The Renormalization of Geometric Operators and Background Independent Field Quantization in Quantum Gravity

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt

dem Fachbereich für Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität Mainz

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Mainz, den 12. September 2020

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D77 / Dissertation der Johannes Gutenberg-Universität Mainz

Erster Gutachter:
Zweiter Gutachter:
Dritter Gutachter:
Datum der mündlichen Prüfung: 01. Oktober 2021

Dedicated to the memory of my mother.

Abstract. The construction of a theory of quantum gravity is among the most challenging pursuits of modern-day physics. From the theoretician's point of view, there are some broad features that any theory of quantum gravity should exhibit. These include renormalizability, unitarity and Background Independence. However, because of the scarce experimental data available on the nature of the gravitational interaction at high energies, the specific realization of these requirements is rather unclear. Therefore, it is inevitable to increase the variety of theoretical approaches towards quantum gravity. These can be parted into two main categories: Those that employ discrete structures at the fundamental level, such as Loop Quantum Gravity or Causal Dynamical Triangulations, and the continuum-based approaches such as the Asymptotic Safety scenario based upon the Effective Average Action, where the ultraviolet completion of quantum gravity is realized via a non-trivial fixed point of the renormalization group flow.

Although each of these approaches' physical properties have been explored to some extent, still only little is known about their relationship to each other. A contact point that seems natural is the comparison of their geometric features at high energies. In the first part of this thesis, we derive suitable geometric features for the continuum-based approaches. Based on the functional renormalization group equation for gravity, we derive a novel flow equation that governs the evolution of renormalized composite operators. This evolution becomes encoded into that of the composite operators' anomalous dimensions. We show that their values in the fixed-point regime can be interpreted as quantum corrections to the classical scaling dimensions of the composite operators. As a main application, we calculate for the first time the scaling dimension at the ultraviolet fixed point of the volume operator for submanifolds embedded into spacetime, within the Einstein-Hilbert truncation as well as the truncation corresponding to higher-derivative gravity at one loop. In the former case, we observe dimensional reduction phenomena: The scaling dimension in the ultraviolet becomes much smaller than its classical value. This unveils the genuinely fractal nature of spacetime, and subsets of it, in the ultraviolet. In the latter case, we find that precisely at the ultraviolet fixed point the quantum corrections to the scaling dimension vanish, because of the asymptotic freedom of higher-derivative gravity. However, its fractal nature still is unveiled slightly away from the fixed point, where we, depending on the dimension of the submanifold, find that the effective scaling dimension either increases or decreases.

In the second part of this thesis, we propose a novel quantization scheme for fields in contact with dynamical gravity, including quantum gravity itself. This scheme is subject to three essential requirements: Background Independence, the use of gravity-coupled approximants, and N-type cutoffs. Therewith we require that Background Independence is already implemented at the level of the regularized precursor of a quantum field theory, i.e., its "approximants". We realize this via the employment of cutoffs of the N-type, which constitute a metric-independent regularization scheme. We initiate the exploration of this quantization scheme by applying it to a scalar field in classical spacetimes, and then to quantum gravity itself, and determining the possible self-consistent spherical background geometries. These turn out to possess striking physical properties. In particular, they embody a solution to the notorious cosmological constant problem which, in the traditional approaches, arises due to the field's quantum vacuum fluctuations.

Der Mensch an sich selbst, insofern er sich seiner gesunden Sinne bedient, ist der größte und genaueste physikalische Apparat, den es geben kann, und das ist eben das größte Unheil der neuern Physik, daß man die Experimente gleichsam vom Menschen abgesondert hat und bloß in dem, was künstliche Instrumente zeigen, die Natur erkennen, ja, was sie leisten kann, dadurch beschränken und beweisen will.

Ebenso ist es mit dem Berechnen. Es ist vieles wahr, was sich nicht berechnen läßt, sowie sehr vieles, was sich nicht bis zum entschiedenen Experiment bringen läßt.

Johann Wolfgang von Goethe (1749–1832);
Naturwissenschaftliche Schriften, Gedankenspäne

Contents

CHAPTER 1

Introduction

1.1. The pursuit of quantum gravity and the results of this thesis

At the beginning of the 20th century two revolutions shattered the world of physics. In 1900 M. Planck proclaimed a finite quantum of action h which gave birth to quantum physics, leading all the way from the wave mechanics of 1920s to the Standard Model of particle physics whose current form was completed with the detection of the Higgs boson in 2012.

Moreover, in 1905 A. Einstein proclaimed the speed of light c to be constant and the upper limit for the speed of signal propagation. This led to special relativity, putting space and time on equal grounds. Einstein conducted probably the greatest intellectual achievement of the 20th century by expanding special relativity to General Relativity in 1915, based upon the equivalence principle. So far, General Relativity has been confirmed in an abundant number of experiments, most recently with the detection of gravitational waves in 2016.

Although logically fully unrelated, the ideas of quantum physics and relativity intertwined in many respects during their development, as W. Pauli explains in the preface to the 1958 Italian and English edition of his 1921 book (article) "Relativitätstheorie" [1]:

> "Es gibt eine Ansicht, nach der die Relativitätstheorie der Endpunkt der 'klassischen Physik' ist, d.h. Physik im Stil von Newton-Faraday-Maxwell und beherrscht durch die 'deterministische' Form von raumzeitlicher Kausalität, während später der neue quantenmechanische Stil der Naturgesetze in Kraft getreten ist. Dieser Gesichtspunkt scheint mir nur teilweise richtig, und er wird dem großen Einfluß Einsteins, des Schöpfers der Relativitätstheorie, auf die allgemeine Denkweise der heutigen Physiker nicht genügend gerecht. Durch ihre erkenntnistheoretische Analyse der Folgen der Endlichkeit der Lichtgeschwindigkeit (und damit aller Signalgeschwindigkeiten) war die spezielle Relativitätstheorie der erste Schritt weg von naiver Veranschaulichung. Der Begriff des Bewegungszustandes des 'lichttragenden Äthers', wie das hypothetische Medium einst genannt wurde, mußte aufgegeben werden, nicht nur, weil er sich

als unbeobachtbar erwies, sondern weil er als Element eines mathematischen Formalismus überflüssig wurde, dessen gruppentheoretische Eigenschaften von ihm nur gestört würden.

Durch die Ausweitung der Transformationsgruppe in der allgemeinen Relativitätstheorie konnte Einstein auch die Vorstellung von ausgezeichneten Trägheitssystemen als unvereinbar mit den gruppentheoretischen Eigenschaften der Theorie eliminieren. Ohne diese allgemein kritische Einstellung, welche naive Veranschaulichung zugunsten einer begrifflichen Analyse der Beziehung zwischen Beobachtungsdaten und den mathematischen Größen in einem theoretischen Formalismus aufgab, wäre der Aufbau der modernen Form der Quantenmechanik nicht möglich gewesen. In der 'komplementären' Quantentheorie führte die erkenntnistheoretische Analyse der Endlichkeit des Wirkungsquantums zu einem weiteren Abrücken von naiven Veranschaulichungen. In diesem Falle mußten sowohl der klassische Feldbegriff als auch der Bahnbegriff von Partikeln (Elektronen) in Raum und Zeit zugunsten rationaler Verallgemeinerungen aufgegeben werden. Wieder wurden diese Begriffsbildungen nicht allein aus dem Grunde verworfen, weil die Bahnen unbeobachtbar sind, sondern auch, weil sie überflüssig wurden und die Symmetrie stören würden, welche der dem mathematischen Formalismus der Theorie zugrundeliegenden Transformationsgruppe eigen ist."

Today, the experimentally confirmed share of physics is still parted into this twofoldness: On the one hand, there are *quantum field theories on Minkowski space* that culminate in the Standard Model of particle physics. It can describe three of the four known fundamental forces – the electromagnetic, the weak and the strong force – with tremendous precision. Especially, it is a highly accurate description of Nature at small distance scales where these forces dominate. On the other hand, there is *General Relativity* which models the remaining gravitational force in terms of the curvature of the dynamical spacetime. General Relativity is a highly accurate description of Nature at large distance scales. There are much too many accounts on each of these two fields of physics,[1] thus, in order to avoid this great deal of redundancy let us only focus on the aspects required to motivate the content of this thesis.

The dichotomy of quantum physics, where spacetime is static, and General Relativity, where spacetime is dynamical, already exhibits the incompleteness of this twofold framework of physics: It cannot be that spacetime is dynamical and static, as well. Therewith also comes the *problem of time*: in quantum

[1]Outstanding historical accounts on each fields' history have been written by A. Pais [2, 3].

physics, time is an (external) absolute element, i.e., it is *not* described by an operator, while in General Relativity it is dynamical. The resulting quest for a quantum theory of gravity (*quantum gravity*) consequently is one of the main pending challenges of modern physics. It must be emphasized that with a theory of quantum gravity one not necessarily wants to unify all four fundamental forces of Nature to build a "theory of everything" but rather the desideratum is a theory of gravity that is applicable at all distance scales, i.e., especially at small distances in the "quantum world".

Essentially, physics is defined as the mathematical description of Nature which a fortiori turns it into an experimental science. So as the need for quantum gravity is obvious from the point of view of theoretical physics, where would one expect experimental signatures of quantum gravity?

A theory of quantum gravity necessarily will include the following constants: the smallest quantum of action h, viz. the reduced Planck constant $\hbar = h/2\pi$, the speed of light c and Newton's constant G which describes the strength of the gravitational interaction. From these constants, one can build units of length, time and mass – called the Planck units – which are typical scales at which one would expect experimental signatures of the quantum gravitational theory. In four spacetime dimensions, these are $\ell_{\mathrm{Pl}} = (\hbar G/c^3)^{1/2} \approx 10^{-35}$m, $t_{\mathrm{Pl}} = (\hbar G/c^5)^{1/2} \approx 10^{-44}$s and $m_{\mathrm{Pl}} = (\hbar c/G)^{1/2} \approx 10^{-5}$g, respectively. From the point of view of General Relativity as a perturbative effective field theory, the tremendously large Planck mass suppresses any gravitational interaction of particles that are scattered in realizable particle accelerators. To illustrate this suppression of gravitational effects, there is the famous analogy that in order to create a particle with a Planck mass the corresponding accelerator needed to be of the size of the Milky Way. However, it must be emphasized that in a non-perturbative treatment Quantum Gravity might exhibit infrared effects, similar to confinement in QCD for example, which could be tested experimentally.

How does one attempt building a theory of quantum gravity? The obvious starting point should be General Relativity, which should be the limit of any theory of quantum gravity at large distance scales. That General Relativity itself is incomplete can be seen in its predictions of *singularities*, e.g. at the center of black holes or at the instant of the Big Bang. Such singularities are unphysical and should therefore be resolved by any theory of quantum gravity.

On the other hand, there are aspects of General Relativity, which should be passed on to quantum gravity. For example, the principle of *Background Independence* should be a main desideratum of quantum gravity. In General Relativity, the principle of Background Independence refers to the fact that the spacetime structure which is realized in Nature, in form of the metric $g_{\mu\nu}$, is not part of the theory's definition but rather determined dynamically by Einstein's equation $G_{\mu\nu}[g] = 8\pi G T_{\mu\nu}[g]$, where $G_{\mu\nu}[g]$ is the Einstein tensor and $T_{\mu\nu}[g]$ the stress-energy tensor of the matter inhabited on spacetime. For a theory of quantum gravity, the principle of Background Independence can be rephrased as the requirement that "*none of the theory's basic rules and assumptions, and none of its predictions, therefore, may depend on any special metric that has been fixed a priori. All metrics of physical relevance must result from the intrinsic quantum gravitational dynamics*" [4].

Part 4 of this thesis is devoted to Background Independence as a first principle. In this part, based on the author's publications [5, 6], we develop a novel scheme for the nonpertubative analysis of quantum fields that are coupled to gravity as well as quantum gravity itself. The novel part of this scheme can be pinned down to the introduction of "N-cutoffs" which regularize the theory via a dimensionless cutoff parameter N. By means of this technical tool, it is possible to quantize matter fields as well as gravity in terms of sequences of "gravity-coupled approximants", thereby rigorously obeying the principle of Background Independence.

In certain simple examples, we explore the physical implications of this quantization scheme; especially, we find the striking result that in these cases, the *cosmological constant problem* does not occur at all.

The cosmological constant problem is one of the major riddles that occur when one tries to bring together quantum physics and General Relativity. Loosely speaking, it refers to the fact that if one treats the stress-energy tensor quantum mechanically, the quantum fields' vacuum energies sum up to gigantic contributions to the effective cosmological constant whose value in this case differs from the observed value "by some 120 orders of magnitude" [7]. Inserted into the semi-classical Einstein equation the gigantic effective cosmological constant predicted by quantum physics would lead to a spacetime curvature so large that the resulting universe, in W. Pauli's words, "could not even reach to the moon" [8, 9].

However, all the quantum-physical arguments entering the considerations about the cosmological constant problem rely on background-dependent calculations. With the novel quantization scheme presented in Part 4 of this thesis, which fully obeys Background Independence, we show that the quantum-mechanical contributions to the cosmological constant in fact cause diametrically opposed effects: The more modes of a field are quantized, i.e., the more vacuum fluctuations contribute to the effective cosmological constant, the *larger* the radius of spacetime becomes, until, when the field is fully quantized, spacetime ultimately becomes perfectly *flat* (rather than not even reaching to the moon).

Another open question that arises when bringing together quantum physics and General Relativity is about the *entropy of black holes*. With semi-classical considerations, J. Bekenstein and S. Hawking showed that black holes, and similar spacetimes such as de Sitter space, possess a thermodynamical entropy, which consequently was named after them. It is a longstanding expectation towards a theory of quantum gravity that it should be able to explain what are the microscopic states of the black hole that the Bekenstein-Hawking entropy "counts". In this regard, another quite intriguing property of the novel quantization scheme presented in Part 4 is that it offers a natural interpretation for these microstates. Namely, we will identify the regularized quantum field (and the spacetime metric) with an approximation by a quantum-mechanical system of finitely many degrees of freedom. Then, it will be easy to see that the Bekenstein-Hawking entropy of de Sitter space "counts" precisely these degrees of freedom.

Next, let us present the existing, viable approaches towards quantum gravity. The rivalry amongst them should not be taken too seriously. In fact, A. Ashtekar found that due to the scarce experimental data "the most promising way of enhancing our chances at success is to increase the amount of variety" [10, 11]. Furthermore, he points out four essential questions that any theory of quantum gravity should address:

Firstly, *non-perturbative methods* are essential for any theory of quantum gravity. The standard methods of perturbative renormalization within traditional quantum field theory turned out to fail when applied to General Relavity [12, 13]. Actually, General Relativity is perturbatively renormalizable as an effective field theory, however infinitely many parameters, that must be taken from experiment, arise. Thus, within perturbative renormalization, General Relativity loses its predictive power.

Secondly, conceptual issues of quantum gravity must be addressed. This applies for instance to the problem of time described above.

Thirdly, Background Independence should be rigorously implemented. This raises especially technical questions, such as the regularization of quantum operators in absence of a background geometry.

Fourthly, it must be emphasized that the geometric structure of spacetime at the Planck scale need not necessarily be a smooth (semi-)Riemannian manifold.

String theory is a quantum gravity candidate that still relies on background-dependent methods. Its main idea is that the fundamental degrees of freedom are one-dimensional open or closed strings. These strings' vibrations can then be interpreted as particles whereby the strings' spectrum also includes the graviton. In this restricted sense string theory is a theory of gravity. To be well-defined, string theory relies on a critical spacetime dimension of $d = 11$. This can be regarded as its main drawback: the way back to our familiar $d = 4$ spacetime dimensions is rather arbitrary, whereby string theory loses most of its predictive power. Furthermore, string theory relies on the concept of supersymmetry which has no reasonable chances of experimental signs, anymore.

There are several approaches that employ Background Independence manifestly because they do not make use of any background at all:

Loop Quantum Gravity [14, 15] uses, loosely speaking, instead of points in spacetime "loops" that are located in three-dimensional space. Therewith one can for example describe the flux of some field through the area encompassed by a loop, fully analogously to the flux of the magnetic field in electrodynamics. This formulation has the technical advantage that any mathematical problem related to a single point in spacetime can be avoided. However, the loops at best should not be visualized as located in a rigid spacetime. Rather, the framework fully implements Background Independence and predicts spacetime to form out of the loops, or graphs called "spin networks". Especially, within this framework it has been shown that areas and volumes are quantized which intuitively leads to the discrete building blocks of spacetime, often referred to as "atoms of spacetime".

Within the *statistical mechanics-based approaches to quantum gravity* [16–21] a particularly promising candidate is the method of *Causal Dynamical Triangulation.* This method attempts to derive spacetime from first principles, whereby spacetime is modelled as discrete simplices that are glued together in a causal way. These simplices do not have a physical meaning and should be regarded as pure approximations. Using Monte Carlo simulations it has been found that Causal Dynamical Triangulation predicts spacetime at large distance scales to be a four-dimensional de Sitter space. This is a striking result because the spacetime dimension is a prediction of the theory, rather than an input. More precisely, it has been found that the spectral dimension of spacetime is four at large distance scales and decreases to two at small distance scales.

Furthermore, there are continuum-based approaches to quantum gravity that also obey to the principle of Background Independence. It is rigorously implemented via the *background field method*:

Asymptotically Safe quantum gravity is a concept which was proposed by S. Weinberg in the late 1970s [22]. By and large, the idea that gravity might be asymptotically safe can be phrased as follows: If there exists a *non-Gaussian fixed point* (i.e., a fixed point where the theory is not a free one) in the space of all couplings of the theory, then the couplings of the theory can be tuned such that at large energy scales they approach this fixed point. This is a sufficient condition in order to avoid unphysical ultraviolet divergences.

More precisely, the Asymptotic Safety scenario relies on Wilson's notion of non-perturbative renormalization [23, 24], formulated by means of the *functional renormalization group* based upon the *effective average action.* The basic input data required by this framework are the field content as well as the symmetries the fields shall be subject to. All resulting action functionals of the fields that are invariant under the proposed symmetries build the *theory space.* As these functionals are parametrized by the couplings of the theory, one synonymously refers to the space of all couplings as the theory space. The effective average action, which introduces a scale dependence for the action functionals of interest, satisfies a number of functional identities, in particular a functional renormalization group equation, the *Wetterich equation* [25], developed in the early 1990s and soon after applied to gravity by M. Reuter [26]. The solutions of the Wetterich equation correspond to trajectories in theory space. In the Asymptotic Safety scenario, the RG flow in theory space determined by the Wetterich equation possesses a non-Gaussian fixed point. In this case, there exists a subset of the theory space consisting of all the points that are "pulled" into the fixed point when moving to larger and larger scales, called the UV-*critical surface*. The trajectories lying on it are free from ultraviolet divergences (i.e., "safe") and thus correspond to *fundamental theories*. Lastly, the dimension of the UV-critical surface corresponds to the degree of predictivity of the theory, because it is precisely the number of measurements required to fix a specific trajectory on that surface.

Higher-derivative gravity [27] is a framework that may not be considered a viable candidate for a fundamental theory of quantum gravity, nevertheless it is a framework that intrigues many physicists as an arena to probe aspects of quantum gravity. The name higher-derivative gravity stems from the fact that the underlying action functional of the theory is built only from operators which are of fourth order in the derivatives. (Rather than at most of second order, as in the Einstein-Hilbert action.) Stunningly, higher-derivative gravity is perturbatively renormalizable; however, the theory is not unitary and therefore was quickly discarded. Recently, the theory again attracted quite some attention because of promising attempts to restore its unitarity. Many features of higher-derivative gravity are universal such that they can also be analyzed by means of the non-perturbative Wetterich equation.

It is a longstanding desire to relate and quantitatively compare the discrete and continuum-based approaches. Envisaged is to bring quantum gravity into a similar condition as the theoretical side of quantum chromodynamics, a part of the Standard Model. There, many aspects of the theory could be double-checked by means of discrete approaches (lattice QCD), on the one hand, and by means of continuum-based frameworks, on the other hand.

This thesis is fully devoted to the continuum-based approaches to quantum gravity. On a technical level, the ultimate goal of quantum gravity is to calculate the functional integral $Z = \int \mathcal{D}\widehat{g}_{\mu\nu}\, \mathrm{e}^{-S[\widehat{g}_{\mu\nu}]}$, where $S[\widehat{g}_{\mu\nu}]$ is the action of the quantum metric $\widehat{g}_{\mu\nu}$. This also is the starting point of this thesis: In Part 1 we will begin with setting up this functional integral for the continuum-based approaches, thereby employing the background field method. Part 2 is devoted to the framework of the functional renormalization group for quantum gravity.

On this basis, in Part 3, follows a study of geometric operators in quantum gravity, within the Asymptotic Safety approach as well as within higher-derivative gravity. This study is based on the author's publications [28–30]. Particularly, in this part we study the scaling behavior of the volume of submanifolds embedded into the quantum spacetime at high energies. Thereby, we are going to observe dimensional reduction phenomena which are typical for quantum theories of gravity. The results presented in Part 3 will be crucial for the comparison of the different approaches towards quantum gravity, when results of similar calculations in Loop Quantum Gravity or Causal Dynamical Triangulation become available.

1.2. The structure of this thesis

This thesis is structured in five parts as follows. Each part begins with a synoptic chapter, in which the main results and statements are briefly presented; furthermore, in case new research results based on the author's work are presented, this chapter includes a conclusion and outlook on future prospects.

The opening Part 1 lays the overall foundation for the analyses to follow. Chapter 3 warms up the detail-oriented reader and outlines the quantization of a massive scalar field on a fixed, classical and compact Riemannian manifold. Later, in Part 4, we will extensively recourse to these results which is why Chapter 3 actually serves more than the pure purpose of a warm up.

Then, in Chapter 4 we set up the path integral quantization for quantum gravity. Thereby, we rely on the background field technique which makes the rigorous employment of Background Independence possible. Finally, two special cases are presented: the path integral based on the Einstein-Hilbert action as well as the path integral for higher-derivative gravity.

The results as presented in Part 1 are still unregularized and it is fair to say that the ultimate objective of a quantum field theory approach to gravity is to give a mathematical meaning to these results, via forms of regularization and renormalization. This is where this thesis splits into two different paths: one path is followed in Parts 2 and 3, while another path is openend in Part 4. These two paths form two separate entities which is why the content of this thesis can be either read as Part 1 to Part 3 or as Part 1 followed by Part 4.

The first path, to which Part 2 of this thesis gives a comprehensive introduction, is the Asymptotic Safety approach. In this approach, Wilson's notion of non-perturbative renormalization is formulated via a distinct functional renormalization group equation that is constructed in Chapter 6.

While the study of approximative solutions to this equation is not the main focus of Part 2, we thoroughly introduce those two approximate solutions that will be made use of in Part 3. The first one is the single-metric "Einstein-Hilbert truncation" that is analyzed in Chapter 7. The second one, presented in Chapter 8, is the truncation that corresponds to higher-derivative gravity. Higher-derivative gravity is special because it can also be renormalized perturbatively. In this sense, the results of Chapter 8 are universal because it has been shown that they coincide with the results obtained via perturbative renormalization.

Finally, Chapter 9 is a last preparatory chapter for Part 3 and introduces the renormalization of composite operators via the functional renormalization group equation. Thereby, we introduce an important characteristic of the renormalization behavior of composite operators, namely its anomalous dimension.

Part 3 is based on the author's publications [28–30] and analyzes the renormalization behavior of geometric operators in Quantum Gravity by means of the functional renormalization group equation for composite operators constructed in Chapter 9. This analysis opens a new line of research in the framework of Asymptotically Safe Quantum Gravity. It is particularly important for the comparison of continuum-based approaches with discrete approaches towards Quantum Gravity. Furthermore, it paves the way towards the hard problem of constructing suitable observables within Quantum Gravity.

In Chapter 11 we study geometric operators in the Asymptotic Safety scenario for quantum gravity. Therefore, we employ the approximative solution of the functional renormalization group equation given in form of the Einstein-Hilbert truncation presented in Chapter 7. The analysis mainly deals with the discussion of the anomalous dimensions of the volume of a submanifold embedded into the quantized spacetime, as well as that of the geodesic length and the geodesic ball.

In Chapter 12 follows an analogous study, this time within the framework of higher-derivative gravity which is interpreted as an approximative solution of the functional renormalization group equation.

Part 4 is based on the author's publications [5, 6] and is logically independent of Parts 2 and 3. Rather, a novel method for the regularization and renormalization of the quantum gravitational path integral of Part 1 by means of "N-cutoffs" is proposed, tested and applied to the cosmological constant problem. Chapter 15 describes this novel framework and puts emphasis on the rigorous implementation of Background Independence. Further, it explains how quantized fields (including gravity) arise as the limit of sequences of quantum-mechanical systems with finitely many degrees of freedom. We call these systems "approximants".

In Chapters 16 and 17 we identify two different candidates for approximants of a quantized scalar field and especially analyze the properties of the self-consistent background geometry that arises due to Background Independence of the framework.

In Chapter 18 we put forward these explorations and transfer the novel framework of quantization also to gravity itself.

The overall studies of Chapters 16 to 18 are complemented by a sequel on Weyl transformations and their anomalous Ward identities, presented in Chapter 19. There, the results of the previous chapters are looked at from a different point of view.

Finally, Part 5 contains the appendix and closes this thesis. Appendices A, B, C and E set up the necessary mathematical background required for the calculations appearing throughout all parts. Appendix D discusses the path integral measure for quantum fields defined on an arbitrary background manifold. These results are especially important for Part 4 and constitute an interesting collection of results. Last not least, Appendix F collects all calculations that have been outsourced from the main text body.

Part 1

The path integral quantization of gravity

CHAPTER 2

Summary of Part 1

The objective of Part 1 is to formally set up the gravitational path integral and to derive the effective action obtained from it. That these results are "formal", means here that they are still up to regularization and renormalization procedures which will be the crucial themes for the remaining parts to follow. Thereby, the guiding principle of Part 1 is the rigorous implementation of Background Independence.

The opening Chapter 3 can be considered a warm up for this purpose and illustrates most of the required steps by taking the example of a scalar field A which is defined on a classical, compact and d-dimensional Riemannian manifold with metric $\bar{g}_{\mu\nu}$ of Euclidean signature $(+ + \cdots +)$. The overall dynamics is determined by an action functional $S[A;\bar{g}] \coloneqq S_{\text{EH}}[\bar{g}] + S_{\text{M}}[A;\bar{g}]$ which is split into two parts. The first part, the Einstein-Hilbert action given by Eq. (3.2), accounts only for the dynamics of the background metric $\bar{g}_{\mu\nu}$ and is not of much relevance for the moment. (However, in later applications it will be essential.) The second part is the action (3.3) for the scalar field A which we assume to not describe any self-interactions:

$$S_{\text{M}}[A;\bar{g}] \coloneqq \frac{1}{2}\int \mathrm{d}^d x \sqrt{\bar{g}}\, A \left[-\Box_{\bar{g}} + \mu^2 + \xi \bar{R}(\bar{g})\right] A\,.$$

Because the action is quadratic in the field A, one sometimes refers to it as a *Gaussian scalar field*. Here, $\Box_{\bar{g}}$ is the Laplacian operator (more precisely, Laplace-Beltrami operator) built from the background metric $\bar{g}_{\mu\nu}$. The constants μ and ξ describe the mass of the scalar field and its coupling to gravity in form of the scalar curvature $\bar{R}$, respectively.

After formally quantizing the scalar field $A \mapsto \widehat{A}$, we then analyze the generating functional for the connected Green's functions (3.9),

$$\exp\left\{W[J;\bar{g}]\right\} \coloneqq \int \mathcal{D}_{\bar{g}}\widehat{A}\, \exp\left\{-S[\widehat{A};\bar{g}] + \int \mathrm{d}^d x \sqrt{\bar{g}(x)}J(x)\widehat{A}(x)\right\}\,,$$

where J can be regarded a source of the scalar field A. By applying functional derivatives with respect to J, this path integral generates all connected Green's functions. Importantly, we then demonstrate how these can be expanded in terms of eigenfunctions of the operator $-\Box_{\bar{g}} + \mu^2 + \xi \bar{R}(\bar{g})$. These expansions will play a crucial role in Part 4. Also note that the measure itself depends on the background metric $\bar{g}_{\mu\nu}$. This is an important detail required for the diffeomorphism invariance of the path integral which is often neglected. Furthermore, the formal effective action (3.22) for the scalar field A is obtained by Legendre-transforming the functional $W[J;\bar{g}]$,

$$\Gamma[\bar{A};\bar{g}] \coloneqq -W[J[\bar{A}];\bar{g}] + \int \mathrm{d}^d x \sqrt{\bar{g}(x)}\, \bar{A}(x) J[\bar{A}](x)\,.$$

In the remainder of Chapter 1, we then perform a saddle point expansion of e^W around the solution to the classical equations of motion which leads to the one-loop gravitational effective action (3.38),

$$\Gamma[\bar{g}] \coloneqq \Gamma[0;\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + \frac{1}{2}\,\mathrm{Tr}\ln\left(-\Box_{\bar{g}} + \mu^2 + \xi \bar{R}\right) + O(2\text{ loops})\,.$$

It can be said that the quantization of the scalar field in this manner follows Background Independence because the background metric $\bar{g}_{\mu\nu}$ is left fully arbitrary during the whole process and is still to be dynamically determined.

In case of gravity itself, which is the content of Chapter 4, it is not that straightforward to built a path integral that obeys Background Independence. Namely, there are two obstacles which did not occur in case of the scalar field:

Firstly, in case of gravity there exists an additional symmetry given by the diffeomorphism invariance of the theory. This gauge invariance of the theory would lead to a diverging path integral because one would integrate infinitely often over physically identical states. Luckily, this issue can be easily fixed by means of the *Faddeev-Popov trick*, fully analogously to Yang Mills theories. Thereby, one fixes the gauge once and for all whereby so-called ghosts fields are introduced that complement the theory. After the implementation of the Faddeev-Popov trick, the theory of the actual physical field and the unphysical ghost fields becomes invariant under Becchi-Rouet-Stora-Tyutin (BRST) transformation. This invariance compensates for the loss of gauge invariance.

Secondly, it is not possible to write down an action functional for metric fluctuations without somehow resorting to a background metric. Thus, the question arises how a theoretical framework that explicitly makes use of background structures can be in accordance with Background Independence. In fact, the answer is simple: Indeed, it can be in accordance with Background Independence, namely by noting that *all* backgrounds together are equivalent to no background, at all. This means, in the construction of the path integral, the background metric $\bar{g}_{\mu\nu}$ must never be specified and furthermore it must be proven that the framework is invariant under changes of this background metric ("background gauge transformations" $\delta^{(B)}$). This procedure is called the *background field method*.

In order to implement the background field technique, we split the full metric $g_{\mu\nu}$, which we assume to be of Euclidean signature, into a background part and a fluctuation part: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The fluctuation field then is formally quantized $h_{\mu\nu} \mapsto \widehat{h}_{\mu\nu}$, whereby it is complemented by the ghost fields C^{μ}, $\bar{C}_{\mu}$ and b_{μ}. After employing the Faddeev-Popov trick, the bare action then is given by Eq. (4.15), i.e.,

$$S[\widehat{h}, \bar{C}, C, b; \bar{g}] \coloneqq S_{\mathrm{cl}}[\bar{g} + \widehat{h}] + S_{\mathrm{GF}}[\widehat{h}; \bar{g}] + S_{\mathrm{gh},1}[\widehat{h}, \bar{C}, C; \bar{g}] + S_{\mathrm{gh},2}[b; \bar{g}] \,.$$

Here, S_{cl} denotes the classical action for the metric field which is assumed to be diffeormorphism invariant a priori. Infinitesimally, general coordinate transformations of the metric amount to applying a Lie derivative L_V to it, where V is some vector field. Since the metric is split into background and fluctuation part, it must therefore be clarified how to distribute the L_V over these parts. This is a crucial technical step for the application of the background field method. We do so as follows: the classical general coordinate transformations are promoted to *quantum gauge transformations* via $\delta^{(Q)}\widehat{g}_{\mu\nu} \coloneqq L_V\widehat{g}_{\mu\nu}$ and $\delta^{(Q)}\bar{g}_{\mu\nu}$. This symmetry of the classical action then is fixed by means of the gauge-fixing action S_{GF}. Then, according to the procedure of Faddeev and Popov, the action must be additionally supplemented by the ghost field actions $S_{\mathrm{gh},1}$ and $S_{\mathrm{gh},2}$ for the ghost fields C^{μ} and $\bar{C}_{\mu}$ as well as b_{μ}, respectively.

In this way, the full bare action possesses two "symmetries": first, it is invariant under the *background gauge transformations* $\delta^{(B)}$ which are defined by $\delta^{(B)} = L_V$, i.e., the Lie derivative is equally distributed among all fields, including the background metric. This realizes Background Independence. Moreover, the full bare action is invariant under the BRST transformations which is a

consequence of the Faddeev-Popov method. All these properties are explicitly proven.
The generating functional for the connected Green's function (4.17) then is defined analogously to the case of the scalar field,

$$\exp\{W[t^{\mu\nu},\sigma^{\mu},\bar{\sigma}_{\mu},d^{\mu};\bar{g}_{\mu\nu}]\} := \int \mathcal{D}\mu[\widehat{h},\bar{C},C,b;\bar{g}]\,\exp\Bigg\{-S[\widehat{h},\bar{C},C,b;\bar{g}] + \int \mathrm{d}^d x\sqrt{\bar{g}}\,\Big[t^{\mu\nu}\widehat{h}_{\mu\nu}+\bar{\sigma}_{\mu}C^{\mu}+\sigma^{\mu}\bar{C}_{\mu}+d^{\mu}b_{\mu}\Big]\Bigg\}.$$

Here $t^{\mu\nu}$, σ^{μ}, $\bar{\sigma}_{\mu}$ and d^{μ} are sources for the metric fluctuation and ghost fields. Again, the measure explicitly depends on the background metric and is constructed in such a way that the whole path integral is invariant under BRST transformations as well as background gauge transformations. The formal effective action then is defined as the Legendre transform of this path integral.

The rest of Chapter 4 focuses on constructing the bare action as well as the effective action for two given classical actions: on the one hand, for the Einstein-Hilbert action, and on the other hand, for the action of higher-derivative gravity. As for the scalar field, we also perform a saddle point expansion of e^{W} around the solutions to the equations of motion for the bare action, in order to determine the gravitational one-loop effective action. Thereby, the bare action takes the form (here exemplified for the Einstein-Hilbert action where one has $b_{\mu}\equiv 0$)

$$S[\widehat{h},\bar{C},C;\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}] + O(2\text{ loops})\,.$$

The one-loop term $S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]$ is quadratic in the metric fluctuations $\widehat{h}_{\mu\nu}$ which is why this structure enables us to interpret the one-loop expansion of the bare action as a matter action for the Gaussian "graviton field" $\widehat{h}_{\mu\nu}$.

Furthermore, the gravitational one-loop effective action in general then reads

$$\begin{aligned}\Gamma[\bar{g}] &:= \Gamma[0,0,0,0;\bar{g}]\\ &= S_{\mathrm{h.-d.}}[\bar{g}] + \frac{1}{2}\,\mathrm{Tr}_{ST^2}\ln\Big[\mathscr{U}[0;\bar{g}]^{\cdot\cdot}{}_{\cdot\cdot}\Big]\\ &\qquad - \mathrm{Tr}_V\ln\Big[\mathscr{M}[g,\bar{g}]^{\cdot}{}_{\cdot}\Big] - \frac{1}{2}\,\mathrm{Tr}_V\ln\Big[\frac{1}{\alpha}Y^{\cdot}{}_{\cdot}[\bar{g}]\Big] + O(2\text{ loops})\,.\end{aligned}$$

The operator $\mathscr{U}[0;\bar{g}]$ is the inverse propagator of the theory, while the operators $\mathscr{M}[g,\bar{g}]$ and $Y[\bar{g}]$ are the Faddeev-Popov operators stemming from the ghost part of the bare action. All these operators are derived in detail for the special cases of the Einstein-Hilbert action as well as higher derivative gravity. In later applications, we will have frequent recourse to the results for these operators.

CHAPTER 3

Warm up: a scalar field on a fixed Riemannian d-dimensional manifold

Executive summary. In Euclidean conventions, we quantize a scalar field via a path integral approach on an arbitrary, yet compactly assumed, background manifold. We conduct a mode decomposition of the Green's functions and formally introduce the effective action. Finally, we deduce the effective action at order one-loop.

3.1. Quantization of the scalar field

To begin with, we develop the quantum field theoretical treatment of a scalar field A, generically and at order one loop. Therefore, we model the background spacetime as a d-dimensional, fixed and classical Riemannian manifold M that we assume to be compact and without boundary. M is equipped with the background metric $\bar{g}$ of Euclidean signature $(++\cdots+)$. In general, we will indicate geometric objects arising from the background metric $\bar{g}_{\mu\nu}$ with a "bar", e.g. the Levi-Civita connection $\bar{\Gamma}^{\alpha}_{\mu\nu}$ and its associated covariant derivate $\bar{D}_{\mu}$.

On the compact Riemannian d-dimensional background manifold $(M, \bar{g})$ we consider the action

$$S[A;\bar{g}] \coloneqq S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{M}}[A;\bar{g}]\,. \tag{3.1}$$

scalar field A. Its first part, accounting for the background geometry, is the Einstein-Hilbert action

$$S_{\mathrm{EH}}[\bar{g}] \coloneqq \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{\bar{g}}\,(-\bar{R}+2\Lambda) \tag{3.2}$$

with Newton's constant G and the cosmological constant Λ. The latter part is the matter action for the scalar field A,

$$S_{\mathrm{M}}[A; \bar{g}] \coloneqq \frac{1}{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\, A \left[-\Box_{\bar{g}} + \mu^2 + \xi \bar{R}(\bar{g}) \right] A \,, \tag{3.3}$$

in which $-\Box_{\bar{g}} = -\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$ is the negative Laplacian while μ and ξ are constants. This is exactly the action (D.13), analyzed in the appendix, with

$$\mathscr{K}[\bar{g}] = -\Box_{\bar{g}} + \mu^2 + \xi \bar{R}(\bar{g}) \,. \tag{3.4}$$

The eigenfunctions $\{\chi_{n,m}\}$ of $\mathscr{K}[\bar{g}]$, an elliptic operator, possess the properties discussed in appendix A.1.2; the eigenvalue problem reads

$$\left(\mathscr{K}[\bar{g}] \right)^{\mathrm{diff}}_x \chi_{n,m}(x) = \mathscr{F}_n \chi_{n,m}(x) \,. \tag{3.5}$$

Here, we explicitly let the index n run over the set $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ to denote by $\chi_{0,m}$ the eigenfunctions with eigenvalue zero in the case of $\mu = 0 = \xi$, if there are any, otherwise we let n run over $\mathbb{N} = \{1, 2, 3 \dots\}$.[1] Furthermore, m is the index accounting for the degeneracy D_n of the eigenvalue $\mathscr{F}_n$. Especially, the set of eigenfunctions

$$\left\{ \chi_{n,m} \;\middle|\; n \in \mathbb{N}_0 \,,\; m \in \{1, 2, \dots, D_n\} \right\} \tag{3.6}$$

forms a basis of $L^2(M, \bar{g})$, the space of square-integrable functions on $(M, \bar{g})$, such that we can expand the scalar field A as in (D.15),

$$A(x) = \sum_{n=0}^{\infty} \sum_{m=1}^{D_n} a_{n,m} \chi_{n,m}(x) \,. \tag{3.7}$$

From this sum, we split the potential zero modes of $\mathscr{K}[\bar{g}]$ for a seperate treatment:

$$A(x) = \sum_{m=1}^{D_0} a_{0,m} \chi_{0,m}(x) + \sum_{n=1}^{\infty} \sum_{m=1}^{D_n} a_{n,m} \chi_{n,m}(x) \eqqcolon A_0(x) + \tilde{A}(x) \,. \tag{3.8}$$

We will refer to A_0 as the "zero" mode of A (in inverted commata) as it is only required to be a zero mode in the case $\mu = 0 = \xi$.

With this construction, we can give a meaning to the expectation value associated to the quantized field $A \mapsto \widehat{A}$ by evaluating the generating functional of

[1] $\mathscr{F}_0 = 0$ for $\mu = 0 = \xi$ implies that in general, for arbitrary μ and ξ, $\mathscr{F}_0$ need not necessarily be zero (but potentially is).

the connected Green's functions $W[J;\bar{g}]$ (also called the *Schwinger* functional) defined by

$$\exp\{W[J;\bar{g}]\} := \int \mathcal{D}_{\bar{g}}\widehat{A}\,\exp\left\{-S[\widehat{A};\bar{g}] + \int \mathrm{d}^d x\sqrt{\bar{g}(x)}J(x)\widehat{A}(x)\right\}, \quad (3.9)$$

where the scalar field $\widehat{A}$ has been coupled to a source J. The $\bar{g}$-dependent measure $\mathcal{D}_{\bar{g}}\widehat{A}$ is explicitly constructed in appendix D.1 and given by

$$\mathcal{D}_{\bar{g}}\widehat{A} := \mathcal{D}\Big[\bar{g}^{1/4}\widehat{A}\Big] = \prod_x \bar{g}^{1/4}(x)\mathrm{d}\widehat{A}(x)\,. \quad (3.10)$$

Importantly, we will absorb the integration over the "zero" modes $a_{0,m}$ w.l.o.g. into the normalization constant,[2] such that $\tilde{A}$ remains as the sole integration variable. Likewise, we split off the "zero" modes from the source J,

$$J(x) =: J_0 + \tilde{J}(x)\,; \quad (3.11)$$

then the functional $\mathrm{e}^{W[\tilde{J};\bar{g}]}$ can be explicitly calculated using (D.18) and completing the square,[3]

$$\exp\left\{W[\tilde{J};\bar{g}]\right\} = \exp\{-S_{\mathrm{EH}}[\bar{g}]\}\,\mathrm{Det}\,(\mathscr{K}[\bar{g}])^{-1/2}\times \\ \times\exp\left\{\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}(x)}\int \mathrm{d}^d y\sqrt{\bar{g}(y)}J(x)\langle x|\mathscr{K}[\bar{g}]^{-1}|y\rangle J(y)\right\}. \quad (3.12)$$

For the time being, we are interested in the $\tilde{J}$-dependence (and not the $\bar{g}$-dependence) of W such that we may absorb $\mathrm{e}^{-S_{\mathrm{EH}}[\bar{g}]}\,\mathrm{Det}\,(\mathscr{K}[\bar{g}])^{-1/2}$ into the normalization constant of e^W to find

$$W[\tilde{J};\bar{g}] = \frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}(x)}\int \mathrm{d}^d y\sqrt{\bar{g}(y)}J(x)\langle x|\mathscr{K}[\bar{g}]^{-1}|y\rangle J(y)\,. \quad (3.13)$$

We define the Green's function, the connected two-point function, as

$$G_{xy} := \langle x|G|y\rangle := G(x,y) \\ := \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 W[\delta J;\bar{g}]}{\delta\tilde{J}(x)\delta\tilde{J}(y)}\bigg|_{\tilde{J}=0} = \langle x|\mathscr{K}[\bar{g}]^{-1}|y\rangle\,. \quad (3.14)$$

On the other hand, we find

[2]The integration over these "zero" modes leads to a Gaussian integral and/or to a delta function (the letter especially in the case $\mu = 0 = \xi$). In both cases, this part of the path integral can be absorbed into its normalization constant.
[3]If we had not split off the zero modes, this result would be ill-defined.

$$\begin{aligned}
&\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 W[\delta J;\bar{g}]}{\delta\tilde{J}(x)\delta\tilde{J}(y)}\bigg|_{\tilde{J}=0} = \\
&\quad=\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2}{\delta\tilde{J}(x)\delta\tilde{J}(y)}\ln\int\mathcal{D}_{\bar{g}}\widehat{\tilde{A}}\,\mathrm{e}^{-S[\widehat{\tilde{A}};\bar{g}]+\int\mathrm{d}^dx\sqrt{\bar{g}}\tilde{J}\widehat{\tilde{A}}}\bigg|_{\tilde{J}=0} \\
&\quad=\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta\tilde{J}(x)}\left[\mathrm{e}^{-W[\tilde{J};\bar{g}]}\int\mathcal{D}_{g}\widehat{\tilde{A}}\,\widehat{\tilde{A}}(y)\,\widehat{\tilde{A}}(y)\mathrm{e}^{-S[\widehat{\tilde{A}};\bar{g}]+\int\mathrm{d}^dx\sqrt{\bar{g}}\tilde{J}\widehat{\tilde{A}}}\right]_{\tilde{J}=0} \\
&\quad=\mathrm{e}^{-W[\tilde{J};\bar{g}]}\int\mathcal{D}_{\bar{g}}\widehat{\tilde{A}}\,\widehat{\tilde{A}}(x)\widehat{\tilde{A}}(y)\,\mathrm{e}^{-S[\widehat{\tilde{A}};\bar{g}]+\int\mathrm{d}^dx\sqrt{\bar{g}}\tilde{J}\widehat{\tilde{A}}}\bigg|_{\tilde{J}=0} \\
&\quad=\frac{\int\mathcal{D}_{\bar{g}}\widehat{\tilde{A}}\,\widehat{\tilde{A}}(x)\widehat{\tilde{A}}(y)\,\mathrm{e}^{-S[\widehat{\tilde{A}};\bar{g}]}}{\int\mathcal{D}_{\bar{g}}\widehat{\tilde{A}}\;\mathrm{e}^{-S[\widehat{\tilde{A}};\bar{g}]}} \\
&\quad=:\langle\widehat{\tilde{A}}(x)\widehat{\tilde{A}}(y)\rangle\,.
\end{aligned}$$

Thus, we can express the (unregularized) Green's function as

$$G(x,y)=\langle x|\mathscr{K}[\bar{g}]^{-1}|y\rangle=\langle\widehat{\tilde{A}}(x)\widehat{\tilde{A}}(y)\rangle\,, \tag{3.15}$$

which amounts to the relation[4]

$$\mathscr{K}[\bar{g}]^{\mathrm{diff}}_x G(x,y)=\langle x|\tilde{\mathbb{1}}|y\rangle\,, \tag{3.16}$$

where $\tilde{\mathbb{1}}=\sum_{n=1}^{\infty}\sum_{m=1}^{D_n}|nm\rangle\langle nm|$, the identity operator on the subspace spanned by $\{\chi_{n,m}\}_{n\geq 1}$ obtained from excluding the "zero" modes. Therewith, we can also express the (unregularized) *expectation value of the kinetic term* via the Green's function,

$$\begin{aligned}
\left\langle(\partial\widehat{\tilde{A}})^2(x)\right\rangle &:= \left\langle g^{\mu\nu}(x)\partial_\mu\widehat{\tilde{A}}(x)\partial_\nu\widehat{\tilde{A}}(x)\right\rangle \\
&=\lim_{y\to x}\bar{g}^{\mu\nu}(x)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\left\langle\widehat{\tilde{A}}(x)\widehat{\tilde{A}}(y)\right\rangle \\
&=\lim_{y\to x}\bar{g}^{\mu\nu}(x)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}G(x,y)\,.
\end{aligned} \tag{3.17}$$

Furthermore, we can rewrite (3.16) using the completeness relation (A.30),

$$\sum_{n=0}^{\infty}\sum_{m=1}^{D_n}\chi_{n,m}(x)\chi^*_{n,m}(y)=\frac{\delta(x-y)}{\sqrt{\bar{g}(y)}}\,. \tag{3.18}$$

[4]Here, the identity operator can be expressed as $\mathbb{1}=\sum_{n=0}^{\infty}\sum_{m=1}^{D_n}|nm\rangle\langle nm|$, cf. appendix A.1.3.

Splitting off the "zero" modes, we find

$$\sum_{n=0}^{\infty}\sum_{m=1}^{D_n}\chi_{n,m}(x)\chi^*_{n,m}(y)=\mathscr{K}[\bar{g}]^{\mathrm{diff}}_x\sum_{n=1}^{\infty}\sum_{m=1}^{D_n}\frac{\chi_{n,m}(x)\chi^*_{n,m}(y)}{\mathscr{F}_n} \tag{3.19}$$

and therewith

$$\begin{aligned}\mathscr{K}[\bar{g}]^{\mathrm{diff}}_x G(x,y)&=\mathscr{K}[\bar{g}]^{\mathrm{diff}}_x\sum_{n=1}^{\infty}\sum_{m=1}^{D_n}\frac{\chi_{n,m}(x)\chi^*_{n,m}(y)}{\mathscr{F}_n}\\&=\frac{\delta(x-y)}{\sqrt{\bar{g}(y)}}-\sum_{m=1}^{D_0}\chi_{0,m}(x)\chi^*_{0,m}(y)\,.\end{aligned} \tag{3.20}$$

Having obtained these basic results, we are not endangered by pitfalls anymore, therefore from now on, for the sake of simplicity, we will simply write $\widehat{A}$ *instead of* $\widehat{\widetilde{A}}$*, keeping in mind that we have excluded the "zero" modes.*

Next, we introduce the *effective action* which in perturbation theory plays the role of the generating functional of the one-particle-irreducible graphs. It is a functional of the scalar field[5] $\bar{A}$, the normalized field expectation value of $\widehat{A}$, i.e.,

$$\bar{A}(x)\coloneqq\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[J;\bar{g}]}{\delta J(x)}\equiv\langle\widehat{A}(x)\rangle\,. \tag{3.21}$$

When this equation is solved for J, we indicate notationally $J=J[\bar{A}]$. Therewith, the effective action (EA) is defined as the Legendre transform of the functional $W[J;\bar{g}]$,

$$\Gamma[\bar{A};\bar{g}]\coloneqq-W[J[\bar{A}];\bar{g}]+\int\mathrm{d}^dx\sqrt{\bar{g}(x)}\,\bar{A}(x)J[\bar{A}](x)\,. \tag{3.22}$$

Note that Eq. (3.21) is inverted by $\delta\Gamma[\bar{A};\bar{g}]/\delta\bar{A}(x)=\sqrt{\bar{g}(x)}J(x)$ and that the effective action fulfills the functional integro-differential equation

$$\exp\left\{-\Gamma[\bar{A};\bar{g}]\right\}=\int\mathcal{D}_{\bar{g}}\widehat{A}\,\exp\left\{-S[\widehat{A};\bar{g}]+\int\mathrm{d}^dx\,(\widehat{A}-\bar{A})(x)\frac{\delta\Gamma[\bar{A};\bar{g}]}{\delta\bar{A}(x)}\right\}\,. \tag{3.23}$$

[5]We point out that the "bar" over $\bar{A}$ is not meant to indicate an association with the metric $\bar{g}$. Simply put, we do not wish to deviate from the standard use of the "bar" for the expectation value at this point.

3.2. The one-loop effective action

To simplify the derivation of the one-loop effective action, we summarize the exponent on the right-hand side (RHS) of the generating functional of the connected Green's functions (3.9) to a single J-dependent action

$$S[A, J; \bar{g}] := S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{M}}[A; \bar{g}] - \int \mathrm{d}^d x \sqrt{\bar{g}(x)}\, J(x) A(x)\,, \tag{3.24}$$

i.e., Eq. (3.9) now can be compactly written as

$$\exp\{W[J; \bar{g}]\} = \int \mathcal{D}_{\bar{g}} \widehat{A}\, \exp\left\{-S[\widehat{A}, J; \bar{g}]\right\}\,. \tag{3.25}$$

Essential for deriving the one loop (1L)-approximation of the EA is the field A_{cl} which is the solution of the classical equations of motion, i.e.,

$$\left.\frac{\delta S[\widehat{A}, J; \bar{g}]}{\delta \widehat{A}}\right|_{\widehat{A}=A_{\mathrm{cl}}} = 0\,. \tag{3.26}$$

We can expand $S[\widehat{A}, J; \bar{g}]$ around A_{cl} using the functional Taylor series (C.5):

$$\begin{aligned} S[\widehat{A}, J; \bar{g}] =& S[A_{\mathrm{cl}}, J; \bar{g}] \\ &+ \frac{1}{2} \int \mathrm{d}^d x \int \mathrm{d}^d y \left(\widehat{A} - A_{\mathrm{cl}}\right)(x) \left.\frac{\delta^2 S[\widehat{A}, J; \bar{g}]}{\delta \widehat{A}(x) \delta \widehat{A}(y)}\right|_{\widehat{A}=A_{\mathrm{cl}}} \left(\widehat{A} - A_{\mathrm{cl}}\right)(y) + \cdots \end{aligned} \tag{3.27}$$

From Eq. (3.3) it is easy to see that

$$\begin{aligned} \frac{\delta^2 S[\widehat{A}, J; \bar{g}]}{\delta \widehat{A}(x) \delta \widehat{A}(y)} =& \sqrt{\bar{g}(x)}\, \left(-\Box^x_{\bar{g}} + \mu^2 + \xi \bar{R}(x)\right) \delta(x-y) \\ =& \sqrt{\bar{g}(x)} \sqrt{\bar{g}(y)}\, \langle x| - \Box_{\bar{g}} + \mu^2 + \xi \bar{R} |y\rangle\,. \end{aligned} \tag{3.28}$$

with $\langle x|y\rangle = \delta(x-y)/\sqrt{\bar{g}(y)}$ (see appendix A.1). Next, we plug this expansion into Eq. (3.25), where it will amount to an expansion in $\hbar$,[6] and shift the integration variable as $\widehat{A} \mapsto \widehat{A} - A_{\mathrm{cl}}$ to obtain

[6]To keep track of the order of $\hbar$, although we use units in which $\hbar \equiv 1$, we point out that the Lorentzian counterpart of the generating functional of the connected Green's functions is given by $\exp \frac{\mathrm{i}}{\hbar} W = \int \mathcal{D}(\,\bullet\,) \exp \frac{\mathrm{i}}{\hbar} S$. After scaling the integration variable as $\widehat{A} \mapsto \hbar^{1/2} \widehat{A}$, the expansion in $\hbar$ emerges clearly [31, p. 455]

$$\exp\{W[J;\bar{g}]\} = \exp\{-S[A_{\mathrm{cl}},J;\bar{g}]\}\times$$
$$\times\int\mathcal{D}_{\bar{g}}\widehat{A}\,\exp\left\{-\frac{1}{2}\int\mathrm{d}^d x\sqrt{\bar{g}(x)}\,\widehat{A}(x)(-\Box_{\bar{g}}+\mu^2+\xi\bar{R})\widehat{A}(x)+O(2\text{ loops})\right\}. \tag{3.29}$$

At order $\hbar$, we have arrived at an Gaussian integral over the scalar field $\widehat{A}$ that we can evaluate using Eq. (D.25) to find the expansion

$$\exp\{W[J;\bar{g}]\} = \exp\{-S[A_{\mathrm{cl}},J;\bar{g}]\}\,\mathrm{Det}\,(-\Box_{\bar{g}}+\mu^2+\xi\bar{R})^{-1/2}+O(2\text{ loops})\,, \tag{3.30}$$

respectively, after solving for W,

$$W[J;\bar{g}] = -S[A_{\mathrm{cl}},J;\bar{g}]-\frac{1}{2}\ln\mathrm{Det}\,(-\Box_{\bar{g}}+\mu^2+\xi\bar{R})+O(2\text{ loops})\,. \tag{3.31}$$

Yet, however, we cannot plug this result into the EA as it is defined in terms of $\bar{A}$, and not A_{cl}. Thus, we must link both these fields in terms of an expansion in $\hbar$. First, it is clear that

$$\bar{A} = A_{\mathrm{cl}}+O(\hbar) \quad\Leftrightarrow\quad A_{\mathrm{cl}} = \bar{A}+O(\hbar)\,. \tag{3.32}$$

Hence, a functional Taylor expansion of the classical action (3.24) in A_{cl} around $\bar{A}$ reads

$$S[A_{\mathrm{cl}},J;\bar{g}] = S[\bar{A},J;\bar{g}]+\int\mathrm{d}^d x\left(A_{\mathrm{cl}}-\bar{A}\right)(x)\frac{\delta S[\widehat{A},J;g]}{\delta\widehat{A}(x)}\bigg|_{\widehat{A}=\bar{A}}+O(\hbar)\,. \tag{3.33}$$

As $A_{\mathrm{cl}}-\bar{A}$ is of order $\hbar$ and

$$\frac{\delta S[\widehat{A},J;\bar{g}]}{\delta\widehat{A}}\bigg|_{\widehat{A}=A_{\mathrm{cl}}} = \frac{\delta S[\widehat{A},J;\bar{g}]}{\delta\widehat{A}}\bigg|_{\widehat{A}=A_{\mathrm{cl}}}+O(\hbar) = O(\hbar)\,, \tag{3.34}$$

it follows that

$$S[A_{\mathrm{cl}},J;\bar{g}] = S[\bar{A},J;\bar{g}]+O(2\text{ loops})\,. \tag{3.35}$$

Consequently the 1L-expansion of $W[J;\bar{g}]$ reads

$$W[J;\bar{g}] = -S[\bar{A},J;\bar{g}]-\frac{1}{2}\ln\mathrm{Det}\,(-\Box_{\bar{g}}+\mu^2+\xi\bar{R})+O(2\text{ loops})\,. \tag{3.36}$$

Plugging this expansion into the EA and using the identity $\ln\mathrm{Det}[\,\cdot\,] = \mathrm{Tr}\ln[\,\cdot\,]$, we have arrived at the 1L-expansion of the EA,[7]

$$\Gamma[\bar{A};\bar{g}] = S_{\mathrm{EH}}[\bar{g}]+S_{\mathrm{M}}[\bar{A};\bar{g}]+\frac{1}{2}\mathrm{Tr}\ln\,(-\Box_{\bar{g}}+\mu^2+\xi\bar{R})+O(2\text{ loops})\,. \tag{3.37}$$

[7]Note that the trace $\mathrm{Tr}=\mathrm{Tr}_S$ is taken on the Hilbert space of scalar fields, cf. appendix A.1.

Henceforth, we agree on explicitly dropping the terms $O(2\text{ loops})$ and denote by $\Gamma[\bar{A};\bar{g}]$ its 1L-approximation. Furthermore, we are especially interested in the functional

$$\boxed{\Gamma[\bar{g}] := \Gamma[0;\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + \frac{1}{2}\,\mathrm{Tr}\ln\left(-\Box_{\bar{g}} + \mu^2 + \xi\bar{R}\right)} \tag{3.38}$$

and additionally abbreviate, also using Eq. (3.4),

$$\boxed{\Gamma_{1\mathrm{L}}[\bar{g}] := \frac{1}{2}\,\mathrm{Tr}\ln\left(-\Box_{\bar{g}} + \mu^2 + \xi\bar{R}\right) \equiv \frac{1}{2}\,\mathrm{Tr}\ln\left[\mathscr{K}[\bar{g}]\right]}\,. \tag{3.39}$$

Note that the trace still must be regularized! Lastly, the functional integro-differential Eq. (3.23) for the one-loop approximation simplifies to

$$\exp\left\{-\Gamma[\bar{g}]\right\} = \int \mathcal{D}_{\bar{g}}A\, \exp\left\{-S[A;\bar{g}]\right\}\,, \tag{3.40}$$

respectively

$$\exp\left\{-\Gamma_{1\mathrm{L}}[\bar{g}]\right\} = \int \mathcal{D}_{\bar{g}}A\, \exp\left\{-S_{\mathrm{M}}[A;\bar{g}]\right\}\,. \tag{3.41}$$

CHAPTER 4

Quantization of metric fluctuations

Executive summary. In Euclidean conventions, we quantize gravity via a path integral approach and formally introduce the effective action for gravity. We thereby employ the background field method to rigorously ensure Background Independence of the construction and further make use of the Faddeev-Popov trick. We demonstrate the construction for quantum gravity based on the classical Einstein-Hilbert action as well as for higher-derivative gravity. Moreover, we explicitly deduce the one-loop effective action for these special cases.
What is new? Proof of the classical BRST invariance of the construction for an arbitrary weight function $Y^{\mu\nu}[g]$ in the gauge-fixing action. Expression for the one-loop effective action for non-vanishing fluctuation fields given by Eq. (4.69).

4.1. The background field method

The classical graviton field "lives" on a generic Euclidean d-dimensional background manifold with metric $\bar{g}_{\mu\nu}$ and usually is interpreted as the fluctuation around this background metric:[1]

$$g_{\mu\nu} := \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{4.1}$$

where we have introduced the full metric $g_{\mu\nu}$. The classical dynamics of this full metric are determined by means of some classical action $S_{\text{cl}}[g]$ that is a priori assumed to be invariant under general coordinate transformations. Infinitesimally these are given by (cf. appendix F.1)

$$\delta g_{\mu\nu} = L_V g_{\mu\nu} = V^\rho \partial_\rho g_{\mu\nu} + V^\rho \partial_\mu g_{\rho\nu} + V^\rho \partial_\nu \phi_{\mu\rho} = D_\mu V_\nu + D_\nu V_\mu \tag{4.2}$$

where L is the Lie derivative and V^μ an infinitesimal vector field.

[1]Other parametrizations are possible, the most general being the exponential parametrization [32]. Here, different from Chapter 3, the background manifold is *not* compact.

In this thesis we apply two different ansätze for the classical action. On the one hand we will make use of the *Einstein-Hilbert action*

$$S_{\mathrm{EH}}[g] \coloneqq 2\kappa^2 \int \mathrm{d}^d x \sqrt{g}\,(-R + 2\Lambda)\,, \tag{4.3}$$

entailing the coupling constant $\kappa^2 \coloneqq 1/(32\pi G)$ and the consmological constant Λ. On the other hand, we will employ the *higher-derivative action*

$$\begin{aligned} S_{\mathrm{h.-d.}}[g] &\coloneqq \int \mathrm{d}^d x \sqrt{g}\ \left[a R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + b R_{\mu\nu} R^{\mu\nu} + c R^2 \right] \\ &= \int \mathrm{d}^d x \sqrt{g}\ \left[\frac{1}{2 f_2^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{1}{f_1^2} E - \frac{1}{6 f_0^2} R^2 \right] \end{aligned} \tag{4.4}$$

of the curvature invariants $I_1 = R_{\mu\nu}R^{\mu\nu}$, $I_2 = R^2$ and $I_3 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ that contain only fourth-order derivatives and in the second version of the action functional are parametrized as follows: The coupling f_0 weighs the squared scalar curvature I_2 while the couplings f_1 and f_2 weigh, respectively, the integrand of the Gauss-Bonnet term $E = I_3 - 4I_1 + I_2$ and the squared Weyl tensor

$$C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = I_3 - \frac{4}{d-2} I_1 + \frac{2}{(d-1)(d-2)} I_2\,. \tag{4.5}$$

The couplings a, b and c are related to the couplings f_0, f_1 and f_2 by [33]

$$a = \frac{1}{2f_2^2} + \frac{1}{f_1^2}\ ,\ b = -\frac{2}{(d-2)f_2^2} - \frac{4}{f_1^2}\ ,\ c = \frac{1}{f_1^2} - \frac{1}{6f_0^2} + \frac{1}{(d-1)(d-2)f_2^2}\,. \tag{4.6}$$

In *four dimensions*, $d = 4$, it can be shown that the Gauss-Bonnet term is undynamical [34], $\frac{\delta}{\delta g_{\mu\nu}} \int \mathrm{d}^4 x \sqrt{g}\, E = 0$, such that in this case it can be removed from the action functional which then is parametrized by two couplings only. Eliminating the coupling f_1 by setting $f_1^2 = -2f_2^2$ turns out to be particularly convenient because thereby E is removed from the action functional by substracting it from the the squared Weyl tensor:

$$\frac{1}{2f_2^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \frac{1}{f_1^2} E = \frac{1}{2f_2^2} (C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} - E) = \frac{1}{2f_2^2} \left(2I_1 - \frac{2}{3} I_2 \right)\,. \tag{4.7}$$

In terms of the couplings a, b and c this procedure corresponds to setting

$$a = 0\ ,\ b = \frac{1}{f_2^2}\ ,\ c = -\frac{1}{3}\left(\frac{1}{f_2^2} + \frac{1}{2f_0^2} \right)\,. \tag{4.8}$$

Thus, the higher-derivative action in $d = 4$ w.l.o.g. is given by [27]

$$S_{\text{h.-d.}}[g] \coloneqq \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{f_2^2}\left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) - \frac{1}{6f_0^2}R^2\right]. \tag{4.9}$$

This action functional is written in such a form that the only term manifestly breaking the Weyl symmetry $g_{\mu\nu} \mapsto \mathrm{e}^f g_{\mu\nu}$, with some arbitrary function f, is the "R^2"-term parametrized by the coupling f_0^2. Therewith, we can make out two special cases: Firstly, the limit $f_0^2 \to \infty$, sometimes referred to as *Weyl's higher-derivative gravity*, in which the action becomes manifestly invariant under Weyl transformations. Secondly, the limit $f_2^2 \to \infty$ in which the action reduces to a surface term that has no degrees of freedom. (Later this property will be resembled in the propagator that becomes singular in the limit $f_2^2 \to \infty$.)

Lastly, note that as the volume element, the scalar, Ricci and Riemann curvature are well-defined globally, the Einstein-Hilbert action as well as the higher-derivative action clearly are invariant under general coordinate transfomations.

For a quantum treatment we also employ the background field technique [35–37] and interpret $h_{\mu\nu}$ as the expectation value of the quantum field $\widehat{h}_{\mu\nu}$ that, fully analogous to the classical field, is the quantum fluctuation around the background metric $\bar{g}_{\mu\nu}$:

$$\widehat{g}_{\mu\nu} \coloneqq \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}\,. \tag{4.10}$$

The expectation value of the full quantum metric $\widehat{g}_{\mu\nu}$ defined herein thus is given by $g_{\mu\nu}$. As before, we will indicate geometric objects arising from the background metric $\bar{g}_{\mu\nu}$ with a "bar", e.g. a covariant derivate $\bar{D}_\mu$. Geometric objects arising from from the full quantum metric $\widehat{g}_{\mu\nu}$ are denoted without specific indication, i.e., D_μ is the covariant derivative associated to the Levi-Civita connection $\Gamma^\alpha_{\mu\nu}$ given by $\widehat{g}_{\mu\nu}$ et cetera.
The *bare* action $S_{\text{cl}}[\widehat{g}]$ of course is still assumed to be invariant under general coordinate transformations. We promote these to *quantum gauge transformations* $\delta^{(Q)}$ of the quantum field $\widehat{h}_{\mu\nu}$ by defining

$$\delta^{(Q)}\widehat{h}_{\mu\nu} \coloneqq L_V\widehat{g}_{\mu\nu} \quad \text{and} \quad \delta^{(Q)}\bar{g}_{\mu\nu} \coloneqq 0 \tag{4.11}$$

where V^μ still is an infinitesimal vector field.

We obtain the expectation value via a path integral approach, making use of the Faddeev-Popov trick that can be straightforwardly applied to gravity [26, 36, 38–42]. This trick consists in fixing the quantum gauge transformation in order to avoid the multiple contribution of physically equivalent gauge configurations to the path integral. We can do so by supplementing the bare action with the gauge-fixing action

$$S_{\mathrm{GF}}[\widehat{h};\bar{g}] \coloneqq \frac{1}{2\alpha}\int \mathrm{d}^d x\sqrt{\bar{g}}\, F_\mu(\widehat{h};\bar{g})Y^{\mu\nu}[\bar{g}]F_\nu(\widehat{h};\bar{g}) \tag{4.12}$$

for the gauge-fixing condition $F_\mu(\widehat{h};\bar{g}) = 0$, where α is a gauge-fixing parameter and $Y^{\mu\nu}[\bar{g}]$ is a "weight function" in form of either a fixed tensor structure or a differential operator built from covariant derivatives such that $S_{\mathrm{GF}}[\widehat{h};\bar{g}]$ contains covariant derivatives of the same order as $S_{\mathrm{cl}}[\widehat{g}]$. In this case, we require the operator $Y^{\mu\nu}[\bar{g}]$ to behave symetrically under partial differentiations: $\int \mathrm{d}^d x\sqrt{\bar{g}}\, A_\mu Y^{\mu\nu}[\bar{g}]^{\mathrm{diff}} B_\nu = \int \mathrm{d}^d x\sqrt{\bar{g}}\, B_\mu Y^{\mu\nu}[\bar{g}]^{\mathrm{diff}} A_\nu$, with A_μ and B_μ ordinary covariant vector fields.[2] To account for having fixed the gauge, we must further supplement the bare action with the Faddeev-Popov action for the Faddeev-Popov ghosts C^μ and $\bar{C}_\mu$ that results from the gauge-fixing condition $F_\mu(\widehat{h};\bar{g}) = 0$,

$$\begin{aligned} S_{\mathrm{gh},1}[\widehat{h},\bar{C},C;\bar{g}] &\coloneqq -\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu \bar{g}^{\mu\nu}\delta^{(Q)}F_\nu(\widehat{h};\bar{g})\bigg|_{V=C} \\ &= -\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu \bar{g}^{\mu\nu}\frac{\partial F_\nu(\widehat{h};\bar{g})}{\partial \widehat{h}_{\alpha\beta}} L_C(\bar{g}_{\alpha\beta}+\widehat{h}_{\alpha\beta})\,, \end{aligned} \tag{4.13}$$

where L is the Lie derivative. In case that $Y^{\mu\nu}[\bar{g}]$ is a differential operator, its contribution to the gauge-fixing procedure must be accounted for by, yet again, further supplementing the bare action with a second Faddeev-Popov action of a third ("real") ghost b_μ [43–45],

$$S_{\mathrm{gh},2}[b;\bar{g}] = \frac{1}{2\alpha}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\, b_\mu Y^{\mu\nu}[\bar{g}] b_\nu\,. \tag{4.14}$$

[2]To antizipate the reason for this condition: On the one hand, it will ensure the BRST invariance of the bare action; on the other hand it is required to obtain $\bar{g}_{\nu\rho}(y)\frac{\delta^2 S_{\mathrm{gh},2}[b;\bar{g}]}{\delta b_\mu(x)\delta b_\rho(y)} = \sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}\langle x,\mu| - Y[\bar{g}]|y,\nu\rangle$ later on.

The Faddeev-Popov construction is marked with the breaking of gauge invariance. This downside is compensated by the invariance of the full bare action

$$S[\widehat{h}, \bar{C}, C, b; \bar{g}] := S_{\mathrm{cl}}[\bar{g} + \widehat{h}] + S_{\mathrm{GF}}[\widehat{h}; \bar{g}] + S_{\mathrm{gh},1}[\widehat{h}, \bar{C}, C; \bar{g}] + S_{\mathrm{gh},2}[b; \bar{g}]\,, \quad (4.15)$$

under the classical BRST transformations

$$\begin{aligned} \delta_\varepsilon \widehat{h}_{\mu\nu} &:= \varepsilon L_C \widehat{g}_{\mu\nu} = \varepsilon L_C (\bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}) \\ \delta_\varepsilon \bar{g}_{\mu\nu} &:= 0 \\ \delta_\varepsilon C^\mu &:= \varepsilon C^\nu \partial_\nu C^\mu \\ \delta_\varepsilon \bar{C}_\mu &:= \varepsilon \alpha^{-1} Y^\nu{}_\mu[\bar{g}] F_\nu(\widehat{h}; \bar{g}) \\ \delta_\varepsilon b_\mu &:= 0\,. \end{aligned} \quad (4.16)$$

Here, ε is an anticommuting and x-independent parameter. Alternatively, one could have defined the anticommuting BRST operator s to act as $s\widehat{h}_{\mu\nu} := L_C \widehat{g}_{\mu\nu}$ et cetera. Importantly, note that the BRST operation is nilpotent (for the last operation only on-shell, however). A proof of its nilpotence and of the invariance $\delta_\varepsilon S[\widehat{h}, \bar{C}, C; \bar{g}] = 0$ can be found in appendices F.2 and F.3.

After these preparations, we can finally become more precise and define the generating functional for the connected Green's function [39], also called the *Schwinger* functional, as

$$\boxed{\begin{aligned} \exp\{W[t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu, d^\mu; \bar{g}_{\mu\nu}]\} := \int \mathcal{D}\mu[\widehat{h}, \bar{C}, C, b; \bar{g}] \exp\Bigg\{ &- S[\widehat{h}, \bar{C}, C, b; \bar{g}] \\ &+ \int \mathrm{d}^d x \sqrt{\bar{g}}\, \Big[t^{\mu\nu} \widehat{h}_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu + d^\mu b_\mu \Big] \Bigg\}. \end{aligned}} \quad (4.17)$$

Here, we have coupled the fields $\chi = (\chi^1, \chi^2, \chi^3, \chi^4)^T := (\widehat{h}, \bar{C}, C, b)^T$ to the sources $J = (J_1, J_2, J_3, J_4) := (t, \sigma, \bar{\sigma}, d)$. By introducing the J-dependent action[3]

$$\tilde{S}[\chi; J; \bar{g}] := S[\widehat{h}, \bar{C}, C, b; \bar{g}] - S_{\mathrm{source}}[\widehat{h}, \bar{C}, C, b; t, \sigma, \bar{\sigma}, d; \bar{g}] \quad (4.18)$$

[3]This shorthand notation also allows the utilization of DeWitt's notation: $J_i \phi^i = \int \mathrm{d}^d x \sqrt{\bar{g}}\, \left[t^{\mu\nu} h_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu + d^\mu b_\mu \right]$. However, we will hardly use it in this thesis.

with

$$S_{\text{source}}[\widehat{h}, \bar{C}, C, b; t, \sigma, \bar{\sigma}, d; \bar{g}] := \int \mathrm{d}^d x \sqrt{\bar{g}} \left[t^{\mu\nu} \widehat{h}_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu + d^\mu b_\mu \right] , \tag{4.19}$$

the Schwinger functional can be compactly written in this shorthand notation:

$$\exp\left\{ W[J; \bar{g}] \right\} = \int \mathcal{D}\mu[\chi; \bar{g}] \, \exp\left\{ -\tilde{S}[\chi, J; \bar{g}] \right\} . \tag{4.20}$$

Furthermore, we specify the measure to the BRST-invariant measure given by Eqs. (D.7), (D.8) and (D.10):

$$\begin{aligned} \mathcal{D}\mu[\widehat{h}, \bar{C}, C, b; \bar{g}] &= \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \mathcal{D}_{\bar{g}} C^\mu \mathcal{D}_{\bar{g}} \bar{C}_\mu \mathcal{D}_{\bar{g}} b_\mu \\ &= \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8} - \frac{3d-2}{4}} \prod_{\mu \geq \nu} \mathrm{d}\widehat{h}_{\mu\nu}(x) \prod_\alpha \mathrm{d}C^\alpha(x) \mathrm{d}\bar{C}_\alpha(x) \mathrm{d}b_\alpha(x) , \end{aligned} \tag{4.21}$$

or, if $Y^{\mu\nu}[\bar{g}]$ does not contain derivatives and hence $b_\mu \equiv 0$,

$$\begin{aligned} \mathcal{D}\mu[\widehat{h}, \bar{C}, C; \bar{g}] &= \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \mathcal{D}_{\bar{g}} C^\mu \mathcal{D}_{\bar{g}} \bar{C}_\mu \\ &= \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8} - \frac{d}{2}} \prod_{\mu \geq \nu} \mathrm{d}\widehat{h}_{\mu\nu}(x) \prod_\alpha \mathrm{d}C^\alpha(x) \mathrm{d}\bar{C}_\alpha(x) . \end{aligned} \tag{4.22}$$

The Schwinger functional by construction fails to be invariant under the quantum gauge transformations $\delta^{(Q)}$ as $\delta^{(Q)} S_{\text{GF}}[\widehat{h}; \bar{g}] \neq 0$. On the one hand, its gauge invariance can be restored in form of its invariance under the classical BRST transformations, $\delta_\varepsilon W[J; \bar{g}] = -\langle \delta_\varepsilon S_{\text{source}}[\chi, J; \bar{g}] \rangle = 0$. All ingredients of $W[J; \bar{g}]$ but the source action are BRST invariant per construction; this condition towards the source action is a *modified Ward identity*. One the other hand, the use of the background field method further induces the invariance of the Schwinger functional (for any gauge-fixing condition) under *background gauge transformations* $\delta^{(B)}$ which are defined by

$$\begin{aligned} \delta^{(B)} \widehat{h}_{\mu\nu} &:= L_V \widehat{h}_{\mu\nu} \\ \delta^{(B)} \bar{g}_{\mu\nu} &:= L_V \bar{g}_{\mu\nu} \\ \delta^{(B)} C^\mu &:= L_V C^\mu \\ \delta^{(B)} \bar{C}_\mu &:= L_V \bar{C}_\mu \\ \delta^{(B)} b_\mu &:= L_V b_\mu \end{aligned} \tag{4.23}$$

where, again, V^μ is an infinitesimal vector field. Note that $\delta^{(Q)}\widehat{g}_{\mu\nu} = \delta^{(B)}\widehat{g}_{\mu\nu} = L_V\widehat{g}_{\mu\nu}$ but the distribution of the Lie derivative over the background field $\bar{g}_{\mu\nu}$ and the quantum field $\widehat{h}_{\mu\nu}$ differs. A proof of the invariance $\delta^{(B)}\mathrm{e}^{W[J;\bar{g}]} = 0$ can be found in appendix F.4.

By means of the generating functional $\mathrm{e}^{W[J;\bar{g}]}$ we introduce the J-dependent expectation values

$$\begin{aligned} h^J_{\mu\nu}(x) &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta t^{\mu\nu}(x)} \equiv \langle \widehat{h}_{\mu\nu}(x)\rangle^J \\ \bar{\xi}^J_\mu(x) &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta \sigma^{\mu}(x)} \equiv \langle \bar{C}_{\mu}(x)\rangle^J \\ \xi^\mu_J(x) &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta \bar{\sigma}_{\mu}(x)} \equiv \langle C^{\mu}(x)\rangle^J \\ \zeta^J_\mu(x) &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta d^{\mu}(x)} \equiv \langle b_{\mu}(x)\rangle^J \,. \end{aligned} \tag{4.24}$$

Again, we can absorb these definitions into a shorthand notation by introducing $\phi = (\phi^1,\phi^2,\phi^3,\phi^4)^T := (h,\bar{\xi},\xi,\zeta)^T$:

$$\phi^i_J(x) = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W[J;\bar{g}]}{\delta J_i(x)} \equiv \langle \chi^i(x)\rangle^J \,. \tag{4.25}$$

We drop the super- and subscript J sometimes for notational convenience, $\phi \equiv \phi_J$, and sometimes to indicate the special case $\phi = \phi_{J=0}$ which is the *actual* expectation value $\phi = \langle\chi\rangle$. It will always be clear from context, which of these two cases we apply – the latter precludes the former. In the next definition for example, we stick with the former notation.

We solve Eq. (4.24) for J to obtain $J = J[\phi;\bar{g}]$ and define the EA as the Legendre transform of the functional $W[J;\bar{g}]$,

$$\Gamma[\phi;\bar{g}] := J_i[\phi;\bar{g}]\phi^i - W[J[\phi;\bar{g}];\bar{g}]\,, \tag{4.26}$$

or more precisely,

$$\boxed{\begin{aligned}\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}] = \int \mathrm{d}^d x\sqrt{\bar{g}}\ &\left[t^{\mu\nu}[\phi;\bar{g}]h_{\mu\nu} + \sigma^\mu[\phi;\bar{g}]\bar{\xi}^\mu + \bar{\sigma}_\mu[\phi;\bar{g}]\xi^\mu + d^\mu[\phi;\bar{g}]\zeta_\mu\right] \\ &- W[J[\phi;\bar{g}];\bar{g}]\end{aligned}}\,. \tag{4.27}$$

It is a straightforward calculation (see appendix F.5) to show that Eq. (4.24) is inverted by the relations

$$\begin{aligned} t^{\mu\nu}(x) &= + \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta h_{\mu\nu}(x)} \\ \sigma^{\mu}(x) &= - \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta \bar{\xi}_{\mu}(x)} \\ \bar{\sigma}_{\mu}(x) &= - \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta \xi^{\mu}(x)} \\ d^{\mu}(x) &= - \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta \zeta_{\mu}(x)} \end{aligned} \tag{4.28}$$

which in shorthand notation can be absorbed into

$$J_i(x) = \frac{(-1)^{|\phi^i|}}{\sqrt{\bar{g}(x)}} \frac{\delta\Gamma[\phi;\bar{g}]}{\delta\phi^i(x)}\,. \tag{4.29}$$

Here, $|\phi^i|$ denotes the Graßmann parity of the variable ϕ^i, i.e., $|\phi^i| = 0$ for ϕ^i even (here only $\phi^1 = h$) and $|\phi^i| = 1$ for ϕ^i odd (here $\phi^2 = \bar{\xi}$, $\phi^3 = \xi$ and $\phi^4 = \zeta$). Therewith, it is easy to verify that the effective action (4.27) fulfills the functional integro-differential equation

$$\begin{aligned} \exp\left\{-\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]\right\} = \int \mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\mathcal{D}_{\bar{g}}C^{\mu}\mathcal{D}_{\bar{g}}\bar{C}_{\mu}\mathcal{D}_{\bar{g}}b_{\mu} \exp\Bigg\{ &- S[\widehat{h},\bar{C},C,b;\bar{g}] \\ + \int \mathrm{d}^d x \Bigg[(\widehat{h}_{\mu\nu} - h_{\mu\nu})(x) \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta h_{\mu\nu}(x)} &- \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta\xi^{\mu}(x)}(C^{\mu} - \xi^{\mu})(x) \\ - \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta\bar{\xi}_{\mu}(x)}(\bar{C}_{\mu} - \bar{\xi}_{\mu})(x) &- \frac{\delta\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}]}{\delta\zeta_{\mu}(x)}(b_{\mu} - \zeta_{\mu})(x) \Bigg] \Bigg\}\,. \end{aligned} \tag{4.30}$$

4.2. Quantization of the Einstein-Hilbert action

For the application to the quantization of the Einstein-Hilbert action $S_{\mathrm{EH}}[\widehat{g}]$ defined by Eq. (4.3) we will solely work with the linear, and thus convenient, gauge-fixing condition

$$F_\mu(\widehat{h};\bar{g}) = \sqrt{2}\mathscr{F}_\mu^{\alpha\beta}[\bar{g}]\,\widehat{h}_{\alpha\beta} \quad \text{with} \quad \mathscr{F}_\mu^{\alpha\beta}[\bar{g}] = \delta_\mu^\beta \bar{g}^{\alpha\gamma}\bar{D}_\gamma - \beta\bar{g}^{\alpha\beta}\bar{D}_\mu\,, \tag{4.31}$$

i.e., $F_\mu(\widehat{h};\bar{g}) = \sqrt{2}(\bar{D}^\alpha \widehat{h}_{\alpha\mu} - \beta\bar{D}_\mu \widehat{h}_\alpha{}^\alpha)$. Therewith the gauge-fixing action (4.12) contains second-order covariant derivatives, just as the Einstein-Hilbert action (4.3); thence to obtain an applicable gauge-fixing action it is sufficient to consider the weight function

$$Y^{\mu\nu}[\bar{g}] = \kappa^2\bar{g}^{\mu\nu}\,. \tag{4.32}$$

With this gauge-fixing condition and weight function, the gauge-fixing action reads

$$S_{\mathrm{GF}}[\widehat{h};\bar{g}] = \frac{1}{\alpha}\kappa^2\int \mathrm{d}^dx\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}(\mathscr{F}_\mu^{\alpha\beta}[\bar{g}]\,\widehat{g}_{\alpha\beta})(\mathscr{F}_\nu^{\rho\sigma}[\bar{g}]\,\widehat{g}_{\rho\sigma}) \tag{4.33}$$

and the Faddeev-Popov action becomes (see appendix F.6 for details)

$$S_{\mathrm{gh},1}[\widehat{h},\bar{C},C;\bar{g}] \equiv S_{\mathrm{gh}}[\widehat{h},\bar{C},C;\bar{g}] = -\sqrt{2}\int \mathrm{d}^dx\sqrt{\bar{g}}\,\bar{C}_\mu \mathscr{M}[\widehat{g},\bar{g}]^\mu{}_\nu C^\nu\,, \tag{4.34}$$

where we have introduced the Faddeev-Popov operator

$$\mathscr{M}[\widehat{g},\bar{g}]^\mu{}_\nu = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(\widehat{g}_{\rho\nu}D_\sigma + \widehat{g}_{\sigma\nu}D_\rho) - 2\beta\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda\widehat{g}_{\sigma\nu}D_\rho\,. \tag{4.35}$$

As can easily be seen, the integral over the ghost fields C^μ and $\bar{C}_\mu$ contained in the path integral (4.17) with the above gauge-fixing condition is simply the exponentiation of Det $(\mathscr{M}[\widehat{g},\bar{g}]^{\boldsymbol{\cdot}}{}_{\boldsymbol{\cdot}})$. Furthermore, we usually employ the *harmonic gauge* which is given for $\alpha = 1$ and $\beta = 1/2$.

As $Y^{\mu\nu}[\bar{g}]$ does not contain derivatives, we can overall set $b_\mu \equiv 0$ such that the full action (4.15) reduces to

$$S[\widehat{h},\bar{C},C;\bar{g}] \coloneqq S_{\mathrm{EH}}[\bar{g}+\widehat{h}] + S_{\mathrm{GF}}[\widehat{h};\bar{g}] + S_{\mathrm{gh}}[\widehat{h},\bar{C},C;\bar{g}]\,. \tag{4.36}$$

and Eq. (4.18) reduces to, with $\chi \equiv (\chi^1,\chi^2,\chi^3)^T = (\widehat{h},\bar{C},C)^T$ and $J \equiv (J_1,J_2,J_3) = (t,\sigma,\bar{\sigma})$,

$$\tilde{S}[\chi;J;\bar{g}] \coloneqq S[\widehat{h},\bar{C},C;\bar{g}] - \int \mathrm{d}^dx\sqrt{\bar{g}}\,\left[t^{\mu\nu}\widehat{h}_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu\bar{C}_\mu\right]\,. \tag{4.37}$$

4.2.1. *The one-loop expansion*

It is clear that in order to obtain one-loop expressions for correlators as, say,

$$\begin{aligned}\left\langle\widehat{h}_{\mu\nu}(y)\widehat{h}^{\rho\sigma}(x)\right\rangle &= \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}I^{\rho\sigma\alpha\beta}[\bar{g}]\frac{\delta^2 W[t,\sigma,\bar{\sigma};\bar{g}]}{\delta t^{\mu\nu}(y)\delta t^{\alpha\beta}(x)}\bigg|_{(t,\sigma,\bar{\sigma})=0} \\ \left\langle\bar{C}_\mu(y)C^\nu(x)\right\rangle &= \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 W[t,\sigma,\bar{\sigma};\bar{g}]}{\delta\sigma^\mu(y)\delta\bar{\sigma}_\nu(x)}\bigg|_{(t,\sigma,\bar{\sigma})=0},\end{aligned} \tag{4.38}$$

we have to expand the action $\tilde{S}[\chi;J;\bar{g}]$ on the RHS of Eq. (4.17). To do so, we introduce the field $\chi_{\text{cl}} = (\chi^1_{\text{cl}},\chi^2_{\text{cl}},\chi^3_{\text{cl}})^T$ as the solution to the classical equations of motion of the action (4.37),

$$\frac{\delta\tilde{S}[\chi;J;\bar{g}]}{\delta\chi^i}\bigg|_{\chi^i=\chi^i_{\text{cl}}} = 0\,. \tag{4.39}$$

Then, we expand the action $\tilde{S}[\chi;J;\bar{g}]$, with the gauge-fixing action (4.33) and Faddeev-Popov action (4.34), in the variable χ around χ_{cl} up to second order in χ using Eq. (C.5):

$$\begin{aligned}\tilde{S}[\chi;J;\bar{g}] = \tilde{S}[\chi_{\text{cl}};J;\bar{g}] &+ \frac{1}{2}\sum_{i,j}\int \mathrm{d}^d x\int \mathrm{d}^d y(\chi-\chi_{\text{cl}})^i(x) \\ &\times \frac{\delta^2\tilde{S}[\chi;J;\bar{g}]}{\delta\chi^j(x)\delta\chi^i(y)}\bigg|_{\chi=\chi_{\text{cl}}}(\chi-\chi_{\text{cl}})^j(y) + O(\chi^3)\,.\end{aligned} \tag{4.40}$$

Inserting this expansion up to second order in χ into the exponent on the RHS of Eq. (4.17) amounts to an expansion up to first order in $\hbar$ (here, we had set $\hbar \equiv 1$, see Footnote 6 of Chapter 3). Thus, the expansion (4.40) in this role leads to a one-loop approximation of the functional e^W.

Moreover, it is not difficult to realize that if one is not interested in correlators mixing the metric fluctuation $\widehat{h}_{\mu\nu}$ and the ghost fields $\bar{C}_\mu$ and C^μ – which we will not be – it is in fact sufficient to expand $S[\chi;\bar{g}]$ around the trivial solution $\chi_{\rm cl}=0$:

$$S[\chi;\bar{g}] = S_{\rm EH}[\bar{g}] + \frac{1}{2}\int \mathrm{d}^d x \int \mathrm{d}^d y\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\rm EH}[\bar{g}+\widehat{h}] + S_{\rm GF}[\widehat{h};\bar{g}])}{\delta \widehat{h}_{\mu\nu}(x)\delta\widehat{h}_{\rho\sigma}(y)}\bigg|_{\widehat{h}=0} \widehat{h}_{\rho\sigma}(y) + \int \mathrm{d}^d x \int \mathrm{d}^d y\, \bar{C}_\rho(x) \frac{\delta^2 S_{\rm gh}[\chi;\bar{g}]}{\delta C^\mu(x)\delta \bar{C}_\rho(y)}\bigg|_{\chi=0} C^\mu(y) + O(\chi^3)\,. \tag{4.41}$$

The former second-order term in this expansion leads us to the important definition of the *inverse propagator* at vanishing metric fluctuation $\mathscr{U}[0;\bar{g}]$ given by

$$\boxed{\int \mathrm{d}^d x \int \mathrm{d}^d y\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\rm EH}[\bar{g}+\widehat{h}] + S_{\rm GF}[\widehat{h};\bar{g}])}{\delta \widehat{h}_{\mu\nu}(x)\delta\widehat{h}_{\rho\sigma}(y)}\bigg|_{\widehat{h}=0} \widehat{h}_{\rho\sigma}(y) =: \int \mathrm{d}^d x\sqrt{\bar{g}}\;\widehat{h}_{\mu\nu}(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\rm diff}\,\widehat{h}^{\rho\sigma}\,.} \tag{4.42}$$

Meanwhile, the latter second-order term in the expansion takes on a familiar form:

$$\int \mathrm{d}^d x \int \mathrm{d}^d y\, \bar{C}_\rho(x) \frac{\delta^2 S_{\rm gh}[\chi;\bar{g}]}{\delta C^\mu(x)\delta \bar{C}_\rho(y)}\bigg|_{\chi=0} C^\mu(y) = -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu \mathscr{M}[\bar{g},\bar{g}]^\mu{}_\nu C^\nu \equiv +S_{\rm gh}[0,\bar{C},C;\bar{g}]\,. \tag{4.43}$$

Interestingly, this expansion, which amounts to the linearization of the theory given by the Einstein-Hilbert action as the bare action $S_{\rm cl}[\widehat{g}]$, allows for the interpretation of its second-order terms as a classical matter action for the (classical) graviton field $h_{\mu\nu}$ and the (classical) ghost fields $\bar{\xi}_\mu$ and ξ^μ:

$$S_{\rm M}[h,\bar{\xi},\xi;\bar{g}] := S_{\rm graviton}[h;\bar{g}] + S_{\rm gh}[0,\bar{\xi},\xi;\bar{g}]\,, \tag{4.44}$$

where we interpret the term of order $h_{\mu\nu}^2$ in the linearized theory as a matter action for the fundamental graviton field $h_{\mu\nu}$:

$$S_{\rm graviton}[h_{\bullet\bullet};\bar{g}_{\bullet\bullet}] := \frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\;h_{\mu\nu}(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\rm diff}\,I[\bar{g}]^{\rho\sigma\alpha\beta}h_{\alpha\beta}\,, \tag{4.45}$$

with $I[\bar{g}]^{\rho\sigma\alpha\beta} = \frac{1}{2}(\bar{g}^{\rho\alpha}\bar{g}^{\sigma\beta} + \bar{g}^{\rho\beta}\bar{g}^{\sigma\alpha})$. The operator $\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}$ will be explicitly stated in the next subsection (for a gauge-fixing condition of type (4.31) it is given by Eq. (4.75) and further specified to the harmonic gauge by Eq. (4.77)). Furthermore, it is evident that the matter action for the classical ghost fields $\bar{\xi}_\mu$ and ξ^μ is determind by the Faddeev-Popov action (4.34) at $h_{\mu\nu} = 0$:

$$S_{\mathrm{gh}}[0, \bar{\xi}, \xi; \bar{g}] = -\sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{\xi}_\mu \mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu \xi^\nu \,, \tag{4.46}$$

where the Faddeev-Popov operator $\mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu$ also is explicitly stated in the next subsection (for a general gauge is given by Eq. (4.74) and for the harmonic gauge by Eq. (4.76)). All in all, the expansion (4.41) thus has become, neglecting higher-order terms,

$$S[h, \bar{\xi}, \xi; \bar{g}] = S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{M}}[h, \bar{\xi}, \xi; \bar{g}] + O(2 \text{ loops}) \,, \tag{4.47}$$

which resembles the interpretation of fluctuation and ghost fields as matter fields on a classical background spacetime. Next, we will finally assemble the generating functional of the connected Green's functions (4.17) at one-loop,

$$\begin{aligned} \exp\{W[t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu; \bar{g}_{\mu\nu}]\} \coloneqq \int \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \mathcal{D}_{\bar{g}} C^\mu \mathcal{D}_{\bar{g}} \bar{C}_\mu \exp\Bigg\{ &- S[\widehat{h}, \bar{C}, C; \bar{g}] \\ &+ \int \mathrm{d}^d x \sqrt{\bar{g}} \left[t^{\mu\nu} \widehat{h}_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu \right] \Bigg\}, \end{aligned} \tag{4.48}$$

by plugging the expansion (4.47) into the RHS which now is, in fact, of such a form that we can fully analytically perform the path integrals:

$$\begin{aligned} &\exp\{W[t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu; \bar{g}_{\mu\nu}]\} \\ &= \exp\{-S_{\mathrm{EH}}[\bar{g}]\} \int \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \exp \int \mathrm{d}^d x \sqrt{\bar{g}} \left\{ -\frac{1}{2} \widehat{h}_{\mu\nu} \mathscr{U}[0; \bar{g}]^{\mu\nu}{}_{\rho\sigma} \widehat{h}^{\rho\sigma} + t^{\mu\nu} \widehat{h}_{\mu\nu} \right\} \\ &\quad \times \int \mathcal{D}_{\bar{g}} C^\mu \mathcal{D}_{\bar{g}} \bar{C}_\mu \exp \int \mathrm{d}^d x \sqrt{\bar{g}} \left\{ \bar{C}_\mu \left(\sqrt{2} \mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu \right) C^\nu + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu \right\}. \end{aligned} \tag{4.49}$$

In order to apply the Gaussian path integrals (D.35) and (D.37), we shift the integrations variables as

$$\begin{aligned} \widehat{h}_{\mu\nu} &\mapsto \widehat{h}_{\mu\nu} - t_{\rho\sigma} (\mathscr{U}[0; \bar{g}]^{-1})^{\rho\sigma}{}_{\mu\nu} \\ \bar{C}_\mu &\mapsto \bar{C}_\mu + \bar{\sigma}_\nu (\mathscr{M}[\bar{g}, \bar{g}]^{-1})^\nu{}_\mu \\ C^\mu &\mapsto C^\mu - (\mathscr{M}[\bar{g}, \bar{g}]^{-1})^\mu{}_\nu \sigma^\nu \end{aligned} \tag{4.50}$$

which leads to, using the linearity of each path integral and partial integration when "completing the square",

$$\begin{aligned}\exp\left\{W[t^{\mu\nu},\sigma^{\mu},\bar{\sigma}_{\mu};\bar{g}_{\mu\nu}]\right\} = \exp\left\{-S_{\mathrm{EH}}[\bar{g}]\right\}\mathrm{Det}\left(\mathscr{U}[0;\bar{g}]^{\cdot\cdot}{}_{\cdot\cdot}\right)^{-1/2}\mathrm{Det}\left(\sqrt{2}\mathscr{M}[\bar{g},\bar{g}]^{\cdot}{}_{\cdot}\right)\\ \times\exp\left\{\frac{1}{2}\int\mathrm{d}^dx\sqrt{\bar{g}}\,t_{\mu\nu}\left(\mathscr{U}[0;\bar{g}]^{-1}\right)^{\mu\nu}{}_{\rho\sigma}t^{\rho\sigma}\right.\\ \left.+\int\mathrm{d}^dx\sqrt{\bar{g}}\,\bar{\sigma}_{\mu}\left((\sqrt{2}\mathscr{M}[\bar{g},\bar{g}])^{-1}\right){}^{\mu}{}_{\nu}\sigma^{\nu}\right\}.\end{aligned} \tag{4.51}$$

Here, we are only interested in the $(t,\sigma,\bar{\sigma})$-depence of $W[t,\sigma,\bar{\sigma};\bar{g}]$ and thus, for the moment, we may absorve everything else into the normalization constant:

$$\begin{aligned}W[t,\sigma,\bar{\sigma};\bar{g}] =& \frac{1}{2}\int\mathrm{d}^dx\sqrt{\bar{g}}\,t_{\mu\nu}\left(\mathscr{U}[0;\bar{g}]^{-1}\right)^{\mu\nu}{}_{\rho\sigma}t^{\rho\sigma}\\ &+\int\mathrm{d}^dx\sqrt{\bar{g}}\,\bar{\sigma}_{\mu}\left((\sqrt{2}\mathscr{M}[\bar{g},\bar{g}])^{-1}\right){}^{\mu}{}_{\nu}\sigma^{\nu}\,.\end{aligned} \tag{4.52}$$

Therewith, we find that the expectation values, given at the introduction of this subsection, in the one-loop approximation of e^{W} also are given by

$$\begin{aligned}\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 W[t,\sigma,\bar{\sigma};\bar{g}]}{\delta t^{\mu\nu}(y)\delta t^{\rho\sigma}(x)}\bigg|_{(t,\sigma,\bar{\sigma})=0} &= \left((\mathscr{U}[0;\bar{g}]^{-1})_{\rho\sigma\,\alpha\beta}\right)^{\mathrm{diff}}_{y}\langle y,\alpha,\beta|x,\mu,\nu\rangle\\ \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 W[t,\sigma,\bar{\sigma};\bar{g}]}{\delta\sigma^{\mu}(y)\delta\bar{\sigma}_{\nu}(x)}\bigg|_{(t,\sigma,\bar{\sigma})=0} &= \left((\sqrt{2}^{-1}\mathscr{M}[\bar{g},\bar{g}]^{-1})^{\nu}{}_{\alpha}\right)^{\mathrm{diff}}_{y}\langle y,\alpha|x,\mu\rangle\,,\end{aligned} \tag{4.53}$$

where $\langle y,\alpha,\beta|x,\mu,\nu\rangle = I^{\alpha\beta}_{\mu\nu}\delta(y-x)/\sqrt{\bar{g}(x)}$ and $\langle y,\alpha|x,\mu\rangle = \delta^{\alpha}_{\mu}\delta(y-x)/\sqrt{\bar{g}(x)}$. Consequently, we have identified those expectation values at one-loop as

$$\begin{aligned}\left\langle\widehat{h}_{\mu\nu}(y)\widehat{h}^{\rho\sigma}(x)\right\rangle &= \left((\mathscr{U}[0;\bar{g}]^{-1})^{\rho\sigma}{}_{\alpha\beta}\right)^{\mathrm{diff}}_{y}\langle y,\alpha,\beta|x,\mu,\nu\rangle\\ &= \langle y,\rho,\sigma|\mathscr{U}[0;\bar{g}]^{-1}|x,\mu,\nu\rangle\\ \left\langle\bar{C}_{\mu}(y)C^{\nu}(x)\right\rangle &= \left((\sqrt{2}^{-1}\mathscr{M}[\bar{g},\bar{g}]^{-1})^{\nu}{}_{\alpha}\right)^{\mathrm{diff}}_{y}\langle y,\alpha|x,\mu\rangle\\ &= \langle y,\nu|(\sqrt{2}\mathscr{M}[\bar{g},\bar{g}])^{-1}|x,\mu\rangle\,.\end{aligned} \tag{4.54}$$

Nota bene. As for the scalar field, we technichally could express these expectation values in a basis of eigenfunctions of the operator $\mathscr{U}[0;\bar{g}]$ and $\mathscr{M}[\bar{g},\bar{g}]$, respectively. However, we will only do so in the following applications, where the background manifold is specified to the d-dimensional sphere and the operators $\mathscr{U}[0;\bar{g}]$ and $\mathscr{M}[\bar{g},\bar{g}]$ simplify substentially.

4.2.2. *The one-loop effective action*

To derive the gravitational one-loop effective action we essentially follow the same steps as in Sections 3.1: We introduce the field $\chi_{\rm cl} = (\chi^1_{\rm cl}, \chi^2_{\rm cl}, \chi^3_{\rm cl})^T$ defined by Eq. (4.39) and expand the action $\tilde{S}[\chi; J; \bar{g}]$ – with the gauge-fixing and Faddeev-Popov action specified to Eq. (4.33) and Eq. (4.34) – in Eq. (4.17) in the variable χ around $\chi_{\rm cl}$ up to first order in $\hbar$ using Eq. (C.5), respectively Eq. (4.40),

$$\exp\{W[J;\bar{g}]\} = \int \mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\mathcal{D}_{\bar{g}}C^\mu\mathcal{D}_{\bar{g}}\bar{C}_\mu \, \exp\Bigg\{ - \tilde{S}[\chi_{\rm cl}; J; \bar{g}] \\ - \frac{1}{2}\sum_{i,j}\int {\rm d}^d x \int {\rm d}^d y (\chi - \chi_{\rm cl})^i(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\chi^j(x)\delta\chi^i(y)}\bigg|_{\chi=\chi_{\rm cl}} (\chi - \chi_{\rm cl})^j(y) + \cdots \Bigg\}. \tag{4.55}$$

After shifting the integration variables as $\chi \mapsto \chi - \chi_{\rm cl}$, the second term in the exponent reads spelled out:

$$\frac{1}{2}\sum_{i,j}\int {\rm d}^d x \int {\rm d}^d y \, \chi^i(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\chi^j(x)\delta\chi^i(y)}\bigg|_{\chi=\chi_{\rm cl}} \chi^j(y) \\ = \frac{1}{2}\int {\rm d}^d x \int {\rm d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\widehat{h}_{\mu\nu}(x)\delta\widehat{h}_{\rho\sigma}(y)}\bigg|_{\chi=\chi_{\rm cl}} \widehat{h}_{\rho\sigma}(y) \\ + \int {\rm d}^d x \int {\rm d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\widehat{h}_{\mu\nu}(y)\delta\bar{C}_\rho(x)}\bigg|_{\chi=\chi_{\rm cl}} \bar{C}_\rho(y) \\ + \int {\rm d}^d x \int {\rm d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\widehat{h}_{\mu\nu}(y)\delta C^\rho(x)}\bigg|_{\chi=\chi_{\rm cl}} C^\rho(y) \\ + \int {\rm d}^d x \int {\rm d}^d y \, \bar{C}_\nu(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta C^\mu(x)\delta\bar{C}_\nu(y)}\bigg|_{\chi=\chi_{\rm cl}} C^\mu(y) \,. \tag{4.56}$$

Let us compute these four terms independently. When employing the gauge fixing action (4.33) and Faddeev-Popov action (4.34), we construe the first term as the definition of the operators $\mathscr{U}[\widehat{h}_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho\sigma}$ and $\mathscr{U}[\widehat{h}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho\sigma}$ on ST^2 in the following sense:

$$
\begin{aligned}
&\int \mathrm{d}^d x \int \mathrm{d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta \widehat{h}_{\mu\nu}(x) \delta \widehat{h}_{\rho\sigma}(y)} \bigg|_{\chi=\chi_{\mathrm{cl}}} \widehat{h}_{\rho\sigma}(y) \\
&\boxed{\begin{aligned} &= \int \mathrm{d}^d x \int \mathrm{d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\mathrm{EH}}[\bar{g} + \widehat{h}] + S_{\mathrm{GF}}[\widehat{h}; \bar{g}] + S_{\mathrm{gh}}[\widehat{h}, \bar{C}, C; \bar{g}])}{\delta \widehat{h}_{\mu\nu}(x) \delta \widehat{h}_{\rho\sigma}(y)} \bigg|_{\chi=\chi_{\mathrm{cl}}} \widehat{h}_{\rho\sigma}(y) \\ &=: \int \mathrm{d}^d x \sqrt{\bar{g}} \; \widehat{h}_{\mu\nu} (\mathscr{U}'[\widehat{h}_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}} \; \widehat{h}^{\rho\sigma} \end{aligned}}
\end{aligned}
\tag{4.57}
$$

and

$$
\boxed{\begin{aligned} &\int \mathrm{d}^d x \int \mathrm{d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\mathrm{EH}}[\bar{g} + \widehat{h}] + S_{\mathrm{GF}}[\widehat{h}; \bar{g}])}{\delta \widehat{h}_{\mu\nu}(x) \delta \widehat{h}_{\rho\sigma}(y)} \bigg|_{\chi=\chi_{\mathrm{cl}}} \widehat{h}_{\rho\sigma}(y) \\ &\qquad =: \int \mathrm{d}^d x \sqrt{\bar{g}} \; \widehat{h}_{\mu\nu} (\mathscr{U}[\widehat{h}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}} \; \widehat{h}^{\rho\sigma} \, . \end{aligned}}
\tag{4.58}
$$

Especially note that $\mathscr{U}'[\widehat{h}_{\mathrm{cl}}, 0, 0; \bar{g}]^{\mu\nu}{}_{\rho\sigma} = \mathscr{U}[\widehat{h}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho\sigma}$. We introduce two further auxiliary operators, $\mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho}$ and $\mathscr{X}_2[h_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho}$, in order to condense the off-diagonal terms:

$$
\begin{aligned}
&\int \mathrm{d}^d x \int \mathrm{d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta \widehat{h}_{\mu\nu}(x) \delta \bar{C}_{\rho}(y)} \bigg|_{\chi=\chi_{\mathrm{cl}}} \bar{C}_{\rho}(y) \\
&\qquad = -\sqrt{2} \int \mathrm{d}^d x \int \mathrm{d}^d y \, \widehat{h}_{\mu\nu}(x) \sqrt{\bar{g}(y)} \left[\frac{\delta}{\delta \widehat{h}_{\mu\nu}(y)} (\mathscr{M}[\widehat{g}, \bar{g}] C)^{\rho}(x) \right]_{\chi=\chi_{\mathrm{cl}}} \bar{C}_{\rho}(y) \\
&\qquad =: - \int \mathrm{d}^d x \sqrt{\bar{g}} \, \widehat{h}_{\mu\nu} \mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\alpha} \bar{g}^{\rho\alpha} \bar{C}_{\rho}
\end{aligned}
\tag{4.59}
$$

and

$$\int \mathrm{d}^d x \int \mathrm{d}^d y \, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta \widehat{h}_{\mu\nu}(x) \delta C^\rho(y)}\bigg|_{\chi=\chi_{\mathrm{cl}}} C^\rho(y)$$
$$=: - \int \mathrm{d}^d x \sqrt{\bar{g}} \, \widehat{h}_{\mu\nu} \mathscr{X}_2[h_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_\rho C^\rho \, . \quad (4.60)$$

Both operators, $\mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_\rho$ and $\mathscr{X}_2[h_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_\rho$, are maps between the Hilbert spaces $V \to ST^2$. In terms of their *dual* operators $\mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{*\rho}{}_{\mu\nu}$ and $\mathscr{X}_2[h_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{*\rho}{}_{\mu\nu}$ we may also think of those operators as maps $ST^2 \to V$ as

$$\mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{*\rho}{}_{\mu\nu} \coloneqq I_{\mu\nu\alpha\beta}[\bar{g}] \bar{g}^{\rho\sigma} \mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{\alpha\beta}{}_\sigma \, , \quad (4.61)$$

and likewise for $\mathscr{X}_2[h_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{*\rho}{}_{\mu\nu}$. Also, note the identities $\mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, 0; \bar{g}] = 0$ and $\mathscr{X}_2[\widehat{h}_{\mathrm{cl}}, 0; \bar{g}] = 0$. Lastly, the term exhibiting derivatives with respect to both ghost fields yields

$$\int \mathrm{d}^d x \int \mathrm{d}^d y \, \bar{C}_\rho(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta C^\mu(x) \delta \bar{C}_\rho(y)}\bigg|_{\chi=\chi_{\mathrm{cl}}} C^\mu(y) = -\sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}} \, \bar{C}_\mu \mathscr{M}[\widehat{g}_{\mathrm{cl}}, \bar{g}]^\mu{}_\nu C^\nu$$
$$\equiv +S_{\mathrm{gh}}[\widehat{h}_{\mathrm{cl}}, \bar{C}, C; \bar{g}] \quad (4.62)$$

where we have additionally defined $\widehat{g}_{\mathrm{cl}} \coloneqq \bar{g} + \widehat{h}_{\mathrm{cl}}$. Inserting these expressions into the expanded exponent of the Schwinger functionals yields

$$\exp\{W[J; \bar{g}]\} = \exp\left\{-\tilde{S}[\chi_{\mathrm{cl}}; J; \bar{g}]\right\}$$
$$\times \int \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \exp\left\{ -\frac{1}{2} \int \mathrm{d}^d x \sqrt{\bar{g}} \; \widehat{h}_{\mu\nu} (\mathscr{U}'[\chi_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}} \; \widehat{h}^{\rho\sigma} \right\}$$
$$\times \int \mathcal{D}_{\bar{g}} C^\mu \mathcal{D}_{\bar{g}} \bar{C}^\mu \exp\left\{ + \int \mathrm{d}^d x \sqrt{\bar{g}} \, \widehat{h}_{\mu\nu} \mathscr{X}_1[\widehat{h}_{\mathrm{cl}}, C_{\mathrm{cl}}; \bar{g}]^{\mu\nu\rho} \bar{C}_\rho \right.$$
$$+ \int \mathrm{d}^d x \sqrt{\bar{g}} \, \widehat{h}_{\mu\nu} \mathscr{X}_2[h_{\mathrm{cl}}, \bar{C}_{\mathrm{cl}}; \bar{g}]^{\mu\nu}{}_\rho C^\rho$$
$$\left. + \sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}} \, \bar{C}_\mu \mathscr{M}[\widehat{g}_{\mathrm{cl}}, \bar{g}]^\mu{}_\nu C^\nu + \cdots \right\} . \quad (4.63)$$

In order two perform the integral over the ghost fields, we shift the integration variables according to[4]

$$\begin{aligned}\bar{C}_\mu &\mapsto \bar{C}_\mu + \frac{1}{\sqrt{2}}\widehat{h}_{\alpha\beta}\mathscr{X}_2[h_{\text{cl}}, \bar{C}_{\text{cl}}; \bar{g}]^{\alpha\beta}{}_\rho(\mathscr{M}[\widehat{g}_{\text{cl}}, \bar{g}]^{-1})^\rho{}_\mu \\ C^\mu &\mapsto C^\mu - \frac{1}{\sqrt{2}}(\mathscr{M}[\widehat{g}_{\text{cl}}, \bar{g}]^{-1})^\mu{}_\rho\widehat{h}_{\alpha\beta}\mathscr{X}_1[\widehat{h}_{\text{cl}}, C_{\text{cl}}; \bar{g}]^{\alpha\beta\rho}\,.\end{aligned} \tag{4.64}$$

This leads to, using Eq. (D.37),

$$\begin{aligned}\exp\{W[J;\bar{g}]\} = {}& \exp\left\{-\tilde{S}[\chi_{\text{cl}}; J; \bar{g}]\right\}\,\text{Det}(\mathscr{M}[\widehat{g}_{\text{cl}}, \bar{g}]^{\boldsymbol{\cdot}}{}_{\boldsymbol{\cdot}}) \\ & \times \int \mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\,\exp\Bigg\{-\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\;\widehat{h}_{\mu\nu}\Bigg[(\mathscr{U}'[\chi_{\text{cl}};\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}} \\ & \qquad + \sqrt{2}\mathscr{X}_2[h_{\text{cl}}, \bar{C}_{\text{cl}}; \bar{g}]^{\mu\nu}{}_\tau(\mathscr{M}[\widehat{g}_{\text{cl}}, \bar{g}]^{-1})^\tau{}_\kappa\mathscr{X}_1[\widehat{h}_{\text{cl}}, C_{\text{cl}}; \bar{g}]_{\rho\sigma}{}^\kappa\Bigg]\,\widehat{h}^{\rho\sigma}\Bigg\} \\ & + O(2\text{ loops})\,.\end{aligned} \tag{4.65}$$

Here, we have absorbed a power of $\sqrt{2}$ appearing in front of $\mathscr{M}$ inside the determinant into the normalization constant. What is left is a Gaussian integral over $\widehat{h}_{\mu\nu}$ that we can perform by means of Eq. (D.35):

$$\begin{aligned}\exp\{W[J;\bar{g}]\} = {}& \exp\left\{-\tilde{S}[\chi_{\text{cl}}; J; \bar{g}]\right\}\,\text{Det}(\mathscr{M}[\widehat{g}_{\text{cl}}, \bar{g}]^{\boldsymbol{\cdot}}{}_{\boldsymbol{\cdot}}) \\ & \times \text{Det}\Bigg(\mathscr{U}'[\chi_{\text{cl}};\bar{g}]^{\boldsymbol{\cdot\cdot}}{}_{\boldsymbol{\cdot\cdot}} \\ & \qquad + \sqrt{2}\mathscr{X}_2[h_{\text{cl}}, \bar{C}_{\text{cl}}; \bar{g}]^{\boldsymbol{\cdot\cdot}}{}_\tau(\mathscr{M}[\widehat{g}_{\text{cl}}, \bar{g}]^{-1})^\tau{}_\kappa\mathscr{X}_1[\widehat{h}_{\text{cl}}, C_{\text{cl}}; \bar{g}]^{*\kappa}{}_{\boldsymbol{\cdot\cdot}}\Bigg)^{-1/2} \\ & + O(2\text{ loops})\,.\end{aligned} \tag{4.66}$$

Along the very same lines as when discussing the scalar field in Section 3.2, it easy to see that when keeping track of $\hbar$ one has

$$\phi = (\widehat{\chi}_{\text{cl}}) + O(2\text{ loops})\,, \tag{4.67}$$

with $\phi \equiv (\phi^1, \phi^2, \phi^3)^T = (h, \bar{\xi}, \xi)^T$ as $b_\mu \equiv 0$ and hence $\zeta_\mu \equiv 0$.

[4]Demonstratively, this amounts to completing the square as $\bar{C}\mathscr{M}C + (h\mathscr{X}_2)C + (h\mathscr{X}_1)\bar{C} = (\bar{C} + h\mathscr{X}_2\mathscr{M}^{-1})\mathscr{M}(C - \mathscr{M}^{-1}h\mathscr{X}_1) + (h\mathscr{X}_2)\mathscr{M}^{-1}(h\mathscr{X}_1)$.

Therewith, we can plug our expansion into the EA (4.27):

$$\begin{aligned}\Gamma[h,\bar{\xi},\xi;\bar{g}] = &\int \mathrm{d}^d x\sqrt{\bar{g}}\ \big[t^{\mu\nu}[\phi;\bar{g}]h_{\mu\nu} + \sigma^{\mu}[\phi;\bar{g}]\bar{\xi}^{\mu} + \bar{\sigma}_{\mu}[\phi;\bar{g}]\xi^{\mu}\big]\\ &+ S[h,\bar{\xi},\xi;\bar{g}] - \int \mathrm{d}^d x\sqrt{\bar{g}}\ \big[t^{\mu\nu}[\phi;\bar{g}]h_{\mu\nu} + \bar{\sigma}_{\mu}[\phi;\bar{g}]\xi^{\mu} + \sigma^{\mu}[\phi;\bar{g}]\bar{\xi}_{\mu}\big]\\ &+ \frac{1}{2}\ln\mathrm{Det}\Big(\mathscr{U}'[\phi;\bar{g}]^{\bullet}{}_{\bullet}\\ &\qquad + \sqrt{2}\mathscr{X}_2[h,\bar{\xi};\bar{g}]^{\bullet\bullet}{}_{\tau}(\mathscr{M}[g,\bar{g}]^{-1})^{\tau}{}_{\kappa}\mathscr{X}_1[\widehat{h}_{\mathrm{cl}},C_{\mathrm{cl}};\bar{g}]^{*\kappa}{}_{\bullet\bullet}\Big)\\ &- \ln\mathrm{Det}(\mathscr{M}[g,\bar{g}]^{\bullet}{}_{\bullet}) + O(2\text{ loops})\,,\end{aligned} \tag{4.68}$$

where we have defined the expectation value of the full metric $g_{\mu\nu} \coloneqq \bar{g}_{\mu\nu} + h_{\mu\nu}$. Finally, we have deduced the EA at 1L,

$$\begin{aligned}\Gamma[h,\bar{\xi},\xi;\bar{g}] = &S[h,\bar{\xi},\xi;\bar{g}] - \mathrm{Tr}_V \ln\big[\mathscr{M}[g,\bar{g}]^{\bullet}{}_{\bullet}\big]\\ &+ \frac{1}{2}\mathrm{Tr}_{ST^2}\ln\Big[\mathscr{U}'[\phi;\bar{g}]^{\bullet\bullet}{}_{\bullet\bullet}\\ &\qquad + \sqrt{2}\mathscr{X}_2[h,\bar{\xi};\bar{g}]^{\bullet\bullet}{}_{\tau}(\mathscr{M}[g,\bar{g}]^{-1})^{\tau}{}_{\kappa}\mathscr{X}_1[\widehat{h}_{\mathrm{cl}},C_{\mathrm{cl}};\bar{g}]^{*\kappa}{}_{\bullet\bullet}\Big]\\ &+ O(2\text{ loops})\,.\end{aligned} \tag{4.69}$$

Henceforth, we refrain from explicitly denoting the terms $O(2\text{ loops})$ when it is clear that we refer to 1L-expressions. In the following application we are especially interested in the EA at vanishing quantum fluctuation and vanishing ghost fields; then, the EA heavily simplifies:

$$\boxed{\begin{aligned}\Gamma[\bar{g}] \coloneqq \Gamma[0,0,0;\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + &\frac{1}{2}\mathrm{Tr}_{ST^2}\ln\big[\mathscr{U}[0;\bar{g}]^{\bullet\bullet}{}_{\bullet\bullet}\big]\\ &- \mathrm{Tr}_V\ln\big[\mathscr{M}[\bar{g},\bar{g}]^{\bullet}{}_{\bullet}\big]\ .\end{aligned}} \tag{4.70}$$

Also, we equip the 1L-term of $\Gamma[\bar{g}]$, i.e., the term of order $\hbar$, with its own definition:

$$\boxed{\Gamma_{1\mathrm{L}}[\bar{g}] \coloneqq \frac{1}{2}\mathrm{Tr}_{ST^2}\ln\big[\mathscr{U}[0;\bar{g}]^{\bullet\bullet}{}_{\bullet\bullet}\big] - \mathrm{Tr}_V\ln\big[\mathscr{M}[\bar{g},\bar{g}]^{\bullet}{}_{\bullet}\big]}\,. \tag{4.71}$$

It is straightforward to see that the functional integro-differential Eq. (4.30) in the present one-loop approximation boils down to

$$\exp\{-\Gamma[\bar{g}]\} = \int \mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\mathcal{D}_{\bar{g}}C^{\mu}\mathcal{D}_{\bar{g}}\bar{C}_{\mu}\,\mathrm{e}^{-S[\widehat{h},\bar{C},C;\bar{g}]}\,, \tag{4.72}$$

where the action $S[\widehat{h}, \bar{C}, C; \bar{g}]$ is given by its one-loop approximation (4.47), or equivalently

$$\exp\{-\Gamma_{1\mathrm{L}}[\bar{g}]\} = \int \mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\mathcal{D}_{\bar{g}}C^{\mu}\mathcal{D}_{\bar{g}}\bar{C}_{\mu}\, \mathrm{e}^{-S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]} \tag{4.73}$$

with the matter action (4.44).

The two operators entering $\Gamma[\bar{g}]$ are the Faddeev-Popov operator in this special case $g_{\mu\nu} = \bar{g}_{\mu\nu}$,

$$(\mathscr{M}[\bar{g}, \bar{g}]^{\mu}{}_{\nu})^{\mathrm{diff}} = \delta^{\mu}_{\nu}\bar{D}^2 + \bar{D}_{\nu}\bar{D}^{\mu} - 2\beta\bar{D}^{\mu}\bar{D}_{\nu}\,, \tag{4.74}$$

as well as the operator $\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}$ that is meticulously calculated in appendix F.7 and in this case reads

$$\begin{aligned}
\kappa^{-2}\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)^{\mathrm{diff}}_{\mathrm{EH}} = {} & \left[d\left(1-2\frac{\beta^2}{\alpha}\right)(\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma}\right]\bar{D}^2 \\
& + \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\big[\delta^{\mu}_{\sigma}\bar{D}^{\nu}\bar{D}_{\rho} + \delta^{\nu}_{\sigma}\bar{D}^{\mu}\bar{D}_{\rho} \\
& \qquad + \delta^{\mu}_{\rho}\bar{D}^{\nu}\bar{D}_{\sigma} + \delta^{\nu}_{\rho}\bar{D}^{\mu}\bar{D}_{\sigma}\big] \\
& + \frac{1}{2}\left(2\frac{\beta}{\alpha}-1\right)\big[\bar{g}^{\mu\nu}\bar{D}_{\sigma}\bar{D}_{\rho} + \bar{g}^{\mu\nu}\bar{D}_{\rho}\bar{D}_{\sigma} \\
& \qquad + \bar{g}_{\rho\sigma}\bar{D}^{\mu}\bar{D}^{\nu} + \bar{g}_{\rho\sigma}\bar{D}^{\nu}\bar{D}^{\mu}\big] \\
& + \left[\frac{d}{2}(\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma}\right]\big(2\Lambda - \bar{R}\big) \\
& + \big[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\big] - \big[\bar{R}_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu} + \bar{R}_{\sigma}{}^{\mu}{}_{\rho}{}^{\nu}\big] \\
& - \frac{1}{2}\big[\delta^{\nu}_{\sigma}\bar{R}^{\mu}{}_{\rho} + \delta^{\nu}_{\rho}\bar{R}^{\mu}{}_{\sigma} + \delta^{\mu}_{\sigma}\bar{R}^{\nu}{}_{\rho} + \delta^{\mu}_{\rho}\bar{R}^{\nu}{}_{\sigma}\big]\,.
\end{aligned} \tag{4.75}$$

In the *harmonic gauge*, $\alpha = 1$ and $\beta = 1/2$, these two operators take a particularly simple form. Using $\bar{g}^{\mu\rho}[\bar{D}_{\nu}, \bar{D}_{\rho}]X_{\nu} = \bar{g}^{\mu\rho}(-\bar{R}^{\nu}{}_{\sigma\rho\nu}X^{\sigma}) = \bar{R}^{\mu}{}_{\nu}X^{\nu}$ the Faddeev-Popov operator then becomes

$$(\mathscr{M}[\bar{g}, \bar{g}]^{\mu}{}_{\nu})^{\mathrm{diff}} = \delta^{\mu}_{\nu}\bar{D}^2 + \bar{R}^{\mu}{}_{\nu}\,. \tag{4.76}$$

Furthermore, all off-diagonal terms in the operator $\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)_{\text{EH}}$ cancel such that it simplifies to

$$\begin{aligned}\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)^{\text{diff}}_{\text{EH}} = \;& \kappa^2\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma}\right](\bar{D}^2 - \bar{R} + 2\Lambda) \\ &+ \kappa^2\left[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\right] - \kappa^2\left[\bar{R}_\rho{}^\mu{}_\sigma{}^\nu + \bar{R}_\sigma{}^\mu{}_\rho{}^\nu\right] \\ &- \frac{1}{2}\kappa^2\left[\delta^\nu_\sigma\bar{R}^\mu{}_\rho + \delta^\nu_\rho\bar{R}^\mu{}_\sigma + \delta^\mu_\sigma\bar{R}^\nu{}_\rho + \delta^\mu_\rho\bar{R}^\nu{}_\sigma\right] \;.\end{aligned} \tag{4.77}$$

Adhering to the harmonic gauge, we further specify the background metric $\bar{g}_{\mu\nu}$ to that of a *maximally symmetric spacetime* whose scalar curvature $\bar{R}(x) \equiv \bar{R}$ is constant. Then the Ricci and Riemann tensor can be expressed through this constant (see [46, Eq. (13.2.4-5)]):

$$\begin{aligned}\bar{R}_{\mu\nu} &= \frac{1}{d}\bar{g}_{\mu\nu}\bar{R} \\ \bar{R}_{\mu\nu\alpha\beta} &= \frac{1}{d(d-1)}\bar{R}\left(g_{\sigma\nu}g_{\mu\rho} - g_{\rho\nu}g_{\mu\sigma}\right) \;.\end{aligned} \tag{4.78}$$

Consequently, the curvature terms in Eq. (4.77) can be summed up as follows:

$$\begin{aligned}\left[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\right] &= 2\,\bar{R}\,(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma} \\ \left[\bar{R}_\rho{}^\mu{}_\sigma{}^\nu + \bar{R}_\sigma{}^\mu{}_\rho{}^\nu\right] &= 2\left[\frac{1}{d-1}\bar{R}\,(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma} - \frac{1}{d(d-1)}\bar{R}\,I^{\mu\nu}{}_{\rho\sigma}\right] \\ \left[\delta^\nu_\sigma\bar{R}^\mu{}_\rho + \delta^\nu_\rho\bar{R}^\mu{}_\sigma + \delta^\mu_\sigma\bar{R}^\nu{}_\rho + \delta^\mu_\rho\bar{R}^\nu{}_\sigma\right] &= \frac{4}{d}\,\bar{R}\,I^{\mu\nu}{}_{\rho\sigma} \;.\end{aligned} \tag{4.79}$$

When inserting these relations into Eq. (4.77), we have determined the operator $(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})_{\text{EH}}$ in the harmonic gauge on maximally symmetric background manifold,

$$\begin{aligned}\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)^{\text{diff}}_{\text{EH}} &= \kappa^2\left[I^{\mu\nu}{}_{\rho\sigma} - (\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}\right]\left(-\bar{D}^2 - 2\Lambda + c_I\bar{R}\right) \\ &\quad - \kappa^2\frac{d-2}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}\left(-\bar{D}^2 - 2\Lambda + c_{\text{trace}}\bar{R}\right)\end{aligned} \tag{4.80}$$

with

$$c_I = \frac{d(d-3)+4}{d(d-1)} \quad \text{and} \quad c_{\text{trace}} = \frac{d-4}{d} \;. \tag{4.81}$$

Likewise, the Faddeev-Popov operator (4.76) in the harmonic gauge on maximally symmetric background manifold reads

$$(\mathscr{M}[\bar{g},\bar{g}]^{\mu}{}_{\nu})^{\text{diff}} = \delta^{\mu}_{\nu}\left[\bar{D}^2 + \frac{1}{d}\bar{R}\right] . \tag{4.82}$$

4.3. Quantization of the higher-derivative action

4.3.1. *With generic spacetime dimension d*

As a further application, we consider the quantization of the higher-derivative action given by Eq. (4.4),

$$S_{\text{h.-d.}}[\widehat{g}] = \int \mathrm{d}^d x\sqrt{\widehat{g}}\,\left[aR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + bR_{\mu\nu}R^{\mu\nu} + cR^2\right] . \tag{4.83}$$

Again, let us choose the gauge-fixing condition (4.31), i.e., $F_\mu(\widehat{h};\bar{g}) = \sqrt{2}(\bar{D}^\alpha\widehat{h}_{\alpha\mu} - \beta\bar{D}_\mu\widehat{h}_\alpha{}^\alpha)$. This time, however, the bare action $S_{\text{h.-d.}}[\widehat{g}]$ contains fourth-order derivatives such we can allow the weight function $Y^{\mu\nu}[\bar{g}]$ to be an operator built from second-order covariant derivatives. Therewith, the gauge fixing action (4.12) contains fourth-order covariant derivatives like the bare action does. A generic choice for the weight function is[5] [33]

$$Y^{\mu\nu}[\bar{g}] = \bar{g}^{\mu\nu}\bar{D}^2 + \gamma\bar{D}^\mu\bar{D}^\nu - \delta\bar{D}^\nu\bar{D}^\mu\,. \tag{4.84}$$

With this weight function and the gauge-fixing condition (4.31), the gauge-fixing action reads

$$\begin{aligned} S_{\text{GF}}[\widehat{h};\bar{g}] = \frac{1}{\alpha}\int \mathrm{d}^d x\sqrt{\bar{g}}\,&(\bar{D}^\alpha\widehat{h}_{\alpha\mu} - \beta\bar{D}_\mu\widehat{h}_\alpha{}^\alpha) \\ &\times\left[\bar{g}^{\mu\nu}\bar{D}^2 + \gamma\bar{D}^\mu\bar{D}^\nu - \delta\bar{D}^\nu\bar{D}^\mu\right](\bar{D}^\beta\widehat{h}_{\beta\nu} - \beta\bar{D}_\nu\widehat{h}_\beta{}^\beta)\,. \end{aligned} \tag{4.85}$$

[5] For an even more general gauge-fixing action, one could also include terms proportional to $\bar{R}_{\mu\nu}$ or $\bar{R}\,\bar{g}_{\mu\nu}$ into the weight function.

