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ORIGINAL PAPER

A Two-Step Stochastic Model of Anaerobic Digestion

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Abstract
This paper deals with a stochastic model of the so-called Anaerobic Model number 2. Here the maximal growth rate of the acidogenic microorganisms is affected by a white noise in environment. As far as the mathematical analysis of this model is concerned, we first use the $Itô$ formula to investigate the existence and uniqueness of the global positive solution. Then, we explore and derive the conditions of persistence and extinction of the microorganisms. Finally, by performing some numerical simulations, we illustrate the theoretical results obtained.

Keywords Stochastic model · Anaerobic digestion · White noise · Extinction · Persistence · $Itô$ formula

Introduction

Anaerobic digestion is a biochemical process where microorganisms transform organic material in absence of oxygen into biogas and other byproducts such as bio-fertilizers. From microbiological and technological points of view, anaerobic digestion is generally considered as a four steps process which are hydrolysis, acidogenesis, acetogenesis and methanogenesis. In the first stage, complex organic molecules are broken down into a simple substrate. During acidogenesis, acidogenic bacteria convert the simple substrate into alcohols and volatile fatty acids; then volatile fatty acids and alcohols are used up by the acetogenic bacteria and they are converted into acetic acid plus carbon dioxide and hydrogen. In the last stage, methanogenic bacteria convert acetic acids into methane and carbon dioxide. Several works on the analysis of this process have been done and their founda-

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tion is the Anaerobic Digestion Model No.1 (ADM1) of the IWA Working Group for the Mathematical Modeling of Anaerobic Digestion Processes. It was introduced in 2002 and it is the most complete model [21]. However, it is too complex to allow a mathematical analysis of its nonlinear dynamics and only numerical research is available. In an approach to formally analyze and study this model, several sub-models were proposed and analyzed. More realistic models of anaerobic digestion are two-step models. An important contribution on the modelling of anaerobic digestion as a two-step process is the model presented in [6], hereafter denoted as AM2 model, and studied in [5, 6]. It has been shown that under certain circumstances, this very simple two-step model is able to adequately capture the main dynamical behavior of the full anaerobic digestion model ADM1 [8, 20]. Moreover, it has been shown that the simple AM2 model can support online control, optimization and supervision strategies, through the synthesis of state observers and control feedback laws [2]. It is to be noted that the AM2 represents well the process at the macroscopic level where the bacteria populations are large and the substrate concentration is relatively high. It is therefore efficient for some applications; it may give good results in suitable environmental conditions and a perfectly stirred and homogeneous bioreactor. In this case the stochastic effects can be neglected according to the law of large numbers. However, it is not the case in real conditions since some perturbations can appear due to indefinite operating conditions and to small population sizes, especially in the case of multi-species such as in an anaerobic digestion system. The AM2 model generally assumes deterministic parameters. However, real-world bioreactors experience various sources of operational and environmental disturbances, such as temperature fluctuations, substrate variability, and hydraulic shocks. These perturbations motivate the incorporation of stochastic elements into the model to better reflect system uncertainties. Earlier works, such as the one by Abdelkader and collaborators [4], proposed stochastic versions of AM2 using approaches like Taylor series expansions and discrete Poisson or Normal approximations. However, they often lack a clear biological justification for the choice of stochastic variables or comprehensive dynamic analyses. In this work, we focus on the stochastic modeling of AM2 by introducing white noise perturbations into the maximal growth rate of acidogenic microorganisms. This choice is biologically motivated by the high sensitivity of acidogenic growth rates to environmental fluctuations, which critically influence the overall system dynamics. We rigorously analyze the stochastic model, proving the existence of a unique global positive solution and deriving persistence and extinction conditions. Our study advances the field by clearly linking the stochastic variable selection to biological and operational realities and by providing new insights into the effects of noise on anaerobic digestion stability and performance. We also outline directions for future research, including extension to other parameters and noise types such as Lévy noise, to further enhance the model's realism and applicability.

The remainder of the paper is outlined as follows. We begin by constructing, in section 2, our stochastic model. In Sect. 3, after establishing the basic properties of our model, we establish some sufficient conditions for the extinction and persistence in the mean of the microorganisms in the bioreactor. The simulation results as well as a comparison with the deterministic model are given in section 4. We conclude the work in section 5.

Formulation of the Model

The Deterministic Anaerobic Model Number 2 (AM2)

Among the different steps of the anaerobic digestion process aforementioned, the AM2 model is based on the following two biological reactions:

1-Acidogenesis: q_1

$$k_1 S_1 \longrightarrow k_2 S_2 + X_1 + CO_2;$$

2-Methanogenesis: q_2

$$k_3 S_2 \longrightarrow CH_4 + X_2 + CO_2;$$

where $q_1 = \mu_1(S_1)X_1$ and $q_2 = \mu_2(S_2)X_2$ denote respectively the velocities of the reactions of acidogenesis and methanogenesis and μ_i, $i = 1, 2$, the specific growth rates of X_i on S_i, S_1 represents the concentration of simple substrate (organic material), S_2 represents the concentration of volatile fatty acids, X_1 the concentration of acidogenic bacteria, X_2 the concentration of methanogenic bacteria and k_i, $i = 1, 2, 3$, are stoichiometric coefficients. The model equations are obtained by the mass balance principle given as follows:

$$d\varepsilon/dt = \underbrace{\varepsilon\ incoming - \varepsilon\ outgoing}_{Physical\ part} + \underbrace{\varepsilon\ produced - \varepsilon\ consumed/dead}_{biological\ part}$$

where ε denote either S_1, X_1, S_2 or X_2 at time t.

Based on the previous reactions and the mass balance principle, the AM2 model was described by Bernard et al. [7] by the following ordinary differential equations system

$$\begin{cases} \dot{S_1} = D(S_{1in} - S_1) - k_1\mu_1(S_1)X_1, \\ \dot{X_1} = (\mu_1(S_1) - \alpha D)X_1, \\ \dot{S_2} = D(S_{2in} - S_2) + k_2\mu_1(S_1)X_1 - k_3\mu_2(S_2)X_2, \\ \dot{X_2} = (\mu_2(S_2) - \alpha D)X_2, \end{cases} \tag{1}$$

where each function of time denotes the concentration of the respective species or substrates as already defined; S_{1in} and S_{2in} are the input concentrations of the substrates S_1 and S_2, respectively; D is the dilution rate and $\alpha \in [0, 1]$ is the part of X_1 and X_2 that has left the bioreactor. We consider as in [5] that $\mu_1(S_1)$ is the Monod type growth function and $\mu_2(S_2)$ is the Haldane type growth function given by:

$$\mu_1(S_1) = m_1 S_1/(a_1 + S_1) \quad and \quad \mu_2(S_2) = m_2 S_2/(a_2 + S_2 + KS_2^2).$$

The dynamics of (1) is completely determined by the break even λ_1, λ_2^1 and λ_2^1, where $m_1\lambda_1/(a_1 + \lambda_1) = \alpha D$ and $\lambda_2^1 < \lambda_2^2$ are solutions of the quadratic equation $\alpha DKS_2^2 + (\alpha D - m_2)S_2 + \alpha D a_2 = 0$. This system admits six possible equilibria,

(i) $E_0\ (S_{1in}, 0, S_{2in}, 0)$

(ii) $E_1^1(S_{1in}, 0, \lambda_2^1, X_2^{1*})$
(iii) $E_2^1(S_{1in}, 0, \lambda_2^2, X_2^{2*})$
(iv) $E_1^2(\lambda_1, X_1^*, \lambda_2^1, X_2^{1*})$
(v) $E_2^2(\lambda_1, X_1^*, \lambda_2^2, X_2^{2*})$
(vi) $E_2^0(\lambda_1, X_1^*, S_{2in}^*, 0)$;

where $X_1^* = \frac{(S_{1in}-\lambda_1)}{\alpha k_1}$, $S_{2in}^* = S_{2in} + \frac{k_2}{k_1}(S_{1in} - \lambda_1)$ and $X_2^{i*} = (\frac{S_{2in}^* - \lambda_2^i)}{\alpha k_3}$, i=1,2.

