\sqrt{x_1a_1}+\sqrt{(1-x_1)(1-a_1)} = \int_0^1 \sqrt{\bar A_3'(x)}\,\mathrm{d}x, \tag{1.2.20}$$

as we can see in Figure 1.9. Thus, we can write (1.2.19) as

$$m_N^{A_3,\mathrm{lead}} = \begin{cases} \sqrt{2\ln 2}\,N, & \text{if } A_3(x)\le x \text{ for all } x\in[0,1],\\ \sqrt{2\ln 2}\,N\displaystyle\int_0^1\sqrt{\bar A_3'(x)}\,\mathrm{d}x, & \text{if } \bar A_3(x) > x \text{ for all } x\in(0,1), \end{cases} \tag{1.2.21}$$

which coincides with the general result in (1.2.14) in the case $A = A_3$.

In both cases, the following calculations are based on Section 10.1.1. of [25]. We determine $(m_N^{A_3})_{N\in\mathbb{N}}$ such that for each $y\in\mathbb{R}$,

$$\lim_{N\uparrow\infty}\mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N^{A_3}(\sigma)\ge m_N^{A_3}+y\right) \tag{1.2.22}$$

exists and is in $(0,1)$. We only compute the leading order $\left(m_N^{A_3,\mathrm{lead}}\right)_{N\in\mathbb{N}}$ of $(m_N^{A_3})_{N\in\mathbb{N}}$ and use slightly more heuristic arguments than for the REM in Section 1.2.1. In this sense, we assume for simplicity that x_1N and $(1-x_1)N$ are both integer. Let $(z_\eta)_{\eta\in\{-1,1\}^{x_1N}}$ and $(z_\sigma)_{\sigma\in\{-1,1\}^N}$ be two independent families of i.i.d. standard Gaussians. Then

$$\left(H_N^{A_3}(\sigma)\right)_{\sigma\in\{-1,1\}^N} \overset{\mathrm{d}}{=} \left(\sqrt{a_1N}\,z_{\sigma|_{x_1N}}+\sqrt{(1-a_1)N}\,z_\sigma\right)_{\sigma\in\{-1,1\}^N}, \tag{1.2.23}$$

since the right-hand side is a centred Gaussian process with the same covariances as $\left(H_N^{A_3}(\sigma)\right)_{\sigma\in\{-1,1\}^N}$. For each particle $\sigma \in \{-1,1\}^N$, the random variable $\sqrt{a_1 N}\, z_{\sigma|_{x_1 N}}$ describes the position of $\sigma|_{x_1 N}$, the ancestor of σ at time $x_1 N$. If σ is at extremal height $M_N^{A_3}$ at time N, then its ancestor $\sigma|_{x_1 N}$ is either at the extremal height $\tilde{m}_N$ of time $x_1 N$, which is

$$\tilde{m}_N = \tilde{m}_N(x_1, a_1) = \sqrt{2x_1 a_1 \ln 2}\, N, \tag{1.2.24}$$

by the same argumentation as in (1.2.3)–(1.2.6). Or, $\sigma|_{x_1 N}$ is at a certain height below $\tilde{m}_N$, say at height $\sqrt{\lambda}\,\tilde{m}_N$ for $\lambda \in [0,1)$. Determining the most likely $\lambda \in [0,1]$ also answers Question 2 for the two-level GREM.

For $y \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N^{A_3}(\sigma) \geq m_N^{A_3} + y\right)\\
&= \mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} \sqrt{a_1 N}\, z_{\sigma|_{x_1 N}} + \sqrt{(1-a_1)N}\, z_\sigma \geq m_N^{A_3} + y\right)\\
&\approx \max_{\lambda\in[0,1]} \mathbb{P}\left(\exists\, \eta \in \{-1,1\}^{x_1 N} : \sqrt{a_1 N}\, z_\eta \approx \sqrt{\lambda}\tilde{m}_N, \max_{\substack{\sigma\in\{-1,1\}^N,\\ \sigma|_{x_1 N}=\eta}} \sqrt{(1-a_1)N}\, z_\sigma \geq m_N^{A_3} - \sqrt{a_1 N}\, z_\eta + y\right)\\
&\leq \sum_{\eta\in\{-1,1\}^{x_1 N}} \max_{\lambda\in[0,1]} \mathbb{P}\left(\sqrt{a_1 N}\, z_\eta \approx \sqrt{\lambda}\tilde{m}_N\right) \mathbb{P}\left(\max_{\substack{\sigma\in\{-1,1\}^N,\\ \sigma|_{x_1 N}=\eta}} \sqrt{(1-a_1)N}\, z_\sigma \geq m_N^{A_3} - \sqrt{\lambda}\tilde{m}_N + y\right),
\end{aligned} \tag{1.2.25}$$

where we used a union bound and the independence of all appearing random variables in the last step. Thus, finding $\lambda^* \in [0,1]$ which maximises the right-hand side of (1.2.25) also provides an upper bound on (1.2.22). Determining the first order of $m_N^{A_3}$ on this upper bound will provide the right guess for λ^*. A matching lower bound can be found by the Paley-Zygmund inequality, proceeding as in Section 2.2.2, (86) ff., of Kistler's lecture notes in [68].

By Lemma 1.2.1,

$$\mathbb{P}\left(\sqrt{a_1 N}\, z_\eta \approx \sqrt{\lambda}\,\tilde{m}_N\right) \approx \mathbb{P}\left(\sqrt{a_1 N}\, z_\eta \geq \sqrt{\lambda}\,\tilde{m}_N\right) = \mathbb{P}\left(z_\eta > \sqrt{2x_1 \lambda N \ln 2}\right) = 2^{-x_1\lambda N} O(N^{-1/2}) \leq 2^{-x_1 \lambda N}, \tag{1.2.26}$$

for each $\eta \in \{-1,1\}^{x_1 N}$. Inserting (1.2.26) into (1.2.25) gives

$$\mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N^{A_3}(\sigma) \geq m_N^{A_3} + y\right) \leq \max_{\lambda\in[0,1]} 2^{x_1(1-\lambda)N} \mathbb{P}\left(\max_{\substack{\sigma\in\{-1,1\}^N,\\ \sigma|_{x_1 N}=\eta}} \sqrt{(1-a_1)N}\, z_\sigma \geq m_N^{A_3} - \sqrt{\lambda}\tilde{m}_N + y\right), \tag{1.2.27}$$

for any $\eta \in \{-1,1\}^{x_1 N}$. We aim to find $m_N^{A_3}$ so that the right-hand side of (1.2.27) has a nontrivial limit. It follows from (1.2.26) that

$$\mathbb{E}\left[\#\left\{\eta \in \{-1,1\}^{x_1 N} : \sqrt{a_1 N}\, z_\eta \geq \sqrt{\lambda}\, \tilde{m}_N\right\}\right] = \sum_{\eta \in \{-1,1\}^{x_1 N}} \mathbb{P}\left(\sqrt{a_1 N}\, z_\eta \geq \sqrt{\lambda}\, \tilde{m}_N\right) = 2^{x_1(1-\lambda)N}(1+o(1)). \tag{1.2.28}$$

Thus, we interpret the right-hand side of (1.2.27) in the following way: Each of the approximately $2^{x_1(1-\lambda)N}$ particles, which are above level $\sqrt{\lambda}\, \tilde{m}_N$ at time $x_1 N$, produces $2^{(1-x_1)N}$ offspring. Thus, there are in total $2^{(1-\lambda x_1)N}$ particles at time N, whose ancestor at time $x_1 N$ was above level $\sqrt{\lambda}\, \tilde{m}_N$. The height gains of each of these particles between time $x_1 N$ and N are i.i.d. Gaussians with mean 0 and variance $(1-a_1)N$. To ensure that the right-hand side of (1.2.27) is of a nontrivial constant order, we need to find $m_N^{A_3}$ so that there is one of these $2^{(1-\lambda x_1)N}$ height gains which exceeds $m_N^{A_3} - \sqrt{\lambda}\, \tilde{m}_N + y$. Equivalently,

$$\lim_{N\uparrow\infty} \max_{\lambda\in[0,1]} \mathbb{P}\left(\max_{1\leq k\leq 2^{(1-\lambda x_1)N}} \sqrt{(1-a_1)N}\, z_k \geq m_N^{A_3} - \sqrt{\lambda}\, \tilde{m}_N + y\right) \tag{1.2.29}$$

needs to be nontrivial, where $(z_k)_{1\leq k\leq 2^{(1-\lambda x_1)N}}$ is a family of i.i.d. standard Gaussian random variables. For each $\lambda \in [0,1]$, we get as in (1.2.3)–(1.2.6) that

$$\lim_{N\uparrow\infty} \mathbb{P}\left(\max_{1\leq k\leq 2^{(1-\lambda x_1)N}} \sqrt{(1-a_1)N}\, z_k \geq m_N^{A_3} - \sqrt{\lambda}\, \tilde{m}_N + y\right) \in (0,1), \tag{1.2.30}$$

if

$$m_N^{A_3} - \sqrt{\lambda}\tilde{m}_N = \sqrt{2(1-\lambda x_1)(1-a_1)\ln 2}\, N. \tag{1.2.31}$$

Thus, to get

$$\lim_{N\uparrow\infty} \max_{\lambda\in[0,1]} \mathbb{P}\left(\max_{1\leq k\leq 2^{(1-\lambda x_1)N}} \sqrt{(1-a_1)N}\, z_k \geq m_N^{A_3} - \sqrt{\lambda}\, \tilde{m}_N + y\right) \in (0,1), \tag{1.2.32}$$

we need to choose the leading order of $m_N^{A_3}$ as

$$m_N^{A_3,\mathrm{lead}} := \max_{\lambda\in[0,1]} g(\lambda) N, \tag{1.2.33}$$

where

$$g(\lambda) := \sqrt{\lambda}\, \frac{\tilde{m}_N}{N} + \sqrt{2(1-\lambda x_1)(1-a_1)\ln 2} = \sqrt{2\lambda x_1 a_1 \ln 2} + \sqrt{2(1-\lambda x_1)(1-a_1)\ln 2}. \tag{1.2.34}$$

The derivative

$$g'(\lambda) = \frac{\sqrt{2x_1 a_1 \ln 2}}{2\sqrt{\lambda}} - \frac{x_1\sqrt{2(1-a_1)\ln 2}}{2\sqrt{1-\lambda x_1}} \tag{1.2.35}$$

has a root in $\lambda^* = \frac{a_1}{x_1}$. We first turn to the case $a_1 < x_1$. Then, $\lambda^* < 1$. One can easily check that for $\lambda \in (0,\lambda^*)$, $g'(\lambda)$ is positive and for $\lambda \in (\lambda^*,1)$, $g'(\lambda)$ is negative. Furthermore, $g'(\lambda) \uparrow \infty$

as $\lambda \downarrow 0$ and $g'(\lambda) \downarrow -\infty$ as $\lambda \uparrow 1$. Thus, g is maximal in λ^* with

$$g(\lambda^*) = a_1\sqrt{2\ln 2} + (1-a_1)\sqrt{2\ln 2} = \sqrt{2\ln 2}. \tag{1.2.36}$$

This means that (1.2.19) holds in the case $a_1 < x_1$: The leading order of $(m_N^{A_3})_{N\in\mathbb{N}}$ is $\sqrt{2\ln 2}\,N$, which is the same as the leading order of $(m_N^{\mathrm{REM}})_{N\in\mathbb{N}}$. The particles at extremal height $\sqrt{2\ln 2}\,N$ at time N were around height

$$\sqrt{\lambda^*}\,\tilde{m}_N = \sqrt{2\ln 2}\,a_1 N \tag{1.2.37}$$

at time $x_1 N$. This is strictly below the extremal height $\tilde{m}_N = \sqrt{2a_1x_1\ln 2}\,N$ at time x_1N. At or above the optimal height $\sqrt{2\ln 2}\,a_1N$ at time x_1N, there are approximately $2^{x_1(1-\lambda^*)N} = 2^{(x_1-a_1)N}$ particles by (1.2.28). So at the optimal height, there are exponentially many potential candidates to reach extremal height at time N – in contrast to the subexponential amount of particles at height $\sqrt{2a_1x_1\ln 2}\,N$. Thus, in the case $a_1 < x_1$ at time x_1N, the benefit of exponentially more particles at lower heights outweighs the loss in height up to the optimal height $\sqrt{2\ln 2}\,a_1N$. This is illustrated in Figure 1.10.

If $a_1 > x_1$, we cannot choose $\lambda^* = \frac{a_1}{x_1}$ since it is above 1. Since $g'(\lambda)$ is positive for $\lambda \in (0,1)$, the maximum of g on $[0,1]$ lies on the boundary in 1 with

$$g(1) = \sqrt{2\ln 2}\left(\sqrt{x_1a_1} + \sqrt{(1-x_1)(1-a_1)}\right). \tag{1.2.38}$$

So if $a_1 > x_1$, (1.2.19) also holds: The leading order of $(m_N^{A_3})_{N\in\mathbb{N}}$ is

$$\sqrt{2\ln 2}\,N\left(\sqrt{x_1a_1} + \sqrt{(1-x_1)(1-a_1)}\right). \tag{1.2.39}$$

One can easily verify that this is strictly below $\sqrt{2\ln 2}\,N$, the leading order for the REM and in the case $a_1 < x_1$. In the case $a_1 > x_1$ we have $\lambda^* = 1$, so the particles at extremal height $\sqrt{2\ln 2}\,N$ at time N also were at extremal height $\tilde{m}_N$ at time x_1N. This is illustrated in Figure 1.11.

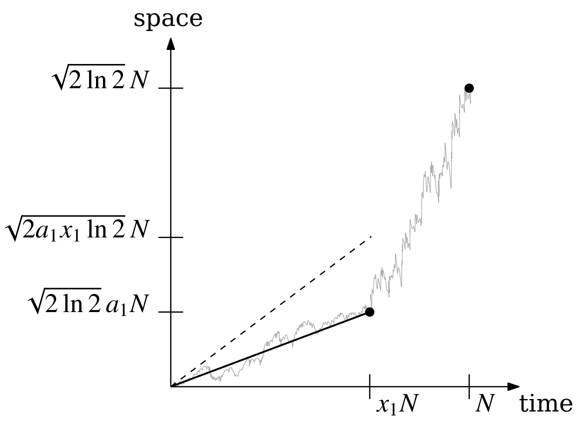

Figure 1.10: If $a_1 < x_1$, then extremal particles at time N are at height $\sqrt{2\ln 2}\,N$. At time x_1N, they are strictly below the extremal height of that time (dashed line).

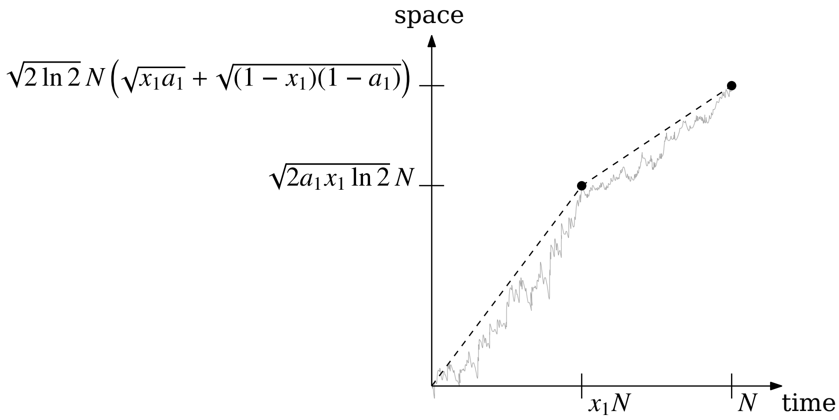

Figure 1.11: If $a_1 > x_1$, then a particle which is extremal at time N is also likely to reach extremal height (dashed line) at time x_1N.

We now inspect the case $a_1 = x_1$: Here $A(x_1) = x_1$, so it neither holds that $A(x) < x$ for all $x \in (0,1)$ nor that $\bar{A}(x) > x$ for all $x \in (0,1)$. We see that g is maximal in 1 as in the case $a_1 > x_1$, so extremal particles at time N in case $a_1 = x_1$ follow the same strategy at time x_1N as those in case $a_1 > x_1$: At time x_1N, they are at extremal height $\tilde{m}_N$ as well. The leading order of $m_N^{A_3}$ is $g(1)N$. However, if $a_1 = x_1$, we have $g(1)N = \sqrt{2\ln 2}\,N$. This means that (1.2.19) also holds if $a_1 = x_1$: The leading order of $m_N^{A_3}$ is the same as for the REM and as in the case $a_1 < x_1$. Thus, the case $a_1 = x_1$ constitutes a boundary between weak correlation regime ($a_1 < x_1$) and the concave regime ($a_1 > x_1$) regarding the leading order of $m_N^{A_3}$ and the typical behaviour at time x_1N of extremal particles at time N.

1.2.5 The free energy of the REM and the CREM

In this section, we study the free energy of the REM and the CREM and explain the behaviour of these models above and below the phase transition in terms of spin glass magnetism. The free energy of the REM is given by

$$F_N^{\text{REM}}(\beta) = -\frac{1}{N}\mathbb{E}\left[\ln Z_{\beta,N}^{\text{REM}}\right], \qquad \beta \geq 0, N \in \mathbb{N}, \tag{1.2.40}$$

where

$$Z_{\beta,N}^{\text{REM}} = \sum_{\sigma\in\{-1,1\}^N} \exp\left(\beta H_N^{\text{REM}}(\sigma)\right). \tag{1.2.41}$$

Its limit satisfies

$$F^{\text{REM}}(\beta) = \lim_{N\uparrow\infty} F_N^{\text{REM}}(\beta) = \begin{cases} -\frac{\beta^2}{2} - \ln 2, & \text{if } \beta \leq \beta_c, \\ -\sqrt{2\ln 2}\,\beta, & \text{if } \beta > \beta_c, \end{cases} \tag{1.2.42}$$

where $\beta_c = \sqrt{2\ln 2}$ is called *critical inverse temperature*. For two different versions of a proof of (1.2.42), we refer to [58, Proposition 6.1] and [25, Theorem 9.1.2]. This phase transition in β_c is one example of significant phenomena of more complex spin glass models (such as the SK model) surprisingly appearing in the REM despite its simple structure.

For $\beta \le \beta_c$, we have

$$F^{\mathrm{REM}}(\beta) = -\tfrac{\beta^2}{2} - \ln 2 = \lim_{N\uparrow\infty} -\frac{1}{N} \ln \mathbb{E}\left[Z_{\beta,N}^{\mathrm{REM}}\right], \tag{1.2.43}$$

since

$$\mathbb{E}\left[Z_{\beta,N}^{\mathrm{REM}}\right] = \sum_{\sigma\in\{-1,1\}^N} \mathbb{E}\left[\mathrm{e}^{-\beta H_N^{\mathrm{REM}}(\sigma)}\right] = \sum_{\sigma\in\{-1,1\}^N} \mathrm{e}^{\frac{\beta^2 N}{2}} = 2^N \mathrm{e}^{\frac{\beta^2 N}{2}}, \tag{1.2.44}$$

where we have used that for a Gaussian random variable Y with mean 0 and variance $v > 0$,

$$\mathbb{E}\left[\mathrm{e}^Y\right] = \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi v}} \mathrm{e}^{-\frac{z^2}{2v}} \mathrm{e}^z = \mathrm{e}^{\frac{v}{2}} \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi v}} \mathrm{e}^{-\frac{(z-v)^2}{2v}} = \mathrm{e}^{\frac{v}{2}}. \tag{1.2.45}$$

We deduce from (1.2.43) that

$$\lim_{N\uparrow\infty} \frac{\ln \mathbb{E}\left[Z_{\beta,N}^{\mathrm{REM}}\right]}{\mathbb{E}\left[\ln Z_{\beta,N}^{\mathrm{REM}}\right]} = 1, \tag{1.2.46}$$

so replacing $Z_{\beta,N}^{\mathrm{REM}}$ by $\mathbb{E}\left[Z_{\beta,N}^{\mathrm{REM}}\right]$ in (1.2.40) does not change the limiting free energy. This means that the system is in a phase analogous to the disordered phase of spin glasses in which the spin glass does not exhibit magnetisation. The energy levels are close to each other, so no spin configuration is clearly favoured over others. This is also emphasised by Theorem 9.3.1 in [25]: For $\beta \le \beta_c$, with a certain mapping $g_N\colon (0,1] \to \{-1,1\}^N$, the measure $\mu_{\beta,N}^{\mathrm{REM}} \circ g_N$ converges weakly as $N \uparrow \infty$ to $\lambda_{[0,1]}$, the Lebesgue measure on $[0,1]$.

For $\beta > \beta_c$, we have

$$\tfrac{1}{\beta} F^{\mathrm{REM}}(\beta) = -\lim_{N\uparrow\infty} \tfrac{m_N^{\mathrm{REM}}}{N}. \tag{1.2.47}$$

This means that replacing in (1.2.40) $Z_{\beta,N}^{\mathrm{REM}}$ by its largest summand $\exp\left(\beta \max_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma)\right)$ does not change the limiting free energy in this phase. The Gibbs measure $\mu_{\beta,N}^{\mathrm{REM}}$ concentrates on the energetically most favoured spin configurations such as $\tilde{\sigma} = \mathrm{argmax}_{\sigma\in\{-1,1\}^N} H_N^{\mathrm{REM}}(\sigma)$. The model is in the analogue of the spin glass phase. In this phase, $\sum_{\sigma\in\{-1,1\}^N} \delta_{\mu_{\beta,N}^{\mathrm{REM}}(\sigma)}$, the point process of the Gibbs masses, converges weakly to the *Poisson-Dirichlet* process. We refer to [24, Theorem 8.10] for a proof and more details. The multi-level version of the Poisson-Dirichlet process are called *Ruelle cascades*. These are the subject of Section 1.4.3. For high inverse temperatures β, the point process of a certain rescaling of the Gibbs masses of the GREM converges weakly to the Ruelle cascades, see [25, Theorem 10.1.14].

An explicit formula for the GREM free energy was proven by Capocaccia et al. in [37]. Bovier and Kurkova generalised this result to the CREM in [32]. Namely, for the free energy of the

CREM with speed function A, which is defined by $\tilde{F}_N^A(\beta) := -\frac{1}{N}\mathbb{E}\left[\ln Z_{\beta,N}^A\right]$ for $\beta \geq 0$ and $N \in \mathbb{N}$, we have

$$\boxed{\tilde{F}^A(\beta) := \lim_{N\uparrow\infty} \tilde{F}_N^A(\beta) = \frac{\beta^2}{2}(\bar{A}(x(\beta)) - 1) - (1 - x(\beta))\ln 2 - \sqrt{2\ln 2}\beta \int_0^{x(\beta)} \sqrt{\bar{A}'(x)}\,\mathrm{d}x,} \tag{1.2.48}$$

where $x(\beta) = \sup\left(x \in (0,1) : \bar{A}'(x) \geq \frac{2\ln 2}{\beta^2}\right)$. Note that if $A(x) \leq x$ for all $x \in [0,1]$, then $\bar{A}(x) = x$ for all $x \in [0,1]$. In this case, $x(\beta) = \mathbb{1}_{\beta \geq \beta_c}(\beta)$, where $\beta_c = \sqrt{2\ln 2}$, so

$$\lim_{N\uparrow\infty} \tilde{F}_N^A(\beta) = \begin{cases} -\frac{\beta^2}{2} - \ln 2, & \text{if } \beta \leq \beta_c, \\ -\sqrt{2\ln 2}\beta, & \text{if } \beta > \beta_c. \end{cases} \tag{1.2.49}$$

Thus, the free energy of the CREM with speed function A satisfying $A(x) \leq x$ for all $x \in [0,1]$ coincides with the free energy of the REM. More generally, $x(\beta) = 0$ holds for all $\beta < \sqrt{\frac{2\ln 2}{\bar{A}'(0)}}$. In this case,

$$\tilde{F}^A(\beta) = -\frac{\beta^2}{2} - \ln 2 = -\frac{1}{N}\ln\mathbb{E}\left[Z_{\beta,N}^A\right], \tag{1.2.50}$$

for each $N \in \mathbb{N}$, proceeding in the last step as in (1.2.44). This means that the model is in the analogue of the disordered phase of spin glasses. We have $x(\beta) = 1$ for all $\beta > \sqrt{\frac{2\ln 2}{\bar{A}'(1)}}$. Then,

$$\tilde{F}^A(\beta) = -\sqrt{2\ln 2}\beta \int_0^{x(\beta)} \sqrt{\bar{A}'(x)}\,\mathrm{d}x = -\lim_{N\uparrow\infty} \frac{m_N^A}{N}, \tag{1.2.51}$$

so the model is in the spin glass phase.

To be consistent with the notation of Chapter 3 and the literature it is based on, we write the free energy of the CREM in the form

$$\begin{aligned} F^A(t) &:= \lim_{N\uparrow\infty} F_N^A(t), \\ F_N^A(t) &:= -\frac{1}{N}\mathbb{E}\left[\ln \textstyle\sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\,H_N^A(\sigma) - Nt\right)\right], \end{aligned} \tag{1.2.52}$$

for $t \geq 0$. Since $F_N^A(t) = \tilde{F}_N^A(\sqrt{2t}) + t$, it holds that

$$F^A(t) = t\bar{A}(x(t)) - (1 - x(t))\ln 2 - 2\sqrt{t\ln 2}\int_0^{x(t)} \sqrt{\bar{A}'(x)}\,\mathrm{d}x, \tag{1.2.53}$$

where $x(t) = \sup\left(x \in (0,1) : \bar{A}'(x) \geq \frac{\ln 2}{t}\right)$. If $A(x) \leq x$ for all $x \in [0,1]$, then $x(t) = \mathbb{1}_{t \geq \ln 2}(t)$, so

$$F^A(t) = \begin{cases} -\ln 2, & \text{if } t \leq \ln 2, \\ t - 2\sqrt{t\ln 2}, & \text{if } t > \ln 2. \end{cases} \tag{1.2.54}$$

We refer to Section 1.4 for an overview the approach of Chapter 3 to study the free energy of the CREM. Our focus now shifts to (variable speed) branching Brownian motion. This process is closely related to the CREM, and the central object of study in Chapter 2.

1.3 (Variable speed) branching Brownian motion

1.3.1 Branching Brownian motion

Branching Brownian motion (BBM, also called *standard* BBM) can be constructed in the following way: A particle starts in position 0 at time 0 and moves according to a standard Brownian motion, see Figure 1.12. After an exponential time with mean 1, it splits into two particles (or it has a more general offspring distribution). Each offspring particle independently performs Brownian motions, starting from the position of its parent, see Figure 1.13, and independently follows the same splitting rule as its parent. This also applies to any further offspring, see Figure 1.14.

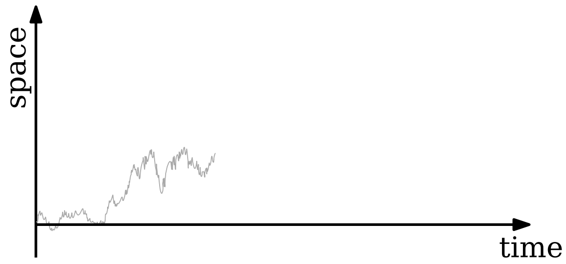

Figure 1.12: BBM before the first split consists of one particle performing Brownian motion.

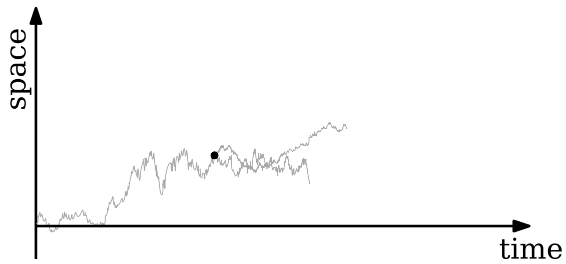

Figure 1.13: After an exponential time with mean 1, the particle splits into two. Each particle independently performs Brownian motion.

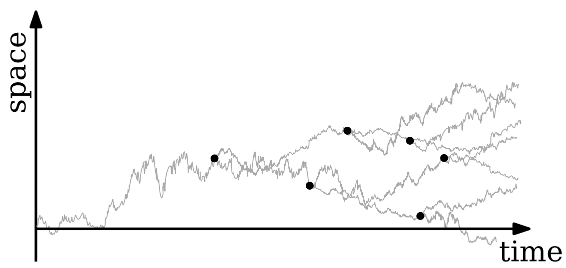

Figure 1.14: Each particle independently follows the same splitting and movement rules as the first particle.

Alternatively, we can construct BBM as in (1.2.12) as a Gaussian process on a tree. In the case of BBM, the underlying tree is random; specifically it is the *binary Galton-Watson tree*. This is a continuous-time tree constructed by independent exponential splitting times with mean 1. We denote the leaves of a Galton-Watson tree at time t by $(\ell_k(t))_{k\le n(t)}$. The time of the most recent common ancestor of the leaves $\ell_i(t), \ell_j(t)$ is denoted by $\ell_i(t) \wedge \ell_j(t)$. Conditional on the Galton-Watson tree at time t, $(x_k(t))_{k\le n(t)}$ is a centred Gaussian process with covariances

$$\mathbb{E}\left[x_i(t)\, x_j(t)\right] = \ell_i(t) \wedge \ell_j(t), \qquad i, j \le n(t). \tag{1.3.1}$$

Comparing this with (1.2.12), we see that the only difference in the construction of BBM to that of the BRW on the binary tree with Gaussian increments is the underlying tree.

By using common notation, an ambiguity emerges for the parameter t. In the context of the free energy of the CREM in (1.2.52) and Chapter 3, t denotes the inverse temperature. In the context of BBM, t denotes the time. The number of particles at time $t \ge 0$ is denoted by $n(t)$. We have $\mathbb{E}[n(t)] = \mathrm{e}^t$; for a proof see [26, Lemma 5.3]. The particle positions at time t are denoted by $(x_k(t))_{k\le n(t)}$.

Moreover, BBM is related to the *Fisher-Kolmogorov-Petrovsky-Piskunov* (F-KPP) equation,

$$\frac{\partial}{\partial_t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial_x^2} u(t,x) + u(t,x)(1 - u(t,x)). \tag{1.3.2}$$

This reaction-diffusion equation was introduced in 1937 by Fisher [67] and independently in the same year by Kolmogorov, Petrovsky and Piskunov [82]. Solutions of the F-KPP equation can be represented in terms of BBM:

Lemma 1.3.1. *Let $(x_k(t))_{k\le n(t)}$ be a BBM and $g\colon \mathbb{R} \to [0,1]$. For*

$$v(t,x) := \mathbb{E}\left[\prod_{k=1}^{n(t)} g(x - x_k(t))\right], \tag{1.3.3}$$

the function $u(t,x) := 1 - v(t,x)$ solves the F-KPP equation (1.3.2) *with initial condition $u(0,x) = 1 - g(x)$.*

This is also called the *McKean representation*. It is credited to the work of McKean [91], which however is preceded by works of Ikeda, Nagasawa and Watanabe [75–77] and Skorohod [107].

1.3.2 The extreme values of BBM

As in Sections 1.2.1 and 1.2.3, we study the behaviour of extremal particles of BBM. For this purpose we set $g(x) = \mathbb{1}_{x\ge 0}(x)$ and notice that

$$v(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} g\left(x - x_k(t)\right)\right] = \mathbb{E}\left[\prod_{k=1}^{n(t)} \mathbb{1}_{x - x_k(t)\ge 0}\right] = \mathbb{E}\left[\prod_{k=1}^{n(t)} \mathbb{1}_{x_k(t)\le x}\right] = \mathbb{P}\left(\max_{k\le n(t)} x_k(t) \le x\right). \tag{1.3.4}$$

By Lemma 1.3.1, $u(t,x) = 1 - \mathbb{P}(\max_{k\leq n(t)} x_k(t) \leq x)$ is a solution of the F-KPP equation (1.3.2) with initial condition $u(0,x) = 1 - g(x) = \mathbb{1}_{x<0}(x)$.

We call a solution u of the F-KPP equation (1.3.2) a *travelling wave solution* at speed $\lambda \geq 0$ if

$$\frac{\partial}{\partial_t} u(t, x + \lambda t) = 0. \tag{1.3.5}$$

Kolmogorov et al. have proven in [82] that a travelling wave solution u of (1.3.2) at speed $\lambda \geq \sqrt{2}$ satisfies

$$u(t, x + \lambda t) = w_\lambda(x), \tag{1.3.6}$$

where w_λ is a solution of

$$\frac{1}{2}\frac{\partial^2}{\partial_x} w_\lambda(x) + \lambda \frac{\partial}{\partial_x} w_\lambda(x) + w_\lambda(x)(1 - w_\lambda(x)) = 0. \tag{1.3.7}$$

It has been shown in [82] that for $\lambda \geq \sqrt{2}$, this differential equation has a unique solution (up to translation) which satisfies certain criteria, see [26, Lemma 5.7] for details.

The following theorem gives a description of the asymptotics of F-KPP solutions in terms of travelling wave solutions at speed $\lambda = \sqrt{2}$. This is based on Bramson's more general results [35, Theorem A and B].

Theorem 1.3.2 [26, Theorem 5.8]**.** *Let u with $0 \leq u(0,\cdot) \leq 1$ be a solution of the F-KPP equation* (1.3.2)*. Then there is a function $m(t)$ and a solution $w_{\sqrt{2}}$ of* (1.3.7) *satisfying the conditions of [26, Lemma 5.7] such that*

$$u(t, x + m(t)) \to w_{\sqrt{2}}(x), \tag{1.3.8}$$

uniformly in x, as $t \uparrow \infty$, if and only if $u(0,\cdot)$ is "steep enough", see (ii) and (iii) of Proposition 2.3.1 for a detailed description. Furthermore, if there exists $b > \sqrt{2}$ such that $\lim_{x\uparrow\infty} \mathrm{e}^{bx} u(0,x) = 0$, then we may choose

$$m(t) = \sqrt{2}\, t - \frac{3}{2\sqrt{2}} \ln t. \tag{1.3.9}$$

From this theorem and (1.3.4) follows that for

$$m^{\mathrm{BBM}}(t) = \sqrt{2}\, t - \frac{3}{2\sqrt{2}} \ln t, \tag{1.3.10}$$

we have

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} x_k(t) - m^{\mathrm{BBM}}(t) \leq y\right) = 1 - w_{\sqrt{2}}(y), \tag{1.3.11}$$

for $y \in \mathbb{R}$. To obtain a probabilistic description of the right-hand side of (1.3.11), we define the *derivative martingale* as

$$Z(t) := \sum_{k=1}^{n(t)} \left(\sqrt{2}\,t - x_k(t)\right) \mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,t - x_k(t)\right)}. \tag{1.3.12}$$

The derivative martingale almost surely has a positive limit as $t \uparrow \infty$, which we denote by Z. Lalley and Sellke [84] have shown that there exists a constant $C > 0$ such that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\le n(t)} x_k(t) - m^{\mathrm{BBM}}(t) \le y\right) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}}\right], \tag{1.3.13}$$

for $y \in \mathbb{R}$. This means that the limiting law of the maximum of BBM is a randomly shifted Gumbel distribution. This shift is described by the derivative martingale limit Z.

The function $m^{\mathrm{BBM}}(t)$ describes the leading order of the maximum and the logarithmic correction. We now compare this to the REM with e^t particles: Let $(x_k^{\mathrm{indep.}}(t))_{k\le \mathrm{e}^t}$ be e^t independent Gaussian random variables with mean 0 and variance t. We proceed as in (1.2.3)–(1.2.6): First, we set

$$m^{\mathrm{indep.}}(t) = \sqrt{2}\,t - \frac{1}{2\sqrt{2}} \ln t. \tag{1.3.14}$$

Since

$$\frac{\left(m^{\mathrm{indep.}}(t)+y\right)^2}{2t} = t - \tfrac{1}{2}\ln t + \sqrt{2}\,y + o(1), \tag{1.3.15}$$

we have by Lemma 1.2.1 that

$$\mathrm{e}^t \mathbb{P}\left(x_1^{\mathrm{indep.}}(t) > m^{\mathrm{indep.}}(t) + y\right) = \mathrm{e}^t \frac{\sqrt{t}}{m^{\mathrm{indep.}}(t)+y} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\left(m^{\mathrm{indep.}}(t)+y\right)^2}{2t}} (1+o(1)) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\sqrt{2}y}(1+o(1)). \tag{1.3.16}$$

Thus,

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\le n(t)} x_k^{\mathrm{indep.}}(t) - m^{\mathrm{indep.}}(t) \le y\right) = \lim_{t\uparrow\infty}\left(1 - \frac{\mathrm{e}^t\mathbb{P}\left(x_1^{\mathrm{indep.}}(t) > m^{\mathrm{indep.}}(t)+y\right)}{\mathrm{e}^t}\right)^{\mathrm{e}^t} = \mathrm{e}^{-\frac{1}{2\sqrt{\pi}}\mathrm{e}^{-\sqrt{2}y}}. \tag{1.3.17}$$

The right-hand side of (1.3.17) is a distribution function of a Gumbel distribution. It does not contain Z – unlike the distribution function on the right-hand side of (1.3.13).

The extremal process of BBM at time t is denoted by $\mathcal{E}_t := \sum_{k\le n(t)} \delta_{x_k(t)-m(t)}$. The limiting extremal process of BBM was first described independently by Arguin, Bovier and Kistler [10] and by Aïdékon, Berestycki, Brunet and Shi [3]. It holds

$$\lim_{t\uparrow\infty} \mathcal{E}_t = \sum_{i,j\in\mathbb{N}} \delta_{\eta_i + \Delta_j^{(i)}}, \tag{1.3.18}$$

with convergence in law, where the points η_i are the atoms of a Poisson point process $\mathcal{P}_Z$ with random intensity measure $CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$. The points $\Delta_j^{(i)}$ are the atoms of i.i.d. point processes $\Delta^{(i)}$, $i \in \mathbb{N}$, which arise as the limit in law as $t \uparrow \infty$ of

$$\sum_{j \le n(t)} \delta_{\bar{x}_j(t) - \max_{k \le n(t)} \bar{x}_k(t)}, \tag{1.3.19}$$

where $(\bar{x}_k(t))_{k \le n(t)}$ are the positions at time t of standard BBM conditioned on the event $\max_{k \le n(t)} x_k(t) \ge \sqrt{2}\, t$. Subag and Zeitouni investigated in [110] under which conditions a point process is a *decorated Poisson point process*, which means that it is of the same structure as the right-hand side of (1.3.18).

Two extremal particles of BBM at time t must have branched off very early or very late, as it is shown in [8, Theorem 2.1]. If these particles branched off early, they have followed independent Brownian paths for most of the time. In the limit, we consider them as two distinct atoms of $\mathcal{P}_Z$. This is a reasonable interpretation because a Poisson point process similar to $\mathcal{P}_Z$ also emerges as limiting extremal process of the REM and of the GREM if the speed function A lies below the identity function. Two extremal particles branching off late have followed the same path for most of the time, so they belong to one atom p_i of $\mathcal{P}_Z$ but distinct atoms of $\Delta^{(i)}$.

The reason for this branching behaviour of extremal particles of BBM at time t lies in the typical paths which these particles have likely taken before time t. A description of such paths is also called a *localisation* of the extremal particles. One such localisation is the effect of *entropic repulsion* which is described in [8, Theorem 2.3]: For t, r large enough, extremal particles of BBM at time t are at time $s \in (r, t - r)$ in their first order at extremal height $m_s^{\mathrm{BBM}} \approx \frac{s}{t} m_t^{\mathrm{BBM}} \approx \sqrt{2}\, s$. However, in the subleading order of the height at this time s, there is a deviation of order $\sqrt{s \wedge (t - s)}$ from the extremal height. This is illustrated in Figure 1.15.

This effect of entropic repulsion is also central in the explanation of the differences between the logarithmic corrections $\frac{3}{2\sqrt{2}} \ln t$ of $m^{\mathrm{BBM}}(t)$ in (1.3.10) and $\frac{1}{2\sqrt{2}} \ln t$ of $m^{\mathrm{indep.}}(t)$ in (1.3.14): By entropic repulsion as in [8, Theorem 2.3], for $y \in \mathbb{R}$ and $r > 0$ large enough,

$$\lim_{t \uparrow \infty} \mathbb{P}\left(\max_{k \le n(t)} x_k(t) - m^{\mathrm{BBM}}(t) > y \right) \approx \lim_{t \uparrow \infty} \mathbb{P}\left(\max_{\substack{k \le n(t) \\ k \in \mathcal{L}_{t,r}}} x_k(t) - m^{\mathrm{BBM}}(t) > y \right), \tag{1.3.20}$$

where

$$\mathcal{L}_{t,r} = \left\{ k \le n(t) \colon x_k(s) \le \sqrt{2}\, s \;\; \forall s \in (r, t - r) \right\}. \tag{1.3.21}$$

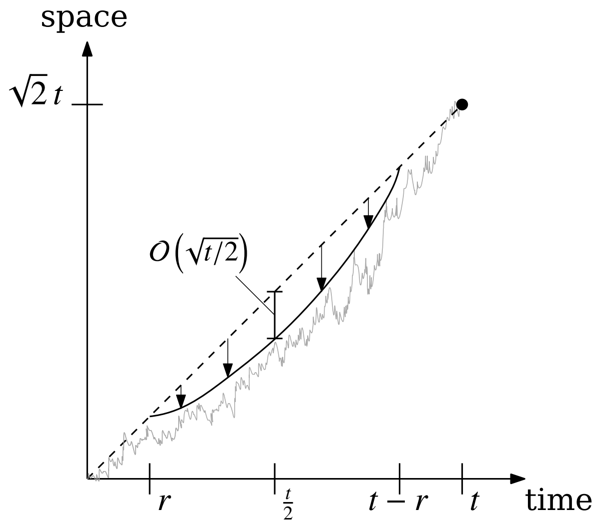

Figure 1.15: The effect of entropic repulsion illustrated for an extremal particle (grey line) at time t of standard BBM. At times $s \in (r, t-r)$, it does not cross the black line, which is of order $\sqrt{s \wedge (t-s)}$ below the maximum at time s (dashed line).

By Markov's inequality and the many-to-one lemma (see e.g. [72]),

$$\begin{aligned}\mathbb{P}\Bigg(\max_{\substack{k\le n(t)\\ k\in\mathcal{L}_{t,r}}} x_k(t) - m^{\mathrm{BBM}}(t) > y\Bigg) &= \mathbb{P}\Big(\textstyle\sum_{k\le n(t)} \mathbb{1}_{\mathcal{L}_{t,r}}(k)\, \mathbb{1}_{x_k(t)>m^{\mathrm{BBM}}(t)+y} \ge 1\Big)\\ &\le \mathbb{E}\Big[\textstyle\sum_{k\le n(t)} \mathbb{1}_{x_k(t)>m^{\mathrm{BBM}}(t)+y,\, x_k(s)\le\sqrt{2}\,s\ \forall s\in(r,t-r)}\Big]\\ &= \mathrm{e}^t\,\mathbb{P}\Big(B_t > m_t^{\mathrm{BBM}} + y, B_s < \sqrt{2}\,s\ \ \forall s\in(r,t-r)\Big), \end{aligned} \tag{1.3.22}$$

where $(B_s)_{s>0}$ denotes a Brownian motion. Let $(\mathfrak{z}^t_{0,0}(s))_{s\in[0,t]}$ be a Brownian bridge from 0 to 0 in time t. Then, for $s \in [0,t]$,

$$\mathfrak{z}^t_{0,0}(s) \overset{\mathrm{d}}{=} B_s - \tfrac{s}{t} B_t \tag{1.3.23}$$

and $(\mathfrak{z}^t_{0,0}(s))_{s\in[0,t]}$ is independent of B_t. Thus,

$$\begin{aligned}\mathbb{P}\Big(B_t > m_t^{\mathrm{BBM}} + y, B_s < \sqrt{2}\,s\,\forall s\in(r,t-r)\Big) &\approx \mathbb{P}\Big(B_t \approx \sqrt{2}\,t, B_s - \sqrt{2}\,s < 0\,\forall s\in(r,t-r)\Big)\\ &\approx \mathbb{P}\Big(B_t \approx \sqrt{2}\,t, \mathfrak{z}^t_{0,0}(s) < 0\,\forall s\in(r,t-r)\Big)\\ &= \mathbb{P}\Big(B_t \approx \sqrt{2}\,t\Big)\,\mathbb{P}\Big(\mathfrak{z}^t_{0,0}(s) < 0\,\forall s\in(r,t-r)\Big)\\ &\approx \mathbb{P}\Big(B_t > m_t^{\mathrm{BBM}} + y\Big)\,\mathbb{P}\Big(\mathfrak{z}^t_{0,0}(s) < 0\,\forall s\in(r,t-r)\Big). \end{aligned} \tag{1.3.24}$$

The probability regarding $\mathfrak{z}^t_{0,0}$ in the last line of (1.3.24) can be computed by a so-called *ballot theorem*[3].

[3]The name "ballot theorem" is due to the interpretation as the probability that in the course of the vote counting, the winner of an election is always ahead.

Lemma 1.3.3 [35, Lemma 2.2]. *For any $x, y > 0$ holds*

$$\mathbb{P}\left(\forall_{0\le s\le t}\colon \mathfrak{z}^t_{0,0}(s) \le (sx + (t-s)y)/t)\right) = 1 - \mathrm{e}^{-2xy/t} \le 2\,\frac{xy}{t}, \tag{1.3.25}$$

and asymptotic equality holds if $xy = o(t)$.

One follows from Lemma 1.3.3 that

$$\mathbb{P}\left(\mathfrak{z}^t_{0,0}(s) < 0\ \forall s \in (r, t-r)\right) \approx O\left(\tfrac{1}{t}\right). \tag{1.3.26}$$

Inserting (1.3.26) and (1.3.24) into (1.3.22) gives

$$\begin{aligned}\mathbb{P}\Bigg(\max_{\substack{k\le n(t)\\ k\in\mathcal{L}_{t,r}}} x_k(t) - m^{\mathrm{BBM}}(t) > y\Bigg) &\le \mathrm{e}^t\mathbb{P}\left(B_t > m_t^{\mathrm{BBM}} + y, B_s < \sqrt{2}\,s\ \forall s \in (r, t-r)\right)\\ &\approx \tfrac{\mathrm{e}^t}{t}\mathbb{P}\left(B_t > m_t^{\mathrm{BBM}} + y\right).\end{aligned} \tag{1.3.27}$$

By Lemma 1.2.1,

$$\mathbb{P}\left(B_t > m_t^{\mathrm{BBM}} + y\right) \approx \tfrac{1}{\sqrt{t}}\mathrm{e}^{-\frac{\left(m_t^{\mathrm{BBM}}+y\right)^2}{2t}} \approx t\mathrm{e}^{-t-\sqrt{2}\,y}, \tag{1.3.28}$$

since

$$\frac{\left(m_t^{\mathrm{BBM}}+y\right)^2}{2t} = t - \tfrac{3}{2}\ln t + \sqrt{2}\,y + o(1). \tag{1.3.29}$$

Inserting (1.3.28) into (1.3.27) gives

$$\mathbb{P}\Bigg(\max_{\substack{k\le n(t)\\ k\in\mathcal{L}_{t,r}}} x_k(t) - m^{\mathrm{BBM}}(t) > y\Bigg) \le O(1). \tag{1.3.30}$$

This means that the factor 3 instead of 1 in the subleading order of m_t^{BBM} produces the factor t on the right-hand side of (1.3.28). This factor t cancels out the factor $\frac{1}{t}$, which comes from the localisation of extremal particles of BBM.

1.3.3 Variable speed BBM

Recall that a speed function $A\colon [0,1] \to [0,1]$ is a nondecreasing and right-continuous function with $A(0) = 0$ and $A(1) = 1$. We define *variable speed BBM* (VSBBM) at time $t > 0$ with speed function A as a centred Gaussian process $(x_k^A(t))_{k\le n(t)}$ on the Galton-Watson tree with covariances (conditional on the tree up to time t)

$$\mathbb{E}\left[x_i^A(t)\,x_j^A(t)\right] = tA\left(\tfrac{\ell_i(t)\wedge\ell_j(t)}{t}\right), \qquad i, j \le n(t). \tag{1.3.31}$$

Comparing (1.3.31) and (1.2.13), we see that VSBBM is the continuous-time analogue of the CREM.

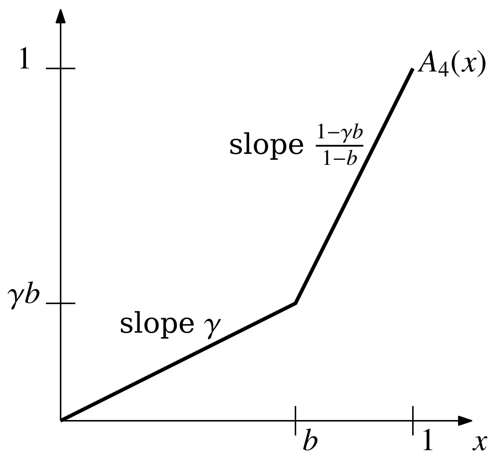

Figure 1.16: An illustration of the speed function A_4, which consists of two linear pieces.

VSBBM at time t can also be constructed with the same branching mechanism as in Figures 1.12–1.14. If a particle splits at time $s \in [0,t]$, its children fluctuate as independent copies of

$$\left(B_{tA(\tilde{s}/t)} - B_{tA(s/t)}\right)_{\tilde{s}\in[s,t]}, \tag{1.3.32}$$

which is a *time-changed* Brownian motion. Recall that $(B_t)_{t\geq 0}$ denotes Brownian motion.

As for the CREM, we give some examples of speed functions:

1. $A_1\colon x \mapsto \mathbb{1}_{x=1}(x)$.
   Then, $\mathbb{E}\left[x_j^{A_1}(t)\, x_k^{A_1}(t)\right] = t\mathbb{1}_{j=k}(j,k)$. Thus, $(x_k^{A_1}(t))_{k\leq n(t)}$ is a family of i.i.d. Gaussians with variance t, the continuous-time analogue of the REM.

2. $A_2\colon x \mapsto x$.
   Then, $\mathbb{E}\left[x_j^{A_2}(t)\, x_k^{A_2}(t)\right] = \ell_j(t) \wedge \ell_k(t)$. Thus, $(x_k^{A_2}(t))_{k\leq n(t)}$ is standard BBM, the continuous time analogue of the BRW on the binary tree with standard Gaussian increments.

3. For $\gamma > 0$, $b \in (0, 1 \wedge \frac{1}{\gamma})$, we define a piecewise-linear speed function A_4 by

   $$A_4\colon x \mapsto \begin{cases} \gamma x, & x \in [0,b), \\ \gamma b + \frac{1-\gamma b}{1-b}(x-b), & x \in [b,1]. \end{cases} \tag{1.3.33}$$

   We refer to Figure 1.16 for an illustration. Particles at time $s \in [0, bt)$ fluctuate as $B_{tA_4(s/t)} = B_{\gamma s}$. Note that $B_{\gamma s} \overset{\mathrm{d}}{=} \sqrt{\gamma} B_s$. This means that if $A_4'(s/t) > 1$, then the particles at time s fluctuate at a higher speed than a Brownian motion. If $A_4'(s/t) < 1$, these particles fluctuate less than a Brownian motion. This observation extends to any (almost everywhere) differentiable speed function A.

Note that the time-change $s \mapsto tA(s/t)$ in (1.3.31) and (1.3.32) depends on the time of the endpoint t in general. This is not the case for standard BBM, since $tA_2(s/t) = s$. If standard

BBM is sampled up to some time $t_1 > 0$ and one wants to extend the sampling up to some time $t_2 > t_1$, one can simply continue the process at time t_1 with the same rules as before. This does not apply to any other variable speed BBM due to the dependence of $s \mapsto tA(s/t)$ on t. Such a VSBBM has to be seen as a family $\left\{x_k^A(s), k \leq n(s)\right\}_{s\leq t,\, t>0}$ also indexed by the time horizon t.

1.3.4 The extreme values of certain examples of variable speed BBM

As in Section 1.2.3, we pose the following questions for a variable speed BBM $(x_k^A(t))_{k\leq n(t)}$ with speed function A:

1. What is the leading order of the maximum of $(x_k^A(t))_{k\leq n(t)}$ as $t \uparrow \infty$? What about subleading orders?
2. What is the limiting law of the maximum under a proper rescaling?
3. What is the best strategy before time t for a particle to reach extremal height at time t?
4. What is the limiting extremal process?

To answer Questions 1 and 2, we aim to find $(m_t^A)_{t>0}$ such that for all $y \in \mathbb{R}$,

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} x_k^A(t) - m_t^A \leq y\right) \tag{1.3.34}$$

exists and is in $(0,1)$. We show with Gaussian comparison techniques that m_t^A is monotonous regarding the speed function A. For this purpose, let A_{low} and A_{high} be two speed functions with $A_{\mathrm{low}}(x) \leq A_{\mathrm{high}}(x)$ for all $x \in [0,1]$. Conditioning $(x_k^{A_{\mathrm{low}}}(t))_{k\leq n(t)}$ and $(x_k^{A_{\mathrm{high}}}(t))_{k\leq n(t)}$ to have the same underlying Galton-Watson tree up to time t enables the Gaussian comparison of these two processes: Denoting by $\mathcal{F}_{\mathrm{tree}(t)}$ the σ-algebra containing the randomness of the Galton-Watson tree up to time t, we have by Slepian's Lemma [26, Lemma 3.7, Corollary 3.10] that

$$\mathbb{P}\left(\max_{k\leq n(t)} x_k^{A_{\mathrm{low}}}(t) \leq y \,\middle|\, \mathcal{F}_{\mathrm{tree}(t)}\right) \leq \mathbb{P}\left(\max_{k\leq n(t)} x_k^{A_{\mathrm{high}}}(t) \leq y \,\middle|\, \mathcal{F}_{\mathrm{tree}(t)}\right), \tag{1.3.35}$$

for $y \in \mathbb{R}$. This implies

$$\begin{aligned}
\mathbb{P}\left(\max_{k\leq n(t)} x_k^{A_{\mathrm{low}}}(t) \leq y\right) &= \mathbb{E}\left[\mathbb{P}\left(\max_{k\leq n(t)} x_k^{A_{\mathrm{low}}}(t) \leq y \,\middle|\, \mathcal{F}_{\mathrm{tree}(t)}\right)\right] \\
&\leq \mathbb{E}\left[\mathbb{P}\left(\max_{k\leq n(t)} x_k^{A_{\mathrm{high}}}(t) \leq y \,\middle|\, \mathcal{F}_{\mathrm{tree}(t)}\right)\right] \\
&= \mathbb{P}\left(\max_{k\leq n(t)} x_k^{A_{\mathrm{high}}}(t) \leq y\right),
\end{aligned} \tag{1.3.36}$$

so that $m_t^{A_{\mathrm{high}}} \leq m_t^{A_{\mathrm{low}}}$. This means that the lower the speed function A, the weaker the correlations of $(x_k^A(t))_{k\leq n(t)}$, so the higher the limiting order of the maximum m_t^A.

The case $A(x) < x$ for all $x \in (0,1)$ is called the *weak correlation regime*. A full description of the asymptotic behaviour of the maximum of $(x_k^A(t))_{k\le n(t)}$ in this setting has been given by Bovier and Hartung [28]. We give an informal summary of their main results: Let $(x_k^A(t))_{k\le n(t)}$ be a VSBBM with speed function A satisfying $A(x) < x$ for all $x \in (0,1)$ (and some regularity conditions). Then,

- for $m_t^A = m_t^{\mathrm{REM}} = \sqrt{2}\,t - \frac{1}{2\sqrt{2}}\ln t$, we have

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\le n(t)} x_k^A(t) - m_t^A \le y\right) = \mathbb{E}\left[\mathrm{e}^{-C_{\mathrm{end}} Y_{\mathrm{begin}} \mathrm{e}^{-\sqrt{2}y}}\right], \tag{1.3.37}$$

  where C_{end} is a constant depending on $A'(1)$ and Y_{begin} is the limit of the so-called *McKean martingale*, which depends on $A'(0)$.

- the limiting extremal process has the same structure as that of standard BBM in (1.3.18). However, the random intensity of the Poisson process is $\sqrt{2}\,C_{\mathrm{end}} Y_{\mathrm{begin}} \mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$ and the analogue of the process in (1.3.19) is conditional on $\max_{k\le n(t)} x_k(t) \ge \sqrt{2A'(1)}\,t$.

This means that in the weak correlation regime, the limiting distribution of the maximum is a randomly shifted Gumbel distribution, which is different from that of standard BBM in (1.3.13). As $t \uparrow \infty$, the only influence of A on $\max_{k\le n(t)} x_k^A(t)$ lies in the slopes of A in 0 and 1.

We describe a localisation of an extremal particle at time t of VSBBM in the weak correlation regime: For r, t large enough, such an extremal particle lies at time $s \in (r, t-r)$ within at most $\sqrt{t(A(s/t) \wedge (1 - A(s/t)))}$ distance to $\sqrt{2}A(s/t)t$, see Proposition 2.1 of [28]. Figure 1.17 provides an illustration of this localisation in the two-speed case.

This localisation coincides with that for the GREM with $A(x) < x$ for all $x \in (0,1)$, see Figure 1.10. There, extremal particles follow paths which lie substantially below the current maximum since at lower levels there are exponentially more candidates. This outweighs the temporary loss of height if $A(x) < x$ for all $x \in (0,1)$, as we have seen in Section 1.2.3. The optimal height at time s of extremal particles at time t for VSBBM in the weak correlation regime is $\sqrt{2}A(s/t)t$. This coincides, up to the rescaling for 2^N instead of e^t particles, with $\sqrt{2\ln 2}\,a_1 N = \sqrt{2\ln 2}\,A(x_1)N$, the analogous quantity at time $x_1 N$ for the two-level GREM with $A(x) < x$ for all $x \in (0,1)$.

We have seen for the GREM that the identity function, $A(x) = x$ for all $x \in [0,1]$, constitutes a boundary for the behaviour of extremal particles. In this case, the leading order of the maximum is $\sqrt{2\ln 2}\,N$, the same as in the case $A(x) \le x$ for all $x \in [0,1]$. However, in both cases $A(x) = x$ or $\bar{A}(x) > x$ for all $x \in (0,1)$, extremal particles at time N are extremal (in their leading order) at times below N.

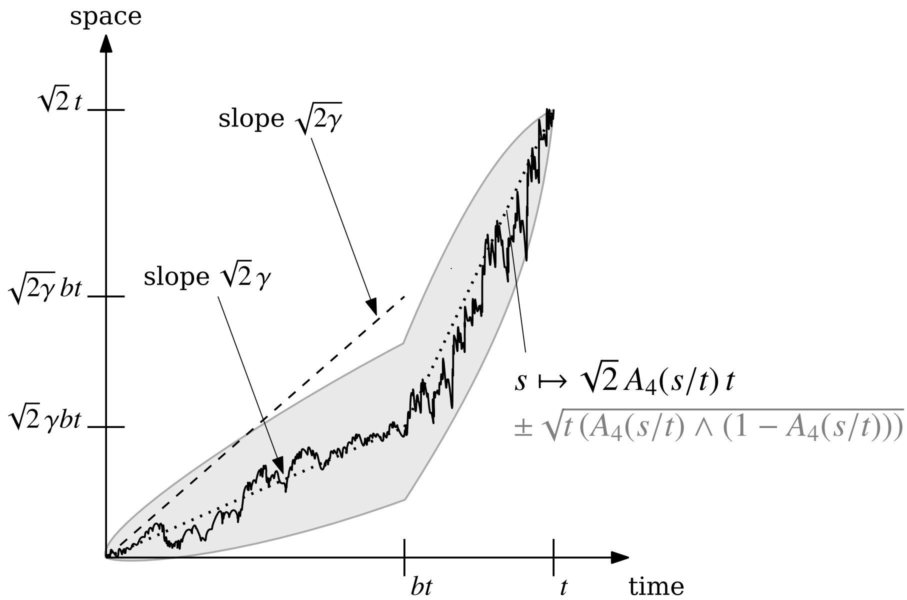

Figure 1.17: The dotted line is $s \mapsto \sqrt{2}A_4(s/t)\,t$, where A_4 is defined as in (1.3.33) as a two-speed function with slope $\gamma \in (0,1)$ on $(0,b) \subset (0,1)$. An extremal particle (black path) of VSBBM with speed function A_4 follows this line with fluctuations (grey area) at most of order $\sqrt{t(A_4(s/t) \wedge (1 - A_4(s/t)))}$. This particle lies below the maximum at time $s \in (0,bt)$, depicted by the dashed line.

We now turn to VSBBM, where $\bar{A}(x) > x$ for $x \in (0,1)$. Fang and Zeitouni have shown in [62] (see [89] for a refinement) that if A is smooth and strictly concave,

$$m_t^A = \sqrt{2}\,t \int_0^1 \sqrt{A'(x)}\,\mathrm{d}x + O(t^{1/3}). \tag{1.3.38}$$

This means that the first order of m_t^A coincides with that of the CREM in (1.2.21) up to the rescaling for e^t instead of 2^N particles.

In [27], Bovier and Hartung studied two-speed BBM with speed function A_4 defined in (1.3.33). In the case $\gamma > 1$, they proved that the limiting extremal process is a concatenation of two rescaled extremal processes of standard BBM, which is also called *BBM cascade*. This gives rise to a similar localisation of extremal particles as for the GREM: An extremal particle at time t of such a two-speed BBM also was extremal (up to entropic repulsion) at the time of the speed change bt.

The previous results on the extremes of VSBBM are summarised in the phase diagram given in Figure 1.18. The extremes of other variants of BBM have also been studied, such as for BBM with absorption [19, 112], BBM in $\mathbb{R}^d$ [21, 78, 109], BBM with self-repulsion [30] or BBM with a spatially inhomogeneous branching rate [73]. (VS)BBM belongs to the class of *log-correlated fields*, where the correlations between “particles” are logarithmic compared to the size of the system. Also for other models of this class, deformations with different speed functions analogous to VSBBM have been studied, see [61, 90, 98] for the branching random walk and [13, 64–66] for the discrete Gaussian free field in dimension two.

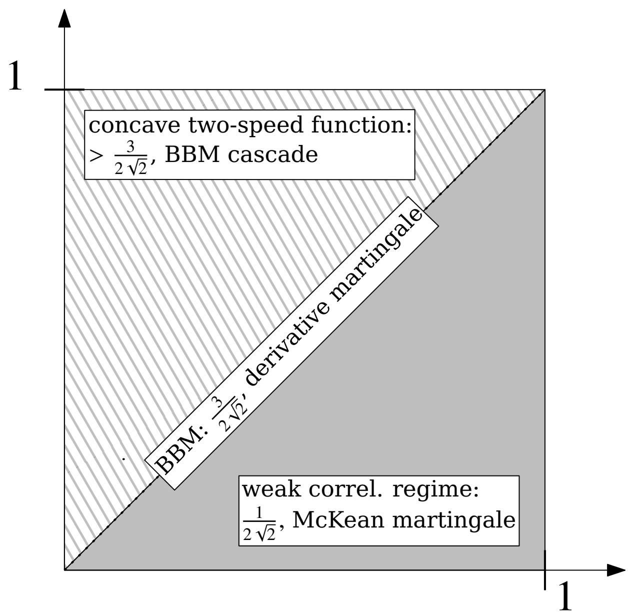

Figure 1.18: A phase diagram for VSBBM, depicting the prefactor of the logarithmic correction and the process which appears in the limiting extremal process.

Fels and Hartung [66] have shown for that model in the weak correlation regime that the limiting extremal process is a decorated Poisson process, as for VSBBM. There are other log-correlated fields, where the order of the maximum exhibits similarities to that of BBM. Examples of those are random approximations of the Riemann zeta function on the critical axis on a short random interval [7, 11, 12, 38], cover times of a two-dimensional torus by Brownian motion [18, 52] or characteristic polynomials of random matrices, see [6, 48, 101] or, for an overview, [5, 14].

1.3.5 Outlook to results

In the previous subsection, standard BBM has been identified as the boundary case where the correlations start to influence the first order of the maximum. The subleading order of m_t^A is discontinuous between the weak correlation regime $\left(\frac{1}{2\sqrt{2}} \ln t\right)$ and standard BBM $\left(\frac{3}{2\sqrt{2}} \ln t\right)$. These observations raise the following questions:

1. How can we interpolate between the weak correlation regime and standard BBM?
2. How can we interpolate between standard BBM and VSBBM with $\bar{A}(x) > x$ for all $x \in (0, 1)$?

These questions were answered by Bovier and Hartung [29] for certain two-speed functions; see also [79] for an answer to Question 1 for the BRW with Gaussian increments. Bovier and Hartung choose a family of speed functions $(A_t^{\pm})_{t>0}$ which depends on the time-horizon t. Recall the end of Section 1.3.3: A VSBBM is a family of processes indexed by the time-horizon t, so a choice of a family of speed functions depending on t is possible.

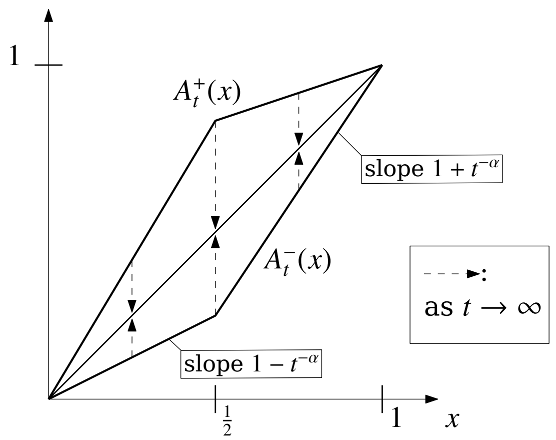

Figure 1.19: The speed functions A_t^+ and A_t^-. The arrows illustrate that these speed functions converge to the identity function as $t \uparrow \infty$.

To answer Question 1 for two-speed functions, Bovier and Hartung investigate the family $(A_t^-)_{t>0}$, which satisfies

$$\tfrac{\partial}{\partial_x} A_t^-(x) = \begin{cases} 1 - t^{-\alpha}, & 0 < x < \frac{1}{2}, \\ 1 + t^{-\alpha}, & \frac{1}{2} < x < 1, \end{cases} \tag{1.3.39}$$

for each $t > 0$, where $\alpha \in (0, \frac{1}{2})$. This is a family of two-speed functions, which approximate the identity function from below, as $t \uparrow \infty$, at a certain speed controlled by the parameter α. To answer Question 2 for two-speed functions, $(A_t^+)_{t>0}$ is chosen as the counterpart of $(A_t^-)_{t>0}$, converging to the identity function from above. Namely, we have

$$\tfrac{\partial}{\partial_x} A_t^+(x) = \begin{cases} 1 + t^{-\alpha}, & 0 < x < \frac{1}{2}, \\ 1 - t^{-\alpha}, & \frac{1}{2} < x < 1, \end{cases} \tag{1.3.40}$$

for each $t > 0$. See Figure 1.19 for an illustration of both families of speed functions.

Let $\left(x_k^\pm(t)\right)_{k \le n(t), t>0}$ be a VSBBM with speed functions $(A_t^+)_{t>0}$ (the + case) or $(A_t^-)_{t>0}$ (the − case). We set

$$m_t^\pm := \begin{cases} \sqrt{2}\, t \int_0^1 \sqrt{\tfrac{\partial}{\partial_x} A_t^+(x)}\, \mathrm{d}x - \frac{3}{2\sqrt{2}}(2 - 2\alpha) \ln t, & \text{in the } + \text{case}, \\ \sqrt{2}\, t - \frac{1+4\alpha}{2\sqrt{2}} \ln t, & \text{in the } - \text{case}. \end{cases} \tag{1.3.41}$$

The main results of [29] are:

- For a constant $C > 0$ which is described in more detail in [29, Theorem 1.1],

$$\lim_{t \uparrow \infty} \mathbb{P}\left(\max_{k \le n(t)} x_k^\pm(t) - m_t^\pm \le y \right) = \begin{cases} \mathbb{E}\left[\mathrm{e}^{-\frac{2}{\sqrt{\pi}} CZ \mathrm{e}^{-\sqrt{2} y}} \right], & \text{in the } + \text{case}, \\ \mathbb{E}\left[\mathrm{e}^{-CZ \mathrm{e}^{-\sqrt{2} y}} \right], & \text{in the } - \text{case}. \end{cases} \tag{1.3.42}$$

- The limiting extremal process of $\left(x_k^\pm(t)\right)_{k \le n(t), t>0}$ is the same as that of standard BBM, up to the constant factor $\frac{2}{\sqrt{\pi}}$ in front of CZ in the + case.

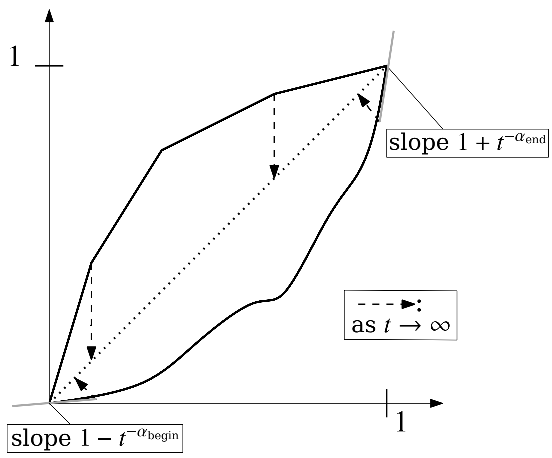

Figure 1.20: An illustration of time-dependent speed functions which satisfy Assumption 2.1.1 above the identity function (dotted line) or 2.1.2 below the identity function.

This means that in the − case, the logarithmic correction interpolates between that of the weak correlation regime, $\frac{1}{2\sqrt{2}} \ln t$, and that of standard BBM, $\frac{3}{2\sqrt{2}} \ln t$. In the + case, the first order of the maximum has the same form as that in (1.3.38). However, the extremal process in the +-case has a different structure than that of the two-speed case above the identity from [27], which we mentioned in the previous section.

To gain a better understanding of the interpolation between the different regimes, it is helpful to prove a generalisation of the results of [29]. This is done in Chapter 2 for general speed functions approximating the identity function from below, called *Case B*. Chapter 2 is excerpted from a joint paper [4] with Bovier, Gros and Hartung, which also contains a generalisation of the + case called *Case A*. The proofs in Case A will be included in Gros's dissertation and are therefore excluded from Chapter 2. We present here the results for both cases to provide a complete overview of the results of [4].

For a precise description of the speed functions $(A_t)_{t>0}$ of Chapter 2, we refer to Assumptions 2.1.1 and 2.1.2. Here, we provide an illustration by Figure 1.20 and by the following informal description:

- In the case above the identity (Case A), for each $t > 0$, A_t is a concave speed function with $A_t(x) > x$ for all $x \in (0, 1)$ consisting of $M \in \mathbb{N}$ linear pieces. As $t \uparrow \infty$, A_t converges to the identity function. There are certain conditions on the differences of neighbouring slopes.
- In the case below the identity (Case B), there exist $\alpha_{\text{begin}}, \alpha_{\text{end}} \in (0, \frac{1}{2})$ so that for each $t > 0$, $A_t'(0) = 1 - t^{-\alpha_{\text{begin}}}$, $A_t'(1) = 1 + t^{-\alpha_{\text{end}}}$ and $A_t(x) < x$ for all $x \in (0, 1)$. There are certain conditions which ensure that $(A_t)_{t>0}$ can be approximated by piecewise linear speed functions and that A_t maintains a distance of order $\approx t^{-1/2}$ from the identity function except in the beginning and the end of $[0, 1]$.

The central results of Chapter 2, Theorem 2.1.3 and 2.1.4, are summarised in the following theorem. This thesis only contains the proofs for Case B.