The Faddeev-Popov action is independent of the choice of weight function and depends only on the gauge-fixing condition. Thus, as in the previous section, the Faddeev-Popov action is given by Eq. (4.34),

$$S_{\text{gh},1}[\widehat{h},\bar{C},C;\bar{g}] = -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu \mathscr{M}[\widehat{g},\bar{g}]^\mu{}_\nu C^\nu\,, \tag{4.86}$$

with the Faddeev-Popov operator given by Eq. (4.35). Lastly, the second ghost-field action (4.14) for the third ghost field b_μ becomes

$$\begin{aligned} S_{\text{gh},2}[b;\bar{g}] &= \frac{1}{2\alpha}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,b_\mu Y^{\mu\nu}[\bar{g}]b_\nu \\ &= \frac{1}{2\alpha}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,b_\mu\left(\bar{g}^{\mu\nu}\bar{D}^2+\gamma\bar{D}^\mu\bar{D}^\nu-\delta\bar{D}^\nu\bar{D}^\mu\right)b_\nu\,. \end{aligned} \tag{4.87}$$

All these ingredients together add up to the full action (4.15),

$$S[\widehat{h},\bar{C},C,b;\bar{g}] \coloneqq S_{\text{h.-d.}}[\bar{g}+\widehat{h}]+S_{\text{GF}}[\widehat{h};\bar{g}]+S_{\text{gh},1}[\widehat{h},\bar{C},C;\bar{g}]+S_{\text{gh},2}[b;\bar{g}]\,, \tag{4.88}$$

that leads to the Schwinger functional (4.17),

$$\begin{aligned} \exp\{W[t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu,d^\mu;\bar{g}_{\mu\nu}]\} \coloneqq \int \mathcal{D}\mu[\widehat{h},\bar{C},C,b;\bar{g}]\,\exp\Bigg\{ &-S[\widehat{h},\bar{C},C,b;\bar{g}] \\ &+\int \mathrm{d}^d x\sqrt{\bar{g}}\,\Big[t^{\mu\nu}\widehat{h}_{\mu\nu}+\bar{\sigma}_\mu C^\mu+\sigma^\mu\bar{C}_\mu+d^\mu b_\mu\Big]\Bigg\}\,. \end{aligned} \tag{4.89}$$

In order to deduce the one-loop expansion of the Schwinger functional, we follow the very same steps as previously in Subsection 4.2.1. In fact, if we had not to deal with a weight function that implies $b_\mu \not\equiv 0$, following the procedure for the one-loop expansion of Subsection 4.2.1 would yield in structurally identical results (i.e., we would only have to replace the bare and gauge-fixing action in the definition of the operator $\mathscr{U}'$ and $\mathscr{U}$). Therefore, let us observe what effect the third ghost field b_μ has on the one-loop expansion: Again, we expand the exponent on the RHS of the Schwinger functional, i.e., $\tilde{S}[\chi;J;\bar{g}]$ (cf. Eq. (4.18)), with $\chi = (\chi^1,\chi^2,\chi^3,\chi^4)^T \coloneqq (\widehat{h},\bar{C},C,b)^T$ and $J=(J_1,J_2,J_3,J_4)\coloneqq(t,\sigma,\bar{\sigma},d)$ around the solution to its equations of motion (4.39) (now with $i=1,2,3,4$). In the second-order term of this expansion,

$$\frac{1}{2}\sum_{i,j}\int \mathrm{d}^d x \int \mathrm{d}^d y\, \chi^i(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\chi^j(x)\delta\chi^i(y)}\bigg|_{\chi=\chi_{\rm cl}} \chi^j(y)\,, \tag{4.90}$$

the only b_μ-dependent term will be the term $i = j = 4$. Thus, this expansion on the RHS of the Schwinger functional leads to the fermionic Gaussian path integral (D.36)

$$\int \mathcal{D}_{\bar{g}} b_\mu \, \exp\left\{-\frac{1}{2\alpha}\int \mathrm{d}^d x \sqrt{\bar{g}}\, b_\mu Y^\mu{}_\nu[\bar{g}] b^\nu\right\} = \mathrm{Det}\left(\frac{1}{\alpha} Y^{\bullet}{}_{\bullet}[\bar{g}]\right)^{+1/2} \tag{4.91}$$

that, importantly, is fully independent of the remaining path integral over $(\widehat{h}, \bar{C}, C)$. (Also, we have absorbed a power of 1/2 into the normalization constant.) By analogously following Subsection 4.2.1 with this remaining part to the one-loop effective action Eq. (4.69) yields the one-loop effective action for higher-derivative gravity,

$$\begin{aligned}\Gamma[h,\bar{\xi},\xi,\zeta;\bar{g}] = & S[h,\bar{\xi},\xi,\zeta;\bar{g}] - \mathrm{Tr}_V \ln\left[\mathscr{M}[g,\bar{g}]^{\bullet}{}_{\bullet}\right] - \frac{1}{2}\mathrm{Tr}_V \ln\left[\frac{1}{\alpha}Y^{\bullet}{}_{\bullet}[\bar{g}]\right] \\ & + \frac{1}{2}\mathrm{Tr}_{ST^2}\ln\Bigg[\left(\mathscr{U}'[h,\bar{\xi},\xi;\bar{g}]^{\bullet\bullet}{}_{\bullet\bullet}\right)_{\rm h.-d.} \\ & \qquad + \sqrt{2}\mathscr{X}_2[h,\bar{\xi};\bar{g}]^{\bullet\bullet}{}_\tau (\mathscr{M}[g,\bar{g}]^{-1})^\tau{}_\kappa \mathscr{X}_1[\widehat{h}_{\rm cl}, C_{\rm cl};\bar{g}]^{*\kappa}{}_{\bullet\bullet}\Bigg] \\ & + O(2\text{ loops})\,,\end{aligned} \tag{4.92}$$

where $\mathscr{X}_1[h,\xi;\bar{g}]$ and $\mathscr{X}_2[h,\bar{\xi};\bar{g}]$ are defined in Subsection 4.2.1 and the operator $\left(\mathscr{U}'[h,\bar{\xi},\xi;\bar{g}]^{\bullet\bullet}{}_{\bullet\bullet}\right)_{\rm h.-d.}$ is defined by

$$\int \mathrm{d}^d x \int \mathrm{d}^d y\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 \tilde{S}[\chi; J; \bar{g}]}{\delta\widehat{h}_{\mu\nu}(x)\delta\widehat{h}_{\rho\sigma}(y)}\bigg|_{\chi=\chi_{\rm cl}} \widehat{h}_{\rho\sigma}(y)$$

$$\boxed{\begin{aligned} &= \int \mathrm{d}^d x \int \mathrm{d}^d y\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\rm h.-d.}[\bar{g}+\widehat{h}] + S_{\rm GF}[\widehat{h};\bar{g}] + S_{\rm gh}[\widehat{h},\bar{C},C;\bar{g}])}{\delta\widehat{h}_{\mu\nu}(x)\delta\widehat{h}_{\rho\sigma}(y)}\bigg|_{\chi=\chi_{\rm cl}} \widehat{h}_{\rho\sigma}(y) \\ &=: \int \mathrm{d}^d x \sqrt{\bar{g}}\; \widehat{h}_{\mu\nu} (\mathscr{U}'[\widehat{h}_{\rm cl},\bar{C}_{\rm cl},C_{\rm cl};\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\rm diff}_{\rm h.-d.}\; \widehat{h}^{\rho\sigma}\,.\end{aligned}}$$

(4.93)

Note that the one-loop effective action depends on ζ_μ only on tree level. Later, we will only apply the one-loop effective action at vanishing field expectations values,

$$\boxed{\begin{aligned}\Gamma[\bar{g}] &:= \Gamma[0,0,0,0;\bar{g}] \\ &= S_{\mathrm{h.-d.}}[\bar{g}] + \frac{1}{2}\,\mathrm{Tr}_{ST^2}\ln\Big[\,(\mathscr{U}[0;\bar{g}]^{\cdot\cdot}{}_{\cdot\cdot})_{\mathrm{h.-d.}}\Big] \\ &\qquad - \mathrm{Tr}_V \ln\Big[\mathscr{M}[g,\bar{g}]^{\cdot}{}_{\cdot}\Big] - \frac{1}{2}\,\mathrm{Tr}_V\ln\Big[\frac{1}{\alpha}Y^{\cdot}{}_{\cdot}[\bar{g}]\Big]\end{aligned}} \tag{4.94}$$

with

$$\boxed{\begin{aligned}&\int \mathrm{d}^d x \int \mathrm{d}^d y\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2(S_{\mathrm{h.-d.}}[\bar{g}+\widehat{h}] + S_{\mathrm{GF}}[\widehat{h};\bar{g}])}{\delta\widehat{h}_{\mu\nu}(x)\delta\widehat{h}_{\rho\sigma}(y)}\bigg|_{\widehat{h}=0} \widehat{h}_{\rho\sigma}(y) \\ &\qquad =: \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu}(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}}_{\mathrm{h.-d.}}\, \widehat{h}^{\rho\sigma}\,.\end{aligned}} \tag{4.95}$$

The operator $\mathscr{U}[0;\bar{g}]_{\mathrm{h.-d.}}$ for a general background metric can be obtained in a cumbersome and lengthy calculation that we leave out of this thesis because $\mathscr{U}[0;\bar{g}]_{\mathrm{h.-d.}}$ in this generality will not be required here. At least, let us comment on its general structure: Setting the gauge-fixing parameters to the values

$$\alpha = -\frac{2}{4a+b} \quad , \quad \beta = \frac{b+4c}{4(c-a)} \quad , \quad \gamma = \frac{2a-2c}{4a+b} \quad \text{and} \quad \delta = 1\,, \tag{4.96}$$

the "off-diagonal terms" contained in $\mathscr{U}[0;\bar{g}]_{\mathrm{h.-d.}}$, i.e., $\bar{g}^{\mu\nu}\bar{D}_\alpha\Box_{\bar{g}}\bar{D}_\beta$, $\bar{g}^{\alpha\beta}\bar{D}_\mu\Box_{\bar{g}}\bar{D}_\nu$, $\bar{D}_\mu\bar{D}_\nu\bar{D}_\alpha\bar{D}_\beta$ and $\bar{g}^{\nu\beta}\bar{D}_\mu\Box\bar{g}\bar{D}_\alpha$, [33] and the operator takes the form

$$(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}}_{\mathrm{h.-d.}} = K[\bar{g}]^{\mu\nu}{}_{\alpha\beta}\left\{I^{\alpha\beta}{}_{\rho\sigma}\Box^2_{\bar{g}} + (V^{\kappa\tau})[\bar{g}]^{\alpha\beta}{}_{\rho\sigma}\bar{D}_\kappa\bar{D}_\tau + W[\bar{g}]^{\alpha\beta}{}_{\rho\sigma}\right\}\,, \tag{4.97}$$

where $\Box_{\bar{g}} = \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ and $I^{\mu\nu}{}_{\rho\sigma}$ is given by Eq. (A.24). The explicit form of the tensors $K[\bar{g}]^{\mu\nu}{}_{\alpha\beta}$, $(V^{\kappa\tau})[\bar{g}]^{\alpha\beta}{}_{\rho\sigma}$ and $W[\bar{g}]^{\alpha\beta}{}_{\rho\sigma}$ can be found e.g. in [33, 47].

As an example for the operator $\mathscr{U}[0;\bar{g}]_{\text{h.-d.}}$ in a general gauge, we determine it on a *flat* d-dimensional background (for an explicit derivation see appendix F.8; also cf. [33]),

$$\begin{aligned}
(\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}_{\text{h.-d.}} = {} & \left(\frac{b}{2}+2a\right) I^{\mu\nu}{}_{\rho\sigma}\Box^2 \\
& + \left[\left(\frac{b}{2}+2c\right) - 2\frac{\beta^2}{\alpha}(1+\gamma-\delta)\right]\delta^{\mu\nu}\delta_{\rho\sigma}\Box^2 \\
& - \left[\frac{b+4a}{4}+\frac{1}{2\alpha}\right]\big\{\delta^\mu_\rho\partial^\nu\Box\partial_\sigma+\delta^\mu_\sigma\partial^\nu\Box\partial_\rho \\
& \qquad + \delta^\nu_\rho\partial^\mu\Box\partial_\sigma+\delta^\nu_\sigma\partial^\mu\Box\partial_\rho\big\} \\
& - \left[\frac{b+4c}{2} - 2\frac{\beta}{\alpha}(1+\gamma-\delta)\right]\big\{\delta^{\mu\nu}\partial_\rho\Box\partial_\sigma \\
& \qquad + \delta_{\rho\sigma}\partial^\mu\Box\partial^\nu\big\} \\
& + \left[2a+b+2c+2\frac{\delta-\gamma}{\alpha}\right]\partial^\mu\partial^\nu\partial_\rho\partial_\sigma\,,
\end{aligned} \tag{4.98}$$

with $\Box=\delta^{\mu\nu}\partial_\nu\partial_\mu$. Note that for the gauge-fixing parameters (4.96) the last three – i.e., the "off-diagonal" – terms cancel.

4.3.2. *Spacetime dimension $d=4$*

In four dimensions, $d=4$, the specific gauge-fixing parameters (4.96) can be expressed in terms of the couplings f_0^2 and f_2^2 in Eq. (4.8):

$$\alpha=-2f_2^2\quad,\quad \beta=\frac{f_0^2+2f_2^2}{2(2f_0^2+f_2^2)}\quad,\quad \gamma=\frac{2f_0^2+f_2^2}{3f_0^2}\quad\text{and}\quad \delta=1\,. \tag{4.99}$$

On the other hand, for a general gauge and on a four-dimensional flat background the operator (4.98) can also be expressed in terms of the couplings f_0 and f_2 given by Eq. (4.8) (again, see appendix F.8 for details):

$$\boxed{\begin{aligned}\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{d=4}_{\text{h.-d.}} = -\Box^2\bigg\{ & -\frac{1}{2f_2^2}\mathbb{P}^{(2)}+\frac{1}{\alpha}\mathbb{P}^{(1)} \\ & +\left(\frac{1}{f_0^2}+\frac{6\beta^2}{\alpha}\right)\mathbb{P}^{(0,ss)}+\frac{2(\beta-1)^2}{\alpha}\mathbb{P}^{(0,ww)} \\ & +\frac{2\sqrt{3}\beta(\beta-1)}{\alpha}\left[\mathbb{P}^{(0,sw)}+\mathbb{P}^{(0,ws)}\right]\bigg\},\end{aligned}} \tag{4.100}$$

where we have set $\delta-\gamma=0$, i.e., chosen the gauge-fixing parameters to deviate from Eq. (4.96) (although α and β still are arbitrary), in order to write $\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{d=4}_{\text{h.-d.}}$ in terms of the projectors $\mathbb{P}^{(2)}$, $\mathbb{P}^{(1)}$, $\mathbb{P}^{(0,ss)}$ and $\mathbb{P}^{(0,ww)}$. These are spin projectors that project a symmertric rank-2 tensor field onto the respective one of its four irreducible representations of the Lorentz group. The projectors are labeled accordingly: $\mathbb{P}^{(2)}$ projects onto a spin-2 representation, $\mathbb{P}^{(1)}$ onto a spin-1 representation and $\mathbb{P}^{(0,ss)}$ as well as $\mathbb{P}^{(0,ww)}$ onto a spin-0 representation; the "projectors" $\mathbb{P}^{(0,sw)}$ and $\mathbb{P}^{(0,ws)}$ map each spin-0 representation onto the other. In appendix A.2.1 this field decomposition and the corresponding projectors are constructed explicity. Using Eq. (A.70), the operator $\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{d=4}_{\text{h.-d.}}$ can be easily inverted. This inverse is the *propagator of higher-derivative gravity* in four dimensions and reads (cf. [48])

$$\boxed{\begin{aligned}\left(\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{d=4}_{\text{h.-d.}}\right)^{-1} = -\frac{1}{\Box^2}\bigg\{ & -2f_2^2\mathbb{P}^{(2)}+\alpha\left[\mathbb{P}^{(1)}+\frac{1}{2(\beta-1)^2}\mathbb{P}^{(0,ww)}\right] \\ & +f_0^2\bigg[\mathbb{P}^{(0,ss)}+\frac{3\beta^2}{(\beta-1)^2}\mathbb{P}^{(0,ww)} \\ & +\frac{\sqrt{3}\beta}{1-\beta}\left(\mathbb{P}^{(0,sw)}+\mathbb{P}^{(0,ws)}\right)\bigg]\bigg\}.\end{aligned}} \tag{4.101}$$

4.3.3. *Weyl-squared gravity in $d = 4$*

As explained earlier already, the limit $f_0^2 \to \infty$ removes the Weyl symmetry-breaking "R^2"-term from the classical action (4.9). As explained earlier already, the limit $f_0^2 \to \infty$ removes the Weyl symmetry-breaking "R^2"-term from the classical action (4.9). Thus, in this limit we must also gauge-fix the Weyl (conformal) symmetry of the action, in which now the sole coupling f_2^2 parametrizes the Weyl tensor. We can do so by additionally imposing the trace of the metric fluctuation to vanish: $\bar{g}^{\mu\nu}\widehat{h}_{\mu\nu} \equiv \widehat{h} \equiv 0$ [49]. Practically, this means that in the bare action (4.88) we must, on the one hand, take the limit $f_0^2 \to \infty$ and, on the other hand, make the replacement $\widehat{h}_{\mu\nu} \mapsto \big(I_{\mathrm{ST}^2\mu\nu}{}^{\alpha\beta} - (P_{\mathrm{tr.}})[\bar{g}]_{\mu\nu}{}^{\alpha\beta}\big)\widehat{h}_{\alpha\beta}$, i.e., we must "project out" the trace of the metric fluctuation. Its corresponding component in field space reduces accordingly to the set of all diffeomorphism-invariant symmetric and traceless rank-2 tensors; importantly, the identity of this space is given by $\mathbb{1}_{\mathrm{ST}^2} - \mathbb{P}_{\mathrm{tr.}}[\bar{g}]$.

Let us take a look at the implications of this substitution, compared to the case $f_0^2 < \infty$. In the gauge-fixing action (4.85), the gauge-fixing condition (4.31) reduces to

$$F_\mu(\widehat{h};\bar{g}) = \sqrt{2}\bar{D}^\alpha \left(I_{\mathrm{ST}^2\alpha\mu}{}^{\kappa\tau} - P_{\mathrm{tr.}}[\bar{g}]_{\alpha\mu}{}^{\kappa\tau}\right)\widehat{h}_{\alpha\tau}\,. \tag{4.102}$$

Although to beginn with, it might seem that the gauge-fixing parameter β has become superfluous, it in fact has been determined to the value[6]

$$\beta = \frac{1}{4}\,, \tag{4.103}$$

which is a specification that results precisely in the above gauge-fixing condition. Further, the inverse propagator at vanishing fluctuation field of Weyl-squared gravity is

$$\mathscr{U}[0;\bar{g}]^{d=4}_{\mathrm{Weyl}} := \left(\mathbb{1}_{\mathrm{ST}^2} - \mathbb{P}_{\mathrm{tr.}}[\bar{g}]\right)\mathscr{U}[0;\bar{g}]^{d=4}_{\mathrm{h.-d.}}\Big|_{f_0^2\to\infty,\ \beta=1/4}\,, \tag{4.104}$$

where $\mathscr{U}[0;\bar{g}]^{d=4}_{\mathrm{h.-d.}}$ is defined by Eq. (4.95).

[6] In fact, $\beta = 1/d$ in a d-dimensional spacetime, but note that the action is only conformally invariant for $d = 4$.

For example, with the remaing gauge-fixing parameters α, γ and δ specified to the values of Eq. (4.99), i.e., $\alpha = -2f_2^2$, $\gamma = 2/3$ and $\delta = 1$, $\mathscr{U}[0;\bar{g}]^{d=4}_{\text{Weyl}}$ takes the form

$$\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)^{d=4,\text{diff}}_{\text{Weyl}} = \left(\mathbb{1}_{\text{ST}^2} - \mathbb{P}_{\text{tr.}}[\bar{g}]\right)^{\mu\nu}{}_{\gamma\delta} K_{\text{Weyl}}[\bar{g}]^{\gamma\delta}{}_{\alpha\beta} \times \\ \times \left\{ I^{\alpha\beta}{}_{\rho\sigma} \Box^2_{\bar{g}} + (V^{\kappa\tau}_{\text{Weyl}})[\bar{g}]^{\alpha\beta}{}_{\rho\sigma} \bar{D}_\kappa \bar{D}_\tau + W_{\text{Weyl}}[\bar{g}]^{\alpha\beta}{}_{\rho\sigma} \right\} . \quad (4.105)$$

The tensors $K_{\text{Weyl}}[\bar{g}]^{\gamma\delta}{}_{\alpha\beta}$, $(V^{\kappa\tau}_{\text{Weyl}})[\bar{g}]^{\alpha\beta}{}_{\rho\sigma}$ and $W_{\text{Weyl}}[\bar{g}]^{\alpha\beta}{}_{\rho\sigma}$ can e.g. be found in [49].

On a flat background, we can obtain $\mathscr{U}[0;\bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}_{\text{Weyl}}$ for an arbitrary gauge-fixing parameter α and for $\gamma - \delta = 0$ from Eq. (4.100):

$$\mathscr{U}[0;\bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}_{\text{Weyl}} = \left(\mathbb{1}_{\text{ST}^2} - \mathbb{P}_{\text{tr.}}[\bar{g}_{\mu\nu} = \delta_{\mu\nu}]\right) \mathscr{U}[0;\bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}_{\text{h.-d.}} \Big|_{f_0^2 \to \infty,\ \beta = 1/4} \\ = -\Box^2 \left\{ -\frac{1}{2f_2^2}\mathbb{P}^{(2)} + \frac{1}{\alpha}\left[\mathbb{P}^{(1)} + \frac{3}{8}\mathbb{P}^{(0,ss)} + \frac{9}{8}\mathbb{P}^{(0,ww)} \right.\right. \\ \left.\left. - \frac{3\sqrt{3}}{8}\mathbb{P}^{(0,sw)} - \frac{9\sqrt{3}}{8}\mathbb{P}^{(0,ws)}\right]\right\} . \quad (4.106)$$

Its inverse, the *propagator of Weyl-squared gravity*, is given by the condition $\left(\mathscr{U}[0;\bar{g}_{\mu\nu} = \delta_{\mu\nu}]_{\text{Weyl}}\right)^{-1} \mathscr{U}[0;\bar{g}_{\mu\nu} = \delta_{\mu\nu}]_{\text{Weyl}} = \mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}$ and reads

$$\left(\mathscr{U}[0;\bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}{}_{\text{Weyl}}\right)^{-1} \\ = -\frac{1}{\Box^2}\left\{ -2f_2^2\mathbb{P}^{(2)} + \alpha\left[\mathbb{P}^{(1)} - \frac{2}{3\sqrt{3}}\mathbb{P}^{(0,ws)} + \frac{2}{3}\mathbb{P}^{(0,ww)}\right]\right\} . \quad (4.107)$$

Unfortunately, at two loops the conformal symmetry is anomalous [49], and consequently radiative corrections will also generate a propagating scalar mode. Nevertheless, Weyl-squared gravity is still relevant for situations in which the conformal symmetry is approximately realized [50].

Part 2

The functional renormalization group equation (FRGE) for quantum gravity

CHAPTER 5

Summary of Part 2

The content of Part 2 interlinks the preparatory construction of the gravitational path integral in Part 1 with the application that follows in Part 3. Essentially, this interlink consists of the explicit construction of the functional renormalization group equation, the presentation of some of its approximate solutions as well as its application to the renormalization of composite operators. The overall composition of Part 2 is tuned to prepare the specific ingredients which will be required in Part 3.

Chapter 6 begins with the explicit construction of the functional renormalization group equation and explains the philosophy of the Asymptotic Safety scenario for quantum gravity. We add to the action appearing in the gravitational path integral a scale-dependent cutoff action $\Delta S_k[\widehat{h}, \bar{C}, C, b; \bar{g}]$ which serves the purpose of an infrared cutoff and regularizes the infrared divergences of the path integral by surpressing the integration below the scale k^2. It is important to note that for the functional renormalization group equation to be well-defined, the construction does not require an ultraviolet regularization at the time. Fully analogous to the formal definition of the effective action by means of the ordinary, unregularized gravitational path integral, one then obtains a scale-dependent effective action, called the *effective average action* $\Gamma_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]$. Here, the first four arguments are the expectation values of the metric fluctuation as well as of the ghost fields. The functional space of which these four fields are elements of is called *field space.*

One can straightforwardly show that the effective average action constructed in this way fulfills the functional renormalization group equation (6.13),

$$\begin{aligned} k\partial_k\Gamma_k[\phi;\bar{g}] = \frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[(k\partial_k\mathscr{R}_{k11}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{11}\right] \\ -\mathrm{Tr}_V\left[(k\partial_k\mathscr{R}_{k23}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{32}\right] \\ -\frac{1}{2}\,\mathrm{Tr}_V\left[(k\partial_k\mathscr{R}_{k44}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{44}\right], \end{aligned}$$

with $\phi = (h, \bar{\xi}, \xi, \zeta)$. The operators $\Gamma_k^{(2)}[\phi;\bar{g}]$ and $\mathscr{R}_k[\bar{g}]$ are operators on field space, namely the Hessians obtained by applying any two functional derivatives to the effective average action and the cutoff action, respectively. Although the cutoff action itself only serves the purporse of an infrared regulator of the gravitational path integral, the functional renormalization group equation is well-defined in the infrared and the ultraviolet, too. The reason is that the scale derivative $k\partial_k\mathscr{R}_k[\bar{g}]$ vanishes in the infrared and the ultraviolet which fully regulates the traces on the RHS.

The idea of the Asymptotic Safety scenario for quantum gravity is as follows: For a moment, let us forget about the gravitational path integral and the struggle to give a physical meaning to it. Rather, let us take the above functional renormalization group equation by the hand and try to find a solution $\Gamma_k[\phi;\bar{g}]$ of it. The space this equation is defined on is the *theory space*, i.e., the space of all diffeomorphism-invariant functionals of the fields. Generally, we may assume that this space has the structure of a vector space, so we can expand

$$\Gamma_k[\phi;\bar{g}] = \sum_{i=1}^{\infty} \bar{u}_i(k) P_i[\phi;\bar{g}]\,.$$

Then, the functional renormalization group equation is nothing but (infinitely many) coupled ordinary differential equation for the dimensionful couplings $\{\bar{u}_i(k)\}$. Hence we can identify theory space with the space of all couplings. Moreover, we can consider a theory to be fully renormalized if we find a trajectory in the space of all couplings that is well-defined for all values of k. Usually, this trajectory will be emdedded into some finite-dimensional hypersurface of theory space. The number of this finite dimension corresponds to the number of free parameters of the theory that must be fixed by experiment. Particularly, this makes the Asymptotic Safety scenario predictive. The search for an ultraviolet-finite trajectory in theory space is fully equivalent to the search for an ultraviolet *fixed point* in theory space, out of which the ultraviolet-finite trajectories originate. (The direction of the flow is the direction of lowering k.)

Unfortunately, it is only possible to find approximative solutions of the functional renormalization group equation. An important ansatz for finding approximative solutions are truncations of the theory space, i.e., to study the equation on a space spanned by finitely many basis functionals $\{P_i[\phi;\bar{g}] \mid i = 1, 2, \ldots, N\}$. This amounts to the ansatz

$$\Gamma_k[\phi;\bar{g}] = \sum_{i=1}^{N} \bar{u}_i(k) P_i[\phi;\bar{g}]\,.$$

Before showing examples for such truncative solutions, it is then explicitly shown how the functional renormalization group equation must be further modified such that it emulates the evolution of the one-loop effective action of some given bare action. We will refer to this modification as the "one-loop approximation of the functional renormalization group equation".

Chapter 7 presents the first approximative solution of the functional renormalization group equation that we will later have recourse to, the *single metric Einstein-Hilbert truncation* [26]. Here, "single metric" refers to the fact that we consider the special case $g_{\mu\nu} = \bar{g}_{\mu\nu}$ of a vanishing metric fluctuation. Further "Einstein-Hilbert truncation" refers to the employed two-dimensional truncation of theory space which is spanned by the functionals $\int \mathrm{d}^d x \sqrt{g}$ and $\int \mathrm{d}^d x \sqrt{g} R$, i.e., by the functionals from which the Einstein-Hilbert action is built. Accordingly, the ansatz for the effective average action (7.2) is built from these two functionals. The two (dimensionless) running couplings that parametrize this ansatz are Newton's constant g_k and the cosmological constant λ_k. Their running is given by ordinary differential equations (the "RG equations") which we explicitly derive from the functional renormalization group equation. We then show in four spacetime dimensions that the flow of these RG equations possesses a fixed point which we numerically calculate.

Furthermore we repeat this analysis for the "one-loop approximation" of the functional renormalization group equation. To these solutions we refer to as the *simplified Einstein-Hilbert flow.* Furthermore, we study this simplified flow in the presence of matter fields. In both cases, a fixed point in four spacetime dimensions exists that we numerically calculate.

Chapter 8 treats a further approximative solution of the functional renormalization group equation. Namely, we show how the latter can be used to calculate the one-loop RG equations (respectively their RHS called the "beta functions") for *higher-derivative gravity in four spacetime dimensions* which is defined via the classical action

$$S_{\text{h.-d.}}[g] := \int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{f_2^2}\left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) - \frac{1}{6f_0^2}R^2\right] .$$

We then show with what ansatz the one-loop approximation of the funtional renormalization group equation must be solved in order to obtain the one-loop beta functions for the couplings f_0^2 and f_2^2. It is important to note that these beta functions are universal, i.e., independent of the employed renormalization scheme. Furthermore, we also consider the scale-invariant special case of *Weyl-squared gravity* which amounts to the limit $f_0 \to \infty$.

It follows a detailed discussion of the solutions to the one-loop RG equations, whereby it turns out to be useful to introduce the new variable $\omega := f_2^2/(2f_0^2)$. We show that higher-derivative as well as Weyl-squared gravity is *asymptotically free* in the coupling f_2^2. This essentially means that $f_2^2 \to 0$ as $k \to \infty$. Moreover, it is shown that the higher-derivative gravity possesses two non-Gaussian fixed points of the variables f_2^2 and ω, with one being fully attractive in the ultraviolet and the other being a saddle point. Thereby one has $f_2^2 = 0$ for both fixed points because of asymptotic freedom, such that by setting ω to the fixed-point value, one obtains a perturbative framework that is fully controlled by f_2^2.

Lastly, Chapter 8 concludes with a discussion of the corresponding RG equations in $4-\varepsilon$ dimensions, which are not universal anymore and moreover gauge-dependent, i.e., not necessarily physical. We discuss how these RG equations can be employed in a meaningful way in order to probe the behavior of results away from four spacetime dimensions.

Part 3 concludes with Chapter 9 which is an extensive discussion of how to renormalize composite operators via the functional renormalization group equation. There are various reasons that motivate the treatment of composite operators within the functional renormalization group approach. These include the following: Firstly, composite operators play a crucial role for the construction of observables for quantum gravity. These are particularly hard to construct because they are required to be diffeomorphism invariant, for instance, one may consider correlation functions at a fixed geodesic length. The geodesic length, however, usually is not included into truncations of theory space which are given by (quasi-)local operators. Thus, a way out of this problem is to renormalize the geodesic length as a composite operator. Secondly, in a theory of quantum gravity, it is rather natural to ask how geometric quantities, such as the volume of some submanifold, behave at the quantum level. Such geometric properties are crucial for the comparison of different approaches to quantum gravity, especially for the comparison of continuum-based with discrete approaches.

The point of origin for this chapter is a set (or basis) of n bare composite operators $\mathscr{O}_1[\hat{h}, \bar{C}, C, b; \bar{g}](x), \ldots, \mathscr{O}_n[\hat{h}, \bar{C}, C, b; \bar{g}](x)$ that we wish to renormalize. We then explicitly show that this is possible by coupling the bare operators to arbitrary sources and incorporating them into the gravitational path integral. Then it is rather straightforward to derive the *composite-operator functional renormalization group equation* Eq. (9.19) for the renormalized operators $[\mathscr{O}_i]_k [\phi; \bar{g}](x)$, $i = 1, \ldots, n$. This flow equation possesses a double-layer structure: it entails these renormalized composite operators, on the one hand, and the effective average action $\Gamma_k[\phi; \bar{g}]$, on the other hand, thus, two approximations are required to actually find solutions to this flow equation. We call the approximations for the effective average action and the renormalized composite operators *first and second truncation*, respectively.

The most important example is that of geometric operators which are composite operators that do not depend on the ghost fields. Their k dependence is governed by the flow equation (9.29),

$$\sum_{j=1}^{n} \bar{\gamma}_{ij}(k) \mathscr{O}_j[g, \bar{g}](x) = -\frac{1}{2} \operatorname{Tr}_{ST^2} \Bigg[\Big(\partial_t \mathscr{R}_k^{\text{grav}}[\bar{g}] \Big) \Big(\big(\Gamma_k^{(2)} \big)_{11}[g, \bar{g}] + \mathscr{R}_k^{\text{grav}}[\bar{g}] \Big)^{-1} \\ \times \mathscr{O}_i^{(2)}[g, \bar{g}](x) \Big(\big(\Gamma_k^{(2)} \big)_{11}[g, \bar{g}] + \mathscr{R}_k^{\text{grav}}[\bar{g}] \Big)^{-1} \Bigg] .$$

Here, $\bar{\gamma}(k)$ is the dimensionful anomalous-dimension matrix, into which the renormalization behavior of the composite operators is fully encoded. In general, the anomalous-dimension matrix will be an $(n \times n)$-matrix if the second truncation is n-dimensional. Further, it is a function of the running couplings that parametrize the first truncation.

Finally, we outline a scaling argument by which we can identify the anomalous-dimension matrix with the scaling dimensions of the composite operators in the ultraviolet, i.e., in the fixed-point regime. For example, in case of a one-dimensional second truncation, we have the scaling property (9.62),

$$[\mathscr{O}]_{k\to\infty}[g,\bar{g}](r) \sim r^{d-\gamma(u^*)} .$$

Here, the geometric operator is assumed not to depend on a spacetime point itself, but rather on a characteristic length scale. Further, it has canonical mass dimension $-d$ and u^* is the fixed point in theory space obtained by means of the first truncation of the effective average action. (Also, γ denotes the dimensionless anomalous dimension.)

CHAPTER 6

The FRGE

Executive summary. We explicitly construct the effective average action and the functional renormalization group equation for quantum gravity. On this basis, we explain the concept of Asymptotic Safety and outline suitable approximation schemes for the effective average action. Moreover, we show how the functional renormalization group equation can be used to derive the one-loop beta functions for a given bare action.
What is new? The rigorous discussion of the "one-loop approximation" of the functional renormalization group equation.

6.1. Introduction

The effective action for quantum gravity, as given by its general – yet formal – definition (4.27) is ill-defined: It still involves ultraviolet (UV)- and infrared (IR)-divergences. For example, the trace appearing in the one-loop expansion (4.70) of the effective action associated to the Einstein-Hilbert action contains notorious UV divergences that will play a special role in Chapter 18. Conventionally, one gives a meaning to the effective action, and therewith to the path integral (4.17) as well, by regularization techniques that *renormalize* the originally divergent quantities. In the conventional, perturbative, approach towards renormalization each divergent "bare" quantity is thwarted by a so-called "counterterm" to give a renormalized quantity, essentially a parameter. Each of these parameters' value must be taken from experiment. In case of the quantization of the Einstein-Hilbert action, infinitely many perturbative parameters must be renormalized, resulting in a loss of predictivety. The theory is not perturbatively renormalizable.

In the following, we introduce an alternative method to the conventional process of perturbative renormalization, given by the functional renormalization group equation (FRGE). This equation is an exact, non-perturbative, differential equation for a scale-dependent version of the effective action, the effective average action (EAA). Remarkably, the FRGE only requires the presence of an IR regularization to be well defined (its structure is such that the IR regularization implies an UV regularization, too). This property goes along with the fact that the FRGE, while constructed with the help of a bare action, is itself fully independent of a bare action. However, when reimplementing an – for the FRGE itself superfluous – UV regularization, one can reconstruct a bare action that then depends on the chosen UV regularization scheme. Importantly, the FRGE might admit IR- and UV-well defined solutions that are predictive, i.e., require only a finite set of parameters to be taken from experiment. Especially intriguing is the existence of such solutions that correspond to theories that are otherwise perturbatively non-renormalizable. Indeed, in the case of quantum gravity strong indications for the existence of such a solution have been found.

6.2. Construction of the FRGE

In order to construct the FRGE from the path integral (4.17), let us equip this path integral explicitly with an IR regularization, in form of a (low momentum) cutoff, while assuming only implicitly its UV regularization. Therefore, we add to the full action $S[\widehat{h},\bar{C},C,b;\bar{g}]$ given by Eq. (4.15) the following IR-cutoff functional for the gravitational field $\widehat{h}_{\mu\nu}$ and the ghosts $\bar{C}_\mu$, C^μ and b_μ:

$$\begin{aligned}\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}] \coloneqq \frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\widehat{h}_{\mu\nu}\,\mathscr{R}_k^{\mathrm{grav}\,\mu\nu}{}_{\rho\sigma}[\bar{g}]^{\mathrm{diff}}\,\widehat{h}^{\rho\sigma} \\ +\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu\,\mathscr{R}_k^{\mathrm{gh},1\,\mu}{}_{\nu}[\bar{g}]^{\mathrm{diff}} C^\nu \\ +\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,b_\mu\,\mathscr{R}_k^{\mathrm{gh},2\,\mu}{}_{\nu}[\bar{g}]^{\mathrm{diff}} b^\nu\,.\end{aligned} \tag{6.1}$$

Fully analogously to Eq. (4.17) we therewith define

$$\boxed{\begin{aligned}&\exp\big\{W_k[t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu,d^\mu;\bar{g}_{\mu\nu}]\big\}\\ &\quad:=\int\mathcal{D}\mu[\widehat{h},\bar{C},C,b;\bar{g}]\,\exp\Bigg\{-S[\widehat{h},\bar{C},C,b;\bar{g}]-\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]\\ &\qquad\qquad+\int\mathrm{d}^d x\sqrt{\bar{g}}\,\Big[t^{\mu\nu}\widehat{h}_{\mu\nu}+\bar{\sigma}_\mu C^\mu+\sigma^\mu\bar{C}_\mu+d^\mu b_\mu\Big]\Bigg\}\\ &\quad=\int\mathcal{D}\mu[\widehat{h},\bar{C},C,b;\bar{g}]\,\exp\Bigg\{-\tilde{S}[\widehat{h},\bar{C},C,b;t,\sigma,\bar{\sigma},d;\bar{g}]-\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]\Bigg\}\,,\end{aligned}} \tag{6.2}$$

where $\tilde{S}[\widehat{h},\bar{C},C,b;t,\sigma,\bar{\sigma},d;\bar{g}]$ is given by Eq. (4.18). The measure is again defined by Eq. (4.21) and in the case of $b_\mu = 0$ (and thus $d^\mu = 0$) is traded for the measure Eq. (4.22). This k-dependent version of the Schwinger functional is also invariant under the background transformations (4.23), $\delta^{(B)}W_k[J;\bar{g}] = 0$, which is proven in appendix F.4; whereas its invariance under the BRST transformations (4.16) requires the *modified Ward identities*[1] $\delta_\varepsilon W_k[J;\bar{g}] = -\langle\delta_\varepsilon\Delta_k S[\chi;\bar{g}]\rangle - \langle\delta_\varepsilon S_{\text{source}}[\chi,J;\bar{g}]\rangle = 0$ to hold (with S_{source} given by Eq. (4.19)).

The cutoff action $\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]$ fulfills thee purpose of an IR cutoff at the scale k that surpresses eigenmodes p^2 of the negative Laplacian $-\bar{D}^2 =: -\Box_{\bar{g}}$ with $p^2 \ll k^2$ while those eigenmodes with $p^2 \gg k^2$ are not surpressed and fully "integrated out". The cutoff operators $\mathscr{R}_k^{\text{grav}}[\bar{g}]$, $\mathscr{R}_k^{\text{gh},1}[\bar{g}]$ and $\mathscr{R}_k^{\text{gh},2}[\bar{g}]$ all have the general structure

$$\mathscr{R}_k[\bar{g}] = \mathscr{Z}_k[\bar{g}]k^{2\gamma}R^{(0)}\big((-\bar{D}^2/k^2)^\gamma\big)\,, \tag{6.3}$$

where $\mathscr{Z}_k[\bar{g}]$ is a tensor structure that depends on the scale k through the running couplings of the respective theory and the power γ is chosen such that $\mathscr{R}[\bar{g}]$ has the same canonical mass dimension as the respective inverse propagator. E.g., for $\mathscr{R}_k^{\text{grav}}[\bar{g}]$ one chooses $\gamma = 1$ when considering the Einstein-Hilbert action (4.3) and $\gamma = 2$ when considerung the higher-derivative action (4.4). Also, w.l.o.g. we may require $\mathscr{Z}_k^{\text{gh},1}[\bar{g}]^\mu{}_\nu \sim \delta^\mu_\nu$ and as well $\mathscr{Z}_k^{\text{gh},2}[\bar{g}]^\mu{}_\nu \sim \delta^\mu_\nu$. On momentum space, the dimensionless function $R^{(0)} : \mathbb{R}_\geq \to [0,1]\,,\, z \mapsto$

[1]Later, this modified Ward identity can be formulated as a condition that the effective average action must fulfill, cf. Section 9.1.

$R^{(0)}(z)$, is required to (at best smoothly) interpolate between $R^{(0)}(0) = 1$ and $\lim_{z\to\infty} R^{(0)}(z) = 0$. A convenient choice of the specific "cutoff profile" is e.g.

$$R^{(0)}(z;s) = \frac{sz}{\mathrm{e}^{sz}-1} \tag{6.4}$$

$$\text{or} \quad R^{(0)}(z) = (1-z)\theta(1-z)\,, \tag{6.5}$$

with θ the Heaviside step function. The former is called the *exponential cutoff*. It depends parametrically on s which is a pratical feature to investigate the cutoff-dependence of results. The latter specification of $R^{(0)}$ is called the *optimized cutoff* [51, 52]. Replacing the conventional Schwinger functional W with W_k, the classical fields $\phi = (h, \bar{\xi}, \xi, \zeta)^T$ given by Eq. (4.24) now are given by $\phi \equiv \phi_k[J;\bar{g}]$, i.e.,

$$\begin{aligned}
h_{\mu\nu}(x) \equiv h_{k\mu\nu}[J;\bar{g}](x) &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W_k[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta t^{\mu\nu}(x)} \equiv \langle \widehat{h}_{\mu\nu}(x)\rangle \\
\bar{\xi}_\mu(x) \equiv \bar{\xi}_{k\mu}[J;\bar{g}](x) &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W_k[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta \sigma^{\mu}(x)} \equiv \langle \bar{C}_{\mu}(x)\rangle \\
\xi^\mu(x) \equiv \xi_k{}^{\mu}[J;\bar{g}] &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W_k[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta \bar{\sigma}_{\mu}(x)} \equiv \langle C^{\mu}(x)\rangle \\
\zeta_\mu(x) \equiv \zeta_{k\mu}[J;\bar{g}] &:= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W_k[t,\sigma,\bar{\sigma},d;\bar{g}]}{\delta d^{\mu}(x)} \equiv \langle b_{\mu}(x)\rangle\,.
\end{aligned} \tag{6.6}$$

Again, we therewith express the source functions $J = (t, \sigma, \bar{\sigma}, d)$ as functionals of ϕ and $\bar{g}$; therewith we further define the analog of the effective action (4.27) as

$$\begin{aligned}
\tilde{\Gamma}_k[h,\bar{\xi},\xi,\zeta;\bar{g}] = \int \mathrm{d}^d x\sqrt{\bar{g}}\ \big[&t_k{}^{\mu\nu}[\phi;\bar{g}]h_{\mu\nu} + \sigma_k{}^{\mu}[\phi;\bar{g}]\bar{\xi}^{\mu} + \bar{\sigma}_{k\mu}[\phi;\bar{g}]\xi^{\mu} + d_k{}^{\mu}[\phi;\bar{g}]\zeta_\mu\big] \\
&- W_k[J_k[\phi;\bar{g}];\bar{g}]\,.
\end{aligned} \tag{6.7}$$

However, the functional $\tilde{\Gamma}_k[h,\bar{\xi},\xi,\zeta;\bar{g}]$ appears only as an auxiliary functional on our way to define the EAA by

$$\boxed{\Gamma_k[\phi;\bar{g}] := \tilde{\Gamma}_k[\phi;\bar{g}] - \Delta_k S[\phi;\bar{g}]\,.} \tag{6.8}$$

Note that in the limit $k \to 0$, in which the cutoff functional $\Delta_k S[\phi; \bar{g}]$ vanishes, i.e., in which the cutoff is fully removed, we recover the conventional effective action (4.27):

$$\lim_{k\to 0} \Gamma_k[\phi; \bar{g}] = \Gamma[\phi; \bar{g}] \,. \tag{6.9}$$

Furthermore, note that the relations (6.6) are now inverted by

$$\begin{aligned}
t^{\mu\nu}(x) \equiv {t_k}^{\mu\nu}[\phi; \bar{g}](x) &= + \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta h_{\mu\nu}(x)} \\
\sigma^{\mu}(x) \equiv {\sigma_k}^{\mu}[\phi; \bar{g}](x) &= - \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta \bar{\xi}_{\mu}(x)} \\
\bar{\sigma}_{\mu}(x) \equiv \bar{\sigma}_{k\mu}[\phi; \bar{g}](x) &= - \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta \xi^{\mu}(x)} \\
d^{\mu}(x) \equiv {d_k}^{\mu}[\phi; \bar{g}](x) &= - \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta \zeta_{\mu}(x)} \,,
\end{aligned} \tag{6.10}$$

or, in summary,

$$(J_a)_k[\phi; \bar{g}](x) = \frac{(-1)^{|\phi^a|}}{\sqrt{\bar{g}(x)}} \frac{\delta \tilde{\Gamma}_k[\phi; \bar{g}]}{\delta \phi^a(x)} \,, \tag{6.11}$$

i.e., in Eq. (4.28) one must replace Γ bei $\tilde{\Gamma}_k$ and not Γ_k. Consequently, the functional integro-differential equation (4.30) now reads

$$\begin{aligned}
\exp\left\{-\Gamma_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]\right\} = \int \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \mathcal{D}_{\bar{g}} C^{\mu} \mathcal{D}_{\bar{g}} \bar{C}_{\mu} \mathcal{D}_{\bar{g}} b_{\mu} \exp\Bigg\{ &- S[\widehat{h}, \bar{C}, C, b; \bar{g}] \\
+ \int \mathrm{d}^d x \Bigg[(\widehat{h}_{\mu\nu} - h_{\mu\nu})(x) \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta h_{\mu\nu}(x)} &- \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta \xi^{\mu}(x)} (C^{\mu} - \xi^{\mu})(x) \\
- \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta \bar{\xi}_{\mu}(x)} (\bar{C}_{\mu} - \bar{\xi}_{\mu})(x) &- \frac{\delta \tilde{\Gamma}_k[h, \bar{\xi}, \xi, \zeta; \bar{g}]}{\delta \zeta_{\mu}(x)} (b_{\mu} - \zeta_{\mu})(x) \Bigg] \Bigg\} \\
\times \exp\left\{ -\Delta_k S[\widehat{h} - h, \bar{C} - \bar{\xi}, C - \xi, b - \zeta; \bar{g}] \right\} \,.
\end{aligned} \tag{6.12}$$

Note that still, the well-definedness of $\Gamma_k[\phi; \bar{g}]$ and of Eq. (6.12) relies on the implicitly assumed UV regularization.

The effective average action $\Gamma_k[\phi;\bar{g}]$ constructed from the bare action as above can be shown to fulfill the *functional renormalization group equation*, also called the *Wetterich equation* [25, 26],

$$\boxed{\begin{aligned}\partial_t\Gamma_k[\phi;\bar{g}] = &\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[(\partial_t\mathscr{R}_{k11}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{11}\right]\\ &-\mathrm{Tr}_V\left[(\partial_t\mathscr{R}_{k23}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{32}\right]\\ &-\frac{1}{2}\,\mathrm{Tr}_V\left[(\partial_t\mathscr{R}_{k44}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{44}\right],\end{aligned}} \tag{6.13}$$

where $t \coloneqq \ln k$ is the renormalization group time and $\partial_t \equiv \partial/\partial t = k\,\partial/\partial k$. In the case $b_\mu \equiv 0$ one fully analogously obtains, now with $\phi \equiv (h,\bar{\xi},\xi)^T$,

$$\boxed{\begin{aligned}\partial_t\Gamma_k[\phi;\bar{g}] = &\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[(\partial_t\mathscr{R}_{k11}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{11}\right]\\ &-\mathrm{Tr}_V\left[(\partial_t\mathscr{R}_{k23}[\bar{g}])\left(\left[\Gamma_k^{(2)}[\phi;\bar{g}]+\mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{32}\right].\end{aligned}} \tag{6.14}$$

An elabortate derivation of the FRGE can be found in appendix F.11. Here, the operators $\mathscr{R}_k[\phi;\bar{g}]$ and $\Gamma_k^{(2)}[\phi;\bar{g}]$ are operators on *field space*, which essentially is the Hilbert space of which the set of fields $\chi = (\widehat{h}_{\mu\nu},\bar{C}_\mu,C^\mu,b_\mu)^T$ is an element of,[2] given by (*no* summation over a and b intended)

$$\langle x,\ldots|\mathscr{R}_{kab}[\bar{g}]|y,\ldots\rangle \coloneqq \frac{(-1)^{|\phi^b|}}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}I_{ab}[\bar{g}]\frac{\delta^2\Delta_k S[\phi;\bar{g}]}{\delta\phi^a(x)\delta\phi^b(y)}\,, \tag{6.15}$$

$$\langle x,\ldots|\left(\Gamma_k^{(2)}\right)_{ab}[\phi;\bar{g}]|y,\ldots\rangle \coloneqq \frac{(-1)^{|\phi^b|}}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}I_{ab}[\bar{g}]\frac{\delta^2\Gamma_k[\phi;\bar{g}]}{\delta\phi^a(x)\delta\phi^b(y)}\,. \tag{6.16}$$

[2]This Hilbert space is given by the tensor product $ST^2\otimes(V_0\oplus V_1)^*\otimes(V_0\oplus V_1)\otimes(V_0\oplus V_1)^*$, where the Hilbert spaces ST^2 and V are defined in appendix A.1 and $V = V_0\oplus V_1$ denotes the $\mathbb{Z}_2$-graded Hilbert space V of anticommuting Graßmann fields. Further, V^* denotes the dual space of V.

The dots "…" and $I_{ab}[\bar{g}]$ symbolically account for the index structure that must be adapted accordingly, e.g.

$$\begin{aligned}
\langle x,\mu,\nu|\mathscr{R}_{k11}[\bar{g}]|y,\rho,\sigma\rangle &= \langle x,\mu,\nu|\mathscr{R}_k{}^{\text{grav}}[\bar{g}]|y,\rho,\sigma\rangle \\
\langle x,\mu|\mathscr{R}_{k23}[\bar{g}]|y,\nu\rangle &= \frac{-1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2\Delta_k S[\phi;\bar{g}]}{\delta\bar{\xi}_\mu(x)\delta\xi^\nu(y)}\,, \\
&= \langle x,\mu|\mathscr{R}_k{}^{\text{gh},1}[\bar{g}]|y,\nu\rangle \\
\langle x,\mu|\mathscr{R}_{k44}[\bar{g}]|y,\nu\rangle &= \frac{-1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\bar{g}_{\nu\alpha}(y)\frac{\delta^2\Delta_k S[\phi;\bar{g}]}{\delta b_\mu(x)\delta b_\alpha(y)}\,, \\
&= \langle x,\mu|\mathscr{R}_k{}^{\text{gh},2}[\bar{g}]|y,\nu\rangle\,.
\end{aligned} \tag{6.17}$$

Also, note that the off-diagonal elements in fields space of these operators, $\mathscr{R}_k^{ab}[\phi;\bar{g}]$ and $(\tilde{\Gamma}_k^{(2)})_{ab}[\phi;\bar{g}]$ with $a \neq b$, are operators that eventually map between disjoint Hilbert spaces.

A striking feature of the FRGE (6.13), or respectively (6.14), is that the traces appearing on the RHS are in fact IR- and UV-finite as they stand, i.e., even when removing the implicitly assumed UV cutoff. This is simply due to the fact that while the cutoff operator $\mathscr{R}_k[\phi;\bar{g}]$, being added to the exponent on the RHS of Eq. (6.2), regulates only IR modes by surpressing their integration, its k-derivative $\partial_t\mathscr{R}_k[\phi;\bar{g}]$, appearing multiplicatively under the trace in Eq. (6.13), regulates IR as well as UV modes. This is due to the fact that the graph of $\partial_t\mathscr{R}_k[\phi;\bar{g}]$ consists only of a small peak around k^2, left and right of this peak the graph quickly approaches zero. Hence, only modes in a small band around k^2 contribute to the trace in Eq. (6.13). This goes along with the fact that the EAA depends on a bare action – that would require and the presence of an explicit UV cutoff – only implicitly via the EAA $\Gamma_k[\phi;\bar{g}]$. These key features thus allow for a philosophical U-turn in the approach towards quantum gravity. Instead of trying to give a meaning to the path integral (4.17) for a *given* bare action, we may forget about the bare action alltogether and take the FRGE (6.13), or respectively (6.14), a differential equation that try to solve for the EAA $\Gamma_k[\phi;\bar{g}]$. For a given solution $\Gamma_k[\phi;\bar{g}]$, the bare action $S[\chi;\bar{g}]$ can then be *reconstructed* by means of Eq. (6.12) when explicitly implementing an UV regularization scheme (i.e., with EAA as the input and the bare action as the output).

Most commonly, the effective average action is considered as a functional of the full metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ given by

$$\boxed{\Gamma_k[g, \bar{g}, \bar{\xi}, \xi, \zeta] := \Gamma_k[g - \bar{g}, \bar{\xi}, \xi, \zeta; \bar{g}]\,.} \tag{6.18}$$

Therewith we further define (in slight abuse of notation):

$$\begin{aligned} \Gamma_k[g, \bar{g}] &:= \Gamma_k[g, \bar{g}, 0, 0, 0] \\ \text{and} \quad \Gamma_k[g] &:= \Gamma_k[g, g]\,. \end{aligned} \tag{6.19}$$

Note that if we transition into these conventions we may use $\delta/\delta h_{\mu\nu}(x) \equiv \delta/\delta g_{\mu\nu}(x)$.

6.3. Beta functions and Asymptotic Safety

With this change of strategy, we must clarify on what space the FRGE, as a differential equation in the renormalization group time t, is actually defined on; i.e., what is the set of functionals out of which solutions to the FRGE might arise? We call this space *theory space* whose definition in general terms is not far to seek: on the one hand, we have chosen the *field space* $\{x \mapsto \phi(x)\}$ of fields defined on the, respectively chosen, background spacetime with metric $\bar{g}$ (here: $\phi = (h, \bar{\xi}, \xi, \zeta)^T$), and on the other hand, we have chosen a *symmetry group* G (here the group of spacetime diffeomorphisms). Then, theory space is the set of all G-invariant, regular functionals $\{x \mapsto \phi(x)\} \to \mathbb{R}$. On this theory space, the FRGE has the structure of a "flow equation", respectively an equation for integral curves, schematically $k\partial_k\Gamma_k[\cdots] = \mathfrak{B}\{\Gamma_k[\cdots]\}$. We will call $\mathfrak{B}$ the *beta functional*. Different initial conditions $\Gamma_{k=\Lambda}[\cdots] =$ "initial functional"$[\cdots]$ for this flow equation lead to (possibly) different solutions. Each solution $\Gamma_k[\cdots]$ is a one-parameter family of functionals in theory space and thence is referred to as a *renormalization group (RG) trajectory*. The *RG flow*, the set of all RG trajectories in theory space is determind by the pair ("theory space", $\mathfrak{B}$). Further, note that for this interpretation it is crucial that the tangent space of the theory space is isomorphic to theory space itself.[3] This condition is trivially fulfilled in the case that the theory space possesses the structure of a vector space which is a condition that we, w.l.o.g. for all our purposes, will apply.

[3]For details on the geometry of theory space see [53].

In this case, theory space thus is equipped with a basis of functionals[4]

$$\big\{P_i[\phi;\bar{g}] \;\big|\; i=1,2,\dots,\infty\big\}\,, \tag{6.20}$$

in which the EAA can be expanded:

$$\Gamma_k[\phi;\bar{g}] = \sum_{i=1}^{\infty} \bar{u}_i(k) P_i[\phi;\bar{g}]\,. \tag{6.21}$$

The coefficients $\{\bar{u}_i\}$ are the dimensionful running couplings of the theory. The left-hand side (LHS) of the FRGE then is given by

$$\partial_t \Gamma_k[\phi;\bar{g}] = \sum_{i=1}^{\infty} \left(\partial_t \bar{u}_i(k)\right) P_i[\phi;\bar{g}]\,, \tag{6.22}$$

while we define the expansion of its RHS in this basis as

$$\mathfrak{B}\left\{\sum_{i=1}^{\infty} \bar{u}_i(k) P_i[\phi;\bar{g}]\right\} =: \sum_{i=1}^{\infty} \bar{\beta}_i\left(\bar{u}_1(k), \bar{u}_2(k), \dots; k\right) P_i[\phi;\bar{g}]\,, \tag{6.23}$$

where the coefficients $\{\bar{\beta}_i\}$ are called the *dimensionful beta functionals*. A comparison of the coefficients on the LHS and RHS yields

$$\partial_t \bar{u}_i(k) = \bar{\beta}_i\left(\bar{u}_1(k), \bar{u}_2(k), \dots; k\right) \quad , \quad i=1,2,\dots\,. \tag{6.24}$$

These *exact* RG equations, that can be regarded as a generalization of the perturbative Callen-Symanzik equations[4], are infinitely many ordinary differential equations; their solution corresponds to "solving the theory".

We can remove the explicit k-dependence of the dimensionful beta functions $\{\bar{\beta}_i\}$ by going over to the *dimensionless running couplings*

$$u_i(k) := k^{-d_i} \bar{u}_i(k)\,, \tag{6.25}$$

with $d_i := [\bar{u}_i]$ the canonical mass dimension. Therewith the RG equations become

$$\partial_t u_i(k) = -d_i u_i(k) + k^{-d_i} \bar{\beta}_i\left(k^{d_1} u_1(k), k^{d_2} u_2(k), \dots; k\right)\,. \tag{6.26}$$

[4] Sometimes, we refer to these functionals as "operators", emphasizing the operator-nature of their counterparts $P_i[\chi;\bar{g}]$.

From dimensional analysis it is clear that $k^{-d_i}\bar{\beta}_i\left(k^{d_1}u_1(k), k^{d_2}u_2(k), \ldots; k\right) = \bar{\beta}\left(u_1(k), u_2(k), \ldots; 1\right)$ such that with the *dimensionless beta functions*

$$\beta_i\left(u_1(k), u_2(k), \ldots\right) \coloneqq -d_i u_i(k) + \bar{\beta}_i\left(u_1(k), u_2(k), \ldots; 1\right) \tag{6.27}$$

we have obtained the dimensionless RG equations

$$\partial_t u_i(k) = \beta_i\left(u_1(k), u_2(k), \ldots\right) . \tag{6.28}$$

We also write this equation as $\partial_t u(k) = \beta(u(k))$, in terms of the vectors $u(k) = (u_1(k), u_2(k), \ldots)$ and $\beta = (\beta_1, \beta_1, \ldots)$. As the dimensionless beta functions do not possesses an explicit k-dependence, these ordinary differential equations thus build an *autonomous system*. It is important to emphasize that only complete RG trajectories, i.e., those for which $k \mapsto u_i(k)$ is regular for all $k \in [0, \infty)$, correspond to a quantum field theory.

In order to solve Eq. (6.28), an *"initial" condition* of the form

$$u(\mu) = u^{(R)} \tag{6.29}$$

is required to select a specific trajectory in theory space. This initial condition can be regarded as "renormalized" couplings at the scale μ. If we infinitesimally shift this scale, $\mu' = \mu + \varepsilon$, we must shift the "renormalized" coupplings accordingly, $u^{(R)\prime} = u(\mu') = u(\mu) + \varepsilon\partial_\mu u(\mu) = u^{(R)} + \varepsilon\partial_\mu u(\mu)$, in order to stay among the same trajectory [54, 55]. To study the parametric dependence of the RG equations (6.28) on the initial condition(s) (6.29), let us denote by $u(k; \mu, u^{(R)})$ the corresponding parametric solution. As the initial conditions $u(\mu) = u^{(R)}$ and $u(\mu') = u^{(R)\prime}$ select the same trajectory (solution), we have

$$\begin{aligned} u\big(k; \mu, u^{(R)}\big) &= u\Big(k; \mu', u^{(R)\prime}\Big) \\ &= u\big(k; \mu, u^{(R)}\big) + \varepsilon\left[\partial_\mu + \sum_i \left(\partial_\mu u_i(\mu)\right)\frac{\partial}{\partial u_i^{(R)}}\right] u\big(k; \mu, u^{(R)}\big) \end{aligned} \tag{6.30}$$

and thus, with $\mu\partial_\mu u_i(\mu) = \beta_i(u(\mu)) = \beta_i(u^{(R)})$,

$$\left[\mu\partial_\mu + \sum_i \beta_i(u^{(R)})\frac{\partial}{\partial u_i^{(R)}}\right] u_j\big(k; \mu, u^{(R)}\big) = 0 \quad ; \quad j = 1, 2, \ldots . \tag{6.31}$$

We can use this linear partial differential equation to investigate the dependence of the RG trajectories on the initial condition. Also, this reasoning applies to the corresponding EAA

$$\Gamma_k[\phi;\bar{g}]\big(\mu, u^{(R)}\big) = \sum_{i=1}^{\infty} \bar{u}_i\big(k;\mu,u^{(R)}\big)\, P_i[\phi;\bar{g}]\,, \tag{6.32}$$

with $\bar{u}_i(k;\mu,u^{(R)}) = k^{d_i} u_i(k;\mu,u^{(R)})$, yielding [54, 55]

$$\left[\mu\partial_\mu + \sum_i \beta_i(u^{(R)}) \frac{\partial}{\partial u_i^{(R)}}\right] \Gamma_k[\phi;\bar{g}]\big(\mu, u^{(R)}\big)\ . \tag{6.33}$$

A crucial feature in analyzing the RG flow are *fixed points* $u^* = (u_1^*, u_2^*, \dots)$ of the beta functions $\{\beta_i\}$:

$$\beta_i(u^*) = 0 \quad \text{for all} \quad i = 1, 2, \dots\ . \tag{6.34}$$

A fixed point is called *Gaussian* if $u^* = 0$ and otherwise *non-Gaussian*. Taylor-expanding the RG equations around u^* yields

$$\begin{aligned}\partial_t u_i(k) &= \beta_i(u) \\ &= \sum_j (\partial_j\beta_i)(u^*)\, \big[u_j(k) - u_j^*\big] + \cdots \\ &=: \sum_j B_{ij}(u^*)\, \big[u_j(k) - u_j^*\big] + \cdots\ .\end{aligned} \tag{6.35}$$

In this linear approximation, the RG equations are solved by

$$u_i(k) = u_i^* + \sum_I C_I V_i^I \left(\frac{k_0}{k}\right)^{\theta_I}\,, \tag{6.36}$$

where k_0 as well as C_I are constants and V^I is a right-eigenvector of the matrix B with eigenvalue $-\theta_I$, i.e., $\sum_j B_{ij}(u^*)V_j^I = -\theta_I V_i^I$. This can easily be checked:

$$\begin{aligned}\partial_t u_i(k) &= k\sum_I C_I V_i^I k_0^{\theta_I}(-\theta_I)k^{-\theta_I-1} \\ &= \sum_I C_I(-V_i^I\theta_I)\left(\frac{k_0}{k}\right)^{\theta_I} \\ &= \sum_j B_{ij}(u^*)\sum_I C_I V_j^I\left(\frac{k_0}{k}\right)^{\theta_I} \\ &= \sum_j B_{ij}(u^*)\left[u_j(k)-u_j^*\right] .\end{aligned} \tag{6.37}$$

The matrix $B(u^*)$ is called the *stability matrix* of the fixed point u^*. For a Gaussian fixed point one generally has $B(u^* = 0) = 0$, whereas for a non-Gaussian fixed point one generally has $B(u^*) \neq B^T(u^*)$ such that its negative eigenvalues θ_I, called the *critical exponents* or *scaling exponents*, might be complex. Furthermore, note that $B(u^*)$ can be decomposed according to

$$\begin{aligned}B_{ij}(u^*) = \partial_j\beta_i(u^*) &= -d_i\delta_{ij} + \frac{\partial}{\partial u_j}\bar{\beta}_i(u_i;1)\bigg|_{u=u^*} \\ &=: -D_{ij} + \bar{B}_{ij}(u^*) ,\end{aligned} \tag{6.38}$$

with $D_{ij} = d_i\delta_{ij}$ and $\bar{B}_{ij}(u^*) = \partial_j\bar{\beta}_i(u^*;1)$. The matrix D simply states the canonical mass dimensions of the running couplings $\{\bar{u}_i(k)\}$, which are also the classical scaling dimensions[5] of their dimensionless counterparts $\{u_i(k)\}$. On the other hand, the eigenvalues of the matrix $\bar{B}(u^*)$ determine the quantum corrections to the classical scaling of the couplings $\{u_i(k)\}$, which is why we call $\bar{B}(u^*)$ the *anomalous-dimension matrix* and its eigenvalues the *anomalous dimensions*. For instance, assume that $B(u^*)$ is a full-rank diagonalizable matrix, and let A be the matrix that diagonalizes it. Then we have $\sum_{l,m} A_{il}B_{lm}(u^*)A^{-1}_{mj} = -\theta_i\delta_{ij}$. It is clear that A also diagonalizes $\bar{B}(u^*)$, i.e., $\sum_{l,m} A_{il}\bar{B}_{lm}(u^*)A^{-1}_{mj} = -\eta_i\delta_{ij}$, where $\{\eta_i\}$ are the negative eigenvalues of $\bar{B}(u^*)$ (correspondingly, $\{-\eta_i\}$ are the anomalous dimensions). It follows that $\theta_i = d_i + \eta_i$, i.e., the full scaling dimension θ_i of the coupling $u_i(k)$ decomposes into the classical scaling dimension

[5]We call a the scaling dimension of the function $y(x)$ if the differential equation $(x\partial_x + a)y(x) = 0$ holds, which is solved by $y(x) \sim x^{-a}$.

d_i and its quantum correction given by the (negative) anomalous dimension η_i.

When lowering k (direction UV $\to$ IR), we classify the "eigendirection" V^I of an RG trajectory as follows:

- *relevant* if $\operatorname{Re}\theta_I > 0$ (grows when lowering k)
- *irrelevant* if $\operatorname{Re}\theta_I < 0$ (shrinks when lowering k)
- *marginal* if $\operatorname{Re}\theta_I = 0$ (constant when lowering k).

Correspondingly we can decompose the tangent space at u^* of theory space in relevant, irrelevant and marginal subspaces that are spannend by the set of the respective V^I. The relevant directions play a particularly special role because these determine the *UV critical hypersurface* $\mathscr{S}_{\text{UV}}$ of u^*. $\mathscr{S}_{\text{UV}}$ is defined as the set of all points in theory space that are pulled into the fixed point u^* by the inverse RG flow (i.e., $k \to \infty$). Its dimension $\Delta_{\text{UV}} \coloneqq \dim \mathscr{S}_{\text{UV}}$ is given by the number of relevant directions, $\Delta_{\text{UV}} = \#\big\{\theta_I \bigm| \operatorname{Re}\theta_I > 0\big\}$. In general, a trajectory in $\mathscr{S}_{\text{UV}}$ thus is parametrized by

$$u_i(k) = u_i^* + \sum_{I \text{ with } \operatorname{Re}\theta_I > 0} C_I V_i^I \left(\frac{k_0}{k}\right)_I^{\theta} . \tag{6.39}$$

In this framework, the *non-perturbative renormalization* of a quantum field theory means finding a complete RG trajectory in theory space, i.e., the limits $k \to \infty$ in the UV and $k \to 0$ in the IR must exist.

The *asymptotically safe* solution to the "UV problem" relies on the existence of a non-Gaussian fixed point and refers to the existence of a Δ_{UV}-dimensional parametric family of "asymptorically safe" trajectories, i.e., those with

$$\lim_{k\to\infty} \Gamma_k[\cdots] = \Gamma_*[\cdots] = \sum_{i=1}^{\infty} \bar{u}_i^* \, P_i[\cdots] . \tag{6.40}$$

Every of these trajectories corresponds to a quantum field theory with fixed values for Δ_{UV} "renormalized", i.e., physical, parameters. These must be determined experimentally. Unfortunately, the Asymptotic Safety construction comes with a catch, indeed: To construct a basis of theory space such that the (infinitely many) equation $\beta_i(u^*) = 0$, let alone the infinitely many ordinary differential equations (6.28), are actually solvable can be a hard problem. In many cases, it is reasonable to begin the search for a fixed point in theory space by screening subspaces of theory space for fixed points. These subspaces are called *truncations*. Note that surely these can also be spanned by infinitely many functionals. An example of a finite truncation is e.g. $\{P_i[\phi;\bar{g}] \mid i = 1, 2, \dots, N\}$ with $N < \infty$. Projecting the RG equations (6.28) onto this truncation, we have

$$\partial_t u_i(k) = \beta_i\left(u_1(k), u_2(k), \dots, u_N(k)\right) \quad , \quad i = 1, \dots, N \,. \tag{6.41}$$

Here, the beta functions are obtained by neglecting all terms on the RHS, i.e., in

$$\mathfrak{B}\left\{\sum_{i=1}^{N} \bar{u}_i(k) P_i[\phi;\bar{g}]\right\} , \tag{6.42}$$

that are proportional to $P_j[\phi;\bar{g}]$ with $j \geq N+1$. The strategy in screening truncations for fixed points is as follows: if a given truncation is shown to exhibit a fixed point one must enlarge the truncation to check if the fixed point still exists in the enlarged truncation. Otherwise, the fixed point could potentially be only a truncation artefact.

6.4. The ansatz for neglecting the evolution of the ghost fields

An important *ansatz* in solving the FRGE consists in surpressing the evolution of the ghosts,

$$\Gamma_k[g,\bar{g},\bar{\xi},\xi,\zeta] = \Gamma_k[g,\bar{g}] + S_{\mathrm{gh},1}[g-\bar{g},\bar{\xi},\xi;\bar{g}] + S_{\mathrm{gh},2}[\zeta;\bar{g}] \,. \tag{6.43}$$

This ansatz amounts to projecting the FRGE (6.13) onto the subspace of theory space given by $\zeta = \bar{\xi} \equiv 0 \equiv \xi$:

$$\begin{aligned}
\partial_t\Gamma_k[g;\bar{g}] = \frac{1}{2}\,\mathrm{Tr}_{ST^2}&\left[\left(\partial_t\mathscr{R}_{k11}\right)[\bar{g}]\left(\left[\Gamma_k^{(2)}[g,\bar{g},0,0,0] + \mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{11}\right] \\
&- \mathrm{Tr}_V\left[\left(\partial_t\mathscr{R}_{k23}\right)[\bar{g}]\left(\left[\Gamma_k^{(2)}[g,\bar{g},0,0,0] + \mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{32}\right] \\
&- \frac{1}{2}\,\mathrm{Tr}_V\left[\left(\partial_t\mathscr{R}_{k44}\right)[\bar{g}]\left(\left[\Gamma_k^{(2)}[g,\bar{g},0,0,0] + \mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{44}\right] .
\end{aligned} \tag{6.44}$$

It is easy to see that the operator $\Gamma_k^{(2)}[g,\bar{g},0,0,0]$ in field space takes a particularly simple form when employing the ansatz (6.43):

$$\Gamma_k^{(2)}[g,\bar{g},0,0,0] = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \tag{6.45}$$

with

$$\begin{aligned} a_{11} &= (\Gamma_k^{(2)})_{11}[g,\bar{g}] \\ a_{23} &= (S^{(2)}_{\text{gh},1})_{23}[g-\bar{g},0,0;\bar{g}] \\ a_{32} &= (S^{(2)}_{\text{gh},1})_{32}[g-\bar{g},0,0;\bar{g}] \\ a_{44} &= (S^{(2)}_{\text{gh},2})_{44}[0;\bar{g}]\,. \end{aligned} \tag{6.46}$$

These operators follow a generalization of Eq. (6.16): we define the operator $F^{(2)}[g,\bar{g},\bar{\xi},\xi,\zeta]$, that is associated to a functional $F[g,\bar{g},\bar{\xi},\xi,\zeta]$, via

$$\langle x,\ldots|\left(F^{(2)}\right)_{ab}[g,\bar{g},\bar{\xi},\xi,\zeta]|y,\ldots\rangle \coloneqq \frac{(-1)^{|\phi^b|}}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} I_{ab}[\bar{g}] \frac{\delta^2 F[g,\bar{g},\bar{\xi},\xi,\zeta]}{\delta\phi^a(x)\delta\phi^b(y)}\,. \tag{6.47}$$

Also the cutoff operator in field space, $\mathscr{R}_k[\bar{g}]$, is, in full generality as given by Eq. (6.15), of the above matrix form,

$$\mathscr{R}_k[\bar{g}] = \begin{pmatrix} \mathscr{R}_k^{\text{grav}}[\bar{g}] & 0 & 0 & 0 \\ 0 & 0 & \mathscr{R}_k^{\text{gh},1}[\bar{g}] & 0 \\ 0 & -\mathscr{R}_k^{\text{gh},1}[\bar{g}] & 0 & 0 \\ 0 & 0 & 0 & \mathscr{R}_k^{\text{gh},2}[\bar{g}] \end{pmatrix}. \tag{6.48}$$

As a matrix of this form is easily inverted, even for non-commuting entries, namely by

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 & 0 \\ 0 & 0 & c^{-1} & 0 \\ 0 & b^{-1} & 0 & 0 \\ 0 & 0 & 0 & d^{-1} \end{pmatrix}, \tag{6.49}$$

the FRGE thus becomes

$$\boxed{\begin{aligned}\partial_t\Gamma_k[g;\bar g] = &\frac{1}{2}\operatorname{Tr}_{ST^2}\left[\left(\partial_t\mathscr{R}_k^{\text{grav}}[\bar g]\right)\left((\Gamma_k^{(2)})_{11}[g,\bar g]+\mathscr{R}_k^{\text{grav}}[\bar g]\right)^{-1}\right]\\ &-\operatorname{Tr}_V\left[\left(\partial_t\mathscr{R}_k^{\text{gh},1}[\bar g]\right)\left((S_{\text{gh},1}^{(2)})_{23}[g-\bar g,0,0;\bar g]+\mathscr{R}_k^{\text{gh},1}[\bar g]\right)^{-1}\right]\\ &-\frac{1}{2}\operatorname{Tr}_V\left[-\left(\partial_t\mathscr{R}_k^{\text{gh},2}[\bar g]\right)\left((S_{\text{gh},2}^{(2)})_{44}[0;\bar g]+\mathscr{R}_k^{\text{gh},2}[\bar g]\right)^{-1}\right].\end{aligned}} \tag{6.50}$$

Again, by proceeding fully analogously, we must discard the last term in the case of $b_\mu \equiv 0 \equiv \zeta_\mu$. With Eq. (4.14) we have, in full generality,

$$(S_{\text{gh},2}^{(2)})_{44}[0;\bar g] = \frac{1}{\alpha}Y[\bar g]\,. \tag{6.51}$$

If we specify the gauge-fixing condition to Eq. (4.31), which amounts to considering the ghost acion (4.34), we find further that

$$(S_{\text{gh},1}^{(2)})_{23}[g-\bar g,0,0;\bar g] = -\sqrt{2}\mathscr{M}[g,\bar g]\,. \tag{6.52}$$

With these further specifications, the FRGE for the above ansatz reads

$$\boxed{\begin{aligned}\partial_t\Gamma_k[g,\bar g] = &\frac{1}{2}\operatorname{Tr}_{ST^2}\left[\left(\partial_t\mathscr{R}_k^{\text{grav}}[\bar g]\right)\left((\Gamma_k^{(2)})_{11}[g,\bar g]+\mathscr{R}_k^{\text{grav}}[\bar g]\right)^{-1}\right]\\ &-\operatorname{Tr}_V\left[\left(\partial_t\mathscr{R}_k^{\text{gh},1}[\bar g]\right)\left(-\sqrt{2}\mathscr{M}[g,\bar g]+\mathscr{R}_k^{\text{gh},1}[\bar g]\right)^{-1}\right]\\ &-\frac{1}{2}\operatorname{Tr}_V\left[\left(\partial_t\mathscr{R}_k^{\text{gh},2}[\bar g]\right)\left(\frac{1}{\alpha}Y[\bar g]+\mathscr{R}_k^{\text{gh},2}[\bar g]\right)^{-1}\right].\end{aligned}} \tag{6.53}$$

6.5. The one-loop approximation of the FRGE

Rigorously obtaining a one-loop approximation of the FRGE (6.13), or respectively (6.14), is per se not possible: the EAA $\Gamma_k[\phi;\bar g]$ is meant to be the outcome from solving the equation, so there is literally nothing we could expand in $\hbar$ for an expansion in loops. Of course, formally expanding $\Gamma_k[\phi;\bar g]$ in a series of vertices is a possible ansatz for solving the FRGE, however, to have at hand a one-loop approximation of the FRGE itself, i.e., an FRGE whose RHS amounts to one-loop beta functions only, is not possible. What is possible, on the other hand, is to construct such an equation using a reference to some bare action: Just as we had derived the exact FRGE from constructing the EAA by means of a bare action, we can review this whole process again and thereby implement the one-loop approximation thanks to the presence of a bare action. The "FRGE" obtained in this way is *not* a closed equation for the EAA, unlike when constructing the exact FRGE. Inspired by this one-loop approximation involving a bare action, we will then propose an approximation scheme of the exact FRGE that mimics this one-loop approximation, yet still is a closed equation for the EAA. As is done frequently in the literature, we will call this approximation the "one-loop approximation of the FRGE" – although it a priori does not amount to an expansion in $\hbar$. Lastly, we will point out in what sense this terminology is legitimate, indeed.

So let us resort to the conventional one-loop effective action for quantum gravity as a guiding light. Examplarily consider the theory of higher-derivative gravity whose one-loop effective action (4.92) we already had deduced in Section 4.3. We can express this one-loop effective action through the operator $S^{(2)}[\phi;\bar{g}]$ associated to the full (bare) action $S[\chi;\bar{g}]$ given by Eq. (4.88). It is straighforward to see that this operator in field space possesses the structure

$$S^{(2)}[\phi;\bar{g}] = \begin{pmatrix} S^{(2)}_{11} & S^{(2)}_{12} & S^{(2)}_{13} & 0 \\ S^{(2)}_{21} = -\big(S^{(2)}_{12}\big)^* & 0 & S^{(2)}_{23} & 0 \\ S^{(2)}_{31} = -\big(S^{(2)}_{13}\big)^* & S^{(2)}_{32} = -\big(S^{(2)}_{23}\big)^* & 0 & 0 \\ 0 & 0 & 0 & S^{(2)}_{44} \end{pmatrix} [\phi;\bar{g}]\,, \tag{6.54}$$

where $(S^{(2)}_{ab})^*$ denotes the dual operator (see Subsection 4.2.2), and is explicitly determined by the matrix elements

$$
\begin{aligned}
\langle x,\mu,\nu|S^{(2)}_{11}[\phi;\bar g]|y,\rho,\sigma\rangle &= \frac{1}{\sqrt{\bar g(x)}\sqrt{\bar g(y)}} I_{\alpha\beta\rho\sigma}[\bar g]\frac{\delta^2 S[\phi;\bar g]}{\delta h_{\mu\nu}(x)\delta h_{\alpha\beta}(y)}\\
&\equiv \langle x,\mu,\nu|\mathscr{U}'[\phi;\bar g]|y,\rho,\sigma\rangle\,;\\
\langle x,\mu,\nu|S^{(2)}_{12}[\phi;\bar g]|y,\rho\rangle &= \frac{-1}{\sqrt{\bar g(x)}\sqrt{\bar g(y)}}\bar g_{\rho\alpha}\frac{\delta^2 S[\phi;\bar g]}{\delta h_{\mu\nu}(x)\delta\bar\xi_\alpha(y)}\\
&\equiv \langle x,\mu,\nu|\mathscr{X}_1[h,\xi;\bar g]|y,\rho\rangle\,;\\
\langle x,\mu,\nu|S^{(2)}_{13}[\phi;\bar g]|y,\rho\rangle &= \frac{-1}{\sqrt{\bar g(x)}\sqrt{\bar g(y)}}\frac{\delta^2 S[\phi;\bar g]}{\delta h_{\mu\nu}(x)\delta\xi^\rho(y)}\\
&\equiv \langle x,\mu,\nu|\mathscr{X}_2[h,\bar\xi;\bar g]|y,\rho\rangle\,;\\
\langle x,\mu|S^{(2)}_{32}[\phi;\bar g]|y,\nu\rangle &= \frac{-1}{\sqrt{\bar g(x)}\sqrt{\bar g(y)}}\bar g^{\mu\alpha}(x)\bar g_{\nu\beta}(y)\frac{\delta^2 S[\phi;\bar g]}{\delta C^\alpha(x)\delta\bar C_\beta(y)}\\
&\equiv \langle x,\mu|\sqrt{2}\mathscr{M}[g,\bar g]|y,\nu\rangle\,;\\
\langle x,\mu|S^{(2)}_{44}[\phi;\bar g]|y,\nu\rangle &= \frac{-1}{\sqrt{\bar g(x)}\sqrt{\bar g(y)}}\bar g_{\nu\alpha}(y)\frac{\delta^2 S[\phi;\bar g]}{\delta b_\mu(x)\delta b_\alpha(y)}\\
&\equiv \langle x,\mu|\frac{1}{\alpha}Y[\bar g]|y,\nu\rangle\,;
\end{aligned}
\tag{6.55}
$$

with $S^{(2)}_{21}[\phi;\bar g] = -S^{(2)}_{12}[\phi;\bar g]^* = -\mathscr{X}_1[h,\xi;\bar g]^*$ and likewise for $S^{(2)}_{31}[\phi;\bar g]$. Therewith, the one-loop effective action (4.92) reads

$$
\begin{aligned}
\Gamma[\phi;\bar g] = S[\phi;\bar g] &+ \frac{1}{2}\,\mathrm{Tr}_{ST^2}\ln\left[S^{(2)}_{11}[\phi;\bar g] - S^{(2)}_{13}[\phi;\bar g]\Big(S^{(2)}_{32}[\phi;\bar g]\Big)^{-1}S^{(2)}_{21}[\phi;\bar g]\right]\\
&- \mathrm{Tr}_V\ln\left[S^{(2)}_{32}[\phi;\bar g]\right] - \frac{1}{2}\,\mathrm{Tr}_V\ln\left[S^{(2)}_{44}[\phi;\bar g]\right].
\end{aligned}
\tag{6.56}
$$

By means of this example, *it is obvious that this is the form of the one-loop effective action for any bare action* $S[\chi;\bar g]$ *of the form Eq.* (4.15).

According to the procedure outlined previously in Section 6.2, it is clear that we obtain a one-loop approximation of the functional $\tilde\Gamma_k[\phi;\bar g]$, as contructed from the bare action, by implementing the replacement on the RHS of the above conventional one-loop effective action, $S[\chi;\bar g]\mapsto S[\chi;\bar g]+\Delta_k S[\chi;\bar g]$. Thereby, an important subtlety occurs: Previously, we had defined the cutoff operator, that is encoded into $\Delta_k S[\chi;\bar{g}]$, by

$$\mathscr{R}_k[\bar{g}] = \mathscr{Z}_k[\bar{g}] k^{2\gamma} R^{(0)}\big((-\bar{D}^2/k^2)^\gamma\big) \,, \tag{6.57}$$

where $\mathscr{Z}_k[\bar{g}]$ depends on k through the running couplings of the theory. For example, if we were to solve the exact FRGE by making use of the ansatz $\Gamma_k[\phi;\bar{g}] = \sum_{i\in I} \bar{u}_i(k) P_i[\phi;\bar{g}]$ we would give $\mathscr{Z}_k[\bar{g}]$ the structure $\mathscr{Z}_k[\bar{g}] \equiv \mathscr{Z}(u(k))[\bar{g}]$. However, in the one-loop approximation of the effective action, the couplings of the bare action are *not* running (yet). Hence, it is not far to seek that if the bare action has the structure $S[\phi;\bar{g}] = \sum_{i\in I} \bar{u}_i P_i[\phi;\bar{g}]$ we may use $\mathscr{Z}(u)[\bar{g}]$ for the structure of the cutoff operator. Thus, for notational clarity, let us define this cutoff operator by

$$\widetilde{\mathscr{R}_k}[\bar{g}] = \mathscr{Z}[\bar{g}] k^{2\gamma} R^{(0)}\big((-\bar{D}^2/k^2)^\gamma\big) \,, \tag{6.58}$$

and by $\widetilde{\Delta_k S}[\chi;\bar{g}]$ the cutoff action in which it is encoded. Thus by implementing the replacement $S[\chi;\bar{g}] \mapsto S[\chi;\bar{g}] + \widetilde{\Delta_k S}[\chi;\bar{g}]$ which goes along with $S^{(2)}[\chi;\bar{g}] \mapsto S^{(2)}[\chi;\bar{g}] + \widetilde{\mathscr{R}_k}[\bar{g}]$, we arrive at the EAA – again, as constructed from the bare action – at one-loop,

$$\begin{aligned}
\Gamma_k[\phi;\bar{g}] &\coloneqq \tilde{\Gamma}_k[\phi;\bar{g}] - \Delta_k S[\phi;\bar{g}] \\
&= S\,[\phi;\bar{g}] + \frac{1}{2}\,\mathrm{Tr}_{ST^2} \ln \Bigg[S^{(2)}_{11}[\phi;\bar{g}] + \widetilde{\mathscr{R}}_{k11}[\bar{g}] \\
&\qquad\qquad - S^{(2)}_{13}[\phi;\bar{g}] \Big(S^{(2)}_{32}[\phi;\bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big)^{-1} S^{(2)}_{21}[\phi;\bar{g}] \Bigg] \\
&\quad - \mathrm{Tr}_V \ln \Big[S^{(2)}_{32}[\phi;\bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big] - \frac{1}{2}\,\mathrm{Tr}_V \ln \Big[- S^{(2)}_{44}[\phi;\bar{g}] + \widetilde{\mathscr{R}}_{k44}[\bar{g}] \Big] \\
&\quad + O(2 \text{ loops}) \\
&=: S[\phi;\bar{g}] + \Gamma^{1\mathrm{L}}_k[\phi;\bar{g}] + O(2 \text{ loops})\,.
\end{aligned} \tag{6.59}$$

In the deep UV, at a scale Λ for $k \to \infty$, the one-loop result constitutes a good approximation of the EAA such that $\Gamma_{k=\Lambda}[\phi;\bar{g}] = S[\phi;\bar{g}] + \Gamma^{1\mathrm{L}}_{k=\Lambda}[\phi;\bar{g}]$ can be regarded as an initial condition for the FRGE. Equivalently, we could have obtained this one-loop approximation by expanding the exponent on the RHS of Eq. (6.12) as a Volterra series (cf. Eq. (C.5)). Importantly, note that due to the presence of a bare action this equation requires an UV cutoff to be well-defined. This fact gives rise to the *reconstruction problem*, i.e., the problem of reconstructing the bare action from a given solution $\Gamma_k[\phi; \bar{g}]$ of the FRGE (6.13) [56].

Taking the derivative with respect to the renormalization group time $t = \ln k$ yields the equation

$$\begin{aligned}
\partial_t \Gamma_k[\phi; \bar{g}] = &+ \frac{1}{2} \operatorname{Tr}_{ST^2} \Bigg\{ \Bigg[S^{(2)}_{11}[\phi; \bar{g}] + \widetilde{\mathscr{R}}_{k11}[\bar{g}] \\
&\qquad - S^{(2)}_{13}[\phi; \bar{g}] \Big(S^{(2)}_{32}[\phi; \bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big)^{-1} S^{(2)}_{21}[\phi; \bar{g}] \Bigg]^{-1} \\
&\quad \times \Bigg[\partial_t \widetilde{\mathscr{R}}_{k11}[\bar{g}] + S^{(2)}_{13}[\phi; \bar{g}] \Big(S^{(2)}_{32}[\phi; \bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big)^{-1} \Big(\partial_t \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big) \\
&\qquad \times \Big(S^{(2)}_{32}[\phi; \bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big)^{-1} S^{(2)}_{21}[\phi; \bar{g}] \Bigg] \Bigg\} \\
&- \operatorname{Tr}_V \Bigg[\Big(S^{(2)}_{32}[\phi; \bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big)^{-1} \partial_t \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Bigg] \\
&- \frac{1}{2} \operatorname{Tr}_V \Bigg[\Big(S^{(2)}_{44}[\phi; \bar{g}] + \widetilde{\mathscr{R}}_{k44}[\bar{g}] \Big)^{-1} \partial_t \widetilde{\mathscr{R}}_{k44}[\bar{g}] \Bigg] \\
&+ O(2 \text{ loops}) \, .
\end{aligned} \tag{6.60}$$

Note that as for the exact FRGE, the traces on the RHS are also UV finite thanks to the presence of k-derivatives $\partial_t \widetilde{\mathscr{R}}_k[\bar{g}]$ of the cutoff operator. On the subspace $\zeta = \bar{\xi} \equiv 0 \equiv \xi$ this one-loop equation boils down to

$$\begin{aligned}
\partial_t \Gamma_k[g, \bar{g}] = &+ \frac{1}{2} \operatorname{Tr}_{ST^2} \Bigg[\Big(S^{(2)}_{11}[g - \bar{g}, 0, 0, 0; \bar{g}] + \widetilde{\mathscr{R}}_{k11}[\bar{g}] \Big)^{-1} \partial_t \widetilde{\mathscr{R}}_{k11}[\bar{g}] \Bigg] \\
&- \operatorname{Tr}_V \Bigg[\Big(S^{(2)}_{32}[g - \bar{g}, 0, 0, 0; \bar{g}] + \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Big)^{-1} \partial_t \widetilde{\mathscr{R}}_{k32}[\bar{g}] \Bigg] \\
&- \frac{1}{2} \operatorname{Tr}_V \Bigg[\Big(S^{(2)}_{44}[g - \bar{g}, 0, 0, 0; \bar{g}] + \widetilde{\mathscr{R}}_{k44}[\bar{g}] \Big)^{-1} \partial_t \widetilde{\mathscr{R}}_{k44}[\bar{g}] \Bigg] \\
&+ O(2 \text{ loops}) \, .
\end{aligned} \tag{6.61}$$

For example, in case of higher derivative gravity, this reduced one-loop equation reads

$$\begin{aligned}\partial_t\Gamma_k[g,\bar{g}] = &+\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}_{\mathrm{h.-d.}}[g-\bar{g};\bar{g}]+\widetilde{\mathscr{R}_k^{\mathrm{grav}}}[\bar{g}]\right)^{-1}\partial_t\widetilde{\mathscr{R}_k^{\mathrm{grav}}}[\bar{g}]\right]\\ &-\mathrm{Tr}_V\left[\left(-\sqrt{2}\mathscr{M}[g,\bar{g}]+\widetilde{\mathscr{R}_k^{\mathrm{gh},1}}[\bar{g}]\right)^{-1}\partial_t\widetilde{\mathscr{R}_k^{\mathrm{gh},1}}[\bar{g}]\right]\\ &-\frac{1}{2}\,\mathrm{Tr}_V\left[\left(\frac{1}{\alpha}Y[\bar{g}]+\widetilde{\mathscr{R}_k^{\mathrm{gh},2}}[\bar{g}]\right)^{-1}\partial_t\widetilde{\mathscr{R}_k^{\mathrm{gh},2}}[\bar{g}]\right]\\ &+O(2\text{ loops})\,.\end{aligned}\tag{6.62}$$

The operator $\mathscr{U}_{\mathrm{h.-d.}}[g-\bar{g};\bar{g}]$ was initially defined by Eq. (4.95) (in this definition, however, we had set $g-\bar{g}=0$ for practical purposes). This one-loop approximation of the FRGE hence corresponds to setting $(\Gamma_k^{(2)})_{11}[g,\bar{g}] = S_{11}^{(2)}[g-\bar{g},0,0,0;\bar{g}] = \mathscr{U}_{\mathrm{h.-d.}}[g-\bar{g};\bar{g}]$ in Eq. (6.50).

We emphasize that one fully analogously obtains for the case $b_\mu \equiv 0 \equiv \zeta_\mu$, when employing the one-loop effective action (4.70) determined by the Einstein-Hilbert action (4.3) as part of the bare action, with $\mathscr{U}_{\mathrm{EH}}[g-\bar{g};\bar{g}]$ given by Eq. (4.58) that

$$\begin{aligned}\partial_t\Gamma_k[g,\bar{g}] = &+\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}_{\mathrm{EH}}[g-\bar{g};\bar{g}]+\widetilde{\mathscr{R}_k^{\mathrm{grav}}}[\bar{g}]\right)^{-1}\partial_t\widetilde{\mathscr{R}_k^{\mathrm{grav}}}[\bar{g}]\right]\\ &-\mathrm{Tr}_V\left[\left(-\sqrt{2}\mathscr{M}[g,\bar{g}]+\widetilde{\mathscr{R}_k^{\mathrm{gh},1}}[\bar{g}]\right)^{-1}\partial_t\widetilde{\mathscr{R}_k^{\mathrm{gh},1}}[\bar{g}]\right]\\ &+O(2\text{ loops})\,.\end{aligned}\tag{6.63}$$

Unlike the exact FRGE, these one-loop approximations are not closed equations for the EAA $\Gamma_k[\phi;\bar{g}]$ and still depend on the bare action $S[\phi;\bar{g}]$. Next, let us consider the general one-loop approximation Eq. (6.60), that we are stuck with. It is tempting to ask what might happen if we emulated this equation by a closed equation for $\Gamma_k[\phi;\bar{g}]$. The following "renormalization group improvement" of Eq. (6.60) is naturally apparent: Simply replace the operator $S^{(2)}[\phi;\bar{g}]$ on the RHS by $\Gamma_k^{(2)}[\phi;\bar{g}]$ and *after* evaluating the k-derivative on the RHS reinstall $\widetilde{\mathscr{R}_k}[\bar{g}] \mapsto \mathscr{R}_k[\bar{g}]$ (which amounts to directly making this replacement and neglegting the k-derivative of $\mathscr{Z}_k$). However, it is immediately obvious that this procedure leads nowhere meaningfull because the closed equation for $\Gamma_k[\phi;\bar{g}]$ obtained in this way clearly is *not* an approximation of the general FRGE (6.13).[6] Therefore, let us move on and perform the "renormalization group improvement" with the one-loop approximation (6.61) on the subspace $\zeta = \bar{\xi} \equiv 0 \equiv \xi$. On this subspace, we would try to solve the (general) FRGE with the ansatz (6.43). Thus, regarding the first procedure, we replace the operator $S^{(2)}[\phi;\bar{g}]$ on the RHS by $\Gamma_k^{(2)}[\phi;\bar{g}]$ and then further employ the ansatz (6.43). For the latter procedure, we introduce the auxiliary cutoff operator

$$\mathscr{R}_{k,k'}[\bar{g}] := \mathscr{Z}_{k'}[\bar{g}] k^{2\gamma} R^{(0)}\big((-\bar{D}^2/k^2)^\gamma\big) \,, \tag{6.64}$$

where the scale k' has no special meaning and only serves the purpose that ∂_t does not act on $\mathscr{Z}_k$. Therewith, the above described "renormalization group improvement" of Eq. (6.61) reads

$$\boxed{\begin{aligned}\partial_t \Gamma_k[g,\bar{g}] = &+\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[\Big((\Gamma_k^{(2)})_{11}[g,\bar{g}] + \mathscr{R}_{k11}[\bar{g}]\Big)^{-1} \partial_t(\mathscr{R}_{k,k'})_{11}[\bar{g}]\right]_{k'=k} \\ &- \mathrm{Tr}_V\left[\Big(S^{(2)}_{32}[g-\bar{g},0,0,0;\bar{g}] + \mathscr{R}_{k32}[\bar{g}]\Big)^{-1} \partial_t(\mathscr{R}_{k,k'})_{32}[\bar{g}]\right]_{k'=k} \\ &- \frac{1}{2}\,\mathrm{Tr}_V\left[\Big(S^{(2)}_{44}[g-\bar{g},0,0,0;\bar{g}] + \mathscr{R}_{k44}[\bar{g}]\Big)^{-1} \partial_t(\mathscr{R}_{k,k'})_{44}[\bar{g}]\right]_{k'=k} \,,\end{aligned}} \tag{6.65}$$

respectively when employing the gauge-fixing condition (4.31),

$$\boxed{\begin{aligned}\partial_t \Gamma_k[g,\bar{g}] = &+\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[\Big((\Gamma_k^{(2)})_{11}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1} \partial_t \mathscr{R}_{k,k'}^{\mathrm{grav}}[\bar{g}]\right]_{k'=k} \\ &- \mathrm{Tr}_V\left[\Big(-\sqrt{2}\mathscr{M}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{gh},1}[\bar{g}]\Big)^{-1} \partial_t \mathscr{R}_{k,k'}^{\mathrm{gh},1}[\bar{g}]\right]_{k'=k} \\ &- \frac{1}{2}\,\mathrm{Tr}_V\left[\Big(\frac{1}{\alpha} Y[\bar{g}] + \mathscr{R}_k^{\mathrm{gh},2}[\bar{g}]\Big)^{-1} \partial_t \mathscr{R}_{k,k'}^{\mathrm{gh},2}[\bar{g}]\right]_{k'=k} \,.\end{aligned}} \tag{6.66}$$

[6] For example, the trace over the Hilbert space ST^2 would contain $\partial_t \mathscr{R}_{k32}[\bar{g}]$, where ∂_t does not act on $\mathscr{Z}_k$, which cannot be the case.

Indeed, this equation is an approximation of the general FRGE (6.13), namely precisely Eq. (6.50) (respectively (6.53)) with the only difference that here on the RHS ∂_t does not act on $\mathscr{Z}_k$. Although not related to an approximation in $\hbar$ per se, we will call Eq. (6.65) the *one-loop approximation of the FRGE* (6.50) (respectively (6.53)). Note that again, in the case of $b_\mu \equiv 0 \equiv \zeta_\mu$ we must discard the last trace in these approximations.

Let us argue why it is legitimate to call this approximation a one-loop approximation. Say, we are given a bare action of the form of Eq. (4.15),

$$S[\widehat{h}, \bar{C}, C, b; \bar{g}] := S_{\text{cl}}[\bar{g} + \widehat{h}] + S_{\text{GF}}[\widehat{h}; \bar{g}] + S_{\text{gh}}[\widehat{h}, \bar{C}, C; \bar{g}] + S_{\text{gh},2}[b; \bar{g}] \,, \quad (6.67)$$

with some classical action $S_{\text{cl}}[\widehat{g}] = \sum_{i\in I} \bar{u}_i P_i[\widehat{g}]$ in which the dimensionful couplings $\{\bar{u}_i\}$ parametrize some basis functionals $P_i[\widehat{g}]$. Through the specific choice of the gauge-fixing parameters entailed in the gauge-fixing action and the actions for the ghosts fields, these may depend on the bare couplings, as well: $S_{\text{GF}}[\widehat{h}; \bar{g}] = S_{\text{GF}}[\widehat{h}; \bar{g}](\{\bar{u}_i\})$, $S_{\text{gh}}[\widehat{h}, \bar{C}, C; \bar{g}] = S_{\text{gh}}[\widehat{h}, \bar{C}, C; \bar{g}](\{\bar{u}_i\})$ and $S_{\text{gh},2}[b; \bar{g}] = S_{\text{gh},2}[b; \bar{g}](\{\bar{u}_i\})$. Furthermore, let us specify the gauge-fixing condition to Eq. (4.31). For this setting, *we can calculate one-loop beta functions* $\{\beta_{u_i}(u)\}$ *of the dimensionless couplings solving the FRGE* (6.66) *at* $g \equiv \bar{g}$ *on the truncated theory space spannend by* $\{P_i[g]\}_{i\in I}$ *with the ansatz*[7]

$$\Gamma_k[g, \bar{g}] := \sum_{i\in I} \bar{u}_i(k) P_i[g] + S_{\text{GF}}[\widehat{h}; \bar{g}](\{\bar{u}_i(k)\}) \,, \quad (6.68)$$

where we simply gave a k-dependence to the couplings entailed in the "classical" and gauge fixing action. In Eq. (6.66), we thereby must also give a k-dependence to the operators $\mathscr{M}[g, \bar{g}] = \mathscr{M}[g, \bar{g}](\{\bar{u}_i(k)\})$ and $Y[\bar{g}] = Y[\bar{g}](\{\bar{u}_i(k)\})$, that results from giving a k-dependence to the couplings entailed in the ghost actions; furthermore, the required cutoff operators then have the structure $\mathscr{R}_{k,k'}[\bar{g}] = \mathscr{Z}[\bar{g}](\{\bar{u}_i(k')\}) k^{2\gamma} R^{(0)}\big((-\bar{D}^2/k^2)^\gamma\big)$. The RHS of Eq. (6.66) at $g \equiv \bar{g}$, expanded in the basis $\{P_i[g]\}_{i\in I}$, will then lead to beta functions $\{\beta_{u_i}(u)\}$ that we apparently can, with the reasoning of this chapter in the back of our mind, interpret as one-loop beta functions obtained from the above bare action. Especially, we emphasize that in case these one-loop beta functions are universal we will obtain precisely these – as from any other renormalization scheme. In Chapter 8 we will demonstrate this procedure using the example of higher-derivative gravity in $d = 4$.

[7]Note that $\Gamma_k[g, g] := \sum_{i\in I} \bar{u}_i(k) P_i[g]$.

6.6. The gravitational fixed point: a quick review of the state of affairs

Until recently, the main line of research in asymptotically safe quantum gravity has been devoted to the search for a gravitational fixed point. The previous examples illustrate results of first stages of this line of research. We discussed these particular examples in detail because because they build the foundation for applications following in the subsequent chapters. Here, let us give a quick review of the state of affairs in the search for the gravitational fixed point. We will not discuss the respective findings in detail as they are located somewhat outside the common thread of this thesis; in this case, we refer to the extensive literture quoted.

The search for the gravitational fixed point so far has been restricted to the FRGE (6.50) which neglects the evolution of the ghost fields and is solved by the ansatz (6.43). In this case the EAA reduces to the functional $\Gamma_k[g,\bar{g}]$. Solutions to this FRGE on truncations of theory space $\{\int \mathrm{d}^d x\sqrt{g}, \int \mathrm{d}^d x\sqrt{\bar{g}}R, \dots\}$ of increasing complexity have been obtained. In the case of $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$, e.g. by including higher curvature terms [47, 57–63], the Goroff-Sagnotti counterterm [64], and polynomials of the Ricci scalar of high order [65–68]. Even solutions on functional, i.e., infinite dimensional, truncations have been found [69–83]. Furthermore, solutions for ansätze that take into account the bimetric character of the flow have been obtained [84–87]. All this solutions consitently show a strong indication for the existence of the gravitational fixed point, which very likely rejects the possibility of the fixed point being a simple truncation artefact. Another possible approximation scheme for the EAA with which to solve the FRGE is its expansion in terms of vertices, schematically defined by

$$\langle x,\dots|\Gamma_k^{(m,n)}[g,\bar{g}]|y,\dots\rangle \coloneqq I[\bar{g}]\frac{\delta^{m+n}\Gamma_k}{\delta\bar{g}^m(x)\delta g^n(y)}\,, \tag{6.69}$$

where $I[\bar{g}]$ aligns the tensor structure of the RHS to that of the RHS. Also solutions obtained from ansätze of expansions in vertices are consistent with the previous results towards an asymptocially safe fixed point [88–94]. Moreover, the gravitational fixed point persists also in presence of a suitable matter content, see e.g. [95–99].

CHAPTER 7

The single metric Einstein-Hilbert truncation

Executive summary. We explicitly discuss the single metric Einstein-Hilbert truncation which is the truncation of theory space spanned by the functionals $\int \mathrm{d}^d x \sqrt{g}$ and $\int \mathrm{d}^d x \sqrt{g} R$. With an appropriate ansatz for the effective average action we obtain the RG equations for Newton's constant and the cosmological constant. We discuss the resulting flow and show that it exhibits a non-Gaussian fixed point. Moreover, we repeat this study for the simplified Einstein-Hilbert flow, with and without the presence of matter fields, that results from the one-loop approximation of the FRGE.

7.1. The full Einstein-Hilbert flow

Consider the field space $\phi = (h, \bar{\xi}, \xi)^T$ together with the diffeomorphism group as the theory space. Let us therewith restrict ourselves to analyzing the FRGE (6.53), that has been obtained by projecting the general FRGE (6.14) onto the subspace $\bar{\xi} \equiv 0 \equiv \xi$ and employing the gauge-fixing condition (4.31). Further, we will only analyze *single metric truncations* which here amounts to settng $g \equiv \bar{g}$. Thus, we have at hand the FRGE

$$\begin{aligned} \partial_t \Gamma_k[g,g] = & \frac{1}{2} \operatorname{Tr}_{ST^2} \left[\left(\partial_t \mathscr{R}_k^{\text{grav}}[g] \right) \left((\Gamma_k^{(2)})_{11}[g,g] + \mathscr{R}_k^{\text{grav}}[g] \right)^{-1} \right] \\ & - \operatorname{Tr}_V \left[\left(\partial_t \mathscr{R}_k^{\text{gh},1}[g] \right) \left(-\sqrt{2} \mathscr{M}[g,g] + \mathscr{R}_k^{\text{gh},1}[g] \right)^{-1} \right]. \end{aligned} \tag{7.1}$$

Note that we still must consider ansätze of the form $\Gamma_k[g,\bar{g}]$ whose variation we must perform in order to obtain $(\Gamma_k^{(2)})_{11}[g,g]$.

The single-metric *Einstein-Hilbert truncation* [26] is a two-dimensional truncation of the theory space spanned by the basis functionals $(g, \bar{g}) \mapsto \int \mathrm{d}^d x \sqrt{g(x)}$ and $(g, \bar{g}) \mapsto \int \mathrm{d}^d x \sqrt{g(x)}\, R[g](x)$. A standard ansatz for solving Eq. (7.1) on this truncated space is

$$\begin{aligned}\Gamma_k[g, \bar{g}] = {} & 2\kappa^2 Z_{Nk} \int \mathrm{d}^d x \sqrt{g} \left(-R[g] + 2\bar{\lambda}_k\right) \\ & + \frac{\kappa^2 Z_{Nk}}{\alpha} \int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} (\mathscr{F}_\mu^{\alpha\beta}[\bar{g}]\, g_{\alpha\beta})(\mathscr{F}_\nu^{\rho\sigma}[\bar{g}]\, g_{\rho\sigma})\end{aligned} \tag{7.2}$$

with $\mathscr{F}_\mu^{\alpha\beta}[\bar{g}]$ as given by Eq. (4.31). This ansatz is nothing but the classical Einstein-Hilbert action (4.3) together with the gauge-fixing action (4.33) after substituting $\kappa^2 \mapsto \kappa^2 Z_{Nk}$ and $\Lambda \mapsto \bar{\lambda}_k$. These are the k-dependent running couplings that parametrize the Einstein-Hilbert truncation. Furthermore, we employ henceforth the harmonic gauge and set $\alpha = 1$ and $\beta = 1/2$.

The only ingredient left in order to evaluate the RHS of the flow equation is the operator $(\Gamma_k^{(2)})_{11}[g, g]$ for the ansatz (7.2). Fortunately, it turns out that we had already calculated said operator in Subsection 4.2.2 and that it is given by Eq. (4.77) after substituting $\kappa^2 \mapsto \kappa^2 Z_{Nk}$ and $\Lambda \mapsto \bar{\lambda}_k$:

$$(\Gamma_k^{(2)})_{11}[g, g] = \mathscr{U}[0, g \equiv \bar{g}]_{\mathrm{E.-H.}}\Big|_{\kappa^2 \mapsto \kappa^2 Z_{Nk} \text{ and } \Lambda \mapsto \bar{\lambda}_k} . \tag{7.3}$$

Let us agree on refraining from explicitly denoting the substitution of these couplings, in slight abuse of notation.

Next, we can make use of the paramount feature of the gravitational FRGE – its background independence. Thus, we may choose a background with metric $\bar{g} \equiv g$ that suits our needs. The Einstein-Hilbert truncation is spanned by functionals depending only on the volume element and the scalar curvature so that employing a maximally symmetric background manifold (cf. 4.2.2) is sufficient in order to perform the projection of the RHS of the FRGE onto that truncation: The scalar curvature is the only magnitude of curvature for these spaces and we do not need to distinguish between R^2 or $R_{\mu\nu}R^{\mu\nu}$ etc.[1] Therewith, the operators $(\Gamma_k^{(2)})_{11}[g,g]$ and $\mathscr{M}[g,g]$ simplify to Eqs. (4.80) and (4.82):

$$\begin{aligned}\left((\Gamma_k^{(2)})_{11}[g,g]^{\mu\nu}{}_{\rho\sigma}\right)^{\text{diff}} &= \kappa^2 Z_{Nk}\left[I^{\mu\nu}{}_{\rho\sigma} - (P_{\text{tr.}}[g])^{\mu\nu}{}_{\rho\sigma}\right]\left(-D^2 - 2\bar{\lambda}_k + c_I R\right) \\ &\quad - \kappa^2 Z_{Nk}\frac{d-2}{2}(P_{\text{tr.}}[g])^{\mu\nu}{}_{\rho\sigma}\left(-D^2 - 2\bar{\lambda}_k + c_{\text{trace}}R\right) , \\ (\mathscr{M}[g,g]^{\mu}{}_{\nu})^{\text{diff}} &= \delta^{\mu}_{\nu}\left[D^2 + \frac{1}{d}R\right] ;\end{aligned} \tag{7.4}$$

with[2]

$$c_I = \frac{d(d-3)+4}{d(d-1)} \quad \text{and} \quad c_{\text{trace}} = \frac{d-4}{d} . \tag{7.5}$$

Lastly, we must choose the specific form (6.3) of the cutoff operators $\mathscr{R}_k^{\text{grav}}[g]$ and $\mathscr{R}_k^{\text{gh},1}[g]$. Therein, to set $\gamma = 1$ is a necessity while it is furthermore convenient to set

$$\begin{aligned}\mathscr{Z}_k^{\text{grav}}[g] &= \kappa^2 Z_{Nk}\left[(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g]) - \frac{d-2}{2}\mathbb{P}_{\text{tr.}}[g]\right] \\ \text{and} \quad \mathscr{Z}_k^{\text{gh},1}[g] &= \sqrt{2}\,\mathbb{1}_V .\end{aligned} \tag{7.6}$$

All in all, we therewith have

$$\begin{aligned}(\Gamma_k^{(2)})_{11}[g,g] + \mathscr{R}_k^{\text{grav}}[g] &= \kappa^2 Z_{Nk}\,(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g])\left[\mathscr{A}_k(-D^2) + c_I R\right] \\ &\quad - \kappa^2 Z_{Nk}\frac{d-2}{2}\mathbb{P}_{\text{tr.}}[g]\left[\mathscr{A}_k(-D^2) + c_{\text{trace}}R\right] \\ \text{and} \quad -\mathscr{M}[g,g] + \mathscr{R}_k^{\text{gh},1}[g] &= \sqrt{2}\,\mathbb{1}_V\left[\mathscr{A}_{0k}(-D^2) + c_V R\right]\end{aligned} \tag{7.7}$$

with $c_V = -1/d$ and the definitions

$$\begin{aligned}\mathscr{A}_k(-D^2) &\coloneqq -D^2 + k^2 R^{(0)}(-D^2/k^2) - 2\bar{\lambda}_k , \\ \mathscr{A}_{0k}(-D^2) &\coloneqq -D^2 + k^2 R^{(0)}(-D^2/k^2) .\end{aligned} \tag{7.8}$$

[1]Note that although the scalar curvature is constant, i.e., "x-independent", for maximally symmetric spaces, we can still distinguish between the operators $\int \mathrm{d}^d x \sqrt{g(x)}$ and $\int \mathrm{d}^d x \sqrt{g(x)}\,R$ as surely $R = R[g]$ is still a functional of g. In other words, we could say that subsequently, all equations hold "for all R".

[2]The projectors I, the identity, and $P_{\text{tr.}}$ can be found in appendix A.2.2.

It can easily be checked that these operators are inverted by

$$\begin{aligned}\Big[(\Gamma_k^{(2)})_{11}[g,g]+\mathscr{R}_k^{\text{grav}}[g]\Big]^{-1} &= \frac{1}{\kappa^2 Z_{Nk}}\left(\mathbb{1}_{ST^2}-\mathbb{P}_{\text{tr.}}[g]\right)\Big[\mathscr{A}_k(-D^2)+c_I R\Big]^{-1}\\ &\quad-\frac{1}{\kappa^2 Z_{Nk}}\frac{2}{d-2}\mathbb{P}_{\text{tr.}}[g]\Big[\mathscr{A}_k(-D^2)+c_{\text{trace}}R\Big]^{-1}\end{aligned} \tag{7.9}$$

and

$$\Big[-\mathscr{M}[g,g]+\mathscr{R}_k^{\text{gh},1}[g]\Big]^{-1} = \frac{1}{\sqrt{2}}\mathbb{1}_V\Big[\mathscr{A}_{0k}(-D^2)+c_V R\Big]^{-1}. \tag{7.10}$$

By further introducing the abbreviations

$$\begin{aligned}\mathscr{N}_k(-D^2) &:= \frac{1}{2Z_{Nk}}\partial_t\Big[Z_{Nk}k^2 R^{(0)}(-D^2/k^2)\Big]\\ &= \Big[1-\frac{1}{2}\eta_N(k)\Big]k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}(-D^2/k^2)\\ \text{and}\quad \mathscr{N}_{0k}(-D^2) &:= \frac{1}{2}\partial_t\Big[k^2R^{(0)}(-D^2/k^2)\Big]\\ &= k^2R^{(0)}(-D^2/k^2)+D^2R^{(0)\prime}(-D^2/k^2)\,,\end{aligned} \tag{7.11}$$

where $\eta_N(k)=-\partial_t \ln Z_{Nk}$ is the *(negative) anomalous dimension*[3] of the operator $\int \mathrm{d}^d x\sqrt{g}\,R[g]$, the RHS of the FRGE simplifies to

$$\begin{aligned}\text{RHS of Eq. (7.1)} &= \mathrm{Tr}_{ST^2}\left[\left(\mathbb{1}_{ST^2}-\mathbb{P}_{\text{tr.}}[g]\right)\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)+c_I R}\right]\\ &\quad+\mathrm{Tr}_{ST^2}\left[\mathbb{P}_{\text{tr.}}[g]\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)+c_{\text{trace}}R}\right]\\ &\quad-2\,\mathrm{Tr}_V\left[\mathbb{1}_V\frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)+c_V R}\right].\end{aligned} \tag{7.12}$$

To actually project this RHS onto the Einstein-Hilbert truncation, we must neglect all terms $O(R^2)$. Hence, we expand the denominators in the traces according to

$$\begin{aligned}\frac{1}{\mathscr{A}+cR} = \frac{1}{\mathscr{A}(1+c\mathscr{A}^{-1}R)} &= \frac{1}{\mathscr{A}}(1-\mathscr{A}^{-1}R)+O(R^2)\\ &= \mathscr{A}^{-1}-c\mathscr{A}^{-2}R+O(R^2)\,,\end{aligned} \tag{7.13}$$

[3] In the conventions for the anomalous dimension of this thesis, $\partial_\tau \ln Z_{Nk}$ would correspond to the anomalous dimension of the operator $\int \mathrm{d}^d x\sqrt{g}\,R[g]$. Hence, the add-on "negative".

which yields

$$
\begin{aligned}
\text{RHS of Eq. (7.1)} = {} & \mathrm{Tr}_{ST^2}\left[\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\mathrm{tr.}}[g]\right)\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)}\right] \\
& + \mathrm{Tr}_{ST^2}\left[\mathbb{P}_{\mathrm{tr.}}[g]\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)}\right] - 2\,\mathrm{Tr}_V\left[\mathbb{1}_V\frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)}\right] \\
& - R\Bigg\{c_I\,\mathrm{Tr}_{ST^2}\left[\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\mathrm{tr.}}[g]\right)\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}\right] \\
& \quad + c_{\mathrm{trace}}\,\mathrm{Tr}_{ST^2}\left[\mathbb{P}_{\mathrm{tr.}}[g]\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}\right] \\
& \quad - 2c_V\,\mathrm{Tr}_V\left[\mathbb{1}_V\frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)^2}\right]\Bigg\} + O(R^2)\,.
\end{aligned}
\tag{7.14}
$$

We can evaluated the traces with help of the heat kernel techniques developed in appendix E. Using Eq. (E.4) and the linearity of the trace we obtain

$$
\begin{aligned}
\text{RHS of Eq. (7.1)} = {} & \left(\frac{1}{4\pi}\right)^{\frac{d}{2}}\Bigg\{\mathrm{tr}\left(I_{ST^2}\right)\Bigg[Q_{\frac{d}{2}}[\mathscr{N}_k/\mathscr{A}_k]\int \mathrm{d}^d x\sqrt{g} \\
& \qquad + \frac{1}{6}Q_{\frac{d}{2}-1}\left[\mathscr{N}_k/\mathscr{A}_k\right]\int \mathrm{d}^d x\sqrt{g}R\Bigg] \\
& - 2\,\mathrm{tr}\left(I_V\right)\Bigg[Q_{\frac{d}{2}}[\mathscr{N}_{0k}/\mathscr{A}_{0k}]\int \mathrm{d}^d x\sqrt{g} \\
& \qquad + \frac{1}{6}Q_{\frac{d}{2}-1}[\mathscr{N}_{0k}/\mathscr{A}_{0k}]\int \mathrm{d}^d x\sqrt{g}R\Bigg]\,, \\
& - \mathrm{tr}\left(I_{ST^2} - P_{\mathrm{tr.}}[g]\right)c_I\,Q_{\frac{d}{2}}\left[\mathscr{N}_k/\mathscr{A}_k^2\right]\int \mathrm{d}^d x\sqrt{g}R \\
& - \mathrm{tr}\left(P_{\mathrm{tr.}}[g]\right)c_{\mathrm{trace}}\,Q_{\frac{d}{2}}\left[\mathscr{N}_k/\mathscr{A}_k^2\right]\int \mathrm{d}^d x\sqrt{g}R \\
& + 2\,\mathrm{tr}\left(I_V\right)c_V\,Q_{\frac{d}{2}}\left[\mathscr{N}_{0k}/\mathscr{A}_{0k}^2\right]\int \mathrm{d}^d x\sqrt{g}R\Bigg\} \\
& + O(R^2)\,.
\end{aligned}
\tag{7.15}
$$

The small traces appearing here are given by $\operatorname{tr} I_{ST^2} = d(d+1)/2$, $\operatorname{tr} P_{\text{tr.}}[g] = (1/d) g_{\mu\nu} g^{\mu\nu} = 1$ and $\operatorname{tr} I_V = \delta^\mu_\mu = d$. Furthermore, the LHS of Eq. (7.1) for the ansatz (7.2) clearly is given by[4]

$$\text{LHS of Eq. (7.1)} = -2\kappa^2 \partial_t Z_{Nk} \int \mathrm{d}^d x \sqrt{g} R + 4\kappa^2 \partial_t (Z_{Nk} \bar{\lambda}_k) \int \mathrm{d}^d x \sqrt{g}\,, \tag{7.16}$$

such that by comparing the coefficients of the truncation's basis functionals $\int \mathrm{d}^d x \sqrt{g}$ and $\int \mathrm{d}^d x \sqrt{g} R$ on the LHS with those of the RHS we can read of the RG equations for the dimensionful running coupling Z_{Nk} and $\bar{\lambda}_k$:

$$4\kappa^2 \partial_t (Z_{Nk} \bar{\lambda}_k) = \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left[\frac{d(d+1)}{2} Q_{\frac{d}{2}}[\mathscr{N}_k / \mathscr{A}_k] - 2d\, Q_{\frac{d}{2}}[\mathscr{N}_{0k} / \mathscr{A}_{0k}]\right] \tag{7.17}$$

and

$$\begin{aligned} -2\kappa^2 \partial_t Z_{Nk} = \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \Bigg[& \frac{d(d+1)}{2} \frac{1}{6} Q_{\frac{d}{2}-1}[\mathscr{N}_k / \mathscr{A}_k] - 2d \frac{1}{6}\, Q_{\frac{d}{2}-1}[\mathscr{N}_{0k} / \mathscr{A}_{0k}] \\ & - \left(\frac{(d+2)(d-1)}{2} c_I + c_{\text{trace}}\right) Q_{\frac{d}{2}}[\mathscr{N}_k / \mathscr{A}_k^2] \\ & + 2d\, c_V\, Q_{\frac{d}{2}}\left[\mathscr{N}_{0k} / \mathscr{A}_{0k}^2\right] \Bigg]\,. \end{aligned} \tag{7.18}$$

Next, it is advantageous to re-express the dimensionful "Q-functionals" on the RHS in terms of dimensionless quantities. These are the *threshold functions*

$$\begin{aligned} \Phi^p_n(w) &:= \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z) - z R^{(0)\prime}(z)}{(z + R^{(0)}(z) + w)^p} \\ \text{and} \quad \tilde{\Phi}^p_n(w) &:= \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z)}{(z + R^{(0)}(z) + w)^p}\,. \end{aligned} \tag{7.19}$$

Using the optimized cutoff, the threshold functions can in fact be analytically evaluated [38, 100–102]. One finds

$$\begin{aligned} \Phi^p_{n\,\text{opt}}(w) &= \frac{1}{\Gamma(n+1)} \frac{1}{(1+w)^p} \\ \text{and} \quad \tilde{\Phi}^p_{n\,\text{opt}}(w) &= \frac{1}{\Gamma(n+2)} \frac{1}{(1+w)^p}\,, \end{aligned} \tag{7.20}$$

[4] Note that $S_{\text{GF}}[0; g] \equiv 0$.

with Γ the ordinary Gamma function. Furthermore, it is not difficult to verify that the threshold functions in general are related to the "Q-functionals" by

$$\begin{aligned} Q_n\left[\mathscr{N}_k/\mathscr{A}_k^m\right] &= k^{2+2(n-m)}\left[\Phi_n^m\left(-2\bar{\lambda}_k/k^2\right)-\frac{1}{2}\eta_N(k)\tilde{\Phi}_n^m\left(-2\bar{\lambda}_k/k^2\right)\right], \\ Q_n[\mathscr{N}_{0k}/\mathscr{A}_{0k}^m] &= k^{2+2(n-m)}\Phi_n^m(0)\,. \end{aligned} \tag{7.21}$$

Therewith the dimensionful RG equations (7.17) and (7.18) become

$$\begin{aligned} \partial_t(Z_{Nk}\bar{\lambda}_k) = \frac{1}{\kappa^2}\left(\frac{1}{4\pi}\right)^{\frac{d}{2}}k^d\Bigg\{\frac{d(d+1)}{8}\Bigg[&\Phi^1_{\frac{d}{2}}\left(-2\bar{\lambda}_k/k^2\right) \\ &-\frac{1}{2}\eta_N(k)\tilde{\Phi}^1_{\frac{d}{2}}\left(-2\bar{\lambda}_k/k^2\right)\Bigg]-\frac{d}{2}\,\Phi^1_{\frac{d}{2}}(0)\Bigg\} \end{aligned} \tag{7.22}$$

and

$$\begin{aligned} \partial_t Z_{Nk} = -\frac{1}{\kappa^2}\left(\frac{1}{4\pi}\right)^{\frac{d}{2}}k^{d-2}\Bigg\{\frac{d(d+1)}{24}\Bigg[&\Phi^1_{\frac{d}{2}-1}\left(-2\bar{\lambda}_k/k^2\right) \\ &-\frac{1}{2}\eta_N(k)\tilde{\Phi}^1_{\frac{d}{2}-1}\left(-2\bar{\lambda}_k/k^2\right)\Bigg] \\ -\frac{d(d-1)}{4}\Bigg[&\Phi^2_{\frac{d}{2}}\left(-2\bar{\lambda}_k/k^2\right) \\ &-\frac{1}{2}\eta_N(k)\tilde{\Phi}^2_{\frac{d}{2}}\left(-2\bar{\lambda}_k/k^2\right)\Bigg] \\ -\frac{1}{6}d\,\Phi^1_{\frac{d}{2}-1}(0)-\Phi^2_{\frac{d}{2}}(0)\Bigg\}\,. \end{aligned} \tag{7.23}$$

From these equations we can obtain the dimensionless RG equations that are formulated in terms of the dimensionless running couplings

$$g_k := \frac{k^{d-2}}{32\pi\kappa^2 Z_{Nk}} \quad \text{and} \quad \lambda_k := k^{-2}\bar{\lambda}_k\,. \tag{7.24}$$

From the definition of the dimensionless running Newton's constant we immediately obtain the first dimensionless RG equation

$$\boxed{\begin{aligned} \partial_t g_k &= \left[(d-2)+\eta_N(\lambda_k,g_k)\right]g_k \\ &=: \beta_g(\lambda_k,g_k)\,. \end{aligned}} \tag{7.25}$$

The other dimensionless RG equation is obatined by rewriting Eq. (7.22) into

$$\boxed{\begin{aligned}\partial_t \lambda_k = & - \left[2 - \eta_N(\lambda_k, g_k)\right] \lambda_k \\ & + g_k\, 32\pi \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left\{ \frac{d(d+1)}{8} \left[\Phi^1_{\frac{d}{2}}(-2\lambda_k) - \frac{1}{2}\eta_N(\lambda_k, g_k) \tilde{\Phi}^1_{\frac{d}{2}}(-2\lambda_k) \right] \right. \\ & \qquad \left. - \frac{d}{2}\, \Phi^1_{\frac{d}{2}}(0) \right\} \\ =: & \ \beta_\lambda(\lambda_k, g_k)\,. \end{aligned}} \tag{7.26}$$

This system of ordinary differential equations, that we sometimes will refer to as the *full Einstein-Hilbert flow*, depends on the (negative) anomalous dimension $\eta_N(\lambda_k, g_k) \equiv \eta_N(k) = -\partial_t \ln Z_{Nk}$ that is determined by Eq. (7.23); namely, this equation can be easily rewritten into

$$\eta_N(k) \equiv \eta_N(\lambda_k, g_k) = \frac{g_k B_1(\lambda_k)}{1 - g_k B_2(\lambda_k)}\,, \tag{7.27}$$

with

$$\begin{aligned} B_1(\lambda_k) &= 32\pi \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left[\frac{d(d+1)}{24} \Phi^1_{\frac{d}{2}-1}(-2\lambda_k) - \frac{d(d-1)}{4} \Phi^2_{\frac{d}{2}}(-2\lambda_k) \right. \\ &\qquad \left. - \frac{1}{6} d\Phi^1_{\frac{d}{2}-1}(0) - \Phi^2_{\frac{d}{2}}(0) \right], \\ B_2(\lambda_k) &= 32\pi \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left[- \frac{d(d+1)}{48} \tilde{\Phi}^1_{\frac{d}{2}-1}(-2\lambda_k) + \frac{d(d-1)}{8} \tilde{\Phi}^2_{\frac{d}{2}}(-2\lambda_k) \right]. \end{aligned} \tag{7.28}$$

7.2. The simplified Einstein-Hilbert flow

Next, let us determine the corresponding beta functions obtained from the "one-loop approximation" (6.66) at $g \equiv \bar{g}$. These are given by Eqs. (7.22) and (7.23) with all k-derivatives of $\mathscr{Z}_k$ discarded; this amounts to setting $\tilde{\phi}^m_n \equiv 0$ and therewith we can thus always obtain "one-loop approximations" of exact results calculated in the single metric Einstein-Hilbert truncation. (Note that this implies $B_2 \equiv 0$.) When considering this "one-loop approximation" it is advisible to superimpose another approximation, an expansion in λ_k. For instance, the traced equations of motion of the Einstein-Hilbert action (cf. Eq. (18.12)) suggest that close to the mass shell one has $R \sim \lambda_k$; therefore we may restrict the expansion in λ_k to leading order which here implicates the substitution $\phi_n^m(-2\lambda_k) \mapsto \phi_n^m(0)$ in the beta functions. The flow associated to this further approximation of the beta functions is called the *simplified Einstein-Hilbert flow* [102]. These beta functions are given by[5]

$$\boxed{\begin{aligned} \partial_t g_k &= (d-2)g_k + B_1(0)g_k^2 \\ &=: \beta_g^{1\mathrm{L}}(\lambda_k, g_k) \end{aligned}} \tag{7.29}$$

and

$$\boxed{\begin{aligned} \partial_t \lambda_k =& -\left[2 - g_k B_1(0)\right]\lambda_k + 32\pi \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left[\frac{d(d+1)}{8} - \frac{d}{2}\right] \Phi^1_{\frac{d}{2}}(0) g_k \\ =:& \ \beta_\lambda^{1\mathrm{L}}(\lambda_k, g_k)\,. \end{aligned}} \tag{7.30}$$

These RG equations contain the "one-loop" (negative) anomalous dimension $\eta_N^{1\mathrm{L}}(\lambda_k, g_k) = B_1(0)g_k$. For example, in four dimensions, $d = 4$, the beta functions become, when employing the optimized cutoff,

$$\begin{aligned} \beta_\lambda^{1\mathrm{L}}(\lambda_k, g_k) &= \frac{1}{2\pi} g_k - \left(2 + \frac{11}{3\pi} g_k\right)\lambda_k \\ \beta_g^{1\mathrm{L}}(\lambda_k, g_k) &= \left(2 - \frac{11}{3\pi} g_k\right) g_k\,. \end{aligned} \tag{7.31}$$

7.3. Numerical analysis for $d = 4$

Lastly, we numerically analyze the flow equations of the single metric Einstein-Hilbert truncation in four spacetime dimensions. First note that the beta functions of both, the "full" as well as the "simplified" Einstein-Hilbert flow, possess a Gaussian fixed point. Furthermore, one can numerically show that also

[5]Note that we label these beta functions as "1L" although on the one hand, there is no clear association to an expansion in $\hbar$ (cf. the comment in the previous section), and on the other hand, we have made a further approximation of the "one-loop approximation" (6.66).

a non-Gaussian fixed point (λ_*, g_*) of the full Einstein-Hilbert flow exists, i.e., $\beta_g(\lambda_*, g_*) = 0 = \beta_\lambda(\lambda_*, g_*)$. Evaluated with the exponential cutoff (6.4) with $s = 1$ and the optimized cutoff (6.5), respectively, these are given by

$$(\lambda_*^{\text{exp}}, g_*^{\text{exp}}) = (0.3590, 0.2723) \tag{7.32}$$

$$\text{and} \quad (\lambda_*^{\text{opt}}, g_*^{\text{opt}}) = (0.1932, 0.7073)\,. \tag{7.33}$$

To determine the number of relevant, irrelevant or marginal "eigendirections" of this fixed point, we caculate the eigenvalues of the corresponding stability matrix

$$B(\lambda_*, g_*) = \begin{pmatrix} \frac{\partial\beta_\lambda}{\partial\lambda} & \frac{\partial\beta_\lambda}{\partial g} \\ \frac{\partial\beta_g}{\partial\lambda} & \frac{\partial\beta_g}{\partial g} \end{pmatrix}(\lambda_*, g_*)\,. \tag{7.34}$$

For the fixed point evaluated with the exponential cutoff for $s = 1$ the stability matrix has the complex conjugated eigenvalues $-1.4198 \pm 3.96282\text{i}$ such that the real part of the corresponding critical exponents (simply the negative eigenvalues) are real. Also with the optimized cutoff on finds two relevant directions, the conjugated eigenvalues being $-1.47531 \pm 3.04322\text{i}$. Hence the full Einstein-Hilbert flow, which takes place in the two-dimensional parameter space (λ, g), possesses a crictical hypersurface of dimension $\Delta_{\text{UV}} = 2$.