These equilibria are only biologically meaningful if each of the components is nonnegative. E_2^1 and E_2^2, when they exist, are called interior equilibria. For more details, please refer to [5].

Our Model Variables

We are interested only in the interactions between the substrates and the microorganisms, so our model variables are

- $S_1(t)$: Concentration of simple substrate at unit time t;
- $X_1(t)$: Concentration of acidogenic microorganisms at unit time t;
- $S_2(t)$: Concentration of volatiles fatty acids at unit time t;
- $X_2(t)$: Concentration of methanogenic microorganisms at unit time t.

Main Assumptions

In order to study this model in a realistic context, we make the following assumptions:

- The chemostat is considered to operate in continuous mode, this means that the flow rate inlet is equal to the outlet flow;
- Recruitment is made in the concentration of simple substrate S_1 and in the concentration of volatile fatty acids S_2;
- There is randomness involved in the maximal growth rate of the acidogenic microorganisms.

To better understand the model formulation and the construction of the mathematical equations, it is necessary to identify and give biological meanings to the specific model parameters.

Parameters and Derivation of Equations of Our Stochastic Model

Our model considers two stages of anaerobic digestion, acidogenesis and methanogenesis. In the first stage a recruitment is made within the concentration of simple substrates S_1 at a rate D resulting in a simple substrate at the chemostat inlet S_{1in}. Simple substrates are broken down by acidogenic microorganisms X_1. The microorganisms use the energy from the simple substrates S_1 to grow, and produce volatile fatty acids S_2 as byproducts. During the process, their growth is influenced by temperature shifts, substrate variability and pH disturbances, hence their maximal growth rate changes to a random variable $\widetilde{m}_1$. We choose to incorporate a white noise to model this effect, the expression of $\widetilde{m}_1$ is therefore

$\widetilde{m}_1 = m_1 + \sigma dB(t)$, where $B(t)$ is a white noise with intensity σ. In the last stage, methanogenic microorganisms X_2 convert the volatile fatty acids S_2 to biogas.

This yields the following stochastic anaerobic digestion model:

$$\begin{cases} dS_1 = \left[D(S_{1in} - S_1) - k_1 \frac{m_1 S_1 X_1}{a_1 + S_1}\right] dt - k_1 \frac{\sigma S_1 X_1}{a_1 + S_1} dB(t), \\ dX_1 = \left(\frac{m_1 S_1}{a_1 + S_1} - \alpha D\right) X_1 dt + \frac{\sigma S_1 X_1}{a_1 + S_1} dB(t), \\ dS_2 = \left[D(S_{2in} - S_2) + k_2 \frac{m_1 S_1 X_1}{a_1 + S_1} - k_3 \frac{m_2 S_2}{a_2 + S_2 + K S_2^2}\right] dt + k_2 \frac{\sigma S_1 X_1}{a_1 + S_1} dB(t), \\ dX_2 = \left(\frac{m_2 S_2}{a_2 + S_2 + K S_2^2} - \alpha D\right) X_2 dt. \end{cases} \tag{2}$$

Notations: Table 1 below describes all the parameters used in the model.

Analysis of the Model

Preliminary Knowledge

In this paper, we denote $\mathbb{R}^4_+ = \{(x, y, z, k) \in \mathbb{R}^4 : x > 0,\ y > 0,\ z > 0,\ and\ k > 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

Definition 1 (i) The population $X(t)$ is said to be strongly persistent in the mean if $\lim_{t\to\infty} inf \frac{1}{t} \int_0^t X(s) ds > 0$;

(ii) The microorganism $X(t)$ is said to be extinct if $\lim_{t\to\infty} \sup \frac{1}{t} \int_0^t X(s) < 0$ almost surely;

(iii) The microorganism $X(t)$ is said to be exponentially extinct if there exists a negative constant C such that $\lim_{t\to\infty} \sup \frac{lnX(t)}{t} < C$ almost surely.

It is worth noting that exponential extinction implies extinction.

Table 1 Parameters and their description

Parameter	Description
S_{1in}	Input concentration of the simple substrate
S_{2in}	Input concentration of the volatiles fatty acids
a_1	Half-saturation constant associated to the substrate S_1
a_2	Half-saturation constant associated to the substrate S_2
D	Dilution rate
k_1	Yield coefficient of acidogenesis
k_2	Yield coefficient of acidogenesis
k_3	Yield coefficient of methanogenesis
σ	Intensity of the noise
m_1	Maximal growth rate of acidogenic microorganisms
m_2	Maximal growth rate of methanogenic microorganisms
K	Inhibition constant associated to S_2
α	Parameter allowing us to decouple the HRT (Hydraulic Retention Time) and the SRT (Solid Retention Time)

Existence and Uniqueness of the Positive Solution of Model(2)

To investigate the dynamics of the stochastic anaerobic digestion model (2), we need to show at least that this model has a unique positive solution. The following Theorem proves that model (2) has a unique positive solution for given positive initial data.

Theorem 1 *For any initial value* $(S_1(0), X_1(0), S_2(0), X_2(0)) \in \mathbb{R}^4_+$, *there exists a unique positive solution* $(S_1(t), X_1(t), S_2(t), X_2(t))$ *to system (2) for* $t \geq 0$, *and the solution will remain in* $\mathbb{R}^4_+$ *with probability one.*

Proof See Appendix A.1. □

Remark 1 Let us set $Z(t) = S_1(t) + (k_1 - k_2)X_1(t) + S_2(t) + k_3X_2(t)$. From system (2), since $\alpha \in [0, 1]$, we have

$$dZ(t) \leq [D(S_{1in} + S_{2in}) - \alpha D(Z(t)]\, dt.$$

This implies that

$$\limsup_{t\to\infty} (S_1(t) + (k_1 - k_2)X_1(t) + S_2(t) + k_3X_2(t)) \leq \frac{(S_{1in} + S_{2in})}{\alpha}.$$

Therefore

$$\begin{aligned}
\limsup_{t\to\infty} S_1(t) &\leq \frac{(S_{1in} + S_{2in})}{\alpha},\\
\limsup_{t\to\infty} X_1(t) &\leq \frac{(S_{1in} + S_{2in})}{\alpha(k_1 - k_2)},\\
\limsup_{t\to\infty} S_2(t) &\leq \frac{(S_{1in} + S_{2in})}{\alpha},\\
\limsup_{t\to\infty} X_2(t) &\leq \frac{(S_{1in} + S_{2in})}{\alpha k_3},
\end{aligned}$$

a.s.

Thus every solution of system (2) is inside the region $\Gamma = \{(S_1, X_1, S_2, X_2) \in \mathbb{R}^4_+ : 0 < S_1, S_2 \leq \frac{(S_{1in}+S_{2in})}{\alpha}; 0 < X_1 \leq \frac{(S_{1in}+S_{2in})}{\alpha(k_1-k_2)}; 0 < X_2 \leq \frac{(S_{1in}+S_{2in})}{\alpha k_3}\}$, for every time t large enough.

Persistence

In this section, we discuss the persistence for the acidogenic and methanogenic microorganisms modeled by (2).

Theorem 2 *Let $(S_1(t), X_1(t), S_2(t), X_2(t)) \in \mathbb{R}^4_+$ be a solution of system (2). If $m_1 > \alpha D$, $m_2 \geq \alpha D$ and $S_{1in} > \lambda_1 + \frac{\sigma^2 S^2_{\alpha in}}{2(m_1-\alpha D)(a_1+S_{\alpha in})}$, then the following inequalities hold:*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t X_1(s)ds \geq \frac{m_1-\alpha D}{\alpha k_1.m_1}(S_{1in}-\lambda_1-\frac{\sigma^2 S^2_{\alpha in}}{2(m_1-\alpha D)(a_1+S_{\alpha in})}) > 0,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t X_2(s)ds \geq$$
$$\left(\frac{m_2 S_{2in}}{a_2+K_{in}} - \alpha D\right) + \frac{k_2(m_1-\alpha D)}{\alpha k_3 k_1.m_1}(S_{1in}-\lambda_1-\frac{\sigma^2 S^2_{\alpha in}}{2(m_1-\alpha D)(a_1+S_{\alpha in})}) > 0,$$

where $K_{in} = S_{\alpha in} + KS^2_{\alpha in}$, with $S_{\alpha in} = (S_{1in} + S_{2in})/\alpha$.

