Viscoelastic phase separation: Well-posedness and numerical analysis

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He who loves practice without theory
is like the sailor who boards ship
without a rudder and compass
and never knows where he may cast.

Leonardo da Vinci

Foreword: This is the final version of the PhD thesis. It has undergone minor revision after the first submission in October 2021. Thanks to the reporter comments, some typographical and grammatical and even some mathematical errors have been removed.

April 2022, Mainz

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Abstract

Viscoelastic phase separation describes dynamically asymmetric demixing of polymer solutions after a deep quench, i.e., after a sudden decrease in the temperature. The dynamic asymmetry of the polymer solution gives rise to new and more complex phenomena than the standard phase separation of binary fluids. Coupled with the incomplete timescale separation, the process forms a complex multiscale problem. Phenomenological continuum mechanical models for standard phase separation, e.g., model H, are insufficient to capture the enriched dynamics observed in experiments for viscoelastic phase separation. Hence, a key difficulty is the derivation of more complex models which resemble experimental data and preserve fundamental principles of physics, e.g., conservation of mass, momentum, and the second law of thermodynamics. For a suitable phenomenological model, we consider mathematical well-posedness of the problem, i.e., existence, uniqueness and stability of solutions. The models are complex nonlinear parabolic systems of partial differential equations with an energy-dissipative structure based on a non-convex free energy functional. The key difficulties arise due to a strongly nonlinear cross-diffusive coupling of one subsystem and a logarithmic type of free energy for another subsystem. We prove the global-in-time existence of dissipative weak solutions in two and three space dimensions using the energy method. Additionally, we employ relative energy methods to derive an abstract stability result. As an application, this approach yields the weak-strong uniqueness principle for dissipative weak solutions. For the numerical approximation and the corresponding error analysis, it is suitable to derive numerical methods which preserve the second law of thermodynamics also on the discrete level. Key difficulties are the correct discretisation of the convective terms on the discrete level and suitable time integration methods for the non-convex energy. For the semi-discretisation of a reduced model, we consider conforming inf-sup finite elements in space and analyse the corresponding semi-discrete problem. The thermodynamic properties are preserved by the Galerkin method, hence using a discrete version of the nonlinear stability estimate allows us to deduce the optimal second-order accuracy in a transparent and structured way. In the fully discrete case, we employ a variational time discretisation via a Petrov-Galerkin method on the semi-discrete system. Time-discrete thermodynamic structure is preserved and together with a fully discrete stability estimate the corresponding error analysis is derived. This allows us to deduce here the optimal second-order accuracy in space and time using realistic smoothness assumptions. Theoretical error estimates of the semi-discrete and fully discrete scheme are illustrated by a series of numerical experiments.

Kurzfassung

Viskoelastische Phasenseparation beschreibt die dynamische asymmetrische Entmischung einer Polymerlösung nach dem Abschrecken, einer plötzlichen Abnahme der Temperatur. Die dynamische Asymmetrie der Polymerlösung ruft neue und komplexere Effekte hervor als die normale Phasenseparation von binären Flüssigkeiten. In Kombination mit der unvollständigen Skalenseparation formt der Prozess ein komplexes Mehrskalensystem. Kontinuumsmechanische Modelle für die Phasenseparation, z.B. das Modell H, sind nicht ausreichend, um die erweiterte Dynamik zu beschreiben, die in Experimenten zur viskoelastischen Phasenseparation beobachtet worden ist. Eine Schlüsselschwierigkeit ist die Konstruktion eines erweiterten Modells, welches die experimentellen Daten reproduzieren kann und konsistent mit den Grundprinzipien der Physik ist. Dies beinhaltet insbesondere die Erhaltung von Masse, Impuls und den zweiten Hauptsatz der Thermodynamik. Für ein solches Modell werden wir die mathematische Wohlgestellheit des Problems, also Existenz, Eindeutigkeit und Stabilität von Lösungen, untersuchen. Die betrachteten Modelle sind komplexe, nichtlineare, parabolische Systeme von partiellen Differenzialgleichungen mit einer energiedissipativen Struktur basierend auf einem nicht-konvexen Energiefunktional. Die Schlüsselschwierigkeit hier entsteht zum einen aufgrund der Kreuzdiffusionsstruktur eines Teilproblems und zum anderen durch die logarithmische freie Energie eines anderen Teilproblems. Wir beweisen die Existenz von globalen dissipativen schwachen Lösungen, in zwei und drei Raumdimensionen, mittels Energiemethoden. Weiter verwenden wir die relative Energiemethode, um ein abstraktes Stabilitätsresultat zu erhalten. Eine Konsequenz ist die schwach-starke Eindeutigkeit für dissipative schwache Lösungen. In der numerischen Approximation und der zugehörigen Fehleranalyse ist es nützlich, dass das Verfahren den zweiten Hauptsatz der Thermodynamik erhält. Die Schlüsselschwierigkeit hier ist die richtige Diskretisierung der konvektiven Terme im diskreten und eine geeignete Zeitintegrationsmethode für den nicht-konvexen Teil der Energie. Für die Teildiskretisierung eines reduzierten Modells betrachten wir konforme inf-sup stabile Finite Elemente Methoden im Raum und analysieren diese. Die thermodynamischen Eigenschaften werden unter Galerkinprojektion erhalten und wir verwenden eine diskrete Version des Stabilitätsresultats um die optimale Approximationsordnung zwei mittels eines transparenten und strukturierten Zugangs zu beweisen. Für eine volle Diskretisierung verwenden wir eine variationelle Zeitdiskretisierung mittels Petrov-Galerkinmethoden auf Basis der Teildiskretisierung im Raum. Die diskrete thermodynamische Struktur ist erhalten und mithilfe eines volldiskreten Stabilitätsresultats führen wir die Fehleranalyse durch. Wir beweisen die optimale Approximationsordnung zwei in Raum und Zeit mittels realistischen Glattheitsannahmen. Die theoretische Analyse der Teil- und Volldiskretisierung wird mittels numerischer Experimente illustriert.

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1 Introduction

The process of mixing and demixing of multiphase flows, i.e., simultaneous flow of at least two thermodynamic phases, is of special interest in chemistry and physics. While the theoretical understanding of mixing is quite far developed, the opposite process, i.e., demixing of multiphase flows, is less understood in general. In the context of two fluid components, for instance water and oil, one speaks about binary fluids and the demixing process is called phase separation. A phenomenological macroscopic description for the phase separation of binary fluids is given by the *model H* of Hohenberg and Halperin [77] and reads

System S.1 (Model H)**.**

$$\begin{aligned}
\partial_t\phi + \mathbf{u}\cdot\nabla\phi &= \operatorname{div}\Big(b(\phi)\nabla\mu\Big),\\
\mu &= -\gamma\Delta\phi + f'(\phi),\\
\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}\Big(\eta(\phi)\mathrm{D}\mathbf{u}\Big) - \nabla p + \nabla\phi\mu,\\
0 &= \operatorname{div}(\mathbf{u})\,.
\end{aligned}$$

This model consists of the Cahn-Hilliard equation for the evolution of the volume fraction ϕ, describing the fraction of one of the two phases. The Cahn-Hilliard equation is a gradient flow, where the chemical potential μ is the variational derivative of the relevant functional, which models the internal behaviour of the mixture. The flow is modelled by the incompressible Navier-Stokes equation for the velocity $\mathbf{u}$ and the pressure p, where the density is set equal to one here. Both equations are coupled, in the sense that in the Cahn-Hilliard equation the volume fraction is transported by means of the velocity, while the volume fraction exhibits the Korteweg stress [7] on the flow, i.e., $\nabla\phi\mu$. Note that this stress can be rewritten as a symmetric second-order tensor by redefining the pressure p, i.e., the stress can be written as a symmetric matrix, in agreement with momentum conservation. The parameter functions $b(\cdot), \eta(\cdot)$ are volume fraction dependent functions that represent the mobility, i.e., diffusion, and the viscosity of the mixture, which are always non-negative. The missing functions and parameters $f(\cdot)$ and γ are related to the free energy functional, which is given by

$$E(\phi,\mathbf{u}) := \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}x. \tag{1.1}$$

Here Ω is the spatial domain. The mixing potential $f(\cdot)$ models the internal behaviour of the mixture and is typically chosen as a non-convex double-well potential, such that the two wells, i.e., minima of f, represent the energetically favourable states for the mixture. The interface width γ penalizes jumps such that the transition between the two separated states is smooth but sufficiently small. As mentioned in the beginning, μ is the variational derivative of $E(\phi, \mathbf{u})$ with respect to ϕ.

The model is compatible with the second law of thermodynamics, which — in absence of external forces — implies that the free energy has to be non-increasing over time. This property will further be labelled *thermodynamic consistency.* For System S.1, using suitable boundary conditions, the temporal evolution of the free energy (1.1) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\phi, \mathbf{u}) = -\int_\Omega |b^{1/2}(\phi)\nabla\mu|^2 + |\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}|^2 \,\mathrm{d}x \leq 0.$$

This illustrates that the model S.1 is dissipative, i.e., in general, the free energy will decrease over time. The model is also well understood from a mathematical point of view, i.e., existence [24], uniqueness [24] of suitable solutions and also the numerical analysis [46] has been established.

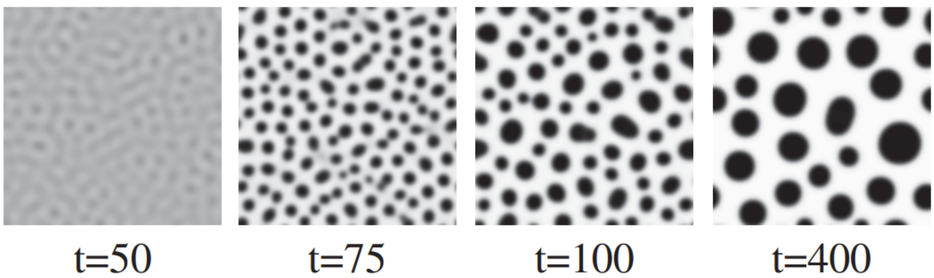

Figure 1.1.: Numerical simulation: Pattern evolution for model H. Taken from [118]. ©IOP Publishing. Reproduced with permission. All rights reserved. Permission conveyed through Copyright Clearance Center, Inc.

Figure 1.1 shows a typical simulation for the above model. One starts from an almost uniformly distributed state and after some time small droplet of the minority phase start to appear and grow in space.

This observation perfectly matches with the underlying physical understanding. In the context of simple binary fluids one speaks about dynamical symmetry [117], which implies that the dynamical properties of both components are very similar, i.e., the characteristic time scales of both components are comparable. Furthermore, it is known that the mean droplets size scales like $t^{1/3}$ [36], which implies a very slow algebraic decay to equilibrium. In the context of the above simulations, this implies that it will take more and more time to form bigger droplets. A typical property of soft matter systems is that the process has an incomplete scale separation, i.e., the regimes corresponding to characteristic time scales cross each other, which makes it difficult to relate observations to a single process.

However, when leaving the case of simple binary fluids for instance by considering a polymer solution, i.e., the mixture of polymer and solvent(fluid), dynamic symmetry does not hold anymore, see [120]. In Figure 1.2, we display snapshots of the demixing of a polymer solution taken from real experiments. One can observe that, after some initial phase, the so-called frozen phase, a phase inversion happens, i.e., the observable dominant phase changes rapidly and a transient network of the polymer is formed. After some time, the network starts to relax and breaks out in the small clusters.

One can see that qualitative observations do not match with the results of model H, i.e., Figure 1.1. Hence, we conclude that some effects of the polymer solution cannot be explained within model H, and therefore we need a more complex model.

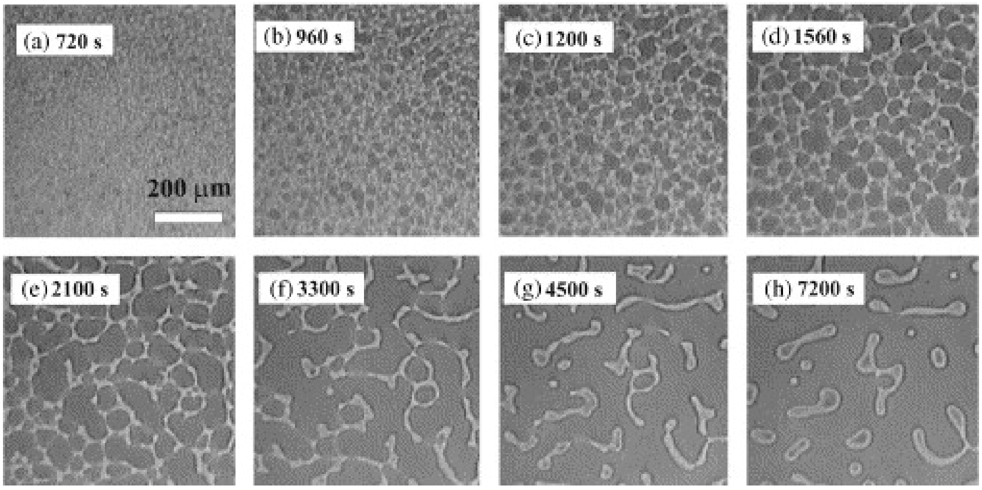

Figure 1.2.: Phase separation of the polystyrene-polyvinylmethylether (PS-PVME) mixture, observed by video phase-contrast microscopy. Taken from [118]. ©IOP Publishing. Reproduced with permission. All rights reserved. Permission conveyed through Copyright Clearance Center, Inc.

Let us shortly discuss the physical background of a polymer solution within the context of phase separation. Polymer solutions are a typical example of a *soft matter system*, in which temperature effects balance other effects, like diffusion, demixing and many more. Hence, the mixing properties of polymer solutions crucially depend on the temperature regime. This leads to the division into the so-called "good" and "poor" solvent conditions. While "good" solvent conditions correspond to temperature regimes at which contacts between solvent particles and the macromolecules of the polymer are more favourable than self-contacts, i.e., a mixing process, for "poor" solvent conditions it is the opposite case, i.e., self-contacts are more favourable, which induces a demixing. Of common interest is to understand the relevant dynamics of demixing after a quench, i.e., a sharp temperature drop, from "good" to "poor" solvent conditions.

In the polymer solution, the new effects are related to what is called *dynamical asymmetry*, see [119, 120]. This means that the dynamic properties, like diffusion and relaxation, are completely different for the solvent and the polymer phase. Note that these dynamic properties give rise to different time scales where the associated effects are prominent, and hence we deal with a multiple time-scale problem. However, additionally to this asymmetry, the problem exhibits an incomplete scale separation, i.e., the time scales are not separated, but rather overlap and from dynamic regions. Within these regions, it is rather difficult from observations to deduce which effect triggers a certain phenomenon.

We will start with reviewing the modelling of the described process. In order to model these new effects, it is natural to build on model H and add viscoelastic effects, based on polymer chain deformation effects, to the model. Such approaches have been considered in several works, see for instance [48, 76, 102, 121]. However, many of them could not reproduce experimental observations during simulations. The basic idea of our model goes back to Tanaka [119]. In order to incorporate the observed effects, he proposed, in addition to a viscoelastic equation associated with the deformation of the polymer chains, another viscoelastic equation for what he calls "bulk stress". The bulk stress is assumed to be a pressure which is directly coupled to the diffusion process of the system. In detail, Tanaka proposed the bulk stress as an isotropic stress tensor, associated with the volume deformation via the relative velocity, i.e., the phase velocity differences. Note that such stress in general can only exists as long as the relative velocity does not coincide with the mixture velocity, which is assumed to behave almost incompressible. Furthermore, Tanaka coined the phenomenon *viscoelastic phase separation* and introduced relevant regimes of this process, see Figure 2.1. Tanaka's model seems to reproduce the experimental observation quite well, see [118, Figure 20].

Zhou et al. [132] questioned the thermodynamic consistency of Tanaka's model. We want to note here that Tanaka's model is in fact not thermodynamically consistent with the typical choice of free energy. Since this violates basic physical principles and is the basis of mathematical discussion, Zhou et al. re-derived a similar model via variational arguments, i.e., the generalized virtual work principle [45]. This ensures thermodynamic consistency, and they provided numerical simulations of their new model, which again seems to describe the new effects. At this stage, it is natural to ask if the numerical method with Zhou et al. employed is also consistent with the second law of thermodynamic. Furthermore, the mathematical well-posedness and convergence analysis of the numerical methods is not discussed. The model in Zhou et al. is of mixed parabolic-hyperbolic type and reads

System S.2 (Viscoelastic model H)**.**

$$
\begin{aligned}
\partial_t \phi + \mathbf{u} \cdot \nabla \phi &= \operatorname{div} \Big(b(\phi) \nabla \mu - b^{1/2}(\phi) \nabla (A(\phi) q) \Big) , \\
\mu &= -\gamma \Delta \phi + f'(\phi), \\
\partial_t q + \mathbf{u} \cdot \nabla q &= -\kappa_1(\phi) q + \operatorname{div} \Big(\nabla (A(\phi) q - b^{1/2}(\phi) \nabla \mu) \Big) , \\
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \operatorname{div} (\eta(\phi) \mathrm{D} \mathbf{u}) + \operatorname{div} (\mathbf{B}) + \mu \nabla \phi, \\
0 &= \operatorname{div} (\mathbf{u}) , \\
\partial_t \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} &= (\nabla \mathbf{u}) \mathbf{B} + \mathbf{B} (\nabla \mathbf{u})^\top - \kappa_2(\phi) \mathbf{B} + B(\phi) \mathrm{D} \mathbf{u}.
\end{aligned}
$$

We observe that in addition to model H, the system contains an evolution equation for the bulk stress q coupled directly to the Cahn-Hilliard equation. Furthermore, the Navier-Stokes equation is coupled to an evolution equation for the viscoelastic stress tensor $\mathbf{B}$, which together form the so-called Oldroyd-B model, see [41].

Strasser et al. [116] proposed efficient and energy-stable numerical methods for the simulation of System S.2. Hence, this ensures that the numerical algorithm preserves the second law of thermodynamics. However, the mathematical analysis of well-posedness and rigorous approximability by numerical methods for this type of model was open.

Let us start reviewing the viscoelastic model H, i.e., System S.2. The model is obtained using the *generalized virtual work principle* of Groot et al. [45]. One starts from a set of not completely determined evolution equations for the chosen state variables and prescribed free energy. In order to close the system, phenomenological relations are chosen such that the system fulfils the second law of thermodynamics. We refer to [28], where we have derived the phenomenological closure used in [132] by model reduction. Such a derivation can be interpreted in the framework *General Equation for Non-Equilibrium Reversible-Irreversible Coupling* (GENERIC) of Grmela and Öttinger [65, 66]. This already indicates that thermodynamic consistency is built-in into the derivation process.

Following [132] the free energy of the System S.2 is given by

$$E_{free}(\phi, q, \mathbf{u}, \mathbf{B}) = \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}q^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}\mathrm{tr}(\mathbf{B})\,\mathrm{d}x, \tag{1.2}$$

where f is a mixing potential, typically non-convex. Using suitable boundary conditions, it can be shown that the following energy-dissipation identity holds for System S.2

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{free}(\phi, q, \mathbf{u}, \mathbf{B}) = -\int_\Omega & |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + |\kappa_1^{1/2}(\phi)q|^2 \\ & + |\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}|^2 + \frac{\kappa_2(\phi)}{2}\mathrm{tr}(\mathbf{B})\,\mathrm{d}x.\end{aligned} \tag{1.3}$$

From a physical point of view, the model is thermodynamically consistent assuming that $\mathrm{tr}(\mathbf{B})$ and the relaxation time $\kappa_2(\phi)$ are non-negative. By adding a suitable identity to $\mathbf{B}$ we can change variables to the so-called conformation tensor $\mathbf{C}$. This state variable is preferable since it is rooted in kinetic theory and should always be a positive-(semi) definite matrix. We note that the Oldroyd-B model arises as the simplest model from kinetic theory, see [86], i.e., considering linear spring potential. In general, for modelling complex effects in polymer solutions, a more sophisticated free energy is proposed, see for instance [114, 115]. The proposed free energy therein depends on a contribution from the spring potential, which is typical of logarithmic Warner type, see [20, 21]. Additionally, there should be a contribution from the inherent entropy of the springs, which is typically modelled by a $\log(\det(\cdot))$ term. We can observe that such a term is completely missing in the above free energy (1.2), cf. [79].

Furthermore, the structure of the energy-dissipation identity heavily influences the mathematical analysis. The well-known energy techniques rely on the a priori estimates following this energy-dissipation identity. Technically, we can replace the problem with a suitable approximation and pass to the limit if the necessary bounds for the approximations are available. However, by looking at the above identity, we can identify several difficulties.

The main issues from a mathematical point of view are twofold. First, due to the highly nonlinear cross-diffusive coupling, one cannot extract independent information on gradient bounds on μ or q. The second problem is the information on the Oldroyd-B equation itself. First, because without a priori information on the definiteness, the energy is not sufficient. This problem can be relaxed by switching variables to the mentioned conformation tensor $\mathbf{C}$. However, the second problem is that the coupling terms to the Navier-Stokes equation are quite ill-behaving. These coupling terms are part of the upper-convected derivative, which is a frame-invariant time derivative for matrices. A common approach in the literature is replacing them with another frame-invariant derivative, which however destroys the energy-dissipation structure. Furthermore, without gradient bounds on the conformation tensor there are only restricted results available, see [37, 87, 92].

Outline and structure of the thesis:
In this thesis, we will study an enhanced version of the System S.2, see [28] for comparison, and prove well-posedness and rigorous error analysis of the model and further related models. These models are derived and constructed to be thermodynamically consistent. In contrast to System S.2, the enhanced version will be a completely parabolic system with an energy-dissipative structure. The thesis will be divided into two main parts.

Well-posedness:
The first part of the thesis is divided into three chapters, i.e., Chapter 2-4, and will consider the well-posedness, i.e., existence, uniqueness, and stability.

1. In Chapter 2 we will introduce the enhanced model and related simplifications. We give an overview of the available results in the literature and identify theoretical gaps, which we will consider. Furthermore, we use this as an outline and informally present the results of the first part of the thesis. Afterwards, we will show that the model we propose is *thermodynamically consistent*.

2. Chapter 3 is devoted to the existence results for the relevant models. We use standard tools of nonlinear parabolic theory, i.e., *Galerkin approximation* and *energy* based arguments, together with *variational*, i.e., weak, formulations. We show the existence of *dissipative global-in-time weak solution* in two and three space dimensions.

3. Chapter 4 provides a *nonlinear stability estimates* for the problems, and as an application, we study continuous dependence on the initial data. Here we apply the *relative energy* techniques, i.e., constructing "distances" based on the energy, to measure differences between solutions. The results of this chapter will be an *abstract stability* result and the *weak-strong uniqueness principle*. Finally, we conclude the first part of the thesis on well-posedness and give an outlook.

Numerical analysis:
The second part of the thesis consists of five chapters, i.e., Chapter 5 -9, and will consider numerical approximations and their rigorous error analysis for a simplified model.

1. In Chapter 5, we will recall the available results on the numerical approximation for the considered model and again outline the results presented in this thesis. Furthermore, we recall some well-known results from approximation theory, including *finite element methods* and *variational time discretisation*.

2. Chapter 6 provides a semi-discrete numerical method by stable conforming finite elements. Using the *Galerkin projection* in space allows us to recover the stability estimate of Chapter 4 to prove *optimal second convergence* of this method in a simple and transparent way.

3. In Chapter 7, we consider a full discretisation by employing *variational Petrov-Galerkin* methods in time. In a similar spirit as in Chapter 6, we prove *optimal second-order convergence* in space and *time* under realistic smoothness assumptions. Hence, we establish rigorous numerical approximability for the simplified model.

4. Chapter 8 illustrates the convergence results of the two preceeding chapters by convergences tests, and we consider some simulations regarding the process of viscoelastic phase separation.

5. In Chapter 9 we will conclude the second part of the thesis on numerical analysis and give an outlook.

Finally, the thesis is completed by an appendix which for completeness provides an extensive list of notation, nomenclature and results which are used in the thesis, applying almost standard notation.

Part I: Well-posedness

2 Mathematical models

In this chapter, we will briefly introduce and motivate the enhanced model and the associated simplifications which we will consider in this work. Furthermore, we will comment on the available results in the literature. Furthermore, we will outline all the results we obtained in this context and emphasize which will be presented in this thesis. Afterwards, we provide the formal proofs for thermodynamic consistency and some additional structures. This energetic structure from the second law of thermodynamics, which we will call energy-dissipative structure, is hidden in the differential equation and contains the most relevant information for mathematical treatment.

2.1. Modelling

In order to resolve the theoretical issues with System S.2 mentioned in Chapter 1, we will replace the Oldroyd-B model with a more nonlinear but mathematically more stable diffusive Peterlin model, see [72, 96]. The Peterlin model used here employs a quadratic spring potential, see [96] for a generalisation.

The problem with the gradient bound in the cross-diffusion part is resolved by considering a diffusive variant of the bulk stress equation. This can be understood as a regularisation. However, the work of Süli et al. [15] implies that such models should indeed contain diffusion which is related to diffusion of the polymer chains. From a mathematical point of view, this type of model reduction by neglecting the diffusion, seems to be counter-productive. We additionally add a regularisation term in the Cahn-Hilliard equation. This is in principle not necessary but facilitates the theoretical arguments. Our viscoelastic phase separation model is similar to the one in our publications [31, 32] and reads

System S.3 (Full Model)**.**

$$
\begin{aligned}
\partial_t \phi + \mathbf{u}\cdot\nabla\phi &= \operatorname{div}\left((1+\varepsilon_0)b(\phi)\nabla\mu\right) - \operatorname{div}\Big(b^{1/2}(\phi)\nabla\Big(A(\phi)q\Big)\Big),\\
\mu &= -\gamma\Delta\phi + f'(\phi),\\
\partial_t q + \mathbf{u}\cdot\nabla q &= -\kappa_1(\phi)q + A(\phi)\Delta(A(\phi)q) - A(\phi)\operatorname{div}\Big(b^{1/2}(\phi)\nabla\mu\Big) + \varepsilon_1\Delta q,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}\left(\eta(\phi)\mathrm{D}\mathbf{u}\right) - \nabla p + \operatorname{div}\left(\operatorname{tr}(\mathbf{C})\mathbf{C}\right) + \nabla\phi\mu,\\
0 &= \operatorname{div}\mathbf{u},\\
\partial_t \mathbf{C} + (\mathbf{u}\cdot\nabla)\mathbf{C} &= (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top + \kappa_2(\phi)\Big(\Phi(\operatorname{tr}(\mathbf{C}))\,\mathbf{I} - \chi(\operatorname{tr}(\mathbf{C}))\,\mathbf{C}\Big) + \varepsilon_2\Delta\mathbf{C}.
\end{aligned}
$$

The equations are assumed to hold on a bounded domain $\Omega \subset \mathbb{R}^d$ in the space dimensions $d = 2, 3$. To avoid technicalities with boundary conditions, we assume that

(A0) $\Omega = \mathbb{T}^d$, where $\mathbb{T}^d$ denotes the d-dimensional torus.

To be precise, ϕ denotes the volume fraction of the polymer, i.e., the percentage of polymer in the solution, which is evolved by a Cahn-Hilliard equation. The term $q\mathbf{I}$ denotes the bulk stress introduced by Tanaka, cf. [118], which is modelled by a convection-diffusion equation and forms a strong coupled cross-diffusion system with the Cahn-Hilliard equation. The viscoelastic flow is model by the Peterlin model via the velocity $\mathbf{u}$ and the pressure p together with the conformation tensor $\mathbf{C}$, which is related to the chain deformation during the flow and is assumed to be a symmetric and positive definite matrix. The conformation tensor can be related to a part of the elastic stress by the equation $\mathbf{T} = \operatorname{tr}(\mathbf{C})\mathbf{C}$. We can directly observe that the system is of parabolic nature.

The total energy of the system S.3 is given by

$$
E_{total}(\phi, q, \mathbf{u}, \mathbf{C}) = \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}q^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{4}\operatorname{tr}(\mathbf{C})^2 - \frac{1}{2}\operatorname{tr}(\ln\mathbf{C})\,\mathrm{d}x. \tag{2.1}
$$

We will show that the above energy again decays monotonically, cf. Theorem 2.3.1. We will see that all the mathematical issues with the System S.2 model are formally cured. However, we need to ensure the positive definiteness of $\mathbf{C}$ to make sense of the total energy.

To make the analysis more easily accessible, we present two simplified models. The reason for division into submodels is that we can study the mathematical properties of simpler systems and can consider specific mathematical problems without considering other problematic contributions. Indeed, we will prove all mathematical results for the simplified models and by combination derive results for the full model, if applicable. As already outlined, the main key points are first the cross-diffusive coupling between the Cahn-Hilliard and the bulk stress equation, while the second key point is the logarithmic type energy and the associated positive definiteness of the conformation tensor.

Hence, the first simplified model is related to the diffusive effects, and called *Cahn-Hilliard-Navier-Stokes-Bulk* (CHNSQ) system, and is given by

System S.4 (CHNSQ model)**.**

$$
\begin{aligned}
\partial_t\phi + \mathbf{u}\cdot\nabla\phi &= \operatorname{div}\Big((1+\varepsilon_0)b(\phi)\nabla\mu\Big) - \operatorname{div}\Big(b^{1/2}(\phi)\nabla\Big(A(\phi)q\Big)\Big),\\
\mu &= -\gamma\Delta\phi + f'(\phi),\\
\partial_t q + \mathbf{u}\cdot\nabla q &= -\kappa_1(\phi)q + A(\phi)\Delta\Big(A(\phi)q\Big) - A(\phi)\operatorname{div}\Big(b^{1/2}(\phi)\nabla\mu\Big) + \varepsilon_1\Delta q,\\
\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}\Big(\eta(\phi)\mathrm{D}\mathbf{u}\Big) - \nabla p + \nabla\phi\mu,\\
0 &= \operatorname{div}\mathbf{u}.
\end{aligned}
$$

This simplified model is complemented with the following total energy

$$E_{total}(\phi, q, \mathbf{u}) = \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}q^2 + \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}x.$$

In order to study the viscoelastic flow and the positive definiteness property of the conformation tensor, we introduce the second simplified model. This model is characterized by the Peterlin model and given by

System S.5 (Peterlin model)**.**

$$\begin{aligned}
\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \operatorname{div}(\eta \mathrm{D}\mathbf{u}) - \nabla p + \operatorname{div}\operatorname{tr}(\mathbf{C})\mathbf{C}, \\
\operatorname{div}\mathbf{u} &= 0, \\
\partial_t\mathbf{C} + (\mathbf{u}\cdot\nabla)\mathbf{C} &= (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top + \Phi(\operatorname{tr}(\mathbf{C}))\,\mathbf{I} - \chi(\operatorname{tr}(\mathbf{C}))\,\mathbf{C} + \varepsilon_2\Delta\mathbf{C}.
\end{aligned}$$

For this system, the total energy is given by

$$E_{total}(\mathbf{u}, \mathbf{C}) = \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{4}\operatorname{tr}(\mathbf{C})^2 - \frac{1}{2}\operatorname{tr}(\ln\mathbf{C}) \,\mathrm{d}x.$$

The above simplified models can also be interpreted as suitable approximations related to different regimes of the viscoelastic phase separation. The CHNSQ model corresponds to the initial phase of the viscoelastic phase separation for a small velocity field. While the effects of the Peterlin model corresponds to the late phase after the network breaking, where the velocity is the dominant variable. This perfectly fits into the different regimes introduced by Tanaka in [118], see Figure 2.1. We may associate the CHNSQ model with the regimes (a)-(e), while the viscoelastic effects modelled by the Peterlin model will be most relevant in (e)-(f). Of course, the multiphase character of the Cahn-Hilliard equation has to be taken into account.

2.2. Outline of the main results

In Table 2.1 we will sketch the relevant well-posedness results for the above models that can be found in the literature in the case of strictly positive coefficient functions and polynomial-type potentials. Furthermore, we only present results and global-in-time weak or strong solutions. Hence, the row *Existence* refers to the existence of global-in-time weak/strong solutions, the row *WSU* is an abbreviation for weak-strong uniqueness of the weak/strong solutions and finally *Stability* refers to the stability of the mentioned solution with respect to perturbations.

Let us note that there is, of course, plenty of literature on the Cahn-Hilliard equation and even more on the incompressible Navier-Stokes equation. For Navier-Stokes, the existence in two and three space dimensions is known, and the results are obtained by Leray [88]. The uniqueness of weak solution in three space dimensions is still open, see [40]. In principle, further results on the local existence of strong solutions and stability are known, see for instance the monographs [25, 122] and the review article [19]. For a review article of the relative energy methods in fluid dynamics, we refer to [127].

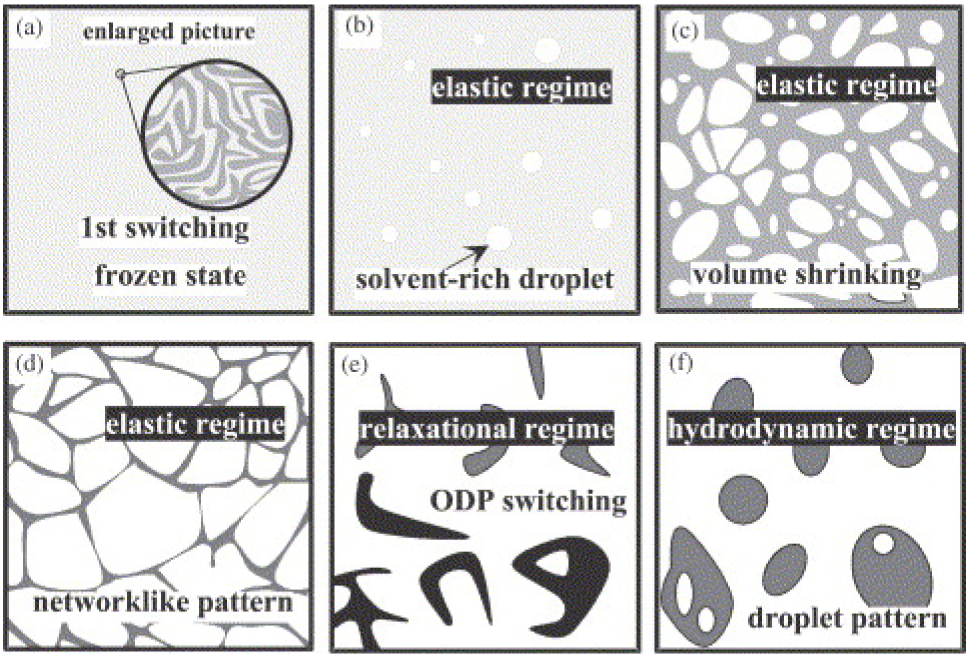

Figure 2.1.: Dynamical regimes across viscoelastic phase separation. Taken from [118]. ©IOP Publishing. Reproduced with permission. All rights reserved. Permission conveyed through Copyright Clearance Center, Inc.

2D	Model H	CHNSQ	Oldroyd-B	Peterlin	Full Model
Existence	✓,[24]	✓	[41, 15]	✓,[30], [96]	✓, [31, 32]
WSU	✓, [24]*	✓, [33]	[41]*	✓, [33], [96]*	✓, [33]
Stability	✓	✓	-	✓	✓
3D					
Existence	✓, [24]	✓	-	✓, [30], [97]	✓
WSU	✓	✓	-	conditional [30]	conditional
Stability	✓	✓	-	conditional	conditional

Table 2.1.: Road map of results. Colorcode: ✓ for results which will be presented in the thesis; ✓ to emphasise our works; ✓ denote results, which can be obtained with the proofs presented here almost verbatim. Here * emphasises that in these cases even uniqueness is known. The three "conditional" results can be obtained using the presented techniques, together with the results in [30].

For the Cahn-Hilliard equation, the existence of weak solutions goes back to Elliott et al., see [50]. See also [2, 24, 39, 103] and the references therein for several theoretical extensions. The case of partially vanishing coefficients and logarithmic potentials is also considered by Elliott et al. [50] and we will comment on that in more detail.

2.3. Thermodynamic consistency

In this section, we provide formal proof for the thermodynamic consistency of System S.3. In order to establish the relevant energy-dissipation identity, we assume that the parameter functions are sufficiently smooth, and the following conditions hold

$$\varepsilon_i, \gamma \geq 0, b(\phi), \eta(\phi), \kappa_1(\phi), \kappa_2(\phi) \geq 0 \text{ and } \chi(\mathrm{tr}(\mathbf{C})) = \Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) \geq 0 \tag{2.2}$$

Furthermore, every model is subjected to suitable initial conditions, for instance System S.3 to $(\phi, q, \mathbf{u}, \mathbf{C})|_{t=0} = (\phi_0, q_0, \mathbf{u}_0, \mathbf{C}_0)$. Recall assumptions (A0), i.e., that Ω is the d-dimensional torus. In principle, the following results translate almost verbatim for a sufficiently smooth boundary with the following boundary conditions

$$\partial_n\phi|_{\partial\Omega} = \partial_n\mu|_{\partial\Omega} = \partial_n q|_{\partial\Omega} = 0, \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \partial_n\mathbf{C}|_{\partial\Omega} = \mathbf{0}. \tag{2.3}$$

Theorem 2.3.1. *Let $(\phi, \mu, q, \mathbf{u}, p, \mathbf{C})$ denote a (sufficiently) smooth solution of System S.3 on $[0, T]$ with conformation tensor $\mathbf{C}(t)$ positive definite for all $t \in [0, T]$. Furthermore, assume (A0) holds, all parameter functions are (sufficiently) smooth, the conditions* (2.2) *hold and recall that $\mathbf{T} = \mathrm{tr}(\mathbf{C})\mathbf{C}$. Then following holds for all $t \in (0, T)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{total}(\phi, q, \mathbf{u}, \mathbf{C}) = -D_{total} \leq 0, \tag{2.4}$$

with

$$\begin{aligned} D_{total} = &\int_\Omega \varepsilon_0|b^{1/2}(\phi)\nabla\mu|^2 + |\kappa_1^{1/2}(\phi)q|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + \varepsilon_1|\nabla q|^2 \,\mathrm{d}x \\ &+ \int_\Omega \eta(\phi)|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 + \frac{\varepsilon}{2}\sum_{i=1}^d |\mathbf{C}^{-1/2}\partial_{x_i}\mathbf{C}\mathbf{C}^{-1/2}|^2 \,\mathrm{d}x \\ &+ \int_\Omega \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}\left(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}\right) \,\mathrm{d}x. \end{aligned}$$

Proof. We recall the definition of the total energy (2.1) as

$$\begin{aligned} E_{total}(\phi, q, \mathbf{u}, \mathbf{C}) &= \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \int_\Omega \frac{1}{2}q^2 + \int_\Omega \frac{1}{2}|\mathbf{u}|^2 + \int_\Omega \frac{1}{4}\mathrm{tr}(\mathbf{C})^2 - \frac{1}{2}\mathrm{tr}(\ln\mathbf{C}) \,\mathrm{d}x \\ &= (i) + (ii) + (iii) + (iv). \end{aligned}$$

In order to establish the energy-dissipation identity, we differentiate every term of (2.1) with respect to time and insert the partial differential equations, i.e., System S.3 suitably. Note that no boundary term appear when integrating by parts, since we work with the d-dimensional torus, i.e., $\Omega = \mathbb{T}^d$. Let us consider the first integral which yields

$$(i) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) \,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega \gamma \nabla\phi \cdot \partial_t \nabla\phi + f'(\phi)\partial_t\phi \,\mathrm{d}x \\
&= \int_\Omega -\gamma\Delta\phi\partial_t\phi + f'(\phi)\partial_t\phi \,\mathrm{d}x = \int_\Omega \mu\partial_t\phi \,\mathrm{d}x \\
&= \int_\Omega -\mathbf{u}\cdot\nabla\phi\mu + \mathrm{div}\Big((1+\varepsilon_0)b(\phi)\nabla\mu\Big)\mu - \mathrm{div}\Big(b^{1/2}(\phi)\nabla(A(\phi)q)\Big)\mu \,\mathrm{d}x, \\
&= -\,\mathbf{c}(\mathbf{u};\phi,\mu) - \int_\Omega (1+\varepsilon_0)\Big|b^{1/2}(\phi)\nabla\mu\Big|^2 - b^{1/2}(\phi)\nabla(A(\phi)q)\cdot\nabla\mu \,\mathrm{d}x. \qquad (2.5)
\end{aligned}
$$

Here we applied the definition of the convective term $\mathbf{c}(\cdot;\cdot,\cdot)$ according to (A.31).

$$
\begin{aligned}
(ii) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|q|^2 \,\mathrm{d}x = \int_\Omega q\partial_t q \,\mathrm{d}x \\
&= \int_\Omega -(\mathbf{u}\cdot\nabla q)q - \kappa_1(\phi)q^2 + \Delta(A(\phi)q)A(\phi)q - \mathrm{div}\Big(n(\phi)\nabla\mu\Big)A(\phi)q + \varepsilon_1 q\Delta q \,\mathrm{d}x \\
&= -\,\mathbf{c}(\mathbf{u};q,q) - \int_\Omega \kappa_1(\phi)q^2 + |\nabla(A(\phi)q)|^2 - n(\phi)\nabla\mu\cdot\nabla(A(\phi)q) + \varepsilon_1|\nabla q|^2 \,\mathrm{d}x \\
&= -\int_\Omega |\kappa_1^{1/2}(\phi)q|^2 + |\nabla(A(\phi)q)|^2 - b^{1/2}(\phi)\nabla\mu\cdot\nabla(A(\phi)q) + \varepsilon_1|\nabla q|^2 \,\mathrm{d}x. \qquad (2.6)
\end{aligned}
$$

Here we used the definition of the convective term $\mathbf{c}(\mathbf{u};q,q)$ and the associated skew-symmetric property yielding $\mathbf{c}(\mathbf{u};q,q) = 0$, see (A.32).

$$
\begin{aligned}
(iii) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}x = \int_\Omega \mathbf{u}\cdot\partial_t\mathbf{u} \,\mathrm{d}x \\
&= \int_\Omega -(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} - \mathrm{div}\left(\eta(\phi)\mathrm{D}\mathbf{u}\right)\cdot\mathbf{u} + \mathrm{div}(\mathbf{T})\cdot\mathbf{u} + \mu\nabla\phi\cdot\mathbf{u} \,\mathrm{d}x \\
&= -\,\mathbf{c}(\mathbf{u};\mathbf{u},\mathbf{u}) + \mathbf{c}(\mathbf{u};\phi,\mu) - \int_\Omega \eta(\phi)\mathrm{D}\mathbf{u} : \nabla\mathbf{u} + \mathbf{T} : \nabla\mathbf{u} \,\mathrm{d}x \\
&= \mathbf{c}(\mathbf{u};\phi,\mu) - \int_\Omega \Big|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\Big|^2 + \mathbf{T} : \nabla\mathbf{u} \,\mathrm{d}x. \qquad (2.7)
\end{aligned}
$$

As for the last equation, we cancel out the first term by skew-symmetry, i.e., (A.33). Furthermore, we use that $\mathrm{D}\mathbf{u} : \nabla\mathbf{u} = \mathrm{D}\mathbf{u} : (\mathrm{D}\mathbf{u} + \mathrm{W}\mathbf{u}) = \mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{u}$, where $\mathrm{W}\mathbf{u}$ denotes the skew-symmetric part of the velocity gradient. For the tensorial part, i.e., (iv) we split the contribution and obtain for the first part

$$
\begin{aligned}
(iv)_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{4}\mathrm{tr}(\mathbf{C})^2 \,\mathrm{d}x = \int_\Omega \frac{1}{2}\mathrm{tr}(\mathbf{C})\partial_t\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x \\
&= \int_\Omega -\mathbf{u}\cdot\nabla\mathrm{tr}(\mathbf{C}) : \mathrm{tr}(\mathbf{C}) + \mathrm{tr}(\mathbf{C})\mathbf{C} : \nabla\mathbf{u} - \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C})^2 \\
&\quad + \frac{d}{2}\kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) + \frac{\varepsilon_2}{2}\mathrm{tr}(\mathbf{C})\Delta\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x \\
&= -\,\mathbf{c}(\mathbf{u};\mathbf{C},\mathbf{C}) + \int_\Omega \mathbf{T} : \nabla\mathbf{u} \,\mathrm{d}x - \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C})^2 \\
&\quad + \frac{d}{2}\kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) - \frac{\varepsilon_2}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 \,\mathrm{d}x \\
&= \int_\Omega \mathbf{T} : \nabla\mathbf{u} \,\mathrm{d}x - \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\left(\mathrm{tr}(\mathbf{C})^2 - d\right) - \frac{\varepsilon_2}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 \,\mathrm{d}x. \qquad (2.8)
\end{aligned}
$$

Here we used the skew-symmetry of $\mathbf{c}(\cdot;\mathbf{C},\mathbf{D})$, i.e., (A.34) and used the definition of $\chi(\mathrm{tr}(\mathbf{C}))$, see (2.2). For the next term we compute

$$\begin{aligned}
(iv)_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\mathrm{tr}(\ln(\mathbf{C}))\,\mathrm{d}x = \int_\Omega \mathbf{C}^{-1} : \partial_t\mathbf{C}\,\mathrm{d}x \\
&= \int_\Omega -(\mathbf{u}\cdot\nabla)\mathbf{C} : \mathbf{C}^{-1} + \mathrm{tr}(\nabla\mathbf{u}\mathbf{C}\mathbf{C}^{-1}) - \frac{d}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C})) \\
&\quad + \frac{1}{2}\kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}^{-1}) + \frac{\varepsilon_2}{2}\Delta\mathbf{C} : \mathbf{C}^{-1}\,\mathrm{d}x \\
&= \int_\Omega \mathrm{div}\,(\mathbf{u})\,\mathrm{tr}(\ln(\mathbf{C})) + \mathrm{tr}(\nabla\mathbf{u}) - \frac{\varepsilon_2}{2}\nabla\mathbf{C} : \nabla\mathbf{C}^{-1} \\
&\quad - \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\left(d - \frac{\mathrm{tr}(\mathbf{C}^{-1})}{\mathrm{tr}(\mathbf{C})}\right)\,\mathrm{d}x \\
&= -\int_\Omega \frac{\varepsilon_2}{2}\nabla\mathbf{C} : \nabla\mathbf{C}^{-1} + \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\left(d - \frac{\mathrm{tr}(\mathbf{C}^{-1})}{\mathrm{tr}(\mathbf{C})}\right)\,\mathrm{d}x. \qquad (2.9)
\end{aligned}$$

Here we applied the Jacobi formula (A.13), divergence-freedom of the velocity $\mathbf{u}$ and the definition of $\chi(\mathrm{tr}(\mathbf{C}))$, see (2.2). Summing up (2.5)-(2.9) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_{total}(\phi,q,\mathbf{u},\mathbf{C}) \\
&= -\int_\Omega \varepsilon_0|b^{1/2}(\phi)\nabla\mu|^2\,\mathrm{d}x + |\kappa_1^{1/2}(\phi)q|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + \varepsilon_1|\nabla q|^2\,\mathrm{d}x \\
&\quad - \int_\Omega \left|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\right|^2 + \frac{\varepsilon}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 - \frac{\varepsilon}{2}\nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_\Omega \kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\left(\mathrm{tr}(\mathbf{C})^2 + \frac{\mathrm{tr}(\mathbf{C}^{-1})}{\mathrm{tr}(\mathbf{C})} - 2d\right)\,\mathrm{d}x \\
&= (*).
\end{aligned}$$

We will estimate all negative terms from above by zero. Computation of the gradient of the inverse and further rearrangement, by recalling that $\mathbf{T} = \mathrm{tr}(\mathbf{C})\mathbf{C}$ yields

$$(*) \le -\frac{\varepsilon_2}{2}\int_\Omega \sum_{i=1}^d \nabla\mathbf{C} : \mathbf{C}^{-1}\partial_{x_i}\mathbf{C}\mathbf{C}^{-1}\,\mathrm{d}x - \frac{1}{2}\int_\Omega \kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}\left(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}\right)\,\mathrm{d}x.$$

The first term yields by definition of the Frobenius inner product and decomposition into the matrix square root

$$-\frac{\varepsilon_2}{2}\int_\Omega \sum_{i=1}^d \nabla\mathbf{C} : \mathbf{C}^{-1}\partial_{x_i}\mathbf{C}\mathbf{C}^{-1}\,\mathrm{d}x = -\frac{\varepsilon}{2}\int_\Omega \sum_{i=1}^d |\mathbf{C}^{-1/2}\partial_{x_i}\mathbf{C}\mathbf{C}^{-1/2}|^2\,\mathrm{d}x \le 0.$$

For the second term we recall that for a symmetric positive definite matrix, in our case $\mathbf{T} = \mathrm{tr}(\mathbf{C})\mathbf{C}$, cf. (A.9) we have

$$\mathrm{tr}\left(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}\right) \ge 0.$$

Together this implies $(*) \le 0$ and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{total}(\phi,q,\mathbf{u},\mathbf{C}) \le 0.$$

This identity can also be obtained by formally testing the weak formulation (3.64) with

$$\left(\mu, \partial_t\phi, q, \mathbf{u}, \frac{1}{2}\mathrm{tr}(\mathbf{C})\mathbf{I} - \frac{1}{2}\mathbf{C}^{-1}\right),$$

respectively. □

The derivation of the above energy-dissipation identity is based on variational arguments. We see that by multiplying the time derivative of the respective variable by the variational derivative of the total energy with respect to the corresponding variable. The time derivative is substituted with the remaining part of the equations. However, after rearrangement, and integration over the domain Ω this is exactly the same as testing the variational formulation with the corresponding variational derivative.

In the following, we will present several additional energy-type inequalities, which are relevant for the analysis but have no direct physical interpretation as the total energy.

Energy-type inequality:
In the above proof, we already derived another relevant equality of the energy functional

$$E(\phi, q, \mathbf{u}, \mathbf{C}) := \int_\Omega \left(\frac{\gamma}{2}|\nabla\phi|^2 + f(\phi)\right) + \int_\Omega \frac{1}{2}q^2 + \int_\Omega \frac{1}{2}|\mathbf{u}|^2 + \int_\Omega \frac{1}{4}|\mathrm{tr}(\mathbf{C})|^2 \,\mathrm{d}x, \tag{2.10}$$

which is summarized in the following lemma.

Lemma 2.3.2. *Consider the same conditions as in Theorem 2.3.1, except the positive definiteness of the conformation tensor* $\mathbf{C}$*. Then for a (sufficiently) smooth solution* $(\phi, \mu, q, \mathbf{u}, p, \mathbf{C})$ *of System S.3 the following evolution holds for the reduced total energy*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi, q, \mathbf{u}, \mathbf{C}) = -D + R. \tag{2.11}$$

with

$$\begin{aligned} D = &\int_\Omega \varepsilon_0\left|b^{1/2}(\phi)\nabla\mu\right|^2 + |\kappa_1^{1/2}(\phi)q|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + \varepsilon_1|\nabla q|^2 \,\mathrm{d}x \\ &+ \int_\Omega \left|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\right|^2 + \frac{\varepsilon_2}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 + \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))|\mathrm{tr}(\mathbf{C})|^2 \,\mathrm{d}x, \\ R = &d\int_\Omega \kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x. \end{aligned}$$

Proof. The results follows by summing only (2.5)-(2.8), i.e., not considering $(iv)_2$. □

Two-dimensional additional energy-type inequality:
In the special case of two space dimensions, we can propose another energy functional, which contains suitable information of the solution and is given by

$$E_{2d}(\phi, q, \mathbf{u}, \mathbf{C}) = \int_\Omega \left(\frac{\gamma}{2}|\nabla\phi|^2 + f(\phi)\right) + \int_\Omega \frac{1}{2}q^2 + \int_\Omega \frac{1}{2}|\mathbf{u}|^2 + \int_\Omega \frac{1}{4}|\mathbf{C}|^2 \,\mathrm{d}x. \tag{2.12}$$

Let us consider the evolution of (2.12). The difference in both energies is only due to the conformation tensor $\mathbf{C}$.

Lemma 2.3.3. *Consider the same conditions as in Theorem 2.3.1, except the positive definiteness of the conformation tensor. For (sufficiently) smooth solution* $(\phi, \mu, q, \mathbf{u}, p, \mathbf{C})$ *of System S.3 in two space dimensions, the following equality holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{2d}(\phi, q, \mathbf{u}, \mathbf{C}) = -D_{2d} + R_{2d} \tag{2.13}$$

with

$$\begin{aligned} D_{2d} &= \int_\Omega \varepsilon_0 \left| b^{1/2}(\phi)\nabla\mu \right|^2 + |\kappa_1^{1/2}(\phi) q|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + \varepsilon_1 |\nabla q|^2 \,\mathrm{d}x \\ &\quad + \int_\Omega \left|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\right|^2 + \frac{\varepsilon}{2}|\nabla\mathbf{C}|^2 + \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))|\mathbf{C}|^2 \,\mathrm{d}x, \\ R_{2d} &= \int_\Omega \kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x. \end{aligned}$$

Proof. The new term from the conformation tensor $\mathbf{C}$ yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{4}|\mathbf{C}|^2 \,\mathrm{d}x &= \int_\Omega \frac{1}{2}\mathbf{C} : \partial_t \mathbf{C} \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega -(\mathbf{u}\cdot\nabla)\mathbf{C} : \mathbf{C} + [(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top] : \mathbf{C} \\ &\quad - \kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))|\mathbf{C}|^2 + \kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) - \varepsilon_2|\nabla\mathbf{C}|^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega 2\mathrm{tr}(\mathbf{C})\mathbf{C} : \nabla\mathbf{u} - \kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))|\mathbf{C}|^2 + \kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) - \varepsilon_2|\nabla\mathbf{C}|^2 \,\mathrm{d}x. \end{aligned} \tag{2.14}$$

For the second equality we used (A.12) and skew symmetry of the trilinear form (A.34). Combination of (2.14) with (2.5)-(2.7) yields

$$\begin{aligned} &\frac{d}{dt}E_{2d}(\phi, q, \mathbf{u}, \mathbf{C}) = -D_{2d} + R_{2d} \\ &= -\int_\Omega \varepsilon_0 \left| b^{1/2}(\phi)\nabla\mu \right|^2 + |\kappa_1^{1/2}(\phi) q|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + \varepsilon_1 |\nabla q|^2 \,\mathrm{d}x \\ &\quad - \int_\Omega \left|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\right|^2 + \frac{\varepsilon}{2}|\nabla\mathbf{C}|^2 + \frac{1}{2}\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))|\mathbf{C}|^2 \,\mathrm{d}x + \int_\Omega \kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x. \end{aligned}$$

□

In contrast to the total energy, the additional functionals E and E_{2d} do not need to decay in time, but an application of the Gronwall lemma, cf. Lemma A.3.1 yields a suitable estimate for $\mathrm{tr}(\mathbf{C})$ and $\mathbf{C}$ without requiring positive definiteness.

Remark 2.3.4. A functional equality with similar estimates as in (2.13) is not available in three space dimensions. This results from the fact that (A.12) is a consequence of the Cayley-Hamilton theorem, cf. (A.11). In three space dimensions, we obtain instead

$$(\mathrm{tr}(\mathbf{C})\mathbf{C} - \mathbf{C}^2) : \nabla\mathbf{u} = \mathbf{C}_{adj} : \nabla\mathbf{u}$$

with $\mathbf{C}_{adj} = \det(\mathbf{C})\mathbf{C}^{-1}$ as the adjoint matrix. In general, this is not zero, and without this property we cannot establish a direct estimate as in two space dimensions.

Energy of the simplified models:
The submodels System S.4 and System S.5, imply corresponding energy laws. Indeed, for the CHNSQ model, i.e., System S.4, all three energy functionals, omitting the contribution of $\mathbf{C}$, are the same and we summarize the result in the following lemma.

Lemma 2.3.5. *Let* $(\phi, \mu, q, \mathbf{u}, p)$ *denote a (sufficiently) smooth solution of System S.4 on* $[0, T]$. *Furthermore, assume (A0) holds, all parameter functions are (sufficiently) smooth and the conditions* (2.2) *hold. Then the following holds for all* $t \in (0, T)$

$$\frac{d}{dt}E(\phi, q, \mathbf{u}) = -\int_\Omega \varepsilon_0 |b^{1/2}(\phi)\nabla\mu|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + |\kappa_1^{1/2}(\phi)q|^2 + \varepsilon_1|\nabla q|^2 + \left|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\right|^2 \,\mathrm{d}x. \tag{2.15}$$

Proof. This follows from the proof of Theorem 2.3.1 by neglecting the contributions of the conformation tensor $\mathbf{C}$. □

For the Peterlin model, we obtain a simplified version of the total energy and two additional energy laws, which are again summarized in a lemma.

Lemma 2.3.6. *Let* $(\mathbf{u}, p, \mathbf{C})$ *denote a (sufficiently) smooth solution of System S.5 on* $[0, T]$ *with conformation tensor* $\mathbf{C}(t)$ *positive definite for all* $t \in [0, T]$. *Furthermore, assume (A0) holds, all parameter functions are (sufficiently) smooth and the conditions* (2.2) *hold. Then the following holds for all* $t \in (0, T)$

$$\frac{d}{dt}E_{total}(\mathbf{u}, \mathbf{C}) = -\int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{1}{2}\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}\left(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}\right) + \frac{\varepsilon_2}{2}|\nabla \mathrm{tr}(\mathbf{C})|^2 + \frac{\varepsilon_2}{2}\sum_{i=1}^{d}|\mathbf{C}^{-1/2}\partial_{x_i}\mathbf{C}\mathbf{C}^{-1/2}|^2 \,\mathrm{d}x, \tag{2.16}$$

$$\frac{d}{dt}E(\mathbf{u}, \mathbf{C}) = -\int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla \mathrm{tr}(\mathbf{C})|^2 + \frac{1}{2}\chi(\mathrm{tr}(\mathbf{C}))|\mathrm{tr}(\mathbf{C})|^2 - \frac{d}{2}\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x, \tag{2.17}$$

$$\frac{d}{dt}E_{2d}(\mathbf{u}, \mathbf{C}) = -\int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla\mathbf{C}|^2 + \frac{1}{2}\chi(\mathrm{tr}(\mathbf{C}))|\mathbf{C}|^2 - \frac{1}{2}\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}) \,\mathrm{d}x. \tag{2.18}$$

Proof. The result simply follows from the proof of Theorem 2.3.1, Lemma 2.3.2 and Lemma 2.3.3 by neglecting the contributions of the volume fraction ϕ and the bulk stress q. □

Before we proceed, let us recap the results derived so far. We have built an enhanced version of the viscoelastic model H, i.e., System S.2, by System S.3. Furthermore, we have provided formal proof that the model is indeed thermodynamic consistent and presented additionally energy-like equalities. These additional equalities will help us in the proofs in the next chapter but have no direct physical meaning.

3 Global weak solutions

In this chapter, we will prove the existence of dissipative global-in-time weak solutions by using the energy-dissipative structure of the models. The chapter is structured as follows. In the first and second sections, we consider the CHNSQ model, i.e., System S.4. We first introduce a notion of weak solutions and state the corresponding existence result in two and three space dimensions, see Section 3.1, which is proven in Section 3.2. The proof is realized by a suitable Galerkin approximation based on energy arguments. Afterwards in the third and fourth sections, we consider the Peterlin model, i.e., System S.5, in two and three space dimensions. As before, we first state the weak formulation and the corresponding existence result in Section 3.3, while the proof is given in Section 3.4. Here we use a mixed Galerkin and semigroup approach to obtain the result. This approach is based on the ideas of [97]. Also, among other results, the existence result of System S.5 can be found in our recent work [30]. Next, we consider the full model, i.e., System S.3 in Section 3.5, where the existence result simply follows by combining the proofs for the CHNSQ and the Peterlin model. In Section 3.6, we comment on the weak formulation in space and time, to derive a suitable setup for the following chapter.

Notation and basic assumptions:
We mainly use standard notation; see appendix A for details. First, we recall assumption (A0), i.e., that Ω is the d-dimensional torus. This implies that the dual spaces of the Sobolev space $H^1(\Omega)$ can be identified with the negative Sobolev space, i.e., $(H^1(\Omega))^* = H^{-1}(\Omega)$. We denote by $\langle \cdot, \cdot \rangle$ the $L^2(\Omega)$ inner product or the dual paring $H^{-1}(\Omega) \times H^1(\Omega)$ or $H^1(\Omega) \times H^{-1}(\Omega)$; the precise meaning will be clear from the context. The spaces $L^2_{\mathrm{div}}(\Omega)^d, H^1_{\mathrm{div}}(\Omega)^d$ correspond to the standard spaces in fluid dynamics, i.e., the space of divergence-free functions in $L^2(\Omega)^d$ and $H^1(\Omega)^d$, respectively, cf. Appendix A. We denote the dual space of $H^1_{\mathrm{div}}(\Omega)^d$ by $H^{-1}_{\mathrm{div}}(\Omega)^d$. In the following, we abbreviate $\int_0^t \int_\Omega$ by $\int_{\Omega_t}$ for all $t \in (0, T)$.

The basic assumptions on the coefficient functions are collected below.

Assumptions 3.0.1 (Regular case)**.**

(A0) *The domain Ω is the d-dimensional torus, i.e., $\Omega = \mathbb{T}^d$.*

(A1) *The mobility function $b \in C^1(\mathbb{R})$ with $0 < b_1 \leq b(s) \leq b_2$ for all $s \in \mathbb{R}$ and $\|b'\|_{0,\infty} \leq b_3$.*

(A2) *The potential* f *is* $C^4(\mathbb{R})$ *with* $f(s), f''(s) \geq -f_1$ *for* $f_1 \geq 0$ *and the growth conditions* $|f^{(k)}(s)| \leq f_2^{(k)} + f_3^{(k)}|s|^{4-k}$ *for* $0 \leq k \leq 4$ *for* $f_2^{(k)}, f_3^{(k)} \geq 0$.

(A3) *The bulk modulus* $A \in C^2(\mathbb{R})$ *with* $0 \leq A_1 \leq A(s) \leq A_2$ *and* $\|A^{(k)}\|_{0,\infty} \leq A_{k+2}$ *for* $k = 1, 2$.

(A4) *The relaxation times* $\kappa_1, \kappa_2 \in C^1(\mathbb{R})$ *with* $0 < \kappa_{i,1} \leq \kappa_i(s) \leq \kappa_{i,2}$ *and* $\|\kappa_i'\|_{0,\infty} \leq \kappa_{i,3}$

(A5) *The interface width and the diffusion coefficients are positive, i.e.,* $\gamma, \varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$.

(A6) *The viscosity* $\eta \in C^1(\mathbb{R})$ *with* $0 < \eta_1 \leq \eta(s) \leq \eta_2$ *for all* $s \in \mathbb{R}$ *and* $\|\eta'\|_{0,\infty} \leq \eta_3$.

(A7) *The generalized relaxation times satisfy* $\chi(\mathrm{tr}(\mathbf{C})) = \mathrm{tr}(\mathbf{C})^2 + a|\mathrm{tr}(\mathbf{C})|$ *and* $\Phi(\mathrm{tr}(\mathbf{C})) = \mathrm{tr}(\mathbf{C}) + a$ *for* $a \geq 0$.

3.1. Weak solutions for System S.4

In this section, we will define a suitable concept of weak solutions for System S.4 and state the corresponding existence result. The main difficulty in the existence will arise from the strong nonlinear coupling between the volume fraction ϕ and the bulk stress q due to the cross-diffusive nature.

Definition 3.1.1. Let (A0)–(A6) hold and $T > 0$ is a given time. The quadruple $(\phi, \mu, q, \mathbf{u})$ is called a weak solution of System S.4 on $\Omega \times (0, T)$, if

$$\begin{aligned}
\phi &\in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) =: \mathbb{W}(0,T),\\
\mu &\in L^2(0,T;H^1(\Omega)) =: \mathbb{Q}(0,T)\\
q &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap W^{1,4/3}(0,T;H^{-1}(\Omega)) =: \mathbb{M}(0,T),\\
\mathbf{u} &\in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)^d) \cap L^2(0,T;H^1_{\mathrm{div}}(\Omega)^d) \cap W^{1,p}(0,T;H^{-1}_{\mathrm{div}}(\Omega)^d) =: \mathbb{X}(0,T),
\end{aligned} \tag{3.1}$$

with $p = 2$ in two space dimensions and $p = \frac{4}{3}$ in three space dimensions, and

$$\begin{aligned}
\langle \partial_t \phi, \psi \rangle - \mathbf{c}(\mathbf{u}; \psi, \phi) + \langle (1+\varepsilon_0) b(\phi) \nabla \mu - b^{1/2}(\phi) \nabla (A(\phi) q), \nabla \psi \rangle &= 0,\\
\langle \mu, \xi \rangle - \gamma \langle \nabla \phi, \nabla \xi \rangle - \langle f'(\phi), \xi \rangle &= 0,\\
\langle \partial_t q, \zeta \rangle - \mathbf{c}(\mathbf{u}; \zeta, q) + \langle \kappa_1(\phi) q, \zeta \rangle + \varepsilon_1 \langle \nabla q, \nabla \zeta \rangle &\\
+ \langle \nabla (A(\phi) q) - b^{1/2}(\phi) \nabla \mu, \nabla (A(\phi) \zeta) \rangle &= 0,\\
\langle \partial_t \mathbf{u}, \mathbf{v} \rangle - \mathbf{c}(\mathbf{u}; \mathbf{v}, \mathbf{u}) + \langle \eta(\phi) \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v} \rangle + \mathbf{c}(\mathbf{v}; \mu, \phi) &= 0,
\end{aligned} \tag{3.2}$$

for any test function $(\psi, \xi, \zeta, \mathbf{v}) \in [H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)^d]$ and almost every $t \in (0, T)$.