A flow diagram of the full Einstein-Hilbert flow is illustrated in Figure 7.1. Therein, the direction of the flow is defined to be the direction of *decreasing* k, which is the direction of *increasing coarse graining*. Especially important are the three different types of trajectories emanating from the non-Gaussian fixed point: trajectories of "type Ia" run, for $k \to 0$, towards negative values for λ while trajectories of "type IIIa" run towards positive values for λ. Furthermore, there are trajectories of "type IIa", called the *separatrix*, that connect the non-Gaussian fixed point with the Gaussian fixed point for $k \to 0$. (For a full classification of the flow's trajectories as well as the explicit form of the linearized solution see e.g. [103].)

Likewise one can show that also non-Gaussian fixed point $(\lambda_*^{1\text{L}}, g_*^{1\text{L}})$ of the simplified Einstein-Hilbert flow exists, i.e., $\beta_g^{1\text{L}}(\lambda_*^{1\text{L}}, g_*^{1\text{L}}) = 0 = \beta_\lambda^{1\text{L}}(\lambda_*^{1\text{L}}, g_*^{1\text{L}})$. Again, evaluated with the exponential cutoff (6.4) with $s = 1$ and the optimized cutoff (6.5), respectively, these are given by

$$(\lambda_*^{\text{exp},1\text{L}}, g_*^{\text{exp},1\text{L}}) = (0.1613, 0.8432) \tag{7.35}$$

$$\text{and} \quad (\lambda_*^{\text{opt},1\text{L}}, g_*^{\text{opt},1\text{L}}) = (0.0682, 1.7136)\,. \tag{7.36}$$

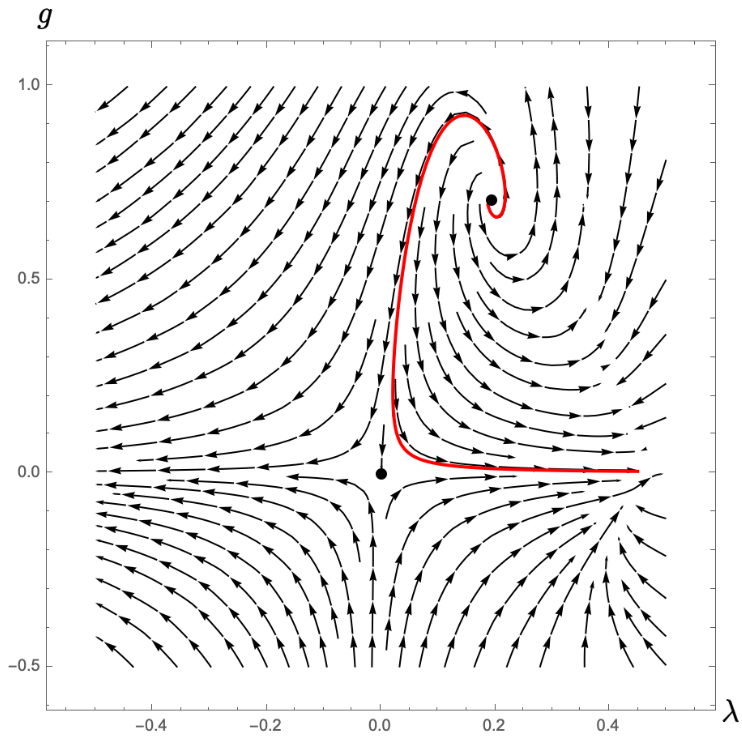

FIGURE 7.1. Flow diagram of the full Einstein-Hilbert flow for $d = 4$ in the (λ, g)-parameter space, obtained by employing the optimized cutoff. The arrows point in the direction of decreasing k. The plotted points are the Gaussian fixed point and the non-Gaussian fixed point (7.33). Highlighted in red is a trajectory of "type IIIa".

The stability matrix of this fixed points has eigenvalues -2 and -4, independent of the specific cutoff profile implemented. Thus, also the simplified Einstein-Hilbert flow possesses a crictical hypersurface of dimension $\Delta_{\text{UV}} = 2$. A flow diagram of the full Einstein-Hilbert flow is illustrated in Figure 7.2

7.4. ADDENDUM: THE SIMPLIFIED EINSTEIN-HILBERT FLOW WITH FREE MATTER FIELDS

As an accessory analysis, let us analyze how the RG equations of the simplified Einstein-Hilbert flow in $d = 4$ are modified when matter is present. This analysis runs slights off the common thread through this thesis, but will be required for

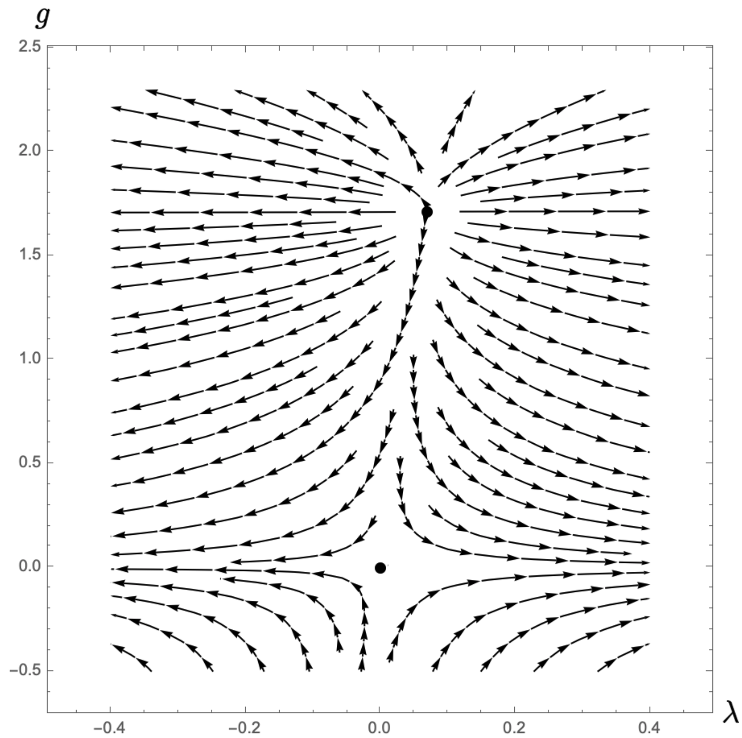

FIGURE 7.2. Flow diagram of the simplified Einstein-Hilbert flow for $d = 4$ in the (λ, g)-parameter space, obtained by employing the optimized cutoff. The arrows point in the direction of decreasing k. The plotted points are the Gaussian fixed point and the non-Gaussian fixed point (7.36).

a small application later on. Therefore, let us consider N_S scalar fields $\{\phi^i\}$, N_D spin-1/2 fermionic fields $\{\psi^i\}$ and N_V abelian ($U(1)$) gauge fields $\{A^i_\mu\}$. All these fields shall be massless such their bare actions, without self-interaction, are given by

$$
\begin{aligned}
S_S[\{\phi^i\};\bar{g}] &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\sum_{i+1}^{N_S}\partial_\mu\phi^i\,\partial_\nu\phi^i\,,\\
S_D[\{\psi^i\};\bar{g}] &= \int \mathrm{d}^4x\sqrt{\bar{g}}\sum_{i=1}^{N_D}\bar{\psi}^i\bar{\not{D}}\psi^i\,,\\
S_V[\{A^i_\mu,\bar{c}^i,c^i\};\bar{g}] &= \frac{1}{4}\int \mathrm{d}^4x\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\sum_{i=1}^{N_V}F^i_{\mu\alpha}F^i_{\nu\beta}\\
&\quad+\frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\sum_{i=1}^{N_V}\left(\bar{g}^{\mu\nu}\bar{D}_\mu A^i_\nu\right)^2\\
&\quad+\int \mathrm{d}^4x\sqrt{\bar{g}}\sum_{i=1}^{N_V}\bar{c}^i(-\bar{D}^2)c^i\,.
\end{aligned}
\tag{7.37}
$$

Here, $\bar{\not{D}} = \gamma^\mu\bar{D}_\mu$ is the *Dirac operator* and $F^i_{\mu\nu} = \bar{D}_\mu A^i_\nu - \bar{D}_\nu A^i_\mu = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$ is the *field strength* of the abelian gauge field, that also is split into background part and fluctuation as $A^i_\mu = \bar{A}^i_\mu + a^i_\mu$. Due to the $U(1)$-invariance of the fields $\{A^i_\mu\}$ we have supplied their action functionals with a gauge-fixing action for the gauge-fixing condition $\bar{D}_\mu A^i_\nu = 0$ as well as the resulting Faddeev-Popov action for the ghosts $\bar{c}^i$ and c^i. In this thesis, we do not explicitly discuss the treatment of matter fields in the functional renormalization group (FRG) formalism; however the procedure is fully analogous to the treatment of metric fluctuations, that are discussed in detail in this thesis. (For details on the role of matter in the FRG approach towards quantum gravity see e.g. [97, 102, 104].) Note that here we have not attributed any couplings to the matter fields. This is because we are only interested in how the numbers N_S, N_D and N_V will modify the beta functions of the simplified Einstein-Hilbert flow for the couplings λ_k and g_k, if we include the matter contributions to the ansatz for solving the FRGE.

It is not difficult to obtain the required FRGE: To obtain the simplified Einstein-Hilbert flow, we have relied on Eq. (6.66) (with the last term on the

RHS discarded). If we enlarge the field space to also include the above matter fields the corresponding generalization of Eq. (6.66) will clearly read

$$\begin{aligned}
\partial_t \Gamma_k[g,\bar{g}] = &+ \frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[\left((\Gamma_k^{(2)})_{11}[g;\bar{g}] + \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\right)^{-1} \partial_t \mathscr{R}_{k,k'}^{\mathrm{grav}}[\bar{g}]\right]_{k'=k} \\
&- \mathrm{Tr}_V\left[\left(-\sqrt{2}\mathscr{M}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{gh},1}[\bar{g}]\right)^{-1} \partial_t \mathscr{R}_{k,k'}^{\mathrm{gh},1}[\bar{g}]\right]_{k'=k} \\
&+ \frac{1}{2} N_S\,\mathrm{Tr}_S\left[\left(S_S^{(2)}[\bar{g}] + \mathscr{R}_k^S[\bar{g}]\right)^{-1} \partial_t \mathscr{R}_k^S[\bar{g}]\right] \\
&- N_D\,\mathrm{Tr}_{\mathrm{spin}-1/2}\left[\left(S_D^{(2)}[\bar{g}] + \mathscr{R}_k^D[\bar{g}]\right)^{-1} \partial_t \mathscr{R}_k^D[\bar{g}]\right] \\
&+ \frac{1}{2} N_V\,\mathrm{Tr}_V\left[\left((S_V^{(2)})_{11}[\bar{g}] + \mathscr{R}_k^V[\bar{g}]\right)^{-1} \partial_t \mathscr{R}_k^V[\bar{g}]\right] \\
&- N_V\,\mathrm{Tr}_S\left[\left((S_V^{(2)})_{32}[\bar{g}] + \mathscr{R}_k^S[\bar{g}]\right)^{-1} \partial_t \mathscr{R}_k^S[\bar{g}]\right],
\end{aligned} \tag{7.38}$$

where

$$\begin{aligned}
\langle x|S_S^{(2)}[\bar{g}]|y\rangle &= \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} \frac{\delta^2 S_S[\{\phi^i\};\bar{g}]}{\delta\phi^i(x)\delta\phi^i(y)} \\
&= \langle x| -\bar{D}^2|y\rangle\,, \\
{}^F\langle x|S_D^{(2)}[\bar{g}]|y\rangle^F &= \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} \frac{\delta^2 S_D[\{\psi^i\};\bar{g}]}{\delta\bar{\psi}^i(x)\delta\psi^i(y)} \\
&= {}^F\langle x| -\bar{\not{D}}|y\rangle^F\,,
\end{aligned} \tag{7.39}$$

where "F" indicates that the operator is defined on the Hilbert space of spin-1/2 fermions,[6]

$$\begin{aligned}
\langle x,\mu|(S_V^{(2)})_{11}[\bar{g}]|y,\nu\rangle &= \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} \bar{g}_{\rho\nu}(y) \frac{\delta^2 S_V[\{A^i,\bar{c}^i,c^i\};\bar{g}]}{\delta a_\mu^i(x)\delta a_\rho^i(y)}\bigg|_{a^i=0} \\
&= \langle x,\mu|\left[\mathbb{1}_V(-\bar{D}^2) + \mathrm{Ric}\right]|y,\nu\rangle\,,
\end{aligned} \tag{7.40}$$

[6]This Hilbert space is not discussed in appendix A.1 because it is only sporadically required, i.e., actually only here, in this section.

where the operator Ric on V is defined by $\langle x,\mu|\mathrm{Ric}|X\rangle = (R^\mu{}_\nu X^\nu)(x)$, and lastly

$$\begin{aligned}\langle x|(S_V^{(2)})_{32}[\bar{g}]|y\rangle &= \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2 S_V[\{A^i,\bar{c}^i,c^i\};\bar{g}]}{\delta c^i(x)\delta\bar{c}^i(y)} \\ &= \langle x|-\bar{D}^2|y\rangle\,.\end{aligned} \tag{7.41}$$

In the derivation of the of the FRGE we have made use of the fact that no action contains any couplings that could run. Therewith, the evolution of the matter fields is trivially suppressed and these hence contribute to the FRGE only with their bare actions. That the "one-loop approximation" of the FRGE for the enlarged field space, that we consider here, is of the specific form (7.38) is evident from the discussion of Section 6.5. Further, as the matter fields contribute only with their bare actions, we have w.l.o.g. set their cutoff operators to $\mathscr{R}_{k,k'}[\bar{g}] \equiv \mathscr{R}_k[\bar{g}]$. Lastly, we must specify these. For the scalar and vector fields it is straightforward to set

$$\begin{aligned}\mathscr{R}_k^S[\bar{g}] &= \mathbb{1}_S\, k^2 R^{(0)}(-\bar{D}^2/k^2) \\ \mathscr{R}_k^V[\bar{g}] &= \mathbb{1}_V\, k^2 R^{(0)}(-\bar{D}^2/k^2)\,.\end{aligned} \tag{7.42}$$

The definition of an appropriate cutoff operator for spin-1/2 fields is rich in details, that we will not bring up here, for potential pitfalls that might occur see e.g. [104]. The eligible choice that we will employ is

$$\begin{aligned}\mathscr{R}_k^D[\bar{g}] &= \mathbb{1}_{\text{spin}-1/2} R_k^D(\bar{\not{D}}) \\ \text{with}\quad R_k^D(\bar{\not{D}}) &= \bar{\not{D}} + \sqrt{\bar{\not{D}} + k^2 R_k^{(0)}(\bar{\not{D}}^2/k^2)}\,.\end{aligned} \tag{7.43}$$

Note that in $d = 4$ the squared Dirac operator is related to the ordinary Laplacian by $\bar{\not{D}}^2 = -\bar{D}^2 + \bar{R}/4$ (called the Schrödinger-Lichnerowicz formula [105, 106]).[7]

[7] In the terminology of [102], cutoff profiles (for $\gamma = 1$) of the form "$R_k^{(0)}(-\bar{D}^2/k^2)$" are referred to as "type I", while profiles of the form "$R_k^{(0)}\left(\frac{-\bar{D}^2+E}{k^2}\right)$", with E an endomorphism, are referred to as "type II". (Note that this terminology is fully unrelated to the classification of RG trajectories.) Except for the spin-1/2 fields here, we solely employ cutoffs of "type I" in this thesis.

To finally solve the FRGE (7.38) we will again use the single-metric ansatz (7.2) and specify the metric $g \equiv \bar{g}$ to that of an maximally symmetric space. Furthermore, we again evaluate all theshold functions at 0. This leads to

$$\begin{aligned} 4\kappa^2 \partial_t \left(Z_{Nk} \bar{\lambda}_k \right) &= 2k^4 \frac{1}{(4\pi)^2} \Phi_2^1(0) \\ &\qquad + \text{matter contributions from} \int \mathrm{d}^4x \sqrt{g} \\ -2\kappa^2 \partial_t Z_{Nk} &= k^2 \frac{1}{(4\pi)^2} \left[\frac{1}{3} \Phi_1^1(0) - 8\Phi_2^2(0) \right] \\ &\qquad + \text{matter contributions from} \int \mathrm{d}^4x \sqrt{g} R \,. \end{aligned} \tag{7.44}$$

These equations are written such that we can literally add on the RHS the coefficients of the operators $\int \mathrm{d}^4x \sqrt{g}$ and $\int \mathrm{d}^4x \sqrt{g} R$ from the expansion of the RHS of Eq. (7.38). Next, let us calculate these contributions individually (with $g \equiv \bar{g}$):

For the scalar part:

$$\begin{aligned} &\frac{N_S}{2} \operatorname{Tr}_S \left[\left(S_S^{(2)}[g] + \mathscr{R}_k^S[g] \right)^{-1} \partial_t \mathscr{R}_k^S[g] \right] \\ &\quad = \frac{N_S}{2} \operatorname{Tr}_S \left[\mathbb{1}_S \left[-D^2 + k^2 R^{(0)}(-D^2/k^2) \right]^{-1} \partial_t \left(k^2 R^{(0)}(-D^2/k^2) \right) \right] \\ &\quad = \frac{N_S}{2} \operatorname{Tr}_S \left[\mathbb{1}_S \frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)} \right] \\ &\quad = N_S \frac{1}{(4\pi)^2} \left\{ Q_2 \left[\frac{\mathscr{N}_{0k}}{\mathscr{A}_{0k}} \right] \int \mathrm{d}^4x \sqrt{g} + \frac{1}{6} Q_1 \left[\frac{\mathscr{N}_{0k}}{\mathscr{A}_{0k}} \right] \int \mathrm{d}^4x \sqrt{g} R \right\} + O(R^2) \\ &\quad = N_S \frac{1}{(4\pi)^2} \left\{ k^4 \Phi_2^1(0) \int \mathrm{d}^4x \sqrt{g} + \frac{1}{6} k^2 \Phi_1^1(0) \int \mathrm{d}^4x \sqrt{g} R \right\} + O(R^2) \,, \end{aligned} \tag{7.45}$$

For the spin-1/2 part:[8]

$$
\begin{aligned}
-N_D\ \mathrm{Tr}_{\mathrm{spin}-1/2}&\left[\left(S_D^{(2)}[\bar g]+\mathscr{R}_k^D[\bar g]\right)^{-1}\partial_t\mathscr{R}_k^D[\bar g]\right]\\
&= -N_D\,\mathrm{Tr}_{\mathrm{spin}-1/2}\left[\mathbb{1}_{\mathrm{spin}-1/2}\left(-\not{D}+\not{D}+\sqrt{\not{D}^2+k^2R^{(0)}(\not{D}^2/k^2)}\right)^{-1}\right.\\
&\qquad\qquad\left.\times\,\partial_t\sqrt{\not{D}^2+k^2R^{(0)}(\not{D}^2/k^2)}\right]\\
&= -N_D\,\mathrm{Tr}_{\mathrm{spin}-1/2}\left[\mathbb{1}_{\mathrm{spin}-1/2}\frac{\mathscr{N}_{0k}(\not{D}^2)}{\mathscr{A}_{0k}(\not{D}^2)}\right]\\
&= -N_D\frac{1}{(4\pi)^2}\left\{4Q_2\left[\frac{\mathscr{N}_{0k}}{\mathscr{A}_{0k}}\right]\int \mathrm{d}^4x\sqrt{g}-\frac{1}{3}Q_1\left[\frac{\mathscr{N}_{0k}}{\mathscr{A}_{0k}}\right]\int\mathrm{d}^4x\sqrt{g}R\right\}+O(R^2)\\
&= -N_D\frac{1}{(4\pi)^2}\left\{4k^4\Phi_2^1(0)\int\mathrm{d}^4x\sqrt{g}-\frac{1}{3}k^2\Phi_1^1(0)\int\mathrm{d}^4x\sqrt{g}R\right\}+O(R^2)\,,
\end{aligned}
\tag{7.46}
$$

For the vector part associated to the gauge field:

$$
\begin{aligned}
\frac{N_V}{2}\,\mathrm{Tr}_V&\left[\left((S_V^{(2)})_{11}[\bar g]+\mathscr{R}_k^V[\bar g]\right)^{-1}\partial_t\mathscr{R}_k^V[\bar g]\right]\\
&= \frac{N_V}{2}\,\mathrm{Tr}_V\left[\left[\mathbb{1}_V\left(-D^2+k^2R^{(0)}(-D^2/k^2)\right)+\mathrm{Ric}\right]^{-1}\right.\\
&\qquad\qquad\left.\times\,\mathbb{1}_V\partial_t\left(k^2R^{(0)}(-D^2/k^2)\right)\right]\\
&= N_V\,\mathrm{Tr}_V\left[\frac{\mathbb{1}_V\mathscr{N}_{0k}(-D^2)}{\mathbb{1}_V\mathscr{A}_{0k}(-D^2)+\mathrm{Ric}}\right]
\end{aligned}
\tag{7.47}
$$

[8] In the third step, we make use of the expansion of heat kernel for the squared Dirac operator. This expansion can be obtained analogously to the prescription in appendix E for the ordinary Laplacian. The corresponding coefficients can be found e.g. in Table 3.1 of [102].

Here, we can use the fact that for a maximally symmetric spacetime one has $R^{\mu}{}_{\nu} = \frac{1}{d}\delta^{\mu}_{\nu} R$ and thus $\mathrm{Ric} = \frac{1}{d} R\, \mathbb{1}_V$. Hence, we can expand

$$
\begin{aligned}
&N_V \operatorname{Tr}_V\left[\frac{\mathbb{1}_V \mathscr{N}_{0k}(-D^2)}{\mathbb{1}_V \mathscr{A}_{0k}(-D^2) + \mathrm{Ric}}\right] \\
&\quad = N_V \operatorname{Tr}_V\left[\mathbb{1}_V \frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2) + R/4}\right] \\
&\quad = N_V \left\{ \operatorname{Tr}_V\left[\mathbb{1}_V \frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)}\right] - \frac{1}{4} R \left[\mathbb{1}_V \frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)^2}\right] \right\} + O(R^2) \\
&\quad = N_V \frac{1}{(4\pi)^2} \left\{ 4k^4 \Phi^1_2(0) \int \mathrm{d}^4x \sqrt{g} + \frac{2}{3} k^2 \Phi^1_1(0) \int \mathrm{d}^4x \sqrt{g} R \right. \\
&\qquad\qquad \left. - k^2 \Phi^2_2(0) \int \mathrm{d}^4x \sqrt{g} R \right\} + O(R^2)
\end{aligned}
\tag{7.48}
$$

For the scalar part associated to the gauge field:

$$
\begin{aligned}
&-N_V \operatorname{Tr}_S\left[\Big((S^{(2)}_V)_{32}[g] + \mathscr{R}^S_k[g] \Big)^{-1} \partial_t \mathscr{R}^S_k[g]\right] \\
&\quad = -N_V \operatorname{Tr}_S\left[\mathbb{1}_S \left[-D^2 + k^2 R^{(0)}(-D^2/k^2)\right]^{-1} \partial_t \left(k^2 R^{(0)}(-D^2/k^2)\right)\right] \\
&\quad = -2N_V \operatorname{Tr}_S\left[\mathbb{1}_S \frac{\mathscr{N}_{0k}(-D^2)}{\mathscr{A}_{0k}(-D^2)}\right] \\
&\quad = -2N_V \frac{1}{(4\pi)^2} \left\{ k^4 \Phi^1_2(0) \int \mathrm{d}^4x \sqrt{g} + \frac{1}{6} k^2 \Phi^1_1(0) \int \mathrm{d}^4x \sqrt{g} R \right\} + O(R^2)\,.
\end{aligned}
\tag{7.49}
$$

All in all, the RG equations including the matter contributions are given by, employing the optimized cutoff,

$$\begin{aligned} 4\kappa^2 \partial_t \left(Z_{Nk} \bar{\lambda}_k \right) &= 2k^4 \frac{1}{(4\pi)^2} \left(1 + \frac{1}{2} N_S - 2N_D + N_V \right) \\ -2\kappa^2 \partial_t Z_{Nk} &= k^2 \frac{1}{(4\pi)^2} \left(-\frac{11}{3} + \frac{1}{6} N_S + \frac{1}{3} N_D - \frac{1}{6} N_V \right) . \end{aligned} \tag{7.50}$$

The second equation implicates the (negative) anomalous dimension

$$\boxed{\begin{aligned} \eta_N^{\text{1L,matter}}(\lambda_k, g_k) &= -\frac{1}{Z_{Nk}} \partial_t Z_{Nk} \\ &= \left[-\frac{11}{3\pi} + \frac{1}{6\pi} \left(N_S + 2N_D - N_V \right) \right] g_k \end{aligned}} \tag{7.51}$$

and with Eq. (7.25) we therewith obtain the dimensionless RG equation for g_k

$$\boxed{\begin{aligned} \partial_t g_k &= \left[2 + \eta_N^{\text{1L,matter}}(\lambda_k, g_k) \right] g_k \\ &=: \beta_g^{\text{1L,matter}}(\lambda_k, g_k) . \end{aligned}} \tag{7.52}$$

From the first equation follows the dimensionless RG equation for λ_k,

$$\boxed{\begin{aligned} \partial_t \lambda_k &= -2\lambda_k + \frac{1}{2\pi} \left(1 + \frac{1}{2} N_S - 2N_D + N_V \right) g_k + \eta_N^{\text{1L,matter}}(\lambda_k, g_k) \, \lambda_k \\ &=: \beta_\lambda^{\text{1L,matter}}(\lambda_k, g_k) . \end{aligned}} \tag{7.53}$$

Note that for $N_S = N_D = N_V = 0$ these equations reduce to Eq. (7.31). The corresponding RG flow possesses the fixed point, given by $\beta_\lambda^{\text{1L,matter}}(\lambda_k, g_k) = 0 = \beta_g^{\text{1L,matter}}(\lambda_k, g_k)$,

$$\begin{aligned} \lambda_*^{\text{1L,matter}} &= \frac{3(2 + N_S - 4N_D + 2N_V)}{4(22 - N_S - 2N_D + N_V)} \\ \text{and} \quad g_*^{\text{1L,matter}} &= \frac{12\pi}{22 - N_S - 2N_D + N_V} . \end{aligned} \tag{7.54}$$

If we require a positive fixed-point value of Newton's constant g_* this condition will give a non-trivial constraint on the matter content compatible with the Asymptotic Safety scenario [102]. For example, for the field content of the "Standard Model of particle physics", given by $N_S = 4$, $N_D = 45/2$ and $N_V = 12$, we find this condition unfulfilled for the above fixed point:

$$\lambda_*^{\text{1L,matter}} = 3 \quad \text{and} \quad g_*^{\text{1L,matter}} = -\frac{4\pi}{5} . \tag{7.55}$$

However, we emphasize that one should not over-interpret this finding: the main virtue of the "one-loop approximation" of the FRGE is its simple structure which puts it in the frontline for any analysis of exploratory character in the FRG framework.

CHAPTER 8

The one-loop beta functions of higher-derivative gravity

Executive summary. We make use of the one-loop approximation of the FRGE in order to derive the one-loop beta functions for higher-derivative as well as Weyl-squared gravity in four spacetime dimensions. We discuss the flow of the resulting RG equations and show that the theory is asymptotically free in the coupling parametrizing the squared Weyl tensor. Moreover, we discuss the corresponding one-loop beta functions in $4-\varepsilon$ dimensions.

8.1. Spacetime dimension $d = 4$

As a further example, let us demonstrate how one can obtain one-loop beta functions of higher-derivative gravity in four dimensions, $d = 4$, within the FRG framework, following the procedure explained in the last paragraph of 6.5. The bare action of the theory is given by Eq. (4.15), with $S_{\mathrm{cl}}[\widehat{g}]$ specified to Eq. (4.9), i.e.,

$$S[\widehat{h}, \bar{C}, C, b; \bar{g}] \coloneqq S_{\mathrm{h.-d.}}[\bar{g} + \widehat{h}] + S_{\mathrm{GF}}[\widehat{h}; \bar{g}] + S_{\mathrm{gh,1}}[\widehat{h}, \bar{C}, C; \bar{g}] + S_{\mathrm{gh,2}}[b; \bar{g}] \quad (8.1)$$

with

$$S_{\mathrm{h.-d.}}[\widehat{g}] \coloneqq \int \mathrm{d}^4x \sqrt{\widehat{g}} \left[-\frac{1}{f_2^2} \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right) - \frac{1}{6 f_0^2} R^2 \right] . \quad (8.2)$$

Furthermore, the gauge-fixing action and ghost actions are determined by the gauge-fixing condition (4.31) and weight function (4.84), such that these are given by Eq. (4.85), Eq. (4.34) and Eq. (4.87), respectively. Next, we set the gauge-fixing parameters to the values given in Eq. (4.99). Therewith, the gauge fixing action and ghost actions obtain a parametric dependence on the couplings f_0^2 and f_2^2,

$$\begin{aligned} S_{\mathrm{GF}}[\widehat{h};\bar{g}] &\equiv S_{\mathrm{GF}}[\widehat{h};\bar{g}]\left(f_0^2,f_2^2\right) \,, \\ S_{\mathrm{gh},1}[\widehat{h},\bar{C},C;\bar{g}] &\equiv S_{\mathrm{gh},1}[\widehat{h},\bar{C},C;\bar{g}]\left(f_0^2,f_2^2\right) \,, \\ S_{\mathrm{gh},2}[b;\bar{g}] &\equiv S_{\mathrm{gh},2}[b;\bar{g}]\left(f_0^2,f_2^2\right) \,, \end{aligned} \tag{8.3}$$

and thus do the operators $\mathscr{M}[\widehat{g},\bar{g}] \equiv \mathscr{M}[\widehat{g},\bar{g}]\,(f_0^2,f_2^2)$ and $Y[\bar{g}] \equiv Y[\bar{g}]\,(f_0^2,f_2^2)$.

For this setting, we can obtain one-loop beta functions (respectively RG equations) by giving a k-dependence to the couplings, $f_0^2 \mapsto f_0^2(k)$ and $f_2^2 \mapsto f_2^2(k)$, and solving the FRGE (6.66) at $g \equiv \bar{g}$ with the ansatz

$$\begin{aligned} \Gamma_k[g,\bar{g}] = &\int \mathrm{d}^4x\sqrt{g}\left[-\frac{1}{f_2^2(k)}\left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) - \frac{1}{6f_0^2(k)}R^2\right] \\ &+ S_{\mathrm{GF}}[\widehat{h};\bar{g}]\left(f_0^2(k),f_2^2(k)\right) \end{aligned} \tag{8.4}$$

on the trucated theory space spanned by the operators $g \mapsto \int \mathrm{d}^4x\sqrt{g}R^2$ and $g \mapsto \int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right)$. The "11"-part of the Hessian for this ansatz at $g \equiv \bar{g}$ is obviously given by Eq. (4.97) after enabling the k-dependence of the couplings,

$$\begin{aligned} \left(\Gamma_k^{(2)}\right)_{11}[g,g] &= (\mathscr{U}[0;g])_{\mathrm{h.-d.}}\left(f_0^2(k),f_2^2(k)\right) \\ &= K[g]\left(f_0^2(k),f_2^2(k)\right)\Big\{\mathbb{1}_{ST^2}\Box_g^2 + (V^{\kappa\tau})[g]\big(f_0^2(k),f_2^2(k)\big)D_\kappa D_\tau \\ &\qquad + W[g]\big(f_0^2(k),f_2^2(k)\big)\Big\}\,. \end{aligned} \tag{8.5}$$

Lastly, we must specify the general form (6.3) of the cutoff operators. It is clear that we must set $\gamma = 2$ for the gravitational cutoff operator and $\gamma = 1$ for the cutoff operators of both ghost parts. With the operators at hand, it is further convenient to set

$$\mathscr{Z}_k^{\mathrm{grav}}[g] = K[g]\big(f_0^2(k),f_2^2(k)\big) \;,\; \mathscr{Z}_k^{\mathrm{gh},1}[g] \;= \sqrt{2}\,\mathbb{1}_V \;\text{ and }\; \mathscr{Z}_k^{\mathrm{gh},2}[g] = \frac{1}{2f_2^2(k)}\mathbb{1}_V\,.$$

All in all, the FRGE (6.66) at $g \equiv \bar{g}$ therewith becomes

$$
\begin{aligned}
-\partial_t \left(\frac{1}{f_2^2(k)} \right) \int \mathrm{d}^4x \sqrt{g} \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right) - \partial_t \left(\frac{1}{6 f_0^2(k)} \right) \int \mathrm{d}^4x \sqrt{g}\, R^2 \\
= \frac{1}{2} \operatorname{Tr}_{ST^2} \Bigg[\Big[\mathbb{1}_{ST^2} \Big(\Box_g^2 + k^4 R^{(0)}(D^4/k^4) \Big) + (V^{\kappa\tau})[g] \big(f_0^2(k), f_2^2(k) \big) D_\kappa D_\tau \\
+ W[g] \big(f_0^2(k), f_2^2(k) \big) \Big]^{-1} \mathbb{1}_{ST^2}\, \partial_t \left(k^4 R^{(0)}(D^4/k^4) \right) \Bigg] \\
- \operatorname{Tr}_V \Bigg[\Big[-\mathscr{M}[g,g] \left(f_0^2(k), f_2^2(k) \right) + \mathbb{1}_V k^2\, R^{(0)}(-D^2/k^2) \Big]^{-1} \\
\times \mathbb{1}_V\, \partial_t \left(k^2 R^{(0)}(-D^2/k^2) \right) \Bigg] \\
- \frac{1}{2} \operatorname{Tr}_V \Bigg[\Big[-Y[g] \left(f_0^2(k), f_2^2(k) \right) + \mathbb{1}_V k^2\, R^{(0)}(-D^2/k^2) \Big]^{-1} \\
\times \mathbb{1}_V\, \partial_t \left(k^2 R^{(0)}(-D^2/k^2) \right) \Bigg] .
\end{aligned}
\tag{8.6}
$$

The traces on the RHS now must be expanded in curvature invariants by means of heat kernel methods[1], including the off-diagonal heat kernel, and then must be projected onto the truncation of theory space given by the operators $\int \mathrm{d}^4x \sqrt{g} R^2$ and $\int \mathrm{d}^4x \sqrt{g} \left(\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right)$. Because of its wide extent, we will not present this heat kernel expansion here; see e.g. [45, 47, 102] for an elaborate derivation. After the RHS of the FRGE has been projected onto the above truncation of theory space, one can read off the beta functions for the couplings f_0^2 and f_2^2, which are the coefficients of the basis functionals $\int \mathrm{d}^4x \sqrt{g} R^2$ and $\int \mathrm{d}^4x \sqrt{g} \left(\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right)$, respectively.

[1] Cf. the literature quoted in appendix E.

Then one finds the RG equations, with $t = \ln k$ the renormalization group time,

$$\boxed{\begin{aligned} \partial_t f_2^2(k) &= -\frac{1}{(4\pi)^2}\frac{133}{10} f_2^4(k) \\ &=: \beta_{f_2^2}\big(f_2^2(k)\big)\ , \\ \partial_t f_0^2(k) &= +\frac{1}{(4\pi)^2}\left[\frac{5}{3}\frac{f_2^4(k)}{f_0^4(k)} + 5\frac{f_2^2(k)}{f_0^2(k)} + \frac{5}{6}\right] f_0^4(k) \\ &=: \beta_{f_0^2}\big(f_0^2(k), f_2^2(k)\big)\ . \end{aligned}} \tag{8.7}$$

The RG equations (8.7) are the same as those obtained from dimensional regularization with the $\overline{\text{MS}}$ scheme [33, 37, 49, 107–110]. In fact, these RG equations are *universal* in the sense that they *do not depend on the employed regularization scheme* [45, 47, 111]. Furthermore, these RG equations can, in fact, be shown to be *independent of the specific choice of gauge-fixing parameters* α, β, γ and δ such that in this sense they are *physical.* Note that all these properties only hold for $d = 4$.

Clearly, the beta functions of the RG equations (8.7) possess the *Gaussian fixed point* $(f_{0\,*}^2, f_{2\,*}^2) = 0$. However, this fixed point is attrative only in one direction: As $\beta_{f_2^2}(f_2^2) \leq 0$ for all f_2^2, in the neighborhood of 0 the function f_2^2 is a monotonically decreasing in k, thus f_2^2 is a relevant direction (the fixed point is attractive); and as on the hand $\beta_{f_0^2}(f_0^2, f_2^2) \geq 0$ for all (f_0^2, f_2^2), the function f_0^2 is a monotonically increasing in k, thus f_0^2 is an irrelevant direction (the fixed point is repulsive). Hence, *higher-derivative gravity is asymptotically free in the coupling* f_2^2. In fact, we need not necessarily set $f_{0\,*}^2 = 0$ to obtain a fixed point because the vanishing parameter f_2^2 already implies the vanishing of both beta functions, for all f_0^2. Thus the Gaussian fixed point, although clearly present, does not resemble any information of preferred values of the coupling f_0^2 in the UV for $k \to \infty$. However, by re-formulating the RG equation for f_0^2 in terms of the variable

$$\omega(k) \equiv \omega\big(f_0^2(k), f_2^2(k)\big) := \frac{f_2^2(k)}{2 f_0^2(k)}\,, \tag{8.8}$$

it is indeed possible to find such preferred values (i.e., fixed points) for ω. We therefore firstly re-write the RG equation for f_0^2 into (thereby making use of the RG equation for f_2^2)

$$\begin{aligned}\partial_t\omega(k) &= -\frac{1}{(4\pi)^2}\frac{25+1098\omega(k)+200\omega^2(k)}{60}f_2^2(k) \\ &=: \beta_\omega\big(\omega(k), f_2^2(k)\big) \ .\end{aligned} \tag{8.9}$$

Still, we encounter the "problem" $\beta_\omega(\omega, 0) \equiv 0$ for all ω such that no specific UV values for the coupling ω can be determined. Secondly, it is therefore customary to introduce the renormalization group time τ given by the differential

$$\mathrm{d}\tau(k) := \frac{f_2^2(k)}{(4\pi)^2}\mathrm{d}t(k)\,. \tag{8.10}$$

As the RG equations (8.7) are physical, i.e., independent of the choice of gauge-fixing parameters, we may integrate this differential with f_2^2 given by the straightforward solution to the RG equation $\partial_t f_2^2(k) = \beta_{f_2^2}(f_2^2(k))$, in order to determine τ. This results in

$$\tau(k) = \frac{10}{133}\ln\Big[133\ln k - 10(4\pi)^2\cdot\mathrm{const.}\Big]\,. \tag{8.11}$$

Especially, for the UV limit $t \to \infty$ we find that $\tau \to \infty$ such that we can w.l.o.g. investigate UV properties by means of the new renormalization group time τ.

In terms of τ the RG equations decouple into the independent ordinary differential equations

$$\boxed{\begin{aligned}\partial_\tau f_2^2(k) &= -\frac{133}{10}f_2^2(k) \\ &=: \beta'_{f_2^2}\big(f_2^2(k)\big) \\ \text{and}\quad \partial_\tau\omega(k) &= -\frac{200\omega^2(k)+1098\omega(k)+25}{60} \\ &=: \beta'_\omega(\omega(k))\ .\end{aligned}} \tag{8.12}$$

The beta functions (8.12) possess the non-Gaussian fixed points

$$f_{2\,*}^2 = 0 \quad \text{and} \quad \omega_{*,1/2} = -\frac{549}{200} \pm \frac{7\sqrt{6049}}{200}\,, \tag{8.13}$$

i.e., $\beta'_{f_2^2}(f^2_{2\,*}) = 0 = \beta'_{f_2^2}(\omega_*)$. By setting ω to either fixed-point value, we obtain a perturbative series that is controlled solely by f_2^2. The stability matrix for the system (8.12) is independent of f_2^2 and in general reads

$$B(\omega_*) = \begin{pmatrix} \partial\beta'_{f_2^2}/\partial f_2^2 & \partial\beta'_{f_2^2}/\partial\omega \\ \partial\beta'_{\omega}/\partial f_2^2 & \partial\beta'_{\omega}/\partial\omega \end{pmatrix}(\omega_*) = \begin{pmatrix} -\frac{133}{10} & 0 \\ 0 & -\frac{183}{10} - \frac{20}{3}\omega_* \end{pmatrix}. \tag{8.14}$$

For the first fixed point,

$$\left(f^2_{2\,*}, \omega_{*,1}\right) \approx (0, -0.0229)\,, \tag{8.15}$$

we find $B(\omega_{*,1}) \approx \mathrm{diag}(-13.3, -18.2)$, i.e., the fixed point $(f^2_{2\,*}, \omega_{*,1})$ is *UV stable*, having two relevant directions. The other fixed point,

$$\left(f^2_{2\,*}, \omega_{*,2}\right) \approx (0, -5.4671) \tag{8.16}$$

is a *saddle point* with one relevant and one irrelevant direction, $B(\omega_{*,2}) \approx \mathrm{diag}(-13.3, +18.2)$.

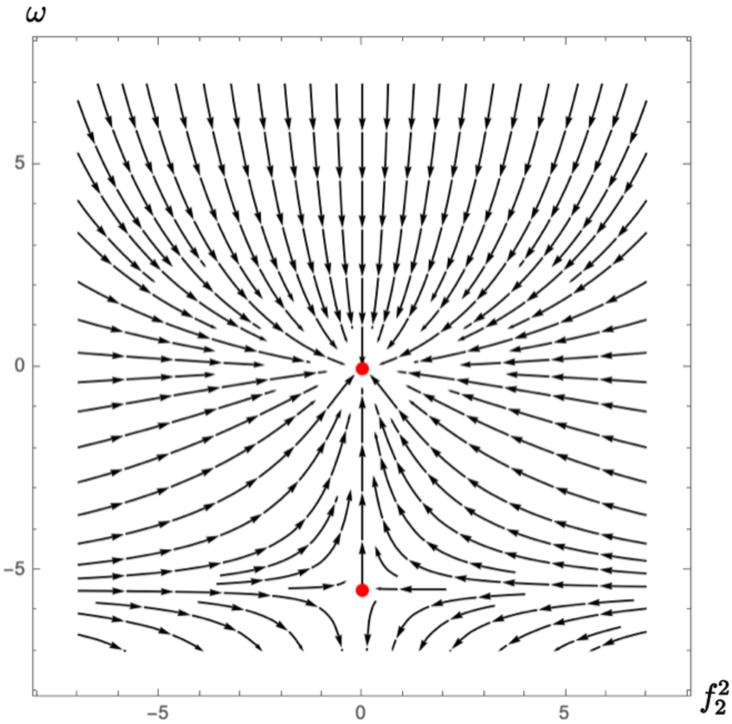

FIGURE 8.1. The flow of higher-derivative gravity for $d = 4$ in the renormalization group time τ, given by the RG equations (8.12). Marked in red are the fixed points (8.15) and (8.16).

Moreover, one can argue that the saddle point $(f^2_{2\,*}, \omega_{*,2})$ lies in an ill-defined region of theory space as follows [45]: Let us consider the ghost operator $\frac{1}{\alpha}Y[g]$ given by Eq. (4.84) with the gauge-fixing parameters specified to Eq. (4.99). Re-expressing the coupling f_0^2 by ω, this yields

$$\begin{aligned}\frac{1}{\alpha}Y^{\mu\nu}[g]^{\text{diff}} &= -\frac{1}{2f_2^2}\left(g^{\mu\nu}D^2 + \frac{2(1+\omega)}{3}D^\mu D^\nu - D^\nu D^\mu\right)\\ &= -\frac{1}{2f_2^2}\left(g^{\mu\nu}D^2 - \frac{1-2\omega}{3}D^\mu D^\nu - R^{\mu\nu}\right).\end{aligned} \tag{8.17}$$

The spectrum of this operator is generally positive provided that

$$\frac{1-2\omega}{3} < 0 \quad \Leftrightarrow \quad \omega > -1\,. \tag{8.18}$$

This condition for a positive ("second") ghost operator is fulfilled among the fixed points only by the UV-attractive fixed point $(f^2_{2\,*}, \omega_{*,1})$.

8.2. Weyl-squared gravity in $d = 4$

In the very same way as in Section 8 we can also obtain the RG equation for Weyl-squared gravity (there is only one, for the coupling f_2^2). Following Subsection **??**, we perform in the ansatz for $\Gamma_k[g,\bar{]}$ (of the previous section) and in the operators $\mathscr{M}[g,\bar{g}]$ and $Y[\bar{g}]$ the substitutions $f_0^2 \to \infty$, $\beta = 1/4$ and $g_{\mu\nu} \mapsto \big(I_{\mu\nu}{}^{\alpha\beta} - (P_{\text{tr.}})[\bar{g}]_{\mu\nu}{}^{\alpha\beta}\big)g_{\alpha\beta}$. When we further specify the remaining gauge-fixing parameters to $\alpha = -2f_2^2$, $\gamma = 2/3$ and $\delta = 1$, this leads to the "11"-component of the Hessian at $g \equiv \bar{g}$,

$$\begin{aligned}\left(\Gamma_k^{(2)}\right)_{11}[g,g] &= (\mathscr{U}[0;g])_{\text{Weyl}}\big(f_2^2(k)\big)\\ &= (\mathbb{1}_{\text{ST}^2} - \mathbb{P}_{\text{tr.}}[g])\,K_{\text{Weyl}}[g]\big(f_2^2(k)\big)\times\\ &\quad\times\left\{\mathbb{1}_{ST^2}\Box_g^2 + (V^{\kappa\tau}_{\text{Weyl}})[g]\big(f_2^2(k)\big)D_\kappa D_\tau + W_{\text{Weyl}}[g]\big(f_2^2(k)\big)\right\}.\end{aligned} \tag{8.19}$$

Analogous to the case $f_0^2 < \infty$ we further set[2]

$$\mathscr{Z}_k^{\text{grav}}[g] = (\mathbb{1}_{\text{ST}^2} - \mathbb{P}_{\text{tr.}}[g])\,K_{\text{Weyl}}[g]\,, \tag{8.20}$$

[2]Kindly remember that in Weyl-squared gravity the first component of field space consists of the symmetric and traceless rank-2 tensors. The identity on this space is $\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[\bar{g}]$.

such that the "one-loop approximation" of the FRGE on the one-dimensional truncation of theory space given by the operator $\int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right)$ becomes

$$
\begin{aligned}
-\partial_t\left(\frac{1}{f_2^2(k)}\right)\int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) \\
=\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[\Big[\mathbb{1}_{ST^2}\Big(\Box_g^2 + k^4R^{(0)}(D^4/k^4)\Big) + (V^{\kappa\tau}_{\mathrm{Weyl}})[g]\big(f_2^2(k)\big)D_\kappa D_\tau \\
+ W_{\mathrm{weyl}}[g]\big(f_2^2(k)\big)\Big]^{-1}\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\mathrm{tr.}}[g]\right)\,\partial_t\left(k^4R^{(0)}(D^4/k^4)\right)\Bigg] \\
-\mathrm{Tr}_V\Bigg[\Big[-\mathscr{M}[g,g]\left(f_2^2(k)\right) + \mathbb{1}_V k^2\,R^{(0)}(-D^2/k^2)\Big]^{-1} \\
\times\,\mathbb{1}_V\,\partial_t\left(k^2R^{(0)}(-D^2/k^2)\right)\Bigg] \\
-\frac{1}{2}\,\mathrm{Tr}_V\Bigg[\Big[-Y[g]\left(f_2^2(k)\right) + \mathbb{1}_V k^2\,R^{(0)}(-D^2/k^2)\Big]^{-1} \\
\times\,\mathbb{1}_V\,\partial_t\left(k^2R^{(0)}(-D^2/k^2)\right)\Bigg]\,.
\end{aligned}
\tag{8.21}
$$

Again, we will not explicitly perform the projection of the RHS onto the operator $\int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right)$ using heat kernel methods, but only state the resulting one-loop RG equation for the coupling f_2^2 that slightly differs from the case $f_0 < \infty$ [49]:

$$
\boxed{
\begin{aligned}
\partial_t f_2^2(k) &= -\frac{1}{(4\pi)^2}\frac{199}{15}f_2^4(k) \\
&=: \beta^{\mathrm{Weyl}}_{f_2^2}\left(f_2^2(k)\right)\,.
\end{aligned}
}
\tag{8.22}
$$

As in the case $f_0^2 < \infty$ one can show that this one-loop RG equation is *universal*, i.e., independent of the employed regularization scheme, and *physical*, i.e., independent of the specific choice of gauge-fixing parameters. Especially, note that Weyl-squared gravity is *asymptotically free* in its sole coupling f_2^2.

8.3. A comment on the one-loop beta functions in $d = 4 - \varepsilon$

Unlike the beta functions of higher-derivative gravity in $d = 4$, the corresponding beta functions in $d = 4 - \varepsilon$ are *not universal*, i.e., regularization-scheme dependent, and *not necessarily physical*, i.e., they are generally gauge-dependent. In the literature, the beta functions of higher-derivative gravity have been determined by means of two non-coinciding regularization schemes. On the one hand, the FRGE in Section 8 can be evaluated using heat kernel methods in an arbitrary dimension d. If we do so and set $d = 4 - \varepsilon$, the beta functions will depend parametrically on ε; especially, the parameter ε need not be small and may take any value resulting in a positive dimension d. The beta functions obtained in this way have been analyzed in [47] and, interestingly, reflect the requirement $\omega > -1$ due to the appearance of "$\ln(1 - \omega)$"-terms. On the other hand, the beta functions in $d = 4 - \varepsilon$ have been obtained by regularizing the conventional effective action with dimensional regularization (using e.g. the $\overline{\text{MS}}$ scheme), i.e., ε here plays the role of the regulator, rather than an external parameter. These beta functions, that are different from those obtained with FRG methods, have been analyzed in [33, 49]. As the beta functions obtained from dimensional regularization are easier to handle, and for ε small numerically yield fixed points that only slightly deviate from those obtained with FRG methods, we will restrict the following discussion of the beta functions in $d = 4 - \varepsilon$ to those obtained with dimensional regularization.

With respect to the renormalization group time τ defined by Eq. (8.10), the RG equations obtained from dimensional regularization read in leading order in ε [33]:

$$\boxed{\begin{aligned}\partial_\tau f_2^2(k) &= -\frac{133}{10} f_2^2(k) + \varepsilon \Bigg[-(4\pi)^2 + \chi(f_2^2(k), \omega(k); \alpha, \beta, \gamma, \delta) \\ &\qquad\qquad - f_2^2(k) \frac{20\omega^2(k) - 302\omega(k) + 5}{60\omega(k)} \Bigg] + O(\varepsilon^2) \\ &=: \beta'_{f_2^2,\varepsilon}\big(f_2^2(k), \omega(k)\big) \\ \text{and}\quad \partial_\tau \omega(k) &= -\frac{200\omega^2(k) + 1098\omega(k) + 25}{60} \\ &\quad + \varepsilon \frac{20\omega^2(k) + 932\omega(k) + 821}{360} + O(\varepsilon^2) \\ &=: \beta'_{\omega,\varepsilon}(\omega(k))\ .\end{aligned}} \tag{8.23}$$

Here, $\chi(f_2^2, \omega; \alpha, \beta, \gamma, \delta)$ is a function that "measures the deviation" from the gauge given by the parameters (4.99), i.e., especially that

$$\chi(f_2^2, \omega; \alpha, \beta, \gamma, \delta)\Big|_{\text{Eq. (4.99)}} = 0\,. \tag{8.24}$$

In this generality, the function χ is not stated in the literature. Note that as the beta function $\beta'_{\omega,\varepsilon}$ is gauge-independent, the gauge-dependence of the system of RG equations (8.23) is solely given by the beta function $\beta'_{f_2^2,\varepsilon}$. This points at the problematic fact that the renormalization group time τ therewith also acquires a gauge-dependence, due to its definition through the coupling f_2^2. Consequently, "asymptotic freedom in the $4-\varepsilon$ theory is not a physical phenomenom, but an artificial occurrence depending on the choice of gauge-fixing condition" [33]. Here, let us make the seemingly natural assumption that $\chi(f_2^2, \omega; \alpha, \beta, \gamma, \delta) \sim f_2^2$. On the basis of this assumption, the fixed points of the system of RG equations (8.23) are *in leading order in* ε in fact *independent of* χ:

$$\begin{aligned}\Big(f^2_{2\,*,\varepsilon}, \omega_{*,1;\varepsilon}\Big) &\approx (-11.8732\varepsilon, -0.0229 + 0.1224\varepsilon) \\ \text{and}\quad \Big(f^2_{2\,*,\varepsilon}, \omega_{*,2;\varepsilon}\Big) &\approx (-11.8732\varepsilon, -5.4671 + 0.5628\varepsilon)\,.\end{aligned} \tag{8.25}$$

As a result, these fixed points resemble, for ε small, the physical property of asymptotic freedom of $d = 4$-case and we may use them to investigate how the fixed-point value of a certain quantity, initially at the fixed point (8.13), changes when slightly moving away from $d = 4$. For this particular purpose, it may also be interesting to ask how this fixed-point value of some quantity changes if we move away from $d = 4$ only on the f_2^2-axis of theory space. Therefore, let us take the limit $\varepsilon \to 0$ of the beta function for ω, $\beta'_{\omega,\varepsilon=0} = \beta'_{\omega}$, which results in the fixed points

$$\begin{aligned} &\left(f^2_{2\,*,\varepsilon}, \omega_{*,1}\right) \approx (-11.8732\varepsilon, -0.0229) \\ \text{and} \quad &\left(f^2_{2\,*,\varepsilon}, \omega_{*,2}\right) \approx (-11.8732\varepsilon, -5.4671)\,. \end{aligned} \tag{8.26}$$

For the limit $f_0^2 \to \infty$ of Weyl-squared gravity, the RG equation with respect to the renormalization group time τ for the coupling f_2^2 in $d = 4 - \varepsilon$ has been calculated using dimensional regularization in [49] and reads, in leading order in ε,

$$\boxed{\begin{aligned} \partial_\tau f_2^2(k) = &-\frac{1}{(4\pi)^2}\frac{199}{15} f_2^4(k) \\ &+\varepsilon\left[-(4\pi)^2 + \chi_{\mathrm{Weyl}}(f_2^2(k);\alpha,\gamma,\delta) + \frac{311}{60} f_2^2(k)\right] + O(\varepsilon^2) \\ =:&\ \beta'_{f_2^2,\varepsilon;\mathrm{Weyl}}\left(f_2^2(k)\right)\,, \end{aligned}} \tag{8.27}$$

where the function $\chi_{\mathrm{Weyl}}(f_2^2;\alpha,\gamma,\delta)$ again measures the deviation from the gauge $\alpha = -2f_2^2$, $\gamma = 2/3$ and $\delta = 1$; i.e., especially $\chi_{\mathrm{Weyl}}(f_2^2; -2f_2^2, 2/3, 1) = 0$. The explicit form the function χ_{Weyl} can be found in [112]. In leading order in ε, the fixed point of this RG equation turns out to be dependent on the gauge parameter α (provided that α is independent of the coupling f_2^2):

$$f^2_{2\,*,\varepsilon;\mathrm{Weyl}} \approx (-11.9030 + 17.8546\alpha)\varepsilon\,. \tag{8.28}$$

Again, this fixed point (in the appropriate gauge) may be used to investigate how quantities, evaluated at the Gaussian fixed point of Weyl-squared gravity, change when one moves away from $d = 4$.

CHAPTER 9

Composite operators

Executive summary. We motivate the study of composite operators within the framework of the functional renormalization group equation for quantum gravity. Then, we explicitly construct an FRGE that governs the k dependence of renormalized composite operators. We show that for geometric operators this renormalization behavior is encoded into the operators' anomalous dimensions. To these, we give a geometrical interpretation in form of quantum corrections to the scaling exponents of the geometric operators.
What is new? The composite-operator FRGE (9.19) for composite operators depending on the ghost fields.

9.1. Motivation

The goal of any quantum-gravitational theory is to give a physical meaning to the path integral (4.17). In the Asymptotic Safety scenario, the physical meaning of this path integral lies in the existence of an UV non-Gaussian fixed point with a finite number of relevant directions. The tool to probe the existence of such a fixed point is the FRG formalism, as described in Section 6.2. So far, the investigations performed among the Asymptotic Safety program show a strong indication for the existence of such a fixed point, cf. Section 6.6

So why study composite operators in the Asymptotic Safety scenario for quantum gravity? The main motivation lies in the fact that, in order to make contact with *quantum-gravitational observables*, knowledge about the *renormalization behavior of geometric operators* may be required that cannot be extracted from the EAA alone. Observables in gravity (i.e., classical or quantum gravity) are challenging to construct because they are required to be diffeomorphism-invariant [113–119]. Therewith, for example, they cannot depend on a single point in spacetime but rather should be considered as the integral of some scalar density over spacetime. Thus, one might need to resort to non-local operators in their construction, which we can illustrate with the following qualitative example for an observable [28, 29]. Consider the correlation function $G(r)$ of two operators $\mathscr{O}_1[\widehat{g}]$ and $\mathscr{O}_2[\widehat{g}]$ at fixed geodesic length r [117, 118]:

$$G(r) = \left\langle \frac{1}{\mathrm{vol}[\widehat{g}]} \int \mathrm{d}^d x \sqrt{\widehat{g}(x)} \int \mathrm{d}^d y \sqrt{\widehat{g}(y)}\, \mathscr{O}_1[\widehat{g}](x) \mathscr{O}_2[\widehat{g}](y) \delta(r - \ell_{\widehat{g}}(x,y)) \right\rangle , \tag{9.1}$$

with $\ell_{\widehat{g}}(x,y)$ the geodesic length (later on, we will give a precise definition of it) and $\mathrm{vol}[\widehat{g}]$ the spacetime volume. This observable essentially depends on $\ell_{\widehat{g}}(x,y)$ which is a non-local operator for which it is realistically impossible to be included into a truncation on which the FRGE is approximated (truncations are usually the linear span of (quasi-)local operators) – hence, we can find a remedy in renormalizing it as a composite operator. Furthermore, we can use this example to illustrate the need for knowledge about the *scaling behavior* of composite operators. In the fixed-point regime, where scale invariance is realized, we expect the scaling behavior $G(r) \sim r^{\Delta}$. In order to obtain the scaling exponent Δ, let us rescale the fixed geodesic distance r by a factor λ,

$$\begin{aligned} G(\lambda r) &= \left\langle \frac{1}{\mathrm{vol}[\widehat{g}]} \int \mathrm{d}^d x \sqrt{\widehat{g}(x)} \int \mathrm{d}^d y \sqrt{\widehat{g}(y)}\, \mathscr{O}_1[\widehat{g}](x) \mathscr{O}_2[\widehat{g}](y) \delta(\lambda r - \ell_{\widehat{g}}(x,y)) \right\rangle \\ &= \mathrm{e}^{-W[\widehat{h},\dots;\bar{g}]} \int \mathcal{D}\mu[\widehat{h},\dots;\bar{g}] \frac{1}{\mathrm{vol}[\widehat{g}]} \int \mathrm{d}^d x \sqrt{\widehat{g}(x)} \int \mathrm{d}^d y \sqrt{\widehat{g}(y)} \\ &\qquad \times \mathscr{O}_1[\widehat{g}](x) \mathscr{O}_2[\widehat{g}](y) \delta(\lambda r - \ell_{\widehat{g}}(x-y)) \\ &= \left\langle \Omega^{\Delta_{\mathrm{vol}} - \Delta_1 - \Delta_2} \frac{1}{\mathrm{vol}[\widehat{g}]} \int \mathrm{d}^d x \sqrt{\widehat{g}(x)} \int \mathrm{d}^d y \sqrt{\widehat{g}(y)} \mathscr{O}_1[\widehat{g}](x) \mathscr{O}_2[\widehat{g}](y) \right. \\ &\qquad \left. \times \left(\lambda r - \Omega^{\Delta_{\ell_g}} \ell_{\widehat{g}}(x-y)\right) \right\rangle . \end{aligned} \tag{9.2}$$

In the second step, we have written out the expectation value, defined in terms of Eq. (4.17). In the third step, we have rescaled the metric by a factor Ω. Thereby, we have assumed that the measure is scale invariant and that the operators are situated in the fixed-point regime, where they all transform homogenously. The respective scaling dimensions of the operators are denoted by Δ_1, Δ_2, Δ_{vol} and

Δ_{ℓ_g}. Still, λ is an arbitrary parameter, so we are free to set $\lambda = -\Omega^{-\Delta_{\ell_g}}$ and eliminate Ω from the equation in favor of λ:

$$
\begin{aligned}
G(\lambda r) = \Bigg\langle \lambda^{\frac{\Delta_1+\Delta_2-\Delta_{\text{vol}}}{\Delta_{\ell_g}}} \frac{1}{\text{vol}[\widehat{g}]} \int \mathrm{d}^d x \sqrt{\widehat{g}(x)} \int \mathrm{d}^d y \sqrt{\widehat{g}(y)} \mathscr{O}_1[\widehat{g}](x) \mathscr{O}_2[\widehat{g}](y) \\
\times \left(\lambda r - \lambda \ell_{\widehat{g}}(x-y)\right) \Bigg\rangle . \\
= \lambda^{\frac{\Delta_1+\Delta_2-\Delta_{\text{vol}}}{\Delta_{\ell_g}}-1} G(r) \, .
\end{aligned}
\tag{9.3}
$$

Consequently, to determine the scaling behavior of the observable $G(r)$, we must determine that of the geodesic length, as well. In quantum gravity, such scaling arguments have been discussed especially in the two-dimensional case [55, 120, 121].

Aside from such qualitative examples, we must admit the longstanding problem of constructing meaningful (four-dimensional) observables in quantum gravity is clearly beyond the scope of this thesis. However, it may also be interesting to study the renormalization behavior of composite operators that are not true, diffeomorphism-invariant, observables. On the one hand, as explained above, one might need to resort to such an operator in order to construct a full-fledged observable. On the other hand, in a theory of quantum gravity, it is natural to study geometric quantities, such as the volume of a submanifold of spacetime, at the quantum level. This can yield general geometric features of the underlying theory for quantum gravity. For example, the effective Hausdoff dimension, the spectral dimension or the walk dimension of spacetime have already been estimated in the asymptotic safety scenario, which typically implies an effective dimensional reduction of spacetime in the fixed-point regime [122–125]. The further study of such geometric operators will be the essence of Chapters 11 and 12. Geometrical properties such as these are of particular interest for the comparison of the Asymptotic Safety approach towards quantum gravity with other approaches, e.g. casual dynamical triangulations or loop quantum gravity. The mentioned dimensional reduction phenomena, for instance, are a common feature of several quantum gravity scenarios [126, 127].

Last not least, the study of composite operators is also motivated by technical aspects, which we again illustrate by an (introductory) example: For the k-dependent Schwinger functional $W_k[j;\bar{g}]$, defined by Eq. (6.2), to be invariant under the classical BRST transformations (4.16), the identity

$$\begin{aligned} 0 &= sW_k[J;\bar{g}] \\ &= -\langle s\Delta_k S[\chi;\bar{g}]\rangle - \langle sS_{\text{source}}[\chi;J;\bar{g}]\rangle \end{aligned} \tag{9.4}$$

must hold. Here, s denotes the anticommuting and nilpotent BRST operator. This identity, however, is rather impractical in the FRG framework which is based on the EAA $\Gamma_k[\phi;\bar{g}]$ because with the above identity, one cannot check whether the EAA is BRST invariant. Therefore, it is desirable to express this identity in terms of the EAA. This can be realized by incorporating the BRST variations $s\widehat{h}_{\mu\nu}$ and sC^μ as *composite operators* into the k-dependent Schwinger functional as follows. Define the functional $W'_k[J;\beta,\tau;\bar{g}]$ analogously to $W_k[J;\bar{g}]$ but with the source action $S_{\text{source}}[\chi;J;\bar{g}]$ replaced by

$$S_{\text{source}}[\chi;J;\beta,\tau;\bar{g}] \coloneqq S_{\text{source}}[\chi;J;\bar{g}] + \int \mathrm{d}^d x \sqrt{\bar{g}}\left(\beta^{\mu\nu} s\widehat{h}_{\mu\nu} + \tau_\mu sC^\mu\right) , \tag{9.5}$$

i.e., we have coupled the BRST variations $s\widehat{h}_{\mu\nu}$ and sC^μ to the sources $\beta^{\mu\nu}$ and τ_μ. Also note that $sS_{\text{source}}[\chi;J;\beta,\tau;\bar{g}] = sS_{\text{source}}[\chi;J;\bar{g}]$ due to the nilpotence of the BRST operator. The EAA $\Gamma'_k[\phi;\beta,\tau;\bar{g}]$ obtained from the functional $W'_k[J;\beta,\tau;\bar{g}]$ will fulfill

$$\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\Gamma'_k[\phi;\beta,\tau;\bar{g}]}{\delta\beta^{\mu\nu}(x)} = -\left\langle s\widehat{h}_{\mu\nu}(x)\right\rangle \quad \text{and} \quad \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\Gamma'_k[\phi;\beta,\tau;\bar{g}]}{\delta\tau_\mu(x)} = -\left\langle sC^\mu(x)\right\rangle \tag{9.6}$$

per construction. With these properties, it is possible to re-formulate Eq. (9.4) as the *modified Ward identity* [26]

$$\begin{aligned} \int \mathrm{d}^d x \Bigg\{ &\frac{\delta(\Gamma'_k[\phi;\beta,\tau;\bar{g}] - S_{\text{GF}}[h;\bar{g}])}{\delta h_{\mu\nu}(x)}\frac{\delta\Gamma'_k[\phi;\beta,\tau;\bar{g}]}{\delta\beta^{\mu\nu}(x)} \\ &+ \frac{\delta(\Gamma'_k[\phi;\beta,\tau;\bar{g}] - S_{\text{GF}}[h;\bar{g}])}{\delta\xi^\mu(x)}\frac{\delta\Gamma'_k[\phi;\beta,\tau;\bar{g}]}{\delta\tau_\mu(x)}\Bigg\} = Y_k\,. \end{aligned} \tag{9.7}$$

Here, we only state Y_k for the case $b_\mu \equiv 0$ [26]:

$$
\begin{aligned}
Y_k = &\operatorname{Tr}_{ST^2}\left[\mathscr{R}_{k11}[\bar{g}]\sum_{a=1}^{3}\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]+\mathscr{R}_k[\bar{g}]\right)^{-1}_{1a}\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]\right)_{a\beta}\right] \\
&-\operatorname{Tr}_V\left[\mathscr{R}_{k23}[\bar{g}]\sum_{a=1}^{3}\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]+\mathscr{R}_k[\bar{g}]\right)^{-1}_{2a}\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]\right)_{a\tau}\right] \\
&-\frac{\sqrt{2}}{\alpha}\operatorname{Tr}_V\left[\mathscr{R}_{k23}[\bar{g}]\mathscr{F}[\bar{g}]\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]+\mathscr{R}_k[\bar{g}]\right)^{-1}_{13}\right],
\end{aligned}
\tag{9.8}
$$

with

$$
\begin{aligned}
\langle x,\ldots|\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]\right)_{a\beta}|y,\mu,\nu\rangle &\coloneqq I_a[\bar{g}]\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2\Gamma'_k[\phi;\beta,\tau;\bar{g}]}{\delta\phi^a(x)\delta\beta^{\mu\nu}(y)}, \\
\langle x,\ldots|\left(\Gamma'^{(2)}_k[\phi;\beta,\tau;\bar{g}]\right)_{a\tau}|y,\nu\rangle &\coloneqq I_{a\mu\nu}[\bar{g}]\frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}}\frac{\delta^2\Gamma'_k[\phi;\beta,\tau;\bar{g}]}{\delta\phi^a(x)\delta\tau_\mu(y)}, \\
\langle x,\mu|\mathscr{F}[\bar{g}]|y,\rho,\sigma\rangle &\coloneqq \mathscr{F}^{\rho\sigma}_\mu[\bar{g}].
\end{aligned}
\tag{9.9}
$$

Here, the tensors $I_a[\bar{g}]$ and $I_{a\mu\nu}[\bar{g}]$ fulfill the role to adapt the tensor structure of the LHS to the RHS.

The modified Ward identity thus is fulfilled by the full EAA $\Gamma_k[\phi;\beta,\tau;\bar{g}]$ per construction. However, this is not the case for approximations of the EAA arising from solving the FRGE on truncations of theory space. Therefore, this modified Ward identity, obtained by means of composite operators, may be used to test the consistency and quality of a given approximation. This would not have been possible with Eq. (9.4) only.

9.2. The composite-operator FRGE

In the subsequent section, we will generalize the incorporation of composite operators into the FRG framework constructed in Section 6.2. The main result of the process will be the *composite-operator FRGE* which describes the renormalization behavior of composite operators as a coevolution with the gravitational EAA (6.13). Especially, this composite-operator FRGE enables us to calculate the *anomalous-dimension matrix* of the composite operators, whose interpretation we will discuss in detail.

To begin with, let us consider n spacetime-dependent composite Operators $\mathscr{O}_1[\hat{h}, \bar{C}, C, b; \bar{g}](x), \ldots, \mathscr{O}_n[\hat{h}, \bar{C}, C, b; \bar{g}](x)$, each acting on either of the Hilbert spaces S, V or ST^2 and each being composed of the background metric $\bar{g}_{\mu\nu}$, its quantum fluctuation $\hat{h}_{\mu\nu}$, as well as the ghost fields C^μ, $\bar{C}_\mu$ and b_μ. Let us couple these operators to arbitrary external sources $\varepsilon \coloneqq (\varepsilon_1, \ldots, \varepsilon_n)$ and define the (modified) k-dependent Schwinger functional $W'_k[J; \varepsilon; \bar{g}]$, with $J = (t, \sigma, \bar{\sigma}, b)$, by

$$\begin{aligned}\exp\big\{W'_k[t, \sigma, \bar{\sigma}, b; \varepsilon_1, \ldots, \varepsilon_n; \bar{g}]\big\} \\ \coloneqq \int \mathcal{D}\mu[\widehat{h}, \bar{C}, C, b; \bar{g}] \exp\Bigg\{ &- \tilde{S}[\widehat{h}, \bar{C}, C, b; t, \sigma, \bar{\sigma}, d; \bar{g}] - \Delta_k S[\widehat{h}, \bar{C}, C, b; \bar{g}] \\ &- \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \sum_{i=1}^{n} \varepsilon_i^a(x) \mathscr{O}_{ia}[\widehat{h}, \bar{C}, C, b; \bar{g}](x) \Bigg\} ,\end{aligned} \tag{9.10}$$

where, with $\chi = (\widehat{h}, \bar{C}, C, b)^T$, $\tilde{S}[\chi; J; \bar{g}]$ is given by Eq. (4.18), $\Delta_k S[\chi; \bar{g}]$ by Eq. (6.1) and the measure is defined by Eq. (4.21) and inthe base $b_\mu \equiv 0$ (and $d^\mu \equiv 0$) by Eq. (4.22). Here, $\varepsilon_i^a(x) \mathscr{O}_{ia}[\chi; \bar{g}](x)$ denotes the sum over the tensor structure which depends on what Hilbert space $\mathscr{O}_i[\chi; \bar{g}](x)$ acts on. For example, if it acted on ST^2, the corresponding source need to be a tensor field ${\varepsilon_i}^{\mu\nu}{}_{\rho\sigma}(x)$ and the above term would read ${\varepsilon_i}^{\mu\nu}{}_{\rho\sigma}(x) {\mathscr{O}_i}^{\rho\sigma}{}_{\mu\nu}[\chi; \bar{g}](x)$. This "modified" k-dependent Schwinger functional is nothing but the ordinary k-dependent Schwinger functional Eq. (6.2) with the substitution (written schematically)

$$\tilde{S} \mapsto \tilde{S} + \sum_i \varepsilon_i \mathscr{O}_i \,. \tag{9.11}$$

Moreover, if we define the expectation value of the i-th composite operator as

$$\begin{aligned}\langle \mathscr{O}_{ia}[\chi; \bar{g}](x) \rangle_k \coloneqq \exp\{-W_k[J; \bar{g}]\} \int \mathcal{D}\mu[\chi; \bar{g}] \mathscr{O}_{ia}[\chi; \bar{g}](x) \\ \times \exp\Big\{ -\tilde{S}[\chi; J; \bar{g}] - \Delta_k S[\chi; \bar{g}] \Big\} ,\end{aligned} \tag{9.12}$$

we immediately obtain the relation

$$\langle \mathscr{O}_{ia}[\chi;\bar{g}](x)\rangle_k = -\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W'_k[J;\varepsilon;\bar{g}]}{\delta\varepsilon_i^a(x)}\bigg|_{\varepsilon=0} . \tag{9.13}$$

Next, let us obtain the effective action that results from the definition (9.10). Therefore, we define the classical fields $\phi = (h, \bar{\xi}, \xi, \zeta)^T$ analogously to Eq. (6.6),

$$h_{\mu\nu}(x) \equiv h_{k\mu\nu}[J;\bar{g}](x) := \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta W'_k[J;\varepsilon;\bar{g}]}{\delta t^{\mu\nu}(x)} \quad \text{et cetera.} \tag{9.14}$$

After solving these relations for the sources J, i.e., $t^{\mu\nu} \equiv t_k^{\mu\nu}[\phi;\varepsilon;\bar{g}]$ etc., the Legendre transform of Eq. (9.10) reads

$$\begin{aligned}\tilde{\Gamma}'_k[\phi;\varepsilon;\bar{g}] = \int\!\mathrm{d}^d x\sqrt{\bar{g}(x)}\,\big[&t_k{}^{\mu\nu}[\phi;\varepsilon;\bar{g}](x)h_{\mu\nu}(x) + \sigma_k{}^{\mu}[\phi;\varepsilon;\bar{g}](x)\bar{\xi}^{\mu}(x)\\ &+ \bar{\sigma}_{k\mu}[\phi;\varepsilon;\bar{g}](x)\xi^{\mu}(x) + d_k{}^{\mu}[\phi;\varepsilon;\bar{g}](x)\zeta_{\mu}(x)\big]\\ &- W'_k[J_k[\phi;\varepsilon;\bar{g}];\varepsilon;\bar{g}] .\end{aligned} \tag{9.15}$$

Analogously to Eq. (6.8), we define the *effective average action* as this Legendre transform with the cutoff functional substracted:

$$\boxed{\Gamma'_k[\phi;\varepsilon;\bar{g}] := \tilde{\Gamma}'_k[\phi;\varepsilon;\bar{g}] - \Delta_k S[\phi;\bar{g}] .} \tag{9.16}$$

Especially note that $\Gamma'_k[\phi;0;\bar{g}] - \Gamma_k[\phi;\bar{g}]$, where the EAA $\Gamma_k[\phi;\bar{g}]$ is given by Eq. (6.8). Furthermore, for the expectation value of the i-th composite operator it directly follows that

$$\boxed{\begin{aligned}\langle \mathscr{O}_{ia}[\chi;\bar{g}](x)\rangle_k &= \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta\Gamma'_k[\phi;\varepsilon;\bar{g}]}{\delta\varepsilon_i^a(x)}\bigg|_{\varepsilon=0}\\ &=: \left[\mathscr{O}_{ia}\right]_k[\phi;\bar{g}](x) .\end{aligned}} \tag{9.17}$$

A crucial point is that by following the derivation of the FRGE (6.13) for the conventional EAA $\Gamma_k[\phi;\bar{g}]$ in appendix F.11 yet again with $\Gamma'_k[\phi;\varepsilon;\bar{g}]$, it is evident that it fulfills the very same FRGE, i.e., with the renormalization group time $t = \ln k$ we have

$$\boxed{\begin{aligned}\partial_t \Gamma'_k[\phi;\varepsilon;\bar{g}] = & \frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[(\partial_t \mathscr{R}_{k11}[\bar{g}])\left(\left[\Gamma'^{(2)}_k[\phi;\varepsilon;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{11}\right] \\ & - \mathrm{Tr}_V\left[(\partial_t \mathscr{R}_{k23}[\bar{g}])\left(\left[\Gamma'^{(2)}_k[\phi;\varepsilon;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{32}\right] \\ & - \frac{1}{2}\,\mathrm{Tr}_V\left[(\partial_t \mathscr{R}_{k44}[\bar{g}])\left(\left[\Gamma'^{(2)}_k[\phi;\varepsilon;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1}\right)^{44}\right].\end{aligned}} \tag{9.18}$$

Here, $\Gamma'^{(2)}_k[\phi;\varepsilon;\bar{g}]$ is defined via Eq. (6.47) and $\mathscr{R}_k[\bar{g}]$ by Eq. (6.15).

By taking a functional derivative of Eq. (9.18) with respect to ε_i^a and then setting $\varepsilon = 0$, we obtain the following *composite-operator FRGE* for the renormalized operator $[\mathscr{O}_{ia}]_k[\phi;\bar{g}](x)$:

$$\begin{aligned}\partial_t [\mathscr{O}_{ia}]_k[\phi;\bar{g}](x) = & -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[\big(\partial_t \mathscr{R}_{k11}[\bar{g}]\big)\Bigg(\left[\Gamma^{(2)}_k[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \\ & \qquad \times [\mathscr{O}_{ia}]^{(2)}_k[\phi;\bar{g}](x)\left[\Gamma^{(2)}_k[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1}\Bigg)^{11}\Bigg] \\ & + \mathrm{Tr}_V\Bigg[\big(\partial_t \mathscr{R}_{k23}[\bar{g}]\big)\Bigg(\left[\Gamma^{(2)}_k[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \\ & \qquad \times [\mathscr{O}_{ia}]^{(2)}_k[\phi;\bar{g}](x)\left[\Gamma^{(2)}_k[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1}\Bigg)^{32}\Bigg] \\ & + \frac{1}{2}\,\mathrm{Tr}_V\Bigg[\big(\partial_t \mathscr{R}_{k44}[\bar{g}]\big)\Bigg(\left[\Gamma^{(2)}_k[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \\ & \qquad \times [\mathscr{O}_{ia}]^{(2)}_k[\phi;\bar{g}](x)\left[\Gamma^{(2)}_k[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1}\Bigg)^{44}\Bigg]\,.\end{aligned} \tag{9.19}$$

Here, we have used the variational rule $\delta A^{-1} = -A^{-1}(\delta A)A^{-1}$ and $\Gamma'_k[\phi;0;\bar{g}] = \Gamma_k[\phi;\bar{g}]$. Further, the operator $[\mathscr{O}_{ia}]^{(2)}_k[\phi;\bar{g}](x)$ in field space is defined via Eq. (6.47). *The composite-operator FRGE* (9.19) *possesses a double-layor structure: to solve it requires an approximation of the EAA* $\Gamma_k[\phi;\bar{g}]$*, on the one hand,*

and an approximation for the renormalized composite operator $[\mathscr{O}_{ia}]_k\,[\phi;\bar{g}](x)$, *on the other hand.* Note that in the case $b_\mu \equiv 0 \equiv \zeta_\mu$, again the last trace of Eq. (9.18), respectively Eq. (9.19), must be discarded.

Lastly, we introduce the notations

$$\Gamma'_k[g,\bar{g},\bar{\xi},\xi,\zeta;\varepsilon] \coloneqq \Gamma'_k[g-\bar{g},\bar{\xi},\xi,\zeta;\varepsilon;\bar{g}] \tag{9.20}$$

and

$$[\mathscr{O}_{ia}]_k\,[g,\bar{g},\bar{\xi},\xi,\zeta](x) \coloneqq [\mathscr{O}_{ia}]_k\,[g-\bar{g},\bar{\xi},\xi,\zeta;\bar{g}](x)\,. \tag{9.21}$$

9.3. Geometric operators and the anomalous-dimension matrix

When we speak of a *geometric composite operator* we refer to composite operators that do not depend on the ghost fields,

$$\mathscr{O}_i[g,\bar{g}](x) \equiv \mathscr{O}_i[g,\bar{g},0,0,0](x)\,, \tag{9.22}$$

and correspondingly the renormalized operator reads $[\mathscr{O}_i]_k[g,\bar{g}](x)$.

In this case, an approximation of the composite-operator FRGE (9.19) that is not far to seek is to approximate $\Gamma_k[\phi;\bar{g}]$ by Eq. (6.43), i.e., to neglect the evolution of the ghost fields. Instead of directly plugging this ansatz into Eq. (9.19), it is more practical to rethink this approximation in terms of the EAA $\Gamma'_k[\phi;\varepsilon;\bar{g}]$. This amounts to the ansatz

$$\boxed{\begin{aligned}\Gamma'_k[g,\bar{g},\bar{\xi},\xi,\zeta;\varepsilon] =& \,\Gamma_k[g,\bar{g}] + S_{\text{gh},1}[g-\bar{g},\bar{\xi},\xi;\bar{g}] + S_{\text{gh},2}[\zeta;\bar{g}] \\ &+ \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \sum_{i=1}^{n} \varepsilon_i^a(x)\,[\mathscr{O}_{ia}]_k\,[g,\bar{g}](x)\,.\end{aligned}} \tag{9.23}$$

This ansatz corresponds to an expansion of $\Gamma'_k[g,\bar{g},\bar{\xi},\xi,\zeta;\varepsilon]$ to first order in ε (provided that the bare composite operator is independent of the ghost fields),

$$\Gamma'_k[g,\bar{g},\bar{\xi},\xi,\zeta;\varepsilon] = \Gamma_k[g,\bar{g},\bar{\xi},\xi,\zeta] + \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \sum_{i=1}^{n} \varepsilon_i^a(x)\,[\mathscr{O}_{ia}]_k\,[g,\bar{g}](x) + \cdots\,, \tag{9.24}$$

and then further specifying $\Gamma_k[g,\bar{g},\bar{\xi},\xi,\zeta]$ to Eq. (6.43). Moreover, we additionally define

$$\boxed{\begin{aligned}\Gamma'_k[g,\bar{g};\varepsilon] &\coloneqq \Gamma_k[g,\bar{g},0,0,0;\varepsilon]\\ &= \Gamma_k[g,\bar{g}] + \int \mathrm{d}^d x\sqrt{\bar{g}(x)}\sum_{i=1}^{n}\varepsilon_i^a(x)\left[\mathscr{O}_{ia}\right]_k[g,\bar{g}](x)\,.\end{aligned}} \tag{9.25}$$

Analogously following Section 6.4, we obtain the following FRGE for $\Gamma'_k[g,\bar{g};\varepsilon]$ (employing the gauge-fixing condition (4.31)):

$$\begin{aligned}\partial_t\Gamma'_k[g,\bar{g};\varepsilon] = &\frac{1}{2}\operatorname{Tr}_{ST^2}\left[\left(\partial_t\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\right)\left(\left(\Gamma'^{(2)}_k\right)_{11}[g,\bar{g};\varepsilon]+\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\right)^{-1}\right]\\ &-\operatorname{Tr}_V\left[\left(\partial_t\mathscr{R}_k^{\mathrm{gh},1}[\bar{g}]\right)\left(-\sqrt{2}\mathscr{M}[g,\bar{g}]+\mathscr{R}_k^{\mathrm{gh},1}[\bar{g}]\right)^{-1}\right]\\ &-\frac{1}{2}\operatorname{Tr}_V\left[\left(\partial_t\mathscr{R}_k^{\mathrm{gh},2}[\bar{g}]\right)\left(\frac{1}{\alpha}Y[\bar{g}]+\mathscr{R}_k^{\mathrm{gh},2}[\bar{g}]\right)^{-1}\right].\end{aligned} \tag{9.26}$$

Again, by taking a functional derivative with respect to $\varepsilon_i^a(x)$ and setting $\varepsilon=0$, we obtain the following approximation of the composite-operator FRGE [54, 55]:

$$\begin{aligned}\partial_t\left[\mathscr{O}_{ia}\right]_k[g,\bar{g}](x) = -\frac{1}{2}\operatorname{Tr}_{ST^2}&\left[\left(\partial_t\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\right)\left(\left(\Gamma^{(2)}_k\right)_{11}[g,\bar{g}]+\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\right)^{-1}\right.\\ &\left.\times\left[\mathscr{O}_{ia}\right]_k^{(2)}[g,\bar{g}](x)\left(\left(\Gamma^{(2)}_k\right)_{11}[g,\bar{g}]+\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\right)^{-1}\right],\end{aligned} \tag{9.27}$$

i.e., in the approximation of neglecting the evolution of the ghost fields no relics of the ghost fields remain (in terms of traces over V) in the geometric-composite-operator FRGE. We could have obtained the same equation if we had plugged the ansatz (6.43) into Eq. (9.19) and set the ghost fields to zero (on both sides of the equation).

Hence, so far we have approximated the EAA $\Gamma_k[\phi;\bar{g}]$ that contributes to the composite-operator FRGE. However, choosing an appropriate ansatz for $\Gamma_k[g,\bar{g}]$ is not yet enough to actually solve the composite operator FRGE. This also requires a suitable approximation of the renormalized composite operator $\left[\mathscr{O}_{ia}\right]_k[g,\bar{g}](x)$. At this point, note that the double-layer structure of the composite-operator FRGE means that it entails two copies of theory space: $\Gamma_k[g,\bar{g}]$ and $[\mathscr{O}_{ia}]_k\,[g,\bar{g}](x)$ each are defined on an distinct copy of it. Thus, with any two ansätze for $\Gamma_k[g,\bar{g}]$ and $[\mathscr{O}_{ia}]_k\,[g,\bar{g}](x)$ also come along two distinct truncations of theory space. Let us refer to these as the *first* and *second truncation*, respectively.

For instance, assume that the bare operators $\mathscr{O}_1[g,\bar{g}](x),\ldots,\mathscr{O}_n[g,\bar{g}](x)$ possesses the same tensor structure and are linearly independent, such that they form a basis of the second truncation. Then the i-th renormalized composite operator reads, expanded in this basis,

$$\boxed{[\mathscr{O}_{ia}]_k\,[g,\bar{g}](x) = \sum_{j=1}^{n} Z_{ij}(k)\mathscr{O}_k[g,\bar{g}](x)\,.} \tag{9.28}$$

By plugging this ansatz into Eq. (9.29) and then multiplying the equation from the left with $Z^{-1}(k)$, we obtain the following composite-operator FRGE [54, 55]:

$$\boxed{\begin{aligned}\sum_{j=1}^{n}\bar{\gamma}_{ij}(k)\mathscr{O}_j[g,\bar{g}](x) = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[&\Big(\partial_t\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,\bar{g}]+\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1}\\ &\times\mathscr{O}_i^{(2)}[g,\bar{g}](x)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,\bar{g}]+\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1}\Bigg]\,,\end{aligned}} \tag{9.29}$$

where

$$\boxed{\bar{\gamma}_{ij}(k) := \sum_{l=1}^{n}\big(Z^{-1}(k)\big)_{il}\,\partial_t Z_{lj}(k)} \tag{9.30}$$

is the (dimensionful) *anomalous-dimension matrix*. (We will justify the name in the upcoming section.) Especially note that on the RHS of Eq. (9.29) only the bare composite operator remains. Thus, given some first truncation with an ansatz for $\Gamma_k[g,\bar{g}]$ and a basis of composite operators $\mathscr{O}_1[g,\bar{g}](x),\ldots,\mathscr{O}_n[g,\bar{g}](x)$, the composite-operator FRGE (9.29) fully encodes the renormalization behavior of these composite operators into the anomalous-dimension matrix $\bar{\gamma}(k)$. Also, if $\{P_i[g,\bar{g}]\}$ is a first truncation and we have at hand the ansatz $\Gamma_k[g,\bar{g}] = \sum_i \bar{u}_i(k) P_i[g,\bar{g}]$, the anomalous-dimension matrix will generally be a function[1]

$$\bar{\gamma}(k) \equiv \bar{\gamma}(\{\bar{u}_i(k)\};k)\,. \tag{9.31}$$

On the other hand, regarding the role of the second truncation, we call the ansatz (9.28) for the renormalized composite operator a *mixing ansatz* if $n \geq 2$ and a *non-mixing ansatz* if $n = 1$. Later, we will restrict our explorative applications to non-mixing ansätze, i.e., such of the form

$$\boxed{[\mathscr{O}]_k[g,\bar{g}](x) = Z(k)\mathscr{O}[g,\bar{g}](x)} \tag{9.32}$$

for a single bare operator $\mathscr{O}[g,\bar{g}](x)$. Its k-dependence is governed by Eq. (9.29) with $n = 1$, i.e.,

$$\boxed{\begin{aligned}\bar{\gamma}(k)\mathscr{O}[g,\bar{g}](x) = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[&\Big(\partial_t \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1} \\ &\times \mathscr{O}^{(2)}[g,\bar{g}](x)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1}\Bigg]\,.\end{aligned}} \tag{9.33}$$

Lastly, we point out that following Section 6.5, the "one-loop approximation" of the composite-operator FRGE (9.29) is given by replacing $\partial_t \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]$ by $\partial_t \mathscr{R}_{k,k'}^{\mathrm{grav}}[\bar{g}]\big|_{k'=k}$, e.g. for a non-mixing ansatz

$$\boxed{\begin{aligned}\bar{\gamma}^{1\mathrm{L}}(k)\mathscr{O}[g,\bar{g}](x) = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[&\Big(\partial_t \mathscr{R}_{k,k'}^{\mathrm{grav}}[\bar{g}]\Big)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1} \\ &\times \mathscr{O}^{(2)}[g,\bar{g}](x)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,\bar{g}] + \mathscr{R}_k^{\mathrm{grav}}[\bar{g}]\Big)^{-1}\Bigg]_{k=k'}\,.\end{aligned}} \tag{9.34}$$

[1]In the case we also have $Z_{ij}(k) \equiv Z_{ij}(\{\bar{u}_i(k)\};k)$ and the partial derivative ∂_t in the definition of $\bar{\gamma}(k)$ must be traded for the total derivative $\mathrm{d}/\mathrm{d}t$ when working with this notation.

9.4. Interpretation of the anomalous-dimension matrix $\bar{\gamma}(k)$

We had already defined the anomalous-dimension matrix in Section 6.3: Given a set of RG equations

$$\partial_t \bar{u}_i(k) = \bar{\beta}_i(\bar{u}(k); k) \tag{9.35}$$

for the dimensionful couplings $\{\bar{u}_i(k)\}$, the anomalous-dimension matrix was defined to be

$$\bar{B}_{ij}(u^*) := \frac{\partial}{\partial u_j} \bar{\beta}_i(u; 1)\bigg|_{u=u^*}, \tag{9.36}$$

where u^* is a fixed point of the corresponding dimensionless RG equations. We argued that the negative eigenvalues of $\bar{B}(u^*)$ are the anomalous scaling dimensions that state the quantum corrections to the classical scaling dimensions of the dimensionless couplings $\{u_i(k)\}$.

In this section, we will firstly show that we may identify $\bar{\gamma}_{ij}(\bar{u}(k); k)$ with $\partial_i \bar{\beta}_j(\bar{u}(k); k)$ if we interpret $\mathscr{O}_1[g, \bar{g}](x), \ldots, \mathscr{O}_n[g, \bar{g}](x)$ as the basis of a *first* truncation. This argument has been developed in [55]. In a second step, we then will show that a dimensionless version of $\bar{\gamma}_{ij}(\bar{u}(k); k)$ evaluated at the fixed point u^* moreover encodes the quantum corrections to the geometrical scaling of the operators, which was shown in [54].

(A) We had motivated the study of composite operators in the FRG formalism particularly due to the need to renormalize operators that one usually cannot include as a basis element into a (first) truncation. However, what happened if we did so? Therefore, let $\mathscr{O}[g, \bar{g}], \ldots \mathscr{O}_n[g, \bar{g}]$ be the basis of a *first* truncation. (W.l.o.g. for this purpose, the operators shall be x-independent.) On this truncation of theory space, let us solve the FRGE (6.53) with the ansatz

$$\Gamma_k[g, \bar{g}] = \sum_{i=1}^{n} \bar{u}_i(k) \mathscr{O}_i[g, \bar{g}] \,. \tag{9.37}$$

Regarding the canonical mass dimensions, let us set $[\mathscr{O}_i[g,\bar{g}]] = -d_i$ such that $[\bar{u}_i(k)] = d_i$ as in Section 6.3 (the action, of course, is dimensionless). The RHS of the FRGE (6.53) then defines the dimensionless beta functions,

$$\begin{aligned}\sum_{j=1}^{n} \bar{\beta}_j(\bar{u}(k);k)\mathscr{O}_j[g,\bar{g}] \\ &= \frac{1}{2}\operatorname{Tr}_{ST^2}\left[\left(\partial_t \mathscr{R}_k^{\text{grav}}[\bar{g}]\right)\left(\sum_{j=1}^{n}\bar{u}_j(k)\mathscr{O}_j^{(2)}[g,\bar{g}] + \mathscr{R}_k^{\text{grav}}[\bar{g}]\right)^{-1}\right] \\ &\quad - \operatorname{Tr}_V\left[\left(\partial_t \mathscr{R}_k^{\text{gh},1}[\bar{g}]\right)\left(-\sqrt{2}\mathscr{M}[g,\bar{g}] + \mathscr{R}_k^{\text{gh},1}[\bar{g}]\right)^{-1}\right] \\ &\quad - \frac{1}{2}\operatorname{Tr}_V\left[\left(\partial_t \mathscr{R}_k^{\text{gh},2}[\bar{g}]\right)\left(\frac{1}{\alpha}Y[\bar{g}] + \mathscr{R}_k^{\text{gh},2}[\bar{g}]\right)^{-1}\right].\end{aligned} \tag{9.38}$$

Taking the derivative with respect to $\bar{u}_i(k)$ yields

$$\begin{aligned}\sum_{j=1}^{n} \frac{\partial}{\partial \bar{u}_i(k)}\bar{\beta}_j(\bar{u}(k);k)\mathscr{O}_j[g,\bar{g}] \\ &= -\frac{1}{2}\operatorname{Tr}_{ST^2}\left[\left(\partial_t \mathscr{R}_k^{\text{grav}}[\bar{g}]\right)\left(\sum_{j=1}^{n}\bar{u}_j(k)\mathscr{O}_j^{(2)}[g,\bar{g}] + \mathscr{R}_k^{\text{grav}}[\bar{g}]\right)^{-1}\right. \\ &\qquad \left. \times\, \mathscr{O}_i^{(2)}[g,\bar{g}]\left(\sum_{j=1}^{n}\bar{u}_j(k)\mathscr{O}_j^{(2)}[g,\bar{g}] + \mathscr{R}_k^{\text{grav}}[\bar{g}]\right)^{-1}\right].\end{aligned} \tag{9.39}$$

This is precisely the RHS of Eq. (9.29) together with the ansatz (9.37) (plus neglecting the x-dependence), i.e., provided that the first *and* second truncation of theory space are spanned by $\mathscr{O}_1[g,\bar{g}],\dots,\mathscr{O}_n[g,\bar{g}]$, we can identify:

$$\boxed{\bar{\gamma}_{ij}(\bar{u}(k);k) = \frac{\partial}{\partial \bar{u}_i(k)}\bar{\beta}_j(\bar{u}(k);k)\,.} \tag{9.40}$$

Next, let us analyze what this equation implies for the anomalous-dimension matrix $\bar{B}(u^*)$. Therefore, note that the ansatz (9.29) implies for the canonical mass dimension of $Z_{ij}(k)$, and therewith of $\bar{\gamma}_{ij}(k)$, that

$$[Z_{ij}(k)] = [\bar{\gamma}_{ij}(k)] \equiv [\bar{\gamma}_{ij}(\bar{u}(k);k)] = -d_i + d_j\,. \tag{9.41}$$

Hence, if we define the matrix $K_{ij}(k) := k^{d_i}\delta_{ij}$ we obtain the *dimensionless* matrix

$$\begin{aligned}\gamma_{ij}(u(k)) &:= \sum_{l,m}^{n} K_{il}(k)\bar{\gamma}_{lm}(\bar{u}(k);k)K^{-1}_{mj}(k) \\ &= k^{d_i}\bar{\gamma}_{ij}(\bar{u}(k);k)k^{-d_j}\,.\end{aligned} \tag{9.42}$$

Also note that for $n = 1$ we have in principal $\gamma(u(k)) \equiv \bar{\gamma}(\bar{u}(k);k)$. Moreover, together with Eq. (9.40) it follows that

$$\begin{aligned}\gamma_{ij}(k) &= \sum_{l,m=1}^{n} K_{il}(k)\frac{\partial}{\partial \bar{u}_l(k)}\bar{\beta}_m(\bar{u}(k);k)K^{-1}_{mj}(k) \\ &= k^{d_i}\frac{\partial}{\partial \bar{u}_i(k)}\bar{\beta}_j(\bar{u}(k);k)k^{-d_j} \\ &= \frac{\partial}{\partial u_i(k)}\bar{\beta}_j(u(k);1)\,.\end{aligned} \tag{9.43}$$

This implies

$$\boxed{\gamma_{ij}(u^*) = \partial_i\bar{\beta}_j(u^*;1) = \bar{B}_{ji}(u^*)\,, \quad \text{i.e.,} \quad \gamma(u^*) = \bar{B}(u^*)^T\,.} \tag{9.44}$$

On the basis of the fact that $\gamma(u^*)$ and $\bar{B}(u^*)$ are related by a simple transposition, it is legitimate to call both the anomalous-dimension matrix because both matrices surely have the same (negative) eigenvalues and thus encode identical quantum corrections to the scaling of the couplings $\{u_i(k)\}$.

(B) The other scaling argument, that we are going to develop now, relies on the study of the dependence of the composite-operator FRGE (9.29) on initial conditions.[2] Therefore, we must take into account that the solution of Eq. (9.29) is obtained in two steps. Firstly, we must choose the first truncation of theory space, let it be spanned by $\{P_i[g,\bar{g}]\}$, and then solve the RG equations $\partial_t u_i(k) = \beta_i(u(k))$ that result from inserting the ansatz $\Gamma_k[g,\bar{g}] = \sum_i \bar{u}_i(k)P_i[g,\bar{g}]$ into the FRGE (6.53). This solution requires the initial conditions $u(\mu) = u^{(R)}$ (respectively $\bar{u}(\mu) = \bar{u}^{(R)}$ for the dimensionful couplings) and we had already shown in Section 6.3 that the parametric solution $\Gamma_k[g,\bar{g}](\mu, u^{(R)})$ fulfills

$$\left[\mu\partial_\mu + \sum_i \beta_i(u^{(R)})\frac{\partial}{\partial u^{(R)}}\right]\Gamma_k[g,\bar{g}](\mu, u^{(R)}) = 0\,. \tag{9.45}$$

[2]Here, we call "$y(x_0) = y_0$" an initial condition even though not necessarily $x_0 = 0$.

Secondly, having obtained this parametric solution we must study, on its basis, the dependence of the solution of the form (9.29) to the composite-operator FRGE (9.28) on the "second-step" initial condition[3]

$$\boxed{Z_{ij}(\mu, \bar{u}(\mu)) \equiv Z_{ij}(\mu, \bar{u}^{(R)}) = Z_{ij}^{(R)}\,.} \tag{9.46}$$

Thus, the parametric solutions to the FRGE (9.29) are essentially given by functions $Z_{ij}(k;\mu,\bar{u}^{(R)},Z^{(R)})$ with $Z^{(R)} = \{Z_{ij}^{(R)}\}$. In Section 6.3 we had argued that the initial conditions $u(\mu) = u^{(R)}$ and $u(\mu') = u^{(R)'}$, that are infinitesimally related by $\mu' = \mu + \varepsilon$ and $u^{(R)'} = u^{(R)} + \varepsilon\partial_\mu u(\mu)$, yield the same solution, i.e., trajectory $u(k)$ in theory space. The very same argument also applies to solutions $Zij(k)$ of the composite-operator FRGE provided that we transform the third initial condition as

$$\begin{aligned} Z_{ij}^{(R)'} &= Z_{ij}(\mu', \bar{u}(\mu')) \\ &= Z_{ij}\big(\mu+\varepsilon, \bar{u}^{(R)} + \varepsilon\partial_\mu \bar{u}(\mu)\big) \\ &= Z_{ij}^{(R)} + \varepsilon \frac{\mathrm{d}}{\mathrm{d}\mu} Z_{ij}(\mu, \bar{u}(\mu))\,. \end{aligned} \tag{9.47}$$

Consequently, we have

$$\begin{aligned} Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big) =& Z_{ij}\Big(k;\mu',\bar{u}^{(R)\prime},Z^{(R)'}\Big) \\ =& Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big) \\ &+ \varepsilon\,\partial_\mu Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big) \\ &+ \varepsilon \sum_l \left(\partial_\mu \bar{u}_l(\mu)\right) \frac{\partial}{\partial \bar{u}_l^{(R)}} Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big) \\ &+ \varepsilon \sum_{m,n} \left(\frac{\mathrm{d}}{\mathrm{d}\mu} Z_{mn}(\mu,\bar{u}(\mu))\right) \frac{\partial}{\partial Z_{mn}^{(R)}} Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big)\,. \end{aligned} \tag{9.48}$$

[3]The initial condition depends on $\bar{u}$ because the $Z_{ij}^{(R)}$ are dimensionful. This is equivalent to studying the dependence of $\Gamma'_k[g,\bar{g};\varepsilon]$ given by Eq. (9.25) on this initial condition [54].

Therefrom follows directly the following linear partial differential equation describing the dependence of solution on its initial conditions:

$$\begin{aligned}&\Bigg\{\mu\partial_\mu + \sum_l \bar{\beta}_l\left(\bar{u}^{(R)};\mu\right)\frac{\partial}{\partial\bar{u}_l^{(R)}}\\&\qquad + \sum_{m,n}\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_{mn}(\bar{u}(\mu);\mu)\right)\frac{\partial}{\partial Z_{mn}^{(R)}}\Bigg\}Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big) = 0\,,\end{aligned} \tag{9.49}$$

respectively,

$$\begin{aligned}&\Bigg\{\mu\partial_\mu + \sum_l \bar{\beta}_l\left(\bar{u}^{(R)};\mu\right)\frac{\partial}{\partial\bar{u}_l^{(R)}}\\&\quad + \sum_{m,n}\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_{mn}(\bar{u}(\mu);\mu)\right)\frac{\partial}{\partial Z_{mn}^{(R)}}\Bigg\}\left[\mathscr{O}_i\right]_k[g,\bar{g}](x)\big(\mu,\bar{u}^{(R)},Z^{(R)}\big) = 0\,.\end{aligned} \tag{9.50}$$

Furthermore, let us assume that the $Z^{(R)}$-dependence of the parametric solution decouples according to

$$Z_{ij}\big(k;\mu,\bar{u}^{(R)},Z^{(R)}\big) = \sum_l Z_{il}^{(R)}\tilde{Z}_{lj}\big(k;\mu,\bar{u}^{(R)}\big)\ . \tag{9.51}$$

Therewith we have that

$$\begin{aligned}&\frac{\partial}{\partial Z_{mn}^{(R)}}\left[\mathscr{O}_i\right]_k[g,\bar{g}](x)\left(\mu,\bar{u}^{(R)},Z^{(R)}\right)\\&\qquad = \frac{\partial}{\partial Z_{mn}^{(R)}}\sum_{l,j} Z_{il}^{(R)}\tilde{Z}_{lj}\big(k;\mu,\bar{u}^{(R)}\big)\,\mathscr{O}_j[g,\bar{g}](x)\\&\qquad = \delta_{mi}\sum_j \tilde{Z}_{nj}\big(k;\mu,\bar{u}^{(R)}\big)\,\mathscr{O}_j[g,\bar{g}](x)\\&\qquad = \delta_{mi}\sum_l\left(Z^{(R)^{-1}}\right)_{nl}\left[\mathscr{O}_l\right]_k[g,\bar{g}](x)\big(k;\mu,\bar{u}^{(R)}\big)\ ,\end{aligned} \tag{9.52}$$

and we can rewrite Eq. (9.50) as, using that $Z^{(R)} = Z(\bar{u}(\mu);\mu)$,

$$\begin{aligned}&\Bigg\{\mu\partial_\mu\delta_{il} + \sum_m \bar{\beta}_m\left(\bar{u}^{(R)};\mu\right)\frac{\partial}{\partial\bar{u}_m^{(R)}}\delta_{il}\\&\quad + \sum_m\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_{im}(\bar{u}(\mu);\mu)\right)Z_{ml}^{-1}(\bar{u}(\mu);\mu)\Bigg\}\left[\mathscr{O}_l\right]_k[g,\bar{g}](x)\big(\mu,\bar{u}^{(R)},Z^{(R)}\big) = 0\,.\end{aligned} \tag{9.53}$$

Here, we can employ the definition of the anomalous-dimension matrix

$$\bar{\gamma}(k) := Z^{-1}(k)\partial_t Z(k) \quad \Leftrightarrow \quad \bar{\gamma}(\bar{u}(k);k) = Z^{-1}(\bar{u}(k);k)\frac{\mathrm{d}}{\mathrm{d}t}Z(\bar{u}(k);k)\,, \tag{9.54}$$

which yields the equation

$$\boxed{\begin{aligned}&\Bigg\{\mu\partial_\mu I + \sum_m \bar{\beta}_m(\bar{u}^{(R)};\mu)\frac{\partial}{\partial \bar{u}_m^{(R)}}I \\ &\qquad + Z^{(R)}\,\bar{\gamma}(u^{(R)};\mu)\,Z^{(R)^{-1}}\Bigg\}[\mathscr{O}]_k[g,\bar{g}](x)\big(\mu,\bar{u}^{(R)},Z^{(R)}\big) = 0\,.\end{aligned}} \tag{9.55}$$

(Written in matrix notation with I the identity matrix.) If we consider this equation in the fixed-point regime for $k,\mu \to \infty$, note that the dimensionful beta functions behave as

$$\bar{\beta}_m(\bar{u}^{(R)};\mu)\Bigg|_{\substack{\mu\to\infty \\ \bar{u}^{(R)}=\bar{u}^*}} = d_m \bar{u}_m^*\,, \tag{9.56}$$

where $\bar{u}_m^* = k^{d_m}u_m^*$ denotes the dimensionful "fixed point".

In a next step, let us assume that the composite operators are independent of the spacetime point x, but rather depend on a characteristic (yet arbitrary) length scale r. For example, the composite operator could describe the volume of an n-sphere in which case r would be its radius. On this basis, we consult another "scaling" equation, next to Eq. (9.55), purely derived from dimensional analysis. Therefore, for the canonical mass dimensions let us apply the conventions in which the metric is dimensionless (cf. appendix A). Then, by expressing the renormalized composite operator $[\mathscr{O}_i]_k[g,\bar{g}](r)\big(\mu,\bar{u}^{(R)},Z^{(R)}\big)$ in terms of its dimensionless analog and thereby taking into account the canonical mass dimensions $[\mathscr{O}_i] = -d_i$, $[Z_{ij}] = -d_i + d_j$ and $[\bar{u}_i] = d_i$, one obtains the identity

$$\begin{aligned}&\Bigg\{\mu\partial_\mu - r\partial_r + \sum_m d_m \bar{u}_m^{(R)}\frac{\partial}{\partial \bar{u}_m^{(R)}} \\ &\quad + \sum_{m,n}(-d_m + d_n)Z_{mn}^{(R)}\frac{\partial}{\partial Z_{mn}^{(R)}} + d_i\Bigg\}[\mathscr{O}_i]_k[g,\bar{g}](r)\big(\mu,\bar{u}^{(R)},Z^{(R)}\big) = 0\,.\end{aligned} \tag{9.57}$$

With the same trick as above, we can trade the partial derivative with respect to $Z^{(R)}_{mn}$ for an inverse matrix $\left(Z^{(R)}\right)^{-1}$, which yields the equation (in matrix notation)

$$\boxed{\begin{aligned}\Big\{\mu\partial_\mu I - r\partial_r I + \sum_m d_m \bar{u}^{(R)}_m \frac{\partial}{\partial \bar{u}^{(R)}} I \\ + Z^{(R)} D \left(Z^{(R)}\right)^{-1}\Big\}[\mathscr{O}]_k[g,\bar{g}](r)\big(\mu, \bar{u}^{(R)}, Z^{(R)}\big) = 0\,,\end{aligned}} \tag{9.58}$$

where the matrix D is given by $D_{ij} = d_i \delta_{ij}$.

Bringing together equations (9.55) and (9.58) in the fixed-point regime for $k, \mu \to \infty$ and $\bar{u}^{(R)} = \bar{u}^*$ directly gives the scaling relation (again in matrix notation)

$$\boxed{\Big\{r\partial_r I + Z^{(R)}\left[-D + \gamma(\bar{u}^{(R)};\mu)\right]\left(Z^{(R)}\right)^{-1}\Big\}[\mathscr{O}]_k[g,\bar{g}](r)\big(\mu, \bar{u}^{(R)}, Z^{(R)}\big)\Big|_{\substack{k,\mu\to\infty \\ \bar{u}^{(R)}=\bar{u}^*}} = 0\,.} \tag{9.59}$$

This equation tells us that the eigenvalues of the matrix

$$-D + \gamma(\bar{u}^{(R)};\mu)\Big|_{\substack{k,\mu\to\infty \\ \bar{u}^{(R)}=\bar{u}^*}} \tag{9.60}$$

are the full scaling dimensions of the operators

$$A^{-1}[\mathscr{O}]_k[g,\bar{g}](r)\big(\mu, \bar{u}^{(R)}, Z^{(R)}\big)\Big|_{\substack{k,\mu\to\infty \\ \bar{u}^{(R)}=\bar{u}^*}} \tag{9.61}$$

in the UV, whereby the matrix A diagonalizes said matrix by an similarity transformation [54].

Particularly special is the case $n = 1$ of a single composite operator. With $d_i \equiv d$ and $\bar{\gamma}_{ij}(\bar{u}(k);k) \equiv \gamma(u(k))$, the operator $[\mathscr{O}]_k[g,\bar{g}](r)$ (which has canonical mass dimension $-d$) scales in the UV as

$$\boxed{[\mathscr{O}]_{k\to\infty}[g,\bar{g}](r) \sim r^{d-\gamma(u^*)}\,.} \tag{9.62}$$

Part 3

Geometric operators in quantum gravity

CHAPTER 10

Summary of Part 3

As explained in Part 2, in a theory of quantum gravity it is rather natural to consider the behavior of geometric quantities at the "quantum level", i.e., in the ultraviolet regime. Part 3 is mainly devoted to study this behavior of the volume V_n of an n-dimensional submanifold embedded into the d-dimensional spacetime for continuum-based approaches towards quantum gravity. In Chapter 9, we had shown that we can do so by renormalizing the geometric operator as a composite operator, which leads to a specific scaling relation in the ultraviolet. Before we summarize the details of the calculations conducted in Part 3, let us firstly demonstrate how such a scaling relation should be interpreted.

(A) Illustration of the concept of a fractional scaling dimension. We say that a function $f(r)$ that depends on some length scale r has *scaling dimension a* if it solves the differential equation

$$\left(r\frac{\partial}{\partial r}+a\right)f(r)=0\,.$$

Surely, its solution reads $f(r)\sim r^{-a}$ which self-explains the term "scaling dimension". For example, the classical n-dimensional volume V_n has the scaling dimension $-n$ as $V_n\sim r^n$. In Part 3, we will see that this scaling relation does not hold in the ultraviolet regime and the main results will be the precise values of the corrected scaling dimensions for certain approximate settings.

The scaling dimension (or equivalently, its negative, up to definition) can be synonymously referred to as the *fractal dimension.* The concept of the fractal dimension, or more general of a fractal, has been worked out in full mathematical rigour [128]. Generally, there exist various ways of defining a fractal dimension, e.g., the Hausdorff dimension, the spectral dimension, the box counting dimension, et cetera. These are not necessarily identical, however they are for certain "classical fractals". Often, the scaling dimension is employed to approximate the Hausdorff dimension, which do not necessarily coincide, as well. Here, in

Part 3, we have at hand a specific scaling relation that we wish to interpret which is why we do not need this level of mathematical rigour and will stick with identifying the fractal dimension with the scaling dimension.

To illustrate the concept of a fractal dimension, let us consider the general scaling relation $V = r^{-a}$ with a being a negative real number. Then V can be thought of as the length of a curve, the area of the surface or generally some volume. If we rescale the length r by a factor $0 < \varepsilon < 1$ the value of V will become smaller by a factor $1/N$, i.e., $V/N = (\varepsilon r)^{-a}$. We conclude that

$$N = \varepsilon^a \quad \Leftrightarrow \quad a = \frac{\ln N}{\ln \varepsilon}\,.$$

As an intuitive example, let us consider the area of a square whose scaling dimension trivially is $a = -2$ as $V = r^2$. If we rescale the side of the square by a factor $\varepsilon = 1/3$, then consequently its area will become smaller by a factor $1/N = 1/9$. In other words, we need the number of $N = \varepsilon^a = (1/3)^{-2} = 9$ rescaled squares in order to cover the area of the original square. If we iterate the process of rescaling by a factor $1/3$ once more we will arive at rescaling the sides of the original square by a factor $\varepsilon = 1/9$. Thus, $N = \varepsilon^a = (1/9)^{-2} = 81$ squares are required to cover the original square. Furthermore, this trivial example also illustrates the concept of *self-similarity* that many fractals exhibit.

It is instructive to use this view on the scaling dimension in order to construct first simple fractals with a non-integer scaling dimension, i.e., *fractional* scaling dimension. Therefore, let us consider the *Cantor set* with a scaling dimension of -0.6309 and the von Koch curve with a scaling dimension of -1.2619. The former is very simple to construct step by step. One starts with a line and then removes its middle third, such that one is left with two lines, $N = 2$, of length $\varepsilon = 1/3$. Then one reiterates this process and removes the middle third of these two lines, such that one is left with four lines, $N = 4$, of length $\varepsilon = 1/9$. The full Cantor set then is obtained by reiterating this process over and over. Thereby, its scaling dimension is constant, namely $a = \ln N/\ln\varepsilon \approx \ln 2/\ln(1/3) \approx \ln 4/\ln(1/9) \approx -0.6309$. Hence, loosely speaking, the Cantor set is a line that becomes “emptier and emptier” as with each iteration step more and more parts are cut from it.

In a similar way we can construct a line that “grows” into two-dimensional

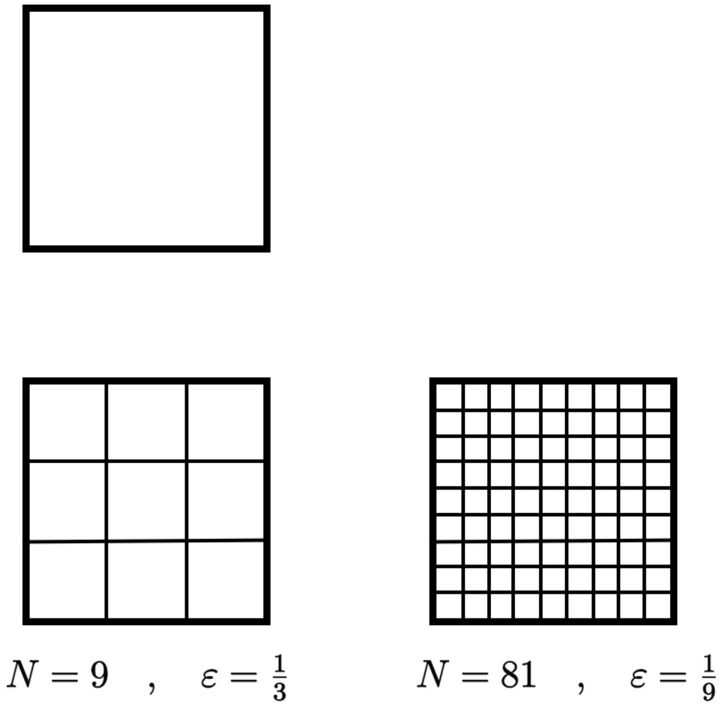

FIGURE 10.1. Rescaling the sides of a square by a factor $0 < \varepsilon < 1$ leads to $N = 1/\varepsilon^2$ squares.

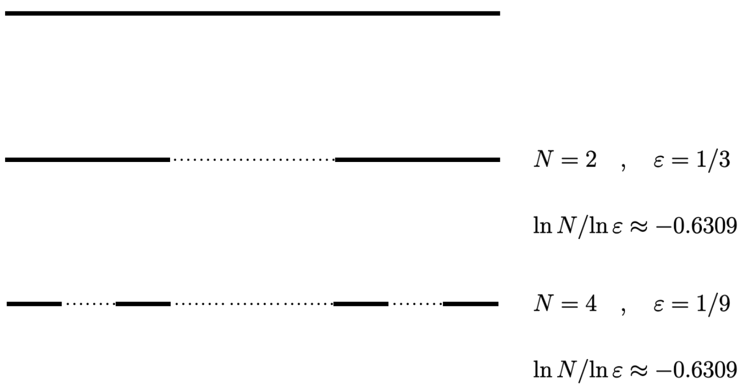

FIGURE 10.2. Construction of the Cantor set which has scaling dimension -0.6309.

space. Again, one starts with a line and removes its middle third. This missing middle third is then augmented by the two sides of an isosceles triangle that would close if the missing middle side was still there. Thus, one has arrived at $N = 4$ lines with length $\varepsilon = 1/3$ of the original line. In turn, if we repeat this process, we arrive at $N = 16$ lines with length $\varepsilon = 1/9$ of the original line. If we repeated this process over and over we would arrive at the von Koch curve. This curve has a constant scaling dimension of $a = \ln N/\ln \varepsilon \approx \ln 4/\ln(1/3) \approx \ln 16/\ln(1/9) \approx -1.2619$, i.e., the curve "grows" into two-dimensional space with each step. We emphasize that it is even possible to construct curves with a scaling dimension of $a = -2$, i.e., curves that fully cover two-dimensional space. (An example is the Peano curve.)

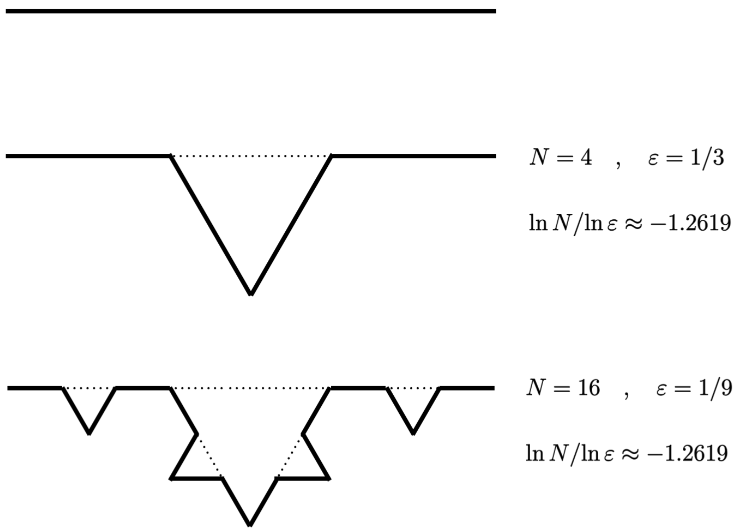

FIGURE 10.3. Construction of the von Koch curve which has scaling dimension -1.2619.