Theorem 1.3.4 [See Theorem 2.1.3 and 2.1.4 in Chapter 2]**.** *Let* $\left(x_k^{A_t}(t)\right)_{k\le n(t),t>0}$ *be a VSBBM with speed functions* $(A_t)_{t>0}$ *as described above (or more precisely in Assumptions 2.1.1 and 2.1.2). Then,*

$$\lim_{t\uparrow\infty}\mathbb{P}\left(\max_{k\le n(t)} x_k^{A_t}(t)-m_t^{A_t}\le y\right)=\mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}}\right], \tag{1.3.43}$$

where in Case A,

$$m_t^{A_t}=\sqrt{2}\,t\int_0^1\sqrt{\tfrac{\partial}{\partial_x}A_t(x)}\,\mathrm{d}x+O(\ln t), \tag{1.3.44}$$

see (2.1.13) *for a precise description of the logarithmic correction. In Case B,*

$$m_t^{A_t}=\sqrt{2}\,t-\frac{1+2(\alpha_{\text{begin}}+\alpha_{\text{end}})}{2\sqrt{2}}\ln t. \tag{1.3.45}$$

Furthermore, in both cases, the limiting extremal process is the same as that of standard BBM.

Note that in Case B, the most important assumptions were regarding the slopes of A_t in 0 and 1. We only demand some distance between A_t and the identity function in Case B to ensure that the effect of entropic repulsion only occurs in the beginning and the end, but not at times in between. This simplifies the computations of Chapter 2. Thus, the following observation in the weak correlation regime carries over to Case B:

As $t\uparrow\infty$*, the only influence of* A_t *on* $\max_{k\le n(t)} x_k^{A_t}(t)$ *lies in* $A_t'(0)$ *and* $A_t'(1)$*.*

A key step in the proof of Theorem 1.3.4 is the localisation of extremal particles in Section 2.4. We refer to Figure 1.21 for an illustration of the localisation in Case A of an extremal particle of three-speed BBM at time t. At the times of the speed changes, the height difference of such an extremal particle to the currently highest particle is of slightly larger order than the height difference induced by entropic repulsion.

We now turn to the localisation results for three-speed BBM in Case B, see also Figure 1.22 for an illustration. In this setting, A_t has slope $1-t^{-\alpha_{\text{begin}}}$ on the first time-interval $(0,b_1(t))$ and slope $1+t^{-\alpha_{\text{end}}}$ on the third and last time-interval $(b_1(t)+b_2(t),t)$. The typical fluctuations at time $s<t$ of an extremal particle at time t from $s\mapsto\sqrt{2}\,tA_t(s/t)$ are at most of order $\sqrt{t(A_t(s/t)\wedge(1-A_t(s/t)))}$, as in the weak correlation regime.

Only at times significantly before the first speed change and significantly after the last speed change, this particle experiences entropic repulsion. This means that the localisation in Case B contains elements from the weak correlation regime (see also Figure 1.17) as well as from standard BBM (entropic repulsion, see also Figure 1.15).

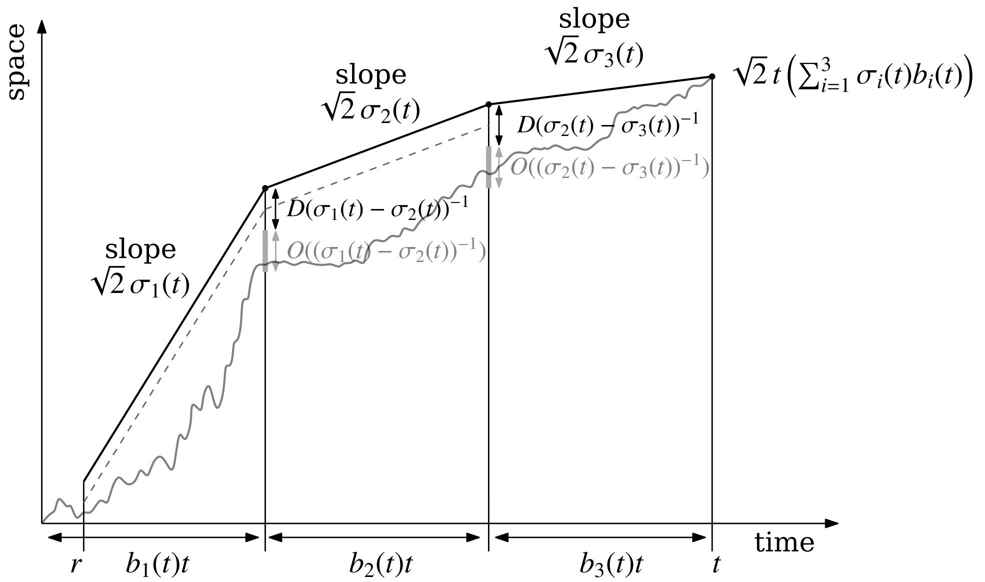

Figure 1.21: The typical path of an extremal particle at time t of three-speed BBM in Case A is depicted by the grey line. The speed function A_t has slope $\sigma_i^2(t)$ on the i-th interval, $i = 1, 2, 3$. The effect of entropic repulsion is depicted by the dashed line.

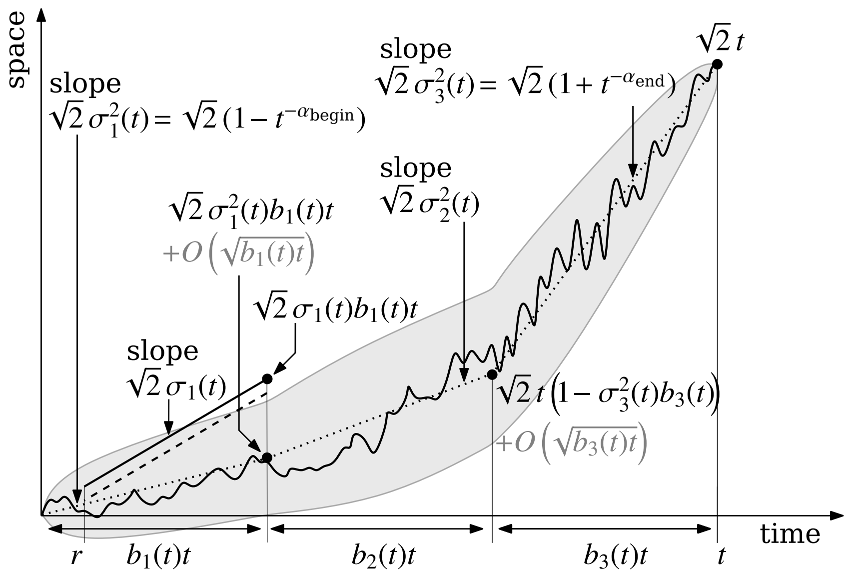

Figure 1.22: An extremal particle at time t of three-speed BBM in Case B follows the function $s \mapsto \sqrt{2}\,tA_t(s/t)$ (dotted line) with fluctuations (grey area) at most of order $\sqrt{t(A_t(s/t) \wedge (1 - A_t(s/t)))}$. The effect of entropic repulsion is depicted by the dashed line.

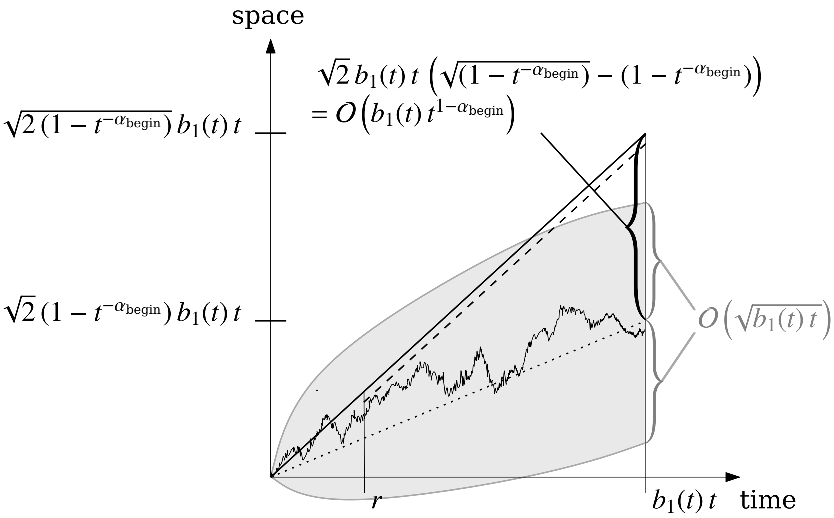

Figure 1.23: This is an extract of Figure 1.22 up to time $b_1(t)\,t$, where in $b_1(t)$, the first speed change of A_t occurs. The fluctuations (grey area) from $s \mapsto \sqrt{2}\,tA_t(s/t)$ (dotted line) ensure that the particle at time $b_1(t)\,t$ does not reach the dashed line, which depicts the effect of entropic repulsion.

We now zoom in on the localisation up to the first speed change $b_1(t)\,t$, see also Figure 1.23. A particle of three-speed BBM in Case B, which is extremal at time t, fluctuates at time $b_1(t)\,t$ at most $O\left(\sqrt{A_t(b_1(t))\,t}\right) = O\left(\sqrt{b_1(t)\,t}\right)$ from $\sqrt{2}\,A_t(b_1(t))\,t$. The height difference of this particle to $\sqrt{2(1-t^{-\alpha_{\text{begin}}})}\,b_1(t)\,t$, the extremal height of time $b_1(t)$, is thus at least

$$\begin{aligned}&\sqrt{2(1-t^{-\alpha_{\text{begin}}}}\,b_1(t)\,t-\sqrt{2}\,A_t(b_1(t))\,t-O\left(\sqrt{b_1(t)\,t}\right)\\&=\sqrt{2(1-t^{-\alpha_{\text{begin}}})}\,b_1(t)\,t-\sqrt{2}\,(1-t^{-\alpha_{\text{begin}}})\,b_1(t)\,t-O\left(\sqrt{b_1(t)\,t}\right)=O(\,b_1(t)\,t^{1-\alpha_{\text{begin}}}),\end{aligned} \tag{1.3.46}$$

with a Taylor approximation of the square root in 1 in the last step. Recall that we assumed that $\alpha_{\text{begin}} \in \left(0, \frac{1}{2}\right)$. Thus, $b_1(t)t^{1-\alpha_{\text{begin}}}$, the height difference from the maximal height at time $b_1(t)t$, is of larger order than $O\left(\sqrt{b_1(t)t}\right)$, the fluctuations from $\sqrt{2}\,A_t(b_1(t))\,t$. This is not the case if we allowed $\alpha_{\text{begin}} > \frac{1}{2}$. Thus, heuristically, in that case the extremal particle at time t would experience entropic repulsion on the whole time-interval $[0, b_1(t)t]$ (except in the very beginning). This means intuitively, that for $\alpha_{\text{begin}} > \frac{1}{2}$, the convergence of $(A_t)_{t>0}$ to A_2, the speed function of standard BBM, is fast enough to see no difference in the localisation in the first (and last) time interval. These will turn out in Chapter 2 to be the only relevant time-intervals for the asymptotics of the maximum.

We investigate how this weakened effect of entropic repulsion for $\alpha_{\text{begin}} \in \left(0, \frac{1}{2}\right)$ affects the logarithmic correction: In the heuristic explanation of the log-correction of standard BBM in (1.3.20)–(1.3.30), we replace the localisation condition "$B_s < \sqrt{2}\,s \;\forall s \in (r, t-r)$" by "$\sqrt{1-t^{-\alpha_{\text{begin}}}}B_s < \sqrt{2(1-t^{-\alpha_{\text{begin}}})}\,s \;\forall s \in (r, b_1(t)\,t]$ *and* $\sqrt{1-t^{-\alpha_{\text{begin}}}}\,B_{b_1(t)t} \approx \sqrt{2}\,(1-t^{-\alpha_{\text{begin}}})b_1(t)\,t$" combined with an analogous condition for $s \in [t - b_3(t)\,t, t-r)$. Instead of

$$\mathbb{P}\left(\mathfrak{z}_{0,0}^{t}(s) < 0 \;\forall s \in (r, t-r)\right) \approx O\left(\tfrac{1}{t}\right), \tag{1.3.47}$$

as in (1.3.26), we will have, denoting by $\mathfrak{z}^{\tilde{t}}_{a,\tilde{a}}$ a Brownian bridge from $a \in \mathbb{R}$ to $\tilde{a} \in \mathbb{R}$ in time $\tilde{t} > 0$,

$$\begin{aligned}&\mathbb{P}\left(\mathfrak{z}^{b_1(t)t}_{0,\sqrt{2}(1-t^{-\alpha_{\text{begin}}})b_1(t)\,t-\sqrt{1-t^{-\alpha_{\text{begin}}}}\,b_1(t)\,t}(s) < 0\ \forall s \in (r, b_1(t)\,t]\right)\\ &\quad\approx \mathbb{P}\left(\mathfrak{z}^{b_1(t)\,t}_{0,b_1(t)\,t^{1-\alpha_{\text{begin}}}}(s) < 0\ \forall s \in (r, b_1(t)\,t]\right) \approx O\left(t^{-\alpha_{\text{begin}}}\right),\end{aligned} \tag{1.3.48}$$

and analogously a term of order $t^{-\alpha_{\text{end}}}$ for the condition regarding $s \in [t - b_3(t)\,t, t - r)$. For standard BBM, the factor $\frac{1}{t}$ changes the prefactor in the log-correction from 1 to 3. The prefactor $1 + 2(\alpha_{\text{begin}} + \alpha_{\text{end}})$ for three-speed VSBBM in Case B comes from (1.3.48). This prefactor interpolates between 1 (weak correlation regime) and 3 (standard BBM) for $\alpha_{\text{begin}}, \alpha_{\text{end}} \in \left(0, \frac{1}{2}\right)$. We have seen that a heuristic explanation for this is the following: The parameters α_{begin} and α_{end} control how fast A_t approaches A_2, the speed function of standard BBM. Thus, they control the point of time when the localisation which also appears in the weak-correlation regime overrules that from standard BBM (entropic repulsion).

We summarise the results on the extremes of VSBBM of Section 1.3.4 and of this section in the phase diagram depicted in Figure 1.24.

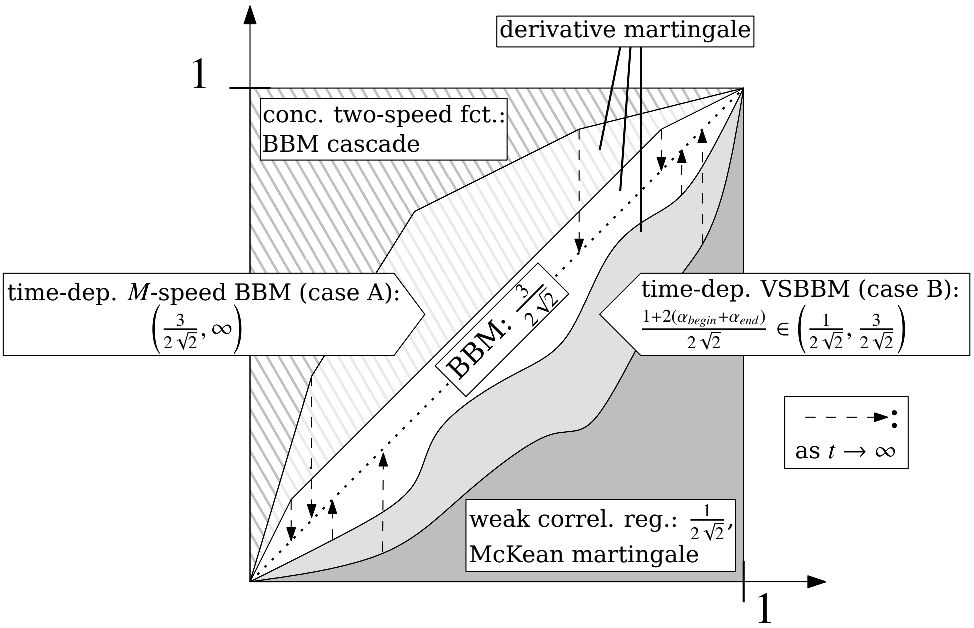

Figure 1.24: A phase diagram for VSBBM which extends Figure 1.18. We have added the cases of time-dependent VSBBM. The case of BBM (white) extends to VSBBM with speed functions $(A_t)_{t>0}$ converging to the identity function so fast that the localisation of standard BBM applies.

1.4 Mean field spin glasses and Hamilton-Jacobi equations

1.4.1 Hamilton-Jacobi equations

We are interested in solutions $f = f(t,x)\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^M \to \mathbb{R}$, $M \in \mathbb{N}$, of the *Hamilton-Jacobi equation* (HJE)

$$\frac{\partial}{\partial_t} f - H(\nabla_x f) = 0, \tag{HJE[M]}$$

where $H\colon \mathbb{R}^M \to \mathbb{R}$ locally Lipschitz continuous is called *nonlinearity* and $\nabla_x f = \left(\frac{\partial}{\partial_{x_1}} f, \ldots, \frac{\partial}{\partial_{x_M}} f\right)$ includes all the partial derivatives except the one w.r.t. t. A solution f of (HJE[M]) with *initial condition* Ψ furthermore satisfies $f(0,\cdot) = \Psi$.

In Chapter 3, we investigate under which conditions the free energy of a CREM can be described by a HJE in a generalised setting. Section 1.4.4 summarises the results of Chapter 3. A similar approach has been used for a variety of models [16, 17, 36, 70, 97] such as the SK model [93, 94, 96]. The textbook [58] by Mourrat and Dominguez provides a good overview on Hamilton-Jacobi equations and their application to the SK model. It is used as a reference several times throughout this section. Additionally, HJEs have been used to analyse problems of statistical interference [42, 44–46, 92, 93], see also Section 4 in [58] for an overview.

We now follow the structure of Section 3 of [58] to discuss the existence and uniqueness of solutions to (HJE[M]). Moreover, we examine their representation by variational formulas. To show the power of the method, we use it in the following Section 1.4.2 to compute the limiting free energy of the Curie-Weiss model. To be able to describe conditions for the uniqueness of a solution of (HJE[M]), we introduce the notion of *viscosity solutions*: Let $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^M \to \mathbb{R}$ be continuous.

- f is called *viscosity subsolution* of (HJE[M]) if for all $(t_*, x_*) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^M$ and all $\phi \in C^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}^M, \mathbb{R})$ so that (t_*, x_*) is a strict local maximum of $f - \phi$, we have

$$\frac{\partial}{\partial_t} \phi(t_*, x_*) - H(\nabla_x \phi)(t_*, x_*) \leq 0. \tag{1.4.1}$$

- f is called *viscosity supersolution* of (HJE[M]) if for all $(t_*, x_*) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^M$ and all $\phi \in C^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}^M, \mathbb{R})$ so that (t_*, x_*) is a strict local minimum of $f - \phi$, we have

$$\frac{\partial}{\partial_t} \phi(t_*, x_*) - H(\nabla_x \phi)(t_*, x_*) \geq 0. \tag{1.4.2}$$

- f is called *viscosity solution* of (HJE[M]) if it is both a viscosity sub- and supersolution of (HJE[M]).

Viscosity solutions of HJEs were first studied in [49, 51, 60], see also [50] for an overview. Note that if $f \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^M, \mathbb{R})$, then for each local extremum (t_*, x_*) of $f - \phi$,

$$\frac{\partial}{\partial_t} (f - \phi)(t_*, x_*) = 0, \tag{1.4.3}$$

and

$$\nabla_x(f-\phi)(t_*, x_*) = 0. \tag{1.4.4}$$

Thus, if $f \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^M, \mathbb{R})$ satisfies (HJE[M]) everywhere, then f is a viscosity solution of (HJE[M]). The *comparison principle* ensures the uniqueness of viscosity solutions up to the initial condition. We state it in the form of Corollary 3.6 in [58].

Proposition 1.4.1 (Comparison principle)**.** *Let* $u\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^M \to \mathbb{R}$ *be a viscosity subsolution of* (HJE[M]) *and* $v\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^M \to \mathbb{R}$ *be a viscosity supersolution of* (HJE[M])*. Then,*

$$\sup_{\mathbb{R}_{\geq 0}\times\mathbb{R}^M} (u-v) = \sup_{\{0\}\times\mathbb{R}^M} (u-v). \tag{1.4.5}$$

Thus, if u is a viscosity subsolution, v a viscosity solution and w a viscosity supersolution of HJE[M] with the same initial condition Ψ, then $u \leq v \leq w$. In particular, two viscosity solutions u and v of (HJE[M]) with the same initial condition Ψ are equal. This demonstrates that the concept of viscosity solutions provides the uniqueness of solutions of (HJE[M]). The strength of this concept is that one does not need to evaluate the partial derivatives of f, so it is applicable to functions which are not in $C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^M, \mathbb{R})$.

For convex nonlinearities or convex initial conditions, viscosity solutions of (HJE[M]) can be represented by the following variational formulas. This dates back to the work of Hopf [74] and Lax [85]. These variational formulas were proven to be viscosity solutions of HJEs in [15, 86, 87].

Theorem 1.4.2 [58, Theorem 3.8 and 3.13, Proposition 3.18]**.**
If H is convex, then the Hopf-Lax formula

$$f(t,x) := \sup_{y\in\mathbb{R}^M} \inf_{p\in\mathbb{R}^M} \Big(\Psi(y) + \langle p,\ x-y\rangle_{\mathbb{R}^M} + tH(p)\Big) \tag{1.4.6}$$

describes the unique viscosity solution of (HJE[M]) *with initial condition* Ψ*. Here,* $\langle\cdot,\ \cdot\rangle_{\mathbb{R}^M}$ *denotes the standard scalar product on* $\mathbb{R}^M$*.*

If Ψ *is convex, then the* Hopf formula

$$f(t,x) := \sup_{p\in\mathbb{R}^M} \inf_{y\in\mathbb{R}^M} \Big(\Psi(y) + \langle p,\ x-y\rangle_{\mathbb{R}^M} + tH(p)\Big) \tag{1.4.7}$$

describes the unique viscosity solution of (HJE[M]) *with initial condition* Ψ*.*

If both H and Ψ *are convex, then* (1.4.6) *and* (1.4.7) *coincide.*

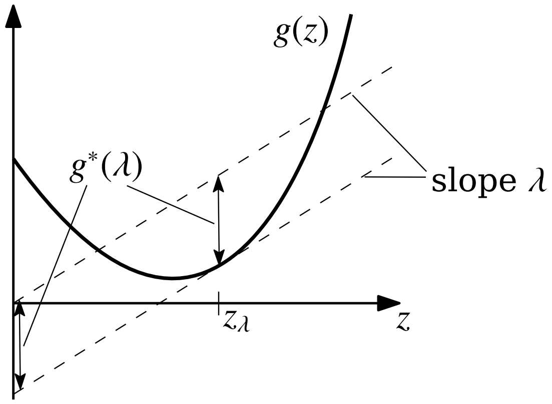

Figure 1.25: The convex dual $g^*(\lambda)$ of a convex function $g\colon \mathbb{R} \to \mathbb{R}$ is the greatest difference between the (dashed) line with slope λ and g. It is assumed in the point $z_\lambda = \sup\{z \in \mathbb{R}\colon g'(z) \le \lambda\}$.

The *convex dual* (or *convex conjugate, Legendre-Fenchel transformation*) of $g\colon \mathcal{H} \to \mathbb{R}$, where $\mathcal{H}$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, is defined by

$$g^*\colon \mathcal{H} \to \mathbb{R} \cup \{\infty\}, \\ \lambda \mapsto \sup_{y \in \mathcal{H}} \Big(\langle \lambda, y \rangle_{\mathcal{H}} - g(y) \Big). \tag{1.4.8}$$

We refer to Figure 1.25 for an illustration of g^* for a convex function $g\colon \mathbb{R} \to \mathbb{R}$.

We rewrite the Hopf-Lax formula (1.4.6) in terms of the convex dual as

$$\sup_{y \in \mathbb{R}^M} \inf_{p \in \mathbb{R}^M} \Big(\Psi(y) + \langle p, x - y \rangle_{\mathbb{R}^M} + tH(p) \Big) = \sup_{y \in \mathbb{R}^M} \Big(\Psi(y) - tH^* \Big(\tfrac{1}{t}(x - y) \Big) \Big). \tag{1.4.9}$$

Analogously, we rewrite the Hopf formula (1.4.7) as

$$\sup_{p \in \mathbb{R}^M} \inf_{y \in \mathbb{R}^M} \Big(\Psi(y) + \langle p, x - y \rangle_{\mathbb{R}^M} + tH(p) \Big) = \sup_{p \in \mathbb{R}^M} \Big(\langle p, x \rangle_{\mathbb{R}^M} + tH(p) - \Psi^*(p) \Big) = (tH - \Psi^*)^*(x). \tag{1.4.10}$$

1.4.2 The Hamilton-Jacobi approach for the free energy of the Curie-Weiss model

In this section, we use the techniques we introduced in the previous Section 1.4.1 to describe the free energy of the Curie-Weiss model as a solution of a HJE.

The Hamiltonian $H_{t,h,N}$ of the *Curie-Weiss model with external field* is defined by

$$\begin{aligned} H_{t,h,N}(\sigma) &:= \frac{t}{N} \sum_{i,j=1}^{N} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i \\ &= N\left(t\left(\tfrac{1}{N} \sum_{i=1}^{N} \sigma_i\right)^2 + h\left(\tfrac{1}{N} \sum_{i=1}^{N} \sigma_i\right)\right), \end{aligned} \tag{1.4.11}$$

for $\sigma \in \{-1,1\}^N$. The parameter $t \geq 0$ is the inverse temperature and $h \in \mathbb{R}$ is the intensity of an external magnetic field. In the case $h > 0$, h affects to which extent positive spins are energetically favoured over negative spins. Note that in the last line of (1.4.11), σ only appears in terms of $\frac{1}{N} \sum_{i=1}^{N} \sigma_i$, which is called *magnetisation density*.

The partition function of this model is denoted by

$$Z_{t,h,N} := \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_{t,h,N}(\sigma)\right) \tag{1.4.12}$$

and the free energy by

$$F_N(t,h) := \frac{1}{N} \ln\left(\tfrac{1}{2^N} Z_{t,h,N}\right). \tag{1.4.13}$$

The limiting free energy satisfies

$$F(t,h) = \lim_{N\uparrow\infty} F_N(t,h) = \sup_{p\in[-1,1]} \left(tp^2 + hp - s(p)\right), \tag{1.4.14}$$

where, with the convention $0 \ln 0 = 0$,

$$s(p) = \tfrac{1-p}{2} \ln(1-p) + \tfrac{1+p}{2} \ln(1+p), \qquad p \in [-1,1]. \tag{1.4.15}$$

For a proof of (1.4.14) with large deviation techniques, we refer to [59, Remark IV.4.2]. We follow the approach from Section 3 of [58] to prove (1.4.14) with the techniques introduced in Subsection 1.4.1.

First, we show that any subsequential limit of F_N is a viscosity solution of the HJE

$$\tfrac{\partial}{\partial_t} f - \left(\tfrac{\partial}{\partial_h} f\right)^2 = 0, \tag{1.4.16}$$

with initial condition $\Psi(h) = \ln\cosh(h)$. For each $N \in \mathbb{N}$, by binomial expansion,

$$Z_{0,h,N} = \sum_{\sigma\in\{-1,1\}^N} \exp\left(h \sum_{i=1}^{N} \sigma_i\right) = \sum_{k=0}^{N} \binom{N}{k} e^{hk} e^{-h(N-k)} = \left(e^h + e^{-h}\right)^N, \tag{1.4.17}$$

so

$$F_N(0,h) = \frac{1}{N}\ln\left(\frac{1}{2^N}Z_{0,h,N}\right) = \ln\left(\frac{e^h+e^{-h}}{2}\right) = \ln\cosh(h). \tag{1.4.18}$$

Thus, any subsequential limit f of F_N satisfies

$$f(0,h) = \Psi(h) = \ln\cosh(h). \tag{1.4.19}$$

We get for the partial derivatives of F_N that

$$\begin{aligned}\frac{\partial}{\partial_t}F_N(t,h) &= \frac{1}{N}\frac{\partial}{\partial_t}\ln\left(\frac{1}{2^N}Z_{t,h,N}\right)\\ &= \frac{1}{N}\frac{1}{Z_{t,h,N}}\frac{\partial}{\partial_t}Z_{t,h,N}\\ &= \frac{1}{N}\frac{1}{Z_{t,h,N}}\sum_{\sigma\in\{-1,1\}^N}\exp\left(H_{t,h,N}(\sigma)\right)\frac{\partial}{\partial_t}H_{t,h,N}(\sigma)\\ &= \frac{1}{Z_{t,h,N}}\sum_{\sigma\in\{-1,1\}^N}\exp\left(H_{t,h,N}(\sigma)\right)\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right)^2\\ &= \left\langle\!\!\left\langle\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right)^2\right\rangle\!\!\right\rangle_{t,h,N},\end{aligned} \tag{1.4.20}$$

where $\langle\!\langle g(\sigma)\rangle\!\rangle_{t,h,N}$ denotes the average of a function $g\colon\ \{-1,1\}^N \to \mathbb{R}$ w.r.t. the Gibbs measure $\mu_{t,h,N}(\sigma) = \frac{1}{Z_{t,h,N}}\exp\left(H_{t,h,N}(\sigma)\right)$. As in (1.4.20), we get that

$$\frac{\partial}{\partial_h}F_N(t,h) = \frac{1}{N}\frac{1}{Z_{t,h,N}}\sum_{\sigma\in\{-1,1\}^N}\exp\left(H_{t,h,N}(\sigma)\right)\frac{\partial}{\partial_h}H_{t,h,N}(\sigma) = \left\langle\!\!\left\langle\frac{1}{N}\sum_{i=1}^N\sigma_i\right\rangle\!\!\right\rangle_{t,h,N}, \tag{1.4.21}$$

and, with the quotient rule, that

$$\frac{1}{N}\frac{\partial^2}{\partial_h^2}F_N(t,h) = \left\langle\!\!\left\langle\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right)^2\right\rangle\!\!\right\rangle_{t,h,N} - \left\langle\!\!\left\langle\frac{1}{N}\sum_{i=1}^N\sigma_i\right\rangle\!\!\right\rangle_{t,h,N}^2 = \frac{\partial}{\partial_t}F_N(t,h) - \left(\frac{\partial}{\partial_h}F_N(t,h)\right)^2. \tag{1.4.22}$$

In particular, the first order partial derivatives of F_N are bounded by 1 for each choice of the parameters N,t,h. The mean-value theorem implies the locally uniform equicontinuity of $(F_N)_{N\in\mathbb{N}}$. Furthermore, we have seen in (1.4.18) that $F_N(0,\cdot)$ does not depend on N, so $(F_N)_{N\in\mathbb{N}}$ is locally uniformly bounded. Thus, by the Arzelà-Ascoli theorem, there exists a subsequence $(F_{N_k})_{k\in\mathbb{N}}$ of $(F_N)_{N\in\mathbb{N}}$ which has a limit f in locally uniform convergence as $k\uparrow\infty$. We first show that f is a viscosity subsolution of (3). Let $(t_*,h_*)\in\mathbb{R}_{\geq 0}\times\mathbb{R}$ and $\phi\in C^\infty(\mathbb{R}_{\geq 0}\times\mathbb{R}^M,\mathbb{R})$ so that (t_*,h_*) is a strict local maximum of $f-\phi$. By the locally uniform convergence of $(F_{N_k})_{k\in\mathbb{N}}$ to f (see [58, Exercise 3.1] for a detailed argumentation), there exists a sequence $(t_k,h_k)\in\mathbb{R}_+\times\mathbb{R}$ so that $(t_k,h_k)\to(t_*,h_*)$ as $k\uparrow\infty$ and $F_{N_k}-\phi$ has a local maximum in (t_k,h_k). The latter condition implies that

$$\frac{\partial}{\partial_t}F_{N_k}(t_k,h_k) = \frac{\partial}{\partial_t}\phi(t_k,h_k), \qquad \frac{\partial}{\partial_h}F_{N_k}(t_k,h_k) = \frac{\partial}{\partial_h}\phi(t_k,h_k) \qquad \text{and} \qquad \frac{\partial^2}{\partial_h^2}F_{N_k}(t_k,h_k) \leq \frac{\partial^2}{\partial_h^2}\phi(t_k,h_k). \tag{1.4.23}$$

From this and (1.4.22) follows that

$$\frac{\partial}{\partial_t}\phi(t_k,h_k) - \left(\frac{\partial}{\partial_h}\phi(t_k,h_k)\right)^2 = \frac{\partial}{\partial_t}F_{N_k}(t_k,h_k) - \left(\frac{\partial}{\partial_h}F_{N_k}(t_k,h_k)\right)^2 = \frac{1}{N_k}\frac{\partial^2}{\partial_h^2}F_{N_k}(t_k,h_k) \le \frac{1}{N_k}\frac{\partial^2}{\partial_h^2}\phi(t_k,h_k). \tag{1.4.24}$$

Taking the limit $k \uparrow \infty$, the right-hand side of (1.4.24) vanishes since $((t_k,h_k))_{k\in\mathbb{N}}$ converges to (t_*,h_*) and ϕ is smooth. This gives

$$\frac{\partial}{\partial_t}\phi(t_*,h_*) - \left(\frac{\partial}{\partial_h}\phi(t_*,h_*)\right)^2 \le 0. \tag{1.4.25}$$

Hence, f is a viscosity subsolution of (1.4.16). One proves analogously that it is also a viscosity supersolution of (1.4.16). With (1.4.20), we conclude that f is a viscosity solution of (1.4.16) with initial condition Ψ. Since Ψ is convex, we get from the Hopf formula (1.4.7) and (1.4.10) that any subsequential limit f of F_N satisfies

$$f(t,h) = \sup_{p\in\mathbb{R}}\left(tp^2 + hp - \Psi^*(p)\right). \tag{1.4.26}$$

Using the definition of the convex dual, one sees that

$$\Psi^*(p) = \begin{cases} s(p), & p \in [-1,1], \\ \infty, & \text{otherwise}, \end{cases} \tag{1.4.27}$$

where s is defined in (1.4.15). Thus, (1.4.26) and (1.4.14) coincide.

1.4.3 The Ruelle cascades

In the previous sections, the limiting free energy $F(t,h)$ of the Curie-Weiss model is endowed with the arguments $t \ge 0$, which is the inverse temperature, and $h \in \mathbb{R}$, which controls the influence of an external magnetic field. In a more general setting, one inspects the *enriched free energy* $F_N(t,h)$, where h is a multidimensional or infinite-dimensional parameter which controls the influence of a term which is added to the Hamiltonian of the model. This term is also called *enrichment*.

In this section, we present a definition of the *Ruelle cascades*, following Section 5.6 of [58]. For the SK model, it has been shown in [93, 94, 96] that an enrichment containing the Ruelle cascades constitutes an enriched free energy which satisfies a HJE. These cascades were first described in [105]. They appear in the limit of the point process of the Gibbs masses not only for the GREM, as mentioned in Section 1.2.5, but also for the SK model, see [99]. Therefore, for the CREM, we choose an enrichment based on Ruelle cascades as well. We introduce this enrichment in the following Subsection 1.4.4.

Let $M \in \mathbb{N}$ and $0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1$. The *Ruelle cascades* (or *Poisson-Dirichlet cascades*) with M levels are $(v_\alpha)_{\alpha\in\mathbb{N}^M}$, the point masses of a random probability measure on $\mathbb{N}^M$ which is constructed iteratively in the following way: We denote by

$$u_{(1)} > u_{(2)} > \dots \tag{1.4.28}$$

the ordered atoms of a Poisson point process on $\mathbb{R}_{>0}$ with intensity $\zeta_1 y^{-1-\zeta_1}\mathrm{d}y$. For each $k \in 1, \dots, M-1$ and each $\gamma \in \mathbb{N}^k$ independently,

$$u_{(\gamma,1)} > u_{(\gamma,2)} > \dots \tag{1.4.29}$$

are sampled as the ordered atoms of a Poisson point process on $\mathbb{R}_{>0}$ with intensity $\zeta_{k+1} y^{-1-\zeta_{k+1}}\mathrm{d}y$. Recall that for $\alpha = (\alpha_1, \dots, \alpha_M)$, we write $\alpha|_k = (\alpha_1, \dots, \alpha_k)$. Then, we set

$$\begin{aligned} w_\alpha &:= \prod_{k=1}^{M} u_{\alpha|_k}, \\ v_\alpha &:= \frac{w_\alpha}{\sum_{\tilde{\alpha}\in\mathbb{N}^M} w_{\tilde{\alpha}}}, \end{aligned} \tag{1.4.30}$$

for $\alpha \in \mathbb{N}^M$ and call $(w_\alpha)_{\alpha\in\mathbb{N}^M}$ the *unnormalised weights* of $(v_\alpha)_{\alpha\in\mathbb{N}^M}$. Since $\sum_{\tilde{\alpha}\in\mathbb{N}^M} w_{\tilde{\alpha}}$ is finite with probability 1 (see for example [58, Lemma 5.23]), $(v_\alpha)_{\alpha\in\mathbb{N}^M}$ constitutes a well-defined probability measure on $\mathbb{N}^M$.

We view $\mathbb{N}^M$ as the leaves of the tree with vertices $\mathbb{T}_M := \varnothing \cup \bigcup_{j=1}^M \mathbb{N}^j$, which only has edges between $\gamma \in \mathbb{N}^k$ and $\tilde{\gamma} \in \mathbb{N}^j$ if $j = k+1$ and $\tilde{\gamma} = (\gamma, n)$ for some $n \in \mathbb{N}$. We assign an edge $\gamma \leftrightarrow (\gamma, n)$ the weight $u_{(\gamma,n)}$ from (1.4.29). Then, v_α is the normalised product of all weights on the path $\varnothing \leftrightarrow \alpha$. We refer to Figure 1.26 for an illustration for $M = 3$.

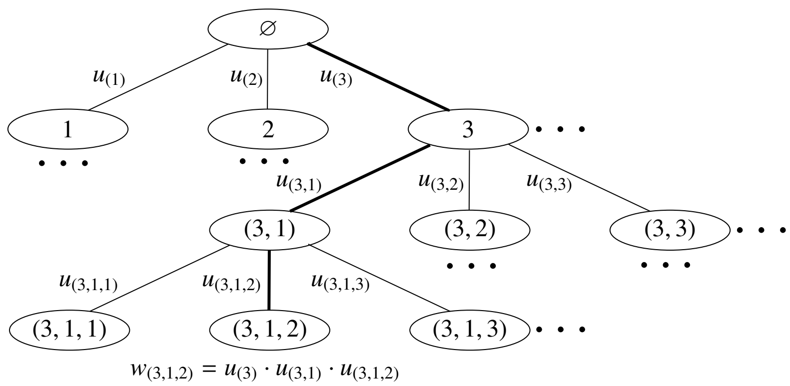

Figure 1.26: The unnormalised three-level Ruelle cascades weight $w_{(3,1,2)}$ is the product of the edge weights along the path $\varnothing \leftrightarrow (3, 1, 2)$, which is highlighted by the thicker edges. The dots indicate the omission of subtrees and vertices.

1.4.4 Outlook to a Hamilton-Jacobi approach for the free energy of the CREM

To study the limiting free energy of the CREM with the Hamilton-Jacobi approach, we introduce the following path spaces:

$$\begin{aligned} Q &:= \{\mathsf{q}\colon [0,1) \to \mathbb{R}_{\geq 0};\ \mathsf{q} \text{ is right-continuous and increasing}\}, \\ Q_p &:= Q \cap L_p([0,1),\mathbb{R}), \qquad \forall\, p \in [1,\infty]. \end{aligned} \tag{1.4.31}$$

We denote the norm on $L_p([0,1),\mathbb{R})$ by $\|\cdot\|_p$. Let $1 \leq p_1 < p_2$, then by Hölder's inequality, for $g \in L_{p_2}([0,1))$,

$$\|g\|_{p_1} = \left(\int_0^1 |g(u)|^{p_1}\,\mathrm{d}u\right)^{1/p_1} \leq \left(\big\| |g|^{p_1}\big\|_{p_2/p_1} \big\|\mathbb{1}_{[0,1)}\big\|_{p_2/(p_2-p_1)}\right)^{1/p_1} = \left(\big\| |g|^{p_1}\big\|_{p_2/p_1}\right)^{1/p_1} = \|g\|_{p_2}. \tag{1.4.32}$$

In particular, $Q_{p_2} \subset Q_{p_1}$ for $1 \leq p_1 < p_2$. This also holds for $p_2 = \infty$.

For $M \in \mathbb{N}$, we denote by $Q^{(M)}$ the set of all $\mathsf{q} \in Q$ which can be written as a step function with M jumps, i.e.

$$\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})}, \tag{1.4.33}$$

where

$$\begin{aligned} &0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1, \\ &0 = q_{-1} < q_0 < \cdots < q_{M-1} < q_M < \infty. \end{aligned} \tag{1.4.34}$$

We prove in Lemma 3.3.2 in Chapter 3 that $\bigcup_{M=0}^{\infty} Q^{(M)}$ is a dense subset of Q_1 w.r.t. $\|\cdot\|_1$. We explicitly define the enriched free energy only for $\mathsf{q} \in Q^{(M)}$. Namely, for $M \in \mathbb{N}$, let $(\zeta_k)_{k=0,\ldots,M+1}$ and $(q_k)_{k=-1,\ldots,M}$ be as in (1.4.34). Let $A\colon [0,1] \to [0,1]$ be a speed function. Recall that for $N \in \mathbb{N}$, the CREM $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ with speed function A is a centred Gaussian process with covariances

$$\mathbb{E}\left[H_N^A(\sigma)\, H_N^A(\tilde\sigma)\right] = NA\left(\tfrac{\sigma\wedge\tilde\sigma}{N}\right), \qquad \sigma,\tilde\sigma \in \{-1,1\}^N. \tag{1.4.35}$$

For $N \in \mathbb{N}$, $t \geq 0$, $\mathsf{q} \in Q^{(M)}$ so that

$$\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})}, \tag{1.4.36}$$

we set

$$F_N(t,\mathsf{q}) := -\frac{1}{N}\mathbb{E}\left[\ln\left(\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)\right],$$
$$H_N(t,\mathsf{q},\sigma,\alpha) := \sqrt{2t}\,H_N^A(\sigma) - Nt + \sqrt{2}\,Y_{\mathsf{q}}(\sigma,\alpha) - Nq_M, \tag{1.4.37}$$

where $(v_\alpha)_{\alpha\in\mathbb{N}^M}$ are the Ruelle cascades with parameters $(\zeta_k)_{k=0,\dots M+1}$ and $\left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}$ is a centred Gaussian process with covariances

$$\mathbb{E}[Y_{\mathsf{q}}(\sigma,\alpha)Y_{\mathsf{q}}(\tilde\sigma,\tilde\alpha)] = (\sigma\wedge\tilde\sigma)\,q_{\alpha\wedge\tilde\alpha}, \qquad \forall\,\sigma,\tilde\sigma\in\{-1,1\}^N,\ \alpha,\tilde\alpha\in\mathbb{N}^M. \tag{1.4.38}$$

Furthermore, we assume that the processes $(v_\alpha)_{\alpha\in\mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ and $\left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}$ are independent. Note that

$$\left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M} \overset{\mathrm{d}}{=} \left(\sum_{i=1}^N \sum_{k=0}^M (q_k-q_{k-1})^{1/2} z_{\sigma|_i,\alpha|_k}\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}, \tag{1.4.39}$$

where each $z_{\sigma|_i,\alpha|_k}$ is from a family of i.i.d. standard Gaussian random variables.

We prove in Proposition 3.3.1 in Chapter 3 that $F_N(t,\cdot)$ is Lipschitz continuous w.r.t. $\|\cdot\|_1$. Also, $\bigcup_{M=0}^\infty Q^{(M)}$ is a dense subset of Q_1 w.r.t. $\|\cdot\|_1$, so F_N has a unique Lipschitz continuous extension to $\mathbb{R}_{\geq 0}\times Q_1$. Namely, for $t\geq 0$ and $\mathsf{q}\in Q_1$,

$$F_N(t,\mathsf{q}) := \lim_{m\uparrow\infty} F_N(t,\mathsf{q}_m), \tag{1.4.40}$$

where $(\mathsf{q}_m)_{m\in\mathbb{N}}$ is a sequence in $\bigcup_{M=0}^\infty Q^{(M)}$ which converges to q w.r.t. $\|\cdot\|_1$. A central result of Chapter 3 is the following formula for the initial condition.

Theorem 1.4.3 [See Theorem 3.1.1 in Chapter 3]**.** *Let* $F_N\colon \mathbb{R}_{\geq 0}\times Q_1\to\mathbb{R}$ *be as in* (1.4.37) *and* (1.4.40)*. For each* $\mathsf{q}\in Q_1$*, we have*

$$\Psi(\mathsf{q}) := \lim_{N\uparrow\infty} F_N(0,\mathsf{q}) = -\ln 2 + \int_0^1 \left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u. \tag{1.4.41}$$

From this follows that Ψ is Lipschitz continuous with Lipschitz constant 1 and convex, see also Corollary 3.5.1 in Chapter 3.

To formulate a Hamilton-Jacobi equation on $\mathbb{R}_{\geq 0}\times Q_2$, we introduce the following notion for derivatives: Let $h\colon Q_2\to\mathbb{R}$ and $\mathsf{q}\in Q_2$. Assume that there exists a unique function $g\in L^2([0,1),\mathbb{R})$ such that for every $\tilde{\mathsf{q}}\in Q_2$, we have

$$h(\mathsf{q}) - h(\tilde{\mathsf{q}}) = \int_0^1 g(u)\,(\mathsf{q}(u)-\tilde{\mathsf{q}}(u))\,\mathrm{d}u + o(\|\mathsf{q}-\tilde{\mathsf{q}}\|_2), \tag{1.4.42}$$

as $\|\mathsf{q}-\tilde{\mathsf{q}}\|_2\downarrow 0$. Then we call g the *Fréchet derivative* of h in q and write $\nabla h = g$. We refer to Section 2.1C of [22] for an introduction to this topic. For each $t\geq 0$ and $f\colon\mathbb{R}_{\geq 0}\times Q_2\to\mathbb{R}$, the Fréchet derivative of $f(t,\cdot)$ in $\mathsf{q}\in Q_2$ is denoted by $\nabla_{\mathsf{q}} f(t,\mathsf{q})$.

We investigate in Chapter 3 the existence and uniqueness of solutions of the HJE

$$\begin{cases} \frac{\partial}{\partial_t} f(t,\mathsf{q}) - \int_0^1 A\left(\left(\nabla_{\mathsf{q}} f(t,\mathsf{q})\right)(u)\right)\, \mathrm{d}u = 0, & \forall\, (t,\mathsf{q}) \in \mathbb{R}_+ \times Q_2, \\ f(0,\mathsf{q}) = \Psi(\mathsf{q}), & \forall\, \mathsf{q} \in Q_2, \end{cases} \tag{$HJE[\mathsf{q}]$}$$

where A is a convex speed function and Ψ is as in Theorem 1.4.3. Furthermore, we compare solutions of (HJE[q]) to the limiting free enriched free energy $\lim_{N\uparrow\infty} F_N$.

Chen and Xia [47] generalised the classical results on viscosity solutions of HJEs on $\mathbb{R}_{\geq 0} \times \mathbb{R}^M$ mentioned in Section 1.4.1 (Comparison principle, Hopf-(Lax) formula) to HJEs on $\mathbb{R}_{\geq 0} \times C$. Here, C denotes a convex cone in an infinite-dimensional Hilbert space satisfying certain conditions. The setting of this thesis is $C = Q_2$.

A priori, we do not know whether for $u \in [0,1)$, $\left(\nabla_{\mathsf{q}} f(t,\mathsf{q})\right)(u)$ lies in the domain $[0,1]$ of A. For this reason, we use the following definition of a *regularisation* to extend the domain of A to $\mathbb{R}$. Let $A\colon [0,1] \to [0,1]$ be a Lipschitz continuous and convex speed function. A function $A_{\text{reg.}}\colon \mathbb{R} \to \mathbb{R}$ is called *regularisation* of A if $A_{\text{reg.}}$ coincides with A on $[0,1]$ and $A_{\text{reg.}}$ is Lipschitz continuous, convex and increasing on $\mathbb{R}$. This is an adaptation of [47, Definition 4.1] to the setting of this thesis, taking into account that speed functions are only defined on $[0,1]$, while analogous functions in [47] are defined on $\mathbb{R}$. The *nonlinearity* $\mathsf{H}\colon L_2([0,1),\mathbb{R}) \to \mathbb{R}$ for a regularisation $A_{\text{reg.}}$ of A is defined by

$$\mathsf{H}(g) = \inf\left\{\int_0^1 A_{\text{reg.}}(\mathsf{q}(u))\, \mathrm{d}u\colon\, \mathsf{q} \in Q_2 \cap \left(g + Q_2^*\right)\right\}, \qquad \forall\, g \in L_2([0,1)), \tag{1.4.43}$$

where

$$Q_2^* := \left\{\mathsf{p} \in L_2([0,1),\mathbb{R})\colon\, \int_0^1 \mathsf{p}(u)\mathsf{q}(u)\, \mathrm{d}u \geq 0\ \forall\, \mathsf{q} \in Q_2\right\}. \tag{1.4.44}$$

We prove in Lemma 3.6.11 that for all $\mathsf{q} \in Q_2$,

$$\mathsf{H}(\mathsf{q}) = \int_0^1 A_{\text{reg.}}(\mathsf{q}(u))\, \mathrm{d}u. \tag{1.4.45}$$

We call $f\colon \mathbb{R}_{\geq 0} \times Q_2 \to \mathbb{R}$ a *viscosity subsolution* of (HJE[q]) if there exists a regularisation $A_{\text{reg.}}$ of A so that f is a viscosity subsolution of

$$\frac{\partial}{\partial_t} f(t,\mathsf{q}) - \mathsf{H}\left(\nabla_{\mathsf{q}} f(t,\mathsf{q})\right) = 0, \qquad \forall\, (t,\mathsf{q}) \in \mathbb{R}_+ \times Q_2, \tag{1.4.46}$$

i.e. for all $(t_*,\mathsf{q}_*) \in \mathbb{R}_{\geq 0} \times Q_2$ and all $\phi \in C^\infty(\mathbb{R}_{\geq 0} \times Q_2, \mathbb{R})$ so that (t_*,q_*) is a strict local maximum of $f - \phi$, we have

$$\frac{\partial}{\partial_t} \phi(t_*,\mathsf{q}_*) - \mathsf{H}\left(\nabla_{\mathsf{q}} \phi(t_*,\mathsf{q}_*)\right) \leq 0. \tag{1.4.47}$$

The generalisation of the definitions of a viscosity (super-)solution is done analogously.

In Section 3.6, we use the results of [47], see also Theorem 3.6.10 for a reformulation in the notation of this thesis, to obtain uniqueness and the following variational representation of viscosity solutions of (HJE[q]). Uniqueness in this context is to be understood in the class of functions $f\colon \mathbb{R}_{\geq 0} \times Q_2 \to \mathbb{R}$ satisfying all of the following:

- $f(t,\cdot)$ is Lipschitz continuous,
- $\sup_{t>0,\, \mathsf{q}\in Q_2} \left|\frac{f(t,\mathsf{q})-f(t,0)}{t}\right| < \infty$,
- for all $t \geq 0$ and all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$,

$$\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\tilde{\mathsf{q}}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2 \quad \Rightarrow \quad f(t,\mathsf{q}) \geq f(t,\tilde{\mathsf{q}}). \tag{1.4.48}$$

Proposition 1.4.4 [See Proposition 3.6.12 in Chapter 3]**.** *Let $A\colon [0,1] \to [0,1]$ be a Lipschitz continuous and convex speed function. Let $A_{reg.}$ be a regularisation of A. Let Ψ be as in Theorem 1.4.3. Then there exists a unique viscosity solution f of (HJE[q]) with $f(0,\cdot) = \Psi$, which is given by the Hopf formula*

$$\begin{aligned} f(t,\mathsf{q}) &= \sup_{\mathsf{p}\in Q_\infty} \inf_{\mathsf{y}\in Q_\infty} \left(\Psi(\mathsf{y}) + \int_0^1 \mathsf{p}(u)(\mathsf{q}(u)-\mathsf{y}(u))\mathrm{d}u + t\int_0^1 A_{reg.}(\mathsf{p}(u))\mathrm{d}u\right) \\ &= \sup_{\substack{\mathsf{p}\in Q_\infty,\\ \|\mathsf{p}\|_\infty \leq 1,\, \|\mathsf{p}\|_1<1}} \left(t\int_0^1 A(\mathsf{p}(u))\mathrm{d}u + \int_0^1 \mathsf{p}(u)\mathsf{q}(u)\mathrm{d}u - \frac{\ln 2}{1-\|\mathsf{p}\|_1}\right), \end{aligned} \tag{1.4.49}$$

for all $(t,\mathsf{q}) \in \mathbb{R}_{\geq 0} \times Q_2$. In particular, f does not depend on the choice of $A_{reg.}$.

Recall from (1.4.8) that the convex dual Ψ_* of Ψ is defined by

$$\Psi_*(\mathsf{p}) = \sup_{\mathsf{q}\in Q_2}\left(\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u - \Psi(\mathsf{q})\right), \tag{1.4.50}$$

for all $\mathsf{p} \in Q_2$. The last line of (1.4.49) follows from the fact that for each $\mathsf{p} \in Q_2$,

$$\begin{aligned} \inf_{\mathsf{y}\in Q_\infty}\left(\Psi(\mathsf{y}) - \int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u\right) &= -\sup_{\mathsf{y}\in Q_\infty}\left(\int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u - \Psi(\mathsf{y})\right) = -\Psi_*(\mathsf{p}) \\ &= \begin{cases} -\infty, & \text{if } \|\mathsf{p}\|_1 \geq 1 \text{ or there exists } u\in[0,1) \text{ with } \mathsf{p}(u)>1,\\ -\frac{\ln 2}{1-\|\mathsf{p}\|_1}, & \text{otherwise, i.e. } \|\mathsf{p}\|_\infty \leq 1 \text{ and } \|\mathsf{p}\|_1 < 1, \end{cases} \end{aligned} \tag{1.4.51}$$

which we prove in Proposition 3.5.3, also using that by Corollary 3.5.4, we can replace "$\mathsf{q} \in Q_2$" by "$\mathsf{q} \in Q_\infty$" in (1.4.50).

Note that for $\mathsf{q}_0 \in Q_2$ with $\mathsf{q}_0(x) = 0$ for all $x \in [0,1)$, it holds $F_N(t,\mathsf{q}_0) = F_N^A(t)$, recalling that F_N^A is the free energy of the CREM with speed function A, see also (1.2.52). The Hamilton-Jacobi approach suggests that the viscosity solution f evaluated in (t,q_0) is a candidate for the limiting free energy. This is true for the CREM with convex and Lipschitz continuous speed function A, as the following central result of Chapter 3 shows.

Theorem 1.4.5 [See Theorem 3.1.2 in Chapter 3]**.** *If A is a Lipschitz continuous convex speed function, then for each $t \geq 0$,*

$$\lim_{N\uparrow\infty} F_N^A(t) = f(t, \mathsf{q}_0), \tag{1.4.52}$$

where f is the unique viscosity solution, which is obtained by Proposition 1.4.4, with $f(0,\cdot) = \lim_{N\uparrow\infty} F_N(0,\cdot)$.

It remains an open question whether Theorem 1.4.5 can be generalised as follows:

Conjecture 1.4.6 [See Conjecture 3.7.1 in Chapter 3]**.** *If A is a convex Lipschitz continuous speed function. The unique viscosity solution f of $(HJE[\mathsf{q}])$ with $f(0,\cdot) = \Psi$ from Proposition 1.4.4 satisfies*

$$f(t,\mathsf{q}) = \lim_{N\uparrow\infty} F_N(t,\mathsf{q}), \qquad \forall\, t \geq 0, \mathsf{q} \in Q_2, \tag{1.4.53}$$

where F_N is the enriched free energy of the CREM with speed function A, see (1.4.37) *and* (1.4.40).

In this thesis, we only provide a proof of the following inequality.

Theorem 1.4.7 [See Theorem 3.7.2 in Chapter 3]**.** *In the setting of Conjecture 1.4.6,*

$$f(t,\mathsf{q}) \leq \liminf_{N\uparrow\infty} F_N(t,\mathsf{q}), \qquad \forall\, t \geq 0, \mathsf{q} \in Q_2. \tag{1.4.54}$$

We conclude this thesis in Section 3.7 with a discussion of the *vector spin glass model*, which satisfies the analogue of Conjecture 1.4.6 for convex nonlinearities, as Chen and Mourrat proved in [43]. They also showed that for general nonlinearities, the limiting enriched free energy is a critical point of the *Hamilton-Jacobi functional*, which is, in the setting of Proposition 1.4.4,

$$\begin{aligned} \mathcal{J}_{t,\mathsf{q}}\colon Q_\infty \times Q_\infty &\to \mathbb{R}, \\ (\mathsf{p},\mathsf{y}) &\mapsto \Psi(\mathsf{y}) + \int_0^1 \mathsf{p}(u)\cdot(\mathsf{q}(u) - \mathsf{y}(u))\mathrm{d}u + t\int_0^1 A_{\mathrm{reg.}}(\mathsf{p}(u))\mathrm{d}u, \end{aligned} \tag{1.4.55}$$

for $t \geq 0$ and $\mathsf{q} \in Q_2$. The validity of a variational formula for general nonlinearities remains an open question.

2 From 1 to infinity: The log-correction for the maximum of variable-speed branching Brownian motion

This chapter is excerpted verbatim from joint work [4] with Anton Bovier[1], Annabell Gros[1] and Lisa Hartung[2]. The contents of Subsection 2.4.1 and 2.5.1 and parts of Section 2.6 will be included in Gros's thesis, so we omit them in this thesis. Besides that the only differences of this chapter to [4] consist of minor changes such as rephrasing references to omitted content. We explain any such change (besides formatting) in a footnote.

2.1 Introduction

2.1.1 Models and background

Variable speed branching Brownian motion [VSBBM] [27–29, 57, 62, 89] is a class of Gaussian processes X indexed by a continuous-time Galton-Watson-tree with branching rate one and offspring distribution $(p_k)_{k\in\mathbb{N}}$, where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$ and $\sum_{k=1}^{\infty} k(k-1)p_k < \infty$. X has mean zero and covariance

$$\mathbb{E}\left[x_i(s)\, x_j(r) \,\middle|\, \mathcal{F}_t^{\text{tree}}\right] = tA(t^{-1}d(x_i(s), x_j(r))), \tag{2.1.1}$$

where $\mathcal{F}_t^{\text{tree}}$ denotes the σ-algebra generated by the Galton-Watson tree up to time t and $d(x_i(s), x_j(r))$ denotes the time of the most recent common ancestor of the particles labelled i and j in the tree. $A\colon [0,1] \to [0,1]$ with $A(0) = 0$ and $A(1) = 1$ is a non-decreasing and right-continuous function, called *speed function*. It is the continuous-time analogon to Derrida's Generalised and Continuous Random Energy Model (GREM/CREM) [31, 32, 56, 79, 80]. VSBBM is a family $(\tilde{X}_t)_{t>0}$ of processes where $\tilde{X}_t = \{\tilde{x}_j^t(s)\colon\ j \le n(t), s \le t\}$ denotes the trajectories of all particles when the time horizon of the process is t. We write $\tilde{x}_j^t(s)$ for the position at time s of the ancestor of a particle labelled j at time t. For simplicity, we write $\tilde{x}_j(t) \equiv \tilde{x}_j^t(t)$. The trajectory $\{\tilde{x}_j^t(s)\colon\ s \le t\}$ for $j \le n(t)$ is denoted by $\tilde{x}_j$.

The case when $A(x) \equiv x$ is standard branching Brownian motion (BBM), the primary example of so-called log-correlated processes, a class of processes that contains, among others, branching random walk and the discrete Gaussian free field in dimension two. Also for these models, deformations with different speed functions analogous to VSBBM have been studied, see [61, 90, 98] for the branching random walk and [64–66] for the discrete Gaussian free field in dimension two.

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The extreme value statistics of standard BBM are by now well understood, see, e.g. [8–10, 33, 35, 40, 41, 84]. The extreme values of variable speed BBM exhibit different behaviour depending on the properties of the speed function.

(i) If $A(x) < x$ for all $x \in (0,1)$, then has been shown in [28] that $A'(0) < 1$ and $A'(1) > 1$ imply that to first sub-leading order,

$$\max_{j \le n(t)} \tilde{x}_j(t) \approx \sqrt{2}\, t - \frac{1}{2\sqrt{2}} \ln(t). \tag{2.1.2}$$

The order of the maximum is the same as in the case of independent particles.

(ii) If $A(x) = x$ for all $x \in [0,1]$, Bramson [33, 35] has proven that

$$\max_{j \le n(t)} \tilde{x}_j(t) \approx m(t) \equiv \sqrt{2}\, t - \frac{3}{2\sqrt{2}} \ln(t). \tag{2.1.3}$$

(iii) If $A(x) > x$ for some $x \in (0,1)$, then, to leading order,

$$\max_{j \le n(t)} \tilde{x}_j(t) \approx \sqrt{2}\, t \int_0^1 \sqrt{\mathrm{conc}(A)'(y)}\, \mathrm{d}y, \tag{2.1.4}$$

where $\mathrm{conc}(A)$ denotes the concave hull of the function A. The sub-leading order depends on the specific form of the covariance. If A is a piecewise linear function with slope σ_1^2 on the interval $[0,b), b \in (0,1)$ and σ_2^2 on $[b,1]$, it has been shown in [27, 61] that the log-correction is

$$-\frac{3}{2\sqrt{2}} \left(\sigma_1 \ln(bt) + \sigma_2 \ln((1-b)t)\right). \tag{2.1.5}$$

If A is strictly concave and continuous, the sub-leading order of the maximum is of order $t^{1/3}$ (see [62]; [89] for a refinement).

Note that, in the first and second cases, the concave hull of A is the identity function. Therefore, (2.1.4) holds in all three cases. We see that standard BBM is on the borderline where correlations begin to affect the properties of the extremes. Moreover, the sub-leading terms are discontinuous at the identity function. To analyse these discontinuities in more detail, [29] considered piecewise linear speed functions A_t such that

$$A_t'(x) = \begin{cases} 1 \pm t^{-\alpha}, & x < 1/2, \\ 1 \mp t^{-\alpha}, & 1/2 < x \le 1. \end{cases} \tag{2.1.6}$$

Another example was studied by Kistler and Schmidt [80]. In the present paper, we generalise the analysis in [29] to a wide class of speed functions. We distinguish between piecewise linear speed functions converging from above and a general class of speed functions converging from below. The case above the identity function is referred to as *Case A*, the other one as case *Case B*.

As explained above, the dependence of the properties of extremes on the speed function is very different in Cases A and B. In particular, in case A, the techniques of proofs are very different in the case when A is piecewise linear and when it is strictly concave. Therefore, in this paper we restrict ourselves to the piecewise linear case. The precise conditions on the speed functions considered are given below in Assumption 2.1.1. In Case B, the dependence on the speed function is only on the slopes of the speed function at zero and one, so in this case there is no need to distinguish between piecewise linear and other speed functions. The corresponding conditions are formulated in Assumption 2.1.2.

The following assumptions describe the class of speed functions we consider in Cases A and B.

Assumption 2.1.1 (Case A; $\mathbf{A_t(s) > s}$). *The family of speed functions $(A_t)_{t>0}$ with $A_t(s) > s$ for all $t > 0, s \in (0,1)$, satisfies:*

(i) The functions $(A_t)_{t>0}$ are piecewise linear and continuous. Their derivatives are given by

$$A_t'(s) = \sum_{k=1}^{\ell} \sigma_k^2(t) \mathbb{1}_{\left(\sum_{j=1}^{k-1} b_j(t), \sum_{j=1}^{k} b_j(t)\right)}(s), \tag{2.1.7}$$

where we call $\sigma_k \colon \mathbb{R}_+ \to \mathbb{R}_+, 1 \le k \le \ell$, velocities *and $b_k \colon \mathbb{R}_+ \to (0,1], 1 \le k \le \ell$, are called* interval lengths*. We assume $\sum_{k=1}^{\ell} b_k(t) = 1$ and $\sum_{k=1}^{\ell} \sigma_k^2(t) b_k(t) = 1$.*

(ii) The functions $(A_t)_{t>0}$ are concave and converge to the identity function, as $t \uparrow \infty$.

(iii) There exists $\beta \in (0, 1/2)$ such that, for all $1 \le k < \ell$, $\sqrt{\min\{b_k(t)t, b_{k+1}(t)t\}} \gg (\sigma_k(t) - \sigma_{k+1}(t))^{-1} \gg t^{\beta}$, as $t \uparrow \infty$.

Here and elsewhere, we use the notation

$$f(t) \ll g(t), \text{as } t \uparrow \infty, \quad \Leftrightarrow \quad \exists \varepsilon > 0 \colon \frac{t^{\varepsilon} f(t)}{g(t)} \downarrow 0, \text{as } t \uparrow \infty, \tag{2.1.8}$$

for functions $f, g \colon \mathbb{R}_+ \to \mathbb{R}$.

To illustrate the assumptions in Case A, consider a two-speed BBM with velocities $\sigma_1^2(t) = 1 + t^{-\alpha_1}$ on the interval $[0, b(t))$ and $\sigma_2^2(t) = 1 - t^{-\alpha_2}$ on $[b(t), 1]$, with $b(t) = 1/(t^{\alpha_2 - \alpha_1} + 1)$ and $\alpha_1, \alpha_2 > 0$. One checks that the assumptions are verified if $\alpha_1 + \alpha_2 < 1$.

Assumption 2.1.2 (Case B; $\mathbf{A_t(s) < s}$). *Let $\alpha_{\text{begin}}, \alpha_{\text{end}} \in (0, 1/2)$. The family of speed functions $(A_t)_{t>0}$ with $A_t(s) < s$, for all $t > 0, s \in (0,1)$, satisfies:*

(a) For each $t > 0$, there exist $b_{\text{begin}}(t) \in (0,1)$ and twice differentiable functions $\underline{B}_t, \overline{B}_t \colon [0,1] \to [0,1]$ with $\underline{B}_t(0) = \overline{B}_t(0) = 0$, for which each of the following hold:

(i) $1 \gg b_{\text{begin}}(t) \gg t^{\alpha_{\text{begin}} - 1/2}$ as $t \uparrow \infty$.

(ii) $\underline{B}_t'(0) = \overline{B}_t'(0) = 1 - t^{-\alpha_{\text{begin}}}$.

(iii) *On* $[0, b_{\text{begin}}(t)]$*, we have* $\underline{B}_t \leq A_t \leq \overline{B}_t$ *and the second derivatives of* $\underline{B}_t, \overline{B}_t$ *are both bounded by* $t^{-\alpha_{\text{begin}}} b_{\text{begin}}(t)^{-1}$ *in the sense of* (2.1.8).

(b) *For each* $t > 0$*, there exist* $b_{\text{end}}(t) \in (0,1)$ *and twice differentiable functions* $\underline{C}_t, \overline{C}_t : [0,1] \to [0,1]$ *with* $\underline{C}_t(1) = \overline{C}_t(1) = 1$*, such that:*

(i) $1 \gg b_{\text{end}}(t) \gg t^{\alpha_{\text{end}} - 1/2}$ *as* $t \uparrow \infty$.

(ii) $\underline{C}'_t(1) = \overline{C}'_t(1) = 1 + t^{-\alpha_{\text{end}}}$.

(iii) *On* $[1 - b_{\text{end}}(t), 1]$*, we have* $\underline{C}_t \leq A_t \leq \overline{C}_t$ *and the second derivatives of* $\underline{C}_t, \overline{C}_t$ *are both bounded by* $t^{-\alpha_{\text{end}}} b_{\text{end}}(t)^{-1}$ *in the sense of* (2.1.8).

(c) $\min_{s \in [b_{\text{begin}}(t), 1 - b_{\text{end}}(t)]} (s - A_t(s)) \gg t^{-1/2}$*, as* $t \uparrow \infty$.

In Case B, the slopes in 0 and 1 are given by $1 - t^{-\alpha_{\text{begin}}}$ and $1 + t^{-\alpha_{\text{end}}}$. The assumptions ensure that A_t can be well approximated by piecewise linear functions in 0 and 1, similarly to the assumptions in [28].

In this paper, we determine the limiting law of the rescaled maximum and the full extremal process in both cases. Recall that, for BBM, see [33, 84],

$$\lim_{t \uparrow \infty} \mathbb{P}\left(\max_{j \leq n(t)} x_j(t) - m(t) \leq y \right) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}} \right], \tag{2.1.9}$$

where $m(t)$ is the same as in (2.1.3), Z is the limit of the *derivative martingale*

$$Z(t) \equiv \sum_{j \leq n(t)} \left(\sqrt{2}\,t - x_j(t) \right) \mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,t - x_j(t) \right)}, \tag{2.1.10}$$

and C is a positive constant.

The extremal process of standard BBM [3, 10] is of the form

$$\lim_{t \uparrow \infty} \sum_{j \leq n(t)} \delta_{x_j(t) - m(t)} = \sum_{k,j} \delta_{\eta_k + \Delta_j^{(k)}}, \tag{2.1.11}$$

where the points η_k are the atoms of a Poisson point process with random intensity measure $CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$. The points $\Delta_j^{(k)}$ are the atoms of i.i.d. point processes $\Delta^{(k)}$, which arise as the limit in law as $t \uparrow \infty$ of

$$\sum_{j \leq n(t)} \delta_{\bar{x}_j(t) - \max_{i \leq n(t)} \bar{x}_i(t)}, \tag{2.1.12}$$

where $\bar{x}(t)$ are the points of standard BBM conditioned on the event $\max_{i \leq n(t)} x_i(t) \geq \sqrt{2}\,t$.

2.1.2 Results

To state our results, we define functions $m^{\pm}$, where the superscript + corresponds to Case A and the superscript − to Case B. Let

$$m^{+}(t) \equiv \sqrt{2}\, t \left(\sum_{k=1}^{\ell} \sigma_k(t) b_k(t) \right) - \frac{3}{2\sqrt{2}} \left(\sum_{k=1}^{\ell} \ln(b_k(t)t) + 2 \sum_{k=1}^{\ell-1} \ln\left(\pi^{1/6} (\sigma_k(t) - \sigma_{k+1}(t)) \right) \right), \tag{2.1.13}$$

and

$$m^{-}(t) \equiv \sqrt{2}\, t - \frac{1 + 2(\alpha_{\text{begin}} + \alpha_{\text{end}})}{2\sqrt{2}} \ln(t). \tag{2.1.14}$$

We notice that m^{+} depends on the details of the speed functions $(A_t)_{t>0}$ while m^{-} depends only on the rate of convergence of the speed functions near 0 and 1. The main results of this paper are the following two theorems.

Theorem 2.1.3. *Let $(\tilde{X}_t)_{t>0}$ be a family of variable speed BBMs with speed functions $(A_t)_{t>0}$ satisfying Assumption 2.1.1 (Case A) or Assumption 2.1.2 (Case B). Let C be the same positive constant as in* (2.1.9) *and Z the limit of the derivative martingale. Then, for all $y \in \mathbb{R}$,*

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{j \le n(t)} \tilde{x}_j(t) - m^{\pm}(t) \le y \right) = \mathbb{E}\left[\exp\left(-CZ\mathrm{e}^{-\sqrt{2}y} \right) \right], \tag{2.1.15}$$

where $m^{\pm} = m^{+}$ in Case A and $m^{\pm} = m^{-}$ in Case B.

Theorem 2.1.4. *Let $(\tilde{X}_t)_{t>0}$ and $m^{\pm}$ be as in Theorem 2.1.3. Let $\Delta_j^{(k)}$ be the atoms of the i.i.d. copies $\Delta^{(k)}$ of the limit of the point process described in* (2.1.12). *Then,*

$$\lim_{t\uparrow\infty} \sum_{j \le n(t)} \delta_{\tilde{x}_j(t) - m^{\pm}(t)} = \sum_{k,j} \delta_{\eta_k + \Delta_j^{(k)}}, \tag{2.1.16}$$

where η_k are the atoms of a Poisson point process with random intensity measure $CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$.