We remark that the weak solution $(\phi, \mu, q, \mathbf{u})$ depends on time, which we suppress for readability whenever there is no confusion, i.e., $(\phi, \mu, q, \mathbf{u}) = (\phi(t), \mu(t), q(t), \mathbf{u}(t))$,

Remark 3.1.2. Every sufficiently smooth solution $(\phi, \mu, q, \mathbf{u}, p)$ of System S.4 is a weak solution in the above sense. This can easily be seen, by using suitable test functions and integration by parts. The pressure p vanishes from the weak formulation since we consider the velocity in the divergence-free space $H^1_{\mathrm{div}}(\Omega)^d$. This again follows from testing and integration by parts.

We proceed to state our first result on the existence of dissipative global-in-time weak solutions.

Theorem 3.1.3. *Let the initial data* $(\phi_0, q_0, \mathbf{u}_0) \in [H^1(\Omega) \times L^2(\Omega) \times L^2_{div}(\Omega)^d]$ *be given. Under assumptions (A0)–(A6) for* $d \in \{2, 3\}$ *and for every* $T > 0$ *there exists a dissipative global-in-time weak solution* $(\phi, \mu, q, \mathbf{u})$ *of System S.4 in the sense of Definition 3.1.1, which satisfies the initial data* $(\phi(0), q(0), \mathbf{u}(0)) = (\phi_0, q_0, \mathbf{u}_0)$ *and furthermore satisfies the energy inequality*

$$E(\phi, q, \mathbf{u})(t) + \int_{\Omega_t} \varepsilon_0 |b^{1/2}(\phi)\nabla\mu|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 + |\kappa_1^{1/2}(\phi)q|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_{\Omega_t} \varepsilon_1 |\nabla q|^2 + \left|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\right|^2 \mathrm{d}x \, \mathrm{d}s \leq E(\phi, q, \mathbf{u})(0) \tag{3.3}$$

for almost every $t \in (0, T)$.

In principle, we call a weak solution that satisfies an energy inequality, a dissipative weak solution. Before proceeding to the proof, let us shortly discuss some extensions.

Remark 3.1.4.

1. In fact, one can prove all results in the case $\varepsilon_0 = 0$, see [31, 32] for the existence result in two space dimensions. While the proof has to be slightly adapted, on which we will comment later, the regularity of the constructed solution only changes in three space dimensions. The regularisation with ε_0 fits into the *boundedness-by-entropy principle* for cross-diffusion systems of Jüngel [82] in the sense that they assume independent gradient bounds. Unfortunately, this technique cannot be applied in our case, since it was only developed for second-order equations with convex entropies.

2. The growth conditions on the potential can be relaxed depending on the space dimensions, again this yields similar existence results, see [31, 32].

3.2. Proof Theorem 3.1.3

The proof presented in this section is inspired by our two-dimensional considerations in [31] and [32].

Before we continue with the proof, let us give a short sketch of the main steps.

1. Approximation and local existence of suitable Galerkin scheme, see Subsection 3.2.1. Here we will briefly describe the construction of the Galerkin system, which severs as a finite-dimensional in space approximation of the original problem. This reduces the problem to a huge system of ordinary differential equations whose local existence can be established using standard theory for ordinary differential equations.

2. In the next step, see Subsection 3.2.2, we will deduce suitable approximation independent bounds on the discrete solution, which also allows us to continue local solutions to global solutions. Here, we will first derive a priori bounds using the

energy estimates. Secondly, we will derive higher estimates for the approximative volume fraction ϕ, using the stability of the Galerkin projection and elliptic regularity. The final set of a priori estimates are bounds for the time derivative of the approximate system, which will be obtained using duality arguments.

3. The third step, see Subsection 3.2.3, is to establish convergence of suitable subsequences. The typical weakly/weakly-* convergences can be directly obtained from the a priori bounds of the second step by using the consequence of Banach-Alaoglu A.3.3, i.e., in reflexive spaces, every bounded sequence has a weakly convergent subsequence. Since our model is nonlinear, we will also need strong convergences, which we obtain by applying compactness arguments via the Aubin-Lions lemma, cf. Lemma A.3.6.

4. The fourth step, see Subsection 3.2.4, is the passage to the limit in the approximative formulations. Here, we show that the constructed sequence does converge to the weak formulation. Hence, proofing the existence of global-in-time weak solutions. This will be done by using the before obtained convergences together with suitable density arguments.

5. The fifth step, see Subsection 3.2.5, is the limit in the energy inequality, i.e., proofing that the weak solution is also dissipative. This will be done again by employing weak and strong convergence arguments together with the generalized lemma of Fatou/lower-semi continuity of norms. This will conclude the proof.

3.2.1. Construction and local existence of Galerkin approximations

In order to prove existence of a weak solution to System S.3 we consider a Galerkin approximation of the weak formulation (3.64). For more details on the construction, we refer to monographs [122, 104, 110]. Let $\psi_j, \mathbf{v}_j, j = 1, \dots, \infty$, be the eigenfunctions of the Laplace operator and Stokes operator respectively, i.e.,

$$-\Delta\psi_j = \lambda_j\psi_j, \qquad -\Delta\mathbf{v}_j + \nabla p = \omega_j\mathbf{v}_j, \quad \operatorname{div}(\mathbf{v}_j) = 0.$$

The eigenfunctions ψ_j are smooth and orthogonal in $H^1(\Omega)$ and $L^2(\Omega)$, while the $\mathbf{v}_j$ are smooth, divergence-free and orthogonal in $H^1_{\mathrm{div}}(\Omega)^d$ and $L^2_{\mathrm{div}}(\Omega)^d$. Without loss of generality we set $\lambda_1 = 0 = \omega_1$, i.e., the constant function 1 is an element of V_m and Q_m, respectively. One can immediately see that

$$H^1(\Omega) = \overline{\operatorname{span}\{\psi_j\}_{j=1}^{\infty}}, \quad H^1_{\mathrm{div}}(\Omega)^d = \overline{\operatorname{span}\{\mathbf{v}_j\}_{j=1}^{\infty}}.$$

The finite-dimensional spaces are then defined by

$$V_m := \operatorname{span}\{\psi_1, \dots, \psi_m\} \qquad Q_m := \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}.$$

Within this set, we then now define orthogonal projections

$$\mathcal{P}_{V_m}(\psi) := \sum_{i=1}^{m}\langle\psi, \psi_i\rangle\psi_i : H^1(\Omega) \to V_m, \quad \mathcal{P}_{Q_m}(\mathbf{v}) := \sum_{i=1}^{m}\langle\mathbf{v}, \mathbf{v}_i\rangle\mathbf{v}_i : H^1_{\mathrm{div}}(\Omega)^d \to Q_m.$$

We then define the m-th Galerkin approximation of (ϕ, μ, q) as an orthogonal projection onto V_m and $\mathbf{u}_m$ as an orthogonal projection onto Q_m via

$$\begin{aligned}\phi_m(x,t) &= \sum_{j=1}^{m} \lambda_{jm}(t)\psi_j(x), \quad \mu_m(x,t) = \sum_{j=1}^{m} \theta_{jm}(t)\psi_j(x),\\ q_m(x,t) &= \sum_{j=1}^{m} \zeta_{jm}(t)\psi_j(x), \quad \mathbf{u}_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)\mathbf{v}_j(x),\\ \phi_{0m} &= \mathcal{P}_{V_m}(\phi_0), \qquad q_{0m} = \mathcal{P}_{V_m}(q_0), \qquad \mathbf{u}_{0m} = \mathcal{P}_{Q_m}(\mathbf{u}_0).\end{aligned}$$

Hence the initial values are also orthogonal projections onto the corresponding finite-dimensional subspace.

The Galerkin approximations satisfies a discrete version of (3.2), which is a system of ordinary differential equations, given by

$$\begin{aligned}\langle \partial_t \phi_m, \psi\rangle - \mathbf{c}(\mathbf{u}_m; \psi, \phi_m) + \langle (1+\varepsilon_0) b(\phi_m)\nabla\mu_m - b^{1/2}(\phi_m)\nabla(A(\phi_m)q), \nabla\psi\rangle &= 0,\\ \langle \mu_m, \xi\rangle - \gamma\langle \nabla\phi_m, \nabla\xi\rangle - \langle f'(\phi_m), \xi\rangle &= 0,\\ \langle \partial_t q_m, \zeta\rangle - \mathbf{c}(\mathbf{u}_m; \zeta, q_m) + \langle \kappa_1(\phi_m) q_m, \zeta\rangle + \varepsilon_1\langle \nabla q_m, \nabla\zeta\rangle &\\ + \langle \nabla(A(\phi_m)q_m) - b^{1/2}(\phi_m)\nabla\mu_m, \nabla(A(\phi_m)\zeta)\rangle &= 0,\\ \langle \partial_t \mathbf{u}_m, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u}_m; \mathbf{v}, \mathbf{u}_m) + \langle \eta(\phi_m)\mathrm{D}\mathbf{u}_m, \mathrm{D}\mathbf{v}\rangle + \mathbf{c}(\mathbf{v}; \mu_m, \phi_m) &= 0,\end{aligned} \tag{3.4}$$

for all $\psi, \xi, \zeta \in V_m$ and $\mathbf{v} \in Q_m$ and all $t \in (0, T)$. The system is subjected to the initial conditions $(\phi_m(0), q_m(0), \mathbf{u}_m(0)) = (\phi_{0m}, q_{0m}, \mathbf{u}_{0m})$. By standard techniques from ordinary differential equations, the solution can b shown to exist up to time T_m.

Lemma 3.2.1. *Let the (A0)-(A6) hold. Then the finite-dimensional Galerkin system* (3.4) *for* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *has a solution up to time* $T_m \le T$ *such that*

$$\begin{aligned}\phi_m &\in C^1([0, T_m]; V_m), \qquad \mu_m \in C^0([0, T_m]; V_m),\\ q_m &\in C^1([0, T_m]; V_m), \qquad \mathbf{u}_m \in C^1([0, T_m]; Q_m).\end{aligned}$$

Proof. The local existence follows from the theory of ordinary differential equations since everything depends continuously on the Galerkin approximations $\phi_m, \mu_m, q_m, \mathbf{u}_m$. However, at first glance, it seems that the system is formally a differential-algebraic equation(DAE) for μ_m. Since in space everything is finite-dimensional and smooth, we can simply insert the definition of μ_m and obtain an ordinary differential equation. □

In the following, we will reproduce the energy inequality to obtain a priori bounds on the approximative solution $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ independent of m to first extend the existence time to T independent of m. This and further a priori bounds will finally allow us to pass to the limit and obtain a weak solution of System S.4.

3.2.2. A priori estimates

In order to obtain all the relevant a priori bounds independent of m, we will first reproduce the energy inequality. Afterwards, we will simply compute several immediate consequences of these bounds. The finally a priori bounds which we will seek are bounds on the time derivatives.

Energy-dissipation bounds:
The energy inequality (2.15) is be obtained by variational arguments, cf. Theorem 2.3.1, by inserting the test functions $(\psi, \xi, \zeta, \mathbf{v}) = (\mu_m, \partial_t\phi_m, q_m, \mathbf{u}_m) \in V_m \times V_m \times V_m \times Q_m$. These are all valid test functions by construction, hence we obtain a discrete energy inequality (2.15) and by integration over time $(0,t)$, $t \leq T_m$, we find the following lemma.

Lemma 3.2.2. *Let (A0)–(A6) hold. Then the Galerkin system* (3.4) *for* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *has a solution up to time* T *independent of* m *and satisfies the energy inequality*

$$\begin{aligned}
&\Big(\int_\Omega \frac{\gamma}{2}|\nabla\phi_m(t)|^2 + f(\phi_m(t)) + \frac{1}{2}|q_m(t)|^2 + \frac{1}{2}|\mathbf{u}_m(t)|^2 \,\mathrm{d}x\Big) \qquad (3.5)\\
&\quad + \int_{\Omega_t} \varepsilon_0|b^{1/2}(\phi_m)\nabla\mu_m|^2 + \Big|b^{1/2}(\phi)\nabla\mu_m - \nabla\Big(A(\phi_m)q_m\Big)\Big|^2 + \varepsilon_1|\nabla q_m|^2 \,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_{\Omega_t} |\kappa_1^{1/2}(\phi_m)q_m|^2 + \Big|\eta^{1/2}(\phi_m)\mathrm{D}\mathbf{u}_m\Big|^2 \,\mathrm{d}x\,\mathrm{d}s\\
&\leq \Big(\int_\Omega \frac{\gamma}{2}|\nabla\phi_m(0)|^2 + f(\phi_m(0)) + \frac{1}{2}|q_m(0)|^2 + \frac{1}{2}|\mathbf{u}_m(0)|^2 \,\mathrm{d}x\Big) \quad \textit{for all } t \in [0,T].
\end{aligned}$$

Proof. After integration up to time T_m and using the Gronwall lemma, i.e., Lemma A.3.1, for the inequality (3.5) implies that we obtain the a priori estimates

$$\begin{aligned}
&\|\nabla\phi_m\|^2_{L^\infty(L^2)} + \|q_m\|^2_{L^\infty(L^2)} + \|\mathbf{u}_m\|^2_{L^\infty(L^2)} + \|\nabla\mu_m\|^2_{L^2(L^2)}\\
&\qquad + \|b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)\|^2_{L^2(L^2)} + \|q_m\|^2_{L^2(H^1)} + \|\mathbf{u}_m\|^2_{L^2(H^1)} \leq C_0.
\end{aligned}$$

Hence we can continue the solution up to time T. The energy inequality and suitable a priori bounds follow directly. □

Note that the constant C_0 in the proof depends on the energy of the initial data and the inverses of the lower bounds for the parametric function b, η, κ_1 and also inversely on $\varepsilon_0, \varepsilon_1$. In order to gain control of ϕ_m in $L^\infty(0,T;L^2(\Omega))$, we insert $\psi = 1$ as test function in $(3.4)_1$, and obtain

$$\langle \partial_t\phi_m, 1\rangle = 0.$$

This implies that the mean value of ϕ_m is constant in time. The desired bound on ϕ_m, i.e., $\phi_m \in L^\infty(0,T;L^2(\Omega))$ follows from Poincare's inequality (A.19).

Next we need an estimate for $\mu_m \in L^2(0,T;L^2(\Omega))$. First we consider the mean value of μ_m by testing $(3.4)_2$ with $\xi = 1$ to obtain

$$\langle \mu_m, 1\rangle = \langle f'(\phi_m), 1\rangle, \quad M(t) := \int_0^T |\langle \mu_m, 1\rangle|^2 \,\mathrm{d}s \leq \int_0^T \|f'(\phi_m)\|^2_{0,1} \,\mathrm{d}s.$$

Using that $p \leq 4$, cf. assumption (A2), we calculate

$$\int_0^T \|f'(\phi_m)\|^2_{0,1} \,\mathrm{d}s \leq c(f) \int_0^T \|\phi_m\|^6_{0,3} \,\mathrm{d}s \leq c\|\nabla\phi_m\|^6_{L^\infty(L^2)} \leq C_0,$$

where we have used the embedding $H^1(\Omega) \subset L^3(\Omega)$ in dimension $d \le 3$, see (A.17). By applying Poincaré's inequality (A.19) we obtain

$$\|\mu_m\|_{0,p} \le c\|\nabla\mu_m\|_{0,2} + C\langle\mu_m, 1\rangle^2 \tag{3.6}$$

for all $p \le 6$ in $d \le 3$. Since the mean value of the chemical μ_m, i.e., $M(t)$, is bounded in $L^2(0,T;L^2(\Omega))$, we obtain $\mu_m \in L^2(0,T;L^p(\Omega))$ for $p \le 6$.

The derived a priori estimates of this part are summarized in the following lemma.

Lemma 3.2.3. *The Galerkin solution* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *of system* (3.4) *satisfies the following a priori bounds independent of* m

$$\begin{aligned}\|\phi_m\|^2_{L^\infty(H^1)} + \|q_m\|^2_{L^\infty(L^2)} + \|\mathbf{u}_m\|^2_{L^\infty(L^2)} + \|\mu_m\|^2_{L^2(H^1)} \\ + \|b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)\|^2_{L^2(L^2)} + \|q_m\|^2_{L^2(H^1)} + \|\mathbf{u}_m\|^2_{L^2(H^1)} \le C_0.\end{aligned} \tag{3.7}$$

Higher order a priori estimates for ϕ:
In this part, we will obtain higher order estimates from the approximation of the chemical potential μ_m. Since the subspace V_m is stable under $-\Delta$ due to construction, we can test the discrete equations with $\Delta\phi_m$, i.e., set $\xi = \Delta\phi_m$ in $(3.4)_2$ and obtain

$$\gamma\|\Delta\phi_m\|^2_{L^2} = \langle f'(\phi_m) - \mu_m, \Delta\phi_m\rangle.$$

Using Hölder's inequality yields and integration over time from 0 to T yields

$$\int_0^T \|\Delta\phi_m\|^2_{L^2}\,\mathrm{d}s \le C(\gamma)\int_0^T \|f'(\phi_m) - \mu_m\|_0^2\,\mathrm{d}s.$$

Since the chemical potential μ_m is already bounded in $L^2(0,T;L^2(\Omega))$ by the a priori estimates (3.7), we only have to consider a suitable bound for $f'(\phi_m)$. By recalling (A1), we compute

$$\int_0^T \|f'(\phi_m)\|_0^2\,\mathrm{d}s \le C(f)\int_0^T \|\phi_m\|^6_{0,6}\,\mathrm{d}s \le C(f)\int_0^T \|\nabla\phi_m\|_0^6\,\mathrm{d}s \le C_0,$$

which is bounded again due to the a priori estimates (3.7). Hence, the right-hand side is bounded by the a priori estimates (3.7) and we conclude that $\Delta\phi_m$ is bounded in $L^2(0,T;L^2(\Omega))$ independent of m. Since (A0) holds, i.e., the domain is smooth, it suffices to control the Laplacian to obtain bounds in $H^2(\Omega)$.

The last remaining goal is to bound $\phi_m \in L^2(0,T;H^3(\Omega))$ independent of m. This will be done by applying the results from the elliptic regularity theory, see [56]. First, we already know that $\mu_m \in L^2(0,T;H^1(\Omega))$. We will show now that $f'(\phi_m)$ is also bounded $L^2(0,T;H^1(\Omega))$ independent of m. Then by using elliptic regularity we immediately obtain the desired regularity. We estimate $f'(\phi_m)$ as follows

$$\begin{aligned}\int_0^T\int_\Omega |\nabla f'(\phi_m)|^2\,\mathrm{d}x\,\mathrm{d}s &= \int_0^T\int_\Omega |f''(\phi_m)\nabla\phi_m|^2\,\mathrm{d}x\,\mathrm{d}s \le \int_0^T \|f''(\phi_m)\|^2_{0,3}\|\nabla\phi_m\|^2_{0,6}\,\mathrm{d}s \\ &\le \int_0^T C(f_2^{(2)} + f_3^{(2)}\|\phi_m\|_1^4)\|\nabla\phi_m\|_1^2\,\mathrm{d}s \\ &\le C(f, \|\phi_m\|^4_{L^\infty(H^1)})\|\nabla\phi_m\|^2_{L^2(H^1)} \le C_0.\end{aligned}$$

The derived estimates are summarized in the following lemma.

Lemma 3.2.4. *The Galerkin solution* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *of* (3.4) *satisfies the additional a priori bounds independent of* m

$$\|\phi_m\|^2_{L^2(H^3)} + \|f'(\phi_m)\|^2_{L^2(H^1)} \le C_0. \tag{3.8}$$

Estimates for the time derivative:
Next, we consider the appropriate space in which the time derivatives are bounded by the a priori estimates. In order to obtain the relevant bounds, we use duality as follows

$$\|\partial_t \phi_m(t)\|_{-1} = \sup_{\psi \in H^1} \frac{\langle \partial_t \phi_m(t), \psi \rangle}{\|\psi\|_1} = \sup_{\psi \in V_m} \frac{\langle \partial_t \phi_m(t), \psi \rangle}{\|\psi\|_1}. \tag{3.9}$$

The last equality holds since the basis functions are orthogonal in $H^1(\Omega)$. In the following, we will insert the discrete variational problem in the numerator and estimate by using Cauchy-Schwarz and Hölder inequality. Hence, this amounts to estimate every inner-product using the dual norm. Finally, (3.9) is integrated over $(0, T)$ to employ the previously derived a priori bounds.

Cahn-Hilliard equation:
We start with the first variational identity, which was given by

$$\begin{aligned}\langle \partial_t \phi_m, \psi \rangle &= \mathbf{c}(\mathbf{u}_m; \psi, \phi_m) - \langle (1+\varepsilon_0) b(\phi_m) \nabla \mu_m - b^{1/2}(\phi_m) \nabla (A(\phi_m) q), \nabla \psi \rangle, \quad \forall \psi \in V_m \\ &= (i) + (ii).\end{aligned}$$

For the first term, we compute

$$\begin{aligned}(i) &\le \|\mathbf{u}_m\|_{0,3} \|\nabla \psi\|_{0,2} \|\phi_m\|_{0,6} \le \|\mathbf{u}_m\|_{0,3} \|\psi\|_1 \|\phi_m\|_{0,6}, \\ \int_0^T \|(i)\|^2_{-1} \,\mathrm{d}s &\le \|\mathbf{u}_m\|^2_{L^2(L^3)} \|\phi_m\|^2_{L^\infty(L^6)}.\end{aligned}$$

For the second term, we estimate

$$\begin{aligned}(ii) &\le (\varepsilon_0 \|b(\phi_m)\nabla \mu_m\|_{0,2} + C\|b^{1/2}(\phi_m)\nabla \mu_m - \nabla(A(\phi_m) q_m)\|_{0,2}) \|\psi\|_1, \\ \int_0^T \|(ii)\|^2_{-1} \,\mathrm{d}s &\le \varepsilon_0 \|b(\phi_m)\nabla \mu_m\|^2_{L^2(L^2)} + C\|b^{1/2}(\phi_m)\nabla \mu_m - \nabla(A(\phi_m) q_m)\|^2_{L^2(L^2)}\end{aligned}$$

Inserting into (3.9) estimating and integrating over time yields

$$\|\partial_t \phi_m\|^2_{-1} \,\mathrm{d}s \le \int_0^t \|(i)\|^2_{-1} + \|(ii)\|^2_{-1} \le C_0,$$

which is bounded by the a priori estimate (3.7).

Bulk stress equation:
Similarly, the following holds

$$\|\partial_t q_m(t)\|_{-1} = \sup_{\zeta \in H^1} \frac{\langle \partial_t q_m(t), \zeta \rangle}{\|\zeta\|_1} = \sup_{\zeta \in V_m} \frac{\langle \partial_t q_m(t), \zeta \rangle}{\|\zeta\|_1}. \tag{3.10}$$

Furthermore, we recall the variational identity for the bulk stress given by

$$\begin{aligned}\langle \partial_t q_m, \zeta\rangle &= \mathbf{c}(\mathbf{u}_m; \zeta, q_m) - \langle \kappa_1(\phi_m) q_m, \zeta\rangle - \varepsilon_1 \langle \nabla q_m, \nabla \zeta\rangle \\ &\quad - \langle \nabla(A(\phi_m) q_m) - b^{1/2}(\phi_m)\nabla\mu_m, \nabla(A(\phi_m)\zeta)\rangle \\ &= (i) + (ii) + (iii) + (iv).\end{aligned}$$

For the first term, we estimate

$$(i) \le \|\mathbf{u}_m\|_{0,3} \|\nabla\psi\|_{0,2} \|q_m\|_{0,6},$$
$$\int_0^T \|(i)\|_{-1}^{4/3}\, \mathrm{d}s \le \int_0^T \|\mathbf{u}_m\|_{0,3}^{4/3} \|q_m\|_{1,2}^{4/3}\, \mathrm{d}s \le \|q_m\|_{L^2(L^2)}^2 + \|\mathbf{u}_m\|_{L^4(L^3)}^4 \le C_0,$$

which again is bounded due (3.7) by using the interpolation inequality (A.25).

Turning to the second and third terms we compute

$$(ii) + (iii) \le \kappa_{1,2} \|q_m\|_0 \|\psi\|_0 + \varepsilon_1 \|\nabla q_m\|_0 \|\nabla\psi\|_0,$$
$$\int_0^T \|(ii) + (iii)\|_{-1}^2\, \mathrm{d}s \le C(\kappa_1, \varepsilon_1) \int_0^T \|q_m\|_0^2 + \|\nabla q_m\|_0^2\, \mathrm{d}s \le C_0,$$

which again is bounded due to (3.7).

For the final term, we again estimate

$$\begin{aligned}(iii) &\le \|b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)\|_{0,2} \|\nabla(A(\phi_m)\psi)\|_{0,2} \\ &\le \|b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)\|_{0,2} (A_2 \|\nabla\psi\|_{0,2} + A_3 \|\nabla\phi_m\|_{0,3} \|\psi\|_{0,6}),\end{aligned}$$

where we applied the upper bounds for $A(\phi)$, cf. (A3), and a Hölder inequality. Using the Sobolev embedding and Young's inequality with $p = 3/2$ and $q = 3$, we find

$$\begin{aligned}\int_0^T \|(iii)\|_{-1}^{4/3}\, \mathrm{d}s &\le C(A) \int_0^T \|b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)\|_{0,2}^{4/3} (1 + \|\nabla\phi_m\|_{0,3}^{4/3})\, \mathrm{d}s \\ &\le C(A) \|b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)\|_{L^2(L^2)}^2 + \|\nabla\phi_m\|_{L^4(L^3)}^4 \le C_0,\end{aligned}$$

due to a priori bounds (3.7) and (3.8) using the interpolation inequality (A.25).

Together, this yields the following bound for the time derivative

$$\int_0^T \|\partial_t q_m\|_{-1}^{4/3}\, \mathrm{d}s \le C \int_0^T \|(i)\|_{-1}^{4/3} + \|(ii)\|_{-1}^2 + \|(iii)\|_{-1}^{4/3}\, \mathrm{d}s \le C_0.$$

Navier-Stokes equations:

For the Navier-Stokes contributions, we obtain a slightly different formula, i.e., we replace $H^1(\Omega)$ by $H^1_{\mathrm{div}}(\Omega)^d$ and V_m by Q_m. Furthermore, we abbreviate the norm of $H^{-1}_{\mathrm{div}}(\Omega)^d$ by $\|\cdot\|_{-1,\mathrm{div}}$. With this notation we obtain

$$\|\partial_t \mathbf{u}_m(t)\|_{-1,\mathrm{div}} = \sup_{\mathbf{v}\in H^1_{\mathrm{div}}(\Omega)^d} \frac{\langle \partial_t \mathbf{u}_m(t), \mathbf{v}\rangle}{\|\mathbf{v}\|_1} = \sup_{\mathbf{v}\in Q_m} \frac{\langle \partial_t \mathbf{u}_m(t), \mathbf{v}\rangle}{\|\mathbf{v}\|_1}. \tag{3.11}$$

We recall the variational identity for the Navier-Stokes equation given by

$$\langle \partial_t \mathbf{u}_m, \mathbf{v}\rangle = \mathbf{c}(\mathbf{u}_m; \mathbf{v}, \mathbf{u}_m) - \langle \eta(\phi_m)\mathrm{D}\mathbf{u}_m, \mathrm{D}\mathbf{v}\rangle - \mathbf{c}(\mathbf{v}; \mu_m, \phi_m)$$

$$= (i) + (ii) + (iii).$$

For the first term we estimate

$$(i) \leq \|\mathbf{u}_m\|_{0,4}^2 \|\nabla \mathbf{v}\|_0.$$

Then using the interpolation inequality (A.22) we find

$$\begin{aligned}\int_0^T \|(i)\|_{-1,\mathrm{div}}^{4/3} \,\mathrm{d}s \leq \int_0^T \|\mathbf{u}_m\|_{0,4}^{8/3} \,\mathrm{d}s \leq \int_0^T \|\mathbf{u}_m\|_{0,2}^{2/3} \|\mathbf{u}_m\|_1^2 \,\mathrm{d}s \\ \leq \|\mathbf{u}_m\|_{L^\infty(L^2)}^{2/3} \|\mathbf{u}_m\|_{L^2(H^1)}^2 \leq C_0,\end{aligned}$$

which is bounded due to the a priori estimate (3.7). Let us note that at this point using the corresponding interpolation inequality in two space dimensions, i.e., (A.21) yields a similar bound with a power 2 instead of a power 4/3.

For the second term, we estimate

$$(ii) \leq \eta_2 \|\mathrm{D}\mathbf{u}_m\|_0 \|\mathrm{D}\mathbf{v}\|_0 \leq C(\eta) \|\nabla \mathbf{u}_m\|_0 \|\nabla \mathbf{v}\|_0,$$

$$\int_0^T \|(ii)\|_{-1,\mathrm{div}}^2 \,\mathrm{d}s \leq C(\eta) \int_0^T \|\nabla \mathbf{u}_m\|_0^2 \,\mathrm{d}s \leq C_0.$$

For the third term, we compute

$$(i) \leq \|\nabla \mu_m\|_0 \|\phi_m\|_{0,3} \|\mathbf{v}\|_{0,6},$$

$$\int_0^T \|(iii)\|_{-1,\mathrm{div}}^2 \,\mathrm{d}s \leq \int_0^T \|\phi_m\|_1^2 \|\nabla \mu_m\|_0^2 \,\mathrm{d}s \leq c \|\phi_m\|_{L^\infty(H^1)}^2 \|\mu_m\|_{L^2(H^1)}^2 \leq C_0,$$

due to the bounds (3.7). Thus, the equation $(3.64)_4$ implies

$$\int_0^T \|\partial_t \mathbf{u}_m\|_{-1,\mathrm{div}}^{4/3} \,\mathrm{d}s \leq C \int_0^T \|(i)\|_{-1,\mathrm{div}}^{4/3} + \|(ii)\|_{-1,\mathrm{div}}^2 + \|(iii)\|_{-1,\mathrm{div}}^2 \,\mathrm{d}s \leq C_0. \tag{3.12}$$

In two space dimensions, the same holds true in $L^2(0,T; H_{\mathrm{div}}^{-1}(\Omega)^d)$. Together, the above parts can be summarized in the following lemma.

Lemma 3.2.5. *Let assumptions (A0)–(A6) hold. Then the Galerkin approximation* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *satisfies the following additionally a priori bounds independent of* m

$$\|\partial_t \phi_m\|_{L^2(H^{-1})}^2 + \|\partial_t q_m\|_{L^{4/3}(H^{-1})}^{4/3} + \|\partial_t \mathbf{u}_m\|_{L^{4/3}(H_{div}^{-1})}^{4/3} \leq C_0. \tag{3.13}$$

3.2.3. Convergent subsequences

We will use the obtained a priori bounds to extract suitable converging subsequences. In principle, from the a priori bounds we will directly obtain weakly/weakly-* convergent subsequences in the corresponding spaces using Banach-Alaoglu A.3.3. However, since the problem involves nonlinear terms, we will also need strong convergence of these sequences. This will be realized by compactness arguments via the Aubin-Lions lemma, cf. Lemma A.3.6, which yields strong convergence in weaker norms.

Consequences of Banach-Alaoglu:
By the Banach-Alaoglu Lemma, see A.3.3, we obtain weakly/weakly-* convergent subsequences directly from the a priori bounds. We will denote the subsequence still by the index m. Hence, we find the following lemma.

Lemma 3.2.6. *Let* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *be the solution of* (3.4) *and let the a priori bounds hold, i.e.,* (3.7)*,* (3.8)*,* (3.13)*. Then we obtain the following weak/weak-* convergences*

$$\begin{aligned} \phi_m &\rightharpoonup^\star \phi \in L^\infty(0,T;H^1(\Omega)), & q_m &\rightharpoonup^\star q \in L^\infty(0,T;L^2(\Omega)), \\ \phi_m &\rightharpoonup \phi \in L^2(0,T;H^3(\Omega)), & q_m &\rightharpoonup q \in L^2(0,T;H^1(\Omega)), \\ \phi_m &\rightharpoonup \phi \in L^{p_1}(0,T;W^{1,p_1}(\Omega)), & q_m &\rightharpoonup q \in L^{p_1}(0,T;L^{p_1}(\Omega)), \\ \partial_t\phi_m &\rightharpoonup \partial_t\phi \in L^2(0,T;H^{-1}(\Omega)), & \partial_t q_m &\rightharpoonup \partial_t q \in L^{4/3}(0,T;H^{-1}(\Omega)), \\ \mu_m &\rightharpoonup \mu \in L^2(0,T;H^1(\Omega)), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \mathbf{u}_m &\rightharpoonup^\star \mathbf{u} \in L^\infty(0,T;L^2_{div}(\Omega)^d), \\ \mathbf{u}_m &\rightharpoonup \mathbf{u} \in L^2(0,T;H^1_{div}(\Omega)^d), \\ \mathbf{u}_m &\rightharpoonup \mathbf{u} \in L^{p_1}(0,T;L^{p_1}(\Omega)^d), \\ \partial_t\mathbf{u}_m &\rightharpoonup \partial_t\mathbf{u} \in L^{4/3}(0,T;H^{-1}_{div}(\Omega)^d). \end{aligned} \tag{3.15}$$

Here the number p_1 *follow from suitable embeddings and is given as follows*

$$p_1 = \begin{cases} 4 & \text{in } d=2 \\ \frac{10}{3} & \text{in } d=3. \end{cases}$$

Proof. The main convergence results follow direct from the a priori bounds, cf. (3.7), (3.8), (3.13), using Banach-Alaoglu. Using the interpolation inequality, see (A.25), we obtain the other uniform bounds for the application of Banach-Alaoglu. □

Aubin-Lions:
To obtain strong convergence of suitable subsequences via compactness arguments, we apply the Aubin-Lions lemma, cf. Lemma A.3.6.

Lemma 3.2.7. *Let* $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ *be the solution of* (3.4) *and let the a priori bounds, i.e.* (3.7)*,* (3.8)*,* (3.13)*, hold. Then we obtain the following strong convergences*

$$\begin{aligned} &\phi_m \to \phi \in L^p(0,T;W^{1,p}(\Omega)) \text{ for } p<p_1, && q_m \to q \in L^p(0,T;L^p(\Omega)) \text{ for } p<p_1, \\ &\phi_m \to \phi \in L^2(0,T;W^{1,p}(\Omega)) \text{ for } p<p_2, && q_m \to q \in L^2(0,T;L^p(\Omega)) \text{ for } p<p_2, \\ &\phi_m \to \phi \text{ a.e. in } \Omega\times(0,T), && q_m \to q \text{ a.e. in } \Omega\times(0,T), \end{aligned}$$

$$\begin{aligned} &\mathbf{u}_m \to \mathbf{u} \in L^2(0,T;L^p(\Omega)) \text{ for } p<p_2, \\ &\mathbf{u}_m \to \mathbf{u} \in L^p(0,T;L^p(\Omega)) \text{ for } p<p_1, \\ &\mathbf{u}_m \to \mathbf{u} \text{ a.e. in } \Omega\times(0,T). \end{aligned}$$

Here the numbers p_1 *and* p_2 *follow from suitable embeddings and are given as follows*

$$p_1 = \begin{cases} 4 & \text{in } d=2 \\ \frac{10}{3} & \text{in } d=3, \end{cases} \qquad p_2 = \begin{cases} \infty & \text{in } d=2 \\ 6 & \text{in } d=3. \end{cases}$$

Proof. We apply the Aubin-Lions lemma, we need suitable sequences of embedded spaces

$$\begin{aligned} &H^2(\Omega) \Subset H^1(\Omega) \subset H^{-1}(\Omega), && H^1(\Omega) \Subset L^2(\Omega) \subset H^{-1}(\Omega) \text{ for } \phi, \\ &H^1(\Omega) \Subset L^2(\Omega) \subset H^{-1}(\Omega) \text{ for } q, && H^1_{\text{div}}(\Omega)^d \Subset L^2_{\text{div}}(\Omega)^d \subset H^{-1}_{\text{div}}(\Omega)^d \text{ for } \mathbf{u}. \end{aligned}$$

□

3.2.4. Passage to the limit

We are now in the position to pass to the limit in the Galerkin approximation (3.4). Let us shortly sketch the main arguments, cf. [110, Section 8.4].

- We multiply the formulation of the Galerkin approximation (3.4) with a smooth function in time $\varphi(s)$ and integrate the variational formulation from 0 to T.
- For this integrated formulation, we will show that we can pass to the limit with $m \to \infty$ in the Galerkin approximation for $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$, but with a fixed test space. This will be the step which we will prove here.
- A standard step, see [110, Theorem 8.28], is to extend the formulation from the discrete test spaces V_m, Q_m to the infinite-dimensional spaces $H^1(\Omega), H^1_{\mathrm{div}}(\Omega)^d$. This is done by a density argument; details are omitted. The main argument is that the test spaces V_k, Q_k are nested with respect to k and as already mentioned dense in $H^1(\Omega), H^1_{\mathrm{div}}(\Omega)^d$, respectively.
- Finally, we obtain a space-time formulation of the weak formulation (3.2). Since the test functions in time, i.e. $\varphi(s)$, is sufficiently smooth by standard approximation arguments, [110], we can convert to the point-wise almost everywhere in time formulation (3.2).

Before going into details, let us discuss the weak limit of the crucial nonlinearity $\nabla(A(\phi_m)q_m)$. Due to our a priori bounds (3.7), we know that

$$\nabla(A(\phi_m)q_m) \rightharpoonup z \in L^2(0,T;L^2(\Omega))$$

for some $z \in L^2(0,T;L^2(\Omega))$. Computation of the gradient yields

$$\nabla(A(\phi_m)q_m) = A(\phi_m)q_m + A'(\phi_m)\nabla\phi_m q_m.$$

The first term can be treated via Lemma A.3.4. We already know that ϕ_m converges almost everywhere in $\Omega \times (0,T)$, see (3.2.7), hence $A(\phi_m)\nabla q_m$ converges weakly to $A(\phi)\nabla q$ in $L^2(0,T;L^2(\Omega))$. For the second term due to a priori bounds (3.7), we already know that it is weakly convergent, and we only have to identify the limit. Since we have strong convergence of $\nabla\phi_m, q_m$ in $L^{10/3-\delta}(0,T;L^{10/3-\delta}(\Omega))$ the product converges in $L^{5/3-\delta}(0,T;L^{5/3-\delta}(\Omega))$ for $0 < \delta \ll 1$ and therefore almost everywhere. Hence, the weak limit must coincide with $q\nabla\phi$. Together we obtain

$$\nabla(A(\phi_m)q_m) \rightharpoonup \nabla(A(\phi)q) \in L^2(0,T;L^2(\Omega)).$$

We will now pass to the limit $m \to \infty$ in (3.4). To do so, we formally multiply by a smooth function $\varphi(s)$ in time and integrate the discrete system (3.4) over the time interval $(0,T)$ which yields

$$\int_0^T \varphi(s)\Big(\langle \partial_t\phi_m, \psi\rangle - \mathbf{c}(\mathbf{u}_m;\psi,\phi_m) + \langle (1+\varepsilon_0)b(\phi_m)\nabla\mu_m, \nabla\psi\rangle \\ - \langle b^{1/2}(\phi_m)\nabla(A(\phi_m)q), \nabla\psi\rangle\Big)\,\mathrm{d}s = 0,$$

$$\int_0^T \varphi(s)\Big(\langle \mu_m, \xi\rangle - \gamma\langle\nabla\phi_m, \nabla\xi\rangle - \langle f'(\phi_m), \xi\rangle\Big)\,\mathrm{d}s = 0,$$

$$\int_0^T \varphi(s)\Big(\langle \partial_t q_m, \zeta\rangle - \mathbf{c}(\mathbf{u}_m; \zeta, q_m) + \langle \kappa_1(\phi_m) q_m, \zeta\rangle + \varepsilon_1 \langle \nabla q_m, \nabla \zeta\rangle \tag{3.16}$$
$$+ \langle \nabla(A(\phi_m) q_m) - b^{1/2}(\phi_m)\nabla\mu_m, \nabla(A(\phi_m)\zeta)\rangle\Big)\, \mathrm{d}s = 0,$$
$$\int_0^T \varphi(s)\Big(\langle \partial_t \mathbf{u}_m, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u}_m; \mathbf{v}, \mathbf{u}_m) + \langle \eta(\phi_m)\mathrm{D}\mathbf{u}_m, \mathrm{D}\mathbf{v}\rangle + \mathbf{c}(\mathbf{v}; \mu_m, \phi_m)\Big)\, \mathrm{d}s = 0,$$

for all $(\psi, \xi, \zeta, \mathbf{v}) \in V_m \times V_m \times V_m \times Q_m$ and all $\varphi \in L^\infty(0,T)$.

The space-time weak formulation of (3.2) restricted to the discrete test spaces reads

$$\int_0^T \varphi(s)\Big(\langle \partial_t \phi, \psi\rangle - \mathbf{c}(\mathbf{u}; \psi, \phi) + \langle (1+\varepsilon_0) b(\phi)\nabla\mu - b^{1/2}(\phi)\nabla(A(\phi)q), \nabla\psi\rangle\Big)\, \mathrm{d}s = 0,$$
$$\int_0^T \varphi(s)\Big(\langle \mu, \xi\rangle - \gamma\langle \nabla\phi, \nabla\xi\rangle - \langle f'(\phi), \xi\rangle\Big)\, \mathrm{d}s = 0,$$
$$\int_0^T \varphi(s)(\langle \partial_t q, \zeta\rangle - \mathbf{c}(\mathbf{u}; \zeta, q) + \langle \kappa_1(\phi) q, \zeta\rangle + \varepsilon_1\langle \nabla q, \nabla\zeta\rangle \tag{3.17}$$
$$+ \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\zeta)\rangle\Big)\, \mathrm{d}s = 0,$$
$$\int_0^T \varphi(s)\Big(\langle \partial_t \mathbf{u}, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u}; \mathbf{v}, \mathbf{u}) + \langle \eta(\phi)\mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v}\rangle + \mathbf{c}(\mathbf{v}; \mu, \phi)\Big)\, \mathrm{d}s = 0,$$

for all $(\psi, \xi, \zeta, \mathbf{v}) \in V_m \times V_m \times V_m \times Q_m$ and all $\varphi \in L^\infty(0,T)$.

In this part we abbreviate $\|\varphi\|_{0,\infty}$ as the $L^\infty(0,T)$-norm of $\varphi(s)$, but only for this variable we make this exception. We note that we can pass to the limit in all linear terms, since this follows directly from weak convergence, see Lemma 3.2.6 and will hence be omitted. Therefore, we only have to deal with nonlinear terms for which have derived suitable strong convergences in Lemma 3.2.7. The terms we consider in the following are simply the differences between the above two formulations.

Cahn-Hilliard equation:
In the Cahn-Hilliard equation $(3.16)_1$ we have to consider

$$P_{0,m} := \int_0^T \int_\Omega \Big[b(\phi_m)\nabla\mu_m - b(\phi)\nabla\mu\Big]\nabla\psi\varphi(s)\, \mathrm{d}x\, \mathrm{d}s.$$

Let us discuss this term in detail and note that many terms later follow the same argument. First, ϕ_m converges almost everywhere in $\Omega \times (0,T)$ and therefore, due to continuity of b, we have almost everywhere convergence of $b(\phi_m)$ to $b(\phi)$. Application of Lemma A.3.4 implies that $g_m := b(\phi_m)\nabla\mu_m$ converges weakly in $L^2(0,T; L^2(\Omega))$ to $g = b(\phi)\nabla\mu$. Indeed, this insight reduces the above problem to

$$\int_0^T \int_\Omega (g - g_m)\nabla\psi\varphi(s)\, \mathrm{d}x\, \mathrm{d}s \to 0,$$

which holds by the definition of weak convergence. This can be seen since $\nabla\psi\varphi$ is bounded in $L^\infty(0,T; L^2(\Omega))$ and by embedding in $L^2(0,T; L^2(\Omega))$.

The next integral can be treated similarly

$$P_{1,m} := \int_0^T \int_\Omega \Big[b^{1/2}(\phi_m)\Big(b^{1/2}(\phi_m)\nabla\mu_m - \nabla\Big(A(\phi_m)q_m\Big)\Big)$$

$$- b^{1/2}(\phi)\Big(b^{1/2}(\phi)\nabla\mu - \nabla\Big(A(\phi)q\Big)\Big)\Big]\nabla\psi\varphi(s)\,\mathrm{d}x\,\mathrm{d}s.$$

Since $b^{1/2}(\phi_m) \to b^{1/2}(\phi)$ a.e. in $\Omega \times (0,T)$ we can apply the weak convergence of $b^{1/2}(\phi_m)\nabla\mu_m - \nabla\Big(A(\phi_m)q_m\Big)$ in $L^2(0,T;L^2(\Omega))$, cf. Lemma A.3.4 to obtain $P_{1,m} \to 0$.

The treatment of the convective term in $(3.16)_1$ is given by

$$\begin{aligned}
P_{2,m} &:= \int_0^T \int_\Omega (\phi_m \mathbf{u}_m - \phi\mathbf{u}) \cdot \nabla\psi\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^T \int_\Omega \phi(\mathbf{u}_m - \mathbf{u})\nabla\psi\varphi(s) + (\phi_m - \phi)\mathbf{u}_m\nabla\psi\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \|\varphi\|_{0,\infty} \int_0^T \|\mathbf{u}_m - \mathbf{u}\|_{0,3}\|\phi\|_{0,6}\|\psi\|_0 + \|\phi_m - \phi\|_{0,6}\|\mathbf{u}_m\|_{0,3}\|\nabla\psi\|_0\,\mathrm{d}s \\
&\le \|\varphi\|_{0,\infty}\|\psi\|_1 \left(\|\mathbf{u}_m - \mathbf{u}\|_{L^2(L^3)}\|\phi\|_{L^2(H^1)} + \|\phi_m - \phi\|_{L^2(L^2)}\|\mathbf{u}_m\|_{L^2(L^3)}\right).
\end{aligned}$$

Recalling (3.14) and (3.15), the integral $P_{2,m}$ goes to zero as $m \to \infty$, due to the strong convergence of $\mathbf{u}_m$ and $\nabla\phi_m$ in $L^2(0,T;L^3(\Omega))$.

Chemical potential:
Next, we consider the limit in the chemical potential, i.e., $(3.16)_2$ where the only nonlinearity is the potential part. Here we have

$$\begin{aligned}
P_{3,m} &:= \int_0^T \int_\Omega (f'(\phi_m) - f'(\phi))\xi\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^T \int_\Omega \int_0^1 f''(\phi_m + \omega(\phi - \phi_m))\,\mathrm{d}\omega(\phi_m - \phi)\xi\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\le (f_2^{(2)} + f_3^{(2)}(\|\phi_m\|_{L^\infty(H^1)} + \|\phi\|_{L^\infty(H^1)})) \int_0^T \|\phi_m - \phi\|_1\|\xi\|_1\|\varphi\|_{0,\infty}\,\mathrm{d}s \to 0.
\end{aligned}$$

The above integral goes to zero, since ϕ_m, ϕ are uniformly bounded in $L^\infty(0,T;H^1(\Omega))$, see (3.7), and ϕ_m converges strongly to ϕ in $L^2(0,T;H^1(\Omega))$, cf. Lemma 3.2.7. Therefore, $P_{2,m} \to 0$ goes to zero as $m \to \infty$.

Bulk stress equation:
Let us consider the limit in the bulk stress equation $(3.16)_3$. The main nonlinearity is

$$\begin{aligned}
P_{4,m} &:= -\int_0^T \int_\Omega \Big(b^{1/2}(\phi_m)\nabla\mu_m - \nabla\Big(A(\phi_m)q_m\Big)\Big)\,\nabla\Big(A(\phi_m)\zeta\Big)\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^T \int_\Omega \Big(b^{1/2}(\phi_m)\nabla\mu_m - \nabla\Big(A(\phi_m)q_m\Big)\Big)\,(A(\phi_m)\nabla\zeta + A'(\phi_m)\nabla\phi_m\zeta)\,\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\
&=: P_{4,1,m} + P_{4,2,m}.
\end{aligned}$$

Since we already know the weak convergence of $b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)$ to its limit $b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)$ in $L^2(0,T;L^2(\Omega))$, we only have to show that $A(\phi_m)\nabla\zeta\varphi$ and $A'(\phi_m)\nabla\phi_m\zeta\varphi$ converge strongly in $L^2(0,T;L^2(\Omega))$. Since $A(\phi_m), A'(\phi_m)$ converges to $A(\phi), A'(\phi)$ a.e. in $\Omega \times (0,T)$, we can conclude that $A(\phi_m)\nabla\zeta\varphi$ converges strongly to $A(\phi)\nabla\zeta\varphi$ in $L^2(0,T;L^2(\Omega))$. This implies $P_{4,1,m} \to P_{4,1}$ as $m \to \infty$. Similarly, by virtue of (3.14), we observe that $A'(\phi_m)\nabla\phi_m\zeta\varphi$ converges strongly to $A'(\phi)\nabla\phi\zeta\varphi$ in $L^2(0,T;L^2(\Omega))$. Hence, we found $P_{4,2,m} \to P_{4,2}$ as $m \to \infty$.

The convective term can be estimated by

$$\begin{aligned} P_{5,m} &:= \int_0^T \int_\Omega (q_m\mathbf{u}_m - q\mathbf{u}) \cdot \nabla\zeta\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^T \int_\Omega q(\mathbf{u}_m - \mathbf{u}) \cdot \nabla\zeta\varphi(s) + (q_m - q)\mathbf{u}_m \cdot \nabla\zeta\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\ &\le \|\varphi\|_{0,\infty}\|\zeta\|_1 \int_0^T \|\mathbf{u}_m - \mathbf{u}\|_{0,3}\|q\|_{0,6} + \|q_m - q\|_{0,3}\|\mathbf{u}_m\|_{0,6}\,\mathrm{d}s \\ &\le \|\varphi\|_{0,\infty}\|\zeta\|_1(\|\mathbf{u}_m - \mathbf{u}\|_{L^2(L^3)}\|q\|_{L^2(H^1)} + \|q_m - q\|_{L^2(L^3)}\|\mathbf{u}_m\|_{L^2(H^1)}). \end{aligned}$$

We observe that $P_{5,m}$ goes to zero due to the strong convergence of $\mathbf{u}_m$ and q_m in $L^2(0,T;L^3(\Omega))$, cf. Lemma 3.2.7, and the bounds on $\mathbf{u}_m, q_m$ in $L^2(0,T;H^1(\Omega))$, see Lemma 3.7. Indeed, this shows $P_{5,m} \to 0$ as $m \to \infty$.

The relaxation term is given by

$$P_{6,m} := \int_0^T \int_\Omega (\kappa_1(\phi_m)q_m - \kappa_1(\phi)q)\zeta\varphi(s)\,\mathrm{d}x\,\mathrm{d}s.$$

Again we can show with Lemma A.3.4 weak convergence of $\kappa_1(\phi_m)q_m$ in $L^2(0,T;L^2(\Omega))$ and therefore $P_{6,m}$ goes to zero as $m \to \infty$.

Navier-Stokes equations:
As a last step we consider the Navier-Stokes equations $(3.16)_4$. The first term is given by

$$P_{7,m} := \int_0^T \int_\Omega (\eta(\phi_m)\mathrm{D}\mathbf{u}_m - \eta(\phi)\mathrm{D}\mathbf{u}) : \mathrm{D}\mathbf{v}\varphi(s)\,\mathrm{d}x\,\mathrm{d}s.$$

Again, with Lemma A.3.4, we obtain the weak convergence of $\eta(\phi_m)\mathrm{D}\mathbf{u}_m$ in $L^2(0,T;L^2(\Omega))$, and therefore the integral $P_{7,m}$ goes to zero as $m \to \infty$, due to weak convergence. Next, we consider the corresponding convective term by

$$\begin{aligned} P_{8,m} &:= \int_0^T \int_\Omega (\mathbf{u}_m \cdot (\mathbf{u}_m \cdot \nabla) - \mathbf{u} \cdot (\mathbf{u} \cdot \nabla))\mathbf{v}\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^T \mathbf{c}(\mathbf{u}_m - \mathbf{u}, \mathbf{v}\varphi(s), \mathbf{u}_m) - \mathbf{c}(\mathbf{u}, \mathbf{v}\varphi(s), \mathbf{u}_m - \mathbf{u})\,\mathrm{d}x\,\mathrm{d}s \\ &\le \|\varphi\|_{0,\infty}\|\mathbf{v}\|_1 \int_0^T \|\mathbf{u}_m - \mathbf{u}\|_{0,3}\|\mathbf{u}_m\|_{0,6} + \|\mathbf{u}_m - \mathbf{u}\|_{0,3}\|\mathbf{u}\|_{0,6}\,\mathrm{d}s \\ &\le \|\varphi\|_{0,\infty}\|\mathbf{v}\|_1\|\mathbf{u}_m - \mathbf{u}\|_{L^2(L^3)}(\|\mathbf{u}_m\|_{L^2(H^1)} + \|\mathbf{u}\|_{L^2(H^1)}). \end{aligned}$$

Again due to the bounds (3.7) and the strong convergence of $\mathbf{u}_m$ in $L^2(0,T;L^3(\Omega))$, cf. Lemma 3.2.7 the integral $P_{9,m}$ goes to zero as $m \to \infty$.

Lastly, we consider the coupling term

$$\begin{aligned} P_{9,m} &:= \int_0^T \int_\Omega (\phi_m\nabla\mu_m - \phi\nabla\mu)\,\mathbf{v}\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^T \int_\Omega [(\phi_m - \phi)\nabla\mu + \nabla(\mu_m - \mu)\phi_m]\,\mathbf{v}\varphi(s)\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

$$\leq \|\mathbf{v}\|_1 \|\varphi\|_\infty \int_0^T \|\phi_m - \phi\|_{0,3} \|\nabla\mu\|_0 \, \mathrm{d}s + \int_0^T \int_\Omega \nabla(\mu_m - \mu)\phi_m \mathbf{v}\varphi(s) \, \mathrm{d}x \, \mathrm{d}s$$
$$\leq \|\phi_m - \phi\|_{L^2(L^3)} \|\nabla\mu\|_{L^2(L^2)} \|\mathbf{v}\|_{0,6} \|\varphi\|_{0,\infty} + \int_0^T \int_\Omega (\mu_m - \mu)\nabla\phi_m \mathbf{v}\varphi(s) \, \mathrm{d}x \, \mathrm{d}s.$$

The first integral of $P_{9,m}$ tends to zero due to the strong convergence of ϕ_m in $L^2(0,T;L^3(\Omega))$, cf. Lemma 3.2.7. The second integral of $P_{9,m}$ goes to zero by weak convergence. Finally $\nabla\phi_m\mathbf{v}$ converges strongly to $\nabla\phi\mathbf{v}$ in $L^2(0,T;L^2(\Omega))$, cf. Lemma 3.2.7.

In summary, we have obtained the following lemma.

Lemma 3.2.8. *Let $(\phi_m, \mu_m, q_m, \mathbf{u}_m)$ be the solution of* (3.4) *and let the a priori bounds* (3.7)*,* (3.8)*,* (3.13) *hold. Furthermore, recall Lemma 3.2.6 and Lemma 3.2.7. Then we can pass to the limit $m \to \infty$, i.e.,* (3.16) *converges to* (3.17)*, and show existence of a global-in-time weak solution $(\phi, \mu, q, \mathbf{u})$ which satisfies Definition 3.1.1.*

As a final step we will show that we can also pass to the limit in the approximation of the energy dissipation identity. Before considering the limit in the energy, let us realize that the initial conditions indeed make sense. First, we observe by application of the second Lions-Aubin embedding (A.29) that our solutions are almost every equal to a continuous function from $[0,T]$ into a suitable space, which is at least the dual space of the associated space test function. Indeed, we can find $\phi \in C([0,T];L^2(\Omega))$, $q \in C([0,T];H^{-1}(\Omega))$, $\mathbf{u} \in C([0,T];H^{-1}_{\mathrm{div}}(\Omega)^d)$. Using test functions in time such that $\varphi(T) = 0$ coupled with standard comparison arguments, see [104], yields

$$\langle \phi_0 - \phi(0), \psi \rangle \varphi(0) = 0, \quad \langle q_0 - q(0), \zeta \rangle \varphi(0) = 0, \quad \langle \mathbf{u}_0 - \mathbf{u}(0), \mathbf{v} \rangle \varphi(0) = 0, \qquad (3.18)$$

for all $(\phi, \zeta, \mathbf{v}) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)^d$. Choosing φ such that $\varphi(0) = 1$, we see that the initial data are obtained in a weak sense. Note that with the third Lions-Aubin embedding (A.30), using the higher regularity of $\phi \in L^2(0,T;H^3(\Omega))$, we can obtain $C([0,T];H^1(\Omega))$. Similarly, in two space dimensions, the velocity is even a continuous function from $[0,T]$ into the space $L^2_{\mathrm{div}}(\Omega)^d$.

Lemma 3.2.9. *The weak solution $(\phi, \mu, q, \mathbf{u})$ of* (3.4) *is continuous in time, i.e.,*

$$\phi \in C([0,T];H^1(\Omega)), q \in C([0,T];H^{-1}(\Omega)), \mathbf{u} \in C([0,T];H^{-1}_{div}(\Omega)^d).$$

Furthermore, the initial data $(\phi_0, q_0, \mathbf{u}_0)$ is attained, i.e., $(\phi(0), q(0), \mathbf{u}(0)) = (\phi_0, q_0, \mathbf{u}_0)$, in the weak sense (3.18)*.*

3.2.5. Limit passage in the energy dissipation identity

In order to pass to the limit in the energy equality (3.5), we recall that for a suitably weakly/weakly-$\star$ convergent sequence $\{g_m\}_{m=1}^\infty$ we have by the generalized lemma of Fatou/lower-semi continuity of norms [55, 58, 57]

$$\|g(t)\|_0 \leq \|g\|_{L^\infty(0,t;L^2(\Omega))} \leq \liminf_{m\to\infty} \|g_m\|_{L^\infty(0,t;L^2(\Omega))}, \qquad (3.19)$$
$$\|g\|_{L^2(0,t;L^2(\Omega))} \leq \liminf_{m\to\infty} \|g_m\|_{L^2(0,t;L^2(\Omega))}$$

for almost every $t \in (0,T)$ Using the weak convergence in Lemma 3.2.6, the strong convergences in Lemma 3.2.7 and by application of Lemma A.3.4 we can show that the dissipation terms converge weakly in $L^2(0,T;L^2(\Omega))$. Hence, the above result can be applied for the dissipation terms $g \in \{b^{1/2}(\phi_m)\nabla\mu_m, \nabla(A(\phi_m)q_m), \kappa_1^{1/2}(\phi_m)q_m, \nabla q_m, \eta^{1/2}(\phi_m)\mathrm{D}\mathbf{u}_m\}$. This implies, for example, that for almost every $t \in (0,T)$

$$\|b^{1/2}(\phi)\nabla\mu\|^2_{L^2(0,t;L^2(\Omega))} \le \liminf_{m\to\infty} \left\|b^{1/2}(\phi_m)\nabla\mu_m\right\|^2_{L^2(0,t;L^2(\Omega))}.$$

With the same reasoning, we can pass to the limit in most of the terms appearing in the energy functional, i.e $\frac{\gamma}{2}\|\nabla\phi_m\|_0^2$, $\frac{1}{2}\|q_m\|_0^2$, $\frac{1}{2}\|\mathbf{u}_m\|_0^2$. For instance this yields

$$\|\nabla\phi(t)\|_0^2 \le \|\nabla\phi\|^2_{L^\infty(0,t;L^2(\Omega))} \le \liminf_{m\to\infty}\|\nabla\phi_m\|^2_{L^\infty(0,t;L^2(\Omega))}$$

almost every $t \in (0,T)$. For the mixing potential we show strong convergence, i.e.,

$$\begin{aligned}\int_\Omega |f(\phi_m) - f(\phi)| \,\mathrm{d}x &\le \int_\Omega \int_0^1 f'(\phi_m + s(\phi - \phi_m))\,\mathrm{d}s(\phi_m - \phi)\,\mathrm{d}x \\ &\le (f_2^{(2)} + f_3^{(2)}(\|\phi_m\|_1 + \|\phi\|_1))\|\phi_m - \phi\|_1 \to 0\end{aligned}$$

for almost every $t \in (0,T)$. Let us recall the discrete energy inequality given by

$$\begin{aligned}&\left(\int_\Omega \frac{\gamma}{2}|\nabla\phi_m(t)|^2 + f(\phi_m(t)) + \frac{1}{2}|q_m(t)|^2 + \frac{1}{2}|\mathbf{u}_m(t)|^2\,\mathrm{d}x\right) \\ &\quad + \int_{\Omega_t} \varepsilon_0|b^{1/2}(\phi_m)\nabla\mu_m|^2 + \left|b^{1/2}(\phi)\nabla\mu_m - \nabla\Big(A(\phi_m)q_m\Big)\right|^2 + \varepsilon_1|\nabla q_m|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_{\Omega_t} |\kappa_1^{1/2}(\phi_m)q_m|^2 + \left|\eta^{1/2}(\phi_m)\mathrm{D}\mathbf{u}_m\right|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le \left(\int_\Omega \frac{\gamma}{2}|\nabla\phi_m(0)|^2 + f(\phi_m(0)) + \frac{1}{2}|q_m(0)|^2 + \frac{1}{2}|\mathbf{u}_m(0)|^2\,\mathrm{d}x\right).\end{aligned}$$

Taking the supremum in time, i.e. $\sup_{t\in[0,T]}$, and $\liminf_{m\to\infty}$, by considering (3.19) allows us to pass to the limit in all terms of the inequality. For the right-hand side, we use that the stability of the orthogonal projections.

Hence, we can pass to the limit in the approximate energy (3.5) and obtain the following lemma.

Lemma 3.2.10. *Let $(\phi,\mu,q,\mathbf{u})$ be a weak solution obtained from Lemma 3.2.8. Then for almost all $t \in (0,T)$ the following inequality holds*

$$\begin{aligned}E(\phi,q,\mathbf{u})(t) &+ \int_{\Omega_t} \varepsilon_0|b^{1/2}(\phi)\nabla\mu|^2 + \left|b^{1/2}(\phi)\nabla\mu - \nabla\Big(A(\phi)q\Big)\right|^2\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_{\Omega_t} \varepsilon_1|\nabla q|^2 + |\kappa_1^{1/2}(\phi)q|^2 + |\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s \le E(\phi,q,\mathbf{u})(0).\end{aligned}$$

Hence, the weak solution obtained in the above sections is a dissipative weak solution.

3.2.6. Comments

Let us shortly comment on suitable extensions of the above results with respect to the parameters ε_i. In the case $\varepsilon_0 = 0$, one has to deduce the bounds for $\nabla\mu_m$ from the cross-diffusive dissipation estimate

$$\int_{\Omega_t} |b^{1/2}(\phi_m)\nabla\mu_m - \nabla(A(\phi_m)q_m)|^2 \,\mathrm{d}x\,\mathrm{d}s \le C_0.$$

Thus, one has first to deduce a suitable space for $\nabla(A(\phi_m)q_m)$ from the existing bounds, which turns out to be only $L^{5/3}(0,T;L^{5/3}(\Omega))$ in three space dimensions. Therefore, the same regularity holds for $\nabla\mu_m$, which is indeed a regularity loss. With this at hand, one can in principle follow the rest of the proof. In summary, the existence proof can be done, but with less regular function spaces.

Note that the case $\varepsilon_1 = 0$ turns out to be more complicated. One has to find estimates for ∇q_m from the bound $\nabla(A(\phi_m)q_m) \in L^2(0,T;L^2(\Omega))$. In principle, this can be done by suitable assumptions for A again in less regular function spaces.

Lastly, the case $\varepsilon_i = 0, i = 1,2$, i.e., the case of System S.2, turns out to be problematic in nature. In this setting, the gradient estimate is only from the cross-diffusion estimate. Therefore, we will not get any independent gradient bound, and the above approach will not work. Indeed, it can happen for suitable constellations of ϕ, μ, q that the diffusion term cancels the other ones, i.e., the behaviour can be conservative in certain regimes.

3.3. Weak solutions for System S.5

Let now us turn to the definition of weak solution for the Peterlin model, i.e., System S.5. In the context of the Peterlin model, the main difficulty will be the construction of weak solutions such that $\mathbf{C}$ is positive definite. In principle, we will follow similar techniques as in the proof of the CHNSQ model. However, for the conformation tensor, other techniques have to be considered, here using parabolic regularity.