That is to say, the acidogenic and the methanogenic microorganisms are persistent in the mean in the chemostat with probability one.

Proof See Appendix A.2. □

Extinction

In this section, we discuss the extinction for the acidogenic and methanogenic microorganisms modeled by (2).

Theorem 3 *Let $(S_1(t), X_1(t), S_2(t), X_2(t)) \in \mathbb{R}^4_+$ be a solution of system (2). If one of the following holds*

(i) *If $S_{1in} \leq \lambda_1 + \frac{\sigma^2 S^2_{1in}}{2(m_1-\alpha D)(a_1+S_{1in})^2}$, $\sigma^2 \leq \frac{m_1(a_1+S_{1in})}{S_{1in}}$ and $\alpha D \geq m_2$,*
(ii) *If $S_{1in} \leq \lambda_1 + \frac{\sigma^2 S^2_{1in}}{2(m_1-\alpha D)(a_1+S_{1in})^2}$, $\sigma^2 \leq \frac{m_1(a_1+S_{1in})}{S_{1in}}$, $\alpha D < m_2$ and $(1-4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2 < 0$,*

then the following inequalities hold:

$$\limsup_{t\to\infty} \frac{lnX_1(t)}{t} \leq \frac{m_1-\alpha D}{a_1+S_{1in}}(S_{1in}-\lambda_1-\frac{\sigma^2 S^2_{1in}}{2(m_1-\alpha D)(a_1+S_{1in})}) < 0,$$

$$\limsup_{t\to\infty} \frac{lnX_2(t)}{t} < -\omega < 0,$$

where ω is a positive constant. That is to say, the microorganisms will go exponentially to extinction in the bioreactor with probability one.

Proof See Appendix A.3. □

Remark 2 Set $\rho_1 = \frac{1}{S_{\alpha in}}\sqrt{2(m_1-\alpha D)(S_{1in}-\lambda_1)(a_1+S_{\alpha in})}$ and $\rho_2 = \frac{1}{S_{1in}}\sqrt{m_1 S_{1in}(a_1+S_{1in})}$. The conditions in Theorem 2 and Theorem 3 can be rewritten as follows:

(i) If $S_{1in} > \lambda_1$ and $m_1 > \alpha D$, when $\sigma < \rho_1$ and $m_2 \geq \alpha D$, both acidogenic and methanogenic microorganisms are persistent in the mean in the bioreactor with probability one (Theorem 2).
(ii) If $\rho_1 < \sigma \leq \rho_2$ and $m_2 \leq \alpha D$, then both acidogenic and methanogenic microorganisms die out in the bioreactor with probability one (Theorem 3).
(iii) If $\rho_1 < \sigma \leq \rho_2$, $m_2 > \alpha D$ and $(1-4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2 < 0$, then both acidogenic and methanogenic microorganisms die out in the bioreactor with probability one (Theorem 3).

Numerical Simulations

In this section we illustrate some theoretical results obtained by performing numerical simulations using the Milstein method which provides a strong order of convergence of 1.0, higher than the Euler-Maruyama scheme [12]. We plot the solution curves of the two systems to distinguish the difference between the stochastic model (2) and the corresponding deterministic model (1). The following basic parameters of the microorganisms and the operating parameters of the chemostat are taken from [6]. We group them together in Table 2 below.

The Milstein scheme was applied to each equation of system (2) and we obtained the following discretization equations to iteratively calculate the approximate solutions of the stochastic system (2) in Matlab:

$$\begin{cases} S_{1i+1} &= S_{1i} + \left[D(S_{1in}-S_{1i}) - k_1\frac{m_1 S_{1i}X_{1i}}{a_1+S_{1i}}\right]\Delta t \\ &\quad -k_1\frac{S_{1i}X_{1i}}{a_1+S_{1i}}\left((\sigma\sqrt{\Delta t}) + \frac{\sigma^2}{2}(\Delta t\varepsilon_{1i}^2 - \Delta t)\right), \\ X_{1i+1} &= X_{1i} + \left(\frac{m_1 S_{1i}X_{1i}}{a_1+S_{1i}} - DX_{1i}\right)\Delta t \\ &\quad +\frac{S_{1i}X_{1i}}{a_1+S_{1i}}\left((\sigma\sqrt{\Delta t}) + \frac{\sigma^2}{2}(\Delta t\varepsilon_{1i}^2 - \Delta t)\right), \\ S_{2i+1} &= S_{2i} + \left[D(S_{2in}-S_{2i}) + k_2\frac{m_1 S_{1i}X_{1i}}{a_1+S_{1i}} - k_3\frac{m_2 S_{2i}}{a_2+S_{2i}+KS_{2i}^2}\right]\Delta t \\ &\quad +k_2\frac{S_{1i}X_{1i}}{a_1+S_{1i}}\left((\sigma\sqrt{\Delta t}) + \frac{\sigma^2}{2}(\Delta t\varepsilon_{1i}^2 - \Delta t)\right), \\ X_{2i+1} &= X_{2i} + \left(\frac{m_2 S_{2i}X_{2i}}{a_2+S_{2i}+KS_{2i}^2} - DX_{2i}\right)\Delta t. \end{cases} \tag{3}$$

The simulations will be perform over the time interval $[0, T]$, $T = 100$ days, with a fixed time step $\Delta t = 1/1000$. It is also worth noting that the choice of the intensity of the noise is based on reference standards and previous studies on stochastic perturbations in microbial growth rates [10, 13, 24].

Table 2 Basic parameters of the microorganisms and operating parameters of the chemostat

Parameters	m_1	a_1	m_2	K	a_2	k_1	k_2	k_3	α
Values	0.6	2.1	1.5	1/50	30	25	250	268	0.5
Unit	d^{-1}	g/l	d^{-1}	mmol/l	d^{-1}	mmol/g	mmol/g	mmol/g	
Source	[19]	[19]	[19]	[19]	[19]	[19]	[19]	[19]	[19]

Numerical Simulations of the Theoretical Results

Figure 1 shows the extinction of the two species X_1 and X_2 due to the increase of substrate inlet concentration and with initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.0, 1.0, 1.0, 0.9)$.

Interpretation: From Fig. 1, we observe the extinction of both acidogenic and methanogenic microorganisms in the stochastic model (2) as well as in the corresponding deterministic model (1). Regarding the stochastic case, this observation is consistent with the conclusion of Theorem 3. To verify the applicability of Theorem 3, we consider the following parameter values: $\sigma = 0.1, D = 0.8$. We define the quantity $B = (1 - 4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2$. Computing the values, we obtain $B = -2400.6$ and $\rho_2 = 0.33$. Since $\sigma < \rho_2$ and $B < 0$, the conditions of Theorem 3 are satisfied, confirming the extinction of the microbial populations as depicted in Fig. 1. From a biological standpoint, this scenario reflects the effect of an increased substrate inlet concentration, as also discussed in [8].

Figure 2 shows the extinction of the two species X_1 and X_2 due to the increase of the dilution rate of the system and with initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.3, 1.0, 1.5, 1.0)$.

Interpretation: From Fig. 2, we observe the extinction of both acidogenic and methanogenic microorganisms in the stochastic model (2), as well as in the corresponding deterministic model (1). Regarding the stochastic case, this behavior is consistent with the conclusion of Theorem 3. To verify the conditions of Theorem 3, we consider the following parameter values and initial condition: $\sigma = 0.1$, $D = 1.2, S_{1in} = 10, S_{2in} = 5$, $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.3, 1.0, 1.5, 1.0)$. We define $B = (1 - 4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2$. With the chosen values, we obtain $B = -3599.85$ and $\rho_2 = 0.84$. Since $\sigma < \rho_2$ and $B < 0$, the extinction of microorganisms observed in Fig. 2 aligns with the conditions of Theorem 3. From a biological perspective, this extinction is mainly due to the increase in the dilution rate.