Observe that in Case A, we can obtain any factor between $\frac{3}{2\sqrt{2}}\ln(t)$ and ∞ in front of the logarithmic correction with an appropriate choice of A_t. More precisely, the logarithmic correction is of the form

$$-\frac{3}{2\sqrt{2}} f(A_t) \ln(t), \tag{2.1.17}$$

where f is a function taking values in $(1, \ell)$. This follows from estimating the terms in m^{+} from above and below with the bounds in Assumption 2.1.1.(iii) and using that $b_k(t) \le 1$ for $1 \le k \le \ell$. In Case B, the logarithmic correction in (2.1.14) only depends on the slope of A_t near 0 and near 1, while the behaviour of A_t away from 0 and 1 is negligible as long as A_t maintains a distance of order $t^{-1/2}$ from the identity function. The prefactor of the logarithmic correction interpolates between $\frac{1}{2\sqrt{2}}$ and $\frac{3}{2\sqrt{2}}$.

Both theorems above follow from the convergence of a class of Laplace functionals. A very nice characterisation of this fact is the following lemma from [20].

Lemma 2.1.5 [20, Lemma 4.4]. *Let $\mathcal{P}_t, \mathcal{P}_\infty$ be point processes on $\mathbb{R}$ such that almost surely, $\mathcal{P}((0,\infty)) < \infty$. The following are equivalent: As $t \uparrow \infty$,*

(i) $\mathcal{P}_t \to \mathcal{P}_\infty$ *and* $\max \mathcal{P}_t \to \max \mathcal{P}_\infty$ *in distribution.*
(ii) $\mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{P}_t(\mathrm{d}y)\right)\right] \to \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{P}_\infty(\mathrm{d}y)\right)\right]$ *for all $\phi \in C^\infty$ which are nondecreasing with support bounded from the left and for which there exists $a \in \mathbb{R}$ such that $\phi(x)$ is constant for $x > a$.*

Note that the fact that the functions ϕ are required to have support bounded only from the left allows to use Bramson's results on the convergence of the F-KPP equations directly and the fact that ϕ can be chosen to be smooth is convenient for applying Gaussian comparison. We prefer, however, to give the proof of Theorem 2.1.3 without using Laplace functionals, since we find this more easy to follow. The proof of Theorem 2.1.4 is then very similar and we only outline the main differences.

2.1.3 Outline of the paper

The remainder of this paper is organised as follows. Section 2.2 provides a collection of relevant notation. Section 2.3 recalls facts on Brownian bridges and the asymptotics of solutions of the F-KPP equation. A crucial step towards the proofs of Theorem 2.1.3 and 2.1.4 is to localise the positions of the ancestral paths of extremal particles. This is done in Section 2.4. Section 2.5 contains the proof of Theorem 2.1.3 and 2.1.4. First, we give a proof of Theorem 2.1.3, which is split between Case A in Subsection 2.5.1 and Case B in Subsection 2.5.2. In both cases, we prove the claim for piecewise linear speed functions. Some technical details are postponed to Appendix 2.6. In Case B, we extend the result to general speed functions with Gaussian comparison techniques. In Subsection 2.5.3, we describe how to modify the proof of Theorem 2.1.3 so it extends to the convergence of Laplace functionals. Applying Lemma 2.1.5 then completes the proof of Theorem 2.1.4.

2.2 Notation

In this section, we introduce notation for ℓ-speed BBM which is used throughout the paper. Denote by

$$a_k(t) \equiv \sum_{j=1}^{k} b_j(t), \quad a_0(t) = 0, \tag{2.2.1}$$

for all $1 \le k \le \ell$, the times at which speed changes occur. In the following we drop the t-dependence of the terms $\sigma_k(t), b_k(t)$ and $a_k(t)$ to shorten the notation.

It is convenient to express ℓ-speed BBM using standard BBMs. To do so, let $\{\tilde{x}_{i_k}^{i_1,\dots,i_{k-1}},$ $k \in \mathbb{N}, i_\ell \in \mathbb{N}\}$ be BBM with variance σ_k^2.

We use multiindices

$$\bar{i}_k \equiv i_1, \dots, i_k \tag{2.2.2}$$

and write $\tilde{x}_{i_1}^{\bar{i}_0} = \tilde{x}_{i_1}$. With this notation, we can rewrite variable speed BBM, as

$$\begin{aligned} &\left\{\tilde{x}_i^t(s) : 1 \le i \le n(t)\right\} \\ &= \left\{\sum_{j=1}^{k-1} \tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) + \tilde{x}_{i_k}^{\bar{i}_{k-1}}(s - a_{k-1}t) : 1 \le i_1 \le n(b_1 t), \dots, 1 \le i_k \le n^{\bar{i}_{k-1}}(b_k t)\right\}, \end{aligned} \tag{2.2.3}$$

for $1 \le k < \ell$ and for $s \in [a_{k-1}t, a_k t)$. The σ-algebra which is generated by all particles of ℓ-speed BBM up to time $s, s \le t$, is called $\mathcal{F}_s$. The path of the particle with position $\sum_{j=1}^{k} \tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t)$ at time $a_k t$ is abbreviated by $\sum_{j=1}^{k} \tilde{x}_{i_j}^{\bar{i}_{j-1}}$.

2.3 Preliminaries

In this section, we recall some results on branching Brownian motion and the F-KPP equation. We start with the fundamental connection between BBM and the F-KPP equation. Let $f : \mathbb{R} \to [0, 1]$ be a function and set

$$v(t, x) \equiv \mathbb{E}\left[\prod_{j=1}^{n(t)} f(x - x_k(t))\right]. \tag{2.3.1}$$

Then $u(t, x) = 1 - v(t, x)$ is the unique solution to the F-KPP equation

$$\partial_t u = \frac{1}{2}\partial_x^2 u + F(u), \tag{2.3.2}$$

with $F(u) = (1-u) - \sum_{k=1}^{\infty} p_k (1-u)^k$ and with initial condition $u(0, x) = 1 - f(x)$.

The following proposition describes the asymptotic behaviour of solutions of the F-KPP equation for large times.

Proposition 2.3.1 [29, 35]**.** *Let u be a solution to the F-KPP equation with initial condition satisfying*

(i) $0 \le u(0,x) \le 1$*;*
(ii) $\exists\, h > 0\colon\ \limsup_{t\uparrow\infty} \frac{1}{t} \ln\left(\int_t^{t(1+h)} u(0,y)\,\mathrm{d}y\right) \le -\sqrt{2}$*;*
(iii) $\exists\, c > 0, M > 0, N > 0\colon\ \int_x^{x+N} u(0,y)\,\mathrm{d}y > c \quad \forall\, x \le -M$*;*
(iv) $\int_0^\infty u(0,y)\, y\, \mathrm{e}^{2y}\,\mathrm{d}y < \infty$.

Then we have, for any function $z\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\uparrow\infty} z(t)/t = 0$,

$$\lim_{t\uparrow\infty} \mathrm{e}^{\sqrt{2}\, z(t)} \mathrm{e}^{(z(t))^2/2t} (z(t))^{-1} u\left(t, z(t) + \sqrt{2}\, t - \tfrac{3}{2\sqrt{2}} \ln(t)\right) = C, \tag{2.3.3}$$

where C is a strictly positive constant depending on the initial condition $u(0,\cdot)$.

Lemma 2.3.2 [35, Proposition 8.2]**.** *For $z\colon \mathbb{R}_+ \to [1,\infty)$ and t large enough, we have*

$$\mathbb{P}\left(\max_{j\le n(t)} x_j(t) > z(t) + \sqrt{2}\, t - \tfrac{3}{2\sqrt{2}} \ln(t)\right) \le C' z(t) \mathrm{e}^{-\sqrt{2}\, z(t)}, \tag{2.3.4}$$

where C' is a strictly positive constant independent of t.

Particles of standard BBM are unlikely to cross the barrier function with slope $\pm\sqrt{2}$.

Lemma 2.3.3. *For any $\varepsilon > 0$, there exists $r_0 < \infty$ such that for all $r > r_0$, for all t large enough,*

$$\mathbb{P}\left(\exists_{j\le n(t)} \exists_{s\in[r,t-r]} \colon |x_j(s)| > \sqrt{2}\, s\right) < \varepsilon, \tag{2.3.5}$$

Proof. In the proof of the convergence of the derivative martingale in [84], it is pointed out that

$$\liminf_{t\uparrow\infty} \min_{j\le n(t)} \left(\sqrt{2}\, t - x_j(t)\right) \uparrow \infty, \text{ a.s.}, \tag{2.3.6}$$

which implies (2.3.5). □

We state a version of Slepian's lemma [108] adapted to variable speed BBM.

Lemma 2.3.4 (Slepian's Lemma)**.** *Let* $(\hat{x}_k(t))_{k\le\hat{n}(t)}$ *and* $(\bar{x}_k(t))_{k\le\bar{n}(t)}$ *be the particle positions at time* t *of variable speed BBMs with speed functions* $\hat{A}$ *and* $\bar{A}$*.*
If $\hat{A} \le \bar{A}$*, then*

$$\mathbb{P}\left(\max_{k\le\hat{n}(t)} \hat{x}_k(t) > y\right) \ge \mathbb{P}\left(\max_{k\le\bar{n}(t)} \bar{x}_k(t) > y\right). \tag{2.3.7}$$

Proof. This follows from [26, Corollary 3.10]. □

We also recall two basic facts about Brownian bridges. We denote by $\mathfrak{z}^t_{a,b}$ a Brownian bridge starting in a and ending in b at time t.

Lemma 2.3.5 [28, Lemma 2.2]**.** *For any* $\gamma > 1/2$ *and for any* $\varepsilon > 0$*, there exists a constant* $r > 0$ *such that*

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\forall_{r\le s\le t-r}\colon |\mathfrak{z}^t_{0,0}(s)| < (s \wedge (t-s))^\gamma\right) > 1-\varepsilon. \tag{2.3.8}$$

Lemma 2.3.6 [35, Lemma 2.2]**.** *For any* $x, y > 0$ *holds*

$$\mathbb{P}\left(\forall_{0\le s\le t}\colon \mathfrak{z}^t_{0,0}(s) \le (sx + (t-s)y)/t\right) = 1 - \mathrm{e}^{-2xy/t} \le 2\,\frac{xy}{t}, \tag{2.3.9}$$

and asymptotic equality holds if $xy = o(t)$*.*

The next lemma allows to restrict events related to maxima to *likely* events.

Lemma 2.3.7 [29, Lemma 3.4]**.** *Let* x_j*,* $j = 1, \ldots, n$*, be path-valued random variables and* $\mathcal{L}$ *be an event on the set of paths such that, for any* $\varepsilon > 0$*,*

$$\mathbb{P}\left(\exists_{j\le n}\colon \{x_j(t) > y\} \wedge \{x_j \in \mathcal{L}\}\right) \ge \mathbb{P}\left(\exists_{j\le n}\colon x_j(t) > y\right) - \varepsilon. \tag{2.3.10}$$

Then

$$\left|\mathbb{P}\left(\max_{j\le n} x_j(t) \le y\right) - \mathbb{P}\left(\max_{j\le n: x_j\in\mathcal{L}} x_j(t) \le y\right)\right| \le \varepsilon. \tag{2.3.11}$$

2.4 Localisation of paths

An essential step in the proof of Theorem 2.1.3 is the control of the particle positions until time $a_{\ell-1}t$.

2.4.1 Localisation of paths in Case A

This subsection is not a part of this thesis.

2.4.2 Localisation of paths in Case B

In[3] this subsection, we prove localisations of an ℓ-speed BBM $(\bar{X}_t)_{t>0}$ with speed functions $(A_t)_{t>0}$ satisfying the following assumption.

Assumption 2.4.1. *Let $\alpha_1, \alpha_\ell \in (0, 1/2)$. We assume that the family of speed functions $(A_t)_{t>0}$ with $A_t(s) < s$ for all $t > 0, s \in (0, 1)$, satisfies:*

(i) The functions $(A_t)_{t>0}$ are piecewise linear and continuous*: Their derivatives are given by*

$$A_t'(s) = \sum_{k=1}^{\ell} \sigma_k^2 \mathbb{1}_{\left(\sum_{j=1}^{k-1} b_j, \sum_{j=1}^{k} b_j\right)}(s), \tag{2.4.1}$$

with velocities $\sigma_k\colon \mathbb{R}_+ \to \mathbb{R}_+, 1 \le k \le \ell$, and interval lengths $b_k\colon \mathbb{R}_+ \to (0, 1]$, $1 \le k \le \ell$. We assume for all $t > 0$ that $\sum_{k=1}^{\ell} b_k = 1$ and $\sum_{k=1}^{\ell} \sigma_k^2 b_k = 1$.

(ii) The velocity on the first interval satisfies $\sigma_1^2 = 1 - t^{-\alpha_1} + o(t^{-\alpha_1})$ and the interval length b_1 satisfies $1 \gg b_1 \gg t^{\alpha_1 - 1/2}$ as $t \uparrow \infty$.

(iii) The velocity on the last interval satisfies $\sigma_\ell^2 = 1 + t^{-\alpha_\ell} + o(t^{-\alpha_\ell})$ and the interval length b_ℓ satisfies $1 \gg b_\ell \gg t^{\alpha_\ell - 1/2}$ as $t \uparrow \infty$.

(iv) Minimum distance to the identity function: $\min_{s \in [b_1, 1-b_\ell]} (s - A_t(s)) \gg t^{-1/2}$ *as $t \uparrow \infty$.*

In Figure 2.1, we illustrate these localisation results for three-speed VSBBM.

We define the set

$$\mathcal{T}_{r_1, r_2, S, \gamma} \equiv \left\{X\colon \forall_{r_1 \le s \le r_2}\colon \left|X(s) + S - \sqrt{2}\, t A_t\left(s/t\right)\right| < \left(A_t\left(s/t\right) \wedge \left(1 - A_t\left(s/t\right)\right)\right)^{\gamma} t^{\gamma}\right\}, \tag{2.4.2}$$

where $0 \le r_1 \le r_2 \le t, S \in \mathbb{R}$ and $\gamma > 0$. Between times $b_1 t$ and $(1 - b_\ell)t$, extremal particles fluctuate like a time-inhomogeneous Brownian bridge with time-inhomogeneity controlled by the speed function.

[3] We moved Assumption 2.4.1 from Subsection 2.5.2 to this subsection and adapted this sentence. This corrects the following error of [4]: In this subsection, we prove localisation results only for $(\bar{X}_t)_{t>0}$ but not for the variable speed BBM $(\tilde{X}_t)_{t>0}$ whose speed functions satisfy Assumption 2.1.2.

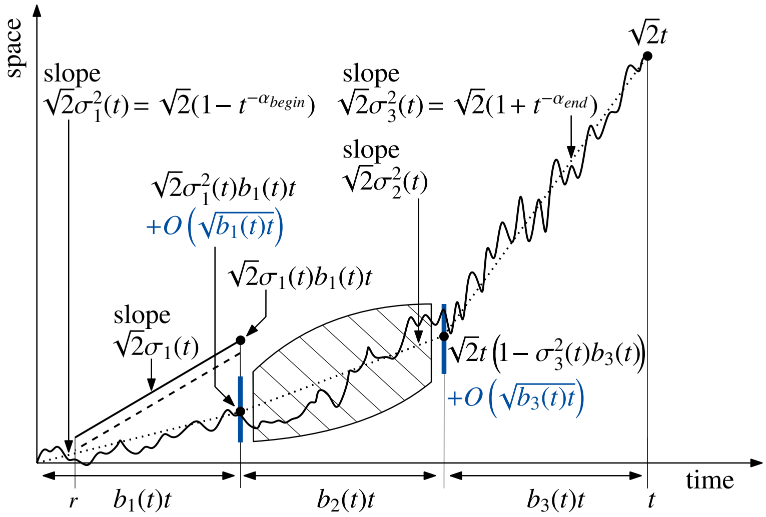

Figure 2.1: Localisation of an extremal particle of three-speed BBM in Case B. The dotted line depicts the function $\sqrt{2}\,A_t(s/t)t$ for $s \in [0,t]$. The dashed box depicts fluctuations of order $\sqrt{\min\{A_t(s/t)t, t - A_t(s/t)t\}}$. The dashed line depicts the effect of entropic repulsion.

Proposition 2.4.2 (Fluctuations in the middle part)**.** *For any $y \in \mathbb{R}$, any $\varepsilon > 0$, any $\gamma > 1/2$ and for all t large enough,*

$$\mathbb{P}\left(\exists_{j \le n(t)} : \left\{\bar{x}_j(t) > m^-(t) - y\right\} \wedge \left\{\bar{x}_j \notin \mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}\right)\right) < \varepsilon. \tag{2.4.3}$$

We first control the localisation at the times of the first and last speed change.

Lemma 2.4.3 (Position at the last speed change)**.** *For any $y \in \mathbb{R}$, any $\varepsilon > 0$, any $\gamma > 1/2$ and for all t large enough,*

$$\mathbb{P}\left(\exists_{j \le n(t)} : \left\{\bar{x}_j(t) > m^-(t) - y\right\} \wedge \left\{\left|\bar{x}_j((1-b_\ell)t) - \sqrt{2}\,(1 - \sigma_\ell^2 b_\ell)t\right| > (\sigma_\ell^2 b_\ell t)^\gamma\right\}\right) < \varepsilon. \tag{2.4.4}$$

Lemma 2.4.4 (Position at the first speed change)**.** *For any $y \in \mathbb{R}$, any $\varepsilon > 0$, any $\gamma > 1/2$ and for all t large enough,*

$$\mathbb{P}\left(\exists_{j \le n(t)} : \left\{\bar{x}_j(t) > m^-(t) - y\right\} \wedge \left\{\left|\bar{x}_j((1-b_\ell)t) - \sqrt{2}\,(1 - \sigma_\ell^2 b_\ell)t\right| > (\sigma_\ell^2 b_\ell t)^\gamma\right\}\right) < \varepsilon. \tag{2.4.5}$$

Proof of Lemma 2.4.3. In[4] the notation of (2.2.3), we write the probability in (2.4.4) as

$$
\begin{aligned}
&\mathbb{P}\Big(\exists_{j\le n(t)}\colon \big\{\bar{x}_j(t) > m^-(t) - y\big\} \wedge \Big\{\Big|\bar{x}_j((1-b_\ell)t) - \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t\Big| > (\sigma_\ell^2 b_\ell t)^\gamma\Big\}\Big) \\
&= \mathbb{P}\Big(\exists_{1\le i_1\le n(b_1 t),\dots,1\le i_\ell \le n^{\bar{i}_{\ell-1}}(b_\ell t)}\colon \\
&\qquad \Big\{\tilde{x}_{i_\ell}^{\bar{i}_{\ell-1}}(b_\ell t) > m^-(t) - \textstyle\sum_{j=1}^{\ell-1} \tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) - y\Big\} \wedge \Big\{\Big|\textstyle\sum_{j=1}^{\ell-1} \tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) - \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t\Big| > (\sigma_\ell^2 b_\ell t)^\gamma\Big\}\Big) \\
&= \mathbb{P}\Big(\textstyle\sum_{1\le i_1\le n(b_1 t),\dots,1\le i_{\ell-1}\le n^{\bar{i}_{\ell-2}}(b_{\ell-1}t)} \mathbb{1}_{\Big\{\max_{1\le i_\ell\le n^{\bar{i}_{\ell-1}}(b_\ell t)} \tilde{x}_{i_\ell}^{\bar{i}_{\ell-1}}(b_\ell t) > m^-(t) - \sum_{j=1}^{\ell-1}\tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) - y\Big\}} \\
&\qquad\qquad \times \mathbb{1}_{\Big\{\Big|\sum_{j=1}^{\ell-1}\tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) - \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t\Big| > (\sigma_\ell^2 b_\ell t)^\gamma\Big\}} \ge 1\Big).
\end{aligned}
\tag{2.4.6}
$$

By the Markov inequality, the many-to-one lemma and the independence of the increments, we get that the right-hand side of (2.4.6) is bounded from above by

$$
\begin{aligned}
&\mathbb{E}\Big[\textstyle\sum_{1\le i_1\le n(b_1 t),\dots,1\le i_{\ell-1}\le n^{\bar{i}_{\ell-2}}(b_{\ell-1}t)} \mathbb{1}_{\Big\{\max_{1\le i_\ell\le n^{\bar{i}_{\ell-1}}(b_\ell t)} \tilde{x}_{i_\ell}^{\bar{i}_{\ell-1}}(b_\ell t) > m^-(t) - \sum_{j=1}^{\ell-1}\tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) - y\Big\}} \\
&\qquad\qquad \times \mathbb{1}_{\Big\{\Big|\sum_{j=1}^{\ell-1}\tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t) - \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t\Big| > (\sigma_\ell^2 b_\ell t)^\gamma\Big\}}\Big] \\
&= \mathrm{e}^{(1-b_\ell)t}\,\mathbb{E}\Big[\mathbb{P}\Big(\max_{i\le n(b_\ell t)} \sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t) - \textstyle\sum_{j=1}^{\ell-1} z_j(b_j t) - y \,\Big|\, \tilde{\mathcal{F}}_{\ell-1}\Big)\mathbb{1}_{\Big\{\Big|\sum_{j=1}^{\ell-1} z_j(b_j t) - \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t\Big| > (\sigma_\ell^2 b_\ell t)^\gamma\Big\}}\Big] \\
&= \mathrm{e}^{(1-b_\ell)t}\int_{J_1} \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_\ell^2 b_\ell)t}}\,\mathrm{e}^{-\frac{\omega^2}{2(1-\sigma_\ell^2 b_\ell)t}}\,\mathbb{P}\Big(\max_{i\le n(b_\ell t)} \sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t) - \omega - y\Big),
\end{aligned}
\tag{2.4.7}
$$

where $\big(x_i^{b_\ell t}(b_\ell t)\big)_{i\le n(b_\ell t)}$ denotes a standard BBM at time $b_\ell t$, z_k, $1\le k\le \ell-1$, denote independent Brownian motions, $\tilde{\mathcal{F}}_{\ell-1} \equiv \sigma(z_1,\dots,z_{\ell-1})$ and

$$
J_1 \equiv \Big(-\infty, \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t - (\sigma_\ell^2 b_\ell t)^\gamma\Big) \cup \Big(\sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t + (\sigma_\ell^2 b_\ell t)^\gamma, \infty\Big). \tag{2.4.8}
$$

We split the range of integration J_1 into $[\sqrt{2}\,(1-b_\ell)t, \infty)$ and $J_2 \equiv J_1 \setminus [\sqrt{2}\,(1-b_\ell)t,\infty)$.
For $\omega \in \big[\sqrt{2}\,(1-b_\ell)t, \infty\big)$, we bound the probability on the right-hand side of (2.4.7) by 1 and the integral by

$$
\mathrm{e}^{(1-b_\ell)t}\int_{\sqrt{2}\,(1-b_\ell)t}^{\infty} \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_\ell^2 b_\ell)t}}\,\mathrm{e}^{-\frac{\omega^2}{2(1-\sigma_\ell^2 b_\ell)t}} \le \mathrm{e}^{(1-b_\ell)t}\,\mathrm{e}^{-\frac{(1-b_\ell)^2 t}{(1-\sigma_\ell^2 b_\ell)}} = o(1), \tag{2.4.9}
$$

by a Gaussian tail bound. For the integral over J_2, we write

$$
z(t,\omega) \equiv \sigma_\ell^{-1}\left(m^-(t) - \omega - y\right) - \sqrt{2}\,b_\ell t, \tag{2.4.10}
$$

[4] The following two sentences rephrase a reference to Subsection 2.4.2. References to this part are slightly reformulated (e.g. “the right-hand side of (2.4.7)”).

and use [29, Lemma 2.3] to get

$$\mathbb{P}\Big(\max_{i\leq n(b_\ell t)} \sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t) - \omega - y\Big) \leq \frac{1}{\sqrt{2\pi}\,\big(\sqrt{2b_\ell t}+z(t,\omega)/\sqrt{b_\ell t}\big)} \mathrm{e}^{-\sqrt{2}\,z(t,\omega)-\frac{z^2(t,\omega)}{2b_\ell t}}$$
$$\leq \frac{1}{2\sqrt{\pi b_\ell t}} \mathrm{e}^{b_\ell t - \frac{(m^-(t)-\omega-y)^2}{2\sigma_\ell^2 b_\ell t}} (1+o(1)). \tag{2.4.11}$$

By (2.4.9) and (2.4.11), the right-hand side of (2.4.7) is not larger than

$$\frac{1}{2\sqrt{\pi b_\ell t}} \mathrm{e}^t \int_{J_2} \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_\ell^2 b_\ell)t}} \mathrm{e}^{-\frac{\omega^2}{2(1-\sigma_\ell^2 b_\ell)t}} \mathrm{e}^{-\frac{(m^-(t)-\omega-y)^2}{2\sigma_\ell^2 b_\ell t}} (1+o(1))$$
$$= \frac{1}{2\sqrt{\pi b_\ell t}} \mathrm{e}^{t-\frac{(m^-(t)-y)^2}{2t}} \int_{J_2} \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_\ell^2 b_\ell)t}} \exp\Big(-\frac{\big(\omega-(1-\sigma_\ell^2 b_\ell)(m^-(t)-y)\big)^2}{2(1-\sigma_\ell^2 b_\ell)\,\sigma_\ell^2 b_\ell t}\Big)(1+o(1))$$
$$= \frac{\mathrm{e}^{\sqrt{2}d}t^{1/2+\alpha_1+\alpha_\ell}}{\sqrt{b_\ell}} \int_{\big(-(\sigma_\ell^2 b_\ell t)^\gamma,(\sigma_\ell^2 b_\ell t)^\gamma\big)^c} \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_\ell^2 b_\ell)t}} \exp\Big(-\frac{\big(\omega-(1-\sigma_\ell^2 b_\ell)\big(m^-(t)-\sqrt{2}\,t-y\big)\big)^2}{2(1-\sigma_\ell^2 b_\ell)\,\sigma_\ell^2 b_\ell t}\Big)(1+o(1)), \tag{2.4.12}$$

which tends to zero as $t\uparrow\infty$ since $m^-(t)-\sqrt{2}\,t-y = O(\ln(t))$. □

Proof of Lemma 2.4.4. Due to Assumption 2.4.1.(iii), we can choose $\gamma > 1/2$ such that

$$1 \gg b_\ell \gg t^{\frac{\alpha_\ell+\gamma-1}{1-\gamma}}. \tag{2.4.13}$$

By Lemma 2.4.3, the probability in (2.4.5) equals

$$\mathbb{P}\Big(\exists_{j\leq n(t)}: \big\{\bar{x}_j(t) > m^-(t)-y\big\} \wedge \Big\{\Big|\bar{x}_j(b_1 t)-\sqrt{2}\,\sigma_1^2 b_1 t\Big| > (\sigma_1^2 b_1 t)^\gamma\Big\}$$
$$\wedge \Big\{\Big|\bar{x}_j((1-b_\ell)t)-\sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t\Big| \leq (\sigma_\ell^2 b_\ell t)^\gamma\Big\}\Big) + O(\epsilon)$$
$$\leq \mathrm{e}^{(1-b_\ell)t}\int_{(I_1)^c} \frac{\mathrm{d}\omega_1}{\sqrt{2\pi\sigma_1^2 b_1 t}} \mathrm{e}^{-\frac{\omega_1^2}{2\sigma_1^2 b_1 t}} \int_{I_2} \frac{\mathrm{d}\omega_2}{\sqrt{2\pi(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} \mathrm{e}^{-\frac{\omega_2^2}{2(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}}$$
$$\times\,\mathbb{P}\Big(\max_{i\leq n(b_\ell t)} \sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t)-\omega_1-\omega_2-y\Big) + O(\epsilon), \tag{2.4.14}$$

where

$$I_1 \equiv \Big(\sqrt{2}\,\sigma_1^2 b_1 t - (\sigma_1^2 b_1 t)^\gamma, \sqrt{2}\,\sigma_1^2 b_1 t + (\sigma_1^2 b_1 t)^\gamma\Big),$$
$$I_2 \equiv \Big(\sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t - \omega_1 - (\sigma_\ell^2 b_\ell t)^\gamma, \sqrt{2}\,(1-\sigma_\ell^2 b_\ell)t - \omega_1 + (\sigma_\ell^2 b_\ell t)^\gamma\Big). \tag{2.4.15}$$

For $\omega_2 \in I_2$, by (2.4.13), we can use Proposition 2.3.1 and get

$$\mathbb{P}\Big(\max_{i\leq n(b_\ell t)} \sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t)-\omega_1-\omega_2-y\Big)$$
$$= \frac{C}{\sqrt{2}} b_\ell t^{1-\alpha_\ell} \mathrm{e}^{b_\ell t - \frac{\left(\sqrt{2}\,t-\omega_1-\omega_2+L(t)-y\right)^2}{2\sigma_\ell^2 b_\ell t}} (1+o(1)), \tag{2.4.16}$$

where

$$L(t) \equiv \frac{3}{2\sqrt{2}}\sigma_\ell \ln(b_\ell t) - \frac{1+2(\alpha_1+\alpha_\ell)}{2\sqrt{2}} \ln(t). \tag{2.4.17}$$

This implies that

$$\begin{aligned}
&\int_{I_2} \frac{d\omega_2}{\sqrt{2\pi(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} e^{-\frac{\omega_2^2}{2(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} \mathbb{P}\left(\max_{i\le n(b_\ell t)} \sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t) - \omega_1 - \omega_2 - y\right)\\
&= \frac{C}{\sqrt{2}} b_\ell t^{1-\alpha_\ell} e^{b_\ell t} e^{-\frac{\left(\sqrt{2}\,t-\omega_1+L(t)-y\right)^2}{2(1-\sigma_1^2 b_1)t}} \int_{I_2} \frac{d\omega_2}{\sqrt{2\pi(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}}\\
&\quad \times \exp\left(-(1-\sigma_1^2 b_1)\frac{\left(\omega_2-(1-\sigma_1^2 b_1)^{-1}(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)\left(\sqrt{2}\,t-\omega_1+L(t)-y\right)\right)^2}{2(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)\sigma_\ell^2 b_\ell t}\right)(1+o(1))\\
&\le \frac{C}{\sqrt{2}} b_\ell t^{1-\alpha_\ell} e^{b_\ell t} e^{-\frac{\left(\sqrt{2}\,t-\omega_1+L(t)-y\right)^2}{2(1-\sigma_1^2 b_1)t}} \frac{\sigma_\ell b_\ell^{1/2}}{(1-\sigma_1^2 b_1)^{1/2}}(1+o(1)).
\end{aligned} \tag{2.4.18}$$

Therefore, (2.4.14) is not larger than

$$\begin{aligned}
&\frac{C}{\sqrt{2}} b_\ell^{3/2} t^{1-\alpha_\ell} e^{-\sqrt{2}(L(t)-y)-\frac{(L(t)-y)^2}{2t}}\\
&\times \int_{\left(-(\sigma_1^2 b_1 t)^\gamma, (\sigma_1^2 b_1 t)^\gamma\right)^c} \frac{d\omega_1}{\sqrt{2\pi(1-\sigma_1^2 b_1)\sigma_1^2 b_1 t}} \exp\left(-\frac{(\omega_1-\sigma_1^2 b_1(L(t)-y))^2}{2(1-\sigma_1^2 b_1)\sigma_1^2 b_1 t}\right)(1+o(1)).
\end{aligned} \tag{2.4.19}$$

The prefactor is of polynomial order and the integral is of order $\exp\left(-\frac{(b_1 t)^{2\gamma-1}}{2}\right)$. By Assumption 2.4.1.(ii), (2.4.19) tends to zero as $t \uparrow \infty$. □

Proof of Proposition 2.4.2. Let γ be close enough to $1/2$ such that (2.4.13) as well as

$$1 \gg b_1 \gg t^{\frac{\alpha_1+\gamma-1}{1-\gamma}} \tag{2.4.20}$$

are satisfied. Choose $\tilde{\gamma}$ s.t. $1/2 < \tilde{\gamma} < \gamma$ and $\tilde{\gamma} < 1$. By Lemmas 2.3.3, 2.4.3, and 2.4.4, the probability in (2.4.3) is equal to

$$\begin{aligned}
&\mathbb{P}\Big(\exists_{j\le n(t)} : \left\{\bar{x}_j(t) > m^-(t) - y\right\} \wedge \left\{\bar{x}_j \in \mathcal{A}_{r,b_1 t,0,\sigma_1} \cap \mathcal{T}_{b_1 t,b_1 t,0,\tilde{\gamma}} \cap \mathcal{T}_{(1-b_\ell)t,(1-b_\ell)t,0,\tilde{\gamma}}\right\}\\
&\quad \wedge \left\{\exists_{b_1 t\le s\le (1-b_\ell)t} : \left|\bar{x}_j(s) - \sqrt{2}\,tA_t(s/t)\right| > \left(A_t(s/t) \wedge (1-A_t(s/t))\right)^\gamma t^\gamma\right\}\Big) + O(\epsilon),
\end{aligned} \tag{2.4.21}$$

for $r, t > 0$ large enough. As in (2.4.6)–(2.4.7)[5], we see that the probability in (2.4.21) is

[5]These two references used to point to Subsection 2.4.1.

bounded from above by

$$
\begin{aligned}
&\mathrm{e}^{(1-b_\ell)t}\,\mathbb{E}\Big[\mathbb{1}_{\bar{x}_1^t\in\mathcal{A}_{r,b_1t,0,\sigma_1}\cap\mathcal{T}_{b_1t,b_1t,0,\tilde{\gamma}}}\\
&\quad\times\mathbb{E}\Big[\mathbb{1}_{\bar{x}_1^t\in\mathcal{T}_{(1-b_\ell)t,(1-b_\ell)t,0,\tilde{\gamma}}}\mathbb{1}_{\exists\, b_1t\le s\le(1-b_\ell)t:\ |\bar{x}_1^t(s)-\sqrt{2}\,tA_t(s/t)|>(A_t(s/t)\wedge(1-A_t(s/t)))^\gamma t^\gamma}\\
&\quad\times\mathbb{P}\Big(\max_{i\le n(b_\ell t)}\sigma_\ell x_i^{b_\ell t}(b_\ell t)>m^-(t)-\bar{x}_1^t((1-b_\ell)t)-y\,\Big|\,\mathcal{F}_{(1-b_\ell)t}\Big)\Big|\,\mathcal{F}_{b_1t}\Big]\Big]\\
&=\mathrm{e}^{(1-b_\ell)t}\int_{I_1}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi\sigma_1^2b_1t}}\mathrm{e}^{-\frac{\omega_1^2}{2\sigma_1^2b_1t}}\,\mathbb{P}\Big(\mathfrak{z}_{0,\frac{\omega_1}{\sigma_1}}^{b_1t}(s)\le\sqrt{2}\,s\ \forall_{s\in[r,b_1t]}\Big)\int_{I_2}\frac{\mathrm{d}\omega_2}{\sqrt{2\pi(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}}\mathrm{e}^{-\frac{\omega_2^2}{2(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}}\\
&\quad\times\mathbb{P}\Big(\exists_{s\in[b_1t,(1-b_\ell)t]}:\ \Big|\mathfrak{z}_{\omega_1,\omega_1+\omega_2}^{(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}\big(A_t(s/t)t-\sigma_1^2b_1t\big)-\sqrt{2}\,tA_t(s/t)\Big|>(A_t(s/t)\wedge(1-A_t(s/t)))^\gamma t^\gamma\Big)\\
&\quad\times\mathbb{P}\Big(\max_{i\le n(b_\ell t)}\sigma_\ell x_i^{b_\ell t}(b_\ell t)>m^-(t)-\omega_1-\omega_2-y\Big),
\end{aligned}
\tag{2.4.22}
$$

with I_1, I_2 as in (2.4.15). We first bound the probabilities involving Brownian bridges. By[6] Lemma 3.4 in [8], for $\gamma > 1/2$ and $\omega_1 \in I_1$,

$$
\begin{aligned}
\mathbb{P}\Big(\mathfrak{z}_{0,\frac{\omega_1}{\sigma_1}}^{b_1t}(s)\le\sqrt{2}\,s\ \forall_{s\in[r,b_1t]}\Big)&\le\tfrac{2\sqrt{r}}{b_1t-r}\Big(\sqrt{2}\,b_1t-\tfrac{\omega_1}{\sigma_1}\Big)(1+o(1))\\
&\le 2\sqrt{r}\Big(\sqrt{2}\,(1-\sigma_1)+\sigma_1^{2\gamma}b_1t^{1-\gamma}\Big)(1+o(1))\\
&=\sqrt{2r}\,t^{-\alpha_1}(1+o(1)).
\end{aligned}
\tag{2.4.23}
$$

For the second probability involving a bridge, the term inside the absolute value equals

$$
\mathfrak{z}_{0,0}^{(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}\Big(A_t(s/t)t-\sigma_1^2b_1t\Big)+\omega_1+\tfrac{A_t(s/t)-\sigma_1^2b_1}{1-\sigma_1^2b_1-\sigma_\ell^2b_\ell}\omega_2-\sqrt{2}\,tA_t(s/t). \tag{2.4.24}
$$

For each $s\in[b_1t,(1-b_\ell)t]$ and ω_1,ω_2 in the ranges of integration,

$$
\omega_1+\tfrac{A_t(s/t)-\sigma_1^2b_1}{1-\sigma_1^2b_1-\sigma_\ell^2b_\ell}\omega_2-\sqrt{2}\,tA_t(s/t)=\,O\Big(\tfrac{1-\sigma_\ell^2b_\ell-A_t(s/t)}{1-\sigma_1^2b_1-\sigma_\ell^2b_\ell}(b_1t)^{\tilde{\gamma}}+\tfrac{A_t(s/t)-\sigma_1^2b_1}{1-\sigma_1^2b_1-\sigma_\ell^2b_\ell}(\sigma_\ell^2b_\ell t)^{\tilde{\gamma}}\Big). \tag{2.4.25}
$$

We will show that, for $s\in[b_1t,(1-b_\ell)t]$ and t large enough,

$$
\tfrac{1-\sigma_\ell^2b_\ell-A_t(s/t)}{1-\sigma_1^2b_1-\sigma_\ell^2b_\ell}(\sigma_1^2b_1t)^{\tilde{\gamma}}+\tfrac{A_t(s/t)-\sigma_1^2b_1}{1-\sigma_1^2b_1-\sigma_\ell^2b_\ell}(\sigma_\ell^2b_\ell t)^{\tilde{\gamma}}\le(A_t(s/t)\wedge(1-A_t(s/t)))^{\tilde{\gamma}}t^{\tilde{\gamma}}(1+o(1)). \tag{2.4.26}
$$

Namely, (2.4.26) holds for $s=b_1t$ and $s=(1-b_\ell)t$. Assumptions 2.4.1.(ii)–(iii) imply that (2.4.26) holds for s s.t. $A_t(s/t)=1/2$. Concavity of $x\mapsto x^{\tilde{\gamma}}$ on $\mathbb{R}_+$ and monotonicity of A_t imply that (2.4.26) holds for all $s\in[b_1t,(1-b_\ell)t]$, and so for those s, (2.4.24) equals

$$
\mathfrak{z}_{0,0}^{(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}\Big(A_t(s/t)t-\sigma_1^2b_1t\Big)+o\Big((A_t(s/t)\wedge(1-A_t(s/t)))^\gamma t^\gamma\Big). \tag{2.4.27}
$$

Thus the one-but-last probability in (2.4.22) is bounded from above by

$$
\mathbb{P}\Big(\exists_{s\in[b_1t,(1-b_\ell)t]}:\ \Big|\mathfrak{z}_{0,0}^{(1-\sigma_1^2b_1-\sigma_\ell b_\ell)t}\big(A_t(s/t)t-\sigma_1^2b_1t\big)\Big|>(A_t(s/t)\wedge(1-A_t(s/t)))^\gamma t^\gamma\Big)(1+o(1)). \tag{2.4.28}
$$

[6] We rephrased a reference to Subsection 2.4.1.

For $s \in [b_1 t, (1-b_\ell)t]$, the fluctuations of the Brownian bridge in (2.4.28) are bounded by those of $\left(\mathfrak{z}_{0,0}^t\left(A_t(s/t)t\right)\right)_{s\in[0,t]}$, and thus (2.4.28) is bounded by

$$\mathbb{P}\left(\exists_{s\in[b_1 t,(1-b_\ell)t]} : \left|\mathfrak{z}_{0,0}^t(A_t(s/t)t)\right| > (A_t(s/t)\wedge(1-A_t(s/t)))^\gamma t^\gamma\right)(1+o(1)). \tag{2.4.29}$$

This probability is by the monotonicity of A_t not larger than

$$\mathbb{P}\left(\exists_{s\in[\tilde{r},t-\tilde{r}]} : \left|\mathfrak{z}_{0,0}^t(s)\right| > (s\wedge(t-s))^\gamma\right) \leq \varepsilon\sqrt{2\pi}/(C\sqrt{r}), \tag{2.4.30}$$

for $\tilde{r}, t > 0$ large enough, by Lemma 2.3.5. With (2.4.23) and (2.4.30) we see that (2.4.22) is smaller than

$$\begin{aligned} &\epsilon\frac{\sqrt{2}}{C}t^{-\alpha_1}\mathrm{e}^{(1-b_\ell)t}\int_{I_1}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi\sigma_1^2 b_1 t}}\mathrm{e}^{-\frac{\omega_1^2}{2\sigma_1^2 b_1 t}}\int_{I_2}\frac{\mathrm{d}\omega_2}{\sqrt{2\pi(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}}\mathrm{e}^{-\frac{\omega_2^2}{2(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} \\ &\qquad\times\mathbb{P}\left(\max_{i\leq n(b_\ell t)}\sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t)-\omega_1-\omega_2-y\right)(1+o(1)). \end{aligned} \tag{2.4.31}$$

For $\omega_1 \in I_1$ and $\tilde{\gamma} > 1/2$, the bound in (2.4.18) is asymptotically sharp, and so the integral over ω_2 in (2.4.31) is equal to

$$\frac{C}{\sqrt{2(1-\sigma_1^2 b_1)}}b_\ell^{3/2}t^{1-\alpha_\ell}\mathrm{e}^{b_\ell t}\exp\left(-\frac{\left(\sqrt{2}\,t-\omega_1+L(t)-y\right)^2}{2(1-\sigma_1^2 b_1)t}\right)(1+o(1)). \tag{2.4.32}$$

Inserting (2.4.32) into (2.4.31) and shifting ω_1 by $\sqrt{2}\,\sigma_1^2 b_1 t$, we see that (2.4.31) is equal to

$$\epsilon b_\ell^{3/2}t^{1-\alpha_1-\alpha_\ell}\mathrm{e}^t\mathrm{e}^{-\frac{(\sqrt{2}t+L(t)-y)^2}{2t}}\int_{-(\sigma_1^2 b_1 t)^{\tilde{\gamma}}}^{(\sigma_1^2 b_1 t)^{\tilde{\gamma}}}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi(1-\sigma_1^2 b_1)\sigma_1^2 b_1 t}}\exp\left(-\frac{\left(\omega_1-\sigma_1^2 b_1(L(t)-y)\right)^2}{2(1-\sigma_1^2 b_1)\sigma_1^2 b_1 t}\right)(1+o(1)). \tag{2.4.33}$$

Note that the integrands in (2.4.19) and (2.4.33) are the same, but the ranges of integration are complementary. We have seen that the integral in (2.4.19) tends to zero as $t \uparrow \infty$, which implies that the Gaussian integral in (2.4.33) tends to one. Recalling the definition of $L(t)$ in (2.4.17), we find that the t-dependent factors outside the integral in (2.4.33) tend to one as well, from which the claim follows. □

The[7] next proposition describes the effect of entropic repulsion on the first time interval $[0, b_1 t]$ for extremal particles.

Proposition 2.4.5 (Entropic repulsion)**.** *Let* $0 < \beta < \alpha_1$*. Then, for any* $y \in \mathbb{R}$ *and for any* $\varepsilon > 0$*, there exists a constant* $\delta \in (0, 1/2)$ *such that, for all* t *large enough,*

$$\mathbb{P}\left(\exists_{j\leq n(t)} : \left\{\bar{x}_j(t) > m^-(t)-y\right\}\wedge\left\{\exists_{t^\beta\leq s\leq b_1 t} : \bar{x}_j(s) > \sqrt{2}\,\sigma_1 s - t^{\beta\delta}\right\}\right) < \varepsilon. \tag{2.4.34}$$

[7] This sentence rephrases a reference to Subsection 2.4.1.

This means that the extremal particles of $(\bar{X}_t)_{t>0}$ lie in $\mathcal{A}_{t^\beta,b_1t,t^{\beta\delta},\sigma_1}$ with high probability. By monotonicity, we can use the superset $\mathcal{A}_{t^\beta,t^\beta,t^{\beta\delta},\sigma_1}\cap\mathcal{A}_{t^\beta,b_1t,t^{\beta\delta/2},\sigma_1}$ of $\mathcal{A}_{t^\beta,b_1t,t^{\beta\delta},\sigma_1}$ instead.

Proof. Let $\gamma > 1/2$ be close enough to 1/2 that (2.4.13) and (2.4.20) are satisfied. By Lemmas 2.3.3, 2.4.3 and 2.4.4, the probability in (2.4.34) is equal to

$$\mathbb{P}\Big(\exists_{j\le n(t)}\colon \Big\{\bar{x}_j(t) > m^-(t) - y\Big\} \wedge \Big\{\bar{x}_j \in \mathcal{T}_{b_1t,b_1t,0,\gamma} \cap \mathcal{T}_{(1-b_\ell)t,(1-b_\ell)t,0,\gamma} \cap \mathcal{A}_{r,b_1t,0,\sigma_1}\Big\} \wedge \Big\{\bar{x}_j(t^\beta) > -\sqrt{2}\,\sigma_1 t^\beta\Big\} \wedge \Big\{\exists_{t^\beta\le s\le b_1t}\colon \bar{x}_j(s) > \sqrt{2}\,\sigma_1 s - t^{\beta\delta}\Big\}\Big) + O(\epsilon), \tag{2.4.35}$$

for $r, t > 0$ large enough. As in (2.4.21)–(2.4.22)[8], we see that the probability in (2.4.35) is bounded from above by

$$\begin{aligned}
&\mathrm{e}^{(1-b_\ell)t}\int_{I_1}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi\sigma_1^2(b_1t)}}\mathrm{e}^{-\frac{\omega_1^2}{2\sigma_1^2(b_1t)}}\int_{I_2}\frac{\mathrm{d}\omega_2}{\sqrt{2\pi(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}}\mathrm{e}^{-\frac{\omega_2^2}{2(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}}\\
&\times\Big[\mathbb{P}\Big(\mathfrak{z}^{b_1t}_{0,\frac{\omega_1}{\sigma_1}-\sqrt{2}\,b_1t}(s)\le 0\,\forall_{s\in[r,b_1t]}\Big) - \mathbb{P}\Big(\mathfrak{z}^{b_1t}_{0,\frac{\omega_1}{\sigma_1}-\sqrt{2}b_1t}(s)\le -\tfrac{t^{\beta\delta}}{\sigma_1}\,\forall_{s\in[t^\beta,b_1t]}\Big)\Big]\\
&\times\mathbb{P}\Big(\max_{i\le n(b_\ell t)}\sigma_\ell x_i^{b_\ell t}(b_\ell t) > m^-(t)-\omega_1-\omega_2-y\Big),
\end{aligned}\tag{2.4.36}$$

with I_1, I_2 as in (2.4.15). We have seen in (2.4.31)–(2.4.33) that replacing the difference of probabilities in (2.4.36) with $t^{-\alpha_1}$ creates a term of order 1. Thus, it suffices to show that

$$\mathbb{P}\Big(\mathfrak{z}^{b_1t}_{0,\frac{\omega_1}{\sigma_1}-\sqrt{2}\,b_1t}(s)\le 0\,\forall_{s\in[r,b_1t]}\Big) - \mathbb{P}\Big(\mathfrak{z}^{b_1t}_{0,\frac{\omega_1}{\sigma_1}-\sqrt{2}b_1t}(s)\le -\tfrac{t^{\beta\delta}}{\sigma_1}\,\forall_{s\in[t^\beta,b_1t]}\Big) \ll t^{-\alpha_1}. \tag{2.4.37}$$

We[9] rewrite this difference of probabilities as

$$\begin{aligned}
&\int_{-\infty}^0 \mathrm{d}z\frac{\exp\Big(-\frac{b_1t}{2(b_1t-t^\beta)t^\beta}\Big(z+\frac{t^\beta}{b_1t}\Big(\sqrt{2}\,b_1t-\frac{\omega_1}{\sigma_1}\Big)\Big)^2\Big)}{\sqrt{2\pi(b_1t-t^\beta)t^\beta/(b_1t)}}\mathbb{P}\Big(\mathfrak{z}^{t^\beta}_{0,z}(s)\le 0\,\forall_{s\in[r,t^\beta]}\Big)\\
&\times\Big[\mathbb{P}\Big(\mathfrak{z}^{b_1t-t^\beta}_{z,\frac{\omega_1}{\sigma_1}-\sqrt{2}\,b_1t}(s)\le 0\,\forall_{s\in[0,b_1t-t^\beta]}\Big) - \mathbb{P}\Big(\mathfrak{z}^{b_1t-t^\beta}_{z,\frac{\omega_1}{\sigma_1}-\sqrt{2}\,b_1t}(s)\le -\tfrac{t^{\beta\delta}}{\sigma_1}\,\forall_{s\in[0,b_1t-t^\beta]}\Big)\Big].
\end{aligned}\tag{2.4.38}$$

The domain of integration in (2.4.38) is split into the three parts

$$J_1 \equiv \big(-\infty, -t^\beta\big), \quad J_2 \equiv \big(-t^\beta, -t^{\beta\delta}/\sigma_1\big), \quad J_3 \equiv \big(-t^{\beta\delta}/\sigma_1, 0\big). \tag{2.4.39}$$

We deal with each part separately:

1. On J_1, we bound all the probabilities in (2.4.38) by 1. Then, since $t^\beta \ll b_1t$, a Gaussian tail estimate shows that (2.4.38) integrated over J_1 is smaller than $\exp(-t^\beta/2)(1+o(1))$.

[8] These two references used to point to Subsection 2.4.1.

[9] The remainder of this proof is similar to parts of Subsection 2.4.1, which were originally referenced here.

2. For $z \in J_2$, by Lemma 2.3.6,

$$\mathbb{P}\left(\mathfrak{z}^{b_1 t - t^\beta}_{z, \frac{\omega_1}{\sigma_1} - \sqrt{2}\, b_1 t}(s) \le 0 \,\forall_{s\in[0, b_1 t - t^\beta]}\right) - \mathbb{P}\left(\mathfrak{z}^{b_1 t - t^\beta}_{z, \frac{\omega_1}{\sigma_1} - \sqrt{2}\, b_1 t}(s) \le -\tfrac{t^{\beta\delta}}{\sigma_1} \,\forall_{s\in[0, b_1 t - t^\beta]}\right)$$
$$= \exp\left(\frac{2z\left(\sqrt{2}\, b_1 t - \frac{\omega_1}{\sigma_1}\right)}{b_1 t - t^\beta}\right)\left[\exp\left(\tfrac{2t^{\beta\delta}}{\sigma_1(b_1 t - t^\beta)}\left(\sqrt{2}\, b_1 t - \tfrac{\omega_1}{\sigma_1} - z - \tfrac{t^{\beta\delta}}{\sigma_1}\right)\right) - 1\right]. \tag{2.4.40}$$

For ω_1 from the integral in (2.4.36), we get as in (2.4.23) that

$$\frac{\sqrt{2}\, b_1 t - \sigma_1^{-1}\omega_1}{b_1 t - t^\beta} = 2^{-1/2} t^{-\alpha_1}(1 + o(1)). \tag{2.4.41}$$

Furthermore, $|z| \le t^\beta$ and $t^\beta \ll \sqrt{b_1 t}$, so the arguments in both exponential functions in (2.4.40) tend to zero as $t \uparrow \infty$. This implies that the right-hand side of (2.4.40) is equal to

$$\tfrac{2t^{\beta\delta}}{\sigma_1(b_1 t - t^\beta)}\left(\sqrt{2}\, b_1 t - \tfrac{\omega_1}{\sigma_1} - z - \tfrac{t^{\beta\delta}}{\sigma_1}\right)(1 + o(1)) \le 2t^{\beta\delta}\frac{\sqrt{2}\, b_1 t - \sigma_1^{-1}\omega_1}{b_1 t - t^\beta}(1 + o(1)) = \sqrt{2}\, t^{\beta\delta - \alpha_1}(1 + o(1)), \tag{2.4.42}$$

where we used (2.4.41) in the last step.

3. For $z \in J_3$, the second probability in the left-hand side of (2.4.40) is zero. By Lemma 2.3.6, the first probability is smaller than

$$\frac{2(-z)\left(\sqrt{2}\, b_1 t - \frac{\omega_1}{\sigma_1}\right)}{b_1 t - t^\beta} \le \frac{2t^{\beta\delta}\left(\sqrt{2}\, b_1 t - \frac{\omega_1}{\sigma_1}\right)}{\sigma_1(b_1 t - t^\beta)} = \sqrt{2}\, t^{\beta\delta - \alpha_1}(1 + o(1)), \tag{2.4.43}$$

where we used (2.4.41) in the last step.

We proceed as in (2.4.23) to bound the first probability in (2.4.38). This gives

$$\mathbb{P}\left(\mathfrak{z}^{t^\beta}_{0,z}(s) \le 0 \,\forall_{s\in[r, t^\beta]}\right) \le \tfrac{2\sqrt{r}}{t^\beta - r}\left(\sqrt{2}\, t^\beta - z\right)(1 + o(1)). \tag{2.4.44}$$

Then, by (2.4.42), (2.4.43) and (2.4.44), (2.4.38) is not larger than

$$\left(e^{-t^\beta/2} + 2\sqrt{2r}\, t^{\beta(\delta - 1/2) - \alpha_1}\int_{-t^{\beta/2}}^0 dx\, e^{-x^2/2}(-x)\right)(1 + o(1)) = O\left(t^{\beta(\delta - 1/2) - \alpha_1}\right) \ll t^{-\alpha_1}, \tag{2.4.45}$$

which completes the proof. □

2.5 Proofs of Theorem 2.1.3 and 2.1.4

To prove Theorem 2.1.3 and 2.1.4, it suffices by Lemma 2.1.5 to prove the convergence of the corresponding Laplace functionals

$$\Psi_t(\phi) \equiv \mathbb{E}\left[\exp\left(-\int \phi(z)\mathcal{E}_t(\mathrm{d}z)\right)\right], \tag{2.5.1}$$

where $\phi \in C^\infty(\mathbb{R})$ is a nondecreasing test function as in Lemma 2.1.5.(ii) and we denote by $\mathcal{E}_t \equiv \sum_{j\leq n(t)} \delta_{\tilde{x}_j(t)-m^\pm(t)}$ the extremal process of a VSBBM whose speed functions $(A_t)_{t>0}$ satisfy Assumption 2.1.1 or 2.1.2. We set $m^\pm(t) = m^+(t)$ if $(A_t)_{t>0}$ satisfies Assumption 2.1.1 and $m^\pm(t) = m^-(t)$ otherwise.

The proof of the convergence of $(\Phi_t(\phi))_{t>0}$ is essentially analogous to the proof of the convergence of the maximum, which formally corresponds to the choice $\phi(z) = 0$, if $z \leq y$ and $\phi(z) = +\infty$, if $z > y$, for $y \in \mathbb{R}$. Writing the proof for the maximum is notationally less heavy and easier on the reader, which is why we give the proof of Theorem 2.1.3 in detail in the following two subsections and only indicate the changes needed for the proof of Theorem 2.1.4 in Subsection 2.5.3.

2.5.1 Proof of Theorem 2.1.3 in Case A

This subsection is not a part of this thesis.

2.5.2 Proof of Theorem 2.1.3 in Case B

In this subsection, we prove Theorem 2.1.3 in Case B. We begin with the case of ℓ-speed BBM and write α_1, α_ℓ instead of $\alpha_{\text{begin}}, \alpha_{\text{end}}$, and b_1, b_ℓ instead of $b_{\text{begin}}, b_{\text{end}}$ to be consistent with Case A. To complete the proof for the more general setting of Theorem 2.1.3, we use Gaussian comparison techniques. Recall[10] that $(\bar{X}_t)_{t>0}$ is an ℓ-speed BBM with speed functions $(A_t)_{t>0}$ satisfying Assumption 2.4.1.

Proposition 2.5.1. *Let C be the positive constant from Proposition 2.3.1 and denote by Z the limit of the derivative martingale. Then, for all $y \in \mathbb{R}$,*

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{j\leq n(t)} \bar{x}_j(t) - m^-(t) \leq y\right) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}}\right], \tag{2.5.2}$$

where

$$m^-(t) = \sqrt{2}\,t - \tfrac{1+2(\alpha_1+\alpha_\ell)}{2\sqrt{2}} \ln(t). \tag{2.5.3}$$

Proof of Proposition 2.5.1. The structure of this proof is identical to that in Case A, which is given in Subsection 2.5.1. Let $\varepsilon > 0$, $r > 0$, $\gamma > 1/2$ and $0 < \beta < \alpha_1$ be such that $t^\beta \ll b_1 t$.

[10] We adapted this sentence and moved Assumption 2.4.1 to Subection 2.4.2.

Lemma 2.3.3 and Propositions 2.4.2 and 2.4.5 imply that

$$\mathbb{P}\left(\exists_{j\le n(t)}\colon \left\{\bar{x}_j(t) > m^-(t) - y\right\} \wedge \left\{\bar{x}_j \notin \mathcal{L}\right\}\right) < \varepsilon, \tag{2.5.4}$$

for all r, t large enough, where

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{b_1 t} \cap \mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}, \\ \mathcal{L}_{b_1 t} &= \mathcal{B}_{t^\beta,t^\beta,0,\sigma_1} \cap \mathcal{A}_{t^\beta,t^\beta,t^{\beta\delta},\sigma_1} \cap \mathcal{A}_{t^\beta,b_1 t,t^{\beta\delta/2},\sigma_1} \cap \mathcal{T}_{b_1 t,b_1 t,0,\gamma}, \end{aligned} \tag{2.5.5}$$

recalling

$$\begin{aligned} \mathcal{A}_{r_1,r_2,S,\sigma} &= \left\{X\colon \forall_{r_1\le s\le r_2}\colon X(s) + S \le \sqrt{2}\,\sigma s\right\}, \\ \mathcal{B}_{r_1,r_2,S,\sigma} &= \left\{X\colon \forall_{r_1\le s\le r_2}\colon X(s) + S > -\sqrt{2}\,\sigma s\right\}. \end{aligned} \tag{2.5.6}$$

By Lemma 2.3.7, Proposition 2.5.1 will follow from

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{j\le n(t)\colon \bar{x}_j\in\mathcal{L}} \bar{x}_j(t) - m^-(t) \le y\right) = \mathbb{E}\left[\exp\left(-CZ\mathrm{e}^{-\sqrt{2}y}\right)\right]. \tag{2.5.7}$$

In[11] the notation of (2.2.3), we write the probability in (2.5.7) as

$$\mathbb{P}\left(\max_{\substack{i_1\le n(b_1 t),\dots,i_\ell\le n^{\bar{i}_{\ell-1}}(b_\ell t),\\ \bar{x}_{i_1}\in\mathcal{L}_{b_1 t},\ \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}\in\mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}}} \bar{x}_{i_\ell}^{\bar{i}_{\ell-1}}(b_\ell t) \le m^-(t) - \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}(b_k t) + y\right). \tag{2.5.8}$$

By the branching property, this probability is equal to

$$\begin{aligned} &\mathbb{E}\Bigg[\prod_{\substack{i_1\le n(b_1 t),\\ \bar{x}_{i_1}\in\mathcal{L}_{b_1 t}}} \mathbb{E}\Bigg[\prod_{\substack{i_2\le n^{i_1}(b_2 t),\dots,i_{\ell-1}\le n^{\bar{i}_{\ell-2}}(b_{\ell-1}t),\\ \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}\in\mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}}} \\ &\times\left(1 - \mathbb{P}\left(\max_{i_\ell\le n^{\bar{i}_{\ell-1}}(b_\ell t)} \bar{x}_{i_\ell}^{\bar{i}_{\ell-1}}(b_\ell t) > m^-(t) - \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}(b_k t) + y \,\middle|\, \mathcal{F}_{(1-b_\ell)t}\right)\right)\Bigg|\mathcal{F}_{b_1 t}\Bigg]\Bigg]. \end{aligned} \tag{2.5.9}$$

Since $\sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}} \in \mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}$, the F-KPP asymptotics from Proposition 2.3.1 imply

$$\begin{aligned} &\mathbb{P}\left(\max_{i_\ell\le n^{\bar{i}_{\ell-1}}(b_\ell t)} \bar{x}_{i_\ell}^{\bar{i}_{\ell-1}}(b_\ell t) > m^-(t) - \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}(b_k t) + y \,\middle|\, \mathcal{F}_{(1-b_\ell)t}\right) \\ &= C\Delta^{\bar{i}_{\ell-1}}(t)\exp\left(b_\ell t - \frac{\left(\Delta^{\bar{i}_{\ell-1}}(t) + \sqrt{2}\,b_\ell t\right)^2}{2b_\ell t}\right)(1 + o(1)), \end{aligned} \tag{2.5.10}$$

with

$$\Delta^{\bar{i}_{\ell-1}}(t) \equiv \frac{1}{\sigma_\ell}\left(m^-(t) - \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}(b_k t) + y\right) - \left(\sqrt{2}\,b_\ell t - \tfrac{3}{2\sqrt{2}}\ln(b_\ell t)\right). \tag{2.5.11}$$

[11] The following two sentences rephrase a reference to Subsection 2.5.1.

Inserting (2.5.10) into (2.5.9) yields

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{\substack{i_2\le n^{i_1}(b_2t),\dots,i_{\ell-1}\le n^{\bar{i}_{\ell-2}}(b_{\ell-1}t),\\ \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}\in\mathcal{T}_{b_1t,(1-b_\ell)t,0,\gamma}}}\Big(1-C\Delta^{\bar{i}_{\ell-1}}(t)\exp\Big(b_\ell t-\frac{\big(\Delta^{\bar{i}_{\ell-1}}(t)+\sqrt{2}\,b_\ell t\big)^2}{2b_\ell t}\Big)(1+o(1))\Big)\Big|\mathcal{F}_{b_1t}\Big]\\
&=\mathbb{E}\Big[\prod_{\substack{i_2\le n^{i_1}(b_2t),\dots,i_{\ell-1}\le n^{\bar{i}_{\ell-2}}(b_{\ell-1}t),\\ \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}\in\mathcal{T}_{b_1t,(1-b_\ell)t,0,\gamma}}}\exp\Big(-C\Delta^{\bar{i}_{\ell-1}}(t)\exp\Big(b_\ell t-\frac{\big(\Delta^{\bar{i}_{\ell-1}}(t)+\sqrt{2}\,b_\ell t\big)^2}{2b_\ell t}\Big)(1+o(1))\Big)\Big|\mathcal{F}_{b_1t}\Big]\\
&=\mathbb{E}\Big[\exp\big(-\mathcal{Y}_{i_1}(t)\big)\Big|\mathcal{F}_{b_1t}\Big](1+o(1)),
\end{aligned}\tag{2.5.12}$$

where

$$\mathcal{Y}_{i_1}(t)\equiv\sum_{\substack{i_2\le n^{i_1}(b_2t),\dots,i_{\ell-1}\le n^{\bar{i}_{\ell-2}}(b_{\ell-1}t),\\ \sum_{k=1}^{\ell-1}\bar{x}_{i_k}^{\bar{i}_{k-1}}\in\mathcal{T}_{b_1t,(1-b_\ell)t,0,\gamma}}}C\Delta^{\bar{i}_{\ell-1}}(t)\exp\Big(b_\ell t-\frac{\big(\Delta^{\bar{i}_{\ell-1}}(t)+\sqrt{2}\,b_\ell t\big)^2}{2b_\ell t}\Big).\tag{2.5.13}$$

The first equality in (2.5.12) holds since, for $i_2,\dots,i_\ell$ from that product, the right-hand side of (2.5.10) tends to zero. The error terms in the second line of (2.5.12) are uniform and we will prove later that $\mathcal{Y}_{i_1}(t)$ converges as $t\uparrow\infty$ for $x_{i_1}\in\mathcal{L}_{b_1t}$. Thus, it is justified in the last line of (2.5.12) to write the error term outside the conditional expectation. We[12] use the inequality

$$1-x\le \mathrm{e}^{-x}\le 1-x+\frac{x^2}{2},\qquad x\ge 0,\tag{2.5.14}$$

to get that

$$1-\mathbb{E}\Big[\mathcal{Y}_{i_1}(t)\Big|\mathcal{F}_{b_1t}\Big]\le\mathbb{E}\Big[\exp\big(-\mathcal{Y}_{i_1}(t)\big)\Big|\mathcal{F}_{b_1t}\Big]\le 1-\mathbb{E}\Big[\mathcal{Y}_{i_1}(t)\Big|\mathcal{F}_{b_1t}\Big]\left(1-\frac{1}{2}\frac{\mathbb{E}\Big[\mathcal{Y}_{i_1}^2(t)\Big|\mathcal{F}_{b_1t}\Big]}{\mathbb{E}\Big[\mathcal{Y}_{i_1}(t)\Big|\mathcal{F}_{b_1t}\Big]}\right).\tag{2.5.15}$$

We postpone the proofs of the following (conditional) first and second moment estimates for $\mathcal{Y}_{i_1}(t)$ to Appendix 2.6.

Lemma 2.5.2. *Let $\gamma>1/2$ be such that*

$$(\sigma_\ell^2 b_\ell t)^\gamma \ll b_\ell t^{1-\alpha_\ell}.\tag{2.5.16}$$

For all i_1 appearing in the product in (2.5.9)*, we have*

$$\mathbb{E}\Big[\mathcal{Y}_{i_1}(t)\Big|\mathcal{F}_{b_1t}\Big]=\frac{C}{\sqrt{2(1-\sigma_1^2 b_1)}}b_\ell^{3/2}t^{1-\alpha_\ell}\exp\Big((1-b_1)t-\frac{\big(\sqrt{2}t-\bar{x}_{i_1}(b_1t)+L(t)+y\big)^2}{2(1-\sigma_1^2b_1)t}\Big)(1+o(1)),\tag{2.5.17}$$

where $L(t)$ is as in (2.4.17).

[12]This sentence rephrases a reference to Subsection 2.5.1.

Lemma 2.5.3. *Let $\gamma > 1/2$ be such that*

$$\min_{s\in[b_1,1-b_\ell]} (s - A_t(s))\, t \gg t^\gamma. \tag{2.5.18}$$

For all i_1 appearing in the product in (2.5.9)*, we have*

$$\mathbb{E}\left[\mathcal{Y}_{i_1}^2(t)\,\middle|\,\mathcal{F}_{b_1 t}\right] \le P(t)\exp\left((1-b_1)t - \frac{\left(\sqrt{2}t - \bar{x}_{i_1}(b_1 t) + L(t) + y\right)^2}{2(1-\sigma_1^2 b_1)t}\right)\mathrm{e}^{-t^{1/2}}(1+o(1)), \tag{2.5.19}$$

with $L(t)$ as in (2.4.17)*. We write P for any term satisfying $P(t) \le t^c$ for some constant $c \in \mathbb{R}$ and for all $t > 0$ large enough.*

By Lemmas 2.5.2 and 2.5.3,

$$\lim_{t\uparrow\infty} \frac{\mathbb{E}\left[\mathcal{Y}_{i_1}^2(t)\,\middle|\,\mathcal{F}_{b_1 t}\right]}{\mathbb{E}\left[\mathcal{Y}_{i_1}(t)\,\middle|\,\mathcal{F}_{b_1 t}\right]} = 0. \tag{2.5.20}$$

From this and (2.5.15) follows that

$$\mathbb{E}\left[\mathrm{e}^{-\mathcal{Y}_{i_1}(t)}\,\middle|\,\mathcal{F}_{b_1 t}\right] = 1 - \mathbb{E}\left[\mathcal{Y}_{i_1}(t)\,\middle|\,\mathcal{F}_{b_1 t}\right](1+o(1)), \tag{2.5.21}$$

so (2.5.9) is equal to

$$\begin{aligned}
&\mathbb{E}\left[\prod_{\substack{i_1\le n(b_1 t),\\ \bar{x}_{i_1}\in\mathcal{L}_{b_1 t}}} \left(1 - \mathbb{E}\left[\mathcal{Y}_{i_1}(t)\,\middle|\,\mathcal{F}_{b_1 t}\right]\right)\right](1+o(1)) \qquad (2.5.22)\\
&= \mathbb{E}\left[\prod_{\substack{i_1\le n(b_1 t),\\ \bar{x}_{i_1}\in\mathcal{L}_{b_1 t}}} \exp\left(-\frac{C}{\sqrt{2(1-\sigma_1^2 b_1)}} b_\ell^{3/2} t^{1-\alpha_\ell} \exp\left((1-b_1)t - \frac{\left(\sqrt{2}t - \bar{x}_{i_1}(b_1 t) + L(t) + y\right)^2}{2(1-\sigma_1^2 b_1)t}\right)\right)\right](1+o(1)).
\end{aligned}$$

In the last step, we used Lemma 2.5.2 and that, for $\bar{x}_{i_1} \in \mathcal{L}_{b_1 t}$, the right-hand side of (2.5.17) tends to zero as $t \uparrow \infty$. We[13] rewrite the expectation on the right-hand side of (2.5.22) by conditioning on $\mathcal{F}_{t^\beta}$ as

$$\mathbb{E}\left[\prod_{\substack{j\le n(t^\beta)\\ \bar{x}_j\in\mathcal{L}_{t^\beta}}} \mathbb{E}\left[\mathrm{e}^{-\tilde{\mathcal{Y}}_j(t)}\,\middle|\,\mathcal{F}_{t^\beta}\right]\right], \tag{2.5.23}$$

[13] We removed a reference to Subsection 2.5.1.

where

$$\tilde{\mathcal{Y}}_j(t) \equiv \sum_{\substack{j^* \le n^j(t^*),\\ \bar{x}^j_{j^*} \in \mathcal{L}_{t^*}}} \frac{C}{\sqrt{2(1-\sigma_1^2 b_1)}} b_\ell^{3/2} t^{1-\alpha_\ell} \exp\left((1-b_1)t - \frac{\left(\sqrt{2}t - \bar{x}^j_{j^*}(t^*) - \bar{x}_j(t^\beta) + L(t) + y\right)^2}{2(1-\sigma_1^2 b_1)t} \right),$$

$$\begin{aligned} \mathcal{L}_{t^\beta} &\equiv \mathcal{A}_{t^\beta, t^\beta, t^{\beta\delta}, \sigma_1} \cap \mathcal{B}_{t^\beta, t^\beta, 0, \sigma_1}, \\ \mathcal{L}_{t^*} &\equiv \mathcal{A}_{0, t^*, t^{\beta\delta/2} + \bar{x}_j(t^\beta) - \sqrt{2}\sigma_1 t^\beta, \sigma_1} \cap \mathcal{T}_{t^*, t^*, \bar{x}_j(t^\beta) - \sqrt{2}\sigma_1 t^\beta, \gamma}. \end{aligned} \tag{2.5.24}$$

Recall that $t^* = b_1 t - t^\beta$ and, independently for each j, $(\bar{x}^j_{j^*}(t^*))_{j^* \le n^j(t^*)}$ are the particles of a BBM with variance σ_1^2 at time t^*. We postpone the proofs of the estimates of the conditional first and second moments to Appendix 2.6.