Definition 3.3.1. Let (A0), (A5)–(A6) hold and for given $T > 0$ the tuple $(\mathbf{u}, \mathbf{C})$ is called a weak solution of System S.5 on $\Omega \times (0,T)$, if

$$\begin{aligned} \mathbf{u} &\in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)^d) \cap L^2(0,T;H^1_{\mathrm{div}}(\Omega)^d) \cap W^{1,p}(0,T;H^{-1}_{\mathrm{div}}(\Omega)^d) = \mathbb{X}(0,T), \\ \mathbf{C} &\in L^\infty(0,T;L^2_S(\Omega)^{d\times d}) \cap L^2(0,T;H^1(\Omega)^{d\times d}) \cap W^{1,4/3}(0,T;H^{-1}(\Omega)^{d\times d}) =: \mathbb{T}(0,T), \end{aligned} \tag{3.20}$$

with $p = 2$ in two space dimensions and $p = \frac{4}{3}$ in three space dimensions and

$$\begin{aligned} &\langle \partial_t \mathbf{u}, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u};\mathbf{v},\mathbf{u}) + \langle \eta \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v}\rangle + \langle \mathrm{tr}(\mathbf{C})\mathbf{C}, \nabla\mathbf{v}\rangle = 0 \\ &\langle \partial_t \mathbf{C}, \mathbf{D}\rangle - \mathbf{c}(\mathbf{u};\mathbf{D},\mathbf{C}) - \langle (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top, \mathbf{D}\rangle + \varepsilon_2\langle \nabla\mathbf{C}, \nabla\mathbf{D}\rangle \\ &\qquad = -\langle \chi(\mathrm{tr}(\mathbf{C}))\mathbf{C}, \mathbf{D}\rangle + \langle \Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\mathbf{D})\rangle, \end{aligned} \tag{3.21}$$

for any test function $(\mathbf{v}, \mathbf{D}) \in [H^1_{\mathrm{div}}(\Omega)^d \times H^1_S(\Omega)^{d\times d}]$ and almost every $t \in (0,T)$.

Let us shortly comment on the symmetry properties for the test functions $\mathbf{D}$. In principle, one can state the weak solution first on the full space $H^1(\Omega)^{d\times d}$. However, choosing $\mathbf{D}$ suitably, one can observe that the variational formulation of $\mathbf{C}_{ij}$ is exactly the same as for $\mathbf{C}_{ji}$. Since we will always assume symmetric initial data symmetry follows immediately, and it suffices to work on the symmetric test space, i.e., $H^1_S(\Omega)^{d\times d}$.

For the Peterlin model in two space dimensions an existence result is already available in [96] and after minor modification can be stated as follows.

Theorem 3.3.2 (Two space dimensions, $d = 2$, [96]). *Let the initial data* $(\mathbf{u}_0, \mathbf{C}_0) \in L^2_{div}(\Omega)^2 \times L^2_S(\Omega)^{2\times 2}$ *be given. Under assumptions (A0), (A4)–(A7) for* $d = 2$ *and for every* $T > 0$ *there exists a dissipative global-in-time weak solution of System S.5 in the sense of Definition 3.3.1 that satisfies the initial data, i.e.,* $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0)$ *and furthermore satisfies the energy-type inequality*

$$\begin{aligned} &E_{2d}(\mathbf{u},\mathbf{C})(t) + \int_{\Omega_t} \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla\mathbf{C}|^2 + \frac{1}{2}\chi(\mathrm{tr}(\mathbf{C}))|\mathbf{C}|^2 \,\mathrm{d}x\,\mathrm{d}s \\ \le &E_{2d}(\mathbf{u},\mathbf{C})(0) + \frac{1}{2}\int_{\Omega_t} \Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C})\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{3.22}$$

for almost all $t \in (0, T)$.

Remark 3.3.3. We note that the above mathematical energy is relevant for the proof. However, indeed the following energy-type inequality

$$\begin{aligned} &E(\mathbf{u},\mathbf{C})(t) + \int_{\Omega_t} \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 + \frac{1}{2}\chi(\mathrm{tr}(\mathbf{C}))|\mathrm{tr}(\mathbf{C})|^2 \,\mathrm{d}x\,\mathrm{d}s \\ \le &E(\mathbf{u},\mathbf{C})(0) + \frac{1}{2}\int_{\Omega_t} d\Phi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C})\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.23}$$

holds also for almost every $t \in (0, T)$, which is tied to the total energy (2.1).

A quick inspection of the above result indicates that in two space dimensions, it is possible to construct weak solutions without investigating the positive definiteness of $\mathbf{C}$. However, without positive definiteness, we cannot expect to make sense of the total energy (2.1). Furthermore, it seems that in three space dimensions this is inevitable to construct weak solutions with this property. Indeed, the positive definiteness is a relevant physical condition, and therefore we present another proof which will also include the three-dimensional case as well as positive definiteness under some conditions. We mention that the basic idea of the proof is inspired by [97]. However, positive definiteness was not considered there. The results in this section are based on our recent work [30].

Theorem 3.3.4 (Two/Three space dimensional, $d = 2, 3$). *Let the initial data* $(\mathbf{u}_0, \mathbf{C}_0) \in L^2_{div}(\Omega)^d \times L^2_{SPD}(\Omega)^{d\times d}$ *be given. Under assumptions (A0), (A4)–(A7) for* $d \in \{2, 3\}$ *with* $\mathbf{C}_0$ *such that* $\mathrm{tr}(\ln(\mathbf{C}_0)) \in L^1(\Omega)$ *and for every* $T > 0$ *there exists a dissipative global-in-time weak solution of System S.5 in the sense of Definition 3.3.1 that satisfies the initial data, i.e.,* $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0)$ *and furthermore satisfies the energy inequality* (3.23). *Moreover, if* $a > 0$ *the conformation tensor*

$\mathbf{C}$ *is positive definite and enjoys the further regularity*

$$\operatorname{tr}(\log \mathbf{C}) \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{3.24}$$
$$\mathbf{C}^{-1/2}\nabla\mathbf{C}\mathbf{C}^{-1/2} \in L^2(0,T;L^2(\Omega)), \quad \operatorname{tr}(\mathbf{C}^{-1}), \operatorname{tr}(\mathbf{C}^{-1})\operatorname{tr}(\mathbf{C}) \in L^1(0,T;L^1(\Omega))$$

and the total energy inequality

$$\begin{aligned} E_{total}(\mathbf{u},\mathbf{C})(t) &+ \int_{\Omega_t} \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla \operatorname{tr}(\mathbf{C})|^2 + \frac{\varepsilon_2}{2}\left|\mathbf{C}^{-1/2}\nabla\mathbf{C}\mathbf{C}^{-1/2}\right|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_{\Omega_t} \frac{1}{2}\chi(\operatorname{tr}(\mathbf{C}))\operatorname{tr}(\mathbf{T}+\mathbf{T}^{-1}-2\mathbf{I})\,\mathrm{d}x\,\mathrm{d}s \le E_{total}(\mathbf{u},\mathbf{C})(0) \end{aligned} \tag{3.25}$$

holds for almost every $t \in (0,T)$. *In the special case of two space dimensions also the energy inequality* (3.22) *holds.*

3.4. Existence proof of Theorem 3.3.4

Let us now turn to the existence proof of Theorem 3.3.4. Before starting the proof, let us shortly mention the main steps.

1. In the first step, see Subsection 3.4.1, we will recall the energy and total energy inequalities from Chapter 2 and derive formal a priori bounds using the Gronwall lemma.

2. In the second step, see Subsection 3.4.2, we introduce a Galerkin approximation for the velocity $\mathbf{u}$ while the conformation tensor is the solution of the parabolic problem for finite-dimensional $\mathbf{u}$. Local existence follows from the theory of ordinary differential equations and parabolic regularity. Further, we obtain global existence using energy type arguments.

3. The third step, see Subsection 3.4.3, consists of deriving suitable approximation independent a priori bounds on the approximative solutions. The first observation is that smooth solutions for the conformation tensor are symmetric and positive definite. Using this information, the first set of a priori bounds follow from the energy inequalities of the system. The second set of a priori estimates are again the bounds on the time derivative, which we obtain from duality.

4. The fourth step, see Subsection 3.4.4, is to establish convergence of suitable subsequences. The typical weak/weak-* convergence can be directly obtained from the a priori bounds of the second step by using the consequence of Banach-Alaoglu. Since our model is nonlinear we will also need strong convergences, which we obtain use compactness arguments via the Aubin-Lions lemma.

5. The fifth step, see Subsection 3.4.5, is the passage to the limit in the approximate formulations. Here, we show that the constructed sequence does converge to the weak formulation. Hence, proving the existence of global-in-time weak solutions.

This will be done by using the previous convergence results together with suitable density arguments.

6. The final step, see Subsection 3.4.6, is the limit in the energy inequalities, i.e., proofing that the weak solution is dissipative. This will be done again by employing a weak and strong convergence argument together with the generalized lemma of Fatou/lower-semi continuity of norms. For the limit in the total energy inequality (3.25) we first have to prove positive definiteness of the conformation tensor in the limit, which is done via strong convergence arguments.

3.4.1. Formal a priori bounds

In order to treat System S.5, we consider several formal energy type estimates. First recall the energy equality (2.17) which can also be obtained by taking the inner product of S.5_1 with $\mathbf{u}$ and S.5_2 with $\text{tr}(\mathbf{C})\mathbf{I}/2$ and integrating over the domain Ω and reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{u},\mathbf{C}) + \int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla\text{tr}(\mathbf{C})|^2 + \frac{1}{2}\left(|\text{tr}(\mathbf{C})|^4 + a|\text{tr}(\mathbf{C})|^3\right)\,\mathrm{d}x \\ = \int_\Omega \frac{1}{2}\left(|\text{tr}(\mathbf{C})|^2 + a\text{tr}(\mathbf{C})\right)\,\mathrm{d}x. \qquad (3.26)\end{aligned}$$

Estimating the last integral of (3.26) by the Hölder inequality we find after applying the Gronwall lemma, cf. Lemma A.3.1 the inequality

$$\|\mathbf{u}\|^2_{L^\infty(L^2)} + \|\text{tr}(\mathbf{C})\|^2_{L^\infty(L^2)} + \|\mathrm{D}\mathbf{u}\|^2_{L^2(L^2)} + \|\nabla\text{tr}(\mathbf{C})\|^2_{L^2(L^2)} + \|\text{tr}(\mathbf{C})\|^2_{L^4(L^4)} \le C_0. \quad (3.27)$$

Here C_0 depends on the final time and bounds of parametric functions and coefficients. These yields uniform bounds in the following spaces

$$\begin{aligned}\mathbf{u} &\in L^\infty(0,T;L^2_{\text{div}}(\Omega)^d) \cap L^2(0,T;H^1_{\text{div}}(\Omega)^d), \qquad (3.28)\\ \text{tr}(\mathbf{C}) &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^4(0,T;L^4(\Omega)).\end{aligned}$$

Since for a smooth solution the matrix $\mathbf{C}$ is positive definite, see [96], we find also that $\mathbf{C} \in L^4(0,T;L^4(\Omega))$, due to the norm equivalence in Lemma A.2.1. Now we can take the Frobenius inner product of S.5_2 with $\mathbf{C}/2$ and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{4}|\mathbf{C}|^2\right) + \frac{\varepsilon_2}{2}\int_\Omega |\nabla\mathbf{C}|^2 + \frac{1}{2}\left(|\text{tr}(\mathbf{C})\mathbf{C}|^2 + a|\text{tr}(\mathbf{C})||\mathbf{C}|^2\right)\,\mathrm{d}x \\ \le \int_\Omega \frac{1}{2}(\text{tr}(\mathbf{C})^2 + a\text{tr}(\mathbf{C}))\,\mathrm{d}x + \int_\Omega (\nabla\mathbf{u}\mathbf{C}) : \mathbf{C}\,\mathrm{d}x. \qquad (3.29)\end{aligned}$$

The first integral of (3.29) can be treated as in (3.26). The second integral of (3.29) can be bounded by

$$\int_\Omega (\nabla\mathbf{u}\mathbf{C}) : \mathbf{C}\,\mathrm{d}x \le \|\nabla\mathbf{u}\|_2\|\mathbf{C}\|_4^2 \le c\|\nabla\mathbf{u}\|_2^2 + \|\mathbf{C}\|_4^4 \le C_0. \qquad (3.30)$$

Using the Gronwall lemma, cf. Lemma A.3.1 implies bounds in the space

$$\mathbf{C} \in L^\infty(0,T;L^2_S(\Omega)^{d\times d}) \cap L^2(0,T;H^1(\Omega)^{d\times d}). \qquad (3.31)$$

Recall the total energy equality in Chapter 2, i.e., (2.16). As already discussed, the total energy law can be obtained by taking the inner product of S.5$_1$ with $\mathbf{u}$ and S.5$_2$ with $\mathrm{tr}(\mathbf{C})\mathbf{I}/2 - \mathbf{C}^{-1}/2$. With this and an application of (A.13) we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{total}(\mathbf{u},\mathbf{C})+&\int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 - \frac{\varepsilon_2}{2}\nabla\mathbf{C} : \nabla\mathbf{C}^{-1}\;\mathrm{d}x \\ &+\int_\Omega \frac{1}{2}\Big(|\mathrm{tr}(\mathbf{C})|^4 + a|\mathrm{tr}(\mathbf{C})|^3 - d|\mathrm{tr}(\mathbf{C})|^2 + ad\mathrm{tr}(\mathbf{C})\Big)\;\mathrm{d}x \\ &-\int_\Omega \frac{1}{2}\Big(d|\mathrm{tr}(\mathbf{C})|^2 + ad|\mathrm{tr}(\mathbf{C})| - \mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{C}^{-1}) - a\mathrm{tr}(\mathbf{C}^{-1})\Big)\;\mathrm{d}x = 0.\end{aligned} \tag{3.32}$$

First we will expand the diffusion term involving the inverse matrix by

$$\begin{aligned}\int_\Omega \nabla\mathbf{C} : \nabla\left(\mathbf{C}^{-1}\right)\;\mathrm{d}x &= -\sum_{i=1}^{d}\int_\Omega \frac{\partial\mathbf{C}}{\partial x_i} : \mathbf{C}^{-1}\frac{\partial\mathbf{C}}{\partial x_i}\mathbf{C}^{-1}\;\mathrm{d}x \\ &= -\sum_{i=1}^{d}\int_\Omega \mathrm{tr}\left(\mathbf{C}^{-1/2}\frac{\partial\mathbf{C}}{\partial x_i}\mathbf{C}^{-1/2}\mathbf{C}^{-1/2}\frac{\partial\mathbf{C}}{\partial x_i}\mathbf{C}^{-1/2}\right)\;\mathrm{d}x \\ &=: -\left\|\mathbf{C}^{-1/2}\nabla\mathbf{C}\mathbf{C}^{-1/2}\right\|_2^2 \le -\frac{1}{d}\|\nabla\mathrm{tr}(\log\mathbf{C})\|_2^2.\end{aligned} \tag{3.33}$$

Here we used the cyclic property of the trace, symmetry of $\mathbf{C}$ and existence of a square root $\mathbf{C}^{1/2}$ which follows from positive definiteness and (A.14). Rewriting (3.32) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{total}(\mathbf{u},\mathbf{C}) + \int_\Omega \eta|\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2}|\nabla\mathrm{tr}(\mathbf{C})|^2 + \frac{\varepsilon_2}{2}\left|\mathbf{C}^{-1/2}\nabla\mathbf{C}\mathbf{C}^{-1/2}\right|^2\;\mathrm{d}x \\ + \frac{1}{2}\int_\Omega\Big(\mathrm{tr}(\mathbf{C})^2 + a\mathrm{tr}(\mathbf{C})\Big)\mathrm{tr}(\mathbf{T}+\mathbf{T}^{-1}-2\mathbf{I})\;\mathrm{d}x \le 0.\end{aligned} \tag{3.34}$$

As before, we integrate over time and apply the Gronwall lemma. Since $\mathbf{C}$ is symmetric positive definite, using (A.8) and the bounds (3.28), we obtain from (3.34) the additional information

$$\mathrm{tr}(\log\mathbf{C}) \in L^\infty(0,T;L^1(\Omega)), \tag{3.35}$$

$$a\mathrm{tr}(\mathbf{C}^{-1}) \in L^1(0,T;L^1(\Omega)), \quad \mathrm{tr}(\mathbf{C}^{-1})\mathrm{tr}(\mathbf{C}) \in L^1(0,T;L^1(\Omega)), \tag{3.36}$$

$$\nabla\mathrm{tr}(\log\mathbf{C}) \in L^2(0,T;L^2(\Omega)), \quad \mathbf{C}^{-1/2}\nabla\mathbf{C}\mathbf{C}^{-1/2} \in L^2(0,T;L^2(\Omega)). \tag{3.37}$$

While all the above calculations and estimates are purely formal, we will make them rigorous in what follows.

3.4.2. Approximation and Estimates

The goal of this subsection is to derive an approximation scheme based on a Galerkin method of $\mathbf{u}$. Similarly to the existence proof of the CHNSQ model, we are introducing $\mathbf{v}_j, j = 1,\ldots$ as the eigenfunctions of the Stokes operator, cf. Subsection 3.2.1. As before, these functions are smooth, divergence-free and orthogonal in $L^2_{\mathrm{div}}(\Omega)^d$ and $H^1_{\mathrm{div}}(\Omega)^d$. Furthermore, we introduce the space $Q_m := \mathrm{span}\{\mathbf{v}_j,\ldots,\mathbf{v}_m\}$ and the associated orthogonal projection by

$$\mathcal{P}_{Q_m}(\mathbf{v}) = \sum_{j=1}^{m}\langle\mathbf{v},\mathbf{v}_j\rangle\mathbf{v}_j : H^1_{\mathrm{div}}(\Omega)^d \to Q_m.$$

Then we define the m-th Galerkin approximation of $\mathbf{u}$ by

$$\mathbf{u}_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)\mathbf{v}_j(x), \quad \mathbf{u}_{0m} = \mathcal{P}_{Q_m}(\mathbf{u}_0). \tag{3.38}$$

Furthermore, $\mathbf{C}_m(\mathbf{u}_m)$ denotes the solution of the parabolic problem S.5_2 for $\mathbf{C}_m$. Together, the approximate system is given by

$$\begin{aligned} &\langle \partial_t \mathbf{u}_m, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u}_m; \mathbf{v}, \mathbf{u}_m) + \langle \eta \mathrm{D}\mathbf{u}_m, \mathrm{D}\mathbf{v}\rangle + \langle \mathrm{tr}(\mathbf{C}_m)\mathbf{C}_m, \nabla \mathbf{v}\rangle = 0, \\ &\partial_t \mathbf{C}_m + (\mathbf{u}_m \cdot \nabla)\mathbf{C}_m - (\nabla \mathbf{u}_m)\mathbf{C}_m + \mathbf{C}_m(\nabla \mathbf{u}_m)^\top \\ &\quad + \chi(\mathrm{tr}(\mathbf{C}_m))\mathbf{C}_m - \Phi(\mathrm{tr}(\mathbf{C}_m))\mathbf{I} - \varepsilon_2 \Delta \mathbf{C}_m = 0. \end{aligned} \tag{3.39}$$

for all $\mathbf{v} \in Q_m$. The system is subjected to the initial data $(\mathbf{u}_m(0), \mathbf{C}_m(0)) = (\mathbf{u}_{0m}, \mathbf{C}_0)$.

Due to the standard theory of ordinary differential equations, there exists a velocity $\mathbf{u}_m$ locally up to time T_m. Further by parabolic regularity, see Prüss and Simonet [106], since the velocity $\mathbf{u}_m$ is smooth, we obtain a symmetric solution $\mathbf{C}_m \in C^1((0,T_M]; C^2(\Omega))$. Using the energy inequality (3.26) to show uniform bounds for the velocity, hence existence can be extended independently of m to time T. We mention that $(\mathbf{u}_m, \mathbf{C}_m)$ is sufficiently smooth to obtain the energy inequality (3.26). Again, parabolic regularity ensures the existence of $\mathbf{C}_m \in C^1((0,T]; C^2(\Omega))$. Let us summarize the results obtained so far.

Lemma 3.4.1. *For every $m > 0$ there exists a solution $(\mathbf{u}_m, \mathbf{C}_m)$ of the discrete system* (3.39) *up to time T independent of m in the following spaces*

$$\mathbf{u}_m \in C^1([0,T]; Q_m), \qquad \mathbf{C}_m \in C^1((0,T]; C^2(\Omega)).$$

3.4.3. A priori bounds

The above bounds are not independent of m, hence we will now derive m independent bounds. Similarly to the proof of the CHNSQ model, we will first deduce m independent a priori bounds via the energy inequality. Afterwards, we will deduce further a priori bounds for the time derivative.

A priori energy bounds:
Since $\mathbf{C}_m$ is positive definite for every m, see [97], we reproduce the energy inequalities (3.26) and (3.29) to gain a priori bounds. Together, we summarize the a priori bounds in the following lemma.

Lemma 3.4.2. *Let $(\mathbf{u}_m, \mathbf{C}_m)$ be the approximative solution of* (3.39). *Then the conformation tensor $\mathbf{C}_m$ is symmetric positive definite and the following m independent a priori bounds hold*

$$\begin{aligned} &\|\mathbf{u}_m\|^2_{L^\infty(L^2)} + \|\mathbf{C}_m\|^2_{L^\infty(L^2)} + \|\mathbf{C}\|^2_{L^2(H^1)} + \|\mathbf{C}\|^4_{L^4(L^4)} \\ &\qquad + \|\chi(\mathrm{tr}(\mathbf{C}_m))\mathbf{C}_m\|^{4/3}_{L^{4/3}(L^{4/3})} + \|\Phi(\mathrm{tr}(\mathbf{C}_m))\|^2_{L^2(L^2)} \le C_0 \end{aligned} \tag{3.40}$$

Proof. Integration of discrete versions of (3.26) and (3.29) yields the a priori bounds for $\mathbf{u}_m$ and $\mathbf{C}_m$. Note that we used the positive definiteness of $\mathbf{C}_m$ to derive these bounds. The a priori bounds for $\chi(\mathrm{tr}(\mathbf{C}_m))\mathbf{C}_m$ and $\Phi(\mathrm{tr}(\mathbf{C}_m))$ follow from simple calculations. □

Furthermore, considering a discrete version of the total energy estimate, i.e (3.34), we can obtain more independent bounds, which we will summarize in an extra lemma.

Lemma 3.4.3. *Let* $(\mathbf{u}_m, \mathbf{C}_m)$ *be the approximative solution of* (3.39). *Then the following* m *independent additional a priori bounds hold*

$$\left\|\operatorname{tr}(\mathbf{C}_m)^2 - 2\operatorname{tr}(\log \mathbf{C}_m)\right\|_{L^\infty(L^1)} + \|\nabla \operatorname{tr}(\log \mathbf{C}_m)\|_{L^2(L^2)} + \left\|\mathbf{C}_m^{-1/2}\nabla \mathbf{C}_m \mathbf{C}_m^{-1/2}\right\|_{L^2(L^2)} \leq C_0,$$
$$a\left\|\operatorname{tr}(\mathbf{C}_m^{-1})\right\|_{L^1(L^1)} + \left\|\operatorname{tr}(\mathbf{C}_m)\operatorname{tr}(\mathbf{C}_m^{-1})\right\|_{L^1(L^1)} \leq C(a). \tag{3.41}$$

Here $C(a)$ *depends inversely on* a, *i.e., it blows up for* $a \to 0$.

By the above bounds, we can take the weak limit for the logarithm which yields

$$\operatorname{tr}(\log \mathbf{C}_m) \rightharpoonup \overline{\operatorname{tr}(\log \mathbf{C}_m)} \in L^2(0,T;H^1(\Omega)). \tag{3.42}$$

As soon as we know that the limit tensor $\mathbf{C}$ is positive definite, we can identify the limit, i.e., neglect the bar.

Note that we will not employ the bounds which are obtained from the total energy inequality, i.e., Lemma 3.4.3 to pass to the limit in the equations. These bounds will only be relevant for the limit in the total energy.

Bounds on the time derivative:
In this part, we will derive a priori bounds on the time derivative. To this end, we will follow the same strategy as for the CHNSQ model for the Navier-Stokes equation, i.e., recall (3.11).

$$\|\partial_t \mathbf{u}_m(t)\|_{-1,\mathrm{div}} = \sup_{\mathbf{v}\in H^1_{\mathrm{div}}(\Omega)^d} \frac{\langle \partial_t \mathbf{u}_m(t), \mathbf{v}\rangle}{\|\mathbf{v}\|_1} = \sup_{\mathbf{v}\in Q_m} \frac{\langle \partial_t \mathbf{u}_m(t), \mathbf{v}\rangle}{\|\mathbf{v}\|_1}. \tag{3.43}$$

Following the computations of for the CHNSQ model using the Cauchy-Schwarz and the Hölder inequality we obtain

$$\int_0^T \|\partial_t \mathbf{u}_m\|^p_{-1,\mathrm{div}}\, \mathrm{d}s \leq c \int_0^T \|\mathrm{D}\mathbf{u}_m\|^p_{0,2} + \|\operatorname{tr}(\mathbf{C}_m)\mathbf{C}_m\|^p_{0,2} + \|\mathbf{u}_m\|^{2p}_{0,4}\, \mathrm{d}s. \tag{3.44}$$

Using the obtained regularity (3.40), the first two terms are bounded for $p \leq 2$. For the third term, we observe that in two space dimensions, due to (A.23) we can take $p \leq 2$. However, in the three-dimensional case, (A.23) implies that $p \leq 4/3$.

Next we consider the evolution equation for the conformation tensor in the operator form given by

$$\partial_t \mathbf{C}_m + \varepsilon_2 \Delta \mathbf{C}_m = \mathbf{F}_m, \tag{3.45}$$
$$\mathbf{F}_m := -(\mathbf{u}_m \cdot \nabla)\mathbf{C}_m + (\nabla \mathbf{u}_m)\mathbf{C}_m + \mathbf{C}_m(\nabla \mathbf{u}_m)^T - \chi(\operatorname{tr}(\mathbf{C}_m))\mathbf{C}_m + \Phi(\operatorname{tr}(\mathbf{C}_m))\mathbf{I}.$$

Note that $\mathbf{C}_m$ is not defined as a Galerkin approximation, hence we will directly estimate the dual norm and calculate for which index p we can show $\mathbf{F}_m \in L^p(0,T;H^{-1}(\Omega)^{d\times d})$. This yields the following inequality

$$\int_0^T \|\mathbf{F}_m\|^p_{-1}\, \mathrm{d}s \leq \int_0^T \|\mathbf{u}_m\|^p_{0,3}\|\mathbf{C}_m\|^p_{0,6} + \|\mathbf{C}\|^p_{0,4}\|\nabla \mathbf{u}_m\|^p_{0,2}$$

$$+ \|\chi(\mathrm{tr}\,(\mathbf{C}_m))\mathbf{C}_m\|_{0,4/3}^p + \|\Phi(\mathrm{tr}\,(\mathbf{C}_m))\|_{0,2}^p \,\mathrm{d}s. \tag{3.46}$$

The last two integral are already bounded by (3.40) and imply $p \leq 4/3$. The first integral can be treated with (A.23) which yields $\mathbf{u}_m \in L^4(0,T;L^3(\Omega)^d)$. Suitable application of Young's inequality implies again $p \leq 4/3$. For the second term, we employ again Young's inequality and use (3.40) to obtain at least $p \leq 4/3$. By bootstrapping and parabolic regularity, we obtain

$$\partial_t \mathbf{C}_m \in L^{4/3}(0,T;H^{-1}(\Omega)^{d\times d}), \quad \mathbf{C}_m \in L^{4/3}(0,T;H^1(\Omega)^{d\times d}).$$

Let us summarize the results of this part in the following lemma.

Lemma 3.4.4. *Let* $(\mathbf{u}_m, \mathbf{C}_m)$ *be the solution of the approximate system* (3.39) *and let the a priori bounds* (3.40) *hold. Then we obtain the* m *independent a priori bounds*

$$\|\partial_t \mathbf{u}_m\|_{L^{4/3}(H^{-1}_{div})}^{4/3} + \|\partial_t \mathbf{C}_m\|_{L^{4/3}(H^{-1})}^{4/3} \leq C_0. \tag{3.47}$$

3.4.4. Convergent subsequences

In the same spirit of the CHNSQ model, we will derive convergence of suitable subsequences. As before, we will use the same index m also for the subsequences. Using Banach-Alaoglu, i.e., lemma A.3.3, we will immediately find weakly/weakly-* convergent subsequences. However, as we have to deal with nonlinear terms, strong convergence in suitable norms is necessary. This will be realized by using compactness arguments via the Aubin-Lions lemma, cf. Lemma A.3.6.

The consequences of Banach-Alaoglu Lemma are summarized in the following lemma and are directly obtained from a priori bounds, i.e., (3.40), (3.47).

Lemma 3.4.5. *Let* $(\mathbf{u}_m, \mathbf{C}_m)$ *satisfy the a priori bounds* (3.40) *and* (3.47). *Then the following weak/weak-* convergences hold*

$$\begin{aligned}
\mathbf{u}_m &\rightharpoonup^\star \mathbf{u} \in L^\infty(0,T;L^2_{div}(\Omega)^d), & \mathbf{C}_m &\rightharpoonup^\star \mathbf{C} \in L^\infty(0,T;L^2(\Omega)^{d\times d}), \\
\mathbf{u}_m &\rightharpoonup \mathbf{u} \in L^2(0,T;H^1_{div}(\Omega)^d), & \mathbf{C}_m &\rightharpoonup \mathbf{C} \in L^2(0,T;H^1(\Omega)^{d\times d}), \\
\mathbf{u}_m &\rightharpoonup \mathbf{u} \in L^{10/3}(0,T;L^{10/3}(\Omega)^d), & \mathbf{C}_m &\rightharpoonup \mathbf{C} \in L^4(0,T;L^4(\Omega)^{d\times d}), \\
\partial_t \mathbf{u}_m &\rightharpoonup \partial_t \mathbf{u} \in L^{4/3}(0,T;H^{-1}_{div}(\Omega)^d), & \partial_t \mathbf{C}_m &\rightharpoonup \partial_t \mathbf{C} \in L^{4/3}(0,T;H^{-1}(\Omega)^{d\times d}).
\end{aligned} \tag{3.48}$$

Application of the Aubin-Lions lemma, cf. Lemma A.3.6, with the following compact and continuous embeddings $H^1_{\mathrm{div}}(\Omega)^d \Subset L^2_{\mathrm{div}}(\Omega)^d \subset H^{-1}_{\mathrm{div}}(\Omega)^d$ and $H^1(\Omega)^{d\times d} \Subset L^2(\Omega)^{d\times d} \subset H^{-1}(\Omega)^{d\times d}$, we obtain the following lemma on strong convergence.

Lemma 3.4.6. *Let* $(\mathbf{u}_m, \mathbf{C}_m)$ *satisfy the a priori bounds* (3.40) *and* (3.47). *Then the following strong convergences hold*

$$\begin{aligned}
&\mathbf{u}_m \to \mathbf{u} \in L^2(0,T;L^p(\Omega)^d) \text{ for } p < 6, && \mathbf{C}_m \to \mathbf{C} \in L^2(0,T;L^p(\Omega)^{d\times d}) \text{ for } p < 6, \\
&\mathbf{u}_m \to \mathbf{u} \text{ a.e. in } \Omega \times (0,T), && \mathbf{C}_m \to \mathbf{C} \text{ a.e. in } \Omega \times (0,T).
\end{aligned}$$

3.4.5. Passage to the limit

In this subsection, we use the obtained convergences to pass to the limit in the Galerkin approximation $\mathbf{u}_m$ for $\mathbf{u}$ and the parabolic solution $\mathbf{C}_m(\mathbf{u}_m)$, cf. (3.39). The technique is similar to Subsection 3.2.4, but we will recall the main ideas by suitable adjustments.

- Convert the parabolic equation for the conformation tensor $\mathbf{C}_m$, i.e., $(3.39)_2$ to the variational formulation on the space $H^1_S(\Omega)^{d\times d}$. This yields no additionally difficulty, since $\mathbf{C}_m$ is smooth.
- We multiply the weak formulation of the approximation (3.39) with a smooth function in time $\varphi(s)$ and integrate the variational formulation from 0 to T.
- For this integrated formulation we will show that we can pass to the limit with $m \to \infty$ in the approximation for $(\mathbf{u}_m, \mathbf{C}_m)$ but with a fixed test space for the velocity. This will be the step which we will prove here. Note that $\mathbf{C}_m$ does not arise from a finite-dimensional approximation, but from parabolic regularity. Hence, this part is directly considered on the infinite-dimensional space $H^1_S(\Omega)^{d\times d}$. This is the step which we will provide.
- The next step is to extend the formulation from the discrete test spaces Q_m to the infinite-dimensional spaces $H^1_{\mathrm{div}}(\Omega)^d$, which again is omitted, cf. Subsection 3.2.4.
- Finally, we obtain a space-time formulation of the weak formulation (3.21). Since the test functions in time, i.e. $\varphi(s)$, is sufficiently smooth by standard approximation arguments, we can convert to the point-wise almost everywhere in time formulation (3.21).

The space-time formulation for the discrete problem then reads

$$\int_0^T \varphi(s)\Big(\langle \partial_t \mathbf{u}_m, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u}_m;\mathbf{v},\mathbf{u}_m) + \langle \eta \mathrm{D}\mathbf{u}_m, \mathrm{D}\mathbf{v}\rangle + \langle \mathrm{tr}(\mathbf{C}_m)\mathbf{C}_m, \nabla \mathbf{v}\rangle\Big)\,\mathrm{d}s = 0, \tag{3.49}$$

$$\begin{aligned}\int_0^T \varphi(s)\Big(&\langle \partial_t \mathbf{C}_m, \mathbf{D}\rangle - \mathbf{c}(\mathbf{u}_m;\mathbf{D},\mathbf{C}_m) - \langle (\nabla \mathbf{u})\mathbf{C}_m + \mathbf{C}_m(\nabla\mathbf{u})^\top, \mathbf{D}\rangle \\ &+ \langle \chi(\mathrm{tr}(\mathbf{C}_m))\mathbf{C}_m, \mathbf{D}\rangle - \langle \Phi(\mathrm{tr}(\mathbf{C}_m)), \mathrm{tr}(\mathbf{D})\rangle + \varepsilon_2\langle \nabla \mathbf{C}_m, \nabla \mathbf{D}\rangle\Big)\,\mathrm{d}s = 0.\end{aligned}$$

for all $\varphi \in L^\infty(0,T)$ and all $(\mathbf{v},\mathbf{D}) \in Q_m \times H^1_S(\Omega)^{d\times d}$. In addition, the space-time formulation of the weak solution (3.21) reads

$$\int_0^T \varphi(s)\Big(\langle \partial_t \mathbf{u}, \mathbf{v}\rangle - \mathbf{c}(\mathbf{u};\mathbf{v},\mathbf{u}) + \langle \eta \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v}\rangle + \langle \mathrm{tr}(\mathbf{C})\mathbf{C}, \nabla \mathbf{v}\rangle\Big)\,\mathrm{d}s = 0, \tag{3.50}$$

$$\begin{aligned}\int_0^T \varphi(s)\Big(&\langle \partial_t \mathbf{C}, \mathbf{D}\rangle - \mathbf{c}(\mathbf{u};\mathbf{D},\mathbf{C}) - \langle (\nabla \mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^\top, \mathbf{D}\rangle \\ &+ \langle \chi(\mathrm{tr}(\mathbf{C}))\mathbf{C}, \mathbf{D}\rangle - \langle \Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\mathbf{D})\rangle + \varepsilon_2\langle \nabla \mathbf{C}, \nabla \mathbf{D}\rangle\Big)\,\mathrm{d}s = 0.\end{aligned}$$

for all $\varphi \in L^\infty(0,T)$ and all $(\mathbf{v},\mathbf{D}) \in Q_m \times H^1_S(\Omega)^{d\times d}$.

Here we will focus on the limiting process in the main nonlinearities of (3.49). The terms which we consider in the following are simply differences of the formulations (3.49) and (3.49).

Navier-Stokes:
First, we start with the Navier-Stokes equation and observe that we have the same weak/weak-* and strong convergences as in the proof of Theorem 3.1.3, see Lemma 3.4.5 and Lemma 3.4.6. Therefore, the only term which is not present in the CHNSQ model has to be considered. This term is the coupling term to the Peterlin equation and is treated via

$$\begin{aligned} P_{1,m} &:= \int_0^T \int_\Omega (\mathrm{tr}(\mathbf{C}_m)\mathbf{C}_m - \mathrm{tr}(\mathbf{C})\mathbf{C}) : \nabla \mathbf{v}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s \\ &= \int_0^T \int_\Omega (\mathrm{tr}(\mathbf{C}_m - \mathbf{C}))\mathbf{C}_m : \nabla \mathbf{v}\varphi(s) + \mathrm{tr}(\mathbf{C})(\mathbf{C}_m - \mathbf{C}) : \nabla \mathbf{v}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s \\ &\leq \int_0^T \left(\|\mathbf{C}_m - \mathbf{C}\|_{0,4} \|\mathrm{tr}(\mathbf{C})\|_{0,4} + \|\mathrm{tr}(\mathbf{C}_m - \mathbf{C})\|_{0,4} \|\mathbf{C}_m\|_{0,4} \right) \|\nabla \mathbf{v}\|_{0,2} \|\varphi\|_{0,\infty} \,\mathrm{d}s \\ &\leq c \left(\|\mathbf{C}_m - \mathbf{C}\|^2_{L^2(L^4)} \|\mathrm{tr}(\mathbf{C})\|^2_{L^2(L^4)} + \|\mathrm{tr}(\mathbf{C}_m - \mathbf{C})\|^2_{L^2(L^4)} \|\mathbf{C}_m\|^2_{L^2(L^4)} \right). \end{aligned}$$

Since $\mathrm{tr}(\mathbf{C}_m), \mathbf{C}_m$ are strongly convergent to $\mathrm{tr}(\mathbf{C}), \mathbf{C}$ in $L^2(0,T;L^4(\Omega))$, cf. (3.48), we see that $P_{1,m} \to 0$ as $m \to \infty$.

Conformation tensor:
Let us now turn to the conformation tensor equation we first consider

$$P_{2,m} := \int_0^T \int_\Omega \chi(\mathrm{tr}(\mathbf{C}_m))\mathbf{C}_m : \mathbf{D}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s. \tag{3.51}$$

The integrand of $P_{2,m}$ is bounded in at least in $L^r(0,T;L^r(\Omega))$ for $\frac{1}{r} = \frac{3}{4} + \frac{1}{6}$, which yields $r = 12/11 > 1$, see. (3.48). Since the integrand is continuous and convergent a.e. in $\Omega \times (0,T)$, cf. (3.48), by the Vitali lemma, i.e., Lemma A.3.5, (3.51) converges to its limit P_2. Further, we consider

$$P_{3,m} := \int_0^T \int_\Omega \Phi(\mathrm{tr}(\mathbf{C}_m))\mathrm{tr}(\mathbf{D})\varphi(s) \,\mathrm{d}x \,\mathrm{d}s.$$

Here the same reason, as for $P_{2,m}$ applies with a better, r and therefore via the Vitali lemma we pass to the limit.

Let us consider the next term which is the convective term of the conformation tensor equation we have the following estimate

$$\begin{aligned} P_{4,m} &:= \int_0^T \int_\Omega \Big[\mathbf{C}_m(\mathbf{u}_m \cdot \nabla) - \mathbf{C}(\mathbf{u} \cdot \nabla) \Big] : \mathbf{D}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s \\ &= \int_0^T \int_\Omega \Big[\mathbf{C}_m((\mathbf{u}_m - \mathbf{u}) \cdot \nabla) + (\mathbf{C}_m - \mathbf{C})(\mathbf{u} \cdot \nabla) \Big] : \mathbf{D}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s \\ &\leq \|\mathbf{D}\|_1 \|\varphi\|_{0,\infty} \int_0^T \|\mathbf{C}_m\|_{0,6} \|\mathbf{u} - \mathbf{u}_m\|_{0,3} + \|\mathbf{u}\|_{0,6} \|\mathbf{C} - \mathbf{C}_m\|_{0,3} \,\mathrm{d}s \\ &\leq \|\mathbf{D}\|_1 \|\varphi\|_{0,\infty} (\|\mathbf{C}_m\|_{L^2(L^6)} \|\mathbf{u} - \mathbf{u}_m\|_{L^2(L^3)} + \|\mathbf{u}\|_{L^2(L^6)} \|\mathbf{C} - \mathbf{C}_m\|_{L^2(L^3)}). \end{aligned}$$

Again $P_{4,m}$ goes to zero, as $m \to \infty$ due to the a priori bounds (3.40) and the strong convergence of $\mathbf{u}_m, \mathbf{C}_m$ in $L^2(0,T;L^3(\Omega))$, cf. (3.48).

As the last term, we consider the upper convective derivative

$$P_{5,m} := \int_0^T \int_\Omega \Big[(\nabla \mathbf{u}_m)\mathbf{C}_m - (\nabla \mathbf{u})\mathbf{C} + \mathbf{C}_m(\nabla \mathbf{u}_m)^T - \mathbf{C}(\nabla \mathbf{u})^T \Big] : \mathbf{D}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s$$

$$= \int_0^T \int_\Omega \Big[(\nabla \mathbf{u}_m - \nabla \mathbf{u})\mathbf{C}_m + \nabla \mathbf{u}(\mathbf{C}_m - \mathbf{C}) \\ + \mathbf{C}_m(\nabla \mathbf{u}_m - \nabla \mathbf{u})^T + (\mathbf{C}_m - \mathbf{C})(\nabla \mathbf{u})^T \Big] : \mathbf{D}\varphi(s) \,\mathrm{d}x \,\mathrm{d}s.$$

Thanks to the strong convergences of $\mathbf{C}_m$ in $L^2(0,T;L^4(\Omega))$ and the weak convergence of $\nabla \mathbf{u}_m$ in $L^2(0,T;L^2(\Omega))$, cf. (3.48), $P_{3,m} \to 0$, as $m \to \infty$.

The terms not considered here are linear and therefore treated by the corresponding weak convergences, cf. (3.48). Hence, we have passed to the limit in all terms of the weak formulation, which yields the existence of a global-in-time weak solution.

Lemma 3.4.7. *Let* $(\mathbf{u}_m, \mathbf{C}_m)$ *be a solution of the approximative system* (3.39)*. Furthermore, let the a priori bounds, i.e.,* (3.40)*,* (3.47)*, and the obtained convergence results, i.e., Lemma 3.4.5 and Lemma 3.4.6, hold. Then we can pass to the limit* $m \to \infty$*, i.e.,* (3.49) *converges to* (3.50) *and show existence of a global-in-time weak solution* $(\mathbf{u}, \mathbf{C})$ *which satisfies the Definition 3.3.1.*

Indeed, similarly to the first submodel, we obtain using the Lions-Aubin lemma, cf. Lemma A.3.6, that the weak solution $(\mathbf{u}, \mathbf{C})$ is continuous in time and initial data is attained in the weak sense via

$$\langle \mathbf{u}_0 - \mathbf{u}(0), \mathbf{v} \rangle = 0, \ \forall \mathbf{v} \in H^1_{\mathrm{div}}(\Omega)^d, \qquad \langle \mathbf{C}_0 - \mathbf{C}(0), \mathbf{D} \rangle = 0, \ \forall \mathbf{D} \in H^1(\Omega)^{d\times d}. \quad (3.52)$$

Lemma 3.4.8. *The weak solution* $(\mathbf{u}, \mathbf{C})$ *is continuous in time, i.e.,*

$$\mathbf{u} \in C([0,T]; H^{-1}_{div}(\Omega)^d), \qquad \mathbf{C} \in C([0,T]; H^{-1}(\Omega)^{d\times d}). \quad (3.53)$$

Furthermore, the initial data $(\mathbf{u}_0, \mathbf{C}_0)$ *is attained, i.e.,* $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}_0, \mathbf{C}_0)$*, in the weak sense* (3.52)*.*

3.4.6. Energy-dissipation and positive definiteness of the conformation tensor

In this subsection, we will consider the limiting process in the energy inequalities. Indeed, this will show that the weak solution we constructed is a dissipative weak solution. For the energy inequality, i.e., (3.26), this follows the same lines as for the CHNSQ model, cf. Subsection 3.2.6. Furthermore, for the limit in the total energy inequality (3.25), we need to prove that the limiting conformation tensor $\mathbf{C}$ is positive definite almost everywhere in $\Omega \times (0,T)$.

Energy inequality:
First, we consider the limit in the discrete version of (3.26). We observe that due to the convergence given by (3.48) we can apply the same techniques as for System S.4, see Subsection 3.2.5. We obtain immediately the following result.

Lemma 3.4.9. *The constructed weak solution* $(\mathbf{u}, \mathbf{C})$ *satisfies for almost every* $t \in (0,T)$ *the following inequality*

$$E(\mathbf{u}, \mathbf{C})(t) + \int_{\Omega_t} \eta |\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2} |\nabla \mathrm{tr}(\mathbf{C})|^2 + \frac{1}{2}\left(|\mathrm{tr}(\mathbf{C})|^4 + a|\mathrm{tr}(\mathbf{C})|^3 \right) \,\mathrm{d}x \,\mathrm{d}s \\ \leq \int_{\Omega_t} \frac{1}{2}(|\mathrm{tr}(\mathbf{C})|^2 + a\mathrm{tr}(\mathbf{C})) \,\mathrm{d}x \,\mathrm{d}s + E(\mathbf{u}, \mathbf{C})(0). \quad (3.54)$$

Proof. Considering the discrete energy inequality (3.26), we use the the generalized lemma of Fatou/lower-semi continuity of norms, (3.19), to pass to the limit and almost all terms. For the remaining integral over time and space on the right-hand side, we use the strong convergence of $\mathbf{C}_m$, cf. (3.48), to pass the limit in the first integral on the right-hand side of (3.54). The limit for the special case of two space dimensions follows from the same arguments. □

Positive definiteness:
In what follows, we want to prove a similar limit for the discrete version of (2.16). Here we follow the ideas in [15, 14, 16, 13]. In order to identify the limit correctly, we first need to prove the positive definiteness of the limit $\mathbf{C}$, since all approximations $\mathbf{C}_m$ are positive definite by construction. Recalling (3.41) we already have

$$\int_{\Omega_T} \operatorname{tr}(\mathbf{C}_m^{-1}) \, \mathrm{d}x \, \mathrm{d}s \leq c(a), \tag{3.55}$$

where the constant $c(a)$ depends inversely on a, i.e., it blows up for $a \to 0$. Estimate (3.55) implies, by using the positive definiteness of $\mathbf{C}_m$ the following estimates

$$\int_{\Omega_T} \left|\mathbf{C}_m^{-1}\right| \mathrm{d}x \, \mathrm{d}s, \quad \int_{\Omega_T} \operatorname{tr}(\mathbf{C}_m^{-1}) \, \mathrm{d}x \, \mathrm{d}s \leq c(a). \tag{3.56}$$

With these estimates at hand, we can prove the following lemma by contradiction.

Lemma 3.4.10. *Let $a > 0$ and the estimates (3.56) hold. Further, let $\mathbf{C}$ be the limit of the sequence of positive definite solutions $\mathbf{C}_m$ of (3.45). Then the limit $\mathbf{C}$ is positive definite a.e. in $\Omega \times [0,T)$. If $a = 0$ we can conclude positive semi-definiteness of the limit solution $\mathbf{C}$ a.e. in $\Omega \times (0,T)$.*

Proof. Assume the existence of a set D of non-zero measure with $D \subset \Omega \times (0,T)$ such that $\mathbf{C}$ is not positive definite. By construction $\mathbf{C}$ is the limit of positive definite sequence $\mathbf{C}_m$ which yields that $\mathbf{C}$ is positive semi-definite, due to the strong convergence of $\mathbf{C}_m$ in $L^2(0,T;L^2(\Omega)^{d\times d})$, cf. (3.48). This implies that $\mathbf{C}$ has at least one zero eigenvalue in D, i.e., there exists a vector function $\mathbf{v} \in L^\infty(0,T;L^\infty(\Omega))^d$ such that $|\mathbf{v}| = 1$ in D and zero else, such that $\mathbf{v}^T\mathbf{C}\mathbf{v} = 0$ a.e. in $\Omega \times (0,T) := \Omega_T$. We estimate the measure of D by

$$\begin{aligned}
|D| = \int_D |\mathbf{v}| \, \mathrm{d}x \, \mathrm{d}s = \int_{\Omega_T} |\mathbf{v}| \, \mathrm{d}x \, \mathrm{d}s &= \int_{\Omega_T} \left|\mathbf{C}_m^{-1/2}\mathbf{C}_m^{1/2}\mathbf{v}\right| \mathrm{d}x \, \mathrm{d}s \\
&\leq \left(\int_{\Omega_T} \left|\mathbf{C}_m^{-1/2}\right|^2 \mathrm{d}x \, \mathrm{d}s\right)^{\frac{1}{2}} \left(\int_{\Omega_T} \left|\mathbf{C}_m^{1/2}\mathbf{v}\right|^2 \mathrm{d}x \, \mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\Omega_T} \left|\mathbf{C}_m^{-1}\right| \mathrm{d}x \, \mathrm{d}s\right)^{\frac{1}{2}} \left(\int_{\Omega_T} \left|\mathbf{v}^T\mathbf{C}_m\mathbf{v}\right| \mathrm{d}x \, \mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq c(a) \left(\int_{\Omega_T} \left|\mathbf{v}^T\mathbf{C}_m\mathbf{v}\right| \mathrm{d}x \, \mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.57}$$

We can see that if $a > 0$ then $c(a)$ is bounded and the right side of the inequality (3.57) converges as $m \to \infty$ since $\mathbf{C}_m$ is converging strongly to $\mathbf{C}$ in $L^2(0,T;L^2(\Omega)^{d\times d})$, cf. Lemma 3.4.6. However, by assumptions $\mathbf{v}^T\mathbf{C}\mathbf{v} = 0$ a.e. in $\Omega \times (0,T)$, i.e., we find $|D| = 0$, which is a contradiction and yields that $\mathbf{C}$ is positive definite a.e.. □

Total energy inequality:
In the case of $a > 0$ we can proceed to pass to the limit in the discrete version of the total energy inequality (3.34). To pass to the limit in the total energy inequality, we rewrite the functionals of (3.41) into

$$\operatorname{tr}(\mathbf{C}_m^{-1}) = \operatorname{tr}(g_1 \circ \mathbf{C}_m), \quad \operatorname{tr}(\mathbf{C}_m)\operatorname{tr}(\mathbf{C}_m^{-1}) = g_2 \circ \mathbf{C}_m, \quad \operatorname{tr}(\log \mathbf{C}_m) = \operatorname{tr}(g_3 \circ \mathbf{C}_m), \tag{3.58}$$

where $g_i : (0, \infty) \to \mathbb{R}, i = 1, \dots, 3$ are continuous functions. Since $\mathbf{C}_m, \mathbf{C}$ are symmetric positive definite a.e. in $\Omega \times (0, T)$ and converge for a.e. (x, t) in $\Omega \times (0, T)$, cf. (3.48), it follows from [63, Exercise 2.37] that

$$g_1 \circ \mathbf{C}_m \longrightarrow g_1 \circ \mathbf{C}, \quad g_2 \circ \mathbf{C}_m \longrightarrow g_2 \circ \mathbf{C}, \quad g_3 \circ \mathbf{C}_m \longrightarrow g_3 \circ \mathbf{C} \text{ a.e. in } \Omega \times (0, T). \tag{3.59}$$

Using the convergence of (3.59) on $(3.41)_2$ we can conclude that $\overline{\operatorname{tr}(\log \mathbf{C}_m)} = \operatorname{tr}(\log \mathbf{C})$ a.e. in $\Omega \times (0, T)$, i.e.,

$$\operatorname{tr}(\log \mathbf{C}_m) \rightharpoonup \operatorname{tr}(\log \mathbf{C}) \in L^2(0, T; H^1(\Omega)). \tag{3.60}$$

For the bounds of $(3.41)_1$ we consider the following. Since $\mathbf{C}_m$ is positive definite a.e. in $\Omega \times (0, T)$ we conclude by virtue of $\mathbf{C}_m \mathbf{C}_m^{-1} = \mathbf{I}$ that $\mathbf{C}_m^{-1}$ is symmetric positive definite a.e. in $\Omega \times (0, T)$. Consequently, we obtain $\operatorname{tr}(\mathbf{C}_m^{-1}) > 0$ a.e. in $\Omega \times (0, T)$. Application of the generalized lemma of Fatou/lower-semi continuity of norms, see [55, 58, 57], yields

$$\int_{\Omega_t} \operatorname{tr}(\mathbf{C}^{-1}) \, \mathrm{d}x \, \mathrm{d}s = \int_{\Omega_t} \overline{\operatorname{tr}(\mathbf{C}_m^{-1})} \, \mathrm{d}x \, \mathrm{d}s \le \liminf_{m \to \infty} \int_{\Omega_t} \operatorname{tr}(\mathbf{C}_m^{-1}) \, \mathrm{d}x \, \mathrm{d}s, \tag{3.61}$$

where we used again $\overline{\operatorname{tr}(\mathbf{C}_m^{-1})} = \operatorname{tr}(\mathbf{C}^{-1})$ a.e. in $\Omega \times (0, T)$. Similarly, we can find

$$\int_{\Omega_t} \operatorname{tr}(\mathbf{C}^{-1})\operatorname{tr}(\mathbf{C}) \, \mathrm{d}x \, \mathrm{d}s = \int_{\Omega_t} \overline{\operatorname{tr}(\mathbf{C}_m^{-1})\operatorname{tr}(\mathbf{C}_m)} \, \mathrm{d}x \, \mathrm{d}s \le \liminf_{m \to \infty} \int_{\Omega_t} \operatorname{tr}(\mathbf{C}_m^{-1})\operatorname{tr}(\mathbf{C}_m) \, \mathrm{d}x \, \mathrm{d}s. \tag{3.62}$$

The last functional to be treated is given by

$$\int_{\Omega_t} \left| \mathbf{C}_m^{-1/2} \nabla \mathbf{C}_m \mathbf{C}_m^{-1/2} \right|^2 \, \mathrm{d}x \, \mathrm{d}s.$$

The limit is obtained in [18] with the same estimates, and we omit it.

With the convergences in (3.48), (3.60), (3.61), (3.62) we can pass to the limit in the total energy inequality and obtain summarize the result in the following lemma.

Lemma 3.4.11. *Let $(\mathbf{u}, \mathbf{C})$ be a weak solution constructed from Lemma 3.4.7. Furthermore, let $a > 0$ and $\mathbf{T} = \operatorname{tr}(\mathbf{C})\mathbf{C}$. Then for almost all $t \in (0, T)$ the following inequality holds*

$$\begin{aligned} E_{total}(\mathbf{u}, \mathbf{C})(t) + \int_{\Omega_t} \eta |\mathrm{D}\mathbf{u}|^2 + \frac{\varepsilon_2}{2} |\nabla \operatorname{tr}(\mathbf{C})|^2 + \frac{\varepsilon_2}{2} \left| \mathbf{C}^{-1/2} \nabla \mathbf{C} \mathbf{C}^{-1/2} \right|^2 \, \mathrm{d}x \, \mathrm{d}s \\ + \int_{\Omega_t} \frac{1}{2} \chi(\operatorname{tr}(\mathbf{C})) \operatorname{tr}(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}) \, \mathrm{d}x \, \mathrm{d}s \le E_{total}(\mathbf{u}, \mathbf{C})(0). \end{aligned} \tag{3.63}$$

Hence, the weak solutions are dissipative weak solutions.

3.5. Weak solutions for System S.3

We now consider the full model, i.e., System S.3 and introduce a suitable concept of weak solution. In the Sections 3.2 and 3.4 we proved existence of global-in-time dissipative weak solutions via variational arguments based on suitable energy inequalities. In principle, the full system exhibits the same a priori bounds from the energy. Hence, we will deduce existence of suitable solutions by combination of both proofs.

In order to state the definition of weak solution, recall the appropriate definitions for the simplified model, i.e., Definition 3.1.1 and Definition 3.3.1, and the abbreviations for the solutions spaces, i.e., $\mathbb{W}(0,T), \mathbb{Q}(0,T), \mathbb{M}(0,T), \mathbb{X}(0,T), \mathbb{T}(0,T)$, cf. (3.1), (3.20).

Definition 3.5.1. Let (A0)–(A7) hold and for given $T > 0$ the quintuple $(\phi, \mu, q, \mathbf{u}, \mathbf{C})$ is called a weak solution of System S.3, if $\phi \in \mathbb{W}(0,T)$, $\mu \in \mathbb{Q}(0,T)$, $q \in \mathbb{M}(0,T)$, $\mathbf{u} \in \mathbb{X}(0,T)$, $\mathbf{C} \in \mathbb{T}(0,T)$ and

$$
\begin{aligned}
\langle \partial_t \phi, \psi \rangle - \mathbf{c}(\mathbf{u}; \psi, \phi) + \langle (1+\varepsilon_0) b(\phi) \nabla \mu - b^{1/2}(\phi) \nabla (A(\phi) q), \nabla \psi \rangle &= 0 \\
\langle \mu, \xi \rangle - \gamma \langle \nabla \phi, \nabla \xi \rangle - \langle f'(\phi), \xi \rangle &= 0 \\
\langle \partial_t q, \zeta \rangle - \mathbf{c}(\mathbf{u}; \zeta, q) + \langle \kappa_1(\phi) q, \zeta \rangle + \varepsilon_1 \langle \nabla q, \nabla \zeta \rangle & \\
+ \langle \nabla (A(\phi) q) - b^{1/2}(\phi) \nabla \mu, \nabla (A(\phi) \zeta) \rangle &= 0 \qquad (3.64) \\
\langle \partial_t \mathbf{u}, \mathbf{v} \rangle - \mathbf{c}(\mathbf{u}; \mathbf{v}, \mathbf{u}) + \langle \eta(\phi) \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v} \rangle + \langle \mathrm{tr}(\mathbf{C}) \mathbf{C}, \nabla \mathbf{v} \rangle + \mathbf{c}(\mathbf{v}; \mu, \phi) &= 0 \\
\langle \partial_t \mathbf{C}, \mathbf{D} \rangle - \mathbf{c}(\mathbf{u}; \mathbf{D}, \mathbf{C}) - \langle (\nabla \mathbf{u}) \mathbf{C} + \mathbf{C} (\nabla \mathbf{u})^\top, \mathbf{D} \rangle + \varepsilon_1 \langle \nabla \mathbf{C}, \nabla \mathbf{D} \rangle & \\
= -\langle \kappa_2(\phi) \chi(\mathrm{tr}(\mathbf{C})) \mathbf{C}, \mathbf{D} \rangle + \langle \kappa_2(\phi) \Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\mathbf{D}) \rangle &,
\end{aligned}
$$

holds for any test function $(\psi, \xi, \zeta, \mathbf{v}, \mathbf{D}) \in [H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)^d \times H^1_S(\Omega)^{d \times d}]$ and almost every $t \in (0,T)$.

With this definition we can state the main result of this section.

Theorem 3.5.2 (Full model S.3)**.** *Let the initial data* $(\phi_0, q_0, \mathbf{u}_0, \mathbf{C}_0) \in H^1(\Omega) \times L^2(\Omega) \times L^2_{div}(\Omega)^d \times L^2_{SPD}(\Omega)^{d \times d}$ *be given. Under assumptions (A0)–(A7) and for* $d \in \{2,3\}$ *with* $\mathbf{C}_0$ *such that* $\mathrm{tr}(\ln(\mathbf{C}_0)) \in L^1(\Omega)$ *and for every* $T > 0$ *there exists a dissipative global-in-time weak solution of the viscoelastic phase separation model, i.e., System S.3, in the sense of Definition 3.5.1 which satisfies the initial data, i.e.,* $(\phi(0), q(0), \mathbf{u}(0), \mathbf{C}(0)) = (\phi_0, q_0, \mathbf{u}_0, \mathbf{C}_0)$ *and furthermore satisfies the energy inequality*

$$
\begin{aligned}
E(\phi, q, \mathbf{u}, \mathbf{C})(t) &+ \int_{\Omega_t} \varepsilon_0 |b^{1/2}(\phi) \nabla \mu|^2 + |b^{1/2}(\phi) \nabla \mu - \nabla (A(\phi) q)|^2 + \varepsilon_1 |\nabla q|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \int_{\Omega_t} |\kappa_1^{1/2}(\phi) q|^2 + \left| \eta^{1/2}(\phi) \mathrm{D}\mathbf{u} \right|^2 + \frac{\varepsilon_2}{2} |\nabla \mathbf{C}|^2 + \frac{1}{2} |\kappa_2^{1/2}(\phi) \chi(\mathrm{tr}(\mathbf{C}))^{1/2} \mathbf{C}|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&\le E(\phi, q, \mathbf{u}, \mathbf{C})(0) + \frac{1}{2} \int_{\Omega_t} h_2(\phi) \chi(\mathrm{tr}(\mathbf{C})) \, \mathrm{d}x \, \mathrm{d}s. \qquad (3.65)
\end{aligned}
$$

Furthermore, if $a > 0$ *the weak solution enjoys the additional regularity* (3.24) *and* *the integrated total energy inequality*

$$
\begin{aligned}
E_{total}(\phi, q, \mathbf{u}, \mathbf{C})(t) &+ \int_{\Omega_t} \varepsilon_0 |b^{1/2}(\phi)\nabla\mu|^2 + |b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_{\Omega_t} \varepsilon_1 |\nabla q|^2 + |\kappa_1^{1/2}(\phi)q|^2 + \left|\eta(\phi)^{1/2}\mathrm{D}\mathbf{u}\right|^2 + \frac{\varepsilon_2}{2}\left|\mathbf{C}^{-1/2}\nabla\mathbf{C}\mathbf{C}^{-1/2}\right|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_{\Omega_t} \frac{\varepsilon_2}{2}|\nabla \mathrm{tr}(\mathbf{C})|^2 + \frac{1}{2}h(\phi)\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{T} + \mathbf{T}^{-1} - 2\mathbf{I}) \,\mathrm{d}x\,\mathrm{d}s \qquad (3.66) \\
&\le E_{total}(\phi, q, \mathbf{u}, \mathbf{C})(0)
\end{aligned}
$$

holds for almost all $t \in (0, T)$*. Here again in two space dimensions and additionally the integrated inequality involving* E_{2d}*, i.e.,* (2.13) *holds.*

Proof. The proof of the above existence result follows almost verbatim by combination of the proofs for Theorem 3.1.3 and Theorem 3.3.4, respectively. For completeness, we summarize the main steps.

- Approximation of the system via Galerkin approximation based on orthogonal eigenfunctions for $(\phi, \mu, q, \mathbf{u})$. While $\mathbf{C}$ is resolved as the solution of the parabolic problem. Local existence holds, and via bootstrapping and energy arguments existence of a global approximation follows. As before, the approximation $\mathbf{C}_m$ is smooth and therefore symmetric positive definite.
- Reproducing suitable energy inequalities to obtain the a priori bounds from the energy-dissipative structure. Afterwards, we need to obtain higher regularity for ϕ_m, cf. Subsection 3.2.2. Finally, we have to deduce bounds on the time derivatives. The crucial point is that we obtain the same a priori bounds as for the simplified models, hence nothing new has to be estimated. This holds since by (A4) $\kappa_2(\phi)$ is bounded from above and below by a positive constant.
- Passage to the limits in the formulation and the energy inequalities. This follows immediately since the only new terms are the $\kappa_2(\phi)$ appearing in the equation of the conformation tensor $\mathbf{C}$. However, since we have suitable strong convergence for ϕ, this yields no difficulty, and we can pass to the limit.

□

With this we have proven the main results of this chapter, i.e., the existence of dissipative global-in-time weak solutions for System S.3 and the two simplified system, i.e., System S.4 and System S.5. In the last section of this chapter, we will discuss space-time formulations, which is a preparatory step for the next chapter.

3.6. Space-Time formulation

Before concluding this chapter, let us briefly discuss space-time weak formulations. As we have already seen in the proofs is that we have actually constructed weak solutions for space-time formulations, where we used $L^\infty(0, T, H^1(\Omega))$ or $L^\infty(0, T; H^1_{\mathrm{div}}(\Omega)^d)$ test functions. This allowed us to convert between a space-time weak formulation and the point-wise, almost everywhere in time, space formulation. However, in the following chapter we will indeed use space-time weak formulation, which employ test functions from $L^p(0,T,H^1(\Omega))$ and $L^p(0,T,H^1_{\text{div}}(\Omega)^d)$.

We can simply use density arguments to relax the regularity assumptions on the test functions. In general, this amounts to test functions such that the dual pairing with the time derivative makes sense.

But, if we would like to exchange the convective form, cf. (A.32)-(A.34), $\mathbf{c}(\mathbf{u},\psi,g)$ to $-\mathbf{c}(\mathbf{u},g,\psi)$, i.e., using partial integration to transfer the gradient from ψ to g, for $g\in\{\phi,q,\mathbf{u},\mathbf{C}\}$ we have to make sure that $\mathbf{c}(\mathbf{u},g,\psi)$ remains bounded and meaningful. Note that in the case of the Cahn-Hilliard equation, due to the high regularity of $\phi\in\ L^2(0,T;H^3(\Omega))$, there is no additional regularity needed. However, for the remaining variables, we have to require $(\zeta,\mathbf{v},\mathbf{D})\in L^4(0,T;H^1(\Omega)\times H^1_{\text{div}}(\Omega)^d\times H^1_S(\Omega)^{d\times d})$. Note that these assumptions could be relaxed depending on the space dimension.