The fractal dimension can also be illustrated by random fractals appearing in Nature. For instance, in a famous paper, B. Mandelbrot discussed the length of the British coast [129], whereby he assumed that the length of a coastline is approximated by means of N measuring rods that cover the coastline. If we then rescale the measuring rods by a factor $0 < \varepsilon < 1$, surely the number N of rods required to cover the coastline will increase, until ultimately, for the limit $\varepsilon \to 0$, the length of the coastline becomes infinite. This counterintuitive problem is "solved" by the concept of the fractal dimension: The fractal dimension of the British coast can be estimated by -1.25 which tells us how much the measured length of the coastline increases as we decrease the lengths of the measuring rods.

To give further examples, the surface of Broccoli has an estimated fractal dimension of -2.7 [130] or the surface of the alveoli of the human lung have an estimated fractal dimension of -2.97 [131] which means that the surface of the lung essentially is three-dimensional.

(B) The anomalous scaling of the volume operator in the ultraviolet. In Chapter 9 we outlined how to renormalize composite operators within the framework of the functional renormalization group for quantum gravity. In practise, to calculate the renormalization of a given set of bare composite operators, two approximations are necessary, which we referred to as *first* and *second truncations*. The first truncation is the truncation of theory space, the space of invariant functionals that define the theory, with a corresponding ansatz for the effective average action Γ_k. The second truncation is an ansatz for a truncated basis of bare composite operators. Generally, with such two given approximations, the renormalization effects of each of these composite operators will intertwine with those of the others, i.e., "mix".

The study of composite operators in quantum gravity via this framework constitutes a brand new line of research. To the initiation of this line of research, the author contributed with the publications [28–30], on which Part 3 is based upon. Of course, when one begins to undertake the study of a new line of research, it is suggestive to study its most basic settings. Here, in Part 3, we therefore restrict the approximations given by the *second truncation* to *non-mixing ansätze*, i.e., we consider only a single composite operator (in other words, a one-dimensional basis of composite operators). For this composite operator, we always choose *geometric operators*, i.e., those which are independent of the ghost fields and do only depend on the background metric and the metric fluctuation. Mainly, we consider the volume operator which is given in form of the volume of an n-dimensional submanifold that is embedded into the d-dimensional spacetime. Also regarding the *first truncation*, we consider rather simple approximations: In Chapter 11 we consider the Einstein-Hilbert truncation, while in Chapter 12 we consider higher-derivative gravity.

Subsequently, let us consider the renormalization of the volume operator V_n within this framework. In Chapter 9 we demonstrated that its renormalization is given by its anomalous dimension $\gamma_n(u(k))$, where $u(k)$ is the set of running couplings that parametrize the *first* truncation, e.g., in case of the Einstein-Hilbert truncation $u(k)$ contains the running of Newton's constant and of the cosmological constant. This anomalous dimension can be calculated by field-theorical methods but can be interpreted in geometric terms. Namely, let r be a length scale on which the volume V_n depends (e.g., radius of the n-sphere). Then in the ultraviolet, for $k \to \infty$, one has the scaling relation (9.62),

$$V_n(r) \sim r^{n-\gamma_n(u^*)} .$$

Here, u^* is the non-Gaussian gravitational fixed point obtained with the ansatz of the first truncation. Moreover, n is the negative of the canonical mass dimension of the volume operator, $[V_n] = -n$. Thus, the volume classically scales as $V_n(r) \sim r^n$, which is why it is n-dimensional. However, the above scaling relation tells us that what was classically an n-dimensional subvolume of spacetime will be distorted at ultraviolet scales, and effectively be $n - \gamma_n(u^*)$ dimensional (i.e., has scaling dimension $-n + \gamma_n(u^*)$ in our conventions). In other words, the Asymptotic Safety scenario for quantum gravity predicts that classical volumes become genuinely *fractal* at the quantum level, whereby we observe the phenomenon of either a dimensional increase or a dimensional reduction in the ultraviolet, depending on the value of the anomalous dimension: If it is positive we observe a dimensional reduction, while if it is negative we observe a dimensional increase. In terms of the above illustrative examples, this phenomenon can be visualized as follows: Say, we consider a straight line, which classically is one-dimensional. If its anomalous dimension is positive in the ultraviolet, its scaling dimension will be reduced and thus have a magnitude of below one, similar to the Cantor set. If its anomalous dimension is negative, the magnitude of its scaling dimension will increase and thus be above one, similar to the von Koch curve.

Furthermore, it is rather natural to express the scaling of a quantum volume in terms of a quantum length, which, following the above scaling relation, is given by $V_n \sim V_1{}^{(n-\gamma_n(u^*))/(1-\gamma_1(u^*))}$, where V_1 is the length operator.

In Chapter 11 we explicitly calculate the anomalous dimension γ_n within the Einstein-Hilbert truncation. For an n-dimensional volume emdedded into a d-dimensional spacetime, it is given by Eq. (11.29) which is one of the main results of this chapter. Importantly, this formula does not depend on the specific shape of the volume, but rather on its classical dimension n. Additionally, we obtain a formula for the corresponding one-loop approximation of the anomalous dimension which is given by Eq. (11.31). In a next step, we numerically evaluate these formulae in $d = 4$ and $d = 3$ spacetime dimensions at the ultraviolet fixed-point which leads to the values of the anomalous dimension presented in Tables 1 and 2 of this chapter. Strikingly, all obtained fixed-point values for γ_n are positive such that in four as well as three spacetime dimensions we observe an *dimensional reduction of volumes in the ultraviolet*. The values of the full anomalous dimension are somewhat larger than their one-loop approximations. More precisely, for $d = 4$ one finds the fixed-point values of the full anomalous dimensions $\gamma_{n=4} \approx 4$ and $\gamma_n \approx n - 1$ for $n = 1, 2, 3$. The first value means that the effective scaling dimension of spacetime is approximately zero which suggests that spacetime might be emptier at small distance scales than one would naively expect. The other values imply that the length of a curve, the area of a surface and the three-volume all scale as approximately one-dimensional objects at small distance scales.

Lastly, we also calculate the one-loop anomalous dimensions of V_n in the presence of matter fields. Here, the sign of the anomalous dimension depends on the specific matter content.

However, it must be emphasized that these results are still of an explorative character, given the rough approximations employed to evaluate the composite-operator FRGE. Generally, one would expect severe corrections to these results when repeating the analysis for more elaborate approximations for the first and second truncation.

Chapter 11 closes with the discussion of the renormalization of a further geometric operator, the geodesic length. Thereby, the discussion of boundary conditions plays a crucial role because of the dependence on the metric. With the approximations assumed here, it turns out that the anomalous dimension of the geodesic length is the same as that of the length of a curve. For more refined approximations, especially for mixing ansätze, we however expect these two anomalous dimensions to differ from each other.

Additionally, we outline an argument to show that the geodesic ball in this framework does not renormalize which implies that the Hausdorff dimension of spacetime is d.

In Chapter 12 we repeat the first analysis of the previous chapter for the framework of higher-derivative gravity at one-loop in four spacetime dimensions. I.e., we calculate the one-loop approximation of the anomalous dimension γ_n of the volume operator V_n by means of the one-loop approximation of the composite-operator FRGE, whereby we specify the first truncation to that of higher-derivative gravity, respectively Weyl-squared gravity. The resulting formulae for γ_n are given by Eq. (12.23) and Eq. (12.26), and constitute the main results of this chapter. In the physical gauge, the anomalous dimension vanishes because both, higher-derivative as well as Weyl-squared gravity, are asymptotically free in the coupling parametrizing the Weyl tensor. However, slightly away from the fixed-point regime where scale invariance still is realized, the anomalous dimension does not vanish and its sign depends on the value of the couplings. Thus we observe an effective dimensional increase as well as a reduction of volumes in the ultraviolet which seems to be a distinct feature of higher derivative gravity. Lastly, we also discuss the fixed-point values of the anomalous dimension in $d = 4 - \varepsilon$ spacetime dimensions. These are given in Table 12.3 and for $\varepsilon > 0$ we mostly observe the effect of a dimensional increase.

CHAPTER 11

Geometric operators in the Asymptotic Safety scenario for quantum gravity

Executive summary. We employ the composite-operator FRGE in order to calculate the anomalous dimension γ_n of an n-dimensional volume that is embedded into the d-dimensional quantized spacetime. Thereby, the first truncation is that of the Einstein-Hilbert truncation and the second truncation amounts to the non-mixing ansatz of the sole volume operator. The resulting formula for γ_n is evaluated numerically and its values at the gravitational fixed point are calculated. It is shown that these quantum corrections to the scaling dimension of the volume operator result in an effective dimensional reduction in the ultraviolet regime. Moreover, we show that within these approximations the anomalous dimension of the geodesic length is the same as that of the length of an ordinary curve. Additionally, we show that the geodesic ball in this setting does not renormalize, which is why the Hausdorff dimension of spacetime is given by its classical dimension.
What is new? All results of this chapter represent novel research results.
Based upon: References [28, 29].

11.1. The composite-operator FRGE on the basis of the Einstein-Hilbert truncation

In this chapter, we consider geometric composite operators of the form $\mathscr{O}[g, g]$, i.e., spacetime-point independent operators at vanishing quantum fluctuation such that we the background metric reads $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$. To study the renormalization behavior of the bare operator $\mathscr{O}[g, g]$, we restrict this analysis to a one-dimensional second truncation. This means that the renormalized operator is given by the *non-mixing ansatz*

$$[\mathscr{O}]_k[g,g] = Z(k)\mathscr{O}[g,g]\,. \tag{11.1}$$

Following Section 9.3, the k-dependence of $[\mathscr{O}]_k[g,g]$ is governed by the composite-operator FRGE (9.33) at $g \equiv \bar{g}$,

$$\begin{aligned}\bar{\gamma}(k)\mathscr{O}[g,g] = -\frac{1}{2}\operatorname{Tr}_{ST^2}\Bigg[\Big(\partial_t \mathscr{R}_k^{\text{grav}}[g]\Big)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,g] + \mathscr{R}_k^{\text{grav}}[g]\Big)^{-1} \\ \times \mathscr{O}^{(2)}[g,g]\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,g] + \mathscr{R}_k^{\text{grav}}[g]\Big)^{-1}\Bigg]\,,\end{aligned} \tag{11.2}$$

where $\bar{\gamma}(k) \coloneqq Z(k)^{-1}\partial_t Z(k)$ is the *anomalous dimension* of the renormalized operator. To solve this composite-operator FRGE, we will furthermore specify the first truncation of theory space to the Einstein-Hilbert truncation we had expounded in Chapter 7. In the Einstein-Hilbert truncation, we can still distinguish between its definining two basis elements (operators) $\int \mathrm{d}^d x\sqrt{g}$ and $\int \mathrm{d}^d x \sqrt{g}R$ when specifying the metric $g_{\mu\nu}$ to that of a maximally symmetric spacetime. Hence, *in this chapter the metric $g_{\mu\nu}$ will be that of a maximlly symmetric spacetime*. Furthermore, following Chapter 7, we employ the ansatz (7.2) for the gravitational EAA $\Gamma_k[g,\bar{g}]$. On a maximally symmetric background, its 11-component of the Hessian at $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$ is given by Eq. (**??**),

$$\begin{aligned}\Big[\big(\Gamma_k^{(2)}\big)_{11}[g,g] + \mathscr{R}_k^{\text{grav}}[g]\Big]^{-1} &= \frac{1}{\kappa^2 Z_{Nk}}\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g]\right)\left[\mathscr{A}_k(-D^2) + c_I R\right]^{-1} \\ &\quad - \frac{1}{\kappa^2 Z_{Nk}}\frac{2}{d-2}\mathbb{P}_{\text{tr.}}[g]\left[\mathscr{A}_k(-D^2) + c_{\text{trace}}R\right]^{-1}\,.\end{aligned} \tag{11.3}$$

Lastly, we specify the tensor structure of the gravitational cutoff operator to Eq. (7.6), i.e., the cutoff operator reads

$$\begin{aligned}&\mathscr{R}_k^{\text{grav}}[g] = \mathscr{Z}_k^{\text{grav}}[g]\,k^2 R^{(0)}(-D^2/k^2)\,, \\ \text{with}\quad &\mathscr{Z}_k^{\text{grav}}[g] = \kappa^2 Z_{Nk}\left[\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g]\right) - \frac{d-2}{2}\mathbb{P}_{\text{tr.}}[g]\right]\,.\end{aligned} \tag{11.4}$$

With these specifications, we composite-operator FRGE we are going to employ becomes

$$
\begin{aligned}
&\bar{\gamma}(k)\mathscr{O}[g,g] \\
&\quad = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[\kappa^2\partial_t\Big(Z_{Nk}\,k^2\,R^{(0)}(-D^2/k^2)\Big)\left[(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g])-\frac{d-2}{2}\mathbb{P}_{\mathrm{tr.}}[g]\right] \\
&\qquad\qquad \times\Bigg(\frac{1}{\kappa^2 Z_{Nk}}\left(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g]\right)\left[\mathscr{A}_k(-D^2)+c_I R\right]^{-1} \\
&\qquad\qquad\qquad -\frac{1}{\kappa^2 Z_{Nk}}\frac{2}{d-2}\mathbb{P}_{\mathrm{tr.}}[g]\left[\mathscr{A}_k(-D^2)+c_{\mathrm{trace}}R\right]^{-1}\Bigg) \\
&\qquad \times\mathscr{O}^{(2)}[g,g]\Bigg(\frac{1}{\kappa^2 Z_{Nk}}\left(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g]\right)\left[\mathscr{A}_k(-D^2)+c_I R\right]^{-1} \\
&\qquad\qquad -\frac{1}{\kappa^2 Z_{Nk}}\frac{2}{d-2}\mathbb{P}_{\mathrm{tr.}}[g]\left[\mathscr{A}_k(-D^2)+c_{\mathrm{trace}}R\right]^{-1}\Bigg)\Bigg]
\end{aligned}
\tag{11.5}
$$

Making use of the cyclicity of the trace, we can rewrite this equation into

$$
\boxed{
\begin{aligned}
\bar{\gamma}(k)\mathscr{O}[g,g] = -\,\mathrm{Tr}_{ST^2}\Bigg[\mathscr{O}^{(2)}[g,g]\Bigg(&\frac{1}{\kappa^2 Z_{Nk}}\big(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g]\big)\left[\mathscr{A}_k(-D^2)+c_I R\right]^{-1} \\
&\times\mathscr{N}_k(-D^2)\big[\mathscr{A}_k(-D^2)+c_I R\big]^{-1} \\
&-\frac{1}{\kappa^2 Z_{Nk}}\frac{2}{d-2}\mathbb{P}_{\mathrm{tr.}}[g]\left[\mathscr{A}_k(-D^2)+c_{\mathrm{trace}}R\right]^{-1} \\
&\times\mathscr{N}_k(-D^2)\left[\mathscr{A}_k(-D^2)+c_{\mathrm{trace}}R\right]^{-1}\Bigg)\Bigg].
\end{aligned}
}
\tag{11.6}
$$

11.2. The volume of an n-dimensional submanifold

11.2.1. *Definition of the volume operator $V_n[g,g]$*

The first operator whose anomalous dimension we are going to calculate and analyze with help of the FRGE (11.6) is the volume of an n-dimensional submanifold N that is embedded in the d-dimensional spacetime manifold M (of course,

$N \subset M$ and $n \leq d$ holds). Here, we can already observe that this operator is *not a true observable* – neither in classical gravity nor in any approach towards quantum gravity – because embedded submanifolds break diffeomorphism invariance. However, as explained in Section 9.1, the study of the renormalization behavior of such geometric operators is natural to a gravitational theory and may point out some of its general geometric features. Furthermore, similar geometric operators have been defined in other approaches to quantum gravity, e.g. loop quantum gravity or causal dynamical triangulations [119, 132]. Thus, studying the renormalization behavior of the volume of an n-dimensional submanifold paves the way to a possible comparison of the predictions of various quantum gravity models.

Here, for calculational simplification, we want to work in local expressions – so let us at first clarify our notation: To derive local expressions, let $x = (x^1, \cdots, x^d) : U \subset M \to U' \subset \mathbb{R}^d$ be a chart of M and $u = (u^1, \cdots, u^n) : \tilde{U} \subset N \to \tilde{U}' \subset \mathbb{R}^n$ be a chart of N. By pulling back the metric g on M with the inclusion map $\imath : N \hookrightarrow M$, a metric $\imath^* g$ on N is induced. For $q \in U \cap \tilde{U}$ this induced metric is locally given by

$$\begin{aligned} g_{ab}(q) &\coloneqq (\imath^* g)_q (\partial/\partial u^a|^q, \partial/\partial u^b|^q) \\ &= g_q \left(\mathrm{d}\imath_q(\partial/\partial u^a|^q), \mathrm{d}\imath_q(\partial/\partial u^b|^q) \right) \\ &= g_{\mu\nu}(\imath(q))\, \mathrm{d}x^\mu|^q (\mathrm{d}\imath_q(\partial/\partial u^a|^q))\ \mathrm{d}x^\nu|^q \big(\mathrm{d}\imath_q(\partial/\partial u^b|^q)\big) \\ &= g_{\mu\nu}(q) \frac{\partial}{\partial u^a|^q}(x^\mu \circ \imath) \frac{\partial}{\partial u^b|^q}(x^\nu \circ \imath)\,. \end{aligned} \tag{11.7}$$

As $N \subset M$ is an embedded submanifold the inclusion map $\imath : N \hookrightarrow M$ is an immersion, i.e., $\mathrm{d}\imath_q : T_q N \to T_q M$ is injective for all $q \in M$. This means that we can identify the tangent space $T_q N$ with the subset $\mathrm{d}\imath_q(T_q N) \subset T_q M$. Consequently, we can also identify the canonical vector fields $(\partial/\partial u^1, \cdots, \partial/\partial u^n) : \tilde{U} \subset N \to TN$ as vector fields on $\tilde{U}$ mapping to TM such that we can write

$$\begin{aligned} g_{ab}(q) &= g_{\mu\nu}(q)\, \mathrm{d}x^\mu|^q (\partial/\partial u^a|^q)\ \mathrm{d}x^\nu|^q \big(\partial/\partial u^b|^q\big) \\ &= g_{\mu\nu}(q) \frac{\partial x^\mu}{\partial u^a}\bigg|^q \frac{\partial x^\nu}{\partial u^b}\bigg|^q . \end{aligned}$$

What would make our calculations much more handier now is if we detached this formula for the induced metric from the submanifold $N \subset M$. That is, we want to replace the q-dependence by some dependence on n-dimensional coordinates. Therefore recall that the canonical tangent vectors of T_qM are defined by $\partial x^\mu/\partial u^a|^q := \partial_a(x^\mu \circ u^{-1})_{u(q)}$ where ∂_a denotes the a-th partial derivative in $\mathbb{R}^n$. This definition gives us the hint to notationally replace the coordinates $u(q) \in \mathbb{R}^n$ by $u \in \mathbb{R}^n$ and the local parametrization $x \circ u^{-1} : \mathbb{R}^n \to \mathbb{R}^d$ by $x : \mathbb{R}^n \to \mathbb{R}^d$. Rewriting the formula for the induced metric as

$$g_{ab} \circ u^{-1} \circ u(q) = g_{\mu\nu} \circ x^{-1} \circ x \circ u^{-1} \circ u(q)\, \partial_a(x^\mu \circ u^{-1})_{u(q)}\, \partial_b(x^\nu \circ u^{-1})_{u(q)}$$

it becomes clear that we furthermore must notationally replace $g_{ab} \circ u^{-1} : \mathbb{R}^n \to \mathbb{R}$ by $g_{ab} : \mathbb{R}^n \to \mathbb{R}$ and $g_{\mu\nu} \circ x^{-1} : \mathbb{R}^d \to \mathbb{R}$ by $g_{\mu\nu} : \mathbb{R}^d \to \mathbb{R}$. Implementing these four substitutions we arrive at

$$\boxed{g_{ab}(u) = g_{\mu\nu}(x(u)) \frac{\partial x^\mu(u)}{\partial u^a} \frac{\partial x^\nu(u)}{\partial u^b}\,.} \tag{11.8}$$

To summarize the rederivation of this formula, we simply expressed the induced metric in terms of real analysis with the help of charts. Additionally, we will denote by $g(u)$ the $n \times n$-Matrix corresponding to $g_{ab}(u)$.

Instead of studying the full volume $\int_N \omega_N$ of the submanifold N, where ω_N denotes the volume form given by the induced metric, we will restrict ourselves to the integration domain $U \cap \tilde{U}$, i.e., we will consider the operator

$$\boxed{V_n[g, \bar{g}] \equiv V_n[g, \bar{g}](U \cap \tilde{U}) := \int_{U \cap \tilde{U}} \omega_N = \int_{\tilde{U}''} \mathrm{d}^n u \sqrt{\det g(u)}\,,} \tag{11.9}$$

where $\tilde{U}'' := u(U \cap \tilde{U}) \subset \mathbb{R}^n$, and set additionally $g \equiv \bar{g}$.

11.2.2. *Calculation of the Hessian of* $V_n[g, g]$

This local volume is an operator we can easily study thanks to (11.8). In order to calculate its Hessian given by Eq. (6.47), i.e.,

$$\boxed{\langle x, \mu, \nu | V_n^{(2)}[g, \bar{g}] | y, \rho, \sigma \rangle = I[\bar{g}]_{\rho\sigma\alpha\beta} \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}} \int_{\tilde{U}''} \mathrm{d}^n u \, \frac{\delta^2 \sqrt{\det g(u)}}{\delta g_{\mu\nu}(x) \delta g_{\alpha\beta}(y)}\,,} \tag{11.10}$$

we express $g_{\mu\nu}$ in terms of g_{ab} and use the chain rule such that we can rewrite a functional derivative with respect to $g_{\mu\nu}$ into a functional derivative with respect to g_{ab},

$$\begin{aligned}\frac{\delta}{\delta g_{\mu\nu}(y)} &= \int \mathrm{d}^n u \frac{\delta g_{ab}(u)}{\delta g_{\mu\nu}(y)} \frac{\delta}{\delta g_{ab}(u)} \\ &= \int \mathrm{d}^n u \frac{\partial x^\mu(u)}{\partial u^a} \frac{\partial x^\nu(u)}{\partial u^b} \delta^{(d)}(x(u)-y) \frac{\delta}{\delta g_{ab}(u)} \,.\end{aligned}$$

Therewith, we can calculate the Hessian step by step, starting with the first functional derivative of the integrand $\sqrt{\det g(u)}$ of (11.9),

$$\begin{aligned}\frac{\delta\sqrt{\det g(u)}}{\delta g_{\mu\nu}(y)} &= \frac{1}{2}\frac{1}{\sqrt{\det g(u)}} \int \mathrm{d}^n u' \frac{\partial x^\mu(u')}{\partial u'^a} \frac{\partial x^\nu(u')}{\partial u'^b} \delta^{(d)}(x(u')-y) \frac{\delta \det(g(u))}{\delta g_{ab}(u')} \\ &= \frac{1}{2}\frac{1}{\sqrt{\det g(u)}} \frac{\partial x^\mu(u)}{\partial u^a} \frac{\partial x^\nu(u)}{\partial u^b} \mathrm{adj}^T(g(u))_{ab} \delta^{(d)}(x(u)-y) \,.\end{aligned}$$

Here, we have used Jacobi's formula for the variation of the determinant,

$$\frac{\delta \det(g(u))}{\delta g_{ab}(u')} = \mathrm{adj}^T(g(u))_{ab} \delta^{(n)}(u-u') \,. \qquad (11.11)$$

where $\mathrm{adj}(A)$ denotes the adjunct of the (square) matrix A which is the transpose of its cofactor matrix. The adjunct matrix is related to the inverse matrix,[1] that we will denote with upper indices as usual, by a factor of the determinant,

$$g^{ab}(u) := (g(u)^{-1})_{ab} = \frac{1}{\det g(u)} \mathrm{adj}(g(u))_{ab} \,. \qquad (11.12)$$

As the adjunct of a symmeric matrix is symmetric as well, i.e., $\mathrm{adj}(g(u))_{ab} = \mathrm{adj}^T(g(u))_{ab}$, we can express the functional derivative of $\sqrt{\det g(u)}$ by means of the inverse g^{ab} as

$$\frac{\delta\sqrt{\det g(u)}}{\delta g_{\mu\nu}(y)} = \frac{1}{2}\sqrt{\det g(u)} \frac{\partial x^\mu(u)}{\partial u^a} \frac{\partial x^\nu(u)}{\partial u^b} g^{ab}(u) \delta^{(d)}(x(u)-y) \,. \qquad (11.13)$$

[1] It should be clarified that the formula (11.8) cannot be inverted in the sense that $x(u)$ cannot be uniquely solved for u. However, the inverse of $g(u)$ is naturally well-defined as $\det g(u) \neq 0$.

Next, it is straight forward to build the second functional derivative of $\sqrt{\det g(u)}$ using the product rule,

$$\begin{aligned}\frac{\delta^2\sqrt{\det g(u)}}{\delta g_{\alpha\beta}(z)\delta g_{\mu\nu}(y)} =& \frac{1}{2}\frac{\delta\sqrt{\det g(u)}}{\delta g_{\alpha\beta}(z)}\frac{\partial x^\mu(u)}{\partial u^a}\frac{\partial x^\nu(u)}{\partial u^b}g^{ab}(u)\delta^{(d)}(x(u)-y)\\ &+\frac{1}{2}\sqrt{\det g(u)}\frac{\partial x^\mu(u)}{\partial u^a}\frac{\partial x^\nu(u)}{\partial u^b}\frac{\delta g^{ab}(u)}{\delta g_{\alpha\beta}(z)}\delta^{(d)}(x(u)-y)\,.\end{aligned} \tag{11.14}$$

In the first term, we can use our result (11.13) for the first functional derivative while in the second term we can again use the chain rule to calculate the functional derivative of g^{ab},

$$\frac{\delta g^{ab}(u)}{\delta g_{\alpha\beta}(z)} = \int \mathrm{d}^n u' \frac{\partial x^\alpha(u')}{\partial u'^c}\frac{\partial x^\beta(u')}{\partial u'^d}\delta^{(d)}(x(u')-z)\frac{\delta(g^{-1}(u))_{ab}}{\delta g_{cd}(u')} \tag{11.15}$$

with

$$\begin{aligned}\frac{\delta(g^{-1}(u))_{ab}}{\delta g_{cd}(u')} &= -g^{ae}(u)\frac{\delta g_{ef}(u)}{\delta g_{cd}(u')}g^{fb}(u)\\ &= -(g^{ac}(u)g^{db}(u)+g^{ad}(u)g^{cb}(u))\delta^{(n)}(u-u')\,.\end{aligned}$$

Plugging in these results into the RHS of (11.14) we obtain the second functional derivative of $\sqrt{\det g(u)}$,

$$\begin{aligned}\frac{\delta^2\sqrt{\det g(u)}}{\delta g_{\alpha\beta}(z)\delta g_{\mu\nu}(y)} =& \frac{1}{4}\sqrt{\det g(u)}\frac{\partial x^\alpha(u)}{\partial u^c}\frac{\partial x^\beta(u)}{\partial u^d}\frac{\partial x^\mu(u)}{\partial u^a}\frac{\partial x^\nu(u)}{\partial u^b}\delta^{(d)}(x(u)-y)\\ &\times\delta^{(d)}(x(u)-z)\Big[g^{ab}(u)g^{cd}(u)-g^{ac}(u)g^{db}(u)-g^{ad}(u)g^{cb}(u)\Big]\,.\end{aligned} \tag{11.16}$$

11.2.3. *Calculation of the anomalous dimension* $\gamma_n(\lambda_k, g_k)$

After having calculated the Hessian $V_n^{(2)}[g, \bar{g}]$ we are ready to evaluate the RHS of the composite-operator FRGE (11.6),

$$\bar{\gamma}_n(k) V_n[g,g] = -\frac{1}{\kappa^2 Z_{Nk}} \operatorname{Tr}_{ST^2} \left[V_n^{(2)}[g,g] \left(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g] \right) \frac{\mathscr{N}_k(-D^2)}{[\mathscr{A}_k(-D^2) + c_I R]^2} - \frac{2}{d-2} V_n^{(2)}[g,g] \mathbb{P}_{\text{tr.}}[g] \frac{\mathscr{N}_k(-D^2)}{[\mathscr{A}_k(-D^2) + c_{\text{trace}} R]^2} \right]. \tag{11.17}$$

Here, $\bar{\gamma}_n(k)$ is the anomalous dimension of the renormalized operator $[V_n]_k[g,g]$. As $\bar{\gamma}_n(k)$ is dimensionless for non-mixing ansätze, later we will be able to re-write the RHS in terms of the dimensionless couplings λ_k and g_k given by Eq. (7.24), such that we have $\gamma_n(\lambda_k, g_k) \equiv \bar{\gamma}_n(k)$. Before we begin with the actual calculation, we already note that we may discard all curvature terms on the RHS, because we must project the RHS onto the operator $V_n[g,g]$ which itself is curvature-independent.[2] Therefore, firstly we remove the explicit R-dependence on the RHS by expanding it to zeroth order in R (i.e., we set $R = 0$):

$$\bar{\gamma}_n(k) V_n[g,g] = -\frac{1}{\kappa^2 Z_{Nk}} \operatorname{Tr}_{ST^2} \left[V_n^{(2)}[g,g] \left(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g] \right) \frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2} - \frac{2}{d-2} V_n^{(2)}[g,g] \mathbb{P}_{\text{tr.}}[g] \frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2} \right]. \tag{11.18}$$

[2] In fact, this means that we could w.l.o.g. specify the metric to that of flat space, $g_{\mu\nu} = \delta_{\mu\nu}$, and then evaluate the trace on the RHS in momentum space. However, for us it is more convenient to evaluate the trace with heat kernel methods that we already have developed and utilized in Chapter 7 for the Einstein-Hilbert truncation.

Note that via the traces over functions of the Laplacian, the RHS still entails an implicit R-dependence. Let us calculate the first term on the RHS for itself:

$$\begin{aligned}
&\mathrm{Tr}_{ST^2}\left[V_n^{(2)}[g,g]\,(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g])\,\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}\right]\\
&\quad=\int \mathrm{d}^d x_1\sqrt{g(x_1)}\int \mathrm{d}^d x_2\sqrt{g(x_2)}\langle x_1,\mu,\nu|V_n^{(2)}[g,g]|x_2,\rho,\sigma\rangle\\
&\qquad\qquad\times\langle x_2,\rho,\sigma|\,(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g])\,\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x_1,\mu,\nu\rangle\\
&\quad=\int \mathrm{d}^d x_1\int \mathrm{d}^d x_2\, I[g]_{\rho\sigma\alpha\beta}\int \mathrm{d}^n u\frac{\delta\sqrt{\det g(u)}}{\delta g_{\mu\nu}(x_1)\delta g_{\alpha\beta}(x_2)}\\
&\qquad\qquad\times\langle x_2,\rho,\sigma|\,(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g])\,\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x_1,\mu,\nu\rangle\\
&\quad=\frac{1}{4}\int \mathrm{d}^n u\sqrt{\det g(u)}\,\frac{\partial x_\rho(u)}{\partial u^c}\frac{\partial x_\sigma(u)}{\partial u^d}\frac{\partial x^\mu(u)}{\partial u^a}\frac{\partial x^\nu(u)}{\partial u^b}(u)\\
&\qquad\qquad\times\left[g^{ab}(u)g^{cd}-g^{ac}(u)g^{db}(u)-g^{ad}(u)g^{cb}(u)\right]\\
&\qquad\qquad\times\langle x(u),\rho,\sigma|\,(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g])\,\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x(u),\mu,\nu\rangle\,.
\end{aligned}\tag{11.19}$$

In the last step, we have used Eq. (11.16). Next, we express the remaining matrix element as a differential operator acting on a delta function,

$$\begin{aligned}
&\langle x(u),\rho,\sigma|\,(\mathbb{1}_{ST^2}-\mathbb{P}_{\mathrm{tr.}}[g])\,\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x(u),\mu,\nu\rangle\\
&\quad=\lim_{z\to x(u)}\Big(I^{\rho\sigma}{}_{\mu\nu}-P[g]^{\rho\sigma}{}_{\mu\nu}(x(u))\Big)\,\frac{\mathscr{N}_k(-D^2)_{x(u)}}{\mathscr{A}_k(-D^2)^2_{x(u)}}\frac{\delta(x(u)-z)}{\sqrt{g(z)}}\,.
\end{aligned}\tag{11.20}$$

At this point, the following traces are required:

$$\begin{aligned}
&I_{ST^2}{}^{\rho\sigma}{}_{\mu\nu}\frac{\partial x_\rho(u)}{\partial u^c}\frac{\partial x_\sigma(u)}{\partial u^d}\frac{\partial x^\mu(u)}{\partial u^a}\frac{\partial x^\nu(u)}{\partial u^b}\left(g^{ab}g^{cd}-g^{ac}g^{db}-g^{ad}g^{cb}\right)(u)\\
&\qquad=\frac{1}{2}(g_{ac}g_{bd}+g_{bc}g_{ad})(u)\left(g^{ab}g^{cd}-g^{ac}g^{db}-g^{ad}g^{cb}\right)(u)=-n^2
\end{aligned}\tag{11.21}$$

and

$$
\begin{aligned}
P_{\text{tr.}}[g]^{\rho\sigma}{}_{\mu\nu}(x(u)) \frac{\partial x_\rho(u)}{\partial u^c} \frac{\partial x_\sigma(u)}{\partial u^d} \frac{\partial x^\mu(u)}{\partial u^a} \frac{\partial x^\nu(u)}{\partial u^b} \left(g^{ab}g^{cd} - g^{ac}g^{db} - g^{ad}g^{cb}\right)(u) \\
= \frac{1}{d} g_{cd}(u) g_{ab}(u) \left(g^{ab}g^{cd} - g^{ac}g^{db} - g^{ad}g^{cb}\right)(u) = \frac{1}{d}(n^2 - 2n)\,.
\end{aligned}
\tag{11.22}
$$

Here, the partial derivatives $\partial x/\partial u$ combine with I_{ST^2} and $P_{\text{tr.}}$ in such a way that a tensor depending solely on the metric g_{ab} is formed and the trace yields a simple number. Therewith, we immediately obtain

$$
\begin{aligned}
&\mathrm{Tr}_{ST^2}\left[V_n^{(2)}[g,g]\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g]\right)\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}\right] \\
&\quad = \frac{1}{4}\int \mathrm{d}^n u \sqrt{\det g(u)}\left[-\frac{(d+1)n^2 - 2n}{d}\langle x(u)|\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x(u)\rangle\right]\,.
\end{aligned}
\tag{11.23}
$$

Fully analogously follows for the second term of Eq. (11.18) that

$$
\begin{aligned}
&\mathrm{Tr}_{ST^2}\left[-\frac{2}{d-2}V_n^{(2)}[g,g]\mathbb{P}_{\text{tr.}}[g]\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}\right] \\
&\quad = \frac{1}{4}\int \mathrm{d}^n u \sqrt{\det g(u)}\left[-\frac{4n - 2n^2}{2d - d^2}\langle x(u)|\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x(u)\rangle\right]\,.
\end{aligned}
\tag{11.24}
$$

All in all, we therewith have boiled down the Eq. (11.18) to

$$
\begin{aligned}
\bar{\gamma}_n(k) V_n[g,g] = \frac{1}{\kappa^2 Z_{Nk}}\frac{1}{4}\left[\frac{(d+1)n^2 - 2n}{d} + \frac{4n - 2n^2}{2d - d^2}\right] \\
\times \int \mathrm{d}^n u \sqrt{\det g(u)}\,\langle x(u)|\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x(u)\rangle\,.
\end{aligned}
\tag{11.25}
$$

In order to finally project the RHS onto the operator $V_n[g,g]$, let us expand the matrix element $\langle x(u)|\mathscr{N}_k(-D^2)/\mathscr{A}_k(-D^2)^2|x(u)\rangle$ using formula (E.5) for the early time expansion of the untraced heat kernel,

$$
\langle x(u)|\frac{\mathscr{N}_k(-D^2)}{\mathscr{A}_k(-D^2)^2}|x(u)\rangle = \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} Q_{\frac{d}{2}}\left[\mathscr{N}_k/\mathscr{A}_k^2\right] + O(R)\,. \tag{11.26}
$$

Fortunately, the term in zeroth order in the curvature is $x(u)$-independent, such that – as we must discard all curvature terms on the RHS – we arrive at the result

$$\bar{\gamma}_n(k)V_n[g,g] = \frac{1}{\kappa^2 Z_{Nk}}\frac{1}{4}\left[\frac{(d+1)n^2-2n}{d}+\frac{4n-2n^2}{2d-d^2}\right] \times \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} Q_{\frac{d}{2}}\left[\mathcal{N}_k/\mathcal{A}_k^2\right] V_n[g,g]\,, \quad (11.27)$$

from which we can read off the anomalous dimension. Thereby, let us re-express the "Q-functional" in terms of the threshold functions via Eq. (7.21):

$$\bar{\gamma}_n(k) = \frac{1}{\kappa^2 Z_{Nk}}\frac{1}{4}\left(\frac{1}{4\pi}\right)^{\frac{d}{2}}\left[\frac{(d+1)n^2-2n}{d}+\frac{4n-2n^2}{2d-d^2}\right] \times k^{d-2}\left[\Phi^2_{\frac{d}{2}}(-2\bar{\lambda}_k/k^2)-\frac{1}{2}\eta_N(k)\tilde{\Phi}^2_{\frac{d}{2}}(-2\bar{\lambda}_k/k^2)\right]\,. \quad (11.28)$$

By furthermore switching to the dimensionless couplings λ_k and g_k defined by Eq. (7.24) we arrive at the final result for the (dimensionless) anomalous dimension:

$$\boxed{\gamma_n(\lambda_k,g_k) \equiv \bar{\gamma}(k) = 2\left(\frac{1}{4\pi}\right)^{\frac{d}{2}-1}\left[\frac{(d+1)n^2-2n}{d}+\frac{4n-2n^2}{2d-d^2}\right]g_k \times\left[\Phi^2_{\frac{d}{2}}(-2\lambda_k)-\frac{1}{2}\eta_N(\lambda_k,g_k)\tilde{\Phi}^2_{\frac{d}{2}}(-2\lambda_k)\right]\,.} \quad (11.29)$$

This equation is the main result of this chapter. It must be emphasized that Eq. (11.29) does not depend on the parametrization $x(u)$ – i.e., this anomalous dimension does not depend on the specific type of n-dimensional submanifold whose volume we consider. The only information of it entailed in γ_n is its dimension n so it is worthwhile to give the factor containing the dependence on the dimensions n its on definition,

$$f(d,n) := \left[\frac{(d+1)n^2-2n}{d}+\frac{4n-2n^2}{2d-d^2}\right]\,. \quad (11.30)$$

Let us point out some specific values regarding the choice of n. Setting $f(d,1) = (d-3)/(d-2)$ and $f(d,d) = d(d+1)$, we have reproduced the results for these special cases that were already calculated in [55]. Interestingly, the anomalous dimension of the area of a surface, i.e., the case of $n = 2$, is d-independent as $f(d,2) \equiv 4$. Further values of interest are $f(d,3) = 9 + 3/(d-2)$ and $f(d,4) = 8(2 + 1/(d-2))$.

Lastly, we define the one-loop anomalous dimension $\gamma_n^{1\text{L}}(\lambda_k, g_k)$ as the anomalous dimension (11.29) along the simplified Einstein-Hilbert flow that we had introduced in Section 7.2. As explained in this section, we therefore must set $\Phi \equiv 0$ and evaluate the remaining threshold function Φ at $\lambda_k \equiv 0$ in Eq. (11.29), which yields

$$\gamma_n^{1\text{L}}(g_k) = 2\left(\frac{1}{4\pi}\right)^{\frac{d}{2}-1}\left[\frac{(d+1)n^2 - 2n}{d} + \frac{4n - 2n^2}{2d - d^2}\right] g_k \, \Phi^2_{d/2}(0) \,. \qquad (11.31)$$

Equivalently, we could have obtained this one-loop approximation of the anomalous dimension with the one-loop approximation (9.34) of the composite-operator FRGE and then setting $\lambda_k \equiv 0$.

11.2.4. *The fixed-point scaling of* $[V_n]_k[g, g]$ *in* $d = 4$

Suppose that r is some characteristic length scale of the n-dimensional volume $V_n[g, g](r)$, e.g. the radius of an n-sphere. (In fact, we can w.l.o.g. consider the volume of an n-sphere because the anomalous dimension does not depend on the specific geometry of the submanifold but only its dimension.) In Section 9.4 we had shown that in this case the renormalized operator $[V_n]_k[g, g](r)$ in the UV scales according to Eq. (9.62), i.e.,

$$[V_n]_{k\to\infty}[g, g](r) \equiv \langle V_n[g, g](r)\rangle_{k\to\infty} \sim r^{n-\gamma_n(\lambda_*, g_*)} \,. \qquad (11.32)$$

Here, (λ_*, g_*) are the non-Gaussian fixed-points of the Einstein-Hilbert flow; further we have used that the canonical mass dimension of the n-dimensional volume is $[V_n[g, g](r)] = -n$. Remarkably, note that also the spacetime dimension d in which the submanifold is embedded enters the anomalous dimension and thus its scaling properties – which is classically not the case. For $n = 1$ the volume of the submanifold of course is the length of a curve, hence let us write $\ell[g, g](r) \coloneqq V_1[g, g](r)$. Therewith we can express the quantum scaling of an n-dimensional volume in the UV in terms of that of a quantum length:

$$\langle V_n[g, g](r)\rangle_{k\to\infty} \sim \left(\langle \ell[g, g](r)\rangle_{k\to\infty}\right)^{\frac{n-\gamma_n(\lambda_*, g_*)}{1-\gamma_1(\lambda_*, g_*)}} \,. \qquad (11.33)$$

Especially, if $\gamma_n(\lambda_*, g_*) > 0$ we observe an *effective dimensional reduction of the scaling properties of spacetime in the UV*. Subsequently, we will explicitly compute the fixed point values of the anomalous dimension of submanifolds embedded in $(d = 4)$-dimensional spacetime. Indeed, we will find $\gamma_n(\lambda_*, g_*) > 0$.

In Section 7.3 we had obatined the non-Gaussian fixed points of the full Einstein-Hilbert flow as well as of the simplified Einstein-Hilbert flow using the exponential cutoff (6.4) as well as the optimized cutoff (6.5). For the full flow, these are given by Eqs. (7.32) and (7.33), while for the simplified flow by Eqs. (7.35) and (7.36). The values of the full anomalous dimension $\gamma_n(\lambda_*, g_*)$ and its one-loop approximation $\gamma_n^{1L}(g_*^{1L})$ in the fixed-point regime are then obtained by numerically evaluating Eq. (11.29) and Eq. (11.31) at these fixed-point values. Note that the threshold functions depend on the cutoff shape, as well. Table 1 depicts the values we obtained for $n = 1, 2, 3, 4$. The values for $n = 1$ and $n = 4$ are reproductions of the values obtained in [55], while the values for $n = 2$ and $n = 3$ are new research results. Remarkably, the values obtained with the

TABLE 1. Fixed-point values of the anomalous dimension in $d = 4$.

	$\gamma_n^{\text{opt}}(\lambda_*^{\text{opt}}, g_*^{\text{opt}})$	$\gamma_n^{\text{exp}}(\lambda_*^{\text{exp}}, g_*^{\text{exp}})$	$\gamma_n^{\text{opt},1\text{L}}(g_*^{\text{opt},1\text{L}})$	$\gamma_n^{\text{exp},1\text{L}}(g_*^{\text{exp},1\text{L}})$
$n = 1$	0.0997	0.1006	0.0682	0.0671
$n = 2$	0.7973	0.8044	0.5455	0.5368
$n = 3$	2.0930	2.1116	1.4318	1.4091
$n = 4$	3.9867	4.0221	2.7273	2.6840

optimized and the exponential cutoff at $s = 1$ differ only marginally. Generally, these results show a small relative error due to their cutoff dependence. To give a few estimates of this error:

- for the s-dependence of the exponential cutoff:

$$\max_{i,j\in\{0.7,1,1.5\}} \left| \frac{\gamma_{\sigma_n}^{\text{exp,s=i}}(\lambda_*^{\text{exp,s=i}}, g_*^{\text{exp,s=i}}) - \gamma_{\sigma_n}^{\text{exp,s=j}}(\lambda_*^{\text{exp,s=j}}, g_*^{\text{exp,s=j}})}{\gamma_{\sigma_n}^{\text{exp,s=1}}(\lambda_*^{\text{exp,s=1}}, g_*^{\text{exp,s=1}})} \right| = 3.2\%$$

- for the s-dependence of the exponential cutoff at one-loop:

$$\max_{i,j\in\{0.7,1,1.5\}} \left| \frac{\gamma_{\sigma_n}^{\text{exp,1L,s=i}}(\lambda_*^{\text{exp,1L,s=i}}, g_*^{\text{exp,1L,s=i}}) - \gamma_{\sigma_n}^{\text{exp,1L,s=j}}(\lambda_*^{\text{exp,1L,s=j}}, g_*^{\text{exp,1L,s=j}})}{\gamma_{\sigma_n}^{\text{exp,1L,s=1}}(\lambda_*^{\text{exp,1L,s=1}}, g_*^{\text{exp,1L,s=1}})} \right| = 1.2\%$$

- for the difference between exponential and optimized cutoff:

$$\left| \frac{\gamma_{\sigma_n}^{\text{opt}}(\lambda_*^{\text{opt}}, g_*^{\text{opt}}) - \gamma_{\sigma_n}^{\text{exp,s=1}}(\lambda_*^{\text{exp,s=1}}, g_*^{\text{exp,s=1}})}{\gamma_{\sigma_n}^{\text{exp,s=1}}(\lambda_*^{\text{exp,s=1}}, g_*^{\text{exp,s=1}})} \right| = 0.9\%$$

- for the difference between exponential and optimized cutoff at one-loop:

$$\left| \frac{\gamma_{\sigma_n}^{\text{opt,1L}}(\lambda_*^{\text{opt,1L}}, g_*^{\text{opt,1L}}) - \gamma_{\sigma_n}^{\text{exp,1L,s=1}}(\lambda_*^{\text{exp,1L,s=1}}, g_*^{\text{exp,1L,s=1}})}{\gamma_{\sigma_n}^{\text{exp,1L,s=1}}(\lambda_*^{\text{exp,1L,s=1}}, g_*^{\text{exp,1L,s=1}})} \right| = 1.6\% \,.$$

Let us discuss the quantum scaling corrections based on the full Einstein-Hilbert flow. An astonishing result is that *for* $n = 4$ *and* $d = 4$ *the classical and quantum contribution almost perfectly cancel, resulting in the effective scaling dimension zero*, which "could suggest that at very small distance scales (fixed-point regime) spacetime is actually much more empty than one would naively expect" [55]. Here, what is new, is the observation that *in four spacetime dimensions the scaling dimension* $-(n - \gamma_n(\lambda_*, g_*))$ *of lower dimensional volumes* $(n < d)$ *universally approximates* -1, i.e., $\langle V_n[g, g](r)\rangle_{k\to\infty} \sim \langle \ell[g, g](r)\rangle_{k\to\infty}$ for $n = 1, 2, 3$ and $d = 4$.

Moreover, if we solve the RG equations (7.25) and (7.26) for the full Einstein-Hilbert flow we can insert the solution into Eq. (11.29). Therewith, we obtain $\gamma_n(\lambda_k, g_k)$ as a function of k and we can also study its values away from the fixed-point regime; in this case, however, we cannot interpret the anomalous dimension as a quantum scaling correction at the scale k because the scaling argument we had developed leading to the scaling relation (9.62) is applicable only in the fixed-point regime. Figure 11.1 exemplifies this and shows the (full) anomalous dimension along a trajectory of type IIIa: along the sprirally approach towards the the fixed point in theory space, the value of $\gamma_n(\lambda_k, g_k)$ even surpasses n – which we can *not* interpret as an effective positive scaling dimension at scale k.

Regarding the one-loop approximations, based on the simplified Einstein-Hilbert flow, of the quantum scaling corrections, we observe the same trend although their magnitude is smaller.

How accurate are these results? Because neither the full nor the simplified Einstein-Hilbert flow take into account the bimetric nature of the gravitational EAA [4], we must emphasize that the approximations for the anomalous dimensions obtained here might be rather crude. For instance, more precise values for the (negative) anomalous dimension η_N can be obtained in more refined truncations, where its value has been shown to be smaller than its single metric

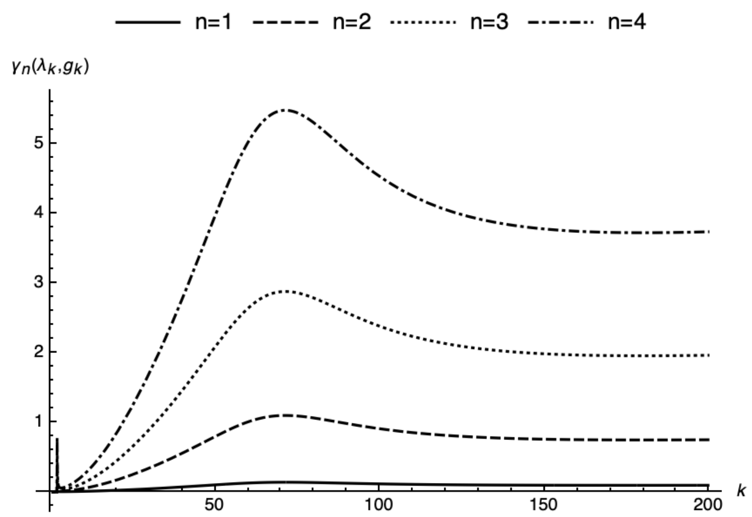

FIGURE 11.1. The anomalous dimension $\gamma_n^{\text{opt}}(\lambda_k^{\text{opt}}, \gamma_k^{\text{opt}})$ as a function of k along a trajectory of type IIIa of the full Einstein-Hilbert flow. The precise trajectory is that marked in red Figure 7.1. The RG equations as well as the anomalous dimension have been solved/calculated using the optimized cutoff.

absolute value of 2 [102]. This suggests that especially the "one-loop approximations", in which contributions from η_N are neglected, may be unreasonable. In fact, we can already give an estimate on the implications that this affair may have for the anomalous dimensions based on the full Einstein-Hilbert flow. Therefore, we will evaluate our single-metric result (11.29) at the fixed-point values found (using the optimized cutoff) in a "level-2" truncation and stated in table 7.5 in [102]: $\lambda_*^{\text{lev. 2}} = -0.049$, $g_*^{\text{lev. 2}} = 1.579$ and $\eta_{N_*}^{\text{lev. 2}} = 0.540$. This yields anomalous dimensions, employing the optimized cutoff, $\gamma_1^{\text{lev. 2}} \approx 0.0474$, $\gamma_2^{\text{lev. 2}} \approx 0.3794$ and $\gamma_3^{\text{lev. 2}} \approx 0.9959$. Compared to the values in table 1 obtained based on the single-metric full Einstein-Hilbert truncation, these fixed-point values for the anomalous dimension are severely smaller.

Lastly, we report the "one-loop" approximation of the anomalous dimension (11.31) evaluated along the simplified Einstein-Hilbert flow with free matter fields present. For N_S scalar fields, N_D spin-1/2 fermionic fields and N_V

abelian $U(1)$-gauge fields, its corresponding non-Gaussian fixed point is given by Eq. (7.54), such that the anomalous dimension reads[3]

$$\gamma_n^{\text{opt,1L}}(g_*^{\text{1L,matter}}) = \frac{12\pi}{22 - N_S - 2N_D + N_V} \frac{n(3n-2)}{8\pi} \,. \tag{11.34}$$

As the one-loop anomalous dimension is proportional to g_*, note that if the considered matter content induces a positive (negative) fixed-point value of Newton's constant, $\gamma_n^{\text{opt,1L}}(g_*^{\text{1L,matter}})$ will be accordingly positive (negative) as well. However, in the Asymptotic Safety scenario for quantum gravity one requires a positive fixed-point value of Newton's constant. In this case $\gamma_n^{\text{opt,1L}}(g_*^{\text{1L,matter}})$ is positive, yet again causing a reduction of the effective scaling dimension of the volume of the n-dimensional submanifold in the UV.

In the Section 9.1, to motivate the study of composite operators, we had briefly mentioned that such effective dimensional reduction phenomena have already been observed in various contexts in the Asymptotic Safety scenario for quantum gravity. More precisely, the spectral and the walk dimensions of the spacetime manifold have been predicted to effectively reduce to two dimensions in the UV limit [122, 123]. Meanwhile the Hausdorff dimension is still equal to the topological dimension, i.e., $d_{\text{H}} = 4$ [123].[4] Here, despite our rough approximations for the first and second truncation of theory space, our results underline the fact that an effective dimensional reduction may indeed constitute a general feature of the Asymptotic Safety scenario for quantum gravity – or rather quantum gravity in general because similar dimensional reduction phenomena have been observed in various scenarios for quantum gravity [126]. Hence, it would be highly interesting to conduct the study of anomalous scaling dimensions of volumes of submanifolds with other theories of gravity. Particularly, the first order formalism [133, 134], extended theories of gravity [135–137], and theories on foliated spacetimes [138–142] show compatibility with the Asymptotic Safety scenario.

[3]This result differs from our result in [29] because there we had employed at cutoff of "type II" (in the terminology of [102]) for all matter fields, which we have here, i.e., in Section 7.4, employed only for the spin-1/2 fermionic fields. For the remaining matter fields we have employed a cutoff of "type I".
[4]We shall come back to the Hausdorff dimension in Section 11.4.

11.2.5. *The fixed-point scaling of* $[V_n]_k[g,g]$ *in* $d=3$

In $d=3$ the fixed points in the Einstein-Hilbert flow are, using the optimized cutoff, $(\lambda_*^{\text{opt}}, g_*^{\text{opt}}) = (0.0629, 0.1989)$ for the full flow and $(\lambda_*^{\text{opt,1L}}, g_*^{\text{opt,1L}}) = (0, 3\pi/40) = (0, 0.2356)$ for the simplified flow. Using the exponential cutoff evaluated at $s=1$ the fixed points read $(\lambda_*^{\text{exp}}, g_*^{\text{exp}}) = (0.1407, 0.1326)$ for the full flow and $(\lambda_*^{\text{exp,1L}}, g_*^{\text{exp,1L}}) = (0, \sqrt{\pi}) = (0, 0.1772)$ for the simplified flow. Table 2 shows the values of the anomalous dimensions γ_n at the fixed point in $d=3$ for the respective cutoff profiles. Particularly, in $d=3$ the one-loop fixed-point values of γ_n^{1L} become cutoff-independent:

$$\gamma_n^{\text{1L}} \overset{d=3}{=} \frac{n(n-1)}{5}\,. \tag{11.35}$$

TABLE 2. Fixed-point values of γ_n for $d=3$. The first two columns show the one-loop result (simplified flow) obtained via the optimized and the exponential cutoff. The third and fourth columns display the results for the full Einstein-Hilbert flow.

	$\gamma_n^{\text{opt,1L}}(g_*^{\text{opt,1L}})$	$\gamma_n^{\text{exp,1L}}(g_*^{\text{exp,1L}})$	$\gamma_n^{\text{opt}}(\lambda_*^{\text{opt}}, g_*^{\text{opt}})$	$\gamma_n^{\text{exp}}(\lambda_*^{\text{exp}}, g_*^{\text{exp}})$
$n=1$	0	0	0	0
$n=2$	0.4	0.4	0.5303	0.5692
$n=3$	1.2	1.2	1.5908	1.7076

11.3. THE GEODESIC LENGTH

11.3.1. *Definition of the operator* ℓ_g

In Section 9.1, we had already demonstrated the need for knowledge of the renormalization and scaling behavior of the geodesic length ℓ_g. In this section we give a first contribution to this "knowledge" by calculating the anomalous dimension of ℓ_g with the composite-operator FRGE (11.6), i.e., with the first truncation specified to the Einstein-Hilbert truncation and the second truncation given by the single operator ℓ_g.

In this setting, we had calculated and discussed the anomalous dimension of the length $\ell[g,g] \equiv V_1[g,g]$ of a curve $c : [0,1] \to M$, given by the equation

$$\ell[g,g] = \int_0^1 \mathrm{d}\sigma \sqrt{g_{\mu\nu}(c(\sigma))\,\frac{\mathrm{d}c^\mu(\sigma)}{\mathrm{d}\sigma}\,\frac{\mathrm{d}c^\nu(\sigma)}{\mathrm{d}\sigma}}\,, \tag{11.36}$$

in the previous section. We had found that the anomalous dimension $\gamma_1(\lambda_k, g_k)$ in this setting has the same value for *any* curve $c(\sigma)$. Hence, one might ask why it is necessary to discuss the anomalous dimension of the geodesic length separately, given that, after all, the geodesic is some parametrized curve $c(\sigma)$ itself. The answer to this question is that here, we do not identify some given parametrized curve as a geodesic but rather *define* the geodesic as the solution $c[g](\sigma)$ to the *geodesic equation*

$$\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}c[g]^\mu(\sigma) + \Gamma^\mu_{\alpha\beta}(c[g](\sigma))c[g]^\alpha(\sigma)c[g]^\beta(\sigma)\,. \tag{11.37}$$

From this point of view, it is obvious that $\gamma_1(\lambda_k, g_k)$ is the anomalous dimension of any given curve whose parametrization does not depend on the metric $g_{\mu\nu}$, which is not the case for the geodesic $c[g](\sigma)$. Thus, here we consider the *geodesic length* to be the operator

$$\ell_g = \int_0^1 \mathrm{d}\sigma \sqrt{g_{\mu\nu}(c[g](\sigma))\,\frac{\mathrm{d}c[g]^\mu(\sigma)}{\mathrm{d}\sigma}\,\frac{\mathrm{d}c[g]^\nu(\sigma)}{\mathrm{d}\sigma}}\,, \tag{11.38}$$

where $c[g](\sigma)$ is a solution to the geodesic equation. It is clear that in general the anomalous dimensions of the geodesic length ℓ_g and of the length of an arbitrary curve $\ell[g,g]$, each obtained with the respective non-mixing ansatz, will differ because the additional g-dependence of the geodesic length gives rise to a more complicated Hessian (i.e., to more graviton vertices) on the RHS of the composite-operator FRGE (9.33).

Moreover, the solution of geodesic equation becomes unique only after specifying either *boundary* or *initial conditions*. Thus, when we analyze the renormalization behavior of the geodesic length ℓ_g we must do so on the basis of either of these supplementary conditions. Generally, we expect a different renormalization behavior for different supplementary conditions which is why we restrict the subsequent analysis to the following ones (note that each is a set of $2d$ conditions):

Boundary value problem. At first we consider the boundary conditions given by externally prescribed (i.e., g-independent) starting and end points,

$$\begin{cases} c[g]^\mu(0) = c^\mu_{\mathrm{Q}} \\ c[g]^\mu(L) = c^\mu_{\mathrm{P}}\,, \end{cases} \tag{11.39}$$

where L is a given distance travelled along c. In this case, we also write $\ell_g \equiv \ell_g(c_{\mathrm{Q}}, c_{\mathrm{P}})$ and refer to the geodesic length as the *geodesic distance of the two points* (as we did in Section 9.1). Furthermore, we assume c_{Q} and c_{P} to be sufficiently close such that no caustics appear and the solution of the geodesic equation is unique.

Initial value problem. In this case, the solution of the geodesic equation becomes unique by externally prescribing an initial point c_{Q} and an initial velocity χ, i.e.,

$$\begin{cases} c[g]^\mu(0) = c^\mu_{\mathrm{Q}} \\ \mathrm{d}c[g]^\mu(\sigma)/\mathrm{d}\sigma\big|_{\sigma=0} = \chi^\mu\,. \end{cases} \tag{11.40}$$

Normalized initial value problem at fixed geodesic length. Here, to make the solution unique, we again externally prescribe an initial point c_{Q}. But rather than fixing an initial velocity vector, we only externally prescribe a normalized initial velocity χ_0, i.e an *initial direction.* Then, we still must impose one further condition. Therefore, we require the geodesic length itself to be equal to the externally prescribed value r:

$$\begin{cases} c[g]^\mu(0) = c^\mu_{\mathrm{Q}} \\ \frac{\mathrm{d}c[g]^\mu(\sigma)/\mathrm{d}\sigma\big|_{\sigma=0}}{\left|\mathrm{d}c[g]^\mu(\sigma)/\mathrm{d}\sigma\big|_{\sigma=0}\right|} = \chi^\mu_0 \\ \ell_g(c_{\mathrm{Q}}, c[g](L)) \equiv r\,. \end{cases} \tag{11.41}$$

Such supplementary conditions involving a fixed geodesic length have occured in the literature in order to define correlators of the form "$\langle\phi(x)\phi(y)\rangle$" [143, 144].

11.3.2. *The anomalous dimension of ℓ_g for the boundary value problem*

In [28, 29], we have calculated the anomalous dimension of the geodesic length for a given boundary condition by iteratively solving the geodesic equation order by order in the metric fluctuation $h_{\mu\nu}$. Here, let us instead expand ℓ_g in Riemann normal coordinates as performed in the appendix of [145]: Let x_*^μ be the coordinates of a point $q \in M$ and x^μ normal coordinates around q. On a sufficiently small neighborhood of q we can choose coordinates y^μ such that for ε small

$$x^\mu = x_*^\mu + \varepsilon y^\mu \,. \tag{11.42}$$

Thus, ε corresponds to the typical length scale of the neighborhood on which the normal coordinates are defined. In these coordinates, we have $g_{\mu\nu}(x_*) = \delta_{\mu\nu}$ by construction and the Taylor series of the metric $g_{\mu\nu}$ at q can be shown to take the form

$$g_{\mu\nu}(x) = g_{\mu\nu}(x_*) - \frac{1}{3} R_{\mu\alpha\nu\beta}(x_*) x^\alpha x^\beta + O(\varepsilon^3) \,. \tag{11.43}$$

Next, let c be a curve parametrized by its arc length s and let us expand c in s,

$$c^\mu(s) = c_0^\mu + c_1^\mu s + \frac{1}{2} c_2^\mu s^2 + O(s^3) \,. \tag{11.44}$$

In Riemann normal coordinates we obtain by inserting this expansion into the geodesic equation (note that the series in s is cut by restricting its coefficients to be of order ε)

$$c^\mu(s) = c_0^\mu + c_1^\mu s + \frac{1}{3} R^\mu{}_{\alpha\beta\rho}(x_*) c_0^\rho c_1^\alpha c_1^\beta s^2 + O(\varepsilon^3) \,. \tag{11.45}$$

The coefficients c_0^μ and c_1^μ are determined by the boundary condition (11.39). With $\xi^\mu := c_{\mathrm{P}}^\mu - c_{\mathrm{Q}}^\mu$ and reparametrizing the geodesic c using the parameter $\sigma = s/L$, $\sigma \in [0,1]$, its expansion in σ will read

$$c^\mu(\sigma) = c_{\mathrm{Q}}^\mu + \sigma \xi^\mu - \frac{\sigma(1-\sigma)}{3} R^\mu{}_{\alpha\beta\rho}(x_*) c_{\mathrm{Q}}^\rho \xi^\alpha \xi^\beta + O(\varepsilon^3) \,. \tag{11.46}$$

By inserting the expansion (11.46) into the geodesic length ℓ_g and expanding $g_{\mu\nu}$ as in (11.43) it is easy to find the expansion

$$\ell_g^2 = g_{\mu\nu}(x_*) \xi^\mu \xi^\nu - \frac{1}{3} R_{\mu\alpha\nu\beta}(x_*) c_{\mathrm{Q}}^\alpha c_{\mathrm{Q}}^\beta \xi^\mu \xi^\nu + O(\varepsilon^3) \,. \tag{11.47}$$

The expansion of the operator ℓ_g in ε then is obtained therefrom by expanding the squareroot as $\sqrt{1+x} = 1 + x/2 + O(x^3)$,

$$\boxed{\ell_g = \sqrt{g_{\mu\nu}(x_*)\xi^\mu\xi^\nu}\left(1 - \frac{1}{6}R_{\mu\alpha\nu\beta}(x_*)c^\alpha_{\mathrm{Q}}c^\beta_{\mathrm{Q}}\xi^\mu\xi^\nu + O(\varepsilon^3)\right)\,.} \tag{11.48}$$

To calculate the anomalous dimension γ_{ℓ_g} of ℓ_g with the composite-operator FRGE (11.6), we must at first derive the Hessian $\ell_g^{(2)}$ defined via Eq. (6.47). To shorten our notation, we will write $\delta_{\mu\nu}a^\mu b^\nu = a \cdot b$, $\delta_{\mu\nu}a^\mu a^\nu = a^2$ et cetera. With these abbreviations, the Hessian is determined through the second-order functional derivative of ℓ_g,

$$\begin{aligned}
\frac{\delta^2 l_g}{\delta g_{\kappa\tau}(z)\delta g_{\alpha\beta}(y)} =& -\frac{1}{4}\frac{1}{\xi^3}\xi^\alpha\xi^\beta\xi^\kappa\xi^\tau\delta(x_*-y)\delta(x_*-z)\left(1-\frac{1}{6}c^\nu_{\mathrm{Q}}c^\sigma_{\mathrm{Q}}\xi^\mu\xi^\rho R_{\mu\nu\rho\sigma}(x_*)\right)\\
&-\frac{1}{12}\frac{1}{\sqrt{\xi^2}}\xi^\alpha\xi^\beta\delta(x_*-y)c^\nu_{\mathrm{Q}}c^\sigma_{\mathrm{Q}}\xi^\mu\xi^\rho\frac{\delta R_{\mu\nu\rho\sigma}(x_*)}{\delta g_{\kappa\tau}(z)}\\
&-\frac{1}{12}\frac{1}{\sqrt{\xi^2}}\xi^\kappa\xi^\tau\delta(x_*-z)c^\nu_{\mathrm{Q}}c^\sigma_{\mathrm{Q}}\xi^\mu\xi^\rho\frac{\delta R_{\mu\nu\rho\sigma}(x_*)}{\delta g_{\alpha\beta}(y)}\\
&-\frac{1}{6}\sqrt{\xi^2}c^\nu_{\mathrm{Q}}c^\sigma_{\mathrm{Q}}\xi^\mu\xi^\rho\frac{\delta^2 R_{\mu\nu\rho\sigma}(x_*)}{\delta g_{\kappa\tau}(z)\delta g_{\alpha\beta}(y)} + O(\varepsilon^3)\,.
\end{aligned} \tag{11.49}$$

With the Hessian arising thereof, we must project the RHS of the composite-operator FRGE (11.6) onto the operator ℓ_g. Hence, we must find some means to keep track of the terms proportional to ℓ_g in (11.49); especially we want to ensure that there are no terms proportional to ℓ_g hidden in the terms $O(\varepsilon^3)$. Therefore, note that when iteratively determining the coefficients c_n of the expansion (11.44) in case of the geodesic, one finds that $c_n \sim \varepsilon^n$ for $n \geq 2$ [145]. Next, from $c_1 = \xi/\ell_g$ one directly finds $c_2 \sim \xi^2$ and therewith also $c_n \sim \varepsilon^n \sim \xi^n$ holds by the iterative construction of the coefficients c_n. Thus, by counting powers of ξ we can keep track of the terms proportional to ℓ_g in (11.49).

As $\varepsilon^n \sim \xi^n$, only the first term of the expansion (11.49) is proportional to ℓ_g as with the remaining terms it is not possible to build the first term of the expansion (11.48) (which is $\sqrt{\xi^2}$). Thus, we immediately obtain the traces

$$\begin{aligned} I[g]_{\kappa\tau\alpha\beta}(x_*)\frac{\delta^2 \ell_g}{\delta g_{\kappa\tau}(z)\delta g_{\alpha\beta}(y)}\bigg|_{\ell_g} &= -\frac{1}{4}\delta(x_* - y)\delta(x_* - z)\ell_g \\ &= d\, P_{\text{tr.}}[g]_{\kappa\tau\alpha\beta}(x_*)\frac{\delta^2 \ell_g}{\delta g_{\kappa\tau}(z)\delta g_{\alpha\beta}(y)}\bigg|_{\ell_g} . \end{aligned}$$

Aside from these traces, the remaining calculation of the anomalous dimension is fully analogous to the procedure of Subsection 11.2.3. With the above traces we thus have

$$\begin{aligned} \bar{\gamma}_{\ell_g}(k)\ell_g &= -\frac{1}{\kappa^2 Z_{Nk}}\,\mathrm{Tr}_{ST^2}\Bigg[\ell_g^{(2)}\left(\mathbb{1}_{ST^2} - \mathbb{P}_{\text{tr.}}[g]\right)\frac{\mathscr{N}_k(-D^2)}{[\mathscr{A}_k(-D^2) + c_I R]^2} \\ &\qquad - \frac{2}{d-2}\ell_g^{(2)}\mathbb{P}_{\text{tr.}}[g]\frac{\mathscr{N}_k(-D^2)}{[\mathscr{A}_k(-D^2) + c_{\text{trace}} R]^2}\Bigg]\Bigg|_{\ell_g} \\ &= \frac{1}{\kappa^2 Z_{Nk}}\frac{1}{4}\left[\frac{d-1}{d} + \frac{2}{2d - d^2}\right]\left(\frac{1}{4\pi}\right)^{\frac{d}{2}} Q_{\frac{d}{2}}\left[\mathscr{N}_k/\mathscr{A}_k^2\right]\ell_g\,. \end{aligned} \tag{11.50}$$

Reading of the anomalous dimension $\bar{\gamma}_{\ell_g}(k) \equiv \gamma_{\ell_g}(\lambda_k, g_k)$ from this equation, it turns out that this is precisely the previously found anomalous dimension $\gamma_1(\lambda_k, g_k)$ for the length of a curve, given by Eq. (11.29) with $n = 1$. Hence, *for the approximations employed here to solve the composite-operator* FRGE, *i.e., with the first truncation specified to the Einstein-Hilbert truncation and a non-mixing ansatz for the second truncation, the anomalous dimensions of the geodesic distance of two points and that of the length of a curve coincide:* $\gamma_{\ell_g}(\lambda_k, g_k) = \gamma_1(\lambda_k, g_k)$. Nevertheless, we must generally expect $\gamma_{\ell_g}(\lambda_k, g_k) \neq \gamma_1(\lambda_k, g_k)$ for more refined approximations, in particular for mixing ansätze.

11.3.3. *The anomalous dimension of ℓ_g for the initial value problem*

To rewrite the geodesic length ℓ_g in terms of the initial condition (11.40), recall that the integrand $\sqrt{g_{\mu\nu}(c[g](\sigma))\,\mathrm{d}c[g]^\mu(\sigma)/\mathrm{d}\sigma\,\mathrm{d}c[g]^\nu(\sigma)/\mathrm{d}\sigma}$ is in fact a constant of motion. Using the initial condition (11.40), we have that

$$\begin{aligned} \ell_g &= \int_0^1 \mathrm{d}\sigma \sqrt{g_{\mu\nu}(c[g](\sigma))\,\frac{\mathrm{d}c[g]^\mu(\sigma)}{\mathrm{d}\sigma}\,\frac{\mathrm{d}c[g]^\nu(\sigma)}{\mathrm{d}\sigma}} \\ &= \int_0^1 \mathrm{d}\sigma \sqrt{g_{\mu\nu}(c[g](0))\,\frac{\mathrm{d}c[g]^\mu(\sigma)}{\mathrm{d}\sigma}\bigg|_{\sigma=0}\,\frac{\mathrm{d}c[g]^\nu(\sigma)}{\mathrm{d}\sigma}\bigg|_{\sigma=0}} \\ &= \sqrt{g_{\mu\nu}(c_{\mathrm{Q}})\chi^\mu\chi^\nu}\,. \end{aligned} \tag{11.51}$$

Thus, the only g-dependence of the geodesic length ℓ_g obtained with the initial condition (11.40) is the explicit g-dependence, while the implicit g-dependence of the curve itself has been removed and traded for the g-independent initial coordinates and velocities. Therewith, it is obvious that *for non-mixing ansätze, the anomalous dimension of the geodesic length obtained with a pure initial condition is identical to that of the length of an ordinary parametrized curve,* $\gamma_{\ell_g} = \gamma_1$. Also, this result holds for any chosen first truncation. However, we yet again must expect both anomalous dimensions to differ from another once we consider mixing ansätze.

11.3.4. *The anomalous dimension of ℓ_g for the normalized initial value problem*

The third supplementary condition of Eq. (11.41) states that the geodesic length is identical to an externally prescribed, g-independent length: $\ell_g \equiv r$. This makes the calculation of the anomalous dimension trivial because of the resulting vanishing Hessian. Hence, in this case we find $\gamma_{\ell_g} \equiv 0$ which means that ℓ_g *is not influenced by gravitational fluctuations at all* and its scaling dimension is the canonical mass dimension at all scales.

This last example shows that when studying the scaling properties of the geodesic length, one must carefully define via which supplementary conditions the geodesic trajectory is to be obtained.

11.4. The geodesic ball and the geodesic sphere

Let $q \in M$ be a point of the spacetime manifold. The geodesic ball of radius r (fixed and g-independent) with respect to the metric $g_{\mu\nu}$ is defined as

$$B_q[g](r) := \mathrm{vol}\left\{ c(1) \;\middle|\; c \text{ geodesic with } c(0) = q \text{ and } g_q(\dot{c}(0), \dot{c}(0)) \leq r \right\}. \tag{11.52}$$

Correspondingly, the geodesic sphere of radius r is the subset of $B_q[g](r)$ in which the inequality becomes an equality,

$$S_q[g](r) := \mathrm{vol}\left\{ c(1) \;\middle|\; c \text{ geodesic with } c(0) = q \text{ and } g_q(\dot{c}(0), \dot{c}(0)) = r \right\}. \tag{11.53}$$

Here, we are especially interestend in the small r limit in which can deduce the *Hausdorff dimension* d_{H} by means of the scaling relation [123]

$$\lim_{r\to 0} \langle B_q[g](r) \rangle_k \sim r^{d_{\mathrm{H}}}\,. \tag{11.54}$$

To find this small r limit, we expand $B_q[g](r)$ in a power series in r around flat space, according to theorem 3.1. of [146],

$$B_q[g](r) = B_{\mathrm{flat}}(r) \left\{ 1 + \frac{R[g](q)}{6(d+2)} r^2 + O(r^4) \right\} \tag{11.55}$$

where $R[g](q)$ is the scalar curvature at q, the terms $O(r^4)$ are higher curvature terms and

$$B_{\mathrm{flat}}(r) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} r^d \tag{11.56}$$

is the volume of the d-dimensional Euclidean ball. Naively, in the limit $r \to 0$ we find

$$\lim_{r\to 0} \langle B_q[g](r) \rangle_k = \lim_{r\to 0} B_{\mathrm{flat}}(r) \tag{11.57}$$

which is independent of $g_{\mu\nu}$. Thus, *in the limit $r \to 0$ (the volume of) the geodesic ball does not renormalize and the Hausdorff dimension is $d_{\mathrm{H}} = d$ in this case.* This is an important result which confirms the result already obtained in [123] via a different argument.

However, we must still clarify whether it has been justified to neglect the expectation values of the curvature terms in the expansion or whether their behavior in the UV for $k \to \infty$ could spoil our naively taken limit $r \to 0$. We can estimate the UV behavior of these terms with the following kind of "mean field" approximation:

$$\langle R[g](q)\rangle_k \approx R[\bar{g}^{\mathrm{SC}}_{\mu\nu}(k)]\,. \tag{11.58}$$

The k-dependent metric $\bar{g}^{\mathrm{SC}}_{\mu\nu}$ is the self-consistent background metric [147], which is defined by the condition $\langle g_{\mu\nu}\rangle_k = \bar{g}^{\mathrm{SC}}_{\mu\nu}(k)$. In fixed-point regime, this metric scales as

$$\bar{g}^{\mathrm{SC}}_{\mu\nu}(k) \sim k^{-2}\,, \tag{11.59}$$

and thus the expectation value of the scalar curvature in the fixed-point regime scales as

$$\langle R[g](q)\rangle_k \sim k^2\,. \tag{11.60}$$

Especially, for $k \to \infty$ the curvature becomes singular which thwarts our naive limit $r \to 0$. On the other hand, in the expansion of the geodesic ball we in fact encounter the term

$$\langle R[g](q) r^{d+2}\rangle_k \sim r^d (kr)^2\,. \tag{11.61}$$

Hence, the higher curvature terms in the expansion of the geodesic ball are negligible in the limit $r \to 0$ if $kr \ll 1$. This condition requires the radius r of the geodesic ball to be within the range of length scales that have already been integrated out by the RG flow: $0 \ll r \ll k^{-1}$. Indeed, this is a *physical requirement* because only in this way the geodesic ball is affected by all the relevant modes. Therewith, we have justified our above naive $r \to 0$ limit to be legitimate.

Lastly, we point out that one fully analogously obtains the small r behavior of the geodesic sphere $S_q[g](r)$ by means of its expansion in r around flat space [146]:

$$S_q[g](r) = S_{\mathrm{flat}}(r)\left\{1 + \frac{R[g](q)}{6d} r^2 + O(r^4)\right\} \tag{11.62}$$

with the volume of the $(d-1)$-dimensional Euclidean sphere

$$S_{\mathrm{flat}}(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{(d-1)}\,. \tag{11.63}$$

Therewith we find that

$$\lim_{r\to 0}\langle S_q[g](r)\rangle_k \sim r^{(d-1)} \tag{11.64}$$

is independent of $g_{\mu\nu}$ as well and thus does not renormalize.

CHAPTER 12

Geometric operators in four-dimensional higher-derivative gravity

Executive summary. Within the framework of four dimensional higher-derivative gravity and Weyl-squared gravity we use the one-loop approximation of the composite-operator FRGE in order to calculate the one-loop anomalous dimensions γ_n of the volume operator, which describes an n-dimensional volume embedded into the four dimensional quantized spacetime. The resulting anomalous dimension is gauge-dependent, whereby we employ the physical gauge in which only the gauge-invariant modes propagate. Then, the anomalous dimension vanishes at the gravitational fixed-point because higher-derivative gravity, as well as Weyl-squared gravity, is asymptotically free in the coupling parametrizing the Weyl tensor. However, slightly away from the fixed-point in the ultraviolet regime, γ_n is non-zero which results in a fractal scaling dimension of the volume operator. Lastly, we calculate and discuss the values of γ_n in $d = 4 - \varepsilon$ spacetime dimensions.
What is new? All results of this chapter represent novel research results.
Based upon: Reference [30].

In this chapter, we also study the renormalization behavior of the composite geometric operator $V_n[g, g]$, the volume of an n-dimensional submanifold, that we had constructed in Subsections 11.2.1 and 11.2.2. Again, we will employ a non-mixing ansatz for the second truncation such the renormalization behavior of $V_n[g, g]$ is fully encoded into its anomalous dimension. What is different from the previous section that instead of the asymptotically safe Einstein-Hilbert truncation we will employ the asymptotically free truncation of theory space given by higher-derivative gravity at one loop in $d = 4$ as the first truncation (cf. Section 4.3 and Chapter 8). Therewith, the technical aim of this section is to calculate the RHS of the composite-operator FRGE (9.34) at $g \equiv \bar{g}$ and with $\mathscr{O}[g,g] = V_n[g,g]$, i.e.,

$$\boxed{\begin{aligned}\bar{\gamma}_n^{1\mathrm{L}}(k) V_n[g,g] = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[\Big(\partial_t \mathscr{R}^{\mathrm{grav}}_{k,k'}[g]\Big)\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,g] + \mathscr{R}_k^{\mathrm{grav}}[g]\Big)^{-1} \\ \times V_n^{(2)}[g,g]\Big(\big(\Gamma_k^{(2)}\big)_{11}[g,g] + \mathscr{R}_k^{\mathrm{grav}}[g]\Big)^{-1}\Bigg]_{k=k'}.\end{aligned}} \tag{12.1}$$

Furthermore, as the geometric operator $V_n[g,g]$ is curvature-independent, we will w.l.o.g. specify the (background) metric to that of flat space, $g_{\mu\nu} = \delta_{\mu\nu}$. Consequently, the Hessian of the EAA $\Gamma_k[g,g]$, in general given by Eq. (8.5). projected onto flat space reads

$$\left(\Gamma_k^{(2)}\right)_{11}[g,g]\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} = \mathscr{U}\left[0; g_{\mu\nu} = \delta_{\mu\nu}\right]^{d=4}_{\mathrm{h.-d.}}\left(f_0^2(k), f_2^2(k)\right), \tag{12.2}$$

where the inverse propagator $\mathscr{U}\left[0; g_{\mu\nu} = \delta_{\mu\nu}\right]^{d=4}_{\mathrm{h.-d.}}$ is given by Eq. (4.100). Also, we will study the limit $f_0^2 \to \infty$ of Weyl-squared gravity, in which the Hessian of the EAA correspondingly reads

$$\left(\Gamma_k^{(2)}\right)_{11}[g,g]\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} = \mathscr{U}\left[0; g_{\mu\nu} = \delta_{\mu\nu}\right]^{d=4}_{\mathrm{Weyl}}\left(f_2^2(k)\right), \tag{12.3}$$

where the inverse propagator $\mathscr{U}\left[0; g_{\mu\nu} = \delta_{\mu\nu}\right]^{d=4}_{\mathrm{Weyl}}$ is given by Eq. (4.106). Lastly, we must specify the structure of the cutoff operator $\mathscr{R}_{k,k'}[g]$ given by Eq. (6.64). While we obviously should set $\gamma = 2$ in this equation, such that the cutoff operator is of the same order as the Hessian of the EAA, it is moreover convenient to set the tensor structure $\mathscr{Z}_k[g]$ to

$$\begin{aligned}\mathscr{Z}_k^{\mathrm{h.-d.}}[g_{\mu\nu} = \delta_{\mu\nu}] = -\Bigg\{ & -\frac{1}{2f_2^2(k)}\mathbb{P}^{(2)} + \frac{1}{\alpha}\mathbb{P}^{(1)} \\ & + \left(\frac{1}{f_0^2(k)} + \frac{6\beta^2}{\alpha}\right)\mathbb{P}^{(0,ss)} + \frac{2(\beta-1)^2}{\alpha}\mathbb{P}^{(0,ww)} \\ & + \frac{2\sqrt{3}\beta(\beta-1)}{\alpha}\left[\mathbb{P}^{(0,sw)} + \mathbb{P}^{(0,ws)}\right]\Bigg\},\end{aligned} \tag{12.4}$$

when generically considerung higher derivative gravity (f_0^2 finite) and to

$$\mathscr{Z}_k^{\mathrm{Weyl}}[g_{\mu\nu}=\delta_{\mu\nu}] = -\Bigg\{ -\frac{1}{2f_2^2(k)}\mathbb{P}^{(2)} + \frac{1}{\alpha}\Bigg[\mathbb{P}^{(1)} + \frac{3}{8}\mathbb{P}^{(0,ss)} + \frac{9}{8}\mathbb{P}^{(0,ww)} - \frac{3\sqrt{3}}{8}\mathbb{P}^{(0,sw)} - \frac{9\sqrt{3}}{8}\mathbb{P}^{(0,ws)}\Bigg]\Bigg\}, \tag{12.5}$$

when considering Weyl squared gravity ($f_0^2 \to \infty$). Further, we will restrict this analysis to employing the optimized cutoff (6.5) as the cutoff profile $R^{(0)}$.

12.1. The one-loop anomalous dimension of the volume operator in higher-derivative gravity

With the ingredients we have set up so far, the inverse propagator appearing in Eq. (12.1) reads

$$\left[\left(\Gamma_k^{(2)}\right)_{11}[g,g] + \mathscr{R}_k^{\mathrm{grav}}[g,g]\right]_{g_{\mu\nu}=\delta_{\mu\nu}} = \left[\Box^2 + k^4 R^{(0)}(\Box^2/k^4)\right]\mathscr{Z}_k^{\mathrm{h.-d.}}[g_{\mu\nu}=\delta_{\mu\nu}]\,, \tag{12.6}$$

with $\Box = \delta^{\mu\nu}\partial_\mu\partial_\nu$. This operator obviously is inverted by the propagator

$$\left[\left(\Gamma_k^{(2)}\right)_{11}[g,g] + \mathscr{R}_k^{\mathrm{grav}}[g,g]\right]^{-1}_{g_{\mu\nu}=\delta_{\mu\nu}} = \frac{1}{[\Box^2 + k^4 R^{(0)}(\Box^2/k^4)]}\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}\,, \tag{12.7}$$

where $\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}$ can be read off from Eq. (4.101):

$$\begin{aligned}\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1} = -\Bigg\{ & -2f_2^2(k)\mathbb{P}^{(2)} + \alpha\left[\mathbb{P}^{(1)} + \frac{1}{2(\beta-1)^2}\mathbb{P}^{(0,ww)}\right] \\ & + f_0^2(k)\Bigg[\mathbb{P}^{(0,ss)} + \frac{3\beta^2}{(\beta-1)^2}\mathbb{P}^{(0,ww)} \\ & + \frac{\sqrt{3}\beta}{1-\beta}\left(\mathbb{P}^{(0,sw)} + \mathbb{P}^{(0,ws)}\right)\Bigg]\Bigg\}.\end{aligned} \tag{12.8}$$

Thus, the one-loop composite-operator FRGE (12.1) becomes

$$\bar{\gamma}_n^{1\mathrm{L}}(k) V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}} = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\Bigg[\partial_t\left[k^4 R^{(0)}(\Box^2/k^4)\right]\,\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]\frac{1}{[\Box^2+k^4R^{(0)}(\Box^2/k^4)]}\,\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1} \times V_n^{(2)}[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}}\frac{1}{[\Box^2+k^4R^{(0)}(\Box^2/k^4)]}\,\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}\Bigg]. \tag{12.9}$$

As on flat space all operators involved commute, we are free to rewrite the RHS as

$$\bar{\gamma}_n^{1\mathrm{L}}(k)\, V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}} = -\frac{1}{2}\,\mathrm{Tr}_{ST^2}\left[V_n^{(2)}[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}}\,\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}\,\frac{\partial_t\left[k^4R^{(0)}(\Box^2/k^4)\right]}{[\Box^2+k^4R^{(0)}(\Box^2/k^4)]^2}\right]. \tag{12.10}$$

The Hessian $V^{(2)}[g,g]$ of the geometric operator is given by Eq. (11.10) together with Eq. (11.16). Let us introduce the auxiliary tensor

$$T^{\mu\nu}{}_{\rho\sigma}(u) = \frac{\partial x_\rho(u)}{\partial u^c}\frac{\partial x_\sigma(u)}{\partial u^d}\frac{\partial x^\mu(u)}{\partial u^a}\frac{\partial x^\nu(u)}{\partial u^b} \times\left[g^{ab}(u)g^{cd}(u)-g^{ac}(u)g^{db}(u)-g^{ad}(u)g^{cb}(u)\right], \tag{12.11}$$

by means of which the matrix elements of the Hessian $V^{(2)}[g,g]$ become

$$\langle x_1,\mu,\nu|V^{(2)}[g,g]|x_2,\rho,\sigma\rangle = \frac{1}{\sqrt{g(x_1)}\sqrt{g(x_2)}}\,\frac{1}{4}\int \mathrm{d}^n u\sqrt{\det g(u)}\,T^{\mu\nu}{}_{\rho\sigma}(u) \times\delta^{(d)}(x(u)-x_1)\delta^{(d)}(x(u)-x_2)\,. \tag{12.12}$$

Therewith, Eq. (12.10) becomes

$$\begin{aligned}\bar{\gamma}_n^{1\mathrm{L}}(k) V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= -\frac{1}{2}\int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \mathrm{d}^4 x_3 \frac{1}{4}\int \mathrm{d}^n u \sqrt{\det g(u)}\, T^{\mu\nu}{}_{\rho\sigma}(u) \\ &\quad \times \delta^{(d)}(x(u)-x_1)\,\delta^{(d)}(x(u)-x_2) \\ &\quad \times \langle x_2,\rho,\sigma|\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}|x_3,\alpha,\beta\rangle \\ &\quad \times \langle x_3,\alpha,\beta|\frac{\partial_t\left[k^4 R^{(0)}(\Box^2/k^4)\right]}{\left[\Box^2 + k^4 R^{(0)}(\Box^2/k^4)\right]^2}|x_1,\mu,\nu\rangle \\ &= -\frac{1}{8}\int \mathrm{d}^n u \sqrt{\det g(u)}\,\langle x(u),\mu,\nu|\mathbb{T}(u)\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1} \\ &\quad \times \frac{\partial_t\left[k^4 R^{(0)}(\Box^2/k^4)\right]}{\left[\Box^2 + k^4 R^{(0)}(\Box^2/k^4)\right]^2}|x(u),\mu,\nu\rangle\,, \end{aligned} \tag{12.13}$$

with $(\mathbb{T}(u)\phi)^{\mu\nu} := T^{\mu\nu}{}_{\rho\sigma}(u)\phi^{\rho\sigma}$. As we have chosen the background metric to be flat, we also are free to evaluate the traces in momentum space, by means of the unity operator $\mathbb{1}_{ST^2} = \int \mathrm{d}^4 p\,|p,\rho,\sigma\rangle\langle p,\rho,\sigma|$,

$$\begin{aligned}\bar{\gamma}_n^{1\mathrm{L}}(k) V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= -\frac{1}{8}\int \mathrm{d}^n u \sqrt{\det g(u)} \int \mathrm{d}^4 p\, \frac{\partial_t\left[k^4 R^{(0)}(p^4/k^4)\right]}{\left[p^4 + k^4 R^{(0)}(p^4/k^4)\right]^2} \\ &\quad \times \langle x(u),\mu,\nu|\mathbb{T}(u)\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}|p,\rho,\sigma\rangle\langle p,\rho,\sigma|x(u),\mu,\nu\rangle\,. \end{aligned} \tag{12.14}$$

Here, we can use that

$$\langle p,\rho,\sigma|x(u),\mu,\nu\rangle = I_{ST^2}{}^{\rho\sigma}{}_{\mu\nu}\frac{\mathrm{e}^{-\mathrm{i}x(u)\cdot p}}{\sqrt{2\pi}^4} \tag{12.15}$$

and that

$$\begin{aligned}&\langle x(u),\mu,\nu|\mathbb{T}(u)\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}|p,\rho,\sigma\rangle \\ &\qquad = T^{\mu\nu}{}_{\alpha\beta}(u)\left(\mathscr{Z}_k^{\mathrm{h.-d.}}[\delta_{\mu\nu}]^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}(p)\frac{\mathrm{e}^{\mathrm{i}x(u)\cdot p}}{\sqrt{2\pi}^4}\,, \end{aligned} \tag{12.16}$$

where the projectors entailed in $\left(\mathscr{Z}_k^{\text{h.-d.}}[\delta_{\mu\nu}]^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}(p)$ are now all expressed through momenta, o.e. $L^{\mu\nu} = p^\mu p^\nu/p^2$ et cetera. Hence we have arrived at

$$\bar{\gamma}_n^{1\text{L}}(k)V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}} = -\frac{1}{8}\frac{1}{(2\pi)^4}\int \mathrm{d}^n u\,\sqrt{\det g(u)}\int \mathrm{d}^4 p\,\frac{\partial_t\left[k^4 R^{(0)}(p^4/k^4)\right]}{\left[p^4 + k^4 R^{(0)}(p^4/k^4)\right]^2} \times T^{\mu\nu}{}_{\alpha\beta}(u)\left(\mathscr{Z}_k^{\text{h.-d.}}[\delta_{\mu\nu}]^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}(p)\,. \tag{12.17}$$

Next, we will show that the tensor trace $T^{\mu\nu}{}_{\alpha\beta}(u)\left(\mathscr{Z}_k^{\text{h.-d.}}[\delta_{\mu\nu}]^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}(p)$ is infact independent of u and p when appearing under an momentum integral. Therefore, we employ symmetric integration under the integral $\int \mathrm{d}^4 p$ [148]:

$$\begin{cases} p_\mu p_\nu \mapsto \frac{1}{4}p^2\delta_{\mu\nu} \\ p_\mu p_\nu p_\rho p_\sigma \mapsto \frac{1}{24}p^4\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\,. \end{cases} \tag{12.18}$$

We find that under the integral $\int \mathrm{d}^4 p$ the following relations hold (the explicit calculations of the relations can be found in appendix F.12):

$$\begin{aligned} T^{\mu\nu}{}_{\rho\sigma}(u)P^{(2)\rho\sigma}{}_{\mu\nu}(p) &= \frac{10}{72}n(2-5n)\,, \\ T^{\mu\nu}{}_{\rho\sigma}(u)P^{(1)\rho\sigma}{}_{\mu\nu}(p) &= \frac{1}{12}n(2-5n)\,, \\ T^{\mu\nu}{}_{\rho\sigma}(u)P^{(0,ss)\rho\sigma}{}_{\mu\nu}(p) &= \frac{1}{72}n(11n-26)\,, \\ T^{\mu\nu}{}_{\rho\sigma}(u)P^{(0,ww)\rho\sigma}{}_{\mu\nu}(p) &= -\frac{1}{24}n(n+2)\,, \\ T^{\mu\nu}{}_{\rho\sigma}(u)\left[P^{(0,sw)} + P^{(0,ws)}\right]^{\rho\sigma}{}_{\mu\nu}(p) &= \frac{1}{\sqrt{3}\,12}n(7n-10)\,, \\ T^{\mu\nu}{}_{\rho\sigma}(u)P^{(0,ws)\rho\sigma}{}_{\mu\nu}(p) &= \frac{1}{\sqrt{3}\,24}n(7n-10)\,. \end{aligned} \tag{12.19}$$

Thus, with Eq. (12.8) we have

$$\begin{aligned} &T^{\mu\nu}{}_{\alpha\beta}(u)\left(\mathscr{Z}_k^{\text{h.-d.}}[\delta_{\mu\nu}]^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}(p) \\ &\quad = 2f_2^2(k)\frac{10}{72}n(2-5n) - \alpha\frac{1}{12}n(2-5n) - f_0^2(k)\frac{1}{72}n(11n-26) \\ &\quad\quad + \frac{6\beta^2 f_0^2(k) + \alpha}{2(\beta-1)^2}\frac{1}{24}n(n+2) + \frac{\sqrt{3}f_0^2(k)\beta}{\beta-1}\frac{1}{\sqrt{3}\,12}n(7n-10)\,. \end{aligned} \tag{12.20}$$

Therewith, we can already read off the one-loop anomalous dimension $\bar{\gamma}_n^{1\text{L}}(k) \equiv \gamma_n^{1\text{L}}\left(f_0^2(k), f_2^2(k)\right)$ from Eq. (12.17):

$$
\begin{aligned}
\gamma_n^{1\text{L}}\big(f_0^2(k), f_2^2(k)\big) = &-\frac{1}{8}\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\, \frac{\partial_t\left[k^4 R^{(0)}(p^4/k^4)\right]}{\left[p^4 + k^4 R^{(0)}(p^4/k^4)\right]^2} \\
&\times \Bigg\{ f_2^2(k)\frac{20}{72}n(2-5n) \\
&\qquad + \alpha\left[-\frac{1}{12}n(2-5n) + \frac{1}{48(\beta-1)^2}n(n+2)\right] \\
&\qquad + f_0^2(k)\left[-\frac{1}{72}n(11n-26) + \frac{6\beta^2}{2(\beta-1)^2}\frac{1}{24}n(n+2)\right. \\
&\qquad\qquad \left.+ \frac{\beta}{(\beta-1)}\frac{1}{12}n(7n-10)\right]\Bigg\}\,.
\end{aligned}
\tag{12.21}
$$

Let us evaluate the momentum integral in polar coordinates with $P = |p|$ using the optimized cutoff (6.5):

$$
\begin{aligned}
\int \mathrm{d}^4p\, \frac{\partial_t\left[k^4 R^{(0)}(p^4/k^4)\right]}{\left[p^4 + k^4 R^{(0)}(p^4/k^4)\right]^2} &= 2\pi^2 \int \mathrm{d}P\, \frac{P^3 k\partial_k (k^4 - P^4)\theta(k^4 - P^4)}{[P^4 + (k^4 - P^4)\theta(k^4-P^4)]^2} \\
&= 2\pi^2 \int_0^k \mathrm{d}P\, \frac{P^3 4k^4}{k^8} \\
&= 2\pi^2\,.
\end{aligned}
\tag{12.22}
$$

All in all, we thus have arrived at the final result for the anomalous dimension:

$$
\boxed{
\begin{aligned}
&\gamma_n^{1\text{L}}\big(f_0^2(k), f_2^2(k)\big) \\
&\quad = -\frac{1}{4(4\pi)^2}\Bigg\{ f_2^2(k)\frac{20}{72}n(2-5n) + \alpha\left[-\frac{1}{12}n(2-5n) + \frac{1}{48(\beta-1)^2}n(n+2)\right] \\
&\qquad\qquad + f_0^2(k)\left[-\frac{1}{72}n(11n-26) + \frac{6\beta^2}{2(\beta-1)^2}\frac{1}{24}n(n+2)\right. \\
&\qquad\qquad\qquad\qquad \left.+ \frac{\beta}{(\beta-1)}\frac{1}{12}n(7n-10)\right]\Bigg\}\,.
\end{aligned}
}
\tag{12.23}
$$

12.2. The one-loop anomalous dimension of the volume operator in Weyl-squared gravity

The calculation of the anomalous dimension based on Weyl-squared gravity is the very same as based on higher-derivative gravity, with the sole difference that the tensor structure of the inverse propagator must be replaced by Eq. (12.5). We can read off its inverse from Eq. (4.107):

$$\mathscr{Z}_k^{\text{weyl}}[g_{\mu\nu} = \delta_{\mu\nu}]^{-1} = -\left\{ -2f_2^2(k)\mathbb{P}^{(2)} + \alpha\left[\mathbb{P}^{(1)} - \frac{2}{3\sqrt{3}}\mathbb{P}^{(0,ws)} + \frac{2}{3}\mathbb{P}^{(0,ww)}\right]\right\}. \tag{12.24}$$

If we trade $\mathscr{Z}_k^{\text{h.-d.}}[\delta_{\mu\nu}]^{-1}$ in Eq. (12.17) for $\mathscr{Z}_k^{\text{weyl}}[\delta_{\mu\nu}]^{-1}$ we can take a shortcut towards calculating the anomalous dimension $\bar{\gamma}_n^{\text{1L,Weyl}}(k)$ of $V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}}$ based on Weyl-squared gravity. Therefore, the only remaining task is to calculate the tensor trace $T^{\mu\nu}{}_{\alpha\beta}(u)(\mathscr{Z}_k^{\text{Weyl}}[\delta_{\mu\nu}]^{-1})^{\alpha\beta}{}_{\rho\sigma}(p)$. Using symmetric integration and the relations (12.19), we find that under the integral $\int \mathrm{d}^4 p$ one has

$$\begin{aligned} &T^{\mu\nu}{}_{\alpha\beta}(u)\left(\mathscr{Z}_k^{\text{h.-d.}}[\delta_{\mu\nu}]^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}(p) \\ &\quad = 2\, f_2^2(k)\frac{10}{72}n(2-5n) + \alpha\left[-\frac{1}{12}n(2-5n) + \frac{1}{54}n(7n-10) + \frac{1}{36}n(n+2)\right]. \end{aligned} \tag{12.25}$$

Thus, by following the procedure of the previous subsection, we find the one-loop anomalous dimension $\bar{\gamma}_n^{\text{1L,Weyl}}(k) \equiv \gamma_n^{\text{1L,Weyl}}\left(f_2^2(k)\right)$,

$$\boxed{\begin{aligned} \gamma_n^{\text{1L,Weyl}}\left(f_2^2(k)\right) = -\frac{1}{4(4\pi)^2}\Bigg\{ & 2\, f_2^2(k)\frac{10}{72}n(2-5n) \\ & + \alpha\left[-\frac{1}{12}n(2-5n) + \frac{1}{54}n(7n-10) + \frac{1}{36}n(n+2)\right]\Bigg\}. \end{aligned}} \tag{12.26}$$

12.3. Discussion of the one-loop anomalous dimensions of the volume operator

Suppose $V_n[g,g]$ depends on some characteristic length scale r. If we then take the expectation value of $V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}}(r)$ with respect to (6.2), with the bare action specified to Eq. (4.88), Eq. (9.62) tells us that in the fixed-point regime we will find the scaling relation

$$\left\langle V_n[g,g]\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}(r)\right\rangle_{k\to\infty} \sim r^{n-\gamma_n^{1\mathrm{L}}\left(f^2_{0\,*},f^2_{2\,*}\right)}\,, \tag{12.27}$$

respectively in the limit $f_0^2 \to \infty$,

$$\left\langle V_n[g,g]\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}(r)\right\rangle_{k\to\infty} \sim r^{n-\gamma_n^{1\mathrm{L,Weyl}}\left(f^2_{2\,*}\right)}\,. \tag{12.28}$$

As we did when studying the scaling behavior of $V_n[g,g](r)$ in the Einstein-Hilbert truncation, we can also express this scaling relation in terms of the length $\ell[g,g] \coloneqq V_1[g,g]$ of a curve:

$$\left\langle V_n[g,g]\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}(r)\right\rangle_{k\to\infty} \sim \left\langle \ell[g,g]\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}(r)\right\rangle_{k\to\infty}^{\frac{n-\gamma_n^{1\mathrm{L}}(f^2_{0\,*},f^2_{2\,*})}{1-\gamma_1^{1\mathrm{L}}(f^2_{0\,*},f^2_{2\,*})}}\,, \tag{12.29}$$

and likewise for the case $f_0^2 \to \infty$.

Before we analyze the values of the anomalous dimension in the fixed-point regime, a few general comments are in order. We observe that to the anomalous dimension (12.23) in higher-derivative gravity, as well as to that of Weyl-squared gravity given by Eq. (12.26), all propagating modes of the gauge-fixed propagator (4.101), respectively (4.107), contribute. This includes both gauge-invariant spin-2 and scalar modes as well as the unphysical vector and pseudoscalar modes (cf. Subsection A.2.2).

In Subsection 11.2.1 we had already noted that $V_n[g,g]$ is not a true observable as it clearly breaks diffeomorphism invariance (on the "bulk" manifold). Its anomalous dimension (12.23) (or (12.26), respectively) reflects this fact in its gauge dependence. In order to construct an actual observable, we could combine $V_n[g,g]$ with some observable amplitude such that the gauge dependencies of the respective anomalous dimensions cancel. This procedure has been explored in two-dimensional quantum gravity, where calculations are usually performed in the conformal gauge [149–152]. However, here we restrict our analysis to a much simpler workaround: It is clear that in the *physical gauge* $\alpha = \beta = 0$ (or $\alpha = 0$ in case of Weyl-squared gravity) only the gauge-invariant, i.e., physical, modes propagate [153]. This gauge is often associated to the unique "Vilkovisky-de Witt effective action" which refers to the construction of an gauge-independent effective action [154]. If the anomalous dimension $\gamma_n^{1L}(f_0^2(k), f_2^2(k))$ or $\gamma_n^{1L,\mathrm{Weyl}}(f_2^2(k))$ it gauge-independent they should clearly vanish in the UV for $k \to \infty$ because higher-derivative gravity as well as Weyl-squared gravity is asymptotically free in the coupling f_2^2 (to which the anomalous dimension should be proportional if it were gauge-invariant). Therefore, it is important to point out that the anomalous dimension in the physical gauge vanishes in the UV (at the Gaussian fixed point). To see this, let us express the anomalous dimension in the physical gauge in terms of the couplings f_2^2 and ω given by Eq. (8.8):

$$\boxed{\gamma_n^{1L}\big(f_2^2(k), \omega(k)\big) = \frac{n}{(4\pi)^2}\frac{f_2^2(k)}{576}\left[40(5n-2) + \frac{1}{\omega(k)}(11n-26)\right]}\,, \tag{12.30}$$

and

$$\boxed{\gamma_n^{1L,\mathrm{Weyl}}\big(f_2^2(k)\big) = \frac{1}{(4\pi)^2}\frac{10}{144}(5n^2-2n)f_2^2(k)\,.} \tag{12.31}$$

As $f_2^2 \to 0$ for $k \to \infty$, the anomalous dimension in the physical gauge clearly vanishes in the UV. (In case of higher-derivative gravity we could say in other words: It vanishes at either of the fixed points (8.15) or (8.16).)

As soon as we slightly move away from the UV-fixed point, we find ourselves in a regime where scale invariance is approximately realized, characterized by an effective fractial scaling dimension of the operator $V_n[g,g]\big|_{g_{\mu\nu}=\delta_{\mu\nu}}$ that still scales according to Eq. (12.27) or Eq. (12.28), respectively. In case of higher-derivative gravity, one finds that, given $f_2^2 > 0$, $\gamma_n^{1L}(f_2^2, \omega)$ takes positive or negative values in this regime depending on n. For $n = 1, 2$ (legnths and areas), $\gamma_n^{1L}(f_2^2, \omega)$ is positive for $1/\omega < (80 - 200n)/(-26 + 11n)$, while for $n = 3, 4$ (three- and four-volumes) it is positive for $1/\omega > (80 - 200n)/(-26 + 11n)$. Hence, the effective scaling dimension of e.g. the length of a curve can decrease or increase with respect to its classical scaling dimension, while the opposite happens for e.g. a three-volume. We can identify this as a distinct feature of the fractal geometry of higher-derivative gravity as other models of quantum gravity usually only show dimensional reduction phenomena. In case of Weyl-squared gravity, it is easy to see that the sign of $\gamma_n^{1L,\mathrm{Weyl}}(f_2^2)$ in the physical gauge goes one-to-one with the sign of f_2^2.

Lastly, let us analyze the values of the anomalous dimension for $d = 4 - \varepsilon$. We will restrict the shift $4 \mapsto 4 - \varepsilon$ to the f_2^2-axis of theory space (see Section 8.3). In this case, higher-derivative gravity is equipped with the non-Gaussian fixed points $(f^2_{2\,*,\varepsilon}, \omega_{*,1})$ and $(f^2_{2\,*,\varepsilon}, \omega_{*,2})$ given by Eq. (8.26). The first one is UV-stable while the second one is a saddle point. The corresponding fixed point (8.28) for Weyl-squared gravity is gauge-dependent and in the physical gauge given by $f^2_{2\,*,\varepsilon;\mathrm{Weyl}} \approx -11.9030\varepsilon$. Table 12.3 depicts the values of the anomalous dimension in the physical gauge at these fixed points.

TABLE 1. Fixed-point values of the one-loop anomalous dimension in $d = 4 - \varepsilon$.

	$\gamma_n^{1\mathrm{L}}(f^2_{2\,*,\varepsilon}, \omega_{*,1})$	$\gamma_n^{1\mathrm{L}}(f^2_{2\,*,\varepsilon}, \omega_{*,2})$	$\gamma_n^{1\mathrm{L,Weyl}}(f^2_{2\,*,\varepsilon;\mathrm{Weyl}})$
$n = 1$	-0.1012ε	-0.0160ε	-0.0157ε
$n = 2$	-0.1291ε	-0.0837ε	-0.0838ε
$n = 3$	-0.0839ε	-0.2031ε	-0.2041ε
$n = 4$	$+0.0348\varepsilon$	-0.3742ε	-0.3769ε

Conclusively, in Subsection 11.2.1 we had argued that the study of the renormalization behavior of n-dimensional volumes might be particularly useful for the comparison of the geometrical features of various approaches towards quantum gravity. In this chapter, we have already calculated the anomalous dimension of some n-dimensional volume for two different approaches: in the previous section for the Asymptotic Safety approach, thereby restricting the analysis to the Einstein-Hilbert truncation of theory space, and here, in this section, for the asymptotically free theory of higher-derivative gravity. Comparing our results for both approaches, we find striking differences. In the Asymptotic Safety approach, represented by the Einstein-Hilbert truncation, the effective scaling dimension of the n-dimensional volume in the UV (at the non-Gaussian fixed point) was dictated by severe quantum corrections, given by the fixed-point values of the anomalous dimension. For $d = n = 4$ we even found that the classical scaling dimension and its quantum correction almost perfectly annihilate. On the other hand, for higher-derivative gravity, the effective scaling of the n-dimensional volume in the UV was found to be purely classical due to the theory being asymptotically free in the coupling f_2^2. This suggests that the asymptotically free higher-derivative theory of quantum gravity and asymptotically safe quantum gravity are *two distinct universality classes*, each being characterized by different fractal scaling properties of the respective geometrical operators.

CHAPTER 13

Conclusion and outlook

The main novel results of Part 3 are the formulae and fixed-point values for the anomalous dimension of a generalized volume operator within two frameworks. On the one hand, we employ the framework of Asymptotic Safety scenario for quantum gravity which we approximate by the Einstein-Hilbert truncation. On the other hand, we employ the framework of higher-derivative gravity, where we restrict the analysis to a one-loop approximation. As the fixed-point values of said anomalous dimension depict severe quantum corrections to the classical scaling dimension in the former scenario, while they vanish due to Asymptotic Freedom in the latter scenario, the obtained results suggest that these two scenarios belong to *two distinct universality classes*.

Generally, it is to expect that for more refined ansätze for the first truncation, the anomalous dimension might obtain large corrections to the preliminary values obtained here. This is because each truncation of theory space on which the FRGE is approximated constitutes for itself a non-perturbative excerpt of a fundamental quantum gravitational theory. Thus by repeating the analysis for larger and larger first truncations might lead to fully different results rather than to smaller and smaller corrections (as for example in perturbation theory when increasing the number of loops for a given analysis). Hence, to verify the obtained results for more refined first truncations is essential in order to find out whether these are true aspects of the theory or rather pure truncation artifacts.

A similar reasoning holds for the second truncation. The overall analyses of Part 3 should be repeated for mixing ansätze of a basis of several composite (geometric) operators. Again, one should therefrom expect large corrections to the values obtained here. Especially, the anomalous dimension of the geodesic length is expected to differ from that of the length of a curve in that case.[1]

[1] For a second truncation given by the operators $\int \mathrm{d}^d x \sqrt{g}\, R^n$, $n \geq 2$, the anomalous-dimension matrix has already been calculated [155, 156].

Despite severe corrections to the results presented in Part 3 can be expected from a repeated analysis with more refined approximations, the results of Part 3 nevertheless represent a highly important first step towards the study of geometric operators in quantum gravity.

Firstly, the results are important for the comparison of the different approaches towards quantum gravity. In the discrete as well as in the continuum-based approaches similar dimensional reduction phenomena have been observed so far. Here, we calculated a further type of dimensional reduction, in form of the fixed-point scaling of the volume operator, for the first time within the continuum-based approaches. It would be intriguing if the corresponding geometric operators analyzed within the discrete approaches scaled similarly. Such studies within the discrete approaches are thus interesting prospects for the future.

Secondly, the results presented here also make an important first step towards the construction of observables within quantum gravity. All formulae for the anomalous dimension presented in Part 3 depend on the gauge-fixing parameters and hence are not full-fledged observables. The construction of a full-fledged observable for quantum gravity clearly is beyond the scope of this thesis, however, it is clear that geometric operators will play a crucial role in the construction of a suitable observable for quantum gravity. As so far no results for renormalized composite operators in quantum gravity were known, the results of Part 3 contribute to the pursuit of formulating suitable observables for quantum gravity.

Part 4

Background Independent field quantization with sequences of gravity-coupled approximants

CHAPTER 14

Summary of Part 4

In Part 4, based on the author's publications [5] and [6], we propose a further novel line of research within the continuum-based approaches towards quantum gravity, which explores the consequences of *Background Independence* for the concepts of *regularization* and *renormalization*.

The principle of Background Independence requires the (background) metric of spacetime to be determined by a dynamical law, rather than fixed a priori by hand. Thus, we necessarily must consider quantized fields on a curved background manifold which is subject to Background Independence. It is suggestive to thereby separate the treatment of quantized matter fields on a curved background from quantum gravity itself, i.e., the quantization of spacetime.

To rigoruosly implement the principle of Background Independence in both these quantum field theoretical settings, we propose three essential requirements:

(R1) Background Independence

(R2) Gravity-coupled approximants

(R3) N-type cutoffs .

These requirements imply a *novel framework for the quantization of fields.* They are not logically independent. Rather **(R2)** is an interpretation of a regularized quantum field that obeys **(R1)**, while **(R3)** is a specific instruction for the realization of **(R2)**.

(A) Sore points of background-dependent quantization frameworks. Before summarizing these requirements in more detail, let us first illustrate some of the problems that occur when gravity-coupled fields are quantized in a framework where these requirements obviously do not hold.

Essentially, a quantum field theory is a quantum system whose degrees of freedom are parametrized by the points of a smooth manifold which typically models spacetime. Thus a quantum field theory can be regarded as a quantum mechanical system in the limit of infinitely many degrees of freedom. However, this limit usually does not straightforwardly allow for a physical interpretation, because of the notorious ultraviolet divergences which are due to the "too many" degrees of freedom of the system. Its physical interpretation then comes along with its regularization and renormalization.

A typical regulator employed in background-dependent quantum field theories, such as those of the Standard Model of particle physics which is defined on the rigid Minkowski space, is the *momentum space cutoff*. Typically, it has the dimension of mass and therefore defines a scale: modes of the field with momenta below this mass scale are retained, while the other field's modes are discarded. Therewith, the definition of the cutoff scale requires a metric on the background manifold (to define the field's momenta). By fixing a metric to introduce a momentum cutoff, obviously the principle of Background Independence is violated.

A typical problem that occurs when fields are quantized in this manner and only then are coupled to gravity is one major aspect of the cosmological constant problem [7, 157–159], namely the gravitational field generated by the zero point oscillations of quantum fields. To estimate this gravitational field one can reason as follows[1]: One quantizes a free and massless field on Minkowski space, which is assumed to follow the dispersion relation $\omega(\mathbf{p}) = |\mathbf{p}|$. Then one identifies the field with a set of harmonic oscillators, each contributing its zero point energy $\frac{1}{2}\hbar\omega$ to that of the vacuum state:

$$\varrho_{\text{vac}} = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}|\,.$$

This vacuum energy is quartically UV divergent and when regulated via a momentum cutoff $|\mathbf{p}| \leq \mathscr{P}$ reads

$$\varrho_{\text{vac}} \sim \mathscr{P}^4\,.$$

Although this vacuum energy is that of a field living on Minkowski space, it is then argued that ϱ_{vac}, like any other form of energy, should contribute to the curvature of spacetime. The curvature of spacetime is determined by the metric of spacetime, which in turn is dynamically determined by Einstein's equation. Thus to account for the vacuum fluctuations of the above quantized field, we

[1] W. Pauli is credited for this argument [9, 158].

should add $\Delta\Lambda = (8\pi G)\varrho_{\text{vac}} = 8\pi c G \mathscr{P}^4$ to the cosmological constant appearing in Einstein's equation. Note the violation of the principle of Background Independence: In some parts of Einstein's equation, namely in $\Delta\Lambda$, the metric has been explicitly fixed to that of Minkowski space. In other words, we employ the vacuum energy of quantized fields living on Minkowski space to determine the geometry of *another* spacetime. Intuitively, this might seem a valid approximation of the quantized field. However, later we will see that actually it is not, because of the paramount role of Background Independence.

Finally, it turns out that for every plausible scale $\mathscr{P}$, the curvature produced by $\Delta\Lambda$ is by far too large to be consistent with observation. For example, if the cutoff is specified to Planck scale, $\mathscr{P} = m_{\text{Pl}}$, the calculation produces a curvature which is about 10^{120} times larger than the value from modern-day cosmological observations.

According to a variant of this reasoning, Einstein's equation contains, besides $\Delta\Lambda$, also a bare cosmological constant, Λ_{b}, whose value is then tuned in dependence on $\mathscr{P}$ in such a way that the sum $\Lambda_{\text{obs}} = \Lambda_{\text{b}}(\mathscr{P}) + \Delta\Lambda(\mathscr{P})$ equals precisely the observed value. This version of the argument avoids making a false prediction (any prediction, in fact), but at the expense of an enormous naturalness problem. To achieve the desired value of Λ_{obs}, the bare quantity $\Lambda_{\text{b}}(\mathscr{P})$ must be consequently fine-tuned with a precision of 120 digits.

Often, the concept of *supersymmetry* was dealt with as a possible way to resolve this form of the cosmological constant problem. It was assumed that the contributions of each field and its "superpartner" to the energy of the vacuum state more or less perfectly cancel, leading to a serverely smaller value of ϱ_{vac}. However, as supersymmetry is unlikely to exist, and if it does then only in a broken form, this avenue of escape from the cosmological constant problem is blocked.

(B) Quantization and Background Independence. Next, let us summarize the requirements **(R2)** and **(R3)** that lead to a quantization framework which realizes Background Independence.

We refer to an *approximant* of a state of a quantum field as a quasi-physical system which is built from finitely many degrees of freedom that are coupled to gravity. I.e., an approximant, denoted as $\Psi_{\mathrm{f}} \otimes$ metric, consists of a quantum mechanical state Ψ_{f} with $\mathrm{f} < \infty$ degrees of freedom, together with a classical background metric. The requirement **(R1)** of Background Independence means that we must employ only those approximants whose metric is determined by the dynamical backreaction of the state Ψ_{f}. As this *self-consistency* condition is fulfilled by these gravity-coupled approximants, we denote them symbolically as $\Psi_{\mathrm{f}}^{\mathrm{SC}} \otimes$ self-consistent metric. Therewith, the requirement **(R2)** further means that a quantum field theory can only be identified with the limit $\mathrm{f} \to \infty$ of a self-consistent approximant.

The main result of Part 4 is the elucidation of the above version of the cosmological constant problem for certain simple quantum field theoretical settings. For these, we construct self-consistent approximants and show that in the limit $\mathrm{f} \to \infty$ the curvature of spacetime does *not* become disproportionately large. Thereby, we put a particular emphasis on the fact that the limit $\mathrm{f} \to \infty$ and the evaluation of the self-consistency condition do *not* commute.

Moreover, we require the (self-consistent) approximants to be physically realizable systems which narrows the number of candidates for a possible regulator. For instance, neither dimensional regularization nor the zeta function technique allow for the interpretation of the regularized system as a viable quantum system. Additionally, the regularization scheme must be in accordance with Background Independence which rules out all metric-dependent regulators, e.g., a momentum cutoff.

A simple regularization scheme, which is in accordance with Background Independence, is the employment of a *cutoff of the N-type*. In some way, N-cutoffs are similar to momentum cutoffs: both organize the field modes in such a way that a certain subset of the modes is retained while the complementary subset of the modes is discarded. In case of a momentum cutoff, the cutoff threshold is set by a mass scale which leads to a metric-dependent selection of which modes to retain and discard. In case of an N-cutoff, however, this selection process is fully independent of metric whereby the cutoff threshold is set by a natural (or positive real) number N. To avoid details at this point, the concept of an N-cutoff is at best illustrated by a simple example:

Consider a Gaussian field on the round 2-sphere with radius r. Then the field's modes can be staggered in terms of the eigenvalues of the negative Laplacia $-\Box_g$, where $g_{\mu\nu}$ is the r-dependent background metric. This spectrum is well known and given by $l(l+1)/r^2$, $l = 0, 1, 2, \ldots$. Now we can implement an N-cutoff by demanding that all field modes with $l \leq N$ are to be retained, while those with $l > N$ are to be discarded. This selection process of field modes is clearly independent of the metric and can w.l.o.g. be extended to more complex systems. Thus, in this seemingly simple manner, quantum field theories can be regularized in accordance with Background Independence.

Lastly, we note that in case an N-cutoff is implemented, the degrees of freedom f of the resulting quantum mechanical system also become a function of N.

(C) The quantum systems discussed in Part 4. The opening Chapter 15 of Part 4 explains in detail the framework of quantization via gravity-coupled approximants, as outlined above. Next, we summarize the self-consistent approximants constructed in Part 4 as well as their properties.

In Chapter 16 we begin the exploration of self-consistent, gravity-coupled approximants by considering a massive scalar field on a classical, curved background manifold. We assume the latter to be compact and without boundary and in all applications specify it to a four-dimensional Euclidean sphere $S^4(L)$, whose metric is parametrized by its radius L. Then by evaluating the self-consistency condition, i.e., the backreaction of the regularized quantum system with $\mathrm{f}(N)$ degrees of freedom on the metric, we obtain the self-consistent spherical background geometries, given by their radii $L^{\mathrm{SC}}(N)$.

Thus, in order to construct a self-consistent approximant, we first must regularize the scalar field via an N-cutoff which leads to a quantum mechanical system with $\mathrm{f}(N)$ degrees of freedom. This system then contributes via its effective stress-energy tensor to the RHS of a semiclassical version of Einstein's equation, which determines the self-consistent background radii $L^{\mathrm{SC}}(N)$. In Chapter 16 we identify a *first type of approximants* by promoting the scalar field's classical stress-energy tensor to an operatorial relation for the quantized scalar field. Its expectation value then is regularized via an N-cutoff which on $S^4(L)$ results in the self-consistency condition (16.55),

$$R(L) \equiv \frac{12}{L^2} = 4\Lambda_{\mathrm{b}} + \frac{3G}{\pi L^4}\left[\mathrm{f}(N) + \sum_{n=1}^{N} D_n \frac{\mu^2}{\mathscr{E}_n(L) + \mu^2 + \xi R(L)}\right].$$

Here, Λ_b is the bare cosmological constant, G is Newton's constant, and $\mathscr{E}_n$ and D_n are the eigenvalues and their degeneracies of $-\Box_g$. Moreover, μ is the mass of the scalar field and ξ is a constant coupling the scalar field to the scalar curvature R. We also show that the degrees of freedom of the regularized system are given by

$$\mathrm{f}(N) = \sum_{n=1}^{N} D_n^{(d=4)} = \frac{1}{12}\left[N^4 + 8N^3 + 23N^2 + 28N\right] .$$

We then discuss the solutions of this self-consistency condition in dependence on the parameters Λ_b, μ and ξ. It turns out that in any of these cases, the *cosmological constant problem does not occur.* If the scalar field was quantized in a background-dependent way, the radius appearing on the RHS of the self-consistency condition would be a rigid, fixed radius. This leads to a form of the cosmological constant problem because the resulting "self-consistent" radius *diverges* in the limit $N \to \infty$, i.e., the radius would shrink as more and more field modes are quantized. On the other hand, with Background Independence strictly implemented, let us for instance consider the case $\mu = 0$ and $\Lambda_\mathrm{b} = 0$. Then the self-cosistent radius is given by Eq. (16.71),

$$L^{\mathrm{SC}}(N)^2 = \frac{G}{4\pi}\,\mathrm{f}(N) = \frac{GN^4}{48\pi}\left\{1 + O\left(\frac{1}{N}\right)\right\} .$$

Hence, the radius of the self-consistent S^4-background geometry *grows* as more and more field modes are quantized, until the background manifold becomes *perfectly flat* in the limit $N \to \infty$. This means that the cosmological constant problem is fully absent.

A further striking result of these self-consistent S^4-geometries is that they allow for a natural explanation of the microscopic degrees of freedom which the thermodynamic Bekenstein-Hawking entropy "counts". Therefore, we consider a four-dimensional de Sitter space whose Bekenstein-Hawking entropy is given by $\mathscr{S} = \frac{\pi}{G}L^2$. When we specify the radius in that formula to the above self-consistent radii arising from the quantization of a massless scalar field at bare cosmological constant, we find that

$$\mathscr{S}(N) = \frac{1}{4}\mathrm{f}(N) .$$

This means that, up to a factor 1/4, the thermodynamical entropy $\mathscr{S}(N)$ precisely amounts to the number of degrees of freedom $\mathrm{f}(N)$ of the self-consistent approximant of the scalar field. This proves a "Λ-$\mathscr{N}$-connection" which was speculated about in the literature [160, 161].

In Chapter 17 we work out a *second type of approximants* for the quantized scalar field. These arise from the point of view of the effective gravitational action $\Gamma[g]$ for the background metric $g_{\mu\nu}$, which is the restriction of the full effective action to a vanishing scalar field, $\Gamma[g] \equiv \Gamma[A; g]|_{A=0}$. From this point of view, the self-consistency condition is given by the equations of motion for the background metric $g_{\mu\nu}$, i.e., $\delta\Gamma[g]/\delta g_{\mu\nu} = 0$. To be able to solve this self-consistency condition, we make use of the one-loop approximation of the effective action which is of the form $\Gamma[g] = S_{\text{EH}}[g] + \Gamma_{1\text{L}}[g]$. Then, the self-consistency condition assumes the form of a (semi-classical) Einstein equation whose RHS is given by an effective stress-energy momentum tensor, induced from the one-loop term $\Gamma_{1\text{L}}[g]$. It turns out, that this effective stress-energy tensor differs from that of the first type of approximants. For a massless scalar field, for instance, the resulting self-consistency condition (17.43) reads

$$\frac{12}{L^2} = 4\Lambda_{\text{b}} - \frac{3G}{\pi}\,\frac{1}{L^4}\,\mathrm{f}(N)\,.$$

The resulting self-consistent radii $L^{\text{SC}}(N)$ exhibit the same physical properties when removing the cutoff as for the first type of approximants: For every $\Lambda_{\text{b}} > 0$, there exists an N-sequence of self-consistent radii $L^{\text{SC}}(N)$ that *grows* when more and more modes are added, until ultimately in the limit $N \to \infty$, the self-consistent radius becomes infinite and the underlying S^4-geometry becomes *perfectly flat*. Hence, also for the second type of approximants, the cosmological constant problem is fully absent in these settings.

Moreover, we also show the origin of the difference between the first and second type of approximants. Namely, their difference can be rooted in the metric dependence of the path integral measure. In many quantum field theoretical considerations, the dynamics of the background metric is not of importance which results in a neglect of the metric dependence of the path integral measure. Then, "semi-classical" considerations usually agree with those obtained from the point of view of the effective action. However, by the example of Chapters 16 and 17, it is shown that there indeed is a difference between these two "quantization techniques".

In Chapter 18 we leave the setting of a quantized scalar field on a classical background manifold and apply the quantization framework by N-sequences of approximants to quantum gravity itself.

The analysis itself mainly follows the previous two chapters on the scalar field, but is much richer in technical detail. For the construction of a *first type of approximants* of quantized metric fluctuations, we come back to the one-loop approximations of the gravitational path integral developed in Part 1. There, a saddle point expansion of the bare action brought it into the form $S = S_{\mathrm{EH}} + S_{\mathrm{M}}$, where the Einstein-Hilbert action solely depends on the background metric, while the next term is quadratic in the metric fluctuation field which is why we may interpret S_{M} as the matter action for a Gaussian graviton field. Thus, the self-consistency condition for the background metric amounts to a semi-classical Einstein equation whose RHS is given by the expectation value of the stress-energy tensor obtained from the matter action S_{M}. When regularized via an N-cutoff, the overall degrees of freedom of the approximants split into those of the graviton field and those of the ghost fields, which in $d = 4$ spacetime dimensions read

$$\begin{aligned} \mathrm{f}_{\mathrm{grav}}(N) &= \frac{1}{12}(10N^4 + 80N^3 + 158N^2 - 8N - 180) \\ \mathrm{f}_{\mathrm{ghosts}}(N) &= \frac{1}{12}(8N^4 + 64N^3 + 160N^2 + 128N)\,. \end{aligned}$$

The explicit form of the self-cosistency condition obtained from this first type of approximants is rather intricate, so we will not discuss them in detail here.