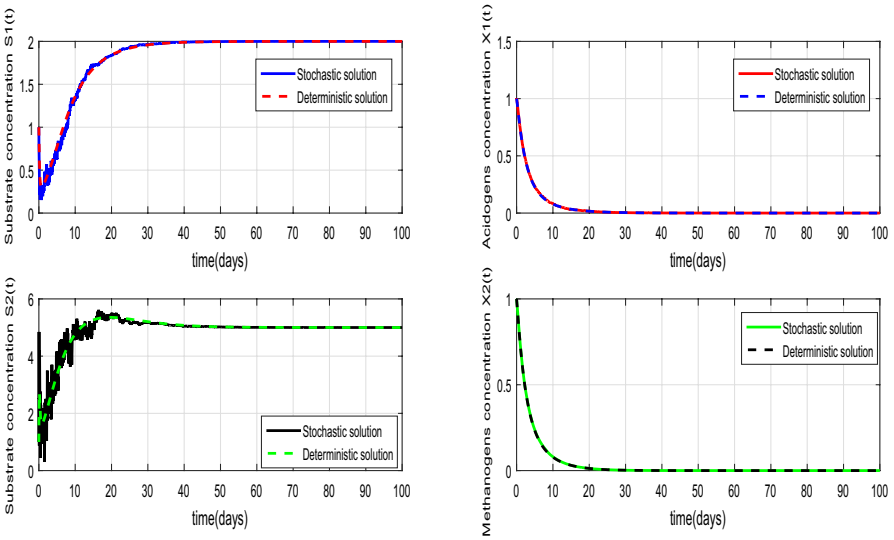

Fig. 1 Deterministic and stochastic solution of AM2 with the parameter $\alpha = 0.5$, $\sigma = 0.1, D = 0.8, S_{1in} = 3, S_{2in} = 5$ and with initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.0, 1.0, 1.0, 0.9)$

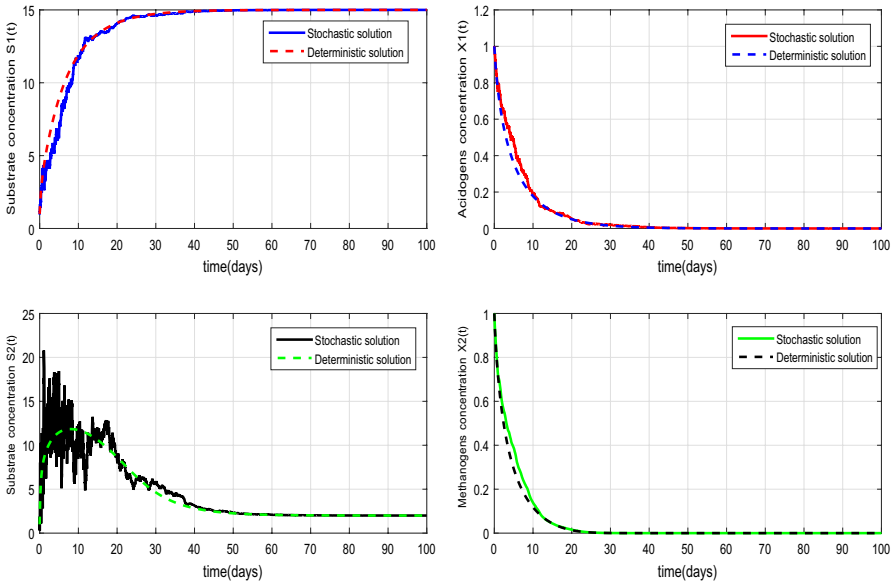

Fig. 2 Deterministic and stochastic solution of AM2 with the parameter $\sigma = 0.1, D = 1.2, S_{1in} = 10, S_{2in} = 5$ and initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.3, 1.0, 1.5, 1.0)$

Figure 3 shows the persistence of the two species X_1 and X_2, with initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.3, 1.0, 1.5, 1.0)$.

Interpretation: From Fig. 3, we observe that both acidogenic and methanogenic microorganisms are persistent in the stochastic model (2), as well as in the corresponding deterministic model (1). Regarding the stochastic case, this observation is consistent with the conclusion of Theorem 2. To verify the conditions stated in Theorem 2, we consider the following parameter values and initial condition: $\sigma = 0.1,\ D = 0.8, S_{1in} = 10, S_{2in} = 5$, $(S_1(0), X_1(0), S_2(0), X_2(0)) = (1.3, 1.0, 1.5, 1.0)$. We compute the relevant threshold values and obtain $\lambda_1 = 4.2$, $\rho_1 = 0.30$ and $\rho_2 = 0.84$. The conditions $S_{1in} > \lambda_1$, $m_1 > \alpha D$, $\sigma < \rho_1$ and $m_2 \geq \alpha D$ are all satisfied. Therefore, Theorem 2 applies, confirming that both microbial populations are persistent in the mean, as shown in Fig. 3. From a biological standpoint, this result indicates that when stochastic fluctuations remain moderate and when the substrate inlet concentrations S_{1in} and S_{2in} are within appropriate ranges, the acidogenic and methanogenic populations remain stable in the bioreactor, thus allowing continuous production of biogas.

Figure 4 shows the extinction of the two species X_1 and X_2 due to the noise and with initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (0.7, 0.6, 0, 1.0)$

Interpretation: From Fig. 4, we observe that the noise negatively impacts the persistence of both acidogenic and methanogenic microorganisms. While the deterministic model predicts persistence, the stochastic model leads to extinction. This highlights the destabilizing effect of stochastic perturbations. To verify the applicability of Theorem 3, we consider the following parameter values and initial condition: $\sigma = 0.9$, $D = 0.8, S_{1in} = 10, S_{2in} = 5,\quad (S_1(0), X_1(0), S_2(0), X_2(0)) = (1.3, 1.0, 1.5, 1.0)$. We

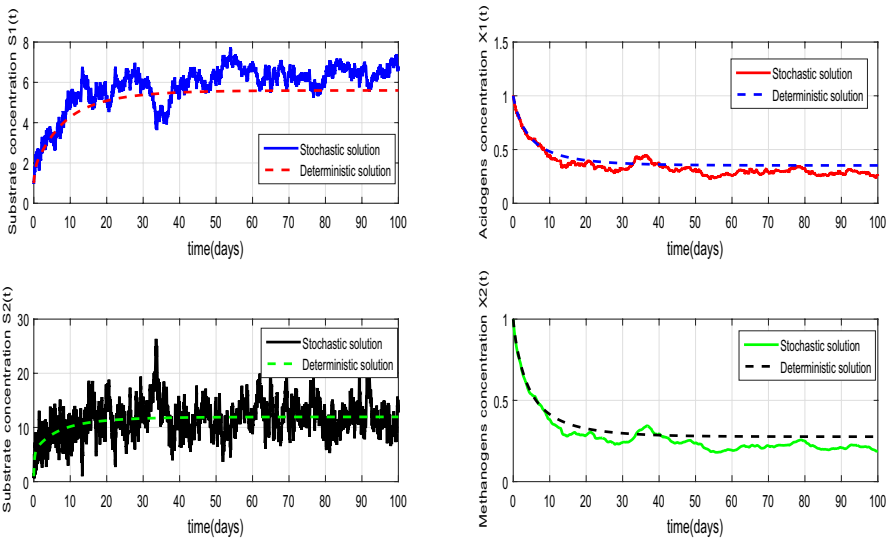

Fig. 3 Deterministic and stochastic solution of AM2 with the parameter $\sigma = 0.1$, $D = 0.8, S_{1in} = 10, S_{2in} = 5$.