Lemma 2.5.4. *Let $\gamma > 1/2$ be such that*

$$(\sigma_1^2 b_1 t)^\gamma \ll b_1 t^{1-\alpha_1}. \tag{2.5.25}$$

For all j appearing in the product in (2.5.23)*, we have*

$$\mathbb{E}\left[\tilde{\mathcal{Y}}_j(t) \,\middle|\, \mathcal{F}_{t^\beta} \right] = C e^{-\sqrt{2}y} \left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta) - t^{\beta\delta/2} \right) e^{-\sqrt{2}\left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta)\right)} (1 + o(1)). \tag{2.5.26}$$

Lemma 2.5.5. *Let γ be as in Lemma 2.5.4. For all j appearing in the product in* (2.5.23)*, we have*

$$\mathbb{E}\left[\tilde{\mathcal{Y}}_j^2(t) \,\middle|\, \mathcal{F}_{t^\beta} \right] \le P(t)\, e^{-\sqrt{2}\left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta)\right)} e^{-\sqrt{2}\, t^{\beta\delta/2}} (1 + o(1)). \tag{2.5.27}$$

By Lemmas 2.5.4 and 2.5.5,

$$\frac{\mathbb{E}\left[\tilde{\mathcal{Y}}_j^2(t) \,\middle|\, \mathcal{F}_{t^\beta} \right]}{\mathbb{E}\left[\tilde{\mathcal{Y}}_j(t) \,\middle|\, \mathcal{F}_{t^\beta} \right]} \le C^{-1} e^{\sqrt{2}y} P(t) \left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta) - t^{\beta\delta/2} \right)^{-1} e^{-\sqrt{2}\, t^{\beta\delta/2}}, \tag{2.5.28}$$

which, for $\bar{x}_j \in \mathcal{L}_{t^\beta}$, converges to 0 as $t \uparrow \infty$. We proceed as in (2.5.20)–(2.5.22), to get that

$$\mathbb{E}\left[\prod_{\substack{j \le n(t^\beta)\\ \bar{x}_j \in \mathcal{L}_{t^\beta}}} \mathbb{E}\left[e^{-\tilde{\mathcal{Y}}_j(t)} \,\middle|\, \mathcal{F}_{t^\beta} \right] \right] = \mathbb{E}\left[\exp\left(-C e^{-\sqrt{2}y} \sum_{\substack{j \le n(t^\beta)\\ \bar{x}_j \in \mathcal{L}_{t^\beta}}} \left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta) - t^{\beta\delta/2} \right) e^{-\sqrt{2}\left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta)\right)} \right) \right] (1 + o(1)). \tag{2.5.29}$$

The sum in (2.5.29) converges to the limit of the derivative martingale.

Lemma 2.5.6. *With the notation from above,*

$$\sum_{\substack{j \le n(t^\beta)\\ \bar{x}_j \in \mathcal{L}_{t^\beta}}} \left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta) - t^{\beta\delta/2} \right) e^{-\sqrt{2}\left(\sqrt{2}\sigma_1 t^\beta - \bar{x}_j(t^\beta)\right)} \to Z, \tag{2.5.30}$$

in probability as $t \uparrow \infty$.

Proof.[14] We refer to the proof of [29, Lemma 3.6] to get that

$$\sum_{\substack{j\le n(t^\beta)\\ \bar{x}_j\in\mathcal{L}_{t^\beta}}}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)\right)\mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)\right)}\to Z, \tag{2.5.31}$$

in probability as $t\uparrow\infty$. It remains to show that

$$\lim_{t\uparrow\infty} t^{\beta\delta}\sum_{j\le n(t^\beta)}\mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)\right)}=0, \tag{2.5.32}$$

in probability for $\delta<1/2$. This follows from the convergence of the Seneta-Heyde scaling of the additive martingale of the branching random walk [83, Theorem 6.1], which can be transferred to the BBM setting (see [83, Subsection 6.5]). □

As almost surely,

$$-C\mathrm{e}^{-\sqrt{2}y}\sum_{\substack{j\le n(t^\beta)\\ \bar{x}_j\in\mathcal{L}_{t^\beta}}}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)-t^{\beta\delta/2}\right)\mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)\right)}<0, \tag{2.5.33}$$

Lemma 2.5.6 implies that

$$\begin{aligned}&\lim_{t\uparrow\infty}\mathbb{P}\left(\max_{j\le n(t):\ \bar{x}_j\in\mathcal{L}}\bar{x}_j(t)-m^-(t)\le y\right)\\ &=\lim_{t\uparrow\infty}\mathbb{E}\left[\exp\left(-C\mathrm{e}^{-\sqrt{2}y}\sum_{\substack{j\le n(t^\beta)\\ \bar{x}_j\in\mathcal{L}_{t^\beta}}}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)-t^{\beta\delta/2}\right)\mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,\sigma_1 t^\beta-\bar{x}_j(t^\beta)\right)}\right)\right]=\mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}}\right].\end{aligned} \tag{2.5.34}$$

This completes the proof of Proposition 2.5.1 up to the proofs of Lemmas 2.5.4 and 2.5.5. □

To complete the proof of Theorem 2.1.3 in Case B, we use the fact that we can approximate the speed functions in that setting from above and below by the speed functions described in the following lemma. We deduce from Proposition 2.5.1 that variable speed BBMs with these speed functions have the same limiting law of the maximum.

Lemma 2.5.7. *Let $\alpha_{\text{begin}},\alpha_{\text{end}}\in(0,1/2)$. Let $(A_t)_{t>0}$ be a family of speed functions which satisfy Assumption 2.1.2 for $\alpha_{\text{begin}},\alpha_{\text{end}}$. Then, there exist $\ell\in\mathbb{N}$ and families of speed functions $(\underline{A}_t)_{t>0},(\overline{A}_t)_{t>0}$ which satisfy for each $t>0$ large enough:*

(i) $\underline{A}_t(s)\le A_t(s)\le\overline{A}_t(s)$ *for all* $s\in[0,1]$.

[14] This proof is done in greater detail than in [4]. This avoids a reference to Section 2.5.1.

(ii) The functions $\underline{A}_t$ and $\overline{A}_t$ are piecewise linear and continuous with slopes

$$\underline{A}_t'(s) = \sum_{k=1}^{\ell} \underline{\sigma}_k^2 \mathbb{1}_{\left(\sum_{j=1}^{k-1} \underline{b}_j, \sum_{j=1}^{k} \underline{b}_j\right)}(s), \quad \text{for } s \in (0,1),$$
$$\overline{A}_t'(s) = \sum_{k=1}^{\ell} \overline{\sigma}_k^2 \mathbb{1}_{\left(\sum_{j=1}^{k-1} \overline{b}_j, \sum_{j=1}^{k} \overline{b}_j\right)}(s), \quad \text{for } s \in (0,1), \tag{2.5.35}$$

where $\underline{\sigma}_k, \underline{b}_k$ (and analogously $\overline{\sigma}_k, \overline{b}_k$) depend on t and satisfy

$$\sum_{k=1}^{\ell} \underline{\sigma}_k^2 \underline{b}_k = 1, \qquad \sum_{k=1}^{\ell} \underline{b}_k = 1 \quad \text{and} \quad \underline{b}_k > 0 \text{ for all } k = 1, \dots, \ell. \tag{2.5.36}$$

(iii) $\underline{A}_t$ and $\overline{A}_t$ satisfy Assumption 2.4.1.(ii)–(iv) with $\alpha_1 = \alpha_{\text{begin}}, \alpha_\ell = \alpha_{\text{end}}$.

Proof. Let $t > 0$. We start with the construction of the first piece of the piecewise linear speed functions $\underline{A}_t$ and $\overline{A}_t$. A_t satisfies Assumption 2.1.2.(a) with corresponding lower and upper bounds $\underline{B}_t$ and $\overline{B}_t$ on $[0, b_{\text{begin}}]$, so we get for all $s \in [0, b_{\text{begin}}]$ that

$$A_t(s) \le \overline{B}_t(s) \le \overline{B}_t(0) + s\overline{B}_t'(0) + \tfrac{s^2}{2} \max_{z \in [0, b_{\text{begin}}]} \overline{B}_t''(z)$$
$$\le \left(1 - t^{-\alpha_{\text{begin}}} + \tfrac{b_{\text{begin}}}{2} \max_{z \in [0, b_{\text{begin}}]} \overline{B}_t''(z)\right) s \equiv \overline{\sigma}_1^2 s, \tag{2.5.37}$$

and

$$A_t(s) \ge \left(1 - t^{-\alpha_{\text{begin}}} - \tfrac{b_{\text{begin}}}{2} \min_{z \in [0, b_{\text{begin}}]} \underline{B}_t''(z)\right) s \equiv \underline{\sigma}_1^2 s. \tag{2.5.38}$$

We set $\underline{b}_1 = \overline{b}_1 = b_{\text{begin}}/2$ and $\underline{A}_t(s) = \underline{\sigma}_1^2 s, \overline{A}_t(s) = \overline{\sigma}_1^2 s$ for all $s \in [0, b_{\text{begin}}/2]$. $\underline{A}_t$ and $\overline{A}_t$ satisfy Assumption 2.4.1.(ii) and $\underline{A}_t(s) \le A_t(s) \le \overline{A}_t(s)$ for all $s \in [0, b_{\text{begin}}/2]$. Analogously, A_t satisfies Assumption 2.1.2.(b) with corresponding lower and upper bounds $\underline{C}_t$ and $\overline{C}_t$ on $[1 - b_{\text{end}}, 1]$, so we set $\underline{b}_\ell = \overline{b}_\ell = b_{\text{end}}/2$ and, for all $s \in [1 - b_{\text{end}}/2, 1]$,

$$1 - \underline{A}_t(s) = \underline{\sigma}_\ell^2 (1 - s),$$
$$1 - \overline{A}_t(s) = \overline{\sigma}_\ell^2 (1 - s), \tag{2.5.39}$$

where

$$\underline{\sigma}_\ell^2 \equiv 1 + t^{-\alpha_{\text{end}}} - \tfrac{b_{\text{end}}}{2} \min_{z \in [1 - b_{\text{end}}, 1]} \underline{C}_t''(z),$$
$$\overline{\sigma}_\ell^2 \equiv 1 + t^{-\alpha_{\text{end}}} + \tfrac{b_{\text{end}}}{2} \max_{z \in [1 - b_{\text{end}}, 1]} \overline{C}_t''(z). \tag{2.5.40}$$

Then, $\underline{A}_t$ and $\overline{A}_t$ satisfy Assumption 2.4.1.(iii) and $\underline{A}_t(s) \le A_t(s) \le \overline{A}_t(s)$ for all $s \in [1 - b_{\text{end}}/2, 1]$. We turn to the construction of $\underline{A}_t$ and $\overline{A}_t$ on $[b_{\text{begin}}/2, 1 - b_{\text{end}}/2]$. For $k = 2, \dots, \ell - 1$, we can choose $\underline{\sigma}_k, \underline{b}_k$ freely, as long as (2.5.36) is satisfied and $\underline{A}_t(s) \le A_t(s)$ for $s \in [b_{\text{begin}}/2, 1 - b_{\text{end}}/2]$.

The latter condition implies that $\underline{A}_t$ satisfies Assumption 2.4.1.(iv). Since A_t satisfies Assumption 2.1.2.(c), there exists $\varepsilon > 0$ such that

$$\min_{s\in[b_{\text{begin}},1-b_{\text{end}}]} (s - A_t(s)) \geq t^{-1/2+\varepsilon}, \tag{2.5.41}$$

for t large enough. For $s \in [b_{\text{begin}}, 1 - b_{\text{end}}]$, we set

$$\overline{A}_t(s) = s - t^{-1/2+\varepsilon}. \tag{2.5.42}$$

This ensures that on $[b_{\text{begin}}, 1-b_{\text{end}}]$, $\overline{A}_t$ is a piecewise linear upper bound of A_t, which satisfies Assumption 2.4.1.(iv). Then, we set $\overline{A}_t$ to be continuous in $b_{\text{begin}}/2$ as well as b_{begin} with $\overline{A}_t'(s) = \overline{\sigma}_2^2$ for $s \in (b_{\text{begin}}/2, b_{\text{begin}})$, where

$$\overline{\sigma}_2^2 \equiv \tfrac{2}{b_{\text{begin}}} \left(\overline{A}_t(b_{\text{begin}}) - \overline{A}_t(b_{\text{begin}}/2)\right) = \tfrac{2}{b_{\text{begin}}} \left((1 - \overline{\sigma}_1^2/2)b_{\text{begin}} - t^{-1/2+\varepsilon}\right). \tag{2.5.43}$$

By Assumption 2.1.2.(a).(i), $\overline{\sigma}_2^2 \in (1, \infty)$ for ε close to 0 and t large enough. Analogously, we set the slope $\overline{\sigma}_4^2$ on $(1 - b_{\text{end}}, 1 - b_{\text{end}}/2)$ as

$$\overline{\sigma}_4^2 \equiv \tfrac{2}{b_{\text{end}}} \left(\overline{A}_t(1 - b_{\text{end}}/2) - \overline{A}_t(1 - b_{\text{end}})\right), \tag{2.5.44}$$

which lies in $(0, 1)$ for ε close to 0 and t large. Thus, $\overline{A}_t$ is a piecewise linear and continuous upper bound of A_t, which satisfies Assumption 2.4.1.(iv) on $[b_{\text{begin}}/2, 1 - b_{\text{end}}/2]$. □

Remark. In the proof of Lemma 2.5.7, it becomes clear that it is always possible to choose $\ell = 5$.

Proof of Theorem 2.1.3. Let $(\underline{A}_t)_{t>0}$ and $(\overline{A}_t)_{t>0}$ be the speed functions from Lemma 2.5.7 corresponding to $(A_t)_{t>0}$, α_{begin} and α_{end}. Let $(\underline{X}_t)_{t>0}, (\overline{X}_t)_{t>0}$ be variable speed BBMs with speed functions $(\underline{A}_t)_{t>0}, (\overline{A}_t)_{t>0}$. Since $(\underline{A}_t)_{t>0}$ and $(\overline{A}_t)_{t>0}$ satisfy Assumption 2.4.1, Proposition 2.5.1 implies that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{j\leq n(t)} \underline{x}_j(t) - m^-(t) \leq y\right) = \lim_{t\uparrow\infty} \mathbb{P}\left(\max_{j\leq n(t)} \overline{x}_j(t) - m^-(t) \leq y\right) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}}\right]. \tag{2.5.45}$$

Since $\underline{A}_t \leq A_t \leq \overline{A}_t$ for all $t > 0$, we get with Gaussian comparison (see Lemma 2.3.4) and (2.5.45) that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{j\leq n(t)} \tilde{x}_j(t) - m^-(t) \leq y\right) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}}\right]. \tag{2.5.46}$$

This completes the proof of Theorem 2.1.3 in Case B. □

2.5.3 Proof of Theorem 2.1.4 in Case B[15]

Let[16] $\tilde{X}$ be a VSBBM with piecewise linear speed functions satisfying Assumption 2.4.1. We compute, for $y \in \mathbb{R}$ and for $\phi \in C^\infty(\mathbb{R})$ which are nondecreasing with support bounded from the left and for which there exists $a \in \mathbb{R}$ such that $\phi(x)$ is constant for $x > a$,

$$
\begin{aligned}
&\Psi_t(\phi(\cdot - y)) \qquad (2.5.47)\\
&= \mathbb{E}\left[\mathrm{e}^{-\sum_{j=1}^{n(t)} \phi\left(\tilde{x}_j(t) - m^\pm(t) - y\right)} \right]\\
&= \mathbb{E}\left[\exp\left(- \sum_{i_1 \le n(b_1 t)} \sum_{i_2 \le n^{i_1}(b_2 t)} \cdots \sum_{i_\ell \le n^{\bar{i}_{\ell-1}}(b_\ell(t))} \phi\left(\sum_{j=1}^{\ell} \sigma_j x_{i_j}^{\bar{i}_{j-1}}(b_j t) - m^\pm(t) - y \right) \right) \right]\\
&= \mathbb{E}\left[\prod_{i_1 \le n(b_1 t)} \mathbb{E}\left[\prod_{i_2 \le n^{i_1}(b_2 t)} \ldots \mathbb{E}\left[\prod_{i_\ell \le n^{\bar{i}_{\ell-1}}(b_\ell(t))} \exp\left(-\phi\left(\sum_{j=1}^{\ell} \sigma_j x_{i_j}^{\bar{i}_{j-1}}(b_j t) - m^\pm(t) - y \right) \right) \middle| \mathcal{F}_{a_{\ell-1} t} \right] \ldots \middle| \mathcal{F}_{a_1 t} \right] \right],
\end{aligned}
$$

where $x_{i_j}^{\bar{i}_{j-1}}, 1 \le j \le \ell$, denote standard BBMs. We want to show that the innermost conditional expectation is a solution to the F-KPP equation and use the asymptotics of these solutions. Recalling that the velocities $\sigma_j, 1 \le j \le \ell$, depend on t, we set

$$
f^t(z) = \mathrm{e}^{-\phi(-\sigma_\ell(t) z)}, \qquad v^t(s, z) = \mathbb{E}\left[\prod_{i_\ell \le n^{\bar{i}_{\ell-1}}(s)} f^t\left(z - x_{i_\ell}(s)\right) \right]. \tag{2.5.48}
$$

For fixed t, the function $1 - v^t$ is a solution to the F-KPP equation with initial conditions $1 - v^t(0, x) = 1 - f^t(x)$. Since the initial conditions depend on the time horizon t, we need the following generalisation of Proposition 2.3.1.

Proposition 2.5.8 [29, Proposition 5.2]. *Let u^t be a family of solutions to the F-KPP equation with initial data satisfying*

$$
u^t(0, x) \to u(0, x), \tag{2.5.49}
$$

pointwise and monotone, for $x \in \mathbb{R}$ as $t \uparrow \infty$, where $u(0, x)$ satisfies the conditions in Proposition 2.3.1(i)–(iv). Then, for any function $z\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\uparrow\infty} z(t)/t = 0$,

$$
\lim_{t\uparrow\infty} \mathrm{e}^{\sqrt{2} z(t)} \mathrm{e}^{(z(t))^2/(2t)} (z(t))^{-1} u^t\left(t, z(t) + \sqrt{2}\, t - \tfrac{3}{2\sqrt{2}} \ln(t)\right) = C, \tag{2.5.50}
$$

where C is the constant from Proposition 2.3.1 and u is the solution of the F-KPP equation with initial condition $u(0, \cdot)$.

[15] We added “in Case B” in the title.

[16] We removed a reference to Case A in this sentence.

For $t \uparrow \infty$, $1 - f^t(x) \to 1 - \mathrm{e}^{-\phi(-x)}$ pointwise and monotone for $x \geq 0$. Therefore, we can apply Proposition 2.5.8 to $u^t \equiv 1 - v^t$. Hence[17]

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i_\ell \leq n^{\bar{i}_{\ell-1}}(b_\ell(t))} \exp\left(-\phi\left(\sum_{j=1}^{\ell} \sigma_j\, x_{i_j}^{\bar{i}_{j-1}}(b_j t) - m^-(t) - y\right)\right)\middle|\mathcal{F}_{a_{\ell-1}t}\right] \\
&= v^t\left(b_\ell t, \tfrac{1}{\sigma_\ell(t)}\left(m^-(t) + y - \sum_{j=1}^{\ell} \tilde{x}_{i_j}^{\bar{i}_{j-1}}(b_j t)\right)\right) \\
&= \left(1 - C(\phi)\Delta^{\bar{i}_{\ell-1}}(t)\,\mathrm{e}^{-\sqrt{2}\,\Delta^{\bar{i}_{\ell-1}}(t) - \frac{\Delta^{\bar{i}_{\ell-1}}(t)^2}{b_\ell t}}\right)(1 + o(1)),
\end{aligned} \tag{2.5.51}$$

where $C(\phi)$ is the constant from Proposition 2.3.1 for the initial condition $u(0, x) = 1 - \mathrm{e}^{-\phi(-x)}$ and $\Delta^{\bar{i}_{\ell-1}}$ is defined in (2.5.11).

From now on, following the same computations as in the proof of Theorem 2.1.3, we get that for variable speed BBMs with piecewise linear speed functions

$$\lim_{t\uparrow\infty} \Psi_t(\phi(\cdot - y)) = \mathbb{E}\left[\mathrm{e}^{-C(\phi)Z\mathrm{e}^{-\sqrt{2}y}}\right]. \tag{2.5.52}$$

The Laplace functional in (2.1.16) corresponds to the point processes in the right-hand side of (2.1.16), see [10]. [18]

It remains to show convergence of the extremal process for variable speed BBM $(\tilde{X}_t)_{t>0}$ with general (not necessarily piecewise linear) speed functions $(A_t)_{t>0}$, which satisfy Assumption 2.1.2. For each $t > 0$, given the underlying Galton-Watson tree of $\tilde{X}_t$, let $\underline{X}_t$ and $\overline{X}_t$ be independent Gaussian processes on this tree with mean 0 and speed functions $\underline{A}_t, \overline{A}_t$ from Lemma 2.5.7. By (2.5.52),

$$\lim_{t\uparrow\infty} \underline{\Psi}_t(\phi(\cdot - y)) = \lim_{t\uparrow\infty} \overline{\Psi}_t(\phi(\cdot - y)) = \mathbb{E}\left[\mathrm{e}^{-C(\phi)Z\mathrm{e}^{-\sqrt{2}y}}\right], \tag{2.5.53}$$

where

$$\begin{aligned}
\underline{\Psi}_t(\phi(\cdot - y)) &\equiv \mathbb{E}\left[\exp\left(-\sum_{j=1}^{n(t)} \phi\left(\underline{x}_j(t) - m^-(t) - y\right)\right)\right], \\
\overline{\Psi}_t(\phi(\cdot - y)) &\equiv \mathbb{E}\left[\exp\left(-\sum_{j=1}^{n(t)} \phi\left(\overline{x}_j(t) - m^-(t) - y\right)\right)\right].
\end{aligned} \tag{2.5.54}$$

Thus, it suffices to prove, for $t > 0$, that

$$\mathbb{E}\left[F\left((\underline{x}_j(t))_{j\leq n(t)}\right)\middle|\mathcal{F}_t^{\text{tree}}\right] \leq \mathbb{E}\left[F\left((\tilde{x}_j(t))_{j\leq n(t)}\right)\middle|\mathcal{F}_t^{\text{tree}}\right] \leq \mathbb{E}\left[F\left((\overline{x}_j(t))_{j\leq n(t)}\right)\middle|\mathcal{F}_t^{\text{tree}}\right], \tag{2.5.55}$$

---

[17] This sentence rephrases a reference to Subsection 2.5.1. We also changed $m^{\pm}$ to m^- in (2.5.51).

[18] We removed a sentence with a reference to Case A.

where $\mathcal{F}_t^{\text{tree}}$ is the σ-algebra of the Galton-Watson tree of $\tilde{X}_t$, $\underline{X}_t$ and $\overline{X}_t$ and

$$F\colon \mathbb{R}^{n(t)} \to \mathbb{R}, \quad z \mapsto \exp\Bigg(-\sum_{j=1}^{n(t)} \phi\Big(z_j - m^-(t) - y\Big)\Bigg). \tag{2.5.56}$$

Note that inside the conditional expectations in (2.5.55), we view $n(t)$ as a constant. Since ϕ is nondecreasing, we get for all $x \in \mathbb{R}^{n(t)}$ and $1 \le i_1, i_2 \le n(t)$, $i_1 \neq i_2$, that

$$\tfrac{\partial^2}{\partial x_{i_1} \partial x_{i_2}} F(x) = \phi'(x_{i_1} - m^-(t) - y)\, \phi'(x_{i_2} - m^-(t) - y)\, F(x) \ge 0. \tag{2.5.57}$$

By Kahane's theorem, see for example [26, Theorem 3.5], (2.5.55) follows from (2.5.57). This completes the proof of Theorem 2.1.4 in Case B. □

2.6 Appendix

We[19] prove the moment estimates from Subsection 2.5.2. The proofs of moment estimates of Subsection 2.5.1 are omitted in this thesis.

Proof of Lemma 2.5.2. Recall that

$$\mathcal{Y}_{i_1}(t) = \sum_{\substack{i_2 \le n^{i_1}(b_2 t), \dots, i_{\ell-1} \le n^{\bar{i}_{\ell-2}}(b_{\ell-1} t), \\ \sum_{k=1}^{\ell-1} \bar{x}_{i_k}^{\bar{i}_{k-1}} \in \mathcal{T}_{b_1 t, (1-b_\ell)t, 0, \gamma}}} C \Delta^{\bar{i}_{\ell-1}}(t) \exp\left(b_\ell t - \frac{\left(\Delta^{\bar{i}_{\ell-1}}(t) + \sqrt{2}\, b_\ell t\right)^2}{2 b_\ell t} \right), \tag{2.5.13}$$

where

$$\Delta^{\bar{i}_{\ell-1}}(t) + \sqrt{2}\, b_\ell t = \tfrac{1}{\sigma_\ell} \left(\sqrt{2}\, t - \sum_{k=2}^{\ell-1} \bar{x}_{i_k}^{\bar{i}_{k-1}}(b_k t) - \bar{x}_{i_1}(b_1 t) + L(t) + y \right). \tag{2.6.1}$$

Since $\sum_{k=2}^{\ell-1} \bar{x}_{i_k}^{\bar{i}_{k-1}}(b_k t)$ is centred with variance $(1 - \sigma_1^2 b_1 - \sigma_\ell^2 b_\ell)t$,

$$\begin{aligned} \mathbb{E}\left[\mathcal{Y}_{i_1}(t) \middle| \mathcal{F}_{b_1 t} \right] = {} & C e^{(1-b_1)t} \int_I \frac{d\omega}{\sqrt{2\pi(1-\sigma_1^2 b_1 - \sigma_\ell^2 b_\ell)t}} e^{-\frac{\omega^2}{2(1-\sigma_1^2 b_1 - \sigma_\ell^2 b_\ell)t}} \\ & \times \tfrac{1}{\sigma_\ell} \left(\sqrt{2}\,(1 - \sigma_\ell b_\ell)t - \omega - \bar{x}_{i_1}(b_1 t) + L(t) + y \right) e^{-\frac{\left(\sqrt{2}\, t - \omega - \bar{x}_{i_1}(b_1 t) + L(t) + y\right)^2}{2\sigma_\ell^2 b_\ell t}} \\ & \times \mathbb{P}\left(\left(\mathfrak{z}^{(1-\sigma_1^2 b_1 - \sigma_\ell^2 b_\ell)t}_{\bar{x}_{i_1}(b_1 t), \bar{x}_{i_1}(b_1 t)+\omega} (A_t(s/t)t - \sigma_1^2 b_1 t) \right)_{s \in [b_1 t, 1 - b_\ell t]} \in \mathcal{T}_{b_1 t, (1-b_\ell)t, 0, \gamma} \right), \end{aligned} \tag{2.6.2}$$

where

$$I \equiv \left(\sqrt{2}\,(1 - \sigma_\ell^2 b_\ell)t - \bar{x}_{i_1}(b_1 t) - (\sigma_\ell^2 b_\ell t)^\gamma, \sqrt{2}\,(1 - \sigma_\ell^2 b_\ell)t - \bar{x}_{i_1}(b_1 t) + (\sigma_\ell^2 b_\ell t)^\gamma \right). \tag{2.6.3}$$

For $\bar{x}_{i_1} \in \mathcal{L}_{b_1 t}$ and ω in the range of integration of (2.6.2), we find, as in (2.4.24)–(2.4.30), that

$$\lim_{t \uparrow \infty} \mathbb{P}\left(\left(\mathfrak{z}^{(1-\sigma_1^2 b_1 - \sigma_\ell^2 b_\ell)t}_{\bar{x}_{i_1}(b_1 t), \bar{x}_{i_1}(b_1 t)+\omega} (A_t(s/t)t - \sigma_1^2 b_1 t) \right)_{s \in [b_1 t, 1 - b_\ell t]} \in \mathcal{T}_{b_1 t, (1-b_\ell)t, 0, \gamma} \right) = 1, \tag{2.6.4}$$

and

$$\begin{aligned} \tfrac{1}{\sigma_\ell} \left(\sqrt{2}\,(1 - \sigma_\ell b_\ell)t - \omega - \bar{x}_{i_1}(b_1 t) + L(t) + y \right) &= \sqrt{2}\,(\sigma_\ell - 1) b_\ell t + O((\sigma_\ell^2 b_\ell t)^\gamma) \\ &= \tfrac{1}{\sqrt{2}} b_\ell t^{1-\alpha_\ell} (1 + o(1)). \end{aligned} \tag{2.6.5}$$

[19]We rephrased the introductory sentences of this section.

Inserting (2.6.4) and (2.6.5) into (2.6.2), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{Y}_{i_1}(t)\,\middle|\,\mathcal{F}_{b_1 t}\right] \\
&= \frac{C}{\sqrt{2}} b_\ell t^{1-\alpha_\ell} \mathrm{e}^{(1-b_\ell)t} \mathrm{e}^{-\frac{\left(\sqrt{2}t-\bar{x}_{i_1}(b_1 t)+L(t)+y\right)^2}{2(1-\sigma_1^2 b_1)t}} \int_I \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} \qquad (2.6.6) \\
&\quad \times \exp\left(-(1-\sigma_1^2 b_1)\frac{\left(\omega-(1-\sigma_1^2 b_1)^{-1}(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)\left(\sqrt{2}\,t-\bar{x}_{i_1}(b_1 t)+L(t)+y\right)\right)^2}{2(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)\sigma_\ell^2 b_\ell t}\right)(1+o(1)).
\end{aligned}$$

For $\gamma > 1/2$ and $\bar{x}_{i_1} \in \mathcal{L}_{b_1 t}$, the integral in (2.6.6) equals

$$\frac{\sigma_\ell b_\ell^{1/2}}{(1-\sigma_1^2 b_1)^{1/2}}(1+o(1)). \qquad (2.6.7)$$

Inserting (2.6.7) into (2.6.6) completes the proof. □

Proof of Lemma 2.5.3. We[20] write

$$\mathbb{E}\left[\mathcal{Y}_{i_1}^2(t)\,\middle|\,\mathcal{F}_{b_1 t}\right] = (T1) + (T2), \qquad (2.6.8)$$

with

$$\begin{aligned}
(T1) &\equiv C^2 \mathrm{e}^{2b_\ell t} \mathbb{E}\Bigg[\sum_{\substack{i_2 \le n^{i_1}(b_2 t),\dots,\\ i_{\ell-1} \le n^{\bar{i}_{\ell-2}}(b_{\ell-1} t)}} \mathbb{1}_{\sum_{k=1}^{\ell-1} \bar{x}_{i_k}^{\bar{i}_{k-1}} \in \mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}} \left(\Delta^{\bar{i}_{\ell-1}}(t)\right)^2 \mathrm{e}^{-\frac{\left(\Delta^{\bar{i}_{\ell-1}}(t)+\sqrt{2}\,b_\ell t\right)^2}{b_\ell t}} \Bigg| \mathcal{F}_{b_1 t}\Bigg], \\
(T2) &\equiv C^2 \mathrm{e}^{2b_\ell t} \mathbb{E}\Bigg[\sum_{\substack{i_2, j_2 \le n^{i_1}(b_2 t),\dots, i_{\ell-1} \le n^{\bar{i}_{\ell-2}}(b_{\ell-1} t),\\ j_{\ell-1} \le n^{\bar{j}_{\ell-2}}(b_{\ell-1} t), \bar{i}_{\ell-1} \neq \bar{j}_{\ell-1}}} \mathbb{1}_{\sum_{k=1}^{\ell-1} \bar{x}_{i_k}^{\bar{i}_{k-1}},\, \sum_{k=1}^{\ell-1} \bar{x}_{j_k}^{\bar{j}_{k-1}} \in \mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma}} \\
&\quad \times \Delta^{\bar{i}_{\ell-1}}(t) \Delta^{\bar{j}_{\ell-1}}(t) \exp\left(-\frac{\left(\Delta^{\bar{i}_{\ell-1}}(t)+\sqrt{2}\,b_\ell t\right)^2 + \left(\Delta^{\bar{j}_{\ell-1}}(t)+\sqrt{2}\,b_\ell t\right)^2}{2b_\ell t}\right) \Bigg| \mathcal{F}_{b_1 t}\Bigg]. \qquad (2.6.9)
\end{aligned}$$

Using that $\mathcal{T}_{b_1 t,(1-b_\ell)t,0,\gamma} \subset \mathcal{T}_{(1-b_\ell)t,(1-b_\ell)t,0,\gamma}$, we obtain, as in (2.6.1)–(2.6.2), by the many-to-one lemma that

$$\begin{aligned}
(T1) &\le C^2 \mathrm{e}^{(1-b_1+b_\ell)t} \int_I \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} \mathrm{e}^{-\frac{\omega^2}{2(1-\sigma_1^2 b_1-\sigma_\ell^2 b_\ell)t}} \qquad (2.6.10) \\
&\quad \times \frac{1}{\sigma_\ell^2}\left(\sqrt{2}\,(1-\sigma_\ell b_\ell)t - \omega - \bar{x}_{i_1}(b_1 t) + L(t) + y\right)^2 \mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega-\bar{x}_{i_1}(b_1 t)+L(t)+y\right)^2}{\sigma_\ell^2 b_\ell t}},
\end{aligned}$$

with I as in (2.6.3). We have seen in (2.6.5) that for ω in the range of integration of (2.6.10),

$$\frac{1}{\sigma_\ell^2}\left(\sqrt{2}\,(1-\sigma_\ell b_\ell)t - \omega - \bar{x}_{i_1}(b_1 t) + L(t) + y\right)^2 \le P(t). \qquad (2.6.11)$$

[20]This sentence rephrases a reference to omitted parts of this section.

Recall that P is universal notation for any function satisfying $P(t) \le t^c$ for some constant $c > 0$ and all $t > 0$ large enough. The exponential terms inside the integral in (2.6.10) are equal to

$$\exp\left(-\frac{\left(\sqrt{2}\,t-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2(1-\sigma_1^2b_1)t}\right)\exp\left(-\frac{\left(\sqrt{2}\,t-\omega-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2\sigma_\ell^2b_\ell t}\right)$$
$$\times\exp\left(-(1-\sigma_1^2b_1)\frac{\left(\omega-\left(1-\sigma_1^2b_1\right)^{-1}\left(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell\right)\left(\sqrt{2}\,t-\bar{x}_{i_1}(b_1t)+L(t)+y\right)\right)^2}{2(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)\sigma_\ell^2b_\ell t}\right). \tag{2.6.12}$$

We insert (2.6.12) back into the right-hand side of (2.6.10), bound the last exponential term by 1 and shift the integral by $\sqrt{2}\,(1-\sigma_\ell^2b_\ell)t-\bar{x}_{i_1}(b_1t)$. This gives

$$(T1) \le P(t)\mathrm{e}^{(1-b_1+b_\ell)t}\exp\left(-\frac{\left(\sqrt{2}\,t-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2(1-\sigma_1^2b_1)t}\right)\int_{-(\sigma_\ell^2b_\ell t)^\gamma}^{(\sigma_\ell^2b_\ell t)^\gamma}\frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_1^2b_1-\sigma_\ell^2b_\ell)t}}\exp\left(-\frac{\left(\sqrt{2}\,\sigma_\ell^2b_\ell t-\omega+L(t)+y\right)^2}{2\sigma_\ell^2b_\ell t}\right). \tag{2.6.13}$$

For ω in the range of the integral in (2.6.13),

$$b_\ell t-\frac{\left(\sqrt{2}\,\sigma_\ell^2b_\ell t-\omega+L(t)+y\right)^2}{2\sigma_\ell^2b_\ell t} \le (1-\sigma_\ell^2)b_\ell t+\sqrt{2}\,(\sigma_\ell^2b_\ell t)^\gamma+\sqrt{2}L(t)+y = -b_\ell t^{1-\alpha_\ell}(1+o(1)). \tag{2.6.14}$$

Assumption 2.4.1.(iii) implies that $b_\ell t^{1-\alpha_\ell} \gg t^{1/2}$, so $(T1)$ is bounded by the right-hand side of (2.5.19). Dropping the condition $\mathcal{T}_{b_1t,(1-b_\ell)t,0,\gamma}$ except at the endpoint and at the time of the branching gives via the many-to-two lemma

$$(T2) \le C^2\mathrm{e}^{2(1-b_1)t}\int_{b_1t}^{(1-b_\ell)t}\mathrm{d}s\,\mathrm{e}^{-(s-b_1t)}\int_{I_1}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi\left(A_t(s/t)-\sigma_1^2b_1\right)t}}\mathrm{e}^{-\frac{\omega_1^2}{2\left(A_t(s/t)-\sigma_1^2b_1\right)t}}$$
$$\times\left(\int_{I_2}\frac{\mathrm{d}\omega_2}{\sqrt{2\pi\left(1-\sigma_\ell^2b_\ell-A_t(s/t)\right)t}}\mathrm{e}^{-\frac{\omega_2^2}{2\left(1-\sigma_\ell^2b_\ell-A_t(s/t)\right)t}}\right. \tag{2.6.15}$$
$$\left.\times\tfrac{1}{\sigma_\ell}\left(\sqrt{2}\,(1-\sigma_\ell b_\ell)t-\omega_1-\omega_2-\bar{x}_{i_1}(b_1t)+L(t)+y\right)\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega_1-\omega_2-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2\sigma_\ell^2b_\ell t}}\right)^2,$$

where

$$f_{t,\gamma}(s) \equiv \left(A_t(s/t)\wedge(1-A_t(s/t))\right)^\gamma t^\gamma, \tag{2.6.16}$$
$$I_1 \equiv \left(\bar{x}_{i_1}(b_1t)-\sqrt{2}\,tA_t(s/t)-f_{t,\gamma}(s),\bar{x}_{i_1}(b_1t)-\sqrt{2}\,tA_t(s/t)+f_{t,\gamma}(s)\right),$$
$$I_2 \equiv \left(\bar{x}_{i_1}(b_1t)+\omega_1-\sqrt{2}\,(1-\sigma_\ell^2b_\ell)t-(\sigma_\ell^2b_\ell t)^\gamma,\bar{x}_{i_1}(b_1t)+\omega_1-\sqrt{2}\,(1-\sigma_\ell^2b_\ell)t+(\sigma_\ell^2b_\ell t)^\gamma\right).$$

In the ω_2-integral in (2.6.16),

$$\tfrac{1}{\sigma_\ell}\left(\sqrt{2}\,(1-\sigma_\ell b_\ell)t-\omega_1-\omega_2-\bar{x}_{i_1}(b_1t)+L(t)+y\right) \le P(t), \tag{2.6.17}$$

and the exponential terms are equal to

$$\exp\left(-\frac{\left(\sqrt{2}\,t-\omega_1-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2(1-A_t(s/t))t}\right)\exp\left(-\frac{(1-A_t(s/t))\left(\omega_2-\frac{1-\sigma_\ell^2 b_\ell-A_t(s/t)}{1-A_t(s/t)}\left(\sqrt{2}\,t-\omega_1-\bar{x}_{i_1}(b_1t)+L(t)+y\right)\right)^2}{2\left(1-\sigma_\ell^2 b_\ell-A_t(s/t)\right)\sigma_\ell^2 b_\ell t}\right). \tag{2.6.18}$$

The first factor in (2.6.18) does not depend on ω_2 and the integral of the second factor over ω_2 is Gaussian and thus can be bounded by $P(t)$. This shows that the ω_2-integral in (2.6.16) is bounded by

$$P(t)\,\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega_1-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2(1-A_t(s/t))t}}. \tag{2.6.19}$$

Inserting this bound into (2.6.16) and then proceeding as in (2.6.12)–(2.6.13), we obtain

$$\begin{aligned}(T2) &\le P(t)\,\mathrm{e}^{2(1-b_1)t}\int_{b_1t}^{(1-b_\ell)t}\mathrm{d}s\,\mathrm{e}^{-(s-b_1t)}\int_{I_1}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi\left(A_t(s/t)-\sigma_1^2 b_1\right)t}}\mathrm{e}^{-\frac{\omega_1^2}{2\left(A_t(s/t)-\sigma_1^2 b_1\right)t}}\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega_1-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{(1-A_t(s/t))t}}\\ &\le P(t)\,\mathrm{e}^{(2-b_1)t}\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2\left(1-\sigma_1^2 b_1\right)t}}\int_{b_1t}^{(1-b_\ell)t}\mathrm{d}s\,\mathrm{e}^{-s}\int_{-f_{t,\gamma}(s)}^{f_{t,\gamma}(s)}\frac{\mathrm{d}\omega_1}{\sqrt{2\pi\left(A_t(s/t)-\sigma_1^2 b_1\right)t}}\mathrm{e}^{-\frac{\left(\sqrt{2}\,(1-A_t(s/t))t-\omega_1+L(t)+y\right)^2}{2(1-A_t(s/t))t}}.\end{aligned} \tag{2.6.20}$$

Bounding ω_1 by $f_{t,\gamma}(s)$, we see that the ω_1-integral is not larger than

$$P(t)\,\mathrm{e}^{-\frac{\left(\sqrt{2}\,(1-A_t(s/t))t-f_{t,\gamma}(s)+L(t)+y\right)^2}{2(1-A_t(s/t))t}}, \tag{2.6.21}$$

so we have

$$(T2) \le P(t)\,\mathrm{e}^{(1-b_1)t}\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\bar{x}_{i_1}(b_1t)+L(t)+y\right)^2}{2\left(1-\sigma_1^2 b_1\right)t}}\int_{b_1t}^{(1-b_\ell)t}\mathrm{d}s\,\mathrm{e}^{tA_t(s/t)-s+\sqrt{2}f_{t,\gamma}(s)}(1+o(1)). \tag{2.6.22}$$

For all s in the range of the integral in (2.6.22), $f_{t,\gamma}(s) \le t^\gamma$ and by (2.5.18), $tA_t(s/t) - s \le -t^{\tilde{\gamma}}$ for some $\tilde{\gamma} > \gamma$ and t large enough. Thus, the integrand in (2.6.22) is bounded from above by $\mathrm{e}^{-t^{\tilde{\gamma}}}(1 + o(1))$ for t large enough. We conclude that $(T2)$ is not larger than the right-hand side of (2.5.19). □

Proof of Lemma 2.5.4. By the many-to-one lemma,

$$\begin{aligned}\mathbb{E}\left[\tilde{\mathcal{Y}}_j(t)\,\middle|\,\mathcal{F}_{t^\beta}\right] &= \frac{C}{\sqrt{2(1-\sigma_1^2 b_1)}}b_\ell^{3/2}t^{1-\alpha_\ell}\mathrm{e}^{t-t^\beta}\int_J\frac{\mathrm{d}\omega}{\sqrt{2\pi\sigma_1^2 t^*}}\mathrm{e}^{-\frac{\omega^2}{2\sigma_1^2 t^*}}\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega-\bar{x}_j(t^\beta)+L(t)+y\right)^2}{2(1-\sigma_1^2 b_1)t}}\\ &\quad\times\mathbb{P}\left(\mathfrak{z}^{t^*}_{\sigma_1^{-1}\bar{x}_j(t^\beta),\,\sigma_1^{-1}\left(\bar{x}_j(t^\beta)+\omega\right)}(s) < \sqrt{2}\,s - t^{\beta\delta/2}\ \forall_{s\in[0,t^*]}\right),\end{aligned} \tag{2.6.23}$$

where

$$J \equiv \left(\sqrt{2}\,\sigma_1^2 b_1 t - \bar{x}_j(t^\beta) - (\sigma_1^2 b_1 t)^\gamma,\ \sqrt{2}\,\sigma_1^2 b_1 t - \bar{x}_j(t^\beta) + (\sigma_1^2 b_1 t)^\gamma\right). \tag{2.6.24}$$

By Lemma 2.3.6, the probability in (2.6.23) satisfies

$$\begin{aligned}
&\mathbb{P}\left(\mathfrak{z}^{t^*}_{\sigma_1^{-1}\bar{x}_j(t^\beta),\,\sigma_1^{-1}(\bar{x}_j(t^\beta)+\omega)}(s) < \sqrt{2}\,s - t^{\beta\delta/2}\ \forall_{s\in[0,t^*]}\right)\\
&= \mathbb{P}\left(\mathfrak{z}^{t^*}_{\sigma_1^{-1}(\bar{x}_j(t^\beta)+t^{\beta\delta/2})-\sqrt{2}\,t^\beta,\,\sigma_1^{-1}(\bar{x}_j(t^\beta)+\omega+t^{\beta\delta/2})-\sqrt{2}\,b_1 t}(s) < 0\ \forall_{s\in[0,t^*]}\right)\\
&= 1-\exp\left(-\tfrac{2}{t^*}\left(\sqrt{2}\,t^\beta - \tfrac{\bar{x}_j(t^\beta)+t^{\beta\delta/2}}{\sigma_1}\right)\left(\sqrt{2}\,b_1 t - \tfrac{\bar{x}_j(t^\beta)+\omega+t^{\beta\delta/2}}{\sigma_1}\right)\right)\\
&= \sqrt{2}\,t^{-\alpha_1}\left(\sqrt{2}\,\sigma_1 t^\beta - \bar{x}_j(t^\beta) - t^{\beta\delta/2}\right)(1+o(1)).
\end{aligned} \tag{2.6.25}$$

In the last step we used that for $\bar{x}_j \in \mathcal{L}_{t^\beta}$,

$$0 \le \sqrt{2}\,t^\beta - \tfrac{\bar{x}_j(t^\beta)+t^{\beta\delta/2}}{\sigma_1} = O(t^\beta), \tag{2.6.26}$$

and that for ω in the range of integration in (2.6.23),

$$\sqrt{2}\,b_1 t - \tfrac{\bar{x}_j(t^\beta)+\omega+t^{\beta\delta/2}}{\sigma_1} = 2^{-1/2} b_1 t^{1-\alpha_1}(1+o(1)). \tag{2.6.27}$$

Inserting (2.6.25) into (2.6.23), we get that $\mathbb{E}\left[\tilde{\mathcal{Y}}_j(t)\,\middle|\,\mathcal{F}_{t^\beta}\right]$ is up to an error term $(1+o(1))$ equal to

$$\begin{aligned}
&C b_\ell^{3/2} t^{1-\alpha_1-\alpha_\ell}\left(\sqrt{2}\,\sigma_1 t^\beta - \bar{x}_j(t^\beta) - t^{\beta\delta/2}\right)\mathrm{e}^{t-t^\beta}\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\bar{x}_j(t^\beta))+L(t)+y\right)^2}{2(t-\sigma_1^2 t^\beta)}}\\
&\times\int_J \frac{\mathrm{d}\omega}{\sqrt{2\pi(1-\sigma_1^2 b_1)\sigma_1^2 t^*}}\exp\left(-\tfrac{t-\sigma_1^2 t^\beta}{2\sigma_1^2(1-\sigma_1^2 b_1)tt^*}\left(\omega-\tfrac{\sigma_1^2 t^*}{t-\sigma_1^2 t^\beta}\left(\sqrt{2}\,t-\bar{x}_j(t^\beta)+L(t)+y\right)\right)^2\right).
\end{aligned} \tag{2.6.28}$$

A Gaussian tail bound shows that for $\gamma > 1/2$ and $\bar{x}_j \in \mathcal{L}_{t^\beta}$, the integral in (2.6.28) is of order 1. Recalling the definition of $L(t)$ in (2.4.17), we find that (2.6.28) is equal to the right-hand side of (2.5.26). □

Proof of Lemma 2.5.5. The structure of this proof is the same as that of Lemma 2.5.3, which we will refer to for explanations. We write

$$\mathbb{E}\left[\tilde{\mathcal{Y}}_j^2(t)\,\middle|\,\mathcal{F}_{t^\beta}\right] = (T1) + (T2), \tag{2.6.29}$$

where

$$(T1) \equiv \left(\frac{Cb_\ell^{3/2}t^{1-\alpha_\ell}}{\sqrt{2(1-\sigma_1^2 b_1)}}\right)^2 \mathrm{e}^{2(1-b_1)t}\,\mathbb{E}\left[\sum_{\substack{j^*\le n^j(t^*),\\ \bar{x}^j_{j^*}\in\mathcal{L}_{t^*}}} \mathrm{e}^{-\frac{\left(\sqrt{2}t-\bar{x}^j_{j^*}(t^*)-\bar{x}_j(t^\beta)+L(t)+y\right)^2}{(1-\sigma_1^2 b_1)t}}\,\middle|\,\mathcal{F}_{b_1 t}\right],$$

$$(T2) \equiv \left(\frac{Cb_\ell^{3/2}t^{1-\alpha_\ell}}{\sqrt{2(1-\sigma_1^2 b_1)}}\right)^2 \mathrm{e}^{2(1-b_1)t}\,\mathbb{E}\left[\sum_{\substack{j^*,k^*\le n^j(t^*),\ j^*\neq k^*,\\ \bar{x}^j_{j^*},\,\bar{x}^j_{k^*}\in\mathcal{L}_{t^*}}} \mathrm{e}^{-\frac{\left(\sqrt{2}t-\bar{x}^j_{j^*}(t^*)-\bar{x}_j(t^\beta)+L(t)+y\right)^2}{2(1-\sigma_1^2 b_1)t}} \times \mathrm{e}^{-\frac{\left(\sqrt{2}t-\bar{x}^j_{k^*}(t^*)-\bar{x}_j(t^\beta)+L(t)+y\right)^2}{2(1-\sigma_1^2 b_1)t}}\,\middle|\,\mathcal{F}_{b_1 t}\right]. \tag{2.6.30}$$

As in (2.6.10), we obtain

$$(T1) \le P(t)\,\mathrm{e}^{2(1-b_1)t+t^*}\int_J \frac{\mathrm{d}\omega}{\sqrt{2\pi\sigma_1^2 t^*}}\mathrm{e}^{-\frac{\omega^2}{2\sigma_1^2 t^*}}\,\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega-\bar{x}_j(t^\beta)+L(t)+y\right)^2}{(1-\sigma_1^2 b_1)t}}, \tag{2.6.31}$$

with J as in (2.6.24). Proceeding as in (2.6.12)–(2.6.14), we see that the right hand side of (2.6.31) is bounded by

$$P(t)\,\mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,\sigma_1 t^\beta - x_j(t^\beta)\right)}\mathrm{e}^{-b_1 t^{1/2}}(1+o(1)). \tag{2.6.32}$$

We get, as in (2.6.16), that

$$(T2) \le P(t)\,\mathrm{e}^{2(1-b_1)t}\int_0^{t^*}\mathrm{d}s\,\mathrm{e}^{t^*+s}\int_{-\infty}^{\sqrt{2}\,\sigma_1(b_1 t-s)-t^{\beta\delta/2}-x_j(t^\beta)} \frac{\mathrm{d}\omega_1}{\sqrt{2\pi\sigma_1^2(t^*-s)}}\mathrm{e}^{-\frac{\omega_1^2}{2\sigma_1^2(t^*-s)}} \times\left(\int_{J-\omega_1}\frac{\mathrm{d}\omega_2}{\sqrt{2\pi\sigma_1^2 s}}\mathrm{e}^{-\frac{\omega_2^2}{2\sigma_1^2 s}}\,\mathrm{e}^{-\frac{\left(\sqrt{2}\,t-\omega_1-\omega_2-\bar{x}_j(t^\beta)+L(t)+y\right)^2}{2(1-\sigma_1^2 b_1)t}}\right)^2, \tag{2.6.33}$$

where $\omega_2 \in J-\omega_1$ denotes $\omega_2+\omega_1\in J$. As in (2.6.18)–(2.6.22), we see that the right-hand side of (2.6.33) is not larger than

$$P(t)\,\mathrm{e}^{-\sqrt{2}\left(\sqrt{2}\,\sigma_1 t^\beta - x_j(t^\beta)\right)}\mathrm{e}^{-\sqrt{2}\,t^{\beta\delta/2}}(1+o(1)).$$ □

3 A Hamilton-Jacobi approach for the free energy of the CREM

This chapter is based on joint work with Fu-Hsuan Ho[1] and Justin Ko[2], which remains to be published. We would like to thank Lisa Hartung and Pascal Maillard for insightful discussions and helpful advice.

3.1 Notation and outline

For the convenience of the reader, we recall notation and central results which were introduced in Chapter 1, particularly in Section 1.4.4.

A function $A\colon [0,1] \to [0,1]$ is called *speed function* if it is right-continuous, increasing and satisfies $A(0) = 0$, $A(1) = 1$. For $N \in \mathbb{N}$, the *continuous random energy model* (CREM) with speed function A is a centred Gaussian process $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ with covariances

$$\mathbb{E}\left[H_N^A(\sigma)\, H_N^A(\tilde\sigma)\right] = NA\left(\tfrac{\sigma\wedge\tilde\sigma}{N}\right), \qquad \sigma,\tilde\sigma \in \{-1,1\}^N. \tag{3.1.1}$$

The *overlap* $\sigma\wedge\tilde\sigma$ in (3.1.1) is defined as

$$\sigma\wedge\tilde\sigma := \max\{i = 0,\dots,N\colon \sigma|_i = \tilde\sigma|_i\}, \tag{3.1.2}$$

where for $\sigma = (\sigma_1,\dots,\sigma_N) \in \{-1,1\}^N$, we write $\sigma|_0 = \varnothing$ and $\sigma|_i = (\sigma_1,\dots,\sigma_i)$ for $i = 1,\dots,N$. The limiting free energy of the CREM with speed function A is, for $t \geq 0$,

$$\begin{aligned}\lim_{N\uparrow\infty} F_N^A(t) &= \lim_{N\uparrow\infty} -\tfrac{1}{N}\mathbb{E}\left[\ln\left(\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\left(\sqrt{2t}\,H_N^A(\sigma) - Nt\right)\right)\right]\\ &= t\bar A(x(t)) - (1-x(t))\ln 2 - 2\sqrt{t\ln 2}\int_0^{x(t)}\sqrt{\bar A'(x)}\,\mathrm{d}x,\end{aligned} \tag{3.1.3}$$

where $x(t) = \sup\left(x\in(0,1)\colon \bar A'(x) \geq \frac{\ln 2}{t}\right)$. We refer to [32] for a proof. If $A(x) \leq x$ for all $x\in[0,1]$ (and in particular if A is a convex speed function), then $x(t) = \mathbb{1}_{t\geq\ln 2}(t)$, so

$$\lim_{N\uparrow\infty} F_N^A(t) = \begin{cases} -\ln 2, & \text{if } t\leq \ln 2,\\ t - 2\sqrt{t\ln 2}, & \text{if } t > \ln 2.\end{cases} \tag{3.1.4}$$

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The *Ruelle cascades* $(v_\alpha)_{\alpha\in\mathbb{N}^M}$ with parameters $(\zeta_k)_{k=1,\dots,M}$, where

$$0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1, \tag{3.1.5}$$

are constructed in the following way: Let $u_{(1)} > u_{(2)} > \dots$ be the ordered atoms of a Poisson point process on $\mathbb{R}_{>0}$ with intensity $\zeta_1 y^{-1-\zeta_1}\,\mathrm{d}y$. For each $k \in 1,\dots,M-1$ and each $\gamma \in \mathbb{N}^k$ independently, $u_{(\gamma,1)} > u_{(\gamma,2)} > \dots$ are sampled as the ordered atoms of a Poisson point process on $\mathbb{R}_{>0}$ with intensity $\zeta_{k+1} y^{-1-\zeta_{k+1}}\,\mathrm{d}y$. Then, we set

$$\begin{aligned} w_\alpha &\coloneqq \prod_{k=1}^{M} u_{\tilde{\alpha}|_k}, \\ v_\alpha &\coloneqq \frac{w_\alpha}{\sum_{\tilde{\alpha}\in\mathbb{N}^M} w_\alpha}, \end{aligned} \tag{3.1.6}$$

for $\alpha = (\alpha_1,\dots,\alpha_M) \in \mathbb{N}^M$, recalling the notation $\alpha|_k = (\alpha_1,\dots,\alpha_k)$ for $k = 1,\dots,M$. We call $(w_\alpha)_{\alpha\in\mathbb{N}^M}$ the *unnormalised weights* of $(v_\alpha)_{\alpha\in\mathbb{N}^M}$. Since $\sum_{\tilde{\alpha}\in\mathbb{N}^M} w_\alpha$ is finite with probability 1 (see for example [58, Lemma 5.23]), $(v_\alpha)_{\alpha\in\mathbb{N}^M}$ is well-defined.

We introduce the following path spaces:

$$\begin{aligned} Q &\coloneqq \{\mathsf{q}\colon [0,1) \to \mathbb{R}_{\geq 0};\ \mathsf{q} \text{ is right-continuous and increasing}\}, \\ Q_p &\coloneqq Q \cap L_p([0,1),\mathbb{R}), \qquad \forall\, p \in [1,\infty]. \end{aligned} \tag{3.1.7}$$

Let $1 \leq p_1 < p_2$, then by Hölder's inequality, for $g \in L_{p_2}([0,1))$,

$$\|g\|_{p_1} = \left(\int_0^1 |g(u)|^{p_1}\,\mathrm{d}u\right)^{1/p_1} \leq \left(\left\||g|^{p_1}\right\|_{p_2/p_1} \left\|\mathbb{1}_{[0,1)}\right\|_{p_2/(p_2-p_1)}\right)^{1/p_1} = \left(\left\||g|^{p_1}\right\|_{p_2/p_1}\right)^{1/p_1} = \|g\|_{p_2}. \tag{3.1.8}$$

In particular, $Q_{p_2} \subset Q_{p_1}$ for $1 \leq p_1 < p_2$. This also holds for $p_2 = \infty$.

For $M \in \mathbb{N}$, we denote by $Q^{(M)}$ the set of all $\mathsf{q} \in Q$ which can be written as a step function with M jumps, i.e.

$$\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})}, \tag{3.1.9}$$

where

$$\begin{aligned} &0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1, \\ &0 = q_{-1} < q_0 < \cdots < q_{M-1} < q_M < \infty. \end{aligned} \tag{3.1.10}$$

Furthermore, the superset of $Q^{(M)}$, which also allows

$$0 = q_{-1} \leq q_0 \leq \cdots \leq q_M < \infty, \tag{3.1.11}$$

is denoted by $Q^{(M)}_{\leq}$.

Let $M, N \in \mathbb{N}$, $t \geq 0$, A be a speed function and $\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})} \in Q^{(M)}$. Let $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$ be the Ruelle cascades with parameters $(\zeta_k)_{k=0,\ldots M+1}$ and let $\left(H_N^A(\sigma)\right)_{\sigma \in \{-1,1\}^N}$ be the CREM with speed function A. Let $\left(Y_{\mathsf{q}}(\sigma, \alpha)\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M}$ be a centred Gaussian process with covariances

$$\mathbb{E}[Y_{\mathsf{q}}(\sigma, \alpha) Y_{\mathsf{q}}(\tilde{\sigma}, \tilde{\alpha})] = (\sigma \wedge \tilde{\sigma})\, q_{\alpha \wedge \tilde{\alpha}}, \qquad \forall\, \sigma, \tilde{\sigma} \in \{-1,1\}^N,\ \alpha, \tilde{\alpha} \in \mathbb{N}^M, \tag{3.1.12}$$

where $\alpha \wedge \tilde{\alpha} := \max\{k = 0, \ldots, M : \alpha|_k = \tilde{\alpha}|_k\}$. Note that

$$\left(Y_{\mathsf{q}}(\sigma, \alpha)\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M} \overset{\mathrm{d}}{=} \left(\sum_{i=1}^{N} \sum_{k=0}^{M} (q_k - q_{k-1})^{1/2} z_{\sigma|_i, \alpha|_k}\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M}, \tag{3.1.13}$$

where each $z_{\sigma|_i, \alpha|_k}$ is from a family of i.i.d. standard Gaussian random variables. We assume that the processes $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma \in \{-1,1\}^N}$ and $\left(Y_{\mathsf{q}}(\sigma, \alpha)\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M}$ are independent. The *enriched free energy* F_N of the CREM with speed function A is defined by

$$\begin{aligned} F_N(t, \mathsf{q}) &:= -\frac{1}{N} \mathbb{E}\left[\ln\left(\sum_{\alpha \in \mathbb{N}^M} \nu_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right)\right)\right], \\ H_N(t, \mathsf{q}, \sigma, \alpha) &:= \sqrt{2t}\, H_N^A(\sigma) - Nt + \sqrt{2}\, Y_{\mathsf{q}}(\sigma, \alpha) - N q_M. \end{aligned} \tag{3.1.14}$$

We prove in Proposition 3.3.1 that $F_N(t, \cdot)$ is Lipschitz continuous w.r.t. $\|\cdot\|_1$ and in Lemma 3.3.2 that $\bigcup_{M=0}^{\infty} Q^{(M)}$ is dense in Q_1 w.r.t. $\|\cdot\|_1$. This implies that F_N has a unique extension to $\mathbb{R}_{\geq 0} \times Q_1$. Namely, for $t \geq 0$ and $\mathsf{q} \in Q_1$,

$$F_N(t, \mathsf{q}) := \lim_{m \uparrow \infty} F_N(t, \mathsf{q}_m), \tag{3.1.15}$$

where $(\mathsf{q}_m)_{m \in \mathbb{N}}$ is a sequence in $\bigcup_{M=0}^{\infty} Q^{(M)}$ which converges to q w.r.t. $\|\cdot\|_1$.

The first central result of this chapter is the following theorem.

Theorem 3.1.1. *Let* $F_N \colon \mathbb{R}_{\geq 0} \times Q_1 \to \mathbb{R}$ *be as in* (3.1.14) *and* (3.1.15)*. For each* $\mathsf{q} \in Q_1$*, we have*

$$\Psi(\mathsf{q}) := \lim_{N \uparrow \infty} F_N(0, \mathsf{q}) = -\ln 2 + \int_0^1 \left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u. \tag{3.1.16}$$

Recall that $h \colon Q_2 \to \mathbb{R}$, is *Fréchet-differentiable* in $\mathsf{q} \in Q_2$ if there exists a unique function $\nabla h \in L^2([0,1), \mathbb{R})$, called *Fréchet derivative*, such that for every $\tilde{\mathsf{q}} \in Q_2$, we have

$$h(\mathsf{q}) - h(\tilde{\mathsf{q}}) = \int_0^1 \nabla h(u)\, (\mathsf{q}(u) - \tilde{\mathsf{q}}(u))\, \mathrm{d}u + o(\|\mathsf{q} - \tilde{\mathsf{q}}\|_2), \tag{3.1.17}$$

as $\|\mathsf{q} - \tilde{\mathsf{q}}\|_2 \downarrow 0$. Let $f \colon \mathbb{R}_{\geq 0} \times Q_2 \to \mathbb{R}$ and $t \geq 0$, then the Fréchet derivative of $f(t, \cdot)$ in $\mathsf{q} \in Q_2$ is denoted by $\nabla_{\mathsf{q}} f(t, \mathsf{q})$.

We study solutions of the Hamilton-Jacobi equation

$$\frac{\partial}{\partial_t} f(t,\mathsf{q}) - \int_0^1 A\left(\left(\nabla_{\mathsf{q}} f(t,\mathsf{q})\right)(u)\right) \mathsf{d}u = 0, \qquad \forall\, (t,\mathsf{q}) \in \mathbb{R}_+ \times Q_2, \qquad (HJE[\mathsf{q}])$$

with initial condition $f(0,\cdot) = \Psi$, where A is a Lipschitz continuous and convex speed function and Ψ is as in Theorem 3.1.1. The notion of a viscosity solution of $(HJE[\mathsf{q}])$ is defined in Definition 3.6.9. For such solutions, existence, uniqueness and a representation by a variational formula is proven in Proposition 3.6.12. Uniqueness in this context is to be understood in the class of functions $f\colon \mathbb{R}_{\geq 0} \times Q_2 \to \mathbb{R}$ satisfying the following:

- $f(t,\cdot)$ is Lipschitz continuous,
- $\sup_{t>0,\, \mathsf{q}\in Q_2} \left|\frac{f(t,\mathsf{q})-f(t,0)}{t}\right| < \infty$,
- for all $t \geq 0$ and all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$,

$$\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\tilde{\mathsf{q}}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2 \quad \Rightarrow \quad f(t,\mathsf{q}) \geq f(t,\tilde{\mathsf{q}}). \qquad (3.1.18)$$

The second central result of this chapter shows that the unique viscosity solution of $(HJE[\mathsf{q}])$ evaluated in $\mathsf{q}_0 \equiv 0 \in Q^{(0)}$ coincides limiting free energy of the CREM.

Theorem 3.1.2. *Let A be a convex Lipschitz continuous speed function. Let Ψ be as in Theorem 3.1.1. The unique viscosity solution f of $(HJE[\mathsf{q}])$ with $f(0,\cdot) = \Psi$ satisfies*

$$f(t,\mathsf{q}_0) = \lim_{N\uparrow\infty} F_N^A(t), \qquad (3.1.19)$$

for all $t \geq 0$, where F_N^A is the free energy of the CREM, see (3.1.3)*.*

The rest of this chapter has the following structure: Section 3.2 provides a collection of preliminary results. Section 3.3 contains the proof of the Lipschitz continuity of $F_N(t,\cdot)$. A crucial step towards the proof of Theorem 3.1.1 is to study level sets of independent copies of the branching random walk (BRW) with standard Gaussian increments. This is done in Section 3.4. That section can be read independently of the rest of this chapter (besides references to Section 3.2). The proof of Theorem 3.1.1 is done in Section 3.5. In Section 3.6, we define the notion of a viscosity solution of $(HJE[\mathsf{q}])$ and prove existence, uniqueness and a representation by a variational formula for such solutions when A is Lipschitz continuous and convex. Furthermore, we prove Theorem 3.1.2. In Section 3.7, we discuss a possible generalisation of Theorem 3.1.2.

3.2 Preliminaries

In this section, we collect preliminary results. The following lemma describes elementary Gaussian tail asymptotics.

Lemma 3.2.1 [See e.g. 63, Chapter VII, Lemma 2]**.** *For $X \sim \mathcal{N}(0, \sigma^2)$, $\sigma > 0$ and $u > 0$,*

$$\mathbb{P}(X > u) = \frac{\sigma}{\sqrt{2\pi}u} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \left(1 + O\left(\frac{\sigma^2}{u^2}\right)\right), \tag{3.2.1}$$

as $\frac{\sigma}{u} \to 0$. Furthermore, dropping the error term gives an upper bound for $\mathbb{P}(X > u)$.

Note that for a Gaussian random variable Z with mean 0 and variance $v > 0$,

$$\mathbb{E}\left[\mathrm{e}^Z\right] = \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi v}} \mathrm{e}^{-\frac{z^2}{2v}} \mathrm{e}^z = \mathrm{e}^{\frac{v}{2}} \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi v}} \mathrm{e}^{-\frac{(z-v)^2}{2v}} = \mathrm{e}^{\frac{v}{2}}. \tag{3.2.2}$$

Lemma 3.2.2 [35, Lemma 2.2]**.** *Let $t > 0$ and $\left(\mathfrak{z}_{0,0}^t(s)\right)_{s \in [0,t]}$ be a Brownian bridge from 0 to 0 in time t. For any $x, y > 0$ holds*

$$\mathbb{P}\left(\forall_{0 \le s \le t} \colon \mathfrak{z}_{0,0}^t(s) \le (sx + (t-s)y)/t\right) = 1 - \mathrm{e}^{-2xy/t} \le 2\,\frac{xy}{t}, \tag{3.2.3}$$

and asymptotic equality holds if $xy = o(t)$.

The following lemma is a helpful tool to compute Gibbs averages. It is an application of the general Gaussian integration by parts formula (see e.g. [58, Theorem 4.5]) to Gibbs averages of Gaussian processes.

Lemma 3.2.3 (Gibbs-Gaussian integration by parts)**.** *Let $\mathcal{S}$ be a complete and separable metric space. Let $(y(\sigma))_{\sigma \in \mathcal{S}}$ and $(z(\sigma))_{\sigma \in \mathcal{S}}$ be centred Gaussian processes. Let μ be a finite measure on $\mathcal{S}$. For each $n \in \mathbb{N}$, the n-fold Gibbs average w.r.t. $(y(\sigma))_{\sigma \in \mathcal{S}}$ and μ is defined by*

$$\left\langle\!\left\langle g(\sigma^{(1)}, \ldots, \sigma^{(n)}) \right\rangle\!\right\rangle_n := \frac{\int_{\mathcal{S}^n} g(\sigma^{(1)}, \ldots, \sigma^{(n)}) \prod_{i=1}^n \exp\left(y(\sigma^{(i)})\right) \mathrm{d}\mu(\sigma^{(i)})}{\left(\int_{\mathcal{S}} \exp\left(y(\sigma)\right)\, \mathrm{d}\mu(\sigma)\right)^n}, \tag{3.2.4}$$

where $g \colon \mathcal{S}^n \to \mathbb{R}$ is a bounded and measurable function.

Then, for such a function g and for each $n \in \mathbb{N}$,

$$\mathbb{E}\left\langle\!\left\langle g(\sigma^{(1)}, \ldots, \sigma^{(n)})\, z(\sigma^{(1)}) \right\rangle\!\right\rangle_n = \mathbb{E}\left\langle\!\left\langle g(\sigma^{(1)}, \ldots, \sigma^{(n)}) \left(\sum_{i=1}^n \mathbb{E}\left[z(\sigma^{(1)}) y(\sigma^{(i)})\right] - n\,\mathbb{E}\left[z(\sigma^{(1)}) y(\sigma^{(n+1)})\right]\right) \right\rangle\!\right\rangle_{n+1}. \tag{3.2.5}$$

Proof. We refer to [58, Theorem 4.6]. □

The following lemma gives a criterion for the exchange of derivation and integration.

Lemma 3.2.4 [See e.g. 81, Theorem 6.28]. *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $I \subset \mathbb{R}$ be an nonempty open interval. Let $g\colon I \times \Omega \to \mathbb{R}$ satisfy all of the following:*

1. *For each $x \in I$, $\omega \mapsto g(x, \omega)$ is Lebesgue integrable w.r.t. μ.*
2. *For μ-almost all $\omega \in \Omega$, the map $I \to \mathbb{R}, x \mapsto g(x, \omega)$ is differentiable. For such ω, we denote the derivative of $g(\cdot, \omega)$ by $g'(\cdot, \omega)$.*
3. *There exists a map $h\colon \Omega \to \mathbb{R}_{\geq 0}$ which is Lebesgue integrable w.r.t. μ and satisfies $|g'(x, \cdot)| \leq h$ μ-almost everywhere for all $x \in I$.*

Then, for each $x \in I$, $g'(x, \cdot)$ is Lebesgue integrable w.r.t. μ. Furthermore, the function $G\colon I \to \mathbb{R}, x \mapsto \int g(\cdot, \omega)\, \mathrm{d}\mu(\omega)$ is differentiable with derivative

$$G'(x) = \int g'(x, \omega)\, \mathrm{d}\mu(\omega). \tag{3.2.6}$$

By the following formula, certain averages over Ruelle cascades can be computed recursively.

Proposition 3.2.5 [See e.g. 58, Theorem 5.25]. *Let $M \in \mathbb{N}$ and let $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$ be the Ruelle cascades with parameters*

$$0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1. \tag{3.2.7}$$

Independent of $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$, let $(\omega_\gamma)_{\gamma \in \mathbb{T}_M}$ and $(\tilde{\omega}_j)_{j=0,\ldots,M}$ be independent families of i.i.d. uniform random variables on $[0,1]$. Let $X_M\colon [0,1]^{M+1} \to \mathbb{R}$ be a measurable function. We set

$$X_k := X_k(\tilde{\omega}_0, \ldots, \tilde{\omega}_k) := \tfrac{1}{\zeta_{k+1}} \ln \mathbb{E}[\exp(\zeta_{k+1} X_{k+1}(\tilde{\omega}_0, \ldots, \tilde{\omega}_{k+1})) \mid \tilde{\omega}_0, \ldots, \tilde{\omega}_k], \tag{3.2.8}$$

recursively for $k = M-1, M-2, \ldots, 0$. Then,

$$\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha \in \mathbb{N}^M} \nu_\alpha \exp\left(X_M(\omega_\varnothing, \omega_{\alpha|_1}, \ldots, \omega_{\alpha|_{M-1}}, \omega_\alpha)\right)\right)\right] = \mathbb{E}\left[X_0(\tilde{\omega}_0)\right]. \tag{3.2.9}$$

Note that the right-hand side of (3.2.9) can be expanded to

$$\mathbb{E}\left[X_0(\tilde{\omega}_0)\right] = \mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\ldots\mathbb{E}\Big[\mathbb{E}[\exp(\zeta_M X_M(\tilde{\omega}_0, \ldots, \tilde{\omega}_M)) \,\Big|\, \mathcal{F}_{M-1}]^{\frac{\zeta_{M-1}}{\zeta_M}} \,\Big|\, \mathcal{F}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}} \ldots \Big|\, \mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}} \Big|\, \mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg], \tag{3.2.10}$$

where we write $\mathcal{F}_k = \sigma(\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_k)$ for k $= 0, \ldots, M-1$.