Again we obtain a *second type of approximants* from the point of view of the effective action. Therefore, we also employ the one-loop approximation of the effective action $\Gamma[g]$ obtained in Part 1. In a way fully analogous to the scalar field, the one-loop term of $\Gamma[g]$ leads to an induced effective stress-energy tensor which differs from that of the “semi-classical” stress-energy tensor precisely by a contribution of the gravitational path integral measure.

We then show that when regulated via an N-cutoff, the radii of the self-consistent S^4-background geometries are solutions of the equation

$$0 = 4\Lambda_{\mathrm{b}} L^4 - 12L^2 - \frac{3G}{\pi}\left\{ \mathrm{f}_{\mathrm{grav}}(N) \pm \mathrm{f}_{\mathrm{ghosts}}(N) \right\} + \text{further } L\text{-dependent terms}\,.$$

Here, the “+” and the “−” refer to the self-consistency conditions obtained from the first and second type of approximants, respectively. We then show that also in case of quantum gravity, the self-consistent S^4-geometries are free of the appearance of the cosmological constant problem: We show that in both cases, "+" and "−", for a non-vanishing and finite bare cosmological constant, there exist N-sequences of self-consistent radii that *grow* as N becomes larger, i.e., the radius of the background 4-sphere becomes larger as N becomes larger, which for $N \to \infty$ leads to a fully flat background manifold. Again, also in case of quantum gravity, the self-consistent S^4-background geometries are free of the cosmological constant problem.

However, we argue that the second type of approximants, i.e., those obtained from the effective action, constitute a more natural candidate for physical approximants of quantum gravity. This is because only there, the degrees of freedom of the graviton field, $\mathrm{f}_{\mathrm{grav}}(N)$, and those of the ghost fields, $\mathrm{f}_{\mathrm{ghosts}}(N)$, in leading order in N naturally combine to $\mathrm{f}_{\mathrm{grav}}(N) - \mathrm{f}_{\mathrm{ghosts}}(N) = 2N^4 + \cdots$, i.e., to the two propagating degrees of freedom of the graviton.

The final Chapter 18 of this thesis is an addendum on the difference between the first and second type of approximants which we constructed. We note that this difference is rooted in the metric dependence of the path integral measure. In Chapter 18, we work out the construction of the first and second type of approximants for a set of general fields. Thereby, we show how the Weyl transformations' anomalous Ward identities, resulting from the overall dependence of the path integral on the background metric, are related to the difference between first and second type of approximants. This is moreover demonstrated for the field content we had already discussed, a scalar field as well as a graviton field accompanied by its ghost fields.

CHAPTER 15

The framework: outline and motivation

Executive summary. We propose a novel framework for the quantization of fields which are in contact with dynamical gravity. This framework is subject to three essential requirements: Background Independence, the use of gravity-coupled approximants, which should constitute physically realizable quantum systems, and the regularization scheme of cutoffs of the N-type, which are characterized by a dimensionless number N.
What is new? The proposal that Background Independence should be already implemented at the level of the regularized quantum field. The regularization tool of N-cutoffs.
Based upon: Reference [5].

In this part, we present a novel scheme for the quantization of matter fields, respectively metric fluctuations, which are coupled to classical gravity. Here, classical gravity refers to the dynamics of a Euclidean background metric $\bar{g}$ of a Riemannian background manifold $(M, \bar{g})$, that in all applications will be assumed compact and without boundary. This scheme satisfies three essential requirements:

(R1) Background Independence

(R2) Gravity-coupled approximants

(R3) N-type cutoffs .

They are not logically independent, but rather **(R2)** relies on **(R1)** while **(R3)** can be regarded as a realization of **(R2)**. In the subsequent three sections we discuss each requirement separately. Then, in the remaining chapters of this part, we implement these requirements in two different sample models: first, in the quantization of a scalar field on a compact Riemannian background, and second in the quantization of gravity in the form of metric fluctuations around a compact Riemannian background.

15.1. First requirement: Background Independence

Background Independence is the paramount feature of classical general relativity and therefore is de rigueur for a quantum gravitational theory. The modern approaches towards quantum gravity [162–164] have incorporated the desideratum of Background Independence in two different ways: on the one hand, there are approaches which literally do not employ a gravitational background in any way, and on the other hand, there are approaches which self-consistently fix a gravitational background by invoking the fundamental dynamical laws [165]. In Chapters 3 and 4, we have already exemplified the latter in terms of the background field technique [35]: In order to Background-Independently quantize fields, we had constructed the quantum fields on a fixed, yet arbitrary, background, given by the background metric field, and then afterwards proven that the (still to be renormalized) theory of these quantum fields is invariant under background gauge transformations. Especially, the following applications of the subsequently presented quantization scheme have recourse to Chapters 3 and 4. This why we can already regard Background Independence as a fulfilled desideratum from here on.

15.2. Second requirement: Gravity-coupled approximants

We define an *approximant* as a quasi-physical system which is built from finitely many quantum degrees of freedom that are coupled to gravity. Thus, the state of an approximant $\mathsf{App}(\mathrm{f})$ is given by a quantum mechanical state Ψ_{f} with $\mathrm{f} < \infty$ degrees of freedom together with a classical (background) metric:

$$\mathsf{App}(\mathrm{f}) = \Psi_{\mathrm{f}} \otimes \text{metric}\,. \tag{15.1}$$

Generally, the quantum mechanical state Ψ_{f} is the state of some matter system; in case of quantized gravity the quantum metric fluctuations around the classical background metric can be interpreted as the graviton field. Further, by *gravity-coupled approximants* we refer to those approximants whose metric is determined by the backreaction of the state Ψ_{f}. The backreaction of the f quantum degrees of freedom on the metric they inhabit is a *self-consistency* condition whichs obeys requirement **(R1)**. Thus, we could also call gravity-coupled approximants SC approximants, symbolically

$$\mathsf{App}^{\mathrm{SC}}(\mathrm{f}) = \Psi_{\mathrm{f}}^{\mathrm{SC}} \otimes \text{self-consistent metric}\,. \tag{15.2}$$

The requirement **(R2)** of gravity-coupled approximants means the following constraint: We allow as *regularized* quantum field theories only gravity-coupled approximants. This restriction has the severe consequence that the quantum field theory (QFT) arising in the limit $\mathrm{f} \to \infty$ always comes along *in combination with a self-consistently determined metric*,

$$\mathsf{App}^{\mathrm{SC}}(\mathrm{f}) \xrightarrow{\mathrm{f}\to\infty} \Psi_{\mathrm{QFT}} \otimes \text{ self-consistent metric}\,. \tag{15.3}$$

The self-consistently determined metric hereby is a solution to semiclassical Einstein equation with the appropriate stress-energy tensor $T_{\mu\nu}[\Psi_{\mathrm{QFT}}]$ on its RHS.

The use of gravity-coupled approximants is motivated by several considerations: Firstly, in practice one always has to resort to approximate calculations in some way. Hence, the resulting approximants should be up to representing physically realizable systems in their own right.

Secondly, gravity-coupled approximants are subject to classical General Relativity because they fulfill the self-consistency condition. This condition is rather natural if one considers a matter QFT on a curved background spacetime or especially Quantum Gravity itself.

Thirdly, gravity-coupled approximants should not be confined to a technical tool for regularizing a QFT. It could very well be that experimentalists observe some finite value $\mathrm{f}_{\mathrm{Obs}}$ at which Nature is to be described by the approximant $\mathsf{App}(\mathrm{f}_{\mathrm{Obs}})$, rather than by the limit $\mathrm{f} \to \infty$ (cf. Section 1.5 of [4]).

Fourthly, the limit $\mathrm{f} \to \infty$ and imposing the backreaction on the metric do not commute. Let us expound this with help of the illustration in Figure 15.1: The top-left box depicts an approximant $\Psi_{\mathrm{f}} \otimes g_{\mu\nu}$, composed of the quantum state Ψ_{f} with f degrees of freedom and an arbitrary metric $g_{\mu\nu}$. The standard approach towards the limit $\mathrm{f} \to \infty$ now consists in assuming a rigid spacetime (RS) with *fixed and arbitrary metric* $g_{\mu\nu}^{\mathrm{RS}}$. Following the top horizontal arrow this leads to a QFT on a rigid spacetime given by the approximant $\Psi_\infty \otimes g_{\mu\nu}^{\mathrm{RS}}$. Then after imposing the backreation, following the downward arrow, we calculate the induced cosmological constant Λ_{ind} and find that it is formally infinite in absence of a cutoff. (If the bare parameters are allowed to depend on f one will equivalently find the naturalness problem of an infinite finetuning of the cosmological constant.) This phenomenologically unacceptable state is illustrated in the lower right box.

On the other hand, if start over at the top-left box with the approximant $\Psi_{\text{f}} \otimes g_{\mu\nu}$ and first impose the backreaction on the metric, following the downward arrow, we will arrive at a gravity-coupled approximant $\Psi_{\text{f}}^{\text{SC}} \otimes (g_{\text{f}}^{\text{SC}})_{\mu\nu}$. Only then we take the limit $\text{f} \to \infty$ and arrive at the state $\Psi_{\infty}^{\text{SC}} \otimes (g_{\infty}^{\text{SC}})_{\mu\nu}$ of a QFT on a self-consistent spacetime. Its metric is selected by the self-consistency condition, adhering to the first requirement of Background Independence. As we will demonstrate in the applications in the following chapters, the QFT on a self-consistent spacetime can exhibit features tremendously different the QFT obtained on a rigid spacetime.

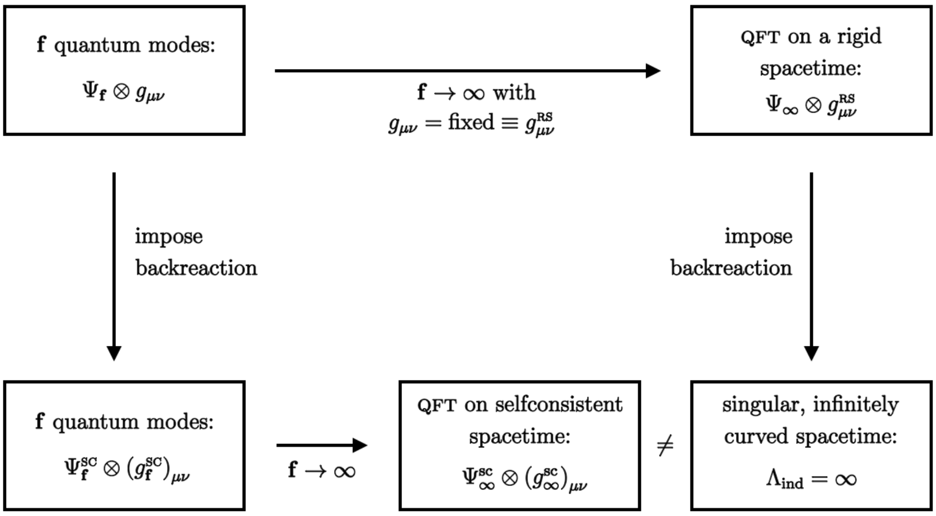

FIGURE 15.1. Inclusion of the gravitational backreaction does not commute with the limit $\text{f} \to \infty$.

15.3. Third requirement: N-type cutoffs

Up to now the concept of a gravity-coupled approximant is still unspecific. The question arises how to construct suitable and useful gravity-coupled approximants. In the following, we will outline a possible construction by imposing a cutoff which is of "N-type". The name derives from the fact that in typical examples the regularization parameter is a positive integer, $N \in \mathbb{N}$, but other cases will occur as well. By implementing an N-type cutoff the degrees of freedom of the quantum system become a function of N, $\mathbf{f} \equiv \mathrm{f}(N)$. In the case $N \in \mathbb{N}$, regularized quantum field theories are represented by ordered sequences of gravity-coupled approximants, $\{\mathsf{App}^{\mathrm{SC}}(N)\ ,\ N = 0, 1, 2, \dots\}$. The removal of the regulator, corresponding in the standard case to, say, sending a lattice constant to zero, amounts to following one such sequence for increasing N.

(A) Definition of a cutoff of the N-type. Let $(M, \bar{g})$ be a Riemannian manifold with metric $\bar{g}_{\mu\nu}$ of Euclidean signature, and let $\mathfrak{F}$ be some function space on $(M, \bar{g})$. We assume w.l.o.g. $\mathfrak{F}$ to be the linear span of a basis $\mathfrak{B} = \{w_\alpha(\,\bullet\,) \mid \alpha \in I\}$, where I is an index set,

$$\mathfrak{F} = \operatorname{span}\mathfrak{B}\,. \tag{15.4}$$

For a dimensionless parameter $N \in \mathbb{N}$ (or $N \in [0, \infty) =: \mathbb{R}^+$ which we, however, will not employ in the following applications) we define a subset of indices $I_N \subset I$ and the subbasis

$$\mathfrak{B}_N \coloneqq \big\{w_\alpha(\,\bullet\,) \;\big|\; \alpha \in I_N\big\} \tag{15.5}$$

such that the following properties are fulfilled:

$$\mathscr{B}_0 = \varnothing \quad , \quad \mathscr{B}_\infty = \mathscr{B} \quad \text{and} \quad N_2 > N_1 \Rightarrow \mathscr{B}_{N_2} \supset \mathscr{B}_{N_1}\,. \tag{15.6}$$

Importantly, note that no momentum scale is involved in the definition of the family of subbases $\{\mathfrak{B}_N\}_{N\in\mathbb{N}}$. Next, we define the subspace $\mathfrak{F}_N \subset \mathfrak{F}$ as the linear span of the basis $\mathfrak{B}_N$,

$$\mathfrak{F}_N \coloneqq \operatorname{span}\mathfrak{B}_N\,. \tag{15.7}$$

Then the one-parameter family $\{\mathfrak{F}_N\}_{N\in\mathbb{N}}$ is called *a cutoff of the N-type* or, short, an *N-cutoff.*

(B) Properties of N-cutoffs. Cutoffs of the N-type cope with the important technical problem of regularizing the path integral

$$Z[\bar{g}] \coloneqq \int_{\mathfrak{F}} \mathcal{D}(\phi; \bar{g})\, \mathrm{e}^{-S[\phi;\bar{g}]}\,, \tag{15.8}$$

that manifestly depends on the background metric $\bar{g}_{\mu\nu}$. (Here, $\phi \in \mathfrak{F}$.) A possible regularized counterpart of this path integral is obtained by restricting the domain of integration to $\mathfrak{F}_N$:

$$Z_N[\bar{g}] \coloneqq \int_{\mathfrak{F}_N} \mathcal{D}(\phi;\bar{g})\, \mathrm{e}^{-S[\phi;\bar{g}]}\,. \tag{15.9}$$

We can regard the functional $Z_N[\bar{g}]$ as a partition function which describes an approximant with a finite number $\mathrm{f} \equiv \mathrm{f}(N)$ degrees of freedom, given by modes of the field $\phi|_{\mathfrak{F}_N}$. This interpretation is legitimate since we may always choose $\mathfrak{F}_N$ "sufficiently small" such that this desired property of $Z_N[\bar{g}]$ holds.

The regulator's free parameter N does not imply a momentum or length scale that would seperate modes of ϕ retained in $\mathfrak{F}_N$ from those discarded. Furthermore, no metric is required to impose an N-cutoff. The only required ingredients are appropriately chosen subsets of indices $\{I_N\}_{N\in\mathbb{N}}$.

Furthermore, we point out that the concepts of *regularization* and *renormalization* must be looked upon separately. Especially, an N-cutoff is a mere regulator. However, it is clear that it is generally possible to contruct sequences $Z_N[\bar{g}]$, $N = 0, 1, \ldots$, that converge to some limit. If this is (im-)possible, we may call the theory under consideration to be (non-)renormalizable.

(C) Eigenbases of metric dependent operators. Let $\mathscr{K}[\bar{g}]$ be a self-adjoint positive operator acting on $\mathfrak{F}$, e.g. the negative Laplacian $\mathscr{K}[\bar{g}] = -\Box_{\bar{g}} \equiv -\bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$, and consider its eigenvalue problem,

$$\mathscr{K}[\bar{g}]w_\alpha[\bar{g}](x) = \lambda_\alpha[\bar{g}]w_\alpha[\bar{g}](x) \quad , \quad \alpha \in I\,, \tag{15.10}$$

with the eigenvalues $\lambda_\alpha[\bar{g}]$ and the eigenfunctions $w_\alpha[\bar{g}]$. We assume the spectrum of $\mathscr{K}[\bar{g}]$ discrete, e.g. by assuming M to be compact. The eigenfunctions then w.l.o.g. form a basis $\mathfrak{B}[\bar{g}] = \{w_\alpha[\bar{g}](\,\bullet\,) \mid \alpha \in I\}$ of $\mathfrak{F}$. It is clear that we may choose metric-independent subsets of indices $I_N \subset I$ and define

$$\mathfrak{B}_N[\bar{g}] = \big\{w_\alpha[\bar{g}](\,\bullet\,) \bigm| \alpha \in I_N\big\}\,, \tag{15.11}$$

such that the resulting one-parameter family of subspaces

$$\big\{\mathfrak{F}_N[\bar{g}] = \operatorname{span}\mathfrak{B}_N[\bar{g}]\big\}_{N\in\mathbb{N}} \tag{15.12}$$

is an N-cutoff. Here, we emphasize that although each subspace $\mathfrak{F}_N[\bar{g}]$ depends on the metric the requirements for an N-cutoff are still met because the subsets of indices $\{I_N\}$ are metric-independent. Furthermore, we emphasize that N-cutoffs constructed from eigenbases of metric-dependent operators are not restricted to those with descrete spectra. However, if the spectrum if continuous the N-cutoff will be given by a continuous parameter $N \in \mathbb{R}^+$.

(D) N-cutoffs vs. $\mathscr{P}$-cutoffs. We complete the discussion of N-cutoffs by expounding a prominent family of counterexamples that do not comply with the third requirement: cutoff of the "$\mathscr{P}$-type". We exemplify these by returning to the above eigenvalue problem, whereby we assume, for convenience, that the eigenvalues are non-degenerate. For the metric $\bar{g}_{\mu\nu}$ fixed, we solve the eigenvalue $\lambda = \lambda_\alpha[\bar{g}]$ for the label, $\alpha = \alpha[\bar{g}](\lambda)$. Therewith, one in practice often uses the eigenvalues to enumerate the eigenfunctions,

$$\mathfrak{B}[\bar{g}] = \big\{ W_\lambda[\bar{g}](\,\bullet\,) \;\big|\; \lambda \in \operatorname{spec}(\mathscr{K}) \big\} \tag{15.13}$$

with the reparametrized mode functions

$$W_\lambda[\bar{g}](x) \coloneqq w_\alpha[\bar{g}](x)\Big|_{\alpha=\alpha[\bar{g}](\lambda)} \,. \tag{15.14}$$

It is not far to seek to define the subbases $\{\mathfrak{B}_{\mathscr{P}}\}_{\mathscr{P}\in\mathbb{R}^+}$ by fixing a momentum scale $\mathscr{P}^2$,

$$\mathfrak{B}_{\mathscr{P}} = \big\{ W_\lambda(\,\bullet\,) \;\big|\; \lambda \le \mathscr{P}^2 \big\} \,. \tag{15.15}$$

The corresponding "$\mathscr{P}$-cutoff" given by

$$\big\{ \mathfrak{F}_{\mathscr{P}} = \operatorname{span} \mathfrak{B}_{\mathscr{P}} \big\}_{\mathscr{P}\in\mathbb{R}^+} \tag{15.16}$$

however does not meet the third requirement, i.e., it cannot be uniquely mapped to an N-cutoff. The reason is obvious: Due to the substitution $\alpha \to \alpha[g](\lambda)$ the *enumeration* of the basis functions has become explicitly metric-dependent, i.e., likewise would be the subsets of indices $\{I_N\}$ had we somehow managed to related $\mathscr{P}$ to N. Also, we note that λ and $\mathscr{P}$, unlike α and N, are dimensionful with canonical mass dimensions $[\lambda] = 2$ and $[\mathscr{P}] = 1$. On the other hand, for a given N-cutoff we can always construct a $\mathscr{P}$-cutoff by $\mathscr{P}^2(N)[\bar{g}] = \lambda_N[\bar{g}]$. All in all, we can conclude that the eigenfunctions used to construct a basis of $\mathfrak{F}$ may be allowed to depend on the metric, but their labeling necessarily must be metric-independent.

15.4. First and second type of approximants

We conclude this chapter with a general remark on two different types of approximants that we may consider for employing the self-consistency condition in order to arrive at a gravity-coupled approximant. In other words, we point out two different candidates for a quantum stress-energy tensor that after the implementation of an N-cutoff become approximants, and whose backreaction on the background metric we may then determine. The explicit construction of these approximants will happen in the subsequent chapters; here we only outline their unregularized "raw-versions".

In Chapters 3 and 4 we had developed the Background Independet quantum field theoretical treatment of a massive scalar field A as well as of the graviton field $h_{\mu\nu}$. The condensate of the one-loop approximations outlined in these chapters was the relation

$$\mathrm{e}^{-\Gamma[\bar{g}]} := \int \prod_i \mathcal{D}\Big(\widehat{\phi}_i;\bar{g}\Big)\ \mathrm{e}^{-S[\{\widehat{\phi}_j\};\bar{g}]} \tag{15.17}$$

with $\bar{g}$ the background metric and e.g. $\{\widehat{\phi}_j\} = \{\widehat{A}\}$ for the quantized scalar field or $\{\widehat{\phi}_j\} = \{\widehat{h}_{\mu\nu}, \bar{C}_\mu, C^\mu\}$ for the quantized graviton field that is supplemented by the ghost fields $\bar{C}_\mu$ and C^μ. Here, $\Gamma[\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + \Gamma_{1\mathrm{L}}[\bar{g}]$ is the one-loop effective action at vanishing field expectation value $\langle\phi_j\rangle = 0$ and $S[\{\phi_j\};\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{M}}[\{\phi_j\};\bar{g}]$ is the sum of the Einstein-Hilbert action of the background metric and the matter action for the classical fields $\{\phi_j\}$.

In Chapters 16, 17 and 18 we then will analyze the backreation of suitable approximants representing the matter fields A and $h_{\mu\nu}$ on the background metric $\bar{g}$, respectively. Importantly, we can do so in two ways. The first way, that we refer to as "*type 1*", is a standard semi-classical treatment of Einstein's equation for the background metric $\bar{g}$: only the matter fields are quantized and one takes the expectation value of Einstein's equation with respect to (15.17),

$$\left\langle \frac{S[\{\widehat{\phi}_j\};\bar{g}]}{\delta g_{\mu\nu}(x)} \right\rangle = 0\,. \tag{15.18}$$

Associated to the matter action $S_{\mathrm{M}}[\{\phi_j\};\bar{g}]$ is the (Euclidean) *stress-energy tensor*

$$T^{\mu\nu}[\{\phi_j\};\bar{g}](x) \coloneqq -\frac{2}{\sqrt{\bar{g}(x)}}\frac{\delta S_{\mathrm{M}}[\{\phi_j\};\bar{g}]}{\delta \bar{g}_{\mu\nu}(x)}\,, \tag{15.19}$$

such that, after promoting the stress-energy tensor to an operatorial relation for $\{\widehat{\phi}_j\}$, the backreaction of "type 1" amounts to the equation

$$\frac{2}{\sqrt{\bar{g}(x)}}\frac{\delta S_{\mathrm{EH}}[\bar{g}]}{\delta \bar{g}_{\mu\nu}(x)} = \left\langle T^{\mu\nu}[\{\widehat{\phi}_j\};\bar{g}](x)\right\rangle\,. \tag{15.20}$$

After imposing an N-cutoff on the RHS, the fields' degrees of freedom $\{\mathrm{f}_j(N)\}$ become encoded into the regularized stress tensor $\left\langle T^{\mu\nu}[\{\widehat{\phi}_j\};\bar{g}](x)\right\rangle_N$ which we thus, together with the background metric $\bar{g}_{\mu\nu}$, identify as a possible candidate for an *approximant*. We will call this candidate a *first type of approximant*. It is self-consistent if and only if it backreacts on the metric via the equation of motion above.

The second way to treat the backreaction of the matter fields, that we refer to as "*type 2*", is more along the lines of ordinary quantum field theory: simply take the equations of motion for the background metric $\bar{g}$ of the one-loop effective action at vanishing field expectation values $\Gamma[\bar{g}]$, i.e.,

$$\frac{\delta\Gamma[\bar{g}]}{\delta \bar{g}_{\mu\nu}(x)} = 0\,. \tag{15.21}$$

To the one-loop term $\Gamma_{1\mathrm{L}}[\bar{g}]$ of the effective action we can associate an *effective stress-energy tensor* $T^{\mu\nu}_{\mathrm{eff}}[\bar{g}]$ by

$$\frac{\delta\Gamma_{1\mathrm{L}}[\bar{g}]}{\delta \bar{g}_{\mu\nu}(x)} \eqqcolon -\frac{1}{2}\sqrt{\bar{g}(x)}\,T^{\mu\nu}_{\mathrm{eff}}[\bar{g}](x)\,, \tag{15.22}$$

such that the backreation of "type 2" amounts to the equation

$$\frac{2}{\sqrt{\bar{g}(x)}}\frac{\delta S_{\mathrm{EH}}[\bar{g}]}{\delta \bar{g}_{\mu\nu}(x)} = T^{\mu\nu}_{\mathrm{eff}}[\bar{g}](x)\,. \tag{15.23}$$

Again, after imposing an N-cutoff on the RHS, we can identify $(T^{\mu\nu}_{\mathrm{eff}})_N[\bar{g}](x)$ with a possible candidate for an approximant, which we call a *second type of approximant*.

Particularly, using Eq. (15.17) it is evident that the calculations of "type 1" and of "type 2" are a priori not identical: When taking a variation with respect to the background metric $\bar{g}$ it becomes clear that $\langle T^{\mu\nu}[\{\phi_j\};\bar{g}](x)\rangle$ and $T^{\mu\nu}_{\text{eff}}[\bar{g}](x)$ differ exactly by the variation of the $\bar{g}$-dependent measure $\prod_i \mathcal{D}(\widehat{\phi}_i;\bar{g})$. This is not surprising: as always, when promoting a classical field to a quantum operator the prescription is not unique and after all only experiment can tell which prescription is “correct”.

Let us illustrate this difference with a quick schematic example: Consider a covariant tensor field X_μ, i.e., the case $\{\phi_j\} = \{X_\mu\}$. Its matter action may have the structure

$$S_{\text{M}}[X_\mu;\bar{g}] = \frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\, X_\mu \bar{g}^{\rho\sigma}\mathscr{O}[\bar{g}]^\mu{}_\rho X_\sigma \tag{15.24}$$

with $\mathscr{O}[\bar{g}]$ some operator (that is assumed to commute with $\bar{g}_{\mu\nu}$) acting on the Hilbert space of covariant vector fields. Thus, when calculating $\langle T^{\mu\nu}[\widehat{X}_\mu;\bar{g}]\rangle$ there are two variations to perform: firstly, that of the structure $\sqrt{\bar{g}}\,\bar{g}^{\rho\sigma}$ and secondly, that of $\mathscr{O}[\bar{g}]$ itself. Interestingly, the measure $\mathcal{D}(\widehat{X}_\mu;\bar{g})$ given by Eq. (D.8) is designed exactly such that the local structure $\sqrt{\bar{g}}\,\bar{g}^{\rho\sigma}$ drops out when calulating the path integral (15.17), i.e.,

$$\mathrm{e}^{-\Gamma_{1\text{L}}[\bar{g}]} = \int \mathcal{D}\Big(\widehat{X}_\mu;\bar{g}\Big)\ \mathrm{e}^{-S_{\text{M}}[\widehat{X}_\mu;\bar{g}]} = \text{Det}\,\big(\mathscr{O}[\bar{g}]^{\boldsymbol{\cdot}}{}_{\boldsymbol{\cdot}}\big)^{-1/2} \tag{15.25}$$

where we have used Eq. (D.33) and dropped a power of $1/2$ from the result. (For this illustrative purpose, we neglected possible gauge redundancies in the path integral that we would have to account for with employing the Faddeev-Popov trick.) Therewith, we have

$$\Gamma_{1\text{L}}[\bar{g}] = \frac{1}{2}\,\text{Tr}_V\left[\ln\,\mathscr{O}[\bar{g}]^{\boldsymbol{\cdot}}{}_{\boldsymbol{\cdot}}\right]. \tag{15.26}$$

Consequently, in the calculation of $T^{\mu\nu}_{\text{eff}}[\bar{g}]$ one must perform only a variation of $\mathscr{O}[\bar{g}]$, while the structure $\sqrt{\bar{g}}\,\bar{g}^{\rho\sigma}$ has been “removed” from appearing in the one-loop effective action by choosing the correct, invariant path integral measure – illustrating the origin of the difference between the backreation of “type 1” and of “type 2”.

CHAPTER 16

A first type of approximants for a quantized scalar field

Executive summary. We determine a first type of approximants for a quantized scalar field. It is constructed by promoting its classical stress-energy tensor to an operatorial relation which then is regularized via an N-cutoff. The backreaction of this approximant on the background metric amounts to a self-consistency condition which we explicitly solve for the case that the background manifold is a 4-sphere. We show that the resulting self-consistent radii of the 4-sphere possess intriguing physical properties. As more degrees of freedom are added to the quantum system, they become larger, and thus the universe becomes flatter. Hence, in this setting, the cosmological constant problem is absent. Moreover, we show that the N-sequences of self-consistent radii allow for an explanation of the microscopic degrees of freedom which the Bekenstein-Hawking entropy of de Sitter space "counts".
What is new? All research results of this chapter are new.
Based upon: Reference [5].

For notational ease, we denote the background metric in this chapter by $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$.

16.1. The classical field

As already defined in the previous section, we employ the definition (15.19) of the (Euclidean) classical stress-energy tensor related to a matter action S_{M},

$$T^{\mu\nu} := -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}} S_{\mathrm{M}}\,, \tag{16.1}$$

which is equivalent to $\delta S_{\mathrm{M}} = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{g}\, T^{\mu\nu}\delta g_{\mu\nu}$.

The action under consideration is (3.3),[1]

$$S_{\mathrm{M}}[A;g] = \frac{1}{2}\int \mathrm{d}^d x \sqrt{g}\left(g^{\mu\nu}\partial_\mu A\,\partial_\nu A + \mu^2 A^2 + \xi R(g) A^2\right) . \tag{16.2}$$

Let us directly perform the metric variation, $\delta g_{\mu\nu} =: h_{\mu\nu}$, of the action in order to find the stress-energy tensor:

$$\begin{aligned}\delta S_{\mathrm{M}}[A;g] = \frac{1}{2}\int \mathrm{d}^d x \Big\{ &(\delta\sqrt{g})\left[g^{\mu\nu}\partial_\mu A\,\partial_\nu A + (\mu^2 + \xi R(g))A^2\right] \\ &+ \sqrt{g}\left[\delta g^{\mu\nu}\partial_\mu A \partial_\nu A + \xi \delta R A^2\right]\Big\} .\end{aligned} \tag{16.3}$$

With the help of appendix B we find

$$\begin{aligned}\delta S_{\mathrm{M}}[A;g] = \frac{1}{2}\int \mathrm{d}^d x \Big\{ &\frac{1}{2}\sqrt{g} g^{\alpha\beta} h_{\alpha\beta}\left[g^{\mu\nu}\partial_\mu A\,\partial_\nu A + (\mu^2 + \xi R(g))A^2\right] \\ - \sqrt{g} g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} \partial_\mu A \partial_\nu A &- \sqrt{g}\xi\left[R^{\alpha\beta}h_{\alpha\beta} - D_\beta(D_\alpha h^{\alpha\beta} - D^\beta h^\alpha_{\ \alpha})\right] A^2\Big\} ,\end{aligned} \tag{16.4}$$

which can be rearranged to

$$\begin{aligned}\delta S_{\mathrm{M}}[A;g] = -\frac{1}{2}\int \mathrm{d}^d x \Big\{ &-\frac{1}{2} g^{\mu\nu}(\partial A)^2 - \frac{1}{2}(\mu^2 + \xi R) g^{\mu\nu} A^2 + \partial^\mu A\,\partial^\nu A \\ &+ \xi R^{\mu\nu} A^2 - \xi D^\mu D^\nu A^2 + \xi g^{\mu\nu} D^2 A^2 \Big\} h_{\mu\nu} .\end{aligned} \tag{16.5}$$

Therefrom, we can read off the stress-energy tensor:[2]

$$\boxed{\begin{aligned}T^{\mu\nu}[A;g] = \partial^\mu A\,\partial^\nu A &- \frac{1}{2} g^{\mu\nu}(\partial A)^2 - \frac{1}{2} g^{\mu\nu}\mu^2 A^2 \\ &+ \xi\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) A^2 - \xi D^\mu D^\nu A^2 + \xi g^{\mu\nu} D^2 A^2\end{aligned}} . \tag{16.6}$$

To verify the conservation law $D_\mu T^{\mu\nu} = 0$ that holds on-shell only, we define the field A^{OS} as the solution to the classical equation of motion:

$$\left.\frac{\delta S_{\mathrm{M}}[A;g]}{\delta A}\right|_{A=A^{\mathrm{OS}}} = 0 \quad\Leftrightarrow\quad \left[-\Box_g + \mu^2 + \xi R\right] A^{\mathrm{OS}} = 0 . \tag{16.7}$$

---

[1]Where the context is clear, we paranthesize products in a lax way, e.g. write $\partial_\mu A\,\partial_\nu A$ instead of $(\partial_\mu A)\partial_\nu A$.

[2]This is the Euclidean counterpart of the stress-energy stated in [154, p. 45] that has been obtained from the corresponding Lorentzian action.

Straightforwardly we obtain

$$D_\mu T^{\mu\nu}[A^{\mathrm{OS}};g] = (\partial^\mu A^{\mathrm{OS}}) g^{\nu\rho}[D_\mu, D_\rho] A^{\mathrm{OS}} \\ + \xi g^{\nu\rho} \left\{ R^\mu{}_\rho D_\mu + D^\mu [D_\rho, D_\mu] + [D_\rho, D_\mu] D^\mu \right\} A^2_{\mathrm{OS}} . \quad (16.8)$$

As $[D_\mu, D_\rho] f = 0$ and $[D_\rho, D_\mu] D^\mu f = -R^\mu{}_\rho D_\mu f$ for any scalar field f, the conservation law $D_\mu T^{\mu\nu}[A^{\mathrm{OS}};g] = 0$ is fulfilled, as it should be.

The trace of the stress-energy tensor is given by

$$T_\mu{}^\mu[A;g] = g_{\mu\nu} T^{\mu\nu}[A;g] = \left[2(d-1)\xi - \frac{d}{2} + 1 \right] (\partial A)^2 - \frac{d}{2}\mu^2 A^2 \\ + \left(-\frac{d}{2} + 1 \right) \xi R A^2 - 2(d-1)\xi A(-\Box_g) A . \quad (16.9)$$

Using the equation of motion, the traced stress-energy tensor can be rewritten into

$$T_\mu{}^\mu[A^{\mathrm{OS}};g] = \left[2(d-1)\xi - \frac{d-2}{2} \right] (\partial A^{\mathrm{OS}})^2 + \left[-\frac{d}{2} + 2(d-1)\xi \right] \mu^2 A^2_{\mathrm{OS}} \\ + \left[2(d-1)\xi - \frac{d-2}{2} \right] \xi R A^2_{\mathrm{OS}} . \quad (16.10)$$

Thus, on-shell, after using the equation of motion, we find that $T_\mu{}^\mu[A^{\mathrm{OS}};g] = 0$ *if* $\xi = \frac{d-2}{4(d-1)}$ *and* $\mu = 0$ (cf. [166, p. 119]).

Next, we consider the integrated and traced stress-energy tensor. We use this occasion to introduce the operator

$$\boxed{\mathscr{T} := -2 \int \mathrm{d}^d x \, g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)}} \quad (16.11)$$

that we are going to widely use throughout this chapter. With its help the integrated and traced stress-energy tensor of any matter action S_{M} can be written as

$$\int \mathrm{d}^d x \sqrt{g(x)} \, T_\mu{}^\mu(x) = \mathscr{T} S_{\mathrm{M}} . \quad (16.12)$$

Here, after a partial integration (assuming an empty boundary) we find

$$\mathscr{T} S_{\mathrm{M}}[A;g] = \int \mathrm{d}^d x \sqrt{g(x)} \, T_\mu{}^\mu[A;g](x) = \int \mathrm{d}^d x \sqrt{g} \left\{ \left(-\frac{d}{2} + 1 \right) A(-\Box_g) A \right. \\ \left. - \frac{d}{2}\mu^2 A^2 + \left(-\frac{d}{2} + 1 \right) \xi R A^2 \right\} . \quad (16.13)$$

Again, using the equation of motion, this result can be rewritten into

$$\int \mathrm{d}^d x \sqrt{g}\, T_\mu{}^\mu[A^{\mathrm{OS}}; g] = -\int \mathrm{d}^d x \sqrt{g}\, \mu^2 A_{\mathrm{OS}}^2 \,. \tag{16.14}$$

This implies that on-shell, after using the equation of motion, $\int \mathrm{d}^d x \sqrt{g}\, T_\mu{}^\mu[A^{\mathrm{OS}}; g] = 0$ *if* $\mu = 0$ *(for any value of* ξ*!).*

16.2. The quantum system at finite N and backreaction of the metric

(A) The semiclassical Einstein equation. In the setting we are considering here, Einstein's classical equation of motion for the background metric $g_{\mu\nu}$ is given by

$$\frac{\delta S[A; g]}{\delta g_{\mu\nu}(x)}[A; g] = 0\,, \tag{16.15}$$

where the action

$$S[A; g] := S_{\mathrm{EH}}[g] + S_{\mathrm{M}}[A; g] \tag{16.16}$$

consists of the Einstein-Hilbert action (3.2) and the matter action (3.3) for the scalar field A. The respective components of these equations of motion are given by

$$\frac{\delta S_{\mathrm{EH}}[g]}{\delta g_{\mu\nu}(z)} = \frac{1}{16\pi G}\sqrt{g(z)}\, \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu}\right)(z) \tag{16.17}$$

and

$$\frac{\delta S_{\mathrm{M}}[A; g]}{\delta g_{\mu\nu}(z)} = -\frac{1}{2}\sqrt{g(z)}\, T^{\mu\nu}[A; g](z)\,. \tag{16.18}$$

With help of the operator $\mathscr{T}$ defined by Eq. (16.11), the generic integrated and traced equations of motion can be compactly summarized as

$$-2\int \mathrm{d}^d z\, g_{\mu\nu}(z) \frac{\delta S}{\delta g_{\mu\nu}}[A; g] = \mathscr{T} S[A; g] = \mathscr{T} S_{\mathrm{EH}}[g] + \mathscr{T} S_{\mathrm{M}}[A; g] = 0\,. \tag{16.19}$$

On a generic manifold, it is easy to see that

$$\mathscr{T} S_{\mathrm{EH}}[g] = \frac{1}{8\pi G}\int \mathrm{d}^d z \sqrt{g(z)}\, \left[\left(\frac{d}{2} - 1\right) R(z) - d\,\Lambda\right]\,, \tag{16.20}$$

such that on a maximally symmetric background spacetime, where the (constant) scalar curvature is the sole magnitude of curvature, it is in fact sufficient to consider the integrated and traced equations of motion in order to determine the full geometry of the background manifold. These amount to

$$\int \mathrm{d}^d z\,\sqrt{g(z)}\,\left[\left(1-\frac{d}{2}\right)R(z)+d\,\Lambda\right]=8\pi G\int \mathrm{d}^d z\sqrt{g(z)}\,T_\mu{}^\mu[A;g](z)\,. \quad (16.21)$$

We will restrict the treatment of the backreation of the first type of approximant to the corresponding semi-classical equations of motion. These are obtained by quantizing the scalar field, $A\mapsto\hat{A}$, and replacing $T_\mu{}^\mu[A;g](z)$ on the RHS by $\langle\hat{T}^\mu_\mu[g](z)\rangle$:

$$\int \mathrm{d}^d z\,\sqrt{g(z)}\,\left[\left(1-\frac{d}{2}\right)R(z)+d\,\Lambda_{\mathrm{b}}\right]=8\pi G\int \mathrm{d}^d z\sqrt{g(z)}\,\left\langle\hat{T}^\mu_\mu[g](z)\right\rangle\,. \quad (16.22)$$

Here, we have traded the classical cosmological constant for its bare counterpart Λ_{b}. To establish a notion of the expectation value of the quantized traced stress-energy tensor, we *define* the following operator on the Hilbert space of scalar fields:[3]

$$\boxed{\begin{aligned}\hat{T}^\mu_\mu[g]:=T_\mu{}^\mu[\hat{A};g]=\left[2(d-1)\xi-\frac{d-2}{2}\right]&D_\mu\hat{A}\,D^\mu\hat{A}-\frac{d}{2}\mu^2\hat{A}^2\\ &-\frac{d-2}{2}\xi R\hat{A}^2-2(d-1)\xi\hat{A}(-\Box_g)\hat{A}\end{aligned}}\,. \quad (16.23)$$

Its expectation value is given by (still requiring regularization!)

$$\begin{aligned}\left\langle\hat{T}^\mu_\mu[g](x)\right\rangle=\lim_{y\to x}\Bigg\{&\left[2(d-1)\xi-\frac{d-2}{2}\right]D^x_\mu D^\mu_y-\frac{d}{2}\mu^2\\ &-\frac{d-2}{2}\xi R(x)-2(d-1)\xi\left(-\Box^x_g\right)\Bigg\}G(x,y)\,. \end{aligned}\quad (16.24)$$

(B) The quantum system at finite N. Now it is time to regularize the RHS by implementing an N-cutoff which results in the first type of approximant $\langle\hat{T}^\mu_\mu[g](x)\rangle_N$. Therefore, we assume the background manifold to be compact and without boundary. As we had already analyzed the spectral problem of the operator $\mathscr{K}[g]$ given by Eq. (3.4), we can define a suitable cutoff of the N-type $\{L^2(M,g)_N\}_{N\in\mathbb{N}}$ by truncating the basis (3.6) of $L^2(M,g)$ built from

[3] *Formally*, also the metric $\hat{g}_{\mu\nu}$ appears to be an operator on the scalar's Hilbert space; it is rigidly coupled to the matter field and has no dynamical degrees of freedom of its own. Thus, we will stick with denoting it by simply $g_{\mu\nu}$.

eigenfunction of $\mathscr{K}[g]$ at the dimensionless number $N \in \mathbb{N}$, playing the role of an UV cutoff:

$$\mathfrak{B}_N := \left\{ \chi_{n,m} \;\middle|\; n = 1, 2, \dots, N \ , \ m \in \{1, 2, \dots, D_n\} \right\} . \tag{16.25}$$

Note that the index n starts running at $n = 1$ because we had separated of potential zero modes of the scalar field. Consequently, instead of the full scalar field $A \in L^2(M, g)$ we only consider its truncated counterpart restricted to $L^2(M, g)_N$, i.e., its expansion (3.7) is truncated accordingly as

$$A(x) = \sum_{n=1}^{N} \sum_{m=1}^{D_n} a_{n,m} \chi_{n,m}(x) . \tag{16.26}$$

The degrees of freedom of the correspondingly truncated quantum field are then given by the set $\{a_{n,m} \mid n = 1, 2, \dots, N \ , \ m \in \{1, 2, \dots, D_n\}\}$. Their total number is

$$\mathrm{f}(N) = \sum_{n=1}^{N} D_n . \tag{16.27}$$

By further implementing the N-cutoff in Eq. (3.20), in which we therefore simply must cut the sum on the RHS at N, we immediately obtain the desired first type of approximant:

$$\begin{aligned} \left\langle \hat{T}^{\mu}_{\mu}[g](x) \right\rangle_N = \sum_{n=1}^{N} \sum_{m=1}^{D_n} \frac{1}{\mathscr{F}_n} \Bigg\{ & \left[2(d-1)\xi - \frac{d-2}{2} \right] (D_\mu \chi_{n,m})(x)(D^\mu \chi^*_{n,m})(x) \\ & - \frac{d}{2}\mu^2 \chi_{n,m}(x)^2 - \frac{d-2}{2} \xi R(x) \chi_{n,m}(x)^2 \\ & - 2(d-1)\xi \chi^*_{n,m}(x)(-\Box_g)\chi_{n,m}(x) \Bigg\} . \end{aligned} \tag{16.28}$$

More generally, the expectation value $\langle A^2(x) \rangle$ is regulated via the N-cutoff as

$$\left\langle \widehat{A}^2(x) \right\rangle_N = \lim_{y \to x} G(x, y)_N = \sum_{n=1}^{N} \sum_{m=1}^{D_n} \frac{\chi_{n,m}(x)\chi^*_{n,m}(x)}{\mathscr{F}_n} . \tag{16.29}$$

The correspondingly regulated expectation value of the kinetic term is given by

$$\left\langle (\partial \widehat{A})^2(x) \right\rangle_N = \lim_{y \to x} \bar{g}^{\mu\nu}(x) \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} G(x, y)_N \tag{16.30}$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{D_n} \frac{(\bar{D}_\mu \chi_{n,m})(x)(\bar{D}^\mu \chi^*_{n,m})(x)}{\mathscr{F}_n} . \tag{16.31}$$

In the limit $N \to \infty$ we would arrive back at the unregularized results.

Going on-shell at the classical level now, i.e., at the quantum level, corresponds to using the (solved) eigenvalue problem $(-\Box_g + \mu^2 + \xi R)\chi_{n,m} = \mathscr{F}_n \chi_{n,m}$ *under the mode sum.* Exploiting this eigenvalue problem, the expectation value becomes

$$\left\langle \hat{T}^\mu_\mu[g](x) \right\rangle_N = \sum_{n=1}^{N} \sum_{m=1}^{D_n} \frac{1}{\mathscr{F}_n} \Bigg\{ \left[2(d-1)\xi - \frac{d-2}{2} \right] (D_\mu \chi_{n,m})(x)(D^\mu \chi^*_{n,m})(x) + \left[-\frac{d}{2} + 2(d-1)\xi \right] \mu^2 \chi_{n,m}(x)^2 + \left[2(d-1)\xi - \frac{d-2}{2} \right] \xi R(x) \chi_{n,m}(x)^2 - 2(d-1)\xi \mathscr{F}_n \chi_{n,m}(x)^2 \Bigg\} . \tag{16.32}$$

The difference between this expactation value and its classical counterpart (16.10) is noticeable. Therewith, we find for the first type of approximant that "on-shell", after using the solved eigenvalue problem,

$$\left\langle \hat{T}^\mu_\mu[g](x) \right\rangle = -\frac{d-2}{2} \sum_{n=1}^{N} \sum_{m=1}^{D_n} \chi_{n,m}(x)^2 \tag{16.33}$$

if $\xi = \frac{d-2}{4(d-1)}$ and $\mu = 0$.

Let us go back again to the "off-shell" result (16.28). With help of the operator $\mathscr{T}$ defined in (16.11), the integrated expectation value may be written as

$$\boxed{\int \mathrm{d}^d x \sqrt{g(x)} \left\langle \hat{T}^\mu_\mu[g](x) \right\rangle_N = \left\langle \mathscr{T} S_\mathrm{M}[\hat{A}; g] \right\rangle_N} . \tag{16.34}$$

It is obvious that this integrated expectation value can be heavily simplified by exploiting the orthogonality property (A.29) of the eigenfunctions $\{\chi_{n,m}\}$. However, at the moment the x-dependence of the scalar curvature R thwarts this plan and so we rearrange the RHS of Eq. (16.28): After a partial integration one has

$$\int \mathrm{d}^d x \sqrt{g} \left\langle \hat{T}^{\mu}_{\mu}[g] \right\rangle_N = \int \mathrm{d}^d x \sqrt{g} \sum_{n=1}^{N} \sum_{m=1}^{D_n} \frac{1}{\mathscr{F}_n} \chi^*_{n,m} \left\{ -\frac{d-2}{2} \mathscr{K}[g] - \mu^2 \right\} \chi_{n,m}$$
$$= \int \mathrm{d}^d x \sqrt{g} \sum_{n=1}^{N} \sum_{m=1}^{D_n} \left\{ -\frac{d}{2} \chi^2_{n,m} + \chi^*_{n,m} \frac{\mathscr{K}[g] - \mu^2}{\mathscr{K}[g]} \chi_{n,m} \right\} . \tag{16.35}$$

Now, we could make use of the orhtogonality property (A.29) but instead we recognize that the RHS defines two traces over the Hilbert space S of scalar fields, that here are evaluated in the basis $\{\chi_{n,m}\}$ of S:

$$\boxed{\begin{aligned}\int \mathrm{d}^d x \sqrt{g(x)} \left\langle \hat{T}^{\mu}_{\mu}[g](x) \right\rangle_N &= \left\langle \mathscr{T} S_{\mathrm{M}}[\hat{A}; g] \right\rangle_N \\ &= -\frac{d}{2} \operatorname{Tr}_S[\mathbb{1}_S]_N + \operatorname{Tr}_S \left[\frac{\mathscr{K}[g] - \mu^2}{\mathscr{K}[g]} \right]_N\end{aligned}} . \tag{16.36}$$

Here,

$$\operatorname{Tr}_S[\mathscr{O}]_N = \sum_{n=1}^{N} \sum_{m=1}^{D_n} \int \mathrm{d}^d x \sqrt{g(x)}\, \chi^*_{n,m}(x) \mathscr{O}^{\mathrm{diff}}_x \chi_{n,m}(x) \tag{16.37}$$

denotes the restriction of the trace to $L^2(M, g)_N$. Note that

$$\operatorname{Tr}_S[\mathbb{1}_S]_N = \mathrm{f}(N) \tag{16.38}$$

counts the degrees of freedom of the quantum system. Therewith, the back-reation of the first type of approximant on a generic background, restricted to the integrated and traced semi-classical equations of motion (16.22), is given by

$$\int \mathrm{d}^d z \sqrt{g(z)} \left[\left(1 - \frac{d}{2} \right) R(z) + d\, \Lambda_{\mathrm{b}} \right]$$
$$= 8\pi G \left\{ -\frac{d}{2} \operatorname{Tr}_S[\mathbb{1}_S]_N + \operatorname{Tr}_S \left[\frac{\mathscr{K}[g] - \mu^2}{\mathscr{K}[g]} \right]_N \right\} . \tag{16.39}$$

(C) The case $M = S^d(L)$. Next, we specialize for the case $M = S^d(L)$, the d-dimensional sphere of radius L. In this case, the radius L, the Euclidean version of the Hubble length, is the only free remaining parameter of the geometry: If $\gamma_{\mu\nu}$ denotes the dimensionless metric on the unit d-sphere, then $g_{\mu\nu} = L^2 \gamma_{\mu\nu}$

will be a metric on $S^d(L)$. Furthermore, its scalar curvature is a constant, $R(x) \equiv \text{const.}$, and related to the radius L by

$$R(L) = \frac{d(d-1)}{L^2}\,. \tag{16.40}$$

This implies that the eigenfunctions $\chi_{n,m}$ of $\mathscr{K}[g]$ are *identical* to the eigenfunctions $u_{n,m}$ of the negative Laplacian,

$$-\Box_g u_{n,m}(x) = \mathscr{E}_n u_{n,m}(x)\,, \tag{16.41}$$

cf. appendix A.1.3. The eigenvalues then are related by

$$\mathscr{F}_n = \mathscr{E}_n + \mu^2 + \xi R\,. \tag{16.42}$$

In this case, we especially find the Green's function

$$G(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{D_n} \frac{u_{n,m}(x)u^*_{n,m}(y)}{\mathscr{E}_n + \mu^2 + \xi R} \tag{16.43}$$

that we again will apply in the limit $y \to x$ and cosequently regularize by cutting the sum over n at the *dimensionless* UV*-cutoff* N. Then, the $(-\Box_g)$-eigenvalue $\mathscr{E}_N$ is the highest one that is retained in the sum. For a given positive integer N, the corresponding *dimensionful* UV*-cutoff* is therefore

$$\mathscr{P}^2_{\text{UV}}(N) = \mathscr{E}_N = \frac{N(N+d-1)}{L^2} = \frac{N^2}{L^2}\left\{1 + O\left(\frac{1}{N}\right)\right\}\,. \tag{16.44}$$

Note that the leading term is d-independent.

By virtue of the maximally symmetric ansatz $S^d(L)$, it is sufficient to consider the integrated and contracted Einstein equation that fully determines L. Furthermore, we can make use of the fact that its volume $\text{vol}\left[S^d(L)\right] = \int \mathrm{d}^d x \sqrt{g(x)}$ is finite; hence the classical equation of motion (16.21) becomes

$$R = \frac{2d}{d-2}\Lambda - \frac{16\pi G}{(d-2)\text{vol}\left[S^d(L)\right]} \int \mathrm{d}^d z \sqrt{g(z)}\, T_\mu{}^\mu[A;g](z)\,. \tag{16.45}$$

The corresponding *semi-classical* equation of motion (16.39) is obtained by replacing $T_\mu{}^\mu[A;g](z)$ by the first type of approximant $\langle \hat{T}^\mu_\mu[g](z)\rangle_N$, i.e.,

$$R = \frac{2d}{d-2}\Lambda_\text{b} - \frac{16\pi G}{(d-2)\text{vol}\left[S^d(L)\right]} \int \mathrm{d}^d z \sqrt{g(z)}\, \left\langle \hat{T}^\mu_\mu[g](z)\right\rangle_N\,. \tag{16.46}$$

The first type of approximant on the RHS fulfills Eq. (16.36). Therewith, we can re-express it in terms of regularized traces that we can calculate using the eigenbasis $\{u_{n,m}\}$ of $-\Box_g$, thereby exploiting the eigenvalue problem $-\Box_g u_{n,m} = \mathscr{E}_n u_{n,m}$, i.e.,

$$\begin{aligned}
\Theta_N(L) := \left\langle \mathscr{T} S_{\mathrm{M}}[\hat{A}; g] \right\rangle_N &= \int \mathrm{d}^d z \sqrt{g(z)} \left\langle \hat{T}^\mu_\mu[g](z) \right\rangle_N \\
&= -\frac{d}{2} \operatorname{Tr}_S[\mathbb{1}_S]_N + \operatorname{Tr}_S \left[\frac{\mathscr{K}[g] - \mu^2}{\mathscr{K}[g]} \right]_N \\
&= -\frac{d}{2} \sum_{n=1}^{N} D_n + \sum_{n=1}^{N} D_n \frac{\mathscr{E}_n(L) + \xi R(L)}{\mathscr{E}_n(L) + \mu^2 + \xi R(L)} \\
&= -\sum_{n=1}^{N} D_n \left[\frac{d}{2} - 1 + \frac{\mu^2}{\mathscr{E}_n(L) + \mu^2 + \xi R(L)} \right] \\
&= -\left(\frac{d}{2} - 1 \right) \mathrm{f}(N) - \sum_{n=1}^{N} D_n \frac{\mu^2}{\mathscr{E}_n(L) + \mu^2 + \xi R(L)} \,.
\end{aligned} \tag{16.47}$$

We can already point out two special cases regarding the mass dependence. The function $\mu^2 \mapsto \mu^2/(\mathscr{E}_n + \mu^2 + \xi R)$ interpolates between 0 in the limit $\mu \to 0$ and 1 in the limit $\mu \to \infty$ at $N < \infty$. The transition between these limits occurs roughly at $\mu^2 \approx \mathscr{E}_n + \xi R \approx \frac{n^2}{L^2} + \xi \frac{d(d-1)}{L^2}$ (cf. table A.1 in appendix A.1.3). Especially note that

$$\Theta_N(L)\bigg|_{\mu=0} = -\left(\frac{d}{2} - 1 \right) \mathrm{f}(N) \tag{16.48}$$

turns out to be independent of ξ. Likewise, in the limit $\mu \to \infty$ at $N < \infty$ one has

$$\Theta_N(L)\bigg|_{\mu\to\infty} = -\frac{d}{2} \mathrm{f}(N) \,. \tag{16.49}$$

All in all, the equation of motion for the radius L of the sphere $S^d(L)$ is given by, with $R(L) = d(d-1)/L^2$,

$$\boxed{R(L) = \frac{2d}{d-2} \Lambda_{\mathrm{b}} - \frac{16\pi G}{(d-2)\mathrm{vol}\left[S^d(L)\right]} \Theta_N(L) \,.} \tag{16.50}$$

The L-dependence of the LHS is the standard L-dependence stemming from the LHS of Einstein's equation. On the other hand, the new L-depedence of the RHS stems from the dependence of $\langle \mathscr{T} S_{\mathrm{M}}[\hat{A}; g] \rangle$, or the "vacuum energy $\sum \frac{1}{2}\hbar\omega$", on the background geometry. This dependence is absent in Pauli-type calculations.

Entering this formula is the volume of a d-dimensional sphere that is given by

$$\mathrm{vol}\left[S^d(L)\right] = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}\, L^d \,. \tag{16.51}$$

Also required for further analysis are the eigenvalues $\mathscr{E}_n^{(d)}$ of the negative Laplacian $-\Box_g$ acting on scalar fields on $S^d(L)$ and their multiplicities $D_n^{(d)}$. These can be found in table A.1 in appendix A.1.3. We interpret solutions to Eq. (16.50) as a *N-sequences of gravity-coupled approximants*: $N = 0, 1, 2, \ldots, \infty$. Starting from $N = 0$, the classical system, we let successively $N = 1, 2 \ldots$; thus "turning on" modes of the quantum field A on the RHS, with the positive integer N acting as a dimensionless UV cutoff. At every given value of N, we determine the self-consistent radius

$$L \equiv L^{\mathrm{sc}}(N) \equiv L^{\mathrm{sc}}(N; \xi, \mu, G, \Lambda_{\mathrm{b}}) \tag{16.52}$$

of that particular d-sphere which amounts to a *self-consistent background spacetime*, provided it exists. Therewith also comes the self-consistent, quantum-mechanically generated cosmological constant

$$\Lambda^{\mathrm{sc}}(N) := \frac{3}{L^{\mathrm{sc}}(N)^2} \,. \tag{16.53}$$

16.3. N-sequences of self-gravitating quantum systems on $S^4(L)$

In four spacetime dimensions, $d = 4$, the first type of approximant given by Eq. (16.47) becomes

$$\Theta_N(L) = -\mathrm{f}(N) - \sum_{n=1}^{N} D_n \, \frac{\mu^2}{\mathscr{E}_n + \mu^2 + \xi R} \tag{16.54}$$

and the semi-classical equation of motion for the radius L of the 4-sphere $S^4(L)$ reads

$$R(L) \equiv \frac{12}{L^2} = 4\Lambda_{\mathrm{b}} + \frac{3G}{\pi L^4} \left[\mathrm{f}(N) + \sum_{n=1}^{N} D_n \frac{\mu^2}{\mathscr{E}_n(L) + \mu^2 + \xi R(L)} \right] . \tag{16.55}$$

Here, we have used of the volume of the 4-sphere that simplifies to

$$\mathrm{vol}\big[S^4(L)\big] = \frac{2\pi^2\sqrt{\pi}L^4}{\Gamma(5/2)} = \frac{8}{3}\pi^2 L^4 =: \sigma_4 L^4 \,. \tag{16.56}$$

The eigenvalues and corresponding multiplicities of $-\Box_g$ acting on scalars in $d = 4$ are given by

$$\mathscr{E}_n^{(d=4)}(L) = \frac{n(n+3)}{L^2} \quad \text{and} \quad D_n^{(d=4)} = \frac{(2n+3)(n+2)(n+1)}{6} \,. \tag{16.57}$$

For $d = 4$ one can easily check using mathematical induction that the degrees of freedom of the quantum system at N are

$$\mathrm{f}(N) = \sum_{n=1}^{N} D_n^{(d=4)} = \frac{1}{12}\left[N^4 + 8N^3 + 23N^2 + 28N\right] \,. \tag{16.58}$$

Earlier we had pointed out the special cases of Eq. (16.47) in the limits $\mu = 0$ and $\mu \to \infty$ (at $N < \infty$). Here, these two limits amount to

$$R(L) \equiv \frac{12}{L^2} \overset{\mu\to 0}{=} 4\Lambda_\mathrm{b} + \frac{3G}{\pi L^4}\,\mathrm{f}(N) \tag{16.59}$$

$$R(L) \equiv \frac{12}{L^2} \overset{\mu\to \infty}{=} 4\Lambda_\mathrm{b} + 2\frac{3G}{\pi L^4}\,\mathrm{f}(N) \,. \tag{16.60}$$

To analize the remaining sum for $\mu \neq 0, \infty$; we apply a partial fraction decomposition to the expression

$$D_n^{(d=4)} \frac{\mu^2}{\mathscr{E}_n^{(d=4)} + \mu^2 + \xi R} = \frac{(\mu L)^2}{6} \frac{(2n+3)(n+2)(n+1)}{n(n+3) + 12\xi + (\mu L)^2} \,. \tag{16.61}$$

Introducing the abbreviation $z \coloneqq (\mu L)^2 + 12\xi$, the decomposition reads

$$D_n^{(d=4)} \frac{\mu^2}{\mathscr{E}_n^{(d=4)} + \mu^2 + \xi R} = \frac{z - 12\xi}{6}\Bigg[(2n+3) + (6-3z)\frac{1}{n(n+3)+z} + (4-2z)\frac{n}{n(n+3)+z}\Bigg] \,. \tag{16.62}$$

Inserting this decomposition into Eq. (16.55) restructures the RHS in such a way that the quartically, quadratically and logarithmically divergent as well as convergent terms in the limit $N \to \infty$ are revealed:

$$
\begin{aligned}
R(L) \equiv \frac{12}{L^2} = \; 4\Lambda_{\rm b} + \frac{3G}{\pi L^4}\Bigg\{ & \mathrm{f}(N) && \text{(quartically div.)} \\
& + \frac{z-12\xi}{6}\Bigg[N(N+4) && \text{(quadratically div.)} \\
& + (4-2z)\sum_{n=1}^{N} \frac{n}{n(n+3)+z} && \text{(logarithmically div.)} \\
& + (6-3z)\sum_{n=1}^{N} \frac{1}{n(n+3)+z}\Bigg]\Bigg\} && \text{(conv.)}\,.
\end{aligned}
$$

Here, we have evaluated the sum $\sum_{n=1}^{N}(2n+3) = N(N+4)$. The conventional approach towards renormalizing the RHS is to discard the divergent terms, e.g. by introducing suitable counterterms. The approach we follow, however, is piecewisely quantizing the system by increasing N step-by-step. Therefore, we maintain the RHS as it is (partially divergent). Besides, the convergent part in the bracket on the RHS is

$$
\lim_{N\to\infty} \sum_{n=1}^{N} \frac{6-3z}{n(n+3)+z} = 6 \begin{cases} 1 - \frac{1}{z} - 2\pi \frac{z-2}{\sqrt{9-4z}} \tan\left[\frac{\pi}{2}\sqrt{9-4z}\right] & \text{for } z \leq 9/4 \\ 1 - \frac{1}{z} - 2\pi \frac{z-2}{\sqrt{4z-9}} \tanh\left[\frac{\pi}{2}\sqrt{4z-9}\right] & \text{for } z > 9/4 \end{cases}, \tag{16.63}
$$

where we can cross the threshold $z = 9/4$ using $\tan \mathrm{i}x = \mathrm{i}\tanh x$. Note that care must be taken when taking the limit $N \to \infty$ as is does not commute with the limit $z \to 0$.

Before explicitly analyzing self-consistent S^4-geometries, i.e., solutions of Eq. (16.55), we can already observe an interesting property of these solutions. In general, the eigenfunctions of $-\Box_g$ acting on the 4-sphere are harmonics labeled by four integer quantum numbers, $u_{n,m} = Y_{nl_1l_2m}$. The main index $n = 0, 1, 2, \ldots$ determines the eigenvalue while the degeneracy index m now is traded for the triple of integers (l_1, l_2, m), with $n \leq l_1 \leq l_2 \leq |m|$. The harmonics of the 4-sphere have the structure

$$
Y_{nl_1l_2m}(\zeta,\eta,\vartheta,\varphi) \propto {}_4\bar{P}_n^{l_1}(\zeta)\,{}_3\bar{P}_{l_1}^{l_2}(\eta)\,{}_2\bar{P}_{l_2}^{m}(\vartheta)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\varphi}\,, \tag{16.64}
$$

where $(\zeta,\eta,\vartheta,\varphi)$ are angular coordinates on the 4-sphere and ${}_i\bar{P}_k^j$ denote generalized associated Legendre functions [167].

The harmonics of the self-consistent S^4 geometry determined by a gravity-coupled approximant of an N-cutoff will be restricted to those with $n = 1, 2, \ldots, N$. One can show that this truncated basis of S^4-harmonics possesses the "resolving power" [168]

$$\Delta\alpha \approx \frac{\pi}{N}\,, \tag{16.65}$$

which is the accuracy with which they can display angular separations. Correspondingly, the minimum proper distance they can resolve is $\Delta\ell \approx \pi L/N$. These geometric properties make the self-consistent $S^4(L^{\mathrm{SC}}(N))$ reminiscent of a *fuzzy sphere* [169].

In the following, we are going to explicitly solve the self-consistency condition (16.55) for the special case of a *massless scalar field*, i.e., the case $\mu = 0$, and analyze the properties of the resulting self-consistent geometry given by the radius $L^{\mathrm{SC}}(N)$. To contrast these results with those obtained within the standard approach from a *background-dependent calculation*, let us first discuss these, also for the case $\mu = 0$. The background-dependent calculation amounts to evaluating the RHS of Eq. (16.55) on a RS. This means that on the RHS we must replace the dynamical radius L by a rigid, fixed radius L^{RS}:

$$\begin{aligned} R[S^4(L(N))] \equiv \frac{12}{L(N)^2} &= 4\Lambda_{\mathrm{b}} + \frac{3G}{\pi (L^{\mathrm{RS}})^4}\,\mathrm{f}(N) \\ &=: 4\Lambda_{\mathrm{tot}}(N)\,. \end{aligned} \tag{16.66}$$

Here we have defined the total cosmological constant $\Lambda_{\mathrm{tot}}(N)$ that behaves as $\Lambda_{\mathrm{tot}}(N) \sim N^4 \to \infty$ for sufficiently large N. Consequently, when the cutoff is removed for $N \to \infty$, the radius $L(N)$ approaches zero and thus the curvature diverges:

$$R[S^4(L(N))] \sim N^4 \xrightarrow{N\to\infty} \infty\,. \tag{16.67}$$

This is an epitome of the cosmological constant problem that arises from summing up vacuum energies propagating on a rigid spacetime to only thereafter solve Einstein's equation for the background geometry [7, 157–159]. If the bare cosmological constant Λ_{b} is independent of N the total cosmological constant Λ_{tot} will become unacceptably large for any given cutoff scale N; or if Λ_{b} is granted to depend on N then it must be tremendously finetuned in order to match cosmological observations, which turns the cosmological constant problem into a naturalness problem.

Moreover, on $S^4(L^{\mathrm{RS}})$ the N-cutoff induces the UV cutoff scale $\mathscr{P}(N)$ via

$$\mathscr{P}^2(N) := \mathscr{E}_N(L^{\mathrm{RS}}) = \frac{N(N+3)}{(L^{\mathrm{RS}})^2}\,. \tag{16.68}$$

As we shall see later on its seemingly trivial behavior

$$\mathscr{P}(N) \xrightarrow{N\to\infty} \infty \tag{16.69}$$

should not be taken for granted at all. Thus it is important for us to keep in mind that for the background-dependent calculation $\mathscr{P}$ is a monotonically increasing function of N.

16.3.1. *Self-consistent approximants: $\mu = 0$ and $\Lambda_{\mathrm{b}} = 0$*

We start with a remark on the case $G = 0$ in which no matter effects are present. Then Eq. (16.55) heavily simplifies to $12/L^2 = 0$ implying $L^{\mathrm{SC}} = \infty$. This is the expected result: In absence of any matter, the maximally symmetric solution is flat space $\mathbb{R}^4$, here obtained as "$S^4(\infty)$". The result is obtained for $N = 0$, i.e., for no quantum mechanical degrees of freedom, $\mathrm{f}(0) = 0$. This case can thus can be considered as the *classical initial point.*

For $G \neq 0$ the self-consistency condition (16.55) for the spherical background geometry then becomes

$$\frac{12}{L^2} = \frac{3G}{\pi}\frac{1}{L^4}\,\mathrm{f}(N)\,, \tag{16.70}$$

such that the self-consistent radius (Hubble length) L^{SC} is given by

$$L^{\mathrm{SC}}(N)^2 = \frac{G}{4\pi}\,\mathrm{f}(N) = \frac{GN^4}{48\pi}\left\{1 + O\left(\frac{1}{N}\right)\right\}\,. \tag{16.71}$$

Then, the self-consistent scalar curvature amounts to[4]

$$R^{\mathrm{SC}}(N) := R\left[S^4(L^{\mathrm{SC}}(N))\right] = \frac{12(4\pi)}{G\,\mathrm{f}(N)} = \frac{144(4\pi)}{GN^4}\left\{1 + O\left(\frac{1}{N}\right)\right\}\,, \tag{16.72}$$

[4]Note that $R^{\mathrm{SC}} \sim 1/G$ is non-analytic in G and therewith clearly of non-perturbative character.

and correspondingly the self-consistent cosmological constant reads

$$\Lambda^{\mathrm{SC}}(N) = \frac{12\pi}{G\,\mathrm{f}(N)}\,. \tag{16.73}$$

For every N there exists one – and only one – self-consistent S^4-background. Quite remarkably, its radius $L^{\mathrm{SC}}(N)$ *grows* when more quantized modes are added, cf. Figure 16.1. At $N = 0$, the classical system ($\hbar = 0$) amounts to a vanishing "Hubble length", $L^{\mathrm{SC}}(0) = 0$, and infinite curvature, $R^{\mathrm{SC}}(0) = \infty$. Adding quantized modes to it ($N = 1, 2, 3, \dots$), the 4-sphere grows and ultimately becomes locally flat at $N \to \infty$: $\lim_{N\to\infty} L^{\mathrm{SC}}(N) = \infty$ and therewith $\lim_{N\to\infty} R^{\mathrm{SC}}(N) = 0$.

To conclude: The fully quantized system, with the cutoff removed (limit $N \to \infty$ taken), admits a self-consistent background geometry which is *perfectly flat*, $S^4(\infty) \cong \mathbb{R}^4$. This is in sharp contradiction to the usual way of taking the UV limit in background-dependent calculations which yields an infinitely curved spacetime, thus creating one form of the cosmological constant problem.

The absolute dimensionful scale of each self-consistent geometry at N is set by the Planck units, with $G = \ell_{\mathrm{Pl}}^2 = 1/m_{\mathrm{Pl}}^2$. Then we have

$$L^{\mathrm{SC}}(N) = \ell_{\mathrm{Pl}}\sqrt{\mathrm{f}(N)/4\pi} \quad \text{and} \quad \Lambda^{\mathrm{SC}}(N) = 12\pi m_{\mathrm{Pl}}^2/\,\mathrm{f}(N)\,. \tag{16.74}$$

Given the fundamental status of the Planck mass, it therewith becomes clear that for the above solution it is *not possible* to construct N-sequences that result in the limit $N \to \infty$ in a QFT with a non-zero observed cosmological constant $\Lambda^{\mathrm{SC}}(N \to \infty) \equiv \Lambda_{\mathrm{obs}} \neq 0$. On the other hand, if we assume that Eq. (16.73) represents a valid law of Nature and experimentalists measure a non-zero, positive cosmological constant Λ_{obs}, then this will necessarily imply that *the physically realized universe carries only finitely many quantum mechanical degrees of freedom* $\mathrm{f}(N_{\mathrm{obs}})$ where $N_{\mathrm{obs}} < \infty$ is fixed by the experimentalists' measurement:

$$\Lambda^{\mathrm{SC}}(N_{\mathrm{obs}}) \stackrel{!}{=} \Lambda_{\mathrm{obs}}\,. \tag{16.75}$$

Let us see how the solution (16.71) relates the dimensionless UV cutoff N to the induced dimensionful cutoff

$$\mathscr{P}_{\mathrm{UV}}^2(N) = \frac{N(N+3)}{L^{\mathrm{SC}}(N)^2} \tag{16.76}$$

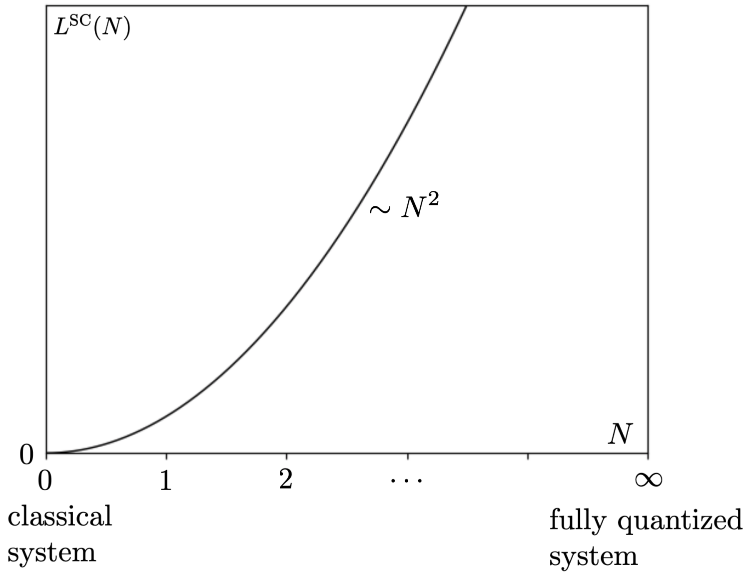

FIGURE 16.1. The self-consistent radius $L^{SC}(N)$ for $\Lambda_b = 0$ and matter constribution on the dynamical $S^4(L)$.

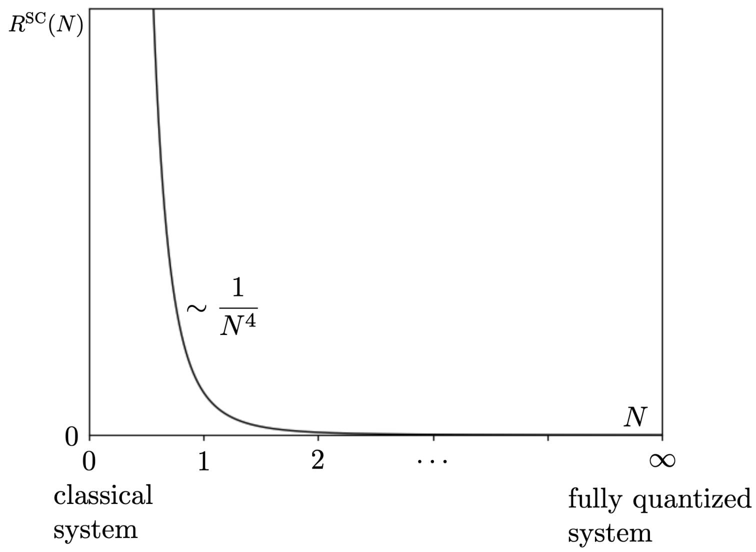

FIGURE 16.2. The self-consistent scalar curvature $R^{SC}(N)$ for $\Lambda_b = 0$ and matter constribution on the dynamical $S^4(L)$.

which, for each N, refers to a *different unit of mass*, namely $1/L^{SC}(N)$. We have

$$\mathscr{P}^2_{UV}(N) = \frac{12\pi}{G}\frac{N(N+3)}{\mathrm{f}(N)} = \frac{48\pi}{G}\frac{1}{N^2}\left\{1 + O\left(\frac{1}{N}\right)\right\} \xrightarrow{N\to\infty} 0\,. \tag{16.77}$$

Strangely enough, this is a *decreasing* function of N, cf. Figure 16.3. Quite paradoxically, the fully quantized system, i.e., the one with N"="∞, has *vanishing dimensionful* UV cutoff when the units are given by the corresponding self-consistent background metric on $S^4(L=\infty)$.

If, instead, we employ a *fixed* length unit L^{RS} in order to set the scale for $\mathscr{P}_{\mathrm{UV}}$, then we will find the standard – opposite – behavior, as shown above.

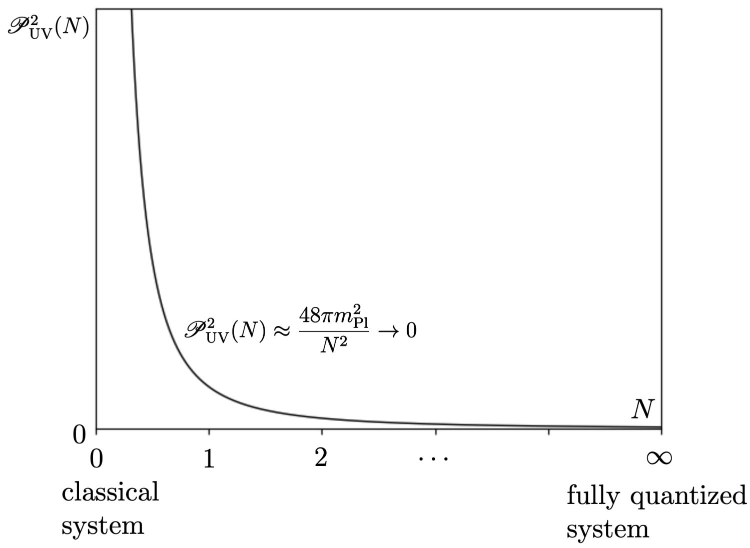

FIGURE 16.3. The self-consistent dimensionful UV cutvoff $\mathscr{P}^{\mathrm{SC}}_{\mathrm{UV}}(N)^2$ for $\Lambda_{\mathrm{b}} = 0$ and matter constribution on the dynamical $S^4(L)$.

16.3.2. *Self-consistent approximants: $\mu = 0$ and $\Lambda_{\mathrm{b}} \neq 0$*

Next, we analize the special case of Eq. (16.55) for a massless scalar field, $\mu = 0$, this time with nonzero bare cosmological constant, $\Lambda_{\mathrm{b}} \neq 0$:

$$\frac{12}{L^2} = 4\Lambda_{\mathrm{b}} + \frac{3G}{\pi}\,\mathrm{f}(N)\frac{1}{L^4}\,. \tag{16.78}$$

This amounts to a quadratic equation for L^2,

$$0 = \frac{2}{3}\Lambda_{\mathrm{b}}L^4 - 2L^2 + \frac{G}{2\pi}\,\mathrm{f}(N)\,, \tag{16.79}$$

that is graphically illustrated in Figure 16.4. The nonzero bare cosmological constant, $\Lambda_\mathrm{b} \neq 0$, causes a "singular perturbation" of the equation for L^2, giving rise to a new branch of solutions. The general solution reads

$$\left(L_\pm^{\mathrm{SC}}(N)\right)^2 = \frac{3}{2\Lambda}\left[1 \pm \sqrt{1 - \frac{G\Lambda_\mathrm{b}}{3\pi}\,\mathrm{f}(N)}\right]. \tag{16.80}$$

$\left(L_\pm^{\mathrm{SC}}(N)\right)^2$ is real provided that $G\Lambda_\mathrm{b}\mathrm{f}(N)/3\pi \leq 1$. For $G\Lambda_\mathrm{b} > 0$ and constant, this condition is violated for N large enough.

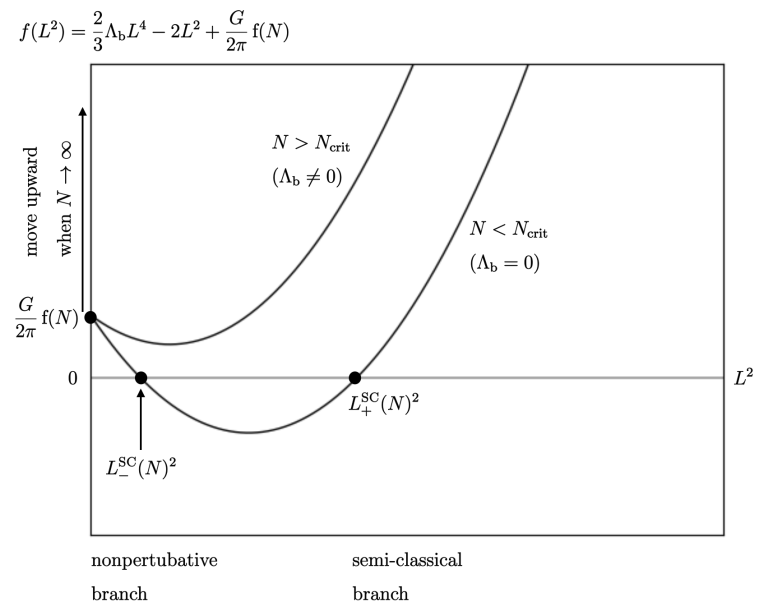

FIGURE 16.4. Illustration of the quadratic equation leading to the two branches of solutions, nonperturbative and semi-classical, determined by $f(L_\pm^{\mathrm{SC}}(N)^2) = 0$.

At first, we comment on the solution for the *classical system*: $N = 0$. In this case, no quantized modes of the scalar field A "live" on the 4-sphere. Its self-consistent radius (16.80) is

$$L_\pm^{\mathrm{SC}}(0)^2 = \frac{3}{2\Lambda_\mathrm{b}}[1 \pm 1] = \begin{cases} 3/\Lambda_\mathrm{b} & \text{for "+"} \\ 0 & \text{for "−"} \end{cases}. \tag{16.81}$$

$L_+^{\mathrm{SC}}(0) = \sqrt{3/\Lambda_\mathrm{b}}$ is the familiar classical relationship connecting the radius ("Euclidean Hubble length") to the cosmological constant. If $\Lambda_\mathrm{b} \neq 0$, which we assume, $S^4(L_+^{\mathrm{SC}}(0))$ is a nondegenerate manifold. On the other hand, $L_-^{\mathrm{SC}}(0) = 0$ has vanishing radius and infinite curvature. Hence, it is usually not considered a meaningful solution. Therefore, we shall refer to the "+"-branch of the solutions as the "*(semi-)classical branch*".

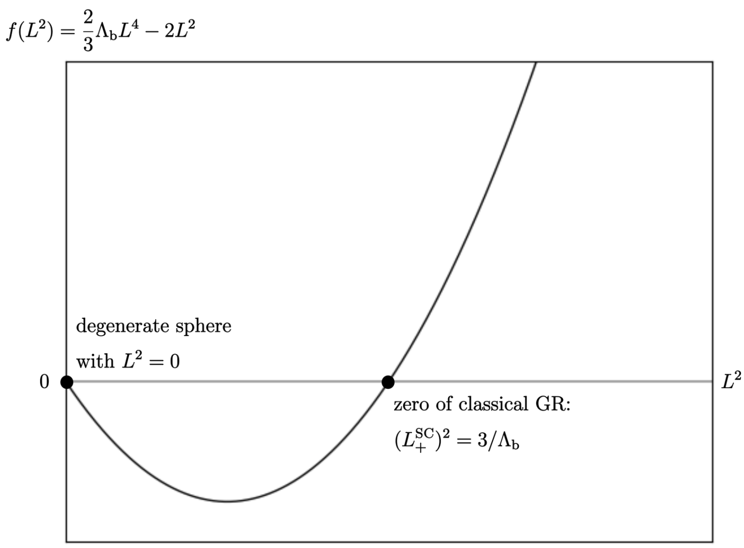

FIGURE 16.5. Illustration of the quadratic equation determining the classical system for $\Lambda_{\mathrm{b}} = 0$: at $N = 0$ one has the vertical intercept $\mathrm{f}(N = 0) = 0$.

Secondly, we note that our earlier solutions for $\Lambda_{\mathrm{b}} = 0$ are recovered by taking the limit $\Lambda_{\mathrm{b}} \to 0$ on the "−"-branch of Eq. (16.80):

$$\begin{aligned}\left(L_\pm^{\mathrm{SC}}(N)\right)^2 &= \frac{3}{2\Lambda_{\mathrm{b}}}\left[1 \pm \left\{1 - \frac{1}{2}\frac{G\Lambda}{3\pi}\,\mathrm{f}(N) + O(\Lambda_{\mathrm{b}}^2)\right\}\right] \\ &= \frac{3}{2\Lambda_{\mathrm{b}}}(1 \pm 1) - (\pm 1)\frac{G\,\mathrm{f}(N)}{4\pi} + O(\Lambda_{\mathrm{b}}^2) = \begin{cases} 3/\Lambda_{\mathrm{b}} \to \infty & \text{for "+"} \\ G\,\mathrm{f}(N)/4\pi & \text{for "−"} \end{cases}.\end{aligned} \tag{16.82}$$

For want of a better name, we refer to the "−"-branch of Eq. (16.80) as the "*nonperturbative branch*".

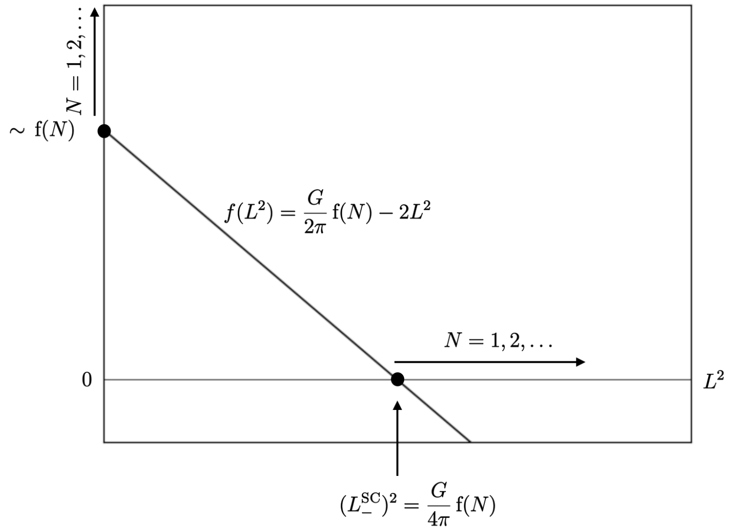

FIGURE 16.6. Illustration of the limit $\Lambda_\mathrm{b} \to 0$ in which the nontrivial solution of $f(L^2) = 0$ amounts to the self-consistent radius shown in Figure 16.1.

Furthermore, if we generally assume that $G > 0$ and $\Lambda_\mathrm{b} > 0$ are N-independent, nonzero and positive, the requirement

$$\frac{G\Lambda_\mathrm{b}}{3\pi}\,\mathrm{f}(N) \leq 1 \tag{16.83}$$

will be violated for all $N > N_\mathrm{crit}$ where the critical number N_crit satisfies

$$\frac{G\Lambda_\mathrm{b}}{3\pi}\,\mathrm{f}(N_\mathrm{crit}) = 1\,, \tag{16.84}$$

respectively, as this equation might not have an integer solution for N_crit,

$$\frac{G\Lambda_\mathrm{b}}{3\pi}\,\mathrm{f}(N_\mathrm{crit}) = \max_N \left\{ \frac{G\Lambda_\mathrm{b}}{3\pi}\,\mathrm{f}(N) \;\middle|\; \frac{G\Lambda_\mathrm{b}}{3\pi}\,\mathrm{f}(N) \leq 1 \right\}. \tag{16.85}$$

If $G\Lambda_\mathrm{b}/3\pi \ll 1$, the critical number N_crit will be very large, $N_\mathrm{crit} \gg 1$, and so we may employ the asymptotic form of $\mathrm{f}(N)$:

$$\frac{G\Lambda_\mathrm{b}}{36\pi} N_\mathrm{crit}^4 \left\{ 1 + O\left(\frac{1}{N_\mathrm{crit}}\right) \right\} = 1\,, \tag{16.86}$$

thus

$$N_\mathrm{crit} \approx \left(\frac{36\pi}{G\Lambda_\mathrm{b}}\right)^{1/4}. \tag{16.87}$$

For, say, $G\Lambda_{\mathrm{b}} = 10^{-120}$ one has thus $N_{\mathrm{crit}} \approx 10^{30}$. Therewith, let us rewrite Eq. (16.80) approximating $\mathrm{f}(N) \approx N^4/12$ and $G\Lambda_{\mathrm{b}}/36\pi \approx 1/N_{\mathrm{crit}}^4$:

$$\left(L_{\pm}^{\mathrm{SC}}(N)\right)^2 = \frac{3}{2\Lambda_{\mathrm{b}}}\left[1 \pm \sqrt{1 - \left(\frac{N}{N_{\mathrm{crit}}}\right)^4}\,\right]. \tag{16.88}$$

Thus, the self-consistent radius for the semi-classical and nonperturbative branch, respectively, is bound as

$$\left(L_{+}^{\mathrm{SC}}(N)\right)^2 \in \frac{3}{2\Lambda_{\mathrm{b}}}[0,1] \quad \text{and} \quad \left(L_{-}^{\mathrm{SC}}(N)\right)^2 \in \frac{3}{2\Lambda_{\mathrm{b}}}[1,2]\,. \tag{16.89}$$

We iterpret this circumstance as for too many modes, *there exists no S^4-type self-consistent background (but perhaps a more complicated one) if* $\Lambda_{\mathrm{b}} \neq 0$. Figure 16.7 clearly illustrates this situation.

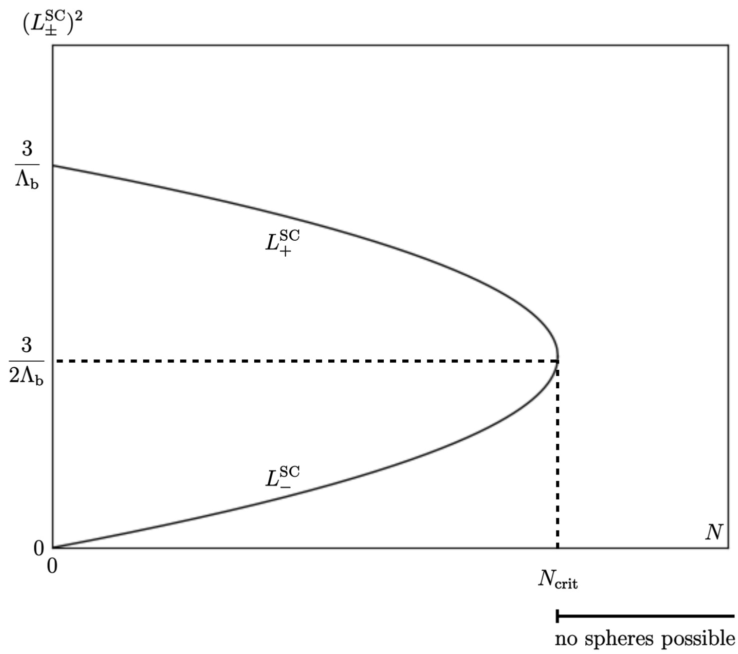

FIGURE 16.7. A bare cosmological constant bounds the two branches associated to the self-consistent radii $L_{+}^{\mathrm{SC}}(N)$ and $L_{-}^{\mathrm{SC}}(N)$. Consequently, there exists a positive integer N_{crit} such that for $N > N_{\mathrm{crit}}$ no self-consistent S^4-background can be realized.

Lastly, it is astonishing to note that the "−"-branch of the general solution Eq. (16.80) even admits a *negative bare cosmological constant*, $\Lambda_\mathrm{b} < 0$:

$$L_-^\mathrm{SC}(N)^2 = \frac{3}{2|\Lambda_\mathrm{b}|}\left[\sqrt{1+\frac{|\Lambda_\mathrm{b}|G}{3\pi}\,\mathrm{f}(N)}-1\right]. \tag{16.90}$$

This solution of astonishing because the classical Einstein equation does not admit an S^4 solution for a negative cosmological constant. Again, the N-sequence of self-consistent radii, which are purely due to quantum effects and thus starting at $N = 1$, grow with increasing N, approaching a flat space in the limit $N \to \infty$.

16.4. Micro states of de Sitter space

De Sitter space is the Lorentzian counterpart of the 4-sphere and possesses an intrinsic entropy $\mathscr{S}$, the Bekenstein-Hawking entropy. This entropy is determined by the cosmological constant, or, equivalently, by the Hubble length $L \equiv \sqrt{3/\Lambda}$:

$$\mathscr{S} = \frac{3\pi}{G\Lambda} \equiv \frac{\pi}{G}L^2\,. \tag{16.91}$$

If we let $\mathscr{A} = 4\pi L^2$ denote the area of the de Sitter horizon then there is yet another way to express the entropy, $\mathscr{S} = \mathscr{A}/4G$. Particularly, $\mathscr{S}$ is known only as a purely thermodynamic quantity, i.e., the longstanding question of what are the underlying microscopic degrees of freedom that $\mathscr{S}$ counts is still unanswered [170]. Following the Euclidean approach towards black holes and thermal spacetimes [171–173], in which the 4-sphere appears as a saddle point of the semi-classical $g_{\mu\nu}$-evolution, we may use our previously found self-consistent radii to explore the intrinsic entropy of de Sitter space for a given N-cutoff:

$$\mathscr{S}(N) \coloneqq \frac{\pi}{G}L^\mathrm{SC}(N)^2\,. \tag{16.92}$$

To interpret $\mathscr{S}(N)$ let us return to the case of a massless scalar field and vanishing bare cosmological constant. If we plug in the self-consitent radii (16.71) that we found for this case, we find a rather striking result:

$$\mathscr{S}(N) = \frac{1}{4}\,\mathrm{f}(N)\,. \tag{16.93}$$

This means that, up to a factor 1/4, *the thermodynamical entropy* $\mathscr{S}(N)$ *is precisely the number of degrees of freedom* $\mathrm{f}(N)$ *of the quantum system* from whose backreaction the spherical (or de Sitter) spacetime arose. Using Eq. (16.74), the area of the resulting Hubble spheres $\mathscr{A}(N)$ becomes proportional to the Planck length squared,

$$\mathscr{A}(N) = \mathrm{f}(N)\ell_{\mathrm{Pl}}^2 . \tag{16.94}$$

This result is in accordance with the intuitive picture that horizon surface of a “fuzzy” de Sitter space is a fuzzy 2-sphere whose angular resolving power $\Delta\alpha \approx \pi/N$ yields the approximate proper distance of neighboring points

$$\Delta\ell \equiv L^{\mathrm{SC}}(N)\Delta\alpha \approx \sqrt{\pi/48}\, N\,\ell_{\mathrm{Pl}} \left\{1 + O\left(\frac{1}{N}\right)\right\} . \tag{16.95}$$

Following our train of thoughts let us further assume that experiment had provided us some finite observed value for the cosmological constant, $\Lambda_{\mathrm{obs}} > 0$. Then in the case $\mu = 0$ and $\Lambda_{\mathrm{b}} = 0$ it follows from Equations (16.75) and (16.71) that Λ_{obs} determines a finite value N_{obs}:

$$\frac{1}{4}\,\mathrm{f}(N_{\mathrm{obs}}) = \frac{3\pi}{G\Lambda_{\mathrm{obs}}} \equiv \mathscr{S}_{\mathrm{obs}} . \tag{16.96}$$

Thus, in this framework *universes with an observed non-zero, positive cosmological constant* Λ_{obs} *are described only by a finite number of degrees of freedom* $\mathrm{f}(N_{\mathrm{obs}})$ *which is determined by the Bekenstein-Hawking entropy* $\mathscr{S}_{\mathrm{obs}}$ *of de Sitter space.* This is precisely an incarnation of the conjectured “Λ-$\mathscr{N}$-connection” and the “$\mathscr{N}$-bound” [160, 161]. This connection refers to the hypothesis that the observed entropy $\mathscr{S}_{\mathrm{obs}}$ of universes with a positive cosmological constant Λ and arbitrary matter content is bounded by some value $\mathscr{N}$, i.e., $\mathscr{S}_{\mathrm{obs}} \leq 3\pi/G\Lambda \equiv \mathscr{N}$. Here, this bound is given by 1/4th of the number of degrees of freedom.

Although presumably rather inadequate, it is tempting to specify this analysis for the observed values of the real universe: A Hubble radius of $L^{\mathrm{SC}}(N_{\mathrm{obs}}) \approx 10^{60}\ell_{\mathrm{Pl}}$ yields $N_{\mathrm{obs}} \approx 10^{30}$ and thus $\mathscr{S}(N_{\mathrm{obs}}) \approx 10^{120}$. This corresponds to an angular uncertainty of $\delta\alpha \approx 10^{-30}$, respectively a minimum proper length of $\delta\ell \approx 10^{30}\ell_{\mathrm{Pl}} \approx 10^{-3}$cm. Interestingly, similar estimates have been obtained with independent arguments based upon the functional renormalization group [168].

CHAPTER 17

A second type of approximants for a quantized scalar field

Executive summary. We determine a second type of approximants for a quantized scalar field. It is constructed from the point of view of the one-loop effective gravitational action and is shown to differ from the first type of approximants by a contribution from the metric dependence of the path integral. In four spacetime dimensions, we explicitly solve the resulting self-consistency condition for the case of spherical background geometries. Therewith, we demonstrate that the resulting N-sequences of self-consistent radii are free from the cosmological constant problem.
What is new? All research results of this chapter are new.
Based upon: Reference [5].

For notational ease, we denote the background metric in this chapter by $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$.

17.1. Backreaction of the metric on the second type of stress tensor candidate

The generic equations of motion for the background metric $g_{\mu\nu}$ given by the (unregularized) one-loop effective action $\Gamma[g]$, defined by Eq. (3.38), read

$$\frac{\delta\Gamma[g]}{\delta g_{\mu\nu}(z)} = \frac{\delta S_{\mathrm{EH}}[g]}{\delta g_{\mu\nu}(z)} + \frac{\delta\Gamma_{1\mathrm{L}}[g]}{\delta g_{\mu\nu}(z)} = 0\,, \tag{17.1}$$

where the first summand is given by (16.17). Again, we restrict the treatment of the backreaction of "type 2" to the integrated and traced equations of motion:

$$\mathscr{T}\Gamma[g] = -2\int \mathrm{d}^d z\, g_{\mu\nu}(z)\frac{\delta}{\delta g_{\mu\nu}(z)}\Gamma[g] = 0\,, \tag{17.2}$$

such that the remaining task is to calculate

$$\begin{aligned}\mathscr{T}\Gamma_{1\mathrm{L}}[g] &= -\int \mathrm{d}^d z\, g_{\mu\nu}(z)\frac{\delta}{\delta g_{\mu\nu}(z)}\,\mathrm{Tr}\ln\left(-\Box_g+\mu^2+\xi R\right)\\ &= -\,\mathrm{Tr}\left[\frac{1}{-\Box_g+\mu^2+\xi R}\int \mathrm{d}^d z\, g_{\mu\nu}(z)\frac{\delta}{\delta g_{\mu\nu}(z)}(-\Box_g+\xi R)\right]\\ &=\frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{-\Box_g+\mu^2+\xi R}\mathscr{T}(-\Box_g+\xi R)\right].\end{aligned} \tag{17.3}$$

When applying these three steps to a regularized version of the effective action some further commentary is required. We have used the standard rule $\delta\,\mathrm{Tr}[\ln\mathscr{K}] = \mathrm{Tr}[\mathscr{K}^{-1}\delta\mathscr{K}]$ that for an arbitrary derivation δ is valid only thanks to the cyclicality of the trace operation.[1] While this holds for the unregularized (full) trace, this property might not transfer to the regularized trace $\mathrm{Tr} = \mathrm{Tr}_{\mathrm{reg}}$. Particularly, $\mathrm{Tr}[AB]_{\mathrm{reg}} = \mathrm{Tr}[BA]_{\mathrm{reg}}$ might *not* be realized when employing an N-cutoff by expressing the trace in terms of the truncated eigenbasis (16.25) of the operator $\mathscr{K}$. However, subsequently we will assume the cyclicality to hold also for the regularized trace and, later on, outline in a second, different, calculation why this assumption is fulfilled. (Also, subsequently we will leave open what is the specific regulator. Surely, later we will come back to the N-cutoff.)

Without further ado, as the trace is taken on the Hilbert space of scalar fields, $\mathrm{Tr} = \mathrm{Tr}_S$, we must evaluate $\mathscr{T}(-\Box_g f)(x)$ and $\mathscr{T}R(x)$ where $f(x) = \langle x|f\rangle$ is an

[1] While strictly speaking up to the definition of the derivative, in standard mathematical practice and conventions the following property of the *left* derivative of a composition of supersmooth functions (functions of commuting as well as anti-commuting variables) holds: The derivative of the inner fuction stands *left* of the derivative of the outer function, cf. theorem 4.4.2 of [174] and eq. (2.15) of [175]. Here, as $\mathscr{T}$ surely is a left derivation, the derivative of the inner function, i.e., $\mathscr{T}\mathscr{K}$ can be placed on the right side only thanks to the cyclicality of the trace.

arbitrary scalar field. Varying the negative Laplacian, we find, using appendix B,

$$
\begin{aligned}
&\int \mathrm{d}^d z\, g_{\mu\nu}(z) \frac{\delta}{\delta g_{\mu\nu}(z)} (-\Box_g f)(x) \\
=&\int \mathrm{d}^d z\, g_{\mu\nu}(z) \Big\{ \quad g^{\alpha\sigma} g^{\beta\tau} I^{\mu\nu}_{\alpha\beta} \delta(x-z) \left(\partial_\sigma \partial_\tau f - \Gamma^\rho_{\sigma\tau} \partial_\rho f\right) \\
&\qquad + \frac{1}{2} g^{\sigma\tau} g^{\rho\beta} D^x_\sigma \delta(x-z) I^{\mu\nu}_{\tau\beta} \\
&\qquad + \frac{1}{2} g^{\sigma\tau} g^{\rho\beta} D^x_\tau \delta(x-z) I^{\mu\nu}_{\sigma\beta} \\
&\qquad - \frac{1}{2} g^{\sigma\tau} g^{\rho\beta} D^x_\beta \delta(x-z) I^{\mu\nu}_{\sigma\tau} \Big\} \\
=&g_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} \big(\partial_\mu \partial_\nu f - \Gamma^\rho_{\mu\nu} \partial_\rho f\big)(x) \\
=&g^{\mu\nu} \big(\partial_\mu \partial_\nu f - \Gamma^\rho_{\mu\nu} \partial_\rho f\big)(x) \\
=&-(-\Box_g f)(x)\,,
\end{aligned}
\tag{17.4}
$$

where in the second step we have used partial integration and the fact that we integrate over an manifold with empty boundary. The other variation we must perform is

$$
\frac{\delta R(x)}{\delta g_{\mu\nu}(z)} = \left[-R^{\mu\nu}(x) + D_\alpha \left(g^{\mu\alpha} g^{\nu\beta} D_\beta - g^{\mu\nu} g^{\alpha\beta} D_\beta\right)\right]_x \delta(x-z)\,, \tag{17.5}
$$

such that

$$
\begin{aligned}
&\int \mathrm{d}^d z\, g_{\mu\nu}(z) \frac{\delta R(x)}{\delta g_{\mu\nu}(z)} \\
&\quad = -R(x) - \int \mathrm{d}^d z\, g_{\mu\nu}(z) \left[D_\alpha \left(g^{\mu\alpha} g^{\nu\beta} D_\beta - g^{\mu\nu} g^{\alpha\beta} D_\beta\right)\right]_z \delta(x-z)\,.
\end{aligned}
\tag{17.6}
$$

The second term is a surface term and as the boundary is empty, we find

$$
\int \mathrm{d}^d z\, g_{\mu\nu}(z) \frac{\delta}{\delta g_{\mu\nu}(z)} R(x) = -R(x)\,. \tag{17.7}
$$

Expressed in terms of $\mathscr{T}$ these variations read

$$
\mathscr{T}(-\Box_g f)(x) = 2(-\Box_g f)(x) \quad \text{and} \quad \mathscr{T} R = -2R\,. \tag{17.8}
$$

Hence the variation of the regularized one-loop effective action reduces to

$$
\boxed{\mathscr{T}\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}} = \mathrm{Tr}\left[\frac{-\Box_g + \xi R}{-\Box_g + \mu^2 + \xi R}\right]_{\mathrm{reg}}}\,. \tag{17.9}
$$

Therewith and with Eq. (16.20), the equation of motion for R,

$$\mathscr{T}\Gamma[g]_{\text{reg}} = \mathscr{T}S_{\text{EH}}[g] + \mathscr{T}\Gamma_{1\text{L}}[g]_{\text{reg}} = 0\,, \tag{17.10}$$

has been determined:

$$\boxed{\frac{1}{8\pi G}\int \mathrm{d}^d z\,\sqrt{g(z)}\,\left[\left(1-\frac{d}{2}\right)R(z) + d\,\Lambda\right] = \mathrm{Tr}\left[\frac{-\Box_g + \xi R}{-\Box_g + \mu^2 + \xi R}\right]_{\text{reg}}}\,. \tag{17.11}$$

Next, we expound the other, in some sense more rigorous, way to deduce this equation. As annouced earlier, it will also resolve the problem attached to the cyclicality of the regularized trace. It makes use of the fact that for any action functional $F[g]$ and its associated Euclidean stress-energy tensor defined by Eq. (15.19), i.e.,

$$T_F^{\mu\nu}[g](x) \coloneqq -\frac{2}{\sqrt{g(x)}}\frac{\delta F[g]}{\delta g_{\mu\nu}(x)}\,, \tag{17.12}$$

the following lemma holds:

$$\begin{aligned}\mathscr{T}F[g] &= \int \mathrm{d}^d x \sqrt{g(x)}\, T_F{}^{\mu}_{\mu}[g](x) \\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha}F\left[\mathrm{e}^{-2\alpha}g\right]\Big|_{\alpha=0}\,,\end{aligned} \tag{17.13}$$

where $\mathscr{T}$ is as defined by Eq. (16.11). Here, we consider the case $F[g] = \Gamma_{1\text{L}}[g]_{\text{reg}}$ and refer to its associated stress-energy tensor, in order to distinguish it from (16.34), as the *effective* stress energy tensor $T_{\text{eff}}^{\mu\nu}[g]$, i.e.,

$$\begin{aligned}\mathscr{T}\Gamma_{1\text{L}}[g]_{\text{reg}} &\eqqcolon \int \mathrm{d}^d x \sqrt{g(x)}\, T_{\text{eff}}{}^{\mu}_{\mu}[g](x) \\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\,\mathrm{Tr}\left[\ln \mathscr{K}[\mathrm{e}^{-2\alpha}g]\right]_{\text{reg}}\Big|_{\alpha=0} \\ &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln \mathscr{K}[\mathrm{e}^{-2\alpha}g]\right]_{\text{reg}}\Big|_{\alpha=0} \\ &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{\mathrm{d}}{\mathrm{d}\alpha}\ln \mathscr{K}[\mathrm{e}^{-2\alpha}g]_{\text{reg}}\Big|_{\alpha=0}\right]\,.\end{aligned} \tag{17.14}$$

The only calculation left to do is that of the *argument* of the trace. This implies that using this lemma, the problem of the previous calculation, associated to the questionable cyclicality of the regularized trace, does *not* occur anymore. On the other hand, the only property of the regularized trace that we are making use of here (we had done so in the previous calculation, too) is $\delta\,\mathrm{Tr}[\,\cdot\,]_{\mathrm{reg}} = \mathrm{Tr}[\delta(\,\cdot\,)]_{\mathrm{reg}}$. This property clearly holds; and so we are good to go on with calculating the argument of the trace.

At first, we apply the (x-independent) Weyl transformation[2] to (the matrix elements of) the operator $\mathscr{K}[g_{\cdot\cdot}] = -\Box_{g_{\cdot\cdot}} + \mu^2 + \xi R(g_{\cdot\cdot})$:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right] = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{-\Box_{\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}} + \mu^2 + \xi R(\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot})\right\}\,. \tag{17.15}$$

Note that the Weyl weight of $g_{\cdot\cdot}$ is -2. As the Christoffel symbols $\Gamma^{\cdot}_{\cdot\cdot} = \frac{1}{2}g^{\cdot\cdot}(\partial g_{\cdot\cdot} + \cdots)$ are uneffected by Weyl transformations, we find that

$$\begin{aligned}\Box_{\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}} &= g^{\mu\nu}D_\mu D_\nu\big|_{g_{\cdot\cdot}\to\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}}\\ &= \mathrm{e}^{+2\alpha}g^{\mu\nu}D_\mu D_\nu\\ &= \mathrm{e}^{+2\alpha}\Box_g\,.\end{aligned} \tag{17.16}$$

As in addition $R(\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}) = \mathrm{e}^{+2\alpha}R(g_{\cdot\cdot})$, we directly arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right] &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\mathrm{e}^{(+2)\alpha}(-\Box_{g_{\cdot\cdot}}) + \mu^2 + \mathrm{e}^{(+2)\alpha}\xi R(g_{\cdot\cdot})\right\}\\ &= 2\left[-\Box_g + \xi R(g)\right]\mathrm{e}^{2\alpha}\,,\end{aligned} \tag{17.17}$$

and furthermore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right]\right|_{\alpha=0} = 2\left[-\Box_g + \xi R(g)\right]\,. \tag{17.18}$$

This implies the vanishing commutator

$$\left[\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right], \mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right]\right] = 0\,. \tag{17.19}$$

As a consequence, we may differentiate $Q(\mathscr{K}[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}])$, where Q is an arbitrary C^1-function, in the naive ("commutative") way with respect to α:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}Q\left(\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right]\right) &= Q'\left(\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right]\right)\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right]\\ &= 2\left[\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right] - \mu^2\right]\mathrm{e}^{2\alpha}Q'\left(\mathscr{K}\left[\mathrm{e}^{(-2)\alpha}g_{\cdot\cdot}\right]\right)\end{aligned} \tag{17.20}$$

[2]For more details on Weyl transformations, see Chapter 19.

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} Q\left(\mathscr{K}\left[\mathrm{e}^{(-2)\alpha} g_{\cdot\cdot}\right]\right)\Bigg|_{\alpha=0} = 2\left[\mathscr{K}\left[g_{\cdot\cdot}\right] - \mu^2\right] Q'\left(\mathscr{K}\left[g_{\cdot\cdot}\right]\right) . \tag{17.21}$$

Here in particular, we have for $Q = \ln$ the operator equation

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \ln\left(\mathscr{K}\left[\mathrm{e}^{(-2)\alpha} g_{\cdot\cdot}\right]\right)\Bigg|_{\alpha=0} = 2\left[\mathscr{K}\left[g_{\cdot\cdot}\right] - \mu^2\right] \mathscr{K}\left[g_{\cdot\cdot}\right]^{-1} . \tag{17.22}$$

This finally is the argument of the trace we wished to calculate such that our overall result becomes

$$\begin{aligned} \mathscr{T}\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}} &=: \int \mathrm{d}^d x \sqrt{g(x)}\, T_{\mathrm{eff}}{}^{\mu}_{\mu}[g](x) \\ &= \frac{1}{2} \operatorname{Tr}\left[\frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \mathscr{K}[\mathrm{e}^{-2\alpha} g] \Bigg|_{\alpha=0} \right]_{\mathrm{reg}} \\ &= \operatorname{Tr}\left[\frac{\mathscr{K}[g] - \mu^2}{\mathscr{K}[g]} \right]_{\mathrm{reg}} \\ &= \operatorname{Tr}\left[\frac{-\Box_g + \xi R}{-\Box_g + \mu^2 + \xi R} \right]_{\mathrm{reg}} , \end{aligned} \tag{17.23}$$

which is exactly Eq. (17.9), our previous result that we wished to put on rigorous grounds.

17.2. The quantum system at finite N: first type vs. second type of approximants

When employing an N-cutoff in order to regularize the (traces appearing in the) one-loop effective action, we identify the second type of approximants with

$$\mathscr{T}\Gamma_{1\mathrm{L}}[g]_N =: \int \mathrm{d}^d x \sqrt{g(x)}\, T_{\mathrm{eff}}{}^{\mu}_{\mu}[g]_N(x) . \tag{17.24}$$

The difference between the result (17.9), respectively (17.11), for the second type of stress tensor candidate and Eq. (16.39) for the first type of stress tensor candidate is noticable. After employing N-cutoff following the previous section, it constitutes in the difference between (17.9) and (16.36):[3]

$$\boxed{\begin{aligned}\mathscr{T}\Gamma[g]_N - \left\langle \mathscr{T} S[\hat{A};g]\right\rangle_N &= \mathscr{T}\Gamma_{1\mathrm{L}}[g]_N - \left\langle \mathscr{T} S_{\mathrm{M}}[\hat{A};g]\right\rangle_N \\ &= \frac{d}{2}\mathrm{Tr}_S[\mathbb{1}_S]_N \\ &= \frac{d}{2}\,\mathrm{f}(N)\,.\end{aligned}} \tag{17.25}$$

The second type of approximant "misses" the term on the RHS in order to match the first type of approximant. Thus phenomenological differences between the two are not far to seek: The "missing" term $\frac{d}{2}\,\mathrm{f}(N)$ for the second type of approximant leads to negative contributions to the bare cosmological costant Λ_b in the case $\mu^2 = 0 = \xi$. This is a crucial difference between the quantum field theoretical equation of motion from which the second type of approximants arise and Eq. (16.39) of the first type of approximants, where there are positive contribitions to Λ_b.

In the following, we will show that the difference (17.25) can be identified as the contribution from the g-dependent measure $\mathcal{D}_g A$ when applying $\mathscr{T}$ to Eq. (3.25). Therefore, we consider Eq. (3.40),

$$\mathrm{e}^{-\Gamma[g]} = \int \mathcal{D}_g A\, \mathrm{e}^{-S[A;g]}\,, \tag{17.26}$$

where $\Gamma[g]$ and $S[A;g]$ are defined by Eq. (3.38) and Eq. (16.16), respectively. For the moment, it is rather impractical regularize this path integral via an N-cutoff, so let us assume that it has been regularized by restricting it to a *finite number of spacetime points*.

The factor $\mathrm{e}^{-S_{\mathrm{EH}}[g]}$ can be canceled out to obtain Eq. (3.41),

$$\mathrm{e}^{-\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}}} = \int \mathcal{D}_g A\, \mathrm{e}^{-S_{\mathrm{M}}[A;g]}\,, \tag{17.27}$$

[3]Recall that traces over the Hilbert space S then are caculated in the truncated eingenbasis (16.25). Further, it is impressive to note that this equation resembles the thermodynamical indentity $U - F = TS$; the difference between internal and free energy is given by the product of temperature and entropy. The latter counts the microscopic degrees of freedom which here also appear on the RHS.

where $\Gamma_{1\mathrm{L}}[g]$ and $S_{\mathrm{M}}[A;g]$ are defined by Eq. (3.39) and Eq. (3.3), respectively. Furthermore, remember that the measure, defined in Eq. (D.6), is given by (for any dimensionality d)

$$\mathcal{D}_g A = \prod_x g^{1/4}(x)\mathrm{d}A(x)\,. \tag{17.28}$$

Applying $\mathscr{T}$ to Eq. (3.41) leads to

$$-\mathrm{e}^{-\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}}}\mathscr{T}\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}} = \int \mathcal{D}_g A\,\big(-\mathscr{T}S_{\mathrm{M}}[A;g]\big)\mathrm{e}^{-S_{\mathrm{M}}[A;g]} + \int \big(\mathscr{T}\mathcal{D}_g A\big)\,\mathrm{e}^{-S_{\mathrm{M}}[A;g]}\,. \tag{17.29}$$

Using the fact that

$$\begin{aligned}\mathrm{e}^{+\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}}}\int \mathcal{D}_g A\,\big(-\mathscr{T}S_{\mathrm{M}}[A;g]\big)\mathrm{e}^{-S_{\mathrm{M}}[A;g]} &= \mathrm{e}^{+\Gamma[g]_{\mathrm{reg}}}\int \mathcal{D}_g A\,\big(-\mathscr{T}S_{\mathrm{M}}[A;g]\big)\mathrm{e}^{-S[A;g]}\\ &= \Big\langle \mathscr{T}S_{\mathrm{M}}[\hat{A};g]\Big\rangle_{\mathrm{reg}}\,,\end{aligned} \tag{17.30}$$

the difference (17.25) can be written as

$$\begin{aligned}\mathscr{T}\Gamma[g]_{\mathrm{reg}} - \Big\langle \mathscr{T}S[\hat{A};g]\Big\rangle_{\mathrm{reg}} &= \mathscr{T}\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}} - \Big\langle \mathscr{T}S_{\mathrm{M}}[\hat{A};g]\Big\rangle_{\mathrm{reg}}\\ &= -\,\frac{1}{\mathrm{e}^{-\Gamma_{1\mathrm{L}}[g]_{\mathrm{reg}}}}\int \big(\mathscr{T}\mathcal{D}_g A\big)\,\mathrm{e}^{-S_{\mathrm{M}}[A;g]}\,.\end{aligned} \tag{17.31}$$

To further calculate $\mathscr{T}\mathcal{D}_g A = \mathscr{T}\prod_x g^{1/4}(x)\mathrm{d}A(x)$ let us again make us of lemma (17.13):

$$\begin{aligned}\mathscr{T}\prod_x g^{1/4}(x) &= \left\{-2\int \mathrm{d}^d x\, g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\right\}\prod_x \det{}^{1/4}\left(g_{\bullet\bullet}(x)\right)\\ &= \lim_{\alpha\to 0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\prod_x \det{}^{1/4}\left(\mathrm{e}^{-2\alpha}g_{\bullet\bullet}(x)\right)\\ &= \lim_{\alpha\to 0}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\prod_x \mathrm{e}^{-\frac{d}{2}\alpha}\right)\cdot\left(\prod_x g^{1/4}(x)\right)\,,\end{aligned} \tag{17.32}$$

where we have used that $\left[\det\left(\mathrm{e}^{-2\alpha}g_{\bullet\bullet}\right)\right]^{1/4} = \left[\mathrm{e}^{-2d\alpha}\det\left(g_{\bullet\bullet}\right)\right]^{1/4}$. Using furthermore that

$$\prod_x \mathrm{e}^{-\frac{d}{2}\alpha} = \mathrm{e}^{-\frac{d}{2}\alpha\sum_x 1}\,, \tag{17.33}$$

we arrive at

$$\mathscr{T}\prod_x g^{1/4}(x) = \left[-\frac{d}{2}\sum_x 1\right]\prod_x g^{1/4}(x)\,. \tag{17.34}$$

At this point we must clarify what is meant by "$\sum_x 1$". As we have regularized the path integral by discretizing spacetime, i.e., restricting spacetime to finitely many points $\{x_j \mid j = 1, 2, \ldots, J\}$, the sum $\sum_x 1 = J$ simply states the number of these spacetime points. The lattice points in field space $\{A(x_j) \mid j = 1, 2, \ldots, J\}$ can be expanded as in Eq. (D.15); thereby we can employ an N-cutoff,

$$A(x_j) = \sum_{n=1}^{N}\sum_{m=1}^{D_n} a_{n,m}\chi_{n,m}(x_j)\,. \tag{17.35}$$

This linear map establishes a bijection between the sets $\{A(x_j) \mid j = 1, 2, \ldots, J\}$ and $\{a_{n,m} \mid n = 1, \ldots, N\,;\ m = 1, \ldots, D_n\}$, and thus we can identify the number of spacetime points with the degrees of freedom of the quantum system, i.e., we can identify the discretization-based cutoff with the N-cutoff:

$$\left(\sum_x 1\right)_{\text{reg}} = \text{Tr}_S[\mathbb{1}_S]_{\text{reg}} \equiv \text{Tr}_S[\mathbb{1}_S]_N = \text{f}(N)\,. \tag{17.36}$$

As a consequence, we have

$$\mathscr{T}\mathcal{D}_g A = -\frac{d}{2}\text{Tr}_S[\mathbb{1}_S]_N\,\mathcal{D}_g A\,. \tag{17.37}$$

Inserting this transformation behavior of the measure into Eq. (17.31), we have re-derived Eq. (17.25):

$$\mathscr{T}\Gamma[g]_N - \left\langle \mathscr{T}S[\hat{A};g]\right\rangle_N = \frac{d}{2}\text{Tr}_S[\mathbb{1}_S]_N\,. \tag{17.38}$$

Therefore, the difference between the two ways – for the first and second type of approximants – of calculating the backreation of the scalar field A on the background metric g can be rooted in the contribution of the g-dependent measure.

17.3. N-sequences of self-gravitating quantum systems on $S^4(L)$

When specifying Eq. (17.11) to the sphere $S^d(L)$, there is not much work left: we can again use that the volume vol $\left[S^d(L)\right] = \int \text{d}^d x \sqrt{g(x)}$ is finite and the scalar curvature R is constant. Additionally, we calculate the trace in the truncated eigenbasis $\{|nm\rangle\}_{n=1,\dots,N}$ of the negative Laplacian $-\Box_g$, cf. Eq. (16.25) and appendix A.1.3. Therewith, we regularize the trace through the dimensionless UV cutoff N:

$$\begin{aligned}\mathscr{T}\Gamma_{1\mathrm{L}}[g]_N = \mathrm{Tr}\left[\frac{-\Box_g + \xi R}{-\Box_g + \mu^2 + \xi R}\right]_N &= \sum_{n=1}^{N}\sum_{m=1}^{D_n}\langle nm|\frac{-\Box_g + \xi R}{-\Box_g + \mu^2 + \xi R}|nm\rangle \\ &= \sum_{n=1}^{N} D_n \frac{\mathscr{E}_n + \xi R}{\mathscr{E}_n + \mu^2 + \xi R}\,. \end{aligned} \tag{17.39}$$

The final result for the equation of motion for the radius L of the d-sphere, with $R(L) = d(d-1)/L^2$, is

$$\boxed{R(L) = \frac{2d}{d-2}\Lambda_\mathrm{b} - \frac{16\pi G}{(d-2)\mathrm{vol}\left[S^d(L)\right]}\Theta_N^{\mathrm{eff}}(L)\,.} \tag{17.40}$$

Here, we have defined analogously to Eq. (16.47):

$$\Theta_N^{\mathrm{eff}}(L) \coloneqq \mathscr{T}\Gamma_{1\mathrm{L}}[g]_N = \sum_{n=1}^{N} D_n \frac{\mathscr{E}_n(L) + \xi R(L)}{\mathscr{E}_n(L) + \mu^2 + \xi R(L)}\,. \tag{17.41}$$

Subsequently, we will restrict the discussion of the N-sequences arising as solution of Eq. (17.40) to the case off a massless scalar field, i.e., $\mu = 0$ and therewith $\xi = 0$, in four spacetime dimensions, $d = 4$. In this case, the second type of approximant

$$\Theta_N^{\mathrm{eff}}(L) \overset{\mu=0}{\equiv} \Theta_N^{\mathrm{eff}} = \mathrm{f}(N) \tag{17.42}$$

becomes independent of L such that, with $R(L) = 12/L^2$, Eq. (17.40) becomes

$$\frac{12}{L^2} = 4\Lambda_\mathrm{b} - \frac{3G}{\pi}\,\frac{1}{L^4}\,\Theta_N^{\mathrm{eff}}\,. \tag{17.43}$$

This equation does not have solutions for a vanishing or a negative bare cosmological constant, in contrast with the first type of approximants. On the other hand, for $\Lambda_{\rm b} > 0$ there exists a complete N-sequence of self-consistent radii, given by

$$\begin{aligned} L^{\rm SC}(N)^2 &= \frac{3}{2\Lambda_{\rm b}}\left[1+\sqrt{1+\frac{G\Lambda_{\rm b}}{3\pi}\,{\rm f}(N)}\right] \\ &= \frac{1}{2}L_{\rm b}^2\left[1+\sqrt{1+\frac{G}{\pi L_{\rm b}^2}\,{\rm f}(N)}\right]\,, \end{aligned} \tag{17.44}$$

where we have defined the bare length $L_{\rm b}$ via $\Lambda_{\rm b} =: 3/L_{\rm b}^2$. As in case of the first type of approximants, the N-sequence of self-consistent spacetimes $S^4(L^{\rm SC}(N))$ has some remarkable properties:

Firstly, we note that a self-consistent S^4 geometry exists for all $N = 0, 1, 2, \ldots$. Hereby, the for 4-sphere for $N = 0$ with radius $L^{\rm SC}(0) = L_{\rm b}$ is standing out as it is purely classical. Such a classical initial point of the N-sequence for $\mu = 0$ and $\Lambda_{\rm b} > 0$ did not exist in the case of the first type of approximants, for which self-consistent S^4 geometries were of purely quantum nature ($N \geq 1$).

Secondly, for $L_{\rm b}$ fixed the N-sequence (17.44) of self-consistent "Hubble" radii $L^{\rm SC}(N)$ monotonically *increases* in the cutoff parameter N. Ultimately, in the limit $N \to \infty$ the self-consistent Hubble radius becomes infinite, i.e., the N-sequence of self-consistent 4-spheres grows until it reaches flat spacetime:

$$S^4(L^{\rm SC}(N)) \xrightarrow{N\to\infty} \mathbb{R}^4\,. \tag{17.45}$$

Note that the flat spacetime, which arises for the fully quantized system, does so without any finetuning. Consequently, the N-sequences of the self-consistent scalar curvature $R^{\rm SC}(N)$ or of the self-consistent cosmological constant $\Lambda^{\rm SC}(N)$ *decrease* in N and ultimately vanishes in the limit $N \to \infty$ of the fully quantized system:

$$R^{\rm SC}(N) = 4\Lambda^{\rm SC}(N) = \frac{12}{L^{\rm SC}(N)^2} \xrightarrow{N\to\infty} 0\,. \tag{17.46}$$

Hence, as in the analogous case for the first type of approximants, the "cosmological constant problem", according to which the effective cosmological constant increases with the amount of quantized vacuum fluctuations from which it arises, does not occur here. This becomes particularly clear in the limit $\mathrm{f} \gg 1$, where $\mathrm{f}(N) \approx \frac{1}{12} N^4$ and thus

$$L^{\mathrm{SC}}(N) \approx \sqrt{L_{\mathrm{b}} \sqrt{\frac{G}{4\pi}\,\mathrm{f}(N)}} \approx \left(\frac{G\hbar}{48\pi}\right)^{1/4} L_{\mathrm{b}}^{1/2}\, N\,. \tag{17.47}$$

Here, we reinstated Planck's constant for a moment. The self-consistent radius $L^{\mathrm{SC}}(N)$ becomes a *linear* function of N when $N \gg 1$. Further, it depends on both G and $\hbar$ in a non-analytic way, showing the *non-perturbative character* of this calculation. We can re-express the dependence on this constants through the Planck length which for $d = 4$ and $c \equiv 1$ reads $\ell_{\mathrm{Pl}} = (\hbar G)^{1/2}$,

$$L^{\mathrm{SC}}(N) \approx \frac{N}{(48\pi)^{1/4}} \sqrt{\ell_{\mathrm{Pl}}\, L_{\mathrm{b}}}\,. \tag{17.48}$$

Here, the dependence of the Planck length is also non-analytic.