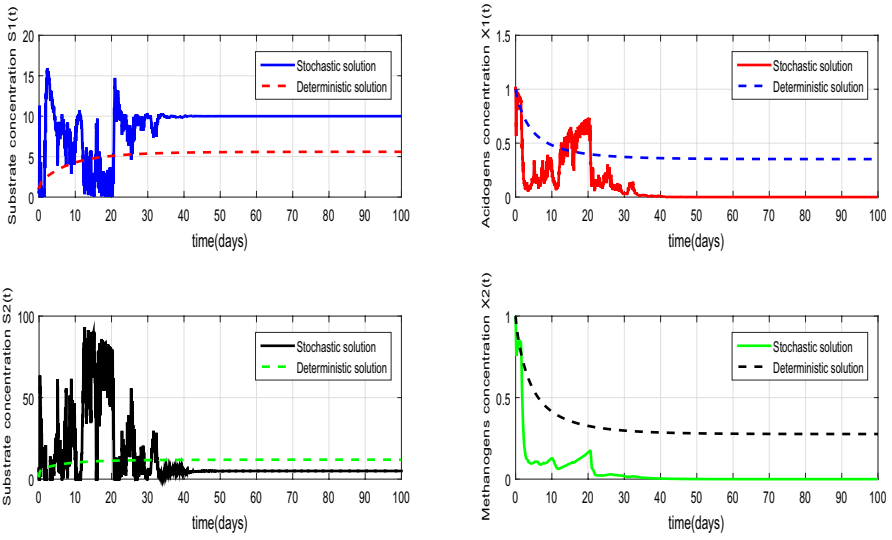

Fig. 4 Deterministic and stochastic solution of AM2 with the parameter $\sigma = 0.9, D = 0.8, S_{1in} = 10, S_{2in} = 5$ and with initial condition $(S_1(0), X_1(0), S_2(0), X_2(0)) = (0.7, 0.6, 0, 1.0)$

define $B = (1 - 4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2$, and compute $B = -3599.85$, $\lambda_1 = 4.2$, $\rho_1 = 0.30$ and $\rho_2 = 0.84$. In the deterministic case, the conditions $S_{1in} > \lambda_1$ and $m_1 > \alpha D$ are satisfied, ensuring microbial persistence. However, for the stochastic system, we observe $\sigma < \rho_2$ and $B < 0$, which by Theorem 3 leads to extinction. Therefore, Fig. 4 illustrates that

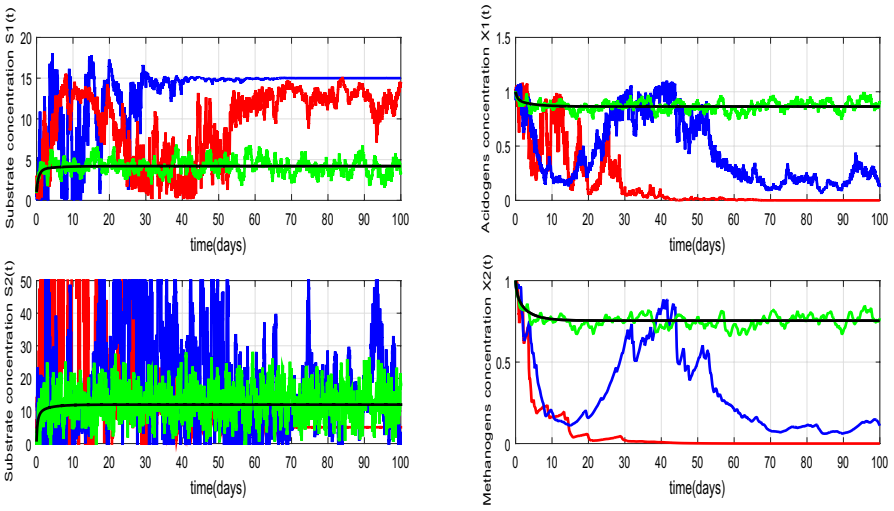

Fig. 5 Solution of stochastic model for $\sigma = 0$ (in black), $\sigma = 0.1$ (in green), $\sigma = 0.4$ (in blue) and $\sigma = 0.9$ (in red) respectively and with parameters values $D = 0.8$, $S_{1in} = 10$, $S_{2in} = 5$

even when the deterministic model predicts persistence, sufficiently large stochastic fluctuations (here $\sigma = 0.9$) can drive the system to extinction. This confirms the significant influence of noise intensity on system stability.

Numerical Simulation of the Impact of the Noise

Here, we perform a sensitivity analysis to explore how variations in noise intensity σ affect the system's behavior with respect to the extinction or persistence of microbial populations. In particular, we vary σ within a realistic range [0, 0.9] and examine the system's qualitative behavior for different initial conditions. The analysis shows that low noise levels $\sigma < 0.4$ have minimal impact, preserving the trends predicted by the deterministic model. However, for $\sigma \geq 0.5$, the stochastic effects become more pronounced, and for $\sigma \geq 0.9$, extinction becomes highly probable, even when the deterministic thresholds suggest persistence (Fig. 5).

Conclusion

In this work, we have constructed and analyzed a stochastic version of the so-called anaerobic model number 2 in which the maximal growth rate of the acidogenic microorganisms (m_1) is influenced by a white noise. We first proved that the stochastic model has a unique global positive solution for any positive initial value and we obtained the following results:

(1) Because of the noise, the persistence condition of the acidogenic population changes, rather than having $S_{1in} > \lambda_1$ as in [5, 6] we have $S_{1in} > \lambda_1 + \frac{\sigma^2 S_{\alpha in}^2}{2(m_1 - \alpha D)(a_1 + S_{\alpha in})^2}$.

(2) When $m_1 < \alpha D$ and $m_2 < \alpha D$, there is extinction of both acidogenic and methanogenic microorganisms in the bioreactor regardless of the size of the noise.

(3) When $\rho_1 < \sigma \leq \rho_2$ and $m_2 \leq \alpha D$ or when $\rho_1 < \sigma \leq \rho_2$, $m_2 > \alpha D$ and $(1 - 4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2 < 0$, then both acidogenic and methanogenic microorganisms die out in the bioreactor with probability one. Theorem 3 shows that the noise can accelerate extinction.
(4) Conversely, when $S_{1in} > \lambda_1$, $m_1 > \alpha D$, $\sigma < \rho_1$ and $m_2 \geq \alpha D$, both acidogenic and methanogenic microorganisms are persistent in the mean in the bioreactor with probability one (Theorem 2).

These results highlight the facts that, beyond substrate concentration limits or dilution rate increases known to cause extinction [8], stochastic fluctuations alone can drive microbial populations to extinction, even when the deterministic system would remain persistent. Moreover, this study provides a biological rationale for selecting the maximal growth rate of acidogenic bacteria as the stochastic variable, motivated by its high sensitivity to environmental and operational perturbations such as temperature and substrate variability. While the current work focuses on white noise influence on the maximal growth of acidogenic bacteria, it opens pathways to examine stochastic effects on other parameters like dilution rate or substrate inflow, which are also critical in reactor performance.

These results underscore the importance of incorporating stochastic modeling in the study of anaerobic digestion processes to better reflect real operational uncertainties. This approach is in line with recent studies that emphasize the role of environmental feedback mechanisms such as leachate recirculation in microbial systems [3, 11], which further motivate the inclusion of noise effects in future modeling efforts. Our work opens promising directions for future research, including stochasticity in other biologically relevant parameters [15, 19, 23], the impact of alternative noise types such as Lévy noise [4], and the coupling with leachate recirculation models to capture more realistic process dynamics.

In summary, this study advances the stochastic modeling of anaerobic digestion by clearly linking stochastic variable selection to biological and operational realities, providing rigorous analysis of microbial persistence and extinction, and setting the stage for further exploration of environmental uncertainties in bioprocesses.