In the following, we present the space-time formulation, which we will use in the following, and remark that following the above discussion all existence results so far remain true.

$$\int_0^t \langle\partial_t\phi,\psi\rangle-\mathbf{c}(\mathbf{u};\psi,\phi)+\langle(1+\varepsilon_0)b(\phi)\nabla\mu-b^{1/2}(\phi)\nabla(A(\phi)q),\nabla\psi\rangle\,\mathrm{d}s=0 \tag{3.67}$$

$$\int_0^t \langle\mu,\xi\rangle-\gamma\langle\nabla\phi,\nabla\xi\rangle-\langle f'(\phi),\xi\rangle\,\mathrm{d}s=0 \tag{3.68}$$

$$\begin{aligned}\int_0^t &\langle\partial_t q,\zeta\rangle+\tilde{\mathbf{c}}(\mathbf{u};q,\zeta)+\langle\kappa_1(\phi)q,\zeta\rangle+\varepsilon_1\langle\nabla q,\nabla\zeta\rangle\\ &+\langle\nabla(A(\phi)q)-n(\phi)\nabla\mu,\nabla(A(\phi)\zeta)\rangle\,\mathrm{d}s=0\end{aligned} \tag{3.69}$$

$$\int_0^t \langle\partial_t\mathbf{u},\mathbf{v}\rangle+\tilde{\mathbf{c}}(\mathbf{u};\mathbf{u},\mathbf{v})+\langle\eta(\phi)\mathrm{D}\mathbf{u},\mathrm{D}\mathbf{v}\rangle+\langle\mathrm{tr}(\mathbf{C})\mathbf{C},\nabla\mathbf{v}\rangle+\mathbf{c}(\mathbf{v};\mu,\phi)\,\mathrm{d}s=0 \tag{3.70}$$

$$\begin{aligned}\int_0^t &\langle\partial_t\mathbf{C},\mathbf{D}\rangle+\tilde{\mathbf{c}}(\mathbf{u};\mathbf{C},\mathbf{D})-\langle(\nabla\mathbf{u})\mathbf{C}+\mathbf{C}(\nabla\mathbf{u})^\top,\mathbf{D}\rangle+\varepsilon_2\langle\nabla\mathbf{C},\nabla\mathbf{D}\rangle\,\mathrm{d}s\\ &=-\int_0^t\langle\kappa_2(\phi)\chi(\mathrm{tr}(\mathbf{C}))\mathbf{C},\mathbf{D}\rangle-\langle\kappa_2(\phi)\Phi(\mathrm{tr}(\mathbf{C})),\mathrm{tr}(\mathbf{D})\rangle\,\mathrm{d}s,\end{aligned} \tag{3.71}$$

for all $\psi,\xi\in L^2(0,T;H^1(\Omega))$, $(\zeta,\mathbf{v},\mathbf{D})\in L^4(0,T;H^{-1}(\Omega)\times H^1_{\text{div}}(\Omega)^d\times H^1_S(\Omega)^{d\times d})$. Here $\tilde{\mathbf{c}}$ denotes another skew-symmetric formulation of $\mathbf{c}$ which is given by

$$\tilde{\mathbf{c}}(\mathbf{u};q,\zeta):=\frac{1}{2}\mathbf{c}(\mathbf{u};q,\zeta)-\frac{1}{2}\mathbf{c}(\mathbf{u};\zeta,q), \tag{3.72}$$

$$\tilde{\mathbf{c}}(\mathbf{u};\mathbf{u},\mathbf{v}):=\frac{1}{2}\mathbf{c}(\mathbf{u};\mathbf{u},\mathbf{v})-\frac{1}{2}\mathbf{c}(\mathbf{u};\mathbf{v},\mathbf{u}), \tag{3.73}$$

$$\tilde{\mathbf{c}}(\mathbf{u};\mathbf{C},\mathbf{D}):=\frac{1}{2}\mathbf{c}(\mathbf{u};\mathbf{C},\mathbf{D})-\frac{1}{2}\mathbf{c}(\mathbf{u};\mathbf{D},\mathbf{C}). \tag{3.74}$$

On the continuous level, there is no difference between $\mathbf{c}$ and $\tilde{\mathbf{c}}$. However, it turns out that $\tilde{\mathbf{c}}$ is skew-symmetric even for non-divergence free velocities $\mathbf{u}$, which will be useful in the second part of the thesis.

Note that due to our regularity assumptions, we have relaxed the conditions on the test functions for μ, i.e. $\xi\in L^2(0,T;H^{-1}(\Omega))$, by using suitable dual paring.

Proposition 3.6.1. *The existence results for the full model, i.e., Theorem 3.5.2 holds also for the space-time weak formulation* (3.67)-(3.71)*.*

Peterlin model:
In the case of the Peterlin model, which will be in the following chapter only considered in two space dimensions, we can make the following further manipulations. We observe that for all symmetric test functions $\mathbf{D} \in H^1_S(\Omega)^{d\times d}$ we have the equivalence

$$\langle (\nabla \mathbf{u})\mathbf{C} + \mathbf{C}(\nabla \mathbf{u})^\top, \mathbf{D}\rangle = 2\langle \nabla \mathbf{u}\mathbf{C}, \mathbf{D}\rangle.$$

At last, we note that using (A.12) in two space dimensions we can rewrite

$$\langle \mathrm{tr}(\mathbf{C})\mathbf{C}, \nabla \mathbf{v}\rangle = \langle \mathbf{C}^2, \nabla \mathbf{v}\rangle, \quad \forall \mathbf{v} \in H^1_{\mathrm{div}}(\Omega)^2.$$

In the following chapter for considerations involving the Peterlin submodel we will therefore work with the formulation

$$\int_0^t \langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \eta\langle \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v}\rangle + \langle \mathbf{C}^2, \nabla \mathbf{v}\rangle \,\mathrm{d}s = 0 \tag{3.75}$$

$$\int_0^t \langle \partial_t \mathbf{C}, \mathbf{D}\rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{C}, \mathbf{D}) - 2\langle (\nabla \mathbf{u})\mathbf{C}, \mathbf{D}\rangle + \varepsilon_1\langle \nabla \mathbf{C}, \nabla \mathbf{D}\rangle \,\mathrm{d}s \tag{3.76}$$

$$= -\int_0^t \langle \chi(\mathrm{tr}(\mathbf{C}))\mathbf{C}, \mathbf{D}\rangle + \langle \Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\mathbf{D})\rangle \,\mathrm{d}s.$$

for all $(\mathbf{v}, \mathbf{D}) \in L^4(0, T; H^1_{\mathrm{div}}(\Omega)^2 \times H^1_S(\Omega)^{2\times 2})$.

Proposition 3.6.2. *The existence results for the Peterlin model in two space dimensions, i.e., Theorem 3.3.4 holds also for the space-time weak formulation* (3.75)-(3.76)*.*

4 Relative energy, stability estimates and weak-strong uniqueness

In the previous chapter, we have observed that the energy functionals (3.65) and (3.66) are crucial for the construction of weak solutions. In this chapter, we will use suitable "distances" based on the energy functional as a tool to study the stability and uniqueness of weak solutions. This will be realized by the so-called *relative energy method* together with a perturbed system approach. Let us shortly sketch the main ideas. If $E(z)$ is a strictly convex energy functional, then a possible relative energy $\mathcal{E}(z|\hat{z})$ is given by

$$\mathcal{E}(z|\hat{z}) := E(z) - E(\hat{z}) - \langle E'(\hat{z}), z - \hat{z} \rangle, \tag{4.1}$$

see also [43, 44]. The convexity of the energy implies several relevant and practical properties of this relative energy. The goal is to derive an inequality for the time evolution of the relative energy, similarly to the energy inequalities, using Gronwall arguments. Instead of a more regular weak solution of our model $\hat{z}$ we will consider a perturbed system in order to derive our results. Finally, this will allow us to control the difference between the weak and the perturbed solutions, roughly only by the initial data and suitable residuals. This will be the abstract stability result. Afterwards, we apply the abstract stability result to study the dependence on initial data. By identifying the residual in this case, we obtain the *weak-strong uniqueness* principle. For simplicity, we will assume that the perturbed solution $\hat{z}$ is sufficiently regular such that every manipulation involving $\hat{z}$ is allowed. We emphasise at this point, that we adopted the name weak-strong uniqueness, despite the fact, that the result is obtained with less regular solutions. The results of this chapter are inspired by our recent preprint on weak-strong uniqueness for the full viscoelastic phase separation model in two space dimensions, see [33]. However, we adapt our recently developed residual approach, see [29], to simplify the whole proof.

Structure of the chapter:

1. In the first section, cf. Section 4.1, we will construct the relative energy for the full model and the related submodels.

2. In Section 4.2 we will introduce a suitable perturbed system for the full model, which will be necessary to obtain the main results of this chapter.

3. Before stating the first results, we will briefly discuss the main idea of the proof in Section 4.3
4. In Section 4.4 we will formulate the abstract stability and weak-strong uniqueness results for the CHNSQ model. The proof of the stability result will be given in Section 4.5, while the proof for the weak-strong uniqueness principle is contained in Section 4.6.
5. In Section 4.7 we will formulate the abstract stability and weak-strong uniqueness results for the Peterlin model in two space dimensions. The proof of the stability result will be given in Section 4.8, while the proof for the weak-strong uniqueness principle is contained in Section 4.9.
6. In Section 4.10 we will formulate the abstract stability and weak-strong uniqueness results for the full model, in two space dimensions. The proof of the stability result follows almost verbatim by the proofs in Section 4.5 and Section 4.5. The necessary adjustments for the proof of the weak-strong uniqueness principle are contained in Section 4.11.
7. Discussion on possible conditional extensions to three space dimensions, convergence to equilibrium and the degenerate problem can be found in Section 4.12.
8. Finally, we will summarize the result of Part I in Section 4.13.

4.1. The relative energy method

We start with the construction and properties of a suitable relative energy. First we recall the energy functional (2.13) for the full model, i.e., System S.3 in two space dimensions as

$$E_{2d}(\phi, q, \mathbf{u}, \mathbf{C}) := \int_\Omega \frac{\gamma}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}q^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{4}|\mathbf{C}|^2 \,\mathrm{d}x. \tag{4.2}$$

For simplicity this will now be denoted by E, instead of E_{2d}. Since most of the energetic contributions are L^2-norms of the corresponding variables, we will briefly derive the relative energy for those contributions as follows

$$\begin{aligned}\mathcal{E}(z|\hat{z}) &= \frac{1}{2}\|z\|_0^2 - \frac{1}{2}\|\hat{z}\|_0^2 - \langle \hat{z}, z - \hat{z}\rangle \\ &= \frac{1}{2}\|z\|_0^2 + \frac{1}{2}\|\hat{z}\|_0^2 - \langle \hat{z}, z\rangle = \frac{1}{2}\|z - \hat{z}\|_0^2.\end{aligned}$$

The above calculations imply that using (4.1) we can introduce the following relative energy functionals

$$\mathcal{E}(q|\hat{q}) := \frac{1}{2}\|q - \hat{q}\|_0^2, \quad \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}) := \frac{1}{2}\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2, \quad \mathcal{E}(\mathbf{C}|\hat{\mathbf{C}}) := \frac{1}{4}\|\mathbf{C} - \hat{\mathbf{C}}\|_0^2.$$

For the mixing energy, we observe that this does not fit into this framework, since f is non-convex. Recalling (A2) we observe that f is λ-convex, i.e., $f(s) + \lambda s^2$ is convex for suitable λ. Therefore, we introduce a penalized relative energy by

$$\mathcal{E}_\alpha(\phi|\hat{\phi}) := \int_\Omega \frac{\gamma}{2}|\nabla(\phi - \hat{\phi})|^2 + f(\phi) - f(\hat{\phi}) - f'(\hat{\phi})(\phi - \hat{\phi}) + \frac{\alpha}{2}|\phi - \hat{\phi}|^2 \,\mathrm{d}x$$

$$= E_{mix,\alpha}(\phi) - E_{mix,\alpha}(\hat{\phi}) - \langle E'_{mix,\alpha}(\hat{\phi}), \phi - \hat{\phi}\rangle. \tag{4.3}$$

In fact, we observe the penalized relative energy corresponds to the relative energy of the convexified energy functional for the mixing energy given by

$$E_{mix,\alpha} := E_{mix}(\phi) + \frac{\alpha}{2}\|\phi\|_0^2.$$

Before stepping forward, we will have a brief look what is a suitable range for α. Using Taylor expansion, we find

$$f(\phi) - f(\hat{\phi}) - f'(\hat{\phi})(\phi - \hat{\phi}) + \frac{\alpha}{2}|\phi - \hat{\phi}|^2 = \frac{f''(\bar{\xi}) + \alpha}{2}|\phi - \hat{\phi}|^2 = (*),$$

where $\bar{\xi}$ is an intermediate value between ϕ and $\hat{\phi}$. Recalling assumption (A2) we can estimate further with the lower bound $-f_1$ for f'' by

$$(*) \geq \frac{\alpha - f_1}{2}|\phi - \hat{\phi}|^2.$$

The above computation shows that $\alpha \geq f_1 > 0$ is enough to obtain a non-negative distance. Hence, we formulate an additional assumption.

(A8) The penalty constant $\alpha \in \mathbb{R}$ is chosen such that $\alpha > f_1 > 0$.

With this assumption, together with the gradient part, the relative energy includes a relative H^1-norm. In total, the full relative energy is given by

$$\mathcal{E}_\alpha(\phi, q, \mathbf{u}, \mathbf{C}|\hat{\phi}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}}) = \mathcal{E}_\alpha(\phi|\hat{\phi}) + \mathcal{E}(q|\hat{q}) + \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}) + \mathcal{E}(\mathbf{C}|\hat{\mathbf{C}}). \tag{4.4}$$

Let us introduce the ordered function set $z = (\phi, q, \mathbf{u}, \mathbf{C})$ and correspondingly $\hat{z} = (\hat{\phi}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}})$ and define the metric space $\mathcal{M}$ such that $z \in \mathcal{M}$ if $\{\phi \in H^1(\Omega), q \in L^2(\Omega), \mathbf{u} \in L^2(\Omega), \mathbf{C} \in L^2(\Omega)\}$. Due to the convexity of the generating functionals, we automatically have the following properties.

Lemma 4.1.1. *Let $z, \hat{z} \in \mathcal{M}$. Then the relative energy $\mathcal{E}_\alpha(z|\hat{z})$ has the following properties*

- $\mathcal{E}_\alpha(z|\hat{z}) \geq 0$ *and* $\mathcal{E}_\alpha(z|\hat{z}) = 0 \Leftrightarrow z = \hat{z}$.
- $\|z_i - \hat{z}_i\|_0^2 \leq C\mathcal{E}_\alpha(z|\hat{z})$ *for* $i = 1, 4$ *and* $\|\phi - \hat{\phi}\|_1^2 \leq C(\gamma, \alpha)\mathcal{E}_\alpha(z|\hat{z})$.

Although the first condition is typical for a metric, the relative energy, formally a Bregman distance, is not a metric. In fact, due to the potential f we lose symmetry and triangle inequality, though all other contributions are metrics on normed spaces.

Similar to the existence proofs in Chapter 3, we will prove the result for the CHNSQ model and the Peterlin model separately and obtain again a full result by combination. Therefore, we introduce the reduced relative energies associated with the CHNSQ model, i.e., System S.4 and Peterlin model, i.e., System S.5, respectively by

$$\mathcal{E}_{1,\alpha}(\phi, q, \mathbf{u}|\hat{\phi}, \hat{q}, \hat{\mathbf{u}}) := \mathcal{E}_\alpha(\phi|\hat{\phi}) + \mathcal{E}(q|\hat{q}) + \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}), \tag{4.5}$$

$$\mathcal{E}_2(\mathbf{u}, \mathbf{C}|\hat{\mathbf{u}}, \hat{\mathbf{C}}) := \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}) + \mathcal{E}(\mathbf{C}|\hat{\mathbf{C}}). \tag{4.6}$$

A similar result as in Lemma 4.1.1 holds for the reduced relative energies. Because the relevant relative energies and properties have been introduced, we can move to the introduction of a perturbed system.

4.2. Perturbed system

In this section, we will introduce a suitable perturbed system of equations for $(\hat{z}, \hat{\mu})$. In principle the idea is that for sufficiently smooth functions, here $(\hat{z}, \hat{\mu})$, by inserting into the variational formulation of the problem, we can interpret these functions as solutions of a perturbed variational formulation.

Let $\hat{z} = (\hat{\phi}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}})$ and $\hat{\mu}$ be given functions, which are sufficiently smooth. Then we define the corresponding residuals $\hat{r}_i, i = 1, \dots, 5$ via the perturbed system

$$\int_0^t \langle \partial_t \hat{\phi}, \psi \rangle - \mathbf{c}(\hat{\mathbf{u}}; \psi, \phi) + \langle \varepsilon_0 b(\phi) \nabla \hat{\mu}, \nabla \psi \rangle \tag{4.7}$$

$$+ \langle b(\phi) \nabla \hat{\mu} - b^{1/2}(\phi) \nabla (A(\phi) \hat{q}), \nabla \psi \rangle \, \mathrm{d}s = \int_0^t \langle \hat{r}_1, \psi \rangle \, \mathrm{d}s, \tag{4.8}$$

$$\int_0^t \langle \hat{\mu}, \xi \rangle - \gamma \langle \nabla \hat{\phi}, \nabla \xi \rangle - \langle f'(\hat{\phi}), \xi \rangle \, \mathrm{d}s = \int_0^t \langle \hat{r}_2, \xi \rangle \, \mathrm{d}s, \tag{4.9}$$

$$\int_0^t \langle \partial_t \hat{q}, \zeta \rangle - \tilde{\mathbf{c}}(\mathbf{u}; \zeta, \hat{q}) + \langle \kappa_1(\phi) \hat{q}, \zeta \rangle + \varepsilon_1 \langle \nabla \hat{q}, \nabla \zeta \rangle \tag{4.10}$$

$$+ \langle \nabla (A(\phi) \hat{q}) - b^{1/2}(\phi) \nabla \hat{\mu}, \nabla (A(\phi) \zeta) \rangle \, \mathrm{d}s = \int_0^t \langle \hat{r}_3, \zeta \rangle \, \mathrm{d}s, \tag{4.11}$$

$$\int_0^t \langle \partial_t \hat{\mathbf{u}}, \mathbf{v} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{v}) + \langle \eta(\phi) \mathrm{D} \hat{\mathbf{u}}, \mathrm{D} \mathbf{v} \rangle \tag{4.12}$$

$$+ \langle \hat{\mathbf{C}} \mathbf{C}, \nabla \mathbf{v} \rangle + \mathbf{c}(\mathbf{v}; \hat{\mu}, \phi) \, \mathrm{d}s = \int_0^t \langle \hat{r}_4, \mathbf{v} \rangle \, \mathrm{d}s, \tag{4.13}$$

$$\int_0^t \langle \partial_t \hat{\mathbf{C}}, \mathbf{D} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{C}}, \mathbf{D}) - 2 \langle (\nabla \hat{\mathbf{u}}) \mathbf{C}, \mathbf{D} \rangle + \varepsilon \langle \nabla \hat{\mathbf{C}}, \nabla \mathbf{D} \rangle \tag{4.14}$$

$$+ \langle \kappa_2(\phi) \chi(\mathrm{tr}(\mathbf{C})) \hat{\mathbf{C}}, \mathbf{D} \rangle - \langle \kappa_2(\phi) \Phi(\mathrm{tr}(\hat{\mathbf{C}})), \mathrm{tr}(\mathbf{D}) \rangle \, \mathrm{d}s = \int_0^t \langle \hat{r}_5, \mathbf{D} \rangle \, \mathrm{d}s, \tag{4.15}$$

for all $(\psi, \xi, \zeta, \mathbf{v}, \mathbf{D}) \in L^2(0, T; H^1(\Omega) \times H^{-1}(\Omega) \times H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)^d \times H^1_S(\Omega)^{d \times d})$ and for almost all $t \in (0, T)$. By testing with $\psi = \hat{\mu} - r_2, \xi = \partial_t \hat{\phi}, \zeta = \hat{q}, \mathbf{v} = \hat{\mathbf{u}}, \mathbf{D} = \hat{\mathbf{C}}$, we obtain the following energetic law

$$E(\hat{z})(t) - E(\hat{z})(0) = - \int_0^t D_{\phi, \mathbf{C}}(\hat{\mu}, \hat{z}) + \mathcal{R}(\hat{\mu}, \hat{z}) \, \mathrm{d}s, \tag{4.16}$$

$$\begin{aligned} D_{\phi, \mathbf{C}}(\hat{\mu}, \hat{z}) =& \varepsilon_0 \| b^{1/2}(\phi) \nabla \hat{\mu} \|_0^2 + \| b^{1/2}(\phi) \nabla \hat{\mu} - \nabla (A(\phi) \hat{q}) \|_0^2 + \| \kappa_1^{1/2}(\phi) \hat{q} \|_0^2 + \varepsilon_1 \| \nabla \hat{q} \|_0^2 \\ &+ \| \eta^{1/2}(\phi) \mathrm{D} \hat{\mathbf{u}} \|_0^2 + \frac{\varepsilon_2}{2} \| \nabla \hat{\mathbf{C}} \|_0^2 + \frac{1}{2} \| \kappa_2^{1/2}(\phi) \chi^{1/2}(\mathrm{tr}(\mathbf{C})) \hat{\mathbf{C}} \|_0^2 \\ &- \frac{1}{2} \langle \kappa_2(\phi) \Phi(\mathrm{tr}(\hat{\mathbf{C}})), \mathrm{tr}(\hat{\mathbf{C}}) \rangle, \\ \mathcal{R}(\hat{\mu}, \hat{z}) =& - \langle r_1, \hat{\mu} \rangle + \langle r_2, \partial_t \hat{\phi} \rangle - \langle r_3, \hat{q} \rangle - \langle r_4, \hat{\mathbf{u}} \rangle - \frac{1}{2} \langle r_5, \hat{\mathbf{C}} \rangle. \end{aligned}$$

Since we want to work on the reduced models, i.e., the CHNSQ model, cf. System S.4 and the Peterlin model, cf. System S.5, we need suitable perturbed systems for both sub systems. However, it is clear that we can easily decompose the perturbed system in analogy to Chapter 2.

Remark 4.2.1. Let us mention that the explicit appearance of a weak solution, i.e., $\phi, \mathbf{u}, \mathbf{C}$ in the perturbed system, can be considered as some kind of linearisation around the weak solution. This choice will allow us to deduce the right dissipation contributions directly, without the introduction of many suitable additions of zero. Furthermore, it reveals some sort of structure, since an analogous linearisation of the weak solution around arbitrary functions does not lead to the right result. Furthermore, if one chooses the variables consistent, the energetic structure is not harmed. We will later see how one can take advantage of this structural property.

4.3. General proof strategies

Before we step into the concrete proofs, let us discuss the main strategy. If the weak solution (z, μ) is smooth enough, we can simply calculate the time derivative of the relative energy. We will do this for a general relative energy and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z|\hat{z}) &= \langle E'(z), \partial_t z\rangle - \langle E'(\hat{z}), \partial_t \hat{z}\rangle - \langle E'(\hat{z}), \partial_t z - \partial_t \hat{z}\rangle - \langle E''(\hat{z})\partial_t \hat{z}, z - \hat{z}\rangle \\ &= \langle E'(z) - E'(\hat{z}), \partial_t z - \partial_t \hat{z}\rangle + \langle E'(z) - E'(\hat{z}) - E''(\hat{z})(z - \hat{z}), \partial_t \hat{z}\rangle. \end{aligned} \tag{4.17}$$

From this point, one would expand the above computation by insert suitable test functions into the corresponding variational formulations and deduce the relative dissipation and the terms to estimate. We note that with such regularity, one can compute the time evolution for every equation separately and combine them at the end. Furthermore, for quadratic energies, the second inner product just vanishes.

However, in our case, we are not allowed to test the weak solution with itself in general, which the above computation would require. Therefore, we try to solve this problem. Let us first present the following abstract lemma.

Lemma 4.3.1. *Let $H(g)$ be a quadratic energy, i.e., $H(g) = \|g\|_0^2$, and let $g, \hat{g}$ given in the following spaces $g \in L^2(0,T;H^1(\Omega)) \cap W^{1,p}(0,T;H^{-1}(\Omega))$, $\hat{g} \in W^{1,q}(0,T;H^1(\Omega)) \cap L^2(0,T;H^{-1}(\Omega))$, with $p^{-1} + q^{-1} = 1, p \leq 2$. Then the following holds*

$$\langle H'(\hat{g}), g - \hat{g}\rangle(t) = \langle H'(\hat{g}), g - \hat{g}\rangle(0) + \int_0^t \langle H'(\hat{g}), \partial_t g - \partial_t \hat{g}\rangle + \langle H''(\hat{g})(g - \hat{g}), \partial_t \hat{g}\rangle \,\mathrm{d}s.$$

Since ϕ is smooth enough, this can be extended to the mixing potential contribution $\int_\Omega f(\phi)$. Also, a similar formula, using suitable dual pairings, can be derived for the quadratic gradient part.

Proof. The proof follows the lines of Lemma 4.1 in [53] and uses standard approximation and density arguments. □

The above lemma will allow us to circumvent the problems arising in the calculations needed to prove (4.17). We will compute this for an abstract relative energy by assuming that there is a suitable energy inequality for z and assume that there is a suitable energetic law like in (4.16), i.e., D, R are defined. We start expanding an abstract relative energy

$$\begin{aligned}\mathcal{E}(z(t)|\hat{z}(t)) &= E(z(t)) - E(\hat{z}(t)) - \langle E'(\hat{z}), z - \hat{z}\rangle(t) \\ &\leq E(z(0)) - E(\hat{z}(0)) - \langle E'(\hat{z}), z - \hat{z}\rangle(0) - \int_0^t D(z) - D(\hat{z}) - \mathcal{R}(\hat{z}) \,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t \langle E'(\hat{z}), \partial_t z - \partial_t \hat{z}\rangle + \langle E''(\hat{z})(z-\hat{z}), \partial_t \hat{z}\rangle \,\mathrm{d}s \\
&\leq \mathcal{E}(z(0)|\hat{z}(0)) - \int_0^t D(z) - D(\hat{z}) - \mathcal{R}(\hat{z}) \,\mathrm{d}s \\
&- \int_0^t \langle E'(\hat{z}), \partial_t z - \partial_t \hat{z}\rangle + \langle E''(\hat{z})(z-\hat{z}), \partial_t \hat{z}\rangle \,\mathrm{d}s.
\end{aligned}
$$

Here we have employed the energy inequality of the dissipative weak solution (3.3) or (3.22), the energetic law of the perturbed solution (4.16) and Lemma 4.3.1. By adding $\pm 2E(\hat{z})$ we can equivalently find

$$
\mathcal{E}(z(t)|\hat{z}(t)) \leq \mathcal{E}(z(0)|\hat{z}(0)) - \int_0^t D(z) + D(\hat{z}) + \mathcal{R}(\hat{z}) \,\mathrm{d}s \tag{4.18}
$$

$$
- \int_0^t \langle E'(\hat{z}), \partial_t z + \partial_t \hat{z}\rangle + \langle E''(\hat{z})(z-\hat{z}), \partial_t \hat{z}\rangle \,\mathrm{d}s, \tag{4.19}
$$

where only the sign change in front of $\hat{D}(\hat{z}), \mathcal{R}(\hat{z})$ and $\partial_t \hat{z}$ in the first inner-product changed. Before we proceed, we recall that in the case of a quadratic energy $E(z) = \frac{1}{2}\|z\|_0^2$ the remainder can be reduced to

$$
\int_0^t \langle E'(\hat{z}), \partial_t z + \partial_t \hat{z}\rangle + \langle E''(\hat{z})(z-\hat{z}), \partial_t \hat{z}\rangle \,\mathrm{d}s = \int_0^t \langle z, \partial_t \hat{z}\rangle + \langle \hat{z}, \partial_t z\rangle \,\mathrm{d}s.
$$

This construction imposes several regularity assumptions on the perturbed solution. Indeed, the perturbed solution must be so regular that it can be taken as a test function in the weak formulation of the original problem and that we can take the weak solution as a test function in the perturbed formulation. Furthermore, we require the perturbed solution so smooth that (4.16) is valid.

In the next section, we will state the stability and weak-strong uniqueness result for the CHNSQ model, i.e., System S.4 which will be derived via the relative energy method.

4.4. Stability estimate & weak-strong uniqueness for System S.4

In this section, we will state the main results on abstract stability and weak-strong uniqueness for the CHNSQ model in two and three space dimensions. The main difficulty arises from dealing with the non-convex energetic structure of the Cahn-Hilliard equation and its cross-diffusive coupling to the bulk stress equation. To this end, we recall the relevant relative energy is here given by (4.5), i.e.,

$$
\begin{aligned}
\mathcal{E}_{1,\alpha}(\phi, q, \mathbf{u}|\hat{\phi}, \hat{q}, \hat{\mathbf{u}}) &= \mathcal{E}_\alpha(\phi|\hat{\phi}) + \mathcal{E}(q|\hat{q}) + \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}) \\
&= \frac{\gamma}{2}\|\nabla(\phi - \hat{\phi})\|_0^2 + \int_\Omega f(\phi) - f(\hat{\phi}) - f'(\hat{\phi})(\phi - \hat{\phi}) \,\mathrm{d}x \\
&+ \frac{\alpha}{2}\|\phi - \hat{\phi}\|_0^2 + \frac{1}{2}\|q - \hat{q}\|_0^2 + \frac{1}{2}\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2.
\end{aligned}
$$

Let us state the main result of this section.

Theorem 4.4.1. *Let (z, μ) with $z = (\phi, q, \mathbf{u})$ be a dissipative weak solution of System S.4 for $d \in \{2, 3\}$ in the sense of Definition 3.1.1. Furthermore, let the set of sufficiently smooth functions $\hat{z} = (\hat{\phi}, \hat{q}, \hat{\mathbf{u}})$ and $\hat{\mu}$, cf. Remark 4.5.1 for the exact regularity requirements, yield the associated residuals $\hat{r}_i$ given by* (4.7)-(4.12)*. Then the following holds*

$$\mathcal{E}_{1,\alpha}(z(t)|\hat{z}(t)) + \int_0^t \mathcal{D}_{1,\phi}(\mu, z|\hat{\mu}, \hat{z})\,\mathrm{d}s \le C_0 \mathcal{E}_{1,\alpha}(z(0)|\hat{z}(0)) \tag{4.20}$$

$$+ C_1 \int_0^t \|r_1\|_{-1}^2 + \|r_2\|_1^2 + \|r_3\|_{-1}^2 + \|r_4\|_{-1}^2\,\mathrm{d}s, \tag{4.21}$$

where $\mathcal{D}_{1,\phi}$ denotes the relative dissipation functional given by

$$\mathcal{D}_{1,\phi}(\mu, z|\hat{\mu}, \hat{z}) = \frac{1}{2}\Big(\|b^{1/2}(\phi)\nabla(\mu - \hat{\mu}) - \nabla(A(\phi)(q - \hat{q}))\|_0^2 + \varepsilon_0\|b^{1/2}(\phi)\nabla(\mu - \hat{\mu})\|_0^2$$
$$+ \|\kappa_1^{1/2}(\phi)(q - \hat{q})\|_0^2 + \varepsilon_2\|\nabla(q - \hat{q})\|_0^2 + \|\eta^{1/2}(\phi)(\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}})\|_0^2\Big).$$

The proof of the theorem will be given in Section 4.5 and the proof of the following lemma will be given in Section 4.6.

As an application of this stability result, we study the stability with respect to the initial data. From this, we can identify the so-called weak-strong uniqueness principle, given by the following corollary.

Corollary 4.4.2. *Let $(\hat{z}, \hat{\mu})$ with $\hat{z} = (\hat{\phi}, \hat{q}, \hat{\mathbf{u}})$ be a more regular dissipative weak solution of System S.4, cf. Remark 4.6.2 for the necessary regularity, existing up to time $T^\dagger \le T$, such that Theorem 4.4.1 holds. Then every dissipative weak solution (z, μ) with $z = (\phi, q, \mathbf{u})$ of System S.4 in the sense of Definition 3.1.1 starting from the same initial data as $(\hat{z}, \hat{\mu})$ coincides with $(\hat{z}, \hat{\mu})$, i.e. $z(t) = \hat{z}(t)$, for almost all $t \in (0, T^\dagger)$ and $\mu \equiv \hat{\mu}$.*

It will become clear from the proof that the necessary regularity can be reduced by either using better inequalities or considering only two space dimensions. Indeed, in two space dimensions, it might be possible to prove a higher regularity estimate for the weak solution itself.

4.5. Relative energy estimates for the CHNSQ model

Let us start with the proof of Theorem 4.4.1. For convenience, we recall the variational identities of the weak solution and the associated perturbed problem. The variational identities of the weak solution are given by

$$\int_0^t \langle \partial_t \phi, \psi \rangle - \mathbf{c}(\mathbf{u}; \psi, \phi) + \langle (1 + \varepsilon_0) b(\phi) \nabla\mu - b^{1/2}(\phi)\nabla(A(\phi)q), \nabla\psi \rangle\,\mathrm{d}s = 0 \tag{4.22}$$

$$\int_0^t \langle \mu, \xi \rangle - \gamma \langle \nabla \phi, \nabla \xi \rangle - \langle f'(\phi), \xi \rangle \,\mathrm{d}s = 0 \tag{4.23}$$

$$\int_0^t \langle \partial_t q, \zeta \rangle + \tilde{\mathbf{c}}(\mathbf{u}; q, \zeta) + \langle \kappa_1(\phi) q, \zeta \rangle + \varepsilon_1 \langle \nabla q, \nabla \zeta \rangle \tag{4.24}$$
$$+ \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\zeta) \rangle \,\mathrm{d}s = 0$$

$$\int_0^t \langle \partial_t \mathbf{u}, \mathbf{v} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \langle \eta(\phi) \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v} \rangle + \mathbf{c}(\mathbf{v}; \mu, \phi) \,\mathrm{d}s = 0 \tag{4.25}$$

for all $(\psi, \xi) \in L^2(0,T; H^1(\Omega) \times H^{-1}(\Omega))$, $(\zeta, \mathbf{v}) \in L^4(0,T; H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)^d)$.
For the perturbed problem, we have the following variational identities

$$\int_0^t \langle \partial_t \hat{\phi}, \psi \rangle - \mathbf{c}(\hat{\mathbf{u}}; \psi, \phi) + \langle \varepsilon_0 b(\phi) \nabla \hat{\mu}, \nabla \psi \rangle \tag{4.26}$$

$$+ \langle b(\phi)\nabla\hat{\mu} - b^{1/2}(\phi)\nabla(A(\phi)\hat{q}), \nabla\psi \rangle \,\mathrm{d}s = \int_0^t \langle \hat{r}_1, \psi \rangle \,\mathrm{d}s, \tag{4.27}$$

$$\int_0^t \langle \hat{\mu}, \xi \rangle - \gamma \langle \nabla \hat{\phi}, \nabla \xi \rangle - \langle f'(\hat{\phi}), \xi \rangle \,\mathrm{d}s = \int_0^t \langle \hat{r}_2, \xi \rangle \,\mathrm{d}s, \tag{4.28}$$

$$\int_0^t \langle \partial_t \hat{q}, \zeta \rangle - \tilde{\mathbf{c}}(\mathbf{u}; \zeta, \hat{q}) + \langle \kappa_1(\phi) \hat{q}, \zeta \rangle + \varepsilon_1 \langle \nabla \hat{q}, \nabla \zeta \rangle \tag{4.29}$$

$$+ \langle \nabla(A(\phi)\hat{q}) - b^{1/2}(\phi)\nabla\hat{\mu}, \nabla(A(\phi)\zeta) \rangle \,\mathrm{d}s = \int_0^t \langle \hat{r}_3, \zeta \rangle \,\mathrm{d}s, \tag{4.30}$$

$$\int_0^t \langle \partial_t \hat{\mathbf{u}}, \mathbf{v} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{v}) + \langle \eta(\phi) \mathrm{D}\hat{\mathbf{u}}, \mathrm{D}\mathbf{v} \rangle + \mathbf{c}(\mathbf{v}; \hat{\mu}, \phi) \,\mathrm{d}s = \int_0^t \langle \hat{r}_4, \mathbf{v} \rangle \,\mathrm{d}s \tag{4.31}$$

for all $(\psi, \xi, \zeta, \mathbf{v}) \in L^2(0,T; H^1(\Omega) \times H^{-1}(\Omega) \times H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)^d)$ and for almost all $t \in (0,T)$. We will start applying the general strategy of the proof, i.e., using (4.19), cf. Section 4.3, to obtain

$$\begin{aligned} \mathcal{E}_{1,\alpha}(z|\hat{z})|_0^t = &- \int_0^t D_{1,\phi}(\mu, z) + D_{1,\phi}(\hat{\mu}, \hat{z}) + \mathcal{R}_1(\hat{\mu}, \hat{z}) \,\mathrm{d}s \\ &\underbrace{- \int_0^t \langle E_1'(\hat{z}), \partial_t z + \partial_t \hat{z} \rangle + \langle E_1''(\hat{z})(z - \hat{z}), \partial_t \hat{z} \rangle \,\mathrm{d}s}_{(*)}\,. \end{aligned} \tag{4.32}$$

We emphasize that the penalty term, since ϕ is sufficiently regular, can be treated by standard calculation, without this expansion. We recall that the dissipation of the weak solution and the "dissipation" of the perturbed solution is given by

$$\begin{aligned} D_{1,\phi}(\mu, z) =& \|b^{1/2}(\phi)\nabla\mu\|_0^2 + \|b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q)\|_0^2 \\ &+ \varepsilon_1 \|\nabla q\|_0^2 + \|\kappa_1^{1/2}(\phi) q\|_0^2 + \|\eta^{1/2}(\phi)\mathrm{D}\mathbf{u}\|_0^2, \\ D_{1,\phi}(\hat{\mu}, \hat{z}) + \mathcal{R}_1(\hat{\mu}, \hat{z}) =& \|b^{1/2}(\phi)\nabla\hat{\mu}\|_0^2 + \|b^{1/2}(\phi)\nabla\hat{\mu} - \nabla(A(\phi)\hat{q})\|_0^2 \\ &+ \varepsilon_1 \|\nabla \hat{q}\|_0^2 + \|\kappa_1^{1/2}(\phi)\hat{q}\|_0^2 + \|\eta^{1/2}(\phi)\mathrm{D}\hat{\mathbf{u}}\|_0^2 \\ &- \langle r_1, \hat{\mu} \rangle + \langle r_2, \partial_t \hat{\phi} \rangle - \langle r_3, \hat{q} \rangle - \langle r_3, \hat{\mathbf{u}} \rangle. \end{aligned} \tag{4.33}$$

Bulk stress equation:
We will start expanding the remainder $(*)$, cf. (4.32), equation-wise and start with the bulk stress contribution, which by testing $\zeta = \hat{q}$ in (4.24) and $\zeta = q$ in (4.29) yields

$$(*)_1 = - \int_0^t \langle q, \partial_t \hat{q} \rangle + \langle \hat{q}, \partial_t q \rangle \,\mathrm{d}s$$

$$= \int_0^t \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\hat{q})\rangle + \langle \nabla(A(\phi)\hat{q}) - b^{1/2}(\phi)\nabla\hat{\mu}, \nabla(A(\phi)q)\rangle$$
$$+ 2\varepsilon_1\langle\nabla q, \nabla\hat{q}\rangle + 2\langle\kappa_1(\phi)q, \hat{q}\rangle \underbrace{+\tilde{\mathbf{c}}(\mathbf{u}; \hat{q}, q) + \tilde{\mathbf{c}}(\mathbf{u}; q, \hat{q})}_{\mathcal{C}_3} + \langle r_3, q\rangle \, \mathrm{d}s.$$

Combination of $(*)_1$ with the corresponding dissipation terms, i.e., $(4.33)_{1,i}, (4.33)_{2,i}$ for $i = 3, 4$ and residual term $(4.33)_{2,8}$ yields

$$P_0 = -\int_0^t \varepsilon_1\|\nabla(q-\hat{q})\|_0^2 + \|\kappa_1^{1/2}(\phi)(q-\hat{q})\|_0^2 + \mathcal{C}_3 + \langle r_3, q-\hat{q}\rangle \tag{4.34}$$
$$+ \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\hat{q})\rangle + \langle \nabla(A(\phi)\hat{q}) - b^{1/2}(\phi)\nabla\hat{\mu}, \nabla(A(\phi)q)\rangle \, \mathrm{d}s.$$

Cahn-Hilliard equation:
Next, we turn to the Cahn-Hilliard part, and we expand

$$(*)_2 = -\int_0^t \gamma\langle\nabla\phi, \nabla\partial_t\hat{\phi}\rangle + \gamma\langle\nabla\hat{\phi}, \nabla\partial_t\phi\rangle + \langle f'(\hat{\phi}), \partial_t\phi + \partial_t\hat{\phi}\rangle$$
$$+ \langle f''(\hat{\phi})(\phi-\hat{\phi}), \partial_t\hat{\phi}\rangle - \alpha\langle\phi-\hat{\phi}, \partial_t\phi - \partial_t\hat{\phi}\rangle \, \mathrm{d}s$$
$$= -\int_0^t \gamma\langle\nabla\phi, \nabla\partial_t\hat{\phi}\rangle + \langle f'(\phi), \partial_t\hat{\phi}\rangle + \gamma\langle\nabla\hat{\phi}, \nabla\partial_t\phi\rangle + \langle f'(\hat{\phi}), \partial_t\phi\rangle$$
$$- \langle f'(\phi) - f'(\hat{\phi}) - f''(\hat{\phi})(\phi-\hat{\phi}), \partial_t\hat{\phi}\rangle - \alpha\langle\phi-\hat{\phi}, \partial_t\phi - \partial_t\hat{\phi}\rangle \, \mathrm{d}s$$

where we added $\pm\langle f'(\phi), \partial_t\hat{\phi}\rangle$. Inserting $\xi = \partial_t\hat{\phi}$ into (4.23) and $\xi = \partial_t\phi$ into (4.28) yields

$$(*)_2 = -\int_0^t \langle\mu, \partial_t\hat{\phi}\rangle + \langle(\mu - r_2), \partial_t\phi\rangle\rangle - \langle f'(\phi) - f'(\hat{\phi}) - f''(\hat{\phi})(\phi-\hat{\phi}), \partial_t\hat{\phi}\rangle$$
$$- \alpha\langle\phi-\hat{\phi}, \partial_t\phi - \partial_t\hat{\phi}\rangle \, \mathrm{d}s = (i) + (ii) + (iii) + (iv). \tag{4.35}$$

In the next step, we will insert suitable test functions into the variational identities to expand $(i) - (iv)$. Since there will be many terms, we treat them pairwise. Let us start with the first two terms by inserting $\psi = \hat{\mu} - r_2$ as test function into (4.22) and $\psi = \mu$ as test function into (4.26) we find

$$(i) + (ii) = -\int_0^t \langle\mu, \partial_t\hat{\phi}\rangle + \langle\hat{\mu} - r_2, \partial_t\phi\rangle \, \mathrm{d}s \tag{4.36}$$
$$= \int_0^t \langle b(\phi)\nabla\mu - b^{1/2}(\phi)\nabla(A(\phi)q), \nabla(\hat{\mu} - r_2)\rangle \tag{4.37}$$
$$+ \langle b(\phi)\nabla\hat{\mu} - b^{1/2}(\phi)\nabla(A(\phi)\hat{q}), \nabla\mu\rangle$$
$$+ 2\varepsilon_0\langle b(\phi)\nabla\mu, \nabla\hat{\mu}\rangle + \varepsilon_0\langle b(\phi)\nabla\mu, \nabla r_2\rangle \tag{4.38}$$
$$\underbrace{-\mathbf{c}(\mathbf{u}; \hat{\mu} - r_2, \phi) - \mathbf{c}(\hat{\mathbf{u}}; \mu, \phi)}_{\mathcal{C}_1} + \langle r_1, \mu\rangle - \langle r_2, \partial_t\phi\rangle \, \mathrm{d}s. \tag{4.39}$$

Combination of P_0, cf. (4.34) and $(i) + (ii)$ together with the dissipation terms $(4.33)_{1,i}, (4.33)_{2,i}$, $i = 1, 2$ and the residual terms $(4.33)_{2,i}$, $i = 6, 7$, yields

$$P_1 := \int_0^t -\varepsilon_1\|\nabla(q-\hat{q})\|_0^2 - (1-\delta)\|b^{1/2}(\phi)\nabla(\mu-\hat{\mu}) - \nabla(A(\phi)(q-\hat{q}))\|_0^2$$

$$- \|\kappa_1^{1/2}(\phi)(q-\hat{q})\|_0^2 - (1-\delta)\varepsilon_0\|b^{1/2}(\phi)\nabla(\mu-\hat{\mu})\|_0^2$$
$$+ \sum_{i=1}^{3}\mathcal{C}_i + \langle r_1, \hat{\mu}-\mu\rangle + \langle r_2, \partial_t\phi - \partial_t\hat{\phi}\rangle + \langle r_3, \hat{q}-q\rangle \,\mathrm{d}s. \tag{4.40}$$

We proceed by estimating the remaining terms related to the residual, and start with

$$\int_0^t \langle r_1, \hat{\mu}-\mu\rangle \,\mathrm{d}s \le C\int_0^t \|r_1\|_{-1}\|\hat{\mu}-\mu\|_1 \,\mathrm{d}s \tag{4.41}$$
$$\le C(\Omega)\int_0^t \|r_1\|_{-1}(|\langle\hat{\mu}-\mu,1\rangle| + \frac{b_1}{b_1}\|\nabla\hat{\mu}-\nabla\mu\|_0)\,\mathrm{d}s \tag{4.42}$$
$$\le C(\Omega)\int_0^t \|r_1\|_{-1}(|\langle\hat{\mu}-\mu,1\rangle| + C(b)\|b^{1/2}(\phi)\nabla(\hat{\mu}-\mu)\|_0)\,\mathrm{d}s. \tag{4.43}$$

The first term can be estimated by inserting into the variational identities, i.e., $\xi = 1$ in (4.23) and (4.28). Recalling (A3) yields

$$\begin{aligned}\langle\hat{\mu}-\mu,1\rangle &= \langle f'(\hat{\phi}) - f'(\phi), 1\rangle - \langle r_2, 1\rangle \\ &\le C(f_2^{(2)} + f_3^{(2)}(\|\phi\|_1 + \|\hat{\phi}\|_1))\|\hat{\phi}-\phi\|_1 + \|r_1\|_{0,1}, \\ |\langle\hat{\mu}-\mu,1\rangle|^2 &\le C(f)\|\hat{\phi}-\phi\|_1^2 + C\|r_2\|_1^2.\end{aligned}$$

To estimate the second residual, we insert $\psi = r_2$ in (4.22) and (4.26) to find

$$\begin{aligned}\int_0^t \langle r_2, \partial_t(\phi-\hat{\phi})\rangle \,\mathrm{d}s &= \int_0^t -2\langle b(\phi)\nabla(\mu-\hat{\mu}) - b^{1/2}(\phi)\nabla(A(\phi)(q-\hat{q})), \nabla r_2\rangle \\ &\quad - 2\varepsilon_0\langle b(\phi)\nabla(\mu-\hat{\mu}), \nabla r_2\rangle \underbrace{+\mathbf{c}(\mathbf{u}; r_2, \phi) - \mathbf{c}(\hat{\mathbf{u}}; r_2, \hat{\phi})}_{\mathcal{C}_4} - \langle r_1, r_2\rangle \,\mathrm{d}s \\ &\le \int_0^t \delta\|b^{1/2}(\phi)\nabla(\mu-\hat{\mu}) - \nabla(A(\phi)(q-\hat{q}))\|_0^2 + \delta\varepsilon_0\|b^{1/2}(\phi)\nabla(\mu-\hat{\mu})\|_0^2 \\ &\quad + C\|r_1\|_{-1}^2 + C(\delta, b_2, \varepsilon_0)\|r_2\|_1^2 + \mathcal{C}_4 \,\mathrm{d}s.\end{aligned}$$

For the third residual, we use Hölder's and Young's inequality to estimate

$$\int_0^t \langle r_3, q-\hat{q}\rangle \,\mathrm{d}s \le \int_0^t \delta\varepsilon_1\|\nabla(q-\hat{q})\|_0^2 + C(\varepsilon_1)\|q-\hat{q}\|_0^2 + C(\delta,\varepsilon_1)\|r_3\|_{-1}^2 \,\mathrm{d}s.$$

We now turn to the remaining integral in $(*)_2$, cf. (4.35), and recall, using (A3), that

$$|f'(\phi) - f'(\hat{\phi}) - f''(\hat{\phi})(\phi-\hat{\phi})| \le (f_2^{(3)} + f_2^{(3)}(|\phi| + |\hat{\phi}|))|\phi-\hat{\phi}|.$$

This allows us to estimate the third term by

$$(iii) \le \int_0^t \|\partial_t\hat{\phi}\|_0\|f'(\phi) - f'(\hat{\phi}) - f''(\hat{\phi})(\phi-\hat{\phi})\|_0 \,\mathrm{d}s \tag{4.44}$$
$$\le \int_0^t \|\partial_t\hat{\phi}\|_0(f_2^{(3)} + f_3^{(3)}(\|\phi\|_1 + \|\hat{\phi}\|_1))\|\phi-\hat{\phi}\|_1^2 \,\mathrm{d}s \tag{4.45}$$
$$\le C(f_2^{(3)}, f_3^{(3)}, \|\phi\|_{L^\infty(H^1)}, \|\hat{\phi}\|_{L^\infty(H^1)})\int_0^t \|\partial_t\hat{\phi}\|_0\mathcal{E}_\alpha(\phi|\hat{\phi})\,\mathrm{d}s \tag{4.46}$$
$$\le C(f)\int_0^t \|\partial_t\hat{\phi}\|_0\mathcal{E}_\alpha(\phi|\hat{\phi})\,\mathrm{d}s. \tag{4.47}$$

The norms of $\phi, \hat{\phi}$ in $C(f)$ are bounded, due to the regularity of the dissipative weak solution for ϕ and the regularity assumption for $\hat{\phi}$, cf. Remark 4.5.1. For the fourth term we obtain by inserting $\psi = \alpha(\phi - \hat{\phi})$ in (4.22) and (4.26)

$$\begin{aligned}
-(iv) =& \alpha \int_0^t \langle b(\phi)\nabla(\mu - \hat{\mu}) - b^{1/2}(\phi)\nabla(A(\phi)(q - \hat{q})), \nabla(\phi - \hat{\phi})\rangle \\
&+ \varepsilon_0 \langle b(\phi)\nabla(\mu - \hat{\mu}), \nabla(\phi - \hat{\phi})\rangle \underbrace{- \mathbf{c}(\mathbf{u} - \hat{\mathbf{u}}; \phi - \hat{\phi}, \phi)}_{\mathcal{C}_2} + \langle r_1, \phi - \hat{\phi}\rangle \, \mathrm{d}s \\
\leq& \int_0^t -\delta \| n(\phi)\nabla(\mu - \hat{\mu}) - \nabla(A(\phi)(q - \hat{q}))\|_0^2 - \delta\varepsilon_0 \| b^{1/2}(\phi)\nabla(\mu - \hat{\mu})\|_0^2 \\
&+ C(\delta, b_2, \varepsilon_0)\|\phi - \hat{\phi}\|_1^2 + \mathcal{C}_2 + C\|r_1\|_{-1}^2 \, \mathrm{d}s.
\end{aligned}$$

Summation of all the estimates yields the following inequality for the cross-diffusive part

$$\begin{aligned}
P_2 \leq& -(2 - 4\delta) \int_0^t \mathcal{D}_{1,\phi}(\mu, q|\hat{\mu}, \hat{q}) \, \mathrm{d}s \\
&+ \int_0^t C(f)\mathcal{E}_{1,\alpha}(z|\hat{z}) + \sum_{i=1}^4 \mathcal{C}_i + C(\delta)\|r_2\|_1^2 + C(\delta) \sum_{i\in\{1,3,4\}} \|r_i\|_{-1}^2 \, \mathrm{d}s.
\end{aligned} \tag{4.48}$$

Navier-Stokes equations:
Let us consider the Navier-Stokes part and recall (4.32). We obtain by inserting $\mathbf{v} = \hat{\mathbf{u}}$ in (4.25) and $\mathbf{v} = \mathbf{u}$ in (4.31)

$$\begin{aligned}
(*)_3 =& -\int_0^t \langle \mathbf{u}, \partial_t \hat{\mathbf{u}}\rangle + \langle \hat{\mathbf{u}}, \partial_t \mathbf{u}\rangle \, \mathrm{d}s \\
=& \int_0^t 2\langle \eta(\phi)\mathrm{D}\mathbf{u}, \mathrm{D}\hat{\mathbf{u}}\rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \hat{\mathbf{u}}) + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{u}) \\
&+ \mathbf{c}(\hat{\mathbf{u}}; \mu, \phi) + \mathbf{c}(\mathbf{u}; \hat{\mu}, \phi) + \langle r_4, \mathbf{u}\rangle \, \mathrm{d}s.
\end{aligned}$$

Combining $(*)_3$ with the remaining dissipation terms $(4.33)_{1,5}$, $(4.33)_{2,5}$, and the remaining residual $(4.33)_{2,9}$ yields

$$\begin{aligned}
P_3 :=& \int_0^t -\|\eta^{1/2}(\phi)(\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}})\|_0^2 + \langle r_4, \hat{\mathbf{u}} - \mathbf{u}\rangle \\
&+ \overbrace{\tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \hat{\mathbf{u}}) + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{u}) + \mathbf{c}(\hat{\mathbf{u}}; \mu, \phi) + \mathbf{c}(\mathbf{u}; \hat{\mu}, \phi)}^{\mathcal{C}_5} \, \mathrm{d}s \\
\leq& \int_0^t -(1 - \delta)\|\eta^{1/2}(\phi)(\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}})\|_0^2 + C(\delta, \eta_2)\|r_4\|_{-1}^2 + C(\delta, \eta_2)\mathcal{E}_{1,\alpha}(z|\hat{z}) + \mathcal{C}_5 \, \mathrm{d}s.
\end{aligned} \tag{4.49}$$

Combining all expansions of (4.32), i.e., (4.48) and (4.49) yields the inequality

$$\begin{aligned}
\mathcal{E}_{1,\alpha}(z|\hat{z})|_0^t + \int_0^t \mathcal{D}_{1,\phi}(\mu, z|\hat{\mu}, \hat{z}) \, \mathrm{d}s \leq& \int_0^t C_1(\delta, f)\mathcal{E}_{1,\alpha}(z|\hat{z}) + \sum_{i=1}^5 \mathcal{C}_i \\
&+ C_2(\delta)\|r_1\|_{-1}^2 + \|r_2\|_1^2 + \|r_3\|_{-1}^2 + \|r_4\|_{-1}^2 \, \mathrm{d}s.
\end{aligned}$$

Let us now consider the convective remainder

$$\int_0^t \sum_{i=1}^5 \mathcal{C}_i \, \mathrm{d}s = \int_0^t \underbrace{-\mathbf{c}(\mathbf{u}; \hat{\mu}, \phi) - \mathbf{c}(\hat{\mathbf{u}}; \mu, \phi)}_{\mathcal{C}_1} \underbrace{- \mathbf{c}(\mathbf{u} - \hat{\mathbf{u}}; \phi - \hat{\phi}, \phi)}_{\mathcal{C}_2}$$

$$\underbrace{+\tilde{\mathbf{c}}(\mathbf{u};q,\hat{q})+\tilde{\mathbf{c}}(\mathbf{u};\hat{q},q)}_{\mathcal{C}_3}\underbrace{+\mathbf{c}(\mathbf{u}-\hat{\mathbf{u}};r_2,\phi)}_{\mathcal{C}_2}$$
$$\underbrace{+\tilde{\mathbf{c}}(\mathbf{u};\mathbf{u},\hat{\mathbf{u}})+\tilde{\mathbf{c}}(\mathbf{u};\hat{\mathbf{u}},\mathbf{u})+\mathbf{c}(\hat{\mathbf{u}};\mu,\phi)+\mathbf{c}(\mathbf{u};\hat{\mu},\phi)}_{\mathcal{C}_5}\ \mathrm{d}s.$$

We observe that due to skew-symmetry of $\tilde{\mathbf{c}}$ the term $\mathcal{C}_3$ and the first part of $\mathcal{C}_5$ are zero, see (A.32) and (A.33). The term $\mathcal{C}_1$ cancels with the last part of $\mathcal{C}_5$. The remaining terms will be estimated via

$$\begin{aligned}
&\int_0^t \mathbf{c}(\mathbf{u}-\hat{\mathbf{u}};r_2,\phi)-\mathbf{c}(\mathbf{u}-\hat{\mathbf{u}};\phi-\hat{\phi},\phi)\ \mathrm{d}s\\
\leq &\int_0^t \|\mathbf{u}-\hat{\mathbf{u}}\|_{0,3}\|\nabla r_2\|_0\|\phi\|_{0,6}+\|\mathbf{u}-\hat{\mathbf{u}}\|_{0,3}\|\phi-\hat{\phi}\|_1\|\phi\|_{0,6}\ \mathrm{d}s\\
\leq &\int_0^t C(\delta,\|\phi\|_1)\mathcal{E}_{1,\alpha}(z|\hat{z})+C(\delta)\|r_2\|_1^2+2\delta\mathcal{D}_\phi(\mathbf{u}|\hat{\mathbf{u}})\ \mathrm{d}s,
\end{aligned}$$

where we applied the interpolation inequality (A.24). Finally, choosing δ small enough yields the desired stability estimate

$$\begin{aligned}
\mathcal{E}_{1,\alpha}(z|\hat{z})|_0^t+\int_0^t \mathcal{D}_{1,\phi}(\mu,z|\hat{\mu},\hat{z})\ \mathrm{d}s &\leq \int_0^t C_1(\phi,\hat{\phi})\mathcal{E}_{1,\alpha}(z|\hat{z})\ \mathrm{d}s\\
&+C_2\int_0^t \|r_1\|_{-1}^2+\|r_2\|_1^2+\|r_3\|_{-1}^2+\|r_4\|_{-1}^2\ \mathrm{d}s
\end{aligned}$$

where $C_1(\phi,\hat{\phi})=C(f_2^{(i)},f_3^{(i)},\|\phi\|_{L^\infty(H^1)},\|\hat{\phi}\|_{L^\infty(H^1)},\|\partial_t\hat{\phi}\|_0)$. In order to apply the Gronwall lemma, cf. Lemma A.3.1 we need $C_1\in L^1(0,T)$. Indeed, all necessary bounds involving the arbitrary function $\hat{\phi}$ are bounded by the regularity assumptions of the theorem. We only need to consider $\|\phi\|_{L^\infty(H^1)}$, which is bounded by the energy estimate, see Definition 3.1.1.

Remark 4.5.1. By inspection of the proof, one can see that we have used the following regularities for the perturbed solution $(\hat{z},\hat{\mu})$

$$\begin{aligned}
&\hat{\phi}\in\mathbb{W}(0,T)\cap W^{1,1}(0,T;L^2(\Omega)), \qquad \hat{\mu}\in\mathbb{Q}(0,T) &(4.50)\\
&\hat{q}\in\mathbb{M}(0,T)\cap L^4(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega)),\\
&\hat{\mathbf{u}}\in\mathbb{X}(0,T)\cap L^4(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega)),
\end{aligned}$$

see (3.1) for the definition of the above spaces.

In the next section, we prove the weak-strong uniqueness principle, i.e., Corollary 4.4.2, which can be obtained as an immediate consequence of the stability estimates. The main idea is that for more regular weak solutions the residuals r_i can be identified quite easily. Afterwards, we only have to bound them suitably by the relative energy and obtain the statement via another application of Gronwall lemma. The weak-strong uniqueness principle can be understood as stability with respect to the initial data.

4.6. Weak-strong uniqueness for the CHNSQ model

Assume that the set of functions $(\hat{\phi}, \hat{\mu}, \hat{q}, \hat{\mathbf{u}})$ is a more regular dissipative weak solution of (3.2), cf. Remark 4.6.2 for the necessary regularity. Then we can identify the residuals r_i by

$$
\begin{aligned}
\int_0^t \langle r_1, \psi\rangle \,\mathrm{d}s =& \int_0^t \langle (1+\varepsilon_0)(b(\phi)-b(\hat{\phi}))\nabla\hat{\mu}, \nabla\psi\rangle - \langle (b^{1/2}(\phi)-b^{1/2}(\hat{\phi}))\nabla(A(\phi)\hat{q}), \nabla\psi\rangle \\
&+ \langle b^{1/2}(\hat{\phi})\nabla((A(\phi)-A(\hat{\phi}))\hat{q}), \nabla\psi\rangle + \mathbf{c}(\hat{\mathbf{u}};\psi,\phi-\hat{\phi}) \,\mathrm{d}s, \\
\int_0^t \langle r_2, \xi\rangle \,\mathrm{d}s =& 0, \\
\int_0^t \langle r_3, \zeta\rangle \,\mathrm{d}s =& \int_0^t \langle (\kappa_1(\phi)-\kappa_1(\hat{\phi}))\hat{q}, \zeta\rangle + \langle \nabla((A(\phi)-A(\hat{\phi}))\hat{q}), \nabla(A(\phi)\zeta)\rangle \\
&+ \langle \nabla(A(\hat{\phi})\hat{q}) - b^{1/2}(\hat{\phi})\nabla\hat{\mu}, \nabla((A(\phi)-A(\hat{\phi}))\zeta)\rangle \\
&- \langle (b^{1/2}(\phi)-b^{1/2}(\hat{\phi}))\nabla\hat{\mu}, \nabla(A(\phi)\zeta)\rangle + \tilde{\mathbf{c}}(\mathbf{u}-\hat{\mathbf{u}};\hat{q},\zeta) \,\mathrm{d}s \\
\int_0^t \langle r_4, \mathbf{v}\rangle \,\mathrm{d}s =& \int_0^t \langle (\eta(\phi)-\eta(\hat{\phi}))\mathrm{D}\hat{\mathbf{u}}, \mathrm{D}\mathbf{v}\rangle + \tilde{\mathbf{c}}(\mathbf{u}-\hat{\mathbf{u}};\hat{\mathbf{u}},\mathbf{v}) + \mathbf{c}(\mathbf{v};\hat{\mu},\phi-\hat{\phi}) \,\mathrm{d}s.
\end{aligned}
$$

We now bound the residuals by relative energy and dissipation terms, such that the Gronwall lemma, cf. Lemma A.3.1 can be used. This gives rise to the next result.

Lemma 4.6.1. *Let $r_i, i = 1, \ldots, 4$ be defined as above. The following bounds for the residuals hold*

$$
\begin{aligned}
\int_0^t \|r_1\|_{-1}^2 \,\mathrm{d}s &\le \int_0^t C_1(z,\hat{z})\mathcal{E}_{1,\alpha}(z|\hat{z}) \,\mathrm{d}s, \\
\int_0^t \|r_3\|_{-1}^2 \,\mathrm{d}s &\le \int_0^t 2\delta\mathcal{D}_{1,\phi}(\mu,z|\hat{\mu},\hat{z}) + C_2(z,\hat{z},\delta)\mathcal{E}_{1,\alpha}(z|\hat{z}) \,\mathrm{d}s, \\
\int_0^t \|r_4\|_{-1}^2 \,\mathrm{d}s &\le \int_0^t 2\delta\mathcal{D}_{1,\phi}(\mu,z|\hat{\mu},\hat{z}) + C_3(\hat{\mathbf{u}},\hat{\mu},\delta)\mathcal{E}_{1,\alpha}(z|\hat{z}) \,\mathrm{d}s.
\end{aligned}
$$

Proof. We will estimate each residual step-by-step.

First residual:
We estimate the first residual by

$$
\begin{aligned}
|\langle r_1, \psi\rangle| \le& \|b(\phi)-b(\hat{\phi})\|_{0,6}\|\nabla\hat{\mu}\|_{1,3}\|\psi\|_1 + \|b^{1/2}(\phi)-b^{1/2}(\hat{\phi})\|_{0,6}\|\nabla(A(\phi)\hat{q})\|_{0,3}\|\psi\|_1 \\
&+ C(b_2)\|\nabla((A(\phi)-A(\hat{\phi}))\hat{q})\|_{0,2}\|\psi\|_1 + \|\hat{\mathbf{u}}\|_{0,3}\|\phi-\hat{\phi}\|_{0,6}\|\psi\|_1.
\end{aligned}
$$

Using the definition of the dual norm, we estimate

$$
\begin{aligned}
\int_0^t \|r_1\|_{-1}^2 \,\mathrm{d}s \le& \int_0^t b_3\|\phi-\hat{\phi}\|_{0,6}^2\|\nabla\hat{\mu}\|_{1,3}^2 + C(b_3)\|\phi-\hat{\phi}\|_{0,6}^2\|\nabla(A(\phi)\hat{q})\|_{0,3}^2 \\
&+ C(b_2)\|\nabla((A(\phi)-A(\hat{\phi}))\hat{q})\|_{0,2}^2 + \|\hat{\mathbf{u}}\|_{0,3}^2\|\phi-\hat{\phi}\|_{0,6}^2 \,\mathrm{d}s \\
\le& (i) + (ii) + (iii) + (iv).
\end{aligned}
$$

Let us expand the second term

$$
(ii) \le \int_0^t C\|\phi-\hat{\phi}\|_{0,6}^2\|\nabla(A(\phi)\hat{q})\|_{0,3}^2 \,\mathrm{d}s
$$

$$\begin{aligned}
&\leq \int_0^t C(A_2\|\nabla\hat{q}\|_{0,3}^2 + A_3\|\nabla\phi\|_{0,3}^2\|\hat{q}\|_{0,\infty}^2)\mathcal{E}_{1,\alpha}(z|\hat{z})\,\mathrm{d}s\\
&\leq C(A)\int_0^t (\|\nabla\hat{q}\|_{0,3}^2 + \|\nabla\phi\|_{0,3}^2\|\hat{q}\|_{0,\infty}^2)\mathcal{E}_{1,\alpha}(z|\hat{z})\,\mathrm{d}s.
\end{aligned}$$

Note that due to the regularity of the dissipative weak solution, cf. Definition 3.1.1, $\|\nabla\phi\|_{L^4(L^3)}$ is bounded, i.e., we require $\hat{q}\in L^\infty(0,T;L^\infty(\Omega))$.