Thirdly, note that for L_{b} (arbitrarily) fixed, we cannot construct N-sequences of self-consistent radii that would result in a finite "observed" value of the Hubble radius $L_{\mathrm{obs}} \equiv (3/\Lambda_{\mathrm{obs}})^{1/2}$. However, we can do so by *finetuning* the bare length L_{b}, i.e., granting it some N-dependence, $L_{\mathrm{b}} \equiv L_{\mathrm{b}}(N)$. By setting Eq. (17.44) equal to L_{obs},

$$L_{\mathrm{obs}}^2 \overset{!}{\equiv} L^{\mathrm{SC}}(N)^2 = \frac{1}{2} L_{\mathrm{b}}(N)^2 \left[1 + \sqrt{1 + \frac{G}{\pi L_{\mathrm{b}}(N)^2}\,\mathrm{f}(N)}\,\right], \tag{17.49}$$

and solving this equation for $L_{\mathrm{b}}(N)$, we find that

$$L_{\mathrm{b}}(N) = \frac{L_{\mathrm{obs}}^2}{\sqrt{L_{\mathrm{obs}}^2 + \frac{G}{4\pi}\,\mathrm{f}(N)}}\,. \tag{17.50}$$

If L_{b} is finetuned in this way, we have constructed an N-sequence of self-consistent radii that in the limit $N \to \infty$ converges to the value L_{obs}, i.e., $L^{\mathrm{ac}}(N) \to L_{\mathrm{obs}}$ as well as $L_{\mathrm{b}}(N) \to 0$. Thereby, the value of L_{obs} can be set freely. Again, this becomes particularly clear in the limit $N \gg 1$ in which

$$L_{\mathrm{b}}(N) \approx (48\pi)^{1/2} \ell_{\mathrm{Pl}} \left(\frac{L_{\mathrm{obs}}}{N \ell_{\mathrm{Pl}}}\right)^2 . \tag{17.51}$$

Again, we have absorbed Newton's constant G (and the Planck constant $\hbar$) into the Planck length ℓ_{Pl}.

Fourthly, we recall that the dimensionless UV cutoff N induces the dimensionful UV cutoff $\mathscr{P}(N)$ by

$$\mathscr{P}(N)^2 := \mathscr{E}_N(L^{\mathrm{SC}}(N)) = \frac{N(N+3)}{L^{\mathrm{SC}}(N)^2}\,. \tag{17.52}$$

As for the background dependent calculation the dimensionfull UV cutoff diverges for $N \to \infty$, analyzing the induced dimensionful UV cutoff $\mathscr{P}(N)$ gives insights towards the difference between the background dependent and the Background Independent calculation. Here, for the latter given by the self-consistent radii (17.44) with L_b fixed we have

$$\mathscr{P}(N)^2 = \frac{2N(N+3)}{L_\mathrm{b}^2}\left[1+\sqrt{1+\frac{12\mathbf{f}(N)}{N_T^4}}\right]^{-1} \tag{17.53}$$

with the abbreviation

$$N_T \equiv (12\pi)^{1/4}\left(\frac{L_\mathrm{b}}{\ell_\mathrm{Pl}}\right)^{1/2}. \tag{17.54}$$

To understand hove the $\mathscr{P}$-cutoff behaves in the limit $N \to \infty$, we approximate it for $N \gg 1$ and re-write it in terms of $m_\mathrm{Pl} \equiv \ell_\mathrm{Pl}^{-1}$,

$$\mathscr{P}(N)^2 \approx (24\pi) m_\mathrm{Pl}^2 \frac{N^2}{N_T^4}\left[1+\sqrt{1+\left(\frac{N}{N_T}\right)^4}\right]^{-1}. \tag{17.55}$$

Therewith, it is easy to that

$$\lim_{N\to\infty} \mathscr{P}(N) = (24\pi)^{1/2}\frac{m_\mathrm{Pl}}{N_T}\,, \tag{17.56}$$

i.e., for the Background Independent calculation, $\mathscr{P}(N)$ never reaches infinity but rather converges to some finite value which is a rather striking result. Figure 17.1 shows a graph of $\mathscr{P}(N)$ for $N_T \gg 1$: For $N = 0$ we find the classical initial point $\mathscr{P}(N = 0) = 0$ of a vanishing cutoff. Then, for N small, $\mathscr{P}(N)$ increases with N until it reaches the "transition" value N_T after which the curve approaches a plateau and $\mathscr{P}(N)$ becomes independent of N. This brings us to the *main conclusion* of this chapter:

In the Background Independent calculation the limits $N \to \infty$ and $\mathscr{P} \to \infty$ are obviously inequivalent. Because per construction, the regulator is only removed fully in the former limit, $N \to \infty$, *it is incorrect to attempt taking the limit $\mathscr{P} \to \infty$ when the gravitational backreaction is taken into account.*

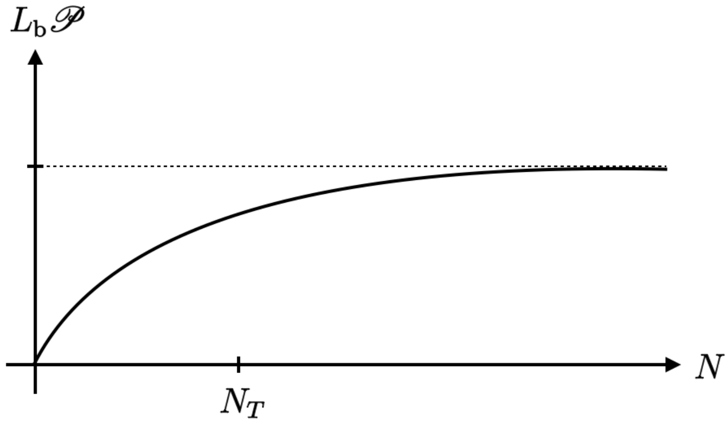

FIGURE 17.1. The dimensionful cutoff scale $\mathscr{P}$ in dependence on N according to Eq. (17.53). The corresponding sequence of approximants assumes a *positive* bare cosmological constant.

If $L_b \gtrsim \ell_{\mathrm{Pl}}$, i.e., if the self-consistent radius of the $(N = 0)$-approximant is of order of the Planck length or larger, then there is no approximant for $0 \leq N < \infty$ that has a dimensionful UV cutoff larger than about the Planck scale, $\mathscr{P}(N) \lesssim m_{\mathrm{Pl}}$. Although the regulator is fully removed and all field modes are integrated out each member of the N-sequence *does not face the problem of transplackian momenta.*

CHAPTER 18

Gravity-coupled approximants for quantized metric fluctuations

Executive summary. We apply the framework for the quantization of fields by gravity-coupled approximants to quantum gravity itself. Therefore, we apply the background field technique and identify a first and second type of approximants for quantized metric fluctuations. The first type of approximants is obtained by interpreting the one-loop term of the bare gravitational action as the matter action for a Gaussian graviton field. Therefrom, we obtain a classical stress-energy tensor which we promote to an operatorial relation that we subsequently regularize by an N-cutoff. The second type of approximants is obtained from the one-loop term of the effective action which induces an effective stress-energy tensor that also is regularized by an N-cutoff. We trace back the difference between the two kinds of approximants to the dependence of the gravitational path integral measure on the background metric. In four spacetime dimensions and for spherical background geometries, we explicitly solve the resulting self-consistency condition for the background metric. We show that for both kinds of approximants, there exist N-sequences of self-consistent "background" radii that are free from the cosmological constant problem.
What is new? All research results of this chapter are new.
Based upon: Reference [6].

18.1. A first type of approximants

(A) The classical graviton field and its backreaction on the background metric. We want to define the first type of approximants for metric fluctuations analogously to those for the scalar field. Therefore, we employ the action functional (4.47) for the (classical) graviton field $h_{\mu\nu}$ and the (classical) ghost fields $\bar{\xi}_\mu$ and ξ^μ,

$$S[h,\bar{\xi},\xi;\bar{g}] = S_{\mathrm{EH}}[\bar{g}] + S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}]\,, \tag{18.1}$$

that we had obtained by linearizing the theory given by the action (4.36). Especially, recall that this linearization corresponds to a one-loop approximation. Associated to the classical matter action (4.44),

$$S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}] \coloneqq S_{\mathrm{graviton}}[h;\bar{g}] + S_{\mathrm{gh}}[0,\bar{\xi},\xi;\bar{g}]\,, \tag{18.2}$$

is the classical stress-energy tensor defined by Eq. (15.19),

$$T^{\mu\nu}[h,\bar{\xi},\xi;\bar{g}](x) \coloneqq -\frac{2}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta\bar{g}_{\mu\nu}(x)}S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}]\,. \tag{18.3}$$

The conservation law $\bar{D}_\mu T^{\mu\nu} = 0$ is induced by the invariance of the action $S[h,\bar{\xi},\xi,\bar{g}]$, as given by its general definition (4.36), under the background gauge transformations (4.23), $\delta^{(B)}S[h,\bar{\xi},\xi,\bar{g}] = 0$ (cf. appendix F.4). Consequently, also every term in the Taylor expansion (4.41), that yields in Eq. (4.47), is $\delta^{(B)}$-invariant, and therewith especially the matter action $S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}]$. Given that $\delta^{(B)}\bar{g}_{\mu\nu} = L_V\bar{g}_{\mu\nu} = \bar{D}_\mu V_\nu + \bar{D}_\nu V_\mu$, we thus have [154, p. 38]

$$\begin{aligned}0 = \delta^{(B)}S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}] &= \int \mathrm{d}^d x \frac{\delta S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}]}{\delta\bar{g}_{\mu\nu}(x)}\delta^{(B)}\bar{g}_{\mu\nu}(x) \\ &= -\int \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\big(\bar{D}_\mu T^{\mu\nu}[h,\bar{\xi},\xi;\bar{g}]\big)(x)V_\nu(x)\,,\end{aligned} \tag{18.4}$$

where have performed a partial integration in the second step. As the infinitesimal vector field V_ν can be chosen fully arbitrary, the conservation law $\bar{D}_\mu T^{\mu\nu}[h,\bar{\xi},\xi;\bar{g}] = 0$ immediately follows. *In fact, we therewith have shown that this conservation law is fulfilled for any stress-energy tensor defined by Eq.* (15.19), *provided that the defining matter action is* $\delta^{(B)}$*-invariant.*

As before for the scalar field, we will restrict the treatment of the backreation of first type (and later also of the second type) of quantum stress tensor candidate to the integrated and traced equations of motion

$$\bar{\mathscr{T}} S[h,\bar{\xi},\xi;\bar{g}] = 0\,, \tag{18.5}$$

where the operator

$$\bar{\mathscr{T}} = -2 \int \mathrm{d}^d x \, \bar{g}_{\mu\nu}(x) \, \delta/\delta\bar{g}_{\mu\nu}(x) \tag{18.6}$$

is defined as in Eq. (16.11) (the "bar" shall indicate that the operator is built from the background metric $\bar{g}$). As outlined in Section 16.2, also here the integrated and traced equations of motion are sufficient when maximally symmetric background spacetimes are under consideration because their scalar curvature is their sole magnitude of curvature. When determined by the action (4.47), these equations of motion amount to

$$\begin{aligned} &\int \mathrm{d}^d z \sqrt{\bar{g}(z)} \left\{ \left(\frac{d}{2} - 1 \right) \bar{R}(z) - d\Lambda \right\} \\ &\qquad = -8\pi G \int \mathrm{d}^d z \sqrt{\bar{g}(z)} \, \bar{g}_{\mu\nu}(z) T^{\mu\nu}[h, \bar{\xi}, \xi; \bar{g}](z) \,, \end{aligned} \tag{18.7}$$

where we have used Eq. (16.20). The RHS is given by (up to the factor $-8\pi G$)

$$\begin{aligned} &\int \mathrm{d}^d z \sqrt{\bar{g}(z)} \, \bar{g}_{\mu\nu}(z) \, T_{\mu\nu}[h, \bar{\xi}, \xi; \bar{g}](z) \\ &\quad = \bar{\mathscr{T}} S_{\mathrm{M}}[h, \bar{\xi}, \xi; \bar{g}] \\ &\quad = \frac{1}{2} \int \mathrm{d}^d x \sqrt{\bar{g}} \; h_{\mu\nu} \bar{\mathscr{T}} \left\{ \left(\mathscr{U}[0; \bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)^{\mathrm{diff}} I[\bar{g}]^{\rho\sigma\alpha\beta} \right\} h_{\alpha\beta} \\ &\qquad - \sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}} \, \bar{\xi}_\mu \bar{\mathscr{T}} \mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu \xi^\nu \,. \end{aligned} \tag{18.8}$$

The required variations are performed in appendix F.9 and are given by

$$\begin{aligned} \bar{\mathscr{T}} \sqrt{\bar{g}(x)} &= - d \sqrt{\bar{g}(x)} \\ \bar{\mathscr{T}} I^{\rho\sigma\alpha\beta}[\bar{g}](x) &= 4 I^{\rho\sigma\alpha\beta}[\bar{g}](x) \\ \bar{\mathscr{T}} \left(\mathscr{U}[0; \bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)^{\mathrm{diff}} A^{\rho\sigma} &= 2 \left(\mathscr{U}[0; \bar{g}]^{\mu\nu}{}_{\rho\sigma}\big|_{\Lambda=0} \right)^{\mathrm{diff}} A^{\rho\sigma} \\ \bar{\mathscr{T}} \left(\mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu\right)^{\mathrm{diff}} X^\nu &= 2 \left(\mathscr{M}[\bar{g}, \bar{g}]^\mu{}_\nu\right)^{\mathrm{diff}} X^\nu \,, \end{aligned} \tag{18.9}$$

where $A^{\rho\sigma}$ and X^{ν} are arbitrary tensor fields. Therewith, the RHS becomes

$$
\begin{aligned}
\int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \bar{g}_{\mu\nu}(z) T^{\mu\nu}[h, \bar{\xi}, \xi; \bar{g}](z) = \bar{\mathscr{T}} S_{\mathrm{M}}[h, \bar{\xi}, \xi; \bar{g}] \\
= \frac{1}{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\; h_{\mu\nu} \Big\{ (4-d) (\mathscr{U}[0; \bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}} \\
+ 2 \left(\mathscr{U}[0; \bar{g}]^{\mu\nu}{}_{\rho\sigma} \big|_{\Lambda=0} \right)^{\mathrm{diff}} \Big\}\, I[\bar{g}]^{\rho\sigma\alpha\beta} h_{\alpha\beta} \\
- \sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{\xi}_{\mu} (2-d) \left(\mathscr{M}[\bar{g}, \bar{g}]^{\mu}{}_{\nu} \right)^{\mathrm{diff}} \xi^{\nu} .
\end{aligned}
\tag{18.10}
$$

(B) The semi-classical Einstein equation. In a next step, we quantize the fields $(h, \bar{\xi}, \xi) \mapsto (\widehat{h}, \bar{C}, C)$ and promote this equation for the integrated and traced stress-energy tensor to an operatorial relation, whose expectation value we subsequently take:

$$
\int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \bar{g}_{\mu\nu}(z) \left\langle \widehat{T}^{\mu\nu}[\bar{g}](z) \right\rangle \coloneqq \left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle . \tag{18.11}
$$

The corresponding semi-classical integrated and traced equations of motion are given by

$$
\begin{aligned}
\int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \left\{ \left(\frac{d}{2} - 1 \right) \bar{R}(z) - d \Lambda_{\mathrm{b}} \right\} \\
= -8\pi G \int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \bar{g}_{\mu\nu}(z) \left\langle \widehat{T}^{\mu\nu}[\bar{g}](z) \right\rangle .
\end{aligned}
\tag{18.12}
$$

Here, we have traded the classical cosmological constant Λ for the bare cosmological constant Λ_{b}. Further, the expectation value is taken with respect to one-loop expansion of the Schwinger functional given by Eq. (4.17), i.e., the action in the exponent on the RHS of Eq. (4.17) is approximated by the expansion (4.47). This result still is up to regularization, i.e., now we should explain how this expectation value is modified when implementing a cutoff of the N-type. However, it will turn out to be more convenient to firstly rephrase the unregularized result before finally implementing the N-cutoff:

$$\left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle = \lim_{y \to x} \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \Bigg\{ \Bigg[\left(2 - \frac{d}{2}\right) \left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)_x^{\mathrm{diff}} + \left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\big|_{\Lambda_{\mathrm{b}}=0} \right)_x^{\mathrm{diff}} \Bigg] I[\bar{g}]^{\rho\sigma\alpha\beta} \left\langle \widehat{h}_{\mu\nu}(y) \widehat{h}_{\alpha\beta}(x) \right\rangle - \sqrt{2}\,(2-d)\left(\mathscr{M}[\bar{g},\bar{g}]^{\mu}{}_{\nu}\right)_x^{\mathrm{diff}} \left\langle \bar{C}_\mu(y) C^\nu(x) \right\rangle \Bigg\}. \tag{18.13}$$

The one-loop expectation values $\langle \widehat{h}_{\mu\nu}(y)\widehat{h}_{\alpha\beta}(x)\rangle$ and $\langle \bar{C}_\mu(y) C^\nu(x)\rangle$ appearing here are precisely those calculated in Eq. (4.54) such that the RHS of the semi-classical equations of motion becomes

$$\left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle = \lim_{y \to x} \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \Bigg\{ \Bigg[\left(2 - \frac{d}{2}\right) \left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)_x^{\mathrm{diff}} + \left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\big|_{\Lambda_{\mathrm{b}}=0} \right)_x^{\mathrm{diff}} \Bigg] \left((\mathscr{U}[0;\bar{g}]^{-1})^{\rho\sigma}{}_{\alpha\beta} \right)_y^{\mathrm{diff}} \langle y,\alpha,\beta | x,\mu,\nu\rangle - \sqrt{2}\,(2-d)\left(\mathscr{M}[\bar{g},\bar{g}]^{\mu}{}_{\nu}\right)_x^{\mathrm{diff}} \left((\sqrt{2}^{-1} \mathscr{M}[\bar{g},\bar{g}]^{-1})^{\nu}{}_{\alpha} \right)_y^{\mathrm{diff}} \langle y,\alpha | x,\mu\rangle \Bigg\}. \tag{18.14}$$

After using that $((\mathscr{U}[0;\bar{g}]^{-1})^{\rho\sigma}{}_{\alpha\beta})_y^{\mathrm{diff}} \langle y,\alpha,\beta|x,\mu,\nu\rangle = \langle y,\rho,\sigma|\mathscr{U}[0;\bar{g}]^{-1}|x,\mu,\nu\rangle$ and $((\mathscr{M}[\bar{g},\bar{g}]^{-1})^{\nu}{}_{\alpha})_y^{\mathrm{diff}} \langle y,\alpha|x,\mu\rangle = \langle y,\nu|\mathscr{M}[\bar{g},\bar{g}]^{-1}|x,\mu\rangle$, we can perform the limit $y \to x$:

$$\left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle = \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \Bigg\{ \langle x,\mu,\nu| \Bigg[\left(2 - \frac{d}{2}\right) \mathscr{U}[0;\bar{g}] + \left(\mathscr{U}[0;\bar{g}]\big|_{\Lambda_{\mathrm{b}}=0} \right) \Bigg] \mathscr{U}[0;\bar{g}]^{-1} |x,\mu,\nu\rangle - (2-d)\langle x,\mu|\mathscr{M}[\bar{g},\bar{g}]\mathscr{M}[\bar{g},\bar{g}]^{-1}|x,\mu\rangle \Bigg\}. \tag{18.15}$$

In this equation, we recognize the traces over the Hilbert spaces ST^2 and V as defined in appendix A.1:

$$\begin{aligned}\int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \bar{g}_{\mu\nu}(z) \left\langle \widehat{T}^{\mu\nu}[\bar{g}](z) \right\rangle &:= \left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle \\ &= \left(2 - \frac{d}{2}\right) \mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right] \\ &\quad + \mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}[0;\bar{g}]\big|_{\Lambda_{\mathrm{b}}=0}\right) \mathscr{U}[0;\bar{g}]^{-1}\right] \\ &\quad - (2-d)\, \mathrm{Tr}_V\left[\mathbb{1}_V\right] .\end{aligned} \tag{18.16}$$

With the result (18.16) for the RHS of the semi-classical integrated and traced equations of motion (18.12), these become

$$\begin{aligned}\int \mathrm{d}^d z \sqrt{\bar{g}(z)} \left\{ \left(\frac{d}{2} - 1\right) \bar{R}(z) - d\Lambda_{\mathrm{b}} \right\} = 8\pi G \Bigg\{ &\left(\frac{d}{2} - 2\right) \mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right] \\ &- \mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}[0;\bar{g}]\big|_{\Lambda_{\mathrm{b}}=0}\right) \mathscr{U}[0;\bar{g}]^{-1}\right] \\ &- (d-2)\, \mathrm{Tr}_V\left[\mathbb{1}_V\right] \Bigg\} .\end{aligned} \tag{18.17}$$

(C) The quantum system at finite N on $S^d(L)$. Finally, we must regularize the first type of stress tensor candidate (18.16) by means of an N-cutoff. We will restrict the implementation of the N-cutoff to the case $M = S^d(L)$, i.e the background spacetime is given by the d-sphere with radius L. On $S^d(L)$ it is particularly simple for us to implement an N-cutoff; namely we can do so by truncating the eigenbases of $-\Box_{\bar{g}}$ of the Hilbert spaces V and ST^2 constructed in appendix A.1.3 and given by (A.42) and (A.52), respectively, at the finite value N:

$$\begin{aligned}\mathfrak{B}^V_N := &\left\{ |nm\rangle^T \;\middle|\; n = 1,2,\ldots,N \;;\; m = 1,2,\ldots,D^T_n \right\} \\ &\times \bigcup \left\{ |nm\rangle^L \;\middle|\; n = 1,2,\ldots,N \;;\; m = 1,2,\ldots,D^L_n \right\} ,\end{aligned} \tag{18.18}$$

and

$$\mathfrak{B}^{ST^2}_N := \left\{ |nm\rangle^{TT} \;\middle|\; n = 2,3,\ldots,N \;;\; m = 1,2,\ldots,D^{TT}_n \right\}$$

$$\times \bigcup \Big\{ |nm\rangle^{L^T,T} \mid n = 2, 3, \dots, N \ ; \ m = 1, 2, \dots, D_n^{L^T,T} \Big\}$$
$$\times \bigcup \Big\{ |nm\rangle^{L^L,T} \mid n = 2, 3, \dots, N \ ; \ m = 1, 2, \dots, D_n^{L^L,T} \Big\}$$
$$\times \bigcup \big\{ |nm\rangle^{\text{trace}} \mid n = 1, 2, \dots, N \ ; \ m = 1, 2, \dots, D_n^{\text{trace}} \big\} \ . \qquad (18.19)$$

Note that we also have excluded the zero mode from the trace (i.e., scalar) part of ST^2. Consequently, when implementing this N-cutoff, we do not consider full vector and symmetric rank-2 tensor fields, but rather only their projections onto the linear spans of the bases $\mathfrak{B}_N^V$ and $\mathfrak{B}_N^{ST^2}$. This means that traces of some operator A_V acting on V or of some operator A_{ST^2} acting on ST^2, given by Eqs. (A.43) and (A.53), when regularized by an N-cutoff read

$$\text{Tr}_V[A_V]_N = \sum_{n=1}^{N} \sum_{m=1}^{D_n^T} {}^T\langle nm|A_V|nm\rangle^T + \sum_{n=1}^{N} \sum_{m=1}^{D_n^L} {}^L\langle nm|A_V|nm\rangle^L \ , \qquad (18.20)$$

and

$$\begin{aligned} \text{Tr}_{ST^2}[A_{ST^2}]_N = &\sum_{n=2}^{N} \sum_{m=1}^{D_n^{TT}} {}^{TT}\langle nm|A_{ST^2}|nm\rangle^{TT} \\ &+ \sum_{n=2}^{N} \sum_{m=1}^{D_n^{L^T,T}} {}^{L^T,T}\langle nm|A_{ST^2}|nm\rangle^{L^T,T} \\ &+ \sum_{n=2}^{N} \sum_{m=1}^{D_n^{L^L,T}} {}^{L^L,T}\langle nm|A_{ST^2}|nm\rangle^{L^L,T} \\ &+ \sum_{n=1}^{N} \sum_{m=1}^{D_n^{\text{trace}}} {}^{\text{trace}}\langle nm|A_{ST^2}|nm\rangle^{\text{trace}} \ . \end{aligned} \qquad (18.21)$$

The degrees of freedom of the quantum system, which are generally given by the degrees of freedom of the graviton f_{grav} and of the ghost fields f_{ghosts}, at the finite N-cutoff then are given by, respectively,

$$\text{f}_{\text{grav}}(N) = \sum_{n=2}^{N} \Big(D_n^{TT} + D_n^{L^T,T} + D_n^{L^L,T} \Big) + \sum_{n=1}^{N} D_n^{\text{trace}} \equiv \text{Tr}_{ST^2}\big[\mathbb{1}_{ST^2}\big]_N \quad (18.22)$$

and

$$\text{f}_{\text{ghosts}}(N) = 2 \sum_{n=1}^{N} \big(D_n^T + D_n^L \big) \equiv 2\,\text{Tr}_V\big[\mathbb{1}_V\big]_N \, . \qquad (18.23)$$

Therewith we can identify the first type of approximants for quantized metric fluctuations with[1]

$$\boxed{\begin{aligned}\Theta_N(L) = \left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle =: \int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \bar{g}_{\mu\nu}(z) \left\langle \widehat{T}^{\mu\nu}[\bar{g}](z) \right\rangle \\ = \left(2 - \frac{d}{2}\right) \mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_N + \mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}[0;\bar{g}]\big|_{\Lambda_{\mathrm{b}}=0}\right) \mathscr{U}[0;\bar{g}]^{-1}\right]_N \\ - (2-d)\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N .\end{aligned}} \tag{18.24}$$

Here the background metric is w.l.o.g. considered to be written in the form $\bar{g}_{\mu\nu} = L^2 \gamma_{\mu\nu}$, with $\gamma_{\mu\nu}$ the dimensionless metric on the unit d-sphere. Furthermore, the scalar curvature of the d-sphere is x-independent, $\bar{R}[S^d(L)] \equiv \text{const.}$, and the volume is finite, given by

$$\mathrm{vol}\left[S^d(L)\right] = \frac{2\pi^{\frac{d+1}{2}} L^d}{\Gamma\left(\frac{d+1}{2}\right)} =: \sigma_d L^d . \tag{18.25}$$

Therewith, the backreaction (18.17) of the first type of approximants $\Theta_N(L)$ on the background metric $\bar{g}_{\mu\nu}$ becomes

$$\boxed{\begin{aligned}\left(\frac{d}{2} - 1\right) \bar{R}(L) = d\,\Lambda_{\mathrm{b}} - \frac{8\pi G}{\sigma_d L^d}\,\Theta_N(L) \\ = d\,\Lambda_{\mathrm{b}} + \frac{8\pi G}{\sigma_d L^d} \Bigg\{ \left(\frac{d}{2} - 2\right) \mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_N \\ - \mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}[0;\bar{g}]\big|_{\Lambda_{\mathrm{b}}=0}\right) \mathscr{U}[0;\bar{g}]^{-1}\right]_N \\ - (d-2)\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N \Bigg\} .\end{aligned}} \tag{18.26}$$

18.2. A second type of approximants

(A) The second stress tensor candidate. To analyze the backreation of the second type of approximants, we consider the equations of motion given by the EA at vanishing quantum fluctuation and vanishing ghost fields. As in this

[1]Here $\Theta_N(L)$ must not be confused with the first type of approximants for scalar field that we had denoted by the same symbol.

case one has $g_{\mu\nu} = \bar{g}_{\mu\nu}$, we will, for ease of notation, *write only* $g_{\mu\nu}$ *instead of* $\bar{g}_{\mu\nu}$ in this section. In general, the equations of motion for the EA (4.70) are given by

$$\frac{\delta\Gamma[g]}{\delta g_{\mu\nu}(x)} = 0\,. \tag{18.27}$$

Again, we confine ourselves to considering the integrated and traced equations of motion that we treat on a generic background manifold:

$$0 = \mathscr{T}\Gamma[g] \quad \Leftrightarrow \quad 0 = \mathscr{T}S_{\text{EH}}[g] + \frac{1}{2}\,\text{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\mathscr{T}\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\right] \\ - \text{Tr}_V\left[\left(\mathscr{M}[g,g]^{-1}\right)^{\cdot}{}_{\mu}\mathscr{T}\mathscr{M}[g,g]^{\mu}{}_{\cdot}\right]\,, \tag{18.28}$$

where $\mathscr{T} = -2\int \mathrm{d}^d x\, g_{\mu\nu}(x)\,\delta/\delta g_{\mu\nu}(x)$ as defined in Eq. (16.11).

Both traces appearing here still are subject to regularization, and surely we later will do so by implementing an N-cutoff. In fact, both traces in Eq. (4.70) are already subject to regularization which is why we have performed the standard rule $\delta\,\text{Tr}\ln[A] = \text{Tr}[A^{-1}\delta A]$ on the already regularized traces. As we had explained in Chapter 17, this rule might not hold when regularizing the traces via a finite mode cutoff (given by cutting its expansion in terms of eigenmodes of the negative Laplacian on a maximally symmetric spacetime). For the traces regulated in this way, the cyclicity property might not hold anymore – which is why the mentioned standard rule might not hold anymore, too, as it makes use of the cyclicity. However, the more basic standard rule $\delta\,\text{Tr}[\,\cdot\,] = \text{Tr}[\delta\,\cdot\,]$ is fulfilled also by the regularized trace: we vary with respect to the metric and the (regularized) traces are metric-independent (cf. their definition in appendix A.1). Thus, by showing that $[\mathscr{T}\mathscr{U},\mathscr{U}] = 0$, we can ensure that any function $Q(\mathscr{U})$ can be differentiated in the ordinary, commutitive way: $\mathscr{T}Q(\mathscr{U}) = Q'(\mathscr{U})\mathscr{T}\mathscr{U}$ (and likewise for $\mathscr{M}$). Particulary, then the rule $\delta\,\text{Tr}_{\text{reg}}\ln[A] = \text{Tr}_{\text{reg}}[A^{-1}\delta A]$ is applicable for $A = \mathscr{U}$ and $A = \mathscr{M}$.

To proceed further, we hence must calculate $\mathscr{T}(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}A^{\rho\sigma}$ as well as $\mathscr{T}(\mathscr{M}[g,g]^{\mu})^{\text{diff}}_{\nu}X^{\nu}$ where, for this purpose, $A^{\rho\sigma}$ and X^{ν} are arbitrary, but g-independent, tensor fields (representing elements of a basis V and ST^2 with respect to which the traces are calculated). As a tedious calculation in appendix F.9 shows, one has

$$\begin{aligned} &\mathscr{T}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma}\right)^{\text{diff}} A^{\rho\sigma} = 2\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma}\Big|_{\Lambda_\text{b}=0}\right)^{\text{diff}} A^{\rho\sigma} \\ \text{and} \qquad &\mathscr{T}\left(\mathscr{M}[g,g]^{\mu}\right)^{\text{diff}}_{\nu} X^{\nu} = 2\left(\mathscr{M}[g,g]^{\mu}\right)^{\text{diff}}_{\nu} X^{\nu}\,. \end{aligned} \tag{18.29}$$

Note that these equations hold for arbitrary gauge fixing parameters α and β. Therewith, the commutators $[\mathscr{T}\mathscr{U},\mathscr{U}] = 0$ and $[\mathscr{T}\mathscr{M},\mathscr{M}] = 0$ obviously are fulfilled and our above step toward calculating the integrated and traced equations of motion has been correct, indeed. These hence are given by

$$\begin{aligned} 0 = \mathscr{T} S_\text{EH}[g] &+ \text{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_\text{b}=0}\right)\right]_\text{reg} \\ &- 2\,\text{Tr}_V\left[\left(\mathscr{M}[g,g]^{-1}\right)^{\cdot}{}_{\mu}\mathscr{M}[g,g]^{\mu}{}_{\cdot}\right]_\text{reg}\,, \end{aligned} \tag{18.30}$$

which together with Eq. (16.20) yields

$$\begin{aligned} 0 = &\int \text{d}^d z \sqrt{g(z)}\,\left[\left(\frac{d}{2}-1\right)R(z) - d\Lambda_\text{b}\right] \\ &+ 8\pi G\,\text{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_\text{b}=0}\right)\right]_\text{reg} - 16\pi G\,\text{Tr}_V\left[\mathbb{1}_V\right]_\text{reg}\,. \end{aligned} \tag{18.31}$$

A byproduct of this general statement is the application of $\mathscr{T}$ to the one-loop term $\Gamma_\text{1L}[g]$, which we identify as the *second type of quantum stress tensor candidate* or the *effective (quantum) stress tensor*,

$$\mathscr{T}\Gamma_\text{1L}[g]_\text{reg} =: \int \text{d}^d x \sqrt{\bar{g}(x)}\,(T_\text{eff})^{\mu}_{\mu}[\bar{g}](x)\,. \tag{18.32}$$

Later, when regularizing via an N-cutoff, we will indentify this second type of stress tensor candidate with the second type of approximants. Generally, we have

$$\boxed{\mathscr{T}\Gamma_\text{1L}[g]_\text{reg} = \text{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_\text{b}=0}\right)\right]_\text{reg} - 2\,\text{Tr}_V\left[\mathbb{1}_V\right]_\text{reg}\,.} \tag{18.33}$$

At this point, note that together with Eq. (18.16) we have obtained the difference

$$\boxed{\bar{\mathscr{T}}\Gamma_\text{1L}[\bar{g}]_\text{reg} - \left\langle \bar{\mathscr{T}} S_\text{M}[\widehat{h},\bar{C},C;\bar{g}]\right\rangle_\text{reg} = \left(\frac{d}{2}-2\right)\text{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_\text{reg} - d\,\text{Tr}_V\left[\mathbb{1}_V\right]_\text{reg}\,.} \tag{18.34}$$

Like in case of backreation of the scalar field A, this difference can tracked back to gravitational measure given by Eq. (4.22), i.e.,

$$\mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\mathcal{D}_{\bar{g}}C^{\mu}\mathcal{D}_{\bar{g}}\bar{C}_{\mu} = \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}} \prod_{\mu\geq\nu} \mathrm{d}\widehat{h}_{\mu\nu}(x) \prod_{\alpha} \mathrm{d}C^{\alpha}(x)\mathrm{d}\bar{C}_{\alpha}(x)\,. \quad (18.35)$$

To see this, let assume this measure to be regularized by restricting it to finitely many spacetime points. Then apply the operator $\bar{\mathscr{T}}$ to Eq. (4.73). This leads directly to

$$\begin{aligned}\bar{\mathscr{T}}\Gamma_{1\mathrm{L}}[\bar{g}]_{\mathrm{reg}} &- \left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle_{\mathrm{reg}} \\ &= \mathrm{e}^{+\Gamma_{1\mathrm{L}}[\bar{g}]_{\mathrm{reg}}} \int \left(\bar{\mathscr{T}} \mathcal{D}_{\bar{g}}\widehat{h}_{\mu\nu}\mathcal{D}_{\bar{g}}C^{\mu}\mathcal{D}_{\bar{g}}\bar{C}_{\mu} \right) \mathrm{e}^{-S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]}\,. \end{aligned} \quad (18.36)$$

With the help of lemma (17.13) we can evaluate the variation of the measure:

$$\begin{aligned}\bar{\mathscr{T}} \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}} &= \lim_{\alpha\to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \prod_x \det\left(\mathrm{e}^{-2\alpha}\bar{g}_{\bullet\bullet}(x)\right)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}} \\ &= \lim_{\alpha\to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \left(\prod_x \mathrm{e}^{-2d\alpha[\frac{(d-4)(d+1)}{8}-\frac{d}{2}]}\right) \left(\prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}}\right) \\ &= \lim_{\alpha\to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \left(\mathrm{e}^{[-\frac{d-4}{2}\frac{d(d+1)}{2}+d^2]\sum_x 1}\right) \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}} \\ &= \left(\left[-\frac{d-4}{2}\frac{d(d+1)}{2}+d^2\right]\sum_x 1\right) \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}} \\ &= \left[-\frac{d-4}{2}\mathrm{Tr}_{\mathrm{ST}^2}[\mathbb{1}_{\mathrm{ST}^2}]_{\mathrm{reg}} + d\,\mathrm{Tr}_V[\mathbb{1}_V]_{\mathrm{reg}}\right] \\ &\quad\times \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}-\frac{d}{2}}\,, \end{aligned} \quad (18.37)$$

where we have used that (here with $f(x) \equiv 1$)

$$\begin{aligned}\mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2} f(\hat{x})\right]_{\mathrm{reg}} &= \int_{\text{“lattice”}} \mathrm{d}^d x \sqrt{\bar{g}(x)}\, \langle x,\mu,\nu|f(\hat{x})|x,\mu,\nu\rangle \\ &= \mathrm{tr}[I_{ST^2}] \int_{\text{“lattice”}} \mathrm{d}^d x\, f(x) \\ &= \frac{d(d+1)}{2}\sum_x f(x) \end{aligned} \quad (18.38)$$

and that

$$\begin{aligned}\mathrm{Tr}_V[\mathbb{1}_V f(\hat{x})]_{\mathrm{reg}} &= \int_{\text{“lattice”}} \mathrm{d}^d x \sqrt{\bar{g}(x)}\, \langle x, \mu | f(\hat{x}) | x, \mu \rangle \\ &= \mathrm{tr}[I_V] \int_{\text{“lattice”}} \mathrm{d}^d x\, f(x) \\ &= d \sum_x f(x)\,. \end{aligned} \tag{18.39}$$

Here, “lattice” refers to discretized spacetime with which we have regularized the path integral. Therewith, we have re-derived precisely Eq. (18.34):

$$\begin{aligned}&\bar{\mathscr{T}}\Gamma_{1\mathrm{L}}[\bar{g}]_{\mathrm{reg}} - \left\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}] \right\rangle_{\mathrm{reg}} \\ &\qquad = \mathrm{e}^{+\Gamma_{1\mathrm{L}}[\bar{g}]_{\mathrm{reg}}} \int \left(\bar{\mathscr{T}} \mathcal{D}_{\bar{g}} \widehat{h}_{\mu\nu} \mathcal{D}_{\bar{g}} C^{\mu} \mathcal{D}_{\bar{g}} \bar{C}_{\mu} \right) \mathrm{e}^{-S_{\mathrm{M}}[\widehat{h}, \bar{C}, C; \bar{g}]} \\ &\qquad = \frac{d-4}{4} \mathrm{Tr}_{ST^2} \left[\mathbb{1}_{ST^2}\right]_{\mathrm{reg}} - d\, \mathrm{Tr}_V \left[\mathbb{1}_V\right]_{\mathrm{reg}}\,. \end{aligned} \tag{18.40}$$

Particularly, note that in Section 17.2 we have shown that we can identify the discretization-based cutoff with an N-cutoff.

(B) The quantum system at finite N on $S^d(L)$. To proceed further, we first specify the gauge fixing conditions to the *harmonic gauge*, $\alpha = 1$ and $\beta = 1/2$, and the metric $g_{\mu\nu}$ to that of an *maximally symmetric background spacetime* such that the operator $\mathscr{U}[0; g]^{\mu\nu}{}_{\rho\sigma}$ is given by Eq. (4.80). It is clear that the operator in this form is inverted by

$$\begin{aligned}\left[\left(\mathscr{U}[0; g]^{-1} \right)^{\mu\nu}{}_{\rho\sigma} \right]^{\mathrm{diff}} &= \kappa^{-2} \left[I^{\mu\nu}{}_{\rho\sigma} - (P_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} \right] \left(-D^2 - 2\Lambda_{\mathrm{b}} + c_I R \right)^{-1} \\ &\quad - \kappa^{-2} \frac{2}{d-2} (P_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} \left(-D^2 - 2\Lambda_{\mathrm{b}} + c_{\mathrm{trace}} R \right)^{-1}\,, \end{aligned} \tag{18.41}$$

i.e., $(\mathscr{U}[0; g]^{-1})^{\mu\nu}{}_{\rho\sigma} \mathscr{U}[0; g]^{\rho\sigma}{}_{\alpha\beta} = I^{\mu\nu}_{\alpha\beta}$.

Next, we explicitly specify the maximally symmetric background to the d-dimensional sphere of radius L, i.e., $S^d(L)$. Therewith, we can finally specify the regulator to a cutoff of the N-type, as described in the previous section.

Especially, the regularized traces now are given by Eqs. (18.20) and (18.21). Then, we identify the *second type of approximants* with Eq. (18.33), i.e.,[2]

$$\boxed{\begin{aligned}\Theta^{\text{eff}}_N(L) &:= \mathscr{T}\Gamma_{1\text{L}}[g]_N \\ &= \text{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_\text{b}=0}\right)\right]_N - 2\,\text{Tr}_V\left[\mathbb{1}_V\right]_N\,.\end{aligned}} \tag{18.42}$$

With the finite volume (18.25) of the d-sphere and its x-independent scalar curvature $R(L)$, the backreaction of $\Theta^{\text{eff}}_N(L)$ on the background metric $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$ becomes

$$\begin{aligned}\left(\frac{d}{2}-1\right)R(L) &= d\,\Lambda_\text{b} - \frac{8\pi G}{\sigma_d L^d}\Theta^{\text{eff}}_N(L) \\ &= d\,\Lambda_\text{b} + \frac{8\pi G}{\sigma_d L^d}\Bigg\{2\,\text{Tr}_V\left[\mathbb{1}_V\right]_N \\ &\qquad - \text{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_\text{b}=0}\right)\right]_N\Bigg\}\,.\end{aligned} \tag{18.43}$$

To prepare the application of the trace formula (18.21), we split the identity $\mathbb{1}_{ST^2}$ according to Eq. (A.86), i.e., $\mathbb{1}_{ST^2} = \mathbb{P}_{TT} + \mathbb{P}_{L^T,T} + \mathbb{P}_{L^L,T} + \mathbb{P}_{\text{tr.}}$. This yields, with $D^2 = \Box_g$,

$$\begin{aligned}&\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\mu\nu}{}_{\rho\sigma}\left(\mathscr{U}[0;g]^{\rho\sigma}{}_{\alpha\beta}\big|_{\Lambda_\text{b}=0}\right)\right]^{\text{diff}} \\ &\qquad = \left(P_{TT} + P_{L^T,T} + P_{L^L,T}\right)^{\mu\nu}{}_{\alpha\beta}\left[\frac{-\Box_g + c_I R}{-\Box_g - 2\Lambda_\text{b} + c_I R}\right] \\ &\qquad\quad + \left(P_{\text{tr.}}\right)^{\mu\nu}{}_{\alpha\beta}\left[\frac{-\Box_g + c_{\text{trace}} R}{-\Box_g - 2\Lambda_\text{b} + c_{\text{trace}} R}\right]\,.\end{aligned} \tag{18.44}$$

Here, P_{TT}, $P_{L^T,T}$ and $P_{L^L,T}$ are the projectors onto the traceless transverse, traceless longitudinal-transverse and traceless longituginal-longitudinal part of the York decomposition (see appendix A.2.2). Therewith, we have finally brought

[2] Again, we use the same symbol as for the second type of approximants for the scalar field. Further, again the metric of the d-sphere is written as $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} = L^2\gamma_{\mu\nu}$.

$(\mathscr{U}[0;g]^{-1})\,(\mathscr{U}[0;g]\big|_{\Lambda_{\rm b}=0})$ into a form such that we can calculate its trace using Eq. (18.21),

$$
\begin{aligned}
\mathrm{Tr}_{ST^2}&\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_{\rm b}=0}\right)\right]_N\\
&=\sum_{n=2}^{N}\sum_{m=1}^{D_n^{TT}}{}^{TT}\langle nm|\frac{-\Box_g+c_IR}{-\Box_g-2\Lambda_{\rm b}+c_IR}|nm\rangle^{TT}\\
&\quad+\sum_{n=2}^{N}\sum_{m=1}^{D_n^{L^T,T}}{}^{L^T,T}\langle nm|\frac{-\Box_g+c_IR}{-\Box_g-2\Lambda_{\rm b}+c_IR}|nm\rangle^{L^T,T}\\
&\quad+\sum_{n=2}^{N}\sum_{m=1}^{D_n^{L^L,T}}{}^{L^L,T}\langle nm|\frac{-\Box_g+c_IR}{-\Box_g-2\Lambda_{\rm b}+c_IR}|nm\rangle^{L^L,T}\\
&\quad+\sum_{n=1}^{N}\sum_{m=1}^{D_n^{\rm trace}}{}^{\rm trace}\langle nm|\frac{-\Box_g+c_{\rm trace}R}{-\Box_g-2\Lambda_{\rm b}+c_{\rm trace}R}|nm\rangle^{\rm trace}\,.
\end{aligned}
\tag{18.45}
$$

Now we are in a position to exploit the eigenvalue problem of the negative Laplacian acting on symmetric rank-2 tensor fields defined on the d-sphere: $-\Box_g(u^J_{n,m})_{\mu\nu}(x)=\mathscr{E}^J_n(u^J_{n,m})(x)$ with $J\in\{(TT),(L^T,T),(L^L,T),\mathrm{trace}\}$. The corresponding eigenvalues and their multiplicities can be found in table A.1 in appendix A.1.3. Applying these eigenvalue problems yields

$$
\begin{aligned}
&\mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]^{-1}\right)^{\cdot\cdot}{}_{\mu\nu}\left(\mathscr{U}[0;g]^{\mu\nu}{}_{\cdot\cdot}\big|_{\Lambda_{\rm b}=0}\right)\right]_N\\
&=\sum_{n=2}^{N}\left\{D_n^{TT}\frac{\mathscr{E}_n^{TT}(L)+c_IR(L)}{\mathscr{E}_n^{TT}(L)-2\Lambda_{\rm b}+c_IR(L)}+D_n^{L^T,T}\frac{\mathscr{E}_n^{L^T,T}(L)+c_IR(L)}{\mathscr{E}_n^{L^T,T}(L)-2\Lambda_{\rm b}+c_IR(L)}\right.\\
&\qquad\left.+D_n^{L^L,T}\frac{\mathscr{E}_n^{L^L,T}(L)+c_IR}{\mathscr{E}_n^{L^L,T}(L)-2\Lambda_{\rm b}+c_IR(L)}\right\}+\sum_{n=1}^{N}D_n^S\frac{\mathscr{E}_n^S(L)+c_{\rm trace}R(L)}{\mathscr{E}_n^S(L)-2\Lambda_{\rm b}+c_{\rm trace}R(L)}\,.
\end{aligned}
\tag{18.46}
$$

In the same way, yet much quicker, we obtain $\mathrm{Tr}_V[\mathbb{1}_V]_N$ calculated using Eq. (18.20):

$$\begin{aligned}\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N &= \sum_{n=1}^{N}\sum_{m=1}^{D_n^T} {}^T\langle nm|\mathbb{1}_V|nm\rangle^T + \sum_{n=1}^{N}\sum_{m=1}^{D_n^L} {}^L\langle nm|\mathbb{1}_V|nm\rangle^L \\ &= \sum_{n=1}^{N}\left(D_n^T + D_n^L\right) .\end{aligned} \tag{18.47}$$

18.3. N-SEQUENCES ON $S^4(L)$

As of yet, we have deduced two equations of motion for the radius L of the d-dimensional sphere $S^d(L)$: Eq. (18.26) obtained with the first type of approximants ("type 1") and Eq. (18.43) obtained with the second type of approximants ("type 2"). For the 4-sphere, $d = 4$, these can be summarized as (with $g_{\mu\nu} \equiv \bar{g}_{\mu\nu}$)

$$\begin{aligned}R[S^4(L)] \equiv \frac{12}{L^2} = 4\Lambda_\mathrm{b} + \frac{8\pi G}{\sigma_4 L^4}\Bigg\{ &-\mathrm{Tr}_{ST^2}\left[\left(\mathscr{U}[0;g]\big|_{\Lambda_\mathrm{b}=0}\right)\mathscr{U}[0;g]^{-1}\right]_N \\ &\mp 2\,\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N\Bigg\},\end{aligned} \tag{18.48}$$

where the "$-$" and "$+$" in front of the last trace refers to the backreaction of "type 1" and "type 2", respectively. In Eqs. (18.46) and (18.47) we had already evaluated both traces in the truncated bases $\mathfrak{B}_N^V$ and $\mathfrak{B}_N^{ST^2}$ of eigenfunctions of the negative Laplacian $-\Box_g$. Therewith, the equation of motion for L reads (note that $\mathscr{E}_n \sim 1/L^2$)

$$\begin{aligned}\frac{12}{L^2} = 4\Lambda_\mathrm{b} + \frac{3G}{\pi}\frac{1}{L^4}\Bigg\{ -\sum_{n=2}^{N}\Bigg[\; & D_n^{TT}\frac{\mathscr{E}_n^{TT}(L) + c_I R(L)}{\mathscr{E}_n^{TT}(L) - 2\Lambda_\mathrm{b} + c_I R(L)} \\ &+ D_n^{L^T,T}\frac{\mathscr{E}_n^{L^T,T}(L) + c_I R(L)}{\mathscr{E}_n^{L^T,T}(L) - 2\Lambda_\mathrm{b} + c_I R(L)} \\ &+ D_n^{L^L,T}\frac{\mathscr{E}_n^{L^L,T}(L) + c_I R(L)}{\mathscr{E}_n^{L^L,T}(L) - 2\Lambda_\mathrm{b} + c_I R(L)}\Bigg] \\ -\sum_{n=1}^{N} D_n^S\frac{\mathscr{E}_n^S(L) + c_\mathrm{trace}R(L)}{\mathscr{E}_n^S(L) - 2\Lambda_\mathrm{b} + c_\mathrm{trace}R(L)} &\mp 2\sum_{n=1}^{N}\left(D_n^T + D_n^L\right)\Bigg\},\end{aligned} \tag{18.49}$$

with $c_I^{d=4} = 2/3$ and $c_{\text{trace}}^{d=4} = 0$. The respective eigenvalues $\mathscr{E}_n^J$ of $-\Box_g$ and their multiplicities D_n^J can be found in table A.1 in appendix A.1.3. Also, we have inserted $\sigma_4 = 8\pi^2/3$. Next, in order to evaluate the sums, we rearrange the RHS which yields

$$\boxed{\begin{aligned}\frac{12}{L^2} = 4\Lambda_\text{b} + \frac{3G}{\pi}\frac{1}{L^4}\Bigg\{ &-\sum_{n=2}^{N}\Big(D_n^{TT} + D_n^{L^T,T} + D_n^{L^L,T}\Big) \\ &-\sum_{n=2}^{N}\Bigg[\ D_n^{TT}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{TT}(L) - 2\Lambda_\text{b} + 8/L^2} \\ &\qquad + D_n^{L^T,T}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{L^T,T}(L) - 2\Lambda_\text{b} + 8/L^2} \\ &\qquad + D_n^{L^L,T}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{L^L,T}(L) - 2\Lambda_\text{b} + 8/L^2}\Bigg] \\ &-\sum_{n=1}^{N} D_n^S - \sum_{n=1}^{N} D_n^S\frac{2\Lambda_\text{b}}{\mathscr{E}_n^S - 2\Lambda_\text{b}} \mp 2\sum_{n=1}^{N}\left(D_n^T + D_n^L\right)\Bigg\}.\end{aligned}} \tag{18.50}$$

Here, Λ_b plays a role similar to the mass μ in the calculation for the scalar field in Section 16.2: The function $\Lambda_\text{b} \mapsto 2\Lambda_\text{b}/(\mathscr{E}_n^J(L) - 2\Lambda_\text{b} + 8/L^2)$ interpolates between zero for $\Lambda_\text{b} = 0$ and -1 for $\Lambda_\text{b} \to \infty$ (for n fixed).[3] However, unlike in case of μ for the scalar field, Λ_b does not interpolate continously between these values as the function has a pole at $2\Lambda_\text{b} = \mathscr{E}_n^J(L) + 8/L^2 = \frac{1}{L^2}(f^J(n) + 8)$ with $f^J(n)$ a positive, monotonically increasing function for all J. Hence the function of Λ_b defined above is positive left of its pole and negative (in the plane below -1) right of its pole. Having had these insights toward its Λ_b-dependence, we will, in the following, analyze Eq. (18.50) for $\Lambda_\text{b} = 0$, $\Lambda_\text{b} \in (0, \infty)$ and $\Lambda_\text{b} \to \infty$ seperately.

[3]Note that in case of the scalar field we encountered an interpolation between 0 and 1 at this place. The difference between the signs becomes apparent when taking a look at defining action functionals of the theories: The action Eq. (3.3) for the scalar field A has the structure "$A(-\Box_g + \mu^2 + \xi R)A$" while the linearized Einstein-Hilbert action for the graviton $h_{\mu\nu}$ has the structure of Eq. (4.80), i.e., "$h(-\Box_{\bar{g}} - 2\Lambda_\text{b} + c\bar{R})h$".

(A) The case $\Lambda_\mathrm{b} = 0$. For the case of a vanishing bare cosmological constant, $\Lambda_\mathrm{b} = 0$, the equation of motion for L reads

$$\begin{aligned} \frac{12}{L^2} &= \frac{3G}{\pi}\frac{1}{L^4}\Bigg\{ -\sum_{n=2}^{N}\left(D_n^{TT} + D_n^{L^T,T} + D_n^{L^L,T}\right) - \sum_{n=1}^{N} D_n^S \mp 2\sum_{n=1}^{N}\left(D_n^T + D_n^L\right) \Bigg\} \\ &= \frac{3G}{\pi}\frac{1}{L^4}\Big\{ -\mathrm{f}_{\mathrm{grav}}(N) \mp \mathrm{f}_{\mathrm{ghosts}}(N)\Big\}\,. \end{aligned} \tag{18.51}$$

Using mathematical induction, the sums appearing on the RHS can be easily proven to yield

$$\begin{aligned} \sum_{n=2}^{N}\left(D_n^{TT} + D_n^{L^T,T} + D_n^{L^L,T}\right) &= \frac{1}{12}(9N^4 + 72N^3 + 135N^2 - 36N - 180)\,, \\ \sum_{n=1}^{N} D_n^S &= \frac{1}{12}(N^4 + 8N^3 + 23N^2 + 28N)\,, \\ \sum_{n=1}^{N}\left(D_n^T + D_n^L\right) &= \frac{1}{12}(4N^4 + 32N^3 + 80N^2 + 64N)\,. \end{aligned} \tag{18.52}$$

(The second sum, here stemming from the trace part of the field decomposition, is identical to the degrees of freedom $\mathrm{f}(N)$ we encountered in the calculation for the scalar field.) Therewith, we have the degrees of freedom[4]

$$\begin{aligned} \mathrm{f}_{\mathrm{grav}}(N) &= \frac{1}{12}(10N^4 + 80N^3 + 158N^2 - 8N - 180) \\ \mathrm{f}_{\mathrm{ghosts}}(N) &= \frac{1}{12}(8N^4 + 64N^3 + 160N^2 + 128N) \end{aligned} \tag{18.53}$$

and the equation of motion for L in leading order in N becomes

$$\frac{12}{L^2} = \frac{3G}{\pi L^4}\frac{1}{12}\begin{Bmatrix} -8-10 & \text{for "type 1"} \\ +8-10 & \text{for "type 2"} \end{Bmatrix} N^4\left[1 + O\left(\frac{1}{N}\right)\right]\,, \tag{18.54}$$

[4]Note that the degrees of freedom are only well-defined for $N \geq 1$ which hints at the purely quantum nature of the approximants. On the other hand, the sum $\sum_{n=1}^{N=0}(\bullet)_n := 0$ is zero per definition such that this formula for the degrees of freedom is not applicable for $N = 0$, where there are zero degrees of freedom of the quantum system, of course.

where the number "10" stems from the graviton's modes while the "8" stems from the ghost fields' modes. Therewith, the self-cosistent radii read

$$L^{\text{SC}}(N)^2 = -\frac{G}{\pi}\begin{Bmatrix} 3/8 & \text{for "type 1"} \\ 1/24 & \text{for "type 2"} \end{Bmatrix} N^4 \left[1 + O\left(\frac{1}{N}\right)\right]. \tag{18.55}$$

Notably, for both types of calculation, the self-consistent radius $L^{\text{SC}}(N)$ is imaginary *for all* N such that in $d = 4$, *the graviton* $h_{\mu\nu}$ *does not permit a self-consitent spherical background for vanishing bare cosmological constant.*

(B) The case $\Lambda_\text{b} \in (0, \infty)$**.** Let us go back again to analyzing the equation of motion for L with a finite and non-vanishing bare cosmological constant Λ_b. With the sums we have already evaluated, Eq. (18.50) becomes

$$\begin{aligned}\frac{12}{L^2} = 4\Lambda_\text{b} - \frac{3G}{\pi}\frac{1}{L^4}\Bigg\{ & \text{f}_\text{grav}(N) \pm \text{f}_\text{ghosts}(N) + 5z\frac{1}{4-z} \\ & + \sum_{n=2}^{N}\Bigg[D_n^{TT}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{TT}(L) - 2\Lambda_\text{b} + 8/L^2} + D_n^{L^T,T}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{L^T,T}(L) - 2\Lambda_\text{b} + 8/L^2} \\ & + D_n^{L^L,T}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{L^L,T}(L) - 2\Lambda_\text{b} + 8/L^2} + D_n^{S}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{S}(L) - 2\Lambda_\text{b}}\Bigg]\Bigg\},\end{aligned} \tag{18.56}$$

where we have introduced the abbreviation $z = 2L^2\Lambda_\text{b}$ and detached the $(n = 1)$-term from the sum over the scalar ("S") modes. Applying a partial fraction decomposition to the remaining sums leads to

$$\begin{aligned}&\sum_{n=2}^{N}\Bigg[D_n^{TT}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{TT}(L) - 2\Lambda_\text{b} + 8/L^2} + D_n^{L^T,T}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{L^T,T}(L) - 2\Lambda_\text{b} + 8/L^2} \\ &\quad + D_n^{L^L,T}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{L^L,T}(L) - 2\Lambda_\text{b} + 8/L^2} + D_n^{S}\frac{2\Lambda_\text{b}}{\mathscr{E}_n^{S}(L) - 2\Lambda_\text{b}}\Bigg] \\ &= \frac{5}{3}z(N-1)(N+5) + \frac{5}{6}z(z-10)\sum_{n=2}^{N}\frac{2n+3}{n(n+3)-z+6} \\ &\quad + \frac{1}{2}z(z-2)\sum_{n=2}^{N}\frac{2n+3}{n(n+3)-z+2} + \frac{1}{3}z(z+2)\sum_{n=2}^{N}\frac{2n+3}{n(n+3)-z}.\end{aligned} \tag{18.57}$$

Therewith we can identify the terms that are quartically, quadratically and logarithmically divergent as well as finite in the limit $N \to \infty$ on the RHS of the equation of motion as

$$\frac{12}{L^2} = 4\Lambda_{\text{b}} - \frac{3G}{\pi}\frac{1}{L^4}\Big\{\text{quartically div. terms} + \text{quadratically div. terms} + \text{logarithmically div. terms} + \text{finite terms}\Big\} \tag{18.58}$$

with:

$$\text{quartically div. t.} = \frac{1}{12}\begin{cases} 18N^4 + 144N^3 + 318N^2 + 120N - 180 & (\text{“type 1”}) \\ 2N^4 + 16N^3 - 2N^2 - 136N - 180 & (\text{“type 2”}) \end{cases}$$

$$\text{quadratically div. terms} = \frac{5}{3}z(N-1)(N+5)$$

$$\begin{aligned} \text{logarithmically div. terms} = \frac{5}{6}z(z-10)\sum_{n=2}^{N}\frac{2n}{n(n+3)-z+6} \\ + \frac{1}{2}z(z-2)\sum_{n=2}^{N}\frac{2n}{n(n+3)-z+2} \\ + \frac{1}{3}z(z+2)\sum_{n=2}^{N}\frac{2n}{n(n+3)-z} \end{aligned} \tag{18.59}$$

$$
\begin{aligned}
\text{finite terms} &= 5z\frac{1}{4-z} + \frac{5}{6}z(z-10)\sum_{n=2}^{N}\frac{3}{n(n+3)-z+6} \\
&\quad + \frac{1}{2}z(z-2)\sum_{n=2}^{N}\frac{3}{n(n+3)-z+2} \\
&\quad + \frac{1}{3}z(z+2)\sum_{n=2}^{N}\frac{3}{n(n+3)-z}\,, \\
&\overset{N\to\infty}{=} 5z\frac{1}{4-z} - 14 + \frac{24}{6-z} + \frac{36}{4-z} + 15z \\
&\quad + \frac{\pi}{2}z\Bigg\{\frac{5(z-10)\tan\left[\frac{\pi}{2}\sqrt{4z-15}\right]}{\sqrt{4z-15}} \\
&\qquad + \frac{3(z-2)\tan\left[\frac{\pi}{2}\sqrt{1+4z}\right]}{\sqrt{1+4z}} \\
&\qquad + \frac{2(z+2)\tan\left[\frac{\pi}{2}\sqrt{9+4z}\right]}{\sqrt{9+4z}}\Bigg\}\,.
\end{aligned}
\tag{18.60}
$$

(In the last step, note that the limits $N \to \infty$ and $z \to 0$ do not commute. Also note that the square root appearing in the argument of the first tangent function is ill-defined for $4z < -15$.)

On the other hand, the remaining sums (18.57) on the RHS of the equation of motion for L can be evaluated by means of the identity

$$\sum_{k=0}^{N}\frac{1}{x+k} = \psi(x+N+1) - \psi(x) \tag{18.61}$$

that is fulfilled by the digamma function[5] [176, 177]

$$\psi(x) := \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)\,, \tag{18.62}$$

[5]The digamma function ψ is meromorphic on $\mathbb{C}\setminus(-\mathbb{N}_0)$ with poles of residue -1 located at $x \in -\mathbb{N}_0$. Restricted to the real part of its domain, it asymptotically behaves as a logarithm: $\lim_{x\to\infty}[\psi(x+1)-\ln(x)] = 0$. Furthermore, note that $\psi(x+N+1)-\psi(x) \in \mathbb{R}$ for all $x \in \mathbb{C}$ and $N \in \mathbb{N}$ as $\psi(z)^* = \psi(z^*) \;\Rightarrow\; \psi(z)+\psi(z^*) = 2\mathrm{Re}\,\psi(z)$.

with Γ the ordinary gamma function. Applying this relation to Eq. (18.57) yields

$$\begin{aligned}\sum_{n=2}^{N}\Bigg[\quad & D_n^{TT}\frac{2\Lambda_\mathrm{b}}{\mathscr{E}_n^{TT}(L)-2\Lambda_\mathrm{b}+8/L^2}+D_n^{L^T,T}\frac{2\Lambda_\mathrm{b}}{\mathscr{E}_n^{L^T,T}(L)-2\Lambda_\mathrm{b}+8/L^2} \\ & +D_n^{L^L,T}\frac{2\Lambda_\mathrm{b}}{\mathscr{E}_n^{L^L,T}(L)-2\Lambda_\mathrm{b}+8/L^2}+D_n^{S}\frac{2\Lambda_\mathrm{b}}{\mathscr{E}_n^{S}(L)-2\Lambda_\mathrm{b}}\Bigg] \\ & =\frac{z}{3}\Bigg[5(N-1)(N+5)\ +\frac{5}{2}(z-10)\Psi_1(N,z) \\ & \qquad +\frac{3}{2}(z-2)\Psi_2(N,z)+(z+2)\Psi_3(N,z)\Bigg],\end{aligned} \tag{18.63}$$

where we have introduced the "Ψ-functions"

$$\begin{aligned}\Psi_i(N,z)=&\,\psi\left(N+\frac{5+\sqrt{4z+q_i}}{2}\right)+\psi\left(N+\frac{5-\sqrt{4z+q_i}}{2}\right) \\ & -\psi\left(\frac{7+\sqrt{4z+q_i}}{2}\right)-\psi\left(\frac{7-\sqrt{4z+q_i}}{2}\right)\end{aligned} \tag{18.64}$$

with $q_1=-15$, $q_2=1$ and $q_3=9$. Note that each Ψ_i is real-valued for all (N,z). All in all the equation of motion for L, Eq. (18.50), therewith becomes

$$\begin{aligned}0=:f_N(L)=&\,4\Lambda_\mathrm{b}L^4-12L^2 \\ & -\frac{3G}{\pi}\Bigg\{\mathrm{f}_\mathrm{grav}(N)\pm\mathrm{f}_\mathrm{ghosts}(N)+5\frac{L^2\Lambda_\mathrm{b}}{2-L^2\Lambda_\mathrm{b}} \\ & \qquad +\frac{2L^2\Lambda_\mathrm{b}}{3}\Bigg[5(N-1)(N+5)+5(L^2\Lambda_\mathrm{b}-5)\Psi_1(N,2L^2\Lambda_\mathrm{b}) \\ & \qquad +3(L^2\Lambda_\mathrm{b}-1)\Psi_2(N,2L^2\Lambda_\mathrm{b})+2(L^2\Lambda_\mathrm{b}+1)\Psi_3(N,2L^2\Lambda_\mathrm{b})\Bigg]\Bigg\}.\end{aligned} \tag{18.65}$$

A few comments are in order. First, it is clear that the real non-negative zeros of the function $f_N(L)$ are the self-consistent radii $L^\mathrm{sc}(N)$ of the S^4-type background manifold. These depend parametrically on G and Λ_b.

Secondly, the question of the existence of such zero points; for given values of N, G and Λ_b; can be answered in the affirmative, reasoning as follows. The digamma function ψ possesses poles at $-k$ with $k \in \mathbb{N}_0$ (cf. Figure 18.1). At each pole, ψ diverges towards $+\infty$ when said pole is approached from the left and towards $-\infty$ when approached from the right. Consequently, as ψ is continuous between the pole at $-k$ and that at $-(k+1)$, it possesses a zero point in each of these intervals. What does this imply for the zeros of $f_N(L)$?

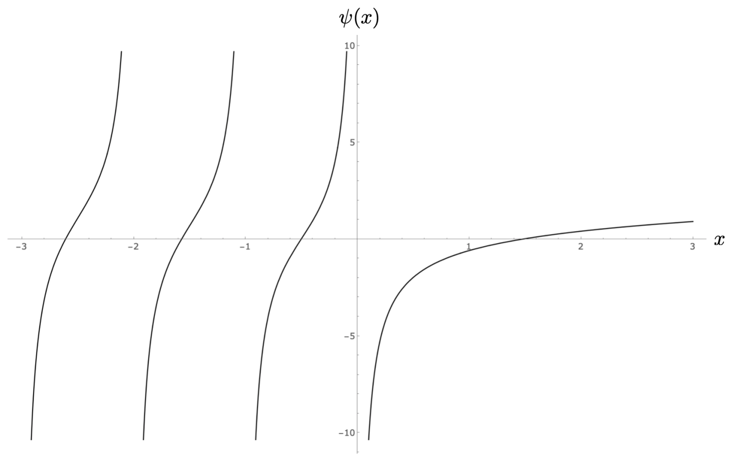

FIGURE 18.1. The graph of the digamma function ψ.

Having fixed N, G and Λ_b; the function f_N possesses poles arising from non-positive arguments of the digamma functions with which the "Ψ-functions" are constructed.[6] These poles are precisely determined by

$$N + \frac{5 - \sqrt{8L^2\Lambda_\mathrm{b} + q_i}}{2} = -k \quad \text{and} \quad \frac{7 - \sqrt{8L^2\Lambda_\mathrm{b} + q_i}}{2} = -k\,, \tag{18.66}$$

with $k \in \mathbb{N}_0$. Clearly, these equations can be solved for non-negative values of L:

$$L^N_{k,i} = \sqrt{\frac{[2(N-k)-5]^2 - q_i}{8\Lambda_\mathrm{b}}} \quad \text{and} \quad L_{k,i} = \sqrt{\frac{[2k-7]^2 - q_i}{8\Lambda_\mathrm{b}}}\,. \tag{18.67}$$

Again, f_N diverges towards $+\infty$ when L approaches the pole at $L^N_{k,i}$, respectively $L_{k,i}$, from the left and towards $-\infty$ when it is approached from the right. Hence, f_N necessarily possesses a zero point for some $L \in [L^N_{k,i}, L^N_{k+1,i}]$ or $L \in [L_{k,i}, L_{k+1,i}]$, respectively (cf. Figure 18.2 for an illustration). Of these

[6]For this discussion, we may ignore the pole located at $L^2\Lambda_\mathrm{b} = 2$ as it arose from seperating the $(n = 1)$-term from the sum earlier on the RHS and therefore could be technically absorbed into the, then modified, "Ψ-functions".

zero points, we are especially interested in the N-dependent ones that amount to N-dependent self-consistent radii of the S^4-type background. In a rough approximation these are given by

$$L^{\mathrm{SC}}_{k,i}(N) \approx \frac{L^N_{k,i} - L^N_{k+1,i}}{2}\,. \tag{18.68}$$

Importantly, for the pair (k, i) fixed to arbitrary values, the sequence of self-consistent radii constructed in this way increases monotonically as N is increased: $L^{\mathrm{SC}}(N) \to \infty$ as $N \to \infty$. Thus, *for non-vanishing and finite bare cosmological constant* Λ_{b}*, there exists, for each pair* (k, i)*, a family of self-consistent radii* $L^{SC}_{k,i}(N)$ *that increases monotonically with* N*. The more modes of the quantum fluctuation* $h_{\mu\nu}$ *are quantized, the larger the self-consistent radius of the* S^4*-type background becomes, until it ultimately, for the fully quantized system* $N \to \infty$*, becomes flat:* $S^4(\infty) \cong \mathbb{R}^4$. This means that among the self-consistent background S^4-geometries arising from Background-Independently quantizing metric fluctuations are S^4-geometries which are *free of the "comological constant problem"*.

Thirdly, we stress that the other families of self-consistent radii can exist. In a logical order, one first specifies the dimensionless cutoff $N = N'$ to only then solve Eq. (18.50) for $L^{\mathrm{SC}}(N')$. If there are several solutions, i.e., several self-consistent radii, at $N = N'$ all of these will be radii of possible realizations of self-consistent S^4-type backgrounds – i.e., nature is free to "choose" among these at each fixed $N = N'$. For instance, increasing k proportionally to N, $k \mapsto k + 1$ as $N \mapsto N + 1$, in the self-consistet radius approximated by Eq. (18.68) results in a *static* self-consistent radius that does not change as N is increased. Further, by overproportionally increasing the parameter k as N increases, one may also construct a family of shrinking self-consistent radii that potentially result in a curvature singularity for $N \to \infty$.

Fourthly, the function f_N might possess further zeros, other than the ones constructed above. These further zeros are due to the fourth-order polynomial structure of f_N in L and occur for certain values of the dimensionless cutoff N and the parameters G and Λ_{b} (cf. Figure 18.3). Especially, note that for N fixed $L^{\mathrm{SC}}_{k,i}(N)$ as approximated by Eq. (18.68) is bound from above by $k = 0$: $L^{\mathrm{SC}}_{k+1,i}(N) \leq L^{\mathrm{SC}}_{k,i}(N)$. However, still a larger self-consitent radius than $L^{\mathrm{SC}}_{0,i}(N)$ might exists due to the fourth-order polynomial structure of f_N that dominates the graph for large values of L.

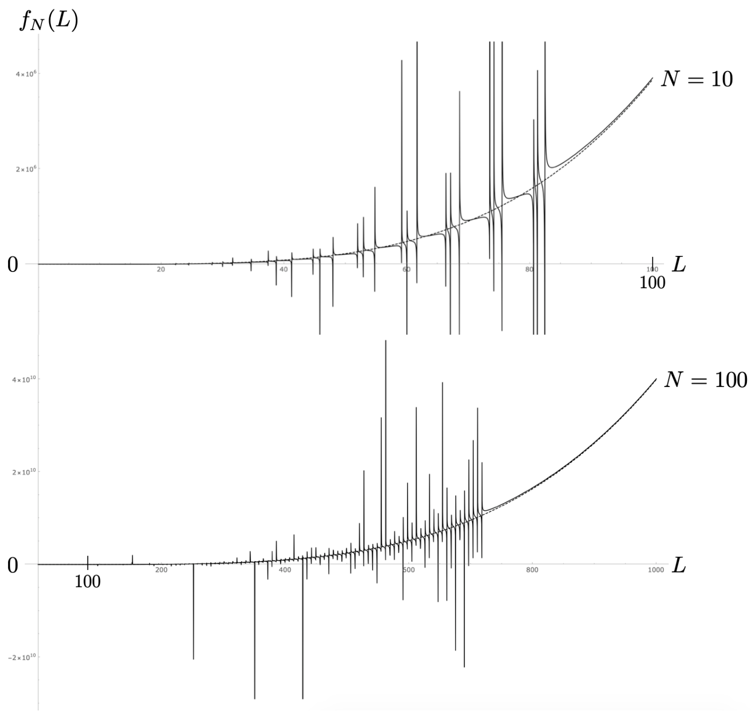

FIGURE 18.2. The graph of the function $f_N(L)$ resulting from the "type 2"-calculation for $N = 10$ and $N = 100$. The parameters G and Λ_b are set to $G = 1$ and $\Lambda_b = 10^{-2}$. The dashed graph for comparison is that of the polynomial $L \mapsto 4\Lambda_b L^4 - 12L^2 - \frac{3G}{\pi}[f_{grav}(N) - f_{ghosts}(N)]$. It is clear to see that this polynomial dictates the trend of the graph from which the poles, resulting from the digamma-functions that f_N entails, emerge. (Due to numeric limitations, the graph at these poles sometimes ends at finite values.) Also clearly illustrated is how the location of the most-right pole $L^N_{k=0,i}$ shifts to the right as N is increased.

Fifthly, we note that *this analysis it applicable to the first type of approximants as well as to the second type of approximants.* However, as the propagating degrees of freedom of the quantum system are known to be given by $f_{grav} - f_{ghosts}$, the second type of approximants appear to be a more natural choice of approximants.

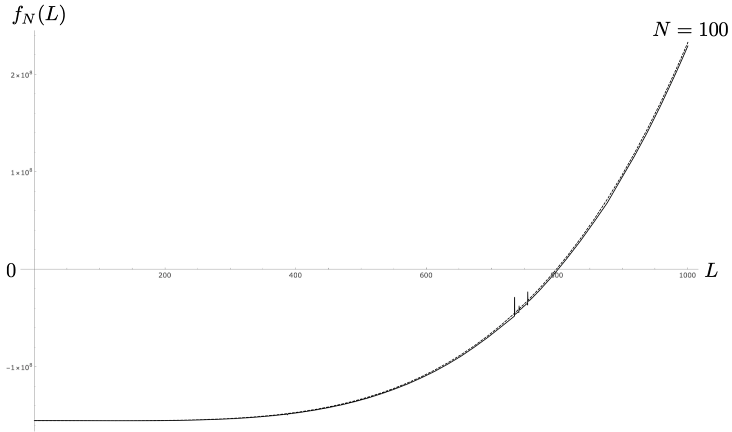

FIGURE 18.3. The graph of the function $f_N(L)$ resulting from the "type 1"-calculation for $N = 100$. The parameters G and $\Lambda_{\rm b}$ are set to $G = 1$ and $\Lambda_{\rm b} = 10^{-1}$. The dashed graph for comparison is that of the polynomial $L \mapsto 4\Lambda_{\rm b}L^4 - 12L^2 - \frac{3G}{\pi}\left[\mathrm{f}_{\rm grav}(N) + \mathrm{f}_{\rm ghosts}(N)\right]$. Clearly visible is the last zero of f_N, that is due to this polynomial's contribution, located right of the last pole of the graph (which due to numerical limitation is depicted only as a small peak).

(C) The case $\Lambda_{\rm b} \in (-\infty, 0)$. In case of non-vanishing negative bare cosmological constant, the self-consistent radii which are due to the "Ψ-functions" are also given by Eq. (18.67), i.e.,

$$L^N_{k,i} = \sqrt{-\frac{[2(N-k)-5]^2 - q_i}{8|\Lambda_{\rm b}|}} \quad \text{and} \quad L_{k,i} = \sqrt{-\frac{[2k-7]^2 - q_i}{8|\Lambda_{\rm b}|}}\,. \tag{18.69}$$

These radii are real if and only if

$$q_i > [2(N-k)-5]^2 \quad \text{viz.} \quad q_i > [2k-7]^2\,. \tag{18.70}$$

Since $q_i \in \{-15, 1, 9\}$ and $N, k \in \mathbb{N}_0$, it is straightforward to see that the argument of the square roots is always negative and hence *there exist no self-consistent radii which are due to the "Ψ-functions", i.e. in other words, which are due to logarithmic divergences.* However, there still may be self-consistent radii arising arising as zeros of $f_N(L)$ which are due to its polynomial structure, i.e., quartic and quadratic divergences. It is a rather difficult task, if not impossible, to proof the existence of these self-consistent radii analytically (without approximations); furthermore they depend parametrically on G and $\Lambda_{\rm b}$. Here, where we only take the very first steps of the program of N-cutoff approximants, we will leave this proof an open task and restrict our analysis to showing examples sequences of self-consistent radii $L^{\text{SC}}(N)$ that grow with N. Figure 18.4 clearly illustrates such a sequence arising from the "type 2"-calculation with a negative bare cosmological constant. For the same values of G and Λ_{b}, Figure 18.5 shows the analogous graph for the calculation of "type 1"; strikingly there exist no self-consistent radii for the calculation of "type 1". This further emphasizes that the calculation of "type 2" is preferable to that of "type 1".

Hence, also in the case of a negative bare cosmological constant, the cosmological constant problem can be avoided and there exist self-consistent background S^4 geometries that in the limit $N \to \infty$ become *perfectly flat*.

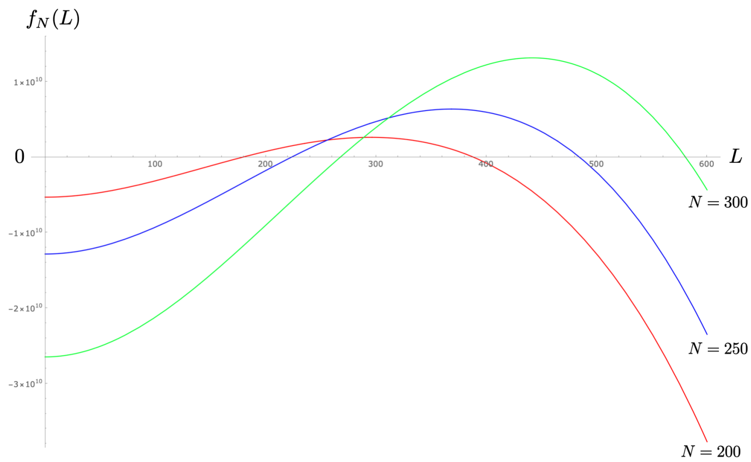

FIGURE 18.4. The graph of the function $f_N(L)$ resulting from the "type 2"-calculation for the parameters $G = 20$ and $\Lambda_{\text{b}} = -1/10$. It is clearly illustrated that the zeros, i.e., self-consistent radii $L^{\text{SC}}(N)$, grow as N becomes larger.

(D) The case $\Lambda_{\text{b}} \to \pm\infty$. For the case of a diverging bare cosmological constant, $\Lambda_{\text{b}} \to \pm\infty$, the equation of motion for L in leading order in N becomes

$$\frac{12}{L^2} = \begin{cases} \pm 4 \cdot \infty - \frac{6G}{\pi} \frac{1}{L^4} \sum_{n=1}^{N} \left(D_n^T + D_n^L \right) & \text{for "type 1"} \\ \pm 4 \cdot \infty + \frac{6G}{\pi} \frac{1}{L^4} \sum_{n=1}^{N} \left(D_n^T + D_n^L \right) & \text{for "type 2"} \end{cases} \tag{18.71}$$

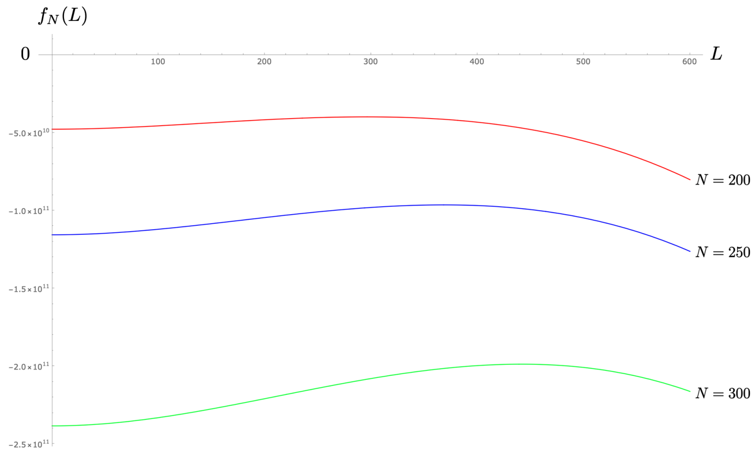

FIGURE 18.5. The graph of the function $f_N(L)$ resulting from the "type 1"-calculation for the parameters $G = 20$ and $\Lambda_{\rm b} = -1/10$. There exist no zero points hence no self-consistent radii in this case.

Hence *for all N, the self-cosistent radius for both calculations, "type 1" and "type 2", becomes singular* in this case, $L^{\rm SC}(N) = 0$, and the curvature diverges, $R^{\rm SC}(N) = \infty$.

CHAPTER 19

Weyl transformations and their Ward identities

Executive summary. For a given set of fields we derive the Weyl transformations' anomalous Ward identities which result from quantizing these fields on a generic Euclidean background manifold. On this basis, we explicitly demonstrate how the anomaly contributes to the difference between first and second kind of approximants, proposed in the previous chapters. By the examples of a scalar field and metric fluctuations, we highlight the general results.
What is new? The relationship between the Weyl transformations' anomaly and the first and second type of approximants.
Based upon: Reference [6].

In the previous sections, we had rooted the difference between the first and second type of approximants (for the scalar as well as the graviton field) in the contribution from the path integral measure. Here, in this short intermezzo on Weyl transformations, we will explicitly show how the anomaly resulting from the non-Weyl-invariance of the measure contributes to this difference.

19.1. Weyl transformations and their anomalous Ward identities

(A) Weyl transformations. On a generic Euclidean background manifold (M, g) consider some action functional $S[\{\phi_j\}; g]$ of a set of fields $\{\phi_j\}$ in which each field ϕ_j is potentially equipped with an arbitrary index structure that we henceforth will not explicitly denote. Associated to the action $S[\{\phi_j\}; g]$ is the stress-energy tensor defined by Eq. (15.19), i.e.,

$$T^{\mu\nu}[\{\phi_j\}; g](x) := -\frac{2}{\sqrt{g(x)}}\frac{\delta S[\{\phi_j\}; g]}{\delta g_{\mu\nu}(x)}\,. \tag{19.1}$$

To each field ϕ_j we assign a *Weyl weight* w_j, that we may choose freely, often it is defined as the canonical mass dimension $w_j = [\phi_j]$ (cf. appendix A for the conventions used here). We define the *Weyl transformation* of the metric field $g_{\mu\nu}$ and each field ϕ_j as

$$g'_{\mu\nu}(x) := \mathrm{e}^{-2\alpha(x)} g_{\mu\nu}(x) \tag{19.2}$$

$$\phi'_j(x) := \mathrm{e}^{w_j \alpha(x)} \phi_j(x) \,. \tag{19.3}$$

Next, we will Taylor-expand the Weyl-transformed action $S[\{\phi_j\}; g]$ in $\alpha(x)$ around $\alpha = 0$:

$$S\left[\{\mathrm{e}^{w_j\alpha}\phi_j\}\,; e^{-2\alpha} g\right] = S\left[\{\phi_j\}\,; g\right] + \int \mathrm{d}^d x \sqrt{g(x)}\, \alpha(x)\, \mathscr{N}\left[\{\phi_j\}\,; g\right](x) + O(\alpha^2) \tag{19.4}$$

where the field $\mathscr{N}[\{\phi_j\}; g]$ has been defined by this expansion. On the other hand, we can first expand the exponential in the transformations (19.2) and (19.3) in α,

$$\begin{aligned} S\left[\{\mathrm{e}^{w_j\alpha}\phi_j\}\,; e^{-2\alpha} g_{\mu\nu}\right] &= S\left[\{(1 + w_j\alpha)\phi_j\}\,; (1 - 2\alpha) g_{\mu\nu}\right] + O(\alpha^2) \\ &= S\left[\{\phi_j + w_j\alpha\phi_j\}\,; g_{\mu\nu} - 2\alpha g_{\mu\nu}\right] + O(\alpha^2)\,, \end{aligned} \tag{19.5}$$

and only then Taylor-expand the action in $w_j\alpha\phi_j$ and $-2\alpha g_{\mu\nu}$ around $\alpha = 0$:

$$\begin{aligned} &S\left[\{\mathrm{e}^{w_j\alpha}\phi_j\}\,; e^{-2\alpha} g_{\mu\nu}\right] \\ &\quad= S\left[\{\phi_j\}\,; g\right] + \sum_j \int \mathrm{d}^d x\, w_j\, \alpha(x)\phi_j(x) \left.\frac{\delta S[\{\phi_j + w_j\alpha\phi_j\}; g_{\mu\nu} - 2\alpha g_{\mu\nu}]}{\delta(w_j\alpha\phi_j)(x)}\right|_{\alpha=0} \\ &\qquad + \int \mathrm{d}^d x\, \left(-2\alpha(x) g_{\mu\nu}(x)\right) \left.\frac{\delta S[\{\phi_j + w_j\alpha\phi_j\}; g_{\mu\nu} - 2\alpha g_{\mu\nu}]}{\delta(-2\alpha g_{\mu\nu})(x)}\right|_{\alpha=0} \\ &\qquad + O(\alpha^2) \\ &\quad= S\left[\{\phi_j\}\,; g\right] + \sum_j \int \mathrm{d}^d x\, w_j\, \alpha(x)\phi_j(x) \frac{\delta S[\{\phi_j\}; g_{\mu\nu}]}{\delta\phi_j(x)} \\ &\qquad -2 \int \mathrm{d}^d x\, \alpha(x) g_{\mu\nu}(x) \frac{\delta S[\{\phi_j\}; g_{\mu\nu}]}{\delta g_{\mu\nu}(x)} + O(\alpha^2) \\ &\quad= S\left[\{\phi_j\}\,; g\right] + \int \mathrm{d}^d x \sqrt{g(x)}\, \alpha(x) \Bigg[\sum_j \frac{w_j}{\sqrt{g(x)}}\, \phi_j(x) \frac{\delta S[\{\phi_j\}; g_{\mu\nu}]}{\delta\phi_j(x)} \\ &\qquad + g_{\mu\nu}(x) T^{\mu\nu}[\{\phi_j\}; g](x) \Bigg] + O(\alpha^2)\,. \end{aligned} \tag{19.6}$$

By comparison with the above definition of the field $\mathscr{N}[\{\phi_j\};g]$ we can read off the identity

$$\mathscr{N}\left[\{\phi_j\};g\right](x) = T_\mu{}^\mu[\{\phi_j\};g](x) + \sum_j \frac{w_j}{\sqrt{g(x)}}\,\phi_j(x)\frac{\delta S[\{\phi_j\};g_{\mu\nu}]}{\delta\phi_j(x)}\,. \qquad (19.7)$$

Furthermore let us define the functional

$$N[\{\phi_j\};g] \coloneqq \int \mathrm{d}^d x\sqrt{g(x)}\,\mathscr{N}\left[\{\phi_j\};g\right](x)\,, \qquad (19.8)$$

for which to define in general, it is sufficient to consider the restriction to $\alpha(x) \equiv$ const. $= \alpha$:

$$S\left[\{\mathrm{e}^{w_j\alpha}\phi_j\};e^{-2\alpha}g\right] = S\left[\{\phi_j\};g\right] + \alpha N\left[\{\phi_j\};g\right] + O(\alpha^2)\,. \qquad (19.9)$$

On the other hand, said functional is determined by

$$\begin{aligned} N\left[\{\phi_j\};g\right] = &\int \mathrm{d}^d x\sqrt{g(x)}\,T_\mu{}^\mu[\{\phi_j\};g](x) \\ &+ \sum_j w_j \int \mathrm{d}^d x\,\phi_j(x)\frac{\delta S[\{\phi_j\};g_{\mu\nu}]}{\delta\phi_j(x)}\,. \end{aligned} \qquad (19.10)$$

With help of the operator $\mathscr{T} \coloneqq -2\int \mathrm{d}^d x\sqrt{g(x)}\,g_{\mu\nu}(x)\delta/\delta g_{\mu\nu}(x)$, the integrated and traced stress-energy tensor can be expressed as

$$\int \mathrm{d}^d x\sqrt{g(x)}\,T_\mu{}^\mu[\{\phi_j\};g](x) = \mathscr{T}S\left[\{\phi_j\};g\right]\,, \qquad (19.11)$$

which leads the way to an alternative derivation of Eq. (19.10) employing lemma (17.13), cf. appendix F.10.

(B) Ward identities. Next, we define the *effective action* $\Gamma[g]$ associated to the action $S[\{\phi_j\};g]$ by[1]

$$\mathrm{e}^{-\Gamma[g_{\bullet\bullet}]} \coloneqq \int \prod_i \mathcal{D}\left(\phi_i;g_{\bullet\bullet}\right)\,\mathrm{e}^{-S[\{\phi_j\};g_{\bullet\bullet}]} \qquad (19.12)$$

where $\mathcal{D}(\phi_j;g_{\bullet\bullet}) \equiv \mathcal{D}_g\phi_j$ is the g dependent path integral measure of the field ϕ_j, cf. appendix D.1. The Weyl transformation of this measure is given by its Jacobian $J_j[\alpha;g]$ that we parametrize as follows:

$$\mathcal{D}\left(\phi'_j;g'_{\bullet\bullet}\right) = J_j[\alpha;g]\mathcal{D}\left(\phi_j;g_{\bullet\bullet}\right) =: \mathrm{e}^{-\int \mathrm{d}^d x\sqrt{g(x)}\,\alpha(x)\mathscr{A}_j(x)}\mathcal{D}\left(\phi_j;g_{\bullet\bullet}\right)\,. \qquad (19.13)$$

[1]In this paragraph, we indicate the placement of indices (where these need not be placed) by bullets because this placement is essential to perform Weyl transformations correctly.

Therewith, applying the Weyl transformations (19.2) and (19.3) to the EA, we find

$$\begin{aligned}\exp\left\{-\Gamma[e^{-2\alpha(\cdot)}g_{\cdot\cdot}]\right\} &= \int\prod_i \mathcal{D}\left(\phi_i';\mathrm{e}^{-2\alpha(\cdot)}g_{\cdot\cdot}\right)\,\exp\left\{-S[\{\phi_j'\};e^{-2\alpha(\cdot)}g_{\cdot\cdot}]\right\}\\ &= \int\prod_i \mathrm{e}^{-\int\mathrm{d}^dx\sqrt{g(x)}\,\alpha(x)\mathscr{A}_i(x)}\mathcal{D}\left(\phi_j;g_{\cdot\cdot}\right)\,\exp\Bigg\{-S[\{\phi_j\};g]\\ &\quad -\int\mathrm{d}^dx\sqrt{g(x)}\alpha(x)\mathscr{N}[\{\phi_j\};g](x)+O(\alpha^2)\Bigg\}\\ &= \int\prod_i \mathcal{D}\left(\phi_j;g_{\cdot\cdot}\right)\mathrm{e}^{-S[\{\phi_j\};g]}\,\exp\Bigg\{\\ &\quad -\int\mathrm{d}^dx\sqrt{g(x)}\alpha(x)\Big[\mathscr{A}_i(x)+\mathscr{N}[\{\phi_j\};g](x)+O(\alpha^2)\Big]\Bigg\}\,.\end{aligned} \tag{19.14}$$

Furthermore expanding the exponential in the last step gives

$$\begin{aligned}&\exp\left\{-\left(\Gamma[e^{-2\alpha(\cdot)}g_{\cdot\cdot}]-\Gamma[g_{\cdot\cdot}]\right)\right\}\\ &\quad=\frac{1}{\mathrm{e}^{-\Gamma[g_{\cdot\cdot}]}}\int\prod_i \mathcal{D}\left(\phi_j;g_{\cdot\cdot}\right)\mathrm{e}^{-S[\{\phi_j\};g_{\cdot\cdot}]}\Bigg\{\\ &\quad\quad 1-\sum_i\int\mathrm{d}^dx\sqrt{g(x)}\alpha(x)\Big[\mathscr{A}_i(x)+\mathscr{N}[\{\phi_j\};g](x)\Big]+O(\alpha^2)\Bigg\}\\ &\quad=: 1-\sum_i\int\mathrm{d}^dx\sqrt{g(x)}\alpha(x)\Big[\mathscr{A}_i(x)+\Big\langle\mathscr{N}[\{\hat{\phi}_j\};g](x)\Big\rangle\Big]+O(\alpha^2)\,,\end{aligned} \tag{19.15}$$

where we have have quantized each field $\phi\mapsto\hat{\phi}_i$ and the expectation value has been defined by the last step, i.e., it is calculated with respect to $\int\prod_i\mathcal{D}\left(\phi_i;g\right)\,\mathrm{e}^{-S[\{\phi_j\};g]}$. On the other hand, an ordinary Taylor expansion of the EA gives

$$\Gamma[e^{-2\alpha(\cdot)}g_{\cdot\cdot}]-\Gamma[g_{\cdot\cdot}]=-2\int\mathrm{d}^dx\,\alpha(x)g_{\mu\nu}(x)\frac{\delta\Gamma[g]}{\delta g_{\mu\nu}(x)}+O(\alpha^2)\,. \tag{19.16}$$

Let us define the *effective stress-energy tensor* $T^{\mu\nu}_{\text{eff}}[g]$ by

$$\frac{\delta\Gamma[g]}{\delta g_{\mu\nu}(x)}=:-\frac{1}{2}\sqrt{g(x)}\,T^{\mu\nu}_{\text{eff}}[g](x)\,. \tag{19.17}$$

Therewith, the ordinary Taylor expansion amounts to

$$\exp\left\{ -\left(\Gamma[e^{-2\alpha(\cdot)}g_{\cdot\cdot}] - \Gamma[g_{\cdot\cdot}]\right) \right\} = 1 - \int \mathrm{d}^d x \sqrt{g(x)}\, \alpha(x) {T_{\mathrm{eff}}}^{\mu}_{\mu}[g](x) + O(\alpha^2)\,. \tag{19.18}$$

Combining both these expansions, we can read off the following identity from term of order α:

$$\int \mathrm{d}^d x \sqrt{g(x)}\, \alpha(x) {T_{\mathrm{eff}}}^{\mu}_{\mu}[g](x) = \sum_i \int \mathrm{d}^d x \sqrt{g(x)} \alpha(x) \left[\mathscr{A}_i(x) + \left\langle \mathscr{N}[\{\hat{\phi}_j\}; g](x) \right\rangle \right]. \tag{19.19}$$

The function α, however, is fully arbitrary. Thus, this identity is equivalent to

$${T_{\mathrm{eff}}}^{\mu}_{\mu}[g](x) = \sum_i \mathscr{A}_i(x) + \left\langle \mathscr{N}[\{\hat{\phi}_j\}; g](x) \right\rangle\,. \tag{19.20}$$

This is the *anomalous Weyl-Ward identity* and we call the first term on the RHS the *anomaly*. Its integrated version reads

$$\begin{aligned} \mathscr{T}\Gamma[g] &=: \int \mathrm{d}^d x \sqrt{g(x)}\, {T_{\mathrm{eff}}}^{\mu}_{\mu}[g](x) \\ &= \sum_i \int \mathrm{d}^d x \sqrt{g(x)}\, \mathscr{A}_i(x) + \left\langle N[\{\hat{\phi}_j\}; g](x) \right\rangle\,. \end{aligned} \tag{19.21}$$

(Note that this equation still requires regularization!)

(C) ${T_{\mathrm{eff}}}^{\mu}_{\mu}$ **vs.** $\langle \widehat{\text{classical}\, T^{\mu}_{\mu}} \rangle$**.** Let us promote Eq. (19.7) to an operatorial relation whose expectation value we subsequently take:

$$\begin{aligned} \left\langle {T_\mu}^\mu[\{\hat{\phi}_j\}; g](x) \right\rangle &= \left\langle \mathscr{N}[\{\hat{\phi}_j\}; g](x) \right\rangle \\ &\quad - \sum_j \frac{w_j}{\sqrt{g(x)}} \left\langle \hat{\phi}_j(x) \frac{\delta S[\{\phi_j\}; g_{\mu\nu}]}{\delta \phi_j(x)} \bigg|_{\phi_j = \hat{\phi}_j} \right\rangle\,. \end{aligned} \tag{19.22}$$

From this operatorial relation and Eq. (19.20) we can deduce the difference

$$\begin{aligned} &{T_{\mathrm{eff}}}^{\mu}_{\mu}[g](x) - \left\langle {T_\mu}^\mu[\{\hat{\phi}_j\}; g](x) \right\rangle \\ &\quad = \sum_j \left[\mathscr{A}_j(x) + \frac{w_j}{\sqrt{g(x)}} \left\langle \hat{\phi}_j(x) \frac{\delta S[\{\phi_j\}; g_{\mu\nu}]}{\delta \phi_j(x)} \bigg|_{\phi_j = \hat{\phi}_j} \right\rangle \right] \end{aligned} \tag{19.23}$$

whose integrated version amounts to

$$\mathscr{T}\Gamma[g] - \left\langle \mathscr{T} S[\{\hat{\phi}_j\}; g] \right\rangle = \sum_j \Bigg[\int \mathrm{d}^d x \sqrt{g(x)}\, \mathscr{A}_j(x) + w_j \int \mathrm{d}^d x \left\langle \hat{\phi}_j(x) \frac{\delta S[\{\phi_j\}; g_{\mu\nu}]}{\delta \phi_j(x)} \bigg|_{\phi_j = \hat{\phi}_j} \right\rangle \Bigg] . \tag{19.24}$$

(Again, all terms appearing in this paragraph still require regularization.)

19.2. Example: a real scalar field

As an example, consider a single real scalar field A on a generic Euclidean background manifold (M, g), i.e., $\phi_j = A$, whose action is given by Eq. (3.3),

$$S_{\mathrm{M}}[A; g] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g}\, A \left[-\Box_g + \mu^2 + \xi R(g) \right] A \,. \tag{19.25}$$

Here, let us set the Weyl weight of the field A to its canonical mass dimension, i.e., $w_j \equiv w = [A]$. We can deduce the canonical mass dimension of the scalar field A from the fact that the action has canonical mass dimension zero. Using the convention of dimensionless coordinates (cf. appendix A), we find

$$0 \stackrel{!}{=} \Big[\underbrace{\textstyle\int \mathrm{d}^d x}_{0} \, \underbrace{\sqrt{g}}_{-d} \, \underbrace{g^{\mu\nu}}_{+2} \, \underbrace{\partial_\mu}_{0} A \, \underbrace{\partial_\nu}_{0} A \Big] \quad \Rightarrow \quad [A] = \frac{d-2}{2} \,. \tag{19.26}$$

Consequently, the scalar field $\phi_j = A$ possesses Weyl weight $w_j \equiv w = (d-2)/2$. Therewith we can deduce the functional $N[A; g]$ by means of Eq. (19.9) (with $\alpha \equiv \text{const.}$):

$$\begin{aligned} S_{\mathrm{M}} \left[\mathrm{e}^{\frac{d-2}{2}\alpha} A; \mathrm{e}^{-2\alpha} g \right] &= \frac{1}{2} \int \mathrm{d}^d x \, \sqrt{\det(\mathrm{e}^{-2\alpha} g_{\bullet\bullet})}\, \mathrm{e}^{\frac{d-2}{2}\alpha} A \Bigg[- \mathrm{e}^{2\alpha} g^{\mu\nu} D_\mu D_\nu \\ &\qquad + \mu^2 + \xi R(\mathrm{e}^{-2\alpha} g) \Bigg] \mathrm{e}^{\frac{d-2}{2}\alpha} A \\ &= \int \mathrm{d}^d x \sqrt{g(x)}\, A \left[(-\Box_g + \xi R(g)) + \mathrm{e}^{-2\alpha} \mu^2 \right] A \,. \end{aligned} \tag{19.27}$$

where we have used that $\sqrt{\det(\mathrm{e}^{-2\alpha} g_{\bullet\bullet})} = \mathrm{e}^{-d\alpha}\sqrt{g}$, $R(\mathrm{e}^{-2\alpha} g) = \mathrm{e}^{2\alpha} R(g)$ and that fact that the Christoffel symbols are not affected by Weyl transformations. Hence we have

$$\begin{aligned} S_{\mathrm{M}}\left[\mathrm{e}^{\frac{d-2}{2}\alpha} A; \mathrm{e}^{-2\alpha} g\right] - S_{\mathrm{M}}[A; g] &= \frac{1}{2}\int \mathrm{d}^d x \sqrt{g(x)}\, A(\mathrm{e}^{-2\alpha} - 1)\mu^2 A \\ &= -\alpha\mu^2 \int \mathrm{d}^d x \sqrt{g} A^2 + O(\alpha^2) \\ &\stackrel{!}{=} \alpha N[A; g] + O(\alpha^2) \end{aligned} \tag{19.28}$$

from which immediately follows that

$$N[A; g] = -\mu^2 \int \mathrm{d}^d x \sqrt{g(x)}\, A(x)^2 \,. \tag{19.29}$$

Next, let us deduce the anomaly $\mathscr{A}(x)$ from Weyl transforming the measure defined by Eq. (D.6),

$$\mathcal{D}(A; g) \coloneqq \prod_x \det(g_{\bullet\bullet})^{1/4}\, \mathrm{d}A(x) \,. \tag{19.30}$$

Thereby let us assume that the measure is regularized by discretizing spacetime, i.e., restricting it to finitely many points. This leads to

$$\begin{aligned} \mathcal{D}(A'; g') &= \prod_x \det\left(\mathrm{e}^{-2\alpha(x)} g_{\bullet\bullet}\right)^{1/4} \mathrm{d}\left(\mathrm{e}^{\frac{d-2}{2}\alpha(x)} A(x)\right) \\ &= \prod_x \mathrm{e}^{-\frac{2d\alpha(x)}{4}} (\det g_{\bullet\bullet})^{1/4}\, \mathrm{e}^{\frac{d-2}{2}\alpha(x)} \mathrm{d}A(x) \\ &= \prod_x \mathrm{e}^{-\alpha(x)} (\det g_{\bullet\bullet})^{1/4}\, \mathrm{d}A(x) \\ &= \mathrm{e}^{-\sum_x \alpha(x)}\, \mathcal{D}(A; g) \\ &\stackrel{!}{=} \mathrm{e}^{-\int \mathrm{d}^d x \sqrt{g(x)}\, \alpha(x) \mathscr{A}(x)}\, \mathcal{D}(A; g) \,. \end{aligned} \tag{19.31}$$

Here, we must clarify the sum in the forelast step. It is a sum over the finitely many points of spacetime. As shown in Section 17.2, we can identify this discretization-based regulator with a cutoff of the N-type. This is why the sum can be rewritten as $\sum_x \alpha(x) = \mathrm{Tr}[\alpha(\hat{x})]_N$ and hence also reads

$$\begin{aligned}\sum_x \alpha(x) &= \int\limits_{\text{“lattice”}} \mathrm{d}^d x \sqrt{g(x)}\, \langle x|\alpha(\hat{x})|x\rangle \\ &= \int\limits_{\text{“lattice”}} \mathrm{d}^d x \sqrt{g(x)}\, \alpha(x)\mathscr{A}(x)\,,\end{aligned} \tag{19.32}$$

where the integration domain “lattice” means the finitely many spacetime points. By setting $\alpha(x) \equiv 1$, we can determine the integrated field $\mathscr{A}$:

$$\begin{aligned}\int\limits_{\text{“lattice”}} \mathrm{d}^d x \sqrt{g(x)}\, \mathscr{A}(x) &= \int\limits_{\text{“lattice”}} \mathrm{d}^d x \sqrt{g(x)}\, \langle x|\mathbb{1}_S|x\rangle \\ &= \mathrm{Tr}_S[\mathbb{1}_S]_N\,.\end{aligned} \tag{19.33}$$

In a further step, we compare the general defintion of the EA, Eq. (19.12), with the EA (3.40) we had defined for the scalar field A in Chapter 3 at order one loop. It is clear that we must replace the general EA $\Gamma[g]$ of the previous section with the 1L-EA (3.39), i.e., $\Gamma_{1\mathrm{L}}[g] = \frac{1}{2}\,\mathrm{Tr}\ln\left(-\Box_g + \mu^2 + \xi R\right) \equiv \frac{1}{2}\,\mathrm{Tr}\ln\mathscr{K}[g]$. Accordingly, when employing an N-cutoff, we have

$$\begin{aligned}\mathscr{T}\Gamma_{1\mathrm{L}}[g]_N &= \left\langle N[\hat{A};g]\right\rangle_N + \int \mathrm{d}^d x \sqrt{g(x)}\, \mathscr{A}_N(x) \\ &= -\,\mu^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \left\langle \hat{A}^2(x)\right\rangle_N + \mathrm{Tr}_S[\mathbb{1}_S]_N\,.\end{aligned} \tag{19.34}$$

Expanding the scalar field A in the truncated basis of eigenfunctions $\{\chi_{n,m}\}_{n=1,\ldots,N}$ of $\mathscr{K}[g] = -\Box_g + \mu^2 + \xi R$, this equation becomes, using Eq. (16.29),

$$\begin{aligned}\mathscr{T}\Gamma_{1\mathrm{L}}[g]_N &= -\,\mu^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \sum_{n=1}^{N}\sum_{m=1}^{D_n} \frac{\chi_{n,m}(x)\chi^*_{n,m}(x)}{\mathscr{F}_n} + \sum_{n=1}^{N} D_n \\ &= \sum_{n=1}^{N} D_n \left[1 - \frac{\mu^2}{\mathscr{F}_n}\right] \\ &= \mathrm{Tr}_S\left[\frac{\mathscr{K}[g] - \mu^2}{\mathscr{K}[g]}\right]_N\end{aligned} \tag{19.35}$$

Furthermore, Eq. (19.24) becomes when employing an N-cutoff, using Eq. (16.29),

$$\begin{aligned}\Big\langle \mathscr{T} S_{\mathrm{M}}[\hat{A};g]\Big\rangle_N &= \Big\langle N[\hat{A};g]\Big\rangle_N - \frac{d-2}{2}\int \mathrm{d}^d x \left\langle \hat{A}(x)\frac{\delta S_{\mathrm{M}}[A;g]}{\delta A(x)}\bigg|_{A=\hat{A}}\right\rangle_N \\ &= -\mu^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \Big\langle \hat{A}^2(x)\Big\rangle_N \\ &\quad - \frac{d-2}{2}\int \mathrm{d}^d x \sqrt{g(x)}\, \Big\langle \hat{A}(x)(\mathscr{K}[g]\hat{A})(x)\Big\rangle_N \\ &= -\mu^2 \int \mathrm{d}^d x \sqrt{g(x)} \sum_{n=1}^{N}\sum_{m=1}^{D_n} \frac{\chi_{n,m}(x)\chi^*_{n,m}(x)}{\mathscr{F}_n} \\ &\quad - \frac{d-2}{2}\int \mathrm{d}^d x \sqrt{g(x)} \sum_{n=1}^{N}\sum_{m=1}^{D_n} \frac{\chi_{n,m}(x)(\mathscr{K}[g]\chi^*_{n,m})(x)}{\mathscr{F}_n} \\ &= -\frac{d}{2}\mathrm{Tr}_S[\mathbb{1}_S]_N + \mathrm{Tr}_S\left[\frac{\mathscr{K}[g]-\mu^2}{\mathscr{K}[g]}\right]_N .\end{aligned} \tag{19.36}$$

Together with the previous equation leads directly to

$$\mathscr{T}\Gamma_{1\mathrm{L}}[g]_N - \Big\langle \mathscr{T} S_{\mathrm{M}}[\hat{A};g]\Big\rangle_N = \frac{d}{2}\,\mathrm{Tr}_S[\mathbb{1}_S]_N\,, \tag{19.37}$$

which precisely is (a regularized version of) Eq. (17.25).

19.3. Example: graviton and ghost fields

As a further example, consider the graviton field $h_{\mu\nu}$ together with the ghost fields $\bar{\xi}_\mu$ and ξ^μ also on a generic Euclidean background manifold $(M,\bar{g})$. Their classical dynamics is determined by the matter action (4.44),

$$S_{\mathrm{M}}[h,\bar{\xi},\xi;\bar{g}] \coloneqq S_{\mathrm{graviton}}[h;\bar{g}] + S_{\mathrm{gh}}[0,\bar{\xi},\xi;\bar{g}]\,, \tag{19.38}$$

in which we specify both, the graviton action (4.45) and the Faddeev-Popov action (4.34), to the harmonic gauge $\alpha = 1$ and $\beta = 1/2$:

$$\begin{aligned} S_{\text{graviton}}[h_{\bullet\bullet};\bar{g}_{\bullet\bullet}] :=&\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\; h_{\mu\nu}(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}\, I[\bar{g}]^{\rho\sigma\alpha\beta}h_{\alpha\beta} \\ =&\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}(x)}\, h_{\mu\nu}\Bigg\{\frac{1}{2}\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma} - \bar{I}^{\mu\nu}{}_{\rho\sigma}\right](\bar{D}^2 - \bar{R} + 2\Lambda) \\ &+ \bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} - \delta^{\nu}_{\sigma}\bar{R}^{\mu}{}_{\rho} - \bar{R}_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu}\Bigg\}\bar{I}^{\rho\sigma\alpha\beta}h_{\alpha\beta} \end{aligned} \tag{19.39}$$

and

$$\begin{aligned} S_{\text{gh}}[0,\bar{\xi},\xi;\bar{g}] =& -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{\xi}_{\mu}\mathscr{M}[\bar{g},\bar{g}]^{\mu}{}_{\nu}\xi^{\nu} \\ =& -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{\xi}_{\mu}\left[\delta^{\mu}_{\nu}\bar{D}^2 + \bar{R}^{\mu}{}_{\nu}\right]\xi^{\nu}\,. \end{aligned} \tag{19.40}$$

We will exploit the freedom of chosing the Weyl weights of the fields $h_{\mu\nu}$, $\bar{\xi}_{\mu}$ and ξ^{μ} by applying the Weyl transformations as follows:

$$\begin{aligned} \bar{g}'_{\mu\nu}(x) =& \mathrm{e}^{-2\alpha(x)}\bar{g}_{\mu\nu}(x) \\ \phi'_j(x) =& \mathrm{e}^{w_j\alpha(x)}\phi_j(x) \end{aligned} \tag{19.41}$$

with

$$\begin{aligned} \phi_j =& h_{\mu\nu} \quad \Rightarrow \quad w_j = \frac{d-6}{2} \\ \phi_j =& \bar{\xi}_{\mu} \text{ or } \xi^{\mu} \quad \Rightarrow \quad w_j = [\phi_j] = \frac{d-2}{2}\,. \end{aligned} \tag{19.42}$$

The Weyl weights of the ghost fields correspond to their canonical mass dimensions, $[\bar{\xi}_{\mu}] = [\xi^{\mu}] = (d-2)/2$, however this is not the case of for graviton field $h_{\mu\nu}$. (The chosen Weyl weight would be its canonical mass dimension if Newton's constant was dimensionless.) Therewith let us obtain the functional $N[h,\bar{\xi},\xi;\bar{g}]$ using Eq. (19.9). With $\alpha(\,\cdot\,) \equiv \alpha$) we therefore expand

$$\begin{aligned} S_{\text{M}}\left[\mathrm{e}^{\frac{d-6}{2}\alpha}h, \mathrm{e}^{\frac{d-2}{2}\alpha}\bar{\xi}, \mathrm{e}^{\frac{d-2}{2}\alpha}\xi; \mathrm{e}^{-2\alpha}\bar{g}\right] =& S_{\text{graviton}}\left[\mathrm{e}^{\frac{d-6}{2}\alpha}h; \mathrm{e}^{-2\alpha}\bar{g}\right] \\ &+ S_{\text{gh}}\left[0, \mathrm{e}^{\frac{d-2}{2}\alpha}\bar{\xi}, \mathrm{e}^{\frac{d-2}{2}\alpha}\xi; \mathrm{e}^{-2\alpha}\bar{g}\right] \end{aligned} \tag{19.43}$$

in the parameter α. Let us expand both terms independently; the former term yields

$$\begin{aligned}
&S_{\text{graviton}}\left[\mathrm{e}^{\frac{d-6}{2}\alpha}h;\mathrm{e}^{-2\alpha}\bar{g}\right]\\
&=\frac{1}{2}\int \mathrm{d}^d x\,\underbrace{\sqrt{\det\left(\mathrm{e}^{-2\alpha}\bar{g}_{\cdot\cdot}\right)}}_{=\mathrm{e}^{-d\alpha}\sqrt{\bar{g}}}\,\mathrm{e}^{\frac{d-6}{2}\alpha}h_{\mu\nu}\Bigg\{\frac{1}{2}\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}-\bar{I}^{\mu\nu}{}_{\rho\sigma}\right]\\
&\quad\times\left(\mathrm{e}^{2\alpha}\bar{D}^2-\mathrm{e}^{2\alpha}\bar{R}+2\Lambda\right)+\mathrm{e}^{2\alpha}\left(\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma}-\delta^{\nu}_{\sigma}\bar{R}^{\mu}{}_{\rho}-\bar{R}_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu}\right)\Bigg\}\mathrm{e}^{4\alpha}\bar{I}^{\rho\sigma\alpha\beta}\mathrm{e}^{\frac{d-6}{2}\alpha}h_{\alpha\beta}\\
&=\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\left\{h_{\mu\nu}\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}h^{\rho\sigma}+\mathrm{e}^{-2\alpha}\Lambda\,h_{\mu\nu}\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}-\bar{I}^{\mu\nu}{}_{\rho\sigma}\right]h^{\rho\sigma}\right\}\\
&=\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\Bigg\{h_{\mu\nu}\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}h^{\rho\sigma}\\
&\qquad\qquad+(1-2\alpha)2\Lambda\,h_{\mu\nu}\frac{1}{2}\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}-\bar{I}^{\mu\nu}{}_{\rho\sigma}\right]h^{\rho\sigma}\Bigg\}+O(\alpha^2)\\
&=S_{\text{graviton}}[h;\bar{g}]-2\alpha\Lambda\int \mathrm{d}^d x\sqrt{\bar{g}}\,h_{\mu\nu}\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}-\bar{I}^{\mu\nu}{}_{\rho\sigma}\right]h^{\rho\sigma}+O(\alpha^2)\,,
\end{aligned}\tag{19.44}$$

while the latter term in fact can be shown to be invariant under Weyl transformations:

$$\begin{aligned}
S_{\text{gh}}&\left[0,\mathrm{e}^{\frac{d-2}{2}\alpha}\bar{\xi},\mathrm{e}^{\frac{d-2}{2}\alpha}\xi;\mathrm{e}^{-2\alpha}\bar{g}\right]\\
&=-\sqrt{2}\int \mathrm{d}^d x\,\mathrm{e}^{-d\alpha}\sqrt{\bar{g}}\,\mathrm{e}^{\frac{d-2}{2}\alpha}\,\bar{\xi}_{\mu}\mathrm{e}^{2\alpha}\left[\delta^{\mu}_{\nu}\bar{D}^2+\bar{R}^{\mu}{}_{\nu}\right]\mathrm{e}^{\frac{d-2}{2}\alpha}\xi^{\nu}\\
&=-\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{\xi}_{\mu}\left[\delta^{\mu}_{\nu}\bar{D}^2+\bar{R}^{\mu}{}_{\nu}\right]\xi^{\nu}\\
&=S_{\text{gh}}[0,\bar{\xi},\xi;\bar{g}]\,.
\end{aligned}\tag{19.45}$$

Thus we have determined the "non-invariance" that amounts to

$$\begin{aligned}
S_{\text{M}}&\left[\mathrm{e}^{\frac{d-6}{2}\alpha}h,\mathrm{e}^{\frac{d-2}{2}\alpha}\bar{\xi},\mathrm{e}^{\frac{d-2}{2}\alpha}\xi;\mathrm{e}^{-2\alpha}\bar{g}\right]-S_{\text{M}}[h,\bar{\xi},\xi;\bar{g}]\\
&=-2\alpha\Lambda\int \mathrm{d}^d x\sqrt{\bar{g}}\,h_{\mu\nu}\left[\frac{d}{2}(\bar{P}_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma}-\bar{I}^{\mu\nu}{}_{\rho\sigma}\right]h^{\rho\sigma}+O(\alpha^2)\\
&\overset{!}{=}\alpha\,N[h,\bar{\xi},\xi;\bar{g}]+O(\alpha^2)\,.
\end{aligned}\tag{19.46}$$

From this equation we can easily read off that

$$\begin{aligned} N[h,\bar{\xi},\xi;\bar{g}] \equiv& N[h;\bar{g}] \\ =& -2\alpha\Lambda \int \mathrm{d}^d x \sqrt{\bar{g}}\, h_{\mu\nu} \left[\frac{d}{2}(\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - \bar{I}^{\mu\nu}{}_{\rho\sigma}\right] h^{\rho\sigma} . \end{aligned} \tag{19.47}$$

Next, we quantize the fields $(h,\bar{\xi},\xi) \mapsto (\widehat{h},\bar{C},C)$ and determine the anomalies $\mathscr{A}_i$ by Weyl transforming the measure for the graviton $\widehat{h}_{\mu\nu}$ and the ghost fields $\bar{C}_\mu$ and C_μ. Again, let us do so seperately and assume that each measure is regularized by discretizing spacetime. The measure for $\widehat{h}_{\mu\nu}$ is given by Eq. (D.10), i.e., (cf. [118, Eq. (2.19)])

$$\mathcal{D}(\widehat{h};\bar{g}) \coloneqq \mathcal{D}\Big[\bar{g}^{\frac{d-4}{4}}\widehat{h}_{\mu\nu}\Big] = \prod_x \bar{g}(x)^{\frac{(d-4)(d+1)}{8}} \prod_{\mu\geq\nu} \mathrm{d}\widehat{h}_{\mu\nu}(x) \,, \tag{19.48}$$

where we have used that the symmetric tensor $\widehat{h}_{\mu\nu}$ possesses $d(d+1)/2$ degrees of freedom. Performing a Weyl transformation on this measure leads to

$$\begin{aligned} \mathcal{D}(\widehat{h}';\bar{g}') =& \mathcal{D}\left(\mathrm{e}^{\frac{d-6}{2}\alpha}\widehat{h};\mathrm{e}^{-2\alpha}\bar{g}\right) \\ =& \prod_x \left(\mathrm{e}^{-2d\alpha(x)}\bar{g}(x)\right)^{\frac{(d-4)(d+1)}{8}} \prod_{\mu\geq\nu} \mathrm{d}\left(\mathrm{e}^{\frac{d-6}{2}\alpha(x)}\widehat{h}_{\mu\nu}(x)\right) \\ =& \prod_x \mathrm{e}^{-\alpha(x)\frac{d(d-4)(d+1)}{4}} g(x)^{\frac{(d-4)(d+1)}{8}} \mathrm{e}^{\alpha(x)\frac{(d-6)d(d+1)}{4}} \prod_{\mu\geq\nu} \mathrm{d}\widehat{h}_{\mu\nu}(x) \\ =& \prod_x \mathrm{e}^{-\frac{d(d+1)}{2}\alpha(x)} \mathcal{D}(\widehat{h};\bar{g}) \\ =& \mathrm{e}^{-\frac{d(d+1)}{2}\sum_x \alpha(x)} \mathcal{D}(\widehat{h};\bar{g}) \\ =& \exp\left\{-\mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\alpha(\hat{x})\right]_N\right\} \mathcal{D}(\widehat{h};\bar{g}) \,. \end{aligned} \tag{19.49}$$

Here, we have identified the discretization-based cutoff with an N-cutoff, as before, i.e.,

$$\begin{aligned}\mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\alpha(\hat{x})\right]_N &= \int_{\text{“lattice”}} \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\langle x,\mu,\nu|\alpha(\hat{x})|x,\mu,\nu\rangle \\ &= \mathrm{tr}[I_{ST^2}] \int_{\text{“lattice”}} \mathrm{d}^d x\,\alpha(x) \\ &= \frac{d(d+1)}{2}\sum_x \alpha(x)\,.\end{aligned} \tag{19.50}$$

By setting $\alpha \equiv 1$, it follows immediately from the definition of the anomalies $\mathscr{A}_j$ that

$$\int_{\text{“lattice”}} \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\mathscr{A}_{\hat{h}}(x) = \mathrm{Tr}_{ST^2}[\mathbb{1}_{ST^2}]_N\,. \tag{19.51}$$

The measure for the ghost fields $\bar{C}_\mu$ and C^μ is given by Eqs. (D.7) and (D.8), i.e.,

$$\mathcal{D}(C;\bar{g})\mathcal{D}(\bar{C};\bar{g}) \coloneqq \mathcal{D}\Big[\bar{g}^{\frac{d+2}{4d}}C^\mu\Big]\,\mathcal{D}\Big[\bar{g}^{\frac{d-2}{4d}}\bar{C}_\mu\Big] = \prod_{x,\mu}\bar{g}(x)^{-d/2}\mathrm{d}C^\mu(x)\mathrm{d}\bar{C}_\mu(x)\,. \tag{19.52}$$

We point out, as we did in appendix D.5, that the ordering of the factors in the product is crucial; here, the ordering is defined in the way the factors stand (no further commuting). Applying a Weyl transformation to the measure for the ghost fields leads to

$$\begin{aligned}\mathcal{D}(C';\bar{g}')\mathcal{D}(\bar{C}';\bar{g}') &= \prod_{x,\mu}\left(\mathrm{e}^{-2d\alpha(x)}\bar{g}(x)\right)^{-d/2}\mathrm{d}\Big(\mathrm{e}^{\frac{d-2}{2}\alpha(x)}C^\mu(x)\Big)\,\mathrm{d}\Big(\mathrm{e}^{\frac{d-2}{2}\alpha(x)}\bar{C}_\mu(x)\Big) \\ &= \prod_{x,\mu}\mathrm{e}^{d^2\alpha(x)}\bar{g}(x)^{-d/2}\mathrm{e}^{-d(d-2)\alpha(x)}\mathrm{d}C^\mu(x)\mathrm{d}\bar{C}_\mu(x) \\ &= \prod_x \mathrm{e}^{2d\alpha(x)}\mathcal{D}(C;\bar{g})\mathcal{D}(\bar{C};\bar{g}) \\ &= \mathrm{e}^{2d\sum_x\alpha(x)}\mathcal{D}(C;\bar{g})\mathcal{D}(\bar{C};\bar{g}) \\ &= \exp\left\{-\left(-2\mathrm{Tr}_V[\mathbb{1}_V\alpha(\hat{x})]_N\right)\right\}\mathcal{D}(C;\bar{g})\mathcal{D}(\bar{C};\bar{g})\,,\end{aligned} \tag{19.53}$$

where we have used that

$$\begin{aligned}\mathrm{Tr}_V[\mathbb{1}_V\alpha(\hat{x})]_N &= \int\limits_{\text{“lattice”}} \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\langle x,\mu|\alpha(\hat{x})|x,\mu\rangle \\ &= \mathrm{tr}[I_V]\int\limits_{\text{“lattice”}} \mathrm{d}^d x\,\alpha(x) \\ &= d\sum_x \alpha(x)\,.\end{aligned} \tag{19.54}$$

Again, by setting $\alpha \equiv 1$ it follows immediately from the definition of the field $\mathscr{A}_j$ that

$$\int\limits_{\text{“lattice”}} \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\left[\mathscr{A}_{\bar{C}}(x) + \mathscr{A}_C(x)\right] = -2\,\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N\,. \tag{19.55}$$

With these ingredients, we are ready to calculate the difference given by Eq. (19.24). Therefore, first note that the general EA defined by Eq. (19.12), which, in this application, is defined by the bare matter action $S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]$, here is precisely is the 1L-EA (4.71),

$$\Gamma_{1\mathrm{L}}[\bar{g}] = \frac{1}{2}\,\mathrm{Tr}_{ST^2}\ln\left[\mathscr{U}[0;\bar{g}]^{\bullet\bullet}{}_{\bullet\bullet}\right] - \mathrm{Tr}_V\ln\left[\mathscr{M}[\bar{g},\bar{g}]^{\bullet}{}_{\bullet}\right]\,. \tag{19.56}$$

Hence, Eq. (19.24), when applying an N-cutoff, here yields

$$\begin{aligned}&\bar{\mathscr{T}}\,\Gamma_{1\mathrm{L}}[\bar{g}]_N - \Big\langle \bar{\mathscr{T}} S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]\Big\rangle_N \\ &= \int\limits_{\text{“lattice”}} \mathrm{d}^d x\sqrt{\bar{g}(x)} \sum_{j\in\{\widehat{h},\bar{C},C\}} \mathscr{A}_j(x) + \frac{d-6}{2}\int \mathrm{d}^d x\left\langle \widehat{h}_{\mu\nu}(x)\frac{\delta S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]}{\delta \widehat{h}_{\mu\nu}(x)}\right\rangle_N \\ &\qquad + \frac{d-2}{2}\int \mathrm{d}^d x\left\langle \bar{C}_\mu(x)\frac{\delta S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]}{\delta \bar{C}_\mu(x)}\right\rangle_N \\ &\qquad + \frac{d-2}{2}\int \mathrm{d}^d x\left\langle C^\mu(x)\frac{\delta S_{\mathrm{M}}[\widehat{h},\bar{C},C;\bar{g}]}{\delta C^\mu(x)}\right\rangle_N \\ &= \mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_N - 2\,\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N + \frac{d-6}{2}\int \mathrm{d}^d x\left\langle \widehat{h}_{\mu\nu}(x)\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\widehat{h}^{\rho\sigma}\right)(x)\right\rangle_N \\ &\qquad - (d-2)\sqrt{2}\int \mathrm{d}^d x\left\langle \bar{C}_\mu(x)\left(\mathscr{M}[\bar{g},\bar{g}]^\mu{}_\nu C^\nu\right)(x)\right\rangle_N \\ &= \mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_N - 2\,\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N + \left(\frac{d}{2}-3\right)\mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_N - (d-2)\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N \\ &= \left(\frac{d}{2}-2\right)\mathrm{Tr}_{ST^2}\left[\mathbb{1}_{ST^2}\right]_N - d\,\mathrm{Tr}_V\left[\mathbb{1}_V\right]_N\,,\end{aligned} \tag{19.57}$$

which is precisely Eq. (18.34). (In the third step, we have applied the one-loop expectation values given by Eq. (4.54).)

CHAPTER 20

Conclusion and outlook

In the final part of this thesis, we considered quantum fields in contact with dynamical gravity and proposed a novel, non-perturbative framework for the analysis of such systems.

This framework is subject to a main principle, Background Independence, which must be rigorously implemented already at the level of the regularized precursors of a quantum field theory, which we refer to as "approximants". The resulting generalized continuum limit of such self-consistent approximants thus results in a quantum field theory whose fields live on a dynamically selected background manifold, in accordance with Background Independence. To realize such limites, we introduced a regularization scheme via cutoffs of the N-type which results in N-sequences of approximants of an increasing number of degrees of freedom. Moreover, we argued that each member of an N-sequence should constitute a (quasi-)physically realizable system. All these considerations together can be summarized in the proposed requirements **(R1,2,3)**.