Appendix A Proofs

A.1. Proof of Theorem 1 Since the coefficients of model (2) are locally Lipschitz continuous, for any given initial value $(S_1(0), X_1(0), S_2(0), X_2(0)) \in \mathbb{R}^4_+$, there is a unique positive local solution $(S_1(t), X_1(t), S_2(t), X_2(t)) \in \mathbb{R}^4_+$ on $[0, \tau_e)$, where τ_e is the explosion time [16]. To show that this solution is global, we need to show that $\tau_e = \infty$ a.s. For this purpose, let's choose $\varepsilon_0 > 0$ such that $S_1(0) > \varepsilon_0$, $X_1(0) > \varepsilon_0$, $S_2(0) > \varepsilon_0$, $X_2(0) > \varepsilon_0$. For any positive $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon_0$. Define the stopping time as follows $\tau_\varepsilon = \inf\{t \in [0, \tau_e) : S_1(t) \leq \varepsilon, X_1(t) \leq \varepsilon, S_2(t) \leq \varepsilon, X_2(t) \leq \varepsilon\}$, where throughout this paper, we set $\inf\{\emptyset\} = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_ε is increasing as $\varepsilon \to 0$. Set $\tau_0 = \lim_{\varepsilon \to 0} \tau_\varepsilon$; whence $\tau_0 \leq \tau_\varepsilon$ a.s.. If we can show $\tau_0 = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(S_1(t), X_1(t), S_2(t), X_2(t)) \in \mathbb{R}^4_+$ for all $t \geq 0$ a.s. In other words, to complete the proof we only need to show that $\tau_0 = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\beta \in (0, 1)$ such that $P\{\tau_0 \leq T\} > \beta$. Hence there is a positive constant $\varepsilon_1 \leq \varepsilon_0$ such that

$$P\{\tau_\varepsilon \leq T\} > \beta \quad for\ any\ positive\ \ \varepsilon \leq \varepsilon_1. \tag{A1}$$

From system (2), the total biomass $Z(t) = S_1(t) + k_1X_1(t) + S_2(t) - k_2X_1(t) + k_3X_2(t)$ satisfies the following relation

$$dZ(t) \leq D((S_{1in} + S_{2in}) - \alpha Z(t))dt.$$

This leads to the fact that, for all $t < \tau_\varepsilon$,

$$Z(t) \leq max\,(Z(0), S_{\alpha in}) := C_0, \tag{A2}$$

where $S_{\alpha in} = \frac{S_{1in}+S_{2in}}{\alpha}$.

We define the function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ by:

$$V(S_1(t), X_1(t), S_2(t), X_2(t)) = -ln\frac{S_1}{C_0}(t) - ln\frac{X_1}{C_0}(t) + S_2(t) - ln\frac{X_2(t)}{C_0}.$$

From equation A2 we have $S_1 \leq C_0$, $X_1(t) \leq \frac{C_0}{(k_1-k_2)}$ and $X_2(t) \leq \frac{C_0}{k_3}$. Since from the law of conservation of matter, $(k_1 - k_2) \geq 1$ and $k_3 \geq 1$, then

$-ln\frac{S_1}{C_0}(t) \geq 0$, $-ln\frac{X_1}{C_0}(t) \geq 0$, $S_2(t) \geq 0$ and $-ln\frac{X_2(t)}{C_0} \geq 0$. So, V is positive definite. Using $Itô$ formula, we get

$$dV = LVdt + \frac{\sigma(k_1X_1 - S_1)}{a_1 + S_1}dB(t) + \frac{k_2\sigma S_1X_1}{(a_1 + S_1)}dB(t),$$

where

$$LV = -\frac{DS_{1in}}{S_1} + DS_{2in} + 2\alpha D + D + \frac{m_1(k_1X_1 - S_1)}{a_1 + S_1} - \frac{m_2(k_3X_2 + S_2)}{a_2 + S_2 + KS_2^2} + k_2\frac{m_1S_1X_1}{(a_1 + S_1)} + \frac{\sigma^2}{2}\left(\frac{k_1^2X_1^2+S_1^2}{(a_1+S_1)^2} + \frac{k_2^2S_1^2X_1^2}{(a_1+S_1)^2}\right).$$

By using relation A2, we obtain

$$\begin{aligned} LV &= -\frac{DS_{1in}}{S_1} + DS_{2in} + 2\alpha D + D + \frac{m_1(k_1X_1 - S_1)}{a_1 + S_1} - \frac{m_2(k_3X_2 + S_2)}{a_2 + S_2 + KS_2^2} + k_2\frac{m_1S_1X_1}{(a_1 + S_1)} \\ &\quad + \frac{\sigma^2}{2}\left(\frac{k_1^2X_1^2+S_1^2}{(a_1+S_1)^2} + \frac{k_2^2S_1^2X_1^2}{(a_1+S_1)^2}\right) \\ &\leq DS_{2in} + 2\alpha D + D + \frac{k_1m_1X_1}{a_1 + S_1} + k_2\frac{m_1S_1X_1}{(a_1 + S_1)} + \frac{\sigma^2}{2}\left(\frac{k_1^2X_1^2+S_1^2}{(a_1+S_1)^2} + \frac{k_2^2S_1^2X_1^2}{(a_1+S_1)^2}\right) \\ &\leq DS_{2in} + 2\alpha D + D + \frac{k_1m_1X_1}{a_1} + k_2m_1X_1 + \frac{\sigma^2}{2}\left(\frac{k_1^2X_1^2+S_1^2}{a_1^2} + \frac{k_2^2S_1^2X_1^2}{a_1^2}\right) \\ &\leq DS_{2in} + 2\alpha D + D + (k_1m_1/a_1 + k_2m_1)\,C_0 + \frac{\sigma^2}{2}\left(\frac{C_0^2}{a_1^2} + \frac{C_0^2}{a_1^2}\right). \end{aligned}$$

Hence we have

$$dV \leq C_1 dt + \left(\frac{\sigma(k_1 X_1 - S_1)}{a_1 + S_1} + \frac{k_2 \sigma S_1 X_1}{(a_1 + S_1)} \right) dB(t),$$

where $C_1 = DS_{2in} + 2\alpha D + D + (k_1 m_1 / a_1 + k_2 m_1) C_0 + \frac{\sigma^2}{2} \left(\frac{C_0^2}{a_1^2} + \frac{C_0^2}{a_1^2} \right)$.

Integrating both sides from 0 to $\tau_0 \wedge T$, and taking expectations, we get

$$\mathbb{E}V(S_1(\tau_0 \wedge T), X_1(\tau_0 \wedge T), S_2(\tau_0 \wedge T), X_2(\tau_0 \wedge T)) \leq V(S_1(0), X_1(0), S_2(0), X_2(0)) + C_1 T.$$

For any positive $\varepsilon \leq \varepsilon_1$, denote the set $\Delta_\varepsilon = \{\tau_\varepsilon \leq T\}$. It then follows from A1 that $P(\Delta_\varepsilon) > \beta$. Note that for every $\upsilon \in \Delta_\varepsilon$, there is at least one of $S_1(\tau_\varepsilon, \upsilon), X_1(\tau_\varepsilon, \upsilon), S_2(\tau_\varepsilon, \upsilon), X_2(\tau_\varepsilon, \upsilon)$ equals ε. Then

$$V(S_1(\tau_\varepsilon), X_1(\tau_\varepsilon), S_2(\tau_\varepsilon), X_2(\tau_\varepsilon)) \geq -\frac{ln\varepsilon}{C_0}.$$

Consequently,

$$\begin{aligned} V(S_1(0), X_1(0), S_2(0), X_2(0)) + C_1 T &\geq E[I_{\Delta_\varepsilon} V(S_1(\tau_\varepsilon \wedge T), X_1(\tau_\varepsilon \wedge T), S_2(\tau_\varepsilon \wedge T), X_2(\tau_\varepsilon \wedge T))] \\ &= P(\Delta_\varepsilon) V(S_1(\tau_\varepsilon), X_1(\tau_\varepsilon), S_2(\tau_\varepsilon), X_2(\tau_\varepsilon)) \\ &> -\beta \frac{ln\varepsilon}{C_0}, \end{aligned}$$

where I_{Δ_ε} is the indicator function of Δ_ε. Letting $\varepsilon \to 0$ leads to the contradiction

$$\infty > V(S_1(0), X_1(0), S_2(0), X_2(0)) + C_1 T = \infty.$$

So we must have $\tau_0 = \infty$ a.s. The proof is thus complete.

A.2. Proof of Theorem 2 We first remark that system (2) has a cascade structure which simplify its study. We will use this structure given by Fig. 6. to prove the result.