Definition 3.2.6. *The* branching random walk *(BRW) on the N-level binary tree with standard Gaussian increments is a centred Gaussian process $(z(\sigma))_{\sigma\in\{-1,1\}^N}$ with covariances*

$$\mathbb{E}\left[z(\sigma)\, z(\tilde{\sigma})\right] = \sigma \wedge \tilde{\sigma}, \qquad \forall\, \sigma, \tilde{\sigma} \in \{-1,1\}^N. \tag{3.2.11}$$

Note that $(z(\sigma))_{\sigma\in\{-1,1\}^N}$ is a CREM whose speed function A is the identity function, i.e. it satisfies $A(x) = x$ for all $x \in [0,1]$. We have

$$(z(\sigma))_{\sigma\in\{-1,1\}^N} \overset{\mathrm{d}}{=} \left(\textstyle\sum_{i=1}^N z_{\sigma|_i}\right)_{\sigma\in\{-1,1\}^N}, \tag{3.2.12}$$

where each $z_{\sigma|_i}$ is from a family of i.i.d. standard Gaussian random variables. Now, we apply Proposition 3.2.5 to the enriched free energy of the CREM.

Lemma 3.2.7. *Let $M, N \in \mathbb{N}$, $t \geq 0$ and $\mathsf{q} = \sum_{k=0}^M q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})} \in Q^{(M)}$, where $(\zeta_k)_{k=1,\ldots,M}$ and $(q_k)_{k=0,\ldots,M}$ are as in* (3.1.10)*. Let F_N be as in* (3.1.14)*. Then,*

$$F_N(t,\mathsf{q}) = t + q_M - \frac{1}{N}\mathbb{E}\left[\ln\left(\mathbb{E}\left[\mathbb{E}\left[\ldots\mathbb{E}\left[\mathbb{E}\left[Z_{t,\mathsf{q}}(z_0,\ldots,z_M)^{\zeta_M}\middle|\mathcal{F}_{M-1}\right]^{\frac{\zeta_{M-1}}{\zeta_M}}\middle|\mathcal{F}_{M-2}\right]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\ldots\middle|\mathcal{F}_1\right]^{\frac{\zeta_1}{\zeta_2}}\middle|\mathcal{F}_0\right]^{\frac{1}{\zeta_1}}\right)\right], \tag{3.2.13}$$

where

$$\begin{aligned} Z_{t,\mathsf{q}}(z_0,\ldots,z_M) &:= \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\, H_N^A(\sigma) + \sqrt{2}\, \textstyle\sum_{j=0}^M (q_j - q_{j-1})^{\frac{1}{2}} z_j(\sigma)\right), \\ \mathcal{F}_k &:= \sigma(z_0,\ldots,z_k), \quad k = 0,\ldots,M-1. \end{aligned} \tag{3.2.14}$$

Furthermore, independent of $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$, the processes $z_1 = (z_1(\sigma))_{\sigma\in\{-1,1\}^N}, \ldots, z_M = (z_M(\sigma))_{\sigma\in\{-1,1\}^N}$ are i.i.d. copies of $z_0 := (z_0(\sigma))_{\sigma\in\{-1,1\}^N}$, a BRW on the N-level binary tree with standard Gaussian increments.

Proof. By the tower property,

$$\begin{aligned} &F_N(t,\mathsf{q}) \\ &= -\frac{1}{N}\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)\right] \\ &= -\frac{1}{N}\mathbb{E}\left[\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\, H_N^A(\sigma) - Nt + \sqrt{2}\, Y_{\mathsf{q}}(\sigma,\alpha) - Nq_M\right)\right)\middle|\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}\right]\right] \\ &= t + q_M - \frac{1}{N}\mathbb{E}\left[\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\, H_N^A(\sigma) + \sqrt{2}\, Y_{\mathsf{q}}(\sigma,\alpha)\right)\right)\middle|\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}\right]\right]. \end{aligned} \tag{3.2.15}$$

To apply Proposition 3.2.5 to the last conditional expectation in (3.2.15), we construct a measurable function $G_M\colon [0,1]^{M+1} \to \mathbb{R}$ so that conditional on $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$,

$$\left(\exp\left(G_M(\omega_\varnothing, \omega_{\alpha|_1}, \dots, \omega_{\alpha|_{M-1}}, \omega_\alpha)\right)\right)_{\alpha\in\mathbb{N}^M} \overset{\mathrm{d}}{=} \left(\textstyle\sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\, H_N^A(\sigma) + \sqrt{2}\, Y_{\mathsf{q}}(\sigma,\alpha)\right)\right)_{\alpha\in\mathbb{N}^M}, \tag{3.2.16}$$

where $(\omega_{\alpha|_k})_{\alpha\in\mathbb{N}^M, k=0,\dots,M}$ is a family of i.i.d. uniform distributions on $[0,1]$. We follow the same construction as in Step 1 of the proof of Proposition 6.3 in [58] which requires the following steps: Let ω be a uniformly distributed random variable on $[0,1]$.

1. There exists a measurable bijection $\iota_N\colon [0,1] \to [0,1]^{2^N}$ so that $\iota_N(\omega)$ is uniformly distributed on $[0,1]^{2^N}$. One can construct ι_N e.g. by splitting the binary representation of $x\in[0,1]$ into 2^N parts.

2. Let Φ be the cumulative distribution function of the standard Gaussian distribution. Let $\Phi_N^{-1}\colon [0,1]^{2^N} \to \mathbb{R}^{2^N}$ be the map which applies Φ^{-1} to each entry of an element of $[0,1]^{2^N}$. Then, $\Phi_N^{-1}\circ\iota_N(\omega)$ is a 2^N-dimensional standard Gaussian vector.

3. Let $(z(\sigma))_{\sigma\in\{-1,1\}^N}$ be a BRW on the N-level binary tree with standard Gaussian increments. Let $\varphi\colon 2^N \to \{-1,1\}^N$ be any bijection. Let $\mathsf{S} = (\varphi(i)\wedge\varphi(j))_{i,j=1,\dots,2^N}$ be the covariance matrix of the Gaussian process $(z(\varphi(i)))_{i=1,\dots,2^N}$. Thus, there exists $\mathsf{R}\in\mathbb{R}^{2^N\times 2^N}$ so that $\mathsf{R}\mathsf{R}^T = \mathsf{S}$. We also denote the corresponding linear map on $\mathbb{R}^{2^N}$ by R. Then, $\mathsf{R}\circ\Phi_N^{-1}\circ\iota_N(\omega) \overset{\mathrm{d}}{=} (z(\varphi(i)))_{i=1,\dots,2^N}$.

4. We set

$$\begin{aligned} \tilde{G}_M\colon [0,1]^{M+1} &\to \mathbb{R}^{2^N}, \\ (u_0,\dots,u_M) &\mapsto \sum_{k=0}^{M} (q_k - q_{k-1})^{1/2}\mathsf{R}\circ\Phi_N^{-1}\circ\iota_N(u_k). \end{aligned} \tag{3.2.17}$$

   Then, for a family of i.i.d. uniform distributions $(\omega_{\alpha|_k})_{\alpha\in\mathbb{N}^M, k=0,\dots,M}$ on $[0,1]$, we have

$$\begin{aligned} \left(\tilde{G}_M(\omega_\varnothing, \omega_{\alpha|_1}, \dots, \omega_{\alpha|_{M-1}}, \omega_\alpha)\right)_{\alpha\in\mathbb{N}^M} &= \left(\textstyle\sum_{k=0}^M (q_k - q_{k-1})^{1/2}\mathsf{R}\circ\Phi_N^{-1}\circ\iota_N(\omega_{\alpha|_k})\right)_{\alpha\in\mathbb{N}^M} \\ &\overset{\mathrm{d}}{=} \left(\textstyle\sum_{k=0}^M (q_k - q_{k-1})^{1/2}\left(z_{\alpha|_k}(\varphi(i))\right)_{i=1,\dots,2^N}\right)_{\alpha\in\mathbb{N}^M}, \end{aligned} \tag{3.2.18}$$

   where each $(z_{\alpha|_k}(\sigma))_{\sigma\in\{-1,1\}^N}$ is from a family of i.i.d. copies of the BRW $(z(\sigma))_{\sigma\in\{-1,1\}^N}$ from step 3. By (3.1.13) and (3.2.12), recalling that each $z_{\sigma|_i,\alpha|_k}$ denotes an element of a family of i.i.d. standard Gaussian random variables,

$$\begin{aligned} \left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N, \alpha\in\mathbb{N}^M} &\overset{\mathrm{d}}{=} \left(\textstyle\sum_{i=1}^N \sum_{k=0}^M (q_k - q_{k-1})^{1/2} z_{\sigma|_i,\alpha|_k}\right)_{\sigma\in\{-1,1\}^N, \alpha\in\mathbb{N}^M} \\ &\overset{\mathrm{d}}{=} \left(\textstyle\sum_{k=0}^M (q_k - q_{k-1})^{1/2} z_{\alpha|_k}(\sigma)\right)_{\sigma\in\{-1,1\}^N, \alpha\in\mathbb{N}^M} \\ &\overset{\mathrm{d}}{=} \left(\pi_{\varphi^{-1}(\sigma)}\circ\tilde{G}_M(\omega_\varnothing, \omega_{\alpha|_1}, \dots, \omega_{\alpha|_{M-1}}, \omega_\alpha)\right)_{\sigma\in\{-1,1\}^N, \alpha\in\mathbb{N}^M}, \end{aligned} \tag{3.2.19}$$

   where $\pi_i\colon \mathbb{R}^{2^N}\to\mathbb{R}$, $i = 1,\dots,2^N$, is the projection to the i-th entry.

5. We set

$$\begin{aligned} G_M\colon [0,1]^{M+1} &\to \mathbb{R}, \\ u &\mapsto \ln\Big(\textstyle\sum_{i=1}^{2^N} \exp\Big(\sqrt{2t}\, H_N^A(\varphi(i)) + \sqrt{2}\,\pi_i \circ \tilde{G}_M(u)\Big)\Big). \end{aligned} \tag{3.2.20}$$

By (3.2.19), G_M satisfies (3.2.16).

We apply Proposition 3.2.5 to the conditional expectation on the right-hand side of (3.2.15) to get that

$$\begin{aligned} &\mathbb{E}\Big[\mathbb{E}\Big[\ln\Big(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\Big(\sqrt{2t}\, H_N^A(\sigma) + \sqrt{2}\, Y_{\mathsf{q}}(\sigma,\alpha)\Big)\Big) \Big| \Big(H_N^A(\sigma)\Big)_{\sigma\in\{-1,1\}^N}\Big]\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[\ln\big(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \exp\big(G_M(\omega_\varnothing, \omega_{\alpha|_1}, \dots, \omega_{\alpha|_{M-1}}, \omega_\alpha)\big)\big) \Big| \Big(H_N^A(\sigma)\Big)_{\sigma\in\{-1,1\}^N}\Big]\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[\ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\dots \mathbb{E}\Big[\mathbb{E}\Big[\exp\big(\zeta_M G_M(\tilde{\omega}_0, \dots, \tilde{\omega}_M)\big) \Big| \tilde{\mathcal{F}}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}} \Big| \tilde{\mathcal{F}}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}} \dots \\ &\qquad\qquad \dots \Big| \tilde{\mathcal{F}}_1\Big]^{\frac{\zeta_1}{\zeta_2}} \Big| \tilde{\mathcal{F}}_0\Big]^{\frac{1}{\zeta_1}}\Big) \Big| \Big(H_N^A(\sigma)\Big)_{\sigma\in\{-1,1\}^N}\Big]\Big] \\ &= \mathbb{E}\Big[\ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\dots \mathbb{E}\Big[\mathbb{E}\Big[\exp\big(\zeta_M G_M(\tilde{\omega}_0, \dots, \tilde{\omega}_M)\big) \Big| \tilde{\mathcal{F}}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}} \Big| \tilde{\mathcal{F}}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}} \dots \Big| \tilde{\mathcal{F}}_1\Big]^{\frac{\zeta_1}{\zeta_2}} \Big| \tilde{\mathcal{F}}_0\Big]^{\frac{1}{\zeta_1}}\Big)\Big], \end{aligned} \tag{3.2.21}$$

with the tower property in the last step, where $\tilde{\omega}_0, \dots, \tilde{\omega}_M$ are i.i.d. uniformly distributed random variables on $[0,1]$ and we write for $k = 0, \dots, M-1$, $\tilde{\mathcal{F}}_k = \sigma(\tilde{\omega}_0, \dots, \tilde{\omega}_k)$. By (3.2.18) and (3.2.20),

$$\exp\big(\zeta_M G_M(\tilde{\omega}_0, \dots, \tilde{\omega}_M)\big) \stackrel{\mathrm{d}}{=} Z_{t,\mathsf{q}}(z_0, \dots, z_M)^{\zeta_M}, \tag{3.2.22}$$

recalling the notation in (3.2.14). Inserting (3.2.22) into (3.2.21) gives

$$\begin{aligned} &\mathbb{E}\Big[\mathbb{E}\Big[\ln\Big(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\Big(\sqrt{2t}\, H_N^A(\sigma) + \sqrt{2}\, Y_{\mathsf{q}}(\sigma,\alpha)\Big)\Big) \Big| \Big(H_N^A(\sigma)\Big)_{\sigma\in\{-1,1\}^N}\Big]\Big] \\ &= \mathbb{E}\Big[\ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\dots \mathbb{E}\Big[\mathbb{E}\Big[Z_{t,\mathsf{q}}(z_0, \dots, z_M)^{\zeta_M} \Big| \mathcal{F}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}} \Big| \mathcal{F}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}} \dots \Big| \mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}} \Big| \mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Big)\Big], \end{aligned} \tag{3.2.23}$$

recalling that $\mathcal{F}_k = \sigma(z_0, \dots, z_k)$ for $k = 0, \dots, M-1$. Thus, inserting (3.2.23) into (3.2.15) completes the proof. □

3.3 Partial derivatives and Lipschitz continuity of the enriched free energy of the CREM

The goal of this section is to prove the following proposition.

Proposition 3.3.1. *Let F_N be the enriched free energy of the CREM defined in* (3.1.14). *For all* $\mathsf{q}, \tilde{\mathsf{q}} \in \bigcup_{M=1}^{\infty} Q_{\leq}^{(M)}$ *and all* $t \geq 0$*,*

$$|F_N(t, \mathsf{q}) - F_N(t, \tilde{\mathsf{q}})| \leq \|\mathsf{q} - \tilde{\mathsf{q}}\|_1. \tag{3.3.1}$$

In particular, there is a unique extension of F_N to the space $\mathbb{R}_{\geq 0} \times Q_1$, which satisfies (3.3.1) *on $\mathbb{R}_{\geq 0} \times Q_1$.*

This and the following Lemma 3.3.2 imply that $F_N(t, \cdot)$ has a unique Lipschitz continuous extension to Q_1, see (3.1.15).

Lemma 3.3.2. *The set $\bigcup_{M=0}^{\infty} Q^{(M)}$ is dense in Q_1 w.r.t. $\|\cdot\|_1$.*

Proof. Let $\mathsf{q} \in Q_1$. It suffices to show that q is Riemann-integrable. Then each step function determined by any Riemann sum of q lies in $\bigcup_{M=0}^{\infty} Q^{(M)}$, since q is increasing. Therefore, q can be approximated pointwise by $(\mathsf{q}_N)_{N \in \mathbb{N}} \subset \bigcup_{M=0}^{\infty} Q^{(M)}$ with $\mathsf{q}_N \leq \mathsf{q}$ for all $N \in \mathbb{N}$. Convergence of $(\mathsf{q}_N)_{N \in \mathbb{N}}$ to q in $L_1([0,1))$ then follows from the dominated convergence theorem.

We now prove that q is Riemann-integrable. Since $\mathsf{q} \in L_1([0,1))$, we have $\int_{[0,1)} \mathsf{q} \, \mathrm{d}\lambda < \infty$, denoting $\mathrm{d}\lambda$ in the Lebesgue integral and $\mathrm{d}u$ in any following Riemann integral. Since q is increasing, it is Riemann-integrable on $[0, a]$ for any $0 < a < 1$ and the Riemann- and Lebesgue-integrals coincide on $[0, a]$. This gives

$$\int_0^1 \mathsf{q}(u) \, \mathrm{d}u = \lim_{a \uparrow 1} \int_0^a \mathsf{q}(u) \, \mathrm{d}u = \lim_{a \uparrow 1} \int_{[0,a]} \mathsf{q} \, \mathrm{d}\lambda = \int_{[0,1)} \mathsf{q} \, \mathrm{d}\lambda < \infty, \tag{3.3.2}$$

so q is Riemann integrable. □

The proof of Proposition 3.3.1 requires a version of the *Ghirlanda-Guerra identities*, which are proven in Proposition 3.3.6, and the computation of "partial derivatives" of F_N in the sense of Proposition 3.3.4. We first prove Lemma 3.3.3, which we use in the proof of Proposition 3.3.4.

Let $M \in \mathbb{N}$ and $\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})} \in Q^{(M)}$, where $(\zeta_k)_{k=1,\dots,M}$ and $(q_k)_{k=0,\dots,M}$ are as in (3.1.10). Let $A\colon [0,1] \to [0,1]$ be a speed function. Recall the definitions of the CREM $\left(H_N^A(\sigma)\right)_{\sigma \in \{-1,1\}^N}$ with speed function A in (3.1.1), of the Ruelle cascades with parameters $(\zeta_k)_{k=0,\dots M+1}$ in (3.1.6) and of $\left(Y_{\mathsf{q}}(\sigma, \alpha)\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M}$ in (3.1.13). We assume that the processes $(v_\alpha)_{\alpha \in \mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma \in \{-1,1\}^N}$

and $\left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}$ are independent. Recall (3.1.14), namely, for $M,N\in\mathbb{N}$, $t\geq 0$ and $\mathsf{q}=\sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})}\in Q^{(M)}$,

$$\begin{aligned} F_N(t,\mathsf{q}) &= -\frac{1}{N}\mathbb{E}\left[\ln\left(\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)\right], \\ H_N(t,\mathsf{q},\sigma,\alpha) &= \sqrt{2t}\,H_N^A(\sigma) - Nt + \sqrt{2}\,Y_{\mathsf{q}}(\sigma,\alpha) - Nq_M. \end{aligned} \tag{3.3.3}$$

Lemma 3.3.3. *It holds*

$$\mathbb{E}\left[\left|\ln\left(\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)\right|\right] < \infty. \tag{3.3.4}$$

Proof. We write

$$X_{t,\mathsf{q}} \coloneqq \sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right). \tag{3.3.5}$$

Our goal is to prove that $\mathbb{E}\left[\left|\ln X_{t,\mathsf{q}}\right|\right]<\infty$. We have

$$\mathbb{E}\left[\left|\ln X_{t,\mathsf{q}}\right|\right] = \mathbb{E}\left[\mathbb{1}_{[1,\infty)}(X_{t,\mathsf{q}})\,\ln X_{t,\mathsf{q}}\right] - \mathbb{E}\left[\mathbb{1}_{(0,1)}(X_{t,\mathsf{q}})\,\ln X_{t,\mathsf{q}}\right] = 2\,\mathbb{E}\left[\mathbb{1}_{[1,\infty)}(X_{t,\mathsf{q}})\,\ln X_{t,\mathsf{q}}\right] - \mathbb{E}\left[\ln X_{t,\mathsf{q}}\right]. \tag{3.3.6}$$

Since $\ln x < x$ for $x>0$ and $X_{t,\mathsf{q}}$ is positive,

$$\mathbb{E}\left[\mathbb{1}_{[1,\infty)}(X_{t,\mathsf{q}})\,\ln X_{t,\mathsf{q}}\right] \leq \mathbb{E}\left[\mathbb{1}_{[1,\infty)}(X_{t,\mathsf{q}})\,X_{t,\mathsf{q}}\right] \leq \mathbb{E}\left[X_{t,\mathsf{q}}\right] = \mathbb{E}\left[\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right]. \tag{3.3.7}$$

By the monotone convergence theorem, we can interchange summation and expectation on the right-hand side of (3.3.7). This and the independence of the processes $(v_\alpha)_{\alpha\in\mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ and $\left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}$ gives

$$\begin{aligned} \mathbb{E}\left[X_{t,\mathsf{q}}\right] &= \sum_{\alpha\in\mathbb{N}^M}\mathbb{E}\left[v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right] \\ &= \exp\left(-Nt-Nq_M\right)\sum_{\alpha\in\mathbb{N}^M}\mathbb{E}\left[v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\left(\sqrt{2t}\,H_N^A(\sigma)+\sqrt{2}\,Y_{\mathsf{q}}(\sigma,\alpha)\right)\right] \\ &= \exp\left(-Nt-Nq_M\right)\sum_{\alpha\in\mathbb{N}^M}\mathbb{E}\left[v_\alpha\right]\sum_{\sigma\in\{-1,1\}^N}\mathbb{E}\left[\exp\left(\sqrt{2t}\,H_N^A(\sigma)+\sqrt{2}\,Y_{\mathsf{q}}(\sigma,\alpha)\right)\right]. \end{aligned} \tag{3.3.8}$$

By the monotone convergence theorem again,

$$\sum_{\alpha\in\mathbb{N}^M}\mathbb{E}\left[v_\alpha\right] = \mathbb{E}\left[\sum_{\alpha\in\mathbb{N}^M} v_\alpha\right] = 1. \tag{3.3.9}$$

By (3.2.2) and the independence of $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ and $\left(Y_{\mathsf{q}}(\sigma,\alpha)\right)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}$, the last expectation on the right-hand side of (3.3.8) satisfies

$$\mathbb{E}\left[\exp\left(\sqrt{2t}\,H_N^A(\sigma)+\sqrt{2}\,Y_{\mathsf{q}}(\sigma,\alpha)\right)\right]=\exp\left(Nt+Nq_M\right). \tag{3.3.10}$$

Inserting (3.3.9) and (3.3.10) into (3.3.8) gives

$$\mathbb{E}\left[X_{t,\mathsf{q}}\right]=\sum_{\sigma\in\{-1,1\}^N}1=2^N<\infty. \tag{3.3.11}$$

In particular, (3.3.7) implies that $\mathbb{E}\left[\mathbb{1}_{[1,\infty)}(X_{t,\mathsf{q}})\,\ln X_{t,\mathsf{q}}\right]<\infty$.

By (3.3.6), it remains to show that $\mathbb{E}\left[\ln X_{t,\mathsf{q}}\right]>-\infty$. We get by Lemma 3.2.7 that

$$\mathbb{E}\left[\ln X_{t,\mathsf{q}}\right]=\mathbb{E}\left[\ln\left(\mathbb{E}\left[\mathbb{E}\left[\dots\mathbb{E}\left[\mathbb{E}\left[Z_{t,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_M}\middle|\mathcal{F}_{M-1}\right]^{\frac{\zeta_{M-1}}{\zeta_M}}\middle|\mathcal{F}_{M-2}\right]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\middle|\mathcal{F}_1\right]^{\frac{\zeta_1}{\zeta_2}}\middle|\mathcal{F}_0\right]^{\frac{1}{\zeta_1}}\right)\right], \tag{3.3.12}$$

recalling that

$$\begin{aligned}Z_{t,\mathsf{q}}(z_0,\dots,z_M)&:=\sum_{\sigma\in\{-1,1\}^N}\exp\left(\sqrt{2t}\,H_N^A(\sigma)+\sqrt{2}\,\textstyle\sum_{j=0}^M(q_j-q_{j-1})^{\frac12}z_j(\sigma)\right),\\ \mathcal{F}_k&:=\sigma(z_0,\dots,z_k),\quad k=0,\dots,M-1.\end{aligned} \tag{3.3.13}$$

Furthermore, independent of $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$, the processes $z_1=(z_1(\sigma))_{\sigma\in\{-1,1\}^N},\dots,z_M=(z_M(\sigma))_{\sigma\in\{-1,1\}^N}$ are i.i.d. copies of $z_0:=(z_0(\sigma))_{\sigma\in\{-1,1\}^N}$, a BRW on the N-level binary tree with standard Gaussian increments. We apply Jensen's inequality to the concave functions $x\mapsto x^{\frac{\zeta_{M-1}}{\zeta_M}},\dots,x\mapsto x^{\frac{\zeta_1}{\zeta_2}}$ in (3.3.12). This and the tower property gives

$$\begin{aligned}\mathbb{E}\left[\ln X_{t,\mathsf{q}}\right]&\geq\frac{1}{z_1}\mathbb{E}\left[\ln\left(\mathbb{E}\left[\mathbb{E}\left[\dots\mathbb{E}\left[\mathbb{E}\left[Z_{t,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_1}\middle|\mathcal{F}_{M-1}\right]\middle|\mathcal{F}_{M-2}\right]\dots\middle|\mathcal{F}_1\right]\middle|\mathcal{F}_0\right]\right)\right]\\&=\frac{1}{z_1}\mathbb{E}\left[\ln\left(\mathbb{E}\left[Z_{t,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_1}\middle|\mathcal{F}_0\right]\right)\right].\end{aligned} \tag{3.3.14}$$

Let $\tilde{\sigma}\in\{-1,1\}^N$. Inserting the bound

$$\begin{aligned}Z_{t,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_1}&=\left(\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\left(\sqrt{2t}\,H_N^A(\sigma)+\sqrt{2}\,\sum_{j=0}^M(q_j-q_{j-1})^{\frac12}z_j(\sigma)\right)\right)^{\zeta_1}\\&\geq\exp\left(\zeta_1\sqrt{2t}\,H_N^A(\tilde{\sigma})+\sqrt{2}\,\textstyle\sum_{j=0}^M(q_j-q_{j-1})^{\frac12}z_j(\tilde{\sigma})\right)\end{aligned} \tag{3.3.15}$$

into (3.3.14), we see that

$$\mathbb{E}\left[\ln X_{t,\mathsf{q}}\right]\geq\frac{1}{z_1}\mathbb{E}\left[\ln\left(\mathbb{E}\left[\exp\left(\zeta_1\sqrt{2t}\,H_N^A(\tilde{\sigma})+\zeta_1\sqrt{2}\,\textstyle\sum_{j=0}^M(q_j-q_{j-1})^{\frac12}z_j(\tilde{\sigma})\right)\middle|\mathcal{F}_0\right]\right)\right]. \tag{3.3.16}$$

By the independence of $H_N^A(\tilde\sigma)$ and $\left(z_j(\tilde\sigma)\right)_{j=1,\dots,M}$ of $\mathcal{F}_0$,

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2t}\,H_N^A(\tilde\sigma)+\zeta_1\sqrt{2}\,\textstyle\sum_{j=0}^M(q_j-q_{j-1})^{\frac12}z_j(\tilde\sigma)\Big)\Big|\mathcal{F}_0\Big]\\&=\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2t}\,H_N^A(\tilde\sigma)+\zeta_1\sqrt{2}\,\textstyle\sum_{j=1}^M(q_j-q_{j-1})^{\frac12}z_j(\tilde\sigma)\Big)\Big]\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2}\,q_0^{\frac12}z_0(\tilde\sigma)\Big)\Big|\mathcal{F}_0\Big],\end{aligned}\tag{3.3.17}$$

where

$$\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2}\,q_0^{\frac12}z_0(\tilde\sigma)\Big)\Big|\mathcal{F}_0\Big]=\exp\Big(\zeta_1\sqrt{2}\,q_0^{\frac12}z_0(\tilde\sigma)\Big).\tag{3.3.18}$$

Furthermore, by independence again and by (3.2.2),

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2t}\,H_N^A(\tilde\sigma)+\zeta_1\sqrt{2}\,\textstyle\sum_{j=1}^M(q_j-q_{j-1})^{\frac12}z_j(\tilde\sigma)\Big)\Big]\\&=\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2t}\,H_N^A(\tilde\sigma)\Big)\Big]\prod_{j=1}^M\mathbb{E}\Big[\exp\Big(\zeta_1\sqrt{2}\,t(q_j-q_{j-1})^{\frac12}z_j(\tilde\sigma)\Big)\Big]\\&=\exp\Big(\zeta_1^2N\,(t+q_M-q_0)\Big).\end{aligned}\tag{3.3.19}$$

Inserting (3.3.17)–(3.3.19) into (3.3.16), we get that

$$\mathbb{E}\left[\ln X_{t,\mathsf{q}}\right]\geq\zeta_1N\,(t+q_M-q_0)+\frac{1}{z_1}\mathbb{E}\Big[\zeta_1\sqrt{2}\,q_0^{\frac12}z_0(\tilde\sigma)\Big]=\zeta_1N\,(t+q_M-q_0)>-\infty.\qquad\square$$

The 1-fold Gibbs average on the spin space $\mathcal{S}_{N,M}=\{-1,1\}^N\times\mathbb{N}^M$ w.r.t. $H_N(t,\mathsf{q},\cdot,\cdot)$ and $(\nu_\alpha)_{\alpha\in\mathbb{N}^M}$ of a bounded and measurable function $g\colon\mathcal{S}_{N,M}\to\mathbb{R}$ is denoted by

$$\langle\!\langle g(\sigma,\alpha)\rangle\!\rangle_{t,\mathsf{q}}:=\frac{\sum_{\alpha\in\mathbb{N}^M}\nu_\alpha\sum_{\sigma\in\{-1,1\}^N}g(\sigma,\alpha)\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)}{\sum_{\alpha\in\mathbb{N}^M}\nu_\alpha\sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)}.\tag{3.3.20}$$

This is the average of the function g w.r.t. the (random) Gibbs measure $\mu_{t,\mathsf{q}}$ with weights

$$\mu_{t,\mathsf{q}}(\hat\sigma,\hat\alpha)=\frac{\nu_{\hat\alpha}\exp\left(H_N(t,\mathsf{q},\hat\sigma,\hat\alpha)\right)}{\sum_{\alpha\in\mathbb{N}^M}\nu_\alpha\sum_{\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)},\qquad(\hat\sigma,\hat\alpha)\in\mathcal{S}_{N,M}.\tag{3.3.21}$$

Note that replacing $(\nu_\alpha)_{\alpha\in\mathbb{N}^M}$ by the unnormalised weights $(w_\alpha)_{\alpha\in\mathbb{N}^M}$ does not change (3.3.20).

The 2-fold Gibbs average $\langle\!\langle g(\sigma,\alpha,\tilde\sigma,\tilde\alpha)\rangle\!\rangle_{t,\mathsf{q},2}$ is denoted by

$$\langle\!\langle g(\sigma,\alpha,\tilde\sigma,\tilde\alpha)\rangle\!\rangle_{t,\mathsf{q},2}:=\frac{\sum_{\alpha,\tilde\alpha\in\mathbb{N}^M}\nu_\alpha\nu_{\tilde\alpha}\sum_{\sigma,\tilde\sigma\in\{-1,1\}^N}g(\sigma,\alpha,\tilde\sigma,\tilde\alpha)\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)+H_N(t,\mathsf{q},\tilde\sigma,\tilde\alpha)\right)}{\sum_{\alpha,\tilde\alpha\in\mathbb{N}^M}\nu_\alpha\nu_{\tilde\alpha}\sum_{\sigma,\tilde\sigma\in\{-1,1\}^N}\exp\left(H_N(t,\mathsf{q},\sigma,\alpha)+H_N(t,\mathsf{q},\tilde\sigma,\tilde\alpha)\right)},\tag{3.3.22}$$

where $g\colon\mathcal{S}_{N,M}^2\to\mathbb{R}$ is a bounded and measurable function. This is the average of g w.r.t. $(\mu_{t,\mathsf{q}})^{\otimes2}$. For $n>2$, the n-fold Gibbs average $\langle\!\langle g(\sigma^{(1)},\alpha^{(1)},\dots,\sigma^{(n)},\alpha^{(n)})\rangle\!\rangle_{t,\mathsf{q},n}$ is defined analogously.

For $M \in \mathbb{N}$, we define the convex cone

$$C_{\leq}^{(M)} := \left\{ q = (q_0, \dots, q_M) \in \mathbb{R}^{M+1} \colon 0 \leq q_0 \leq \dots \leq q_M \right\}. \tag{3.3.23}$$

Its interior is

$$C_{<}^{(M)} := \left\{ q = (q_0, \dots, q_M) \in \mathbb{R}^{M+1} \colon 0 < q_0 < \dots < q_M \right\}. \tag{3.3.24}$$

With Lemma 3.2.3, 3.2.4 and 3.3.3, we can compute the "partial derivatives" of $F_N(t, \mathsf{q})$ in the following sense:

Proposition 3.3.4. *Let $M, N \in \mathbb{N}$ and let $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$ be the Ruelle cascades with parameters $0 = \zeta_0 < \zeta_1 < \zeta_2 < \dots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1$. Let $A \colon [0,1] \to [0,1]$ be a speed function. We set*

$$\begin{aligned} \tilde{F}_N \colon \mathbb{R}_+ \times C_{<}^{(M)} &\to \mathbb{R}, \\ (t, q) = (t, q_0, \dots, q_M) &\mapsto F_N\left(t, \textstyle\sum_{k=0}^M q_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})}\right), \end{aligned} \tag{3.3.25}$$

where F_N is defined in (3.1.14). Then, for each $t > 0$ and each $q = (q_0, \dots, q_M) \in C_{<}^{(M)}$, denoting $\mathsf{q} = \sum_{k=0}^M q_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})}$, we have

$$\begin{aligned} \tfrac{\partial}{\partial_t} \tilde{F}_N(t, q) &= \mathbb{E}\left\langle\!\!\left\langle A\left(\tfrac{\sigma \wedge \tilde{\sigma}}{N}\right) \right\rangle\!\!\right\rangle_{t, \mathsf{q}, 2}, \\ \tfrac{\partial}{\partial_{q_k}} \tilde{F}_N(t, q) &= \mathbb{E}\left\langle\!\!\left\langle \mathbb{1}_{\alpha \wedge \tilde{\alpha} = k}(\alpha, \tilde{\alpha}) \tfrac{\sigma \wedge \tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \mathsf{q}, 2}, \qquad k = 0, \dots, M. \end{aligned} \tag{3.3.26}$$

Proof. Let $M \in \mathbb{N}$, $t > 0$, $q = (q_0, \dots, q_M) \in C_{<}^{(M)}$ and $0 = \zeta_0 < \zeta_1 < \zeta_2 < \dots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1$. We have

$$\tfrac{\partial}{\partial_t} \tilde{F}_N(t, q) = -\frac{1}{N} \frac{\partial}{\partial_t} \mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha \in \mathbb{N}^M} \nu_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha) \right) \right) \right], \tag{3.3.27}$$

where H_N is as in (3.1.14) and $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$ are the Ruelle cascades with parameters $(\zeta_k)_{k=0,\dots,M+1}$.

The next step is to apply Lemma 3.2.4 in order to interchange derivative and expectation in (3.3.27). We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the joint probability space of $(\nu_\alpha)_{\alpha \in \mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma \in \{-1,1\}^N}$ and $\left(Y_{\mathsf{q}}(\sigma, \alpha)\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M}$ and abbreviate a realisation of all these processes by $\omega \in \Omega$. Now, we check the prerequisites 1.–3. of Lemma 3.2.4 for the function

$$\begin{aligned} g \colon \mathbb{R} \times \Omega &\to \mathbb{R}, \\ (t, \omega) &\mapsto \ln\left(\textstyle\sum_{\alpha \in \mathbb{N}^M} \nu_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha) \right) \right). \end{aligned} \tag{3.3.28}$$

1. For each $t > 0$, by Lemma 3.3.3, $\mathbb{E}\left[|g(t, \omega)|\right] < \infty$.

2. Let $t_0 > 0$ and $I = (t_0 - \varepsilon, t_0 + \varepsilon)$, where $\varepsilon \in (0, t_0)$. We prove that for $\mathbb{P}$-almost all realisations ω of $(v_\alpha)_{\alpha \in \mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma \in \{-1,1\}^N}$ and $\left(Y_{\mathsf{q}}(\sigma, \alpha)\right)_{\sigma \in \{-1,1\}^N, \alpha \in \mathbb{N}^M}$, the function $g(\cdot, \omega)$ is differentiable on I. We have

$$\begin{aligned} \tfrac{\partial}{\partial_t} g(t, \omega) &= \tfrac{\partial}{\partial_t} \ln\left(\textstyle\sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right)\right) \\ &= \frac{\tfrac{\partial}{\partial_t} \sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right)}{\sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right)}. \end{aligned} \tag{3.3.29}$$

Note that for $t \in I$ and $\alpha \in \mathbb{N}^M$,

$$\begin{aligned} \sum_{\sigma \in \{-1,1\}^N} \left|\tfrac{\partial}{\partial_t} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right)\right| &= \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right) \left|\tfrac{\partial}{\partial_t} H_N(t, \mathsf{q}, \sigma, \alpha)\right| \\ &= \sum_{\sigma \in \{-1,1\}^N} \left|(2t)^{-1/2} H_N^A(\sigma) - N\right| \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right) \\ &\leq \sum_{\sigma \in \{-1,1\}^N} \left((2(t_0 - \varepsilon))^{-1/2} \left|H_N^A(\sigma)\right| + N\right) \exp\left(H_N^{\max}(t_0, \varepsilon, \mathsf{q}, \sigma, \alpha)\right), \end{aligned} \tag{3.3.30}$$

where

$$H_N^{\max}(t_0, \varepsilon, \mathsf{q}, \sigma, \alpha) = \sqrt{2(t_0 + \varepsilon)} \left|H_N^A(\sigma)\right| - (t_0 - \varepsilon)N + \sqrt{2}\, Y_{\mathsf{q}}(\sigma, \alpha) - N q_M. \tag{3.3.31}$$

One computes with the same methods as in (3.3.8)–(3.3.11) that

$$\mathbb{E}\left[\textstyle\sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \left((2(t_0 - \varepsilon))^{-1/2} \left|H_N^A(\sigma)\right| + N\right) \exp\left(H_N^{\max}(t_0, \varepsilon, \mathsf{q}, \sigma, \alpha)\right)\right] < \infty. \tag{3.3.32}$$

In particular, the argument of this expectation is $\mathbb{P}$-a.s. finite. This and a classical result of analysis (see e.g. Theorem 7.10 of [104]) imply that $\mathbb{P}$-a.s., the function series

$$t \mapsto \sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \tfrac{\partial}{\partial_t} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right) \tag{3.3.33}$$

is uniformly convergent on I. By Theorem 7.17 in [104], we can exchange derivation and summation in the numerator on the right-hand side of (3.3.29) for $\mathbb{P}$-almost all $\omega \in \Omega$. For these ω and $t \in I$, we get that

$$\begin{aligned} &\tfrac{\partial}{\partial_t} \textstyle\sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right) \\ &= \textstyle\sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \tfrac{\partial}{\partial_t} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right) \\ &= \textstyle\sum_{\alpha \in \mathbb{N}^M} v_\alpha \sum_{\sigma \in \{-1,1\}^N} \left((2t)^{-1/2} H_N^A(\sigma) - N\right) \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right). \end{aligned} \tag{3.3.34}$$

Inserting (3.3.34) into (3.3.29), we get for $\mathbb{P}$-almost all $\omega \in \Omega$ that $g(\cdot, \omega)$ is differentiable on I with

$$\frac{\partial}{\partial_t} g(t,\omega) = \frac{\sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} \left((2t)^{-1/2} H_N^A(\sigma) - N\right) \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)}{\sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)}$$
$$= \langle\!\langle (2t)^{-1/2} H_N^A(\sigma) - N \rangle\!\rangle_{t,\mathsf{q}}, \tag{3.3.35}$$

for $t \in I$, where $\langle\!\langle\cdot\rangle\!\rangle_{t,\mathsf{q}}$ is the 1-fold Gibbs average defined in (3.3.20).

3. By (3.3.35), $\mathbb{P}$-a.s. and for all $t \in I$, $\tilde{\alpha} \in \mathbb{N}^M$, $\tilde{\sigma} \in \{-1,1\}^N$,

$$\left|\frac{\partial}{\partial_t} g(t,\omega)\right| \le \frac{\sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} (2t)^{-1/2} \left|H_N^A(\sigma)\right| \exp\left(H_N(t_0,\mathsf{q},\sigma,\alpha)\right)}{\sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)}$$
$$\le \frac{\exp\left(-H_N^{\min}(t_0,\varepsilon,\mathsf{q},\tilde{\sigma},\tilde{\alpha})\right)}{2\nu_{\tilde{\alpha}}(t_0-\varepsilon)} \sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} \left|H_N^A(\sigma)\right| \exp\left(H_N^{\max}(t_0,\varepsilon,\mathsf{q},\sigma,\alpha)\right), \tag{3.3.36}$$

where $H_N^{\max}$ is as in (3.3.31) and

$$H_N^{\min}(t_0,\varepsilon,\mathsf{q},\tilde{\sigma},\tilde{\alpha}) = -\sqrt{2(t_0-\varepsilon)}\left|H_N^A(\tilde{\sigma})\right| - (t_0+\varepsilon)N + \sqrt{2}\, Y_{\mathsf{q}}(\tilde{\sigma},\tilde{\alpha}) - Nq_M. \tag{3.3.37}$$

Elementary computations as in (3.3.8)–(3.3.11) show that the right-hand side of (3.3.36) has finite mean.

Thus, by Lemma 3.2.4, we can exchange derivation and expectation in (3.3.27) to get that

$$\frac{\partial}{\partial_t} \tilde{F}_N(t,q) = -\frac{1}{N}\frac{\partial}{\partial_t}\mathbb{E}\left[\ln\left(\sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)\right]$$
$$= -\frac{1}{N}\mathbb{E}\left[\frac{\partial}{\partial_t}\ln\left(\sum_{\alpha\in\mathbb{N}^M} \nu_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)\right]$$
$$= -\frac{1}{N}\mathbb{E}\left[\langle\!\langle (2t)^{-1/2} H_N^A(\sigma) - N\rangle\!\rangle_{t,\mathsf{q}}\right] = 1 - \frac{1}{N}(2t)^{-1/2}\mathbb{E}\langle\!\langle H_N^A(\sigma)\rangle\!\rangle_{t,\mathsf{q}}. \tag{3.3.38}$$

using (3.3.35) in the second last step.

We want to apply the Gibbs-Gaussian integration by parts formula from Lemma 3.2.3 to the right-hand side of (3.3.38). In the setting of this proof, $\mathcal{S} = \mathcal{S}_{N,M} = \{-1,1\}^N \times \mathbb{N}^M$, $n = 1$, $g \equiv 1$ and μ is a measure on $\mathcal{S}_{N,M}$ with $\mu((\sigma,\alpha)) = \nu_\alpha$ for all $(\sigma,\alpha) \in \mathcal{S}_{N,M}$. By Lemma 3.2.3,

$$(2t)^{-1/2}\mathbb{E}\langle\!\langle H_N^A(\sigma)\rangle\!\rangle_{t,\mathsf{q}} = (2t)^{-1/2}\mathbb{E}\left\langle\!\!\left\langle \mathbb{E}\left[H_N(t,\mathsf{q},\sigma,\alpha)H_N^A(\sigma)\right] - \mathbb{E}\left[H_N(t,\mathsf{q},\sigma,\alpha)H_N^A(\tilde{\sigma})\right]\right\rangle\!\!\right\rangle_{t,\mathsf{q},2}$$
$$= N - N\mathbb{E}\left\langle\!\!\left\langle A\left(\tfrac{\sigma\wedge\tilde{\sigma}}{N}\right)\right\rangle\!\!\right\rangle_{t,\mathsf{q},2}, \tag{3.3.39}$$

where we used in the last step that $(2t)^{-1/2}\mathbb{E}\left[H_N(t,\mathsf{q},\sigma,\alpha)H_N^A(\tilde{\sigma})\right] = NA\left(\frac{\sigma\wedge\tilde{\sigma}}{N}\right)$ for $\sigma,\tilde{\sigma} \in \{-1,1\}^N$ and $\alpha \in \mathbb{N}^M$. Inserting (3.3.39) into (3.3.38) gives

$$\frac{\partial}{\partial_t}\tilde{F}_N(t,q) = \mathbb{E}\left\langle\!\!\left\langle A\left(\tfrac{\sigma\wedge\tilde{\sigma}}{N}\right)\right\rangle\!\!\right\rangle_{t,\mathsf{q},2}. \tag{3.3.40}$$

To compute $\frac{\partial}{\partial_{q_k}}\tilde{F}_N(t,q)$, one justifies the exchange of derivation and summation/expectation as above for $\frac{\partial}{\partial_t}\tilde{F}_N(t,q)$. This gives for $k=0,\dots,M$ that

$$\begin{aligned}
\frac{\partial}{\partial_{q_k}}\tilde{F}_N(t,q) &= -\frac{1}{N}\frac{\partial}{\partial_{q_k}}\mathbb{E}\Big[\ln\Big(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\Big(H_N(t,\mathsf{q},\sigma,\alpha)\Big)\Big)\Big]\\
&= -\frac{1}{N}\mathbb{E}\Big[\frac{\partial}{\partial_{q_k}}\ln\Big(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\Big(H_N(t,\mathsf{q},\sigma,\alpha)\Big)\Big)\Big]\\
&= -\frac{1}{N}\mathbb{E}\left[\frac{\frac{\partial}{\partial_{q_k}}\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\Big(H_N(t,\mathsf{q},\sigma,\alpha)\Big)}{\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\Big(H_N(t,\mathsf{q},\sigma,\alpha)\Big)}\right]\\
&= -\frac{1}{N}\mathbb{E}\left[\frac{\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\frac{\partial}{\partial_{q_k}}\exp\Big(H_N(t,\mathsf{q},\sigma,\alpha)\Big)}{\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N}\exp\Big(H_N(t,\mathsf{q},\sigma,\alpha)\Big)}\right]\\
&= -\frac{1}{N}\mathbb{E}\Big\langle\!\Big\langle \frac{\partial}{\partial_{q_k}}H_N(t,\mathsf{q},\sigma,\alpha)\Big\rangle\!\Big\rangle_{t,\mathsf{q}}\\
&= \mathbb{1}_{k=M}(k) - \frac{\sqrt{2}}{N}\mathbb{E}\Big\langle\!\Big\langle \frac{\partial}{\partial_{q_k}}Y_{\mathsf{q}}(\sigma,\alpha)\Big\rangle\!\Big\rangle_{t,\mathsf{q}}.
\end{aligned}\tag{3.3.41}$$

Recall from (3.1.13) that $\big(Y_{\mathsf{q}}(\sigma,\alpha)\big)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M} \overset{\mathrm{d}}{=} \Big(\sum_{i=1}^N\sum_{k=0}^M(q_k-q_{k-1})^{1/2}z_{\sigma|_i,\alpha|_k}\Big)_{\sigma\in\{-1,1\}^N,\,\alpha\in\mathbb{N}^M}$, where each $z_{\sigma|_i,\alpha|_k}$ is from a family of i.i.d. standard Gaussian random variables which is independent of $(v_\alpha)_{\alpha\in\mathbb{N}^M}$ and $\big(H_N^A(\sigma)\big)_{\sigma\in\{-1,1\}^N}$. By Lemma 3.2.3, we get for $k=0,\dots,M-1$ that

$$\begin{aligned}
\mathbb{E}\Big\langle\!\Big\langle \frac{\partial}{\partial_{q_k}}Y_{\mathsf{q}}(\sigma,\alpha)\Big\rangle\!\Big\rangle_{t,\mathsf{q}} &= \mathbb{E}\Big\langle\!\Big\langle \frac{\partial}{\partial_{q_k}}\textstyle\sum_{i=1}^N\sum_{j=0}^M(q_j-q_{j-1})^{1/2}z_{\sigma|_i,\alpha|_j}\Big\rangle\!\Big\rangle_{t,\mathsf{q}}\\
&= \tfrac{1}{2}\mathbb{E}\Big\langle\!\Big\langle(q_k-q_{k-1})^{-1/2}\textstyle\sum_{i=1}^N z_{\sigma|_i,\alpha|_k}\Big\rangle\!\Big\rangle_{t,\mathsf{q}} - \tfrac{1}{2}\mathbb{E}\Big\langle\!\Big\langle(q_{k+1}-q_k)^{-1/2}\textstyle\sum_{i=1}^N z_{\sigma|_i,\alpha|_{k+1}}\Big\rangle\!\Big\rangle_{t,\mathsf{q}}\\
&= -\tfrac{1}{\sqrt{2}}\mathbb{E}\Big\langle\!\Big\langle(\sigma\wedge\tilde{\sigma})\Big(\mathbb{1}_{\alpha\wedge\tilde{\alpha}\geq k}(\alpha,\tilde{\alpha})-\mathbb{1}_{\alpha\wedge\tilde{\alpha}\geq k+1}(\alpha,\tilde{\alpha})\Big)\Big\rangle\!\Big\rangle_{t,\mathsf{q},2}\\
&= -\tfrac{1}{\sqrt{2}}\mathbb{E}\Big\langle\!\Big\langle(\sigma\wedge\tilde{\sigma})\mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha,\tilde{\alpha})\Big\rangle\!\Big\rangle_{t,\mathsf{q},2},
\end{aligned}\tag{3.3.42}$$

using that

$$(q_j-q_{j-1})^{-1/2}\mathbb{E}\Big[H_N(t,\mathsf{q},\sigma,\alpha)\textstyle\sum_{i=1}^N z_{\tilde{\sigma}|_i,\tilde{\alpha}|_j}\Big] = \sqrt{2}\,(\sigma\wedge\tilde{\sigma})\mathbb{1}_{\alpha\wedge\tilde{\alpha}\geq j}(\alpha,\tilde{\alpha}),\tag{3.3.43}$$

for $\sigma,\tilde{\sigma}\in\{-1,1\}^N$, $\alpha,\tilde{\alpha}\in\mathbb{N}^M$ and $j=0,\dots,M$. In the case $k=0$, we used the convention $q_{-1}=0$. Inserting (3.3.42) into (3.3.41) gives

$$\frac{\partial}{\partial_{q_k}}\tilde{F}_N(t,q) = \mathbb{E}\Big\langle\!\Big\langle\mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha,\tilde{\alpha})\tfrac{\sigma\wedge\tilde{\sigma}}{N}\Big\rangle\!\Big\rangle_{t,\mathsf{q},2},\tag{3.3.44}$$

for $k=0,\dots,M-1$. Analogously, by (3.3.43) and Lemma 3.2.3,

$$\begin{aligned}
\mathbb{E}\Big\langle\!\Big\langle \frac{\partial}{\partial_{q_M}}Y_{\mathsf{q}}(\sigma,\alpha)\Big\rangle\!\Big\rangle &= \tfrac{1}{2}\mathbb{E}\Big\langle\!\Big\langle(q_M-q_{M-1})^{-1/2}\textstyle\sum_{i=1}^N z_{\sigma|_i,\alpha|_M}\Big\rangle\!\Big\rangle_{t,\mathsf{q}}\\
&= \tfrac{N}{\sqrt{2}}\Big(1-\mathbb{E}\Big\langle\!\Big\langle\tfrac{\sigma\wedge\tilde{\sigma}}{N}\mathbb{1}_{\alpha\wedge\tilde{\alpha}=M}(\alpha,\tilde{\alpha})\Big\rangle\!\Big\rangle_{t,\mathsf{q},2}\Big),
\end{aligned}\tag{3.3.45}$$

so

$$\frac{\partial}{\partial_{q_M}}\tilde{F}_N(t,q) = \mathbb{E}\Big\langle\!\Big\langle\mathbb{1}_{\alpha\wedge\tilde{\alpha}=M}(\alpha,\tilde{\alpha})\tfrac{\sigma\wedge\tilde{\sigma}}{N}\Big\rangle\!\Big\rangle_{t,\mathsf{q},2}.$$

□

Proposition 3.3.5. *In the setting of Proposition 3.3.4, for each $t > 0$, the function $\tilde{F}_N(t, \cdot)$ is in $C^1\left(C_<^{(M)}, \mathbb{R}\right)$.*

Proof. Let $t > 0$, $M, N \in \mathbb{N}$, $q = (q_0, \dots, q_M) \in C_<^{(M)}$ and $0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1$. We write $\mathsf{q} = \sum_{k=0}^M q_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})} \in Q^{(M)}$. By Proposition 3.3.4, $\tilde{F}_N(t, \cdot)$ is partially differentiable in q with partial derivatives

$$\frac{\partial}{\partial_{q_k}} \tilde{F}_N(t, q) = \mathbb{E}\left\langle\!\!\left\langle \mathbb{1}_{\alpha \wedge \tilde{\alpha} = k}(\alpha, \tilde{\alpha}) \, \tfrac{\sigma \wedge \tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \mathsf{q}, 2}, \qquad k = 0, \dots, M. \tag{3.3.46}$$

It remains to show that for each $k \in \{0, \dots, M\}$, the partial derivatives in (3.3.46) are continuous in q. For this purpose, let $q^{(n)} \in C_<^{(M)}$ for $n \in \mathbb{N}$ with $\lim_{n\uparrow\infty} \|q - q^{(n)}\|_1 = 0$. We write $\mathsf{q}^{(n)} = \sum_{j=0}^M q_j^{(n)} \mathbb{1}_{[\zeta_j, \zeta_{j+1})} \in Q^{(M)}$ for $n \in \mathbb{N}$. For all $t \geq 0$, all $n \in \mathbb{N}$ and all $k \in \{0, \dots, M\}$, $\langle\!\langle \mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha, \tilde{\alpha}) \frac{\sigma\wedge\tilde{\sigma}}{N} \rangle\!\rangle_{t, \mathsf{q}^{(n)}, 2}$ is bounded from above by 1. Thus, by the dominated convergence theorem, for all $k \in \{0, \dots, M\}$,

$$\begin{aligned}
\lim_{n\uparrow\infty} \frac{\partial}{\partial_{q_k}} \tilde{F}_N(t, q^{(n)}) &= \lim_{n\uparrow\infty} \mathbb{E}\left\langle\!\!\left\langle \mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha, \tilde{\alpha}) \tfrac{\sigma\wedge\tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \mathsf{q}^{(n)}, 2} \\
&= \mathbb{E}\left[\lim_{n\uparrow\infty} \mathbb{E}\left\langle\!\!\left\langle \mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha, \tilde{\alpha}) \tfrac{\sigma\wedge\tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \mathsf{q}^{(n)}, 2} \right] \\
&= \mathbb{E}\left\langle\!\!\left\langle \mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha, \tilde{\alpha}) \tfrac{\sigma\wedge\tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \mathsf{q}, 2} \\
&= \frac{\partial}{\partial_{q_k}} \tilde{F}_N(t, q).
\end{aligned} \tag{3.3.47}$$

In the second last step, we used that the map

$$\begin{aligned}
C_<^{(M)} &\to \mathbb{R}, \\
p = (p_0, \dots, p_M) &\mapsto \left\langle\!\!\left\langle \mathbb{1}_{\alpha\wedge\tilde{\alpha}=k}(\alpha, \tilde{\alpha}) \tfrac{\sigma\wedge\tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \sum_{j=0}^M p_j \mathbb{1}_{[z_j, z_{j+1})}, 2}
\end{aligned} \tag{3.3.48}$$

is clearly continuous for each realisation of all the randomness contained in the Gibbs average $\langle\!\langle \cdot \rangle\!\rangle_{t, \sum_{j=0}^M p_j \mathbb{1}_{[z_j, z_{j+1})}, 2}$. Note that by (3.1.13), this randomness can be described by $(\nu_\alpha)_{\alpha\in\mathbb{N}^M}$, $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ and $\left(z_{\sigma|_i, \alpha|_j}\right)_{\sigma\in\{-1,1\}^N;\, \alpha\in\mathbb{N}^M;\, i=1,\dots,N;\, j=0,\dots,M}$, so it does not depend on p. □

For a bounded and measurable function $g\colon \mathbb{N}^M \to \mathbb{R}$, which does not depend on the σ-spins, the 1-fold Gibbs average in (3.3.20) can be simplified to

$$\langle\!\langle g(\alpha) \rangle\!\rangle_{t, \mathsf{q}} = \frac{\sum_{\alpha\in\mathbb{N}^M} \tilde{w}_{t, \mathsf{q}, \alpha}\, g(\alpha)}{\sum_{\alpha\in\mathbb{N}^M} \tilde{w}_{t, \mathsf{q}, \alpha}}, \tag{3.3.49}$$

where

$$\tilde{w}_{t, \mathsf{q}, \alpha} := w_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t, \mathsf{q}, \sigma, \alpha)\right). \tag{3.3.50}$$

For $n \in \mathbb{N}$ and $g\colon (\mathbb{N}^M)^n \to \mathbb{R}$ bounded and measurable, the n-fold average over the α-spins is

$$\left\langle\!\!\left\langle g(\alpha^{(1)}, \ldots, \alpha^{(n)}) \right\rangle\!\!\right\rangle_{t,\mathsf{q},n} := \frac{\sum_{\alpha^{(1)},\ldots,\alpha^{(n)}\in\mathbb{N}^M} \prod_{i=1}^n \tilde{w}_{t,\mathsf{q},\alpha^{(i)}} g(\alpha^{(1)}, \ldots, \alpha^{(n)})}{\sum_{\tilde{\alpha}^{(1)},\ldots,\tilde{\alpha}^{(n)}\in\mathbb{N}^M} \prod_{i=1}^n \tilde{w}_{t,\mathsf{q},\tilde{\alpha}^{(i)}}}. \tag{3.3.51}$$

We now prove that $(\tilde{\nu}_\alpha)_{\alpha\in\mathbb{N}^M} := \left(\frac{\tilde{w}_{t,\mathsf{q},\alpha}}{\sum_{\tilde{\alpha}\in\mathbb{N}^M} \tilde{w}_{t,\mathsf{q},\tilde{\alpha}}}\right)_{\alpha\in\mathbb{N}^M}$ satisfies the Ghirlanda-Guerra identities. These identities form a universal property which characterises the structure and the multi-overlap distribution of a random probability measure. We refer to Section 2.3–2.5 of [100] for more details.

Proposition 3.3.6 (Ghirlanda-Guerra identities)**.** *Let $M, N \in \mathbb{N}$ and let $(\nu_\alpha)_{\alpha\in\mathbb{N}^M}$ be the Ruelle cascades with parameters $0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1$. Let $t > 0$ and $q = (q_0, \ldots, q_M) \in C_<^{(M)}$. We write $\mathsf{q} = \sum_{k=0}^M q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})}$ and $(\langle\!\langle\cdot\rangle\!\rangle_k)_{k\in\mathbb{N}} = \left(\langle\!\langle\cdot\rangle\!\rangle_{t,\mathsf{q},k}\right)_{k\in\mathbb{N}}$ as in (3.3.51). Let $n \in \mathbb{N}$ and $p = (p_0, \ldots, p_M) \in C_\le^{(M)}$. Let $g\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ be bounded and measureable functions. We set $R^n = (R_{\ell_1,\ell_2})_{\ell_1,\ell_2=1,\ldots,n} = \left(p_{\alpha^{(\ell_1)}\wedge\alpha^{(\ell_2)}}\right)_{\ell_1,\ell_2=1,\ldots,n}$, $R_{1,n+1} = p_{\alpha^{(1)}\wedge\alpha^{(n+1)}}$ for each $\alpha^{(1)}, \ldots, \alpha^{(n+1)} \in \mathbb{N}^M$. Then,*

$$\mathbb{E}\langle\!\langle g(R^n)\, h(R_{1,n+1})\rangle\!\rangle_{n+1} = \frac{1}{n}\mathbb{E}\langle\!\langle g(R^n)\rangle\!\rangle_n \langle\!\langle h(R_{1,2})\rangle\!\rangle_2 + \frac{1}{n}\sum_{\ell=2}^n \mathbb{E}\langle\!\langle g(R^n)\, h(R_{1,\ell})\rangle\!\rangle_n, \tag{3.3.52}$$

and

$$\mathbb{E}\langle\!\langle h(R_{1,2})\rangle\!\rangle_2 = \sum_{j=0}^M h(p_j)(\zeta_{j+1} - \zeta_j). \tag{3.3.53}$$

Proof. This proof has the same structure as that of Theorem 5.28 of [58], substituting $(\nu_\alpha)_{\alpha\in\mathbb{N}^M}$ by $(\tilde{\nu}_\alpha)_{\alpha\in\mathbb{N}^M}$.

Step 1: Reduction to polynomials and $p_0 = 0$. There exists a polynomial of degree at most M which coincides with h in the points $p_0, \ldots, p_M$. Since h is only evaluated in the points $(R_{1,\ell})_{\ell=1,\ldots,n+1}$, which can only assume the values $p_0, \ldots, p_M$, it suffices to prove (3.3.52) and (3.3.53) for polynomial functions $h\colon \mathbb{R} \to \mathbb{R}$. By the linearity of all the terms in (3.3.52) and (3.3.53), we assume without loss of generality that h is the form $x \mapsto h(x) = x^m$, $m \in \mathbb{N}_0$.

We now argue why $p_0 = 0$ can be assumed without loss of generality: With the functions

$$\begin{aligned} \theta_p\colon \{0, \ldots, M\} &\to \mathbb{R}_{\ge 0}, \\ k &\mapsto p_k \end{aligned} \tag{3.3.54}$$

and

$$\begin{aligned} \theta_p^{n\times n}\colon \{0, \ldots, M\}^{n\times n} &\to \mathbb{R}_{\ge 0}^{n\times n}, \\ (S_{\ell_1,\ell_2})_{\ell_1,\ell_2=1,\ldots,n} &\mapsto \left(\theta_p(S_{\ell_1,\ell_2})\right)_{\ell_1,\ell_2=1,\ldots,n}, \end{aligned} \tag{3.3.55}$$

we rewrite (3.3.52) and (3.3.53) as

$$\begin{aligned}\mathbb{E}\left\langle\!\!\left\langle (g\circ\theta_p^{n\times n})(\tilde{R}^n)\cdot(h\circ\theta_p)(\tilde{R}_{1,n+1})\right\rangle\!\!\right\rangle_{n+1} &= \frac{1}{n}\mathbb{E}\left\langle\!\!\left\langle (g\circ\theta_p^{n\times n})(\tilde{R}^n)\right\rangle\!\!\right\rangle_n \left\langle\!\!\left\langle (h\circ\theta_p)(\tilde{R}_{1,2})\right\rangle\!\!\right\rangle_2 \\ &\quad + \frac{1}{n}\sum_{\ell=2}^{n}\mathbb{E}\left\langle\!\!\left\langle (g\circ\theta_p^{n\times n})(\tilde{R}^n)\cdot(h\circ\theta_p)(\tilde{R}_{1,\ell})\right\rangle\!\!\right\rangle_n, \\ \mathbb{E}\langle\!\langle (h\circ\theta_p)(\tilde{R}_{1,2})\rangle\!\rangle_2 &= \sum_{j=0}^{M}(h\circ\theta_p)(j)(\zeta_{j+1}-\zeta_j), \end{aligned} \tag{3.3.56}$$

where $\tilde{R}^n = (\tilde{R}_{\ell_1,\ell_2})_{\ell_1,\ell_2=1,\dots,n} = (\alpha^{(\ell_1)}\wedge\alpha^{(\ell_2)})_{\ell_1,\ell_2=1,\dots,n}$. Since $(g\circ\theta_p^{n\times n})$ and $(h\circ\theta_p)$ are bounded and measurable functions, we can state (3.3.52) and (3.3.53) in a form that does not depend on the choice of $(p_0,\dots,p_M)$. Namely, for each $\tilde{g}\colon\mathbb{R}^{n\times n}\to\mathbb{R}$ and each $\tilde{h}\colon\mathbb{R}\to\mathbb{R}$ which are bounded and measureable functions,

$$\begin{aligned}\mathbb{E}\left\langle\!\!\left\langle \tilde{g}(\tilde{R}^n)\,\tilde{h}(\tilde{R}_{1,n+1})\right\rangle\!\!\right\rangle_{n+1} &= \frac{1}{n}\mathbb{E}\left\langle\!\!\left\langle \tilde{g}(\tilde{R}^n)\right\rangle\!\!\right\rangle_n\left\langle\!\!\left\langle \tilde{h}(\tilde{R}_{1,2})\right\rangle\!\!\right\rangle_2 + \frac{1}{n}\sum_{\ell=2}^{n}\mathbb{E}\left\langle\!\!\left\langle \tilde{g}(\tilde{R}^n)\,\tilde{h}(\tilde{R}_{1,\ell})\right\rangle\!\!\right\rangle_n, \\ \mathbb{E}\langle\!\langle \tilde{h}(\tilde{R}_{1,2})\rangle\!\rangle_2 &= \sum_{j=0}^{M} h(j)(\zeta_{j+1}-\zeta_j). \end{aligned} \tag{3.3.57}$$

If (3.3.52) and (3.3.53) are proven for a specific choice of $p=(p_0,\dots,p_M)\in C_\le^{(M)}$ with $p_0=0$, then the validity of this result can be transferred to any other choice $\tilde{p}=(\tilde{p}_0,\dots,\tilde{p}_M)\in C_\le^{(M)}$. This can be done by concatenating h with $\theta_p\circ\theta_{\tilde{p}}^{-1}$ and g with $\theta_p^{n\times n}\circ\left(\theta_{\tilde{p}}^{n\times n}\right)^{-1}$. Thus, we can assume that $p_0=0$. This is necessary to apply the *Bolthausen-Sznitman invariance* in Step 3 of this proof.

Step 2: Basic computations. For $m=0$, i.e. constant functions h, (3.3.52) and (3.3.53) are trivial. Let $m\ge 1$. We define

$$\left(Z_{p^m}(\alpha)\right) := \sum_{j=1}^{M}\left(p_j^m - p_{j-1}^m\right)^{1/2} z_{\alpha|_j}, \tag{3.3.58}$$

for each $\alpha\in\mathbb{N}^M$, where each $z_{\alpha|_j}$ is from a family of i.i.d. standard Gaussian random variables which are independent from both $(\tilde{w}_\alpha)_{\alpha\in\mathbb{N}^M}$ and $(H_N(t,\mathsf{q},\sigma,\alpha))_{\sigma\in\{-1,1\}^N,\alpha\in\mathbb{N}^M}$. Then,

$$\begin{aligned}\mathbb{E}\langle\!\langle g(R^n)\,Z_{p^m}(\alpha^{(1)})\rangle\!\rangle_n &= \mathbb{E}\left[\textstyle\sum_{\alpha^{(1)},\dots,\alpha^{(n)}\in\mathbb{N}^M}\tilde{v}_{\alpha^{(1)}}\cdots\tilde{v}_{\alpha^{(N)}}\,g(R^n)\,Z_{p^m}(\alpha^{(1)})\right] \\ &= \textstyle\sum_{\alpha^{(1)},\dots,\alpha^{(n)}\in\mathbb{N}^M} g(R^n)\,\mathbb{E}\left[\tilde{v}_{\alpha^{(1)}}\cdots\tilde{v}_{\alpha^{(N)}}Z_{p^m}(\alpha^{(1)})\right], \end{aligned} \tag{3.3.59}$$

where, since $\mathbb{E}\left[\left|Z_{p^m}\right|\right]<\infty$, the dominated convergence theorem justifies the exchange of expectation and sum. By the independence of $\left(Z_{p^m}(\alpha)\right)_{\alpha\in\mathbb{N}^M}$ and $(\tilde{v}_\alpha)_{\alpha\in\mathbb{N}^M}$,

$$\mathbb{E}\left[\tilde{v}_{\alpha^{(1)}}\cdots\tilde{v}_{\alpha^{(N)}}Z_{p^m}(\alpha^{(1)})\right] = \mathbb{E}\left[\tilde{v}_{\alpha^{(1)}}\cdots\tilde{v}_{\alpha^{(N)}}\right]\mathbb{E}\left[Z_{p^m}(\alpha^{(1)})\right] = 0. \tag{3.3.60}$$

Inserting (3.3.60) into (3.3.59) gives

$$\mathbb{E}\langle\!\langle g(R^n)\, Z_{p^m}(\alpha^{(1)})\rangle\!\rangle_n = 0, \tag{3.3.61}$$

which is the first step towards 3.3.52.

Step 3: Bolthausen-Sznitman invariance. We use the Bolthausen-Sznitman invariance to replace $\langle\!\langle\cdot\rangle\!\rangle$ in (3.3.61) by a different Gibbs average. For this purpose, we define the random Gibbs weights

$$\tilde{v}_\alpha^{(\beta,m)} := \frac{\tilde{w}_\alpha \exp\left(\beta Z_{p^m}(\alpha)\right)}{\sum_{\tilde{\alpha}\in\mathbb{N}^M} \tilde{w}_{\tilde{\alpha}} \exp\left(\beta Z_{p^m}(\tilde{\alpha})\right)}, \qquad \alpha \in \mathbb{N}^M, \tag{3.3.62}$$

for each $\beta \geq 0$. Note that $\tilde{v}_\alpha^{(0,m)} = \tilde{v}_\alpha$ for each $\alpha \in \mathbb{N}^M$. The n-fold average $\langle\!\langle\cdot\rangle\!\rangle_n^{(\beta,m)}$ w.r.t. $\left(\tilde{v}_\alpha^{(\beta,m)}\right)_{\alpha\in\mathbb{N}^M}$ is defined by

$$\left\langle\!\!\left\langle f(\alpha^{(1)},\dots,\alpha^{(n)})\right\rangle\!\!\right\rangle_n^{(\beta,m)} := \sum_{\alpha^{(1)},\dots,\alpha^{(n)}\in\mathbb{N}^M} \prod_{i=1}^n \tilde{v}_{\alpha^{(i)}}^{(\beta,m)} f(\alpha^{(1)},\dots,\alpha^{(n)}), \tag{3.3.63}$$

where $f\colon \left(\mathbb{N}^M\right)^n \to \mathbb{R}$ is a bounded and measurable function.

Since $p_0 = 0$, the Bolthausen-Sznitman invariance [58, Lemma 5.27] implies that for each $\beta \geq 0$ there exists a random permutation $\pi\colon \mathbb{N}^M \to \mathbb{N}^M$ which satisfies both of the following:

1. π preserves the tree structure on $\mathbb{N}^M$, i.e. $\pi(\alpha)\wedge\pi(\tilde{\alpha}) = \alpha\wedge\tilde{\alpha}$ for all $\alpha,\tilde{\alpha}\in\mathbb{N}^M$.
2. It holds

$$\left(w_\alpha, Z_{p^m}(\alpha)\right)_{\alpha\in\mathbb{N}^M} \overset{\mathrm{d}}{=} \left(w_{\pi(\alpha)} \exp\left(\beta Z_{p^m}(\pi(\alpha)) - \frac{\beta^2 b_m}{2}\right), Z_{p^m}(\pi(\alpha)) - \beta b_m\right)_{\alpha\in\mathbb{N}^M}, \tag{3.3.64}$$

where $b_m := \sum_{j=1}^M \left(p_j^m - p_{j-1}^m\right)\zeta_j$.