For the third term, we expand the gradient and obtain

$$\nabla((A(\phi)-A(\hat{\phi}))\hat{q}) = (A(\phi)-A(\hat{\phi}))\nabla\hat{q} + \nabla(A(\phi)-A(\hat{\phi}))\hat{q}.$$

Using Taylor's theorem and the above expansion yields

$$\begin{aligned}
(iii) &\leq C(n)\int_0^t A_2\|\phi-\hat{\phi}\|_{0,6}^2\|\nabla\hat{q}\|_{0,3}^2 + A_3\|\phi-\hat{\phi}\|_1^2\|\hat{q}\|_{0,\infty}^2\\
&\qquad + A_4\|\phi+\hat{\phi}\|_{1,3}^2\|\phi-\hat{\phi}\|_{0,6}^2\|\hat{q}\|_{0,\infty}^2\,\mathrm{d}s\\
&\leq C(A)\int_0^t (\|\nabla\hat{q}\|_{0,3}^2 + \|\hat{q}\|_{0,\infty}^2 + \|\phi+\hat{\phi}\|_{1,3}^2\|\hat{q}\|_{0,\infty}^2 + \|\mathbf{u}\|_{0,3}^2)\mathcal{E}_{1,\alpha}(z|\hat{z})\,\mathrm{d}s.
\end{aligned}$$

Similar as for the second term, the bracket is integrable in at least $L^1(0,T^\dagger)$. For the fourth term we simply estimate

$$(iv) \leq \int_0^t \|\hat{\mathbf{u}}\|_{0,3}^2\mathcal{E}_{1,\alpha}(z|\hat{z})\,\mathrm{d}s.$$

All together, the first residual can be bounded by

$$\begin{aligned}
\int_0^t \|r_1\|_{-1}^2\,\mathrm{d}s &\leq C\int_0^t (\|\nabla\hat{\mu}\|_{1,3}^2 + \|\nabla\hat{q}\|_{0,3}^2 + \|\hat{q}\|_{0,\infty}^2 + \|\hat{q}\|_{0,\infty}^2\|\phi+\hat{\phi}\|_{1,3}^2)\mathcal{E}_{1,\alpha}(z|\hat{z})\,\mathrm{d}s\\
&\leq \int_0^t C_1(z,\hat{z})\mathcal{E}_1(z|\hat{z})\,\mathrm{d}s.
\end{aligned}$$

As already discussed C_1 is integrable in $L^1(0,T^\dagger)$.

Third residual:
Let us consider the third residual. We estimate

$$\begin{aligned}
|\langle r_3,\zeta\rangle| &\leq \|\kappa_1(\phi)-\kappa_1(\hat{\phi})\|_{0,6}\|\hat{q}\|_{0,3} + \|b^{1/2}(\phi)-b^{1/2}(\hat{\phi})\|_{0,6}\|\nabla\hat{\mu}\|_{0,3}\|\nabla(A(\phi)\zeta)\|_0\\
&\quad + \|\nabla((A(\phi)-A(\hat{\phi}))\hat{q})\|_0\|\nabla(A(\phi)\zeta)\|_0\\
&\quad + \|b^{1/2}(\hat{\phi})\nabla\hat{\mu} - \nabla(A(\hat{\phi})\hat{q})\|_{0,3}\|\nabla((A(\phi)-A(\hat{\phi}))\zeta)\|_{0,3/2}\\
&\quad + \|\mathbf{u}-\hat{\mathbf{u}}\|_{0,3}\|\nabla\hat{q}\|_0\|\zeta\|_{0,6} + \|\mathbf{u}-\hat{\mathbf{u}}\|_{0,3}\|\nabla\zeta\|_0\|\hat{q}\|_{0,6}.
\end{aligned}$$

Estimating the second, third and fourth terms further, we find

$$\begin{aligned}
(ii) &\leq C(b_2)\|\phi-\hat{\phi}\|_{0,6}\|\nabla\hat{\mu}\|_{0,3}(A_2\|\zeta\|_1 + A_3\|\nabla\phi\|_3\|\zeta\|_{0,6}),\\
(iii) &\leq (A_3\|\nabla\hat{q}\|_{0,3}\|\phi-\hat{\phi}\|_{0,6} + A_3\|\phi-\hat{\phi}\|_1\|\hat{q}\|_{0,\infty}\\
&\qquad + A_4\|\nabla\phi+\nabla\hat{\phi}\|_{0,3}\|\phi-\hat{\phi}\|_{0,6}\|\hat{q}\|_{0,\infty})(A_2\|\zeta\|_1 + A_3\|\nabla\phi\|_3\|\zeta\|_6),\\
(iv) &\leq \|b^{1/2}(\hat{\phi})\nabla\hat{\mu} - \nabla(A(\hat{\phi})\hat{q})\|_{0,3}(A_3\|\phi-\hat{\phi}\|_{0,6}\|\nabla\zeta\|_0 + A_3\|\phi-\hat{\phi}\|_1\|\zeta\|_{0,6}
\end{aligned}$$

$$+ A_4\|\nabla\phi + \nabla\hat{\phi}\|_{0,3}\|\phi - \hat{\phi}\|_{0,6}\|\zeta\|_{0,6}).$$

Using the definition of the dual norm, we obtain

$$\begin{aligned}\int_0^t \|r_3\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t & C\|\phi - \hat{\phi}\|_{0,6}^2\|\hat{q}\|_{0,3}^2 + C\|\phi - \hat{\phi}\|_{0,6}^2\|\nabla\hat{\mu}\|_{0,3}^2(1 + \|\nabla\phi\|_3^2)\\ &+ C(\|\nabla\hat{q}\|_{0,3}^2 + \|\hat{q}\|_{0,\infty}^2 + \|\nabla\phi + \nabla\hat{\phi}\|_{0,3}^2\|\hat{q}\|_{0,\infty}^2)(1 + \|\nabla\phi\|_{0,3}^2)\|\phi - \hat{\phi}\|_1^2\\ &+ C\|b^{1/2}(\hat{\phi})\nabla\hat{\mu} - \nabla(A(\hat{\phi})\hat{q})\|_{0,3}^2(1 + \|\nabla\phi + \nabla\hat{\phi}\|_{0,3}^2)\|\phi - \hat{\phi}\|_1^2\\ &+ \|\mathbf{u} - \hat{\mathbf{u}}\|_{0,3}^2\|\nabla\hat{q}\|_0^2 + \|\mathbf{u} - \hat{\mathbf{u}}\|_{0,3}^2\|\hat{q}\|_{0,6}^2 \,\mathrm{d}s\\ = (i) + (ii) &+ (iii) + (iv) + (v) + (vi).\end{aligned}$$

We observe that $(i) - (iv)$ is already in the right form. For the remainder, we obtain

$$(v) + (vi) \leq \int_0^t 2\delta\mathcal{D}_\phi(\mathbf{u}|\hat{\mathbf{u}}) + C(\delta)\|\hat{q}\|_1^4\mathcal{E}_{1,\alpha}(z|\hat{z}) \,\mathrm{d}s.$$

Using these estimates, we can bound the residual by

$$\begin{aligned}\int_0^t \|r_3\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t & C\Big(\|\hat{q}\|_{0,3}^2 + \|\nabla\hat{\mu}\|_{0,3}^2 + \|\nabla\hat{\mu}\|_{0,3}^2\|\nabla\phi\|_{0,3}^2\\ &+ (\|\nabla\hat{q}\|_{0,3}^2 + \|\hat{q}\|_{0,\infty} + \|\nabla\phi + \nabla\hat{\phi}\|_{0,3}^2\|\hat{q}\|_{0,\infty}^2)(1 + \|\nabla\phi\|_3^2)\\ &+ \|n(\hat{\phi})\nabla\hat{\mu} - \nabla(A(\hat{\phi})\hat{q})\|_{0,3}^2(1 + \|\nabla\phi + \nabla\hat{\phi}\|_{0,3}^2)\\ &+ \|\hat{q}\|_1^4\Big)\mathcal{E}_{1,\alpha}(z|\hat{z}) + 2\delta\mathcal{D}_\phi(\mathbf{u}|\hat{\mathbf{u}}) \,\mathrm{d}s\\ \leq \int_0^t & C_2(z,\hat{z},\delta)\mathcal{E}_{1,\alpha}(z|\hat{z}) + 2\delta\mathcal{D}_{1,\phi}(\mu, z|\hat{\mu},\hat{z}) \,\mathrm{d}s.\end{aligned}$$

For sufficiently smooth $(\hat{z},\hat{\mu})$, C_2 is integrable if $\nabla\phi$ is bounded in $L^4(0,T;L^3(\Omega))$, which is bounded be definition of the weak solution, cf. Definition 3.1.1.

Fourth residual:
We first estimate in standard manner the inner product, defining the fourth residual by

$$\begin{aligned}|\langle r_4, \mathbf{v}\rangle| \leq{}& \|\eta(\phi) - \eta(\hat{\phi})\|_{0,6}\|\mathrm{D}\hat{\mathbf{u}}\|_{0,3}^2 + C\|\mathbf{u} - \hat{\mathbf{u}}\|_{0,3}\|\hat{\mathbf{u}}\|_{0,6}\|\mathbf{v}\|_1\\ &+ C\|\mathbf{u} - \hat{\mathbf{u}}\|_{0,3}\|\nabla\hat{\mathbf{u}}\|_{0,2}\|\mathbf{v}\|_{0,6} + \|\mathbf{v}\|_{0,6}\|\nabla\hat{\mu}\|_{0,3/2}\|\phi - \hat{\phi}\|_{0,6}.\end{aligned}$$

Using the negative norm we find the following inequality

$$\begin{aligned}\int_0^t \|r_4\|_{-1}^2 \,\mathrm{d}s \leq{}& \int_0^t \eta_3\|\phi - \hat{\phi}\|_{0,6}^2\|\mathrm{D}\hat{\mathbf{u}}\|_{0,3}^2 + C\|\mathbf{u} - \hat{\mathbf{u}}\|_{0,3}^2\|\hat{\mathbf{u}}\|_{0,6}^2\\ &+ C\|\mathbf{u} - \hat{\mathbf{u}}\|_{0,3}^2\|\nabla\hat{\mathbf{u}}\|_{0,2}^2 + \|\nabla\hat{\mu}\|_{0,3/2}^2\|\phi - \hat{\phi}\|_{0,6}^2 \,\mathrm{d}s\\ \leq{}& \int_0^t 2\delta\|\eta^{1/2}(\phi)(\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}})\|_0^2 + C(\delta,\eta)\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2(\|\nabla\hat{\mathbf{u}}\|_{0,2}^4 + \|\nabla\hat{\mathbf{u}}\|_{0,3}^2)\\ &+ \|\nabla\hat{\mu}\|_{0,3/2}^2\|\phi - \hat{\phi}\|_{0,6}^2 \,\mathrm{d}s\\ \leq{}& \int_0^t 2\delta\mathcal{D}_{1,\phi}(\mu, z|\hat{\mu},\hat{z}) + C_3(\hat{\mathbf{u}},\hat{\mu},\delta)\mathcal{E}_{1,\alpha}(z|\hat{z}) \,\mathrm{d}s,\end{aligned}$$

here C_3 depends only on bounds of $\hat{\mathbf{u}}$ and $\hat{\mu}$ and therefore is integrable. □

Remark 4.6.2. Reviewing the above estimates, we need additionally to the assumptions of Theorem 4.4.1 the following regularity

$$\begin{aligned} &\hat{\mu} \in L^4(0,T;W^{1,3}(\Omega)), && n(\hat{\phi})\nabla\hat{\mu} - \nabla(A(\hat{\phi})\hat{q}) \in L^4(0,T;L^3(\Omega)), \\ &\hat{q} \in L^\infty(0,T;L^\infty(\Omega)) \cap L^4(0,T;W^{1,3}(\Omega)), && \hat{\mathbf{u}} \in L^2(0,T;W^{1,3}(\Omega)). \end{aligned} \tag{4.51}$$

Using Theorem 4.4.1 together the estimate in Lemma 4.6.1 and choosing δ small enough yields after another application of the Gronwall Lemma A.3.1

$$\mathcal{E}_{1,\alpha}(z(t)|\hat{z}(t)) + \int_0^t \mathcal{D}_{1,\phi}(\mu, z|\hat{\mu}, \hat{z}) \,\mathrm{d}s \leq C\mathcal{E}_{1,\alpha}(z(0)|\hat{z}(0)).$$

Since z and $\hat{z}$ start from the same initial data, we conclude that $\mathcal{E}_{1,\alpha}(z(0)|\hat{z}(0)) = 0$. The properties of the relative energy, cf. Lemma 4.1.1 and the lower bound on the relative dissipation functional concludes the proof Lemma 4.4.2.

4.7. Stability estimates & weak-strong uniqueness for System S.5

In the case of the Peterlin model, i.e., System S.5, we have to restrict ourselves to the two-dimensional case and consider η, κ_2 constant. This is due to technical reasons, and we comment later on extensions into three space dimensions. Again, we recall that the relevant relative energy for the following results is given by (4.6), i.e.,

$$\mathcal{E}_2(\mathbf{u}, \mathbf{C}|\hat{\mathbf{u}}, \hat{\mathbf{C}}) = \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}) + \mathcal{E}(\mathbf{C}|\hat{\mathbf{C}}) = \frac{1}{2}\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2 + \frac{1}{4}\|\mathbf{C} - \hat{\mathbf{C}}\|_0^2.$$

We can develop a stability estimate which is given by the following theorem.

Theorem 4.7.1. *Let $z = (\mathbf{u}, \mathbf{C})$ be a dissipative weak solution of System S.5 for $d = 2$ in the sense of Definition 3.3.1. Furthermore, let the $\hat{z} = (\hat{\mathbf{u}}, \hat{\mathbf{C}})$ be sufficiently smooth, cf. Remark 4.8.1, induce the associated residuals $\hat{r}_i$ given by* (4.12)-(4.14)*. Then the following holds*

$$\mathcal{E}_2(z(t)|\hat{z}(t)) + \int_0^t \mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) \,\mathrm{d}s \leq C_0\mathcal{E}_2(z(0)|\hat{z}(0)) \tag{4.52}$$

$$+ C_1 \int_0^t \|r_4\|_{-1}^2 + \|r_5\|_{-1}^2 \,\mathrm{d}s, \tag{4.53}$$

where $\mathcal{D}_{2,\mathbf{C}}$ denotes the relative dissipation functional given by

$$\mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) = \frac{1}{2}\Big(\eta\|\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}}\|_0^2 + \frac{\varepsilon_2}{2}\|\nabla(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 + \frac{1}{2}\|\chi^{1/2}(\mathrm{tr}(\mathbf{C}))(\mathbf{C} - \hat{\mathbf{C}})\|_0^2\Big).$$

The proof of the theorem will be given in Section 4.8 and the following lemma will be proven in Section 4.9. Similarly to before, we study as an application the stability with respect to the initial data, which yields a corresponding weak-strong uniqueness principle characterized by the following result.

Corollary 4.7.2. *Assume $d = 2$. Let $\hat{z}$ with $\hat{z} = (\hat{\mathbf{u}}, \hat{\mathbf{C}})$ be a regular dissipative weak solution of System S.5, cf. Remark 4.8.1 for the necessary regularity, existing up to time $T^\dagger \leq T$, such that Theorem 4.7.1 holds. Then every dissipative weak solution z with $z = (\mathbf{u}, \mathbf{C})$ of System S.5 in the sense of Definition 3.3.1 starting from the same initial data as $\hat{z}$ coincides with $\hat{z}$, i.e. $z(t) = \hat{z}(t)$, for almost all $t \in (0, T^\dagger)$.*

In the case of the two-dimensional Peterlin model, higher regularity is available for better initial data, see [98]. Hence, in two space dimensions our weak-strong uniqueness principle implies uniqueness of weak solutions for the Peterlin model, see [98].

4.8. Relative energy estimate for the Peterlin model

For convenience, we recall the weak formulation and the associated perturbed problem. The variational identities of the weak problem are given by

$$\int_0^t \langle \partial_t \mathbf{u}, \mathbf{v} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \langle \eta \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v} \rangle + \langle \mathrm{tr}(\mathbf{C})\mathbf{C}, \nabla \mathbf{v} \rangle \,\mathrm{d}s = 0 \tag{4.54}$$

$$\begin{aligned} \int_0^t \langle \partial_t \mathbf{C}, \mathbf{D} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{C}, \mathbf{D}) - 2\langle (\nabla \mathbf{u})\mathbf{C}, \mathbf{D} \rangle + \varepsilon_2 \langle \nabla \mathbf{C}, \nabla \mathbf{D} \rangle \,\mathrm{d}s& \\ = - \int_0^t \langle \chi(\mathrm{tr}(\mathbf{C}))\mathbf{C}, \mathbf{D} \rangle + \langle \Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\mathbf{D}) \rangle \,\mathrm{d}s&, \end{aligned} \tag{4.55}$$

for all $(\mathbf{v}, \mathbf{D}) \in L^4(0, T; H^1_{\mathrm{div}}(\Omega)^2 \times H^1_S(\Omega)^{2\times 2})$. While the variational identities for the perturbed problem are given by

$$\int_0^t \langle \partial_t \hat{\mathbf{u}}, \mathbf{v} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{v}) + \langle \eta \mathrm{D}\hat{\mathbf{u}}, \mathrm{D}\mathbf{v} \rangle + \langle \hat{\mathbf{C}}\mathbf{C}, \nabla \mathbf{v} \rangle \,\mathrm{d}s = \int_0^t \langle \hat{r}_4, \mathbf{v} \rangle \,\mathrm{d}s \tag{4.56}$$

$$\begin{aligned} \int_0^t \langle \partial_t \hat{\mathbf{C}}, \mathbf{D} \rangle + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{C}}, \mathbf{D}) - 2\langle (\nabla \hat{\mathbf{u}})\mathbf{C}, \mathbf{D} \rangle + \varepsilon \langle \nabla \hat{\mathbf{C}}, \nabla \mathbf{D} \rangle& \\ + \langle \chi(\mathrm{tr}(\mathbf{C}))\hat{\mathbf{C}}, \mathbf{D} \rangle - \langle \Phi(\mathrm{tr}(\hat{\mathbf{C}})), \mathrm{tr}(\mathbf{D}) \rangle \,\mathrm{d}s = \int_0^t \langle \hat{r}_5, \mathbf{D} \rangle \,\mathrm{d}s&, \end{aligned} \tag{4.57}$$

for all $(\mathbf{v}, \mathbf{D}) \in L^2(0, T; H^1_{\mathrm{div}}(\Omega)^2 \times H^1_S(\Omega)^{2\times 2})$ and for almost all $t \in (0, T)$.

Proof. We will start expanding the relative energy associated to the Peterlin model in two space dimensions, cf. (4.6), i.e.

$$\mathcal{E}_2(\mathbf{u}, \mathbf{C} | \hat{\mathbf{u}}, \hat{\mathbf{C}}) = \frac{1}{2}\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2 + \frac{1}{4}\|\mathbf{C} - \hat{\mathbf{C}}\|_0^2. \tag{4.58}$$

We again use the abstract calculation of Section 4.3, see (4.19), which implies

$$\begin{aligned} \mathcal{E}_2(z(t)|\hat{z}(t)) - \mathcal{E}_2(z(0)|\hat{z}(0)) = &- \int_0^t D_{2,\mathbf{C}}(z) + D_{2,\mathbf{C}}(\hat{z}) + \mathcal{R}_2(\hat{z}) \,\mathrm{d}s \\ &\underbrace{- \int_0^t \langle E_2'(\hat{z}), \partial_t z - \partial_t \hat{z} \rangle + \langle E_2''(\hat{z})(z - \hat{z}), \partial_t \hat{z} \rangle \,\mathrm{d}s}_{(*)} \end{aligned} \tag{4.59}$$

and we recall the definition of the $D, \mathcal{R}$ by

$$\begin{aligned} D_{2,\mathbf{C}}(z) =& \eta\|\mathrm{D}\mathbf{u}\|_0^2 + \frac{\varepsilon_2}{2}\|\nabla\mathbf{C}\|_0^2 + \frac{1}{2}\|\chi^{1/2}(\mathrm{tr}(\mathbf{C}))\mathbf{C}\|_0^2 - \frac{1}{2}\langle\Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\mathbf{C})\rangle \\ D_{2,\mathbf{C}}(\hat{z}) + \mathcal{R}_2(\hat{z}) =& \eta\|\mathrm{D}\hat{\mathbf{u}}\|_0^2 + \frac{\varepsilon_2}{2}\|\nabla\hat{\mathbf{C}}\|_0^2 + \frac{1}{2}\|\chi^{1/2}(\mathrm{tr}(\mathbf{C}))\hat{\mathbf{C}}\|_0^2 - \frac{1}{2}\langle\Phi(\mathrm{tr}(\hat{\mathbf{C}})), \mathrm{tr}(\hat{\mathbf{C}})\rangle \\ & - \langle r_4, \hat{\mathbf{u}}\rangle - \langle r_5, \hat{\mathbf{C}}\rangle. \end{aligned} \tag{4.60}$$

Navier-Stokes equation:
We will start expanding $(*)$, see (4.59), for the energetic contribution of the Navier-Stokes equation and obtain

$$\begin{aligned} (*)_1 =& -\int_0^t \langle\mathbf{u}, \partial_t\mathbf{u}\rangle + \langle\hat{\mathbf{u}}, \partial_t\mathbf{u}\rangle \,\mathrm{d}s \\ =& \int_0^t 2\langle\eta\mathrm{D}\mathbf{u}, \mathrm{D}\hat{\mathbf{u}}\rangle \underbrace{+\langle\mathbf{C}\mathbf{C}, \nabla\hat{\mathbf{u}}\rangle + \langle\hat{\mathbf{C}}\mathbf{C}, \nabla\mathbf{u}\rangle}_{\mathcal{C}_1} \\ & \underbrace{+\tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \hat{\mathbf{u}}) + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{u})}_{\mathcal{C}_2} - \langle r_4, \mathbf{u}\rangle \,\mathrm{d}s. \end{aligned}$$

We observe immediately that $\mathcal{C}_2$ vanishes due to skew-symmetric, see (A.33). In combination with the corresponding term of the dissipation, i.e. $(4.60)_{1,1}, (4.60)_{2,1}$, the residual term $(4.60)_{2,5}$, we find

$$P_1 := \int_0^t -\eta\|\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}}\|_0^2 + \mathcal{C}_1 - \langle r_4, \mathbf{u} - \hat{\mathbf{u}}\rangle \,\mathrm{d}s. \tag{4.61}$$

Peterlin equation:
Considering now $(*)$ for the contribution from the Peterlin model, we obtain

$$\begin{aligned} (*)_2 =& -\frac{1}{2}\int_0^t \langle\mathbf{C}, \partial_t\hat{\mathbf{C}}\rangle + \langle\hat{\mathbf{C}}, \partial_t\mathbf{C}\rangle \,\mathrm{d}s \\ =& \int_0^t \varepsilon_2\langle\nabla\mathbf{C}, \nabla\hat{\mathbf{C}}\rangle + \langle\chi(\mathrm{tr}(\mathbf{C}))\mathrm{tr}(\mathbf{C}), \mathrm{tr}(\hat{\mathbf{C}})\rangle + \frac{1}{2}\langle r_5, \mathbf{C}\rangle \\ & - \frac{1}{2}\langle\Phi(\mathrm{tr}(\hat{\mathbf{C}})), \mathrm{tr}(\mathbf{C})\rangle - \frac{1}{2}\langle\Phi(\mathrm{tr}(\mathbf{C})), \mathrm{tr}(\hat{\mathbf{C}})\rangle \\ & \underbrace{+\frac{1}{2}\mathbf{c}(\mathbf{u}; \mathbf{C}, \hat{\mathbf{C}}) + \frac{1}{2}\mathbf{c}(\mathbf{u}; \hat{\mathbf{C}}, \mathbf{C})}_{\mathcal{C}_3} \underbrace{-\langle\nabla\mathbf{u}\mathbf{C}, \hat{\mathbf{C}}\rangle - \langle\nabla\hat{\mathbf{u}}\mathbf{C}, \mathbf{C}\rangle}_{\mathcal{C}_4} \,\mathrm{d}s. \end{aligned}$$

As before $\mathcal{C}_3$ vanishes due to skew-symmetry (A.34) and combination with the remaining dissipative terms, i.e., $(4.60)_{1,i}, (4.60)_{2,i}$ for $i \in \{2, 4\}$ and the residual $(4.60)_{2,6}$, yields

$$\begin{aligned} P_2 :=& \int_0^t -\frac{\varepsilon_2}{2}\|\nabla(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 - \frac{1}{2}\|\chi(\mathrm{tr}(\mathbf{C}))^{1/2}(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 + \frac{1}{2}\|\mathrm{tr}(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 \\ & - \langle r_5, \mathbf{C} - \hat{\mathbf{C}}\rangle + \mathcal{C}_4 \,\mathrm{d}s. \end{aligned} \tag{4.62}$$

Note that we have inserted the definition of $\Phi(\mathrm{tr}(\mathbf{C})) = \mathrm{tr}(\mathbf{C}) + a$, cf. (A7), in order to cancel out the terms related to a. Finally, we use the estimates P_1, (4.61) and P_2, (4.62) into (4.59) and obtain

$$\mathcal{E}_2(z|\hat{z})|_0^t + 2\int_0^t \mathcal{D}_{2,\mathbf{C}}(z|\hat{z})(s) \,\mathrm{d}s \le C_0 \int_0^t \mathcal{E}_2(z|\hat{z}) \,\mathrm{d}s$$

$$+\int_0^t \mathcal{C}_1 + \mathcal{C}_4 + \langle r_4, \mathbf{u} - \hat{\mathbf{u}}\rangle + \langle r_5, \mathbf{C} - \hat{\mathbf{C}}\rangle \,\mathrm{d}s.$$

In order to obtain the proposed stability estimate, we have to estimate the terms on the right-hand side suitably. First we start with the residuals which can easily be estimated using the negative Sobolev norms and Young's inequality via

$$\int_0^t \langle r_4, \mathbf{u} - \hat{\mathbf{u}}\rangle + \langle r_5, \mathbf{C} - \hat{\mathbf{C}}\rangle \,\mathrm{d}s \leq \int_0^t \|r_4\|_{-1}\|\mathbf{u} - \hat{\mathbf{u}}\|_1 + \|r_5\|_{-1}\|\mathbf{C} - \hat{\mathbf{C}}\|_1 \,\mathrm{d}s$$
$$\leq \int_0^t 2\delta\mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) + C\mathcal{E}_2(z|\hat{z}) + C(\delta,\eta_2)\|r_4\|_{-1}^2 + C(\delta,\varepsilon_2)\|r_5\|_{-1}^2 \,\mathrm{d}s.$$

Finally, we consider the convective remainder $\mathcal{C}_1 + \mathcal{C}_4$, which vanishes

$$\sum_{i=1}^4 \mathcal{C}_i = \langle \mathbf{C}\mathbf{C}, \nabla\hat{\mathbf{u}}\rangle + \langle \hat{\mathbf{C}}\mathbf{C}, \nabla\mathbf{u}\rangle - \langle \nabla\mathbf{u}\mathbf{C}, \hat{\mathbf{C}}\rangle - \langle \nabla\hat{\mathbf{u}}\mathbf{C}, \mathbf{C}\rangle = 0.$$

Choosing $\delta = \frac{1}{2}$ yields the final inequality

$$\mathcal{E}_2(z|\hat{z})|_0^t + \int_0^t \mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) \,\mathrm{d}s \leq \int_0^t C_1\mathcal{E}_2(z|\hat{z}) + C_2\|r_4\|_{-1}^2 + \|r_5\|_{-1}^2 \,\mathrm{d}s.$$

Using the Gronwall lemma, cf. Lemma A.3.1 concludes the proof of Theorem 4.7.1. □

The regularity assumptions for $(\hat{\mathbf{u}}, \hat{\mathbf{C}})$ follow only from the abstract calculations with the relative energy, see (4.19) and the insertion of suitable test function.

Remark 4.8.1. We find the following set of regularity, recalling that we worked in two space dimensions

$$\hat{\mathbf{u}} \in \mathbb{X}(0,T) \cap L^4(0,T; H^1_{\mathrm{div}}(\Omega)^2), \quad \hat{\mathbf{C}} \in \mathbb{T}(0,T) \cap L^4(0,T; H^1(\Omega)^{2\times 2}).$$

See . (3.20) for the definition of the above spaces.

4.9. Weak-strong uniqueness for the Peterlin model

To prove the weak strong uniqueness principle for the two-dimensional Peterlin model (S.5) we assume that $(\hat{\mathbf{u}}, \hat{\mathbf{C}})$ is a regular dissipative weak solution of (3.75)-(3.76), cf. Remark 4.8.1 for the necessary regularity. We can then identify the residuals as

$$\langle r_3, \mathbf{v}\rangle = \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}, \mathbf{v}) - \tilde{\mathbf{c}}(\hat{\mathbf{u}}; \hat{\mathbf{u}}, \mathbf{v}) + \langle \hat{\mathbf{C}}(\mathbf{C} - \hat{\mathbf{C}}), \nabla\mathbf{v}\rangle, \tag{4.63}$$
$$\langle r_4, \mathbf{D}\rangle = \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{C}}, \mathbf{D}) - \tilde{\mathbf{c}}(\hat{\mathbf{u}}; \hat{\mathbf{C}}, \mathbf{D}) + 2\langle \nabla\hat{\mathbf{u}}(\mathbf{C} - \hat{\mathbf{C}}), \mathbf{D}\rangle \tag{4.64}$$
$$+ \langle (\chi(\mathrm{tr}(\mathbf{C})) - \chi(\mathrm{tr}(\hat{\mathbf{C}}))\hat{\mathbf{C}}, \mathbf{D}\rangle. \tag{4.65}$$

The next step is to estimate the residuals by the relative energy and dissipation terms, such that we can employ the Gronwall lemma, cf. Lemma A.3.1.

Lemma 4.9.1. *Let $z = (u, \mathbf{C})$ be a dissipative weak solution of System S.5 and $\hat{z} = (\hat{\mathbf{u}}, \hat{\mathbf{C}})$ be a more regular weak solution, such that Theorem 4.7.1 is satisfied. Then the following estimates hold*

$$\int_0^t \|r_4\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t 2\delta \mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) + C_4(\hat{\mathbf{u}}, \hat{\mathbf{C}})\mathcal{E}_2(z|\hat{z}) \,\mathrm{d}s,$$
$$\int_0^t \|r_5\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t 2\delta \mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) + C_5(\hat{\mathbf{u}}, \hat{\mathbf{C}})\mathcal{E}_2(z|\hat{z}) \,\mathrm{d}s,$$

where C_4, C_5 are at least integrable in $L^1(0, T^\dagger)$.

Proof. After integration by parts we obtain

$$|\langle r_4, \mathbf{v}\rangle| \leq C\|\mathbf{u} - \hat{\mathbf{u}}\|_{0,4}\|\nabla\hat{\mathbf{u}}\|_{0,2}\|\nabla\mathbf{v}\|_0 + C\|\hat{\mathbf{C}}\|_{0,4}\|\mathbf{C} - \hat{\mathbf{C}}\|_{0,4}\|\nabla\mathbf{v}\|_0.$$

Using the definition of the negative Sobolev norm, we obtain

$$\begin{aligned}
\int_0^t \|r_4\|_{-1}^2 \,\mathrm{d}s &\leq C\int_0^t \|\mathbf{u} - \hat{\mathbf{u}}\|_{0,4}^2\|\nabla\hat{\mathbf{u}}\|_0^2 + \|\hat{\mathbf{C}}\|_{0,4}^2\|\mathbf{C} - \hat{\mathbf{C}}\|_{0,4}^2 \,\mathrm{d}s \\
&\leq \int_0^t \delta\eta\|\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}}\|_0^2 + \delta\frac{\varepsilon_2}{2}\|\nabla(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 + C(\delta, \eta)\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2\|\hat{\mathbf{u}}\|_1^4 \\
&\quad + C\|\hat{\mathbf{C}}\|_{0,4}^4\|\mathbf{C} - \hat{\mathbf{C}}\|_0^2 \,\mathrm{d}s \\
&\leq \int_0^t \delta\mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) + C_4(\hat{\mathbf{u}}, \hat{\mathbf{C}})\mathcal{E}_2(z|\hat{z}) \,\mathrm{d}s,
\end{aligned}$$

where C_4 depends on the norms $\|\hat{\mathbf{u}}\|_{L^4(H^1)}$ and $\|\hat{\mathbf{C}}\|_{L^4(H^1)}$.

In a similar manner, we obtain

$$\begin{aligned}
|\langle r_5, \mathbf{D}\rangle| &\leq C\|\mathbf{u} - \hat{\mathbf{u}}\|_{0,4}\|\nabla\hat{\mathbf{C}}\|_{0,2}\|\nabla\mathbf{D}\|_0 + 2\|\nabla\hat{\mathbf{u}}\|_{0,2}\|\mathbf{C} - \hat{\mathbf{C}}\|_{0,4}\|\mathbf{D}\|_{0,6} \\
&\quad + \|\chi(\mathrm{tr}(\mathbf{C})) - \chi(\mathrm{tr}(\hat{\mathbf{C}}))\|_{0,4/3}\|\hat{\mathbf{C}}\|_{0,6}\|\mathbf{D}\|_{0,12}.
\end{aligned}$$

We further estimate by recalling $\chi(\mathrm{tr}(\mathbf{C})$ from (A7), i.e., $\chi(\mathrm{tr}(\mathbf{C})) = \mathrm{tr}(\mathbf{C})^2 + a\mathrm{tr}(\mathbf{C})$ and usage of the interpolation inequality (A.24) we find that

$$\begin{aligned}
\int_0^t \|r_5\|_{-1}^2 \,\mathrm{d}s &\leq C\int_0^t \|\mathbf{u} - \hat{\mathbf{u}}\|_{0,4}^2\|\hat{\mathbf{C}}\|_1^2 + \|\nabla\hat{\mathbf{u}}\|_0^2\|\mathbf{C} - \hat{\mathbf{C}}\|_{0,4}^2 \\
&\quad + \|\mathrm{tr}(\mathbf{C} - \hat{\mathbf{C}})\|_0^2\|\hat{\mathbf{C}}\|_1^2 + \|\mathrm{tr}(\mathbf{C} + \hat{\mathbf{C}})\|_0^2\|\mathrm{tr}(\mathbf{C} - \hat{\mathbf{C}})\|_4^2\|\hat{\mathbf{C}}\|_1^2 \,\mathrm{d}s \\
&\leq \int_0^t \delta\eta\|(\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}})\|_0^2 + 2\delta\frac{\varepsilon_2}{2}\|\nabla(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 \\
&\quad + C(\|\hat{\mathbf{C}}\|_1^2 + \|\hat{\mathbf{C}}\|_1^4 + \|\hat{\mathbf{C}}\|_1^4\|\mathrm{tr}(\mathbf{C} + \hat{\mathbf{C}})\|_0^4)\mathcal{E}_2(z|\hat{z}) \,\mathrm{d}s \\
&\leq \int_0^t 2\delta\mathcal{D}_{2,\mathbf{C}}(z|\hat{z}) + C_5(\hat{\mathbf{u}}, \hat{\mathbf{C}}, \mathbf{C})\mathcal{E}_2(z|\hat{z}) \,\mathrm{d}s,
\end{aligned}$$

where C_5 depends on the norms $\|\hat{\mathbf{u}}\|_{L^4(H^1)}$, $\|\hat{\mathbf{C}}\|_{L^4(H^1)}$ and $\|\mathrm{tr}(\mathbf{C} + \hat{\mathbf{C}})\|_{L^\infty(L^2)}$. □

A quick inspection of the used regularity implies that the regularity for the stability estimate, i.e., Theorem 4.7.1, is enough.

Use of Theorem 4.7.1 together with the estimate in Lemma 4.9.1 and choosing δ small enough yields directly after application of the Gronwall lemma

$$\mathcal{E}_2(z|\hat{z})(t) + \int_0^t \mathcal{D}_{2,\mathbf{C}}(z|\hat{z})\,\mathrm{d}s \le \mathcal{E}_2(z|\hat{z})(0).$$

Since z and $\hat{z}$ start from the same initial data, we conclude $\mathcal{E}_2(z|\hat{z})(0) = 0$ and therefore the application of the above inequality by using the properties of the relative energy, cf. Lemma 4.1.1 concludes the proof of Corollary 4.7.2.

4.10. Stability estimate & weak-strong uniqueness for System S.3

In this section, we state stability and weak-strong uniqueness result for the full model, i.e., System S.3. In principle, the stability result follows by combination of Theorem 4.4.1 and Theorem 4.7.1 for the full model, i.e., System S.3, in two space dimensions. Furthermore, we consider the weak strong uniqueness principle, which is obtained by only minor adjustments. Let us recall that the relevant relative energy employed here is given by (4.4), i.e.,

$$\begin{aligned}\mathcal{E}_\alpha(\phi, q, \mathbf{u}, \mathbf{C}|\hat{\phi}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}}) &= \mathcal{E}_\alpha(\phi|\hat{\phi}) + \mathcal{E}(q|\hat{q}) + \mathcal{E}(\mathbf{u}|\hat{\mathbf{u}}) + \mathcal{E}(\mathbf{C}|\hat{\mathbf{C}}) \\ &= \frac{\gamma}{2}\|\nabla(\phi - \hat{\phi})\|_0^2 + \int_\Omega f(\phi) - f(\hat{\phi}) - f'(\hat{\phi})(\phi - \hat{\phi})\,\mathrm{d}x \\ &\quad + \frac{\alpha}{2}\|\phi - \hat{\phi}\|_0^2 + \frac{1}{2}\|q - \hat{q}\|_0^2 + \frac{1}{2}\|\mathbf{u} - \hat{\mathbf{u}}\|_0^2 + \frac{1}{4}\|\mathbf{C} - \hat{\mathbf{C}}\|_0^2.\end{aligned}$$

Then the main results of this can be stated as follows.

Theorem 4.10.1. *Let (z, μ) with $z = (\phi, q, \mathbf{u}, \mathbf{C})$ be a dissipative weak solution of System S.3 for $d = 2$ in the sense of Definition 3.5.1. Furthermore, let the set of smooth functions $\hat{z} = (\hat{\phi}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}})$ and $\hat{\mu}$, cf. Remark 4.5.1 and Remark 4.8.1 for the necessary regularity, induce the associated residuals r_i given by (4.7)-(4.14). Then the following holds*

$$\begin{aligned}\mathcal{E}_\alpha(z(t)|\hat{z}(t)) + \int_0^t \mathcal{D}_{\phi,\mathbf{C}}(\mu, z|\hat{\mu}, \hat{z})\,\mathrm{d}s &\le C_0\mathcal{E}_\alpha(z(0)|\hat{z}(0)) && (4.66)\\ &\quad + C_1\int_0^t \|r_1\|_{-1}^2 + \|r_2\|_1^2 + \|r_3\|_{-1}^2 + \|r_4\|_{-1}^2 + \|r_5\|_{-1}^2\,\mathrm{d}s, && (4.67)\end{aligned}$$

where $\mathcal{D}_{\phi,\mathbf{C}}$ denotes the relative dissipation functional given by

$$\begin{aligned}\mathcal{D}_{\phi,\mathbf{C}}(\mu, z|\hat{\mu}, \hat{z}) =& \frac{1}{2}\Big(\|b^{1/2}(\phi)\nabla(\mu - \hat{\mu}) - \nabla(A(\phi)(q - \hat{q}))\|_0^2 + \varepsilon_0\|b^{1/2}(\phi)\nabla(\mu - \hat{\mu})\|_0^2 \\ &+ \|\kappa_1^{1/2}(\phi)(q - \hat{q})\|_0^2 + \varepsilon_1\|\nabla(q - \hat{q})\|_0^2 + \|\eta^{1/2}(\phi)(\mathrm{D}\mathbf{u} - \mathrm{D}\hat{\mathbf{u}})\|_0^2 \\ &+ \frac{\varepsilon_2}{2}\|\nabla(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 + \frac{1}{2}\|\kappa_2^{1/2}(\phi)\chi^{1/2}(\mathrm{tr}(\mathbf{C}))(\mathbf{C} - \hat{\mathbf{C}})\|_0^2\Big).\end{aligned}$$

Proof. The proof follows almost directly by combining the arguments used in the proofs of Theorem 4.4.1 and Theorem 4.7.1, using the full perturbed problem (4.7)-(4.14). Similar to the existence result, the only change is the appearance of $\kappa_2(\phi)$. Since the perturbed problem is linearised around the weak solution, i.e., $\kappa_2(\phi)$ is the prefactor in the weak solution and the perturbed solution, we can conclude the result by recalling the uniform bounds, cf. (A4). For brevity, let us shortly sketch the main steps

1. Using the abstract calculations (4.19) on (4.4).
2. Expanding the relative energy by inserting test functions suitably into the variational formulation. This exactly can be obtained by combining all computations in the stability proofs for the CHNSQ model and the Peterlin model, i.e. Section 4.8 and Section 4.5
3. Estimation of the remaining integrals and the Gronwall lemma. Since κ_2 is bounded the estimates are exactly the same as before.

□

We also obtain a weak-strong uniqueness principle, given by the following corollary, whose proof is given in Section 4.11.

Corollary 4.10.2. *Assume $d = 2$. Let $(\hat{z}, \hat{\mu})$ with $\hat{z} = (\hat{\phi}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}})$ be a more regular dissipative weak solution of System S.3, cf. Remark 4.6.2 and Remark 4.8.1 for the necessary regularity, existing up to time $T^\dagger \leq T$, such that Theorem 4.10.1 holds. Then every dissipative weak solution (z, μ) with $z = (\phi, q, \mathbf{u}, \mathbf{C})$ of System S.3 in the sense of Definition 3.5.1 starting from the same initial data as $(\hat{z}, \hat{\mu})$ coincides with $(\hat{z}, \hat{\mu})$, i.e. $z(t) = \hat{z}(t)$, for almost all $t \in (0, T^\dagger)$ and $\mu \equiv \hat{\mu}$.*

4.11. Weak-strong uniqueness for System S.3

If $(\hat{\phi}, \hat{\mu}, \hat{q}, \hat{\mathbf{u}}, \hat{\mathbf{C}})$ is a more regular weak solution of (3.64) we can identify the residuals r_i as follows. In fact, the residuals $r_i, i = 1, \dots 4$ do not change and the new contribution in the fifth residual is given by

$$\langle r_{5,new}, \mathbf{D}\rangle = \langle \kappa_2(\phi)(\chi(\mathrm{tr}(\mathbf{C})) - \chi(\mathrm{tr}(\hat{\mathbf{C}})))\hat{\mathbf{C}}, \mathbf{D}\rangle + \langle (\kappa_2(\phi) - \kappa_2(\hat{\phi}))\chi(\mathrm{tr}(\hat{\mathbf{C}}))\hat{\mathbf{C}}, \mathbf{D}\rangle.$$

While the first term can be treated as before, since $\kappa_2 \leq \kappa_{2,2}$, we estimate the second term as follows

$$\langle r_{5,new,2}, \mathbf{D}\rangle = \langle (\kappa_2(\phi) - \kappa_2(\hat{\phi}))\chi(\mathrm{tr}(\hat{\mathbf{C}}))\hat{\mathbf{C}}, \mathbf{D}\rangle \leq C(\kappa_{3,2})\|\phi - \hat{\phi}\|_1 \|\hat{\mathbf{C}}\|_{0,4}^3 \|\mathbf{D}\|_{0,12}.$$

Consequently, we obtain in the negative Sobolev norm

$$\int_0^t \|r_{5,new,2}\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t C(\kappa_{3,2})\|\phi - \hat{\phi}\|_1^2 \|\hat{\mathbf{C}}\|_{0,4}^3 \,\mathrm{d}s \leq \int_0^t C(\hat{\mathbf{C}})\mathcal{E}_\alpha(z|\hat{z}) \,\mathrm{d}s.$$

Again this does not yield a new regularity assumption for $\hat{\mathbf{C}}$ and as before we complete the proof by application of the Gronwall lemma, cf. Lemma A.3.1.

4.12. Related work and further applications

Before concluding the first part of the thesis, we will give a short outlook on several related topics and results.

Extensions of the relative energy method:
A similar result is possible in the case $\varepsilon_0 = 0$ again with lower integrability of the weak solution and consequently higher integrability of the perturbed variables. In principle, when estimating a term of the form $\nabla\mu - \nabla\hat{\mu}$ against the relative dissipation, one has to add and subtract $\pm\nabla(A(\phi)(q - \hat{q})$. And of course, one of these factors has to be estimated by the relative dissipation $\nabla q - \nabla\hat{q}$.

In [30] we have proven a conditional weak-strong uniqueness result for the three-dimensional Peterlin model. In principle, combination of the mentioned result together with Corollary 4.4.2 implies a conditional weak-strong uniqueness result for System S.3 in three space dimensions. Let us shortly note, that the relative energy for the elastic contribution in this case is given by $\mathcal{E}_{el}(\mathbf{C}|\hat{\mathbf{C}}) = \frac{1}{4}\|\mathrm{tr}(\mathbf{C} - \hat{\mathbf{C}})\|_0^2 + \frac{\beta}{2}\|\mathbf{C} - \hat{\mathbf{C}}\|_0^2$, for any $\beta > 0$. While the first contribution can be dealt using the energy inequality in any space dimension, the second term has no associated energy inequality in three space dimensions. Exactly, for this term one needs to assume that the weak solution is regular enough to allow the computation, i.e., the condition is a regularity assumption for the conformation tensor of the weak solution $\mathbf{C}$. Furthermore, the stability approach translates also to the three-dimensional case, since the main structure is preserved. Even more, since both relative energies are quadratic, the stability estimate translates verbatim. However, even the stability estimate would be conditional. An open question is whether similar results can be obtained with a relative energy approach based on the total energy of the Peterlin model.

Convergence to equilibrium:
A rather natural question is to ask whether the CHNSQ model or the Peterlin model converges to an equilibrium solution if $t \to \infty$. This is known for the Cahn-Hilliard equation and the Navier-Stokes equation separately. In the Navier-Stokes case, either $\mathbf{u}_\infty = \mathbf{0}$ in the case of Dirichlet boundary conditions and $\mathbf{u}_{\infty,i} = \langle \mathbf{u}_i(0), 1\rangle$ in the periodic case. The convergence to these states is exponential in time and follows directly from the energy-dissipation identity. In the case of the Cahn-Hilliard equation, this is more involved since the related energy functional exhibits infinitely many non-trivial states. However, it can be shown that for every initial datum there is a unique equilibrium solution to which the solution converges, here only algebraic in time, see [111, 3]. The same results hold for model H, see [80]. We believe that the above-mentioned techniques can be extended to the CHNSQ model since the bulk stress exhibits the trivial equilibrium $q_\infty = 0$. Furthermore, for the viscoelastic Peterlin model, such convergence results are not known. In the case of Oldroyd-B and the FENE-P model, the convergence towards the equilibrium is discussed in [79] and is again exponential in time. We expect that the decay to equilibrium can be shown, however, with an algebraic rate.

Degenerate mobilities:

Another topic that has to be mentioned is the so-called degenerate case. In the above consideration, we always assumed that the mobility function $b(\phi)$ is strictly positive, i.e., non-degenerate, and the potential is of polynomial-type. However, since the Cahn-Hilliard equation is a fourth-order partial differential equation, we have a priori no maximum principle available. Nevertheless, from a physical point of view, ϕ is representing the volume fraction which has the natural domain $\phi \in [0,1]$. Even more, the general view is that the pure states, i.e., $\phi \in \{0,1\}$ are not reached if they are not present at the initial time. However, all this is not known to be true in the case of strictly positive mobilities and polynomial-type potentials. To gain a maximum-like principle, one can consider the equation with so-called degenerate mobilities, i.e., $b(\phi) \approx \phi^n(1-\phi)^n\ n \geq 0$ with a suitable potential which can either be smooth or even more singular. In this context, a typically potential is the Flory-Huggins potential, i.e.,

$$f(\phi) = \phi\log(\phi) + (1-\phi)\log(1-\phi) + \chi\phi(1-\phi).$$

In our work [32] we considered the above-mentioned case for System S.3 with $\varepsilon_0 = 0$ in two space dimensions. The result translates immediately to $\varepsilon_0 > 0$. Even more, the approach of [32] would allow treating the three-dimensional case in the same fashion. We refrain from the proof and only state a suitable concept of weak solutions, the main idea of the proof and some additional considerations. The technique extends to known results in the literature, see [50] and [24] for similar results for Cahn-Hilliard and Cahn-Hilliard-Navier-Stokes respectively.

The main problem is a lack a priori estimate for μ, since $b(\phi)$ can vanish in certain parts of the domain. A further issue is to establish suitable L^∞-bounds for ϕ such that possible singularities in f or f' are not reached. We make the following four assumptions:

(D1) We assume $b \in C^1([0,1])$ with $b(s) = 0$ if and only if $s \in \{0,1\}$.

(D2) The potential can be divided into $f = f_c + f_n$, with a convex part $f_c \in C^2(0,1)$ and a smooth concave part $f_n \in C^2([0,1])$.

(D3) We assume the compatibility condition $(bf_c'') \in C([0,1])$.

(D4) We assume that $(Ab^{-1/2})(s), (A'b^{-1/2})(s)$ are uniformly bounded in $L^\infty(\mathbb{R})$.

The assumption $(D4)$ is only due to technical reasons and in principle states that $A(\phi)$ should also go to zero fast enough for $\phi \to 0$, or $\phi \to 1$. From a physical point of view, this seems reasonable, since in the pure phases the degenerate Cahn-Hilliard equation should also reduce to a transport equation. Next, we state the corresponding definition of a weak solution and an existence result, where we have for simplicity neglected the effects of the conformation tensor $\mathbf{C}$. However, a similar result holds also in this case.

Definition 4.12.1. For every $T > 0$ the quadruple $(\phi, \mathbf{J}, q, \mathbf{u})$ is called a weak solution of System S.4 on $\Omega \times (0,T)$ if it satisfies

$$\begin{aligned} &\phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), && q \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\ &\mathbf{u} \in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)^d) \cap L^2(0,T;H^1_{\mathrm{div}}(\Omega)^d), && \mathbf{J} = b^{1/2}(\phi)\tilde{\mathbf{J}}, \tilde{\mathbf{J}} \in L^2(0,T;L^2(\Omega)) \end{aligned}$$

with time derivatives

$$\partial_t\phi \in L^2(0,T;H^{-1}(\Omega)), \quad \partial_t q \in L^{4/3}(0,T;H^{-1}(\Omega)), \quad \partial_t\mathbf{u} \in L^p(0,T;H^{-1}_{\mathrm{div}}(\Omega)^d),$$

where $p = 2$ for $d = 2$ and $p = 4/3$ for $d = 3$. Further, for any test function $(\psi, \zeta, \boldsymbol{\xi}, \mathbf{v}) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \cap L^\infty(\Omega) \times H^1_{\text{div}}(\Omega)^d$ and almost every $t \in (0, T)$ it holds that

$$\begin{aligned}
&\langle \partial_t \phi, \psi \rangle - \mathbf{c}(\mathbf{u}; \psi, \phi) + \varepsilon_0 \langle \mathbf{J}, \nabla \psi \rangle + \langle \mathbf{J} - b^{1/2}(\phi) \nabla (A(\phi) q), \nabla \psi \rangle = 0 \\
&\langle \mathbf{J}, \boldsymbol{\xi} \rangle - \gamma \langle \Delta \phi, \operatorname{div}(b(\phi) \boldsymbol{\xi}) \rangle - \langle b(\phi) f''(\phi) \nabla \phi, \boldsymbol{\xi} \rangle = 0 \\
&\langle \partial_t q, \zeta \rangle + \mathbf{c}(\mathbf{u}; q, \zeta) + \langle \kappa_1(\phi) q, \zeta \rangle + \langle \nabla (A(\phi) q - \tilde{\mathbf{J}}, \nabla (A(\phi) \zeta) + \langle \varepsilon_1 \nabla q, \nabla \zeta \rangle = 0 \\
&\langle \partial_t \mathbf{u}, \mathbf{v} \rangle + \mathbf{c}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \langle \eta(\phi) \mathrm{D}\mathbf{u}, \mathrm{D}\mathbf{v} \rangle + \langle \gamma \Delta \phi \nabla \phi, \mathbf{v} \rangle = 0.
\end{aligned} \tag{4.68}$$

The above definition of a weak solution is much weaker than Definition 3.1.1, i.e., we characterize only the flux $\mathbf{J}$ instead of the chemical potential μ. Also, from the above definition, we observe why (D3) is important. Following [32] one can establish the following existence result.

Theorem 4.12.2. *Let the initial data $(\phi_0, q_0, \mathbf{u}_0) \in H^1(\Omega) \times L^2(\Omega) \times L^2_{div}(\Omega)^d$ be given. Let assumptions (D1)–(D4) hold. Further, let $\phi_0 : \Omega \to [0, 1]$ and $\phi_0 \in H^1(\Omega)$. The potential function f and the entropy function G, cf. [32], fulfil*

$$\int_\Omega \Big(f(\phi_0) + G(\phi_0) \, \mathrm{d}x \Big) < \infty. \tag{4.69}$$

Then for any given $T > 0$ there exists a dissipative global-in-time weak solution $(\phi, q, \mathbf{J}, \mathbf{u})$ of the System S.4 in the sense of Definition 4.12.1. Furthermore, the initial data is attained, i.e., $(\phi(0), q(0), \mathbf{u}(0)) = (\phi_0, q_0, \mathbf{u}_0)$. Moreover,

- *the integrated energy inequality*

$$\begin{aligned}
E(\phi, q, \mathbf{u})(t) &+ \int_{\Omega_t} \varepsilon_0 |\tilde{\mathbf{J}}|^2 + \Big| \tilde{\mathbf{J}} - \nabla (A(\phi) q) \Big|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \int_{\Omega_t} |\kappa_1^{1/2}(\phi) q|^2 + \varepsilon_1 |\nabla q|^2 + \Big| \eta^{1/2}(\phi) \mathrm{D}\mathbf{u} \Big|^2 \, \mathrm{d}x \, \mathrm{d}s \le E(\phi, q, \mathbf{u})(0)
\end{aligned}$$

  holds for almost every $t \in (0, T)$.
- *$\phi(x, t) \in [0, 1]$ for a.e. $(x, t) \in \Omega \times (0, T)$.*

If the mobility function satisfies $b'(0) = b'(1) = 0$, then the set

$$\{(x, t) \in \Omega \times (0, T) \mid \phi(x, t) = 0 \text{ or } \phi(x, t) = 1\} \tag{4.70}$$

has zero measure.

The proof can be sketched as follows:

1. Regularize the mobility b, the potential f and the bulk modulus A such that Theorem 3.1.3 is applicable. This yields a sequence of solutions, labelled by δ.

2. Due to the energy inequality, we gain several a priori estimates for the sequence. However, we lose the estimate for μ_δ. Therefore, the function G_δ is introduced via $G''_\delta(s) = b_\delta^{-1}(s)$. One can consider an entropy estimate based by testing with $G'_\delta(\phi_\delta)$, which yields uniform bounds for the $\phi_\delta \in L^2(0, T; H^2(\Omega))$ and we get $\int_\Omega G_\delta(\phi_\delta)$ is bounded uniformly in $L^\infty(0, T)$.

3. This bounds on G_δ allows us to establish the desired uniform L^∞-bounds on ϕ_δ, which yields in the limit $\phi \in [0,1]$ almost everywhere in Ω_T, see [32] for details.

4. Afterwards one passes to the limit with $\delta \to 0$ in the equations and the energy.

5. Prove of (4.70) by exploiting the singular behaviour of G together with the uniform bounds from the entropy estimate.

Furthermore, we emphasize that the stability estimate does not transfer to the degenerate case, since the lower bound of the mobility occurs in the constant for the Gronwall lemma.

However, in the spirit of the conditional weak-strong uniqueness in [30], let us assume the weak solution component $\phi(x,t) \in [\beta, 1-\beta]$ for almost all $(x,t) \in \Omega \times (0,T)$ with $\beta > 0$, probably dependent on time. In this case, the degenerate weak solution is a regular one in the sense of Definition 3.1.1. Hence, the stability estimate and the weak-strong uniqueness result transfer directly. The condition of compact containment in $[0,1]$ with constant β, is known as the strong separation principle. Unfortunately, it is only known to hold rigorously in one space-dimension or with constant mobility and sufficiently singular potential, see [38, 103].

It would be interesting to study if the refined entropy estimate of Grün et al., see [67, 105], can be transferred to the Cahn-Hilliard equation with the above-mentioned mobilities. Of course, then a further translation to the models studied in the thesis seems possible.

4.13. Conclusion of the theoretical part

Before we proceed with our investigation on numerical approximability in the second part, let us shortly recap the main points we made. First, we have shown that it is possible to construct dissipative weak solutions of the considered models in the relevant space dimensions two and three. This is done and heavily relies on exploiting the energy-dissipative structure behind the equations. The proofs are mainly based on the Galerkin approximation for the CHNSQ model, i.e., System S.4, where the main difficulty is to obtain the necessary bounds from the cross-diffusion part of the model. With these estimates at hand, in general, we proceed by standard energy methods, however adapting to the fact of several nonlinearities in the equations. For the Peterlin model, i.e., System S.5, we have applied a combination of the Galerkin approximation for the velocity $\mathbf{u}$, while we resolve the conformation tensor by parabolic regularity. A crucial point is to construct solutions for the conformation tensor, which are at least positive semi-definite. While from a physical point of view this is necessary to interpret $\mathbf{C}$ as a physical quantity, in the mathematical setting this is not so obvious, at least not on the level of weak solutions. Both results together imply an existence result for the full viscoelastic phase separation model, i.e., System S.3.

In the next step, we have investigated the uniqueness and stability of these dissipative weak solutions. In this context, we have applied the relative energy method to compare the weak solutions with a suitable perturbed problem and corresponding residuals. Using the energy-dissipative structure of the weak solutions yields a rather general stability estimate. The obtained estimates are then applied in the context of dependence of initial data, which at the very end yield a weak-strong uniqueness principle for the corresponding models. Again, the approach only exploits the structure which is hidden in the equations and in fact it is not valid for weak solutions without energy inequality. Finally, we made several comments on possible extensions and related results.

In summary, the whole part can be shortly rephrased as Hadamard well-posedness [73] in a weak sense, i.e., uniqueness and continuous dependence, are replaced by the weak-strong uniqueness principle and the stability estimates, respectively.

Part II: Numerical analysis

5 Numerical methods

In the next chapters of the thesis, we will focus on numerical approximation and the corresponding error analysis of the CHNSQ model, i.e., System S.4. In Chapter 5, we will review the literature on suitable numerical methods for System S.4 and several subsystems of System S.4, i.e. the Cahn-Hilliard equations, the Navier-Stokes equations and the model H, cf. System S.1. Afterwards, we recall relevant tools from the approximation theory focusing on variational discretisation which will be used in the upcoming error analysis. Chapter 6 will consider a semi-discretisation of System S.4 in space by inf-sup stable conforming finite elements based on the weak formulation (3.2). We will apply the nonlinear stability estimate of the preceeding part to analyse the convergence behaviour of the numerical solution. As a next step, we consider a full discretisation by employing a variational method, i.e., the Petrov-Galerkin method, in *time* in Chapter 7. The variational character of the time discretisation allows us to deduce a fully discrete version of the stability estimate, which we will employ in the error analysis. Chapter 8 will illustrate the obtained error estimates and consider typical experiments related to viscoelastic phase separation. Finally, in Chapter 9 we conclude the second part of the thesis on numerical analysis and briefly comment on further directions and applications.

To this end, we recall the CHNSQ model by

System S.4 (CHNSQ model)**.**

$$\begin{aligned}
&\partial_t\phi + \mathbf{u}\cdot\nabla\phi = \operatorname{div}\Big((1+\varepsilon_0)b(\phi)\nabla\mu\Big) - \operatorname{div}\Big(b^{1/2}(\phi)\nabla\Big(A(\phi)q\Big)\Big),\\
&\quad \mu = -\gamma\Delta\phi + f'(\phi),\\
&\partial_t q + \mathbf{u}\cdot\nabla q = -\kappa(\phi)q + A(\phi)\Delta\Big(A(\phi)q\Big) - A(\phi)\operatorname{div}\Big(b^{1/2}(\phi)\nabla\mu\Big) + \varepsilon_1\Delta q,\\
&\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \operatorname{div}\Big(\eta(\phi)\mathrm{D}\mathbf{u}\Big) - \nabla p + \nabla\phi\mu,\\
&\operatorname{div}\mathbf{u} = 0.
\end{aligned}$$

5.1. Review of the literature

The only numerical results concerning the approximation of the CHNSQ system, i.e., System S.4, we are aware of, are proposed by Zhou et al. [132] and Strasser et al. [116]. Both of them consider the whole viscoelastic model H, i.e., System S.2. In [116] several efficient linear energy-stable numerical approximations are presented. Therein, the focus is on time discretisation, while in space finite differences and finite volume approaches are employed. However, the error analysis of this scheme is open. We will use conforming finite element discretisation in space and the Petrov-Galerkin discretisation in time to perform rigorous error analysis. Since for the CHNSQ model, no scheme of this category is available, we will review results for numerical approximations of subsystems contained in the CHNSQ system. Namely, for the Cahn-Hilliard equations, the incompressible Navier Stokes equations and finally the model H, i.e., the Cahn-Hilliard-Navier-Stokes system.

Cahn-Hilliard equations:
When written as a system of second-order differential equations, cf. System S.4, the Cahn-Hilliard equations involves two variables, i.e., the volume fraction ϕ and the chemical potential μ, which can be discretised separately. In the context of finite element methods (FEM), this is known and realized by mixed finite element methods, see [42, 51, 52, 47]. Furthermore, the problem is also considered by the discontinuous Galerkin (DG) methods [83, 129, 93] and by the Fourier spectral methods [75, 89, 90]. We mention the work of Feng [61], where explicit dependence on the interface width γ is considered. Since the Cahn-Hilliard equation is a formal gradient flow, it is desirable to preserve this structure and the corresponding energy-dissipative structure on the discrete level. In the literature, this is termed "energy stability". The main difficulties on the discrete level arise due to the non-convex potential term $f(\phi)$. Application of typical time integration methods, like Runge-Kutta or multi-step schemes, in general, destroy the energy structure without stabilisation. Hence, a large part of the literature focuses on suitable decomposition or expansions of the potential, see for instance [70, 71]. Rather recently, the *Invariant Energy Quadratisation* (IEQ) method and the *scalar auxiliary variable* (SAV) approach was considered, see [113, 6]. Roughly, the main idea is reformulating the problem with another auxiliary variable, such that the non-convex potential is replaced by a new variable with quadratic energy. This can be understood as a relaxation scheme and is capable of dealing with non-convex energies in a stable manner.

Navier-Stokes equations:
The incompressible Navier-Stokes equations are very intensively studied from a numerical point of view. In the framework of finite elements, one of the main issues arises from the correct discretisation of the convective term. An equally prominent issue comes from the saddle point structure of the problem. For the continuous problem, one can simply retract the analysis on the divergence-free subspace where the pressure simply vanishes. However, the construction of such *exact divergence-free* finite-dimensional subspaces, i.e., a suitable finite elements space, is delicate and non-trivial. Due to this reason, a typical choice for discretisation involves also pressure and is also considered via mixed finite elements, i.e., discretising the velocity and the pressure in separate spaces. Since the pressure is only a Lagrange multiplicative enforcing the incompressibility, one has to choose *inf-sup* stable finite element space pairs for the velocity and pressure to preserve this structure, see [9], or one has to stabilize the problem in various ways for instance see [27]. In general, the theory is well-developed, see for instance the monographs [122, 81].

Model H:

In general, model H inherits the main issues from the Cahn-Hilliard and Navier-Stokes equations. However, as we have already seen in the theoretical part the coupling terms between the Cahn-Hilliard and the Navier-Stokes equations, i.e., $\mathbf{u} \cdot \nabla\phi$ and $\phi\nabla\mu$, are again of convective type. However, for the energy-dissipation identity, they will not cancel by skew-symmetry, but by their subtraction. Hence, a suitable discretisation to preserve this cancellation property is necessary. Mixed finite element methods for discretising the model H can be found in [60, 84]. A second order in time Crank-Nicolson discretisation is given in [74], without any error analysis. A Fourier-spectral method was proposed in [95] and extensions of the SAV and IEQ schemes are considered in [64, 91].