The first sub-model corresponds to a simple stochastic chemostat model [9, 14, 24] with randomness involved on the maximal growth rate of microorganisms (in this case the acidogens).

Set $Z_1(t) = S_1(t) + k_1 X_1(t)$, the total biomass of the first sub-system.

We have

$$\begin{aligned} dZ_1(t) &= dS_1(t) + k_1 dX_1(t) \\ &= \left(D(S_{1in} - S_1) - k_1 \frac{m_1 S_1 X_1}{a_1 + S_1} \right) dt - k_1 \frac{\sigma S_1 X_1}{a_1 + S_1} dB \\ &\quad + k_1 \left(\frac{m_1 S_1}{a_1 + S_1} - \alpha D \right) X_1 dt + k_1 \frac{\sigma S_1 X_1}{a_1 + S_1} dB \\ &= (D(S_{1in} - S_1) - \alpha k_1 D X_1) dt. \end{aligned}$$

Integrating both sides from 0 to t yields

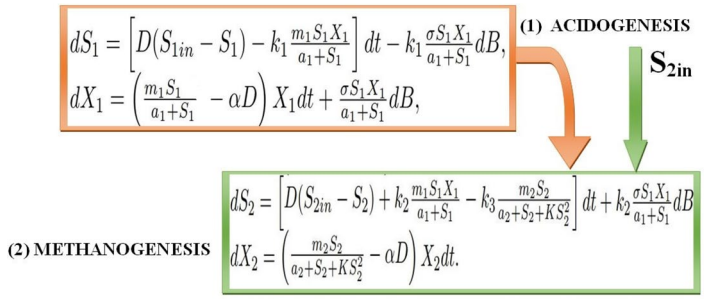

Fig. 6 Cascade structure of system (2)

$$Z_1(t) - Z_1(0) = D(S_{1in}t - \int_0^t S_1(s)ds - \alpha k_1 \int_0^t X_1(s)ds).$$

According to the law of conservation of matter, the left side is always zero, so:

$S_1(t) + k_1 X_1(t) = S_1(0) + k_1 X_1(0)$.

Thus we have

$$\frac{1}{t}\int_0^t S_1(s)ds = S_{1in} - \alpha \frac{k_1}{t}\int_0^t X_1(s)ds. \tag{A3}$$

Define the function $V_1(X_1) = lnX_1$. By the $Itô$ formula we have

$$\begin{aligned} dV_1 &= \frac{1}{X_1}\left(\tfrac{m_1 S_1}{a_1+S_1} - \alpha D\right) X_1 dt + \frac{1}{X_1}\frac{\sigma S_1 X_1}{a_1 + S_1} dB - \frac{\sigma^2}{2}\left(\tfrac{X_1^2 S_1^2}{(a_1+S_1)^2}\right)\frac{1}{X_1^2} dt \\ &= \left(\tfrac{m_1 S_1}{a_1+S_1} - \alpha D - \tfrac{\sigma^2 S_1^2}{2(a_1+S_1)^2}\right) dt + \frac{\sigma S_1}{a_1 + S_1} dB. \end{aligned}$$

Since $S_1(t) \le S_{\alpha in}$, we get

$$\left(\tfrac{m_1 S_1}{a_1+S_1} - \alpha D - \tfrac{\sigma^2 S_1^2}{2(a_1+S_1)^2}\right) \ge \left(\tfrac{m_1 S_1}{a_1+S_{\alpha in}} - \alpha D - \tfrac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2})\right).$$

So we have

$$dV_1 \ge \left(\tfrac{m_1 S_1}{a_1+S_{\alpha in}} - \alpha D - \tfrac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2})\right) + \frac{\sigma S_1}{a_1 + S_1} dB. \tag{A4}$$

Integrating both sides of (A4) from 0 to t and dividing both sides by t yields

$$\frac{m_1}{a_1 + S_{\alpha in}}\frac{1}{t}\int_0^t S_1(s)ds \le \left(\tfrac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2}) + \alpha D\right) + \frac{lnX_1(t) - lnX_1(0) - M(t)}{t}, \tag{A5}$$

where $M(t) = \int_0^t \frac{\sigma S_1(s)}{a_1+S_1(s)} dB(s)$.

Together with A3, we get

$$\frac{m_1}{a_1+S_{\alpha in}}\left(S_{1in}-\frac{\alpha k_1}{t}\int_0^t X_1(s)ds\right)-\left(\frac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2}+\alpha D\right)+\frac{lnX_1(0)+M(t)}{t}\leq\frac{lnX_1(t)}{t}.$$

So, we obtain

$$\frac{lnX_1(t)}{t}\geq\frac{m_1}{a_1+S_{\alpha in}}\left(S_{1in}-\frac{\alpha k_1}{t}\int_0^t X_1(s)ds\right)-\left(\frac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2}+\alpha D\right)+\theta(t), \quad \text{(A6)}$$

where $\theta(t)=\frac{lnX_1(0)}{t}+\frac{M(t)}{t}$.

Using the definition of λ_1 we have

$$\begin{aligned}\frac{lnX_1(t)}{t}\geq&\frac{m_1-\alpha D}{a_1+S_{\alpha in}}(S_{1in}-\lambda_1-\frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2})-\frac{\alpha k_1 m_1}{a_1+S_{\alpha in}}.\frac{1}{t}\int_0^t X_1(s)ds\\&-\frac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2}+\alpha D+\theta(t).\end{aligned} \quad \text{(A7)}$$

Notice that $M(t)$ is a local continuous martingale with

$$M(0)=0\ and\ \lim_{t\to\infty}\sup\frac{<M,M>_t}{t}\leq\frac{\sigma^2 S_{\alpha in}^2}{2(a_1+S_{\alpha in})^2}<\infty. \quad \text{(A8)}$$

By the strong law of large numbers [16], we obtain

$$\lim_{t\to\infty}\frac{M(t)}{t}=0\ a.s.$$

Obviously, $\lim_{t\to\infty}\theta(t)=0$ a.s. Using Lemma 5.1 in [15], it follows from A7 that if $m_1>\alpha D$ and $S_{1in}>\lambda_1+\frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2}$, then

$$\lim_{t\to\infty}\inf\frac{1}{t}\int_0^t X_1(s)ds\geq\frac{m_1-\alpha D}{\alpha k_1 m_1}(S_{1in}-\lambda_1-\frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2})>0. \quad \text{(A9)}$$

We now study the second system in Fig. 6. It is important to note that the input concentration of this system is the sum of S_{2in} and the quantity coming from the first system as shown in Fig. 6. The total biomass of the second sub-system obeys $Z_2(t)=S_2(t)+k_3X_2(t)-k_2X_1(t)$.

We have

$$\begin{aligned}dZ_2(t)=&dS_2(t)+k_3dX_2(t)-k_2dX_1(t)\\=&\left(D(S_{2in}-S_2)+k_2\frac{m_1S_1X_1}{a_1+S_1}-k_3\frac{m_2S_2}{a_2+S_2+KS_2^2}\right)dt\\&+k_2\frac{\sigma S_1X_1}{a_1+S_1}dB(t)+k_3\left(\frac{m_2S_2}{a_2+S_2+KS_2^2}-\alpha D\right)X_2dt\\&-k_2\left(\frac{m_1S_1}{a_1+S_1}-\alpha D\right)X_1dt-k_2\frac{\sigma S_1X_1}{a_1+S_1}dB\\=&(D(S_{2in}-S_2)-\alpha k_3DX_2+\alpha k_2DX_1)dt.\end{aligned}$$

Integrating both sides from 0 to t, we have

$$Z_2(t) - Z_2(0) = D(S_{2in}t - \int_0^t S_2(s)ds + \alpha k_2 \int_0^t X_1(s)ds - \alpha k_3 \int_0^t X_2(s)ds).$$

According to the law of conservation of matter, the left side is always zero, so: $S_2(t) - k_2X_1(t) + k_3DX_2(t) = S_2(0) - k_2X_1(0) + k_3DX_2(0) = Z_2(t) = Z_0(0) = S_{2in}$.