After proposing this quantization framework, we began with probing the properties of the explicit quantum systems that arise from it. Therefore, on the one hand, we considered a quantized scalar field on a classical background manifold, and, on the other hand, quantum gravity itself, by means of the background field technique. We identified two non-identical candidates for N-sequences of approximants, which in turn are inspired from two different versions of the field's operatorial stress-energy tensor that is not unique at the quantum level. As a starting point, we chose to determine the underlying self-consistent spherical geometries, given by a sequence of N-dependent, self-consistent radii. It must be emphasized that we henceforth determined these in *exact* calculations.

When specialized to four spacetime dimensions, the resulting N-sequences of self-consistent radii exhibited striking physical properties. Especially we were able to demonstrate the absence of the cosmological constant problem, whereby *adding further degrees of freedom to the quantum system flattens the universe.* Furthermore, they allow for an interpretation of the microscopic degrees of freedom that form the thermodynamic Bekenstein-Hawking entropy of de Sitter space.

These preliminary investigations of the proposed novel quantization framework should be extended in a number of directions. Especially, more physically realizable approximants should be constructed. Then the convergence properties of the resulting N-sequences should be analyzed, i.e., conditions should be worked out under which N-sequences of approximants converge to quantum field theories of desirable properties.

Regarding the analyzed field content, this will particularly require the treatment of self-interacting matter. Regarding the self-consistent background geometries, it is necessary to expand their analysis to more complicated background structures. In particular, background structures of cosmological models might result in approximants with physically intriguing properties.

Part 5

Appendix

APPENDIX A

Mathematical background

The following appendix gives an overview about the mathematical conventions and notations used in this thesis. As the content of such an appendix can easily get out of control, let us specify a few notational peculiarities right away: Throughtout this thesis, we use natural units in which $\hbar \equiv 1 \equiv c$. In theories of gravity, canonical mass dimensions can follow the conventions of either dimensionless coordinates, i.e. $[x^\mu] = 0$, $[\partial_\mu] = 0$, $[g_{\mu\nu}] = -2$ and $[g^{\mu\nu}] = +2$ etc., or of a dimensionless metric tensor, i.e. $[g_{\mu\nu}] = 0$, $[g^{\mu\nu}] = 0$, $[x^\mu] = -1$, $[\partial_\mu] = +1$. In case that we specify canonical mass dimensions, we always point out which convention we follow. With $x, y \in \mathbb{R}^d$, the delta-function is defined by $\delta(x-y) \coloneqq \prod_{i=1}^d \delta(x^i - y^i)$. Furthermore, det and tr denote the determinant and trace of a matrix, while Det and Tr denote the determinant and trace of an operator (that potentially is equipped with a tensor structure, as well).

A.1. Conventions and notation

A.1.1. *Manifolds and Hilbert spaces*

Where not mentioned elsewise, we model classical spacetime as an d-dimensional Riemannian manifold M with metric g of *Euclidean* signature, i.e. signature $++\cdots+$. In the rare cases of working on *Lorentzian* manifolds we adopt the signature $-+\cdots+$. In special cases, Lorentzian metrics can be converted into a Euclidean metrics applying a *Wick rotation* of the time coordinate in the complex plane. The mathematical aspects of Wick rotations are delicate [178] and as these are hardly made use of throughout this thesis, we will, later on, solely expound the Wick rotations actually used.

We denote by $g_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$, $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ and $R^\sigma{}_{\rho\mu\nu} = -\partial_\nu \Gamma^\sigma_{\mu\rho} + \cdots$ the components of the metric tensor, *scalar* and *Ricci curvature* as well as the

Riemann tensor emerging from g, respectively.[1] Also, D_μ denotes the covariant derivative given by g and we call $\Box_g := g^{\mu\nu} D_\mu D_\nu$ the *Laplacian operator*.

Attached to (M, g) is the *Hilbert space* $L^2(M, g)$ of square-integrable functions with the scalar product

$$(\,\cdot\,,\,\cdot\,)_g : L^2(M,g) \times L^2(M,g) \to \mathbb{C}$$

which is locally given by

$$(f_1, f_2)_g := \int \mathrm{d}^d x \sqrt{g(x)}\, f_1^*(x) f_2(x)\,, \tag{A.1}$$

where $g(x) = \det\left(g_{\mu\nu}(x)\right)$ denotes the determinant of the metric.[2] Let us define another, auxilliary, scalar product by

$$(f_1, f_2)_1 := \int \mathrm{d}^d x\, f_1^*(x) f_2(x)\,, \tag{A.2}$$

which is independent of the metric g. This scalar product is *not* associated to $L^2(M, g)$ but plays a crucial role in evaluating the Gaussian path integral.

Next, we may express $(\,\cdot\,,\,\cdot\,)_g$ employing the *bra-ket* notation:

$$(f_1, f_2)_g := \langle f_1 | f_2 \rangle \tag{A.3}$$

with $|f_2\rangle \in L^2(M, g)$ and $\langle f_1| \in L^2(M, g)^*$, the dual space. Introducing the basis $\{|x\rangle \,|\, x \in \mathbb{R}^d\}$ one has $f(x) := \langle x|f\rangle$ and consequently

$$\begin{aligned}
\langle f_1 | f_2 \rangle &= \int \mathrm{d}^d x \sqrt{g(x)}\, f_1^*(x) f_2(x) \\
&= \int \mathrm{d}^d x \sqrt{g(x)}\, \langle f_1^* | x\rangle \langle x | f_2 \rangle \\
&= \langle f_1 | \int \mathrm{d}^d x \sqrt{g(x)} | x\rangle \langle x | f_2 \rangle
\end{aligned}$$

such that in this setting the unit operator on $L^2(M, g)$ is given by

$$\mathbb{1}_{L^2} = \int \mathrm{d}^d x \sqrt{g(x)}\, |x\rangle\langle x|\,; \tag{A.4}$$

[1] Note that in the common textbooks [179] and [46], the Riemann tensor is defined with the opposed sign.

[2] We denote by g the metric as a geometric object, i.e. a section of a vector bundle, as well the determinant of its local components. In the respective context, it will be clear to what notion we are referring to (mostly the latter).

that is the *completeness* relation of the basis $\{|x\rangle\}$. Its *orthogonality* follows as

$$\langle x|y\rangle = \langle x|\mathbb{1}_{L^2}|y\rangle = \frac{\delta(x-y)}{\sqrt{g(y)}} \tag{A.5}$$

and we can therewith confirm consistency,

$$\langle z|f\rangle = \int \mathrm{d}^d x \sqrt{g(x)}\, \langle z|x\rangle\langle x|f\rangle = f(z)\,.$$

Defining matrix elements of an *operator* A on $L^2(M,g)$ by

$$A_{xy} \coloneqq \langle x|A|y\rangle\,, \tag{A.6}$$

matrix multiplication of A with another operator B and the *operator trace* of A are given by, respectively,

$$(AB)_{xy} = \int \mathrm{d}^d z \sqrt{g(z)}\, A_{xz} B_{zy} \tag{A.7}$$

$$\mathrm{Tr}[A] = \int \mathrm{d}^d z \sqrt{g(z)}\, A_{zz}\,. \tag{A.8}$$

A *differential Operator* A^{diff} associated to the abstract operator A is defined as (for $f \in L^2(M,g)$)

$$\left(A^{\mathrm{diff}} f\right)(x) \coloneqq \int \mathrm{d}^d y \sqrt{g(y)}\, A_{xy} f_y = \langle x|A|f\rangle\,. \tag{A.9}$$

Alternative notations are $\left(A^{\mathrm{diff}} f\right)(x) = \left(A^{\mathrm{diff}(x)} f\right)(x) \equiv \left(A_x^{\mathrm{diff}(x)} f\right)(x)$ where "diff(x)" refers to differentiation with respect to x. The *inverse operator* A^{-1} of A is defined by

$$A^{-1}A = AA^{-1} = \mathbb{1}_{L^2} \tag{A.10}$$

from which we derive the relations

$$\left(AA^{-1}\right)_{xy} = \mathbb{1}_{L^2\,xy} \tag{A.11}$$

$$\int \mathrm{d}^d \sqrt{g(z)}\, A_{xz} \left(A^{-1}\right)_{zy} = \mathbb{1}_{L^2\,xy} \tag{A.12}$$

$$A^{\mathrm{diff}(x)} \langle x|A^{-1}|y\rangle = \frac{\delta(x-y)}{\sqrt{g(y)}}\,. \tag{A.13}$$

So far, we have built a framework to handle *scalars* on a Riemannian manifolds, i.e. the space of square-integrable functions $L^2(M,g)$. In addition to that, we are required to accordingly treat *vector fields* and *symmetric rank-2 tensor fields*. Geometrically, vector fields are elements of ΓTM, the space of sections of the tangent bundle $TM \to M$, while symmetric rank-2 tensor fields are elements of $\Gamma\mathrm{Sym}^2(TM)$, the space of sections of the vector bundle of bilinear mappings on TM, $\mathrm{Sym}^2(TM) \to M$. Locally, $X \in \Gamma TM$ and $\phi \in \Gamma\mathrm{Sym}^2(TM)$ can be expressed, with help of a chart $x : U \subset M \to U' \subset \mathbb{R}^d$, as [179–181]

$$X_q = X^\mu(q) \frac{\partial}{\partial x^\mu|^q} \tag{A.14}$$

$$\phi_q = \phi^{\mu\nu}(q) \frac{\partial}{\partial x^\mu|^q} \otimes \frac{\partial}{\partial x^\nu|^q} \tag{A.15}$$

where $q \in U$.[3] Relying on the fact that vector budles are locally trivial, we further restrict ΓTM to V and $\Gamma\mathrm{Sym}^2(TM)$ to ST^2, which are subspaces identified by the local isomorphisms

$$\Gamma TM|_U \supseteq V|_U \cong L^2(M,g)|_U \otimes \mathbb{R}^d$$

$$\Gamma\mathrm{Sym}^2(TM)|_U \supseteq ST^2|_U \cong L^2(M,g)|_U \otimes \mathbb{R}^{\frac{d(d+1)}{2}} \,.$$

Then we may extend the scalar product (A.1) to the space of vector fields and symmetric rank-2 tensors, respectively,

$$(X,Y)_g := \int \mathrm{d}^d x \sqrt{g(x)}\, g_{\mu\nu}(x) X^{*\mu}(x) Y^\nu(x) \tag{A.16}$$

$$(\phi,\psi)_g := \int \mathrm{d}^d x \sqrt{g(x)}\, \frac{1}{2}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})(x)\phi^{*\mu\nu}(x)\psi^{\alpha\beta}(x)\,, \tag{A.17}$$

where $X, Y \in V$ and $\phi, \psi \in ST^2$ (or complexifications thereof). These local extentions of $(\,\cdot\,,\,\cdot\,)_g$ are obviously well-behaved under coordinate transformations and hence defined globally. Therewith, we identify the spaces of vector fields and of symmetric rank-2 tensor fields as Hilbert spaces. Their canocial bases are given by, respectively

$$\big\{|x,\mu\rangle \,\big|\, x \in \mathbb{R}^d\,,\, \mu \in \{1,\dots,d\}\big\} \tag{A.18}$$

[3]At this short moment in time, we are aware of the ongoing abuse of notation: Physicists usually treat sections of vector bundles locally, i.e. consider mappings *codomain of chart* $\to$ *manifold* $\to$ *total space* that are well-behaved under chart transitions. If x denotes a chart and X a vector field, then such mappings are obtained by $X \circ x^{-1}$. However, we denote this mapping, that we consider only in local treatments, as $x \mapsto X^\mu(x)$ where $x \in \mathbb{R}^d$, and conveniently forget about the interposed chart but keep in mind the transformation behaviour under chart transitions from now on.

$$\big\{|x,\mu,\nu\rangle \,\big|\, x \in \mathbb{R}^d\,;\ \mu \le \nu\,;\ \mu,\nu \in \{1,\ldots,d\}\big\}\ . \tag{A.19}$$

These bases map $X \in V$ and $\phi \in ST^2$ to their local components, $X^\mu(x) = \langle x,\mu|X\rangle$, $X^*_\mu(x) = \langle X|x,\mu\rangle$, $\phi^{\mu\nu}x = \langle x,\mu,\nu|\phi\rangle$ and $\phi^*_{\mu\nu}(x) = \langle\phi|x,\mu,\nu\rangle$, and fulfill the relations:[4]

$$\langle x,\mu|y,\nu\rangle = \delta^\mu_\nu \frac{\delta(x-y)}{\sqrt{g(y)}} \tag{A.20}$$

$$\langle x,\mu,\nu|y,\alpha,\beta\rangle = \frac{1}{2}\big(\delta^\mu_\alpha\delta^\nu_\beta + \delta^\mu_\beta\delta^\nu_\alpha\big)\frac{\delta(x-y)}{\sqrt{g(y)}} \tag{A.21}$$

as well as

$$\sum_\mu \int \mathrm{d}^d x \sqrt{g(x)}\,|x,\mu\rangle\langle x,\mu| = \textit{unity operator on } \Gamma TM \tag{A.22}$$

$$\sum_{\mu,\nu} \int \mathrm{d}^d x \sqrt{g(x)}\,|x,\mu,\nu\rangle\langle x,\mu,\nu| = \textit{unity operator on } \Gamma\mathrm{Sym}^2(TM)\,. \tag{A.23}$$

Regarding the scalar products, we verify that for $X, Y \in V$ and $\phi,\psi \in ST^2$ one has

$$\begin{aligned}\langle X|Y\rangle &= \sum_\mu \int \mathrm{d}^d x \sqrt{g(x)}\,\langle X|x,\mu\rangle\langle x,\mu|Y\rangle \\ &= \int \mathrm{d}^d x \sqrt{g(x)}\,X^*_\mu(x) Y^\mu(x) \\ &\equiv (X,Y)_g\end{aligned}$$

and

$$\begin{aligned}\langle \phi|\psi\rangle &= \sum_{\mu,\nu} \int \mathrm{d}^d x \sqrt{g(x)}\,\langle \phi|x,\mu,\nu\rangle\langle x,\mu,\nu|\psi\rangle \\ &= \int \mathrm{d}^d x \sqrt{g(x)}\,\phi^*_{\mu\nu}(x)\psi^{\mu\nu}(x) \\ &\equiv (\phi,\psi)_g\,.\end{aligned}$$

After having meticulously introduced the geometric framework to work with, let us summarize the precedent outline by introducing a simplified notation. We denote by…

[4] When indices appear in bras and kets, we explicitly denote whether they are summed over.

…S the Hilbert space of scalars with canonical basis $\{|x\rangle\}$. Its unit operator is $\mathbb{1}_S = \int \mathrm{d}^d x \sqrt{g(x)}\, |x\rangle\langle x|$ and the operator A_S acting on S is locally given by $\langle x|A_S|f\rangle = (A_S)_x^{\text{diff}} f(x)$. Its trace reads $\mathrm{Tr}_S[A_S] = \int \mathrm{d}^d x \sqrt{g(x)}\, \langle x|A_S|x\rangle$.

…V the Hilbert space of vector fields with canonical basis $\{|x\mu\rangle\}$. Its unit operator is $\mathbb{1}_V = \sum_\mu \int \mathrm{d}^d x \sqrt{g(x)}\, |x,\mu\rangle\langle x,\mu|$ and the operator A_V acting on V is locally given by $\langle x,\mu|A_V|X\rangle = (A_V X)^\mu(x) = (A_V)^\mu{}_{\nu x}^{\text{diff}} X^\nu(x)$. Therewith, its trace is given by the formula $\mathrm{Tr}_V[A_V] = \sum_\mu \int \mathrm{d}^d x \sqrt{g(x)}\, \langle x,\mu|A_V|x,\mu\rangle$.

…ST^2 the Hilbert space of vector fields with canonical basis $\{|x,\mu,\nu\rangle\}$. Its unit operator is $\mathbb{1}_{ST^2} = \sum_{\mu,\nu} \int \mathrm{d}^d x \sqrt{g(x)}\, |x,\mu,\nu\rangle\langle x,\mu,\nu|$ and the operator A_{ST^2} acting on ST^2 is locally given by $\langle x,\mu,\nu|A_{ST^2}|\phi\rangle = (A_{ST^2}\phi)^{\mu\nu}(x) = (A_{ST^2})^{\mu\nu}{}_{\alpha\beta}{}_x^{\text{diff}} \phi^{\alpha\beta}(x)$. Therewith, its trace is given by $\mathrm{Tr}_{ST^2}[A_{ST^2}] = \sum_{\mu,\nu} \int \mathrm{d}^d x \sqrt{g(x)}\, \langle x,\mu,\nu|A_{ST^2}|x,\mu,\nu\rangle$.

Also, we point out the significant fact that Tr_S, Tr_V as well as Tr_{ST^2} are independent of the metric g due to (A.5), (A.20) and (A.21).

Lastly we introduce the general notation

$$\langle x,\mu_1,\mu_2,\ldots|\mathbb{1}|y,\nu_1,\nu_2,\ldots\rangle =: I^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \frac{\delta(x-y)}{\sqrt{g(y)}}\,,$$

such that especially

$$I_{ST^2}{}^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho\right) \tag{A.24}$$

and $I_V{}^\mu{}_\nu = \delta^\mu_\nu$ and $I_S = 1$. Usually, as the context is clear, we write only I for I_{ST^2}.

A.1.2. *Separable Hilbert spaces*

Often we assume M to be *compact*. This implies the Hilbert space $S = L^2(M,g)$ to be *separable* and equipped with a *generic countable basis* $\{|n\rangle \,|\, n \in \mathbb{N}\}$ of S.

Without loss of generality, this basis can be chosen *orthonormal*, $\langle n|m\rangle = \delta_{nm}$. In terms of the functions $\langle x|n\rangle =: u_n(x)$ this relation reads

$$\delta_{nm} = \langle n|m\rangle = \langle n|\mathbb{1}_S|m\rangle = \int \mathrm{d}^d x \sqrt{g(x)}\, u_n^*(x) u_m(x) \tag{A.25}$$

and the completeness relation $\mathbb{1}_S = \sum_n |n\rangle\langle n|$ amounts to

$$\frac{\delta(x-y)}{\sqrt{g(y)}} = \langle x|y\rangle = \sum_n \langle x|n\rangle\langle n|y\rangle = \sum_n u_n^*(x) u_n(y)\,. \tag{A.26}$$

In this setting the trace of the operator A_S acting on S can be computed as

$$\mathrm{Tr}_S[A_S] = \sum_n \langle n|A_S|n\rangle\,. \tag{A.27}$$

Therewith it is clear that the Hilbert spaces V and ST^2 of vector fields and symmetric rank-2 tensor fields are separable, too. However, in order not to digress in the preliminary, we dub their generic countable bases only in special cases.

Next, we will study the properties of a basis of S that has been constructed in terms of the eigenfunctions of some operator acting on S. Therefore, let $\mathscr{K}$ be some operator acting on S that is *self-adjoint* with respect to (A.1). We assume its eigenvalue problem,

$$\mathscr{K}\chi_n = \mathscr{F}_n \chi_n \tag{A.28}$$

to be solved. Furthermore, we assume $\mathscr{K}$ to have a fully discrete spectrum and to be positive definite; then one finds the sequence

$$\mathscr{F}_1 < \mathscr{F}_2 < \cdots < \mathscr{F}_n \xrightarrow{n\to\infty} \infty$$

and $\big\{\chi_{n,m}\,\big|\, n\in\mathbb{N}\,,\, m\in\{1,\ldots,D_n\}\big\}$, where m is the index accounting for the degeneracy D_n of the eigenvalue $\mathscr{F}_n$, forms an orthonormal and complete basis of S. The orthogonality and completeness realtions, (A.25) and (A.26), now read

$$\delta_{nk}\delta_{ml} = \int \mathrm{d}^d x \sqrt{g(x)} \chi_{n,m}^*(x) \chi_{k,l}(x) \tag{A.29}$$

$$\frac{\delta(x-y)}{\sqrt{g(y)}} = \sum_{n\in\mathbb{N}} \sum_{m=1}^{D_n} \chi_{n,m}^*(x) \chi_{n,m}(y)\,. \tag{A.30}$$

Here, we draw attention to an important lemma: If $\mathscr{K}$ is self-adjoint with respect to (A.1) then

$$\mathscr{L} \coloneqq g^{1/4}(\hat{x})\mathscr{K} g^{-1/4}(\hat{x}) \tag{A.31}$$

is self-adjoint with respect to (A.2). Note that for the sake of clarity, we denoted the position operator in the argument on $g^{1/4}$ with a "hat" – elsewise, we drop the "hat" for operators. The proof is a sequence of equal signs:

$$\begin{aligned}
(f_1, \mathscr{L} f_2)_1 &= \int \mathrm{d}^d x f_1^*(x) g^{1/4}(x) \left(\mathscr{K} g^{-1/4} f_2\right)(x) \\
&= \int \mathrm{d}^d x \sqrt{g}\, g^{-1/2} f_1^*\, g^{1/4} \mathscr{K} g^{-1/4} f_2 \\
&= \left(g^{-1/2} f_1 g^{1/4}, \mathscr{K} g^{-1/4} f_2\right)_g \\
&= \left(\mathscr{K} g^{-1/4} f_1, g^{-1/4} f_2\right)_g \\
&= \int \mathrm{d}^d x\, g^{1/2} \left(\mathscr{K} g^{-1/4} f_1^*\right) g^{-1/4} f_2 \\
&= \int \mathrm{d}^d x \left(g^{1/4} \mathscr{K} g^{-1/4} f_1^*\right) f_2 \\
&= (\mathscr{L} f_1, f_2)_1 \qquad \square
\end{aligned}$$

where in the fourth step we have used that $\mathscr{K}$ is self-adjoint with respect to (A.1).

To diagonalize $\mathscr{L}$, let us consider its eigenvalue problem:

$$\mathscr{L}\psi_n = \mathscr{F}_n' \psi_n \quad \Leftrightarrow \quad \mathscr{K} g^{-1/4}\psi_n = \mathscr{F}_n' g^{-1/4}\psi_n\,.$$

Without loss of generality, we can choose $\psi_n = g^{-1/4}\chi_n$ which implies

$$\mathscr{F}_n' = \mathscr{F}_n\,.$$

Hence, we have found that

$$\text{spectrum}(\mathscr{L}) = \text{spectrum}(\mathscr{K}) = \{\mathscr{F}_n\}\,. \tag{A.32}$$

The eigenfunctions ψ_n of $\mathscr{L}$ occur with the same degree of degeneracy as χ_n (that of $\mathscr{F}_n$) and fullfill

$$\delta_{nk}\delta_{ml} = \int \mathrm{d}^d x\, g^{1/4}(x) g^{1/4}(x) \chi_{n,m}^*(x) \chi_{k,l}(x) = \left(\psi_{n,m}, \psi_{k,l}\right)_1 \tag{A.33}$$

$$\delta(x-y) = g^{1/4}(x)g^{1/4}(y)\sum_n\sum_{m=1}^{D_n}\chi^*_{n,m}(x)\chi_{n,m}(y) = \sum_n\sum_{m=1}^{D_n}\psi^*_{n,m}(x)\psi_{n,m}(y)\,. \tag{A.34}$$

Note that $\{\psi_{n,m}\}$ forms a basis of S, too.

A.1.3. *The special case $M = S^d(L)$*

Sometimes we restrict (background) spacetime to the special case of $M = S^d(L)$, the d-sphere of radius L, and employ the bases of S, V and ST^2 given in terms of the eigenfunctions of the negative Laplacian $-\Box_g = -g^{\mu\nu}D_\mu D_\nu$. The spectrum of the Laplacian on the d-sphere, acting on S, V and ST^2, has been determined rigorously [182–184]. Subsequently, we summarize the required results while introducing compact notations for these.

The eigenvalue problem for the Laplacian acting on *scalars* reads

$$-\Box_g u_{n,m}(x) = \mathscr{E}^S_n u_{n,m}(x)\,, \tag{A.35}$$

where $n = 0, 1, 2, \dots$ and $m = 1, 2, \dots, D^S_n$ accounts for the degeneracy of $\mathscr{E}^S_n$.[5] We introduce the state $|nm\rangle$ in S by

$$\langle x|nm\rangle \coloneqq u_{n,m}(x)\,; \tag{A.36}$$

thence

$$\big\{|nm\rangle \;\big|\; n = 0, 1, 2, \dots\;;\; m = 1, 2, \dots, D^S_n\big\} \tag{A.37}$$

is an orthonormal basis of S and the trace of an operator A_S is given by

$$\boxed{\begin{aligned}\mathrm{Tr}_S[A_S] &= \sum_{n=0}^\infty\sum_{m=1}^{D^S_n}\langle nm|A_S|nm\rangle \\ &= \sum_{n=0}^\infty\sum_{m=1}^{D^S_n}\int \mathrm{d}^d x\sqrt{g(x)}\,u_{n,m}(x)A_{S_x}^{\mathrm{diff}}u_{n,m}(x)\,.\end{aligned}}$$

[5]Here, we explicitly let the index n start running from zero to mark the (single) zero mode – the constant function.

A *vector field* can be decomposed into transverse and longitudinal parts, cf. (A.75). Accordingly, the eigenvalue problem of $-\Box_g$ can be decomposed into transverse and longitudinal eigenfunctions:

$$-\Box_g \left(u^T_{n,m}\right)_\mu(x) = \mathscr{E}^T_n \left(u^T_{n,m}\right)_\mu(x)\,, \tag{A.38}$$

where $D^\mu \left(u^T_{n,m}\right)_\mu = 0$; $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, D^T_n$; as well as

$$-\Box_g \left(u^L_{n,m}\right)_\mu(x) = \mathscr{E}^L_n \left(u^L_{n,m}\right)_\mu(x)\,, \tag{A.39}$$

where $\left(u^L_{n,m}\right)_\mu = D_\mu u_{n,m}$; $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, D^L_n$. We introduce the states $|nm\rangle^T$ and $|nm\rangle^L$ in V by

$$\langle x, \mu | nm\rangle^T := a^T_n \left(u^T_{n,m}\right)^\mu(x) \tag{A.40}$$

and

$$\langle x, \mu | nm\rangle^L := a^L_n \left(u^L_{n,m}\right)^\mu(x) \tag{A.41}$$

where a^T_n and a^L_n are constants chosen such that ${}^I\langle nm|kl\rangle^J = \delta_{IJ}\delta_{nm}\delta_{kl}$; $I, J = T, L$. Then

$$\begin{aligned}&\left\{|nm\rangle^T \;\middle|\; n = 1, 2, \ldots\;;\; m = 1, 2, \ldots, D^T_n\right\}\\ &\qquad\qquad \times \bigcup \left\{|nm\rangle^L \;\middle|\; n = 1, 2, \ldots\;;\; m = 1, 2, \ldots, D^L_n\right\}\end{aligned} \tag{A.42}$$

is an orthonormal basis of V and the trace of an operator A_V is given by

$$\boxed{\mathrm{Tr}_V[A_V] = \sum_{n=1}^{\infty}\sum_{m=1}^{D^T_n} {}^T\langle nm|A_V|nm\rangle^T + \sum_{n=1}^{\infty}\sum_{m=1}^{D^L_n} {}^L\langle nm|A_V|nm\rangle^L}\,, \tag{A.43}$$

or put another way,

$$\mathrm{Tr}_V[A_V] = \sum_{I=T,L}\sum_{n=1}^{\infty}\sum_{m=1}^{D^I_n} \left(a^I_n\right)^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \left(u^I_{n,m}\right)_\mu(x) (A_V)^\mu{}_{\nu x}^{\;\mathrm{diff}} \left(u^I_{n,m}\right)^\nu(x)\,.$$

A covariant *symmetric rank-2 tensor field* can be decomposed into transverse-traceless, longitudinal-transverse-traceless, longitudinal-longitudinal-traceless and trace part, cf. (A.84). Accordingly, the eigenvalue problem of $-\Box_g$ can be decomposed:

$$-\Box_g \left(u^{TT}_{n,m}\right)_{\mu\nu}(x) = \mathscr{E}^{TT}_n \left(u^{TT}_{n,m}\right)_{\mu\nu}(x)\,, \tag{A.44}$$

where $D^\mu \left(u^{TT}_{n,m}\right)_{\mu\nu} = 0 = g^{\mu\nu} \left(u^{TT}_{n,m}\right)_{\mu\nu}$; $n = 2, 3, \ldots$ and $m = 1, 2, \ldots, D^{TT}_n$;

$$-\Box_g \left(u^{L^T,T}_{n,m}\right)_{\mu\nu}(x) = \mathscr{E}^{L^T,T}_n \left(u^{L^T,T}_{n,m}\right)_{\mu\nu}(x)\,, \tag{A.45}$$

where $\left(u^{L^T,T}_{n,m}\right)_{\mu\nu} = D_\mu \left(u^T_{n,m}\right)_\nu + D_\nu \left(u^T_{n,m}\right)_\mu$; $n = 2, 3, \ldots$ and $m = 1, 2, \ldots, D^{L^T,T}_n$;

$$-\Box_g \left(u^{L^L,T}_{n,m}\right)_{\mu\nu}(x) = \mathscr{E}^{L^L,T}_n \left(u^{L^L,T}_{n,m}\right)_{\mu\nu}(x)\,, \tag{A.46}$$

where $\left(u^{L^L,T}_{n,m}\right)_{\mu\nu} = 2D_\mu D_\nu u_n - \frac{2}{d} g_{\mu\nu}\Box_g u_n$; $n = 2, 3, \ldots$ and $m = 1, 2, \ldots, D^{L^L,T}_n$; as well as

$$-\Box_g g_{\mu\nu} u_{n,m}(x) = \mathscr{E}^{\text{trace}}_n u_{n,m}(x)\,, \tag{A.47}$$

where $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots, D^{\text{trace}}_n$. We introduce the states $|nm\rangle^{TT}$, $|nm\rangle^{L^T,T}$, $|nm\rangle^{L^L,T}$ and $|nm\rangle^{\text{trace}}$ in ST^2 by

$$\langle x, \mu, \nu | nm\rangle^{TT} := a^{TT}_n \left(u^{TT}_{n,m}\right)^{\mu\nu}(x)\,, \tag{A.48}$$

$$\langle x, \mu, \nu | nm\rangle^{L^T,T} := a^{L^T,T}_n \left(u^{L^T,T}_{n,m}\right)^{\mu\nu}(x)\,, \tag{A.49}$$

$$\langle x, \mu, \nu | nm\rangle^{L^L,T} := a^{L^L,T}_n \left(u^{L^L,T}_{n,m}\right)^{\mu\nu}(x)\,, \tag{A.50}$$

$$\langle x, \mu, \nu | nm\rangle^{\text{trace}} := a^{\text{trace}}_n g^{\mu\nu}(x) u_{n,m}(x) \tag{A.51}$$

where a^{TT}_n, $a^{L^T,T}_n$, $a^{L^L,T}_n$ and a^{trace}_n are constants chosen such that ${}^I\langle nm|kl\rangle^J = \delta_{IJ}\delta_{nm}\delta_{kl}$; $I, J \in \{(TT), (L^T, T), (L^L, T), \text{trace}\}$. Then

$$\begin{aligned} &\left\{|nm\rangle^{TT} \;\middle|\; n = 2, 3, \ldots \;;\; m = 1, 2, \ldots, D^{TT}_n\right\} \\ &\qquad \times \bigcup \left\{|nm\rangle^{L^T,T} \;\middle|\; n = 2, 3, \ldots \;;\; m = 1, 2, \ldots, D^{L^T,T}_n\right\} \\ &\qquad \times \bigcup \left\{|nm\rangle^{L^L,T} \;\middle|\; n = 2, 3, \ldots \;;\; m = 1, 2, \ldots, D^{L^L,T}_n\right\} \\ &\qquad\qquad \times \bigcup \left\{|nm\rangle^{\text{trace}} \;\middle|\; n = 0, 1, \ldots \;;\; m = 1, 2, \ldots, D^{\text{trace}}_n\right\} \end{aligned} \tag{A.52}$$

is an orthonormal basis of ST^2.

The trace of an operator A_{ST^2} can be calculated by means of this basis,

$$\boxed{\begin{aligned}\mathrm{Tr}_{ST^2}[A_{ST^2}] = &\sum_{n=2}^{\infty}\sum_{m=1}^{D_n^{TT}} {}^{TT}\langle nm|A_{ST^2}|nm\rangle^{TT} \\ &+\sum_{n=2}^{\infty}\sum_{m=1}^{D_n^{L^T,T}} {}^{L^T,T}\langle nm|A_{ST^2}|nm\rangle^{L^T,T} \\ &+\sum_{n=2}^{\infty}\sum_{m=1}^{D_n^{L^L,T}} {}^{L^L,T}\langle nm|A_{ST^2}|nm\rangle^{L^L,T} \\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{D_n^{\mathrm{trace}}} {}^{\mathrm{trace}}\langle nm|A_{ST^2}|nm\rangle^{\mathrm{trace}},\end{aligned}} \tag{A.53}$$

which is equivalent to

$$\begin{aligned}\mathrm{Tr}_{ST^2}[A_{ST^2}] = &\sum_{I}\sum_{n=2}^{\infty}\sum_{m=1}^{D_n^{I}} \left(a_n^I\right)^2 \int \mathrm{d}^d x\sqrt{g}\, \left(u_{n,m}^I\right)_{\mu\nu} (A_{ST^2})^{\mu\nu}{}_{\alpha\beta}{}^{\mathrm{diff}} \left(u_{n,m}^I\right)^{\alpha\beta} \\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{D_n^{\mathrm{trace}}} \left(a_n^{\mathrm{trace}}\right)^2 \int \mathrm{d}^d x\sqrt{g}\, g_{\mu\nu} u_n (A_{ST^2})^{\mu\nu}{}_{\alpha\beta}{}^{\mathrm{diff}} g^{\alpha\beta} u_n\,,\end{aligned}$$

where $I \in \{(TT),(L^T,T),(L^L,T)\}$.

Lastly, we point out that the index n starts running from 0, 1 and 2 for u_n, u_n^T and u_n^{TT}, respectively, per construction [182, 184] (corresponding to the rank of the tensor). All other eigenfunctions are constructed from these three; and consequently one can deduce the respective ranges that n indexes [183].

FIGURE A.1. Spectrum of $-\Box_g$ on $S^d(L)$. Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient.

Eigenfunction	Eigenvalue	Degeneracy	$n =$
scalars:			
u_n	$\mathscr{E}_n^S = \frac{n(n+d-1)}{L^2}$	$D_n^S = \frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}$ $= 2\binom{n+d-2}{d-1} + \binom{n+d-2}{d-2}$	$0, 1, \dots$
vectors:			
$\left(u_n^T\right)_\mu$	$\mathscr{E}_n^T = \frac{n(n+d-1)-1}{L^2}$	$D_n^T = \frac{n(n+d-1)(2n+d-1)(n+d-3)!}{(d-2)!(n+1)!}$ $= n\left[\binom{n+d-1}{d-2} + \binom{n+d-2}{n}\right] + \binom{n+d-3}{n-1}$	$1, 2, \dots$
$\left(u_n^L\right)_\mu$	$\mathscr{E}_n^L = \frac{n(n+d-1)-(d-1)}{L^2}$	$D_n^T \equiv D_n^S$	$1, 2, \dots$
symmetric tensors:			
$\left(u_n^{TT}\right)_{\mu\nu}$	$\mathscr{E}_n^{TT} = \frac{n(n+d-1)-2}{L^2}$	$D_n^{TT} = \frac{(d+1)(d-2)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2(d-1)!(n+1)!}$ $= (d+1)(n-1)\binom{n+d-3}{d-3} + (d+1)(d-2)\binom{n+d-3}{d-1} + \frac{(d+1)(n-1)}{2}\binom{n+d-2}{d-3}$	$2, 3, \dots$
$\left(u_n^{L^T,T}\right)_{\mu\nu}$	$\mathscr{E}_n^{L^T,T} = \frac{n(n+d-1)-(d+2)}{L^2}$	$D_n^{L^T,T} \equiv D_n^T$	$2, 3, \dots$
$\left(u_n^{L^L,T}\right)_{\mu\nu}$	$\mathscr{E}_n^{L^L,T} = \frac{n(n+d-1)-2d}{L^2}$	$D_n^{L^L,T} \equiv D_n^S$	$2, 3, \dots$
$g_{\mu\nu}u_n$	$\mathscr{E}_n^{\text{trace}} = \mathscr{E}_n^S$	$D_n^{\text{trace}} \equiv D_n^S$	$0, 1, \dots$

A.2. Field decomposition

Details on how to decompose a tensor field can be found in many textbooks, e.g. [102, 185, 186]. Subsequently, we introduce the most important properties. In general, we denote a projector onto some subspace of a Hilbert space by $\mathbb{P}$ and by P its associated tensor structure given by

$$\langle x, \mu_1, \mu_2, \ldots |\mathbb{P}| y, \nu_1, \nu_2, \ldots \rangle = (P^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \cdots})^{\mathrm{diff}}_x \frac{\delta(x-y)}{\sqrt{g(y)}} \,.$$

(Usually, we write $P^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \cdots}$ for notational simplicity and keep in mind that we deal with a differential operator.)

A.2.1. *On flat spacetime*

Tensor fields defined on a *flat spacetime*, i.e. Euclidean space with the metric $g_{\mu\nu} = \delta_{\mu\nu}$ or Minkowski space with the metric $g_{\mu\nu} = \eta_{\mu\nu}$, can be decomposed into irreducible representations of the Lorentz group. These are labeled by spin J and parity P, in short J^P [102, p. 21].

A *vector field* X_μ can be decomposed into its *transverse* and *longitudinal* components. These correspond to the following representations of the Lorentz-group, respectively. Firstly, a $(d-1)$-dimensional representation with $J^P = 1^-$ given by the projector

$$L_\mu{}^\nu \coloneqq \frac{p_\mu p^\nu}{p^2} \tag{A.54}$$

in Fourier space (otherwise $L^\mu{}_\nu = \frac{\partial^\mu \partial_\nu}{\partial^2}$) and, secondly, a one-dimensional representation with $J^P = 0^+$ given by the projector

$$T_\mu{}^\nu \coloneqq \delta_\mu^\nu - L_\mu{}^\nu \,. \tag{A.55}$$

This implies that we may split the vector field X_μ as

$$X_\mu = X_\mu^T + \partial_\mu \sigma \coloneqq L_\mu{}^\nu X_\nu + T_\mu{}^\nu X_\nu \tag{A.56}$$

where $\partial^\mu X_\mu^T = 0$ and σ is a scalar field.

A *symmetric rank-2 tensor field* $\phi_{\mu\nu}$ can be decomposed into four irreducible representations of the Lorentz group – firstly, a $(d+1)(d-2)/2$-dimensional representation with $J^P = 2^+$ given by the projector

$$\left(P^{(2)}\right)_{\mu\nu}^{\ \ \alpha\beta} = \frac{1}{2}\left(T^\alpha_\mu T^\beta_\nu + T^\beta_\mu T^\alpha_\nu\right) - \frac{1}{d-1} T_{\mu\nu} T^{\alpha\beta} \tag{A.57}$$

on the transeverse-traceless part of $\phi_{\mu\nu}$:

$$X^{TT}_{\mu\nu} := \left(P^{(2)}\right)_{\mu\nu}^{\ \ \alpha\beta} \phi_{\alpha\beta}\,. \tag{A.58}$$

Secondly, there is a $(d-1)$-dimensional representation with $J^P = 1^-$ given by the projector

$$\left(P^{(1)}\right)_{\mu\nu}^{\ \ \alpha\beta} := \frac{1}{2}\left(T^\alpha_\mu L^\beta_\nu + T^\beta_\mu T^\alpha_\nu + T^\alpha_\nu L^\beta_\mu + T^\beta_\nu L^\alpha_\mu\right) \tag{A.59}$$

whose projection (traceless, not transverse) of $\phi_{\mu\nu}$ can be written in terms of a transverse vector ξ^T_μ:

$$\left(P^{(1)}\right)_{\mu\nu}^{\ \ \alpha\beta} \phi_{\alpha\beta} := \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu\,. \tag{A.60}$$

Lastly, there are two one-dimensional representations, each with $J^P = 0^+$, given by the projectors

$$\left(P^{(0,\mathrm{ss})}\right)_{\mu\nu}^{\ \ \alpha\beta} := \frac{1}{d-1} T_{\mu\nu} T^{\alpha\beta} \tag{A.61}$$

$$\left(P^{(0,\mathrm{ww})}\right)_{\mu\nu}^{\ \ \alpha\beta} := L_{\mu\nu} L^{\alpha\beta} \tag{A.62}$$

whose actions on $\phi_{\mu\nu}$ can be expressed through two scalar fields, s and w, respectively:

$$\left(P^{(0,\mathrm{ss})}\right)_{\mu\nu}^{\ \ \alpha\beta} \phi_{\alpha\beta} = \frac{1}{d} T_{\mu\nu} s \tag{A.63}$$

where $s = \frac{d-1}{d} T^{\alpha\beta}\phi_{\alpha\beta}$ (transverse, not traceless) and

$$\left(P^{(0,\mathrm{ww})}\right)_{\mu\nu}^{\ \ \alpha\beta} \phi_{\alpha\beta} = \frac{1}{d} L_{\mu\nu} w \tag{A.64}$$

where $w = L^{\alpha\beta}\phi_{\alpha\beta}$ (neither transverse nor traceless).

The subsequent relations equally hold for the actual projectors $\mathbb{P}^{(2)}$, $\mathbb{P}^{(1)}$ *etc.* (with I exchanged for $\mathbb{1}$):

With help of the identity I given by Eq. (A.24) we can formulate the *completeness* relation that the projectors defined above fulfill,

$$P^{(2)} + P^{(1)} + P^{(0,ss)} + P^{(0,ww)} = I\,, \tag{A.65}$$

which is equivalent to

$$\phi^{TT}_{\mu\nu} + \left(\partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu\right) + \frac{1}{d} T_{\mu\nu} s + \frac{1}{d} L_{\mu\nu} w = \phi_{\mu\nu}\,. \tag{A.66}$$

Next we note that it is possible to define projectors that entwist the two one-dimensional representations. These are given by

$$\left(P^{(0,\mathrm{sw})}\right)_{\mu\nu}{}^{\alpha\beta} = \frac{1}{\sqrt{d-1}} T_{\mu\nu} L^{\alpha\beta} \tag{A.67}$$

$$\left(P^{(0,\mathrm{ws})}\right)_{\mu\nu}{}^{\alpha\beta} = \frac{1}{\sqrt{d-1}} L_{\mu\nu} T^{\alpha\beta} \tag{A.68}$$

and can also be included in expressing an *orthogonality* relation for the projectors:

$$\left(P^{(I,\mathrm{ab})}\right)_{\mu\nu}{}^{\rho\sigma} \left(P^{(J,\mathrm{cd})}\right)_{\rho\sigma}{}^{\alpha\beta} = \delta_{IJ}\delta_{bc}\left(P^{(J,\mathrm{ad})}\right)_{\mu\nu}{}^{\alpha\beta}\,, \tag{A.69}$$

where $P^{(2)} \equiv P^{(2,00)}$, $P^{(1)} \equiv P^{(1,00)}$, $I, J = 0, 1, 2$ and $a, b, c, d \in \{0, s, w\}$.

It can be easily verified that a linear combination

$$Y = a_2 P^{(2)} + a_1 P^{(1)} + a_{ss} P^{(0,ss)} + a_{ww} P^{(0,ww)} + a_{sw} P^{(0,sw)} + a_{ws} P^{(0,ws)}$$

where $a_2, a_1, a_s, a_w, a_{sw} \in \mathbb{R}$, is inverted by

$$\begin{aligned} Y^{-1} = \frac{1}{a_2} P^{(2)} + \frac{1}{a_1} P^{(1)} &+ \frac{a_{ww}}{a_{ss}a_{ww} - a_{sw}a_{ws}} P^{(0,ss)} + \frac{a_{ss}}{a_{ss}a_{ww} - a_{sw}a_{ws}} P^{(0,ww)} \\ &- \frac{a_{sw}}{a_{ss}a_{ww} - a_{sw}a_{ws}} P^{(0,sw)} - \frac{a_{ws}}{a_{ss}a_{ww} - a_{sw}a_{ws}} P^{(0,ws)}\,, \end{aligned} \tag{A.70}$$

i.e. $YY^{-1} = Y^{-1}Y = I$.

Finally, we introduce the *projector on the trace part* of a symmetric rank-2 tensor,

$$(P_{\mathrm{tr.}})_{\mu\nu}^{\alpha\beta} := \frac{1}{d} g_{\mu\nu} g^{\alpha\beta} \tag{A.71}$$

and note that this definition also holds on curved manifolds (just as the identity (A.24)). With that, one has the relations

$$P_{\text{tr.}} = \frac{d-1}{d} P_{\text{tr.}}^{(0,ss)} + \frac{1}{d} P_{\text{tr.}}^{(0,ww)} + \frac{\sqrt{d-1}}{d} \left(P_{\text{tr.}}^{(0,sw)} + P_{\text{tr.}}^{(0,ws)} \right) ; \qquad \text{(A.72)}$$

$$I - P_{\text{tr.}} = P_{\text{tr.}}^{(2)} + P_{\text{tr.}}^{(1)} + \frac{1}{d} P_{\text{tr.}}^{(0,ss)} + \frac{d-1}{d} P_{\text{tr.}}^{(0,ww)} - \frac{\sqrt{d-1}}{d} \left(P_{\text{tr.}}^{(0,sw)} + P_{\text{tr.}}^{(0,ws)} \right) .$$

A.2.2. *On curved spactime*

On *curved spacetime* the decomposition of a symmetric rank-2 tensor field into irreducible representations of the Lorentz group is not possible.[6] However, a similar decomposition, the *York decomposition*, is still possible [187]. First, a symmetric tensor field $\phi_{\mu\nu}$ is split into its transverse and longitudinal parts:

$$\phi_{\mu\nu} = \phi_{\mu\nu}^T + \phi_{\mu\nu}^L \qquad \text{(A.73)}$$

where $D^\mu \phi_{\mu\nu}^T = 0$. The longitudinal component can be expressed through a vector ξ_μ,

$$\phi_{\mu\nu}^L = D_\mu \xi_\nu + D_\nu \xi_\mu \, . \qquad \text{(A.74)}$$

Then again, the vector ξ_μ can be decomposed into transverse and longitudinal parts:

$$\xi_\mu = \xi_\mu^T + D_\mu \sigma \, , \qquad \text{(A.75)}$$

where $D^\mu \xi_\mu^T = 0$. The longitudinal part of $\phi_{\mu\nu}$ hence becomes

$$\phi_{\mu\nu}^L = D_\mu \xi_\nu^T + D_\nu \xi_\mu^T + 2 D_\mu D_\nu \sigma \, . \qquad \text{(A.76)}$$

Next, we will further decompose $\phi_{\mu\nu}^T$ into its traceless and trace part. Therefore, we write the trace of $\phi_{\mu\nu}$ as

$$\phi := g^{\mu\nu} \phi_{\mu\nu} = g^{\mu\nu} \phi_{\mu\nu}^T + g^{\mu\nu} \phi_{\mu\nu}^L = g^{\mu\nu} \phi_{\mu\nu}^T + 2 D^2 \sigma \, , \qquad \text{(A.77)}$$

where $D^2 = g^{\mu\nu} D_\mu D_\nu$. Hence,

$$g^{\mu\nu} \phi_{\mu\nu}^T = \phi - 2 D^2 \sigma \quad \Leftrightarrow \quad \phi_{\mu\nu}^T = \frac{1}{d} g_{\mu\nu} \left(\phi - 2 D^2 \sigma \right) \qquad \text{(A.78)}$$

[6] This is obviously only possible for tensors whose transformation behaviour under coordinate transformations is given by elements of the Lorentz group.

and

$$\phi^T_{\mu\nu} = \phi^{TT}_{\mu\nu} + \frac{1}{d} g_{\mu\nu} \left(\phi - 2D^2\sigma\right) \tag{A.79}$$

where $g^{\mu\nu}\phi^{TT}_{\mu\nu} = 0$ (transverse-traceless). All together, the York decomposition reads

$$\phi_{\mu\nu} = \phi^{TT}_{\mu\nu} + D_\mu \xi^T_\nu + D_\nu \xi^T_\mu + 2D_\mu D_\nu \sigma + \frac{1}{d} g_{\mu\nu} \left(\phi - 2D^2\sigma\right) \,. \tag{A.80}$$

Nota bene. Applying general coordinate transformations to $\phi_{\mu\nu}$, it is straightforward to see that the fields ξ^T_μ and σ are pure gauge fields while $\phi^{TT}_{\mu\nu}$ and the combination $\phi - 2D^2\sigma$ are invariant and hence physical degrees of freedom [185]. Futhermore in flat space, i.e. $g_{\mu\nu} = \delta_{\mu\nu}$ or $g_{\mu\nu} = \eta_{\mu\nu}$, one finds that $s = \phi - 2D^2\sigma$ and $w = \phi + (d-1)D^2\sigma$ [102].

Often it is convenient to take a look at the York decomposition from a different angle: Define by

$$\phi^{LT}_{\mu\nu} := D_\mu \xi_\nu + D_\nu \xi_\mu - \frac{2}{d} g_{\mu\nu} D^2 \sigma \tag{A.81}$$

the traceless longitudinal part of $\phi_{\mu\nu}$. This directly leads to the decomposition

$$\phi_{\mu\nu} = \phi^{TT}_{\mu\nu} + \phi^{LT}_{\mu\nu} + \frac{1}{d} g_{\mu\nu} \phi \,.$$

Next, we can further decompose the traceless longitudinal part by decomposing the vector ξ_μ according to Eq. (A.75):

$$\phi^{L^T,T}_{\mu\nu} := D_\mu \xi^T_\nu + D_\nu \xi^T_\mu \tag{A.82}$$

is the tranceless longitudinal-transverse part of $\phi_{\mu\nu}$ (where the latter "transverse" refers to the decomposition of ξ_μ) and

$$\phi^{L^L,T}_{\mu\nu} := 2\left(D_\mu D_\nu \sigma - \frac{1}{d} g_{\mu\nu} D^2 \sigma\right) \tag{A.83}$$

is the traceless longitudinal-longitudinal part of $\phi_{\mu\nu}$. Then the York decomposition can be written as

$$\phi_{\mu\nu} = \phi^{TT}_{\mu\nu} + \phi^{L^T,T}_{\mu\nu} + \phi^{L^L,T}_{\mu\nu} + \frac{1}{d} g_{\mu\nu} \phi \,. \tag{A.84}$$

Note that the degrees of freedom of $\phi^{TT}_{\mu\nu}$, $(d+1)(d-2)/2$, of $\phi^{L^T,T}_{\mu\nu}$, $d-1$, and of $\phi^{L^L,T}_{\mu\nu}$ and $\frac{1}{d} g_{\mu\nu}\phi$, each having a single degree of freedom, sum up to $d(d+1)/2$

– corresponding to the decomposition in flat spacetime into irreducible representations of the Lorentz group.

Also on curved spacetimes, projectors on the respective parts of the York decomposition can be constructed. The definition (A.24) of the identity $I^{\mu\nu}_{\rho\sigma}$ on the space symmetric tensors does not depend on the metric can is therefore trivially valid also on curved spacetimes. Likewise the projector on the trace part

$$(P_{\mathrm{tr.}})^{\mu\nu}_{\rho\sigma} \coloneqq \frac{1}{d} g_{\rho\sigma} g^{\mu\nu} \tag{A.85}$$

fulfills its purpose on curved spacetimes: $(P_{\mathrm{tr.}})^{\mu\nu}_{\rho\sigma}\phi_{\mu\nu} = (1/d)g_{\rho\sigma}\phi$. The remaining projectors cannot be adopted as simply from flat space and their construction is much more involving. Following [62] the projector on the traceless longitudinal part is given by[7]

$$(P_{LT})^{\mu\nu}{}_{\rho\sigma} = (P_1)^{\mu\nu}_{\alpha}(P_2^{-1})^{\alpha}_{\beta}(-D_\gamma)(I - P_{\mathrm{tr.}})^{\gamma\beta}{}_{\rho\sigma}$$

where

$$\begin{aligned} (P_1)^{\mu\nu}_{\alpha} &= D^\mu \delta^\nu_\alpha + D^\nu \delta^\mu_\alpha - \frac{2}{d} g^{\mu\nu} D_\alpha \\ (P_2)^{\alpha}_{\beta} &= \left[-D^2 \delta^\alpha_\beta - R^\alpha{}_\beta - \frac{d-2}{d} D^\alpha D_\beta \right] . \end{aligned}$$

Therewith one has $(P_{LT})^{\mu\nu}{}_{\rho\sigma}\phi^{\rho\sigma} = (\phi^{LT})^{\mu\nu}$. In practical situations the appearance of the inverse projector P_2^{-1} makes it difficult to deal with the projector P_{LT}. In this thesis, the projector P_{LT} is only formally required, however. Furthermore, this definition also entails the projector onto the vector part,

$$(P_\xi)^{\alpha}_{\rho\sigma} \coloneqq (P_2^{-1})^{\alpha}_{\beta}(-D_\gamma)(I - P_{\mathrm{tr.}})^{\gamma\beta}{}_{\rho\sigma}$$

that acts as $(P_\xi)^{\alpha}_{\rho\sigma}\phi^{\rho\sigma} = \xi^\alpha$. Hence, by applying the projectors of a contravariant vector onto its longitudinal and transverse part,

$$\begin{aligned} (P_L)^\mu{}_\nu \xi^\nu &\coloneqq D^\mu (D^2)^{-1} D_\nu \xi^\nu = D^\mu \sigma \\ (P_T)^\mu{}_\nu \xi^\nu &\coloneqq (I_V - P_L)^\mu{}_\nu \xi^\nu = (\xi^T)^\mu , \end{aligned}$$

[7] $(P_{LT})^{\mu\nu}{}_{\rho\sigma}$ acts on contravariant second-rank tensors – we could raise and lower its indices using $I^{\mu\nu}_{\rho\sigma}$ such that it acts on covariant tensors. Here, we refrain from doing so in order to not artificially complicate the matter.

we can construct the projector of a contravariant second-rank tensor fields onto its traceless longitudinal-transverse and traceless longitudinal-longitudinal part. These are

$$(P_{L^T,T})^{\mu\nu}{}_{\rho\sigma} := (D^\mu \delta^\nu_\alpha + D^\nu \delta^\mu_\alpha)\,(P_T)^\alpha{}_\beta (P_\xi)^\beta{}_{\rho\sigma}$$
$$(P_{L^L,T})^{\mu\nu}{}_{\rho\sigma} := \left(D^\mu (P_L)^\nu{}_\alpha + D^\nu (P_L)^\mu{}_\alpha - \frac{2}{d} g^{\mu\nu} D_\beta (P_L)^\beta{}_\alpha\right)(P_\xi)^\alpha{}_{\rho\sigma}$$

and act as $(P_{L^T,T})^{\mu\nu}{}_{\rho\sigma}\phi^{\rho\sigma} = D^\mu(\xi^T)^\nu + D^\nu(\xi^T)^\mu$ and $(P_{L^L,T})^{\mu\nu}{}_{\rho\sigma}\phi^{\rho\sigma} = 2D^\mu D^\nu\sigma - (2/d)g^{\mu\nu}D^2\sigma$. Obviously these projectors fulfill

$$P_{LT} = P_{L^T,T} + P_{L^L,T}\,.$$

Lastly, we define the projector on the traceless transverse part by

$$(P_{TT})^{\mu\nu}{}_{\rho\sigma} = (I - P_{LT} - P_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma}\,,$$

that acts as $(P_{TT})^{\mu\nu}{}_{\rho\sigma}\phi^{\rho\sigma} = (\phi^{TT})^{\mu\nu}$, such that we obtain the overall identity

$$I = P_{TT} + P_{L^T,T} + P_{L^L,T} + P_{\mathrm{tr.}}\,. \tag{A.86}$$

Furthermore, it should be said that these projectors fulfilll the defining properties of a set of projectors: $P_I P_J = 0$ for $I \neq J$ and $P_I^2 = P_I$.

A.3. Lorentzian vs. Euclidean action functionals

Here, we wish to briefly illustrate how Euclidean action functionals can be derived from Lorentzian action functionals by a formal analytic continuation ("Wick rotation"). All the omitted subtleties can be found in the standard textbooks [102, 118, 188].

Consider the action functionals of a scalar field A on a four-dimensional curved spacetime with Lorentzian signature:

$$S^{\mathrm{Lor}} = -\frac{1}{2}\int \mathrm{d}^4\sqrt{-g}\,\left(g^{\mu\nu}\partial_\mu A\,\partial_\nu A + (m^2 + \xi R)A^2\right)$$

and the four-dimensional Einstein-Hilbert action

$$S^{\mathrm{Lor}}_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4 x\sqrt{-g}(R - 2\Lambda)\,.$$

Next, assume g to be a static Lorentzian metric of the form

$$g^{\mathrm{Lor}}_{\mu\nu}(x) = \begin{pmatrix} g_{00}(x) & 0 \\ 0 & g_{ij}(x) \end{pmatrix}$$

of signature $- + + \ldots$, i.e. $g_{00} < 0$. Here, static means that with $x^\mu \equiv (t, x^i)$, the components g_{00} and g_{ij} depend on x^i only. In order to map this metric to the Euclidean, we "Wick-rotate" the time t to the imaginary time t_{E} by letting $t = -\mathrm{i}t_{\mathrm{E}}$ and considering t_{E} real thereafter. The formal rules for Wick-rotating the action functionals are $\int \mathrm{d}t = -\mathrm{i}\int \mathrm{d}t_{\mathrm{E}}$, $g^{\mu\nu}_{\mathrm{Lor}}\partial_\mu A\,\partial_\nu A = g^{\mu\nu}_{\mathrm{E}}\partial_\mu A\,\partial_\nu A$ and $R_{\mathrm{Lor}}(g) = R\,(g_{\mathrm{E}})$ where

$$g^{\mathrm{E}}_{\mu\nu} = \begin{pmatrix} -g_{00} & 0 \\ 0 & g_{ij} \end{pmatrix} .$$

Therewith we can determine the Wick-rotated action functionals,

$$\begin{aligned} \mathrm{i}S &= (-\mathrm{i})(-\mathrm{i})\frac{1}{2}\int \mathrm{d}t_{\mathrm{E}} \int \mathrm{d}^3x\sqrt{g_{\mathrm{E}}}\left(g^{\mu\nu}_{\mathrm{E}}\partial_\mu A\,\partial_\nu A + (m^2+\xi R)A^2\right) \\ &= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{g_{\mathrm{E}}}\left(g^{\mu\nu}_{\mathrm{E}}\partial_\mu A\,\partial_\nu A + (m^2+\xi R)A^2\right) \\ &=: -S_{\mathrm{E}}\,, \end{aligned}$$

and

$$\begin{aligned} \mathrm{i}S_{\mathrm{EH}} &= \frac{1}{16\pi G}\int \mathrm{d}t_{\mathrm{E}}\int \mathrm{d}^3x\sqrt{g_{\mathrm{E}}}(R-2\Lambda) \\ &=: -\left(S_{\mathrm{EH}}\right)_{\mathrm{E}} . \end{aligned}$$

APPENDIX B

Metric variations of geometric objects

This appendix is nothing but a list of variations of geometric objects built from the metric $g_{\mu\nu}$. All these variations can be obtained in tedious, yet simple calculations, applying basic algebra and, here and there, commutators of covariant derivatives: $[D_\mu, D_\nu]\phi_\rho = R_{\mu\nu\rho}{}^{\lambda}\phi_\rho$ and $[D_\mu, D_\nu]\phi_{\rho\sigma} = R_{\mu\nu\rho}{}^{\lambda}\phi_{\lambda\sigma} + R_{\mu\nu\sigma}{}^{\lambda}\phi_{\rho\lambda}$. Indices are raised and lowered using $g_{\mu\nu}$.

- $\delta g_{\mu\nu} =: h_{\mu\nu}$
- $\delta g^{\mu\nu} = -h^{\mu\nu}$
- $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}h_{\mu\nu}$
- $\delta^2\sqrt{g} = \sqrt{g}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu}\right)$
- $\delta\Gamma_{\beta\gamma}{}^{\epsilon} = \frac{1}{2}g^{\epsilon\sigma}\left[D_\beta h_{\gamma\sigma} + D_\gamma h_{\beta\sigma} - D_\sigma h_{\beta\gamma}\right]$
- $\delta R_{\alpha\beta\gamma}{}^{\epsilon} = -D_\alpha\delta\Gamma_{\beta\gamma}{}^{\epsilon} + D_\beta\delta\Gamma_{\alpha\gamma}{}^{\epsilon}$
  $= \frac{1}{2}\big[- R_{\alpha\beta\gamma}{}^{\mu}h_\mu{}^{\epsilon} + R_{\alpha\beta\mu}{}^{\epsilon}h_\gamma{}^{\mu} - D_\alpha D_\gamma h_\beta{}^{\epsilon}$
  $+ D_\beta D_\gamma h_\alpha{}^{\epsilon} + D_\alpha D^\epsilon h_{\beta\gamma} - D_\beta D^\epsilon h_{\alpha\gamma}\big]$
- $\delta R_{\mu\nu} = \frac{1}{2}\big[R_{\mu\sigma}h_\nu{}^{\sigma} + R_\nu{}^{\sigma}h_{\sigma\mu} + 2R_{\alpha\mu\nu\gamma}h^{\alpha\gamma}$
  $+ D_\nu D_\alpha h^\alpha{}_\mu - D_\mu D_\nu h_\alpha{}^{\alpha} - D_\alpha D^\alpha h_{\mu\nu} + D_\mu D^\alpha h_{\alpha\nu}\big]$
- $\delta R = -R^{\mu\nu}h_{\mu\nu} + D_\beta(D_\alpha h^{\alpha\beta} - D^\beta h^\alpha_\alpha)$
- $g^{\alpha\gamma}\,\delta^2 R_{\alpha\beta\gamma\delta} = D_\lambda h_{\alpha\delta}D^\lambda h^\alpha_\beta - D_\alpha h^{\lambda\alpha}D_\beta h_{\lambda\delta} - D_\alpha h^{\lambda\alpha}D_\delta h_{\lambda\beta}$
  $+ D_\alpha h^{\lambda\alpha}D_\lambda h_{\beta\delta} - D^\alpha h^\lambda_\beta D_\lambda h_{\alpha\delta} + \frac{1}{2}\big[- D_\lambda h D^\lambda h_{\beta\delta}$
  $+ D_\beta h^{\alpha\lambda}D_\delta h_{\alpha\lambda} + D_\beta h^\lambda_\delta D_\lambda h + D_\delta h^\lambda_\beta D_\lambda h\big]$
- $\delta^2 R = R_{\beta\mu}h^{\beta\gamma}h^\mu_\gamma - R_{\alpha\beta\gamma\mu}h^{\beta\gamma}h^{\alpha\mu}$
  $- 3h^{\beta\gamma}D_\gamma D_\alpha h^\alpha_\beta + 2h^{\beta\gamma}D_\beta D_\gamma h + 2h^{\beta\gamma}D^2 h_{\beta\gamma}$
  $- h^{\beta\delta}D_\alpha D_\beta h^\alpha_\delta - D_\alpha h^\lambda_\beta D_\lambda h^{\alpha\beta} + \frac{3}{2}D^\lambda h_{\alpha\beta}D_\lambda h^{\alpha\beta}$
  $- 2D^\alpha h^\lambda_\alpha D_\beta h^\beta_\lambda + 2D^\alpha h^\lambda_\alpha D_\lambda h - \frac{1}{2}D^\lambda h D_\lambda h$

APPENDIX C

Functional calculus

In this appendix, we review the basic conventions and definitions of functional calculus as well as the properties of the functional derivative.

C.1. Functionals and functional derivatives

Consider a function space $\mathfrak{F}$, e.g. $\mathfrak{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$. In this case, an element $f \in \mathfrak{F}$ is function $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$. A *functional* F then is a function $\mathfrak{F} \to \mathbb{R}$, $f \mapsto F[f]$. Thus a functional is a function whose domain is a function space. A few examples ($n = 1$):

- $F[f] = \int_{x_1}^{x_2} \mathrm{d}x\, f(x)$
- $F_g[f] = \int_{\infty}^{\infty} \mathrm{d}x\, g(f(x), f'(x))$ with $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ fixed
- $F_x[f] = f(x)$ with $x \in \mathbb{R}$ fixed ("evaluation functional") .

To prepare the definition of the functional derivative, we define the delta function

$$\delta_{x_0} : \mathbb{R}^n \to \mathbb{R} \quad , \; x \mapsto \delta_{x_0}(x) := \delta(x - x_0) \tag{C.1}$$

that will play the role of a basis vector in the "x_0-th direction". Therewith the functional derivative of a functional $F : C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ with respect to $f(x_0)$ is defined as

$$\frac{\delta F}{\delta f(x_0)}[f] := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F\left[f + \varepsilon \delta_{x_0}\right] - F[f] \right) . \tag{C.2}$$

This definition is in full analogy with the partial derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$\frac{\partial f}{\partial x_i}(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(f(x + \varepsilon e_i) - f(x) \right) \tag{C.3}$$

where $e_i = (0, 0, \ldots, 1, 0, 0, \ldots, 0)$ (with 1 at the i-th entry). Just like the partial derivative $\partial f / \partial x_i$ again is a function $\mathbb{R}^n \to \mathbb{R}$, the functional derivative

$\delta F/\delta f(x_0)$ again is a functional $C^\infty(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}$. Table 1 depicts the details of this analogy.

TABLE 1. Corresponding objects in the analogy between function and functional.

Function f	Functional F
argument $x \in \mathbb{R}^n$	argument $f \in \mathfrak{F}$
$x = \{x_i \mid i = 1, \ldots, n\}$	$f = \{f(x) \mid x \in \mathbb{R}^n\}$
$i = \{1, \ldots, n\} = N_n$	$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
$x_\bullet : N_n \to \mathbb{R}$	$f : \mathbb{R}^n \to \mathbb{R}$
e_i	δ_{x_0}
δ_{ij}	$\delta(x - x_0)$

An important example is the functional derivative of the "evaluation functional" $F[f] := f(x)$ with $x \in \mathbb{R}^n$ fixed – in the distributional sense this is the delta distribution. We have

$$\frac{\delta F}{\delta f(x_0)}[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(f(x) + \varepsilon \delta_{x_0}(x) - f(x) \right) = \delta_{x_0}(x)\,.$$

Analogous to $\delta x_i/\delta x_j = \delta_{ij}$ we write henceforth $\delta f(x)/\delta f(x_0) = \delta(x - x_0)$.

Another, alternative, yet more general, definition of the functional derivative is to consider $\delta F[f]/\delta f(x)$ to be *defined* by

$$\begin{aligned} \int \mathrm{d}^d x \frac{\delta F[f]}{\delta f(x)} \phi(x) :=& \lim_{\varepsilon \to 0} \frac{F[f + \varepsilon\phi] - F[f]}{\varepsilon} \\ =& \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[f + \varepsilon\phi] \bigg|_{\varepsilon = 0} \\ =:& \delta F(f, \phi)\,. \end{aligned}$$

The more general functional derivative $\delta F(f, \phi)$ can be regarded as a "directional derivative" in the direction of ϕ. It is related to our previous definition by

$$\delta F(f, \delta_{x_0}) = \int \mathrm{d}^d x \frac{\delta F[f]}{\delta f(x)} \delta(x - x_0) = \frac{\delta F[f]}{\delta f(x_0)}\,.$$

Futhermore, the functional derivative can be generalized to functions $f : \mathbb{R}^n \to \mathbb{R}^k$ by componentwise differentiation,

$$\frac{\delta f_\alpha(x)}{\delta f_\beta(x_0)} = \delta_{\alpha\beta}\delta(x - x_0) \tag{C.4}$$

where $\alpha, \beta \in \{1, 2, \ldots, k\}$. Therewith, also the functional Taylor series (more precisely: Volterra series) goes along the lines of the analogy with ordinary calculus:

$$\begin{aligned} F[f+h] = F[f] &+ \sum_\alpha \int \mathrm{d}^n x \frac{\delta F[f]}{\delta f_\alpha(x)} h_\alpha(x) \\ &+ \frac{1}{2!}\sum_{\alpha,\beta} \int \mathrm{d}^n x_1 \int \mathrm{d}^n x_2 \frac{\delta^2 F[f]}{\delta f_\alpha(x_1)\delta f_\beta(x_2)} h_\beta(x_1) h_\alpha(x_2) + \cdots \end{aligned} \tag{C.5}$$

C.2. Properties of the functional derivative

To state the properties of the functional derivative, let $F, F_1, F_2 : \mathfrak{F} \to \mathbb{R}$ be functionals, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $G : \mathfrak{F} \to \mathfrak{F}$ a function-valued functional, d.h. $G[f] \in \mathfrak{F}$ and $(F \circ G)[f] \in \mathbb{R}$. Then the following properties hold:

$$\frac{\delta}{\delta f(x)}\left(\alpha_1 F_1[f] + \alpha_2 F_2[f]\right) = \alpha_1 \frac{\delta F_1}{\delta f(x)}[f] + \alpha_2 \frac{\delta F_2}{\delta f(x)}[f] \quad \text{(linearity)} \tag{C.6}$$

$$\frac{\delta}{\delta f(x)}(F_1 F_2)[f] = \frac{\delta F_1}{\delta f(x)}[f] F_2[f] + F_1[f] \frac{\delta F_2}{\delta f(x)}[f] \quad \text{(product rule)} \tag{C.7}$$

$$\frac{\delta(F \circ G)}{\delta f(x)}[f] = \int \mathrm{d}^n y \, \frac{\delta F[G[f]]}{\delta(G[f])(y)} \frac{\delta(G[f])}{\delta f(x)}(y) \quad \text{(chain rule)} \tag{C.8}$$

$$\frac{\delta}{\delta f(x)} \quad \text{commutes with} \quad \int \mathrm{d}^n y \tag{C.9}$$

$$\frac{\delta}{\delta f(x)} \frac{\partial}{\partial y_i} f(y) = \frac{\partial}{\partial y_i} \frac{\delta f(y)}{\delta f(x)} = \frac{\partial}{\partial y_i} \delta(y - x) \tag{C.10}$$

Proof. All these properties can proven in the same way as their counterparts in ordinary calculus would be proven. As an example, let us prove the chain rule (C.8): Define $v_x \in \mathfrak{F}$ by

$$v_x \coloneqq \frac{1}{\varepsilon}\left(G[f+\varepsilon\delta_x] - G[f]\right) - \frac{\delta G[f]}{\delta f(x)} \xrightarrow{\varepsilon\to 0} 0$$

and $w : \mathfrak{F} \times \mathfrak{F} \to \mathbb{R}$ by

$$w(\rho,\phi) \coloneqq \frac{1}{\varepsilon}\left(F[\rho+\varepsilon\phi] - F[\rho]\right) - \delta F(\rho,\phi) \xrightarrow{\varepsilon\to 0} 0\,.$$

Therewith, we can write

$$G[f+\varepsilon\delta_x] = G[f] + \varepsilon\left(\frac{\delta G[f]}{\delta f(x)} + v_x\right)$$

and

$$F[\rho+\varepsilon\phi] = F[\rho] + \varepsilon\left(\delta F(\rho,\phi) + w(\rho,\phi)\right)\,.$$

From these two equations we can deduce that

$$\begin{aligned} F\left[G[f+\varepsilon\delta_x]\right] = F\left[G[f]\right] + \varepsilon\Bigg\{ & \delta F\left(G[f], \frac{\delta G[f]}{\delta f(x)} + v_x\right) \\ & + w\left(G[f], \frac{\delta G[f]}{\delta f(x)} + v_x\right)\Bigg\}\,, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \frac{F\left[G[f+\varepsilon\delta_x]\right] - F\left[G[f]\right]}{\varepsilon} = & \ \delta F\left(G[f], \frac{\delta G[f]}{\delta f(x)} + v_x\right) \\ & + w\left(G[f], \frac{\delta G[f]}{\delta f(x)} + v_x\right)\,. \end{aligned}$$

In the limit $\varepsilon \to 0$ we find

$$\begin{aligned} \frac{\delta(F\circ G)}{\delta f(x)}[f] &= \delta F\left(G[f], \frac{\delta G[f]}{\delta f(x)}\right) \\ &= \int \mathrm{d}^n y\, \frac{\delta F[G[f]]}{\delta(G[f])(y)}\frac{\delta(G[f])}{\delta f(x)}(y) \end{aligned}$$

which is what we wanted to show.

Lastly, we point out some special cases of the chain rule (C.8). Consider $G[f] := g \circ f$ with $g : \mathbb{R} \to \mathbb{R}$ fixed and $F[f] := f(x_0)$ with $x_0 \in \mathbb{R}^n$ fixed. Then we find

$$\frac{\delta(F \circ G)[f]}{\delta f(x)} = g'(f(x_0))\,\delta(x_0 - x) \tag{C.11}$$

where the prime denotes the ordinary derivative. Therewith we also find that, with G as above and F an arbitrary functional,

$$\frac{\delta(F \circ G)[f]}{\delta f(x)} = \frac{\delta F[g \circ f]}{\delta(g \circ f)(x)}\,g'(f(x)) \tag{C.12}$$

and

$$\frac{\delta(g \circ F)[f]}{\delta f(x)} = \frac{\delta F[f]}{\delta f(x)}\,g'(F[f])\,. \tag{C.13}$$

APPENDIX D

The path integral measure

Path integrals play a key role in this thesis. The catchiest part in their construction, aside from their regularization, is the construction of "correct" path integral measures. The following appendix is a detailed exposition on this issue; we will construct all path integral measures used throughout this thesis and apply these in evaluating Gaussian integrals. Lastly, a technical note: All Gaussian integrals evaluated here can be w.l.o.g. generalized by replacing $f(\hat{x}) \mapsto \hat{f}$.

D.1. Constructing an invariant measure

Given two real scalar fields f and ϕ on flat space, consider the simple path integral

$$\int \mathcal{D}\phi \, \mathrm{e}^{-\int \mathrm{d}^d x \, f(x)\phi^2(x)} \tag{D.1}$$

where the measure is given by $\mathcal{D}\phi \coloneqq \mathcal{N} \prod_x \mathrm{d}\phi(x)$. It is understood that the normalization constant $\mathcal{N}$ is always chosen such that we may ignore irrelevant multplicative constants. This measure is still of an ill-defined nature and must be regularized, e.g. by Fourier transforming the field ϕ and employing a cutoff in momentum space. At the formal level, the integral is easily evaluated [31, p. 188],

$$\begin{aligned}
\int \mathcal{D}\phi \, \mathrm{e}^{-\int \mathrm{d}^d x \, f(x)\phi^2(x)} &= \left(\prod_x \int \mathrm{d}^d \phi(x) \right) \mathrm{e}^{-\sum_x f(x)\phi^2(x)} \\
&= \prod_x \int \mathrm{d}^d \phi(x) \mathrm{e}^{-f(x)\phi^2(x)} \\
&= \prod_x \sqrt{\frac{\pi}{f(x)}} = \mathrm{Det}\left(f(\hat{x}) \right)^{-1/2} .
\end{aligned} \tag{D.2}$$

With f and ϕ defined on a curved manifold (M, g) we wish to evaluate path integrals such as

$$\int \mathcal{D}\phi \, \mathrm{e}^{-\int \mathrm{d}^d x \sqrt{g(x)} f(x) \phi^2(x)} = \mathrm{Det}\left(\sqrt{g(\hat{x})} f(\hat{x})\right)^{-1/2} . \tag{D.3}$$

Due to the transformation behaviour of the scalar density $\sqrt{g(x)}$ it is obvious that the path integral (D.3) is neither diffeomorphism- nor BRST-invariant. However, by defining the path integral with an appropriate measure it is possible to restore its BRST invariance. Eq. (D.3) suggests to introduce the scalar density

$$\tilde{\phi} := g^{1/4}\phi \tag{D.4}$$

and to define the path integral measure $\mathcal{D}\tilde{\phi}$ as $\mathcal{D}_g\phi := \mathcal{D}\left[g^{1/4}\phi\right]$. Then

$$\begin{aligned} \int \mathcal{D}\tilde{\phi} \, \mathrm{e}^{-\int \mathrm{d}^d x \sqrt{g(x)} f(x) \phi^2(x)} &= \int \mathcal{D}\big[g^{1/4}\phi\big] \; \mathrm{e}^{-\int \mathrm{d}^d x f(x) \left(g^{1/4}(x)\phi(x)\right)^2} \\ &= \mathrm{Det}\left(f(\hat{x})\right)^{-1/2} . \end{aligned} \tag{D.5}$$

The BRST-invariance of a genergal path integral $\int \mathcal{D}\tilde{\phi} \, F[\tilde{\phi}]$ has been proven rigorously by showing that the Jacobian of the BRST transformation associated to diffeomorphisms is equal to 1 [189]. From the invariant scalar measure $\mathcal{D}\tilde{\phi}$ it is straightforward to derive the corresponding invariant measures for path integrals over tensors of any rank. The most frequently needed measures are given by [189, p. 260]

$$\mathcal{D}_g\phi := \mathcal{D}\big[g^{1/4}\phi\big] \quad \text{for a scalar field } \phi \tag{D.6}$$

$$\mathcal{D}_g\phi^\mu := \mathcal{D}\Big[g^{\frac{d+2}{4d}}\phi^\mu\Big] \quad \text{for a contravariant vector field } \phi^\mu \tag{D.7}$$

$$\mathcal{D}_g\phi_\mu := \mathcal{D}\Big[g^{\frac{d-2}{4d}}\phi_\mu\Big] \quad \text{for a covariant vector field } \phi_\mu \tag{D.8}$$

$$\mathcal{D}_g\phi^{\mu\nu} := \mathcal{D}\Big[g^{\frac{d+4}{4d}}\phi^{\mu\nu}\Big] \quad \text{for a contravariant tensor field } \phi^{\mu\nu} \tag{D.9}$$

$$\mathcal{D}_g\phi_{\mu\nu} := \mathcal{D}\Big[g^{\frac{d-4}{4d}}\phi_{\mu\nu}\Big] \quad \text{for a covariant tensor field } \phi_{\mu\nu} \, , \tag{D.10}$$

where $\mathcal{D}\phi_\mu := \prod_{x,\mu} \mathrm{d}\phi_\mu(x)$, $\mathcal{D}\phi_{\mu\nu} := \prod_{x,\mu,\nu} \mathrm{d}\phi_{\mu\nu}(x)$ etc.; and therewith $\mathcal{D}_g\phi = \prod_x \left(g^{1/4}(x)\mathrm{d}\phi(x)\right)$ et cetera.[1]

Assuming (D.6) to be invariant, the invariance of the remaining measures can be easily proven: With the help of vielbeins we will transform the tensors under consideration into tensors "without" spacetime indices. Then we can use (D.6) to obtain the invariant measures. The vielbeins are defined by $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \delta_{ab}$ from which one obtains $\sqrt{g} = \det e_\mu{}^a$ as well as $\sqrt{g}^{-1} = \det e^\mu{}_a$ from $g^{\mu\nu} = e^\mu{}_a e^\nu{}_b \delta^{ab}$. Next, we will derive (D.8) and (D.10) explicitly – the remaining measures follow analogously.

The covariant vector field "without" spacetime indices is given by

$$\tilde{\phi}_a := g^{1/4} e^\mu{}_a \phi_\mu \,. \tag{D.11}$$

Therewith, it follows that

$$\begin{aligned}
\mathcal{D}\tilde{\phi}_a &= \mathcal{D}\big[g^{1/4} e^\mu{}_a \phi_\mu\big] \\
&= \prod_x \det\left(g^{1/4}(x) e^\mu{}_a(x)\right) \mathcal{D}\phi_\mu \\
&= \prod_x g^{\frac{d-2}{4}}(x) \mathcal{D}\phi_\mu \\
&= \mathcal{D}\Big[g^{\frac{d-2}{4d}} \phi_\mu\Big]
\end{aligned}$$

is invariant. The covariant tensor field "without" spacetime indices is, in turn, given by

$$\tilde{\phi}_{ab} := g^{1/4} e^\mu{}_a e^\nu{}_b \phi_{\mu\nu} \,, \tag{D.12}$$

from which follows in the same way that

$$\begin{aligned}
\mathcal{D}\tilde{\phi}_{ab} &= \mathcal{D}\big[g^{1/4} e^\mu{}_a e^\nu{}_b \phi_{\mu\nu}\big] \\
&= \prod_x \det\left(g^{1/4}(x) e^\mu{}_a(x) \otimes e^\nu{}_b(x)\right) \mathcal{D}\phi_{\mu\nu} \\
&= \prod_x g^{d^2/4}(x) \left(\det e^\mu{}_a(x)\right)^{2d} \mathcal{D}\phi_{\mu\nu} \\
&= \prod_x g^{\frac{d^2}{4}-d}(x) \mathcal{D}\phi_{\mu\nu} \\
&= \mathcal{D}\Big[g^{\frac{d-4}{4d}} \phi_{\mu\nu}\Big]
\end{aligned}$$

---

[1]For symmetric tensor fields, such as the metric field, we define the measure as $\int \mathcal{D}\phi_{\mu\nu} := \prod_{x,\mu\leq\nu} \int \mathrm{d}\phi_{\mu\nu}(x)$ to avoid double counting.

is invariant.[2] Here, $\phi_{\mu\nu}$ has been vectorized and thus is thought to possess a single index running from 0 to $d^2 - 1$. Furthermore, "$\otimes$" in the second line denotes the *Kronecker product*[3] such that $e^\mu{}_a(x) \otimes e^\nu{}_b(x)$ is regarded as a $d^2 \times d^2$-dimensional matrix; also the determinant in the second line is defined with respect to this concept. Note that in $d = 4$ the measure $\mathcal{D}\phi_{\mu\nu}$ already is the correct invariant measure for functional integration over covariant rank-2 tensors.

Nota bene. Throughout the proof we had considered ϕ, ϕ_μ, ϕ^μ, $\phi_{\mu\nu}$ and $\phi^{\mu\nu}$ to be *bosonic variables*. It is straightforward to show that for *fermionic (Grassmann) variables* the results are identical; however, note that if ϕ is Grassmann one must use

$$\mathcal{D}\tilde{\phi} = \mathcal{D}\big[g^k\phi\big] = \mathrm{Det}(g^k)^{-1}\mathcal{D}\phi\,.$$

This becomes obvious when we remind us of the definition of integration over Grassmann variables, $\int \mathrm{d}\eta\,\eta := 1$. Rescaling η with a constant c one finds that $\mathrm{d}(c\eta) = d\eta/c$.

At last, we point out that in order to define a diffeomorphism-invariant path integral, it is *not* necessary to choose preferred coordinates in function space [191]. We also may define a scalar density of arbitrary weight w by $\tilde{\phi} = g^{-w/2}\phi$ to obtain for the exponent in (D.5)

$$\int \mathrm{d}^d x\,\sqrt{g(x)}f(x)\phi^2(x) = \int \mathrm{d}^d x\, f(x)\left(g^{\frac{w}{2}+\frac{1}{4}}(x)\tilde{\phi}(x)\right)^2 .$$

Consequently, defining the measure accordingly, the path integral

$$\begin{aligned}\int \mathcal{D}\Big[g^{\frac{w}{2}+\frac{1}{4}}\tilde{\phi}\Big]\,\mathrm{e}^{-\int \mathrm{d}^d x\,\sqrt{g(x)}f(x)\phi^2(x)} &= \int \mathcal{D}\Big[g^{\frac{w}{2}+\frac{1}{4}}\tilde{\phi}\Big]\,\mathrm{e}^{-\int \mathrm{d}^d x\, f(x)\left(g^{\frac{w}{2}+\frac{1}{4}}(x)\tilde{\phi}(x)\right)^2} \\ &= \#\,\mathrm{Det}\left(f(\hat{x})\right)^{-1/2}\end{aligned}$$

is invariant, too. Earlier, we made the easiest choice of setting $w = -1/2$. This choice of coordinates in function space is analogous to choosing an orthonormal frame in a vector space. However, note that when substituting $\tilde{\phi} = g^{1/4}\phi$ we arrive back at (D.5).

[2]When considering the metric field $g_{\mu\nu}$, the very same result can be derived by analysing the metric in field space (the "supermetric" or "DeWitt-metric") [118, 190].

[3]The Kronecker $A \otimes B$ product of two $d \times d$-dimensional matrices A and B is a $d^2 \times d^2$-dimensional matrix whose determinant fulfills $\det(A \otimes B) = (\det A)^d(\det B)^d$.

D.2. The Gaussian integral over scalar fields

D.2.1. *The action functional*

Consider a real scalar field A on a Riemannian manifold (M, g) that we, for ease of the subsequent analysis, assume to be compact. We define the action functional for A by

$$S[A; g] \coloneqq \frac{1}{2}\left(A, \mathscr{K}A\right)_g = \frac{1}{2}\int \mathrm{d}^d x\, \sqrt{g(x)} A(x) \left(\mathscr{K}A\right)(x) \tag{D.13}$$

where $\mathscr{K}$ is an operator defined to have the properties outlined in subsection A.1.2. In terms of the scalar density $B \coloneqq g^{1/4}A$ the action reads

$$\begin{aligned} S[A(B); g] &= S\left[g^{-1/4}B; g\right] \\ &= \frac{1}{2}\int \mathrm{d}^d x\, g^{1/2} g^{-1/4} B \mathscr{K} g^{-1/4} B \\ &= \frac{1}{2}\int \mathrm{d}^d x\, B g^{1/4} \mathscr{K} g^{-1/4} B \\ &= \frac{1}{2}\left(B, \mathscr{L}B\right)_1 \end{aligned} \tag{D.14}$$

where $\mathscr{L} \coloneqq g^{1/4}(\hat{x}) \mathscr{K} g^{-1/4}(\hat{x})$.

We can expand A in the eigenbasis $\{\chi_{n,m}\}$ of $\mathscr{K}$,

$$A(x) = \sum_{n,m} a_{n,m} \chi_{n,m}(x)\,. \tag{D.15}$$

Multiplying this equation with $g^{1/4}(x)$ we obtain the expansion of B in the eigenbasis $\{\psi_{n,m}\}$ of $\mathscr{L}$,

$$B(x) = \sum_{n,m} a_{n,m} \psi_{n,m}(x)\,. \tag{D.16}$$

Note that both expansions possess the same coefficients. Inserting these expansions into S we find

$$\begin{aligned} S[A;g] &= \frac{1}{2}\left(A, \mathscr{K} A\right)_g \\ &= \frac{1}{2}\int \mathrm{d}^d x \sqrt{g} \sum_{n,m} a_{n,m}\chi_{n,m}\mathscr{K}\sum_{k,l} a_{k,l}\chi_{k,l} \\ &= \frac{1}{2}\sum_{n,m,k,l} a_{n,m}a_{k,l}\mathscr{F}_k \int \mathrm{d}^d x\sqrt{g}\,\chi_{n,m}\chi_{k,l} \\ &= \frac{1}{2}\sum_{n,m}\mathscr{F}_n a_{n,m}^2 \end{aligned} \tag{D.17}$$

where we have used (A.29). Likewise, using (A.33), we find

$$\begin{aligned} S[A(B);g] &= \frac{1}{2}\left(B, \mathscr{L} B\right)_1 \\ &= \frac{1}{2}\int \mathrm{d}^d x \sum_{n,m} a_{n,m}\psi_{n,m}\mathscr{L}\sum_{k,l} a_{k,l}\psi_{k,l} \\ &= \frac{1}{2}\sum_{n,m,k,l} a_{n,m}a_{k,l}\mathscr{F}_k \int \mathrm{d}^d x\,\psi_{n,m}\psi_{k,l} \\ &= \frac{1}{2}\sum_{n,m}\mathscr{F}_n a_{n,m}^2\,, \end{aligned} \tag{D.18}$$

which confirms that $(B, \mathscr{L}B)_1 = (A, \mathscr{K}A)_g$.

D.2.2. *The transformation formula*

In order to evaluate certain path integrals, it is often useful to expand the scalar field A as in (D.15),

$$A(x) = \sum_n \bar{a}_n\bar{\chi}_n(x)\,, \tag{D.19}$$

where the summation over two indices has been absorbed into a single sum. In section D.1 we have learned that instead of $\int\mathcal{D}A$ we must choose $\int\mathcal{D}B = \int\mathcal{D}\left[g^{1/4}A\right] = \int\mathcal{D}_g A$ as the measure to construct an diffeomorphism-invariant path integral. The scalar density B can be expanded as in (D.16),

$$B(x) = \sum_n \bar{a}_n\bar{\psi}_n(x)\,, \tag{D.20}$$

with $\bar{\psi}_n = g^{1/4}\bar{\chi}_n$. After performing this expansion, one wishes to change the integration variables from $B(x)$ to a_n. To do so, we make use of two fundamental rules. Firstly,

$$\int \mathcal{D}_g A\, F[A] = \int \mathcal{D}A\ \mathrm{Det}\left(\sqrt{g(\hat{x})}\right)^{-1/2} F[A]\,, \tag{D.21}$$

whose proof is trivial on the prevailing level, given that $\mathcal{D}_g A = \prod_x g^{-1/4}(x)\mathrm{d}A(x)$. Secondly, expanding any scalar field C in a basis $\{u_n\}$, $C(x) = \sum_n \alpha_n u_n(x)$, one has

$$\int \mathcal{D}C\, G[C] = \left(\prod_{n'} \int \mathrm{d}c_{\alpha'}\right)\ G\left[\sum_n \alpha_n u_n\right]\,. \tag{D.22}$$

This relation can be easily proven when considering the transition from the integration variable $C(x)$ to α_n is given by $C(x) \equiv C_x = \sum_n J_{xn}\alpha_n$, with $J_{xn} = u_n(x)$ the Jacobian. From the properties of the basis,

$$\sum_n J^*_{xn} J_{yn} = \sum_n u^*_n(x) u_n(y) = \delta(x-y)$$
$$\sum_x J^*_{xn} J_{xm} = \int \mathrm{d}x u^*_n(x) u_m(x) = \delta_{nm}\,,$$

it follows that $J = J^\dagger$ and hence $\det J = 1$. Therewith, it is obvious that $\mathcal{D}C = \prod_x \mathrm{d}C(x) = \prod_n \mathrm{d}\alpha_n$.

Applying these rules to the expansions of the scalar fields A and B amounts to the rules

$$\int \mathcal{D}A\, F[A] = \prod_{n,m} \int \mathrm{d}a_{n,m}\, \mathrm{Det}\, \sqrt{g(\hat{x})}\, F\left[A[a]\right] \tag{D.23}$$

$$\int \mathcal{D}B\, F[B] = \prod_{n,m} \int \mathrm{d}a_{n,m}\, F\left[B[a]\right]\,. \tag{D.24}$$

D.2.3. *The Gaussian integral in configuration space*

In quantum field theory, the by far most important functional integral to evaluate is the Gaussian one. Therefore, let us evaluate the integrals

$$\int \mathcal{D}A\, \mathrm{e}^{-S[A;g]} \quad \text{and} \quad \int \mathcal{D}_g A\, \mathrm{e}^{-S[A;g]} = \int \mathcal{D}B\, \mathrm{e}^{-S[A(B);g]}$$

with S defined as in (D.13) and (D.14). After expanding A and B as in (D.15) and (D.16), we found in equations (D.17) and (D.18) that

$$S[A;g] = \frac{1}{2}\sum_{n,m} \mathscr{F}_n a_{n,m}^2 = S[A(B);g]\,.$$

With (D.24) we can finally evaluate:

$$\begin{aligned}\int \mathcal{D}B\, \mathrm{e}^{-S[A(B);g]} &= \prod_{n,m}\int \mathrm{d}a_{n,m}\, \mathrm{e}^{-\frac{1}{2}\mathscr{F}_n a_{n,m}^2} \\ &= \prod_{n,m}\sqrt{2\pi}\mathscr{F}_n^{-1/2} \\ &= \mathrm{Det}\,\mathscr{K}^{-1/2} \\ &= \mathrm{Det}\,\mathscr{L}^{-1/2}\,.\end{aligned} \tag{D.25}$$

Likewise, we obtain for the other integral

$$\begin{aligned}\int \mathcal{D}A\, \mathrm{e}^{-S[A;g]} &= \mathrm{Det}\,\sqrt{g(\hat{x})}^{-1/2}\prod_{n,m}\int \mathrm{d}a_{n,m}\, \mathrm{e}^{-\frac{1}{2}\mathscr{F}_n a_{n,m}^2} \\ &= \mathrm{Det}\,\sqrt{g(\hat{x})}^{-1/2}\,\mathrm{Det}\,\mathscr{K}^{-1/2} \\ &= \mathrm{Det}\left(\sqrt{g(\hat{x})}\mathscr{K}\right)^{-1/2}\,.\end{aligned} \tag{D.26}$$

Note that in this case – as a consistency check – we derived, starting from a non-diffeomorphism-invariant path integral, a result that is not diffeomorphism-invariant, too. I.e., we are always going to consider $\mathcal{D}_g A$ as the correct measure to integrate over functionals depending on the scalar field A.

D.2.4. *A note on the phase space measure*

Quite interestingly, the results (D.25) and (D.26) can be related to evaluating the *canonical path integral over phase space.* For this single purpose, we will consider the action

$$S^{\mathrm{Lor}}[A;g] \coloneqq -\frac{1}{2}\int \mathrm{d}^d x\sqrt{-g}\,\left[g^{\mu\nu}\partial_\mu A\,\partial_\nu A + \Omega(g)A^2\right]$$

defined on a d-dimensional *Lorentzian* manifold with signature $-++\ldots$, so that we are not required to use an Euclidean Hamilton formalism. Here, Ω is an ordinary function. Defining the Lagrangian as [192, p. 131 ff.]

$$\mathcal{L}^{\rm Lor}[A;g] := -\frac{1}{2}\left[g^{\mu\nu}\partial_\mu A\,\partial_\nu A + \Omega(g)A^2\right]$$

one has

$$S[A;g] = \int \mathrm{d}^d x\sqrt{-g}\,\mathcal{L}\,.$$

In order to derive the associated Hamiltonian density

$$\mathcal{H}[A;g] := p_A\dot{A}(A,p_A) - \mathcal{L}$$

let us employ the ADM decomposition of the metric [188]:

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_c & h_{ab}\end{pmatrix} \tag{D.27}$$

and for the inverse metric,

$$g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^b/N^2 \\ N_c/N^2 & h^{ab} - N^a N^b/N^2\end{pmatrix}\,, \tag{D.28}$$

with N the lapse function and N^a the shift vector (the indices $a, b, c\ldots$ run over $\{1,2,3\}$ and describe vectors on the spatial Cauchy surface). Note that $-g^{00} = 1/N^2$ and that the volume element then reads

$$\sqrt{-g} = N\sqrt{h}\,.$$

The canonical momentum is defined as

$$\begin{aligned} p_A &:= \frac{\partial}{\partial\dot{A}}\left(\sqrt{-g}\mathcal{L}\right) \\ &= -\sqrt{-g}g^{00}\left(\dot{A} + \frac{g^{0a}}{g^{00}}\partial_a A\right)\,. \end{aligned}$$

Applying the ADM decomposition, this expression becomes more handy,

$$p_A = \frac{\sqrt{h}}{N}\left(\dot{A} - N^a\partial_a A\right)\,.$$

Therefrom, we can express $\dot{A}$ in terms of A and p_A,

$$\begin{aligned} \dot{A} &= -\frac{1}{\sqrt{-g}g^{00}}p_A - \frac{g^{0a}}{g^{00}}\partial_a A \\ &= \frac{N}{\sqrt{h}}p_A + N^a\partial_a A\,, \end{aligned}$$

as well as $\mathcal{L}$ in terms of A and p_A,

$$\sqrt{-g}\mathcal{L} = -\frac{1}{2}\left[-\frac{N}{\sqrt{h}}p_A^2 + N\sqrt{h}h^{ab}\partial_a A\,\partial_b A + N\sqrt{h}\,\Omega(g)A^2\right],$$

such that the Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\frac{N}{\sqrt{h}}p_A^2 + p_A N^a\partial_a A + \frac{1}{2}N\sqrt{h}\left[h^{ab}\partial_A A\,\partial_b A + \Omega(g)A^2\right].$$

Having obtained these results, it is easy for us to evaluate the canonical path integral (which is anologous to the elementary quantum-mechanical path integrals[4])

$$\int\mathcal{D}A\int\mathcal{D}p_A\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}^dx\left(p_A\dot{A}-\mathcal{H}\right)} = \int\mathcal{D}A\int\mathcal{D}p_A\,\exp\,\mathrm{i}\int\mathrm{d}^dx\Bigg\{p_A\left(\dot{A}-N^a\partial_a A\right) - \frac{1}{2}\frac{N}{\sqrt{h}}p_A^2 - \frac{1}{2}N\sqrt{h}\left(h^{ab}\partial_a A\,\partial_b A + \Omega(g)A^2\right)\Bigg\}. \quad \text{(D.29)}$$

Note that $\mathcal{D}A = \prod_x \mathrm{d}A(x)$ is g-independent. Using eq. (D.3), we can perform the Gaussian integration over the field momentum $p_A(x)$ to obtain

$$\begin{aligned}\int\mathcal{D}A\int\mathcal{D}p_A\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}^dx\left(p_A\dot{A}-\mathcal{H}\right)} &= \left(\mathrm{Det}\,\frac{N^2(\hat{x})}{\sqrt{-g(\hat{x})}}\right)^{-1/2}\int\mathcal{D}A\,\mathrm{e}^{\mathrm{i}S^{\mathrm{Lor}}[A;g]} \\ &=: \int\mathcal{D}'_gA\,\mathrm{e}^{\mathrm{i}S^{\mathrm{Lor}}[A;g]},\end{aligned} \quad \text{(D.30)}$$

with

$$\begin{aligned}\mathcal{D}'_gA &:= \prod_x\left(\frac{N^2(x)}{\sqrt{-g(x)}}\right)^{-1/2}\mathrm{d}A(x) \\ &= \prod_x\left\{\sqrt{-g^{00}(x)}(-g(x))^{1/4}\mathrm{d}A(x)\right\} = \left(\prod_x\sqrt{-g^{00}(x)}\right)\mathcal{D}_gA\end{aligned}$$

where we have used (D.28) and denoted by $\mathcal{D}_gA$ the Lorentzian counterpart of the measure from the previous subsections.

[4] $\mathcal{D}A\mathcal{D}p_A$ is analogous to the "Liouville measure" $\mathrm{d}q\wedge\mathrm{d}p$ in elementary quantum mechanics.

In order to obtain the invariant (formally fully covariant) Lagrangian path integral (D.25), this result suggests that $\mathcal{D}A\mathcal{D}\left[Np_A\right] \equiv \mathcal{D}A\mathcal{D}\left[p_A/\sqrt{-g^{00}}\right]$ is the correct phase space measure [191], i.e.

$$\int \mathcal{D}A \int \mathcal{D}\left[Np_A\right] \mathrm{e}^{\mathrm{i}\int \mathrm{d}^d x \left(p_A \dot{A} - \mathcal{H}\right)} = \int \mathcal{D}_g A \, \mathrm{e}^{\mathrm{i}S[A;g]} =: Z[g]\,. \tag{D.31}$$

with $\mathcal{D}_g A := \prod_x \left(-g(x)\right)^{-1/4} \mathrm{d}A(x)$, as in the previous subsections, is the final result for the correct path integral in phase space.

On the other hand, we may firstly ackowledge the result (D.25) to be defined over an "un-foliated" curved spacetime. Then the result (D.30), and therewith (D.31), follows from (D.25) for *any* foliation, chosen to be applied to (D.25), in the legit gauge $N \equiv 1$.

D.3. The Gaussian integral over vector fields

When we wish to perform a path integral over a covariant (real) vector field ϕ_μ, we ought to choose the invariant measure (D.8), $\mathcal{D}_g\phi_\mu := \mathcal{D}\left[g^{\frac{d-2}{4d}}\phi_\mu\right]$. A Gaussian path integral over vector fields thus is

$$\int \mathcal{D}\left[g^{\frac{d-2}{4d}}\phi_\mu\right] \exp\left\{-\int \mathrm{d}^d x \sqrt{g}\phi_\mu \left(\mathscr{O}_V \phi\right)^\mu\right\} \tag{D.32}$$

where $\mathscr{O}_V$ is an Operator on V, the Hilbert space of vector fields defined in section A.1, that acts as $\left(\mathscr{O}_V\phi\right)^\mu(x) = \left(\mathscr{O}_V\right)^\mu{}_{\nu x}{}^{\mathrm{diff}}\,\phi^\nu(x)$. Now the path integral (D.32) can be easily performed, using (D.25) and taking into account the additional vector structure,

$$\begin{aligned}
&\int \mathcal{D}_g\phi_\mu \exp\left\{-\int \mathrm{d}^d x \sqrt{g}\phi_\mu \left(\mathscr{O}_V\phi\right)^\mu\right\} \\
&= \int \mathcal{D}\left[g^{\frac{d-2}{4d}}\phi_\mu\right] \exp\left\{-\int \mathrm{d}^d x \sqrt{g(x)}\phi_\mu(x) (\mathscr{O}_V)^\mu{}_{\rho x}{}^{\mathrm{diff}} \left(g^{\rho\nu}\phi_\nu\right)(x)\right\} \\
&= \mathrm{Det}\left(g(\hat{x})^{\frac{d-2}{4d}}\delta^\mu_\nu\right) \mathrm{Det}\left(\sqrt{g(\hat{x})}(\mathscr{O}_V)^\mu{}_\rho g^{\rho\nu}(\hat{x})\right)^{-1/2} \\
&= \mathrm{Det}\, g(\hat{x})^{\frac{d-2}{4}}\, \mathrm{Det}\, g(\hat{x})^{-\frac{d}{4}}\, \mathrm{Det}\left(g(\hat{x})^{-1}\right)^{-1/2} \mathrm{Det}\left((\mathscr{O}_V)^\mu{}_\rho\right)^{-1/2} \\
&= \mathrm{Det}\left((\mathscr{O}_V)^\bullet{}_\bullet\right)^{-1/2}\,.
\end{aligned} \tag{D.33}$$

In the third step, we have partially perfomed the determinants with respect to the vector structure. This important result confirms that that the measure (D.8) has been constructed indeed in a correct way, i.e. to be diffeomorphism-invariant.

D.4. The Gaussian integral over symmetric rank-2 tensor fields

The Gaussian path integral over a (real) covariant symmetric rank-2 tensor field $\phi_{\mu\nu}$ in turn is given by

$$\int \mathcal{D}\Big[g^{\frac{d-4}{4d}}\phi_{\mu\nu}\Big]\ \exp\Big\{-\int \mathrm{d}^d x\sqrt{g}\phi_{\mu\nu}\left(\mathscr{O}_{ST^2}\phi\right)^{\mu\nu}\Big\} \tag{D.34}$$

where $\mathscr{O}_{ST^2}$ is an Operator on ST^2, the Hilbert space of symmetric rank-2 tensor fields defined in section A.1, that acts as $\left(\mathscr{O}_{ST^2}\phi\right)^{\mu\nu}(x) = \left(\mathscr{O}_V\right)^{\mu\nu}{}_{\alpha\beta_x}{}^{\text{diff}}\,\phi^{\alpha\beta}(x)$. To perfom this integral, we will regard the symmetric pair $(\mu\nu)$ of indices in $\phi_{\mu\nu}$ to have been (half-)vectorized into a single index, such that the sum over μ and ν in (D.34), each from 0 to $d-1$, becomes a single sum from 0 to $d(d+1)/2$ (avoiding double counting), and the index structure of $\left(\mathscr{O}_V\right)^{\mu\nu}{}_{\alpha\beta}$ can be interpreted as a $d(d+1)/2 \times d(d+1)/2$-matrix. Therewith, we have transformed the integral (D.34) into the form of the integral (D.32) – that we already have evaluated – and find the Gaussian path integral over symmetric rank-2 tensor fields,

$$\begin{aligned}
&\int \mathcal{D}_g\phi_{\mu\nu}\ \exp\Big\{-\int \mathrm{d}^d x\sqrt{g}\phi_{\mu\nu}\left(\mathscr{O}_{ST^2}\phi\right)^{\mu\nu}\Big\}\\
&= \int \mathcal{D}\Big[g^{\frac{d-4}{4d}}\phi_{\mu\nu}\Big]\ \exp\Big\{-\int \mathrm{d}^d x\sqrt{g(x)}\phi_{\mu\nu}(x)\left(\mathscr{O}_{ST^2}\right)^{\mu\nu}{}_{\rho\sigma_x}{}^{\text{diff}}\left(I^{\rho\sigma\alpha\beta}\phi_{\alpha\beta}\right)(x)\Big\}\\
&= \mathrm{Det}\left[g(\hat{x})^{\frac{(d-4)}{4d}\frac{d(d+1)}{2}}\right]\mathrm{Det}\left(\sqrt{g(\hat{x})}\left(\mathscr{O}_{ST^2}\right)^{\mu\nu}{}_{\rho\sigma}I^{\rho\sigma\alpha\beta}(\hat{x})\right)^{-1/2}\\
&= \mathrm{Det}\left[g(\hat{x})^{\frac{(d-4)}{4}\frac{(d+1)}{2}}\right]\mathrm{Det}\left(\sqrt{g(\hat{x})}\,I^{\rho\sigma\alpha\beta}(\hat{x})\right)^{-1/2}\mathrm{Det}\left(\left(\mathscr{O}_{ST^2}\right)^{\mu\nu}{}_{\rho\sigma}\right)^{-1/2}
\end{aligned}$$

where $I^{\rho\sigma\alpha\beta} := \frac{1}{2}\left(g^{\rho\alpha}g^{\sigma\beta} + g^{\rho\beta}g^{\sigma\alpha}\right)$ and the determinant of the tensor structure is taken as the determinant of the $d(d+1)/2 \times d(d+1)/2$-matrix constructed from the respective tensor. Using the fact that $\det A = \det\left[(\det A)^{1/n}\,\mathbb{1}_{n\times n}\right]$ for an $n \times n$-matrix A, we obtain (cf. [118, p. 59ff])

$$\mathrm{Det}\left(\sqrt{g(\hat{x})}\,I^{\rho\sigma\alpha\beta}(\hat{x})\right) = \mathrm{Det}\left[\sqrt{g(\hat{x})}\,g^{-2/d}(\hat{x})\frac{1}{2}\left(\delta^{\rho\alpha}\delta^{\sigma\beta} + \delta^{\rho\beta}\delta^{\sigma\alpha}\right)\right]$$

$$
\begin{aligned}
&= \mathrm{Det}\left[\left(g^{-2/d}(\hat{x})g^{1/2}(\hat{x})\right)^{\frac{d(d+1)}{2}}\right] \\
&= \mathrm{Det}\left[g(\hat{x})^{\frac{(d-4)}{2d}\frac{d(d+1)}{2}}\right] .
\end{aligned}
$$

As an immediate consequence, the Gaussian integral over symmetric rank-2 tensor field is determined:

$$
\int \mathcal{D}\Big[g^{\frac{d-4}{4d}}\phi_{\mu\nu}\Big] \exp\left\{-\int \mathrm{d}^d x \sqrt{g}\phi_{\mu\nu}\left(\mathscr{O}_{ST^2}\phi\right)^{\mu\nu}\right\} = \mathrm{Det}\left(\left(\mathscr{O}_{ST^2}\right)^{\bullet\bullet}{}_{\bullet\bullet}\right)^{-1/2} . \tag{D.35}
$$

Interestingly, *requiring* this result to hold for $\mathscr{O}_{ST^2} = I$ sometimes is used to define the measure indirectly, in order to bypass the ill-definedness of the product over all spacetime-points in the defintion of the measure used here [193].

D.5. The Gaussian integral over fermionic variables

Along the same lines, Gaussian integrals over fermionic fields on curved spacetimes can be obtained. Here, the starting point is the path integral over the fermionic scalar field η on flat space [31]:

$$
\int \mathcal{D}\eta \, \exp\left\{-\int \mathrm{d}^d x \, \eta(x)\left(\mathscr{O}_S\eta\right)(x)\right\} = \mathrm{Det}\left(\mathscr{O}_S\right)^{+1/2} ,
$$

where $\mathscr{O}_S$ is an operator acting on scalar fields. This path integral is a cosequence of the definitions $\int \mathrm{d}\eta := 0$ and $\int \mathrm{d}\eta\,\eta := 1$ that define the integration of fermionic variables. Remarkbly, the sign of the exponent, $-1/2$ or $+1/2$, entrenches the difference between bosonic and fermionic Gaussian path integrals. With the previous discussion on bosonic variables in the back of our mind and using eqs. (D.6), this Gaussian path integral for the fermionic scalar field η defined on a curved spacetime becomes

$$
\int \mathcal{D}_g\eta \, \exp\left\{-\int \mathrm{d}^d x \sqrt{g(x)}\, \eta(x)\left(\mathscr{O}_S\eta\right)(x)\right\} = \mathrm{Det}\left(\mathscr{O}_S\right)^{+1/2} ,
$$

with $\mathcal{D}_g\eta = \mathcal{D}\big[g^{1/4}\eta\big] = \mathrm{Det}\; g(\hat{x})^{-1/4}\mathcal{D}\eta$. Likewise, the Gaussian integral over a fermionic vector field η_μ reads

$$
\int \mathcal{D}_g\eta_\mu \, \exp\left\{-\int \mathrm{d}^d x \sqrt{g(x)}\, \eta_\mu(x)\left(\mathscr{O}_V\eta\right)^\mu(x)\right\} = \mathrm{Det}\left(\mathscr{O}_V{}^{\bullet}{}_{\bullet}\right)^{+1/2} \tag{D.36}
$$

where $\mathscr{O}_V$ is an operator acting on vector fields. Using Eq. (D.8), the employed measure is $\mathcal{D}_g\eta_\mu = \mathcal{D}\Big[g^{\frac{d-2}{4d}}\eta_\mu\Big] = \mathrm{Det}\; g(\hat{x})^{-\frac{d-2}{4}}\mathcal{D}\eta_\mu$.

Also, we are in need of deriving Gaussian integrals over *complex fermionic* scalar and vector fields. Therefore, let η and $\bar{\eta}$ be conjugate complex feriomic scalar fields and η^μ and $\bar{\eta}_\mu$ conjugate complex fermionic vector fields. In the definition of fermionic path integrals, there is a sign ambiguity given by the ordering of the factors in the product $\prod_x \mathrm{d}\eta(x)\mathrm{d}\bar{\eta}(x)$. Therefore, it is crucial to specify the ordering with which to the measure is defined. Here, we will employ the "natural" ordering

$$\mathcal{D}\eta\mathcal{D}\bar{\eta} := \prod_x \mathrm{d}\eta(x)\mathrm{d}\bar{\eta}(x)$$

which results in the "hyperbolic" Gaussian integrals[5] [31]

$$\int \mathcal{D}\eta\mathcal{D}\bar{\eta}\;\exp\left\{+\int \mathrm{d}^d x\,\bar{\eta}(x)(\mathscr{O}_S\eta)(x)\right\} = \mathrm{Det}\,\mathscr{O}_S\,,$$

as well as

$$\int \mathcal{D}\eta^\mu\mathcal{D}\bar{\eta}_\mu\;\exp\left\{+\int \mathrm{d}^d x\,\bar{\eta}_\mu(x)\left(\mathscr{O}_V\eta\right)^\mu(x)\right\} = \mathrm{Det}\left(\mathscr{O}_V{}^{\bullet}{}_{\bullet}\right)\,.$$

Applying the measures (D.6), (D.7) and (D.8), we immediately find the corresponding Gaussian integrals over the respective fields defined on a curved spacetime,

$$\begin{aligned}
&\int \mathcal{D}_g\eta\,\mathcal{D}_g\bar{\eta}\;\exp\left\{+\int \mathrm{d}^d x\sqrt{g(x)}\,\bar{\eta}(x)(\mathscr{O}_S\eta)(x)\right\}\\
&\qquad = \int \mathcal{D}\big[g^{1/4}\eta\big]\,\mathcal{D}\big[g^{1/4}\bar{\eta}\big]\;\exp\left\{+\int \mathrm{d}^d x\sqrt{g(x)}\,\bar{\eta}(x)(\mathscr{O}_S\eta)(x)\right\}\\
&\qquad = \mathrm{Det}\,\mathscr{O}_S\,,
\end{aligned}$$

and

$$\int \mathcal{D}_g\eta^\mu\,\mathcal{D}_g\bar{\eta}_\mu\;\exp\left\{+\int \mathrm{d}^d x\sqrt{g(x)}\,\bar{\eta}_\mu(x)\left(\mathscr{O}_V\eta\right)^\mu(x)\right\}$$

[5] For the ordering $\mathcal{D}\eta\mathcal{D}\bar{\eta} = \prod_x \mathrm{d}\bar{\eta}(x)\mathrm{d}\eta(x)$, the fermionic Gaussian integral is $\int\mathcal{D}\eta\mathcal{D}\bar{\eta}\;\exp\left\{-\int \mathrm{d}^d x\,\bar{\eta}(x)(\mathscr{O}_S\eta)(x)\right\} = \mathrm{Det}\,\mathscr{O}_S$. The corresponding Gaussian integral over complex bosonic scalars, a and $\bar{a}$, is $\int\mathcal{D}\bar{a}\mathcal{D}a\;\exp\left\{-\int \mathrm{d}^d x\,\bar{a}(x)(\mathscr{O}_S a)(x)\right\} = \mathrm{Det}\,\mathscr{O}^{-1}$.

$$= \int \mathcal{D}\Big[g^{\frac{d-2}{4d}}\eta^\mu\Big]\,\mathcal{D}\Big[g^{\frac{d+2}{4d}}\bar{\eta}_\mu\Big]\;\exp\left\{+\int \mathrm{d}^d x\sqrt{g(x)}\,\bar{\eta}_\mu(x)\,(\mathscr{O}_V\eta)^\mu\,(x)\right\} \quad \text{(D.37)}$$
$$= \mathrm{Det}\,(\mathscr{O}_V\,\dot{.}\,)\;.$$

As for the real-valued fermionic fields, the only technicality to take into account is the Jacobian arising from the transformation of the fermionic variables, that for the case of vector fields is given by

$$\mathcal{D}\Big[g^{\frac{d-2}{4d}}\eta_\mu\Big]\,\mathcal{D}\Big[g^{\frac{d+2}{4d}}\bar{\eta}^\mu\Big] = \mathrm{Det}\;g(\hat{x})^{-\frac{d-2}{4}}\;\mathrm{Det}\;g(\hat{x})^{-\frac{d+2}{4}}\mathcal{D}\eta^\mu\mathcal{D}\bar{\eta}_\mu$$
$$= \mathrm{Det}\left(\sqrt{g(\hat{x})}\delta^\mu_\nu\right)^{-1}\mathcal{D}\eta^\mu\mathcal{D}\bar{\eta}_\mu\,.$$

APPENDIX E

The evaluation of traces using heat kernel techniques

Traces of the form $\mathrm{Tr}\,[\mathbb{1}W(-D^2)]$; where $\mathbb{1}$ is the identity on the space the trace is taken on, e.g. (A.23) for the Hilbert space ST^2, and W is a scalar function; can be evaluated by means of *heat kernel techniques*. These have been extensively studied in the literature [40, 194–199]. Here, we may skip the details and restrict our calculation to applying the formula for the local *early time expansion* of the heat kernel

$$\lim_{y\to x}\langle x,\mu_1,\mu_2,\ldots|\mathbb{1}\mathrm{e}^{sD^2}|y,\mu_1,\mu_2,\ldots\rangle = \left(\frac{1}{4\pi s}\right)^{\frac{d}{2}}\sum_{n=0}^{\infty} s^n\,\mathrm{tr}\,a_n(x)\,, \tag{E.1}$$

where tr denotes the trace of the tensor structure of the *Seeley-DeWitt coefficients a_n*. In fact, the applications in this thesis only demand the first two of these coefficients[1]

$$\begin{aligned} a_0(x) &\equiv I \\ \text{and}\quad a_1(x) &= \frac{1}{6}R(x)I\,, \end{aligned}$$

where I denotes the identity of the tensor structure of the Hilbert space in cosideration, i.e. specifically ${I_{ST^2}}^{\mu\nu}{}_{\rho\sigma}$ given by Eq. (A.24), ${I_V}^{\mu}{}_{\nu} = \delta^\mu_\nu$ and $I_S = 1$. To actually apply formula (E.1), we Fourier transform the trace according to[2]

$$\mathrm{Tr}\left[\mathbb{1}W(-D^2)\right] = \int_{-\infty}^{\infty}\mathrm{d}s\,\tilde{W}(s)\,\mathrm{Tr}\left[\mathbb{1}\mathrm{e}^{-\mathrm{i}sD^2}\right]\,,$$

which requires the function W to decrease sufficiently fast in the direction of $\pm\infty$. With $s\mapsto -\mathrm{i}s$ we obtain

$$\mathrm{Tr}\left[\mathbb{1}W(-D^2)\right] = \int_{-\infty}^{\infty}\mathrm{d}s\,\tilde{W}(s)\int\mathrm{d}^dx\sqrt{g(x)}\langle x,\mu_1,\mu_2,\ldots|\mathbb{1}\mathrm{e}^{-\mathrm{i}sD^2}|x,\mu_1,\mu_2,\ldots\rangle$$

[1]The coefficients a_n with $n\geq 3$ are at least squared in the curvature and thus can be neglected in the Einstein-Hilbert truncation.
[2]Alternatively, we could use a Laplace transform.

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \mathrm{d}s\, \tilde{W}(s) \int \mathrm{d}^d x \sqrt{g(x)} \left(\frac{1}{4\pi s}\right)^{\frac{d}{2}} \sum_{n=0}^{\infty} s^n \,\mathrm{tr}\, a_n(x) \\
&= \mathrm{tr}(I) \int_{-\infty}^{\infty} \mathrm{d}s\, \tilde{W}(s) \left(\frac{\mathrm{i}}{4\pi s}\right)^{\frac{d}{2}} \int \mathrm{d}^d x \sqrt{g(x)} \left[1 - \frac{1}{6}\mathrm{i}sR(x)\right] \\
&\quad + O(R^2)\,.
\end{aligned}
\tag{E.2}
$$

Can we also use this formula to calculate $\mathrm{Tr}\,[\mathbb{P}\, W(-D^2)]$ where $\mathbb{P}$ is a projector on a subspace of the Hilbert space the trace is taken over? Videlicat, we can interpret $\mathbb{P}$ as the identity of this very subspace (the isomorphism is obvious) and, for example, consider $\mathrm{Tr}_V\,[\mathbb{P}_T\, W(-D^2)]$ where $\mathbb{P}_T$ is the projector on the transverse part of a vector field (cf. subsection A.2.2). If we denote the corresponding subspace of V by V_T with identity $\mathbb{1}_{V_T}$ we find that $\mathrm{Tr}_V\,[\mathbb{P}_T\, W(-D^2)] = \mathrm{Tr}_{V_T}\,[\mathbb{1}_{V_T}\, W(-D^2)]$, which is from where we could proceed the calculation with the above recipe. Unfortunately, there is an obstacle here that renders this recipe not applicable: The projector $\mathbb{P}_T$ is defined by differential constraints involving covariant derivatives which ultimately gives $\mathrm{Tr}_V\,[\mathbb{P}_T\, W(-D^2)]$ a further curvature depedence. In the expression $\mathrm{Tr}_{V_T}\,[\mathbb{1}_{V_T}\, W(-D^2)]$, this curvature dependence is hidden in the subspace V_T itself because V_T in fact is spanned per definitionem by $\{\mathbb{P}_T|x,\mu\rangle \mid x \in \mathbb{R}^d\,;\, \mu = 1,\ldots,d\}$.[3] On the other hand, it is obvious that the above formula applies only to Hilbert spaces that are not given by differential constraints and thus are "curvature independent". Inspecting the counter-example V_T a little further, we in fact obtain, using the defintion of the projector $\mathbb{P}_T$,

$$
\begin{aligned}
\mathrm{Tr}_{V_T}\left[\mathbb{1}_{V_T}\, W(-D^2)\right] &= \mathrm{Tr}_V\left[\mathbb{P}_T\, W(-D^2)\right] \\
&= \mathrm{Tr}_V\left[\mathbb{1}_V\, W(-D^2)\right] - \mathrm{Tr}_V\left[D^{\bullet}\frac{1}{D^2}D_{\bullet}\, W(-D^2)\right].
\end{aligned}
$$

The first term can be calculated using the standard above procedure while the second term requires the application of *off-diagonal heat kernel* methods [62,

[3]The structure of this discussion is similar to the question whether the trace is independent of the metric $g_{\mu\nu}$, although it is calculated e.g. for scalar fields as $\mathrm{Tr}_S[A] = \int \mathrm{d}^d x \sqrt{g(x)}\langle x|A|x\rangle$. Surely, the trace *is* independent of the metric, namely because the states are g-dependend themselves: $\langle x|y\rangle = \delta(x-y)/\sqrt{g(y)}$.

200] that are not required in this thesis. Thus, we can conclude that we indeed may caculate $\mathrm{Tr}\,[\mathbb{P}\,W(-D^2)]$ with the above formula, provided that $\mathbb{P}$ is *not* defined by differential constraints. The only example appearing in this thesis for such a projector, aside from the identity, is the projector $\mathbb{P}_{\mathrm{tr.}}$ on the trace part of a symmetric rank-2 tensor field, given by $P_{\mathrm{tr.}}[g]^{\mu\nu}_{\alpha\beta} = \frac{1}{d} g^{\mu\nu} g_{\alpha\beta}$. We thence have

$$
\begin{aligned}
&\mathrm{Tr}_{ST^2}\left[\mathbb{P}_{\mathrm{tr.}}\,W(-D^2)\right] \\
&\quad = \underbrace{\mathrm{tr}(P_{\mathrm{tr.}})}_{=1} \int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) \left(\frac{\mathrm{i}}{4\pi s}\right)^{\frac{d}{2}} \int \mathrm{d}^d x \sqrt{g(x)} \left[1 - \frac{1}{6}\mathrm{i}sR(x)\right] + O(R^2)\,.
\end{aligned}
$$

Next, we can still work on Eq. (E.2) in order to organize the intermediate result:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) \left(\frac{\mathrm{i}}{4\pi s}\right)^{\frac{d}{2}} \int \mathrm{d}^d x \sqrt{g(x)} \left[1 - \frac{1}{6}\mathrm{i}sR(x)\right] \\
&\qquad = \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left\{ \int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) \left(\frac{\mathrm{i}}{s}\right)^{\frac{d}{2}} \int \mathrm{d}^d x \sqrt{g(x)} \right. \\
&\qquad\qquad\qquad \left. + \frac{1}{6} \int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) \left(\frac{\mathrm{i}}{s}\right)^{\frac{d}{2}-1} \int \mathrm{d}^d x \sqrt{g(x)} R(x) \right\} \\
&\qquad = \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left\{ Q_{\frac{d}{2}}[W] \int \mathrm{d}^d x \sqrt{g(x)} + \frac{1}{6} Q_{\frac{d}{2}-1}[W] \int \mathrm{d}^d x \sqrt{g(x)} R(x) \right\}
\end{aligned}
$$

with the "Q-functionals"

$$
Q_n[W] \coloneqq \int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) (-\mathrm{i}s)^{-n}\,. \tag{E.3}
$$

This *Mellin transform* can also be expressed through W instead of $\tilde{W}$ by making use of the Gamma function[4] $\Gamma(n) = \mathrm{i}^n \int_0^\infty \mathrm{d}z\, z^{n-1} \mathrm{e}^{-\mathrm{i}z}$:

$$
\begin{aligned}
Q_n[W] &= \int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) (-\mathrm{i}s)^{-n} \frac{1}{\Gamma(n)} (-\mathrm{i})^n \int_0^\infty \mathrm{d}z\, z^{n-1} \mathrm{e}^{\mathrm{i}z} \\
&= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} \mathrm{d}s\,\tilde{W}(s) \int_0^\infty \mathrm{d}z\, \frac{1}{s} \left(\frac{z}{s}\right)^{n-1} \mathrm{e}^{\mathrm{i}z}
\end{aligned}
$$

[4]Note that here $\Gamma(n) \in \mathbb{R}$ and thus $\Gamma(n)^* = \Gamma(n)$.

$$= \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \int_{-\infty}^\infty \mathrm{d}s\, \tilde{W}(s) \mathrm{e}^{\mathrm{i}zs}$$
$$= \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} W(z)\,.$$

In the third step we have substituted $z \mapsto z/s$.

All in all, we have obtained the equation, with W a scalar function,

$$\boxed{\begin{aligned}\mathrm{Tr}\left[\mathbb{A}W(-D^2)\right] = \mathrm{tr}(A)\left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \Bigg\{ & Q_{\frac{d}{2}}[W] \int \mathrm{d}^d x \sqrt{g(x)} \\ & + \frac{1}{6} Q_{\frac{d}{2}-1}[W] \int \mathrm{d}^d x \sqrt{g(x)} R(x) \Bigg\} + O(R^2)\,,\end{aligned}} \tag{E.4}$$

with $\mathbb{A} = \mathbb{1}$ and $A = I$ or on ST^2 also $\mathbb{A} = \mathbb{P}_{\text{tr.}}$ and $A = P$.

Lastly, it is obvious that if we follow the procedure yet again with the *untraced heat kernel*, we will obtain the expansion

$$\begin{aligned}&\lim_{y\to x} \langle x, \mu_1, \mu_2, \ldots |\, \mathbb{A}W(-D^2) | y, \mu_1, \mu_2, \ldots \rangle \\ &\qquad = \mathrm{tr}(A) \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \left\{ Q_{\frac{d}{2}}[W] + \frac{1}{6} Q_{\frac{d}{2}-1}[W]\, R(x) \right\} + O(R^2)\,.\end{aligned} \tag{E.5}$$

APPENDIX F

Outsourced calculations

This appendix is a conglomerate of meticulous calculations that the author decided to dislodge from from the main text for various reasons, mostly because the respective calculation is long and comprehensive. Then the author does not have to encumber the reader with it at that point. Sometimes calculations have been moved in order not to digress from the main line of thought, as well.