Before going into details, we present a small road map concerning error analysis for the CHNSQ model and simplified equations. In Table 5.1 we sketch the relevant references containing a rigorous error analysis results from the literature on the CHNSQ model, the Cahn-Hilliard equations and the model H in the case of strictly positive coefficient functions and polynomial-type potentials.

discretisation	Cahn-Hilliard	Model H	CHNSQ
only time	$[34]_{1,\cdot}$, $[113]_{1,\cdot}$, $[126]_{2,\cdot}$, $[107]_{2,\cdot}$	$[35]_{1,\cdot}$, $[91]_{1,\cdot}$	-
DG	$[83]_{1,k}$, $[93]_{1,k}$	$[94]_{2,k}$	-
Fourier spectral	$[75]_{2,k}$, $[89]_{1,k}$, $[90]_{1,k}$, $[131]_{2,k}$	-	-
mixed FEM	$[52]_{\cdot,2}$, $[47]_{2,2}$, $[130]_{2,k}$, $[29]^{*}_{2,2}$	$[60]_{0,0}$, $[84]_{0,0}$, $[46]_{2,2}$, $\checkmark^{*}_{2,2}$	$\checkmark^{*}_{2,2}$

Table 5.1.: Road map of results. Colourcode: ✓ for results which will be presented in the thesis; ✓ to emphasize our results;. ✓ to denote results, which can be obtained with the proofs presented here almost verbatim. $\checkmark_{s,k}$ denotes a proven convergence rate of order s in time and k in space. Finally, * emphasises that in these cases the error analysis is done considering variable mobility and viscosity, if applicable.

One can see that numerical methods for the Cahn-Hilliard equations are well studied, even in the context of rigorous error analysis. However, in the case of model H, i.e., the Cahn-Hilliard-Navier-Stokes system, rigorous error analysis is not considered that often. We emphasise that rigorous error estimates for a rather general potential involving variable mobility and viscosity are not present in the literature. The goal of the following chapters is to fill this gap and present the rigorous error analysis for the CHNSQ model for variable mobility and viscosity. Of course, the result immediately translates to the simpler model H, cf. System S.1. The techniques and ideas used in the following chapters are based on our study of the Cahn-Hilliard equations in [29].

In the following section, we will introduce relevant notation, techniques, and results for the upcoming numerical analysis. Since we will employ variational methods in space and *time*, we will first introduce the necessary tool for conforming finite element discretisation in space. In time, we will employ the Petrov-Galerkin method, which allows us to preserve the space-time variational character of the problem.

5.2. Conforming finite elements

Conforming finite element methods are particular Galerkin approximations, based on piecewise polynomial approximations over an appropriate partition of the computational domain. The techniques and ideas presented in the following are based on our recent work [29] and extend the results therein to the CHNSQ model.

Domain and Mesh:
Like in our analysis, we will consider as domain Ω the d-dimensional torus, i.e., $\Omega = \mathbb{T}^d$, cf. (A0), and denote by $\langle \cdot, \cdot \rangle$ the inner-product on $L^2(\Omega)$ as well as dual parings on $H^s(\Omega) \times H^{-s}(\Omega)$ or $H^{-s}(\Omega) \times H^s(\Omega)$. This choice of computational domain can be interpreted as a square or cube with periodic boundary conditions. Let $\mathcal{T}_h$ denote conforming partition of $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ into triangles or tetrahedra. We denote by ρ_K and h_K the inner-circle radius and diameter of the element $K \in \mathcal{T}_h$, respectively, and call $h = \max_{K \in \mathcal{T}_h} h_K$ the global mesh size. We assume that $\mathcal{T}_h$ is quasi-uniform, i.e., there exists a constant $\sigma > 0$ such that $\sigma h \leq \rho_K \leq h_K \leq h$ for all $K \in \mathcal{T}_h$. We further assume that the mesh $\mathcal{T}_h$ is periodic in the sense that it can be extended periodically to periodic extensions of the domain Ω. In the following, we will abbreviate the above partition of Ω into triangles or tetrahedra as a triangulation.

We then denote by the space of continuous piecewise quadratic and mean-free piecewise linear polynomials over the mesh $\mathcal{T}_h$ by

$$\mathcal{V}_h := \{v \in H^1(\Omega)^d : v|_K \in P_2(K)^d, \quad \forall K \in \mathcal{T}_h\}, \text{ and} \tag{5.1}$$

$$\mathcal{Q}_h := \{v \in L^2_0(\Omega) \cap H^1(\Omega) : v|_K \in P_1(K), \quad \forall K \in \mathcal{T}_h\}, \tag{5.2}$$

respectively.
Projection operators:
We start by introducing some projection operators and recall the corresponding error estimates. Let $\pi_h^0 : H^1(\Omega) \to \mathcal{V}_h$ denote the L^2-orthogonal projection which can be characterized by

$$\langle \pi_h^0 u - u, v_h \rangle = 0 \qquad \forall v_h \in \mathcal{V}_h. \tag{5.3}$$

By definition, π_h^0 is a contraction in $L^2(\Omega)$. For a quasi-uniform triangulation $\mathcal{T}_h$, π_h^0 is also stable with respect to the H^1-norm, i.e., $\|\pi_h^0 u\|_1 \leq C(\sigma)\|u\|_1$ for all $u \in H^1(\Omega)$; see [54, Lemma 1.131, Proposition 1.134]. Moreover, the following error estimate holds

$$\|u - \pi_h^0 u\|_s \leq Ch^{r-s}\|u\|_r \tag{5.4}$$

for all $-1 \leq s \leq 1$ and $0 \leq r \leq 4$. We will also apply the H^1-elliptic projection $\pi_h^1 : H^1(\Omega) \to \mathcal{V}_h$, which is characterized by the variational problem

$$\langle \nabla(\pi_h^1 u - u), \nabla v_h \rangle + \langle \pi_h^1 u - u, v_h \rangle = 0 \qquad \forall v_h \in \mathcal{V}_h. \tag{5.5}$$

By standard finite element error analysis and duality arguments, one can show that

$$\|u - \pi_h^1 u\|_s \leq Ch^{r-s}\|u\|_r, \tag{5.6}$$

for all $-1 \leq s \leq 1$ and $1 \leq r \leq 3$; see [26, Theorem 5.7.6/5.8.3] for details. Since we assumed quasi-uniformity of the mesh $\mathcal{T}_h$, we can further consider the inverse inequalities [26]

$$\|v_h\|_1 \leq c_{inv} h^{-1} \|v_h\|_0 \qquad \text{and} \qquad \|v_h\|_{0,p} \leq c_{inv} h^{d/p-d/q} \|v_h\|_{0,q} \tag{5.7}$$

which hold for all discrete functions $v_h \in \mathcal{V}_h$ and all $1 \leq q \leq p \leq \infty$ of the quasi-uniform triangulation in dimension d.

Inf-sup stability:
It is well-known that the choice $\mathcal{V}_h^d \times \mathcal{Q}_h$, i.e., the lowest order Taylor-Hood elements, is an inf-sup stable pair [9]. This means that the well-known *discrete inf-sup stability* [22] holds

$$\|w_h\| \leq \frac{1}{\beta} \sup_{0 \neq \mathbf{v}_h \in \mathcal{V}_h^d} \frac{\langle w_h, \operatorname{div} \mathbf{v}_h \rangle}{\|\mathbf{v}_h\|_1}, \forall w_h \in \mathcal{Q}_h. \tag{5.8}$$

Here the constant $\beta > 0$ is strictly positive and under rather general conditions on $\mathcal{T}_h$, i.e., more than two triangles for $d = 2$ and every tetrahedron has an interior vertex for $d = 3$, cf. [22, Theorem 8.8.1/8.8.2], it is independent of h and only depends on the domain Ω and the quasi-uniform structure of the triangulation, i.e., on σ. In the upcoming analysis we will need the subspace of discretely divergence free functions of $\mathcal{V}_h^d$ given by

$$\mathbb{V}_h := \{\mathbf{v}_h \in \mathcal{V}_h^d : \langle \operatorname{div} \mathbf{v}_h, w_h \rangle = 0, \forall w_h \in \mathcal{Q}_h\}. \tag{5.9}$$

Due to the discrete inf-sup stability (5.8), this space is not empty. In principle, this space mimics in a discrete sense the continuous space $H^1_{\text{div}}(\Omega)^d$. Furthermore, we emphasize that functions in $\mathbf{v}_h \in \mathbb{V}_h$ do not satisfy $\operatorname{div} \mathbf{v}_h = 0$ in general. The equality does not even hold elementwise, but it does hold for the whole domain, i.e., $\int_\Omega \operatorname{div} \mathbf{v}_h = 0$. This holds since the discrete divergence is well-defined, hence multiplication with 1 and integration by parts shows the result.

Stokes projection:
Let us introduce the Stokes projector $(\mathbf{P}_h^1, P_h^1) : H^1(\Omega)^d \times L_0^2(\Omega) \to \mathcal{V}_h^d \times \mathcal{Q}_h$ given by

$$\langle \nabla \mathbf{u} - \nabla \mathbf{P}_h^1(\mathbf{u}, p), \nabla \mathbf{v} \rangle + \langle \operatorname{div}\Big(\mathbf{u} - \mathbf{P}_h^1(\mathbf{u}, p)\Big), w_h \rangle + \langle \operatorname{div} \mathbf{v}_h, p - P_h^1(\mathbf{u}, p) \rangle = 0.$$

for all $\mathbf{v}_h \in \mathcal{V}_h$ and $w_h \in \mathcal{Q}_h$.

Lemma 5.2.1. *The Stokes projection $(\mathbf{P}_h^1(\mathbf{u}, p), P_h^1(\mathbf{u}, p))$ is well-defined for functions $(\mathbf{u}, p) \in (H^1(\Omega)^d, L_0^2(\Omega))$ and the following error estimate holds,*

$$\|\mathbf{u} - \mathbf{P}_h^1(\mathbf{u}, p)\|_s + \|p - P_h^1(\mathbf{u}, p)\|_{s-1} + \| \leq C h^{r-s} \|\mathbf{u}\|_r + C h^{r-s} \|p\|_{r-1} \tag{5.10}$$

for all $-1 \leq s \leq 1$ and $1 \leq r \leq 3$.

Proof. The well-posedness of the projector is shown in [22, Proposition 8.2.1] and crucial depends on the discrete inf-sup stability (5.8). The approximation error estimate can be found in [68, Theorem 6] and [26, Theorem 12.6.7]. □

Note that in the case when we are projecting the pair $(\mathbf{u}, 0)$ we set $\mathbf{u}_h = \mathbf{P}_h^1 \mathbf{u}$ and obtain the estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_s \leq Ch^{r-s}\|\mathbf{u}\|_r, \tag{5.11}$$

for all $-1 \leq s \leq 1$ and $1 \leq r \leq 3$.

For the projection of the pressure p_h we introduce in analogy to (5.3) the L^2-projection onto $\mathcal{Q}_h$ denoted by $\pi_{p,h}^0 : L_0^2(\Omega) \to \mathcal{Q}_h$ via

$$\langle p - \pi_{p,h}^0 p, w_h \rangle = 0, \quad \forall w_h \in \mathcal{Q}_h.$$

Moreover, the following error estimate holds

$$\|p - \pi_{p,h}^0 p\|_0 \leq Ch^2 \|p\|_2. \tag{5.12}$$

5.3. Time discretisation

For the approximation in time we will employ the Petrov-Galerkin method as a variational approach. The main difference between the standard Galerkin method and the Petrov-Galerkin method is that in the latter case the ansatz and test space are not required to coincide. We will consider the Petrov-Galerkin ansatz in time of degree 1, i.e., the ansatz functions are piecewise linear polynomials, while the test functions are piecewise constant functions over a suitable partition of $[0, T]$.

Given a step size $\tau = T/N > 0$ we define the discrete time points $t^n := n\tau$ and the corresponding time partition of the interval $[0, T]$ by $\mathcal{I}_\tau := \{0 = t^0, t^1, \ldots, T^N = T\}$. We write $P_k(\mathcal{I}_\tau)$ for the space of piecewise polynomials of degree k over the time partition $\mathcal{I}_\tau$, and denote by $P_k^c(\mathcal{I}_\tau){:=}P_k(\mathcal{I}_\tau) \cap C(0, T)$ the corresponding sub-space of continuous functions. To this end, let

$$I_\tau^1 : H^1(0, T) \to P_1^c(\mathcal{I}_\tau), \qquad I_\tau^1 u(t^n) = u(t^n)$$

denote the piecewise linear interpolation with respect to time. Furthermore, let

$$\bar{\pi}_\tau^0 : L^2(0, T) \to P_0(I_\tau), \qquad \bar{\pi}_\tau^0 u(t) = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} u(t)dt, \qquad t \in (t^{n-1}, t^n),$$

be the L^2-orthogonal projection to piecewise constant functions in time. For later reference, we summarize some important properties of these operators.

Lemma 5.3.1. *For $u \in W^{r,q}(0, T)$, $0 \leq r \leq 1$, $1 \leq p \leq q \leq \infty$, it holds*

$$\|u - \bar{\pi}_\tau^0 u\|_{L^p(0,T)} \leq C\tau^{1/p - 1/q + r}\|u\|_{W^{r,q}(0,T)}. \tag{5.13}$$

For $u \in W^{r,q}(0, T)$ with $1 \leq r \leq 2$ and $1 \leq p \leq q \leq \infty$, it holds

$$\|u - I_\tau^1 u\|_{L^p(0,T)} \leq C\tau^{1/p - 1/q + r}\|u\|_{W^{r,q}(0,T)}. \tag{5.14}$$

Moreover, the interpolation and projection operators commute with differentiation, i.e.,

$$\partial_t(I_\tau^1 u) = \bar{\pi}_\tau^0(\partial_t u). \tag{5.15}$$

Proof. The proof for these standard results can be found, e.g., in [112]. □

Here and in the following, let $\bar{a} = \bar{\pi}_\tau^0 a$ denote the L^2-orthogonal projection onto $P_0(\mathcal{I}_\tau)$.

For the piecewise constant L^2-projection we can show the following estimate for the product error, for the proof see Appendix A, cf. also [29, Lemma 23].

Lemma 5.3.2. *Let $u, v \in H^2(t^{n-1}, t^n)$ then the following holds true*

$$\|\bar{u}\bar{v} - \overline{uv}\|_{L^2(t^{n-1},t^n)} \leq C\tau^2 \|uv\|_{H^2(t^{n-1},t^n)} \leq C\tau^2 \|u\|_{H^2(t^{n-1},t^n)} \|v\|_{H^2(t^{n-1},t^n)}, \tag{5.16}$$

where C is independent on τ.

In order to further quantify errors due to the nonlinear coefficient functions, we provide the following lemma with proof given in the appendix.

Lemma 5.3.3. *Let $\phi \in P_1^c(\mathcal{I}_\tau)$. Then for any $g \in W^{2,p}(0,T)$ with $1 \leq p \leq \infty$, it holds*

$$\|g(\bar{\phi}) - \overline{g(\phi)}\|_{L^p(0,T)} \leq C\tau^2 \|g(\phi)\|_{W^{2,p}(0,T)}, \tag{5.17}$$

with a constant C independent of τ.

6 Semi-discrete problem

In this chapter, we will turn to the semi-discretization of the CHNSQ model in space, for which we consider an conforming Galerkin approximation of the variational principle (6.1)–(6.5) with inf-sup stable finite elements. In the first section of this chapter, i.e., Section 6.1, we state the semi-discretisation which we will analyse in the following and establish the existence of dissipative discrete solutions via the results of Picard-Lindelöf and state the main result on optimal second order convergence. In Section 6.2, we introduce a discrete perturbed problem and generalize the stability result of Section 4.4 to the discrete setting. In Section 6.3, we choose the perturbed solution as suitable projections of the continuous solution and establish the projection errors. In Section 6.4, we conduct the error analysis, which reduces to estimating the residuals by the relative energy, the relative dissipation and approximation error. Finally, in Section 6.5 we establish error estimates for the discrete pressure using the discrete inf-sup stability (5.8).

To numerical methods and their analysis presented in this section rely on the weak formulation of System S.4, which is given by

$$
\begin{aligned}
\langle \partial_t \phi, \psi\rangle - \mathbf{c}(\mathbf{u}; \psi, \phi) + (1+\varepsilon_0)\langle b(\phi)\nabla\mu, \nabla\psi\rangle - \langle b^{1/2}(\phi)\nabla(A(\phi)q), \nabla\psi\rangle &= 0, && (6.1)\\
\langle \mu, \xi\rangle - \gamma\langle \nabla\phi, \nabla\xi\rangle - \langle f'(\phi), \xi\rangle &= 0, && (6.2)\\
\langle \partial_t q, \zeta\rangle + \tilde{\mathbf{c}}(\mathbf{u}; q, \zeta) + \varepsilon_1\langle \nabla q, \nabla\zeta\rangle + \langle \kappa(\phi) q, \zeta\rangle & && (6.3)\\
+ \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\zeta)\rangle &= 0, &&\\
\langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + \langle \eta(\phi)\nabla\mathbf{u}, \nabla\mathbf{v}\rangle - \langle p, \operatorname{div}\mathbf{v}\rangle + \mathbf{c}(\mathbf{v}; \mu, \phi) &= 0, && (6.4)\\
0 &= \langle \operatorname{div}\mathbf{u}, w\rangle, && (6.5)
\end{aligned}
$$

for all $\psi, \xi, \zeta \in H^1(\Omega)$, $\mathbf{v} \in H^1(\Omega)^d, w \in Q := L^2_0(\Omega)$ and for a.a. $t \in (0,T)$.

For simplicity, in contrast to the formulation (3.2) we consider only $\nabla\mathbf{u}$ instead of its symmetric part $\mathrm{D}\mathbf{u}$ and the pressure p is now part of the problem, as discussed in the preceeding chapter. We further recall that $\mathbf{c}(\cdot;\cdot,\cdot)$ denote the convection terms, cf. (A.31), and $\tilde{\mathbf{c}}(\cdot;\cdot,\cdot)$ denote the skew-symmetric variant, cf. (3.72) and (3.73), which are given by

$$
\mathbf{c}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \langle (\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}\rangle, \quad \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \frac{1}{2}\mathbf{c}(\mathbf{u}; \mathbf{v}, \mathbf{w}) - \frac{1}{2}\mathbf{c}(\mathbf{u}; \mathbf{w}, \mathbf{v})
$$

6.1. Convergence result for a semi-discrete approximation

In this section, we will state the semi-discrete problem, establish the existence of dissipative discrete solutions and state the main result of this chapter on optimal second order convergence of the proposed scheme. Let us introduce the approximation spaces

$$\begin{aligned} \mathbb{W}_h(0,T) &:= H^1(0,T;\mathcal{V}_h), & \mathbb{Q}_h(0,T) &:= L^2(0,T;\mathcal{V}_h), \\ \mathbb{X}_h(0,T) &:= H^1(0,T;\mathcal{V}_h^d), & \mathbb{P}_h(0,T) &:= L^2(0,T;\mathcal{Q}_h). \end{aligned}$$

We recall the finite element space $\mathcal{V}_h, \mathcal{Q}_h$ given in (5.1)-(5.2), are piecewise quadratic finite elements and piecewise linear finite elements with zero mean. Furthermore, we recall (A0), i.e. that the domain Ω is the d-dimensional torus. Using these spaces we can state the semi-discrete problem as follows.

Problem P.1 (Semi-discrete scheme)**.** Let $\phi_{h,0} \in \mathcal{V}_h$, $q_{h,0} \in \mathcal{V}_h$, $\mathbf{u}_{h,0} \in \mathcal{V}_h^d$ be given. Find the functions $(\phi_h, \mu_h, q_h, \mathbf{u}_h, p_h) \in \mathbb{W}_h(0,T) \times \mathbb{Q}_h(0,T) \times \mathbb{W}_h(0,T) \times \mathbb{X}_h(0,T) \times \mathbb{P}_h(0,T)$ with $\phi_h(0) = \phi_{h,0}$, $q_h(0) = q_{h,0}$, $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, and such that

$$\begin{aligned} \langle \partial_t \phi_h, \psi_h \rangle &= \mathbf{c}(\mathbf{u}_h; \psi_h, \phi_h) - \varepsilon_0 \langle b(\phi_h) \nabla \mu_h \nabla \psi_h \rangle && (6.6) \\ &\quad - \langle b^{1/2}(\phi_h) \nabla \mu_h - \nabla (A(\phi_h) q_h), b^{1/2}(\phi_h) \nabla \psi_h \rangle, \\ \langle \mu_h, \xi_h \rangle &= \gamma \langle \nabla \phi_h, \nabla \xi_h \rangle + \langle f'(\phi_h), \xi_h \rangle, && (6.7) \\ \langle \partial_t q_h, \zeta_h \rangle &= \tilde{\mathbf{c}}(\mathbf{u}_h; \zeta_h, \bar{q}_h) - \langle \kappa(\phi_h) q_h, \zeta_h \rangle - \varepsilon_1 \langle \nabla q_h, \nabla \zeta_h \rangle && (6.8) \\ &\quad - \langle \nabla (A(\phi_h) q_h) - b^{1/2}(\phi_h) \nabla \mu_h, \nabla (A(\phi_h) \zeta_h) \rangle, \\ \langle \partial_t \mathbf{u}_h, \mathbf{v}_h \rangle &= \tilde{\mathbf{c}}(\mathbf{u}_h; \mathbf{v}_h, \mathbf{u}_h) - \langle \eta(\phi_h) \nabla \mathbf{u}_h, \nabla \mathbf{v}_h \rangle && (6.9) \\ &\quad + \langle p_h, \operatorname{div} \mathbf{v}_h \rangle - \mathbf{c}(\mathbf{v}_h; \mu_h, \phi_h), \\ \langle \operatorname{div} \mathbf{u}_h, \bar{w}_h \rangle &= 0, && (6.10) \end{aligned}$$

holds for all $(\psi_h, \xi_h, \zeta_h, \mathbf{v}_h, w_h) \in \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d \times \mathcal{Q}_h$ and for all $0 \le t \le T$.

Note that the ansatz functions depend on time, while the test functions are time independent. We further recall that $\mathbf{c}(\cdot;\cdot,\cdot)$ denote the convection terms, cf. (A.31), and $\tilde{\mathbf{c}}(\cdot;\cdot,\cdot)$ denote the skew-symmetric variant, cf. (3.72) and (3.73).

Remark 6.1.1. Let us comment on why in the Navier-Stokes and the bulk stress equation the skew-symmetric convection term $\tilde{\mathbf{c}}$ is used, while for the discretisation of the Cahn-Hilliard terms we can use the standard convection term $\mathbf{c}$. On the continuous level, the convection terms which are discretised by $\tilde{\mathbf{c}}$, cancel after integration by parts and divergence-freedom of the velocity $\mathbf{u}$. Indeed, $\mathbf{c}$ and $\tilde{\mathbf{c}}$ are the same. However, on the discrete level divergence-freedom is not obtained in this sense. Hence, we have to make sure that the form still cancels when testing with $\mathbf{u}_h$ or q_h. For the Cahn-Hilliard terms both convective terms cancel each other, i.e., divergence-freedom is not relevant, hence the only relevant aspect is to make sure that the convection term is chosen such that it conserves mass, i.e. $\mathbf{c}(\mathbf{u}_h; \psi_h, \phi_h)$ vanishes for $\psi_h = 1$.

Before we turn to the error analysis, let us first discuss the existence of a discrete solution and the related uniform bounds. To this end, we consider the following lemma.

Lemma 6.1.2. *Let assumptions (A0)–(A6) hold. Then for any initial value $\phi_{0,h} \in \mathcal{V}_h$, $q_{0,h} \in \mathcal{V}_h$, $\mathbf{u}_{0,h} \in \mathcal{V}_h^d$, Problem P.1 has a unique solution $(\phi_h, \mu_h, q_h, \mathbf{u}_h, p_h)$. Moreover, for all $0 \le t \le T$, one has conservation of mass, i.e. $\int_\Omega \phi_h(t)\,\mathrm{d}x = \int_\Omega \phi_{0,h}\,\mathrm{d}x$, as well as the energy-dissipation equality $E(\phi_h, q_h, \mathbf{u}_h)(t) + \int_0^t D_{\phi_h}(\mu_h, q_h, \mathbf{u}_h)(s)\,\mathrm{d}s = E(\phi_{0,h}, q_h, \mathbf{u}_{0,h})$. Furthermore, the following uniform bounds hold*

$$\begin{aligned}&\|\phi_h\|^2_{L^\infty(H^1)} + \|q_h\|^2_{L^\infty(L^2)} + \|\mathbf{u}_h\|^2_{L^\infty(L^2)} + \|\mu_h\|^2_{L^2(H^1)} + \|q_h\|^2_{L^2(H^1)} \\ &+ \|b^{1/2}(\phi_h)\nabla\mu_h - \nabla(A(\phi_h)q_h)\|^2_{L^2(L^2)} + \|\mathbf{u}_h\|^2_{L^2(H^1)} \le C(\|\phi_{h,0}\|_1^2, \|q_{h,0}\|_0^2, \|\mathbf{u}_{h,0}\|_0^2).\end{aligned} \tag{6.11}$$

Proof. The above system consists of three differential variables (ϕ, q, μ) and two algebraic variables (μ, p). In order to apply the Picard-Lindelöf theorem, we eliminate both algebraic variables from the system and reconstruct them after the existence of $(\phi_h, q_h, \mathbf{u}_h)$ is ensured. First, we restrict Problem P.1 to the space of discretely divergence free functions and consider $\mathbf{u}_h \in H^1(0,T;\mathbb{V}_h)$ and $\mathbf{v}_h \in \mathbb{V}_h$. This eliminates the discrete pressure p_h from the equation (6.9). The discrete chemical potential μ_h can be eliminated by simply inserting its definition into (6.6).

Using (A0)–(A6), the existence of a unique solution $(\phi_h, q_h, \mathbf{u}_h) \in C^1(0,T;\mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d)$ can be deduced from the Picard-Lindelöf theorem. The discrete chemical potential $\mu_h \in C(0,T;\mathcal{V}_h)$ can be reconstructed, immediately. Conservation of mass and the energy dissipation identity then follow with similar arguments as for the continuous problem by testing equations (6.6)–(6.9) with $\psi_h = 1$ and $(\psi_h, \xi_h, \zeta_h, \mathbf{v}_h) = (\mu_h, \partial_t\phi_h, q_h, \mathbf{u}_h)$, respectively. Using the obtained bounds, we get the existence of the discrete pressure $p_h \in C(0,T;\mathcal{Q}_h)$ from the discrete inf-sup stability (5.8). The uniform bounds are then obtained from the energy-dissipation identity of the full system by testing (6.6)–(6.10) with $(\psi_h, \xi_h, \zeta_h, \mathbf{v}_h, w_h) = (\mu_h, \partial_t\phi_h, q_h, \mathbf{u}_h, p_h)$. □

In order to simplify the presentation of the subsequent error analysis, we introduce a simplification and formulate an additional assumption.

(A9) We assume $A = 1$.

However, we will only need this simplification in the error analysis and briefly remark when using it. Discussion on the extension to the nonlinear case $A = A(\cdot)$ is given in Chapter 9. For later reference, let us summarize the main assumptions underlying our error analysis for the semi-discretisation.

Assumptions 6.1.3. *Assume that assumptions (A0)–(A6) and (A8)–(A9) hold and let $(\phi, \mu, q, \mathbf{u}, p)$ be a weak solution of (6.1)–(6.5) with the initial data $(\phi_0, q_0, \mathbf{u}_0) \in H^3(\Omega) \times H^2(\Omega) \times H^1_{div}(\Omega)^d \cap H^2(\Omega)$. Furthermore, let $(\phi, \mu, q, \mathbf{u}, p)$ satisfies the following additional regularity assumptions*

$$\begin{aligned}\phi &\in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)), & \mu &\in L^2(0,T;H^3(\Omega)),\\ q &\in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)), & &\\ \mathbf{u} &\in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)), & p &\in L^2(0,T;H^2(\Omega)).\end{aligned}$$

Furthermore, we denote by $(\phi_h, \mu_h, q_h, \mathbf{u}_h, p_h)$ be the solution of Problem P.1 with the initial data $\phi_h(0) = \pi_h^1\phi_0$, $q_h(0) = \pi_h^0 q_0$ and $\mathbf{u}_h(0) = \mathbf{P}_h^1\mathbf{u}_0$.

Remark 6.1.4. We emphasise that from a regularity viewpoint, the regularity assumption $\phi \in H^1(0,T;H^1(\Omega))$ immediately implies more regularity for ϕ, i.e. typical regularity would be $\phi \in L^\infty(0,T;H^3(\Omega))$ or $\phi \in L^2(0,T;H^5(\Omega))$. Similar implications hold also for q and $\mathbf{u}$.

With the existence result at hand, we can state the main result for the semi-discrete problem, characterized by the following theorem.

Theorem 6.1.5. *Let Assumptions 6.1.3 hold. Then the error estimate*

$$\begin{aligned}&\|\phi_h-\phi\|^2_{L^\infty(H^1)}+\|q_h-q\|^2_{L^\infty(L^2)}+\|\mathbf{u}_h-\mathbf{u}\|^2_{L^\infty(L^2)}\\&+\|\mu_h-\mu\|^2_{L^2(H^1)}+\|q_h-q\|^2_{L^2(H^1)}+\|\mathbf{u}_h-\mathbf{u}\|^2_{L^2(H^1)}\le Ch^4\end{aligned}$$

holds, where C is a constant independent of h. Furthermore, the following error estimate for the discrete pressure holds

$$\|p_h-p\|^2_{L^2(L^2)}\le Ch^4.$$

Before we start the proof, let us shortly sketch the main steps

1. In Section 6.2, we will introduce a suitable discrete perturbed system, which using the relative energy methods from Chapter 4 implies a discrete stability result.
2. In Section 6.3, we will consider the approximation errors by choosing the perturbed solution as a suitable projections of the solution $(\phi,\mu,q,\mathbf{u},p)$.
3. In Section 6.4, we will prove Theorem 6.1.5. Using the stability estimate of Section 6.2, together with the choice of perturbed solution from Section 6.3, reduces the error estimate to simply computing and estimating the residuals suitably. In this section, we will use the assumption (A9) for the first time.
4. In Section 6.5 the error analysis for the pressure is conducted using inf-sup stability with the techniques in [81]. In this section also assumption (A9) is used.

6.2. Semi-discrete stability estimate

With similar arguments as for the continuous problem, we will first study the stability of the semi-discrete solution with respect to perturbations. For a given set of functions $(\hat\phi_h,\hat\mu_h,\hat q_h,\hat{\mathbf{u}}_h,\hat p_h)\in\mathbb{W}_h(0,T)\times\mathbb{Q}_h(0,T)\times\mathbb{W}_h(0,T)\times\mathbb{X}_h(0,T)\times\mathbb{P}(0,T)$, we then define semi-discrete residuals $(\hat r_{1,h},\hat r_{2,h},\hat r_{3,h},\hat r_{4,h})\in L^2(0,T;\mathcal{V}_h\times\mathcal{V}_h\times\mathcal{V}_h\times\mathcal{V}_h^d)$ by

$$\begin{aligned}\langle\partial_t\hat\phi_h,\psi_h\rangle&=\mathbf{c}(\hat{\mathbf{u}}_h;\psi_h,\phi_h)-\varepsilon_0\langle b(\phi_h)\nabla\hat\mu_h\nabla\psi_h\rangle\\&\quad-\langle b^{1/2}(\phi_h)\nabla\hat\mu_h-\nabla(A(\phi_h)\hat q_h),b^{1/2}(\phi_h)\nabla\psi_h\rangle+\langle\hat r_{1,h},\psi_h\rangle,\end{aligned}\tag{6.12}$$

$$\langle\hat\mu_h,\xi_h\rangle=\gamma\langle\nabla\hat\phi_h,\nabla\xi_h\rangle+\langle f'(\hat\phi_h),\xi_h\rangle+\langle\hat r_{2,h},\xi_h\rangle,\tag{6.13}$$

$$\begin{aligned}\langle\partial_t\hat q_h,\zeta_h\rangle&=\tilde{\mathbf{c}}(\mathbf{u}_h;\zeta_h,\hat q_h)-\langle\kappa(\phi)\hat q_h,\zeta_h\rangle-\varepsilon_1\langle\nabla\hat q_h,\nabla\zeta_h\rangle\\&\quad-\langle\nabla(A(\phi_h)\hat q_h)-b^{1/2}(\phi_h)\nabla\hat\mu_h,\nabla(A(\phi_h)\zeta_h)\rangle+\langle\hat r_{3,h},\zeta_h\rangle,\end{aligned}\tag{6.14}$$

$$\langle \partial_t \hat{\mathbf{u}}_h, \mathbf{v}_h \rangle = \tilde{\mathbf{c}}(\mathbf{u}_h; \mathbf{v}_h, \hat{\mathbf{u}}_h) - \langle \eta(\phi_h) \nabla \hat{\mathbf{u}}_h, \nabla \mathbf{v}_h \rangle \tag{6.15}$$
$$+ \langle \hat{p}_h, \operatorname{div} \mathbf{v}_h \rangle - \mathbf{c}(\mathbf{v}_h; \hat{\mu}_h, \phi_h) + \langle \hat{r}_{4,h}, \mathbf{v}_h \rangle,$$
$$\langle \operatorname{div} \hat{\mathbf{u}}_h, w_h \rangle = 0, \tag{6.16}$$

for all $\psi_h, \xi_h, \zeta_h \in \mathcal{V}_h$, $\mathbf{v}_h \in \mathcal{V}_h^d$, $w_h \in \mathcal{Q}_h$ and $0 \le t \le T$. The functions $(\hat{\phi}_h, \hat{\mu}_h, \hat{q}_h, \hat{\mathbf{v}}_h, \hat{p}_h)$ can be understood as solutions of the perturbed semi-discrete problem, i.e., (6.12)–(6.16). Similar to the continuous case we measure the distance between the semi-discrete solution of Problem P.1 and the solution of the perturbed problem (6.12)–(6.16) via the relative energy, given by

$$\mathcal{E}_\alpha(\phi_h, q_h, \mathbf{u}_h | \hat{\phi}_h, \hat{q}_h, \hat{\mathbf{u}}_h) := \frac{\gamma}{2} \|\nabla \phi_h - \nabla \hat{\phi}_h\|_0^2 + f(\phi_h) - f(\hat{\phi}_h) - f'(\hat{\phi}_h)(\phi_h - \hat{\phi}_h)$$
$$+ \frac{\alpha}{2} \|\phi_h - \hat{\phi}_h\|_0^2 + \frac{1}{2} \|q_h - \hat{q}_h\|_0^2 + \frac{1}{2} \|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_0^2.$$

Furthermore, we often abbreviate the relative energies related to the single variables by $\mathcal{E}_\alpha(\phi_h|\hat{\phi}_h), \mathcal{E}(q_h|\hat{q}_h), \mathcal{E}(\mathbf{u}_h|\hat{\mathbf{u}}_h)$, respectively. Following the proof of the continuous stability estimate, i.e., Theorem 4.4.1 we can almost verbatim deduce a semi-discrete version. Note that this result does not require assumption (A9).

Lemma 6.2.1. *Let assumptions (A0)–(A6), and (A8) hold. Let $z_h = (\phi_h, q_h, \mathbf{u}_h)$ and the functions $(z_h, p_h, \mu_h) \in \mathbb{W}_h(0,T) \times \mathbb{W}_h(0,T) \times \mathbb{X}_h(0,T) \times \mathbb{P}_h(0,T) \times \mathbb{Q}_h(0,T)$ denote a solution of Problem P.1. Furthermore, let $\hat{z}_h = (\hat{\phi}_h, \hat{q}_h, \hat{\mathbf{u}}_h)$ and given the set of functions $(\hat{z}_h, \hat{p}_h, \hat{\mu}_h) \in \mathbb{W}_h(0,T) \times \mathbb{W}_h(0,T) \times \mathbb{X}_h(0,T) \times \mathbb{P}_h(0,T) \times \mathbb{Q}_h(0,T)$ and the associated residuals $\hat{r}_{i,h}, i = 1, \dots, 4$, which are defined by* (6.12)*–*(6.16)*. Then the estimate*

$$\mathcal{E}_\alpha(z_h|\hat{z}_h)(t) + \int_0^t \mathcal{D}_{\phi_h}(\mu_h, z_h|\hat{\mu}_h, \hat{z}_h)\, \mathrm{d}s \tag{6.17}$$
$$\le e^{c(t)} \mathcal{E}_\alpha(z_h|\hat{z}_h)(0) + C e^{c(t)} \int_0^t \sum_{i \in \{1,3,4\}} \|\hat{r}_{i,h}\|_{-1,h}^2 + \|\hat{r}_{2,h}\|_1^2 \, \mathrm{d}s$$

holds for all $0 \le t \le T$ with $c(t) = c_0 t + c_1 \int_0^t \|\partial_t \hat{\phi}_h\|_0 \,\mathrm{d}s$ such that c_0, c_1, C depend only on the uniform $L^\infty(H^1) \times L^2(H^1)$ bounds for (ϕ_h, μ_h) and $(\hat{\phi}_h, \hat{\mu}_h)$, respectively. We denote by $\|\hat{r}\|_{-1,h} = \sup_{v_h \in \mathcal{V}_h} \frac{\langle \hat{r}, v_h \rangle}{\|v_h\|_1}$. It holds that $\|\hat{r}\|_{-1,h} \le \|\hat{r}\|_{-1} = \sup_{v_h \in H^1(\Omega)} \frac{\langle \hat{r}, v \rangle}{\|v\|_1}$.

Proof. The result follows with the very same arguments as used in the proof of Theorem 4.4.1 and is therefore omitted. In fact, the arguments are easier since we are allowed to test $\partial_t z_h$ with z_h since $\mathcal{V}_h$ is a finite-dimensional space and $(\phi, q, \mathbf{u})$ are C^1 in time, cf. proof of Lemma 6.1.2. □

We can observe that the following lower bounds hold for the relative energy and relative dissipation.

Lemma 6.2.2. *The following lower bounds hold for the relative energy $\mathcal{E}_\alpha$ and relative dissipation $\mathcal{D}_{\phi_h}$*

$$\|\phi_h - \hat{\phi}_h\|_1^2 + \|q_h - \hat{q}_h\|_0^2 + \|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_0^2 \le C \mathcal{E}_\alpha(z_h|\hat{z}_h), \tag{6.18}$$

$$\frac{\varepsilon_0 b_1}{2}\|\nabla(\mu_h - \hat{\mu}_h)\|_0^2 + \frac{\varepsilon_1}{2}\|\nabla(q_h - \hat{q}_h)\|_0^2 + \frac{\eta_1}{2}\|\nabla(\mathbf{u}_h - \hat{\mathbf{u}}_h)\|_0^2 \leq \mathcal{D}_{\phi_h}(\mu_h, z_h|\hat{\mu}_h, z_h). \tag{6.19}$$

Since the difference of mean values of $\mu_h - \hat{\mu}_h$ can be controlled by the relative energy we deduce full control of $\|\mu_h - \hat{\mu}_h\|_1^2$.

Proof. The bounds in (6.18) follow immediately from the construction of the relative energy and (A8). The bounds in (6.19) follow from the lower bounds of the parametric functions, i.e., (A1), (A4), (A6). For the mean value of $\mu_h - \hat{\mu}_h$ we consider the difference of (6.7) and (6.13) and insert $\xi_h = 1$ to obtain

$$\langle \mu_h - \hat{\mu}_h, 1\rangle = \langle f'(\phi_h) - f'(\hat{\phi}_h), 1\rangle \leq C(f_2^{(2)}, f_2^{(3)}(\|\phi_h\|_1, \|\hat{\phi}_h\|_1))\mathcal{E}_\alpha(\phi_h|\hat{\phi}_h).$$

This already completes the proof. □

Similar to the decomposition of the relative energy into separate contributions, we will frequently use the single dissipation functionals $\mathcal{D}_{\phi_h}(\mu_h|\hat{\mu}_h)$, $\mathcal{D}_{\phi_h}(q_h|\hat{q}_h)$ and $\mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h)$. In order to identify the residuals, we will now choose $(\hat{z}_h, \hat{p}_h, \hat{\mu}_h)$ as suitable projections of the smooth solution (z, p, μ).

6.3. Auxiliary results

In this section, we will make a specific choice for the perturbed solution $(\hat{z}_h, \hat{\mu}_h)$ as suitable projections of the true solution. Furthermore, we will obtain the projection errors which will be necessary for the error analysis in Section 6.4. Let $(\phi, \mu, q, \mathbf{u}, p)$ be a weak solution of (6.1)–(6.5). We introduce $\hat{\phi}_h(t)$ as the H^1-elliptic projection, i.e. $\hat{\phi}_h(t) = \pi_h^1\phi(t) \in \mathcal{V}_h$, $\hat{q}(t) = \pi_h^0 q(t) \in \mathcal{V}_h$ by the L^2-projection in space and the pair $(\hat{\mathbf{u}}_h(t), \hat{p}_h(t)) = (\mathbf{P}_h^1\mathbf{u}(t), \pi_{p,h}^0 p(t)) \in \mathcal{V}_h \times \mathcal{Q}_h$ by the Stokes projection with zero pressure and the L^2-projection for the pressure. Finally, we choose $\hat{\mu}_h \in V_h$ by solving the elliptic variational problems

$$\langle \hat{\mu}_h(t) - \mu(t), \xi_h\rangle - \gamma\langle \nabla\hat{\phi}_h(t) - \nabla\phi(t), \nabla\xi_h\rangle - \langle f'(\hat{\phi}_h(t)) - f'(\phi(t)), \xi_h\rangle = 0 \tag{6.20}$$

for all $\xi_h \in \mathcal{V}_h$ and $0 \leq t \leq T$. Existence of a unique solution $\hat{\mu}_h$ follows immediately, since the problem is linear in $\hat{\mu}_h(t)$ and has finite dimensions. For this choice of $(\hat{\phi}_h, \hat{\mu}_h, \hat{q}_h, \hat{\mathbf{u}}_h, \hat{p}_h)$, we have the following result for the approximation error.

Lemma 6.3.1. *Let Assumptions 6.1.3 hold and let $(\hat{\phi}_h, \hat{\mu}_h, \hat{q}_h, \hat{\mathbf{u}}_h, \hat{p}_h)$ be defined as above. Then the following projection errors hold*

$$\begin{aligned}
\|\phi(t) - \hat{\phi}_h(t)\|_1 &\leq Ch^2\|\phi(t)\|_3, & \|\partial_t\phi(t) - \partial_t\hat{\phi}_h(t)\|_{-1,h} &\leq Ch^2\|\partial_t\phi(t)\|_1,\\
\|q(t) - \hat{q}_h(t)\|_1 &\leq Ch^2\|q(t)\|_3, & \|\partial_t q(t) - \partial_t\hat{q}_h(t)\|_{-1,h} &\leq Ch^2\|\partial_t q(t)\|_1,\\
\|\mathbf{u}(t) - \hat{\mathbf{u}}_h(t)\|_1 &\leq Ch^2\|\mathbf{u}(t)\|_3, & \|\partial_t\mathbf{u}(t) - \partial_t\hat{\mathbf{u}}_h(t)\|_{-1,h} &\leq Ch^2\|\partial_t\mathbf{u}(t)\|_1,\\
\|\mu(t) - \hat{\mu}_h(t)\|_1 &\leq C'h^2(\|\mu(t)\|_3 + \|\phi(t)\|_3), & \|p(t) - \hat{p}_h(t)\|_0 &\leq Ch^2\|p(t)\|_2,
\end{aligned}$$

for $0 \leq t \leq T$ with constants $C = C(\Omega)$ and $C' = C'(\Omega, \gamma, f_2^{(2)}, f_3^{(2)}, C_T(\|\phi_0\|_3))$.

Remark 6.3.2. It will become clear in the proof that assumption (A8)–(A9) will not be used in the following proof.

Proof. The estimates for $\phi - \hat{\phi}_h$, $q - \hat{q}_h$ and $\mathbf{u} - \hat{\mathbf{u}}_h$ together with the corresponding time derivatives follow directly from (5.6), (5.4) and (5.11), respectively. The estimate for $p - \hat{p}_h$ follows from (5.12). The bound for the error $\mu - \hat{\mu}_h$ is given in [29] and for completeness we provide it here. We use the triangle inequality to split the error in the chemical potential, i.e., $\mu - \hat{\mu}_h$ into

$$\|\hat{\mu}_h - \mu\|_1 \leq \|\hat{\mu}_h - \pi_h^0 \mu\|_1 + \|\pi_h^0 \mu - \mu\|_1.$$

Using the standard projection error (5.4), the last term can directly be estimated by $\|\pi_h^0 \mu - \mu\|_1 \leq Ch^2 \|\mu\|_3$. For the first term we apply inverse inequality (5.7), such that the discrete error component can be bounded by

$$\|\hat{\mu}_h - \pi_h^0 \mu\|_1 \leq Ch^{-1} \|\hat{\mu}_h - \pi_h^0 \mu\|_0.$$

For the error in the L^2-norm, we can deduce, using the definition of the L^2-projection (5.3), that

$$\|\hat{\mu}_h - \pi_h^0 \mu\|_0^2 = (\hat{\mu}_h - \pi_h^0 \mu, \hat{\mu}_h - \pi_h^0 \mu) = (\hat{\mu}_h - \mu, \hat{\mu}_h - \pi_h^0 \mu),$$

since $\xi_h = \hat{\mu}_h - \pi_h^0 \mu \in \mathcal{V}_h$. We can then use the variational problem (6.20) with this specific choice if test function ξ_h, and observe that

$$\begin{aligned}(\hat{\mu}_h - \hat{\mu}, \xi_h) &= \gamma(\nabla(\hat{\phi}_h - \phi), \nabla \xi_h) + (f'(\hat{\phi}_h) - f'(\phi), \xi_h) \\ &= \gamma(\phi - \hat{\phi}_h, \xi_h) + (f'(\hat{\phi}_h) - f'(\phi), \xi_h).\end{aligned}$$

Here we used the particular choice of $\hat{\phi}_h = \pi_h^1 \phi$ and (5.5), to replace the gradient term in the second step. We note that using this property of the elliptic projection is central to obtain the optimal convergence rate. Proceeding with standard arguments, we then obtain

$$\begin{aligned}(\hat{\mu}_h - \hat{\mu}, \xi_h) &\leq \gamma \|\phi - \hat{\phi}_h\|_0 \|\xi_h\|_0 + \|f'(\hat{\phi}_h) - f'(\phi)\|_0 \|\xi_h\|_0 \\ &\leq C(f_2^{(2)}, f_3^{(2)}, C_T, \Omega) \|\hat{\phi}_h - \phi\|_0 \|\xi_h\|_0.\end{aligned}$$

In the last step, we have used similar arguments as before to estimate the second term, involving the potential f', using the bounds $\|\hat{\phi}_h\|_{0,\infty} + \|\hat{\phi}\|_{0,\infty} \leq C\|\phi\|_2$ for the H^1-projection and the true solution. In summary, we thus obtain

$$\|\hat{\mu}_h - \mu\|_1 \leq Ch^2(\|\mu\|_3 + \|\phi\|_3),$$

with constant C independent of the mesh size and uniform for all $0 \leq t \leq T$. □

Using that $(\phi, \mu, q, \mathbf{u}, p)$ solves (6.1)–(6.5) and the definition of the semi-discrete perturbed solution $(\hat{\phi}_h, \hat{\mu}_h, \hat{q}_h, \hat{\mathbf{u}}_h, \hat{p}_h)$, one can see that (6.12)–(6.16) is satisfied with residuals $\hat{r}_{2,h} = 0$ and

$$\langle \hat{r}_{1,h}, \psi_h \rangle = \langle \partial_t \hat{\phi}_h - \partial_t \phi, \psi_h \rangle + \mathbf{c}(\mathbf{u} - \hat{\mathbf{u}}_h; \psi_h, \phi_h) + \mathbf{c}(\hat{\mathbf{u}}_h; \psi_h, \phi - \phi_h) \tag{6.21}$$

$$
\begin{aligned}
&+ \varepsilon_0\langle b(\phi_h)\nabla(\hat{\mu}_h - \mu), \nabla\psi_h\rangle + \varepsilon_0\langle (b(\phi_h) - b(\phi))\nabla\mu, \nabla\psi_h\rangle \\
&+ \langle b^{1/2}(\phi_h)\nabla\hat{\mu}_h - \nabla(A(\phi_h)\hat{q}_h), b^{1/2}(\phi_h)\nabla\psi_h\rangle \\
&- \langle b^{1/2}(\phi)\nabla\mu - \nabla(A(\phi)q), b^{1/2}(\phi)\nabla\psi_h\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{r}_{3,h}, \zeta_h\rangle =& \langle \partial_t\hat{q}_h - \partial_t q, \zeta_h\rangle + \tilde{\mathbf{c}}(\mathbf{u}_h - \mathbf{u}; \hat{q}_h, \zeta_h) + \tilde{\mathbf{c}}(\mathbf{u}; \hat{q}_h - q, \zeta_h) \\
&+ \langle \kappa(\phi_h)(\hat{q}_h - q), \zeta_h\rangle + \langle (\kappa(\phi_h) - \kappa(\phi))q, \zeta_h\rangle + \varepsilon_0\langle \nabla(q_h - q), \nabla\zeta_h\rangle \\
&+ \langle \nabla(A(\phi_h)\hat{q}_h) - b^{1/2}(\phi_h)\nabla\mu_h, \nabla(A(\phi_h)\zeta_h)\rangle \\
&- \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\zeta_h)\rangle,
\end{aligned}
\tag{6.22}
$$

$$
\begin{aligned}
\langle \hat{r}_{4,h}, \mathbf{v}_h\rangle =& \langle \partial_t\hat{\mathbf{u}}_h - \partial_t\mathbf{u}, \mathbf{v}_h\rangle + \tilde{\mathbf{c}}(\mathbf{u}_h - \mathbf{u}; \hat{\mathbf{u}}_h, \mathbf{v}_h) + \tilde{\mathbf{c}}(\mathbf{u}; \hat{\mathbf{u}}_h - \mathbf{u}, \mathbf{v}_h) \\
&+ \langle \eta(\phi_h)\nabla(\hat{\mathbf{u}}_h - \mathbf{u}), \nabla\mathbf{v}_h\rangle + \langle (\eta(\phi_h) - \eta(\phi))\nabla\mathbf{u}, \nabla\mathbf{v}_h\rangle \\
&- \langle \hat{p}_h - p, \operatorname{div}\mathbf{v}_h\rangle + \mathbf{c}(\mathbf{v}_h; \hat{\mu}_h, \phi_h - \phi) + \mathbf{c}(\mathbf{v}_h; \hat{\mu}_h - \mu, \phi).
\end{aligned}
\tag{6.23}
$$

In the next step, we will estimate the residual between the discrete relative energy, the discrete dissipation and projection errors.

6.4. Error estimates

In this section, we will conduct the error analysis stated in Theorem 6.1.5. Using the above constructions the error analysis reduces to estimating the discrete residuals suitably by the relative energy, the relative dissipation and the projection errors. We remark that this is the first section where assumption (A9), i.e., $A = 1$, is used. We will proceed in estimating the residuals, which yields the following result.

Lemma 6.4.1. *Let Assumptions 6.1.3 and Lemma 6.3.1 hold. Then the following estimates hold*

$$
\begin{aligned}
&\int_0^t \|\hat{r}_{1,h}\|_{-1}^2 \,\mathrm{d}s \le C_1h^4 + \int_0^t C_2\mathcal{E}_\alpha(\phi_h|\hat{\phi}_h)\,\mathrm{d}s, \qquad \hat{r}_{2,h} = 0, \\
&\int_0^t \|\hat{r}_{3,h}\|_{-1}^2 \,\mathrm{d}s \le C_3h^4 + \int_0^t C_4\mathcal{E}_\alpha(\phi_h, \mathbf{u}_h|\hat{\phi}_h, \hat{\mathbf{u}}_h) + 2\delta\mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h)\,\mathrm{d}s, \\
&\int_0^t \|\hat{r}_{4,h}\|_{-1}^2 \,\mathrm{d}s \le C_5h^4 + \int_0^t C_6\mathcal{E}_\alpha(\phi_h, \mathbf{u}_h|\hat{\phi}_h, \hat{\mathbf{u}}_h) + 2\delta\mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h)\,\mathrm{d}s,
\end{aligned}
$$

with constants $C_i, i = 1, ..., 6$ independent of h and $\delta > 0$ sufficiently small.

Proof. **First residual:** We start estimating the first residual, cf. (6.21). Using the definition of the dual norm we find directly

$$
\begin{aligned}
\int_0^t \|\hat{r}_{1,h}\|_{-1}^2 \,\mathrm{d}s \le& \int_0^t \|\partial_t(\hat{\phi}_h - \phi)\|_{-1}^2 + \|\mathbf{u} - \hat{\mathbf{u}}_h\|_{0,3}^2\|\phi_h\|_{0,6}^2 + \|\hat{\mathbf{u}}_h\|_{0,3}^2\|\phi_h - \phi\|_{0,6}^2 \\
&+ (1 + \varepsilon_0)\|b(\phi_h)\nabla(\hat{\mu}_h - \mu)\|_0^2 + \|b(\phi_h) - b(\phi)\|_{0,6}^2\|\nabla\mu\|_{0,3}^2 \\
&+ \|b^{1/2}(\phi_h)\nabla(\hat{q}_h - q)\|_0^2 + \|b^{1/2}(\phi) - b^{1/2}(\phi_h)\|_{0,6}^2\|\nabla q\|_{0,3}^2 \,\mathrm{d}s \\
&= (i) + (ii) + (iii) + (iv) + (v) + (vi) + (vii).
\end{aligned}
$$

We will consider every term separately and estimate all these terms suitably by means of the relative energy, relative dissipation and projection errors.
Let us start with the first term. Definition of $\hat{\phi}_h$ together with Lemma 6.3.1 yields

$$(i) = \int_0^t \|\partial_t(\pi_h^1\phi - \phi)\|_{-1}^2 \,\mathrm{d}s \leq C \int_0^t h^4 \|\partial_t \phi\|_1^2 \,\mathrm{d}s.$$

For the second term we use the uniform $L^\infty(0,T;H^1(\Omega))$ bounds of ϕ_h, i.e., (6.11) and Lemma 6.3.1 to find

$$(ii) \leq C(\phi_h) \int_0^t \|\mathbf{u} - \hat{\mathbf{u}}_h\|_{0,3}^2 \,\mathrm{d}s \leq C \int_0^t h^4 \|\mathbf{u}\|_{0,3}^2 \,\mathrm{d}s.$$

For the third term, we employ the stability of the projection and Lemma 6.3.1 to find

$$\begin{aligned}
(iii) &\leq C \int_0^t \|\mathbf{u}\|_{0,3}^2 \|\phi_h \pm \hat{\phi}_h - \phi\|_{0,6}^2 \,\mathrm{d}s \\
&\leq C \int_0^t \|\mathbf{u}\|_{0,3}^2 (\|\phi_h - \hat{\phi}_h\|_{0,6}^2 + \|\hat{\phi}_h - \phi\|_{0,6}^2) \,\mathrm{d}s \\
&\leq C \int_0^t h^4 \|\mathbf{u}\|_{0,3}^2 \|\phi\|_3^2 + \|\mathbf{u}\|_{0,3}^2 \mathcal{E}_\alpha(\phi_h, \hat{\phi}_h) \,\mathrm{d}s.
\end{aligned}$$

For the next term, we obtain using the bounds for b, i.e., (A1) and Lemma 6.3.1, we get

$$(iv) \leq C \int_0^t \|\nabla(\hat{\mu}_h - \mu)\|_0^2 \,\mathrm{d}s \leq C \int_0^t h^4 (\|\mu\|_{0,3}^2 + \|\phi\|_{0,3}^2) \,\mathrm{d}s.$$

By Taylor's theorem, the bounds on b, i.e., (A1), and Lemma 6.3.1 we deduce

$$(v) \leq C(b_3) \int_0^t \|\phi_h - \phi\|_{0,6}^2 \|\nabla \mu\|_{0,3}^2 \,\mathrm{d}s \leq C \int_0^t \|\mu\|_{1,3}^2 \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h) + h^4 \|\mu\|_{1,3}^2 \|\phi\|_3^2 \,\mathrm{d}s.$$

The sixth term can be treated by the arguments used for the fourth term, and we find

$$(vi) \leq C \int_0^t h^4 \|q\|_3^2 \,\mathrm{d}s.$$

The seventh term follows the same reasoning as the fifth and yields

$$(vii) \leq C \int_0^t \|q\|_{1,3}^2 \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h)(s) + h^4 \|q\|_{1,3}^2 \|\phi\|_3^2 \,\mathrm{d}s.$$

Summing the above estimates together yields

$$\int_0^t \|\hat{r}_{1,h}\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t C_1(z,\mu) h^4 + C_2(z,\mu) \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h) \,\mathrm{d}s$$

with $C_1(z,\mu) = C(\|\partial_t \phi\|_1, \|\phi\|_3, \|\mathbf{u}\|_3, \|\mu\|_3, \|q\|_3)$, $C_2(z,\mu) = C(\|\mu\|_{1,3}, \|\mathbf{u}\|_{0,3}, \|q\|_{1,3})$.
Note that the second residual is zero.
Third residual: We will now consider the third residual, which yields

$$\int_0^t \|\hat{r}_{3,h}\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t \|\partial_t(\hat{q}_h - q)\|_{-1}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,3}^2 \|\hat{q}_h\|_1^2 + \|\mathbf{u}\|_{0,3}^2 \|\hat{q}_h - q\|_{0,6}^2$$

$$\begin{aligned}
&+ \|h^{1/2}(\phi_h)(\hat{q}_h - q)\|_0^2 + \|h^{1/2}(\phi_h) - h^{1/2}(\phi)\|_{0,6}^2 \|\hat{q}_h\|_{0,3}^2 \\
&+ C(\varepsilon_1)\|\nabla(\hat{q}_h - q)\|_0^2 + \|b(\phi_h)\nabla(\hat{\mu}_h - \mu)\|_0^2 + \|b(\phi_h) - b(\phi)\|_{0,6}^2 \|\mu\|_{1,3}^2 \,\mathrm{d}s \\
=&(i) + (ii) + (iii) + (iv) + (v) + (vi) + (vii).
\end{aligned}$$

Consider the first term, which by similar reasoning as in the first residual yields

$$(i) \leq C \int_0^t h^4 \|\partial_t q\|_1^2 \,\mathrm{d}s.$$

The second term yields by using the stability of the L^2-projection, an addition of zero and the interpolation inequality

$$\begin{aligned}
(ii) &\leq C \int_0^t (\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{0,3}^2 + \|\hat{\mathbf{u}}_h - \mathbf{u}_h\|_{0,3}^2)\|q\|_1^2 \,\mathrm{d}s \\
&\leq 2\delta \int_0^t \mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h) \,\mathrm{d}s + C \int_0^t \|q\|_1^2 \mathcal{E}(\mathbf{u}_h|\hat{\mathbf{u}}_h) + h^4 \|q\|_1^2 \|\mathbf{u}\|_{2,3}^2 \,\mathrm{d}s.
\end{aligned}$$

The third term is straightforward using Lemma 6.3.1 and yields

$$(iii) \leq C \int_0^t h^4 \|\mathbf{u}\|_{0,3}^2 \|q\|_3^2 \,\mathrm{d}s.$$

We note that the fourth and fifth terms can be estimated using the bounds on κ, Taylor's theorem and Lemma 6.3.1 which yields

$$(iv) + (v) \leq C \int_0^t h^4 \|q\|_3^2 + h^4 \|q\|_{0,3}^2 \|\phi\|_3^2 + \|q\|_{0,3}^2 \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h) \,\mathrm{d}s.$$

Since the last three terms already appeared in the first residual, we obtain directly

$$(vi) + (vii) + (viii) \leq C \int_0^t h^4(\|q\|_2^2 + \|\mu\|_{0,3}^2 + \|\phi\|_3^2 + \|\mu\|_{1,3}^2 \|\phi\|_3^2) + \|\mu\|_{1,3}^2 \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h) \,\mathrm{d}s.$$

Summing up the estimates yields the following bound for the third residual

$$\int_0^t \|\hat{r}_{3,h}\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t C_3(z,\mu)h^4 + C_4(z,\mu)\mathcal{E}(\phi_h, \mathbf{u}_h|\hat{\phi}_h, \hat{\mathbf{u}}_h) + 2\delta \mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h) \,\mathrm{d}s,$$

where $C_3 = C(\|\partial_t q\|_1, \|\phi\|_3, \|\mathbf{u}\|_3, \|\mu\|_3, \|q\|_3)$ and $C_4 = C(\|\mu\|_{1,3}, \|\mathbf{u}\|_{0,3}, \|q\|_{1,3})$.

Fourth residual: Let us now turn to the final residual. Using the definition of the dual norm, we obtain

$$\begin{aligned}
\int_0^t \|\hat{r}_{4,h}\|_{-1}^2 \,\mathrm{d}s \leq \int_0^t &\|\partial_t(\hat{\mathbf{u}}_h - \mathbf{u})\|_{-1}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{0,3}^2 \|\hat{\mathbf{u}}_h\|_1^2 + \|\mathbf{u}\|_{0,3}^2 \|\hat{\mathbf{u}}_h - \mathbf{u}\|_{0,6}^2 \\
&+ \|\eta^{1/2}(\phi_h)\nabla(\hat{\mathbf{u}}_h - \mathbf{u})\|_0^2 + \|\eta^{1/2}(\phi_h) - \eta^{1/2}(\phi)\|_{0,6}^2 \|\nabla \hat{\mathbf{u}}_h\|_{0,3}^2 \\
&+ \|\hat{p}_h - p\|_0^2 + \|\hat{\mu}_h\|_{0,3/2}^2 \|\phi_h - \phi\|_{0,6}^2 + \|\hat{\mu}_h - \mu\|_0^2 \|\phi\|_{0,3}^2 \,\mathrm{d}s \\
&= (i) + (ii) + (iii) + (iv) + (v) + (vi) + (vii).
\end{aligned}$$

We note that all estimates follow the same structure as the estimates of the other residuals. Therefore, we omit the details and list the estimates

$$(i) \leq C \int_0^t h^4 \|\partial_t \mathbf{u}\|_1^2 \,\mathrm{d}s,$$

$$
\begin{aligned}
(ii) + (iii) &\leq \int_0^t 2\delta \mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h)\, \mathrm{d}s + C\int_0^t h^4 \|\mathbf{u}\|_1^2 \|\mathbf{u}\|_3^2 + \|\mathbf{u}\|_1^4 \mathcal{E}(\mathbf{u}_h|\hat{\mathbf{u}}_h)\, \mathrm{d}s,\\
(iv) + (v) &\leq C\int_0^t h^4 (\|\mathbf{u}\|_3^2 + \|\mathbf{u}\|_{1,3}^2 \|\phi\|_3^2) + \|\mathbf{u}\|_{1,3}^2 \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h)\, \mathrm{d}s,\\
(vi) &\leq C\int_0^t h^4 \|p\|_2^2\, \mathrm{d}s,\\
(vii) + (viii) &\leq C\int_0^t h^4 (\|\mu\|_{1,3}\|\phi\|_3^2 + \|\mu\|_3^2 + \|\phi\|_3^2) + \|\mu\|_{1,3}^2 \mathcal{E}_\alpha(\phi_h|\hat{\phi}_h)\, \mathrm{d}s.
\end{aligned}
$$

Altogether this yields the following bound for the fourth residual

$$
\int_0^t \|\hat{r}_{4,h}\|_{-1}^2\, \mathrm{d}s \leq \int_0^t C_5(z,\mu)h^4 + C_6(z,\mu)\mathcal{E}_\alpha(\phi_h, \mathbf{u}_h|\hat{\phi}_h, \hat{\mathbf{u}}_h) + 2\delta\mathcal{D}_{\phi_h}(\mathbf{u}_h|\hat{\mathbf{u}}_h)\, \mathrm{d}s,
$$

where $C_5(z,\mu) = C(\|\partial_t \mathbf{u}\|_1, \|\phi\|_3, \|\mathbf{u}\|_3, \|\mu\|_3, \|p\|_2)$ and $C_6(z,\mu) = C(\|\mu\|_{1,3}, \|\mathbf{u}\|_{1,3})$. □

With these estimates at hand, we can control the discrete error. Using the semi-discrete stability estimate, i.e., Lemma 6.2.1, the bounds in Lemma 6.4.1 we can estimate the error by the Gronwall lemma, cf. Lemma A.3.1. This yields the following result.

Lemma 6.4.2. *Let Assumptions 6.1.3 and Lemma 6.4.1 hold. Then*

$$
\begin{aligned}
&\|\phi_h - \hat{\phi}_h\|_{L^\infty(H^1)}^2 + \|q_h - \hat{q}_h\|_{L^\infty(L^2)}^2 + \|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{L^\infty(L^2)}^2\\
&\quad + \|\mu_h - \hat{\mu}_h\|_{L^2(H^1)}^2 + \|q_h - \hat{q}_h\|_{L^2(H^1)}^2 + \|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{L^2(H^1)}^2 \leq Ch^4.
\end{aligned}
$$

The proof is omitted. Since the projection errors are already identified in Lemma 6.3.1 we can use the triangle inequality for a suitable norm $\|\cdot\|_*$ to show

$$
\|z_h - z\|_* \leq \|z_h - \hat{z}_h\|_* + \|\hat{z}_h - z\|_*.
$$

While the first error is estimated in Lemma 6.4.2, the second error is the standard projection error and is already given in Lemma 6.3.1. This yields the convergence result up to the pressure convergence.