Thus we have

$$\frac{1}{t}\int_0^t S_2(s)ds = S_{2in} + \frac{\alpha k_2}{t}\int_0^t X_1(s)ds - \frac{\alpha k_3}{t}\int_0^t X_2(s)ds). \tag{A10}$$

Define the function $V_2(X_2) = lnX_2$. By the $Itô$ formula we have

$$\begin{aligned} dV_2 &= \frac{1}{X_2}\left(\frac{m_2S_2}{a_2+S_2+KS_2^2} - \alpha D\right)X_2 \\ &= \left(\frac{m_2S_2}{a_2+S_2+KS_2^2} - \alpha D\right). \end{aligned}$$

Since $S_2(t) \le S_{\alpha in}$, we get

$$\left(\frac{m_2S_2}{a_2+S_2+KS_2^2} - \alpha D\right) \ge \left(\frac{m_2S_2}{a_2+S_{\alpha in}+KS_{\alpha in}^2} - \alpha D\right).$$

So we have

$$dV_2 \ge \left(\frac{m_2S_2}{a_2+K_{in}} - \alpha D\right), \tag{A11}$$

where $K_{in} = S_{\alpha in} + KS_{\alpha in}^2$.

Integrating both sides of A11 from 0 to t and dividing both sides by t yields

$$\frac{m_2}{a_2+K_{in}}.\frac{1}{t}\int_0^t S_2(s)ds \le \alpha D + \frac{lnX_2(t) - lnX_2(0)}{t}. \tag{A12}$$

Together with A10, we get

$$\frac{m_2}{a_2+K_{in}}\left(S_{2in} + \frac{\alpha k_2}{t}\int_0^t X_1(s)ds - \frac{\alpha k_3}{t}\int_0^t X_2(s)ds)\right) \le \alpha D + \frac{lnX_2(t) - lnX_2(0)}{t}. \tag{A13}$$

That is to say

$$\frac{lnX_2(t)}{t} \ge \left(\frac{m_2S_{2in}}{a_2+K_{in}} - \alpha D\right) + \frac{\alpha k_2}{t}\int_0^t X_1(s)ds - \frac{\alpha k_3}{t}\int_0^t X_2(s)ds) - \frac{lnX_2(0)}{t}. \tag{A14}$$

Since when $S_{1in} > \lambda_1 + \frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2}$,

$$\lim_{t\to\infty} \inf \frac{1}{t}\int_0^t X_1(s)ds \geq \frac{m_1-\alpha D}{\alpha k_1.m_1}(S_{1in} - \lambda_1 - \frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2} > 0.$$

it follows from A14 that if $S_{1in} > \lambda_1 + \frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2}$ and $m_2 \geq \alpha D$,

$$\lim_{t\to\infty} \inf \frac{1}{t}\int_0^t X_2(s)ds \geq$$
$$\left(\frac{m_2 S_{2in}}{a_2+K_{in}} - \alpha D\right) + \frac{k_2(m_1-\alpha D)}{\alpha k_3 k_1 m_1}(S_{1in} - \lambda_1 - \frac{\sigma^2 S_{\alpha in}^2}{2(m_1-\alpha D)(a_1+S_{\alpha in})^2}) > 0.$$

A.3. Proof of Theorem 3 Define the function $V_1(X_1) = lnX_1$. By the $Itô$ formula, we have

$$\begin{aligned} dV_1 &= \frac{1}{X_1}\left(\frac{m_1 S_1}{a_1+S_1} - \alpha D\right) X_1 dt + \frac{1}{X_1}\frac{\sigma S_1 X_1}{a_1+S_1} dB - \frac{\sigma^2}{2}\left(\frac{X_1^2 S_1^2}{(a_1+S_1)^2}\right)\frac{1}{X_1^2} dt \\ &= \left(\frac{m_1 S_1}{a_1+S_1} - \alpha D - \frac{\sigma^2 S_1^2}{2(a_1+S_1)^2}\right) dt + \frac{\sigma S_1}{a_1+S_1} dB. \end{aligned} \tag{A15}$$

Set $p = \frac{S_1}{a_1+S_1}$ and $H(p) = -\frac{\sigma^2}{2}p^2 + m_1 p - \alpha D$.

H(*p*) takes its maximum value H_{max} on the interval $[0, \frac{S_{1in}}{a_1+S_{1in}}]$ at $p = \frac{S_{1in}}{a_1+S_{1in}}$, where

$$H_{max} = \frac{(m_1-\alpha D)}{a_1+S_{1in}}(S_{1in} - \lambda_1 - \frac{\sigma^2 S_{1in}^2}{2(m_1-\alpha D)(a_1+S_{1in})}).$$

It follows from the monotonicity of the function $\frac{S_1}{a_1+S_1}$ on $[0, S_{1in}]$ that when $\sigma^2 \leq \frac{m_1(a_1+S_{1in})}{S_{1in}}$ we have $-\frac{\sigma^2}{2}(\frac{S_1}{a_1+S_1})^2 + m_1(\frac{S_1}{a_1+S_1}) - \alpha D \leq \frac{(m_1-\alpha D)}{a_1+S_{1in}}(S_{1in} - \lambda_1 - \frac{\sigma^2 S_{1in}^2}{2(m_1-\alpha D)(a_1+S_{1in})})$.

Integrating both sides of A15 from 0 to *t* yields

$$lnX_1(t) - lnX_1(0) \leq \frac{(m_1-\alpha D)}{a_1+S_{1in}}(S_{1in} - \lambda_1 - \frac{\sigma^2 S_{1in}^2}{2(m_1-\alpha D)(a_1+S_{1in})})t + M(t),$$

where $M(t) = \int_0^t \frac{\sigma S_1(s)}{a_1+S_1(s)} dB(s)$.

Dividing on both sides by t and letting t tends to ∞ we have

$$\lim_{t\to\infty} \sup \frac{lnX_1(t)}{t} \leq \frac{m_1-\alpha D}{a_1+S_{1in}}(S_{1in} - \lambda_1 - \frac{\sigma^2 S_{1in}^2}{2(m_1-\alpha D)(a_1+S_{1in})}) < 0 \quad a.s.$$

and

$$\frac{lnX_2(t) - lnX_2(0)}{t} = \frac{1}{t}\int_0^t \left(\frac{m_2 S_2(s)}{a_2 + S_2(s) + KS_2^2(s)} - \alpha D\right) ds \tag{A16}$$

$$= \frac{1}{t}\int_0^t \left(\frac{-DKS_2^2(s) - (D - m_2)S_2(s) - Da_2}{a_2 + S_2(s) + KS_2^2(s)}\right) ds, \tag{A17}$$

since

$$dX_2 = \left(\frac{m_2 S_2}{a_2 + S_2 + KS_2^2} - \alpha D\right) X_2 dt.$$

Finally we get,

$$\lim_{t\to\infty} \sup \frac{lnX_2(t)}{t} = \lim_{t\to\infty} \sup \frac{1}{t}\int_0^t \left(\frac{-\alpha DKS_2^2(s) - (\alpha D - m_2)S_2(s) - \alpha Da_2}{a_2 + S_2(s) + KS_2^2(s)}\right) ds \quad a.s. \tag{A18}$$

Set

$$H_2(s) = \frac{m_2 S_2(s)}{a_2 + S_2(s) + KS_2^2(s)} - \alpha D.$$

(i) If $\alpha D \geq m_2$ then $H_2(s) = \left(\frac{-\alpha DKS_2^2(s) - (\alpha D - m_2)S_2(s) - \alpha Da_2}{a_2 + S_2(s) + KS_2^2(s)}\right) < -\alpha D < 0.$

(ii) If $\alpha D < m_2$ and $(1 - 4a_2K)(\alpha D)^2 - 2\alpha D m_2 + m_2^2 < 0$ then $H_2(s) \leq \sup_{s\in(0,\infty)} H_2(s) < 0$. Thus as in [19], we conclude that there exists a positive constant ω such that

$$\lim_{t\to\infty} \sup \frac{lnX_2(t)}{t} < -\omega < 0.$$

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Declarations

Conflict of interest The authors declare no conflict of interest.

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