F.1. Derivation of the infinitesimal coordinate transformation

Consider a d-dimensional (semi-)Riemannian manifold (M, g). In a local chart $x : U \subset M \to U' \subset \mathbb{R}^d$ the metric is given by local functions $g_{\mu\nu} : U \to \mathbb{R}$. In abuse of notation, let us think of the functions $g_{\mu\nu}$ as being actually the mappings $g_{\mu\nu} \circ x^{-1}$, i.e. we think of these local functions as $g_{\mu\nu} : \mathbb{R}^d \to \mathbb{R}$. With $x \in \mathbb{R}^d$ consider the infinitesimal coordinate transformation

$$x'^{\mu}(x) := x^{\mu} - V^{\mu}(x)$$

where V^{μ} is an infinitesimal vector field. We insert this coordinate transformation into the argument of $g_{\mu\nu}$ to then expand it around x':

$$\begin{aligned} g_{\mu\nu}(x) = g_{\mu\nu}(x' + V(x)) &= g_{\mu\nu}(x') + \frac{\partial}{\partial x^{\alpha}} g_{\mu\nu}(x') \left(x'^{\alpha} + V^{\alpha}(x) - x'^{\alpha} \right) + \cdots \\ &= g_{\mu\nu}(x') + \left(\frac{\partial}{\partial x^{\alpha}} g_{\mu\nu}(x) \right) V^{\alpha}(x) + O(V^2)\,. \end{aligned}$$

The chart transitions are given by

$$\begin{aligned} \frac{\partial x'^{\mu}}{\partial x^{\nu}} &= \delta^{\mu}_{\nu} - \frac{\partial V^{\mu}(x)}{\partial x^{\nu}}\,; \\ \frac{\partial x^{\mu}}{\partial x'^{\nu}} &= \delta^{\mu}_{\nu} + \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial V^{\mu}(x)}{\partial x^{\alpha}} = \delta^{\mu}_{\nu} + \frac{\partial V^{\mu}(x)}{\partial x^{\nu}} + O(V^2)\,; \end{aligned}$$

such that the metric transforms as

$$\begin{aligned} g'_{\mu\nu}(x') &= g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\mu\nu}(x) + g_{\mu\beta}(x)\frac{\partial V^\beta(x)}{\partial x^\nu} + g_{\alpha\nu}(x)\frac{\partial V^\alpha(x)}{\partial x^\mu} + O(V^2)\,. \end{aligned}$$

Bringing this transformation behaviour together with the above Taylor expansions immediatle yields

$$\begin{aligned} g'_{\mu\nu}(x) &= g_{\mu\nu}(x) + V^\alpha(x)\frac{\partial g_{\mu\nu}(x)}{\partial x^\alpha} + g_{\mu\alpha}(x)\frac{\partial V^\alpha(x)}{\partial x^\nu} + g_{\alpha\nu}\frac{\partial V^\alpha(x)}{\partial x^\mu} + O(V^2) \\ &= g_{\mu\nu}(x) + (L_V g)_{\mu\nu}(x) + O(V^2)\,. \end{aligned}$$

We particularly stress that LHS and RHS are defined at the *same* point x.

F.2. Proof of the nilpotence of the brst operation

We apply successively apply the BRST operations δ_η and δ_ε, with η and ε anticommuting parameters. Note that δ_η and δ_ε are "even", i.e. commuting, however.

Nilpotence applied to $\widehat{h}_{\mu\nu}$:

$$\begin{aligned}
\delta_\eta \delta_\varepsilon \widehat{h}_{\mu\nu} &= \delta_\eta(\varepsilon L_C \widehat{g}_{\mu\nu}) \\
&= \varepsilon \delta_\eta (L_C \widehat{g}_{\mu\nu}) \\
&= \varepsilon \left(L_{\delta_\eta C} \widehat{g}_{\mu\nu} + L_C \delta_\eta \widehat{g}_{\mu\nu} \right) \\
&= \varepsilon \left(L_{\eta C^\tau \partial_\tau C} \widehat{g}_{\mu\nu} + L_C(\eta L_C \widehat{g}_{\mu\nu}) \right) \\
&= \varepsilon \eta \left(L_{\eta C^\tau \partial_\tau C} \widehat{g}_{\mu\nu} - L_C(\eta L_C \widehat{g}_{\mu\nu}) \right) \\
&= 0
\end{aligned}$$

as for an arbitrary covariant tensor field $\phi_{\alpha\beta}$ one has

$$\begin{aligned}
& L_{C^\sigma \partial_\sigma C} \phi_{\alpha\beta} - L_C(L_C \phi_{\alpha\beta}) \\
&= (C^\sigma \partial_\sigma C^\rho) \partial_\rho \phi_{\alpha\beta} + [\partial_\alpha (C^\sigma \partial_\sigma C^\rho)] \phi_{\rho\beta} + [\partial_\beta (C^\sigma \partial_\sigma C^\rho)] \phi_{\alpha\rho} \\
&\quad - C^\sigma \partial_\sigma (L_C \phi_{\alpha\beta}) - (\partial_\alpha C^\sigma) L_C \phi_{\sigma\beta} - (\partial_\beta C^\sigma) L_C \phi_{\alpha\sigma} \\
&= - \Big[\underbrace{C^\sigma C^\rho \partial_\sigma \partial_\rho \phi_{\alpha\beta}}_{=0} + \underbrace{C^\sigma (\partial_\alpha C^\rho) \partial_\sigma \phi_{\rho\beta}}_{(3)} + \underbrace{C^\sigma (\partial_\beta C^\rho) \partial_\sigma \phi_{\alpha\rho}}_{(2)} \\
&\quad \underbrace{- C^\gamma (\partial_\alpha C^\sigma) \partial_\gamma \phi_{\sigma\beta}}_{(3)} + \underbrace{(\partial_\alpha C^\sigma)(\partial_\beta C^\gamma) \phi_{\sigma\gamma}}_{(1)} \underbrace{- C^\gamma (\partial_\beta C^\sigma) \partial_\gamma \phi_{\sigma\alpha}}_{(2)} \\
&\quad \underbrace{- (\partial_\alpha C^\gamma)(\partial_\beta C^\sigma) \phi_{\gamma\alpha}}_{(1)} \Big] \\
&= 0\,.
\end{aligned} \tag{F.1}$$

(The eponymously labeled terms cancel; the first term vanishes due to a symmetric-antisymmetric contraction.)

Nilpotence applied to C^μ:

$$\begin{aligned}\delta_\eta\delta_\varepsilon C^\mu &= \delta_\eta(\varepsilon C^\nu\partial_\nu C^\mu)\\ &= \varepsilon[(\delta_\eta C^\nu)\partial_\nu C^\mu + C^\nu\partial_\nu(\delta_\eta C^\mu)]\\ &= \varepsilon[(\eta C^\rho\partial_\rho C^\nu)\partial_\nu C^\mu + C^\nu\partial_\nu(\eta C^\rho\partial_\rho C^\mu)]\\ &= \varepsilon\eta[(C^\rho\partial_\rho C^\nu)\partial_\nu C^\mu - (C^\nu\partial_\nu C^\rho)\partial_\rho C_\mu - \underbrace{C^\nu C^\rho\partial_\nu\partial_\rho}_{=0} C^\mu]\\ &= 0\,.\end{aligned}$$

Nilpotence applied to $\bar{C}_\mu$:

$$\begin{aligned}\delta_\eta\delta_\varepsilon\bar{C}_\mu &= \delta_\eta[\varepsilon\alpha^{-1}Y^\nu{}_\mu[\bar{g}]F_\nu(\widehat{h};\bar{g})]\\ &= \varepsilon\alpha^{-1}Y^\nu{}_\mu[\bar{g}]\frac{\partial F_\nu(\widehat{h};\bar{g})}{\partial\widehat{h}_{\alpha\beta}}\delta_\eta\widehat{h}_{\alpha\beta}\\ &= \varepsilon\eta\alpha^{-1}Y^\nu{}_\mu[\bar{g}]\frac{\partial F_\nu(\widehat{h};\bar{g})}{\partial\widehat{h}_{\alpha\beta}}L_C\widehat{g}_{\alpha\beta}\,.\end{aligned}$$

Therewith $\delta_\eta\delta_\varepsilon\bar{C}_\mu = 0$ iff we use the equation of motion of the Fadeev-Popov action with respect to $\bar{C}_\mu$:

$$\frac{\delta S_{\rm gh}[\widehat{h},\bar{C},C;\bar{g}]}{\delta\bar{C}_\mu} = 0 \quad\Leftrightarrow\quad \frac{\partial F_\mu(\widehat{h};\bar{g})}{\partial\widehat{h}_{\alpha\beta}}L_C\widehat{g}_{\alpha\beta} = 0\,.$$

F.3. Proof of the BRST invariance of $S_{\rm cl} + S_{\rm GF} + S_{\rm gh}$

The invariance of the bare action itself is trivial as the BRST transformation reduces to a general coordinate transformation of the metric:

$$S_{\rm cl}[\widehat{g} + \delta_\varepsilon \widehat{g}] = S_{\rm cl}[\widehat{g} + \varepsilon L_C \widehat{g}] = S_{\rm cl}[\widehat{g} + L_{\varepsilon C} \widehat{g}] = S_{\rm cl}[\widehat{g}]\,.$$

Next, we insert the BRST transformation into the gauge-fixing condition and expand (all terms $O(\varepsilon^2)$ can be neglected):

$$F_\mu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g}) = F_\mu(\widehat{h}; \bar{g}) + \frac{\partial F_\mu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} \delta_\varepsilon h_{\alpha\beta}\,.$$

Therewith, we apply the BRST transformation to the gauge-fixing action and to the Fadeev-Popov action (again, neglecting terms $O(\varepsilon^2)$):

$$\begin{aligned}
S_{\rm GF}[\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g} + \delta_\varepsilon \bar{g}] &= \frac{1}{2\alpha} \int {\rm d}^d x \sqrt{\bar{g}}\, F_\mu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g}) Y^{\mu\nu}[\bar{g}] F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g}) \\
&= \frac{1}{2\alpha} \int {\rm d}^d x \sqrt{\bar{g}} \left(F_\mu(\widehat{h}; \bar{g}) + \frac{\partial F_\mu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} \delta_\varepsilon h_{\alpha\beta} \right) \\
&\qquad \times Y^{\mu\nu}[\bar{g}] \left(F_\nu(\widehat{h}; \bar{g}) + \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})}{\partial(\widehat{h}_{\rho\sigma} + \delta_\varepsilon \widehat{h}_{\rho\sigma})} \delta_\varepsilon h_{\rho\sigma} \right) \\
&= \frac{1}{2\alpha} \int {\rm d}^d x \sqrt{\bar{g}} \Bigg(F_\mu(\widehat{h}; \bar{g}) Y^{\mu\nu}[\bar{g}] F_\nu(\widehat{h}; \bar{g}) \\
&\qquad + 2 F_\mu(\widehat{h}; \bar{g}) Y^{\mu\nu}[\bar{g}] \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} \delta_\varepsilon h_{\alpha\beta} \Bigg) \\
&= S_{\rm GF}[\widehat{h}; \bar{g}] + \frac{\varepsilon}{\alpha} \int {\rm d}^d x \sqrt{\bar{g}}\, F_\mu(\widehat{h}; \bar{g}) Y^{\mu\nu}[\bar{g}] \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} L_C \widehat{g}_{\alpha\beta}\,.
\end{aligned}$$

Note that if $Y^{\mu\nu}[\bar{g}]$ is a differential operator built from covariant derivatives we made use of its behavior under partial differentiations, $\int {\rm d}^d x \sqrt{\bar{g}}\, A_\mu Y^{\mu\nu}[\bar{g}]^{\rm diff} B_\nu = \int {\rm d}^d x \sqrt{\bar{g}}\, B_\mu Y^{\mu\nu}[\bar{g}]^{\rm diff} A_\nu$, that we had required in its definition.

$$
\begin{aligned}
&S_{\text{gh}}[\widehat{h} + \delta_\varepsilon \widehat{h}, \bar{C} + \delta_\varepsilon \bar{C}, C + \delta_\varepsilon C; \bar{g} + \delta_\varepsilon \bar{g}] \\
&\qquad = -\int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \left(\bar{C}_\mu + \varepsilon \alpha^{-1} Y^\sigma{}_\mu[\bar{g}] F_\sigma(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})\right) \\
&\qquad\quad \times \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} L_{C+\delta_\varepsilon C} \left(\bar{g}_{\alpha\beta} + \widehat{h}_{\alpha\beta} + \varepsilon L_{C+\delta_\varepsilon C} \widehat{g}_{\alpha\beta}\right) \\
&\qquad = -\int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \left(\bar{C}_\mu + \varepsilon \alpha^{-1} Y^\sigma{}_\mu[\bar{g}] F_\sigma(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})\right) \\
&\qquad\quad \times \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}_{\alpha\beta}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} L_{C+\delta_\varepsilon C} \left(\bar{g}_{\alpha\beta} + \widehat{h}_{\alpha\beta} + \varepsilon L_C \widehat{g}_{\alpha\beta}\right) \\
&\qquad = -\int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \left(\bar{C}_\mu + \varepsilon \alpha^{-1} Y^\sigma{}_\mu[\bar{g}] F_\sigma(\widehat{h} + \delta_\varepsilon \widehat{h}; \bar{g})\right) \\
&\qquad\quad \times \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}_{\alpha\beta}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} \left[L_C(\bar{g}_{\alpha\beta} + \widehat{h}_{\alpha\beta}) + L_{\delta_\varepsilon C}(\bar{g}_{\alpha\beta} + \widehat{h}_{\alpha\beta}) - \varepsilon L_C(L_C \widehat{g}_{\alpha\beta})\right] \\
&\qquad = S_{\text{gh}}[\widehat{h}, \bar{C}, C; g] - \int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \Bigg\{ + \bar{C}_\mu \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}_{\alpha\beta}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} \varepsilon L_{C^\rho \partial_\rho C} \widehat{g}_{\alpha\beta} \\
&\qquad\qquad - \bar{C}_\mu \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}_{\alpha\beta}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} \varepsilon L_C(L_C \widehat{g}_{\alpha\beta}) \\
&\qquad\qquad + \frac{\varepsilon}{\alpha} \left[Y^\sigma{}_\mu[\bar{g}] F_\sigma(\widehat{h}; \bar{g})\right] \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}_{\alpha\beta}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} L_C \widehat{g}_{\alpha\beta} \Bigg\} \\
&\qquad \overset{\text{Eq. (F.1)}}{=} S_{\text{gh}}[\widehat{h}, \bar{C}, C; g] - \int \mathrm{d}^d x \sqrt{\bar{g}} \left\{ \frac{\varepsilon}{\alpha} F_\mu(\widehat{h}; \bar{g}) Y^{\mu\nu}[\bar{g}] \frac{\partial F_\nu(\widehat{h} + \delta_\varepsilon \widehat{h}_{\alpha\beta}; \bar{g})}{\partial(\widehat{h}_{\alpha\beta} + \delta_\varepsilon \widehat{h}_{\alpha\beta})} L_C \widehat{g}_{\alpha\beta} \right\}.
\end{aligned}
$$

Again, note that if $Y^{\mu\nu}[\bar{g}]$ contains derivatives then partial integrations have been performed in the last step. With this result, it immediately follows that

$$S_{\text{GF}}[\widehat{h}+\delta_\varepsilon \widehat{h}; \bar{g}+\delta_\varepsilon \bar{g}] + S_{\text{gh}}[\widehat{h}+\delta_\varepsilon \widehat{h}, \bar{C}+\delta_\varepsilon \bar{C}, C+\delta_\varepsilon C; \bar{g}+\delta_\varepsilon \bar{g}] = S_{\text{GF}}[\widehat{h}; \bar{g}] + S_{\text{gh}}[\widehat{h}, \bar{C}, C; \bar{g}]$$

which is what was left to show, as $\delta_\varepsilon S_{\text{gh},2}[b; \bar{g}] = 0$ holds trivially.

F.4. Proof of the invariance of $\mathrm{e}^{W[J;\bar{g}]}$, $\mathrm{e}^{W_k[J;\bar{g}]}$ and $\mathrm{e}^{W_k[J;\beta,\tau;\bar{g}]}$ under background gauge transformations $\delta^{(B)}$

We prepare the proof with a small lemma: Let f be a scalar built from tensors $T_{i\,\nu_1\cdots\nu_l}^{\mu_1\cdots\mu_k}$ for which $\delta^{(B)}(T_i)_{\nu_1\cdots\nu_l}^{\mu_1\cdots\mu_k} = L_V(T_i)_{\nu_1\cdots\nu_l}^{\mu_1\cdots\mu_k}$ holds. Consequently, $\delta^{(B)}f = L_V f$ and

$$\begin{aligned}\delta^{(B)}\int \mathrm{d}^d x\sqrt{\bar{g}}\,f &= \int \mathrm{d}^d x\delta^{(B)}(\sqrt{\bar{g}}\,f)\\ &= \int \mathrm{d}^d x\,\left[(L_V\sqrt{\bar{g}})f + \sqrt{\bar{g}}\,L_V f\right]\\ &= \int \mathrm{d}^d x\,\left[\sqrt{\bar{g}}\,(\bar{D}_\rho V^\rho) + \sqrt{\bar{g}}\,V^\rho\bar{D}_\rho f\right]\\ &= \int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{D}_\rho(V^\rho f)\\ &= 0\,.\end{aligned}$$

The last step holds due Gauss's law; in the third step have used that

$$\begin{aligned}L_V\sqrt{\bar{g}} &= \frac{1}{2}\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}L_V\bar{g}_{\mu\nu}\\ &= \frac{1}{2}\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\left[V^\rho\partial_\rho\bar{g}_{\mu\nu} + (\partial_\mu V^\rho)\bar{g}_{\rho\nu} + (\partial_\nu V^\rho)\bar{g}_{\mu\rho}\right]\\ &= \frac{1}{2}\sqrt{\bar{g}}\,\left[V^\rho\bar{g}^{\mu\nu}\partial_\rho\bar{g}_{\mu\nu} + 2\partial_\rho V^\rho\right]\\ &= \frac{1}{2}\sqrt{\bar{g}}\,\left[2V^\rho\bar{\Gamma}^\mu_{\nu\rho} + 2\partial_\rho V^\rho\right]\\ &= \sqrt{\bar{g}}\bar{D}_\rho V^\rho\,.\end{aligned}$$

As the classical action $S_{\mathrm{cl}}[\widehat{g}]$ in form of the Einstein-Hilbert action (4.3) or of the higher-derivative action (4.4), $S_{\mathrm{GF}}[\widehat{h};\bar{g}]$ as in Eq. (4.12), $S_{\mathrm{gh}}[\widehat{h},\bar{C},C,\bar{g}]$ as in Eq. (4.13), $S_{\mathrm{gh},2}[b;\bar{g}]$ as in Eq. (4.14), $S_{\mathrm{source}}[\widehat{h},\bar{C},C,b;t,\sigma,\bar{\sigma},d;\bar{g}]$ as in Eq. (4.19) and $\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]$ as in Eq. (6.1) precisely are of that form, one immediately has that $\delta^{(B)}S_{\mathrm{cl}}[\widehat{g}] = 0$, $\delta^{(B)}S_{\mathrm{GF}}[\widehat{h};\bar{g}] = 0$, $\delta^{(B)}S_{\mathrm{gh}}[\widehat{h},\bar{C},C,\bar{g}] = 0$, $\delta^{(B)}S_{\mathrm{gh},2}[b;\bar{g}] = 0$ and $\delta^{(B)}S_{\mathrm{source}}[\widehat{h},\bar{C},C,b;t,\sigma,\bar{\sigma},d;\bar{g}] = 0$ as well as $\delta^{(B)}\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}] = 0$. The crucial ingredient here is that $\delta^{(B)}F_\mu(\widehat{h};\bar{g}) = L_V F_\mu(\widehat{h};\bar{g})$ for the background gauge transformations.

The action $S_{\text{source}}[\widehat{g}, \bar{C}, C; t, \sigma, \bar{\sigma}; \beta, \tau; \bar{g}]$ is almost of that form – the term $\tau_\mu C^\nu \partial_\nu C^\mu$ does not transform as a tensor. However, it can still be tamed. Applying the background gauge transformation to it we find by expanding and neglecting terms $O(V^2)$:

$$\begin{aligned}
\tau_\mu C^\nu \partial_\nu C^\mu + \delta^{(B)}(\tau_\mu C^\nu \partial_\nu C^\mu) &= (\tau_\mu + L_V\tau_\mu)(C^\nu + L_V C^\nu)\partial_\nu(C^\mu + L_V C^\mu) \\
&= \tau_\mu C^\nu \partial_\nu C^\mu + V^\sigma(\partial_\sigma \tau_\mu) C^\nu \partial_\nu C^\mu + \underbrace{\tau_\sigma(\partial_\mu V^\sigma) C^\nu \partial_\nu C^\mu}_{(1)} \\
&\quad + \tau_\mu V^\sigma(\partial_\sigma C^\nu)\partial_\nu C^\mu \underbrace{- \tau_\mu C^\sigma(\partial_\sigma V^\nu)\partial_\nu C^\mu}_{(2)} \\
&\quad + \underbrace{\tau_\mu C^\nu(\partial_\nu V^\sigma)\partial_\sigma C^\mu}_{(2)} + \tau_\mu C^\nu V^\sigma \partial_\nu \partial_\sigma C^\mu \\
&\quad \underbrace{- \tau_\mu C^\nu(\partial_\nu C^\sigma)(\partial_\sigma V^\mu)}_{(1)} - \tau_\mu C^\nu C^\sigma \partial_\nu \partial_\sigma V^\mu \\
&= \tau_\mu C^\nu \partial_\nu C^\mu + V^\sigma \partial_\sigma(\tau_\mu C^\nu \partial_\nu C^\mu) - \underbrace{\tau_\mu C^\nu C^\sigma \partial_\nu \partial_\sigma V^\mu}_{=0} \\
&= \tau_\mu C^\nu \partial_\nu C^\mu + L_V(\tau_\mu C^\nu \partial_\nu C^\mu)\,,
\end{aligned}$$

where the eponymously labeled terms cancel. We have used that $L_V\tau_\mu = V^\sigma\partial_\sigma\tau_\mu + \tau_\sigma\partial_\mu V^\sigma$ and $L_V C^\mu = V^\sigma\partial_\sigma C^\mu - C^\sigma\partial_\sigma V^\mu$. Thus we may especially write $C^\nu L_V(\partial_\nu C^\mu) = C^\nu\partial_\nu(L_V C^\mu)$ and therewith have

$$\delta^{(B)}(\sqrt{\bar{g}}\,\tau_\mu C^\nu \partial_\nu C^\mu) = \sqrt{\bar{g}}\,\bar{D}_\rho(V^\rho \tau_\mu C^\nu \partial_\nu C^\mu)\,.$$

It follows that also $\delta^{(B)} S_{\text{source}}[\widehat{g}, \bar{C}, C; t, \sigma, \bar{\sigma}; \beta, \tau; \bar{g}] = 0$.

Lastly, we must show the invariance of the gravitational measure (4.21), respectively (4.22), under $\delta^{(B)}$. It is clear that applying $\delta^{(B)}$ to Eq. (4.21) or Eq. (4.22) amounts to applying a general coordinate transformation. In fact, on can show that any measure

$$\prod_x \bar{g}(x)^\alpha \prod_{\text{index structure}} \mathrm{d}\phi(x)\,,$$

where ϕ is a tensor field with arbitrary index structure and α is an arbitrary parameter, is ivariant under general (infinitesimal) coordinate transformations $x \mapsto x'(x) = x + V(x)$ [118, 201–203] given by

$$\prod_x \bar{g}(x)^\alpha \prod_{\text{index structure}} \mathrm{d}\phi(x) \mapsto \prod_x \det\left(\frac{\partial x'^\mu(x)}{\partial x^\nu}\right)^\beta \bar{g}(x)^\alpha \prod_{\text{index structure}} \mathrm{d}\phi(x)\,.$$

The power β depends on the parameter α and the dimension d but is not of importance here. Calculating the Jacobian yields

$$\begin{aligned}\prod_x \det\left(\frac{\partial x'^\mu(x)}{\partial x^\nu}\right)^\beta &= \prod_x \det(\delta^\mu_\nu + \partial_\nu V^\mu)^\beta(x)\\ &= \prod_x (1 + \partial_\mu V^\mu)^\beta(x)\\ &= \prod_x \mathrm{e}^{\beta\partial_\mu V^\mu(x)}\\ &= \mathrm{e}^{\beta\sum_x \partial_\mu V^\mu(x)}\\ &= \mathrm{e}^{\beta\int \mathrm{d}^d x \partial_\mu V^\mu(x)}\\ &= 1\,,\end{aligned}$$

assuming an empty boundary of the manifold. Hence, the gravitational measure (4.22) also is $\delta^{(B)}$-invariant. As a side note, in the literature the above calculation has been used as a point of suspension of the dicussion regarding the potential irrelevance of the parameter α.

All in all, we thus have shown that $\delta^{(B)}\mathrm{e}^{W[J;\bar{g}]} = 0$, $\delta^{(B)}\mathrm{e}^{W_k[J;\bar{g}]} = 0$ and $\delta^{(B)}\mathrm{e}^{W_k[J;\beta,\tau;\bar{g}]} = 0$.

F.5. Derivation of the relation $J_i = ((-1)^{|\phi^i|}/\sqrt{\bar{g}})\,\delta\Gamma/\delta\phi^i$

We consider only the case $b_\mu \equiv 0$, i.e. that $Y^{\mu\nu}[\bar{g}]$ does not contain derivatives. In the following we abbreviate: $t \equiv t[h, \bar{\xi}, \xi]$, $\sigma \equiv \sigma[h, \bar{\xi}, \xi]$ and $\bar{\sigma} \equiv \bar{\sigma}[h, \bar{\xi}, \xi]$ as well as $W = W[J[\phi]; \bar{g}]$. We first show the relation for $i = 1$ and then for $i = 3$. The calculation for $i = 2$ is fully analogous.

$$\begin{aligned}
\frac{\delta\Gamma[\phi;\bar{g}]}{\delta h_{\alpha\beta}(y)} = \int \mathrm{d}^d x \sqrt{\bar{g}} \Bigg[& \frac{\delta t^{\mu\nu}(x)}{\delta h_{\alpha\beta}(y)} h_{\mu\nu}(x) + t^{\mu\nu}(x) \underbrace{\frac{\delta h_{\mu\nu}(x)}{\delta h_{\alpha\beta}(y)}}_{=I^{\alpha\beta}_{\mu\nu}\delta(x-y)} \\
& + \frac{\delta\bar{\sigma}_\mu(x)}{\delta h_{\alpha\beta}(y)} \xi^\mu(x) + \frac{\delta\sigma^\mu(x)}{\delta h_{\alpha\beta}(y)} \bar{\xi}_\mu(x) \Bigg] \\
- \int \mathrm{d}^d x & \left[\frac{\delta t^{\mu\nu}(x)}{\delta h_{\alpha\beta}(y)} \underbrace{\frac{\delta W}{\delta t^{\mu\nu}(x)}}_{=\sqrt{\bar{g}(x)}h_{\mu\nu}(x)} + \frac{\delta\sigma^\mu(x)}{\delta h_{\alpha\beta}(y)} \underbrace{\frac{\delta W}{\delta\sigma^\mu(x)}}_{=\sqrt{\bar{g}(x)}\bar{\xi}_\mu(x)} + \frac{\delta\bar{\sigma}_\mu(x)}{\delta h_{\alpha\beta}(y)} \underbrace{\frac{\delta W}{\delta\bar{\sigma}_\mu(x)}}_{=\sqrt{\bar{g}(x)}\xi^\mu(x)} \right] \\
= \sqrt{\bar{g}(y)}\, t^{\alpha\beta}(y)\,. &
\end{aligned}$$

In the case $i = 3$ it is crucial to note that when calculating the *left* derivative of a composition of functions of even and odd variables, the inner derivatives stands *left* of the outer derivative (cf. Footnote 1 of Chapter 17).

$$\begin{aligned}
\frac{\delta\Gamma[\phi;\bar{g}]}{\xi^\alpha(y)} = \int \mathrm{d}^d x \sqrt{\bar{g}} \Bigg[& \frac{\delta t^{\mu\nu}(x)}{\delta\xi^\alpha(y)} h_{\mu\nu}(x) + \frac{\delta\bar{\sigma}_\mu(x)}{\delta\xi^\alpha(y)} \xi^\mu(x) \\
& - \bar{\sigma}_\mu(x) \frac{\delta\xi^\mu(x)}{\delta\xi^\alpha(y)} + \frac{\delta\sigma^\mu(x)}{\delta\xi^\alpha(y)} \bar{\xi}_\mu(x) \Bigg] \\
- \int \mathrm{d}^d x & \left[\frac{\delta t^{\mu\nu}(x)}{\delta\xi^\alpha(y)} \frac{\delta W}{\delta t^{\mu\nu}(x)} + \frac{\delta\sigma^\mu(x)}{\delta\xi^\alpha(y)} \frac{\delta W}{\delta\sigma^\mu(x)} + \frac{\delta\bar{\sigma}_\mu(x)}{\delta\xi^\alpha(y)} \frac{\delta W}{\delta\bar{\sigma}_\mu(x)} \right] \\
= - \sqrt{\bar{g}(y)}\, \bar{\sigma}_\alpha(y)\,. &
\end{aligned}$$

F.6. Derivation of the Fadeev-Popov operator $\mathscr{M}[\widehat{g},\bar{g}]^{\mu}{}_{\nu}$

$$
\begin{aligned}
S_{\rm gh}[\widehat{h},\bar{C},C;\bar{g}] &= -\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu \bar{g}^{\mu\nu}\frac{\partial F_\nu}{\partial \widehat{h}_{\alpha\beta}} L_C(\bar{g}_{\alpha\beta}+\widehat{h}_{\alpha\beta}) \\
&= -\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu \bar{g}^{\mu\nu}\sqrt{2}\mathscr{F}_\mu^{\alpha\beta}[\bar{g}] L_C(\bar{g}_{\alpha\beta}+\widehat{h}_{\alpha\beta}) \\
&= -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu\left[\bar{g}^{\mu\beta}\bar{g}^{\alpha\gamma}\bar{D}_\gamma - \beta\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{D}_\nu\right] L_C\widehat{g}_{\alpha\beta}\,.
\end{aligned}
$$

Next, add

$$
\begin{aligned}
0 = &-\beta\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda\left[\widehat{g}_{\sigma\nu}\widehat{g}^{\nu\delta}(\partial_\rho\widehat{g}_{\delta\tau}-\partial_\delta\widehat{g}_{\rho\tau})C^\tau\right] \\
&+\frac{1}{2}\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\left[\widehat{g}_{\rho\nu}\widehat{g}^{\nu\delta}(\partial_\sigma\widehat{g}_{\delta\tau}-\partial_\delta\widehat{g}_{\sigma\tau})C^\tau\right] \\
&+\frac{1}{2}\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\left[\widehat{g}_{\sigma\nu}\widehat{g}^{\nu\delta}(\partial_\rho\widehat{g}_{\delta\tau}-\partial_\delta\widehat{g}_{\rho\tau})C^\tau\right]
\end{aligned}
$$

in form of a "zero" to

$$
\begin{aligned}
\left[\bar{g}^{\mu\beta}\bar{g}^{\alpha\gamma}\bar{D}_\gamma - \beta\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{D}_\nu\right] L_C\widehat{g}_{\alpha\beta} = &+\bar{g}^{\mu\beta}\bar{g}^{\alpha\gamma}\bar{D}_\gamma(C^\rho\partial_\rho\widehat{g}_{\alpha\beta}) \\
&+\bar{g}^{\mu\beta}\bar{g}^{\alpha\gamma}\bar{D}_\gamma((\partial_\alpha C^\rho)\widehat{g}_{\rho\beta}) \\
&+\bar{g}^{\mu\beta}\bar{g}^{\alpha\gamma}\bar{D}_\gamma((\partial_\beta C^\rho)\widehat{g}_{\alpha\rho}) \\
&-\beta\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{D}_\nu(C^\rho\partial_\rho\widehat{g}_{\alpha\beta}) \\
&-\beta\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{D}_\nu((\partial_\alpha C^\rho)\widehat{g}_{\rho\beta}) \\
&-\beta\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{D}_\nu((\partial_\beta C^\rho)\widehat{g}_{\alpha\rho})\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left[\bar{g}^{\mu\beta}\bar{g}^{\alpha\gamma}\bar{D}_\gamma - \beta\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\bar{D}_\nu\right] L_C\widehat{g}_{\alpha\beta} = &\,\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\left[\bar{D}_\lambda(\widehat{g}_{\rho\nu}D_\sigma C^\nu)+\bar{D}_\lambda(\widehat{g}_{\sigma\nu}D_\rho C^\nu)\right] \\
&-2\beta\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda(\widehat{g}_{\sigma\nu}D_\rho C^\nu)\,.
\end{aligned}
$$

Insert this back into $S_{\rm gh}$:

$$
\begin{aligned}
S_{\rm gh}[h,\bar{C},C;\bar{g}] &= -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu\left[\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(\widehat{g}_{\rho\nu}D_\sigma+\widehat{g}_{\sigma\nu}D_\rho)-2\beta\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda\widehat{g}_{\sigma\nu}D_\rho\right]C^\nu \\
&= -\sqrt{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{C}_\mu\mathscr{M}[\widehat{g},\bar{g}]^{\mu}{}_{\nu}C^\nu\,.
\end{aligned}
$$

F.7. Derivation of the operator $(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})_{\mathrm{EH}}$

Here, we explicitly calculate the inverse propagator $\left(\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma}\right)_{\mathrm{EH}}$ for the Einstein-Hilbert action (4.3) with gauge fixing action (4.33) (with unspecified gauge fixing parameters), defined by Eq. (4.42),

$$\int \mathrm{d}^d x \int \mathrm{d}^d y\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\mathrm{EH}}[\bar{g}+\widehat{h}] + S_{\mathrm{GF}}[\widehat{h};\bar{g}])}{\delta \widehat{h}_{\mu\nu}(x) \delta \widehat{h}_{\rho\sigma}(y)}\Bigg|_{\chi=0} \widehat{h}_{\rho\sigma}(y) =: \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu} (\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}}\, \widehat{h}^{\rho\sigma}\,.$$

We define the integrals I_1, I_2 and I_3 as:

$$\begin{aligned}
&\int \mathrm{d}^d y \int \mathrm{d}^d z\, \widehat{h}_{\mu\nu}(x) \frac{\delta^2 (S_{\mathrm{EH}}[\bar{g}+\widehat{h}] + S_{\mathrm{GF}}[\widehat{h};\bar{g}])}{\delta \widehat{h}_{\mu\nu}(y) \delta \widehat{h}_{\rho\sigma}(z)}\Bigg|_{\chi=0} \widehat{h}_{\rho\sigma}(z) \\
&= -2\kappa^2 \int \mathrm{d}^d x \int \mathrm{d}^d y \int \mathrm{d}^d z\, \widehat{h}_{\mu\nu}(y) \left[\frac{\delta^2}{\delta g_{\mu\nu}(y)\delta g_{\rho\sigma}(z)} \left(\sqrt{g(x)} R x\right)\right]_{g=\bar{g}} \widehat{h}_{\rho\sigma}(z) \\
&\quad + 4\Lambda\kappa^2 \int \mathrm{d}^d x \int \mathrm{d}^d y \int \mathrm{d}^d z\, \widehat{h}_{\mu\nu}(y) \left[\frac{\delta^2}{\delta g_{\mu\nu}(y)\delta g_{\rho\sigma}(z)} \sqrt{g(x)}\right]_{g=\bar{g}} \widehat{h}_{\rho\sigma}(z) \\
&\quad + \frac{1}{\alpha}\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}(x)}\, \bar{g}^{\gamma\delta}(x) \int \mathrm{d}^d y \int \mathrm{d}^d z\, \widehat{h}_{\mu\nu}(y) \\
&\qquad \times \left[\frac{\delta^2}{\delta \widehat{h}_{\mu\nu}(y)\delta \widehat{h}_{\rho\sigma}(z)} \left(\mathscr{F}_\gamma^{\alpha\beta}[\bar{g}]_x^{\mathrm{diff}} \widehat{h}_{\alpha\beta}(x)\right) \left(\mathscr{F}_\delta^{\tau\varepsilon}[\bar{g}]_x^{\mathrm{diff}} \widehat{h}_{\tau\varepsilon}(x)\right)\right]_{\widehat{h}=0} \widehat{h}_{\rho\sigma}(z) \\
&=: I_1 + I_2 + I_3\,,
\end{aligned}$$

where we had taken into account that $\mathscr{F}_\mu^{\alpha\beta}[\bar{g}]\widehat{g}_{\alpha\beta} = \mathscr{F}_\mu^{\alpha\beta}[\bar{g}]\widehat{h}_{\alpha\beta}$. Next we bring the integrals I_1, I_2 and I_3 independently into such a form that, at the end, we can read off $\mathscr{U}[0;\bar{g}]$.

We start with I_3, noting that further one has $\mathscr{F}_\mu^{\alpha\beta}[\bar{g}]\widehat{h}_{\alpha\beta} = \bar{D}^\alpha \widehat{h}_{\alpha\mu} - \beta \bar{D}_\mu \widehat{h}_\alpha^\alpha$ and therewith:[1]

$$\begin{aligned}
I_3 = \frac{1}{\alpha} 2\kappa^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \bar{g}^{\gamma\delta}(x) \int \mathrm{d}^d y \int \mathrm{d}^d z\, \widehat{h}_{\mu\nu}(y) \left(\mathscr{F}_\gamma^{\alpha\beta}[\bar{g}]_x^{\mathrm{diff}} I_{\alpha\beta}^{\mu\nu} \delta(x-y)\right) \\
\times \left(\mathscr{F}_\delta^{\tau\varepsilon}[\bar{g}]_x^{\mathrm{diff}} I_{\tau\varepsilon}^{\rho\sigma} \delta(x-z)\right) \widehat{h}_{\rho\sigma}(z)
\end{aligned}$$

[1]Remember also that $\frac{\delta \widehat{h}_{\mu\nu}(x_1)}{\delta \widehat{h}_{\alpha\beta}(x_2)} = I_{\mu\nu}^{\alpha\beta} \delta(x_1 - x_2)$ with $I_{\mu\nu}^{\alpha\beta}$ defined in Eq. (A.24).

$$
\begin{aligned}
&= \frac{1}{\alpha} 2\kappa^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \bar{g}^{\mu\nu}(x) \int \mathrm{d}^d y \int \mathrm{d}^d z\, \widehat{h}_{\alpha\beta}(y) \\
&\qquad \times \left[\left(\delta^\beta_\mu \bar{g}^{\alpha\gamma}(x) \bar{D}^y_\gamma - \beta \bar{g}^{\alpha\beta}(x) \bar{D}^y_\mu \right) \delta(x-y) \right] \\
&\qquad \times \left[\left(\delta^\sigma_\nu \bar{g}^{\rho\tau}(x) \bar{D}^z_\tau - \beta \bar{g}^{\rho\sigma}(x) \bar{D}^z_\nu \right) \delta(x-z) \right] \widehat{h}_{\rho\sigma}(z)\,,
\end{aligned}
$$

where have used that $\bar{D}^x_\mu \delta(x-y) = \partial^x_\mu \delta(x-y) = -\partial^y_\mu \delta(x-y)$. Next, we perform two partial integrations, in y and z respectively, in order two evaluate the delta functions. Yet another partial integration in x yields

$$
\begin{aligned}
I_3 = &-\frac{1}{\alpha} 2\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}(x)}\, \bar{g}^{\alpha\beta}(x) \widehat{h}_{\mu\nu} \Big[(\delta^\nu_\alpha \bar{g}^{\mu\gamma} \bar{D}_\gamma - \beta \bar{g}^{\mu\nu} \bar{D}_\alpha) \\
&\qquad\qquad \times (\delta^\sigma_\beta \bar{g}^{\rho\tau} \bar{D}_\tau - \beta \bar{g}^{\rho\sigma} \bar{D}_\beta) \Big]_x \widehat{h}_{\rho\sigma}(x) \\
= &\frac{1}{\alpha} 2\kappa^2 \int \mathrm{d}^d x \sqrt{g(x)}\, \widehat{h}_{\mu\nu}(x) \Big[-\delta^\nu_\sigma \bar{g}^{\mu\gamma} \bar{D}_\gamma \bar{D}_\rho + \beta \bar{g}^{\nu\beta} \bar{g}^{\mu\gamma} \bar{g}_{\rho\sigma} \bar{D}_\gamma \bar{D}_\beta \\
&\qquad\qquad + \beta \bar{g}^{\mu\nu} \bar{D}_\sigma \bar{D}_\rho - \beta^2 d\, (P_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} \bar{D}^2 \Big]_x \widehat{h}^{\rho\sigma}(x)\,,
\end{aligned}
$$

where $(\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} = (1/d) \bar{g}_{\rho\sigma} \bar{g}^{\mu\nu}$ is the projector on the trace part of $\bar{g}_{\mu\nu}$, see appendix A.2.2.

Next, the integral I_2 can be quickly treated. Using the second variation of the metric determinant (cf. appendix B),

$$
\frac{\delta^2 \sqrt{g(x)}}{\delta g_{\mu\nu}(y) \delta g_{\rho\sigma}(z)} = \frac{\sqrt{g(x)}}{2} \left[\frac{1}{2} g^{\mu\nu} g^{\rho\sigma} - \frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \right] (x) \delta(x-y) \delta(x-z)\,,
$$

one straightforwardly obtains

$$
I_2 = 2\kappa^2 \Lambda \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu} \left[\frac{d}{2} (\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma} \right] \widehat{h}^{\rho\sigma}\,.
$$

Lastly, we determine I_3. To do so, we again use the metric variations stated in appendix B:

$$
\begin{aligned}
\widehat{h}_{\mu\nu}(y) \frac{\delta^2 (\sqrt{g(x)} R(x))}{\delta g_{\mu\nu}(y) \delta g_{\rho\sigma}(z)} \widehat{h}_{\rho\sigma}(z) = \sqrt{g(x)} \delta(x-y) \widehat{h}_{\mu\nu}(y) \Bigg\{ & \\
&+ \frac{1}{2} \left[\frac{1}{2} g^{\mu\nu} g^{\rho\sigma} - \frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \right] R \\
&- g^{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} R_{\alpha\beta} + g^{\mu\nu} g^{\rho\alpha} g^{\sigma\beta} D_\beta D_\alpha \\
&- \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} D^2 + g^{\mu\alpha} g^{\nu\sigma} g^{\rho\beta} R_{\alpha\beta}
\end{aligned}
$$

$$- g^{\mu\alpha} g^{\nu\beta} g^{\rho\delta} g^{\sigma\tau} R_{\delta\alpha\beta\tau} - g^{\mu\sigma} g^{\nu\gamma} g^{\rho\tau} D_\gamma D_\tau + \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} D^2 \Bigg\}_x \delta(x-z) \widehat{h}_{\rho\sigma}(z)\,.$$

Therewith one has

$$\begin{aligned} I_1 = 2\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}(x)}\, \widehat{h}_{\mu\nu}(x) \Bigg\{ & - \frac{1}{2} \left[\frac{d}{2} (P_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma} \right] R \\ & + \bar{g}^{\mu\nu} \bar{R}_{\rho\sigma} - \bar{g}^{\mu\nu} \bar{D}_\sigma \bar{D}_\rho + \frac{d}{2} (P_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} \bar{D}^2 - \bar{g}^{\mu\alpha} \delta^\nu_\sigma \bar{R}_{\alpha\rho} \\ & + \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} \bar{R}_{\rho\alpha\beta\sigma} + \delta^\mu_\sigma \bar{g}^{\nu\gamma} \bar{D}_\gamma \bar{D}_\rho - \frac{1}{2} I^{\mu\nu}{}_{\rho\sigma} \bar{D}^2 \Bigg\}_x \widehat{h}^{\rho\sigma}(x)\,. \end{aligned}$$

Putting these results together, the off-diagonal terms can be summed up by exploiting the symmetry $(\mu\nu) \leftrightarrow (\rho\sigma)$ and partial integration. One finds

$$\begin{aligned} I_1 + I_2 + I_3 = 2\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu} \Bigg\{ & \left(1 - \frac{1}{\alpha}\right) \delta^\nu_\sigma \bar{g}^{\mu\gamma} \bar{D}_\gamma \bar{D}_\rho + \left(2\frac{\beta}{\alpha} - 1\right) \bar{g}^{\mu\nu} \bar{D}_\sigma \bar{D}_\rho \\ & + \frac{1}{2} \left[d \left(1 - 2\frac{\beta^2}{\alpha}\right) (\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma} \right] \bar{D}^2 \\ & + \left[\frac{d}{2} (\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma} \right] \left(\Lambda - \frac{1}{2} \bar{R} \right) \\ & + \bar{g}^{\mu\nu} \bar{R}_{\rho\sigma} - \bar{g}^{\mu\alpha} \delta^\nu_\sigma \bar{R}_{\alpha\rho} + \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} \bar{R}_{\rho\alpha\beta\sigma} \Bigg\} \widehat{h}^{\rho\sigma}\,. \end{aligned}$$

As $I_1 + I_2 + I_3 = \int \mathrm{d}^d x \sqrt{\bar{g}}\; \widehat{h}_{\mu\nu} (\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}}_{\mathrm{EH}} \widehat{h}^{\rho\sigma}$ holds per construction, we can read of the operator $(\mathscr{U}[0;\bar{g}])_{\mathrm{EH}}$:

$$\begin{aligned} \kappa^{-2} (\mathscr{U}[0;\bar{g}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}}_{\mathrm{EH}} = \; & \left[d \left(1 - 2\frac{\beta^2}{\alpha}\right) (\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma} \right] \bar{D}^2 \\ & + \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \big[\delta^\mu_\sigma \bar{D}^\nu \bar{D}_\rho + \delta^\nu_\sigma \bar{D}^\mu \bar{D}_\rho \\ & \qquad + \delta^\mu_\rho \bar{D}^\nu \bar{D}_\sigma + \delta^\nu_\rho \bar{D}^\mu \bar{D}_\sigma \big] \\ & + \frac{1}{2} \left(2\frac{\beta}{\alpha} - 1\right) \big[\bar{g}^{\mu\nu} \bar{D}_\sigma \bar{D}_\rho + \bar{g}^{\mu\nu} \bar{D}_\rho \bar{D}_\sigma \\ & \qquad + \bar{g}_{\rho\sigma} \bar{D}^\mu \bar{D}^\nu + \bar{g}_{\rho\sigma} \bar{D}^\nu \bar{D}^\mu \big] \\ & + \left[\frac{d}{2} (\bar{P}_{\mathrm{tr.}})^{\mu\nu}{}_{\rho\sigma} - I^{\mu\nu}{}_{\rho\sigma} \right] (2\Lambda - \bar{R}) \end{aligned}$$

$$
\begin{aligned}
&+ \left[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\right] \\
&- \frac{1}{2}\left[\delta^{\nu}_{\sigma}\bar{R}^{\mu}{}_{\rho} + \delta^{\nu}_{\rho}\bar{R}^{\mu}{}_{\sigma} + \delta^{\mu}_{\sigma}\bar{R}^{\nu}{}_{\rho} + \delta^{\mu}_{\rho}\bar{R}^{\nu}{}_{\sigma}\right] \\
&- \left[\bar{R}_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu} + \bar{R}_{\sigma}{}^{\mu}{}_{\rho}{}^{\nu}\right] .
\end{aligned}
$$

F.8. Derivation of the operator $(\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma})_{\text{h.-d.}}$

We spread the calculation of the operator $(\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma})_{\text{h.-d.}}$ given by Eq. (4.98) into two parts: **(A)** The variation of $S_{\text{h.-d.}}[\bar{g}+\widehat{h}]$, given by Eq. (4.4), on the one hand and **(B)** that of $S_{\text{GF}}[\widehat{h};\bar{g}]$, given by Eq. (4.85), on the other hand. In a further step **(C)**, we will specify this result to four spacetime dimensions which leads to Eq. (4.100). For that purpose, we introduce two auxiliary operators, $\Omega^{\mu\nu}{}_{\rho\sigma}[\bar{g}]$ and $\Omega_{\text{GF}}{}^{\mu\nu}{}_{\rho\sigma}[\bar{g}]$. The former arises from applying the variation $\delta\bar{g}_{\mu\nu}:=\widehat{h}_{\mu\nu}$ twice to the higher-derivative action evaluated at $\bar{g}$,

$$\delta^2 S_{\text{h.-d.}}[\bar{g}] =: \int \mathrm{d}^d x\sqrt{\bar{g}}\,\widehat{h}_{\mu\nu}\Omega^{\mu\nu}{}_{\rho\sigma}[\bar{g}]^{\text{diff}}\,\widehat{h}^{\rho\sigma}\,;$$

while the latter arises from reformulating the gauge-fixing action $S_{\text{GF}}[\widehat{h};\bar{g}]$ that per se already is quadratic in $\widehat{h}_{\mu\nu}$,

$$S_{\text{GF}}[\widehat{h};\bar{g}] =: \int \mathrm{d}^d x\sqrt{\bar{g}}\,\widehat{h}_{\mu\nu}\Omega_{\text{GF}}{}^{\mu\nu}{}_{\rho\sigma}[\bar{g}]^{\text{diff}}\,\widehat{h}^{\rho\sigma}\,.$$

A direct cosequence of these definitions together with that of the operator $(\mathscr{U}[0;\bar{g}_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma})_{\text{h.-d.}}$, Eq. (4.95), is

$$\int \mathrm{d}^d x\sqrt{\bar{g}}\,\widehat{h}_{\mu\nu}\left(\mathscr{U}[0;\bar{g}_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma}\right)^{\text{diff}}_{\text{h.-d.}}\widehat{h}^{\rho\sigma} = \int \mathrm{d}^d x\sqrt{\bar{g}}\,\widehat{h}_{\mu\nu}\Omega_{\text{GF}}{}^{\mu\nu}{}_{\rho\sigma}[\bar{g}]^{\text{diff}}\,\widehat{h}^{\rho\sigma}\,,$$

respectively,

$$\mathscr{U}[0;\bar{g}]_{\text{h.-d.}} = \Omega[\bar{g}] + 2\,\Omega_{\text{GF}}[\bar{g}]\,. \tag{F.2}$$

(A) With some straightforward algebra the second variation of $S_{\text{h.-d.}}[\bar{g}]$ can be expanded as

$$\begin{aligned}\delta^2 S_{\text{h.-d.}}[\bar{g}] &= \int \mathrm{d}^d x\left\{a\,\delta^2\big(\sqrt{\bar{g}}\,\bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta}\big) + b\,\delta^2\big(\sqrt{\bar{g}}\,\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\big) + c\,\delta^2\big(\sqrt{\bar{g}}\,\bar{R}^2\big)\right\}\\ &= \int \mathrm{d}^d x\Big\{\big(\delta^2\sqrt{\bar{g}}\big)\left[a\,\bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta} + b\,\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + c\,\bar{R}^2\right]\\ &\qquad + 2\left(\delta\sqrt{\bar{g}}\right)\left[a\,\delta\big(\bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta}\big) + b\,\delta\big(\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\big) + c\,\delta\big(\bar{R}^2\big)\right]\\ &\qquad + \sqrt{\bar{g}}\,\left[a\,\delta^2\big(\bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta}\big) + b\,\delta^2\big(\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\big) + c\,\delta^2\big(\bar{R}^2\big)\right]\Big\}.\end{aligned}$$

Further expanding the Riemann, Ricci and scalar curvature in the last term according to $\delta^2\bar{R}^2 = 2\delta(\bar{R}\,\delta\bar{R}) = 2[(\delta\bar{R})^2 + \bar{R}\,\delta^2\bar{R}]$, it is clear that the only surviving term when evaluating this second variation on flat space with $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$, i.e. the only curvature-independent term, is "$2(\delta\bar{R})^2$"; and thus

$$\delta^2 S_{\text{h.-d.}}[\bar{g}]\Big|_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}} = \int \mathrm{d}^d x \underbrace{\sqrt{\bar{g}}\big|_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}}}_{=1} \Big\{ 2a\left(\delta\bar{R}_{\mu\nu\alpha\beta}\right)\left(\delta\bar{R}^{\mu\nu\alpha\beta}\right) + 2b\left(\delta\bar{R}_{\mu\nu}\right)\left(\delta\bar{R}^{\mu\nu}\right) + 2c\left(\delta\bar{R}\right)^2 \Big\}_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}} .$$

The variations required here can be found in appendix B and projected onto flat space read, with $\widehat{h} \equiv \widehat{h}_\alpha{}^\alpha$ and $\Box = \partial_\alpha\partial^\alpha$,

$$\begin{aligned}
\delta\bar{R}_{\mu\nu\alpha\beta}\Big|_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}} &= \frac{1}{2}\Big(- \partial_\mu\partial_\alpha\widehat{h}_{\nu\beta} + \partial_\nu\partial_\alpha\widehat{h}_{\mu\beta} + \partial_\mu\partial_\beta\widehat{h}_{\nu\alpha} - \partial_\nu\partial_\beta\widehat{h}_{\mu\alpha}\Big)\\
\delta\bar{R}_{\mu\nu}\Big|_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}} &= \frac{1}{2}\Big(\partial_\alpha\partial_\nu\widehat{h}_\mu{}^\alpha - \partial_\mu\partial_\nu\widehat{h} - \Box\widehat{h}_{\mu\nu} + \partial_\mu\partial^\alpha\widehat{h}_{\alpha\nu}\Big)\\
\delta\bar{R}\Big|_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}} &= \partial_\beta\partial_\alpha\widehat{h}^{\alpha\beta} - \Box\widehat{h}\,.
\end{aligned}$$

Next, we bring these variations, when squared and integrated, into the desired form. *For the coupling a*:

$$\begin{aligned}
\int \mathrm{d}^d x\ \left(\delta\bar{R}_{\mu\nu\alpha\beta}\right)\delta\bar{R}^{\mu\nu\alpha\beta}\Big|_{\bar{g}=\delta} &= \int \mathrm{d}^d x\,\frac{1}{4}\Big(- \partial^\mu\partial^\alpha\widehat{h}^{\nu\beta} + \partial^\nu\partial^\alpha\widehat{h}^{\mu\beta} + \partial^\mu\partial^\beta\widehat{h}^{\nu\alpha} - \partial^\nu\partial^\beta\widehat{h}^{\mu\alpha}\Big)\\
&\qquad\times\Big(- \partial_\mu\partial_\alpha\widehat{h}_{\nu\beta} + \partial_\nu\partial_\alpha\widehat{h}_{\mu\beta} + \partial_\mu\partial_\beta\widehat{h}_{\nu\alpha} - \partial_\nu\partial_\beta\widehat{h}_{\mu\alpha}\Big)\\
&= \int \mathrm{d}^d x\,\frac{1}{4}\Big(+ \widehat{h}^{\nu\beta}\partial^\alpha\partial^\mu\partial_\mu\partial_\alpha\widehat{h}_{\nu\beta} - \widehat{h}^{\nu\beta}\partial^\alpha\partial^\mu\partial_\nu\partial_\alpha\widehat{h}_{\mu\beta} - \widehat{h}^{\nu\beta}\partial^\alpha\partial^\mu\partial_\mu\partial_\beta\widehat{h}_{\nu\alpha}\\
&\quad + \widehat{h}^{\nu\beta}\partial^\alpha\partial^\mu\partial_\nu\partial_\beta\widehat{h}_{\mu\alpha} - \widehat{h}^{\mu\beta}\partial^\alpha\partial^\nu\partial_\mu\partial_\alpha\widehat{h}_{\nu\beta} + \widehat{h}^{\mu\beta}\partial^\nu\partial^\alpha\partial^\nu\partial_\alpha\widehat{h}_{\mu\beta}\\
&\quad + \widehat{h}^{\mu\beta}\partial^\alpha\partial^\nu\partial_\mu\partial_\beta\widehat{h}_{\nu\alpha} - \widehat{h}^{\mu\beta}\partial^\nu\partial^\alpha\partial_\nu\partial_\beta\widehat{h}_{\mu\alpha} - \widehat{h}^{\nu\alpha}\partial^\beta\partial^\mu\partial_\mu\partial_\alpha\widehat{h}_{\nu\beta}\\
&\quad + \widehat{h}^{\nu\alpha}\partial^\beta\partial^\mu\partial_\nu\partial_\alpha\widehat{h}_{\mu\beta} + \widehat{h}^{\nu\alpha}\partial^\beta\partial_\mu\partial_\mu\partial_\beta\widehat{h}_{\nu\alpha} - \widehat{h}^{\nu\alpha}\partial^\beta\partial^\mu\partial_\nu\partial_\beta\widehat{h}_{\mu\alpha}\\
&\quad + \widehat{h}^{\mu\alpha}\partial^\beta\partial^\nu\partial_\mu\partial_\alpha\widehat{h}_{\nu\beta} - \widehat{h}^{\mu\alpha}\partial^\beta\partial^\nu\partial_\nu\partial_\alpha\widehat{h}_{\mu\beta} - \widehat{h}^{\mu\alpha}\partial^\beta\partial^\nu\partial_\mu\partial_\beta\widehat{h}_{\nu\alpha}\\
&\quad + \widehat{h}^{\mu\alpha}\partial^\beta\partial^\nu\partial_\nu\partial_\beta\widehat{h}_{\mu\alpha}\Big)
\end{aligned}$$

$$
\begin{aligned}
= \int \mathrm{d}^d x \frac{1}{4} \widehat{h}_{\mu\nu} \Big(&+ \delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \partial^\mu\partial^\nu\partial^\alpha\partial^\beta \\
&- \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 + \partial^\mu\partial^\nu\partial^\alpha\partial^\beta - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box \\
&- \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \partial^\mu\partial^\nu\partial^\alpha\partial^\beta + \delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box \\
&+ \partial^\mu\partial^\nu\partial^\alpha\partial^\beta - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 \Big) \widehat{h}_{\alpha\beta} \\
= \int \mathrm{d}^d x \, \widehat{h}_{\mu\nu} \Big(&\delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 - 2\delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \partial^\mu\partial^\nu\partial^\alpha\partial^\beta \Big) \widehat{h}_{\alpha\beta} \, ;
\end{aligned}
$$

For the coupling b:

$$
\begin{aligned}
\int \mathrm{d}^d x \, \left(\delta \bar{R}_{\mu\nu}\right) \delta \bar{R}^{\mu\nu} \Big|_{\bar{g}=\delta} = \int \mathrm{d}^d x \frac{1}{4} &\Big(- \partial_\alpha\partial_\nu \widehat{h}_\mu{}^\alpha + \partial_\mu\partial_\nu \widehat{h} + \Box \widehat{h}_{\mu\nu} - \partial_\mu\partial^\alpha \widehat{h}_{\nu\alpha} \Big) \\
\times &\Big(- \partial_\beta\partial^\nu \widehat{h}^{\mu\beta} + \partial^\mu\partial^\nu \widehat{h} + \Box \widehat{h}^{\mu\nu} - \partial^\mu\partial^\beta \widehat{h}_\beta{}^\nu \Big)
\end{aligned}
$$

$$
\begin{aligned}
= \int \mathrm{d}^d x \frac{1}{4} \Big(&+ \widehat{h}_\mu{}^\alpha \partial_\nu\partial_\alpha\partial_\beta\partial^\nu \widehat{h}^{\mu\beta} - \widehat{h}_\mu{}^\alpha \partial_\nu\partial_\alpha\partial^\mu\partial^\nu \widehat{h} - \widehat{h}_\mu{}^\alpha \partial_\nu\partial_\alpha \Box \widehat{h}^{\mu\nu} \\
&+ \widehat{h}_\mu{}^\alpha \partial_\nu\partial_\alpha\partial^\mu\partial^\beta \widehat{h}_\beta{}^\nu - \widehat{h}\partial_\nu\partial_\mu\partial_\beta\partial^\nu \widehat{h}^{\mu\beta} + \widehat{h}\partial_\nu\partial_\mu\partial^\mu\partial^\nu \widehat{h} \\
&+ \widehat{h}\partial_\nu\partial_\mu \Box \widehat{h}^{\mu\nu} - \widehat{h}\partial_\nu\partial_\mu\partial^\mu\partial^\beta \widehat{h}_\beta{}^\nu - \widehat{h}_{\mu\nu}\Box\partial_\beta\partial^\nu \widehat{h}^{\mu\beta} \\
&+ \widehat{h}_{\mu\nu}\Box\partial^\mu\partial^\nu \widehat{h} + \widehat{h}_{\mu\nu}\Box^2 \widehat{h}^{\mu\nu} - \widehat{h}_{\mu\nu}\Box\partial^\mu\partial^\beta \widehat{h}_\beta{}^\nu \\
&+ \widehat{h}_{\nu\alpha}\partial^\alpha\partial_\mu\partial_\beta\partial^\nu \widehat{h}^{\mu\beta} - \widehat{h}_{\nu\alpha}\partial^\alpha\partial_\mu\partial^\mu\partial^\nu \widehat{h} - \widehat{h}_{\nu\alpha}\partial^\alpha\partial_\mu \Box \widehat{h}^{\mu\nu} \\
&+ \widehat{h}_{\nu\alpha}\partial^\alpha\partial_\mu\partial^\mu\partial^\beta \widehat{h}_\beta{}^\nu \Big) \\
= \int \mathrm{d}^d x \frac{1}{4} \widehat{h}_{\mu\nu} \Big(&+ \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box - \delta^{\mu\nu}\partial^\alpha\partial^\beta\Box - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \partial^\mu\partial^\nu\partial^\alpha\partial^\beta \\
&- \delta^{\mu\nu}\partial^\alpha\partial^\beta\Box + \delta^{\mu\nu}\delta^{\alpha\beta}\Box^2 + \delta^{\mu\nu}\partial^\alpha\partial^\beta\Box - \delta^{\mu\nu}\partial^\alpha\partial^\beta\Box \\
&- \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \delta^{\mu\nu}\partial^\alpha\partial^\beta\Box + \delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box \\
&+ \partial^\mu\partial^\nu\partial^\alpha\partial^\beta - \delta^{\mu\nu}\partial^\alpha\partial^\beta\Box - \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box + \delta^{\mu\alpha}\partial^\nu\partial^\beta\Box \Big) \widehat{h}_{\alpha\beta} \\
= \int \mathrm{d}^d x \, \widehat{h}_{\mu\nu} \frac{1}{4} \Big(&\delta^{\mu\nu}\delta^{\alpha\beta}\Box^2 + \delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 - 2\delta^{\mu\alpha}\partial^\nu\partial^\beta\Box \\
&- 2\delta^{\mu\nu}\partial^\alpha\partial^\beta\Box + 2\partial^\mu\partial^\nu\partial^\alpha\partial^\beta \Big) \widehat{h}_{\alpha\beta} \, ;
\end{aligned}
$$

For the coupling c:

$$
\int \mathrm{d}^d x \, \left(\delta \bar{R}\right) \delta \bar{R} \Big|_{\bar{g}=\delta} = \int \mathrm{d}^d x \left(\partial_\mu\partial_\alpha \widehat{h}^{\alpha\beta} - \Box \widehat{h} \right) \left(\partial_\mu\partial_\nu \widehat{h}^{\mu\nu} - \Box \widehat{h} \right)
$$

$$= \int \mathrm{d}^d x\, \widehat{h}_{\mu\nu} \left(\partial^\mu\partial^\nu\partial^\alpha\partial^\beta - 2\delta^{\mu\nu}\partial^\alpha\partial^\beta\Box + \delta^{\mu\nu}\delta^{\alpha\beta}\Box^2\right) \widehat{h}_{\alpha\beta}\,.$$

(In each second step of these three calculations we have used partial integration.) Therewith, it follows that

$$\begin{aligned}\delta^2 S_{\text{h.-d.}}[\bar{g}]\Big|_{\bar{g}_{\mu\nu}=\delta_{\mu\nu}} = \int \mathrm{d}^d x\, \widehat{h}_{\mu\nu} \Bigg[&\left(\frac{b}{2}+2c\right)\delta^{\mu\nu}\delta^{\alpha\beta}\Box^2 + \left(\frac{B}{2}+2a\right)\delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 \\ &- (b+4a)\delta^{\mu\alpha}\partial^\nu\partial^\beta\Box - (b+4c)\delta^{\mu\nu}\partial^\alpha\partial^\beta\Box \\ &+ (2a+b+2c)\partial^\mu\partial^\nu\partial^\alpha\partial^\beta \Bigg] \widehat{h}_{\alpha\beta}\end{aligned}$$

and thus, with $I^{\mu\nu}{}_{\rho\sigma}$ given by Eq. (A.24),

$$\begin{aligned}\Omega^{\mu\nu}{}_{\rho\sigma}\,[\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{\text{diff}} = &+\left(\frac{b}{2}+2c\right)\delta^{\mu\nu}\delta_{\rho\sigma}\Box^2 + \left(\frac{b}{2}+2a\right) I^{\mu\nu}{}_{\rho\sigma}\Box^2 \\ &- \frac{b+4a}{4}\left[\delta^\mu_\rho\partial^\nu\Box\partial_\sigma + \delta^\mu_\sigma\partial^\nu\Box\partial_\rho + \delta^\nu_\rho\partial^\mu\Box\partial_\sigma + \delta^\nu_\sigma\partial^\mu\Box\partial_\rho\right] \\ &- \frac{b+4c}{2}\left[\delta^{\mu\nu}\partial_\rho\Box\partial_\sigma + \delta_{\rho\sigma}\partial^\mu\Box\partial^\nu\right] \\ &+ (2a+b+2c)\partial^\mu\partial^\nu\partial_\rho\partial_\sigma\,.\end{aligned} \tag{F.3}$$

(B) Next, we re-write the gauge-fixing action (4.85) that is built from the gauge-fixing condition $F_\mu(\widehat{h};\bar{g})=0$ given by Eq. (4.31) and the weight function $Y^{\mu\nu}[\bar{g}]$ given by Eq. (4.84); we do so on a generic background with metric $\bar{g}_{\mu\nu}$ (i.e. here $\widehat{h}=\bar{g}^{\mu\nu}\widehat{h}_{\mu\nu}$, $\Box_{\bar{g}}=\bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ etc.):

$$\begin{aligned}S_{\text{GF}}[\widehat{h};\bar{g}] &= \frac{1}{2\alpha}\int \mathrm{d}^d x\sqrt{\bar{g}}\, F_\mu(\widehat{h};\bar{g}) Y^{\mu\nu}[\bar{g}]^{\text{diff}} F_\nu(\widehat{h};\bar{g}) \\ &= \frac{1}{\alpha}\int \mathrm{d}^d x\sqrt{\bar{g}}\left(\bar{D}_\rho\widehat{h}^\rho{}_\mu - \beta\bar{D}_\mu\widehat{h}\right) Y^{\mu\nu}[\bar{g}]^{\text{diff}}\left(\bar{D}_\sigma\widehat{h}^\sigma{}_\nu - \beta\bar{D}_\nu\widehat{h}\right)\end{aligned}$$

$$
\begin{aligned}
&= \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu} \Bigg[-\frac{1}{\alpha} \bar{D}^\mu \left(\bar{g}^{\nu\beta} \Box_{\bar{g}} + \gamma \bar{D}^\nu \bar{D}^\beta - \delta \bar{D}^\beta \bar{D}^\nu \right) \bar{D}^\alpha \\
&\qquad + \frac{\beta}{\alpha} \bar{g}^{\alpha\beta} \bar{D}^\mu \left(\bar{g}^{\nu\rho} \Box_{\bar{g}} + \gamma \bar{D}^\nu \bar{D}^\rho - \delta \bar{D}^\rho \bar{D}^\nu \right) \bar{D}_\rho \\
&\qquad + \frac{\beta}{\alpha} \bar{g}^{\mu\nu} \bar{D}_\rho \left(\bar{g}^{\rho\beta} \Box_{\bar{g}} + \gamma \bar{D}^\rho \bar{D}^\beta - \delta \bar{D}^\beta \bar{D}^\rho \right) \bar{D}^\alpha \\
&\qquad - \frac{\beta^2}{\alpha} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \bar{D}_\rho \left(\bar{g}^{\rho\sigma} \Box_{\bar{g}} + \gamma \bar{D}^\rho \bar{D}^\sigma - \delta \bar{D}^\sigma \bar{D}^\rho \right) \bar{D}_\sigma \Bigg] \widehat{h}_{\alpha\beta} \\
&= \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu} \Bigg[-\frac{1}{\alpha} \bar{g}^{\nu\beta} \bar{D}^\mu \Box_{\bar{g}} \bar{D}^\alpha - \frac{\gamma}{\alpha} \bar{D}^\mu \bar{D}^\nu \bar{D}^\alpha \bar{D}^\beta + \frac{\delta}{\alpha} \bar{D}^\mu \bar{D}^\nu \bar{D}^\alpha \bar{D}^\beta \\
&\qquad + \frac{\delta}{\alpha} \bar{D}^\mu [\bar{D}^\beta, \bar{D}^\nu] \bar{D}^\alpha + \frac{\beta}{\alpha} \bar{g}^{\alpha\beta} \bar{D}^\mu \Box_{\bar{g}} \bar{D}^\nu + \frac{\gamma\beta}{\alpha} \bar{g}^{\alpha\beta} \bar{D}^\mu \Box_{\bar{g}} \bar{D}^\nu \\
&\qquad + \frac{\gamma\beta}{\alpha} \bar{g}^{\alpha\beta} \bar{D}^\mu [\bar{D}^\nu, \Box_{\bar{g}}] - \frac{\delta\beta}{\alpha} \bar{g}^{\alpha\beta} \bar{D}^\mu \Box_{\bar{g}} \bar{D}^\nu - \frac{\delta\beta}{\alpha} \bar{g}^{\alpha\beta} \bar{D}^\mu \bar{D}^\rho [\bar{D}^\nu, \bar{D}_\rho] \\
&\qquad + \frac{\beta}{\alpha} \bar{g}^{\mu\nu} \bar{D}^\beta \Box_{\bar{g}} \bar{D}^\alpha + \frac{\gamma\beta}{\alpha} \bar{g}^{\mu\nu} \bar{D}^\beta \Box_{\bar{g}} \bar{D}^\alpha + \frac{\gamma\beta}{\alpha} \bar{g}^{\mu\nu} [\Box_{\bar{g}}, \bar{D}^\beta] \bar{D}^\alpha \\
&\qquad - \frac{\delta\beta}{\alpha} \bar{g}^{\mu\nu} \bar{D}^\beta \Box_{\bar{g}} \bar{D}^\alpha - \frac{\delta\beta}{\alpha} \bar{g}^{\mu\nu} [\bar{D}_\rho, \bar{D}^\beta] \bar{D}^\rho \bar{D}^\alpha - \frac{\beta^2}{\alpha} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \Box_{\bar{g}}^2 \\
&\qquad - \frac{\beta^2}{\alpha} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \bar{D}^\sigma [\Box_{\bar{g}}, \bar{D}_\sigma] - \frac{\beta^2\gamma}{\alpha} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \Box_{\bar{g}}^2 + \frac{\delta\beta^2}{\alpha} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \Box_{\bar{g}}^2 \\
&\qquad + \frac{\delta\beta^2}{\alpha} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \bar{D}_\rho [\bar{D}^\sigma, \bar{D}^\rho] \bar{D}_\sigma \Bigg] \widehat{h}_{\alpha\beta} \\
&= \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu} \Bigg[-\frac{1}{\alpha} \bar{g}^{\nu\beta} \bar{D}^\mu \Box_{\bar{g}} \bar{D}^\alpha + \frac{\delta - \gamma}{\alpha} \bar{D}^\mu \bar{D}^\nu \bar{D}^\alpha \bar{D}^\beta \\
&\qquad + 2\frac{\beta}{\alpha}(1 + \gamma - \delta) \bar{g}^{\mu\nu} \bar{D}^\alpha \Box_{\bar{g}} \bar{D}^\beta - \frac{\beta^2}{\alpha}(1 + \gamma - \delta) \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} \Box_{\bar{g}}^2 \\
&\qquad + \text{curvature-dependent terms} \Bigg] \widehat{h}_{\alpha\beta}
\end{aligned}
$$

Therefrom, we can read off the operator $\Omega_{\mathrm{GF}}{}^{\mu\nu}{}_{\rho\sigma}[\bar{g}]$ evaluated at $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ easily:

$$\begin{aligned}\Omega_{\mathrm{GF}}{}^{\mu\nu}{}_{\rho\sigma}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}] = &-\frac{1}{4\alpha}\left[\delta^{\mu}_{\rho}\partial^{\nu}\Box\partial_{\sigma}+\delta^{\mu}_{\sigma}\partial^{\nu}\Box\partial_{\rho}+\delta^{\nu}_{\rho}\partial^{\mu}\Box\partial_{\sigma}+\delta^{\nu}_{\sigma}\partial^{\mu}\Box\partial_{\rho}\right]\\ &+\frac{\delta-\gamma}{\alpha}\partial^{\mu}\partial^{\nu}\partial_{\rho}\partial_{\sigma}+\frac{\beta}{\alpha}(1+\gamma-\delta)\left[\delta^{\mu\nu}\partial_{\rho}\Box\partial_{\sigma}+\delta_{\rho\sigma}\partial^{\mu}\Box\partial^{\nu}\right]\\ &-\frac{\beta^2}{\alpha}(1+\gamma-\delta)\delta^{\mu\nu}\delta_{\rho\sigma}\Box^2\,.\end{aligned} \tag{F.4}$$

Finally, we can bring the result (F.3) from **(A)** and the result (F.4) from **(B)** together; with Eq. (F.2) these lead directly to the operator $\mathscr{U}[0;\bar{g}]_{\mathrm{h.-d.}}$,

$$\begin{aligned}(\mathscr{U}[0;\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma})^{\mathrm{diff}}_{\mathrm{h.-d.}} &= \Omega^{\mu\nu}{}_{\rho\sigma}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}] + 2\,\Omega_{\mathrm{GF}}{}^{\mu\nu}{}_{\rho\sigma}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}]\\ &= \left(\frac{b}{2}+2a\right)I^{\mu\nu}{}_{\rho\sigma}\Box^2\\ &\quad+\left[\left(\frac{b}{2}+2c\right)-2\frac{\beta^2}{\alpha}(1+\gamma-\delta)\right]\delta^{\mu\nu}\delta_{\rho\sigma}\Box^2\\ &\quad-\left[\frac{b+4a}{4}+\frac{1}{2\alpha}\right]\left\{\delta^{\mu}_{\rho}\partial^{\nu}\Box\partial_{\sigma}+\delta^{\mu}_{\sigma}\partial^{\nu}\Box\partial_{\rho}\right.\\ &\qquad\qquad\left.+\delta^{\nu}_{\rho}\partial^{\mu}\Box\partial_{\sigma}+\delta^{\nu}_{\sigma}\partial^{\mu}\Box\partial_{\rho}\right\}\\ &\quad-\left[\frac{b+4c}{2}-2\frac{\beta}{\alpha}(1+\gamma-\delta)\right]\left\{\delta^{\mu\nu}\partial_{\rho}\Box\partial_{\sigma}\right.\\ &\qquad\qquad\left.+\delta_{\rho\sigma}\partial^{\mu}\Box\partial^{\nu}\right\}\\ &\quad+\left[2a+b+2c+2\frac{\delta-\gamma}{\alpha}\right]\partial^{\mu}\partial^{\nu}\partial_{\rho}\partial_{\sigma}\,.\end{aligned}$$

This precisely is Eq. (4.98) and thus what we wished to show.

(C) In four spacetime dimensions, $d = 4$, we may eliminate one coupling thanks to the undynamical Gauss-Bonnet term. The couplings f_0 and f_2 that are related to a, b and c via Eq. (**??**) can be regarded as the most convenient choice; thence subsequently let us rewrite the operator $\Omega^{\mu\nu}{}_{\rho\sigma}\,[\bar{g}_{\mu\nu}=\delta_{\mu\nu}]$ given by Eq. (F.3) in terms of the couplings f_0 and f_2. Therefore, we introduce the auxiliary notation

$$A_{\mu\nu\alpha\beta}\,\mathrm{sym.} := A_{((\mu\nu)(\alpha\beta))}\,,$$

i.e. "sym." indicates that a tensor structure is to be fully symmetrized. One has

$$
\begin{aligned}
\Omega^{\mu\nu\alpha\beta}\left[\bar{g}_{\mu\nu}=\delta_{\mu\nu}\right] = & \Bigg\{ + \left(\frac{b}{2}+2c\right)\delta^{\mu\nu}\delta^{\alpha\beta}\Box^2 + \left(\frac{b}{2}+2a\right)\delta^{\mu\alpha}\delta^{\nu\beta}\Box^2 \\
& - (b+4a)\delta^{\mu\alpha}\partial^\nu\Box\partial^\beta - (b+4c)\delta^{\mu\nu}\partial^\alpha\Box\partial^\beta \\
& + (2a+b+2c)\partial^\mu\partial^\nu\partial^\alpha\partial^\beta \Bigg\}\,\text{sym.} \\
= \Box^2 \Bigg\{ & \frac{1}{2f_2^2}\Bigg[+ \delta^{\mu\nu}\delta^{\alpha\beta} - \frac{4}{3}\delta^{\mu\nu}\delta^{\alpha\beta} + \delta^{\mu\alpha}\delta^{\nu\beta} \\
& - 2\delta^{\mu\alpha}\frac{\partial^\nu\partial^\beta}{\Box} - 2\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box} + \frac{8}{3}\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box} \\
& + 2\partial^\mu\partial^\nu\partial^\alpha\partial^\beta - \frac{4}{3}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta \Bigg] \\
& + \frac{1}{f_0^2}\left[- \frac{1}{3}\delta^{\mu\nu}\delta^{\alpha\beta} + \frac{2}{3}\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box} - \frac{1}{3}\frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\Box^2}\right]\Bigg\}\,\text{sym.} \\
= \Box^2 \Bigg\{ & \frac{1}{2f_2^2}\Bigg[+ \underbrace{\delta^{\mu\alpha}\delta^{\nu\beta}\,\text{sym.}}_{=I^{\mu\nu\alpha\beta}} \underbrace{- 2\left(\delta^{\mu\alpha}\frac{\partial^\nu\partial^\beta}{\Box} - \frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\Box^2}\right)\text{sym.}}_{=P^{(1)\mu\nu\alpha\beta}} \\
& \underbrace{- \frac{1}{3}\left(\delta^{\mu\nu}\delta^{\alpha\beta} - 2\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box} + \frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\Box^2}\right)\text{sym.}}_{=P^{(0,ss)\mu\nu\alpha\beta}} \\
& + \underbrace{\left(\cancel{2\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box}} - \frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\Box^2} - \cancel{2\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box}}\right)\text{sym.}}_{=P^{(0,ww)\mu\nu\alpha\beta}}\Bigg] \\
& + \frac{1}{f_0^2}\underbrace{\left(-\frac{1}{3}\delta^{\mu\nu}\delta^{\alpha\beta} + \frac{2}{3}\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box} - \frac{1}{3}\frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\Box^2}\right)\text{sym.}}_{=-P^{(0,ss)\mu\nu\alpha\beta}}\Bigg\} \\
\overset{(\mathrm{A.65})}{=} & -\Box^2\left(-\frac{1}{2f_2^2}P^{(2)\mu\nu\alpha\beta} + \frac{1}{f_0^2}P^{(0,ss)\mu\nu\alpha\beta}\right),
\end{aligned}
$$

where we have made use of the spin projectors $P^{(2)}$, $P^{(1)}$, $P^{(O,ss)}$ and $P^{(0,ww)}$ that are defined by Eqs. (A.57), (A.59), (A.61) and (A.62). Next, we also rewrite the operator ${\Omega_{\mathrm{GF}}}^{\mu\nu\alpha\beta}[\bar{g}_{\mu\nu} = \delta_{\mu\nu}]$ given by Eq. (F.4) in terms of these projectors. Thereby, for convenience, we set the gauge fixing parameters γ and δ to $\gamma-\delta=0$; then one has

$$\Omega_{\text{GF}}{}^{\mu\nu\alpha\beta}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}] = -\Box^2\left\{-\frac{\beta}{\alpha}\delta^{\mu\nu}\delta^{\alpha\beta} - \frac{1}{\alpha}\delta^{\mu\alpha}\frac{\partial^\nu\partial^\beta}{\Box} + \frac{2\beta}{\alpha}\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box}\right\}.$$

The projector P on the trace-part of a symmetric rank-2 tensor field can be decomposed according to Eq. (A.72):

$$\begin{aligned}\frac{1}{4}\delta^{\mu\nu}\delta^{\alpha\beta} &= P^{\mu\nu\alpha\beta}\\ &= \frac{3}{4}P^{(0,ss)\mu\nu\alpha\beta} + \frac{1}{4}P^{(0,ww)\mu\nu\alpha\beta} + \frac{\sqrt{3}}{4}\left(P^{(0,sw)}+P^{(0,ws)}\right)^{\mu\nu\alpha\beta},\end{aligned}$$

where the "projectors" $P^{(0,sw)}$ and $P^{(0,ws)}$ are given by Eqs. (A.67) and (A.68). Furthermore one has, with Eqs. (A.54) and (A.55),

$$\begin{aligned}\delta^{\mu\alpha}\frac{\partial^\nu\partial^\beta}{\Box}\,\text{sym.} &= \left(T^{\mu\alpha}L^{\nu\beta}+L^{\alpha\beta}L^{\mu\nu}\right)\text{sym.}\\ &= \frac{1}{2}\left(P^{(1)\mu\nu\alpha\beta}+2P^{(0,ww)\mu\nu\alpha\beta}\right)\end{aligned}$$

and

$$\begin{aligned}\delta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\Box}\,\text{sym.} &= \left(T^{\mu\nu}L^{\alpha\beta}+L^{\mu\nu}L^{\alpha\beta}\right)\text{sym.}\\ &= \frac{\sqrt{3}}{2}\left(P^{(0,sw)}+P^{(0,ws)}\right)^{\mu\nu\alpha\beta}+P^{(0,ww)\mu\nu\alpha\beta}.\end{aligned}$$

Therewith the operator $\Omega_{\text{GF}}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}]$ immediately reads

$$\begin{aligned}\Omega_{\text{GF}}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}] = -\Box^2\Bigg\{&\frac{1}{\alpha}\mathbb{P}^{(1)} + 3\frac{\beta^2}{\alpha}\mathbb{P}^{(0,ss)} + \frac{(\beta-1)^2}{\alpha}\mathbb{P}^{(0,ww)}\\ &+\sqrt{3}\frac{\beta(\beta-1)}{\alpha}\left[\mathbb{P}^{(0,sw)}+\mathbb{P}^{(0,ws)}\right]\Bigg\}.\end{aligned}$$

Consequently, the inverse propagator in $d=4$, $\mathscr{U}[0,\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{d=4}_{\text{h.-d.}}$, reads

$$\mathscr{U}[0,\bar{g}_{\mu\nu}=\delta_{\mu\nu}]^{d=4}_{\text{h.-d.}} = \Omega[\bar{g}_{\mu\nu}=\delta_{\mu\nu}] + 2\,\Omega_{\text{GF}}[\bar{g}_{\mu\nu}=\delta_{\mu\nu}]$$

$$= -\Box^2 \Bigg\{ -\frac{1}{2f_2^2}\mathbb{P}^{(2)} + \frac{1}{\alpha}\mathbb{P}^{(1)} + \left(\frac{1}{f_0^2} + \frac{6\beta^2}{\alpha}\right)\mathbb{P}^{(0,ss)} + \frac{2(\beta-1)^2}{\alpha}\mathbb{P}^{(0,ww)} + \frac{2\sqrt{3}\beta(\beta-1)}{\alpha}\left[\mathbb{P}^{(0,sw)} + \mathbb{P}^{(0,ws)}\right] \Bigg\}$$

The operator $\mathscr{U}[0, \bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}_{\mathrm{h.-d.}}$ in this form can be easily inverted by means of Eq. (A.70). The required parameters are

$$a_2 = -\frac{1}{2f_2^2} \quad , \quad a_1 = \frac{1}{\alpha} \quad , \quad a_{ss} = \frac{6\beta^2 f_0^2 + \alpha}{\alpha f_0^2} \quad , \quad a_{ww} = \frac{2(\beta-1)^2}{\alpha} \ ,$$

$$a_{sw} = a_{ws} = \frac{2\sqrt{3}\beta(\beta-1)}{\alpha} \quad , \quad a_{ss}a_{ww} - a_{sw}^2 = \frac{2(\beta-1)^2}{\alpha f_0^2} \ ,$$

$$\frac{a_{ss}}{a_{ss}a_{ww} - a_{sw}^2} = \frac{6\beta^2 f_0^2 + \alpha}{2(\beta-1)^2} \quad , \quad \frac{a_{ww}}{a_{ss}a_{ww} - a_{sw}^2} = f_0^2 \quad , \quad \frac{a_{sw}}{a_{ss}a_{ww} - a_{sw}^2} = \frac{\sqrt{3}f_0^2\beta}{\beta-1} \ ,$$

such that $\left(\mathscr{U}[0, \bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}_{\mathrm{h.-d.}}\right)^{-1}$ – which is the propagator of higher-derivative gravity in $d=4$ – is given by

$$\left(\mathscr{U}[0; \bar{g}_{\mu\nu} = \delta_{\mu\nu}]^{d=4}_{\mathrm{h.-d.}}\right)^{-1} = -\frac{1}{\Box^2}\Bigg\{ -2f_2^2\mathbb{P}^{(2)} + \alpha\left[\mathbb{P}^{(1)} + \frac{1}{2(\beta-1)^2}\mathbb{P}^{(0,ww)}\right] + f_0^2\Bigg[\mathbb{P}^{(0,ss)} + \frac{3\beta^2}{(\beta-1)^2}\mathbb{P}^{(0,ww)} + \frac{\sqrt{3}\beta}{1-\beta}\left(\mathbb{P}^{(0,sw)} + \mathbb{P}^{(0,ws)}\right)\Bigg]\Bigg\} .$$

This is precisely Eq. (4.101) and thus what we wanted to show.

F.9. Calculation of $\mathscr{T}(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}A^{\rho\sigma}$ and $\mathscr{T}(\mathscr{M}[g,g]^{\mu}{}_{\nu})^{\text{diff}}X^{\nu}$

First, we introduce the auxiliary operator

$$\gamma := -\frac{1}{2}\mathscr{T} = \int \mathrm{d}^d x\, g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\,.$$

With its help, the relations we wish to proof now read

$$\gamma\big(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma}\big)^{\text{diff}}A^{\rho\sigma} = -\,\big(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma}\big|_{\Lambda=0}\big)^{\text{diff}}A^{\rho\sigma}$$
$$\gamma\big(\mathscr{M}[g,g]^{\mu}\big)^{\text{diff}}_{\nu}X^{\nu} = -\,\big(\mathscr{M}[g,g]^{\mu}{}_{\nu}\big)^{\text{diff}}X^{\nu}\,.$$

In the following, we verify all the relations needed as ingredients to calculate $\gamma(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}A^{\rho\sigma}$ and $\gamma(\mathscr{M}[g,g]^{\mu}{}_{\nu})^{\text{diff}}X^{\nu}$. These are in summary:

(1) $\gamma g_{\mu\nu} = g_{\mu\nu} \quad , \quad \gamma g^{\mu\nu} = -g^{\mu\nu}$

(2) $\gamma(P_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma} = 0 = \gamma I^{\mu\nu}{}_{\rho\sigma}$

(3) $\gamma\Gamma^{\rho}_{\mu\nu} = 0$

(4) $\gamma R = -R$

(5) $\gamma R_{\mu\nu} = 0 \quad , \quad \gamma R^{\mu\nu} = -2R^{\mu\nu} \quad , \quad \gamma R^{\mu}{}_{\nu} = -R^{\mu}{}_{\nu}$

(6) $\gamma R_{\mu\nu\rho}{}^{\sigma} = 0 \quad , \quad \gamma R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu} = -R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu}$

(7) $\gamma(D_\tau D_\nu A^{\rho\sigma}) = 0$

(8) $\gamma(D^\mu D_\nu A^{\rho\sigma}) = -D^\mu D_\nu A^{\rho\sigma}$ which implies $\gamma(D^2 A^{\rho\sigma}) = -D^2 A^{\rho\sigma}$

(9) $\gamma(D^\mu D^\nu A^{\rho\sigma}) = -2D^\mu D^\nu A^{\rho\sigma}$

(10) $\gamma(D_\tau D_\nu X^{\rho}) = 0$

(11) $\gamma(D^\mu D_\nu X^{\rho}) = -D^\mu D_\nu X^{\rho}$ which implies $\gamma(D^2 X^{\rho}) = -D^2 X^{\rho}$

Aplying these rules to the operator $(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}$ as given by Eq. (4.75) acting on $A^{\rho\sigma}$ and to the operator $(\mathscr{M}[g,g]^{\mu}{}_{\nu})^{\text{diff}}$ as given by Eq. (4.74), the relations $\gamma(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma})^{\text{diff}}A^{\rho\sigma} = -(\mathscr{U}[0;g]^{\mu\nu}{}_{\rho\sigma}|_{\Lambda=0})^{\text{diff}}A^{\rho\sigma}$ as well as $\gamma(\mathscr{M}[g,g]^{\mu}{}_{\nu})^{\text{diff}}X^{\nu} = -(\mathscr{M}[g,g]^{\mu}{}_{\nu})^{\text{diff}}X^{\nu}$ directly follow. It remains to verify the above rules:[2]

[2]Cf. appendix B for the applied metric variantions; especially note that $\frac{\delta g_{\mu\nu}(y)}{\delta g_{\alpha\beta}(x)} = I^{\alpha\beta}_{\mu\nu}\delta(x-y)$.

(1)

$$\int \mathrm{d}^d x\, g_{\alpha\beta}(x) \frac{\delta}{\delta g_{\alpha\beta}(x)} g_{\mu\nu}(y) = \int \mathrm{d}^d x\, g_{\alpha\beta}(x) I^{\alpha\beta}_{\mu\nu} \delta(x-y) = g_{\mu\nu}(y)$$

$$\begin{aligned} \int \mathrm{d}^d x\, g_{\alpha\beta}(x) \frac{\delta}{\delta g_{\alpha\beta}(x)} g^{\mu\nu}(y) &= -\int \mathrm{d}^d x\, g_{\alpha\beta}(x) g^{\mu\rho}(y) g^{\nu\sigma}(y) I^{\alpha\beta}_{\rho\sigma} \delta(x-y) \\ &= -\, g^{\mu\nu}(y) \end{aligned}$$

(2)

$$\begin{aligned} \gamma (P_{\text{tr.}})^{\mu\nu}{}_{\rho\sigma} &= \gamma \left(\frac{1}{d} g^{\mu\nu} g_{\rho\sigma} \right) \\ &= \frac{1}{d} \left(-g^{\mu\nu} g_{\rho\sigma} + g^{\mu\nu} g_{\rho\sigma} \right) = 0 \\ \gamma I^{\mu\nu}{}_{\rho\sigma} &= 0 \quad \text{(holds trivially)} \end{aligned}$$

(3)

$$\begin{aligned} \gamma \Gamma^{\rho}_{\mu\nu}(y) = & \int \mathrm{d}^d x\, g_{\alpha\beta}(x) \frac{1}{2} g^{\rho\sigma}(y) \Big[D^y_\mu \delta(y-x) I^{\alpha\beta}_{\nu\sigma} \\ & \qquad + D^y_\nu \delta(y-x) I^{\alpha\beta}_{\mu\sigma} - D^y_\sigma \delta(y-x) I^{\alpha\beta}_{\mu\nu} \Big] \\ = & -\frac{1}{2} g^{\rho\sigma}(y) \int \mathrm{d}^d x \sqrt{g(x)} \left[D^x_\mu \frac{\delta(y-x)}{\sqrt{g(x)}} I^{\alpha\beta}_{\nu\sigma} \right. \\ & \qquad \left. + D^x_\nu \frac{\delta(y-x)}{\sqrt{g(x)}} I^{\alpha\beta}_{\mu\sigma} - D^x_\sigma \frac{\delta(y-x)}{\sqrt{g(x)}} I^{\alpha\beta}_{\mu\nu} \right] \\ = & \, 0 \quad \text{(after using partial integration)} \end{aligned}$$

(4)

$$\begin{aligned} \gamma R(y) = & \int \mathrm{d}^d x g_{\alpha\beta}(x) \Big[-R^{\mu\nu}(y) + D^\mu_y D^\nu_y - g^{\mu\nu}(y) D^2_y \Big] I^{\alpha\beta}_{\mu\nu} \delta(x-y) \\ = & R(y) + \underbrace{\int \mathrm{d}^d x\, (d-1) D^2_x \delta(x-y)}_{= \int \mathrm{d}^d x \sqrt{g(x)}\, D^x_\mu \left[\frac{(d-1)}{\sqrt{g(x)}} D^\mu_x \delta(x-y) \right] = 0} \\ = & -R(y) \end{aligned}$$

(5)

$$\begin{aligned}
\gamma R_{\mu\nu}(y) =& \int \mathrm{d}^d x\, g_{\alpha\beta}(x) \frac{1}{2}\Bigg[\quad R_\mu{}^\rho(y)\frac{\delta g_{\nu\rho}(y)}{\delta g_{\alpha\beta}(x)} + R_\nu{}^\rho(y)\frac{\delta_{\rho\mu}(y)}{\delta g_{\alpha\beta}(x)} \\
&\qquad +2R^\rho{}_{\mu\nu}{}^\sigma(y)\frac{\delta g_{\rho\sigma}(y)}{\delta g_{\alpha\beta}(x)} + D^y_\nu D^\rho_y \frac{\delta g_{\rho\mu}(y)}{\delta g_{\alpha\beta}(x)} \\
&\qquad -D^2_y \frac{\delta g_{\mu\nu}(y)}{\delta g_{\alpha\beta}(x)} + D^y_\mu D^\rho_y \frac{\delta g_{\rho\nu}(y)}{\delta g_{\alpha\beta}(x)}\Bigg] \\
=& \frac{1}{2}\Bigg[R_{\mu\nu}(y) + R_{\nu\mu}(y) + 2g_{\rho\sigma}(y)R^\rho{}_{\mu\nu}{}^\sigma(y) \\
&+ \underbrace{\int \mathrm{d}^d x\, g_{\rho\mu}(x) D^y_\nu D^\rho_y \delta(x-y)}_{\text{=surface term=0}} - \underbrace{\int \mathrm{d}^d x\, g_{\rho\sigma}(x) D^y_\mu D^y_\nu g^{\rho\sigma}(y)\delta(x-y)}_{\text{=surface term=0}} \\
&- \underbrace{\int \mathrm{d}^d x\, g_{\mu\nu}(x) D^2_y \delta(x-y)}_{\text{=surface term=0}} - \underbrace{\int \mathrm{d}^d x\, g_{\rho\nu}(y) D^y_\mu D^\rho_y \delta(x-y)}_{\text{=surface term=0}}\Bigg] \\
=& \frac{1}{2}\Big[2R_{\mu\nu}(y) - 2R_{\mu\nu}(y)\Big] \\
=& 0 \\
\gamma R^{\mu\nu}(y) =& \gamma\Big[g^{\mu\rho}(y)g^{\nu\sigma}(y)R_{\rho\sigma}(y)\Big] \\
=& -g^{\mu\rho}(y)g^{\nu\sigma}(y)R_{\rho\sigma}(y) - g^{\mu\rho}(y)g^{\nu\sigma}(y)R_{\rho\sigma}(y) \\
=& -2R^{\mu\nu}(y) \\
\gamma R^\mu{}_\nu(y) =& \gamma\Big[g^{\mu\rho}(y)R_{\rho\nu}(y)\Big] = -R^\mu{}_\nu(y)
\end{aligned}$$

(6)

$$\begin{aligned}
\gamma R_{\mu\nu\rho}{}^{\sigma}(y) =& \frac{1}{2}\int \mathrm{d}^d x\, g_{\alpha\beta}(x)\Bigg[\\
&- R_{\mu\nu\rho}{}^{\tau}(y) g^{\sigma\gamma}(y)\frac{\delta g_{\tau\gamma}(y)}{\delta g_{\alpha\beta}(x)} + R_{\mu\nu\tau}{}^{\sigma}(y) g^{\tau\gamma}(y)\frac{\delta g_{\rho\gamma}(y)}{\delta g_{\alpha\beta}(x)} \\
&\underbrace{- D^y_\mu D^y_\rho g^{\sigma\tau}(y)\frac{\delta g_{\nu\tau}(y)}{\delta g_{\alpha\beta}(x)}}_{=\text{surface term}=0} + \underbrace{D^y_\nu D^y_\rho g^{\sigma\tau}(y)\frac{\delta g_{\mu\tau}(y)}{\delta g_{\alpha\beta}(x)}}_{=\text{surface term}=0} \\
&+ \underbrace{D^y_\mu D^\sigma_y \frac{\delta g_{\nu\rho}(y)}{\delta g_{\alpha\beta}(x)}}_{=\text{surface term}=0} - \underbrace{D^y_\nu D^\sigma_y \frac{\delta g_{\mu\rho}(y)}{\delta g_{\alpha\beta}(x)}}_{=\text{surface term}=0}\Bigg] \\
=& \frac{1}{2}\Big[- g_{\tau\gamma}(y) g^{\sigma\gamma}(y) R_{\mu\nu\rho}{}^{\tau}(y) + g_{\rho\gamma}(y) g^{\tau\gamma}(y) R_{\mu\nu\tau}{}^{\sigma}(y)\Big] \\
=& \Big[- R_{\mu\nu\rho}{}^{\sigma}(y) + R_{\mu\nu\rho}{}^{\sigma}(y)\Big] \\
=& 0 \\
\gamma R_{\sigma}{}^{\mu}{}_{\sigma}{}^{\nu}(y) =& \gamma\big[g^{\mu\tau}(y) R_{\rho\tau\sigma}{}^{\nu}(y)\big] = -g^{\mu\tau}(y) R_{\rho\tau\sigma}{}^{\nu}(y) = -R_{\sigma}{}^{\mu}{}_{\sigma}{}^{\nu}(y)
\end{aligned}$$

(7-11)

First, one has

$$D^\mu D_\nu A^{\rho\sigma} = g^{\mu\tau} D_\tau D_\nu A^{\rho\sigma} = g^{\mu\tau}\Big[\partial_\tau D_\nu A^{\rho\sigma} + \Gamma^\rho_{\tau\varepsilon} D_\nu A^{\varepsilon\sigma} + \Gamma^\sigma_{\tau\varepsilon} D_\nu A^{\rho\varepsilon} - \Gamma^\varepsilon_{\nu\tau} D_\varepsilon A^{\rho\sigma}\Big]\,.$$

Next, as $D_\nu A^{\rho\sigma} = \partial_\nu A^{\rho\sigma} + \Gamma^\rho_{\nu\varepsilon} A^{\varepsilon\sigma} + \Gamma^\sigma_{\nu\varepsilon} A^{\rho\varepsilon}$, the terms inside the square brackets depend only via the Christoffel symbols on the metric. Therewith, one directly has:

$$\begin{aligned}
\gamma(D_\tau D_\nu A^{\rho\sigma}) &= 0 \\
\gamma(D^\mu D_\nu A^{\rho\sigma}) &= \gamma(g^{\mu\tau} D_\tau D_\nu A^{\rho\sigma}) = -D^\mu D_\nu A^{\rho\sigma} \\
\gamma(D^\mu D^\nu A^{\rho\sigma}) &= \gamma(g^{\mu\tau} g^{\nu\varepsilon} D_\tau D_\varepsilon A^{\rho\sigma}) = -2 D^\mu D^\nu A^{\rho\sigma}\,.
\end{aligned}$$

(The proofs of $\gamma(D_\tau D_\nu X^\rho) = 0$ and $\gamma(D^\mu D_\nu X^\rho) = -D^\mu D_\nu X^\rho$ are fully analogous.)

F.10. Calculation of Eq. (19.10) using lemma (17.13)

$$
\begin{aligned}
\int \mathrm{d}^d x \sqrt{g(x)}\, T_\mu{}^\mu[\{\phi_j\}; g](x) &= \mathscr{T} S\left[\{\phi_j\}\, ; g\right] \\
&= -2 \int \mathrm{d}^d x g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} S\left[\{\phi_j\}\, ; g\right] \\
&= \lim_{\alpha \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} S\left[\{\phi_j\}\, ; \mathrm{e}^{-2\alpha} g\right] \\
&= \lim_{\alpha \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \left\{ S\left[\{(1 + w_j\alpha - w_j\alpha)\phi_j\}\, ; (1 - 2\alpha) g\right] + O(\alpha^2) \right\} \\
&= \lim_{\alpha \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \Bigg\{ S\left[\{(1 + w_j\alpha)\phi_j\}\, ; (1 - 2\alpha) g\right] \\
&\qquad + \int \mathrm{d}^d x \sum_j -w_j \alpha \phi_j(x) \frac{\delta S[\{\phi_j\}; g]}{\delta \phi_j(x)} + O(\alpha^2) \Bigg\} \\
&= \lim_{\alpha \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \Bigg\{ S\left[\{\phi_j\}\, ; g\right] + \alpha N\left[\{\phi_j\}\, ; g\right] + O(\alpha^2) \\
&\qquad - \alpha \sum_j w_j \int \mathrm{d}^d x\, \phi_j(x) \frac{\delta S[\{\phi_j\}; g]}{\delta \phi_j(x)} + O(\alpha^2) \Bigg\} \\
&= N\left[\{\phi_j\}\, ; g\right] - \sum_j w_j \int \mathrm{d}^d x\, \phi_j(x) \frac{\delta S[\{\phi_j\}; g]}{\delta \phi_j(x)}
\end{aligned}
$$

F.11. Derivation of the functional renormalization group equation

We only show the case $b_\mu \not\equiv 0$; the case $b_\mu \equiv 0$ follows fully analogously by employing the measure (4.22) instead of (4.21). Taking the partial derivative with respect to the renormalization group time $t = \ln k$ of the functional $\tilde{\Gamma}_k[\phi;\bar{g}]$ yields (thereby taking note of the comment on the chain rule for the left derivative of Graßmann variables in Footnote 1 of Chapter 17)

$$\begin{aligned}
\partial_t \tilde{\Gamma}_k[\phi;\bar{g}] = & \int \mathrm{d}^d x \sqrt{\bar{g}} \Big[\partial_t t_k{}^{\mu\nu} h_{\mu\nu} + \partial_t \bar{\sigma}_{k\mu} \xi^\mu + \partial_t \sigma_k{}^\mu \bar{\xi}_\mu + \partial_t d_k{}^\mu \zeta_\mu \Big] \\
& - (\partial_t W_k)\left[J_k[\phi;\bar{g}];\bar{g}\right] \\
& - \int \mathrm{d}^d x \sqrt{\bar{g}} \Big[(\partial_t t_k{}^{\mu\nu}) \underbrace{\frac{\delta W_k\left[J_k[\phi;\bar{g}];\bar{g}\right]}{\delta t_k{}^{\mu\nu}}}_{=\sqrt{\bar{g}} h_{\mu\nu}} + (\partial_t \bar{\sigma}_{k\mu}) \underbrace{\frac{\delta W_k\left[J_k[\phi;\bar{g}];\bar{g}\right]}{\delta \bar{\sigma}_{k\mu}}}_{=\sqrt{\bar{g}}\xi^\mu} \\
& \qquad + (\partial_t \sigma_k{}^\mu) \underbrace{\frac{\delta W_k\left[J_k[\phi;\bar{g}];\bar{g}\right]}{\delta \sigma_k{}^\mu}}_{=\sqrt{\bar{g}}\bar{\xi}_\mu} + (\partial_t d_k{}^\mu) \underbrace{\frac{\delta W_k\left[J_k[\phi;\bar{g}];\bar{g}\right]}{\delta d_{k\mu}}}_{=\sqrt{\bar{g}}\zeta_\mu} \Big] \\
= & - (\partial_t W_k)\left[J_k[\phi;\bar{g}];\bar{g}\right] \\
= & \frac{\int \mathcal{D}\mu[\widehat{h},\bar{C},C,b;\bar{g}]\, \partial_t \Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]\, \mathrm{e}^{-\tilde{S}[\widehat{h},\bar{C},C,b;t,\sigma,\bar{\sigma},d;\bar{g}] - \Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]}}{\int \mathcal{D}\mu[\widehat{h},\bar{C},C,b;\bar{g}]\, \mathrm{e}^{-\tilde{S}[\widehat{h},\bar{C},C,b;t,\sigma,\bar{\sigma},d;\bar{g}] - \Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]}} \\
= & \left\langle \partial_t \Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}] \right\rangle .
\end{aligned}$$

Before proceeding further, let us rewrite the cutoff action (6.1) using Eqs. (A.5), (A.20) and (A.21) (see also appendix A.1)

$$\begin{aligned}
&\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]\\
&= \int \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\int \mathrm{d}^d y\sqrt{\bar{g}(y)}\Bigg\{+\frac{1}{2}\widehat{h}_{\mu\nu}(x)\,\mathscr{R}_k^{\mathrm{grav}\,\mu\nu}{}_{\rho\sigma}[\bar{g}]_x^{\mathrm{diff}}\frac{\delta(x-y)}{\sqrt{\bar{g}(y)}}\,\widehat{h}^{\rho\sigma}(y)\\
&\qquad + \bar{C}_\mu(x)\,\mathscr{R}_k^{\mathrm{gh},1\,\mu}{}_\nu[\bar{g}]_x^{\mathrm{diff}}\frac{\delta(x-y)}{\sqrt{\bar{g}(y)}}C^\nu(y)\\
&\qquad + \frac{1}{2}b_\mu(x)\,\mathscr{R}_k^{\mathrm{gh},2\,\mu}{}_\nu[\bar{g}]_x^{\mathrm{diff}}\frac{\delta(x-y)}{\sqrt{\bar{g}(y)}}b^\nu(y)\Bigg\}.\\
&= \int \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\int \mathrm{d}^d y\sqrt{\bar{g}(y)}\Bigg\{+\frac{1}{2}\widehat{h}_{\mu\nu}(x)\langle x,\mu,\nu|\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]|y,\rho,\sigma\rangle\,\widehat{h}^{\rho\sigma}(y)\\
&\qquad + \bar{C}_\mu(x)\langle x,\mu|\mathscr{R}_k^{\mathrm{gh},1}[\bar{g}]|y,\nu\rangle\,C^\nu(y)\\
&\qquad + \frac{1}{2}b_\mu(x)\,\langle x,\mu|\mathscr{R}_k^{\mathrm{gh},2}[\bar{g}]|y,\nu\rangle\,b^\nu(y)\Bigg\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\partial_t\tilde{\Gamma}_k[\phi;\bar{g}] \qquad\qquad (\text{F.5})\\
&= \Big\langle \partial_t\Delta_k S[\widehat{h},\bar{C},C,b;\bar{g}]\Big\rangle\\
&= \int \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\int \mathrm{d}^d y\sqrt{\bar{g}(y)}\Bigg\{\ \frac{1}{2}\langle x,\mu,\nu|\partial_t\mathscr{R}_k^{\mathrm{grav}}[\bar{g}]|y,\rho,\sigma\rangle\,\Big\langle\widehat{h}_{\mu\nu}(x)\widehat{h}^{\rho\sigma}(y)\Big\rangle\\
&\qquad + \langle x,\mu|\partial_t\mathscr{R}_k^{\mathrm{gh},1}[\bar{g}]|y,\nu\rangle\,\langle\bar{C}_\mu(x)C^\nu(y)\rangle\\
&\qquad + \frac{1}{2}\langle x,\mu|\partial_t\mathscr{R}_k^{\mathrm{gh},2}[\bar{g}]|y,\nu\rangle\,\langle b_\mu(x)b^\nu(y)\rangle\ \Bigg\}.
\end{aligned}$$

Next, we must find a way to express the expectation values above through the EAA $\Gamma_k[\phi;\bar{g}]$. Therefore, we firstly acknowledge that

$$\begin{aligned}
\frac{\delta^2 W_k[J;\bar{g}]}{\delta J_a(x)\delta J_b(y)} &= \frac{\delta^2}{\delta J_a(x)\delta J_b(y)}\ln\exp W_k[J;\bar{g}]\\
&= \frac{\delta}{\delta J_a(x)}\left(\frac{1}{\mathrm{e}^{W_k[J;\bar{g}]}}\frac{\delta \mathrm{e}^{W_k[J;\bar{g}]}}{\delta J_b(y)}\right)\\
&= -\frac{1}{(\mathrm{e}^{W_k[J;\bar{g}]})^2}\frac{\delta \mathrm{e}^{W_k[J;\bar{g}]}}{\delta J_a(x)}\frac{\delta \mathrm{e}^{W_k[J;\bar{g}]}}{\delta J_b(y)} + \frac{1}{(\mathrm{e}^{W_k[J;\bar{g}]})^2}\frac{\delta^2 \mathrm{e}^{W_k[J;\bar{g}]}}{\delta J_a(x)\delta J_b(y)}
\end{aligned}$$

$$= \frac{\delta W_k[J;\bar{g}]}{\delta J_a(x)} \frac{\delta W_k[J;\bar{g}]}{\delta J_b(y)} + \sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)} \left\langle \chi^a(x)\chi^b(y) \right\rangle$$
$$= \sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)} \left\{ \left\langle \chi^a(x)\chi^b(y) \right\rangle - \left\langle \chi^a(x) \right\rangle \left\langle \chi^b(y) \right\rangle \right\}$$
$$= \sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)} \left\{ \left\langle \chi^a(x)\chi^b(y) \right\rangle - \phi^a(x)\phi^b(y) \right\} . \tag{F.6}$$

Secondly, note that Eq. (6.10) inverts Eq. (6.6) in the following sense (here, the index structure is supressed):

$$\begin{aligned} \delta(x-y)\delta^a_b &= \frac{\delta J_a(x)}{\delta J_b(y)} \\ &= \int \mathrm{d}^d z \frac{\delta (\phi^c)_k[J;\bar{g}](z)}{\delta J_b(z)} \frac{\delta (J_a)_k[\phi;\bar{g}]}{\delta \phi^c(z)} \\ &= \int \mathrm{d}^d z \left(\frac{\delta}{\delta J_b(z)} \frac{1}{\sqrt{\bar{g}(z)}} \frac{\delta W_k[J;\bar{g}]}{\delta J_c(z)} \right) \left(\frac{\delta}{\delta \phi^c(z)} \frac{(-1)^{|\phi^a|}}{\sqrt{\bar{g}(z)}} \frac{\delta \tilde{\Gamma}_k[\phi;\bar{g}]}{\delta \phi^a(x)} \right) . \end{aligned}$$

We can re-express this identity by introducing the crucial operators $G_k[J;\bar{g}]$ and $\tilde{\Gamma}^{(2)}_k[\phi;\bar{g}]$ in field space by (we point out that in this definition *no* sum over a and b is intended)

$$\begin{aligned} \langle x,\ldots|G^{ab}_k[J;\bar{g}]|y,\ldots\rangle &\coloneqq \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} I^{ab}[\bar{g}] \frac{\delta^2 W_k[J;\bar{g}]}{\delta J_a(x)\delta J_b(y)} , \\ \langle x,\ldots| \left(\tilde{\Gamma}^{(2)}_k\right)_{ab}[\phi;\bar{g}]|y,\ldots\rangle &\coloneqq \frac{(-1)^{|\phi^b|}}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} I_{ab}[\bar{g}] \frac{\delta^2 \tilde{\Gamma}_k[\phi;\bar{g}]}{\delta \phi^a(x)\delta \phi^b(y)} , \end{aligned}$$

where "…", I_{ab} and I^{ab} symbolically account for the index structure that must be adapted accordingly, e.g.

$$\begin{aligned} \langle x,\mu,\nu|G^{11}_k[J;\bar{g}]|y,\rho,\sigma\rangle &\coloneqq \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} I^{\mu\nu\alpha\beta}[\bar{g}] \frac{\delta^2 W_k[J;\bar{g}]}{\delta t^{\alpha\beta}(x)\delta t^{\rho\sigma}(y)} , \\ \langle x,\mu|G^{23}_k[J;\bar{g}]|y,\nu\rangle &\coloneqq \frac{1}{\sqrt{\bar{g}(x)}\sqrt{\bar{g}(y)}} \delta^\mu_\alpha \delta^\beta_\nu \frac{\delta^2 W_k[J;\bar{g}]}{\delta \sigma^\alpha(x)\delta \bar{\sigma}_\beta(y)} . \end{aligned}$$

Hence, the above identity reads

$$\begin{aligned} \int \mathrm{d}^d z \sqrt{\bar{g}(z)}\, \langle x,\ldots|G^{ab}_k[J;\bar{g}]|z,\ldots\rangle\langle z,\ldots|(\tilde{\Gamma}^{(2)}_k)_{bc}[\phi;\bar{g}]|y,\ldots\rangle &= \delta^a_c \frac{\delta(x-y)}{\sqrt{\bar{g}(y)}} I^a_c \\ &= \delta^a_c \langle x,\ldots|y,\ldots\rangle . \end{aligned}$$

Therewith, especially the following relation holds:

$$\boxed{G_k[J;\bar{g}] = \left(\tilde{\Gamma}_k^{(2)}[\phi;\bar{g}]\right)^{-1}\,,}\,.$$

and furthermore Eq. (F.6) amounts to

$$\left\langle \chi^a(x)\chi^b(y)\right\rangle I_{ab} = \langle x,\ldots|G_k^{ab}[J;\bar{g}]|y,\ldots\rangle + \phi^a(x)\phi^b(y)I_{ab}\,.$$

By plugging this equation back into our intermediate result (F.5) we obtain

$$\begin{aligned}\partial_t\tilde{\Gamma}_k[\phi;\bar{g}] = \int \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\int \mathrm{d}^d y\sqrt{\bar{g}(y)}\Bigg\{ \\ &\frac{1}{2}\langle x,\mu,\nu|\partial_t\mathscr{R}_k^{\text{grav}}[\bar{g}]|y,\rho,\sigma\rangle\Big[\langle y,\rho,\sigma|G_k^{11}[J;\bar{g}]|x,\mu,\nu\rangle + h_{\mu\nu}(x)h^{\rho\sigma}(y)\Big] \\ &+\langle x,\mu|\partial_t\mathscr{R}_k^{\text{gh},1}[\bar{g}]|y,\nu\rangle\Big[\langle y,\nu|G_k^{23}[J;\bar{g}]|x,\nu\rangle + \bar{\xi}_\mu(x)\xi^\nu(y)\Big] \\ &+\frac{1}{2}\langle x,\mu|\partial_t\mathscr{R}_k^{\text{gh},2}[\bar{g}]|y,\nu\rangle\Big[\langle y,\nu|G_k^{44}[J;\bar{g}]|x,\nu\rangle + \zeta_\mu(x)\zeta^\nu(y)\Big]\Bigg\}\,.\end{aligned}$$

It immediately follows that

$$\begin{aligned}\partial_t\Gamma_k[\phi;\bar{g}] &= \partial_t\tilde{\Gamma}_k[\phi;\bar{g}] - \partial_t\Delta_k S[\phi;\bar{g}] \\ &= \int \mathrm{d}^d x\sqrt{\bar{g}(x)}\,\int \mathrm{d}^d y\sqrt{\bar{g}(y)}\Bigg\{ \\ &\quad\frac{1}{2}\langle x,\mu,\nu|\partial_t\mathscr{R}_k^{\text{grav}}[\bar{g}]|y,\rho,\sigma\rangle\langle y,\rho,\sigma|G_k^{11}[J;\bar{g}]|x,\mu,\nu\rangle \\ &\quad+\langle x,\mu|\partial_t\mathscr{R}_k^{\text{gh},1}[\bar{g}]|y,\nu\rangle\langle y,\nu|G_k^{23}[J;\bar{g}]|x,\nu\rangle \\ &\quad+\frac{1}{2}\langle x,\mu|\partial_t\mathscr{R}_k^{\text{gh},2}[\bar{g}]|y,\nu\rangle\langle y,\nu|G_k^{44}[J;\bar{g}]|x,\nu\rangle\Bigg\}\,.\end{aligned}$$

Note that $G_k^{23}[J;\bar{g}] = -G_k^{32}[J;\bar{g}]$. Further applying the definitions (6.15) and (6.16), from which

$$G_k[J;\bar{g}] = \left(\Gamma^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right)^{-1}$$

follows, yields

$$
\begin{aligned}
&\partial_t \Gamma_k[\phi;\bar{g}] \\
&= \int \mathrm{d}^d x \sqrt{\bar{g}(x)} \Bigg\{ \frac{1}{2} \langle x,\mu,\nu| \left(\partial_t \mathscr{R}_{k11}[\bar{g}]\right) \left(\left[\Gamma^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \right)^{11} |x,\mu,\nu\rangle \\
&\qquad\qquad - \langle x,\mu| \left(\partial_t \mathscr{R}_{k23}[\bar{g}]\right) \left(\left[\Gamma^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \right)^{32} |x,\mu\rangle \\
&\qquad\qquad - \frac{1}{2} \langle x,\mu| \left(\partial_t \mathscr{R}_{k44}[\bar{g}]\right) \left(\left[\Gamma^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \right)^{44} |x,\mu\rangle \Bigg\} \\
&= \frac{1}{2} \operatorname{Tr}_{ST^2} \left[\left(\partial_t \mathscr{R}_{k11}[\bar{g}]\right) \left(\left[\Gamma_k^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \right)^{11} \right] \\
&\quad - \operatorname{Tr}_V \left[\left(\partial_t \mathscr{R}_{k23}[\bar{g}]\right) \left(\left[\Gamma_k^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \right)^{32} \right] \\
&\quad - \frac{1}{2} \operatorname{Tr}_V \left[\left(\partial_t \mathscr{R}_{k44}[\bar{g}]\right) \left(\left[\Gamma_k^{(2)}[\phi;\bar{g}] + \mathscr{R}_k[\bar{g}]\right]^{-1} \right)^{44} \right].
\end{aligned}
$$

This is precisely Eq. (6.13) and thus what we wanted to show.

F.12. Derivation of the relations (12.19)

In this section, we solely work in four-dimensional flat spacetime. The derivation (or proof) of the relations (12.19) proceeds as follows: First, we show that under the integral $\int \mathrm{d}^4 p$ using symmetric integration the projectors $P^{(2)}(p)$, $P^{(1)}(p)$ etc. can be expressed in terms of $P_{\mathrm{tr.}}[g_{\mu\nu} = \delta_{\mu\nu}]$ and $I_{ST^2} \equiv I$. Then the relations (12.19) directly follow from the Equations (11.21) and (11.22), i.e.

$$I^{\mu\nu}{}_{\rho\sigma} T^{\rho\sigma}{}_{\mu\nu}(u) = -n^2 \quad \text{and} \quad P_{\mathrm{tr.}}[\delta_{\mu\nu}]^{\mu\nu}{}_{\rho\sigma} T^{\rho\sigma}{}_{\mu\nu}(u) = \frac{1}{d}(n^2 - 2n)\,.$$

Consider the relations (12.19) to be labeled top-down from (1) to (6). Subsequently, we will show each relation separately.

(1)

$$\begin{aligned}
P^{(2)}_{\mu\nu\rho\sigma}(p) &= \frac{1}{2} T_{\mu\rho} T_{\nu\sigma} + \frac{1}{2} T_{\mu\sigma} T_{\nu\rho} - \frac{1}{3} T_{\mu\nu} T_{\rho\sigma} \\
&= \frac{1}{2}\left[\left(\delta_{\mu\rho} - \frac{p_\mu p_\rho}{p^2}\right)\left(\delta_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2}\right) + \left(\delta_{\mu\sigma} - \frac{p_\mu p_\sigma}{p^2}\right)\left(\delta_{\nu\rho} - \frac{p_\nu p_\rho}{p^2}\right) \right] \\
&\quad - \frac{1}{3}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2}\right)
\end{aligned}$$

Using symmetric integration under the integral $\int \mathrm{d}^4 p$ we have

$$\begin{aligned}
P^{(2)}_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\Bigg[& \underbrace{\delta_{\mu\nu}\delta_{\nu\sigma}}_{(i)} - \frac{1}{4}\underbrace{\delta_{\mu\rho}\delta_{\nu\sigma}}_{(ii)} - \frac{1}{4}\underbrace{\delta_{\mu\rho}\delta_{\nu\sigma}}_{(ii)} + \frac{1}{24}(\delta_{\mu\nu}\delta_{\rho\sigma} + \underbrace{\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}}_{(iii)}) \\
& + \underbrace{\delta_{\mu\sigma}\delta_{\nu\rho}}_{(i)} - \frac{1}{4}\underbrace{\delta_{\mu\sigma}\delta_{\nu\rho}}_{(ii)} - \frac{1}{4}\underbrace{\delta_{\mu\sigma}\delta_{\nu\rho}}_{(iii)} + \frac{1}{24}(\delta_{\mu\nu}\delta_{\rho\sigma} + \underbrace{\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}}_{(iii)}) \Bigg] \\
& - \frac{1}{3}\left[\delta_{\mu\nu}\delta_{\rho\sigma} - \frac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma} - \frac{1}{4}\delta_{\rho\sigma}\delta_{\mu\nu} + \frac{1}{24}(\delta_{\mu\nu}\delta_{\rho\sigma} + \underbrace{\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}}_{(i)}) \right] \\
= & \left(\frac{1}{24} - \frac{1}{3}\left(\frac{1}{2} + \frac{1}{24}\right)\right) 4\frac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma} + \left(1 - 2\frac{1}{4} + \frac{1}{12} - \frac{1}{3\cdot 2\cdot 6}\right) I_{\mu\nu\rho\sigma} \\
= & -\frac{5}{9} P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma} + \frac{5}{9} I[\delta]_{\mu\nu\rho\sigma}\,.
\end{aligned}$$

Thus,

$$T^{\mu\nu}{}_{\rho\sigma}(u)P^{(2)\rho\sigma}{}_{\mu\nu}(p) = -\frac{5}{9}\frac{(n^2-2n)}{4} - \frac{5}{9}n^2$$
$$= \frac{10}{72}n(2-5n)\,.$$

(2)

$$P^{(1)}_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\big(T_{\mu\rho}L_{\nu\sigma} + T_{\mu\sigma}L_{\nu\rho} + T_{\nu\rho}L_{\mu\sigma} + T_{\nu\sigma}L_{\mu\rho}\big)$$
$$= \frac{1}{2}\Bigg(\delta_{\mu\rho}\frac{p_\nu p_\sigma}{p^2} - \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4} + \delta_{\mu\sigma}\frac{p_\nu p_\rho}{p^2} - \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}$$
$$+ \delta_{\nu\rho}\frac{p_\mu p_\sigma}{p^2} - \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4} + \delta_{\nu\sigma}\frac{p_\mu p_\rho}{p^2} - \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}\Bigg)$$

Using symmetric integration under the integral $\int \mathrm{d}^4p$ we have

$$P^{(1)}_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\Bigg(\frac{1}{4}\delta_{\mu\rho}\delta_{\nu\sigma} + \frac{1}{4}\delta_{\mu\sigma}\delta_{\nu\rho} + \frac{1}{4}\delta_{\nu\rho}\delta_{\mu\sigma} + \frac{1}{4}\delta_{\nu\sigma}\delta_{\mu\rho}$$
$$- 4\frac{1}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\Bigg)$$
$$= \frac{1}{2}\left(I[\delta]_{\mu\rho\nu\sigma} - \frac{4}{6}P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma} - \frac{4}{12}I[\delta]_{\mu\rho\nu\sigma}\right)$$
$$= \frac{1}{3}I[\delta]_{\mu\rho\nu\sigma} - \frac{1}{3}P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma}\,.$$

Thus,

$$T^{\mu\nu}{}_{\rho\sigma}(u)P^{(1)\rho\sigma}{}_{\mu\nu}(p) = \frac{1}{3}(-n^2) - \frac{1}{3}\frac{1}{4}(n^2-2n)$$
$$= \frac{1}{12}n(2-5n)\,.$$

(3)

$$P^{(0,ss)}_{\mu\nu\rho\sigma}(p) = \frac{1}{3}T_{\mu\nu}T_{\rho\sigma}$$

$$= \frac{1}{3}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2}\right)$$
$$= \frac{1}{3}\left(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\nu}\frac{p_\rho p_\sigma}{p^2} - \delta_{\rho\sigma}\frac{p_\mu p_\nu}{p^2} + \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}\right)$$

Using symmetric integration under the integral $\int \mathrm{d}^4 p$ we have

$$P^{(0,ss)}_{\mu\nu\rho\sigma}(p) = \frac{1}{3}\Bigg(\delta_{\mu\nu}\delta_{\rho\sigma} - \frac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma} - \frac{1}{4}\delta_{\rho\sigma}\delta_{\mu\nu}$$
$$+ \frac{1}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\Bigg)$$
$$= \frac{1}{3}\left(\left(1 - \frac{1}{2} + \frac{1}{24}\right)4P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma} + \frac{1}{12}I_{\mu\nu\rho\sigma}\right)$$
$$= \frac{1}{36}I[\delta]_{\mu\nu\rho\sigma} + \frac{13}{18}P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma}$$

Thus,

$$T^{\mu\nu}{}_{\rho\sigma}(u)P^{(0,ss)\rho\sigma}{}_{\mu\nu}(p) = \frac{1}{36}(-n^2) + \frac{13}{18}\cdot\frac{1}{4}(n^2 - 2n)$$
$$= \frac{1}{72}n(11n - 26)\,.$$

(4)

$$P^{(0,ww)}_{\mu\nu\rho\sigma}(p) = L_{\mu\nu}L_{\rho\sigma}$$
$$= \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}$$

Using symmetric integration under the integral $\int \mathrm{d}^4 p$ we have

$$P^{(0,ww)}_{\mu\nu\rho\sigma}(p) = \frac{1}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)$$
$$= \frac{1}{6}P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma} + \frac{1}{12}I[\delta]_{\mu\nu\rho\sigma}\,.$$

Thus,

$$T^{\mu\nu}{}_{\rho\sigma}(u)P^{(0,ww)\rho\sigma}{}_{\mu\nu}(p) = \frac{1}{12}(-n^2) + \frac{1}{6}\cdot\frac{1}{4}(n^2 - 2n)$$
$$= -\frac{1}{24}n(n+2)\,.$$

(5)

$$\left[P^{(0,sw)} + P^{(0,ws)}\right]_{\mu\nu\rho\sigma}(p) = \frac{1}{\sqrt{3}}\left(T_{\mu\nu}L_{\rho\sigma} + L_{\mu\nu}T_{\rho\sigma}\right)$$
$$= \frac{1}{\sqrt{3}}\left(\delta_{\mu\nu}\frac{p_\rho p_\sigma}{p^2} - \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4} + \delta_{\rho\sigma}\frac{p_\mu p_\nu}{p^2} - \frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}\right)$$

Using symmetric integration under the integral $\int \mathrm{d}^4 p$ we have

$$\left[P^{(0,sw)} + P^{(0,ws)}\right]_{\mu\nu\rho\sigma}(p) = \frac{1}{\sqrt{3}}\left(2\frac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma} - 2\frac{1}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\right)$$
$$= \frac{1}{\sqrt{3}}\left(\left(\frac{2}{4} - \frac{2}{24}\right)4P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma} - \frac{2}{24}\cdot 2I[\delta]_{\mu\nu\rho\sigma}\right)$$
$$= \frac{1}{\sqrt{3}}\left(\frac{5}{3}P_{\mathrm{tr.}}[\delta]_{\mu\nu\rho\sigma} - \frac{1}{6}I[\delta]_{\mu\nu\rho\sigma}\right)$$

Thus,

$$T^{\mu\nu}{}_{\rho\sigma}(u)\left[P^{(0,sw)} + P^{(0,ws)}\right]^{\rho\sigma}{}_{\mu\nu}(p) = \frac{1}{\sqrt{3}}\left(-\frac{1}{6}(-n^2) + \frac{5}{3}\cdot\frac{1}{4}(n^2 - 2n)\right)$$
$$= \frac{1}{\sqrt{3}\,12}n(7n - 10)\,.$$

The proof of (6) is fully equivalent to (5).

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Colophon

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