6.5. Pressure estimate

In this section, we will consider error estimates for the discrete pressure $p_h - \hat{p}_h$ using the discrete inf-sup stability, i.e. (5.8) and the techniques of Ayuso et al., see [11].

The error estimate can be deduced from the discrete inf-sup stability (5.8), i.e., from

$$
\|p_h - \hat{p}_h\|_0 \leq \frac{1}{\beta} \sup_{\mathbf{0}\neq \mathbf{v}_h \in \mathcal{V}_h^d} \frac{\langle p_h - \hat{p}_h, \operatorname{div} \mathbf{v}_h\rangle}{\|\mathbf{v}_h\|_1}.
$$

Rearrangement and insertion of the discrete formulation (6.9) and (6.15) yields

$$
\begin{aligned}
\langle p_h - \hat{p}_h, \operatorname{div} \mathbf{v}_h\rangle &= \langle \partial_t \mathbf{u}_h - \partial_t \hat{\mathbf{u}}_h, \mathbf{v}_h\rangle + \mathbf{c}(\mathbf{u}_h; \mathbf{u}_h - \hat{\mathbf{u}}_h, \mathbf{v}_h)\\
&\quad + \langle \eta(\phi_h)\nabla(\mathbf{u}_h - \hat{\mathbf{u}}_h), \nabla \mathbf{v}_h\rangle + \mathbf{c}(\mathbf{v}_h; \mu_h - \hat{\mu}_h, \phi_h) - \langle \hat{r}_{4,h}, \mathbf{v}_h\rangle.
\end{aligned}
$$

From the error estimate of Theorem 6.1.5 we immediately deduce

$$\begin{aligned}\int_0^t \beta^2\|p_h - \hat{p}_h\|_0^2 \,\mathrm{d}s \le \int_0^t &\|\partial_t \mathbf{u}_h - \partial_t \hat{\mathbf{u}}_h\|_{-1,h}^2 + C\|\mathbf{u}_h\|_{0,\infty}^2\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_1^2 + C(\eta)\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_1^2 \\ &+ \|\phi_h\|_{0,3}^2\|\nabla(\mu_h - \hat{\mu}_h)\|_0^2 + \|\hat{r}_{4,h}\|_{-1}^2 \,\mathrm{d}s \\ \le \int_0^t &\|\partial_t \mathbf{u}_h - \partial_t \hat{\mathbf{u}}_h\|_{-1,h}^2 + C\|\mathbf{u}_h\|_{0,\infty}^2\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_1^2 + Ch^4 \,\mathrm{d}s. \qquad (6.24)\end{aligned}$$

For the second term we add a suitable zero, use the inverse inequality (5.7) with $p = \infty, q = 2, d \le 3$ and obtain

$$\begin{aligned}\int_0^t \|\mathbf{u}_h\|_{0,\infty}^2\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_1^2 &\le \int_0^t \|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{0,\infty}^2\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_1^2 + C\|\hat{\mathbf{u}}_h\|_{0,\infty}\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_1^2 \,\mathrm{d}s \\ &\le h^{-3}\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{L^\infty(L^2)}^2\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{L^2(H^1)}^2 + C(\mathbf{u})\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{L^2(H^1)}^2 \\ &\le Ch^4. \qquad (6.25)\end{aligned}$$

The above inequalities together yield

$$\beta^2 \int_0^t \|p_h - \hat{p}_h\|_0^2 \,\mathrm{d}s \le Ch^4 + C\int_0^t \|\partial_t \mathbf{u}_h - \partial_t \hat{\mathbf{u}}_h\|_{-1,h}^2 \,\mathrm{d}s. \qquad (6.26)$$

Thus, the only term to estimate is $\|\partial_t(\mathbf{u}_h - \hat{\mathbf{u}}_h)\|_{-1,h}^2$. However, the standard way of error estimation would already include the discrete error for the pressure. Therefore, we need another way to obtain the desired convergence result from the discrete weakly-divergence free formulation, i.e., the formulation, where the pressure contribution vanishes. To prove this, we will consider techniques developed by Ayuso et al., Ahmed et al. and John, [11, 5, 81].

Lemma 6.5.1. *Let Lemma 6.4.2 hold. Then the following error estimate holds*

$$\|\partial_t(\mathbf{u}_h - \hat{\mathbf{u}}_h)\|_{L^2(H^{-1})}^2 + \|p_h - p\|_{L^2(L^2)}^2 \le Ch^4.$$

Proof. Let us recall the subspace of discretely divergence free functions of $\mathcal{V}_h^d$ given by $\mathbb{V}_h := \{\mathbf{v}_h \in \mathcal{V}_h^d : \langle \operatorname{div} \mathbf{v}_h, w_h\rangle = 0, \forall w_h \in \mathcal{Q}_h\}$, cf. (5.9). We introduce the linear operator $A_h : \mathbb{V}_h \to \mathbb{V}_h$ defined by

$$\langle A_h\mathbf{u}_h, \mathbf{v}_h\rangle = \langle \nabla\mathbf{u}_h, \nabla\mathbf{v}_h\rangle, \qquad \forall \mathbf{v}_h \in \mathbb{V}_h.$$

Then the following equalities, proven in [5], hold

$$\|A_h^{1/2}\mathbf{u}_h\|_0 = \|\nabla\mathbf{u}_h\|_0, \qquad \|\nabla A_h^{-1/2}\mathbf{u}_h\|_0 = \|\mathbf{u}_h\|_0, \quad \forall \mathbf{u}_h \in \mathbb{V}_h. \qquad (6.27)$$

Following the lines of Ayuso et al., John and John et al. [11, 5, 81], one can the following lemma.

Lemma 6.5.2. *There exists a positive constant C independent of h such that*

$$\|\partial_t(\mathbf{u}_h - \hat{\mathbf{u}}_h)\|_{-1}^2 \le C\|A_h^{-1/2}\partial_t(\mathbf{u}_h - \hat{\mathbf{u}}_h)\|_0. \qquad (6.28)$$

Proof. The proof can be found in [11, Lemma 3.11]. It use the stability of the discrete Leray-projection, i.e. the L^2-projection from $H^1_{\mathrm{div}}(\Omega)^d$ onto $\mathbb{V}_h$, and usage of the norm equalities (6.27) with the inverse inequality. □

We introduce the short notation $\mathbf{e}_h = \mathbf{u}_h - \hat{\mathbf{u}}_h$. It suffices to obtain an error estimate for $\|A_h^{-1/2}\partial_t \mathbf{e}_h\|_0$. To this end we take the difference of (6.9) and (6.15), restricted to the space $\mathbb{V}_h$ and insert $A_h^{-1}\partial_t\mathbf{e}_h$ as test function. We obtain

$$\begin{aligned}\|A_h^{-1/2}\partial_t\mathbf{e}_h\|_0^2 =& - \langle \eta(\phi)\nabla\mathbf{e}_h, \nabla A_h^{-1}\partial_t\mathbf{e}_h\rangle - \tilde{\mathbf{c}}(\mathbf{u}_h;\mathbf{e}_h, A_h^{-1}\partial_t\mathbf{e}_h)\\ &- \langle \phi_h\nabla(\mu_h-\hat{\mu}_h), A_h^{-1}\partial_t\mathbf{e}_h\rangle - \langle \hat{r}_{4,h}, A_h^{-1}\partial_t\mathbf{e}_h\rangle\\ =&(i)+(ii)+(iii)+(iv).\end{aligned}$$

For the first term, we use Hölder's inequality and (6.27) to estimate

$$(i) \le \|\eta(\phi_h)\nabla\mathbf{e}_h\|_0\|\nabla A_h^{-1}\partial_t\mathbf{e}_h\|_0 \le C(\eta)\|\nabla\mathbf{e}_h\|_0\|A_h^{-1/2}\partial_t\mathbf{e}_h\|_0.$$

For the second term, we first expand the trilinear form $\tilde{\mathbf{c}}$ by

$$(ii) = \frac{1}{2}\langle(\mathbf{u}_h\cdot\nabla)\mathbf{e}_h, A_h^{-1}\partial_t\mathbf{e}_h\rangle - \frac{1}{2}\langle\mathbf{e}_h,(\mathbf{u}_h\cdot\nabla)A_h^{-1}\partial_t\mathbf{e}_h\rangle.$$

For the first part, we use the dual paring. For the second part, we directly estimate by using again (6.27) to find

$$(iii) \le C(\|(\mathbf{u}_h\cdot\nabla)\mathbf{e}_h\|_{-1} + \|\mathbf{e}_h\|_{0,2}\|\mathbf{u}_h\|_{0,\infty})\|A_h^{-1/2}\partial_t\mathbf{e}_h\|_0.$$

For the third and fourth terms, we use the dual pairing and again (6.27) to obtain

$$(iii)+(iv) \le (\|\phi_h\nabla(\mu_h-\hat{\mu}_h)\|_{-1} + \|\hat{r}_{4,h}\|_{-1})\|A_h^{-1/2}\partial_t\mathbf{e}_h\|_0.$$

Together this yields the bound

$$\begin{aligned}\|A_h^{-1/2}\partial_t\mathbf{e}_h\|_0 \le& C\|\nabla\mathbf{e}_h\|_0 + C(\|(\mathbf{u}_h\cdot\nabla)\mathbf{e}_h\|_{-1} + \|\mathbf{e}_h\|_{0,2}\|\mathbf{u}_h\|_{0,\infty})\\ &+ \|\phi_h\nabla(\mu_h-\hat{\mu}_h)\|_{-1} + \|\hat{r}_{4,h}\|_{-1}.\end{aligned}$$

We can observe that we have already estimated the square of all this terms, see (6.24) and (6.25). Therefore, we find by (6.28), squaring and integration over time

$$\int_0^t \|\partial_t\mathbf{e}_h\|_{-1}^2 \,\mathrm{d}s \le C\int_0^t \|A_h^{-1/2}\partial_t\mathbf{e}_h\|_0 \,\mathrm{d}s \le Ch^4.$$

The convergence of the discrete pressure follows then immediately from (6.26). This concludes the proof. □

With this, we have proven the rigorous second order convergence of the semi-discretisation in space, cf. Problem P.1, which preserves the energy-dissipative structure.

7 Fully discrete approximation

In this chapter, we will consider a full discretisation of the CHNSQ model in *space* and *time* by using a Petrov-Galerkin method in time on the semi-discretisation in space from Chapter 6. The main result of this chapter is the proof optimal second order error estimates in *space* and *time*.

In Section 7.1, we state the full discretisation of the CHNSQ model, i.e., System S.3, which we will analyse in the following. We establish the existence of dissipative discrete solutions via Brouwer's fixed-point theorem and state the main result on optimal second order convergence in space and time. In Section 7.2, we introduce a fully discrete perturbed problem and prove a fully discrete stability result, by using the discrete relative energy. In Section 7.3, we choose the perturbed solution as suitable projection and interpolation of the continuous solution and establish the projection errors. In Section 7.4, we conduct the error analysis, which reduces to estimating the residuals by the relative energy, the relative dissipation and projection error. Furthermore, we establish error estimates for the discrete pressure using the inf-sup stability. The last section, i.e., Section 7.6, considers uniqueness of discrete solutions. This again is treated using the discrete stability estimate and the proof is very similar to the weak-strong uniqueness proof, cf. Section 4.6.

7.1. Convergence result for the full discretisation

We now introduce the subsequent time discretization, for which we again employ a variational method. We recall that $\tau = T/N$, $N \in \mathbb{N}$, the discrete time points are given by $t^n := n\tau$ and the corresponding partition of $[0,T]$ by $\mathcal{I}_\tau := \{0 = t^0, t^1, \ldots, t^N = T\}$. Furthermore, $\overline{g} = \bar{\pi}_\tau^0 g$ denotes the piecewise constant projection of g with respect to time. For piecewise constant functions in time, this is simply the identity, while for piecewise linear functions this amounts to the mean-value in time per element. We look for approximations

$$\phi_{h,\tau} \in \mathbb{W}_{h,\tau}(0,T), \bar{\mu}_{h,\tau} \in \mathbb{Q}_{h,\tau}(0,T), q_{h,\tau} \in \mathbb{W}_{h,\tau}(0,T), \mathbf{u}_{h,\tau} \in \mathbb{X}_{h,\tau}(0,T), \bar{p}_{h,\tau} \in \mathbb{P}_{h,\tau}(0,T),$$

where the spaces are given by

$$\mathbb{W}_{h,\tau}(0,T) := P_1^c(\mathcal{I}_\tau; \mathcal{V}_h), \qquad \mathbb{Q}_{h,\tau}(0,T) := P_0(\mathcal{I}_\tau; \mathcal{V}_h),$$

$$\mathbb{X}_{h,\tau}(0,T) := P_1^c(\mathcal{I}_\tau;\mathcal{V}_h^d), \qquad \mathbb{P}_{h,\tau}(0,T) := P_0(\mathcal{I}_\tau;\mathcal{Q}_h).$$

To this end, we introduce the fully discrete problem as follows.

Problem P.2 (Fully discrete scheme)**.** Let $\phi_{h,0} \in \mathcal{V}_h, q_{h,0} \in \mathcal{V}_h$, $\mathbf{u}_{h,0} \in \mathcal{V}_h^d$ be given. Find the discrete functions $(\phi_{h,\tau}, \bar{\mu}_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau}, \bar{p}_{h,\tau}) \in \mathbb{W}_{h,\tau}(0,T) \times \mathbb{Q}_{h,\tau}(0,T) \times \mathbb{W}_{h,\tau}(0,T) \times \mathbb{X}_{h,\tau}(0,T) \times \mathbb{P}_{h,\tau}(0,T)$ with $\phi_{h,\tau}(0) = \phi_{h,0}, q_{h,\tau}(0) = q_{h,0}$, $\mathbf{u}_{h,\tau}(0) = \mathbf{u}_{h,0}$ and such that

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \langle \partial_t \phi_{h,\tau}, \bar{\psi}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \mathbf{c}(\bar{\mathbf{u}}_{h,\tau}; \bar{\psi}_{h,\tau}, \phi_{h,\tau}) - \langle \varepsilon_0 b(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau}\nabla\bar{\psi}_{h,\tau}\rangle && (7.1)\\
&\quad - \langle b^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau} - \nabla(A(\bar{\phi}_{h,\tau})\bar{q}_{h,\tau}), b^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\psi}_{h,\tau}\rangle \,\mathrm{d}s,\\
\int_{t^{n-1}}^{t^n} \langle \bar{\mu}_{h,\tau}, \bar{\xi}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \gamma\langle \nabla\phi_{h,\tau}, \nabla\bar{\xi}_{h,\tau}\rangle + \langle f'(\phi_{h,\tau}), \bar{\xi}_{h,\tau}\rangle \,\mathrm{d}s, && (7.2)\\
\int_{t^{n-1}}^{t^n} \langle \partial_t q_{h,\tau}, \bar{\zeta}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \bar{\zeta}_{h,\tau}, \bar{q}_{h,\tau}) - \langle \kappa(\bar{\phi}_{h,\tau})\bar{q}_{h,\tau}, \bar{\zeta}_{h,\tau}\rangle - \varepsilon_1\langle \nabla q_{h,\tau}, \nabla\bar{\zeta}_{h,\tau}\rangle\\
&\quad - \langle \nabla(A(\bar{\phi}_{h,\tau})\bar{q}_{h,\tau}) - b^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau}, \nabla(A(\bar{\phi}_{h,\tau})\bar{\zeta}_{h,\tau})\rangle \,\mathrm{d}s, && (7.3)\\
\int_{t^{n-1}}^{t^n} \langle \partial_t \mathbf{u}_{h,\tau}, \bar{\mathbf{v}}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \bar{\mathbf{v}}_{h,\tau}, \bar{\mathbf{u}}_{h,\tau}) - \langle \eta(\bar{\phi}_{h,\tau})\nabla\bar{\mathbf{u}}_{h,\tau}, \nabla\bar{\mathbf{v}}_{h,\tau}\rangle \,\mathrm{d}s && (7.4)\\
&\quad + \int_{t^{n-1}}^{t^n} \langle \bar{p}_{h,\tau}, \operatorname{div}\bar{\mathbf{v}}_{h,\tau}\rangle - \mathbf{c}(\bar{\mathbf{v}}_{h,\tau}; \bar{\mu}_{h,\tau}, \phi_{h,\tau}) \,\mathrm{d}s,\\
0 &= \int_{t^{n-1}}^{t^n} \langle \operatorname{div}\mathbf{u}_{h,\tau}, \bar{w}_{h,\tau}\rangle \,\mathrm{d}s, && (7.5)
\end{aligned}$$

for all $(\bar{\psi}_{h,\tau}, \bar{\xi}_{h,\tau}, \bar{\zeta}_{h,\tau}, \bar{\mathbf{v}}_{h,\tau}, \bar{w}_{h,\tau}) \in P_0(t^{n-1}, t^n; \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d \times \mathcal{Q}_h)$ and $n \geq 1$.

We remark that the algebraic variables $(\bar{\mu}_{h,\tau}, \bar{p}_{h,\tau})$, i.e., the variables without a time derivative, are considered piecewise constant in time, while the differential variables $(\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau})$ are piecewise linear in time. Recall that $(\bar{\phi}_{h,\tau}, \bar{q}_{h,\tau}, \bar{\mathbf{u}}_{h,\tau})$ then denote the piecewise mean value of $(\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau})$ on $\mathcal{I}_\tau$, i.e., the L^2-projection in time onto piecewise constant functions. We further note that for all $z_{h,\tau} \in P_c^1(\mathcal{I}_\tau;\mathcal{V}_h)$

$$\int_{t^{n-1}}^{t^n} \langle \partial_t z_{h,\tau}, z_{h,\tau} - \bar{z}_{h,\tau}\rangle \,\mathrm{d}s = 0 \text{ for all } n.$$

This follows since $\partial_t z \in P_0(I_\tau;\mathcal{V}_h)$ is piecewise constant in time and using the particular choice of the discrete method, we can show the following result.

Lemma 7.1.1. *Let (A0)–(A6) hold. Then for any $(\phi_{0,h}, q_{h,0}, \mathbf{u}_{0,h}) \in \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d$ and any $h, \tau > 0$ Problem P.2 has a solution. Moreover, every solution $(\phi_{h,\tau}, \bar{\mu}_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau}, \bar{p}_{h,\tau})$ of* (7.1)*–*(7.5) *satisfies for all $0 \leq t^n \leq T$ the identities $\int_\Omega \phi_{h,\tau}(t^n)\,\mathrm{d}x = \int_\Omega \phi_{0,h}\,\mathrm{d}x$ and $E(\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau})(t^n) + \int_0^{t^n} D_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{q}_{h,\tau}, \bar{\mathbf{u}}_{h,\tau})\,\mathrm{d}s = E(\phi_{h,0}, q_{h,0}, \mathbf{u}_{h,0})$. As a direct consequence, we obtain uniform bounds*

$$\begin{aligned}
&\|\phi_{h,\tau}\|_{L^\infty(H^1)}^2 + \|q_{h,\tau}\|_{L^\infty(L^2)}^2 + \|\mathbf{u}_{h,\tau}\|_{L^\infty(L^2)}^2 + \|\bar{\mathbf{u}}_{h,\tau}\|_{L^2(H^1)}^2 + \|\bar{q}_{h,\tau}\|_{L^2(H^1)}^2 && (7.6)\\
&\quad + \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau} - \nabla(A(\bar{\phi}_{h,\tau})\bar{q}_{h,\tau})\|_{L^2(L^2)}^2 + \|\bar{\mu}_{h,\tau}\|_{L^2(H^1)}^2\\
&\quad \leq C(\|\phi_{h,0}\|_1^2, \|q_{h,0}\|_0^2, \|\mathbf{u}_{h,0}\|_0^2).
\end{aligned}$$

Proof. Conservation of mass follows by testing the variational identity (7.1) by $\bar{\psi}_{h,\tau} = 1$. To derive the energy-dissipation identity we test the variational identities (7.1)–(7.5) with the set $(\bar{\psi}_{h,\tau}, \bar{\xi}_{h,\tau}, \bar{\zeta}_{h,\tau}, \bar{\mathbf{v}}_{h,\tau}, \bar{q}_{h,\tau}) = (\bar{\mu}_{h,\tau}, \partial_t \phi_{h,\tau}, \bar{q}_{h,\tau}, \bar{\mathbf{u}}_{h,\tau}, \bar{p}_{h,\tau})$ and obtain

$$E(\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau})|_0^{t^n} + \int_0^{t^n} \varepsilon_0 \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau}\|_0^2 + \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau} - \nabla(A(\bar{\phi}_{h,\tau})\bar{q}_{h,\tau})\|_0^2$$
$$+ \varepsilon_1 \|\nabla \bar{q}_{h,\tau}\|_0^2 + \|\kappa^{1/2}(\bar{\phi}_{h,\tau})\bar{q}_{h,\tau}\|_0^2 + \|\eta^{1/2}(\bar{\phi}_{h,\tau})\nabla\bar{\mathbf{u}}_{h,\tau}\|_0^2 \,\mathrm{d}s = 0.$$

To show the existence of at least one solution, we use an induction argument. Let $\phi_{h,\tau}(t^{n-1}), q_{h,\tau}(t^{n-1}), \mathbf{u}_{h,\tau}(t^{n-1})$ be given. In the n-th time step, we only need to determine

$$\phi_h^n := \phi_{h,\tau}(t^n),\ q_h^n := q_{h,\tau}(t^n),\ \mathbf{u}_h^n := \mathbf{u}_{h,\tau}(t^n),\ \mu_h^{n-1/2} := \bar{\mu}_{h,\tau}(t^n - \tau/2) \in \mathcal{V}_h$$

and $p_h^{n-1/2} := \bar{p}_{h,\tau}(t^n - \tau/2) \in \mathcal{Q}_h$. The above values can be derived either from Problem P.2 or the equivalent nonlinear time stepping scheme which we will consider in Chapter 8, see Problem P.3. Similar to the semi-discrete case, we restrict the solutions to the space of discrete divergence free functions, i.e., $\mathbf{u}_{h,\tau} \in P_1^c(\mathcal{I}_\tau; \mathbb{V}_h)$ and $\bar{\mathbf{v}}_{h,\tau} \in P_0(t^{n-1}, t^n; \mathbb{V}_h)$. Again, the pressure contribution $\bar{p}_{h,\tau}$ vanishes from the problem. From the discrete energy-dissipation identity, the bounds for the coefficients, and the equivalence of norms on finite-dimensional spaces, one can deduce that potential solutions are necessarily bounded.

The existence of a solution for the n-th time step then follows from Brouwer's fixed-point theorem. Finally, the existence of the discrete pressure $p_h^{n-1/2}$ follows from the discrete inf-sup stability (5.8). The uniform bounds for the solution, finally, follow directly from the energy-dissipation identity and using the lower bounds (A1), (A4)–(A6). □

In order to state the main result on error estimates, we introduce the following assumptions.

Assumptions 7.1.2. *Assume (A0)–(A6) and (A8)-(A9) and let* $(\phi, \mu, q, \mathbf{u}, p)$ *denote a weak solution of* (6.1)–(6.5) *with the initial value* $(\phi_0, q_0, \mathbf{u}_0) \in H^3(\Omega) \times H^2(\Omega) \times H^1_{div}(\Omega)^d \cap H^2(\Omega)^d$. *Furthermore, let* $(\phi, \mu, q, \mathbf{u}, p)$ *satisfy the additional regularity*

$$\begin{aligned}
&\phi \in H^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^3_p(\Omega)),\\
&\mu \in H^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; W^{1,3}(\Omega)) \cap L^2(0,T; H^3(\Omega)),\\
&q \in H^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; W^{1,3}(\Omega)) \cap L^2(0,T; H^3(\Omega)),\\
&\mathbf{u} \in H^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; W^{1,3}(\Omega)) \cap L^2(0,T; H^3(\Omega)),\\
&p \in L^2(0,T; H^2(\Omega)), \quad f'(\phi) \in H^2(0,T; H^1(\Omega)),\\
&\quad b(\phi)\nabla\mu, b^{1/2}(\phi)\nabla q, \kappa(\phi)q, \eta(\phi)\nabla\mathbf{u} \in H^2(0,T; L^2(\Omega)).
\end{aligned}$$

Furthermore, we denote by $(\phi_{h,\tau}, \bar{\mu}_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau}, \bar{p}_{h,\tau})$ *be a solution of Problem P.2 with* $\phi_{h,\tau}(0) = \pi_h^1\phi_0$, $q_{h,\tau}(0) = \pi_h^0 q_0$, $\mathbf{u}_{h,\tau}(0) = \mathbf{P}_h^1\mathbf{u}_0$.

With this existence result and the necessary assumptions, we are ready to formulate the convergence result for the fully discrete problem, i.e., Problem P.2.

Theorem 7.1.3. *Let Assumptions 7.1.2 hold. Then for any step size $\tau \leq \tau_0$ sufficiently small and $h > 0$ the following convergence result holds*

$$\max_{t^n \in \mathcal{I}_\tau} \left(\|\phi_{h,\tau} - \phi\|_1^2 + \|q_{h,\tau} - q\|_0^2 + \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_0^2 \right) + \|\bar{\mu}_{h,\tau} - \bar{\mu}\|_{L^2(H^1)}^2 + \|\bar{q}_{h,\tau} - \bar{q}\|_{L^2(H^1)}^2 + \|\bar{\mathbf{u}}_{h,\tau} - \bar{\mathbf{u}}\|_{L^2(H^1)}^2 \leq C(h^4 + \tau^4),$$

where C is independent of τ and h. Under the assumption $\tau = c_p h$, with a constant c_p independent of τ and h, the discrete solution is unique and the following error estimate for the pressure holds,

$$\|\bar{p}_{h,\tau} - \bar{p}\|_{L^2(L^2)}^2 \leq C(h^4 + \tau^4).$$

Remark 7.1.4. Let us interpret the above convergence result. First, the obtained convergence rates are order optimal, i.e. from a pure approximation point of view the rates cannot be enhanced. For the differential variables, i.e. $(\phi, q, \mathbf{u})$, one obtains pointwise error estimates in the energy norms, i.e. H^1 norm for ϕ and L^2 norm for q and $\mathbf{u}$. For the algebraic variables, i.e. (μ, p), and the space derivatives in the dissipation terms, one only obtains error estimates in L^2 norms in time. This will be important for the experimental convergence test later.

Before we step into the proof, let us again give a proof sketch.

1. In Section 7.2 we will introduce a fully discrete perturbed problem and a corresponding relative energy. Using the relative energy techniques, we will prove a fully discrete stability estimate. Since the energy-dissipative structure and all consequences are only valid for every time step, i.e., every element in $\mathcal{I}_\tau$, we have to consider the discrete Gronwall lemma, Lemma A.3.2, instead of the continuous lemma, cf. Lemma A.3.1.

2. In Section 7.3 we introduce a certain choice of the perturbed solution as suitable projection and interpolation of the continuous solution. Furthermore, we will deduce the projection errors in this section.

3. In Section 7.4 we will conduct the error analysis. By the aforementioned constructions, this reduces to estimating the fully discrete residual suitably by the relative energy, relative dissipation and projection and interpolation error. Furthermore, we consider the error analysis for the discrete pressure, which fortunately almost follows the lines of Section 6.4. In analogy to the semi-discrete case, this is the first time assumption (A9) is used.

4. In the last section, i.e., Section 7.6, we will investigate the uniqueness of the full discrete solutions. In contrast to the semi-discrete case where uniqueness follows immediately from Picard-Lindelöf, this is not the case for Brouwer's fixed-point theorem and hence this has to be investigated separately. This will again be considered using the full discrete stability estimate by following the ideas of the weak-strong uniqueness proof, i.e., Section 4.6, by assuming the perturbed solutions to be discrete solutions. Here, assumption (A9) is again used.

7.2. Discrete stability estimate

In this section, we will introduce the fully discrete perturbed system and the corresponding stability result. Let $(\hat{\phi}_{h,\tau}, \hat{\bar{\mu}}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau}, \hat{\bar{p}}_{h,\tau}) \in \mathbb{W}_{h,\tau}(0,T) \times \mathbb{Q}_{h,\tau}(0,T) \times \mathbb{W}_{h,\tau}(0,T) \times \mathbb{X}_{h,\tau}(0,T) \times \mathbb{P}_{h,\tau}(0,T)$ be a set of arbitrary discrete functions, we then define the discrete residuals $\bar{r}_{i,h,\tau} \in \mathbb{Q}_{h,\tau}(0,T)$, $i = 1, \dots, 4$ via

$$
\begin{aligned}
\int_{t^{n-1}}^{t^n} \langle \partial_t \hat{\phi}_{h,\tau}, \bar{\psi}_{h,\tau} \rangle \, \mathrm{d}s = \int_{t^{n-1}}^{t^n} & \mathbf{c}(\hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\psi}_{h,\tau}, \phi_{h,\tau}) - \langle \varepsilon_0 b(\bar{\phi}_{h,\tau}) \nabla \hat{\bar{\mu}}_{h,\tau} \nabla \bar{\psi}_{h,\tau} \rangle \qquad (7.7)\\
& - \langle b^{1/2}(\bar{\phi}_{h,\tau}) \nabla \hat{\bar{\mu}}_{h,\tau} - \nabla (A(\bar{\phi}_{h,\tau}) \hat{\bar{q}}_{h,\tau}), b^{1/2}(\bar{\phi}_{h,\tau}) \nabla \bar{\psi}_{h,\tau} \rangle \\
& + \langle \bar{r}_{1,h,\tau}, \bar{\psi}_{h,\tau} \rangle \, \mathrm{d}s \\
\int_{t^{n-1}}^{t^n} \langle \hat{\bar{\mu}}_{h,\tau}, \bar{\xi}_{h,\tau} \rangle \, \mathrm{d}s = \int_{t^{n-1}}^{t^n} & \gamma \langle \nabla \hat{\phi}_{h,\tau}, \nabla \bar{\xi}_{h,\tau} \rangle + \langle f'(\hat{\phi}_{h,\tau}), \bar{\xi}_{h,\tau} \rangle + \langle \bar{r}_{2,h,\tau}, \bar{\xi}_{h,\tau} \rangle \, \mathrm{d}s \qquad (7.8)\\
\int_{t^{n-1}}^{t^n} \langle \partial_t \hat{q}_{h,\tau}, \bar{\zeta}_{h,\tau} \rangle \, \mathrm{d}s = \int_{t^{n-1}}^{t^n} & \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \bar{\zeta}_{h,\tau}, \hat{\bar{q}}_{h,\tau}) - \langle \kappa(\bar{\phi}_{h,\tau}) \hat{\bar{q}}_{h,\tau}, \bar{\zeta}_{h,\tau} \rangle - \varepsilon_1 \langle \nabla \hat{q}_{h,\tau}, \nabla \bar{\zeta}_{h,\tau} \rangle \\
& - \langle \nabla (A(\bar{\phi}_{h,\tau}) \hat{\bar{q}}_{h,\tau}) - b^{1/2}(\bar{\phi}_{h,\tau}) \nabla \hat{\bar{\mu}}_{h,\tau}, \nabla (A(\bar{\phi}_{h,\tau}) \bar{\zeta}_{h,\tau}) \rangle \\
& + \langle \bar{r}_{3,h,\tau}, \bar{\zeta}_{h,\tau} \rangle \, \mathrm{d}s \qquad (7.9)\\
\int_{t^{n-1}}^{t^n} \langle \partial_t \hat{\mathbf{u}}_{h,\tau}, \bar{\mathbf{v}}_{h,\tau} \rangle \, \mathrm{d}s = \int_{t^{n-1}}^{t^n} & \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \bar{\mathbf{v}}_{h,\tau}, \hat{\bar{\mathbf{u}}}_{h,\tau}) - \langle \eta(\bar{\phi}_{h,\tau}) \nabla \hat{\mathbf{u}}_{h,\tau}, \nabla \bar{\mathbf{v}}_{h,\tau} \rangle \qquad (7.10)\\
& + \langle \hat{\bar{p}}_{h,\tau}, \operatorname{div} \bar{\mathbf{v}}_{h,\tau} \rangle - \mathbf{c}(\bar{\mathbf{v}}_{h,\tau}; \hat{\bar{\mu}}_{h,\tau}, \phi_{h,\tau}) + \langle \bar{r}_{4,h,\tau}, \bar{\mathbf{v}}_{h,\tau} \rangle \, \mathrm{d}s \\
0 = \int_{t^{n-1}}^{t^n} & \langle \operatorname{div} \hat{\mathbf{u}}_{h,\tau}, \bar{w}_{h,\tau} \rangle \, \mathrm{d}s \qquad (7.11)
\end{aligned}
$$

for all $(\bar{\psi}_{h,\tau}, \bar{\xi}_{h,\tau}, \bar{\zeta}_{h,\tau}, \bar{\mathbf{v}}_{h,\tau}, \bar{w}_{h,\tau}) \in P_0(t^{n-1}, t^n; \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d \times \mathcal{Q}_h)$ and for all $n \geq 1$.

In order to obtain a discrete analogue of the stability estimate (6.17) we use a fully discrete version of the relative energy as a measure between the discrete solution of Problem P.2 and the solution of the corresponding perturbed problem (7.7)–(7.11). The relative energy is given by

$$
\begin{aligned}
\mathcal{E}_\alpha(\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau} | \hat{\phi}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau}) := & \frac{\gamma}{2} \|\nabla \phi_{h,\tau} - \nabla \hat{\phi}_{h,\tau}\|_0^2 \\
& + \int_\Omega f(\phi_{h,\tau}) - f(\hat{\phi}_{h,\tau}) - f'(\hat{\phi}_{h,\tau})(\phi_{h,\tau} - \hat{\phi}_{h,\tau}) \\
& + \frac{\alpha}{2} \|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_0^2 + \frac{1}{2} \|q_{h,\tau} - \hat{q}_{h,\tau}\|_0^2 + \frac{1}{2} \|\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}\|_0^2.
\end{aligned}
$$

Furthermore, we often abbreviate the relative energies related to the single variables by $\mathcal{E}_\alpha(\phi_{h,\tau} | \hat{\phi}_{h,\tau}), \mathcal{E}(q_{h,\tau} | \hat{q}_{h,\tau}), \mathcal{E}(\mathbf{u}_{h,\tau} | \hat{\mathbf{u}}_{h,\tau})$, respectively.

Following the semi-discrete case, we can derive a stability estimate. However, since we are discrete in time, we have to resort to the discrete Gronwall lemma, cf. Lemma A.3.2. Since this inequality works different from the continuous version, we will provide full proof of the stability estimate. We will again use the notation notation $z_{h,\tau} = (\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau})$ where $(z_{h,\tau}, \bar{p}_{h,\tau}, \bar{\mu}_{h,\tau})$ denotes the discrete solution of Problem P.2 and we denote $\hat{z}_{h,\tau} = (\hat{\phi}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau})$ where $(\hat{z}_{h,\tau}, \hat{\bar{p}}_{h,\tau}, \hat{\bar{\mu}}_{h,\tau}) = (\hat{\phi}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau}, \hat{\bar{p}}_{h,\tau}, \hat{\bar{\mu}}_{h,\tau})$ is the discrete perturbed solution, cf. (7.7)–(7.11). In total, we find the following result.

Lemma 7.2.1. *Let Assumptions 7.1.2 hold. Then for a step size $\tau \leq \tau_0$ with τ_0 sufficiently small and $h > 0$ we find*

$$\mathcal{E}_\alpha(z_{h,\tau}(t^n)|\hat{z}_{h,\tau}(t^n)) + \int_0^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau})\, \mathrm{d}s \leq e^{ct^n}\mathcal{E}_\alpha(z_{h,\tau}(0)|\hat{z}_{h,\tau}(0)) + Ce^{ct^n}\int_0^{t^n} \sum_{i\in\{1,3,4\}} \|\bar{r}_{i,h,\tau}\|_{-1}^2 + \|\bar{r}_{2,h,\tau}\|_1^2\, \mathrm{d}s$$

for all $t^n \in \mathcal{I}_\tau$ with relative dissipation

$$\begin{aligned}\mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) =& \frac{1}{2}\Big(\varepsilon_0\|b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau})\|_0^2 + \|\kappa^{1/2}(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau})\|_0^2 \\ &+ \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}) - \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau}))\|_0^2 \\ &+ \varepsilon_1\|\nabla(\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau})\|_0^2 + \|\eta^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau})\|_0^2\Big)\end{aligned}$$

with constants $c = c_0 + c_1\|\partial_t\hat{\phi}_{h,\tau}\|_{L^\infty(L^2)}$, and c_0, c_1, C depending only on bounds for the coefficients, the domain Ω, and the uniform bounds for $\phi_{h,\tau}$ and $\hat{\phi}_{h,\tau}$ in $L^\infty(H^1(\Omega))$.

Similar to the semi-discrete case, we derive lower bounds for the relative energy and relative dissipation in the following result.

Lemma 7.2.2. *Let $\mathcal{E}_\alpha$ denote the relative energy and $\mathcal{D}_{\phi_{h,\tau}}$ the relative dissipation. Then*

$$\|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_1^2 + \|q_{h,\tau} - \hat{q}_{h,\tau}\|_0^2 + \|\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}\|_0^2 \leq C\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}), \tag{7.12}$$

$$\begin{aligned}\frac{\varepsilon_0 b_1}{2}\|\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau})\|_0^2 &+ \frac{\varepsilon_1}{2}\|\nabla(\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau})\|_0^2 \\ &+ \frac{\eta_1}{2}\|\nabla(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau})\|_0^2 \leq \mathcal{D}_{\phi_h}(\bar{\mu}_h, \bar{z}_h|\hat{\bar{\mu}}_h, \hat{\bar{z}}_h).\end{aligned} \tag{7.13}$$

Since the difference of mean values of $\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}$ is controlled by the relative energy, one can get full control of $\|\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}\|_1^2$.

Proof. The result follows the lines of the proofs for (6.18) and (6.19) and therefore is omitted. We only recall that $\bar{q}_{h,\tau}$ and $\bar{\mathbf{u}}_{h,\tau}$ denote the mean values in time per element. □

We will frequently use the dissipation functional related to every variable which we denote by $\mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \hat{\bar{\mu}}_{h,\tau})$, $\mathcal{D}_{\phi_{h,\tau}}(\bar{q}_{h,\tau}, \hat{\bar{q}}_{h,\tau})$, $\mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}, \hat{\bar{\mathbf{u}}}_{h,\tau})$.

Proof of Lemma 7.2.1 . First, we compute the time derivative of a general relative energy using the fundamental theorem of calculus, we obtain

$$\begin{aligned}\mathcal{E}(z_{h,\tau}|\hat{z}_{h,\tau})|_{t^{n-1}}^{t^n} &= \int_{t^{n-1}}^{t^n} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z_{h,\tau}|\hat{z}_{h,\tau})\, \mathrm{d}s \\ &= \int_{t^{n-1}}^{t^n} \langle E'(z_{h,\tau}) - E'(\hat{z}_{h,\tau}), \partial_t(z_{h,\tau} - \hat{z}_{h,\tau})\rangle \\ &\quad + \langle E'(z_{h,\tau}) - E'(\hat{z}_{h,\tau}) - E''(\hat{z}_{h,\tau})(z_{h,\tau} - \hat{z}_{h,\tau}), \partial_t\hat{z}_{h,\tau}\rangle\, \mathrm{d}s.\end{aligned} \tag{7.14}$$

Due to the linearity of the relative energy with respect to the solution components, i.e., $\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau}$ we can consider the contribution per equation separately and combine them at the very end. Again we emphasize that for quadratic energy, i.e. $E(z) = \|z\|_0^2$, the second inner-product vanishes.

Bulk stress equation: We start by considering the relative energy related to (7.3) and (7.9). Using (7.14), we find that

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(q_{h,\tau}|\hat{q}_{h,\tau})\,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \langle q_{h,\tau} - \hat{q}_{h,\tau}, \partial_t(q_{h,\tau} - \hat{q}_{h,\tau})\rangle\,\mathrm{d}s \\ &= \int_{t^{n-1}}^{t^n} \langle \bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}, \partial_t(q_{h,\tau} - \hat{q}_{h,\tau})\rangle\,\mathrm{d}s = (*).\end{aligned}$$

Inserting $\zeta_{h,\tau} = \bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau} \in P_0(t^{n-1}, t^n; \mathcal{V}_h)$ into (7.3) and (7.9) yields

$$\begin{aligned}(*) = \int_{t^{n-1}}^{t^n} &-\langle \kappa(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}), (\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})\rangle - \varepsilon_0\|\nabla(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})\|_0^2 \\ &- \langle \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})) - b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau}), \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}))\rangle \\ &+ \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}, \bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}) + \langle \bar{r}_{3,h,\tau}, \bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}\rangle\,\mathrm{d}s. \end{aligned} \tag{7.15}$$

We observe that the first two terms contributed to the relative dissipation $\mathcal{D}_{\phi_{h,\tau}}$. The third term is another part of the relative dissipation, which we will discuss soon. The fourth term vanishes due to skew-symmetry of $\tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}, \bar{\mathbf{v}}, \bar{\mathbf{v}}) = 0$, cf. (A.32) and (3.72) and the last term can be estimated by

$$\int_{t^{n-1}}^{t^n} \langle \bar{r}_{3,h,\tau}, \bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}\rangle\,\mathrm{d}s \leq \int_{t^{n-1}}^{t^n} \delta\varepsilon_1\|\nabla(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})\|_0^2 + C(\delta)\|q_{h,\tau} - \hat{q}_{h,\tau}\|_0^2 + C\|\bar{r}_{3,h,\tau}\|_{-1,h}^2\,\mathrm{d}s.$$

Here and in the following $\delta, C(\delta) > 0$ are constants from Young's inequality, which we will choose at the very end. Together this yields the first inequality

$$\begin{aligned}&\mathcal{E}(q_{h,\tau}|\hat{q}_{h,\tau})\big|_{t^{n-1}}^{t^n} + \int_{t^{n-1}}^{t^n} \|\kappa^{1/2}(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})\|_0^2 + (1-\delta)\varepsilon_1\|\nabla(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})\|_0^2\,\mathrm{d}s \\ &\leq \int_{t^{n-1}}^{t^n} C(\delta)\mathcal{E}(q_{h,\tau}|\hat{q}_{h,\tau}) + C(\delta)\|\bar{r}_{3,h,\tau}\|_{-1,h}^2 \\ &\quad \underbrace{+\langle \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})) - b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau}), \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}))\rangle}_{\mathcal{R}}\,\mathrm{d}s.\end{aligned} \tag{7.16}$$

Cahn-Hilliard equations: Let us now consider the contributions from the Cahn-Hilliard equation, i.e., (7.1)-(7.2) and (7.7)-(7.8). The abstract calculations (7.14) imply

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\,\mathrm{d}s =& \int_{t^{n-1}}^{t^n} \gamma\langle \nabla(\phi_{h,\tau} - \hat{\phi}_{h,\tau}), \nabla\partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\rangle \\ &+ \langle f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau}), \partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\rangle \\ &+ \alpha\langle \phi_{h,\tau} - \hat{\phi}_{h,\tau}, \partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\rangle \\ &+ \langle f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau}) - f''(\hat{\phi}_{h,\tau}), \partial_t\hat{\phi}_{h,\tau}\rangle\,\mathrm{d}s \\ =& I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{7.17}$$

We start with the first two terms. Inserting $\bar{\xi}_{h,\tau} = \partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau}) \in P_0(t^{n-1}, t^n; \mathcal{V}_h)$ into the variational identities (7.2) and (7.8) yields

$$I_1 + I_2 = \int_{t^{n-1}}^{t^n} \langle \bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau}, \partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\rangle\,\mathrm{d}s = (*).$$

Next we insert $\bar{\psi}_{h,\tau} = \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau} \in P_0(t^{n-1}, t^n; \mathcal{V}_h)$ as test function into the identities (7.1) and (7.7) and deduce

$$\begin{aligned}
(*) =& \int_{t^{n-1}}^{t^n} \langle \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau}, \partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\rangle \,\mathrm{d}s \\
=& \int_{t^{n-1}}^{t^n} -\varepsilon_0 \langle b(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}), \nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau})\rangle \\
& - \langle b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}) - \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau})), b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau})\rangle \\
& + \mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau}, \phi_{h,\tau}) + \langle \bar{r}_{1,h,\tau}, \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + \bar{r}_{2,h,\tau}\rangle \,\mathrm{d}s \\
=& (i) + (ii) + (iii) + (iv).
\end{aligned}$$

The first term can be rewritten and estimated via Hölder's and Youngs' inequality as

$$(i) \le \int_{t^{n-1}}^{t^n} (1-\delta)\varepsilon_0 \|b^{1/2}(\phi_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau})\|_0^2 + C(\delta)\|\bar{r}_{2,h,\tau}\|_1^2 \,\mathrm{d}s.$$

For the second term, we consider the reminder in (7.16) and obtain

$$\begin{aligned}
\mathcal{R} + (ii) \le \int_{t^{n-1}}^{t^n} & - (1-\delta)\|b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}) - \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau}))\|_0^2 \\
& + C(\delta)\|\bar{r}_{2,h,\tau}\|_1^2 \,\mathrm{d}s.
\end{aligned}$$

By definition of the dual norm, Poincaré's inequality (A.19), and the bounds for the coefficients, the third term can be further estimated by

$$\begin{aligned}
(iii) \le& \int_{t^{n-1}}^{t^n} \|\bar{r}_{1,h,\tau}\|_{-1} \left(\|\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}\|_1 + \|\bar{r}_{2,h,\tau}\|_1 \right) \,\mathrm{d}s \\
\le& \int_{t^{n-1}}^{t^n} \|\bar{r}_{1,h,\tau}\|_{-1} \left(C|\langle \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}, 1\rangle| + C\|b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau})\|_2 + \|\bar{r}_{2,h,\tau}\|_1 \right) \,\mathrm{d}s \\
\le& \int_{t^{n-1}}^{t^n} C(\delta)\|\bar{r}_{1,h,\tau}\|_{-1}^2 + |\langle \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}, 1\rangle|^2 + 2\delta\, \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}) + \|\bar{r}_{2,h,\tau}\|_1^2 \,\mathrm{d}s.
\end{aligned}$$

In the last step, we use Youngs' inequality to separate the factors with the same arbitrary parameter $\delta > 0$ as before. In order to control the mean value of $\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}$ we can use the variational identities (7.2) and (7.8) with $\bar{\xi}_{h,\tau} = 1 \in P_0(t^{n-1}, t^n; \mathcal{V}_h)$, which leads to

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} |\langle \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}, 1\rangle|^2 \,\mathrm{d}s =& \int_{t^{n-1}}^{t^n} |\langle f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau}) + \bar{r}_{2,h,\tau}, 1\rangle|^2 \,\mathrm{d}s \\
\le& C(\Omega) \int_{t^{n-1}}^{t^n} \|\bar{r}_{2,h,\tau}\|_{0,1}^2 + \|f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau})\|_{0,1}^2 \,\mathrm{d}s.
\end{aligned}$$

From the bounds for the potential f in assumption (A3), we can further deduce that

$$\begin{aligned}
|f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau})| =& \left| \int_0^1 f''(\hat{\phi}_{h,\tau} + s(\phi_{h,\tau} - \hat{\phi}_{h,\tau})) \,\mathrm{d}s (\phi_{h,\tau} - \hat{\phi}_{h,\tau}) \right| \\
\le& \left(f_2^{(2)} + f_3^{(2)}(|\phi_{h,\tau}| + |\hat{\phi}_{h,\tau}|)^2 \right) |\phi_{h,\tau} - \hat{\phi}_{h,\tau}|.
\end{aligned}$$

An application of Hölder's inequality, the continuous embedding of $H^1(\Omega)$ into $L^p(\Omega)$ for $p \le 6$, and the uniform bounds for $\phi_{h,\tau}$ in (7.6), and for $\hat{\phi}_{h,\tau}$, then lead to

$$\|f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau})\|_{0,1}^2 \le \left(C(\Omega) f_2^{(2)} + f_3^{(2)}(\|\phi_{h,\tau}\|_{0,6}^4 + \|\hat{\phi}_{h,\tau}\|_{0,6}^4) \right) \|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_{0,6}^2$$

$$\leq C(\Omega, f_2^{(2)}, f_3^{(2)}, \|\phi_{h,\tau}\|_{L^\infty(H^1)}, \|\hat{\phi}_{h,\tau}\|_{L^\infty(H^1)})\|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_1^2.$$

Using $\|\bar{r}_{2,h,\tau}\|_{0,1}^2 \leq C(\Omega)\|\bar{r}_{2,h,\tau}\|_1^2$ and the lower bound (6.18) for the relative energy, we arrive at

$$\begin{aligned}(iii) \leq \int_{t^{n-1}}^{t^n} & 2\delta\, \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}|\bar{\hat{\mu}}_{h,\tau}) + C(\delta)\|\bar{r}_{1,h,\tau}\|_{-1,h}^2 + C(\Omega)\|\bar{r}_{2,h,\tau}\|_1^2 \\ &+ C(\Omega, f_2^{(2)}, f_3^{(2)}, \|\phi_{h,\tau}\|_{L^\infty(H^1)}, \|\hat{\phi}_{h,\tau}\|_{L^\infty(H^1)}, \gamma)\, \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\, \mathrm{d}s.\end{aligned}$$

Let us consider the third term in (7.17). By inserting $\psi_{h,\tau} = \alpha(\bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau}) \in P_0(t^{n-1}, t^n; \mathcal{V}_h)$ into the variational identities (7.1) and (7.7) we derive

$$\begin{aligned}I_3 =& \alpha \int_{t^{n-1}}^{t^n} \langle \bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau}, \partial_t(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\rangle\, \mathrm{d}s \\ =& \alpha \int_{t^{n-1}}^{t^n} -\varepsilon_0 \langle b(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau}), \nabla(\bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau})\rangle \\ &- \langle b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau}) - \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau})), b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau})\rangle \\ &+ \mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \bar{\hat{\mathbf{u}}}_{h,\tau}; \nabla(\bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau}), \phi_{h,\tau}) + \langle \bar{r}_{1,h,\tau}, \phi_{h,\tau} - \bar{\hat{\phi}}_{h,\tau}\rangle\, \mathrm{d}s.\end{aligned}$$

All terms above except the convection term $\mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \bar{\hat{\mathbf{u}}}_{h,\tau}, \bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau}, \phi_{h,\tau})$ will be estimated via Hölder's and Young's inequality, and we obtain

$$\begin{aligned}I_3 \leq& \int_{t^{n-1}}^{t^n} \delta\varepsilon_0 \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau})\|_0^2 + C(\alpha)\|\bar{r}_{1,h,\tau}\|_{-1}^2 + C(\delta,\alpha)\|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_1^2 \\ &+ \delta\|b^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mu}_{h,\tau} - \bar{\hat{\mu}}_{h,\tau}) - \nabla(A(\bar{\phi}_{h,\tau})(\bar{q}_{h,\tau} - \bar{\hat{q}}_{h,\tau}))\|_0^2 \\ &+ \alpha\mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \bar{\hat{\mathbf{u}}}_{h,\tau}; \nabla(\bar{\phi}_{h,\tau} - \bar{\hat{\phi}}_{h,\tau}), \phi_{h,\tau})\, \mathrm{d}s.\end{aligned}$$

From the bounds in assumption (A3), we can deduce that

$$|f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau}) - f''(\hat{\phi}_{h,\tau})(\phi_{h,\tau} - \hat{\phi}_{h,\tau})| \leq \left(f_2^{(3)} + f_3^{(3)}(|\phi_{h,\tau}| + |\hat{\phi}_{h,\tau}|)\right)|\phi_{h,\tau} - \hat{\phi}_{h,\tau}|^2.$$

Using Hölder's inequality, embedding estimates of H^1 into L^p for $p \leq 6$, and the uniform bounds for $\phi_{h,\tau}$ in (7.6), we can bound the fourth term in (7.17) by

$$\begin{aligned}I_4 \leq& \int_{t^{n-1}}^{t^n} \|\partial_t\hat{\phi}_{h,\tau}\|_0 \|f'(\phi_{h,\tau}) - f'(\hat{\phi}_{h,\tau}) - f''(\hat{\phi}_{h,\tau})(\phi_{h,\tau} - \hat{\phi}_{h,\tau})\|_0\, \mathrm{d}s \\ \leq& \int_{t^{n-1}}^{t^n} \|\partial_t\hat{\phi}_{h,\tau}\|_0 \left(f_2^{(3)} + f_3^{(3)}(\|\phi_{h,\tau}\|_{0,6} + \|\hat{\phi}_{h,\tau}\|_{0,6})\right)\|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_{0,6}^2\, \mathrm{d}s \\ \leq& C(\Omega, f_2^{(3)}, f_3^{(3)}, \|\phi_{h,\tau}\|_{L^\infty(H^1)}, \|\hat{\phi}_{h,\tau}\|_{L^\infty(H^1)}, \gamma)\|\partial_t\hat{\phi}_{h,\tau}\|_{L^\infty(L^2)} \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\, \mathrm{d}s.\end{aligned}$$

Combining all estimates obtained so far, we find the following inequality

$$\begin{aligned}&\mathcal{E}_\alpha(\phi_{h,\tau}, q_{h,\tau}|\hat{\phi}_{h,\tau}, \hat{q}_{h,\tau})|_{t^{n-1}}^{t^n} + (2 - 4\delta)\int_{t^{n-1}}^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{q}_{h,\tau}|\bar{\hat{\mu}}_{h,\tau}, \bar{\hat{q}}_{h,\tau})\, \mathrm{d}s \\ &\leq \int_{t^{n-1}}^{t^n} C_1\mathcal{E}_\alpha(\phi_{h,\tau}, q_{h,\tau}|\hat{\phi}_{h,\tau}, \hat{q}_{h,\tau}) + C_2 \sum_{i\in\{1,3\}} \|\bar{r}_{i,h,\tau}\|_{-1,h}^2 + \|\bar{r}_{2,h,\tau}\|_1^2\end{aligned}$$

$$+ \mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau} + r_{2,h,\tau}, \phi_{h,\tau})$$
$$+ \alpha\mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}; \nabla(\bar{\phi}_{h,\tau} - \hat{\bar{\phi}}_{h,\tau}), \phi_{h,\tau}) \,\mathrm{d}s, \tag{7.18}$$

where $C_1 = c_0 + c_1\|\partial_t\hat{\phi}_{h,\tau}\|_{L^\infty(L^2)}$. Let us note that the constants c_0, c_1 here only depend on uniform bounds of $\phi_{h,\tau}$ and $\hat{\phi}_{h,\tau}$ in $L^\infty(H^1(\Omega))$.

Navier-Stokes equations: Finally, we consider the contributions of the Navier-Stokes equations. By (7.14) we find

$$\int_{t^{n-1}}^{t^n} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mathbf{u}_{h,\tau}|\hat{\mathbf{u}}_{h,\tau}) \,\mathrm{d}s = \int_{t^{n-1}}^{t^n} \langle \mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}, \partial_t(\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau})\rangle \,\mathrm{d}s$$
$$= \int_{t^{n-1}}^{t^n} \langle \bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}, \partial_t(\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau})\rangle \,\mathrm{d}s = (*).$$

Insertion of $\mathbf{v}_{h,\tau} = \bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau} \in P_0(t^{n-1}, t^n; \mathcal{V}_h^d)$ into (7.4) and (7.10) yields

$$(*) = \int_{t^{n-1}}^{t^n} -\langle \eta(\bar{\phi}_{h,\tau})\nabla(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}), \nabla(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau})\rangle$$
$$+ \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}, \bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}) + \langle \bar{p}_{h,\tau} - \hat{\bar{p}}_{h,\tau}, \operatorname{div}\left(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\right)\rangle$$
$$- \mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}, \phi_{h,\tau}) + \langle \bar{r}_{4,h,\tau}, \bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\rangle \,\mathrm{d}s.$$

The first term yields the contribution to the relative dissipation $\mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})$. The second vanishes due to skew-symmetry of the convective form $\tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}, \bar{\mathbf{v}}, \bar{\mathbf{v}}) = 0$, cf. (A.33) and (3.73). The third term vanishes by insertion of $\bar{w}_{h,\tau} = \bar{p}_{h,\tau} - \hat{\bar{p}}_{h,\tau} \in P_0(t^{n-1}, t^n; \mathcal{Q}_h)$ into (7.5) and (7.11). We keep the fourth term for later cancellation and estimate the fifth term by

$$\int_{t^{n-1}}^{t^n} \langle \bar{r}_{4,h,\tau}, \bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\rangle \,\mathrm{d}s$$
$$\leq \int_{t^{n-1}}^{t^n} \delta\|\eta^{1/2}(\bar{\phi}_{h,\tau})\nabla(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau})\|_0^2 + C(\delta)\|\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}\|_0^2 + C(\delta)\|\bar{r}_{4,h,\tau}\|_{-1,h}^2 \,\mathrm{d}s$$
$$\leq \int_{t^{n-1}}^{t^n} \delta\mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})(s) + C(\delta)\mathcal{E}(\mathbf{u}_{h,\tau}|\hat{\mathbf{u}}_{h,\tau})(s) + C(\delta)\|\bar{r}_{4,h,\tau}\|_{-1,h}^2 \,\mathrm{d}s.$$

Together this yields the inequality

$$\mathcal{E}(\mathbf{u}_{h,\tau}, |\hat{\mathbf{u}}_{h,\tau})|_{t^{n-1}}^{t^n} + (2-2\delta)\int_{t^{n-1}}^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau}) \,\mathrm{d}s \leq \int_{t^{n-1}}^{t^n} C_1(\delta)\mathcal{E}(\mathbf{u}_{h,\tau}|\hat{\mathbf{u}}_{h,\tau})$$
$$+ C_2(\delta)\|\bar{r}_{4,h,\tau}\|_{-1,h}^2 - \mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}, \phi_{h,\tau}) \,\mathrm{d}s \tag{7.19}$$

Finally, we combine (7.18) and (7.19) and after cancellation of convective terms in the Cahn-Hilliard equations and the Navier-Stokes equations, i.e., $\mathbf{c}(\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}, \phi_{h,\tau})$, we obtain the inequality

$$\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau})|_{t^{n-1}}^{t^n} + (2-4\delta)\int_{t^{n-1}}^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) \,\mathrm{d}s \tag{7.20}$$
$$\leq \int_{t^{n-1}}^{t^n} C_1(\delta)\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) + C_2(\delta)\sum_{i\in\{1,3,4\}} \|\bar{r}_{i,h,\tau}\|_{-1,h}^2 + \|\bar{r}_{2,h,\tau}\|_1^2$$

$$+\,\mathbf{c}(\bar{\mathbf{u}}_{h,\tau}-\hat{\bar{\mathbf{u}}}_{h,\tau};\bar{r}_{2,h,\tau},\phi_{h,\tau})+\alpha\mathbf{c}(\bar{\mathbf{u}}_{h,\tau}-\hat{\bar{\mathbf{u}}}_{h,\tau};\nabla(\bar{\phi}_{h,\tau}-\hat{\bar{\phi}}_{h,\tau}),\phi_{h,\tau})\,\mathrm{d}s,$$

where $C_1=c_0+c_1\|\partial_t\hat{\phi}_{h,\tau}\|_{L^\infty(L^2)}$. Hence, it remains to estimate the last two terms in (7.20) and start with the first one

$$\begin{aligned}&\int_{t^{n-1}}^{t^n}\mathbf{c}(\bar{\mathbf{u}}_{h,\tau}-\hat{\bar{\mathbf{u}}}_{h,\tau};\bar{r}_{2,h,\tau},\phi_{h,\tau})\,\mathrm{d}s\\ &\le\int_{t^{n-1}}^{t^n}\|\bar{\mathbf{u}}_{h,\tau}-\hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}\|\bar{r}_{2,h,\tau}\|_1\|\phi_{h,\tau}\|_1\,\mathrm{d}s\\ &\le\int_{t^{n-1}}^{t^n}\delta\mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})+C_2(\phi_{h,\tau})\|\bar{r}_{2,h,\tau}\|_1^2+\mathcal{E}(\mathbf{u}_{h,\tau}|\hat{\mathbf{u}}_{h,\tau})\,\mathrm{d}s.\end{aligned}$$

For the second term we can similarly derive

$$\begin{aligned}&\int_{t^{n-1}}^{t^n}\alpha\mathbf{c}(\bar{\mathbf{u}}_{h,\tau}-\hat{\bar{\mathbf{u}}}_{h,\tau};\nabla(\bar{\phi}_{h,\tau}-\hat{\bar{\phi}}_{h,\tau}),\phi_{h,\tau})\,\mathrm{d}s\\ &\le\int_{t^{n-1}}^{t^n}\alpha\|\bar{\mathbf{u}}_{h,\tau}-\hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}\|\phi_{h,\tau}-\hat{\bar{\phi}}_{h,\tau}\|_1\|\phi_{h,\tau}\|_1\,\mathrm{d}s\\ &\le\int_{t^{n-1}}^{t^n}\delta\mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})+C_2(\phi_{h,\tau},\alpha)\mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})+C(\alpha)\mathcal{E}(\mathbf{u}_{h,\tau}|\hat{\mathbf{u}}_{h,\tau})\,\mathrm{d}s.\end{aligned}$$

In total, we obtain by choosing $\delta=1/6$ the final inequality

$$\begin{aligned}&\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau})|_{t^{n-1}}^{t^n}+\int_{t^{n-1}}^{t^n}\mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau},\bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau},\hat{\bar{z}}_{h,\tau})\,\mathrm{d}s\\ &\le C_1\int_{t^{n-1}}^{t^n}\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau})+C_2\sum_{i\in\{1,3,4\}}\|\bar{r}_{i,h,\tau}\|_{-1,h}^2+C\|\bar{r}_{2,h,\tau}\|_1^2\,\mathrm{d}s.\end{aligned}$$

Since the inequality holds for all $t^n\in\mathcal{I}_\tau$ instead of all $t\in(0,T)$ we have to use the discrete Gronwall lemma, cf. Lemma A.3.2. Using uniform bounds in $L^\infty(0,T;H^1(\Omega))$ for $\phi_{h,\tau},\hat{\phi}_{h,\tau}$ we estimate

$$C_1\int_{t^{n-1}}^{t^n}\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau})\,\mathrm{d}s\le\tau c(\gamma)\Big[\mathcal{E}_\alpha(z_{h,\tau}(t^n)|\hat{z}_{h,\tau}(t^n))+\mathcal{E}_\alpha(z_{h,\tau}(t^{n-1})|\hat{z}_{h,\tau}(t^{n-1}))\Big].$$

Under the assumption that $\tau\le 1/(2c(\gamma)c)=:\tau_0$, we can rewrite the inequality into

$$u^n+b^n\le e^\lambda u^{n-1}+d^n,$$

with

$$u^n=\mathcal{E}_\alpha(z_{h,\tau}(t^n)|\hat{z}_{h,\tau}(t^n)),\qquad b^n=e^{\gamma\tau}\int_{t^{n-1}}^{t^n}\mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau},\bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau},\hat{\bar{z}}_{h,\tau})\,\mathrm{d}s,$$
$$d^n=C_2e^{\gamma\tau}\int_{t^{n-1}}^{t^n}\sum_{i\in\{1,3,4\}}\|\bar{r}_{i,h,\tau}\|_{-1}^2+\|\bar{r}_{2,h,\tau}\|_1^2\,\mathrm{d}s.$$

One can observe that $e^{\gamma\tau}=\frac{1+c(\gamma)c\tau}{1-c(\gamma)c\tau}$ which implies $\gamma\approx 2c(\gamma)c$. Then the assertion follows by the discrete Gronwall lemma and the bounds on the relative energy and dissipation functional. □

7.3. Auxiliary results

In this section, we will consider a specific choice for the perturbed solution as suitable interpolation and projection of the continuous solution. Furthermore, we will quantify the projection and interpolation error, i.e., the error between the continuous solution and the discrete perturbed solutions. As discrete approximations $(\hat{\phi}_{h,\tau}, \hat{\bar{\mu}}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau}, \hat{\bar{p}}_{h,\tau}) \in \mathbb{W}_{h,\tau} \times \mathbb{Q}_{h,\tau} \times \mathbb{W}_{h,\tau} \times \mathbb{Z}_{h,\tau} \times \mathbb{P}_{h,\tau}$ for the weak solution of (6.1)–(6.5), we now choose

$$\hat{\phi}_{h,\tau} = I^1_\tau \pi^1_h \phi \quad \text{and} \quad \hat{\bar{\mu}}_{h,\tau} = \bar{\pi}^0_\tau \pi^0_h \mu \quad \text{and} \quad \hat{q}_{h,\tau} = I^1_\tau \pi^0_h q \tag{7.21}$$

$$\text{and} \quad \hat{\mathbf{u}}_{h,\tau} = I^1_\tau \mathbf{P}^1_h \mathbf{u} \quad \text{and} \quad \hat{\bar{p}}_{h,\tau} = \bar{\pi}^0_\tau \pi^0_{p,h} p. \tag{7.22}$$

The above choice uses the linear interpolator I^1_τ and the L^2-projection onto piecewise constant function $\bar{\pi}^0_\tau$ in time. While in space we use the elliptic projection π^1_h the L^2-projection onto $\mathcal{V}_h$, i.e. π^0_h, the Stokes projector $\mathbf{P}^1_h$ and the L^2-projection onto $\mathcal{Q}_h$, i.e. $\pi^0_{p,h}$. Using these choices we obtain the following projection errors results.