By (3.3.64) and by the independence of $\left(Z_{p^m}(\alpha)\right)_{\alpha\in\mathbb{N}^M}$, $(H_N(t,\mathsf{q},\sigma,\alpha))_{\sigma\in\{-1,1\}^N,\alpha\in\mathbb{N}^M}$ and $(\tilde{v}_\alpha)_{\alpha\in\mathbb{N}^M}$,

$$\begin{aligned}&\left(w_\alpha, \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right),\, Z_{p^m}(\alpha)\right)_{\alpha\in\mathbb{N}^M}\\ &\overset{\mathrm{d}}{=} \left(w_{\pi(\alpha)} \exp\left(\beta Z_{p^m}(\pi(\alpha)) - \frac{\beta^2 b_m}{2}\right), \sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right),\, Z_{p^m}(\pi(\alpha)) - \beta b_m\right)_{\alpha\in\mathbb{N}^M}.\end{aligned} \tag{3.3.65}$$

The covariances of $\left(\sum_{\sigma\in\{-1,1\}^N} \exp\left(H_N(t,\mathsf{q},\sigma,\alpha)\right)\right)_{\alpha\in\mathbb{N}^M}$ only depend on the α-overlaps (besides t,q,N). Thus, since these overlaps are invariant under π, the statement in (3.3.65) remains true if we replace $H_N(t,\mathsf{q},\sigma,\alpha)$ on the right-hand side by $H_N(t,\mathsf{q},\sigma,\pi(\alpha))$. This implies

$$\left(\tilde{v}_\alpha, Z_{p^m}(\alpha)\right)_{\alpha\in\mathbb{N}^M} \overset{\mathrm{d}}{=} \left(\tilde{v}_{\pi(\alpha)}^{(\beta,m)}, Z_{p^m}(\pi(\alpha)) - \beta b_m\right)_{\alpha\in\mathbb{N}^M}. \tag{3.3.66}$$

Step 4: Gibbs-Gaussian integration by parts. By (3.3.61) and (3.3.66),

$$\begin{aligned} 0 &= \mathbb{E}\langle\!\langle g(R^n)\, Z_{p^m}(\alpha^{(1)})\rangle\!\rangle_n \\ &= \mathbb{E}\Big[\textstyle\sum_{\alpha^{(1)},\dots,\alpha^{(n)}\in\mathbb{N}^M} \tilde{\nu}_{\alpha^{(1)}}\cdots\tilde{\nu}_{\alpha^{(n)}}\, g(R^n)\, Z_{p^m}(\alpha^{(1)})\Big] \\ &= \mathbb{E}\Big[\textstyle\sum_{\alpha^{(1)},\dots,\alpha^{(n)}\in\mathbb{N}^M} \tilde{\nu}^{(\beta,m)}_{\pi(\alpha^{(1)})}\cdots\tilde{\nu}^{(\beta,m)}_{\pi(\alpha^{(n)})}\, g(R^n)\, \Big(Z_{p^m}(\pi(\alpha^{(1)}))-\beta b_m\Big)\Big]. \end{aligned} \tag{3.3.67}$$

Since π preserves the tree structure, applying it to each component of R^n on the right-hand side of (3.3.67) does not change any value. This gives

$$\begin{aligned} 0 &= \mathbb{E}\Big[\textstyle\sum_{\alpha^{(1)},\dots,\alpha^{(n)}\in\mathbb{N}^M} \tilde{\nu}^{(\beta,m)}_{\alpha^{(1)}}\cdots\tilde{\nu}^{(\beta,m)}_{\alpha^{(n)}}\, g(R^n)\, \Big(Z_{p^m}(\alpha^{(1)})-\beta b_m\Big)\Big] \\ &= \mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\, \Big(Z_{p^m}(\alpha^{(1)})-\beta b_m\Big)\Big\rangle\!\!\Big\rangle^{(\beta,m)}_n \\ &= \beta\,\mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\, \Big(\textstyle\sum_{i=1}^n p^m_{\alpha^{(1)}\wedge\alpha^{(i)}} - n p^m_{\alpha^{(1)}\wedge\alpha^{(n+1)}} - b_m\Big)\Big\rangle\!\!\Big\rangle^{(\beta,m)}_{n+1}, \end{aligned} \tag{3.3.68}$$

with the Gibbs-Gaussian integration by parts formula (see Lemma 3.2.3), using that for $\alpha,\tilde{\alpha}\in\mathbb{N}^M$ we have $\mathbb{E}\big[Z_{p^m}(\alpha)\,Z_{p^m}(\tilde{\alpha})\big] = p^m_{\alpha\wedge\tilde{\alpha}}$. Since all the arguments of $\langle\!\langle\cdot\rangle\!\rangle^{(\beta,m)}_{n+1}$ on the right-hand side of (3.3.68) are invariant under π, we can use (3.3.66) to change $\langle\!\langle\cdot\rangle\!\rangle^{(\beta,m)}_{n+1}$ back to $\langle\!\langle\cdot\rangle\!\rangle_{n+1}$. This gives

$$\begin{aligned} 0 &= \beta\,\mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\, \Big(\textstyle\sum_{i=1}^n p^m_{\alpha^{(1)}\wedge\alpha^{(i)}} - n p^m_{\alpha^{(1)}\wedge\alpha^{(n+1)}} - b_m\Big)\Big\rangle\!\!\Big\rangle_{n+1} \\ &= \beta\,\mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\, \Big(\textstyle\sum_{i=1}^n (R_{1,i})^m - n\,(R_{1,n+1})^m - b_m\Big)\Big\rangle\!\!\Big\rangle_{n+1}. \end{aligned} \tag{3.3.69}$$

Thus, since $R_{1,1} = p_M$,

$$\mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\ (R_{1,n+1})^m\Big\rangle\!\!\Big\rangle_{n+1} = \frac{1}{n}\sum_{i=2}^n \mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\ (R_{1,i})^m\Big\rangle\!\!\Big\rangle_n + \frac{p^m_M - b_m}{n}\mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\Big\rangle\!\!\Big\rangle_n. \tag{3.3.70}$$

With $n = 1$ and $g \equiv 1$, this gives

$$\mathbb{E}\Big\langle\!\!\Big\langle g(R^n)\ (R_{1,2})^m\Big\rangle\!\!\Big\rangle_2 = p^m_M - b_m = \sum_{j=0}^M p^m_j\left(\zeta_{j+1}-\zeta_j\right). \tag{3.3.71}$$

This means that we have proven (3.3.53). From (3.3.70) and (3.3.71) also follows (3.3.52). □

Using the previous propositions, we prove the central result of this section.

Proof of Proposition 3.3.1. This proof has the same structure as that of [58, Proposition 6.3], the analogous result for the SK model. Let $M \in \mathbb{N}$.

Step 1: Averaging over the Ruelle cascades. For $\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})} \in Q^{(M)}$, we have by Lemma 3.2.7 that

$$F_N(t,\mathsf{q}) = q_M + t - \frac{1}{N}\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots\mathbb{E}\Bigg[\mathbb{E}\Big[Z_{t,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_M}\Big|\mathcal{F}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}}\Bigg|\mathcal{F}_{M-2}\Bigg]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\Bigg|\mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}}\Bigg|\mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg], \tag{3.3.72}$$

recalling

$$Z_{t,\mathsf{q}}(z_0,\dots,z_M) := \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\,H_N^A(\sigma) + \sqrt{2}\,\sum_{j=0}^{M}(q_j-q_{j-1})^{\frac{1}{2}} z_j(\sigma)\right),$$
$$\mathcal{F}_k := \sigma(z_0,\dots,z_k), \quad k = 0,\dots,M-1. \tag{3.3.73}$$

Furthermore recall that, independent of $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$, the processes $z_1 = (z_1(\sigma))_{\sigma\in\{-1,1\}^N},\dots,$ $z_M = (z_M(\sigma))_{\sigma\in\{-1,1\}^N}$ are i.i.d. copies of $z_0 := (z_0(\sigma))_{\sigma\in\{-1,1\}^N}$, a BRW on the N-level binary tree with standard Gaussian increments.

Step 2: Repetitions of parameters. Let $\hat{\mathsf{q}} = \sum_{k=0}^{M} \hat{q}_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})} \in Q_{\leq}^{(M)} \setminus Q^{(M)}$, i.e. there exists $k = 0,\dots,M-1$ with $\hat{q}_k = \hat{q}_{k+1}$. Furthermore, there exists $\hat{\mathsf{p}} \in \bigcup_{k=0}^{M-1} Q^{(k)}$ with $\hat{\mathsf{p}}(u) = \hat{\mathsf{q}}(u)$ for all $u \in [0,1]$. In (3.1.14), we have defined F_N only for piecewise constant paths $\mathsf{q} \in Q^{(M)}$ without any repetition in the parameters. Now, we ensure that a naive extension of the domain of $F_N(t,\cdot)$ to $\bigcup_{k=0}^{\infty} Q_{\leq}^{(k)}$ is compatible with (3.1.14) in the sense that $F_N(t,\hat{\mathsf{q}}) = F_N(t,\hat{\mathsf{p}})$ for all $t > 0$. If we define $F_N(t,\hat{\mathsf{q}})$ as in (3.3.72), replacing q by $\hat{\mathsf{q}}$, then the factor $(\hat{q}_{k+1} - \hat{q}_k)^{\frac{1}{2}}$, which appears in $Z_{t,\hat{\mathsf{q}}}(z_0,\dots,z_M)$, is zero. Thus, z_{k+1} does not affect the right-hand side of (3.3.72) and we get

$$\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots\mathbb{E}\Bigg[\mathbb{E}\Big[Z_{t,\hat{\mathsf{q}}}(z_0,\dots,z_M)^{\zeta_M}\Big|\mathcal{F}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}}\Bigg|\mathcal{F}_{M-2}\Bigg]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\Bigg|\mathcal{F}_{k+1}\Bigg]^{\frac{\zeta_{k+1}}{\zeta_{k+2}}}\Bigg|\mathcal{F}_k\Bigg]^{\frac{\zeta_k}{\zeta_{k+1}}}$$
$$= \mathbb{E}\Bigg[\dots\mathbb{E}\Bigg[\mathbb{E}\Big[Z_{t,\hat{\mathsf{q}}}(z_0,\dots,z_M)^{\zeta_M}\Big|\mathcal{F}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}}\Bigg|\mathcal{F}_{M-2}\Bigg]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\Bigg|\mathcal{F}_{k+1}\Bigg]^{\frac{\zeta_k}{\zeta_{k+2}}}. \tag{3.3.74}$$

Repeating this procedure for all repetitions in the parameters of $\hat{\mathsf{q}}$ shows that $F_N(t,\hat{\mathsf{q}}) = F_N(t,\hat{\mathsf{p}})$ for all $t \geq 0$.

Step 3: Extending the partial derivatives. Recall the definitions of $C_{\leq}^{(M)}$, $C_{<}^{(M)}$ and $\tilde{F}_N$ in (3.3.23), (3.3.24) and (3.3.25), respectively. We have computed the partial derivatives of $\tilde{F}_N$ in Proposition 3.3.4 and have shown in Proposition 3.3.5, $\tilde{F}_N(t,\cdot)$ is continuously differentiable on $C_{<}^{(M)}$ for each $t > 0$. Since $C_{<}^{(M)}$ is convex, the partial derivatives extend to the boundary, so $\tilde{F}_N(t,\cdot)$ is continuously differentiable on $C_{\leq}^{(M)}$ for each $t \geq 0$. This is a classical result from analysis, see e.g. [111]. In particular, the formula (3.3.26) for the partial derivatives of $\tilde{F}_N$ in Proposition 3.3.4 is also true for $t \geq 0$ and $q \in C_{\leq}^{(M)}$.

Step 4: Lipschitz continuity. We now compare $F_N(t, \mathsf{q})$ and $F_N(t, \tilde{\mathsf{q}})$ for $\mathsf{q}, \tilde{\mathsf{q}} \in \bigcup_{M=1}^{\infty} Q_{\leq}^{(M)}$ by taking $M \in \mathbb{N}$ and

$$
\begin{aligned}
&0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1, \\
&0 = q_{-1} \leq q_0 \leq \cdots \leq q_M < \infty, \\
&0 = \tilde{q}_{-1} \leq \tilde{q}_0 \leq \cdots \leq \tilde{q}_M < \infty,
\end{aligned}
\tag{3.3.75}
$$

so that

$$
\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})} \qquad \text{and} \qquad \tilde{\mathsf{q}} = \sum_{k=0}^{M} \tilde{q}_k \mathbb{1}_{[\zeta_k, \zeta_{k+1})}. \tag{3.3.76}
$$

For $\lambda \in [0, 1]$, we define

$$
\mathsf{q}^{(\lambda)} := \lambda \mathsf{q} + (1 - \lambda)\tilde{\mathsf{q}} = \sum_{k=0}^{M} (\lambda q_k + (1 - \lambda)\tilde{q}_k) \mathbb{1}_{[\zeta_k, \zeta_{k+1})}. \tag{3.3.77}
$$

Note that $\mathsf{q}^{(\lambda)} \in Q_{\leq}^{(M)}$. By the fundamental theorem of calculus,

$$
|F_N(t, \mathsf{q}) - F_N(t, \tilde{\mathsf{q}})| = \left| \int_0^1 \tfrac{\partial}{\partial_\lambda} F_N(t, \mathsf{q}^{(\tilde{\lambda})}) \,\mathrm{d}\tilde{\lambda} \right| \leq \sup_{\tilde{\lambda} \in [0,1]} \left| \tfrac{\partial}{\partial_\lambda} F_N(t, \mathsf{q}^{(\tilde{\lambda})}) \right|. \tag{3.3.78}
$$

For each $\tilde{\lambda} \in [0, 1]$, the chain rule implies that

$$
\tfrac{\partial}{\partial_\lambda} F_N \left(t, \mathsf{q}^{(\tilde{\lambda})} \right) = \sum_{k=0}^{M} (q_k - \tilde{q}_k) \tfrac{\partial}{\partial_{q_k}} \tilde{F}_N \left(t, q^{(\lambda)} \right), \tag{3.3.79}
$$

where

$$
q^{(\tilde{\lambda})} := (\tilde{\lambda} q_0 + (1 - \tilde{\lambda})\tilde{q}_0, \ldots, \tilde{\lambda} q_M + (1 - \tilde{\lambda})\tilde{q}_M). \tag{3.3.80}
$$

By (3.3.26) and Proposition 3.3.6,

$$
\tfrac{\partial}{\partial_{q_k}} \tilde{F}_N(t, q^{(\tilde{\lambda})}) = \mathbb{E} \left\langle\!\!\left\langle \mathbb{1}_{\alpha \wedge \tilde{\alpha} = k} (\alpha, \tilde{\alpha}) \, \tfrac{\sigma \wedge \tilde{\sigma}}{N} \right\rangle\!\!\right\rangle_{t, \mathsf{q}^{(\tilde{\lambda})}, 2} \leq \mathbb{E} \left\langle\!\!\left\langle \mathbb{1}_{\alpha \wedge \tilde{\alpha} = k} (\alpha, \tilde{\alpha}) \right\rangle\!\!\right\rangle_{t, \mathsf{q}^{(\tilde{\lambda})}, 2} = \zeta_{k+1} - \zeta_k, \tag{3.3.81}
$$

where $\langle\!\langle \cdot \rangle\!\rangle_{t, \mathsf{q}^{(\lambda)}, 2}$ is defined in (3.3.51). Inserting (3.3.79) and (3.3.81) into (3.3.78) gives

$$
|F_N(t, \mathsf{q}) - F_N(t, \tilde{\mathsf{q}})| \leq \sum_{k=0}^{M} |\zeta_{k+1} - \zeta_k| \, |q_k - \tilde{q}_k| = \|\mathsf{q} - \tilde{\mathsf{q}}\|_1. \tag{3.3.82}
$$

Since by Lemma 3.3.2, $\bigcup_{M=1}^{\infty} Q^{(M)}$ is dense in Q_1, (3.3.82) implies the existence of a unique extension of F_N to $\mathbb{R}_{\geq 0} \times Q_1$. This extension satisfies (3.3.82) on its domain. □

3.4 The level sets of the M-BRW with Gaussian increments

Let $M, N \in \mathbb{N}$ and let $(z_1(\sigma))_{\sigma\in\{-1,1\}^N}, \ldots, (z_M(\sigma))_{\sigma\in\{-1,1\}^N}$ be i.i.d. copies of $(z_0(\sigma))_{\sigma\in\{-1,1\}^N}$, a branching random walk on the N-level binary tree with standard Gaussian increments. We write

$$z_j(\sigma) = \sum_{i=1}^{N} z_{j,\sigma|_i}, \qquad j \in \{0, \ldots, M\}, \sigma \in \{-1, 1\}^N, \tag{3.4.1}$$

where $(z_{j,\sigma|_i})_{\sigma\in\{-1,1\}^N;\, i=1,\ldots,N;\, j=0,\ldots,M}$ are i.i.d. standard Gaussians. For $\ell = 1, \ldots, N$, the σ-algebra generated by $(z_{j,\sigma|_i})_{\sigma\in\{-1,1\}^N;\, i=1,\ldots,\ell;\, j=0,\ldots,M}$ is denoted by $\mathcal{F}_\ell$. Let $a = (a_0, \ldots, a_M) \in \mathbb{R}_+^{M+1}$. We define the *additive martingale* (or *McKean martingale*) for the family of branching random walks $(z_j(\sigma))_{\sigma\in\{-1,1\}^N;\, j=0,\ldots,M}$ by

$$Y_a(N) = 2^{-N} \mathrm{e}^{-\frac{1}{2}\|a\|_2^2 N} \sum_{\sigma\in\{-1,1\}^N} \exp\left(\textstyle\sum_{j=0}^{M} a_j z_j(\sigma)\right). \tag{3.4.2}$$

Proposition 3.4.1. *Let $a = (a_0, \ldots, a_M) \in \mathbb{R}_+^{M+1}$ with $\|a\|_2^2 < 2\ln 2$. Then, $(Y_a(N))_{N\in\mathbb{N}}$ is a nonnegative and uniformly integrable $\mathcal{F}_N$-adapted martingale with $\mathbb{E}[Y_a(N)] = 1$ for all $N \in \mathbb{N}$.*

Thus, there exists by the martingale convergence theorem an a.s. limit of $(Y_a(N))_{N\in\mathbb{N}}$, which we denote by $Y_a(\infty)$.

Theorem 3.4.2. *Let $a = (a_0, \ldots, a_M) \in \mathbb{R}_+^{M+1}$ with $\|a\|_2^2 < 2\ln 2$. Then, for*

$$T_a(N) := \left\{\sigma \in \{-1, 1\}^N : z_j(\sigma) \geq a_j N \;\forall_{j=0,\ldots,M}\right\}, \tag{3.4.3}$$

we have

$$\lim_{N\uparrow\infty} \frac{|T_a(N)|}{\mathbb{E}[|T_a(N)|]} = Y_a(\infty) \tag{3.4.4}$$

almost surely.

In this section, we first use the methods of [69] to prove Theorem 3.4.2. Then, we prove Proposition 3.4.1 and the following proposition.

Proposition 3.4.3. *Let $a = (a_0, \ldots, a_M) \in \mathbb{R}_+^{M+1}$ with $\|a\|_2^2 < 2\ln 2$. Then, $\mathbb{P}(Y_a(\infty) = 0) = 0$ and $\mathbb{P}(Y_a(\infty) > 0) = 1$.*

From this we obtain the following corollary.

Corollary 3.4.4. *In the setting of Theorem 3.4.2,*

$$\lim_{N\uparrow\infty} \mathbb{P}(|T_a(N)| = 0) = 0. \tag{3.4.5}$$

Proof. By Theorem 3.4.2 and Proposition 3.4.3,

$$\lim_{N\uparrow\infty} \mathbb{P}\left(|T_a(N)| = 0\right) = \lim_{N\uparrow\infty} \mathbb{P}\left(\frac{|T_a(N)|}{\mathbb{E}[|T_a(N)|]} = 0\right) = \mathbb{P}\left(Y_a(\infty) = 0\right) = 0. \qquad \square$$

Corollary 3.4.4 will be used in the proof of Theorem 3.1.1 in Section 3.5.

For the proof of Theorem 3.4.2, we first establish the following lemmas.

Lemma 3.4.5. *Let $a = (a_0, \ldots, a_M) \in \mathbb{R}_+^{M+1}$ with $\|a\|_2^2 < 2\ln 2$ and let $T_a(N)$ be as in (3.4.3). Let $r = r(N) \uparrow \infty$ as $N \uparrow \infty$ with $r = o(N)$. Then for all $\varepsilon > 0$ exist $c_1, c_2 > 0$ possibly depending on ε such that*

$$\mathbb{P}\left(\frac{\big||T_a(N)| - \mathbb{E}\left[|T_a(N)| \,|\, \mathcal{F}_r\right]\big|}{\mathbb{E}\left[|T_a(N)|\right]} > \varepsilon\right) \le c_1 \mathrm{e}^{-c_2 r}, \tag{3.4.6}$$

for N large enough.

Proof. We adapt the proof of [69, Proposition 1.3] to our setting. Let $\delta > 0$. We set

$$\begin{aligned} T^{\le}_{a,r,\delta}(N) &:= \left\{\sigma \in T_a(N) \colon \textstyle\sum_{i=1}^n z_{j,\sigma|_i} \le (a_j + \delta) n \quad \forall_{n=r,\ldots,N} \,\forall_{j=0,\ldots,M}\right\}, \\ T^{>}_{a,r,\delta}(N) &:= T_a(N) \setminus T^{\le}_{a,r,\delta}(N). \end{aligned} \tag{3.4.7}$$

Since

$$\begin{aligned} &\mathbb{P}\left(\frac{\big||T_a(N)| - \mathbb{E}[|T_a(N)| \,|\, \mathcal{F}_r]\big|}{\mathbb{E}[|T_a(N)|]} > \varepsilon\right) \\ &\le \mathbb{P}\left(\frac{\Big|\big|T^{\le}_{a,r,\delta}(N)\big| - \mathbb{E}\Big[\big|T^{\le}_{a,r,\delta}(N)\big| \,\Big|\, \mathcal{F}_r\Big]\Big|}{\mathbb{E}\Big[\big|T_a(N)\big|\Big]} + \frac{\big|T^{>}_{a,r,\delta}(N)\big|}{\mathbb{E}\Big[\big|T_a(N)\big|\Big]} + \frac{\mathbb{E}\Big[\big|T^{>}_{a,r,\delta}(N)\big| \,\Big|\, \mathcal{F}_r\Big]}{\mathbb{E}\Big[\big|T_a(N)\big|\Big]} > \varepsilon\right) \\ &\le \mathbb{P}\left(\frac{\Big|\big|T^{\le}_{a,r,\delta}(N)\big| - \mathbb{E}\Big[\big|T^{\le}_{a,r,\delta}(N)\big| \,\Big|\, \mathcal{F}_r\Big]\Big|}{\mathbb{E}\Big[\big|T_a(N)\big|\Big]} > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(\frac{\big|T^{>}_{a,r,\delta}(N)\big|}{\mathbb{E}\Big[\big|T_a(N)\big|\Big]} > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(\frac{\mathbb{E}\Big[\big|T^{>}_{a,r,\delta}(N)\big| \,\Big|\, \mathcal{F}_r\Big]}{\mathbb{E}\Big[\big|T_a(N)\big|\Big]} > \frac{\varepsilon}{3}\right) \\ &=: (P1) + (P2) + (P3), \end{aligned} \tag{3.4.8}$$

it suffices to bound each summand $(P1), (P2), (P3)$ by $c_1 \mathrm{e}^{-c_2 r}$ for some $c_1, c_2 > 0$.

1. First moment estimates for $(P2)$ and $(P3)$: We first compute

$$\begin{aligned} \mathbb{E}\left[|T_a(N)|\right] &= \sum_{\sigma \in \{-1,1\}^N} \mathbb{P}\left(z_j(\sigma) \ge a_j N \,\forall_{j=0,\ldots,M}\right) \\ &= 2^N \prod_{j=0}^M \int_{a_j N}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}} \\ &= 2^N \exp\left(-\frac{\|a\|_2^2}{2} N\right) \prod_{j=0}^M \frac{1}{\sqrt{2\pi a_j^2 N}} (1 + o(1)), \end{aligned} \tag{3.4.9}$$

by the independence of $(z_j)_{j=0,\dots,M}$ and by Lemma 3.2.1. Similarly,

$$\mathbb{E}\left[\left|T^{\le}_{a,r,\delta}(N)\right|\right] = \sum_{\sigma\in\{-1,1\}^N} \mathbb{P}\left(\forall_{j=0,\dots,M}: z_j(\sigma) \ge a_j N \text{ and } \textstyle\sum_{i=1}^n z_{j,\sigma|_i} \le (a_j+\delta)n\ \forall_{n=r,\dots,N}\right)$$
$$= 2^N \prod_{j=0}^{M} \int_{a_jN}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}} \mathbb{P}\left(\mathfrak{z}^N_{0,z}(n) \le (a_j+\delta)n\ \forall_{n=r,\dots,N}\right), \tag{3.4.10}$$

where $\mathfrak{z}^N_{0,z}$ denotes a Brownian bridge from 0 to z in time N. For $z < (a_j + \frac{\delta}{4})N$, since $s \mapsto (a_j+\delta)s$ lies above $s \mapsto \frac{\delta r}{2} + (a_j + \frac{\delta}{2})s$ for $s \in [r, N]$,

$$\mathbb{P}\left(\mathfrak{z}^N_{0,z}(n) \le (a_j+\delta)n\ \forall_{n=r,\dots,N}\right) \ge \mathbb{P}\left(\mathfrak{z}^N_{0,z}(s) \le (a_j+\delta)s\ \forall_{s\in[r,N]}\right)$$
$$\ge \mathbb{P}\left(\mathfrak{z}^N_{0,z}(s) \le \frac{\delta r}{2} + (a_j+\frac{\delta}{2})s\ \forall_{s\in[0,N]}\right)$$
$$= 1 - \exp\left(-2\,\frac{\frac{\delta r}{2}\left(\frac{\delta r}{2}+(a_j+\frac{\delta}{2})N - z\right)}{N}\right)$$
$$\ge 1 - \mathrm{e}^{-\frac{\delta^2}{8}r}, \tag{3.4.11}$$

using a Ballot Theorem as in Lemma 3.2.2 for the equality. In the integral in (3.4.10), we bound the probability of the Brownian bridge from below by 0 for $z \ge (a_j + \frac{\delta}{4})N$ and insert (3.4.11) in the remaining part of the integral. This gives

$$\mathbb{E}\left[\left|T^{\le}_{a,r,\delta}(N)\right|\right] \ge 2^N \left(1 - \mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1} \prod_{j=0}^{M} \int_{a_jN}^{(a_j+\frac{\delta}{4})N} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}}. \tag{3.4.12}$$

With the bounds in (3.4.9) and (3.4.12), we get

$$\mathbb{E}\left[\left|T^{>}_{a,r,\delta}(N)\right|\right] = \mathbb{E}\left[|T_a(N)|\right] - \mathbb{E}\left[\left|T^{\le}_{a,r,\delta}(N)\right|\right]$$
$$\le 2^N \left(\prod_{j=0}^{M} \int_{a_jN}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}} - \left(1 - \mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1} \prod_{j=0}^{M} \int_{a_jN}^{(a_j+\frac{\delta}{4})N} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}}\right)$$
$$= 2^N \left(1 - \left(1 - \mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1}\right) \prod_{j=0}^{M} \int_{a_jN}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}}$$
$$+ 2^N \left(1 - \mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1} \left(\prod_{j=0}^{M} \int_{a_jN}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}} - \prod_{j=0}^{M} \int_{a_jN}^{(a_j+\frac{\delta}{4})N} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}}\right). \tag{3.4.13}$$

The second-last line of (3.4.13) is equal to

$$2^N \left(1 - \left(1 - \mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1}\right) \prod_{j=0}^{M} \int_{a_jN}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N}}$$
$$= \left(1 - \left(1 - \mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1}\right) \mathbb{E}\left[|T_a(N)|\right]$$
$$= (M+1)\,\mathrm{e}^{-\frac{\delta^2}{8}r}\, \mathbb{E}\left[|T_a(N)|\right](1+o(1)), \tag{3.4.14}$$

since $r = r(N) \uparrow \infty$ as $N \uparrow \infty$. We bound the last line of (3.4.13) for any $\sigma \in \{-1,1\}^N$ by

$$\begin{aligned}
&2^N\left(1-\mathrm{e}^{-\frac{\delta^2}{8}r}\right)^{M+1}\left(\prod_{j=0}^{M}\int_{a_jN}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi N}}\mathrm{e}^{-\frac{z^2}{2N}}-\prod_{j=0}^{M}\int_{a_jN}^{(a_j+\frac{\delta}{4})N}\frac{\mathrm{d}z}{\sqrt{2\pi N}}\mathrm{e}^{-\frac{z^2}{2N}}\right)\\
&\le 2^N\Big(\mathbb{P}\left(\forall_{j=0,\dots,M}\colon z_j(\sigma)\ge a_jN\right)-\mathbb{P}\left(\forall_{j=0,\dots,M}\colon z_j(\sigma)\in[a_jN,(a_j+\tfrac{\delta}{4})N]\right)\Big)\\
&= 2^N\,\mathbb{P}\left(\forall_{j=0,\dots,M}\colon z_j(\sigma)\ge a_jN,\ \exists_{j=0,\dots,M}\colon z_j(\sigma)>(a_j+\tfrac{\delta}{4})N\right)\\
&\le 2^N\sum_{k=0}^{M}\mathbb{P}\left(\forall_{j=0,\dots,M-1}\colon z_j(\sigma)\ge a_jN,\ z_k(\sigma)>(a_k+\tfrac{\delta}{4})N\right),
\end{aligned}\tag{3.4.15}$$

where for each $k=0,\dots,M$,

$$\begin{aligned}
\mathbb{P}\left(\forall_{j=0,\dots,M-1}\colon z_j(\sigma)\ge a_jN,\ z_k(\sigma)>(a_k+\tfrac{\delta}{4})N\right)&\le\exp\Big(-\tfrac{(a_k+\delta/4)^2}{2}N-\textstyle\sum_{\substack{j=0\\ j\neq k}}^{M}\tfrac{a_j^2}{2}N\Big)\\
&\le\exp\left(-\tfrac{\delta^2}{32}N\right)\exp\Big(-\textstyle\sum_{j=0}^{M}\tfrac{a_j^2}{2}N\Big).
\end{aligned}\tag{3.4.16}$$

Thus, the last line of (3.4.15) satisfies

$$\begin{aligned}
&2^N\sum_{k=0}^{M}\mathbb{P}\left(\forall_{j=0,\dots,M-1}\colon z_j(\sigma)\ge a_jN,\ z_k(\sigma)>(a_k+\tfrac{\delta}{4})N\right)\\
&\le 2^N(M+1)\exp\left(-\tfrac{\delta^2}{32}N\right)\exp\Big(-\textstyle\sum_{j=0}^{M}\tfrac{a_j^2}{2}N\Big)\\
&=(M+1)\exp\left(-\tfrac{\delta^2}{32}N\right)\mathbb{E}\left[|T_a(N)|\right]\prod_{j=0}^{M}\left(2\pi a_j^2N\right)^{1/2}(1+o(1)),
\end{aligned}\tag{3.4.17}$$

where we used (3.4.9) for the equality. We see that the last line of (3.4.17) is of lower order than the last line of (3.4.14). Thus, inserting (3.4.14) and (3.4.15) into (3.4.13) gives

$$\mathbb{E}\left[\left|T^{>}_{a,r,\delta}(N)\right|\right]\le(M+1)\,\mathrm{e}^{-\frac{\delta^2}{8}r}\mathbb{E}\left[|T_a(N)|\right](1+o(1)).\tag{3.4.18}$$

By Markov's inequality and (3.4.18),

$$(P2)=\mathbb{P}\left(\frac{\left|T^{>}_{a,r,\delta}(N)\right|}{\mathbb{E}\left[\left|T_a(N)\right|\right]}>\frac{\varepsilon}{3}\right)\le\frac{\mathbb{E}\left[\left|T^{>}_{a,r,\delta}(N)\right|\right]}{\frac{\varepsilon}{3}\mathbb{E}\left[\left|T_a(N)\right|\right]}\le\frac{3(M+1)}{\varepsilon}\mathrm{e}^{-\frac{\delta^2}{8}r}(1+o(1)).\tag{3.4.19}$$

Proceeding as for $(P2)$ and additionally using the tower property, we get

$$\begin{aligned}
(P3)&=\mathbb{P}\left(\frac{\mathbb{E}\left[\left|T^{>}_{a,r,\delta}(N)\right|\,\middle|\,\mathcal{F}_r\right]}{\mathbb{E}\left[\left|T_a(N)\right|\right]}>\frac{\varepsilon}{3}\right)\\
&\le\frac{\mathbb{E}\left[\mathbb{E}\left[\left|T^{>}_{a,r,\delta}(N)\right|\,\middle|\,\mathcal{F}_r\right]\right]}{\frac{\varepsilon}{3}\mathbb{E}\left[\left|T_a(N)\right|\right]}=\frac{\mathbb{E}\left[\left|T^{>}_{a,r,\delta}(N)\right|\right]}{\frac{\varepsilon}{3}\mathbb{E}\left[\left|T_a(N)\right|\right]}\le\frac{3(M+1)}{\varepsilon}\mathrm{e}^{-\frac{\delta^2}{8}r}(1+o(1)).
\end{aligned}\tag{3.4.20}$$

2. Second moment estimates for $(P1)$*:* By Markov's inequality,

$$(P1) = \mathbb{P}\left(\frac{\left|\left|T_{a,r,\delta}^{\leq}(N)\right| - \mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right| \middle| \mathcal{F}_r\right]\right|}{\mathbb{E}\left[\left|T_a(N)\right|\right]} > \frac{\varepsilon}{3}\right) \leq \frac{9\,\mathbb{E}\left[\left(\left|T_{a,r,\delta}^{\leq}(N)\right| - \mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right| \middle| \mathcal{F}_r\right]\right)^2\right]}{\varepsilon^2\,\mathbb{E}\left[\left|T_a(N)\right|\right]^2}. \tag{3.4.21}$$

The tower property gives

$$\mathbb{E}\left[\left(\left|T_{a,r,\delta}^{\leq}(N)\right| - \mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right| \middle| \mathcal{F}_r\right]\right)^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right|^2 \middle| \mathcal{F}_r\right] - \mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right| \middle| \mathcal{F}_r\right]^2\right]. \tag{3.4.22}$$

Using the branching property, we write

$$\begin{aligned} \left|T_{a,r,\delta}^{\leq}(N)\right| &= \sum_{\sigma\in\{-1,1\}^r} \sum_{\tilde{\sigma}\in\{-1,1\}^{N-r}} I_{\sigma,\tilde{\sigma}}, \\ I_{\sigma,\tilde{\sigma}} &:= \prod_{j=0,\dots,M} \mathbb{1}_{z_j(\sigma)+\tilde{z}_j^{(\sigma)}(\tilde{\sigma})\geq a_j N}\, \mathbb{1}_{z_j(\sigma)+\sum_{i=1}^n \tilde{z}_{j,\tilde{\sigma}|_i}^{(\sigma)} \leq (a_j+\delta)(n+r)\ \forall_{n=0,\dots,N-r}}, \end{aligned} \tag{3.4.23}$$

where $(z_0(\sigma))_{\sigma\in\{-1,1\}^r}$ is a BRW on the binary tree of depth r with standard Gaussian increments. For each $\sigma \in \{-1,1\}^r$, $(\tilde{z}_0^{(\sigma)}(\tilde{\sigma}))_{\tilde{\sigma}\in\{-1,1\}^{N-r}}$ denotes an independent copy of a BRW on the binary tree of depth $N-r$ with standard Gaussian increments. We have $\tilde{z}_0^{(\sigma)}(\tilde{\sigma}) = \sum_{i=1}^{N-r} \tilde{z}_{0,\tilde{\sigma}|_i}^{(\sigma)}$, where all the summands for all $\sigma, \tilde{\sigma}$ are i.i.d. standard Gaussian. There are M independent copies (denoted in the same way except the index $j = 1,\dots,M$ instead of $j = 0$) of all processes mentioned in this column. We have

$$\begin{aligned} &\mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right|^2 \middle| \mathcal{F}_r\right] - \mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right| \middle| \mathcal{F}_r\right]^2 \\ &= \sum_{\sigma_1,\sigma_2\in\{-1,1\}^r} \mathbb{E}\left[\left(\textstyle\sum_{\tilde{\sigma}_1\in\{-1,1\}^{N-r}} I_{\sigma_1,\tilde{\sigma}_1}\right)\left(\textstyle\sum_{\tilde{\sigma}_2\in\{-1,1\}^{N-r}} I_{\sigma_2,\tilde{\sigma}_2}\right) \middle| \mathcal{F}_r\right] \\ &\quad - \sum_{\sigma_1,\sigma_2\in\{-1,1\}^r} \mathbb{E}\left[\left(\textstyle\sum_{\tilde{\sigma}_1\in\{-1,1\}^{N-r}} I_{\sigma_1,\tilde{\sigma}_1}\right) \middle| \mathcal{F}_r\right] \mathbb{E}\left[\left(\textstyle\sum_{\tilde{\sigma}_2\in\{-1,1\}^{N-r}} I_{\sigma_2,\tilde{\sigma}_2}\right) \middle| \mathcal{F}_r\right]. \end{aligned} \tag{3.4.24}$$

If $\sigma_1 \neq \sigma_2$, the branching of the particles $(\sigma_1, \tilde{\sigma}_1)$ and $(\sigma_2, \tilde{\sigma}_2)$ has occurred up to time r, so, conditional on $\mathcal{F}_r$, $I_{\sigma_1,\tilde{\sigma}_1}$ and $I_{\sigma_2,\tilde{\sigma}_2}$ are independent. Thus, the terms with $\sigma_1 \neq \sigma_2$ in (3.4.24) cancel out and (3.4.24) can be reduced to

$$\begin{aligned} &\mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right|^2 \middle| \mathcal{F}_r\right] - \mathbb{E}\left[\left|T_{a,r,\delta}^{\leq}(N)\right| \middle| \mathcal{F}_r\right]^2 \\ &= \sum_{\sigma\in\{-1,1\}^r} \left(\mathbb{E}\left[\left(\textstyle\sum_{\tilde{\sigma}\in\{-1,1\}^{N-r}} I_{\sigma,\tilde{\sigma}}\right)^2 \middle| \mathcal{F}_r\right] - \mathbb{E}\left[\left(\textstyle\sum_{\tilde{\sigma}\in\{-1,1\}^{N-r}} I_{\sigma,\tilde{\sigma}}\right) \middle| \mathcal{F}_r\right]^2\right) \\ &\leq \sum_{\sigma\in\{-1,1\}^r} \mathbb{E}\left[\left(\textstyle\sum_{\tilde{\sigma}\in\{-1,1\}^{N-r}} I_{\sigma,\tilde{\sigma}}\right)^2 \middle| \mathcal{F}_r\right] \\ &= \sum_{\ell=r}^{N} \sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N \\ \sigma_1\wedge\sigma_2=\ell}} \mathbb{P}\left(\sigma_1, \sigma_2 \in T_{a,r,\delta}^{\leq}(N)\right). \end{aligned} \tag{3.4.25}$$

It remains to show that there exists $c_1, c_2 > 0$ such that for the right-hand side of (3.4.25),

$$\sum_{\ell=r}^{N} \sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}} \mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) \leq c_1 \mathbb{E}\left[|T_a(N)|\right]^2 \mathrm{e}^{-c_2 r}(1 + o(1)). \tag{3.4.26}$$

Since the proof of (3.4.26) is lengthy, we postpone it to Lemma 3.4.6. Inserting (3.4.25) and (3.4.26) into (3.4.21) gives

$$(P1) \leq \tfrac{9c_1}{\varepsilon^2} \mathrm{e}^{-c_2 r}(1 + o(1)), \tag{3.4.27}$$

which, combined with the bounds on $(P2)$ and $(P3)$, completes the proof of Lemma 3.4.5 up to the proof of Lemma 3.4.6. □

Lemma 3.4.6. *Let $a = (a_0, \ldots, a_M) \in \mathbb{R}_+^{M+1}$ with $\|a\|_2^2 < 2\ln 2$ and let $r = r(N) \uparrow \infty$ as $N \uparrow \infty$ with $r = o(N)$. Then for all $\varepsilon > 0$ exist $c_1, c_2 > 0$ possibly depending on ε and $\delta > 0$ such that*

$$\sum_{\ell=r}^{N} \sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}} \mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) \leq c_1 \mathbb{E}\left[|T_a(N)|\right]^2 \mathrm{e}^{-c_2 r}(1 + o(1)), \tag{3.4.28}$$

for N large enough. Here, $T_a(N)$ is defined in (3.4.3) *and $T^{\leq}_{a,r,\delta}(N)$ is defined in* (3.4.7).

Proof. For $\ell = r, \ldots, N$ and $\sigma_1, \sigma_2 \in \{-1,1\}^N$ with $\sigma_1 \wedge \sigma_2 = \ell$, we drop the localisation condition in $T^{\leq}_{a,r,\delta}(N)$ except for the level of the branching ℓ. This gives

$$\begin{aligned}\mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) &\leq \mathbb{P}\left(\forall_{j=0,\ldots,M} \forall_{k=1,2}\colon z_j(\sigma_k) \geq a_j N \text{ and } \textstyle\sum_{i=1}^{\ell} z_{j,\sigma_k|_i} \leq (a_j+\delta)\ell\right)\\ &= \prod_{j=0}^{M} \int_{-\infty}^{(a_j+\delta)\ell} \frac{\mathrm{d}z}{\sqrt{2\pi\ell}} \mathrm{e}^{-\frac{z^2}{2\ell}} \left(\int_{a_j N - z}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(N-\ell)}} \mathrm{e}^{-\frac{y^2}{2(N-\ell)}}\right)^2,\end{aligned} \tag{3.4.29}$$

also using the independence of $(z_j)_{j=0,\ldots,M}$ in the last step. We write

$$\gamma \coloneqq \min_{j=0,\ldots,M} \tfrac{a_j - 2\delta}{a_j + \delta} \tag{3.4.30}$$

and choose $\delta > 0$ small enough such that $\gamma > 0$. In (3.4.28), we split the sum over ℓ in $\lfloor \gamma N \rfloor$ to get that

$$\sum_{\ell=r}^{N} \sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}} \mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) = (S1) + (S2), \tag{3.4.31}$$

where

$$(S1) := \sum_{\ell=r}^{\lfloor \gamma N \rfloor -1} \sum_{\substack{\sigma_1,\sigma_2 \in \{-1,1\}^N \\ \sigma_1 \wedge \sigma_2 = \ell}} \mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right),$$

$$(S2) := \sum_{\lfloor \gamma N \rfloor}^{N} \sum_{\substack{\sigma_1,\sigma_2 \in \{-1,1\}^N \\ \sigma_1 \wedge \sigma_2 = \ell}} \mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right). \tag{3.4.32}$$

We first deal with $(S1)$. For $\ell = r, \ldots, \lfloor \gamma N \rfloor - 1$, we have for all $j = 0, \ldots, M$ that $a_j N - z \geq 2\delta N > 0$ if $z \leq (a_j + \delta)\ell$. Thus, we can apply the Gaussian tail bound given in Lemma 3.2.1 to the integral over y in (3.4.29) to get that

$$\int_{a_j N - z}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(N-\ell)}} \mathrm{e}^{-\frac{y^2}{2(N-\ell)}} \leq \frac{\sqrt{N-\ell}}{\sqrt{2\pi}(a_j N - z)} \exp\left(-\frac{(a_j N - z)^2}{2(N-\ell)}\right). \tag{3.4.33}$$

Inserting this into (3.4.29) and bounding $\frac{1}{a_j N - z} \leq \frac{1}{2\delta N}$, we see that if $\sigma_1 \wedge \sigma_2 = \ell$ and $\ell = r, \ldots, \lfloor \gamma N \rfloor - 1$, then

$$\mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) \leq \prod_{j=0}^{M} \int_{-\infty}^{(a_j+\delta)\ell} \frac{\mathrm{d}z}{\sqrt{2\pi\ell}} \frac{N-\ell}{8\pi\delta^2 N^2} \exp\left(-\frac{z^2}{2\ell} - \frac{(a_j N - z)^2}{(N-\ell)}\right). \tag{3.4.34}$$

By a completion of squares,

$$-\frac{z^2}{2\ell} - \frac{(a_j N - z)^2}{(N-\ell)} = -\frac{a_j^2 N^2}{N+\ell} - \frac{N+\ell}{2\ell(N-\ell)} \left(z - \frac{2\ell N}{N+\ell} a_j\right)^2. \tag{3.4.35}$$

We insert (3.4.35) into (3.4.34) and then shift the integral by $\frac{2\ell N}{N+\ell} a_j$ to get that

$$\mathbb{P}\left(\sigma_1, \sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) \leq \prod_{j=0}^{M} \frac{N-\ell}{8\pi\delta^2 N^2} \exp\left(-\frac{a_j^2 N^2}{N+\ell}\right) \int_{-\infty}^{(a_j+\delta)\ell - \frac{2\ell N}{N+\ell} a_j} \frac{\mathrm{d}z}{\sqrt{2\pi\ell}} \exp\left(-\frac{N+\ell}{2\ell(N-\ell)} z^2\right), \tag{3.4.36}$$

if $\sigma_1 \wedge \sigma_2 = \ell \in \{r, \ldots, \lfloor \gamma N \rfloor - 1\}$. For such ℓ,

$$a_j + \delta - \frac{2N}{N+\ell} a_j = \delta - \frac{N-\ell}{N+\ell} a_j < \delta - \frac{1-\gamma}{1+\gamma} a_j \leq \delta - \frac{1 - \frac{a_j - 2\delta}{a_j+\delta}}{1 + \frac{a_j-2\delta}{a_j+\delta}} a_j = \delta - \frac{3\delta a_j}{2a_j - \delta} < -\frac{\delta}{2}, \tag{3.4.37}$$

so the upper integral limit in (3.4.36) is negative. Thus, we can apply the Gaussian tail bound from Lemma 3.2.1 to the integral in (3.4.36) to get that

$$\begin{aligned}
&\int_{-\infty}^{(a_j+\delta)\ell - \frac{2\ell N}{N+\ell} a_j} \frac{\mathrm{d}z}{\sqrt{2\pi\ell}} \exp\left(-\frac{N+\ell}{2\ell(N-\ell)} z^2\right) \\
&= \frac{\sqrt{N-\ell}}{\sqrt{N+\ell}} \int_{-\infty}^{(a_j+\delta)\ell - \frac{2\ell N}{N+\ell} a_j} \frac{\mathrm{d}z \sqrt{N+\ell}}{\sqrt{2\pi\ell(N-\ell)}} \exp\left(-\frac{N+\ell}{2\ell(N-\ell)} z^2\right) \\
&\leq \frac{N-\ell}{N+\ell} \frac{\sqrt{\ell}}{\sqrt{2\pi}} \left(-(a_j+\delta)\ell + \frac{2\ell N}{N+\ell} a_j\right)^{-1} \exp\left(-\frac{N+\ell}{2\ell(N-\ell)} \left((a_j+\delta)\ell - \frac{2\ell N}{N+\ell} a_j\right)^2\right),
\end{aligned} \tag{3.4.38}$$

where

$$\tfrac{N+\ell}{2\ell(N-\ell)}\left((a_j+\delta)\ell-\tfrac{2\ell N}{N+\ell}a_j\right)^2=\tfrac{\ell}{2(N^2-\ell^2)}((N-\ell)a_j-(N+\ell)\delta)^2, \tag{3.4.39}$$

and, by (3.4.37),

$$\left(-(a_j+\delta)\ell+\tfrac{2\ell N}{N+\ell}a_j\right)^{-1}<\tfrac{2}{\delta\ell}. \tag{3.4.40}$$

We insert (3.4.38)–(3.4.40) into (3.4.36) to obtain

$$\mathbb{P}\left(\sigma_1,\sigma_2\in T^{\le}_{a,r,\delta}(N)\right)\le\prod_{j=0}^{M}\frac{(N-\ell)^2}{2(2\pi)^{3/2}\delta^3\ell^{1/2}N^2(N+\ell)}\exp\left(-\tfrac{a_j^2N^2}{N+\ell}-\tfrac{\ell}{2(N^2-\ell^2)}((N-\ell)a_j-(N+\ell)\delta)^2\right), \tag{3.4.41}$$

for $\ell=r,\dots,\lfloor\gamma N\rfloor-1$. We bound

$$\frac{(N-\ell)^2}{2(2\pi)^{3/2}\delta^3\ell^{1/2}N^2(N+\ell)}\le\frac{c(\delta)}{N}, \tag{3.4.42}$$

(where $c(\delta)>0$ is a universal constant for all terms not depending on ℓ or N) and insert (3.4.41) into the part of the sum in (3.4.26) which is indexed by $\ell=r,\dots,\lfloor\gamma N\rfloor-1$. This gives

$$\begin{aligned}(S1)&=\sum_{\ell=r}^{\lfloor\gamma N\rfloor-1}\sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}}\mathbb{P}\left(\sigma_1,\sigma_2\in T^{\le}_{a,r,\delta}(N)\right)\\&\le\sum_{\ell=r}^{\lfloor\gamma N\rfloor-1}\sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}}\prod_{j=0}^{M}\frac{c(\delta)}{N}\exp\left(-\tfrac{a_j^2N^2}{N+\ell}-\tfrac{\ell}{2(N^2-\ell^2)}((N-\ell)a_j-(N+\ell)\delta)^2\right)\\&=c(\delta)N^{-M-1}2^{2N}\sum_{\ell=r}^{\lfloor\gamma N\rfloor-1}2^{-\ell}\prod_{j=0}^{M}\exp\left(-\tfrac{a_j^2N^2}{N+\ell}-\tfrac{\ell}{2(N^2-\ell^2)}((N-\ell)a_j-(N+\ell)\delta)^2\right),\end{aligned} \tag{3.4.43}$$

where we used in the last step that there are $2^{2N-\ell-1}$ pairs $(\sigma_1,\sigma_2)\in(\{-1,1\}^N)^2$ with $\sigma_1\wedge\sigma_2=\ell$. Note that

$$\begin{aligned}-\tfrac{a_j^2N^2}{N+\ell}-\tfrac{\ell}{2(N^2-\ell^2)}((N-\ell)a_j-(N+\ell)\delta)^2&=-a_j^2(N-\ell)-\tfrac{a_j^2\ell^2}{N+\ell}-\tfrac{\ell(N-\ell)}{2(N+\ell)}a_j^2+\delta a_j\ell-\underbrace{\tfrac{\ell(N+\ell)^2}{2(N^2-\ell^2)}\delta^2}_{\le 0}\\&\le-a_j^2N+\ell\left(a_j^2-\tfrac{a_j^2\ell}{N+\ell}-\tfrac{(N-\ell)}{2(N+\ell)}a_j^2+\delta a_j\right)\\&=-a_j^2N+\ell\left(\tfrac{a_j^2}{2}+\delta a_j\right).\end{aligned} \tag{3.4.44}$$

Inserting (3.4.44) into (3.4.43) gives

$$(S1)\le c(\delta)N^{-M-1}2^{2N}\exp\left(-\|a\|_2^2N\right)\sum_{\ell=r}^{\lfloor\gamma N\rfloor-1}2^{-\ell}\exp\left(\tfrac{\|a\|_2^2}{2}\ell+\delta\ell\textstyle\sum_{j=0}^{M}a_j\right), \tag{3.4.45}$$

where by (3.4.9),

$$N^{-M-1}2^{2N}\exp\left(-\|a\|_2^2N\right)=\mathbb{E}\left[|T_a(N)|\right]^2\prod_{j=0}^{M}\sqrt{2\pi a_j^2}\,(1+o(1)). \tag{3.4.46}$$

Since $\|a\|_2^2<2\ln 2$, we can choose δ small enough so that $\tilde{\delta}:=\ln 2-\frac{\|a\|_2^2}{2}-\delta\sum_{j=0}^{M}a_j$ is positive. Then we get for the sum in (3.4.45) that

$$\sum_{\ell=r}^{\lfloor\gamma N\rfloor-1}2^{-\ell}\exp\left(\tfrac{\|a\|_2^2}{2}\ell+\delta\ell\textstyle\sum_{j=0}^{M}a_j\right)=\sum_{\ell=r}^{\lfloor\gamma N\rfloor-1}\mathrm{e}^{-\tilde{\delta}\ell}\le\sum_{\ell=r}^{\infty}\mathrm{e}^{-\tilde{\delta}\ell}=\tfrac{\mathrm{e}^{-\tilde{\delta}r}}{1-\mathrm{e}^{-\tilde{\delta}}}. \tag{3.4.47}$$

Inserting (3.4.46) and (3.4.47) into (3.4.45) gives

$$(S1)\le c(\delta)\,\mathbb{E}\left[|T_a(N)|\right]^2\mathrm{e}^{-\tilde{\delta}r}(1+o(1)), \tag{3.4.48}$$

so we have shown (3.4.26) for the summands indexed by $\ell=r,\dots,\lfloor\gamma N\rfloor-1$.

It remains to bound $(S2)$. For $\ell=\lfloor\gamma N\rfloor,\dots,N$ and $\sigma_1,\sigma_2\in\{-1,1\}^N$ with $\sigma_1\wedge\sigma_2=\ell$, we split the integral over z in (3.4.29) at the point $(a_j-\delta)N$ to get that

$$\begin{aligned}\mathbb{P}\left(\sigma_1,\sigma_2\in T^{\le}_{a,r,\delta}(N)\right)&\le\prod_{j=0}^{M}\int_{-\infty}^{(a_j+\delta)\ell}\frac{\mathrm{d}z}{\sqrt{2\pi\ell}}\left(\mathbb{1}_{z\ge(a_j-\delta)N}+\mathbb{1}_{z<(a_j-\delta)N}\right)\mathrm{e}^{-\frac{z^2}{2\ell}}\left(\int_{a_jN-z}^{\infty}\frac{\mathrm{d}y}{\sqrt{2\pi(N-\ell)}}\mathrm{e}^{-\frac{y^2}{2(N-\ell)}}\right)^2\\&\le\prod_{j=0}^{M}\left(\int_{-\infty}^{(a_j-\delta)N}\frac{\mathrm{d}z}{\sqrt{2\pi\ell}}\mathrm{e}^{-\frac{z^2}{2\ell}}\left(\int_{a_jN-z}^{\infty}\frac{\mathrm{d}y}{\sqrt{2\pi(N-\ell)}}\mathrm{e}^{-\frac{y^2}{2(N-\ell)}}\right)^2+\int_{(a_j-\delta)N}^{(a_j+\delta)\ell}\frac{\mathrm{d}z}{\sqrt{2\pi\ell}}\mathrm{e}^{-\frac{z^2}{2\ell}}\right).\end{aligned} \tag{3.4.49}$$

We have by Lemma 3.2.1 that

$$\int_{(a_j-\delta)N}^{(a_j+\delta)\ell}\frac{\mathrm{d}z}{\sqrt{2\pi\ell}}\mathrm{e}^{-\frac{z^2}{2\ell}}\le\exp\left(-\frac{(a_j-\delta)^2N^2}{2\ell}\right)\le\exp\left(-a_j^2N+\frac{a_j^2\ell}{2}+\gamma^{-1}(1-\tfrac{\gamma}{2})(2a_j-\delta)\delta\ell\right), \tag{3.4.50}$$

using in the last step that for $\delta<2a_j$,

$$\begin{aligned}-\frac{(a_j-\delta)^2N^2}{2\ell}&=\frac{-a_j^2N^2+2a_j\delta N^2-\delta^2N^2+2a_j^2N\ell-a_j^2\ell^2}{2\ell}-a_j^2N+\frac{a_j^2\ell}{2}\\&=(2a_j-\delta)\delta\frac{2N-\ell}{2}-\frac{(a_j-\delta)^2(N-\ell)^2}{2\ell}-a_j^2N+\frac{a_j^2\ell}{2}\\&\le(1-\tfrac{\gamma}{2})(2a_j-\delta)\delta N-a_j^2N+\frac{a_j^2\ell}{2}\\&\le\gamma^{-1}(1-\tfrac{\gamma}{2})(2a_j-\delta)\delta\ell-a_j^2N+\frac{a_j^2\ell}{2}.\end{aligned} \tag{3.4.51}$$

For the integral over $(-\infty,(a_j-\delta)N]$ in (3.4.49), we proceed as in (3.4.33)–(3.4.36) to obtain

$$\begin{aligned}\int_{-\infty}^{(a_j-\delta)N}\frac{\mathrm{d}z}{\sqrt{2\pi\ell}}\mathrm{e}^{-\frac{z^2}{2\ell}}\left(\int_{a_jN-z}^{\infty}\frac{\mathrm{d}y}{\sqrt{2\pi(N-\ell)}}\mathrm{e}^{-\frac{y^2}{2(N-\ell)}}\right)^2 &\le c(\delta)\,\tfrac{N-\ell}{N^2}\exp\left(-\tfrac{a_j^2N^2}{N+\ell}\right)\int_{-\infty}^{(a_j+\delta)\ell-\frac{2\ell N}{N+\ell}a_j}\frac{\mathrm{d}z}{\sqrt{2\pi\ell}}\exp\left(-\tfrac{N+\ell}{2\ell(N-\ell)}z^2\right)\\ &\le c(\delta)\exp\left(-\tfrac{a_j^2N^2}{N+\ell}\right)\\ &\le c(\delta)\exp\left(-a_j^2N+\tfrac{a_j^2\ell}{2}+\tfrac{1-\gamma}{2(1+\gamma)}a_j^2\ell\right),\end{aligned}\tag{3.4.52}$$

since

$$-\tfrac{a_j^2N^2}{N+\ell}=-a_j^2(N-\ell)-\tfrac{a_j^2\ell^2}{N+\ell}=-a_j^2N+\tfrac{a_j^2\ell}{2}+\tfrac{a_j^2(N-\ell)\ell}{2(N+\ell)}\le-a_j^2N+\tfrac{a_j^2\ell}{2}+\tfrac{1-\gamma}{2(1+\gamma)}a_j^2\ell.\tag{3.4.53}$$

We have $\gamma\uparrow 1$ as $\delta\downarrow 0$, so we can choose δ small enough to get for the following terms, which appear in the last lines of (3.4.50) and (3.4.52), that

$$\max_{j=0,\dots,M}\left(\gamma^{-1}(1-\tfrac{\gamma}{2})(2a_j-\delta)\delta\ \wedge\ \tfrac{1-\gamma}{2(1+\gamma)}a_j^2\right)\ell<\tfrac{1}{4(M+1)}\left(2\ln 2-\|a\|_2^2\right)\ell.\tag{3.4.54}$$

Inserting (3.4.50), (3.4.52) and then (3.4.54) into (3.4.49) gives

$$\begin{aligned}\mathbb{P}\left(\sigma_1,\sigma_2\in T_{a,r,\delta}^{\le}(N)\right)&\le c(\delta)\exp\left(\left(2\ln 2-\|a\|_2^2\right)\tfrac{\ell}{4}\right)\prod_{j=0}^{M}\exp\left(-a_j^2N+\tfrac{a_j^2\ell}{2}\right)\\ &=c(\delta)\exp\left(-\|a\|_2^2N\right)\exp\left(\left(2\ln 2+\|a\|_2^2\right)\tfrac{\ell}{4}\right),\end{aligned}\tag{3.4.55}$$

so

$$\begin{aligned}(S2)&=\sum_{\ell=\lfloor\gamma N\rfloor}^{N}\sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}}\mathbb{P}\left(\sigma_1,\sigma_2\in T_{a,r,\delta}^{\le}(N)\right)\\ &\le c(\delta)\exp\left(-\|a\|_2^2N\right)\sum_{\ell=\lfloor\gamma N\rfloor}^{N}\sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}}\exp\left(\left(2\ln 2+\|a\|_2^2\right)\tfrac{\ell}{4}\right)\\ &=c(\delta)\,2^{2N}\exp\left(-\|a\|_2^2N\right)\sum_{\ell=\lfloor\gamma N\rfloor}^{N}\exp\left(\left(2\ln 2-\|a\|_2^2\right)\tfrac{\ell}{4}\right),\end{aligned}\tag{3.4.56}$$

using in the last step that the sum over σ_1,σ_2 has $2^{2N-\ell-1}$ summands. Since $\|a\|_2^2<2\ln 2$,

$$\begin{aligned}\sum_{\ell=\lfloor\gamma N\rfloor}^{N}\exp\left(\left(2\ln 2-\|a\|_2^2\right)\tfrac{\ell}{4}\right)&\le\sum_{\ell=\lfloor\gamma N\rfloor}^{\infty}\exp\left(\left(2\ln 2-\|a\|_2^2\right)\tfrac{\ell}{4}\right)\\ &=\tfrac{1}{1-\exp\left(\frac14\left(2\ln 2-\|a\|_2^2\right)\right)}\exp\left(\left(2\ln 2-\|a\|_2^2\right)\tfrac{\lfloor\gamma N\rfloor}{4}\right).\end{aligned}\tag{3.4.57}$$

Inserting (3.4.57) into (3.4.56) and applying (3.4.9) gives

$$(S2)\le c(\delta)\,\mathcal{O}(N^M)\,\mathbb{E}\left[|T_a(N)|\right]^2\exp\left(\left(2\ln 2-\|a\|_2^2\right)\tfrac{\lfloor\gamma N\rfloor}{4}\right).\tag{3.4.58}$$

Combining (3.4.48) and (3.4.58) and using that $O(N^M)\exp\left(\left(2\ln 2 - \|a\|_2^2\right)\frac{\lfloor \gamma N\rfloor}{4}\right) = o(1)$ if $\|a\|_2^2 < 2\ln 2$, we get

$$\sum_{\ell=r}^{N} \sum_{\substack{\sigma_1,\sigma_2\in\{-1,1\}^N\\ \sigma_1\wedge\sigma_2=\ell}} \mathbb{P}\left(\sigma_1,\sigma_2 \in T^{\leq}_{a,r,\delta}(N)\right) \leq c(\delta)\mathbb{E}\left[|T_a(N)|\right]^2 \mathrm{e}^{-\tilde{\delta}r}(1+o(1)), \tag{3.4.59}$$

which completes the proof of this lemma. □

The following lemma is also used in the proof of Theorem 3.4.2.

Lemma 3.4.7. *Let $a = (a_0,\dots,a_M) \in \mathbb{R}_+^{M+1}$ with $\|a\|_2^2 < 2\ln 2$ and let $T_a(N)$ be as in* (3.4.3). *Let $r = r(N) \uparrow \infty$ as $N \uparrow \infty$ such that there exists $c \in (0,1)$ with $r = o(N^c)$. Then,*

$$\frac{\mathbb{E}\left[|T_a(N)|\,|\,\mathcal{F}_r\right]}{\mathbb{E}\left[|T_a(N)|\right]} \to Y_a(\infty), \tag{3.4.60}$$

almost surely as $N \uparrow \infty$.

Proof of Theorem 3.4.2. In Lemma 3.4.5, we can choose $r = r(N)$ with $r(N) \uparrow \infty$ and $r(N) = o(N)$ as $N \uparrow \infty$ so that the right-hand side of (3.4.6) is summable: A possible choice is $r(N) = \ln(N^2)$. This implies that for such r,

$$\frac{\left||T_a(N)| - \mathbb{E}\left[|T_a(N)|\,|\,\mathcal{F}_r\right]\right|}{\mathbb{E}\left[|T_a(N)|\right]} \to 0, \tag{3.4.61}$$

almost surely as $N \uparrow \infty$. Thus, applying Lemma 3.4.7 completes the proof of Theorem 3.4.2. □

Proof of Lemma 3.4.7. We use notation as in (3.4.1)–(3.4.2) and denote by $(\sigma,\tilde{\sigma})$ the concatenation of σ and $\tilde{\sigma}$ to get that

$$\begin{aligned}
\mathbb{E}\left[|T_a(N)|\,|\,\mathcal{F}_r\right] &= \mathbb{E}\left[\textstyle\sum_{\sigma\in\{-1,1\}^N} \mathbb{1}_{z_j(\sigma)\geq a_jN\,\forall j=0,\dots,M}\,\middle|\,\mathcal{F}_r\right]\\
&= \mathbb{E}\left[\textstyle\sum_{\sigma\in\{-1,1\}^r}\sum_{\tilde{\sigma}\in\{-1,1\}^{N-r}} \mathbb{1}_{\sum_{i=1}^r z_{j,\sigma|_i}+\sum_{i=r+1}^N z_{j,(\sigma,\tilde{\sigma})|_i}\geq a_jN\,\forall j=0,\dots,M}\,\middle|\,\mathcal{F}_r\right]\\
&= \sum_{\sigma\in\{-1,1\}^r}\sum_{\tilde{\sigma}\in\{-1,1\}^{N-r}} \mathbb{P}\left(\textstyle\sum_{i=r+1}^N z_{j,(\sigma,\tilde{\sigma})|_i} \geq a_jN - \sum_{i=1}^r z_{j,\sigma|_i}\ \forall_{j=0,\dots,M}\,\middle|\,\mathcal{F}_r\right).
\end{aligned} \tag{3.4.62}$$

By our choice of $r = r(N)$, there exists $c \in (\frac{1}{2},1)$ such that $r(N) = o(N^{2c-1})$. We rewrite (3.4.62) as

$$\mathbb{E}\left[|T_a(N)|\,|\,\mathcal{F}_r\right] = (S1) + (S2), \tag{3.4.63}$$

where

$$
\begin{aligned}
(S1) &= \sum_{\substack{\sigma\in\{-1,1\}^r\\ \sigma\in L_{a,r,c}}} \sum_{\tilde\sigma\in\{-1,1\}^{N-r}} \mathbb{P}\left(\textstyle\sum_{i=r+1}^N z_{j,(\sigma,\tilde\sigma)|_i} \geq a_j N - \sum_{i=1}^r z_{j,\sigma|_i}\ \forall_{j=0,\dots,M} \,\middle|\, \mathcal{F}_r\right),\\
(S2) &= \sum_{\substack{\sigma\in\{-1,1\}^r\\ \sigma\notin L_{a,r,c}}} \sum_{\tilde\sigma\in\{-1,1\}^{N-r}} \mathbb{P}\left(\textstyle\sum_{i=r+1}^N z_{j,(\sigma,\tilde\sigma)|_i} \geq a_j N - \sum_{i=1}^r z_{j,\sigma|_i}\ \forall_{j=0,\dots,M} \,\middle|\, \mathcal{F}_r\right),\\
L_{a,r,c} &= \left\{\sigma\in\{-1,1\}^r :\ \textstyle\sum_{i=1}^r z_{j,\sigma|_i} < a_j N^c\ \forall_{j=0,\dots,M}\right\}.
\end{aligned}
\tag{3.4.64}
$$

We bound the conditional probability contained in $(S2)$ by 1 to get

$$
(S2) \leq 2^{N-r}\left|\{-1,1\}^r \setminus L_{a,r,c}\right|. \tag{3.4.65}
$$

For any $\varepsilon > 0$, by Markov's inequality,

$$
\mathbb{P}\left(2^{N-r}\left|\{-1,1\}^r \setminus L_{a,r,c}\right| > \varepsilon\right) \leq \frac{2^{N-r}}{\varepsilon}\mathbb{E}\left[\left|\{-1,1\}^r \setminus L_{a,r,c}\right|\right]. \tag{3.4.66}
$$

We write $a_* = \min_{j=0,\dots,M} a_j$ and denote by $(z_{*,\sigma|_i})_{\sigma\in\{-1,1\}^N, i=1,\dots,N}$ a family of i.i.d. standard Gaussians. We have

$$
\begin{aligned}
\mathbb{E}\left[\left|\{-1,1\}^r \setminus L_{a,r,c}\right|\right] &= 2^r\,\mathbb{P}\left(\exists_{j=0,\dots,M}:\ \textstyle\sum_{i=1}^r z_{j,\sigma|_i} > a_j N^c\right)\\
&\leq 2^r (M+1)\,\mathbb{P}\left(\textstyle\sum_{i=1}^r z_{*,\sigma|_i} > a_* N^c\right)\\
&\leq 2^r (M+1)\,\mathrm{e}^{-\frac{a_*^2 N^{2c}}{2r}},
\end{aligned}
\tag{3.4.67}
$$

applying the Gaussian tail bound given in Lemma 3.2.1 in the last step. By (3.4.66), (3.4.67) and our choice of r and c,

$$
\sum_{N=2}^{\infty} \mathbb{P}\left(2^{N-r}\left|\{-1,1\}^r \setminus L_{a,r,c}\right| > \varepsilon\right) \leq \frac{M+1}{\varepsilon}\sum_{N=2}^{\infty} 2^N \mathrm{e}^{-\frac{a_*^2 N^{2c}}{2r}} < \infty. \tag{3.4.68}
$$

Thus, the random variable $2^{N-r}\left|\{-1,1\}^r \setminus L_{a,r,c}\right|$ converges to 0 almost surely as $N\uparrow\infty$. Thus by (3.4.65),

$$
(S2) \to 0, \quad \text{almost surely as } N\uparrow\infty. \tag{3.4.69}
$$

If $\sigma \in L_{a,r,c}$, then we can apply the Gaussian tail asymptotics from Lemma 3.2.1 to the conditional probability appearing in $(S1)$ in (3.4.64). This and the independence of $(z_j)_{j=0,\dots,M}$

imply for any $\sigma \in L_{a,r,c}$, $\tilde{\sigma} \in \{-1,1\}^{N-r}$ that

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=r+1}^{N} z_{j,(\sigma,\tilde{\sigma})|_i} \geq a_j N - \sum_{i=1}^{r} z_{j,\sigma|_i} \ \forall_{j=0,\dots,M} \,\middle|\, \mathcal{F}_r\right) &= \prod_{j=0}^{M} \mathbb{P}\left(\sum_{i=r+1}^{N} z_{j,(\sigma,\tilde{\sigma})|_i} \geq a_j N - \sum_{i=1}^{r} z_{j,\sigma|_i} \,\middle|\, \mathcal{F}_r\right) \\
&= \prod_{j=0}^{M} \frac{\sqrt{N-r}}{\sqrt{2\pi}\left(a_j N - \sum_{i=1}^{r} z_{j,\sigma|_i}\right)} \exp\left(-\frac{\left(a_j N - \sum_{i=1}^{r} z_{j,\sigma|_i}\right)^2}{2(N-r)}\right)(1+o(1)) \\
&= \prod_{j=0}^{M} \frac{1}{\sqrt{2\pi a_j^2 N}} \exp\left(-\frac{a_j^2}{2} N + a_j r \sum_{i=1}^{r} z_{j,\sigma|_i}\right)(1+o(1)).
\end{aligned} \tag{3.4.70}$$

Inserting (3.4.70) into the first line of (3.4.64) gives

$$\begin{aligned}
(S1) &= \sum_{\sigma \in L_{a,r,c}} \sum_{\tilde{\sigma} \in \{-1,1\}^{N-r}} \mathbb{P}\left(\sum_{i=r+1}^{N} z_{j,(\sigma,\tilde{\sigma})|_i} \geq a_j N - \sum_{i=1}^{r} z_{j,\sigma|_i} \ \forall_{j=0,\dots,M} \,\middle|\, \mathcal{F}_r\right) \\
&= 2^{N-r} \sum_{\sigma \in L_{a,r,c}} \prod_{j=0}^{M} \frac{1}{\sqrt{2\pi a_j^2 N}} \exp\left(-\frac{a_j^2}{2} N + a_j r \sum_{i=1}^{r} z_{j,\sigma|_i}\right)(1+o(1)) \\
&= \mathbb{E}\left[|T_a(N)|\right] \sum_{\sigma \in L_{a,r,c}} \exp\left(-r\left(\ln 2 + \frac{\|a\|_2^2}{2}\right) + \sum_{j=0}^{M} a_j \sum_{i=1}^{r} z_{j,\sigma|_i}\right)(1+o(1)),
\end{aligned} \tag{3.4.71}$$

where we used (3.4.9) in the last step. With the notation

$$X_{a,c}(r) := \sum_{\substack{\sigma \in \{-1,1\}^r \\ \sigma \notin L_{a,r,c}}} \exp\left(-r\left(\ln 2 + \frac{\|a\|_2^2}{2}\right) + \sum_{j=0}^{M} a_j \sum_{i=1}^{r} z_{j,\sigma|_i}\right), \tag{3.4.72}$$

we rewrite (3.4.71) to

$$(S1) = \mathbb{E}\left[|T_a(N)|\right] \left(Y_a(r) - X_{a,c}(r)\right)(1+o(1)). \tag{3.4.73}$$

It remains to show that $X_{a,c}(r) \to 0$ almost surely as $N \uparrow \infty$. We postpone this proof to the following Lemma 3.4.8. Since by (3.4.73), Lemma 3.4.8 and (3.4.69),

$$\frac{(S1)}{\mathbb{E}[|T_a(N)|]} \to Y_a(\infty) \quad \text{and} \quad (S2) \to 0, \tag{3.4.74}$$

almost surely as $N \uparrow \infty$, we have shown (3.4.60), which completes the proof of Lemma 3.4.7 up to the proof of Lemma 3.4.8. □

Lemma 3.4.8. *In the setting of Lemma 3.4.7 with $X_{a,c}(r)$ as in* (3.4.72)*, $X_{a,c}(r) \to 0$ almost surely as $N \uparrow \infty$.*

Proof. For any $\varepsilon > 0$, by Markov's inequality and (3.4.67),

$$\begin{aligned}\mathbb{P}\left(X_{a,c}(r) > \varepsilon\right) &\leq \varepsilon^{-1}\mathbb{E}\left[X_{a,c}(r)\right] \\ &= \varepsilon^{-1}\exp\left(-r\frac{\|a\|_2^2}{2}\right)2^{-r}\sum_{\sigma\in\{-1,1\}^r}\mathbb{E}\left[\mathbb{1}_{\sigma\notin L_{a,r,c}}\exp\left(\textstyle\sum_{j=0}^M a_j\sum_{i=1}^r z_{j,\sigma|_i}\right)\right]. \end{aligned} \tag{3.4.75}$$

For each $\sigma \in \{-1,1\}^r$, recalling the notation $a_* = \min_{j=0,\dots,M} a_j$, we get for the expectation in the last line of (3.4.75) that

$$\mathbb{E}\left[\mathbb{1}_{\sigma\notin L_{a,r,c}}\exp\left(\textstyle\sum_{j=0}^M a_j\sum_{i=1}^r z_{j,\sigma|_i}\right)\right] \leq (M+1)\,\mathbb{E}\left[\mathrm{e}^{a_*\sum_{i=1}^r z_{M,\sigma|_i}}\mathbb{1}_{\sum_{i=1}^r z_{M,\sigma|_i}>a_*N^c}\right]\prod_{j=0}^{M-1}\mathbb{E}\left[\mathrm{e}^{a_j\sum_{i=1}^r z_{j,\sigma|_i}}\right]. \tag{3.4.76}$$

By the Gaussian tail bound from Lemma 3.2.1,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{a_*\sum_{i=1}^r z_{M,\sigma|_i}}\mathbb{1}_{\sum_{i=1}^r z_{M,\sigma|_i}>a_*N^c}\right] &= \int_{a_*N^c}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi r}}\mathrm{e}^{-\frac{z^2}{2r}}\mathrm{e}^{a_*z} \\ &= \mathrm{e}^{\frac{1}{2}a_*^2 r}\int_{a_*N^c}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi r}}\mathrm{e}^{-\frac{(z-a_*r)^2}{2r}} \\ &\leq \mathrm{e}^{\frac{1}{2}a_*^2 r}\mathrm{e}^{-\frac{(a_*N^c-a_*r)^2}{2r}}.\end{aligned} \tag{3.4.77}$$

Also, (3.2.2) implies

$$\mathbb{E}\left[\mathrm{e}^{a_j\sum_{i=1}^r z_{j,\sigma|_i}}\right] = \exp\left(r\frac{\|a\|_2^2}{2}\right). \tag{3.4.78}$$

We insert (3.4.77) and (3.4.78) into (3.4.76) to get that

$$\mathbb{E}\left[\mathrm{e}^{a_*\sum_{i=1}^r z_{M,\sigma|_i}}\mathbb{1}_{\sum_{i=1}^r z_{M,\sigma|_i}>a_*N^c}\right] \leq (M+1)\exp\left(-\frac{a_*^2(N^c-r)^2}{2r}\right)\exp\left(r\frac{\|a\|_2^2}{2}\right), \tag{3.4.79}$$

Inserting (3.4.79) into (3.4.75) gives

$$\mathbb{P}\left(X_{a,c}(r) > \varepsilon\right) \leq \varepsilon^{-1}(M+1)\,\mathrm{e}^{-\frac{(a_*N^c-a_*r)^2}{2r}}, \tag{3.4.80}$$

so by our choice of $r = r(N)$, $\mathbb{P}\left(X_{a,c}(r) > \varepsilon\right)$ is summable over N, which implies that $X_{a,c}(r) \to 0$ almost surely as $N \uparrow \infty$. □

Proof of Proposition 3.4.1. Cleary, $(Y_a(N))_{N\in\mathbb{N}}$ is nonnegative. To prove that $(Y_a(N))_{N\in\mathbb{N}}$ is a martingale, note that for $N \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\left[Y_a(N+1)\,|\,\mathcal{F}_N\right] &= 2^{-(N+1)}\mathrm{e}^{-\frac{1}{2}\|a\|_2^2(N+1)} \sum_{\sigma\in\{-1,1\}^N}\sum_{\tilde{\sigma}\in\{-1,1\}} \mathbb{E}\left[\exp\left(\sum_{j=0}^M a_j\left(z_{j,(\sigma\tilde{\sigma})}+\sum_{i=1}^N z_{j,\sigma|_i}\right)\right)\middle|\mathcal{F}_N\right] \\ &= 2^{-(N+1)}\mathrm{e}^{-\frac{1}{2}\|a\|_2^2(N+1)} \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sum_{j=0}^M a_j \sum_{i=1}^N z_{j,\sigma|_i}\right) \\ &\quad\times \sum_{\tilde{\sigma}\in\{-1,1\}} \mathbb{E}\left[\exp\left(\sum_{j=0}^M a_j z_{j,(\sigma\tilde{\sigma})}\right)\middle|\mathcal{F}_N\right].\end{aligned} \tag{3.4.81}$$

Since by independence and by (3.2.2),

$$\mathbb{E}\left[\exp\left(\sum_{j=0}^M a_j z_{j,(\sigma\tilde{\sigma})}\right)\middle|\mathcal{F}_N\right] = \mathbb{E}\left[\exp\left(\sum_{j=0}^M a_j z_{j,(\sigma\tilde{\sigma})}\right)\right] = \mathrm{e}^{-\frac{1}{2}\|a\|_2^2}, \tag{3.4.82}$$

we get

$$\mathbb{E}\left[Y_a(N+1)\,|\,\mathcal{F}_N\right] = 2^{-N}\mathrm{e}^{-\frac{1}{2}\|a\|_2^2 N} \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sum_{j=0}^M a_j \sum_{i=1}^N z_{j,\sigma|_i}\right) = Y_a(N), \tag{3.4.83}$$

so $(Y_a(N))_{N\in\mathbb{N}}$ is an $\mathcal{F}_N$-adapted martingale.

For each $\sigma \in \{-1,1\}^N$, by independence and by (3.2.2),

$$\mathbb{E}\left[\exp\left(\sum_{j=0}^M a_j z_j(\sigma)\right)\right] = \prod_{j=0}^M \mathbb{E}\left[\mathrm{e}^{a_j z_j(\sigma)}\right] = \prod_{j=0}^M \mathrm{e}^{\frac{1}{2}a_j^2 N} = \mathrm{e}^{\frac{1}{2}\|a\|_2^2 N}, \tag{3.4.84}$$

so $\mathbb{E}\left[Y_a(N)\right] = 1$ for all $N \in \mathbb{N}$.

We now prove the uniform integrability of $(Y_a(N))_{N\in\mathbb{N}}$, following the same structure as the proof of [27, Proposition 4.3]. Let $\varepsilon > 0$. We aim to show that there exists $\delta > 0$ such that for all $N \in \mathbb{N}$,

$$\mathbb{E}\left[Y_a(N)\mathbb{1}_{Y_a(N)>\delta}\right] < \varepsilon. \tag{3.4.85}$$

For $A, r > 0$ and $\eta \in \left(\frac{1}{2}, 1\right)$, we write

$$\tilde{Y}_{a,A,r,\eta}(N) := 2^{-N}\mathrm{e}^{-\frac{1}{2}\|a\|_2^2 N} \sum_{\sigma\in\{-1,1\}^N} \mathbb{1}_{\sigma\in L_{A,j}\cap L_{r,\eta,j}\ \forall j=0,\dots,M} \exp\left(\sum_{j=0}^M a_j z_j(\sigma)\right), \tag{3.4.86}$$

with $(z_j(\sigma))_{\sigma\in\{-1,1\}^N;\, j=0,\dots,M}$ as in (3.4.1) and

$$\begin{aligned} L_{A,j} &:= \left\{\sigma\in\{-1,1\}^N : z_j(\sigma) \in \left[a_j N - A\sqrt{N}, a_j N + A\sqrt{N}\right]\right\}, \\ L_{r,\eta,j} &:= \left\{\sigma\in\{-1,1\}^N : \left|\sum_{i=1}^n z_{j,\sigma|_i} - \tfrac{n}{N} z_j(\sigma)\right| \le ((n\wedge(N-n))\vee r)^\eta\ \ \forall_{n=1,\dots,N}\right\}. \end{aligned} \tag{3.4.87}$$

Then, for any $\delta > 0$,

$$\begin{aligned}\mathbb{E}\left[Y_a(N)\mathbb{1}_{Y_a(N)>\delta}\right] &= \mathbb{E}\left[\tilde{Y}_{a,A,r,\eta}(N)\mathbb{1}_{\tilde{Y}_{a,A,r,\eta}(N)>\frac{\delta}{2}}\right] + \mathbb{E}\left[\tilde{Y}_{a,A,r,\eta}(N)\left(\mathbb{1}_{Y_a(N)>\delta} - \mathbb{1}_{\tilde{Y}_{a,A,r,\eta}(N)>\frac{\delta}{2}}\right)\right] \\ &\quad + \mathbb{E}\left[\left(Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right)\mathbb{1}_{Y_a(N)>\delta}\right] \\ &=: (E1) + (E2) + (E3). \end{aligned} \tag{3.4.88}$$

We have

$$(E3) = \mathbb{E}\left[\left(Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right)\mathbb{1}_{Y_a(N)>\delta}\right] \leq \mathbb{E}\left[Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right] \tag{3.4.89}$$

and

$$\begin{aligned}(E2) &= \mathbb{E}\left[\tilde{Y}_{a,A,r,\eta}(N)\left(\mathbb{1}_{Y_a(N)>\delta} - \mathbb{1}_{\tilde{Y}_{a,A,r,\eta}(N)>\frac{\delta}{2}}\right)\right] \\ &\leq \mathbb{E}\left[\tilde{Y}_{a,A,r,\eta}(N)\mathbb{1}_{Y_a(N)-\tilde{Y}_{a,A,r,\eta}(N)>\delta/2}\mathbb{1}_{\tilde{Y}_{a,A,r,\eta}(N)\leq\delta/2}\right] \\ &\leq \tfrac{\delta}{2}\,\mathbb{P}\left(Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N) > \delta/2\right) \\ &\leq \mathbb{E}\left[Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right], \end{aligned} \tag{3.4.90}$$

using Markov's inequality in the last step. Furthermore,

$$(E1) = \mathbb{E}\left[\tilde{Y}_{a,A,r,\eta}(N)\mathbb{1}_{\tilde{Y}_{a,A,r,\eta}(N)>\frac{\delta}{2}}\right] \leq \tfrac{2}{\delta}\,\mathbb{E}\left[(\tilde{Y}_{a,A,r,\eta}(N))^2\mathbb{1}_{\tilde{Y}_{a,A,r,\eta}(N)>\frac{\delta}{2}}\right] \leq \tfrac{2}{\delta}\,\mathbb{E}\left[(\tilde{Y}_{a,A,r,\eta}(N))^2\right]. \tag{3.4.91}$$

Inserting (3.4.89)–(3.4.91) into (3.4.88) gives

$$\mathbb{E}\left[Y_a(N)\mathbb{1}_{Y_a(N)>\delta}\right] \leq 2\,\mathbb{E}\left[Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right] + \tfrac{2}{\delta}\,\mathbb{E}\left[(\tilde{Y}_{a,A,r,\eta}(N))^2\right]. \tag{3.4.92}$$

First, we show that for $A, r > 0$ large enough, for all $\eta > \frac{1}{2}$ and for all $N \in \mathbb{N}$,

$$\mathbb{E}\left[Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right] < \tfrac{\varepsilon}{4}. \tag{3.4.93}$$

We have

$$\mathbb{E}\left[Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right] = 1 - 2^{-N}\mathrm{e}^{-\frac{1}{2}\|a\|_2^2 N}\sum_{\sigma\in\{-1,1\}^N}\prod_{j=0}^{M}\mathbb{E}\left[\mathbb{1}_{\sigma\in L_{A,j}\cap L_{r,\eta,j}}\mathrm{e}^{a_j z_j(\sigma)}\right]. \tag{3.4.94}$$

For each $\sigma \in \{-1,1\}^N$ and each $j = 0, \ldots, M$, we get for the process in the definition of $L_{r,\eta,j}$ in (3.4.87) that

$$\left(\textstyle\sum_{i=1}^n z_{j,\sigma|_i} - \frac{n}{N}z_j(\sigma)\right)_{n=1,\ldots N} \overset{\mathrm{d}}{=} \left(\mathfrak{z}_{0,0}^N(n)\right)_{n=1,\ldots,N}, \tag{3.4.95}$$

where $\mathfrak{z}_{0,0}^N$ is a Brownian bridge (going from 0 to 0) with time horizon N. In particular, $z_j(\sigma)$ is independent of $\left(\mathfrak{z}_{0,0}^N(n)\right)_{n=1,\ldots,N}$, so we get for the expectation on the right-hand side of (3.4.94) that

$$\mathbb{E}\left[\mathbb{1}_{\sigma\in L_{A,j}\cap L_{r,\eta,j}}\mathrm{e}^{a_j z_j(\sigma)}\right] = \mathbb{P}\left(\left|\mathfrak{z}_{0,0}^N(n)\right| \le ((n\wedge(N-n))\vee r)^\eta \ \ \forall_{n=1,\dots,N}\right)\int_{a_jN-A\sqrt{N}}^{a_jN+A\sqrt{N}} \frac{\mathrm{d}z}{\sqrt{2\pi N}}\mathrm{e}^{-\frac{z^2}{2N}}\mathrm{e}^{a_j z}. \tag{3.4.96}$$

The previous probability satisfies

$$\mathbb{P}\left(\left|\mathfrak{z}_{0,0}^N(n)\right| \le ((n\wedge(N-n))\vee r)^\eta \ \ \forall_{n=1,\dots,N}\right) \ge \mathbb{P}\left(\left|\mathfrak{z}_{0,0}^N(s)\right| \le ((s\wedge(N-s))\vee r)^\eta \ \ \forall_{s\in[0,N]}\right). \tag{3.4.97}$$

By [27, Lemma 2.3], for any $\tilde{\varepsilon} > 0$ exists $r_0 > 0$ such that for $r > r_0$,

$$\mathbb{P}\left(\left|\mathfrak{z}_{0,0}^N(s)\right| \le ((s\wedge(N-s))\vee r)^\eta \ \ \forall_{s\in[0,N]}\right) > 1-\tilde{\varepsilon}. \tag{3.4.98}$$

For the integral in (3.4.96), we see that for A large enough,

$$\int_{a_jN-A\sqrt{N}}^{a_jN+A\sqrt{N}} \frac{\mathrm{d}z}{\sqrt{2\pi N}}\mathrm{e}^{-\frac{z^2}{2N}}\mathrm{e}^{a_j z} = \mathrm{e}^{\frac{1}{2}a_j^2N}\int_{a_jN-A\sqrt{N}}^{a_jN+A\sqrt{N}} \frac{\mathrm{d}z}{\sqrt{2\pi N}}\mathrm{e}^{-\frac{(z-a_j)^2}{2N}} = \mathrm{e}^{\frac{1}{2}a_j^2N}\int_{-A}^{A}\frac{\mathrm{d}z}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^2}{2}} \ge (1-\tilde{\varepsilon})\,\mathrm{e}^{\frac{1}{2}a_j^2N}. \tag{3.4.99}$$

Inserting (3.4.96)–(3.4.99) into (3.4.94) shows that for any $\tilde{\varepsilon} > 0$ exists $r, A > 0$ such that

$$\mathbb{E}\left[Y_a(N) - \tilde{Y}_{a,A,r,\eta}(N)\right] \le 1 - 2^{-N}\mathrm{e}^{-\frac{1}{2}\|a\|_2^2N}\sum_{\sigma\in\{-1,1\}^N}\prod_{j=0}^{M}(1-\tilde{\varepsilon})^2\mathrm{e}^{\frac{1}{2}a_j^2N} = 1-(1-\tilde{\varepsilon})^{2(M+1)}, \tag{3.4.100}$$

so we can choose $\tilde{\varepsilon}$ small enough such that (3.4.93) is satisfied.

We aim to prove that $\mathbb{E}\left[(\tilde{Y}_{a,A,r,\eta}(N))^2\right]$ is uniformly bounded over $N \in \mathbb{N}$. If this is proven, we can choose $\delta > 0$ large enough in (3.4.92) to get that for all $N \in \mathbb{N}$,

$$\frac{2}{\delta}\,\mathbb{E}\left[(\tilde{Y}_{a,A,r,\eta}(N))^2\right] < \frac{\varepsilon}{2}. \tag{3.4.101}$$

Inserting this and (3.4.93) into (3.4.92) then completes the proof.

By the independence of $z_0, z_1, \dots, z_N$, summing over all possible overlap values, we get

$$\mathbb{E}\left[\left(\tilde{Y}_{a,A,r,\eta}(N)\right)^2\right] = \sum_{i=0}^{N} 2^{-2N}\mathrm{e}^{-\|a\|_2^2N}\sum_{\substack{\sigma,\tilde{\sigma}\in\{-1,1\}^N\\ \sigma\wedge\tilde{\sigma}=i}}\prod_{j=0}^{M}\mathbb{E}\left[\mathbb{1}_{\sigma,\tilde{\sigma}\in L_{A,j}\cap L_{r,\eta,j}}\mathrm{e}^{a_j(z_j(\sigma)+z_j(\tilde{\sigma}))}\right] =: \sum_{i=0}^{N} T_i(N). \tag{3.4.102}$$

If $\sigma \wedge \tilde{\sigma} = N$, then $\sigma = \tilde{\sigma}$, so

$$\begin{aligned} T_N(N) &= 2^{-2N} \mathrm{e}^{-\|a\|_2^2 N} \sum_{\sigma \in \{-1,1\}^N} \prod_{j=0}^{M} \mathbb{E}\left[\mathbb{1}_{\sigma \in L_{A,j} \cap L_{r,\eta,j}} \mathrm{e}^{2a_j z_j(\sigma)} \right] \\ &\le 2^{-2N} \mathrm{e}^{-\|a\|_2^2 N} \sum_{\sigma \in \{-1,1\}^N} \prod_{j=0}^{M} \mathbb{E}\left[\mathbb{1}_{\sigma \in L_{A,j}} \mathrm{e}^{2a_j z_j(\sigma)} \right], \end{aligned} \tag{3.4.103}$$

dropping the condition $L_{r,\eta,j}$ in the last step. We have

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{\sigma \in L_{A,j}} \mathrm{e}^{2a_j z_j(\sigma)} \right] &= \int_{a_j N - A\sqrt{N}}^{a_j N + \sqrt{N}} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{z^2}{2N} + 2a_j z} \\ &= \mathrm{e}^{\frac{1}{2} a_j^2 N} \int_{a_j N - A\sqrt{N}}^{a_j N + \sqrt{N}} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{(z-a_j)^2}{2N} + a_j z} \\ &\le \mathrm{e}^{\frac{3}{2} a_j^2 N + A a_j \sqrt{N}} \int_{a_j N - A\sqrt{N}}^{a_j N + \sqrt{N}} \frac{\mathrm{d}z}{\sqrt{2\pi N}} \mathrm{e}^{-\frac{(z-a_j)^2}{2N}} \\ &\le \mathrm{e}^{\frac{3}{2} a_j^2 N + A a_j \sqrt{N}}. \end{aligned} \tag{3.4.104}$$

Inserting (3.4.104) into (3.4.103) gives

$$T_N(N) \le 2^{-N} \exp\left(\frac{\|a\|_2^2 N}{2} + \sum_{j=0}^{M} A a_j \sqrt{N} \right) \to 0, \tag{3.4.105}$$

as $N \uparrow \infty$, since $\|a\|_2^2 < 2 \ln 2$. In particular, $(T_N(N))_{N \in \mathbb{N}}$ is bounded.

If $\sigma \wedge \tilde{\sigma} = 0$, $z_j(\sigma)$ and $z_j(\tilde{\sigma})$ are independent for each $j = 0, \ldots, M$. This and the fact that by (3.2.2),

$$\mathbb{E}\left[\mathbb{1}_{\sigma \in L_{A,j} \cap L_{r,\eta,j}} \mathrm{e}^{a_j z_j(\sigma)} \right] \le \mathbb{E}\left[\mathrm{e}^{a_j z_j(\sigma)} \right] = \mathrm{e}^{\frac{a_j^2 N}{2}}, \tag{3.4.106}$$

imply

$$T_0(N) = 2^{-2N} \mathrm{e}^{-\|a\|_2^2 N} \sum_{\substack{\sigma, \tilde{\sigma} \in \{-1,1\}^N \\ \sigma \wedge \tilde{\sigma} = 0}} \prod_{j=0}^{M} \mathbb{E}\left[\mathbb{1}_{\sigma \in L_{A,j} \cap L_{r,\eta,j}} \mathrm{e}^{a_j z_j(\sigma)} \right]^2 \le 2^{-2N} \mathrm{e}^{-\|a\|_2^2 N} \sum_{\substack{\sigma, \tilde{\sigma} \in \{-1,1\}^N \\ \sigma \wedge \tilde{\sigma} = 0}} \prod_{j=0}^{M} \mathrm{e}^{a_j^2 N} = \frac{1}{2}, \tag{3.4.107}$$

for each $N \in \mathbb{N}$, since the previous sum over σ and $\tilde{\sigma}$ has 2^{2N-1} summands.

We inspect the case $\sigma, \tilde{\sigma} \in \{-1,1\}^N$ with $\sigma \wedge \tilde{\sigma} = i$, $i = 1, \ldots, N-1$, and $j = 0, \ldots, M$. Note that

$$L_{A,j} \cap L_{r,\eta,j} \subseteq L_{A,j} \cap \tilde{L}_{i,j}, \tag{3.4.108}$$

where

$$\tilde{L}_{i,j} := \left\{\sigma \in \{-1,1\}^N : \sum_{k=1}^{i} \in I_1(i,N)\right\},$$
$$I_1(i,N) := \left[a_j i - A\tfrac{i}{\sqrt{N}} - ((i \wedge (N-i)) \vee r)^\eta, a_j i + A\tfrac{i}{\sqrt{N}} + ((i \wedge (N-i)) \vee r)^\eta\right]. \tag{3.4.109}$$

Applying (3.4.108) to the expectation containing $\sigma, \tilde{\sigma}$ and j in (3.4.102) gives

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\sigma,\tilde{\sigma}\in L_{A,j}\cap L_{r,\eta,j}} \mathrm{e}^{a_j(z_j(\sigma)+z_j(\tilde{\sigma}))}\right] &\le \mathbb{E}\left[\mathbb{1}_{\sigma,\tilde{\sigma}\in L_{A,j}\cap \tilde{L}_{i,j}} \mathrm{e}^{a_j(z_j(\sigma)+z_j(\tilde{\sigma}))}\right] \\ &= \int_{I_1(i,N)} \frac{\mathrm{d}z_1}{\sqrt{2\pi i}} \mathrm{e}^{-\frac{z_1^2}{2i}} \left(\int_{a_jN-z_1-A\sqrt{N}}^{a_jN-z_1+A\sqrt{N}} \frac{\mathrm{d}z_2}{\sqrt{2\pi(N-i)}} \mathrm{e}^{-\frac{z_2^2}{2(N-i)}+a_j(z_1+z_2)}\right)^2. \end{aligned} \tag{3.4.110}$$

The integral regarding z_2 in (3.4.110) satisfies

$$\begin{aligned}\int_{a_jN-z_1-A\sqrt{N}}^{a_jN-z_1+A\sqrt{N}} \frac{\mathrm{d}z_2}{\sqrt{2\pi(N-i)}} \mathrm{e}^{-\frac{z_2^2}{2(N-i)}+a_j(z_1+z_2)} &= \mathrm{e}^{a_jz_1+\frac{1}{2}a_j^2(N-i)} \int_{a_jN-z_1-A\sqrt{N}}^{a_jN-z_1+A\sqrt{N}} \frac{\mathrm{d}z_2}{\sqrt{2\pi(N-i)}} \mathrm{e}^{-\frac{(z_2-a_j(N-i))^2}{2(N-i)}} \\ &\le \mathrm{e}^{a_jz_1+\frac{1}{2}a_j^2(N-i)}, \end{aligned} \tag{3.4.111}$$

so inserting this into (3.4.110) gives

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\sigma,\tilde{\sigma}\in L_{A,j}\cap L_{r,\eta,j}} \mathrm{e}^{a_j(z_j(\sigma)+z_j(\tilde{\sigma}))}\right] &\le \mathrm{e}^{a_j^2(N-i)} \int_{I_1(i,N)} \frac{\mathrm{d}z_1}{\sqrt{2\pi i}} \mathrm{e}^{-\frac{z_1^2}{2i}+2a_jz_1} \\ &= \mathrm{e}^{a_j^2(N-i/2)} \int_{I_1(i,N)} \frac{\mathrm{d}z_1}{\sqrt{2\pi i}} \mathrm{e}^{-\frac{(z_1-a_ji)^2}{2i}+a_jz_1} \\ &\le \mathrm{e}^{a_j^2(N-i/2)} \mathrm{e}^{a_j^2 i + a_jA\frac{i}{\sqrt{N}}+a_j((i\wedge(N-i))\vee r)^\eta} \int_{I_1(i,N)} \frac{\mathrm{d}z_1}{\sqrt{2\pi i}} \mathrm{e}^{-\frac{(z_1-a_ji)^2}{2i}} \\ &\le \mathrm{e}^{a_j^2(N+i/2)} \mathrm{e}^{a_jA\frac{i}{\sqrt{N}}+a_j((i\wedge(N-i))\vee r)^\eta}. \end{aligned} \tag{3.4.112}$$

By (3.4.102) and (3.4.112), since there are 2^{2N-i-1} summands $(\sigma,\tilde{\sigma})$ with overlap i,

$$\begin{aligned}\sum_{i=1}^{N-1} T_i(N) &= \sum_{i=1}^{N-1} 2^{-2N} \mathrm{e}^{-\|a\|_2^2 N} \sum_{\substack{\sigma,\tilde{\sigma}\in\{-1,1\}^N \\ \sigma\wedge\tilde{\sigma}=i}} \prod_{j=0}^{M} \mathbb{E}\left[\mathbb{1}_{\sigma,\tilde{\sigma}\in L_{A,j}\cap L_{r,\eta,j}} \mathrm{e}^{a_j(z_j(\sigma)+z_j(\tilde{\sigma}))}\right] \\ &\le \sum_{i=1}^{N-1} 2^{-2N} \mathrm{e}^{-\|a\|_2^2 N} \sum_{\substack{\sigma,\tilde{\sigma}\in\{-1,1\}^N \\ \sigma\wedge\tilde{\sigma}=i}} \prod_{j=0}^{M} \mathrm{e}^{a_j^2(N+i/2)} \mathrm{e}^{a_jA\frac{i}{\sqrt{N}}+a_j((i\wedge(N-i))\vee r)^\eta} \\ &= \frac{1}{2} \sum_{i=1}^{N-1} \exp\left(\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right) i + \left(A\tfrac{i}{\sqrt{N}} + ((i\wedge(N-i))\vee r)^\eta\right) \textstyle\sum_{j=0}^{M} a_j\right). \end{aligned} \tag{3.4.113}$$

We split the last sum over i into the sum from $i = 1$ to r and from $i = r+1$ to $N-1$, labelling these sums $(R1)$ and $(R2)$, respectively. Then,

$$\begin{aligned}(R1) &= \sum_{i=1}^{r} \exp\left(\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right) i + \left(A \tfrac{i}{\sqrt{N}} + r^\eta\right) \textstyle\sum_{j=0}^{M} a_j\right) \\ &\leq \exp\left(Ar + r^\eta \textstyle\sum_{j=0}^{M} a_j\right) \sum_{i=0}^{\infty} \exp\left(\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right) i\right) \\ &= \exp\left(Ar + r^\eta \textstyle\sum_{j=0}^{M} a_j\right) \frac{1}{1 - \exp\left(\frac{1}{2}\|a\|_2^2 - \ln 2\right)},\end{aligned} \tag{3.4.114}$$

so $(R1)$ is uniformly bounded over N. To bound $(R2)$, note that if $\|a\|_2^2 < 2\ln 2$, there exists $N_0 \in \mathbb{N}$ such that for all $i \geq N_0$,

$$(A+1)i^\eta \sum_{j=0}^{M} a_j < \frac{i}{2}\left(\ln 2 - \tfrac{1}{2}\|a\|_2^2\right). \tag{3.4.115}$$

Since by (3.4.113), $\left(\sum_{i=1}^{N-1} T_i(N)\right)_{N \leq N_0+2}$ is uniformly bounded, it remains to find a uniform bound for $\sum_{i=1}^{N-1} T_i(N)$ if $N > N_0 + 2$. In this case, we can choose $r \geq N_0$. We have

$$\begin{aligned}(R2) &= \sum_{i=r+1}^{N-1} \exp\left(\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right) i + \left(A \tfrac{i}{\sqrt{N}} + (i \wedge (N-i))^\eta\right) \textstyle\sum_{j=0}^{M} a_j\right) \\ &\leq \sum_{i=r+1}^{N-1} \exp\left(i\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right) + (A+1)i^\eta \textstyle\sum_{j=0}^{M} a_j\right) \\ &\leq \sum_{i=r+1}^{N-1} \exp\left(\tfrac{i}{2}\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right)\right),\end{aligned} \tag{3.4.116}$$

applying (3.4.115) in the last step, which is possible since $r \geq N_0$. Thus,

$$(R2) \leq \sum_{i=0}^{\infty} \exp\left(\tfrac{i}{2}\left(\tfrac{1}{2}\|a\|_2^2 - \ln 2\right)\right) = \frac{1}{1 - \exp\left(\frac{1}{2}\left(\frac{1}{2}\|a\|_2^2 - \ln 2\right)\right)}, \tag{3.4.117}$$

since $\|a\|_2^2 < 2\ln 2$. We have seen in (3.4.114) and (3.4.117) that $(R1)$ and $(R2)$ are uniformly bounded over N. Thus, $\sum_{i=1}^{N-1} T_i(N)$ is uniformly bounded over N. Also, $(T_0(N))_{N\in\mathbb{N}}$ and $(T_N(N))_{N\in\mathbb{N}}$ are bounded, as we have seen in (3.4.105) and (3.4.107). Thus, $\mathbb{E}\left[(\tilde{Y}_{a,A,r,\eta}(N))^2\right]$ is by (3.4.102) uniformly bounded over N, so (3.4.101) holds and the proof is completed. □

Proof of Proposition 3.4.3. This proof has the same structure as that of [39, Lemma 2.11]. For $1 < r < N$, with notation as in (3.4.1)–(3.4.2),

$$\begin{aligned} Y_a(N) &= 2^{-N} e^{-\frac{1}{2}\|a\|_2^2 N} \sum_{\sigma \in \{-1,1\}^N} \exp\left(\sum_{j=0}^M a_j \sum_{i=1}^N z_{j,\sigma|_i}\right) \\ &= 2^{-N} e^{-\frac{1}{2}\|a\|_2^2 N} \sum_{\sigma \in \{-1,1\}^r,\, \tilde{\sigma} \in \{-1,1\}^{N-r}} \exp\left(\sum_{j=0}^M a_j \left(\sum_{i=1}^r z_{j,\sigma|_i} + \sum_{i=r+1}^N z_{j,(\sigma,\tilde{\sigma})|_i}\right)\right) \\ &= 2^{-r} e^{-\frac{1}{2}\|a\|_2^2 r} \sum_{\sigma \in \{-1,1\}^r} \exp\left(\sum_{j=0}^M a_j \sum_{i=1}^r z_{j,\sigma|_i}\right) Y_a^\sigma(N-r), \end{aligned} \tag{3.4.118}$$

where for each $\sigma \in \{-1,1\}^r$, we write $(\sigma, \tilde{\sigma})$ for the concatenation of σ with $\tilde{\sigma}$ and denote

$$Y_a^\sigma(N-r) := 2^{-(N-r)} e^{-\frac{1}{2}\|a\|_2^2 (N-r)} \sum_{\tilde{\sigma} \in \{-1,1\}^{N-r}} \exp\left(\sum_{j=0}^M a_j \sum_{i=r+1}^N z_{j,(\sigma,\tilde{\sigma})|_i}\right). \tag{3.4.119}$$

Since $(Y_a^\sigma(N-r))_{\sigma \in \{-1,1\}^r}$ are i.i.d. copies of $Y_a(N-r)$, we get from (3.4.118) that

$$Y_a(\infty) = 2^{-r} e^{-\frac{1}{2}\|a\|_2^2 r} \sum_{\sigma \in \{-1,1\}^r} \exp\left(\sum_{j=0}^M a_j \sum_{i=1}^r z_{j,\sigma|_i}\right) Y_a^\sigma(\infty), \tag{3.4.120}$$

where $(Y_a^\sigma(\infty))_{\sigma \in \{-1,1\}^r}$ are i.i.d. copies of $Y_a(\infty)$. Thus,

$$\begin{aligned} \mathbb{P}\left(Y_a(\infty) = 0\right) &= \mathbb{P}\left(2^{-r} e^{-\frac{1}{2}\|a\|_2^2 r} \sum_{\sigma \in \{-1,1\}^r} \exp\left(\sum_{j=0}^M a_j \sum_{i=1}^r z_{j,\sigma|_i}\right) Y_a^\sigma(\infty) = 0\right) \\ &= \mathbb{P}\left(\forall_{\sigma \in \{-1,1\}^r} : Y_a^\sigma(\infty) = 0\right) \\ &= \mathbb{P}\left(Y_a(\infty) = 0\right)^{2^r}, \end{aligned} \tag{3.4.121}$$

which implies $\mathbb{P}(Y_a(\infty) = 0) \in \{0, 1\}$. By Proposition 3.4.1, the convergence $Y_a(N) \to Y_a(\infty)$ as $N \uparrow \infty$ is also in L^1, so $\mathbb{E}[Y_a(\infty)] = \lim_{N\uparrow\infty} \mathbb{E}[Y_a(N)] = 1$. Thus, $\mathbb{P}(Y_a(\infty) = 0)$ cannot be one, so it has to be zero. □

3.5 Proof of Theorem 3.1.1

The central result of this section is Theorem 3.1.1, i.e. we show that for $F_N : \mathbb{R}_{\geq 0} \times Q_1 \to \mathbb{R}$, which is defined in (3.1.14) and (3.1.15), and for $\mathsf{q} \in Q_1$,

$$\Psi(\mathsf{q}) := \lim_{N\uparrow\infty} F_N(0,\mathsf{q}) = -\ln 2 + \int_0^1 \left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u. \tag{3.5.1}$$

The convex dual Ψ_* of Ψ is defined by

$$\Psi_*(\mathsf{p}) = \sup_{\mathsf{q}\in Q_2} \left(\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u - \Psi(\mathsf{q})\right), \tag{3.5.2}$$

for all $\mathsf{p} \in Q_2$. We use (3.5.1) to prove an explicit formula for Ψ_* of Ψ in Proposition 3.5.3. Moreover, (3.5.1) directly implies the following corollary regarding the regularity of Ψ.

Corollary 3.5.1. *Ψ is convex and Lipschitz continuous on Q_1 with Lipschitz constant 1.*

Proof. We start by proving Lipschitz continuity. Let $\mathsf{q}, \tilde{\mathsf{q}} \in Q_1$ and assume w.l.o.g. that

$$\zeta := \inf\left\{s \in (0,1) : \mathsf{q}(s) > \tfrac{\ln 2}{u^2}\right\} \leq \inf\left\{s \in (0,1) : \tilde{\mathsf{q}}(s) > \tfrac{\ln 2}{u^2}\right\} =: \tilde{\zeta}. \tag{3.5.3}$$

Then, since $\tilde{\mathsf{q}}(u) < \frac{\ln 2}{u^2}$ for $u \in [0, \tilde{\zeta})$,

$$|\Psi(\mathsf{q}) - \Psi(\tilde{\mathsf{q}})| = \left|\int_\zeta^{\tilde{\zeta}} \mathsf{q}(u) - \tfrac{\ln 2}{u^2}\,\mathrm{d}u + \int_{\tilde{\zeta}}^1 \mathsf{q}(u) - \tilde{\mathsf{q}}(u)\,\mathrm{d}u\right| \leq \int_\zeta^1 |\mathsf{q}(u) - \tilde{\mathsf{q}}(u)|\,\mathrm{d}u \leq \|\mathsf{q} - \tilde{\mathsf{q}}\|_1. \tag{3.5.4}$$

To prove convexity, note that for any $\lambda \in [0,1]$, we have $\lambda\mathsf{q} + (1-\lambda)\tilde{\mathsf{q}} \in Q_1$. Since $s \mapsto (s)_+$ is convex,

$$\begin{aligned}\Psi(\lambda\mathsf{q} + (1-\lambda)\tilde{\mathsf{q}}) &\leq -\ln 2 + \int_0^1 \lambda\left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ + (1-\lambda)\left(\tilde{\mathsf{q}}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u \\ &= \lambda\Psi(\mathsf{q}) + (1-\lambda)\Psi(\tilde{\mathsf{q}}).\end{aligned}$$

□

For $M \in \mathbb{N}$, we define

$$\begin{aligned}\tilde{Q}^{(M)} := \Big\{\mathsf{q} \in Q_{\leq}^{(M)} : \mathsf{q} = \textstyle\sum_{j=0}^M q_j \mathbb{1}_{[\zeta_j,\zeta_{j+1})} \text{ with } 0 \leq q_0 \leq \cdots \leq q_M < \infty \\ \text{and there exists } k \in \{1,\ldots,M\} \text{ with } q_k = \tfrac{\ln 2}{\zeta_k^2}\Big\}.\end{aligned} \tag{3.5.5}$$

Lemma 3.5.2. *For each $M \in \mathbb{N}$, $\tilde{Q}^{(M+1)}$ is dense in $Q^{(M)}$ w.r.t. $\|\cdot\|_1$.*

Proof. Let $\mathsf{q} \in Q^{(M)}$ and $\varepsilon > 0$ be sufficiently small. We distinguish the following three cases, see Figure 3.1 for an illustration:

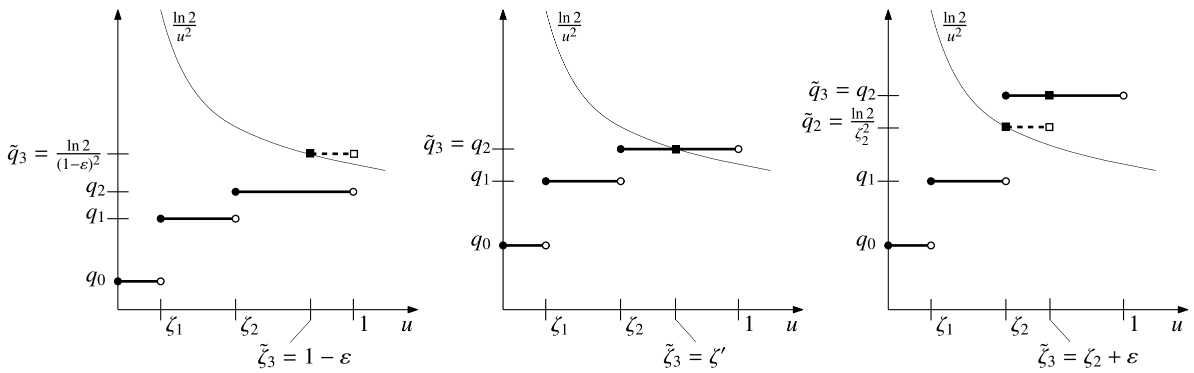

Figure 3.1: Examples of cases 1.–3. for $M = 2$ from left to right. Newly added path segments of $\tilde{\mathsf{q}}$ are dashed with boxes at their boundary. Shared path segments of q and $\tilde{\mathsf{q}}$ are thick with circles at their boundaries.

1. $q_M < \ln 2$. In this case, we define $\tilde{\mathsf{q}} = \sum_{j=0}^{M+1} \tilde{q}_j \mathbb{1}_{[\tilde{\zeta}_j,\tilde{\zeta}_{j+1})} \in \tilde{Q}^{(M+1)}$, where

$$\tilde{q}_j = \begin{cases} q_j, & j = 0,\dots,M, \\ \frac{\ln 2}{(1-\varepsilon)^2}, & j = M+1, \end{cases} \tag{3.5.6}$$

and

$$\tilde{\zeta}_j = \begin{cases} \zeta_j, & j = 0,\dots,M, \\ 1-\varepsilon, & j = M+1, \\ 1, & j = M+2. \end{cases} \tag{3.5.7}$$

Then, we have that $\|\tilde{\mathsf{q}} - \mathsf{q}\|_1 \le \left(\frac{\ln 2}{(1-\varepsilon)^2} - q_M\right)\varepsilon$.

2. There exists $k \in [\![0, M-1]\!]$ such that $\mathsf{q}(\zeta_k) < \frac{\ln 2}{\zeta_k^2}$ and $\mathsf{q}(\zeta_{k+1}) > \frac{\ln 2}{\zeta_{k+1}^2}$, and there exists $\hat{\zeta} \in [\zeta_k, \zeta_{k+1})$ such that $q_k = \frac{\ln 2}{\hat{\zeta}^2}$. Then we define $\tilde{\mathsf{q}} = \sum_{j=0}^{M+1} \tilde{q}_j \mathbb{1}_{[\tilde{\zeta}_j,\tilde{\zeta}_{j+1})} \in \tilde{Q}^{(M+1)}$, where

$$\tilde{q}_j = \begin{cases} q_j, & j = 0,\dots,k, \\ \frac{\ln 2}{\zeta_{k+1}^2}, & j = k+1, \\ q_{j-1}, & j = k+2,\dots,M, \end{cases} \tag{3.5.8}$$

and

$$\tilde{\zeta}_j = \begin{cases} \zeta_j, & j = 0,\dots,k, \\ \hat{\zeta}, & j = k+1, \\ \zeta_{j-1}, & j = k+2,\dots,M. \end{cases} \tag{3.5.9}$$

Then, we have that $\|\tilde{\mathsf{q}} - \mathsf{q}\|_1 = 0$.

3. There exists $k \in [\![0, M-1]\!]$ such that $\mathsf{q}(\zeta_k) < \frac{\ln 2}{\zeta_k^2}$ and $\mathsf{q}(\zeta_{k+1}) > \frac{\ln 2}{\zeta_{k+1}^2}$, and $q_k < \frac{\ln 2}{\zeta^2}$ for all $\zeta \in [\zeta_k, \zeta_{k+1})$. Then we define $\tilde{\mathsf{q}} = \sum_{j=0}^{M+1} \tilde{q}_j \mathbb{1}_{[\tilde{\zeta}_j, \tilde{\zeta}_{j+1})} \in \tilde{Q}^{(M+1)}$, where

$$\tilde{q}_j = \begin{cases} q_j, & j = 0, \dots, k, \\ \frac{\ln 2}{\zeta_{k+1}^2}, & j = k+1, \\ q_{j-1}, & j = k+2, \dots, M, \end{cases} \tag{3.5.10}$$

and

$$\tilde{\zeta}_j = \begin{cases} \zeta_j, & j = 0, \dots, k+1, \\ \zeta_{k+1} + \varepsilon, & j = k+2, \\ \zeta_{j-1}, & j = k+3, \dots, M. \end{cases} \tag{3.5.11}$$

Then, we have that $\|\tilde{\mathsf{q}} - \mathsf{q}\|_1 \le (q_{k+1} - q_k)\varepsilon$.

Thus, we have shown that $\bigcup_{M=0}^{\infty} \tilde{Q}^{(M+1)}$ is dense in $\bigcup_{M=0}^{\infty} Q^{(M)}$. □

Now we prove the central result of this section.

Proof of Theorem 3.1.1. Let $M \in \mathbb{N}$. By Lemma 3.3.2 and Lemma 3.5.2, $\bigcup_{M=0}^{\infty} \tilde{Q}^{(M)}$ is dense in Q_1 w.r.t. $\|\cdot\|_1$. This and the Lipschitz continuity of the right-hand side of (3.5.1) imply that it suffices to prove (3.5.1) for $q \in \tilde{Q}^{(M)}$. Let $k \in \{1, \dots, M\}$ and $\mathsf{q} \in \tilde{Q}^{(M)}$ with $\mathsf{q}_k = \frac{\ln 2}{\zeta_k^2}$. In this setting, since q is increasing,

$$\int_0^1 \left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u = \int_{\zeta_k}^1 \mathsf{q}(u) - \tfrac{\ln 2}{u^2}\, \mathrm{d}u = \ln 2 - \tfrac{\ln 2}{\zeta_k} + \int_{\zeta_k}^1 \mathsf{q}(u)\, \mathrm{d}u, \tag{3.5.12}$$

so we aim to prove that

$$\Psi(\mathsf{q}) = -\tfrac{\ln 2}{\zeta_k} + \int_{\zeta_k}^1 \mathsf{q}(u)\, \mathrm{d}u. \tag{3.5.13}$$

Recall that

$$Y_{\mathsf{q}}(\sigma, \alpha) \overset{\mathrm{d}}{=} \sum_{i=1}^{N} \sum_{k=0}^{M} (q_k - q_{k-1})^{1/2} z_{\sigma|_i, \alpha|_k}, \tag{3.5.14}$$

where each $z_{\sigma|_i, \alpha|_k}$ is from a family of i.i.d. standard Gaussians. For $\mathsf{q} \in \tilde{Q}^{(M)}$, we have by Lemma 3.2.7 that

$$\begin{aligned} \Psi(\mathsf{q}) &= \lim_{N\uparrow\infty} F_N(0, \mathsf{q}) \\ &= q_M - \lim_{N\uparrow\infty} \tfrac{1}{N}\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\mathbb{E}[Z_{0,\mathsf{q}}(z_0, \dots, z_M)^{\zeta_M} \Big| \mathcal{F}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}} \Big| \mathcal{F}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}} \dots \Big| \mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}} \Big| \mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg], \end{aligned} \tag{3.5.15}$$

recalling that

$$Z_{0,\mathsf{q}}(z_0,\dots,z_M) = \sum_{\sigma\in\{-1,1\}^N} \exp\Big(\sqrt{2}\ \textstyle\sum_{j=0}^M (q_j-q_{j-1})^{\frac12} z_j(\sigma)\Big),$$
$$\mathcal{F}_k = \sigma(z_0,\dots,z_k), \quad k=0,\dots,M-1. \tag{3.5.16}$$

Furthermore, $z_1 = (z_1(\sigma))_{\sigma\in\{-1,1\}^N},\dots,z_M = (z_M(\sigma))_{\sigma\in\{-1,1\}^N}$ are i.i.d. copies of $z_0 \coloneqq (z_0(\sigma))_{\sigma\in\{-1,1\}^N}$, a branching random walk on the N-level binary tree with standard Gaussian increments. We prove (3.5.13) by presenting matching upper and lower bounds of Ψ.

Lower bound: Since the function $x\mapsto x^{\zeta_M}$ is concave and thus subadditive,

$$\begin{aligned}
&\mathbb{E}\Big[Z_{0,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_M}\,\Big|\,\mathcal{F}_{M-1}\Big]\\
&=\mathbb{E}\Big[\Big(\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\sum_{j=0}^M(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big)^{\zeta_M}\,\Big|\,\mathcal{F}_{M-1}\Big]\\
&\le \mathbb{E}\Big[\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_M\sum_{j=0}^M(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\,\Big|\,\mathcal{F}_{M-1}\Big]\\
&=\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_M\textstyle\sum_{j=0}^{M-1}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\mathbb{E}\Big[\sqrt{2}\,\zeta_M(q_M-q_{M-1})^{1/2}z_M(\sigma)\Big]\\
&=\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_M\textstyle\sum_{j=0}^{M-1}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\exp\Big(\zeta_M^2(q_M-q_{M-1})N\Big),
\end{aligned}\tag{3.5.17}$$

using (3.2.2) in the last step. Inserting (3.5.17) into (3.5.15) and then proceeding as in (3.5.17) up to level k, we see that

$$\begin{aligned}
\Psi(\mathsf{q}) &= q_M - \lim_{N\uparrow\infty}\tfrac1N\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\mathbb{E}[Z_{0,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_M}\Big|\mathcal{F}_{M-1}\Big]^{\frac{\zeta_{M-1}}{\zeta_M}}\Big|\mathcal{F}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg]\\
&\ge q_M - \zeta_M(q_M-q_{M-1})\\
&\quad-\lim_{N\uparrow\infty}\tfrac1N\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\Big(\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_M\sum_{j=0}^{M-1}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big)^{\frac{\zeta_{M-1}}{\zeta_M}}\Big|\mathcal{F}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\\
&\qquad\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg]\\
&\ge q_M - \sum_{j=k+1}^M \zeta_j(q_j-q_{j-1})\\
&\quad-\lim_{N\uparrow\infty}\tfrac1N\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\Big(\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_{k+1}\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big)^{\frac{\zeta_{k}}{\zeta_{k+1}}}\Big|\mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_{k}}}\dots\\
&\qquad\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg].
\end{aligned}\tag{3.5.18}$$

By subadditivity of $x\mapsto x^{\frac{\zeta_k}{\zeta_{k+1}}}$ and then Jensen's inequality (applied to the concave functions

$x \mapsto x^{\frac{\zeta_{k-1}}{\zeta_k}}$, $x \mapsto x^{\frac{\zeta_{k-2}}{\zeta_k}}$, ..., $x \mapsto x^{\frac{\zeta_1}{\zeta_k}}$ and $x \mapsto \ln x$),

$$
\begin{aligned}
&\mathbb{E}\Big[\ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\Big(\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_{k+1}\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big)^{\frac{\zeta_k}{\zeta_{k+1}}}\Big|\mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}}\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Big)\Big]\\
&\le \mathbb{E}\Big[\ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_k\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big|\mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}}\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Big)\Big]\\
&\le \tfrac{1}{\zeta_k}\ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_k\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big|\mathcal{F}_{k-1}\Big]\dots\Big|\mathcal{F}_0\Big]\Big]\Big)\\
&= \tfrac{1}{\zeta_k}\ln\Big(\mathbb{E}\Big[\textstyle\sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\,\zeta_k\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big)\Big]\Big)\\
&= \tfrac{1}{\zeta_k}\ln\Big(2^N\exp\big(\zeta_k^2 q_k N\big)\Big)\\
&= N\tfrac{\ln 2}{\zeta_k}+\zeta_k q_k N. \qquad (3.5.19)
\end{aligned}
$$

Plugging this into (3.5.18) and then using $q_k = \frac{\ln 2}{\zeta_k^2}$, we obtain

$$
\Psi(\mathsf{q}) \ge q_M - \sum_{j=k+1}^{M}\zeta_j(q_j-q_{j-1}) - \tfrac{\ln 2}{\zeta_k} - \zeta_k q_k = q_M - \sum_{j=k+1}^{M}\zeta_j(q_j-q_{j-1}) - 2\sqrt{q_k\ln 2}. \qquad (3.5.20)
$$

Since

$$
\begin{aligned}
-\tfrac{\ln 2}{\zeta_k} + \int_{\zeta_k}^{1}\mathsf{q}(u)\,\mathrm{d}u &= -\tfrac{\ln 2}{\zeta_k} + \sum_{j=k}^{M}(\zeta_{k+1}-\zeta_k)q_k\\
&= -\tfrac{\ln 2}{\zeta_k} - \zeta_k q_k + q_M - \sum_{j=k+1}^{M}\zeta_j(q_j-q_{j-1})\\
&= q_M - \sum_{j=k+1}^{M}\zeta_j(q_j-q_{j-1}) - \sqrt{q_k\ln 2}, \qquad (3.5.21)
\end{aligned}
$$

we have shown that

$$
\Psi(\mathsf{q}) \ge -\tfrac{\ln 2}{\zeta_k} + \int_{\zeta_k}^{1}\mathsf{q}(u)\,\mathrm{d}u. \qquad (3.5.22)
$$

Thus, to prove (3.5.13), we need to find an upper bound which matches (3.5.22).

Upper bound: If $k \le M-1$, we define the Gibbs measure corresponding to level k as

$$
\begin{aligned}
\mu_{\mathsf{q},k}(\sigma) &:= \tfrac{1}{\tilde{Z}_{\mathsf{q},k}}\exp\Big(\sqrt{2}\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big) \quad \forall\,\sigma\in\{-1,1\}^N,\\
\tilde{Z}_{\mathsf{q},k} &:= \tilde{Z}_{\mathsf{q},k}(z_0,\dots,z_k) := \sum_{\sigma\in\{-1,1\}^N}\exp\Big(\sqrt{2}\sum_{j=0}^{k}(q_j-q_{j-1})^{1/2}z_j(\sigma)\Big). \qquad (3.5.23)
\end{aligned}
$$

We have

$$\begin{aligned}
&\mathbb{E}\left[Z_{0,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_M}\,\middle|\,\mathcal{F}_{M-1}\right]\\
&=\tilde{Z}_{\mathsf{q},k}^{\zeta_M}\mathbb{E}\left[\left(\textstyle\sum_{\sigma\in\{-1,1\}^N}\mu_{\mathsf{q},k}(\sigma)\exp\left(\sqrt{2}\sum_{j=k+1}^M(q_j-q_{j-1})^{1/2}z_j(\sigma)\right)\right)^{\zeta_M}\,\middle|\,\mathcal{F}_{M-1}\right]\\
&=\tilde{Z}_{\mathsf{q},k}^{\zeta_M}\mathbb{E}\left[\mathbb{E}_{\mu_{\mathsf{q},k}}\left[\exp\left(\sqrt{2}\textstyle\sum_{j=k+1}^M(q_j-q_{j-1})^{1/2}z_j(\sigma)\right)\right]^{\zeta_M}\,\middle|\,\mathcal{F}_{M-1}\right].
\end{aligned}\tag{3.5.24}$$

Then, by applying Jensen's inequality to $\mathbb{E}_{\mu_{\mathsf{q},k}}$,

$$\begin{aligned}
&\mathbb{E}_{\mu_{\mathsf{q},k}}\left[\exp\left(\sqrt{2}\textstyle\sum_{j=k+1}^M(q_j-q_{j-1})^{1/2}z_j(\sigma)\right)\right]^{\zeta_M}\\
&\geq\mathbb{E}_{\mu_{\mathsf{q},k}}\left[\exp\left(\sqrt{2}\,\zeta_M\textstyle\sum_{j=k+1}^M(q_j-q_{j-1})^{1/2}z_j(\sigma)\right)\right]\\
&=\sum_{\sigma\in\{-1,1\}^N}\mu_{\mathsf{q},k}(\sigma)\exp\left(\sqrt{2}\,\zeta_M\textstyle\sum_{i=k+1}^M(q_i-q_{i-1})^{1/2}z_i(\sigma)\right).
\end{aligned}\tag{3.5.25}$$

Inserting (3.5.25) into (3.5.24) and then using that $\mathcal{F}_{M-1}=\sigma(z_0,\dots,z_{M-1})$ and that z_M is independent of $\mathcal{F}_{M-1}$, we get

$$\begin{aligned}
&\mathbb{E}\left[Z_{0,\mathsf{q}}(z_0,\dots,z_M)^{\zeta_M}\,\middle|\,\mathcal{F}_{M-1}\right]\\
&\geq\tilde{Z}_{\mathsf{q},k}^{\zeta_M}\mathbb{E}\left[\sum_{\sigma\in\{-1,1\}^N}\mu_{\mathsf{q},k}(\sigma)\exp\left(\sqrt{2}\,\zeta_M\textstyle\sum_{i=k+1}^M(q_i-q_{i-1})^{1/2}z_i(\sigma)\right)\,\middle|\,\mathcal{F}_{M-1}\right]\\
&=\tilde{Z}_{\mathsf{q},k}^{\zeta_M}\sum_{\sigma\in\{-1,1\}^N}\mu_{\mathsf{q},k}(\sigma)\exp\left(\sqrt{2}\,\zeta_M\textstyle\sum_{i=k+1}^{M-1}(q_i-q_{i-1})^{1/2}z_i(\sigma)\right)\mathbb{E}\left[\mathrm{e}^{\sqrt{2}\,\zeta_M(q_M-q_{M-1})^{1/2}z_M(\sigma)}\right]\\
&=\exp\left(\zeta_M^2\,(q_M-q_{M-1})\,N\right)\tilde{Z}_{\mathsf{q},k}^{\zeta_M}\sum_{\sigma\in\{-1,1\}^N}\mu_{\mathsf{q},k}(\sigma)\exp\left(\sqrt{2}\,\zeta_M\textstyle\sum_{i=k+1}^{M-1}(q_i-q_{i-1})^{1/2}z_i(\sigma)\right).
\end{aligned}\tag{3.5.26}$$

Inserting (3.5.26) into (3.5.15) and then proceeding analogously to (3.5.26) up to level k, we get that

$$\begin{aligned}
&\Psi(\mathsf{q})\\
&\leq q_M-\zeta_M(q_M-q_{M-1})\\
&\quad-\lim_{N\uparrow\infty}\frac{1}{N}\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[(\tilde{Z}_{\mathsf{q},k})^{\zeta_{M-1}}\mathbb{E}_{\mu_{\mathsf{q},k}}\left[\exp\left(\sqrt{2}\,\zeta_M\textstyle\sum_{i=k+1}^{M-1}(q_i-q_{i-1})^{1/2}z_i(\sigma)\right)\right]^{\frac{\zeta_{M-1}}{\zeta_M}}\\
&\qquad\qquad\Big|\mathcal{F}_{M-2}\Big]^{\frac{\zeta_{M-2}}{\zeta_{M-1}}}\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg]\\
&\leq q_M-\sum_{j=k+1}^M\zeta_j(q_j-q_{j-1})-\lim_{N\uparrow\infty}\tfrac{1}{N}X_N^{(k)},
\end{aligned}\tag{3.5.27}$$

where

$$X_N^{(k)}\coloneqq\mathbb{E}\Bigg[\ln\Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\dots\mathbb{E}\Big[(\tilde{Z}_{\mathsf{q},k})^{\zeta_k}\Big|\mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}}\dots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Bigg)\Bigg].\tag{3.5.28}$$

In the case $k=M$, $X_N^{(M)}$ is the nested expectation in (3.5.15), i.e. we did not need to apply the steps in (3.5.27) in this case. Let $a = (a_0, \dots, a_k) \in \mathbb{R}_+^{k+1}$ with $\|a\|_2^2 < 2\ln 2$. We set

$$T_a(N) \coloneqq \left\{\sigma \in \{-1,1\}^N : z_j(\sigma) \geq a_j N \;\forall\, j = 0, \dots, k\right\}. \tag{3.5.29}$$

Then, for any $k = 1, \dots, M$,

$$\begin{aligned}
X_N^{(k)} &= \mathbb{E}\Bigg[\left(\mathbb{1}_{|T_a(N)|\geq 1} + \mathbb{1}_{|T_a(N)|=0}\right) \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[(\tilde{Z}_{\mathsf{q},k})^{\zeta_k} \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg] \\
&\geq \mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|\geq 1} \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[\Big(\textstyle\sum_{\sigma\in\{-1,1\}^N} \mathbb{1}_{\sigma\in T_a(N)} \exp\Big(\sqrt{2}\sum_{j=0}^k (q_j - q_{j-1})^{1/2} z_j(\sigma)\Big)\Big)^{\zeta_k} \\
&\qquad\qquad \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg] \\
&\quad + \mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|=0} \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[(\tilde{Z}_{\mathsf{q},k})^{\zeta_k} \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg].
\end{aligned} \tag{3.5.30}$$

For $\sigma \in T_a$ and $j = 0, \dots, k$, we have $z_j(\sigma) \geq a_j$ so the first summand of the last line of (3.5.30) satisfies

$$\begin{aligned}
&\mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|\geq 1} \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[\Big(\textstyle\sum_{\sigma\in\{-1,1\}^N} \mathbb{1}_{\sigma\in T_a(N)} \exp\Big(\sqrt{2}\sum_{j=0}^k (q_j - q_{j-1})^{1/2} z_j(\sigma)\Big)\Big)^{\zeta_k} \\
&\qquad\qquad \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg] \\
&\geq \sqrt{2}\,\mathbb{P}\left(|T_a(N)| \geq 1\right) \sum_{j=0}^k (q_j - q_{j-1})^{1/2} a_j N \\
&\quad + \mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|\geq 1} \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[|T_a(N)|^{\zeta_k} \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg] \\
&\geq \sqrt{2}\,\mathbb{P}\left(|T_a(N)| \geq 1\right) \sum_{j=0}^k (q_j - q_{j-1})^{1/2} a_j N.
\end{aligned} \tag{3.5.31}$$

For the second summand of the last line of (3.5.30), we estimate, for any $\tilde{\sigma} \in \{-1,1\}^N$,

$$\begin{aligned}
&\mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|=0} \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[(\tilde{Z}_{\mathsf{q},k})^{\zeta_k} \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg] \\
&\geq \mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|=0} \ln\Bigg(\mathbb{E}\Bigg[\mathbb{E}\Bigg[\dots \mathbb{E}\Big[\exp\Big(\sqrt{2}\,\zeta_k \textstyle\sum_{j=0}^k (q_j - q_{j-1})^{1/2} z_j(\tilde{\sigma})\Big) \Big| \mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}} \dots \Bigg| \mathcal{F}_1\Bigg]^{\frac{\zeta_1}{\zeta_2}} \Bigg| \mathcal{F}_0\Bigg]^{\frac{1}{\zeta_1}}\Bigg)\Bigg] \\
&\geq \mathbb{E}\Bigg[\mathbb{1}_{|T_a(N)|=0} \frac{1}{\zeta_1} \ln\Bigg(\mathbb{E}\Big[\exp\Big(\sqrt{2}\,\zeta_1 \textstyle\sum_{j=0}^k (q_j - q_{j-1})^{1/2} z_j(\tilde{\sigma})\Big) \Big| \mathcal{F}_0\Big]\Bigg)\Bigg],
\end{aligned} \tag{3.5.32}$$

using Jensen's inequality for the concave functions $x \mapsto x^{\frac{\zeta_{k-1}}{\zeta_k}}, \dots, x \mapsto x^{\frac{\zeta_1}{\zeta_2}}$ in the last step. By the independence of $z_1, \ldots, z_k$ and $\mathcal{F}_0$ and by (3.2.2),

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\sqrt{2}\,\zeta_1 \textstyle\sum_{j=0}^{k}(q_j - q_{j-1})^{1/2} z_j(\tilde{\sigma})\Big)\Big|\mathcal{F}_0\Big] \\
&= \exp\Big(\sqrt{2}\,\zeta_1 q_0^{1/2} z_0(\tilde{\sigma})\Big)\,\mathbb{E}\Big[\exp\Big(\sqrt{2}\,\zeta_1 \textstyle\sum_{j=1}^{k}(q_j - q_{j-1})^{1/2} z_j(\tilde{\sigma})\Big)\Big] \\
&= \exp\Big(\sqrt{2}\,\zeta_1 q_0^{1/2} z_0(\tilde{\sigma})\Big)\,\exp\Big(\zeta_1^2 (q_M - q_0)N\Big). \qquad (3.5.33)
\end{aligned}$$

Inserting (3.5.33) into (3.5.32) gives

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_{|T_a(N)|=0} \ln\Big(\mathbb{E}\Big[\mathbb{E}\Big[\ldots\mathbb{E}\Big[(\tilde{Z}_{\mathsf{q},k})^{\zeta_k}\Big|\mathcal{F}_{k-1}\Big]^{\frac{\zeta_{k-1}}{\zeta_k}}\ldots\Big|\mathcal{F}_1\Big]^{\frac{\zeta_1}{\zeta_2}}\Big|\mathcal{F}_0\Big]^{\frac{1}{\zeta_1}}\Big)\Big] \\
&\geq \zeta_1 (q_M - q_0)\,\mathbb{P}\left(|T_a(N)| = 0\right) N + \mathbb{E}\left[\mathbb{1}_{|T_a(N)|=0}\, q_0^{1/2} z_0(\tilde{\sigma})\right]. \qquad (3.5.34)
\end{aligned}$$

The last summand of the last line of (3.5.34) satisfies

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{|T_a(N)|=0}\, q_0^{1/2} z_0(\tilde{\sigma})\right] \\
&= \mathbb{E}\left[\mathbb{1}_{|T_a(N)|=0} \mathbb{1}_{z_0(\tilde{\sigma}) \geq -N^{2/3}}\, q_0^{1/2} z_0(\tilde{\sigma})\right] + \mathbb{E}\left[\mathbb{1}_{|T_a(N)|=0} \mathbb{1}_{z_0(\tilde{\sigma}) < -N^{2/3}}\, q_0^{1/2} z_0(\tilde{\sigma})\right], \qquad (3.5.35)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{|T_a(N)|=0} \mathbb{1}_{z_0(\tilde{\sigma}) \geq -N^{2/3}}\, q_0^{1/2} z_0(\tilde{\sigma})\right] &\geq -q_0^{1/2} N^{2/3}\,\mathbb{P}\Big(|T_a(N)| = 0, z_0(\tilde{\sigma}) \geq -N^{2/3}\Big) \\
&\geq -q_0^{1/2} N^{2/3}\,\mathbb{P}\left(|T_a(N)| = 0\right), \qquad (3.5.36)
\end{aligned}$$

and, since $z_0(\tilde{\sigma})$ is negative,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{|T_a(N)|=0} \mathbb{1}_{z_0(\tilde{\sigma}) < -N^{2/3}}\, q_0^{1/2} z_0(\tilde{\sigma})\right] &\geq q_0^{1/2}\,\mathbb{E}\left[\mathbb{1}_{z_0(\tilde{\sigma}) < -N^{2/3}}\, z_0(\tilde{\sigma})\right] \\
&= q_0^{1/2} \int_{-\infty}^{-N^{2/3}} \frac{\mathrm{d}y}{\sqrt{2\pi N}} y \mathrm{e}^{-\frac{y^2}{2N}} \\
&= -\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{N^{2/3}}{2}}. \qquad (3.5.37)
\end{aligned}$$

Combining the estimates in (3.5.30)–(3.5.37), we see that

$$\frac{1}{N} X_N^{(k)} \geq \sqrt{2}\,\mathbb{P}\left(|T_a(N)| \geq 1\right) \sum_{j=0}^{k} (q_j - q_{j-1})^{1/2} a_j + \zeta_1 (q_M - q_0)\,\mathbb{P}\left(|T_a(N)| = 0\right) + o(1). \qquad (3.5.38)$$

Thus,

$$\lim_{N\uparrow\infty} \frac{1}{N} X_N^{(k)} \geq \sqrt{2} \sum_{j=0}^{k} (q_j - q_{j-1})^{1/2} a_j, \qquad (3.5.39)$$

since by Corollary 3.4.4,

$$\lim_{N\uparrow\infty} \mathbb{P}\left(|T_a(N)| = 0\right) = 0, \qquad (3.5.40)$$

which implies

$$\lim_{N\uparrow\infty} \mathbb{P}\left(|T_a(N)| \geq 1\right) = 1 - \lim_{N\uparrow\infty} \mathbb{P}\left(|T_a(N)| = 0\right) = 1. \tag{3.5.41}$$

Inserting (3.5.39) into (3.5.27) gives

$$\Psi(\mathsf{q}) \leq q_M - \sum_{j=k+1}^{M} \zeta_j (q_j - q_{j-1}) - \sqrt{2} \sum_{j=0}^{k} (q_j - q_{j-1})^{1/2} a_j. \tag{3.5.42}$$

We have

$$\begin{aligned} &\sup\left(\textstyle\sum_{j=0}^{k}(q_j - q_{j-1})^{1/2} a_j \colon a = (a_0, \ldots, a_k) \in \mathbb{R}_+^{k+1},\ \|a\|_2^2 < 2\ln 2\right) \\ &= \max\left(\textstyle\sum_{j=0}^{k}(q_j - q_{j-1})^{1/2} a_j \colon a = (a_0, \ldots, a_k) \in \mathbb{R}_+^{k+1},\ \|a\|_2^2 \leq 2\ln 2\right) \\ &= 2\sqrt{q_k \ln 2}, \end{aligned} \tag{3.5.43}$$

since by the Cauchy-Schwartz inequality, the maximizer a^* satisfies

$$a_j^* = \sqrt{2\ln 2}\, \frac{(q_j - q_{j-1})^{1/2}}{q_k^{1/2}}, \quad j = 0, \ldots, k. \tag{3.5.44}$$

Thus,

$$\Psi(\mathsf{q}) \leq q_M - \sum_{j=k+1}^{M} \zeta_j (q_j - q_{j-1}) - 2\sqrt{q_k \ln 2}. \tag{3.5.45}$$

By (3.5.21), we have found a matching upper bound to (3.5.22). Thus, (3.5.13) holds for $\mathsf{q} \in \tilde{Q}^{(M)}$ with $\mathsf{q}_k = \frac{\ln 2}{\zeta_k^2}$. □

The formula for Ψ in Theorem 3.1.1 also allows us to compute Ψ_*.

Proposition 3.5.3. *For* $\Psi \colon Q_1 \to \mathbb{R}$ *as in Theorem 3.1.1 and* $\mathsf{p} \in Q_2$*,*

$$\Psi^*(\mathsf{p}) = \begin{cases} \infty, & \text{if } \|\mathsf{p}\|_1 \geq 1 \text{ or there exists } u \in [0,1) \text{ with } \mathsf{p}(u) > 1, \\ \frac{\ln 2}{1 - \|\mathsf{p}\|_1}, & \text{otherwise.} \end{cases} \tag{3.5.46}$$

Proof. Let $\mathsf{p} \in Q_2$. By Theorem 3.1.1 and the definition of convex dual in (3.5.2),

$$\Psi^*(\mathsf{p}) = \ln 2 + \sup_{\mathsf{y} \in Q_2} h_{\mathsf{p}}(\mathsf{y}), \tag{3.5.47}$$

where

$$h_{\mathsf{p}}(\mathsf{y}) \coloneqq \int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u - \int_0^1 \left(\mathsf{y}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u. \tag{3.5.48}$$

If there exists $u_1 = u_1(\mathsf{p}) \in [0,1)$ such that $\mathsf{p}(u_1) > 1$, then we set $\tilde{\mathsf{y}} = y\mathbb{1}_{[u_1,1)}$ with $y \geq \frac{\ln 2}{u_1^2}$. We have

$$\begin{aligned}\Psi^*(\mathsf{p}) &= \ln 2 + \sup_{\mathsf{y}\in Q_2} h_{\mathsf{p}}(\mathsf{y}) \\ &\geq \ln 2 + h_{\mathsf{p}}(\tilde{\mathsf{y}}) \\ &= \int_{u_1}^1 \mathsf{p}(u)y\,\mathrm{d}u - \int_{u_1}^1 y - \tfrac{\ln 2}{u^2}\,\mathrm{d}u + \ln 2 \\ &\geq y\int_{u_1}^1 (\mathsf{p}(u) - 1)\,\mathrm{d}u. \end{aligned} \tag{3.5.49}$$

Since p is increasing, $\mathsf{p}(u) > 1$ for all $u \in [u_1, 1)$, so the integral $\int_{u_1}^1 (\mathsf{p}(u) - 1)\,\mathrm{d}u$ in the last line of (3.5.49) is strictly positive. Thus, the last line of (3.5.49) can be arbitrarily large as $y \to \infty$.

Note that for $\mathsf{y} \in Q_2$, since y is increasing,

$$\int_0^1 \left(\mathsf{y}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u = \int_{u_*(\mathsf{y})}^1 \mathsf{y}(u) - \tfrac{\ln 2}{u^2}\,\mathrm{d}u = \ln 2 - \tfrac{\ln 2}{u_*(\mathsf{y})} + \int_{u_*(\mathsf{y})}^1 \mathsf{y}(u)\,\mathrm{d}u, \tag{3.5.50}$$

where

$$u_*(\mathsf{y}) = \sup\left\{u \in (0,1) \colon \mathsf{y}(u) < \tfrac{\ln 2}{u^2}\right\}, \tag{3.5.51}$$

so

$$h_{\mathsf{p}}(\mathsf{y}) = \int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u - \int_{u_*(\mathsf{y})}^1 \mathsf{y}(u)\,\mathrm{d}u + \tfrac{\ln 2}{u_*(\mathsf{y})} - \ln 2. \tag{3.5.52}$$

If $\|\mathsf{p}\|_1 \geq 1$, we set $\tilde{\mathsf{y}} \equiv y$, where $y \geq \ln 2$. Then $u_*(\tilde{\mathsf{y}}) = \sqrt{\frac{\ln 2}{y}}$. Thus, by (3.5.47) and (3.5.52),

$$\begin{aligned}\Psi^*(\mathsf{p}) = \ln 2 + \sup_{\mathsf{y}\in Q_2} h_{\mathsf{p}}(\mathsf{y}) &\geq \ln 2 + h_{\mathsf{p}}(\tilde{\mathsf{y}}) \\ &= \int_0^1 \mathsf{p}(u)y\,\mathrm{d}u - \int_{u_*(\tilde{\mathsf{y}})}^1 y\,\mathrm{d}u + \tfrac{\ln 2}{u_*(\tilde{\mathsf{y}})} \\ &= y\,(\|\mathsf{p}\|_1 - 1) + 2\sqrt{y\ln 2}. \end{aligned} \tag{3.5.53}$$

Since $\|\mathsf{p}\|_1 \geq 1$, the lower bound above can be arbitrarily large as $y \to \infty$.

Finally, suppose that both $\|\mathsf{p}\|_1 < 1$ and $\mathsf{p} \leq 1$ on $[0,1)$. We first establish the lower bound by choosing the constant path $\tilde{\mathsf{y}} \equiv \frac{\ln 2}{(1-\|\mathsf{p}\|_1)^2}$. Then,

$$u_*(\tilde{\mathsf{y}}) = \frac{\sqrt{\ln 2}}{\sqrt{\frac{\ln 2}{(1-\|\mathsf{p}\|_1)^2}}} = 1 - \|\mathsf{p}\|_1. \tag{3.5.54}$$

By (3.5.47), (3.5.52) and (3.5.54),

$$\Psi^*(\mathsf{p}) \geq h_{\mathsf{p}}(\tilde{\mathsf{y}}) = \frac{\ln 2}{(1-\|\mathsf{p}\|_1)^2} \int_0^1 \mathsf{p}(u)\,\mathrm{d}u - \int_{u_*(\tilde{\mathsf{y}})}^1 \frac{\ln 2}{(1-\|\mathsf{p}\|_1)^2}\,\mathrm{d}u + \frac{\ln 2}{u_*(\tilde{\mathsf{y}})} = \frac{\ln 2}{1-\|\mathsf{p}\|_1}. \tag{3.5.55}$$

It remains to show the matching upper bound. If $\|\mathsf{p}\|_1 = 0$, then the monotonicity of p implies that $\mathsf{p} \equiv 0$. In this case, obviously, $\Psi^*(\mathsf{p}) = \ln 2 = \frac{\ln 2}{1-\|\mathsf{p}\|_1}$. From now on, we assume $\|\mathsf{p}\|_1 \in (0,1)$.

Note that

$$\sup_{y \in Q_2} h_{\mathsf{p}}(\mathsf{y}) = \max\left\{ \sup_{\mathsf{y}\in Q_2, u_*(\mathsf{y})=1} h_{\mathsf{p}}(\mathsf{y}), \sup_{\mathsf{y}\in Q_2, u_*(\mathsf{y})<1} h_{\mathsf{p}}(\mathsf{y}) \right\}, \tag{3.5.56}$$

where $u_*(\mathsf{y})$ is defined in (3.5.51). By this definition, for each $\mathsf{y} \in Q_2$ with $u_*(\mathsf{y}) = 1$, we have $\mathsf{y}(u) \leq \ln 2$ for all $u \in [0,1)$. This and (3.5.52) imply

$$h_{\mathsf{p}}(\mathsf{y}) = \int_0^1 \mathsf{y}(u)\mathsf{p}(u)\,\mathrm{d}u \leq \|\mathsf{p}\|_1 \ln 2. \tag{3.5.57}$$

Thus,

$$\sup_{\mathsf{y}\in Q_2, u_*(\mathsf{y})=1} h_p(\mathsf{y}) \leq \|\mathsf{p}\|_1 \ln 2. \tag{3.5.58}$$

On the other hand, for $\mathsf{y} \in Q_2$ with $u_*(\mathsf{y}) < 1$, we rewrite (3.5.52) as

$$h_{\mathsf{p}}(\mathsf{y}) = \int_0^1 \mathsf{y}(u)\mathsf{p}(u)\,\mathrm{d}u - \int_{u_*(\mathsf{y})}^1 \mathsf{y}(u)\,\mathrm{d}u - \ln 2 + \frac{\ln 2}{u_*(\mathsf{y})} \tag{3.5.59}$$

$$= \int_0^{u_*(\mathsf{y})} \mathsf{y}(u)\mathsf{p}(u)\,\mathrm{d}u + \int_{u_*(\mathsf{y})}^1 \mathsf{y}(u)(\mathsf{p}(u)-1)\,\mathrm{d}u - \ln 2 + \frac{\ln 2}{u_*(\mathsf{y})}. \tag{3.5.60}$$

By the definition of u_* in (3.5.51), it holds for $u \in [0, u_*(\mathsf{y}))$ that $\mathsf{y}(u) < \frac{\ln 2}{u_*(\mathsf{y})^2}$. Also, $\mathsf{y}(u) \geq \frac{\ln 2}{u_*(\mathsf{y})^2}$ for $u \in (u_*(\mathsf{y}), 1)$. Applying this to the right-hand side of (3.4.92) gives, since we assumed that $\mathsf{p} \leq 1$ on $[0,1)$,

$$\begin{aligned} h_p(\mathsf{y}) &\leq \int_0^{u_*(\mathsf{y})} \frac{\ln 2}{u_*(\mathsf{y})^2}\mathsf{p}(u)\,\mathrm{d}u + \int_{u_*(\mathsf{y})}^1 \frac{\ln 2}{u_*(\mathsf{y})^2}(\mathsf{p}(u)-1)\,\mathrm{d}u - \ln 2 + \frac{\ln 2}{u_*(\mathsf{y})} \\ &= \frac{\ln 2}{u_*(\mathsf{y}^2)}\left(\|\mathsf{p}\|_1 - (u_*(\mathsf{y})-1)^2\right) = g(u_*(\mathsf{y})), \end{aligned} \tag{3.5.61}$$

where for $x \in (0,1)$,

$$g(x) \coloneqq \frac{\ln 2}{x^2}\left(\|\mathsf{p}\|_1 - (x-1)^2\right). \tag{3.5.62}$$

Note that g assumes its maximum on $(0,1)$ in $1 - \|\mathsf{p}\|_1$ with maximum value $\frac{\|\mathsf{p}\|_1 \ln 2}{1-\|\mathsf{p}\|_1}$. Thus, taking the supremum over y in (3.5.61) gives

$$\sup_{\mathsf{y}\in Q_2, u_*(\mathsf{y})<1} h_{\mathsf{p}}(\mathsf{y}) \leq \frac{\|\mathsf{p}\|_1 \ln 2}{1-\|\mathsf{p}\|_1} \leq \frac{\|\mathsf{p}\|_1 \ln 2}{1-\|\mathsf{p}\|_1}. \tag{3.5.63}$$

Combining (3.5.56), (3.5.58) and (3.5.63), we get that

$$\sup_{y\in Q_2} h_{\mathsf{p}}(\mathsf{y}) \leq \max\left\{\|\mathsf{p}\|_1 \ln 2, \tfrac{\|\mathsf{p}\|_1 \ln 2}{1-\|\mathsf{p}\|_1}\right\} = \tfrac{\|\mathsf{p}\|_1 \ln 2}{1-\|\mathsf{p}\|_1}, \tag{3.5.64}$$

where the equality in the last step follows from the fact that $1 - \|\mathsf{p}\|_1 \in (0, 1)$. Thus,

$$\Psi^*(\mathsf{p}) = \sup_{y\in Q_2} h_{\mathsf{p}}(\mathsf{y}) + \ln 2 \leq \tfrac{\|\mathsf{p}\|_1 \ln 2}{1-\|\mathsf{p}\|_1} + \ln 2 = \tfrac{\ln 2}{1-\|\mathsf{p}\|_1}.$$

□

Corollary 3.5.4. *In the setting of Proposition 3.5.3, for each* $\mathsf{p} \in Q_2$*,*

$$\Psi^*(\mathsf{p}) = \ln 2 + \sup_{\mathsf{y}\in Q_2} h_{\mathsf{p}}(\mathsf{y}) = \ln 2 + \sup_{\mathsf{y}\in Q_\infty} h_{\mathsf{p}}(\mathsf{y}) = \ln 2 + \sup_{\mathsf{y}\in Q^{(M)}} h_{\mathsf{p}}(\mathsf{y}), \tag{3.5.65}$$

where $h_{\mathsf{p}}(\mathsf{y})$ *is as in* (3.5.48)*, namely*

$$h_{\mathsf{p}}(\mathsf{y}) := \int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u - \int_0^1 \left(\mathsf{y}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u. \tag{3.5.66}$$

Proof. Note that in the proof of Proposition 3.5.3, we always used constant or piecewise constant functions $\tilde{\mathsf{y}}$ for the lower bounds. This is also implicitly the case for the upper bounds: The equation (3.5.57) translates to

$$h_{\mathsf{p}}(\mathsf{y}) \leq h_{\mathsf{p}}(\tilde{\mathsf{y}}), \tag{3.5.67}$$

where $\tilde{\mathsf{y}} \equiv \ln 2$. Also we can write (3.5.61) as

$$h_{\mathsf{p}}(\mathsf{y}) \leq h_{\mathsf{p}}(\tilde{\mathsf{y}}), \tag{3.5.68}$$

where $\tilde{\mathsf{y}} \equiv \frac{\ln 2}{u_*(\mathsf{y})^2}$. Thus, we get the same result as in Proposition 3.5.3 if we take the supremum over $\mathsf{y} \in Q^{(M)}$ or $\mathsf{y} \in Q_\infty$. □

3.6 An infinite-dimensional HJE and its viscosity solution

In this section, we study the existence and uniqueness of solutions of the Hamilton-Jacobi equation

$$\tfrac{\partial}{\partial_t} f(t,\mathsf{q}) - \int_0^1 A\left(\left(\nabla_{\mathsf{q}} f(t,\mathsf{q})\right)(u)\right) \mathrm{d}u = 0, \qquad \forall\,(t,\mathsf{q}) \in \mathbb{R}_+ \times Q_2, \tag{$HJE[\mathsf{q}]$}$$

with $f(0,\cdot) = \Psi$, for a Lipschitz continuous and convex speed function $A\colon [0,1] \to [0,1]$ and $\Psi\colon Q_1 \to \mathbb{R}$ as in Theorem 3.1.1, i.e.

$$\Psi(\mathsf{q}) = -\ln 2 + \int_0^1 \left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u. \tag{3.6.1}$$

for $\mathsf{q} \in Q_1$. In Definition 3.6.9, we introduce the notion of a *viscosity solution* of ($HJE[\mathsf{q}]$). In Proposition 3.6.12, we prove existence, uniqueness and the representation by a variational formula for a viscosity solution of ($HJE[\mathsf{q}]$). This relies on [47, Theorem 4.6], see Theorem 3.6.10 for a restatement in our setting.

The goal of this section is to prove Theorem 3.1.2, i.e. that the unique viscosity solution f of ($HJE[\mathsf{q}]$) with $f(0,\cdot) = \Psi$ satisfies

$$f(t,\mathsf{q}_0) = \lim_{N\uparrow\infty} F_N^A(t), \tag{3.6.2}$$

for all $t \geq 0$, where $\mathsf{q}_0 \equiv 0 \in Q_{\leq}^{(0)}$ and F_N^A is the free energy of the CREM, see (3.1.3).

In the following, we state definitions from [47], which we use in the proof of Proposition 3.6.12.

Definition 3.6.1. *Let $\mathcal{H}$ be a Hilbert space and $C \subset \mathcal{H}$ be a closed cone. A function $G\colon \mathcal{D} \to \mathbb{R} \cup \{\infty\}$, where $\mathcal{D} \subset \mathcal{H}$, is C-*increasing *(over $\mathcal{D}$) if $G(x) \geq G(y)$ holds for all $x, y \in \mathcal{D}$ with $x - y \in C$.*

Note that Q_2 is a closed convex cone on the Hilbert space $L_2([0,1),\mathbb{R})$. Its *dual cone* is the closed convex cone defined by

$$Q_2^* \coloneqq \left\{\mathsf{p} \in L_2([0,1),\mathbb{R})\colon \int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u \geq 0\ \forall \mathsf{q} \in Q_2\right\}. \tag{3.6.3}$$

A function $G\colon Q_2 \to \mathbb{R}$ is Q_2^**-increasing* (over Q_2) if, for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$,

$$\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\tilde{\mathsf{q}}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2 \quad \Rightarrow \quad G(\mathsf{q}) \geq G(\tilde{\mathsf{q}}). \tag{3.6.4}$$

Lemma 3.6.2. *The function* Ψ *from Theorem 3.1.1 is* Q_2^*-increasing *(over* Q_2*).*

Proof. For $M \in \mathbb{N}$, we write

$$Q^{(M)}_{\text{equidist.}} := \left\{ \mathsf{q} \in Q\colon \text{ there exist } 0 < q_0 < \cdots < q_{M-1} \text{ so that } \mathsf{q} = \textstyle\sum_{j=0}^{M-1} q_j \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)} \right\}. \tag{3.6.5}$$

We aim to prove that for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$, we have

$$\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\tilde{\mathsf{q}}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2 \quad \Rightarrow \quad \Psi(\mathsf{q}) \geq \Psi(\tilde{\mathsf{q}}). \tag{3.6.6}$$

Let $M \in \mathbb{N}$. By the following continuity-density argument, it suffices to prove that (3.6.6) holds for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q^{(M)}_{\text{equidist.}}$. The Cauchy-Schwartz inequality implies for all $\mathsf{p}, \mathsf{q}, \tilde{\mathsf{q}} \in Q_2$ that

$$\left| \int_0^1 (\mathsf{q}(u) - \tilde{\mathsf{q}}(u))\, \mathsf{p}(u)\,\mathrm{d}u \right| \leq \int_0^1 \left| \mathsf{q}(u) - \tilde{\mathsf{q}}(u) \right| \mathsf{p}(u)\,\mathrm{d}u \leq \|\mathsf{q} - \tilde{\mathsf{q}}\|_2 \|\mathsf{p}\|_2. \tag{3.6.7}$$

Thus, the function $L_2([0,1)) \ni g \mapsto \int_0^1 g(u)\mathsf{p}(u)\,\mathrm{d}u$ is Lipschitz continuous w.r.t. $\|\cdot\|_2$ for all $\mathsf{p} \in Q_2$. The function Ψ is Lipschitz continuous w.r.t. $\|\cdot\|_1$ by Corollary 3.5.1 and thus by (3.1.8) also Lipschitz continuous w.r.t. $\|\cdot\|_2$. With the same arguments as in the proof of Lemma 3.3.2, one proves that $\bigcup_{k \in \mathbb{N}} Q^{(k)}_{\text{equidist.}}$ is dense in Q_1. Recall from (3.1.8) that $Q_2 \subset Q_1$, so $\bigcup_{k \in \mathbb{N}} Q^{(k)}_{\text{equidist.}}$ is dense in Q_2 as well. Thus, (3.6.6) holds for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$ if it holds for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q^{(M)}_{\text{equidist.}}$.

Let $\mathsf{q}^{(0)} = \sum_{j=0}^{M-1} q_j^{(0)} \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)}, \mathsf{q}^{(1)} = \sum_{j=0}^{M-1} q_j^{(1)} \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)} \in Q^{(M)}_{\text{equidist.}}$ so that

$$\int_0^1 \mathsf{p}(u)\mathsf{q}^{(1)}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\mathsf{q}^{(0)}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2. \tag{3.6.8}$$

We prove that $\Psi\left(\mathsf{q}^{(1)}\right) \geq \Psi\left(\mathsf{q}^{(0)}\right)$.

For each $\lambda \in [0,1]$, we set $\mathsf{q}^{(\lambda)} = \lambda \mathsf{q}^{(1)} + (1-\lambda)\mathsf{q}^{(0)}$. Then, $\mathsf{q}^{(\lambda)} = \sum_{j=0}^{M-1} q_j^{(\lambda)} \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)} \in Q^{(M)}_{\text{equidist.}}$, where $q^{(\lambda)} = \lambda\left(q_0^{(1)}, \ldots, q_{M-1}^{(1)}\right) + (1-\lambda)\left(q_0^{(0)}, \ldots, q_{M-1}^{(0)}\right)$. We set

$$\begin{aligned} \tilde{\Psi}\colon C_<^{(M-1)} &\to \mathbb{R}, \\ (q_0, \ldots, q_{M-1}) &\mapsto \Psi\left(\textstyle\sum_{j=0}^{M} q_j \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)}\right), \end{aligned} \tag{3.6.9}$$

recalling that

$$C_<^{(M-1)} = \left\{ q = (q_0, \ldots, q_{M-1}) \in \mathbb{R}^M \colon 0 < q_0 < \cdots < q_{M-1} \right\}. \tag{3.6.10}$$

By the fundamental theorem of calculus,

$$\Psi\left(\mathsf{q}^{(1)}\right) - \Psi\left(\mathsf{q}^{(0)}\right) = \tilde{\Psi}\left(q_0^{(1)}, \ldots, q_{M-1}^{(1)}\right) - \tilde{\Psi}\left(q_0^{(0)}, \ldots, q_{M-1}^{(0)}\right) = \int_0^1 \tfrac{\partial}{\partial \lambda} \tilde{\Psi}\left(q^{(\tilde{\lambda})}\right) \mathrm{d}\tilde{\lambda}, \tag{3.6.11}$$

where by the chain rule,

$$\tfrac{\partial}{\partial_\lambda}\tilde{\Psi}\left(q^{(\tilde{\lambda})}\right) = \sum_{j=0}^{M-1}\left(q_j^{(1)} - q_j^{(0)}\right)\tfrac{\partial}{\partial_j}\tilde{\Psi}\left(q^{(\tilde{\lambda})}\right). \tag{3.6.12}$$

To keep this proof concise, we postpone the proof that $\nabla\tilde{\Psi}(q) \in C_{\leq}^{(M-1)}$ for all $q \in C_{<}^{(M-1)}$ to the following Lemma 3.6.3. This result implies for each $\lambda \in [0,1]$ that, setting

$$\mathsf{p}^{(\lambda)} := \sum_{j=0}^{M-1}\tfrac{\partial}{\partial_j}\tilde{\Psi}(q^{(\lambda)})\,\mathbb{1}_{\left[\frac{j}{M},\frac{j+1}{M}\right)}, \tag{3.6.13}$$

we have $\mathsf{p}^{(\lambda)} \in Q_{\text{equidist.}}^{(M)} \subset Q_2$. Then, by (3.6.8), for each $\lambda \in [0,1]$,

$$\sum_{j=0}^{M-1}\left(q_j^{(1)} - q_j^{(0)}\right)\tfrac{\partial}{\partial_j}\tilde{\Psi}\left(q^{(\lambda)}\right) = \int_0^1 (\mathsf{q}(u) - \tilde{\mathsf{q}}(u))\,\mathsf{p}^{(\lambda)}(u)\,\mathrm{d}u \geq 0. \tag{3.6.14}$$

Inserting (3.6.12) and (3.6.14) into (3.6.11) gives $\Psi\left(\mathsf{q}^{(1)}\right) \geq \Psi\left(\mathsf{q}^{(0)}\right)$, which completes this proof up to the proof of Lemma 3.6.3. □

Lemma 3.6.3. *Let*

$$\begin{aligned}\tilde{\Psi}\colon C_{<}^{(M-1)} &\to \mathbb{R},\\ (q_0,\ldots,q_{M-1}) &\mapsto \Psi\left(\textstyle\sum_{j=0}^{M} q_j\mathbb{1}_{\left[\frac{j}{M},\frac{j+1}{M}\right)}\right),\end{aligned} \tag{3.6.15}$$

as in (3.6.9). *Let $M \in \mathbb{N}$ and $q = (q_0,\ldots,q_{M-1})$. Then, the gradient $\nabla\tilde{\Psi}(q)$ lies in $C_{\leq}^{(M-1)}$.*

Proof. Let $M \in \mathbb{N}$ and $(q_0,\ldots,q_{M-1}) \in C_{<}^{(M-1)}$. We write $\mathsf{q} := \sum_{j=0}^{M-1} q_j\mathbb{1}_{\left[\frac{j}{M},\frac{j+1}{M}\right)}$. Since q is increasing, by Theorem 3.1.1,

$$\begin{aligned}\tilde{\Psi}(q_0,\ldots,q_{M-1}) = \Psi(\mathsf{q}) &= -\ln 2 + \int_0^1 \left(\mathsf{q}(u) - \tfrac{\ln 2}{u^2}\right)_+ \mathrm{d}u\\ &= -\ln 2 + \int_{x_*(\mathsf{q})}^1 \mathsf{q}(u) - \tfrac{\ln 2}{u^2}\,\mathrm{d}u\\ &= -\tfrac{\ln 2}{x_*(\mathsf{q})} + \int_{x_*(\mathsf{q})}^1 \mathsf{q}(u)\,\mathrm{d}u,\end{aligned} \tag{3.6.16}$$

where $x_*(\mathsf{q}) = \inf\left\{x \in (0,1]\colon \mathsf{q}(x) > \frac{\ln 2}{x^2}\right\}$, setting $x_*(\mathsf{q}) = 1$ if $q_{M-1} \leq \ln 2$. We compute the partial derivatives of $\tilde{\Psi}$ in $(q_0,\ldots,q_{M-1})$:

1. If $x_*(\mathsf{q}) = 1$, then $\tilde{\Psi}(q_0,\ldots,q_{M-1}) = -\ln 2$, so $\frac{\partial}{\partial_{q_j}}\tilde{\Psi}(q_0,\ldots,q_{M-1}) = 0$ for all $j = 0,\ldots M$.

2. If $x_*(\mathsf{q}) = \frac{k}{M}$ for some $k \in \{1, \dots, M-1\}$, then

$$\tilde{\Psi}(q_0, \dots, q_{M-1}) = -\frac{M \ln 2}{k} + \sum_{j=k}^{M-1} \frac{q_j}{M}, \tag{3.6.17}$$

so

$$\frac{\partial}{\partial_{q_j}} \tilde{\Psi}(q_0, \dots, q_{M-1}) = \begin{cases} 0, & \text{if } j < k, \\ \frac{1}{M}, & \text{if } j \geq k. \end{cases} \tag{3.6.18}$$

3. If $x_*(\mathsf{q}) \in \left(\frac{k}{M}, \frac{k+1}{M}\right)$ for some $k \in \{1, \dots, M-1\}$, then $x_*(\mathsf{q}) = \sqrt{\frac{\ln 2}{q_k}}$. This gives

$$\begin{aligned} \tilde{\Psi}(q_0, \dots, q_{M-1}) &= -\sqrt{q_k \ln 2} + \int_{\sqrt{\frac{\ln 2}{q_k}}}^{1} \mathsf{q}(u)\, \mathrm{d}u \\ &= -\sqrt{q_k \ln 2} + q_k \left(\frac{k+1}{M} - \sqrt{\frac{\ln 2}{q_k}}\right) + \sum_{j=k+1}^{M-1} \frac{q_j}{M} \\ &= -2\sqrt{q_k \ln 2} + q_k \frac{k+1}{M} + \sum_{j=k+1}^{M-1} \frac{q_j}{M}. \end{aligned} \tag{3.6.19}$$

Thus,

$$\frac{\partial}{\partial_{q_j}} \tilde{\Psi}(q_0, \dots, q_{M-1}) = \begin{cases} 0, & \text{if } j < k, \\ \frac{k+1}{M} - \sqrt{\frac{\ln 2}{q_k}} = \frac{k+1}{M} - x_*(\mathsf{q}), & \text{if } j = k, \\ \frac{1}{M}, & \text{if } j > k. \end{cases} \tag{3.6.20}$$

In particular, in each case,

$$\nabla \tilde{\Psi}(q_0, \dots, q_{M-1}) \in C_{\leq}^{(M-1)}.$$

□

Definition 3.6.4. *Let $D \geq 1$ and $M_+^{D\times D} \subset \mathbb{R}^{D\times D}$ be the set of positive semidefinite $D \times D$ matrices. Note that $M_+^{D\times D}$ is a closed cone. A function $g\colon M_+^{D\times D} \to \mathbb{R}$ is called* proper *if g is $M_+^{D\times D}$-increasing and, for each $T \in M_+^{D\times D}$, the function*

$$\begin{aligned} g_T\colon M_+^{D\times D} &\to \mathbb{R}, \\ S &\mapsto g(S+T) - g(S), \end{aligned} \tag{3.6.21}$$

is $M_+^{D\times D}$-increasing.

We are interested in the case $D = 1$, where $M_+^{1\times 1} = \mathbb{R}_{\geq 0}$.

Lemma 3.6.5. *A Lebesgue-measurable function* $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ *is proper if and only if it is convex and increasing.*

Proof. The condition of g being $\mathbb{R}_{\geq 0}$-increasing is equivalent to g being increasing on $\mathbb{R}_{\geq 0}$. Assume that g is convex and increasing. We prove that g is proper, i.e. g is increasing and for all $x, y, z \geq 0$ with $x \geq y$,

$$g(x+z) - g(y+z) \geq g(x) - g(y). \tag{3.6.22}$$

Let $x, y, z \geq 0$ with $x \geq y$. For $z = 0$, (3.6.22) is trivial. For $z > 0$, we set $\lambda := \frac{x-y}{x+z-y} \in [0,1]$, which ensures that $\lambda y + (1-\lambda)(x+z) = y + z$ and $\lambda(x+z) + (1-\lambda)y = x$. Then, by the convexity of g,

$$\begin{aligned} g(x+z) - g(y+z) &= g(x+z) - g\big(\lambda y + (1-\lambda)(x+z)\big) \\ &\geq g(x+z) - \lambda g(y) - (1-\lambda) g(x+z) \\ &= \lambda\big(g(x+z) - g(y)\big) \\ &= \lambda g(x+z) + (1-\lambda) g(y) - g(y) \\ &\geq g\big(\lambda(x+z) + (1-\lambda) y\big) - g(y) \\ &= g(x) - g(y). \end{aligned} \tag{3.6.23}$$

Thus, g is proper if it is convex and increasing.

Conversely, assume that g is proper, i.e. g is increasing and satisfies (3.6.22) for all $x, y, z \geq 0$ with $x \geq y$. This implies that for $x \geq y \geq 0$, setting $\tilde{z} = \frac{x-y}{2}, \tilde{x} = \frac{x+y}{2}, \tilde{y} = y$,

$$g(x) - g\left(\tfrac{x+y}{2}\right) = g(\tilde{x} + \tilde{z}) - g(\tilde{y} + \tilde{z}) \geq g(\tilde{x}) - g(\tilde{y}) = g\left(\tfrac{x+y}{2}\right) - g(y). \tag{3.6.24}$$

We have obtained that

$$g\left(\tfrac{x+y}{2}\right) \leq \tfrac{1}{2} g(x) + \tfrac{1}{2} g(y), \tag{3.6.25}$$

for $x \geq y \geq 0$. By symmetry, this also holds for $y \geq x \geq 0$. Thus, g is midpoint-convex and, by assumption, Lebesgue-measurable. A classical result from convex analysis, which was proven independently in [23] and [106], implies the convexity of g. □

Note that the latter proof relies on the fact that "≤" is a total order on $\mathbb{R}_{\geq 0}$, so this technique fails on $M_+^{D\times D}$ for $D > 1$.

In (HJE[q]), we evaluate $A\left(\left(\nabla_{\mathsf{q}} f(t, \mathsf{q})\right)(u)\right)$ for $u \in [0,1)$. However, we do not know a priori whether $\left(\nabla_{\mathsf{q}} f(t, \mathsf{q})\right)(u)$ lies in the domain $[0,1]$ of A. For this reason, we use the following definition of a *regularisation* to extend A to the domain $\mathbb{R}$.

Definition 3.6.6. *Let* $A\colon [0,1] \to [0,1]$ *be a Lipschitz continuous and convex speed function. A function* $A_{reg.}\colon \mathbb{R} \to \mathbb{R}$ *is called* regularisation *of* A *if* $A_{reg.}$ *coincides with* A *on* $[0,1]$ *and* $A_{reg.}$ *is Lipschitz continuous on* $\mathbb{R}$ *and proper on* $\mathbb{R}_{\geq 0}$*.*

This is an adaptation of [47, Definition 4.1] to the setting of this thesis, taking into account that speed functions are only defined on $[0,1]$, while analogous functions in [47] are defined on $\mathbb{R}$.

Lemma 3.6.7. *Let* A *be a Lipschitz continuous and convex speed function. Then, there exists a regularisation* $A_{reg.}$ *of* A*.*

Proof. Let $L > 0$ be a Lipschitz constant of A. Then, the function

$$A_{\text{reg.}}\colon \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto \begin{cases} 0, & \text{if } x < 0, \\ A(x), & \text{if } x \in [0,1], \\ 1 + L(x-1), & \text{if } x > 0, \end{cases} \tag{3.6.26}$$

is clearly increasing, Lipschitz-continuous and coincides with A on $[0,1]$ per construction. Since A is increasing and L-Lipschitz continuous, it is differentiable for Lebesgue-almost all $x \in [0,1]$ with $0 \leq A'(x) \leq L$. Since $A'_{\text{reg.}}(x) = L$ for $x > 1$ and $A'_{\text{reg.}}(x) = 0$ for $x < 0$, $A_{\text{reg.}}$ is convex, so it is by Lemma 3.6.5 proper on $\mathbb{R}_{\geq 0}$. □

Definition 3.6.8. *Let* $A_{reg.}$ *be a regularisation of the Lipschitz continuous and convex speed function* A*. The* nonlinearity H *corresponding to* $A_{reg.}$ *is the function*

$$\begin{aligned} \mathsf{H}\colon\ & L_2([0,1),\mathbb{R}) \to \mathbb{R}, \\ & g \mapsto \inf\left\{\int_0^1 A_{reg.}(\mathsf{p}(u))\,\mathrm{d}u\colon \mathsf{p} \in Q_2 \cap \left(g + Q_2^*\right)\right\}. \end{aligned} \tag{3.6.27}$$

Definition 3.6.9. *We call* $f\colon \mathbb{R}_{\geq 0}\times Q_2 \to \mathbb{R}$ *a* viscosity subsolution (supersolution) *of* $(HJE[\mathsf{q}])$ *if there exists a regularisation* $A_{reg.}$ *of* A *so that* f *is a viscosity subsolution (supersolution) of*

$$\tfrac{\partial}{\partial_t} f(t,\mathsf{q}) - \mathsf{H}\left(\nabla_{\mathsf{q}} f(t,\mathsf{q})\right) = 0, \qquad \forall\, (t,\mathsf{q}) \in \mathbb{R}_+ \times Q_2, \tag{3.6.28}$$

i.e. for all $(t_*,\mathsf{q}_*) \in \mathbb{R}_{\geq 0}\times Q_2$ *and all* $\phi \in C^\infty(\mathbb{R}_{\geq 0} \times Q_2, \mathbb{R})$ *so that* (t_*,q_*) *is a strict local maximum (minimum) of* $f - \phi$*, we have*

$$\tfrac{\partial}{\partial_t}\phi(t_*,\mathsf{q}_*) - \mathsf{H}\left(\nabla_{\mathsf{q}}\phi(t_*,\mathsf{q}_*)\right) \begin{cases} \leq 0, & \textit{in the case of a subsolution,} \\ \geq 0, & \textit{in the case of a supersolution.} \end{cases} \tag{3.6.29}$$

The function f *is a* viscosity solution *of* $(HJE[\mathsf{q}])$ *if it is both a viscosity sub- and supersolution of* $(HJE[\mathsf{q}])$*.*

Once we prove Proposition 3.6.12, we know that the unique viscosity solution of (HJE[q]) with initial condition Ψ does not depend on the choice of the regularisation. Uniqueness of viscosity solutions in this setting is to be understood in the class of functions $f\colon \mathbb{R}_{\geq 0} \times Q_2 \to \mathbb{R}$ satisfying all of the following:

- $f(t,\cdot)$ is Lipschitz continuous,
- $\sup_{t>0,\, \mathsf{q}\in Q_2} \left|\frac{f(t,\mathsf{q})-f(t,0)}{t}\right| < \infty$,
- for all $t \geq 0$, $f(t,\cdot)$ is Q_2^*-increasing.

We now state the parts from Theorem 4.6 of [47], which we use for the proof of Proposition 3.6.12.

Theorem 3.6.10 [47, Theorem 4.6]**.** *Let $A\colon [0,1] \to [0,1]$ be a Lipschitz continuous and proper function. Let $A_{reg.}$ be a regularisation of A. Let $G\colon Q_2 \to \mathbb{R}$ be Q_2^*-increasing with*

$$|G(\mathsf{q}) - G(\tilde{\mathsf{q}})| \leq \|\mathsf{q} - \tilde{\mathsf{q}}\|_1, \tag{3.6.30}$$

for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$. Then, there exists a unique viscosity solution f of (HJE[q]) with $f(0,\cdot) = G$, which is given by the Hopf-Lax formula

$$f(t,\mathsf{q}) = \sup_{\mathsf{y}\in Q_\infty} \inf_{\mathsf{p}\in Q_\infty} \left(G(\mathsf{y}) + \int_0^1 \mathsf{p}(u)\cdot(\mathsf{q}(u) - \mathsf{y}(u))\,\mathrm{d}u + t\int_0^1 A_{reg.}(\mathsf{p}(u))\,\mathrm{d}u \right), \tag{3.6.31}$$

for all $(t,\mathsf{q}) \in \mathbb{R}_{\geq 0} \times Q_2$. If furthermore, G is convex, then f is given by the Hopf formula

$$f(t,\mathsf{q}) = \sup_{\mathsf{p}\in Q_\infty} \inf_{\mathsf{y}\in Q_\infty} \left(G(\mathsf{y}) + \int_0^1 \mathsf{p}(u)\cdot(\mathsf{q}(u) - \mathsf{y}(u))\,\mathrm{d}u + t\int_0^1 A_{reg.}(\mathsf{p}(u))\,\mathrm{d}u \right), \tag{3.6.32}$$

for all $(t,\mathsf{q}) \in \mathbb{R}_{\geq 0} \times Q_2$.

When comparing Theorem 3.6.10 to Theorem 4.6 of [47], note that we used the following Lemma 3.6.11 and the fact that $Q_\infty \subset Q_2$ to simplify the nonlinearity in the variational formulas (3.6.31) and (3.6.32).

Lemma 3.6.11. *Let A be a Lipschitz continuous and convex speed function. For each regularisation $A_{reg.}$ of A and each $\mathsf{q} \in Q_2$, we have*

$$\mathsf{H}(\mathsf{q}) = \int_0^1 A_{reg.}(\mathsf{q}(u))\,\mathrm{d}u. \tag{3.6.33}$$

Proof. We prove that the map

$$\begin{aligned} G\colon Q_2 &\to \mathbb{R}_{\geq 0}, \\ \mathsf{q} &\mapsto \int_0^1 A_{\mathrm{reg.}}(\mathsf{q}(u))\,\mathrm{d}u, \end{aligned} \tag{3.6.34}$$

is Q_2^*-increasing. Then, for each q in Q_2 and for each $\mathsf{p} \in Q_2 \cap \left(\mathsf{q} + Q_2^*\right)$, we have $\mathsf{p} - \mathsf{q} \in Q_2^*$, so $G(\mathsf{p}) \geq G(\mathsf{q})$. This implies that

$$\mathsf{H}(\mathsf{q}) = \inf\left\{G(\mathsf{p})\colon \mathsf{p} \in Q_2 \cap \left(\mathsf{q} + Q_2^*\right)\right\} = G(\mathsf{q}), \tag{3.6.35}$$

which completes the proof.

Let $L > 0$ be the Lipschitz constant of A. Since $|G(\mathsf{q}) - G(\tilde{\mathsf{q}})| \leq L\|\mathsf{q} - \tilde{\mathsf{q}}\|_1$ for all $\mathsf{q}, \tilde{\mathsf{q}} \in Q_2$, we can apply the same continuity-density argument as in (3.6.6)–(3.6.8). Thus, it suffices to prove that for $M \in \mathbb{N}$ and $\mathsf{q}, \tilde{\mathsf{q}} \in Q^{(M)}_{\text{equidist.}}$, where $Q^{(M)}_{\text{equidist.}}$ is as in (3.6.5), it holds

$$\int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\tilde{\mathsf{q}}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2 \quad \Rightarrow \quad G(\mathsf{q}) \geq G(\tilde{\mathsf{q}}). \tag{3.6.36}$$

To prove this, we proceed as in (3.6.11)–(3.6.14): Note that for $\mathsf{q} = \sum_{k=0}^{M-1} q_k \mathbb{1}_{\left[\frac{k}{M}, \frac{k+1}{M}\right)} \in Q^{(M)}_{\text{equidist.}}$, we have $G(\mathsf{q}) = \tilde{G}(q_0, \ldots, q_{M-1})$, where

$$\begin{aligned} \tilde{G}\colon C_<^{(M-1)} &\to \mathbb{R}, \\ (p_0, \ldots, p_{M-1}) &\mapsto \frac{1}{M}\sum_{k=0}^{M-1} A_{\text{reg.}}(p_k). \end{aligned} \tag{3.6.37}$$

Let $\mathsf{q}^{(0)} = \sum_{j=0}^{M-1} q_j^{(0)} \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)}, \mathsf{q}^{(1)} = \sum_{j=0}^{M-1} q_j^{(1)} \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)} \in Q^{(M)}_{\text{equidist.}}$, which satisfy

$$\int_0^1 \mathsf{p}(u)\mathsf{q}^{(1)}(u)\,\mathrm{d}u \geq \int_0^1 \mathsf{p}(u)\mathsf{q}^{(0)}(u)\,\mathrm{d}u \text{ for all } \mathsf{p} \in Q_2. \tag{3.6.38}$$

We prove that $G(\mathsf{q}^{(1)}) \geq G(\mathsf{q}^{(0)})$.

For each $\lambda \in [0,1]$, we set $\mathsf{q}^{(\lambda)} = \lambda\mathsf{q}^{(1)} + (1-\lambda)\mathsf{q}^{(0)}$. Then, $\mathsf{q}^{(\lambda)} = \sum_{j=0}^{M-1} q_j^{(\lambda)} \mathbb{1}_{\left[\frac{j}{M}, \frac{j+1}{M}\right)} \in Q^{(M)}_{\text{equidist.}}$, where $q^{(\lambda)} = \lambda\left(q_0^{(1)}, \ldots, q_{M-1}^{(1)}\right) + (1-\lambda)\left(q_0^{(0)}, \ldots, q_{M-1}^{(0)}\right) \in C_<^{(M-1)}$. Since $A_{\text{reg.}}$ is increasing, it is Lebesgue-almost everywhere differentiable on $\mathbb{R}_{\geq 0}$, so $\tilde{G}$ is Lebesgue-almost everywhere partially differentiable on $C_<^{(M-1)}$. Thus, by the fundamental theorem of calculus,

$$G\left(\mathsf{q}^{(1)}\right) - G\left(\mathsf{q}^{(0)}\right) = \tilde{G}\left(q_0^{(1)}, \ldots, q_{M-1}^{(1)}\right) - \tilde{G}\left(q_0^{(0)}, \ldots, q_{M-1}^{(0)}\right) = \int_0^1 \tfrac{\partial}{\partial_\lambda}\tilde{G}\left(q^{(\tilde{\lambda})}\right)\mathrm{d}\tilde{\lambda}. \tag{3.6.39}$$

This and the same computations as in (3.6.11)–(3.6.14) imply that (3.6.36) holds for $\mathsf{q}, \tilde{\mathsf{q}} \in Q^{(M)}_{\text{equidist.}}$ if $\nabla\tilde{G}(q) \in C_\leq^{(M-1)}$ for Lebesgue-almost all $q \in C_<^{(M-1)}$.

Let $q = (q_0, \ldots, q_{M-1})$ so that the derivative of $A_{\text{reg.}}$ exists in each entry of q. Then, for each $j, k = 0, \ldots, M-1$ with $j < k$, since $A_{\text{reg.}}$ in convex,

$$\tfrac{\partial}{\partial_{q_j}}G(q_0, \ldots, q_{M-1}) = \tfrac{1}{M}A'_{\text{reg.}}(q_j) \leq \tfrac{1}{M}A'_{\text{reg.}}(q_k) = \tfrac{\partial}{\partial_{q_k}}G(q_0, \ldots, q_{M-1}), \tag{3.6.40}$$

so $\nabla\tilde{G}(q) \in C_\leq^{(M-1)}$. This completes the proof. □

Proposition 3.6.12. *Let $A\colon [0,1] \to [0,1]$ be a Lipschitz continuous and convex speed function. Let $A_{reg.}$ be a regularisation of A. Let Ψ be as in Theorem 3.1.1. Then, there exists a unique viscosity solution f of $(HJE[\mathsf{q}])$ with $f(0,\cdot) = \Psi$, which is given by the Hopf formula*

$$\begin{aligned} f(t,\mathsf{q}) &= \sup_{\mathsf{p}\in Q_\infty} \inf_{\mathsf{y}\in Q_\infty} \left(\Psi(\mathsf{y}) + \int_0^1 \mathsf{p}(u)\cdot(\mathsf{q}(u)-\mathsf{y}(u))\,\mathrm{d}u + t\int_0^1 A_{reg.}(\mathsf{p}(u))\,\mathrm{d}u \right) \\ &= \sup_{\substack{\mathsf{p}\in Q_\infty,\\ \|\mathsf{p}\|_\infty\le 1,\,\|\mathsf{p}\|_1<1}} \left(t\int_0^1 A(\mathsf{p}(u))\,\mathrm{d}u + \int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u - \frac{\ln 2}{1-\|\mathsf{p}\|_1} \right), \end{aligned} \tag{3.6.41}$$

for all $(t,\mathsf{q}) \in \mathbb{R}_{\ge 0} \times Q_2$. In particular, f does not depend on the choice of the regularisation $A_{reg.}$.

Proof. By Lemma 3.6.5, A is proper. By Lemma 3.6.2 and Corollary 3.5.1, Ψ is Q_2^*-increasing, convex and satisfies (3.6.30) for all $\mathsf{q},\tilde{\mathsf{q}} \in Q_2$. Thus, by Theorem 3.6.10, there exists a unique viscosity solution f of $(HJE[\mathsf{q}])$ with $f(0,\cdot) = \Psi$, which is given by the Hopf formula

$$f(t,\mathsf{q}) = \sup_{\mathsf{p}\in Q_\infty} \inf_{\mathsf{y}\in Q_\infty} \left(\Psi(\mathsf{y}) + \int_0^1 \mathsf{p}(u)\cdot(\mathsf{q}(u)-\mathsf{y}(u))\,\mathrm{d}u + t\int_0^1 A_{\mathrm{reg.}}(\mathsf{p}(u))\,\mathrm{d}u \right), \tag{3.6.42}$$

for all $(t,\mathsf{q}) \in \mathbb{R}_{\ge 0} \times Q_2$. By Proposition 3.5.3 and Corollary 3.5.4, for each $\mathsf{p} \in Q_2$,

$$\begin{aligned} \inf_{\mathsf{y}\in Q_\infty} \left(\Psi(\mathsf{y}) - \int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u \right) &= -\sup_{\mathsf{y}\in Q_\infty} \left(\int_0^1 \mathsf{p}(u)\mathsf{y}(u)\,\mathrm{d}u - \Psi(\mathsf{y}) \right) \\ &= -\Psi_*(\mathsf{p}) \\ &= \begin{cases} -\infty, & \text{if } \|\mathsf{p}\|_1 \ge 1 \text{ or there exists } u \in [0,1) \text{ with } \mathsf{p}(u) > 1, \\ -\frac{\ln 2}{1-\|\mathsf{p}\|_1}, & \text{otherwise, i.e. } \|\mathsf{p}\|_\infty \le 1 \text{ and } \|\mathsf{p}\|_1 < 1. \end{cases} \end{aligned} \tag{3.6.43}$$

Inserting (3.6.43) into (3.6.42) gives

$$\begin{aligned} f(t,\mathsf{q}) &= \sup_{\substack{\mathsf{p}\in Q_\infty,\\ \|\mathsf{p}\|_\infty\le 1,\,\|\mathsf{p}\|_1<1}} \left(t\int_0^1 A_{\mathrm{reg.}}(\mathsf{p}(u))\,\mathrm{d}u + \int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u - \frac{\ln 2}{1-\|\mathsf{p}\|_1} \right) \\ &= \sup_{\substack{\mathsf{p}\in Q_\infty,\\ \|\mathsf{p}\|_\infty\le 1,\,\|\mathsf{p}\|_1<1}} \left(t\int_0^1 A(\mathsf{p}(u))\,\mathrm{d}u + \int_0^1 \mathsf{p}(u)\mathsf{q}(u)\,\mathrm{d}u - \frac{\ln 2}{1-\|\mathsf{p}\|_1} \right), \end{aligned} \tag{3.6.44}$$

using in the last step that any regularisation $A_{\mathrm{reg.}}$ of A coincides with A on $[0,1]$. □

Finally, we prove the central result of this section.

Proof of Theorem 3.1.2. Recall that $\mathsf{q}_0 \equiv 0 \in \mathcal{Q}_{\leq}^{(0)}$. Since A is a convex speed function, we have $A(x) \leq x$ for all $x \in [0,1]$. If $t = 0$, then by Theorem 3.1.1 and (3.1.4),

$$f(0, \mathsf{q}_0) = \Psi(q_0) = -\ln 2 = \lim_{N\uparrow\infty} F_N^A(0). \tag{3.6.45}$$

Let $t > 0$. By Proposition 3.6.12,

$$\begin{aligned} f(t, \mathsf{q}_0) &= \sup_{\substack{\mathsf{p}\in\mathcal{Q}_\infty, \\ \|\mathsf{p}\|_\infty \leq 1, \|\mathsf{p}\|_1 < 1}} \left(t \int_0^1 A(\mathsf{p}(u))\, \mathrm{d}u - \frac{\ln 2}{1 - \|\mathsf{p}\|_1} \right) \\ &= \sup_{\lambda\in[0,1)} \sup_{\substack{\mathsf{p}\in\mathcal{Q}_\infty, \\ \|\mathsf{p}\|_\infty \leq 1, \|\mathsf{p}\|_1 = \lambda}} \left(t \int_0^1 A(\mathsf{p}(u))\, \mathrm{d}u - \frac{\ln 2}{1-\lambda} \right), \end{aligned} \tag{3.6.46}$$

where we used in the last step that $\|\cdot\|_1 \leq \|\cdot\|_\infty$ on $L_\infty([0,1), \mathbb{R})$. Let $\lambda \in [0,1)$. We set $\mathsf{p}^{(\lambda)} = \mathbb{1}_{[1-\lambda,1)}$. In the case $\lambda = 0$, this is to be understood as $\mathsf{p}^{(0)} \equiv 0$. Since $\mathsf{p}^{(\lambda)} \in \mathcal{Q}_2$ with $\|\mathsf{p}\|_\infty = 1$ and $\|\mathsf{p}\|_1 = \lambda$, we get

$$\sup_{\substack{\mathsf{p}\in\mathcal{Q}_\infty, \\ \|\mathsf{p}\|_\infty \leq 1, \|\mathsf{p}\|_1 = \lambda}} \left(t \int_0^1 A(\mathsf{p}(u))\, \mathrm{d}u \right) \geq t \int_0^1 A(\mathsf{p}^{(\lambda)}(u))\, \mathrm{d}u = \lambda t, \tag{3.6.47}$$

using that $A(0) = 0$ and $A(1) = 1$. Since $A(x) \leq x$ for all $x \in [0,1]$,

$$\int_0^1 A(\mathsf{p}(u))\, \mathrm{d}u \leq \int_0^1 \mathsf{p}(u)\, \mathrm{d}u = \|\mathsf{p}\|_1, \tag{3.6.48}$$

so

$$\sup_{\substack{\mathsf{p}\in\mathcal{Q}_\infty, \\ \|\mathsf{p}\|_\infty \leq 1, \|\mathsf{p}\|_1 = \lambda}} \left(t \int_0^1 A(\mathsf{p}(u))\, \mathrm{d}u \right) \leq \sup_{\substack{\mathsf{p}\in\mathcal{Q}_\infty, \\ \|\mathsf{p}\|_\infty \leq 1, \|\mathsf{p}\|_1 = \lambda}} t\, \|\mathsf{p}\|_1 = \lambda t. \tag{3.6.49}$$

Combining (3.6.47) and (3.6.49), we get

$$\sup_{\substack{\mathsf{p}\in\mathcal{Q}_\infty, \\ \|\mathsf{p}\|_\infty \leq 1, \|\mathsf{p}\|_1 = \lambda}} \left(t \int_0^1 A(\mathsf{p}(u))\, \mathrm{d}u \right) = \lambda t. \tag{3.6.50}$$

Inserting this into (3.6.46) gives

$$f(t, \mathsf{q}_0) = \sup_{\lambda\in[0,1)} h_t(\lambda), \tag{3.6.51}$$

where

$$\begin{aligned} h_t \colon \mathbb{R} \setminus \{1\} &\to \mathbb{R}, \\ \lambda &\mapsto \lambda t - \frac{\ln 2}{1-\lambda}. \end{aligned} \tag{3.6.52}$$

We compute the first and second derivative of h_t:

$$h_t'(\lambda) = t - \frac{\ln 2}{(1-\lambda)^2},$$
$$h_t''(\lambda) = -\frac{2\ln 2}{(1-\lambda)^3}. \tag{3.6.53}$$

Note that h_t'' is negative on $[0,1)$, so any critical points of h_t in $[0,1)$ will be local maxima. One computes that the only roots of h_t' are in $1+\sqrt{\frac{\ln 2}{t}}$, which is not in $[0,1)$, and $\lambda_*(t) = 1-\sqrt{\frac{\ln 2}{t}}$. If $t \geq \ln 2$, then $\lambda_*(t) \in [0,1)$, so

$$f(t,\mathsf{q}_0) = \sup_{\lambda\in[0,1)} h_t(\lambda) = h_t(\lambda_*(t)) = t\left(1-\sqrt{\frac{\ln 2}{t}}\right) - \sqrt{t\ln 2} = t - 2\sqrt{t\ln 2}. \tag{3.6.54}$$

If $t \in (0,\ln 2)$, then one sees that h_t' is negative between its roots and in particular negative on $[0,1)$. Thus, h_t assumes its maximum on $[0,1)$ in 0, which implies

$$f(t,\mathsf{q}_0) = \sup_{\lambda\in[0,1)} h_t(\lambda) = h_t(0) = -\ln 2. \tag{3.6.55}$$

We have obtained in (3.6.54) and (3.6.55) that

$$f(t,\mathsf{q}_0) = \begin{cases} -\ln 2, & \text{if } t \leq \ln 2, \\ t - 2\sqrt{t\ln 2}, & \text{if } t > \ln 2, \end{cases} \tag{3.6.56}$$

which coincides with $\lim_{N\uparrow\infty} F_N^A(t)$ if $A(x) \leq x$, see (3.1.4). □

3.7 Outlook: Towards a generalisation of Theorem 3.1.2

In this section, we discuss a generalisation of Theorem 3.1.2 in the form of the following conjecture.

Conjecture 3.7.1. *Let* $A\colon [0,1] \to [0,1]$ *be a convex Lipschitz continuous speed function. Let* F_N *be the enriched free energy of the CREM with speed function* A*, see* (3.1.3)*. Let* Ψ *be as in Theorem 3.1.1. Then, the unique viscosity solution* f *of* $(HJE[\mathsf{q}])$ *with* $f(0,\cdot) = \Psi$ *from Proposition 3.6.12 satisfies*

$$f(t,\mathsf{q}) = \lim_{N\uparrow\infty} F_N(t,\mathsf{q}), \tag{3.7.1}$$

for all $t \geq 0$ *and* $\mathsf{q} \in Q_2$*.*

In this thesis, we prove one direction of Conjecture 3.7.1 in the form of the following theorem.

Theorem 3.7.2. *In the setting of Conjecture 3.7.1, it holds*

$$f(t,\mathsf{q}) \leq \liminf_{N\uparrow\infty} F_N(t,\mathsf{q}), \tag{3.7.2}$$

for all $t \geq 0$ *and* $\mathsf{q} \in Q_2$*.*

The proof of Theorem 3.7.2 relies on Theorem 3.7.3, whose setting we introduce in the following: For each $N \in \mathbb{N}$, let $\mathcal{H}_N$ be a finite-dimensional Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}_N}$ and let P_N be a probability measure on $\mathcal{H}_N$ whose support is contained in $\bar{B}_{\sqrt{N}}(0)$, the closed ball around $0 \in \mathcal{H}_N$ with radius $\sqrt{N}$. We also call P_N the *reference measure*. Let $A\colon [0,1] \to \mathbb{R}$ be locally Lipschitz. Assume that for each $N \in \mathbb{N}$, there exists a centred Gaussian process $\left(\hat{H}_N^A(\tau)\right)_{\tau\in\operatorname{supp} P_N}$ with covariance

$$\mathbb{E}\left[\hat{H}_N^A(\tau)\hat{H}_N^A(\tilde{\tau})\right] = NA\left(\frac{\langle\tau,\tilde{\tau}\rangle_{\mathcal{H}_N}}{N}\right), \qquad \forall\, \tau,\tilde{\tau} \in \operatorname{supp} P_N. \tag{3.7.3}$$

For each $N \in \mathbb{N}$, let $e_1,\ldots,e_{\dim\mathcal{H}_N}$ be an orthonormal basis of $\mathcal{H}_N$. We define an isometry between $\mathbb{R}^{\dim\mathcal{H}_N}$ and $\mathcal{H}_N$ by

$$\begin{aligned} \varphi_N\colon \mathbb{R}^{\dim\mathcal{H}_N} &\to \mathcal{H}_N, \\ x = (x_1,\ldots,x_{\dim\mathcal{H}_N}) &\mapsto \sum_{i=1}^{\dim\mathcal{H}_N} x_i e_i. \end{aligned} \tag{3.7.4}$$

For $M, N \in \mathbb{N}$, let $(z_\beta)_{\beta\in\bigcup_{i=0}^M \mathbb{N}^M}$ be a family of i.i.d. standard Gaussian vectors on $\mathbb{R}^{\dim \mathcal{H}_N}$. We set

$$(\tilde{z}_\beta)_{\beta\in\bigcup_{i=0}^M \mathbb{N}^M} = \left(\varphi_N(z_\beta)\right)_{\beta\in\bigcup_{i=0}^M \mathbb{N}^M}. \tag{3.7.5}$$

For $t \geq 0$, $\mathsf{q} = \sum_{k=0}^M q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})} \in Q^{(M)}$ with

$$\begin{aligned} &0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{M-1} < \zeta_M < \zeta_{M+1} = 1, \\ &0 = q_{-1} < q_0 < \cdots < q_{M-1} < q_M < \infty, \end{aligned} \tag{3.7.6}$$

and $(v_\alpha)_{\alpha\in\mathbb{N}^M}$ the Ruelle cascades with parameters $(\zeta_k)_{k=0,\dots,M+1}$, we define the enriched free energy of $\left(\hat{H}_N^A(\tau)\right)_{\tau\in\operatorname{supp} P_N}$ by

$$\begin{aligned} &\hat{F}_N(t,\mathsf{q}) \\ &:= -\frac{1}{N}\mathbb{E}\left[\ln\left(\sum_{\alpha\in\mathbb{N}^M} v_\alpha \int \exp\left(\sqrt{2t}\,\hat{H}_N^A(\tau) - NtA\left(\frac{\langle\tau,\tau\rangle_{\mathcal{H}_N}}{N}\right) + \sqrt{2}\left\langle y_{\mathsf{q}}(\alpha),\,\tau\right\rangle_{\mathcal{H}_N} - q_M\langle\tau,\,\tau\rangle_{\mathcal{H}_N}\right)\mathrm{d}P_N(\tau)\right)\right], \end{aligned} \tag{3.7.7}$$

where

$$y_{\mathsf{q}}(\alpha) := \sum_{j=0}^M \left(q_j - q_{j-1}\right)^{1/2}\tilde{z}_{\alpha|_j}, \qquad \forall\,\alpha\in\mathbb{N}^M. \tag{3.7.8}$$

We assume that $(v_\alpha)_{\alpha\in\mathbb{N}^M}$, $\left(\hat{H}_N^A(\tau)\right)_{\tau\in\operatorname{supp} P_N}$ and $\left(y_{\mathsf{q}}(\alpha)\right)_{\alpha\in\mathbb{N}^M}$ are independent for each M, N, q.

One shows as in Proposition 3.3.1 that for each $t \geq 0$, $\hat{F}_N(t,\cdot)$ is Lipschitz continuous w.r.t. $\|\cdot\|_1$. This and Lemma 3.3.2 imply that $\hat{F}_N$ can be continuously extended from $\mathbb{R}_{\geq 0}\times Q^{(M)}$ to $\mathbb{R}_{\geq 0}\times Q_1$.

Theorem 3.7.3 is a reformulation of [47, Theorem 4.13], which in turn is a version of [95, Theorem 3.4], where the definition of a *viscosity (sub-)solution* does not contain any boundary conditions. We refer to [95, Definition 4.1] for further details on these boundary conditions.

Theorem 3.7.3 [47, Theorem 4.13]. *In the setting described in this section, assume that the speed function $A\colon [0,1]\to[0,1]$ is Lipschitz continuous and proper. Also assume that $\hat{F}_N(0,\cdot)\colon Q_2\to\mathbb{R}$ converges pointwise, as $N\uparrow\infty$, to a function $G\colon Q_2\to\mathbb{R}$, which is Q_2^*-increasing with*

$$|G(\mathsf{q}) - G(\tilde{\mathsf{q}})| \leq \|\mathsf{q}-\tilde{\mathsf{q}}\|_1, \tag{3.7.9}$$

for all $\mathsf{q},\tilde{\mathsf{q}}\in Q_2$.

Then, for each $t \geq 0$ and $\mathsf{q} \in Q_2$,

$$\liminf_{N\uparrow\infty} \hat{F}_N(t,\mathsf{q}) \geq \hat{f}(t,\mathsf{q}), \tag{3.7.10}$$

where $\hat{f}$ is the unique viscosity solution of $(HJE[\mathsf{q}])$ with $\hat{f}(0,\cdot) = G$ from Theorem 3.6.10.

Now, we prove the central result of this section.

Proof of Theorem 3.7.2. We first prove that the enriched free energy F_N in (3.1.14) can be written in the form (3.7.7). The following steps are all done for each $N \in \mathbb{N}$. We set $\mathcal{H}_N = \mathbb{R}^{2^{N+1}}$ and denote by $\cdot$ the standard scalar product on $\mathbb{R}^{2^{N+1}}$. Note that $\left|\bigcup_{i=1}^N \{-1,1\}^i\right| = \sum_{i=1}^N 2^i = 2^{N+1}-2$. We identify the first $2^{N+1}-2$ unit vectors $e_1,\dots,e_{2^{n+1}-2} \in \mathbb{R}^{2^{N+1}}$ by $(e_z)_{z\in\bigcup_{i=1}^N\{-1,1\}^i}$ in an arbitrary but fixed order. The map

$$\begin{aligned} \iota_N\colon\ \{-1,1\}^N &\to \mathbb{R}^{2^{N+1}}, \\ \sigma &\mapsto \sum_{i=1}^N e_{\sigma|_i}, \end{aligned} \tag{3.7.11}$$

defines an embedding of $\{-1,1\}^N$ in $\mathbb{R}^{2^{N+1}}$ so that for $\sigma,\tilde{\sigma} \in \{-1,1\}^N$,

$$\iota_N(\sigma)\cdot\iota_N(\tilde{\sigma}) = \sum_{i,j=1}^N e_{\sigma|_i}\cdot e_{\tilde{\sigma}|_j} = \sum_{i,j=1}^N \mathbb{1}_{\sigma|_i=\tilde{\sigma}|_j}(\sigma,\tilde{\sigma}) = \sum_{i=1}^N \mathbb{1}_{\sigma|_i=\tilde{\sigma}|_i}(\sigma,\tilde{\sigma}) = \sigma\wedge\tilde{\sigma}. \tag{3.7.12}$$

We write $\mathcal{S}_N := \iota_N\left(\{-1,1\}^N\right)$ and set P_N to be the uniform distribution on $\mathcal{S}_N$. Then, $\operatorname{supp} P_N = \mathcal{S}_N \subset \bar{B}_{\sqrt{N}}(0)$, where the latter denotes the closed ball around $0 \in \mathbb{R}^{2^{N+1}}$ with radius $\sqrt{N}$. Recall that the CREM $\left(H_N^A(\sigma)\right)_{\sigma\in\{-1,1\}^N}$ with speed function A is a centred Gaussian process with covariance

$$\mathbb{E}\left[H_N^A(\sigma)H_N^A(\tilde{\sigma})\right] = NA\left(\tfrac{\sigma\wedge\tilde{\sigma}}{N}\right), \qquad \forall\, \sigma,\tilde{\sigma} \in \{-1,1\}^N. \tag{3.7.13}$$

Note that $\iota_N\colon\ \{-1,1\}^N \to \mathcal{S}_N$ is a bijection. This enables us to set

$$(\hat{H}_N^A(\tau))_{\tau\in\mathcal{S}_N} := \left(H_N^A(\iota_N^{-1}(\tau))\right)_{\tau\in\mathcal{S}_N}. \tag{3.7.14}$$

Then, $\left(\hat{H}_N^A(\tau)\right)_{\tau\in\mathcal{S}_N}$ is a centred Gaussian process with covariances as in (3.7.3), i.e.

$$\mathbb{E}\left[\hat{H}_N^A(\tau)\hat{H}_N^A(\tilde{\tau})\right] = \mathbb{E}\left[H_N^A(\iota_N^{-1}(\tau))H_N^A(\iota_N^{-1}(\tilde{\tau}))\right] = NA\left(\tfrac{\iota_N^{-1}(\tau)\wedge\iota_N^{-1}(\tilde{\tau})}{N}\right) = NA\left(\tfrac{\tau\cdot\tilde{\tau}}{N}\right), \tag{3.7.15}$$

for all $\tau,\tilde{\tau} \in \mathcal{S}_N$, where we used (3.7.12) in the last step.

For $M \in \mathbb{N}$ and $\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})} \in Q^{(M)}$,we set

$$y_{\mathsf{q}}(\alpha) \coloneqq \sum_{j=0}^{M} \left(q_j - q_{j-1}\right)^{1/2} z_{\alpha|_j}, \qquad \forall\, \alpha \in \mathbb{N}^M, \tag{3.7.16}$$

where each $z_{\alpha|_j} = (z_{\alpha|_j,1}, \ldots, z_{\alpha|_j,2^{N+1}})$ is from a family of i.i.d. standard Gaussian vectors on $\mathbb{R}^{2^{N+1}}$. This means that $(y_{\mathsf{q}}(\alpha) \cdot \tau)_{\alpha\in\mathbb{N}^M,\, \tau\in\mathcal{S}_N}$ is a centred Gaussian process with covariances

$$\begin{aligned}
\mathbb{E}\left[\left(y_{\mathsf{q}}(\alpha)\cdot\tau\right)\left(y_{\mathsf{q}}(\tilde\alpha)\cdot\tilde\tau\right)\right] &= \sum_{j_1,j_2=0}^{M} \left(q_{j_1} - q_{j_1-1}\right)^{1/2}\left(q_{j_2} - q_{j_2-1}\right)^{1/2} \mathbb{E}\left[\left(z_{\alpha|_{j_1}}\cdot\tau\right)\left(z_{\alpha|_{j_2}}\cdot\tilde\tau\right)\right] \\
&= \sum_{j_1,j_2=0}^{M} \left(q_{j_1} - q_{j_1-1}\right)^{1/2}\left(q_{j_2} - q_{j_2-1}\right)^{1/2} \sum_{i_1,i_2=1}^{2^{N+1}} \tau_{i_1}\tilde\tau_{i_2}\mathbb{E}\left[z_{\alpha|_{j_1},i_1} z_{\tilde\alpha|_{j_2},i_2}\right] \\
&= \sum_{j_1,j_2=0}^{M} \left(q_{j_1} - q_{j_1-1}\right)^{1/2}\left(q_{j_2} - q_{j_2-1}\right)^{1/2} \sum_{i_1,i_2=1}^{2^{N+1}} \tau_{i_1}\tilde\tau_{i_2} \mathbb{1}_{\alpha|_{j_1}=\tilde\alpha|_{j_2}}(\alpha,\tilde\alpha)\, \mathbb{1}_{i_1=i_2}(i_1,i_2) \\
&= \sum_{j=0}^{M} \left(q_j - q_{j-1}\right) \mathbb{1}_{\alpha|_j=\tilde\alpha|_j}(\alpha,\tilde\alpha) \sum_{i=1}^{2^{N+1}} \tau_i\tilde\tau_i \\
&= q_{\alpha\wedge\tilde\alpha}\, \tau\cdot\tilde\tau,
\end{aligned} \tag{3.7.17}$$

for each $\alpha, \tilde\alpha \in \mathbb{N}^M$ and each $\tau, \tilde\tau \in \mathcal{S}_N$. In particular, by (3.7.12) and (3.7.17), recalling the definition of Y_{q} in (3.1.12), we get

$$(y_{\mathsf{q}}(\alpha)\cdot\tau)_{\alpha\in\mathbb{N}^M,\,\tau\in\mathcal{S}_N} \overset{\mathrm{d}}{=} \left(Y_{\mathsf{q}}(\iota_N^{-1}(\tau),\alpha)\right)_{\alpha\in\mathbb{N}^M,\,\tau\in\mathcal{S}_N}, \tag{3.7.18}$$

for each $M \in \mathbb{N}$ and each $\mathsf{q} \in Q^{(M)}$. As in (3.7.7), for $M \in \mathbb{N}$ and $\mathsf{q} = \sum_{k=0}^{M} q_k \mathbb{1}_{[\zeta_k,\zeta_{k+1})} \in Q^{(M)}$, we set

$$\hat F_N(t,\mathsf{q}) \coloneqq -\frac{1}{N}\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \int \exp\left(\sqrt{2t}\, \hat H_N^A(\tau) - NtA\left(\frac{\tau\cdot\tau}{N}\right) + \sqrt{2}\, y_{\mathsf{q}}(\alpha)\cdot\tau - q_M\, \tau\cdot\tau\right) \mathrm{d}P_N(\sigma)\right)\right]. \tag{3.7.19}$$

Then,

$$\begin{aligned}
\hat F_N(t,\mathsf{q}) &= -\frac{1}{N}\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \frac{1}{2^N}\sum_{\tau\in S_N} \exp\left(\sqrt{2t}\, \hat H_N^A(\tau) - Nt + \sqrt{2}\, y_{\mathsf{q}}(\alpha)\cdot\tau - Nq_M\right)\right)\right] \\
&= \ln 2 - \frac{1}{N}\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\, \hat H_N^A(\iota_N(\sigma)) - Nt + \sqrt{2}\, y_{\mathsf{q}}(\alpha)\cdot\iota_N(\sigma) - Nq_M\right)\right)\right] \\
&= \ln 2 - \frac{1}{N}\mathbb{E}\left[\ln\left(\textstyle\sum_{\alpha\in\mathbb{N}^M} v_\alpha \sum_{\sigma\in\{-1,1\}^N} \exp\left(\sqrt{2t}\, H_N^A(\sigma) - Nt + \sqrt{2}\, Y_{\mathsf{q}}(\sigma,\alpha) - Nq_M\right)\right)\right] \\
&= F_N(t,\mathsf{q}) + \ln 2,
\end{aligned} \tag{3.7.20}$$

with F_N as in (3.1.14), using (3.7.12), (3.7.14) and (3.7.18) in the second last step. In particular, by Theorem 3.1.1,

$$\lim_{N\uparrow\infty} \hat F_N(0,\mathsf{q}) = \lim_{N\uparrow\infty} F_N(0,\mathsf{q}) + \ln 2 = \Psi(\mathsf{q}) + \ln 2, \tag{3.7.21}$$

for each $\mathsf{q} \in Q_2$. Theorem 3.7.3 implies that

$$\liminf_{N\uparrow\infty} \hat{F}_N(t, \mathsf{q}) \geq \hat{f}(t, \mathsf{q}), \tag{3.7.22}$$

for all $t \geq 0$, $\mathsf{q} \in Q_2$, where $\hat{f}$ is the unique viscosity solution of $(HJE[\mathsf{q}])$ with $\hat{f}(0, \cdot) = \Psi + \ln 2$. Then, the Hopf formula (3.6.32) implies that

$$\hat{f}(t, \mathsf{q}) = f(t, \mathsf{q}) + \ln 2, \tag{3.7.23}$$

for all $t \geq 0$, $\mathsf{q} \in Q_2$, where f is the unique viscosity solution of $(HJE[\mathsf{q}])$ with $f(0, \cdot) = \Psi$ from Proposition 3.6.12. Thus, by (3.7.20), (3.7.22) and (3.7.23), we conclude that for all $t \geq 0$, $\mathsf{q} \in Q_2$,

$$\liminf_{N\uparrow\infty} F_N(t, \mathsf{q}) \geq f(t, \mathsf{q}), \tag{3.7.24}$$

which completes the proof. □

Chen and Mourrat proved in [43, Theorem 1.1] that equality as in (3.7.1) holds in the setting of the Hilbert space $\mathbb{R}^{D\times N}$ with $D, N \in \mathbb{N}$, where the reference measure P_N is a product measure, i.e. $P_N = P_1^{\otimes N}$, where P_1 is a probability measure on $\mathbb{R}^D$. We explain why this is not the case for the embedding of the enriched CREM into $\mathbb{R}^{2^{N+1}}$ as in the proof of Theorem 3.7.2: There, the reference measure P_N is the uniform distribution on the embedding of $\{-1, 1\}^N$, so it is a discrete probability measure with 2^N point masses. To obtain a contradiction, assume that $P_N = P_1^{\otimes 2^{N+1}}$, where P_1 is a probability measure on $\mathbb{R}$. Since P_N is discrete, P_1 must also be a discrete measure. Let $\ell \in \mathbb{N}$ be the number of point masses of P_1. Then, P_N has $\ell^{2^{N+1}}$ point masses, which contradicts the fact that P_N has 2^N point masses. Thus, the embedding of the enriched CREM into $\mathbb{R}^{2^{N+1}}$ as in the proof of Theorem 3.7.2 does not fit the setting of [43].

Because of the similarity of the settings of this thesis and of Chen's and Mourrat's work [43], one might expect that Conjecture 3.7.1 is true. Nevertheless, we cannot directly adapt Chen's and Mourrat's proof methods since their approach fundamentally depends on P_N having a product measure structure. The resolution of this technical barrier and the proof of the matching bound in Conjecture 3.7.1 will be explored in future work.

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