Lemma 7.3.1. *Let Assumptions 7.1.2 hold. For the functions defined in* (7.21)-(7.22) *one has the following projection and interpolation errors*

$$\begin{aligned} &\|\hat{\phi}_{h,\tau} - \phi\|^2_{L^2(H^1)} \le C(h^4 + \tau^4), && \|\hat{\bar{\mu}}_{h,\tau} - \bar{\mu}\|^2_{L^2(H^1)} \le Ch^4, \\ &\|\hat{q}_{h,\tau} - q\|^2_{L^2(H^1)} \le C(h^4 + \tau^4), && \|\nabla(\hat{\bar{q}}_{h,\tau} - \bar{q})\|^2_{L^2(L^2)} \le Ch^4, \\ &\|\hat{\mathbf{u}}_{h,\tau} - \mathbf{u}\|^2_{L^2(H^1)} \le C(h^4 + \tau^4), && \|\nabla(\hat{\bar{\mathbf{u}}}_{h,\tau} - \bar{\mathbf{u}})\|^2_{L^2(L^2)} + \|\hat{\bar{p}}_{h,\tau} - \bar{p}\|^2_{L^2(L^2)} \le Ch^4. \end{aligned}$$

Proof. Let us first consider the error for the piecewise linear approximation. We estimate

$$\begin{aligned} \|I^1_\tau \pi^1_h \phi - \phi \pm \pi^1_h \phi\|^2_{L^2(H^1)} &\le \|I^1_\tau \pi^1_h \phi - \pi^1_h \phi\|^2_{L^2(H^1)} + \|\pi^1_h \phi - \phi\|^2_{L^2(H^1)} \\ &\le C\tau^4 \|\phi\|^2_{H^2(H^1)} + Ch^4 \|\phi\|^2_{L^2(H^3)}. \end{aligned}$$

Similarly, choosing the appropriate projection, these estimates translates to $\hat{q}_{h,\tau}$ and $\hat{\mathbf{u}}_{h,\tau}$. For the piecewise constant approximation, we use the stability of the L^2-projection in time to estimate

$$\|\hat{\bar{\mu}}_{h,\tau} - \bar{\mu}\|^2_{L^2(H^1)} \le C\|\pi^0_h \mu - \mu\|^2_{L^2(H^1)} \le Ch^4 \|\mu\|^2_{L^2(H^3)}.$$

Similar estimates can be done for the pressure and the gradient terms, using $\overline{I^1_\tau g} = \bar{g}$. □

For this particular construction, we can identify the corresponding residuals.

Lemma 7.3.2. *Let Assumptions 7.1.2 hold and let* $(\hat{\phi}_{h,\tau}, \hat{\bar{\mu}}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau}, \hat{\bar{p}}_{h,\tau})$ *be defined as in* (7.21)-(7.22)*. Then* (7.7)–(7.11) *holds with*

$$\begin{aligned} \int_{t^{n-1}}^{t^n} \langle \bar{r}_{1,h,\tau}, \bar{\psi}_{h,\tau} \rangle \,\mathrm{d}s = \int_{t^{n-1}}^{t^n} &\langle \partial_t(\pi^1_h \phi - \phi), \bar{\psi}_{h,\tau} \rangle + \varepsilon_0 \langle b(\bar{\phi}_{h,\tau}) \nabla \hat{\bar{\mu}}_{h,\tau} - b(\phi) \nabla \mu, \nabla \bar{\psi}_{h,\tau} \rangle \\ &+ \langle b^{1/2}(\bar{\phi}_{h,\tau})(b^{1/2}(\bar{\phi}_{h,\tau}) \nabla \hat{\bar{\mu}}_{h,\tau} - \nabla(A(\bar{\phi}_{h,\tau}) \hat{\bar{q}}_{h,\tau})), \nabla \bar{\psi}_{h,\tau} \rangle \\ &- \langle b^{1/2}(\phi)(b^{1/2}(\phi) \nabla \mu - \nabla(A(\phi) q)), \nabla \bar{\psi}_{h,\tau} \rangle \\ &- \mathbf{c}(\hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\psi}_{h,\tau}, \phi_{h,\tau}) + \mathbf{c}(\mathbf{u}; \bar{\psi}_{h,\tau}, \phi) \,\mathrm{d}s, \end{aligned}$$

$$\int_{t^{n-1}}^{t^n} \langle \bar{r}_{2,h,\tau}, \bar{\xi}_{h,\tau}\rangle \,\mathrm{d}s = \int_{t^{n-1}}^{t^n} \langle \hat{\bar{\mu}}_{h,\tau} - I_\tau^1 \mu, \bar{\xi}_{h,\tau}\rangle + \gamma\langle \nabla(\hat{\phi}_{h,\tau} - I_\tau^1\phi), \nabla\bar{\xi}_{h,\tau}\rangle + \langle f'(\hat{\phi}_{h,\tau}) - I_\tau^1 f'(\phi), \bar{\xi}_{h,\tau}\rangle \,\mathrm{d}s,$$

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \langle \bar{r}_{3,h,\tau}, \bar{\zeta}_{h,\tau}\rangle \,\mathrm{d}s = \int_0^t & \langle \partial_t \pi_h^0 q - \partial_t q, \bar{\zeta}_{h,\tau}\rangle + \langle \kappa(\bar{\phi}_{h,\tau})\hat{\bar{q}}_{h,\tau} - \kappa(\phi)\nabla q, \bar{\zeta}_{h,\tau}\rangle \\ & + \varepsilon_1\langle \nabla\hat{q}_{h,\tau} - \nabla q, \nabla\bar{\zeta}_{h,\tau}\rangle + \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \hat{\bar{q}}_{h,\tau}, \bar{\zeta}_{h,\tau}) - \tilde{\mathbf{c}}(\mathbf{u}; q, \bar{\zeta}_{h,\tau}) \\ & + \langle \nabla(A(\bar{\phi}_{h,\tau})\hat{\bar{q}}_{h,\tau}) - b^{1/2}(\bar{\phi}_{h,\tau})\nabla\hat{\bar{\mu}}_{h,\tau}, \nabla(A(\bar{\phi}_{h,\tau})\bar{\zeta}_{h,\tau})\rangle \\ & - \langle \nabla(A(\phi)q) - b^{1/2}(\phi)\nabla\mu, \nabla(A(\phi)\bar{\zeta}_{h,\tau})\rangle \,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \langle \bar{r}_{4,h,\tau}, \bar{\mathbf{v}}_{h,\tau}\rangle \,\mathrm{d}s = \int_0^t & \langle \partial_t \mathbf{P}_h^1 \mathbf{u} - \partial_t \mathbf{u}, \bar{\mathbf{v}}_{h,\tau}\rangle + \langle \eta(\bar{\phi}_{h,\tau})\nabla\hat{\bar{\mathbf{u}}}_{h,\tau} - \eta(\phi)\nabla\mathbf{u}, \nabla\bar{\mathbf{v}}_{h,\tau}\rangle \\ & - \langle \hat{\bar{p}}_{h,\tau} - p, \operatorname{div}(\bar{\mathbf{v}}_{h,\tau})\rangle + \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau}; \hat{\bar{\mathbf{u}}}_{h,\tau}, \bar{\mathbf{v}}_{h,\tau}) - \tilde{\mathbf{c}}(\mathbf{u}; \mathbf{u}, \bar{\mathbf{v}}_{h,\tau}) \\ & - \mathbf{c}(\bar{\mathbf{v}}_{h,\tau}; \hat{\bar{\mu}}_{h,\tau}, \phi_{h,\tau}) + \mathbf{c}(\bar{\mathbf{v}}_{h,\tau}; \mu, \phi) \,\mathrm{d}s.\end{aligned}$$

Proof. The representation of the first, third and fourth residual is obtained as follows. Testing the corresponding continuous variational identity, i.e (6.1), (6.3), (6.4), with $\psi = \bar{\psi}_{h,\tau}$, $\zeta = \bar{\zeta}_{h,\tau}$ and $\mathbf{v} = \bar{\mathbf{v}}_{h,\tau}$ and integration over time yields the continuous contribution. Subtracting the continuous contribution from (7.7), (7.9), (7.10), respectively, yields the representation. The formula for the time derivative error follows from the commuting diagram property (5.15). For the second residual we consider (6.2) and test it at time t^n and t^{n-1} with $\xi = \bar{\xi}_{h,\tau}$. Noting that

$$\int_{t^{n-1}}^{t^n} a(s)\bar{b}(s) \,\mathrm{d}s = \frac{\tau}{2}(a(t^n) + a(t^{n-1}))b(t^{n-1/2})$$

for all $a \in P_1(t^{n-1}, t^n)$ and $\bar{b} \in P_0(t^{n-1}, t^n)$, we obtain

$$\int_{t^{n-1}}^{t^n} \langle I_\tau^1\mu, \bar{\xi}_{h,\tau}\rangle + \gamma\langle \nabla I_\tau^1 \phi_{h,\tau}, \nabla\bar{\xi}_{h,\tau}\rangle + \langle I_\tau^1 f'(\phi), \bar{\xi}_{h,\tau}\rangle \,\mathrm{d}s = 0 \tag{7.23}$$

for all $\bar{\xi}_{h,\tau} \in P_0(t^{n-1}, t^n; \mathcal{V}_h)$. The residual representation is then obtained by subtracting equation (7.23) from identity (7.8). □

7.4. Error estimate

In this section, we will conduct the error analysis. By the constructions in the last two sections, this will reduce to estimating the residuals suitably by the relative energy, the relative dissipation and the projection errors. Furthermore, we consider the discrete pressure error and uniqueness of solutions. Let us recall that this will be the first time, where we use assumption (A9), i.e. $A = 1$.

In the following lemma, we summarize suitable estimates for the residuals.

Lemma 7.4.1. *Let Assumptions 7.1.2 and Lemma 7.3.2 hold. Then*

$$\int_{t^{n-1}}^{t^n} \|\bar{r}_{1,h,\tau}\|_{-1}^2 \,\mathrm{d}s \le C(h^4 + \tau^4) + \int_{t^{n-1}}^{t^n} C\mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau}) \,\mathrm{d}s,$$

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \|\bar{r}_{2,h,\tau}\|_1^2 \,\mathrm{d}s \le C(h^4+\tau^4),\\
&\int_{t^{n-1}}^{t^n} \|\bar{r}_{3,h,\tau}\|_{-1}^2 \,\mathrm{d}s \le C(h^4+\tau^4) + \int_{t^{n-1}}^{t^n} C\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) + 4\delta \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau}) \,\mathrm{d}s,\\
&\int_{t^{n-1}}^{t^n} \|\bar{r}_{4,h,\tau}\|_{-1}^2 \,\mathrm{d}s \le C(h^4+\tau^4) + \int_{t^{n-1}}^{t^n} C\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) + 4\delta \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau}) \,\mathrm{d}s.
\end{aligned}
$$

Proof. As before, we will estimate every residual separately. Let us again recall that $\bar{g}_{h,\tau}$ denotes the piecewise mean value in time for all differential variables, i.e. $(\phi_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau})$ and their perturbed counterpart $(\hat{\phi}_{h,\tau}, \hat{q}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau})$.

First residual: Use of the dual norm yields

$$
\begin{aligned}
\int_{t^{n-1}}^{t^n} \|\bar{r}_{1,h,\tau}\|_{-1}^2 \,\mathrm{d}s \le \int_{t^{n-1}}^{t^n} &\|\partial_t(\pi_h^1\phi - \phi)\|_{-1}^2 + (1+\varepsilon_0)\|b(\bar{\phi}_{h,\tau})\nabla\bar{\mu}_{h,\tau} - \overline{b(\phi)\nabla\mu}\|_0^2\\
&+ \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla\hat{\bar{q}}_{h,\tau} - \overline{b^{1/2}(\phi)\nabla q}\|_0^2 + \|\bar{\phi}_{h,\tau}\hat{\bar{\mathbf{u}}}_{h,\tau} - \overline{\phi\mathbf{u}}\|_{0,2}^2 \,\mathrm{d}s\\
&= (i) + (ii) + (iii) + (iv).
\end{aligned}
$$

Let us estimate the terms separately. The first term can be bounded as in the semi-discrete case by

$$
(i) \le Ch^4\|\phi\|_{L^2(H^1)}^2.
$$

Since all other terms can be treated similarly to the second, we will briefly describe how to estimate the second term.

$$
\begin{aligned}
(ii) \le C\int_{t^{n-1}}^{t^n} &\|b(\bar{\phi}_{h,\tau})\nabla(\hat{\bar{\mu}}_{h,\tau} - \nabla\bar{\mu})\|_0^2 + \|(b(\bar{\phi}_{h,\tau}) - b(\bar{\phi}))\nabla\bar{\mu}\|_0^2\\
&+ \|(b(\bar{\phi}) - \overline{b(\phi)})\nabla\bar{\mu}\|_0^2 + \|\overline{b(\phi)}\nabla\bar{\mu} - \overline{b(\phi)\nabla\mu}\|_0^2 \,\mathrm{d}s\\
= (a) + (b) &+ (c) + (d).
\end{aligned}
$$

The first term can be treated by the stability of the L^2-projection in time, the projection estimates in Lemma 6.3.1 which yields

$$
(a) \le C\int_{t^{n-1}}^{t^n} \|\pi_h^0\mu - \mu\|_1^2 \,\mathrm{d}s \le Ch^4\|\mu\|_{L^2(H^3)}^2.
$$

With the stability of the L^2-projection and Lemma 7.3.1 we deduce that

$$
\begin{aligned}
(b) &\le C\int_{t^{n-1}}^{t^n} \|\phi_{h,\tau} - \phi\|_{0,6}^2\|\mu\|_{1,3}^2 \,\mathrm{d}s\\
&\le Ch^4\|\mu\|_{L^\infty(W^{1,3})}^2\|\phi\|_{L^2(H^3)}^2 + C\tau^4\|\mu\|_{L^\infty(W^{1,3})}^2\|\phi\|_{H^2(H^1)}^2\\
&\quad + \|\mu\|_{L^\infty(W^{1,3})}^2 \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau}) \,\mathrm{d}s.
\end{aligned}
$$

The third term can be treated by Lemma 5.17, and we find

$$
(c) \le C\tau^4\|\mu\|_{L^\infty(W^{1,3})}^2\|b(\phi)\|_{H^2(L^6)}^2.
$$

The last term can be treated by Lemma 5.16 and we obtain

$$
(d) \le C\tau^4\|b(\phi)\nabla\mu\|_{H^2(L^2)}^2.
$$

Together this implies

$$(ii) \leq C_1(\phi,\mu)h^4 + C_2(\phi,\mu)\tau^4 + C(\|\mu\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\,\mathrm{d}s,$$

where the two constants have the following dependence

$$\begin{aligned} C_1(\phi,\mu) &= C(\|\phi\|_{L^2(H^3)}, \|\mu\|_{L^2(H^3)}, \|\mu\|_{L^\infty(W^{1,3})}) \text{ and} \\ C_2(\phi,\mu) &= C(\|\phi\|_{H^2(H^1)}, \|b(\phi)\nabla\mu\|_{H^2(L^2)}, \|\mu\|_{L^\infty(W^{1,3})}). \end{aligned}$$

We observe that in principle, the third, and fourth terms can be treated with the same technique. Using the uniform $L^\infty(H^1)$ bounds on $\phi_{h,\tau}$ we estimate

$$\begin{aligned} (iii) &\leq C_3(\phi,q)h^4 + C_4(\phi,q)\tau^4 + C(\|q\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\,\mathrm{d}s, \\ (iv) &\leq C_5(\phi,\mathbf{u})h^4 + C_6(\phi,\mathbf{u})\tau^4 + C(\|\mathbf{u}\|_{L^\infty(L^3)}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\,\mathrm{d}s, \end{aligned}$$

with the constants

$$\begin{aligned} C_3(\phi,q) &= C(\|q\|_{L^2(H^3)}, \|\phi\|_{L^2(H^3)}, \|q\|_{L^\infty(W^{1,3})}), \\ C_4(\phi,q) &= C(\|\phi\|_{H^2(H^1)}, \|q\|_{H^2(H^1)}, \|b^{1/2}(\phi)\nabla q\|_{H^2(L^2)}, \|q\|_{L^\infty(W^{1,3})}), \\ C_5(\phi,\mathbf{u}) &= C(\|\phi\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^\infty(L^3)}), \\ C_6(\phi,\mathbf{u}) &= C(\|\phi\|_{H^2(H^1)}, \|\mathbf{u}\|_{H^2(H^1)}, \|\mathbf{u}\|_{L^\infty(L^3)}). \end{aligned}$$

Finally, this yields the following bound for the first residual

$$\int_{t^{n-1}}^{t^n} \|\bar{r}_{1,h,\tau}\|_{-1}^2\,\mathrm{d}s \leq C(h^4+\tau^4) + \int_{t^{n-1}}^{t^n} C\mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\,\mathrm{d}s.$$

Second residual: The second residual can be expressed equivalently in strong form as

$$\bar{r}_{2,h,\tau} = (\overline{\pi_h^0\mu} - \overline{I_\tau^1\pi_h^0\mu}) + (\overline{I_\tau^1\phi} - \overline{\hat{\phi}_{h,\tau}}) + (\overline{f'(\hat{\phi}_{h,\tau})} - \overline{I_\tau^1 f'(\phi)}),$$

where $\overline{g} = \bar{\pi}_\tau^0 g$ denotes the piecewise constant projection of g with respect to time. This pointwise representation allows us to estimate

$$\begin{aligned} \int_{t^{n-1}}^{t^n} \|\bar{r}_{2,h,\tau}\|_1^2\,\mathrm{d}s \leq \|\pi_h^0\mu - I_\tau^1\pi_h^0\mu\|_{L^2(H_p^1)}^2 + \|I_\tau^1\phi - \hat{\phi}_{h,\tau}\|_{L^2(H_p^1)}^2 \\ + \|f'(\hat{\phi}_{h,\tau}) - I_\tau^1 f'(\phi)\|_{L^2(H_p^1)}^2 = (i) + (ii) + (iii). \end{aligned}$$

For the first term using the contraction of the L^2-projection in space, we obtain

$$(i) \leq \|\mu - I_\tau^1\mu\|_{L^2(H^1)}^2 \leq C\tau^4\|\mu\|_{H^2(H^1)}^2.$$

For the second term, using the error estimate for the H^1-projection π_h^1 and we find

$$(ii) \leq C\|\phi - \pi_h^1\phi\|_{L^\infty(H^1)}^2 \leq Ch^4\|\phi\|_{L^\infty(H^3)}^2.$$

For the last term, we employ the uniform bounds of ϕ and $\hat{\phi}_{h,\tau}$ in $L^\infty(0,T;W^{1,\infty}(\Omega))$. Therefore, all terms $f^{(k)}(\cdot)$ can be bounded by a constant $C(f)$ and we obtain the estimate

$$\begin{aligned}(iii) &\le \|f'(\hat{\phi}_{h,\tau}) - f'(\phi)\|^2_{L^2(H^1)} + \|f'(\phi) - I^1_\tau f'(\phi)\|^2_{L^2(H^1)} \\ &\le C(f)\|\hat{\phi}_{h,\tau} - \phi\|^2_{L^2(H^2)} + C\tau^4\|f'(\phi)\|^2_{H^2(H^1)} \\ &\le C(f)h^4\|\phi\|^2_{L^2(H^3)} + C(f)\tau^4\|\phi\|^2_{H^2(H^1)} + C\tau^4\|f'(\phi)\|^2_{H^2(H^1)}.\end{aligned}$$

Note that one could compute the norm involving f' to find the necessary regularity assumption on ϕ. Together, this yields the following bound for the second residual

$$\int_{t^{n-1}}^{t^n} \|\bar{r}_{2,h,\tau}\|^2_1 \,\mathrm{d}s \le C(h^4 + \tau^4).$$

Third residual: For the third residual, we find

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|\bar{r}_{3,h,\tau}\|^2_{-1} \,\mathrm{d}s \le C \int_{t^{n-1}}^{t^n} &\|\partial_t(\pi^1_h q - q)\|^2_{-1} + \|\kappa(\bar{\phi}_{h,\tau})\hat{\bar{q}}_{h,\tau} - \overline{\kappa(\phi)q}\|^2_0 \\ &+ \|\nabla(\hat{\bar{q}}_{h,\tau} - \bar{q})\|^2_0 + \|b^{1/2}(\bar{\phi}_{h,\tau})\nabla\hat{\bar{\mu}}_{h,\tau} - \overline{b^{1/2}(\phi)\nabla\mu}\|^2_0 \\ &+ \|\bar{\mathbf{u}}_{h,\tau}\hat{\bar{q}}_{h,\tau} - \overline{\mathbf{u}q}\|^2_0 + \|\bar{\mathbf{u}}_{h,\tau}\nabla\hat{\bar{q}}_{h,\tau} - \overline{\mathbf{u}\nabla q}\|^2_{0,6/5} \,\mathrm{d}s \\ = (i) + (ii) &+ (iii) + (iv) + (v).\end{aligned}$$

Similar estimates as for the first residual, we find the following bound

$$(i) \le Ch^4\|q\|^2_{L^2(H^1)}.$$

The second and fourth terms can be treated by the same arguments as the second term of the first residual and implies

$$\begin{aligned}(ii) + (iv) \le &C_1(\phi,q,\mu)h^4 + C_2(\phi,q,\mu)\tau^4 \\ &+ C(\|\mu\|_{L^\infty(W^{1,3})}, \|q\|_{L^\infty(L^3)}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau}) \,\mathrm{d}s,\end{aligned}$$

with

$$\begin{aligned}C_1(\phi,q,\mu) &= C(\|\phi\|_{L^2(H^3)}, \|q\|_{L^2(H^2)}, \|\mu\|_{L^2(H^2)}, \|\mu\|_{L^\infty(W^{1,3})}, \|q\|_{L^\infty(L^3)}), \\ C_2(\phi,q,\mu) &= C(\|\phi\|_{H^2(H^1)}, \|\kappa(\phi)q\|_{H^2(L^2)}, \|b^{1/2}(\phi)\nabla\mu\|_{H^2(L^2)}, \|\mu\|_{L^\infty(W^{1,3})}, \|q\|_{L^\infty(L^3)}).\end{aligned}$$

For the third term we use the projection errors, i.e., Lemma 7.3.1, and obtain

$$(iii) \le Ch^4\|q\|^2_{L^2(H^3)} + C\tau^4\|q\|^2_{H^2(H^1)}.$$

For the fifth term, we decompose as follows

$$(v) \le \int_{t^{n-1}}^{t^n} \|\bar{\mathbf{u}}_{h,\tau} - \bar{\mathbf{u}}\|^2_{0,3}\|\hat{\bar{q}}_{h,\tau}\|^2_{0,6} + \|\hat{\bar{q}}_{h,\tau} - \bar{q}\|^2_{0,3}\|\bar{\mathbf{u}}\|^2_{0,6} + \|\bar{q}\bar{\mathbf{u}} - \overline{q\mathbf{u}}\|^2_0 \,\mathrm{d}s.$$

A similar argument as before, paired with an interpolation inequality, yields

$$(v) \le C_3(q,\mathbf{u})h^4 + C_4(q,\mathbf{u})\tau^4 + 2\delta \int_{t^{n-1}}^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau}) \,\mathrm{d}s$$

$$+ C(\|q\|_{L^\infty(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}, \mathbf{u}_{h,\tau}|\hat{\phi}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau})\, \mathrm{d}s.$$

Here the constants depend on $C_3(q,\mathbf{u}) = C(\|q\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^2(W^{2,3})}, \|q\|_{L^\infty(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)})$ and $C_4(q,\mathbf{u}) = C(\|q\|_{H^2(L^3)}, \|\mathbf{u}\|_{H^2(H^1)}\|q\|_{L^\infty(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)})$.

For the sixth term in a similar fashion, we first estimate

$$(vi) \leq \int_{t^{n-1}}^{t^n} \|\bar{\mathbf{u}}_{h,\tau} - \bar{\mathbf{u}}\|_{0,3}^2 \|\hat{\bar{q}}_{h,\tau}\|_1^2 + \|\hat{\bar{q}}_{h,\tau} - \bar{q}\|_{0,3}^2 \|\bar{\mathbf{u}}\|_1^2 + \|\nabla \bar{q}\bar{\mathbf{u}} - \overline{\nabla q \mathbf{u}}\|_{0,6/5}^2\, \mathrm{d}s.$$

Using the same arguments, as before, i.e., using Lemma 5.3.2 yields

$$\begin{aligned}(vi) &\leq C_5(q,\mathbf{u})h^4 + C_6(q,\mathbf{u})\tau^4 + 2\delta \int_{t^{n-1}}^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})\, \mathrm{d}s \\ &\quad + C(\|q\|_{L^\infty(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}, \mathbf{u}_{h,\tau}|\hat{\phi}_{h,\tau}, \hat{\mathbf{u}}_{h,\tau})\, \mathrm{d}s.\end{aligned}$$

Here the constants depend on $C_5(q,\mathbf{u}) = C(\|q\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^2(W^{2,3})}, \|q\|_{L^\infty(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)})$ and $C_6(q,\mathbf{u}) = C(\|q\|_{H^2(H^1)}, \|\mathbf{u}\|_{H^2(H^1)}\|q\|_{L^\infty(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)})$. In total, the third residual can therefore be estimated by

$$\int_{t^{n-1}}^{t^n} \|\bar{r}_{3,h,\tau}\|_{-1}^2\, \mathrm{d}s \leq C(h^4 + \tau^4) + \int_{t^{n-1}}^{t^n} C\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) + 4\delta \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})\, \mathrm{d}s.$$

Fourth residual: The fourth residual we estimate

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|\bar{r}_{4,h,\tau}\|_{-1}^2\, \mathrm{d}s &\leq \int_{t^{n-1}}^{t^n} \|\partial_t(\pi_h^1 \mathbf{u} - \mathbf{u})\|_{-1}^2 + \|\eta(\bar{\phi}_{h,\tau})\nabla \hat{\bar{\mathbf{u}}}_{h,\tau} - \overline{\eta(\phi)\nabla \mathbf{u}}\|_0^2 \\ &\quad + \|\hat{\bar{p}}_{h,\tau} - \bar{p}\|_0^2 + \|\hat{\bar{\mathbf{u}}}_{h,\tau}\hat{\bar{\mathbf{u}}}_{h,\tau} - \overline{\mathbf{u}\mathbf{u}}\|_0^2 \\ &\quad + \|\hat{\bar{\mathbf{u}}}_{h,\tau}\nabla \hat{\bar{\mathbf{u}}}_{h,\tau} - \overline{\mathbf{u}\nabla \mathbf{u}}\|_{0,6/5}^2 + \|\bar{\phi}_{h,\tau}\nabla \hat{\bar{\mu}}_{h,\tau} - \overline{\phi \nabla \mu}\|_{0,6/5}^2\, \mathrm{d}s \\ &= (i) + (ii) + (iii) + (iv) + (v) + (vi).\end{aligned}$$

The first and third terms can be bounded by the projection errors, i.e., Lemma 7.3.1 and the stability of the L^2-projection in time via

$$(i) + (iii) \leq Ch^4\|\mathbf{u}\|_{L^2(H^1)}^2 + Ch^4\|p\|_{L^2(H^2)}^2.$$

The second term follows by similar expansions as before and can be estimated by

$$(ii) \leq C_1(\phi,\mathbf{u})h^4 + C_2(\phi,\mathbf{u})\tau^4 + C(\|\mathbf{u}\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau})\, \mathrm{d}s$$

where the constants depend on $C_1(\phi,\mathbf{u}) = C(\|\phi\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^\infty(W^{1,3})})$ and $C_2(\phi,\mathbf{u}) = C(\|\phi\|_{H^2(H^1)}, \|\mathbf{u}\|_{H^2(H^1)}, \|\eta(\phi)\nabla \mathbf{u}\|_{H^2(L^2)}, \|\mathbf{u}\|_{L^\infty(W^{1,3})})$.

The bounds for the fourth and fifth term follow from similar estimates as for the third residual and yields

$$\begin{aligned}(iv) + (v) &\leq C_5(\mathbf{u})h^4 + C_6(\mathbf{u})\tau^4 + 4\delta \int_{t^{n-1}}^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau})\, \mathrm{d}s \\ &\quad + C(\|\mathbf{u}\|_{L^\infty(H^1)}) \int_{t^{n-1}}^{t^n} \mathcal{E}(\mathbf{u}_{h,\tau}|\hat{\mathbf{u}}_{h,\tau})\, \mathrm{d}s.\end{aligned}$$

Here the constants are dependent on the norms via $C_5(\mathbf{u}) = C(\|\mathbf{u}\|_{L^2(H^3)}, \|\mathbf{u}\|_{L^\infty(H^1)})$ and $C_6(\mathbf{u}) = C(\|\mathbf{u}\|_{H^2(H^1)}, \|\mathbf{u}\|_{L^\infty(H^1)})$.

Finally, the sixth term can be estimated by

$$(vi) \leq \int_{t^{n-1}}^{t^n} \|\hat{\phi}_{h,\tau}\|_{0,3}^2 \|\bar{\mu}_{h,\tau} - \bar{\mu}\|_1^2 + \|\bar{\phi}_{h,\tau} - \bar{\phi}\|_{0,3}^2 \|\mu\|_1^2 + \|\bar{\phi}\nabla\bar{\mu} - \overline{\phi\nabla\mu}\|_{0,6/5}^2 \,\mathrm{d}s.$$

Estimation as before yields the bound

$$(vi) \leq C_7(\phi, \mu)h^4 + C_8(\phi, \mu)\tau^4 + C(\|\mu\|_{L^\infty(H^1)})$$

with the space constant $C_7(\phi, \mu) = C(\|\phi\|_{L^2(H^3)}, \|\mu\|_{L^2(H^3)}, \|\mu\|_{L^\infty(H^1)})$ and the time constant $C_8(\phi, \mu) = C(\|\phi\|_{H^2(H^1)}, \|\mu\|_{H^2(H^1)}, \|\mu\|_{L^\infty(H^1)})$.

Therefore, we obtain as full estimate for the residual

$$\|\bar{r}_{4,h,\tau}\|_{-1}^2 \,\mathrm{d}s \leq C(h^4 + \tau^4) + \int_{t^{n-1}}^{t^n} C\mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) + 4\delta\mathcal{D}_{\phi_{h,\tau}}(\bar{\mathbf{u}}_{h,\tau}|\hat{\bar{\mathbf{u}}}_{h,\tau}) \,\mathrm{d}s.$$

□

With these estimates for the residuals at hand, we can proceed and using the discrete Gronwall lemma, cf. Lemma A.3.2, we obtain the following result for the discrete error.

Lemma 7.4.2. *Let the Assumptions 7.1.2, Lemma 7.2.1 and Lemma 7.4.1 hold. Then*

$$\max_{t^n \in \mathcal{I}_\tau} \left(\|\phi_{h,\tau} - \hat{\phi}_{h,\tau}\|_1^2 + \|q_{h,\tau} - \hat{q}_{h,\tau}\|_0^2 + \|\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}\|_0^2 \right)$$
$$+ \|\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau}\|_{L^2(H^1)}^2 + \|\bar{q}_{h,\tau} - \hat{\bar{q}}_{h,\tau}\|_{L^2(H^1)}^2 + \|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_{L^2(H^1)}^2 \leq C(h^4 + \tau^4)$$

holds with a constant C independent on h and τ.

Proof. We use the stability estimate, i.e., Lemma 7.2.1 and the bounds for the discrete residuals Lemma 7.4.1, in order to obtain a inequality for the relative energy. Application of the discrete Gronwall lemma similarly, as in Lemma 7.2.1, yields

$$\mathcal{E}_\alpha(z_{h,\tau}(t^n)|\hat{z}_{h,\tau}(t^n)) + \tfrac{1}{2}\int_0^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) \,\mathrm{d}s \leq C(h^4 + \tau^4), \qquad \forall t^n \in \mathcal{I}_\tau.$$

Using the lower bounds for the relative energy (7.12) and the relative dissipation (7.13) then already concludes the proof. □

The error estimates in Theorem 7.1.3 follow by standard triangle inequality, which requires to estimate the projection error, i.e., Lemma 7.3.1 and the discrete error, i.e., Lemma 7.4.2. Finally, it remains to deduce the error estimate for the discrete pressure.

7.5. Pressure estimate

We will now consider the error for the discrete pressure, i.e., $\bar{p}_{h,\tau} - \hat{\bar{p}}_{h,\tau}$. Inspired by the semi-discrete considerations, we can prove the following lemma.

Lemma 7.5.1. *Let Lemma 7.4.2 hold and additionally assume $\tau = c_p h$, where c_p is a constant independent of h and τ. Then the following error estimate holds*

$$\|\partial_t(\mathbf{u}_{h,\tau} - \hat{\mathbf{u}})\|^2_{L^2(H^{-1})} + \|\bar{p}_{h,\tau} - \bar{p}\|^2_{L^2(L^2)} \leq C(\tau^4 + h^4).$$

Proof. Following the semi-discrete case, we use the discrete inf-sup stability (5.8) together with Lemma 7.4.2 and deduce

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \beta^2 \|\bar{p}_{h,\tau} - \hat{\bar{p}}_{h,\tau}\|_0^2 \,\mathrm{d}s \\
&\leq \int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}_{h,\tau} - \partial_t \hat{\mathbf{u}}_{h,\tau}\|^2_{-1,h} + C\|\bar{\mathbf{u}}_{h,\tau}\|^2_{0,\infty}\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_1^2 \\
&\qquad + C(\eta)\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_1^2 + \|\bar{\phi}_{h,\tau}\|^2_{0,3}\|\nabla(\bar{\mu}_{h,\tau} - \hat{\bar{\mu}}_{h,\tau})\|_0^2 + \|\bar{r}_{4,h,\tau}\|^2_{-1} \,\mathrm{d}s \\
&\leq \int_0^t \|\partial_t \mathbf{u}_h - \partial_t \hat{\mathbf{u}}_h\|^2_{-1,h} + C\|\bar{\mathbf{u}}_{h,\tau}\|^2_{0,\infty}\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_1^2 + C(h^4 + \tau^4) \,\mathrm{d}s.
\end{aligned} \tag{7.24}$$

We estimate the second term as follows

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \|\bar{\mathbf{u}}_{h,\tau}\|^2_{0,\infty}\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_1^2 \,\mathrm{d}s &\leq \int_{t^{n-1}}^{t^n} (\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{0,\infty} + \|\hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{0,\infty})\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_1^2 \,\mathrm{d}s \\
&\leq (\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{L^\infty(L^\infty)} + \|\hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{L^\infty(L^\infty)}) \int_{t^{n-1}}^{t^n} \|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_1^2 \,\mathrm{d}s \\
&\leq C(\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{L^\infty(L^\infty)} + \|\hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{L^\infty(L^\infty)})(h^4 + \tau^4) = (*).
\end{aligned}$$

For the first term we use the inverse inequality (5.7) in space with $p = \infty, q = 2, d \leq 3$. Since the second term is uniformly bounded, we obtain by means of Lemma 7.4.2

$$(*) \leq C(h^{-3}\|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|^2_{L^\infty(L^2)} + 1)(h^4 + \tau^4) \leq Ch^{-3}(h^4 + \tau^4)(h^4 + \tau^4) + C(h^4 + \tau^4).$$

The choice $\tau = ch$ yields altogether the estimate

$$\int_{t^{n-1}}^{t^n} \beta^2 \|\bar{p}_{h,\tau} - \hat{\bar{p}}_{h,\tau}\|_0^2 \,\mathrm{d}s \leq \int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}_{h,\tau} - \partial_t \hat{\mathbf{u}}_{h,\tau}\|^2_{-1,h} \,\mathrm{d}s + C(h^4 + \tau^4).$$

Using the same argumentation as in the proof of Lemma 6.5.1 we can show that

$$\int_{t^{n-1}}^{t^n} \|\partial_t \mathbf{u}_{h,\tau} - \partial_t \hat{\mathbf{u}}_{h,\tau}\|^2_{-1,h} \,\mathrm{d}s \leq C(h^4 + \tau^4).$$

The result then follows from the triangle inequality and the estimates for the projection error, stated Lemma 7.3.1.

□

In the above proof, we have used the natural restriction $\tau = c_p h$ to obtain convergence rates for the pressure. The pressure is reconstructed via the discrete inf-sup stability (5.8), hence there is a unique pressure for every velocity. In the following, we will investigate under which conditions the discrete solution is also unique.

7.6. Uniqueness of discrete solutions

In this section, we will prove the uniqueness of discrete solutions of Problem P.2 by using the discrete stability estimate, i.e., Lemma 7.4.1. This follows the idea of the weak-strong uniqueness proof, i.e., assuming that the discrete perturbed solutions are indeed discrete solutions of the same problem.

In order to obtain uniqueness on the fully discrete level, we will again use the stability estimate of Lemma 7.2.1. However, in contrast to the error estimate, we will now assume that the perturbed solution is itself another solution of Problem P.2. This immediately implies $\bar{r}_{2,h,\tau} = 0$ and the remaining residuals are given by

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \langle \bar{r}_{1,h,\tau}, \bar{\psi}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \mathbf{c}(\hat{\bar{\mathbf{u}}}_{h,\tau}; \bar{\psi}_{h,\tau}, \phi_{h,\tau} - \hat{\phi}_{h,\tau}) \\
&\quad - (1+\varepsilon_0)\langle (b(\bar{\phi}_{h,\tau}) - b(\hat{\bar{\phi}}_{h,\tau}))\nabla\hat{\bar{\mu}}_{h,\tau}, \bar{\nabla}\psi_{h,\tau}\rangle \\
&\quad + \langle (b^{1/2}(\bar{\phi}_{h,\tau}) - b^{1/2}(\hat{\bar{\phi}}_{h,\tau}))\nabla\hat{\bar{q}}_{h,\tau}, \bar{\nabla}\psi_{h,\tau}\rangle \,\mathrm{d}s, \\
\int_{t^{n-1}}^{t^n} \langle \bar{r}_{3,h,\tau}, \bar{\psi}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}; \bar{\zeta}_{h,\tau}, \hat{\bar{q}}_{h,\tau}) - \langle (\kappa(\phi_{h,\tau}) - \kappa(\hat{\phi}_{h,\tau}))\hat{\bar{q}}_{h,\tau}, \bar{\zeta}_{h,\tau}\rangle \\
&\quad + (1+\varepsilon_0)\langle (b^{1/2}(\phi_{h,\tau}) - b^{1/2}(\hat{\phi}_{h,\tau}))\nabla\hat{\bar{\mu}}_{h,\tau}, \bar{\nabla}\zeta_{h,\tau}\rangle \,\mathrm{d}s, \\
\int_{t^{n-1}}^{t^n} \langle \bar{r}_{4,h,\tau}, \bar{\mathbf{v}}_{h,\tau}\rangle \,\mathrm{d}s &= \int_{t^{n-1}}^{t^n} \tilde{\mathbf{c}}(\mathbf{u}_{h,\tau} - \hat{\mathbf{u}}_{h,\tau}; \bar{\mathbf{v}}_{h,\tau}, \hat{\bar{\mathbf{u}}}_{h,\tau}) - \mathbf{c}(\bar{\mathbf{v}}_{h,\tau}; \hat{\bar{\mu}}_{h,\tau}, \phi_{h,\tau} - \hat{\phi}_{h,\tau}) \\
&\quad - \langle (\eta(\phi_{h,\tau}) - \eta(\hat{\phi}_{h,\tau}))\nabla\hat{\bar{\mathbf{u}}}_{h,\tau}, \bar{\nabla}\mathbf{v}_{h,\tau}\rangle \,\mathrm{d}s.
\end{aligned}$$

The above residuals and the following estimates are quite similar to the calculations for weak-strong uniqueness, see Section 4.6.

Lemma 7.6.1. *Let Lemma 7.4.2 hold. Under the condition that $\tau = c_p h$ for a constant c_p independent of h, τ the solution $(\phi_{h,\tau}, \bar{\mu}_{h,\tau}, q_{h,\tau}, \mathbf{u}_{h,\tau}, \bar{p}_{h,\tau})$ of Problem P.2 is unique.*

Proof. The proof is divided into two steps. In the first step, we estimate the residuals. In the second step, we verify several uniform bounds and apply the discrete Gronwall lemma, cf. Lemma A.3.2. We start estimating the first residual by

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \|\bar{r}_{1,h,\tau}\|_{-1}^2 \,\mathrm{d}s &\le C\int_{t^{n-1}}^{t^n} \|\bar{\phi}_{h,\tau} - \hat{\bar{\phi}}_{h,\tau}\|_{0,6}^2 (\|\hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}^2 + \|\nabla\hat{\bar{\mu}}_{h,\tau}\|_{0,3}^2 + \|\nabla\hat{\bar{q}}_{h,\tau}\|_{0,3}^2) \,\mathrm{d}s \\
&\le C(\|\hat{\bar{\mathbf{u}}}_{h,\tau}\|_{L^\infty(L^3)}, \|\hat{\bar{\mu}}_{h,\tau}\|_{L^\infty(W^{1,3})}, \|\hat{\bar{q}}_{h,\tau}\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(\phi_{h,\tau}|\hat{\phi}_{h,\tau}) \,\mathrm{d}s.
\end{aligned}$$

The third residual similarly yields

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \|\bar{r}_{3,h,\tau}\|_{-1}^2 \,\mathrm{d}s &\le C\int_{t^{n-1}}^{t^n} \|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}^2 \|\nabla\hat{\bar{q}}_{h,\tau}\|_{0,3}^2 \\
&\quad + \|\bar{\phi}_{h,\tau} - \hat{\bar{\phi}}_{h,\tau}\|_{0,6}^2 (\|\hat{\bar{q}}_{h,\tau}\|_{0,3}^2 + \|\nabla\hat{\bar{\mu}}_{h,\tau}\|_{0,3}^2) \,\mathrm{d}s \\
&\le C(\|\hat{\bar{\mu}}_{h,\tau}\|_{L^\infty(W^{1,3})}, \|\hat{\bar{q}}_{h,\tau}\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) \,\mathrm{d}s \\
&\quad + \int_{t^{n-1}}^{t^n} 2\delta\mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) \,\mathrm{d}s.
\end{aligned}$$

The fourth residual yields

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|\bar{r}_{4,h,\tau}\|_{-1}^2 \,\mathrm{d}s \leq C \int_{t^{n-1}}^{t^n} & \|\bar{\mathbf{u}}_{h,\tau} - \hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}^2 \|\nabla \hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}^2 \\ & + \|\bar{\phi}_{h,\tau} - \hat{\bar{\phi}}_{h,\tau}\|_{0,6}^2 (\|\nabla \hat{\bar{\mathbf{u}}}_{h,\tau}\|_{0,3}^2 + \|\nabla \hat{\bar{\mu}}_{h,\tau}\|_{0,3}^2) \,\mathrm{d}s \\ \leq & \, C(\|\hat{\bar{\mu}}_{h,\tau}\|_{L^\infty(W^{1,3})}, \|\hat{\bar{\mathbf{u}}}_{h,\tau}\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) \,\mathrm{d}s \\ & + \int_{t^{n-1}}^{t^n} 2\delta \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) \,\mathrm{d}s.\end{aligned}$$

Summing up together the estimates yields

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \sum_{i\in\{1,3,4\}} \|\bar{r}_{i,h,\tau}\|_{-1}^2 \,\mathrm{d}s \leq & \int_{t^{n-1}}^{t^n} 4\delta \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) \,\mathrm{d}s \\ & + C(\|\hat{\bar{\mu}}_{h,\tau}\|_{L^\infty(W^{1,3})}, \|\hat{\bar{\mathbf{u}}}_{h,\tau}\|_{L^\infty(W^{1,3})}, \|\hat{\bar{q}}_{h,\tau}\|_{L^\infty(W^{1,3})}) \int_{t^{n-1}}^{t^n} \mathcal{E}_\alpha(z_{h,\tau}|\hat{z}_{h,\tau}) \,\mathrm{d}s.\end{aligned}$$

In order to be able to apply the discrete Gronwall lemma, we have to prove that $\bar{g}_{h,\tau}$ is uniformly bounded in $L^\infty(0,T;W^{1,3}(\Omega))$ for $\bar{g}_{h,\tau} \in \{\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{\mathbf{u}}}_{h,\tau}, \hat{\bar{q}}_{h,\tau}\}$. To verify this we estimate $\bar{g}_{h,\tau}$ as follows

$$\|\bar{g}_{h,\tau}\|_{L^\infty(W^{1,3})} \leq \|\bar{g}_{h,\tau} - \Pi\bar{g}\|_{L^\infty(W^{1,3})} + \|\Pi\bar{g} - \bar{g}\|_{L^\infty(W^{1,3})} + \|\bar{g}\|_{L^\infty(W^{1,3})}.$$

Here Π denotes either the L^2-projection for μ and q, or the Stokes projection with zero pressure for $\mathbf{u}$. We observe that the second and the third term are uniformly bounded by the projection error and the regularity assumption of μ, q or $\mathbf{u}$, see Lemma 7.3.1. Therefore, we only have to bound the first term. Using the inverse inequality (5.7) with $p = 3, q = 2, d \leq 3$ in space and with $p = \infty, q = 2, d = 1$ in time, as well as the convergence estimates of Lemma 7.4.2, we obtain

$$\|\bar{g}_{h,\tau} - \Pi\bar{g}\|_{L^\infty(W^{1,3})} \leq Ch^{-1/2}\tau^{-1/2}(h^4 + \tau^4).$$

Consequently, for $\tau = c_p h$ the right hand-side is uniformly bounded. As a second step we apply the discrete Gronwall lemma similarly to Lemma 7.2.1 which yields

$$\mathcal{E}_\alpha(z_{h,\tau}(t^n)|\hat{z}_{h,\tau}(t^n)) + \tfrac{1}{2}\int_0^{t^n} \mathcal{D}_{\phi_{h,\tau}}(\bar{\mu}_{h,\tau}, \bar{z}_{h,\tau}|\hat{\bar{\mu}}_{h,\tau}, \hat{\bar{z}}_{h,\tau}) \,\mathrm{d}s \leq 0. \tag{7.25}$$

Together with the lower bounds of the relative energy and dissipation we know that (7.25) is bounded from below by zero and consequently we find

$$\phi_{h,\tau}(t^n) = \hat{\phi}_{h,\tau}(t^n), \quad q_{h,\tau}(t^n) = \hat{q}_{h,\tau}(t^n), \quad \mathbf{u}_{h,\tau}(t^n) = \hat{\mathbf{u}}_{h,\tau}(t^n) \text{ and } \bar{\mu}_{h,\tau} \equiv \hat{\bar{\mu}}_{h,\tau}$$

for all $n \geq 1$, which already concludes the proof of the lemma. □

8 Numerical experiments

In this chapter we consider numerical experiments using the fully discrete scheme of Chapter 7, i.e., Problem P.2. In the first section, i.e., Section 8.1, we will consider a time-stepping formulation which is completely equivalent to Problem P.2. In Section 8.2 we will illustrate the convergence results of the last two chapters, i.e., the semi-discrete convergence result, i.e., Theorem 6.1.5 and the fully discrete convergence result, i.e., Theorem 7.1.3. This will be done by considering a suitable test problem on nested grids in space-time and computation of the experimental convergence order. We test the CHNSQ model with three different realisations of A. In Section 8.3 we consider the applications within the viscoelastic phase separation by conducting a reference experiment again for three different realisations of A. Within this section, we will consider the structure factor as a first estimate and the evolution of suitable time scales of the problem.

8.1. Time-stepping formulation

In this section, we will convert the fully discrete space-time formulation, i.e., Problem P.2, to a more standard but equivalent time-stepping formulation. We set $g_h^n := g_{h,\tau}(t^n)$ and denoted the midpoint approximation of piecewise linear functions in time by

$$\bar{g}_{h,\tau} := g_h^{n-1/2} = (g_h^n + g_h^{n-1})/2,$$

i.e., for $g \in \{\phi, q, \mathbf{u}\}$. For piecewise constant function in time $\bar{g}_{h,\tau} =: g_h^{n-1/2}$ denotes the unknown, i.e., for $g \in \{\mu, p\}$. Using the formulation of Problem P.2 we can compute almost all time integral directly and obtain the following time-stepping formulation of the problem.

Problem P.3 (Time-stepping formulation)**.** Given $(\phi_h^{n-1}, q_h^{n-1}, \mathbf{u}_h^{n-1}) \in \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d$. Find the discrete functions $(\phi_h^n, \mu_h^{n-1/2}, q_h^n, \mathbf{u}_h^n, p_h^{n-1/2}) \in \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d \times \mathcal{Q}_h$ such that

$$\begin{aligned}
\langle \frac{\phi_h^n - \phi_h^{n-1}}{\tau}, \psi_h \rangle - \mathbf{c}(\mathbf{u}_h^{n-1/2}; \psi_h, \phi_h^{n-1/2}) + \varepsilon_0 \langle b(\phi_h^{n-1/2}) \nabla \mu_h^{n-1/2}, \nabla \psi_h \rangle& \\
+ \langle b^{1/2}(\phi_h^{n-1/2}) \nabla \mu_h^{n-1/2} - \nabla (A(\phi_h^{n-1/2}) q_h^{n-1/2}), b^{1/2}(\phi_h^{n-1/2}) \nabla \psi_h \rangle &= 0, \\
\langle \mu_h^{n-1/2}, \xi_h \rangle - \gamma \langle \nabla \phi_h^{n-1/2}, \nabla \xi_h \rangle - \langle \overline{\tilde{f}'(\phi_h^n, \phi_h^{n-1})}, \xi_h \rangle &= 0,
\end{aligned}$$

$$
\begin{aligned}
\langle \frac{q_h^n - q_h^{n-1}}{\tau}, \zeta_h \rangle &+ \tilde{\mathbf{c}}(\mathbf{u}_h^{n-1/2}; q_h^{n-1/2}, \zeta_h) + \langle \kappa(\phi_h^{n-1/2}) q_h^{n-1/2}, \zeta_h \rangle + \varepsilon_1 \langle \nabla q_h^{n-1/2}, \nabla \zeta_h \rangle \\
&+ \langle \nabla (A(\phi_h^{n-1/2}) q_h^{n-1/2}) - b^{1/2}(\phi_h^{n-1/2}) \nabla \mu_h^{n-1/2}, \nabla (A(\phi_h^{n-1/2}) \zeta_h) \rangle = 0, \\
\langle \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau}, \mathbf{v}_h \rangle &+ \tilde{\mathbf{c}}(\mathbf{u}_h^{n-1/2}; \mathbf{u}_h^{n-1/2}, \mathbf{v}_h) + \langle \eta(\phi_h^{n-1/2}) \nabla \mathbf{u}_h^{n-1/2}, \nabla \mathbf{v}_h \rangle - \langle p_h^{n-1/2}, \operatorname{div} \mathbf{v}_h \rangle \\
&+ \mathbf{c}(\mathbf{v}_h; \mu_h^{n-1/2}, \phi_h^{n-1/2}) = 0, \\
0 &= \langle \operatorname{div} \mathbf{u}_h^{n-1/2}, w_h \rangle
\end{aligned}
$$

holds for all $(\psi_h, \xi_h, \zeta_h, \mathbf{v}_h, w_h) \in \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h \times \mathcal{V}_h^d \times \mathcal{Q}_h$.

Here $\overline{\tilde{f}'(\phi_h^n, \phi_h^{n-1})}$ denotes the time average of $f'(\phi_{h,\tau})$ and is given by

$$
\overline{\tilde{f}'(\phi_h^n, \phi_h^{n-1})} := \frac{1}{\tau} \int_{t^{n-1}}^{t^n} f'(\phi_{h,\tau}) \, \mathrm{d}s.
$$

Since $\phi_{h,\tau}$ is uniquely determined by the values ϕ_h^n and ϕ_h^{n-1} this can be computed using a suitable quadrature up to sufficiently high accuracy, hence almost exact. Note that, using low-order quadrature rules, i.e. inexact integration, results in well-known discretisation for the potential, see [69, 116]. However, such an inexact integration, in general, destroys the energy-dissipative structure.

In this notation, the energy-dissipation identity follows by testing $\psi_h = \mu_h^{n-1/2} \in \mathcal{V}_h$, $\xi_h = (\phi_h^n - \phi_h^{n-1})/\tau \in \mathcal{V}_h, \zeta_h = q_h^{n-1/2} \in \mathcal{V}_h, \mathbf{v}_h = \mathbf{u}_h^{n-1/2} \in \mathcal{V}_h^d, w_h = p_h^{n-1/2} \in \mathcal{Q}_h$ and reads

$$
E(\phi_h^n, q_h^n, \mathbf{u}_h^n) - E(\phi_h^{n-1}, q_h^{n-1}, \mathbf{u}_h^{n-1}) = -\tau D_{\phi_h^{n-1/2}}(\mu_h^{n-1/2}, q_h^{n-1/2}, \mathbf{u}_h^{n-1/2}) \tag{8.1}
$$

where the dissipation is given by

$$
\begin{aligned}
&D_{\phi_h^{n-1/2}}(\mu_h^{n-1/2}, q_h^{n-1/2}, \mathbf{u}_h^{n-1/2}) \\
=&\Big(\varepsilon_0 \left\| b^{1/2}(\phi_h^{n-1/2}) \nabla \mu_h^{n-1/2} \right\|_0^2 + \left\| b^{1/2}(\phi_h^{n-1/2}) \nabla \mu_h^{n-1/2} - \nabla (A(\phi_h^{n-1/2}) q_h^{n-1/2}) \right\|_0^2 \\
&+ \left\| \kappa^{1/2}(\phi_h^{n-1/2}) q_h^{n-1/2} \right\|_0^2 + \varepsilon_1 \left\| \nabla q_h^{n-1/2} \right\|_0^2 + \left\| \eta^{1/2}(\phi_h^{n-1/2}) \nabla \mathbf{u}_h^{n-1/2} \right\|_0^2 \Big).
\end{aligned}
$$

Before we consider the numerical tests, let us remark on several properties of the numerical method, i.e., Problem P.3.

1. The discretisation is an implicit time-stepping scheme that is very similar to Crank-Nicolson schemes or implicit midpoint rules, respectively.

2. The *exact* integration of $\overline{\tilde{f}'(\phi_h^n, \phi_h^{n-1})}$ is crucial for the *exact* energy-dissipation identity (8.1).

3. Solving of the nonlinear system per time step is realized by suitable fixed point methods, i.e., Newton method or simple fixed-point iterations. In general, this amounts to solving some linear systems per time step.

8.2. Experimental convergence

In this section, we will illustrate the results of the error analysis numerically. We will consider the same test for three different realisations of the CHNSQ model via Problem P.3. Before going to the example, let us discuss the structure of the three different realisations. We consider the following cases.

Model H: In this case, we assume that $A = 0$. This allows us to completely decouple the bulk stress equation for q. Hence, we consider the Cahn-Hilliard-Navier-Stokes system with an unrelated heat equation for q. Of course, all the results of the preceeding sections translate immediately. This will be relevant to understand the impact of the bulk stress on the model and can furthermore be used to check that indeed this model is not sufficient to describe viscoelastic phase separation.

Constant A: In this case, we consider the setup in our error analysis where (A9) holds. Hence, we can illustrate the rigorous results and compare them with model H to understand the impact of the bulk stress.

Nonlinear A: In the application, the exact choice of A seems to have an important impact. From a theoretical point of view, many results hold even in this case, while the rigorous error analysis can in principle be done, cf. Chapter 9. We will focus on specific choices of A, which is used in the literature and can be interpreted as an interpolation between the two other cases.

In all three cases, we consider the following experiment. While for model H, we neglect all contributions for q, i.e. the initial data is not relevant.

Experiment 8.2.1. We consider the following initial conditions

$$\begin{aligned} \phi_0 &= 0.25\cos(2\pi x)\cos(2\pi y) + 0.5, \\ q_0 &= 0.01\sin(2\pi x)\sin(2\pi y), \\ \mathbf{u}_0 &= 0.25(-\sin(\pi x)^2\sin(2\pi y), \sin(\pi y)^2\sin(2\pi x))^\top \end{aligned}$$

with parameters $\gamma = 0.001, \varepsilon_0 = \varepsilon_1 = 0$. For the nonlinear parametric functions we chose $b(\phi) = 10^{-1}\phi^2(1-\phi)^2 + 10^{-3}$, $f(\phi) = (\phi - 0.01)^2(\phi - 0.99)^2$, $\kappa(\phi) = 10^{-3}(10\phi^2 + 10^{-4})^{-1}$, $\eta(\phi) = 10^{-3}(1 + 2.5 \cdot 10^{-1}(1+\phi)^2)$ and

$$A = \begin{cases} 0, & \text{for model H} \\ 5 \cdot 10^{-4}\Big[1 + \tanh(10[\cot(\pi\phi^*) - \cot(\pi\phi)])\Big] & \text{for CHNSQ} \\ 1 \cdot 10^{-3}, & \text{for CHNSQ.} \end{cases}$$

The evolutionary behaviour of this experiment is shown in Figure 8.1 for the volume fraction ϕ and Figure 8.2 for the magnitude of the velocity $|\mathbf{u}|^2$. We observe that the volume fraction is rotated by the velocity field all the time, until the separation process starts to form connected clusters of the pure phases which are around $\phi \approx 0.01$ and $\phi \approx 0.99$. As time evolves the solution approaches a state where we have one big cluster of each separate phase. The velocity field preserves its rotation form over a long time and starts to disperse in small areas with a high velocity and big areas with a small velocity. The realisation for constant and nonlinear A can be found in the Appendix. Both look very similar. Furthermore, the energy decay in time and the error of mass conservation, mean divergence and the energy-dissipation error for the three different cases as described at the beginning of this section are depicted in Figures 8.3, 8.4, 8.5, respectively. We observe in all three cases that the energy is decaying over time and the errors of the energy-dissipation equality, the mass conservation and the mean divergence are very small being of the order 10^{-15}. The error in the energy-dissipation identity (8.1) is so small compared to the mass error that it is below the mean divergence error.

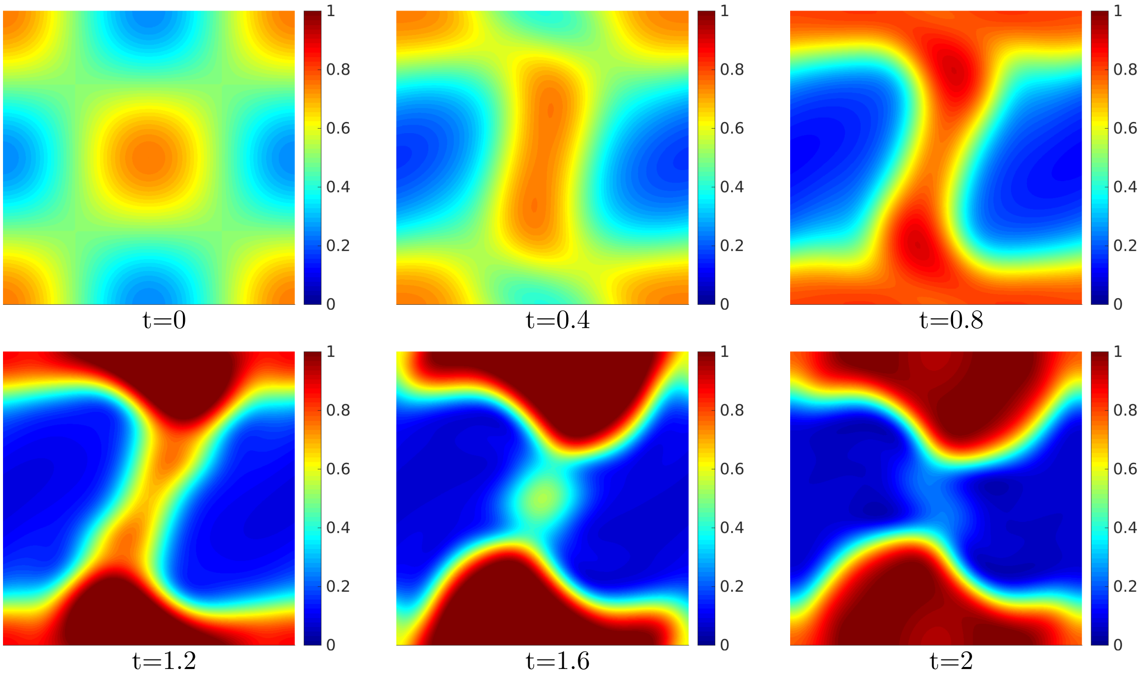

Figure 8.1.: **Model H:** Snapshots of the volume fraction ϕ for Experiment 8.2.1.

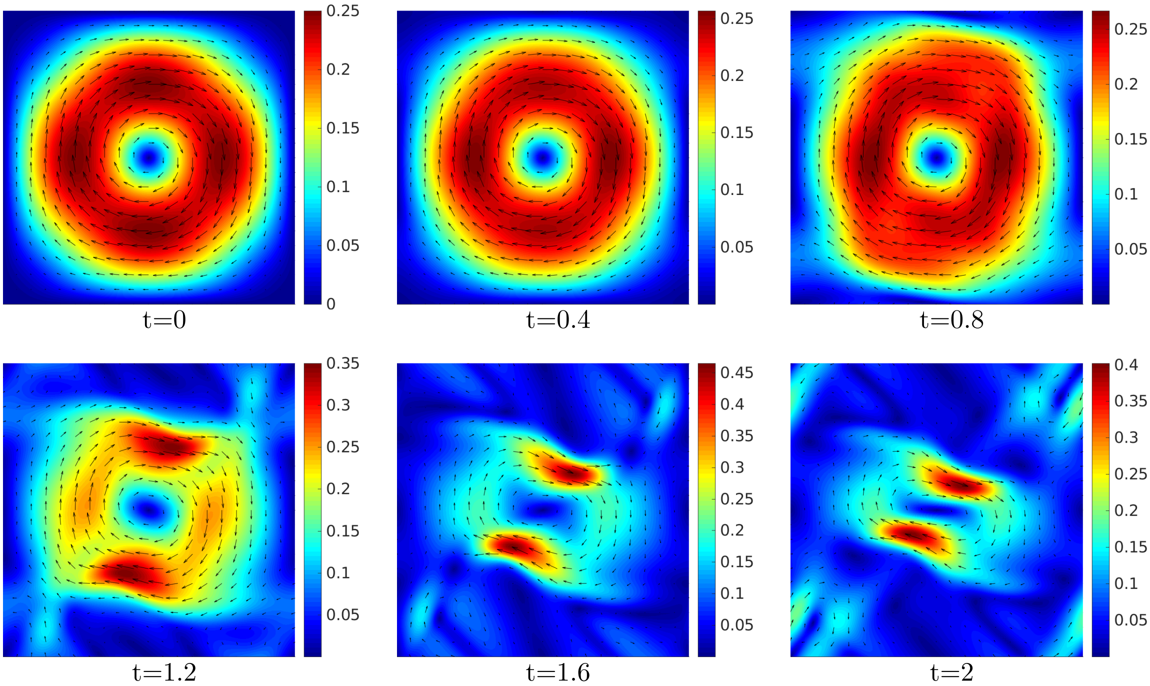

Figure 8.2.: **Model H:** Snapshots of the velocity field $\mathbf{u}$ for Experiment 8.2.1.

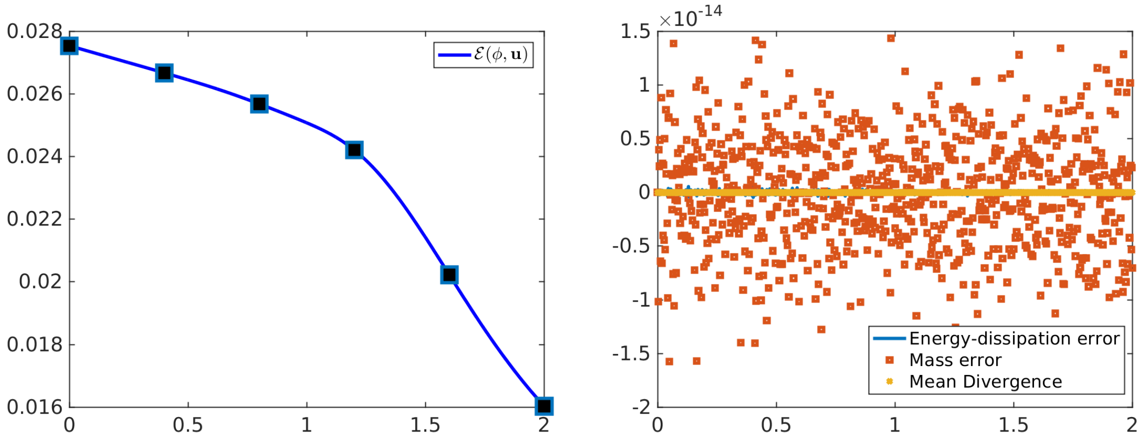

Figure 8.3.: **Model H:** (Left): Evolution of the energy $\mathcal{E}(\phi, \mathbf{u})$ and (Right): Energy-dissipation and mass conservation and mean divergence error for Experiment 8.2.1.

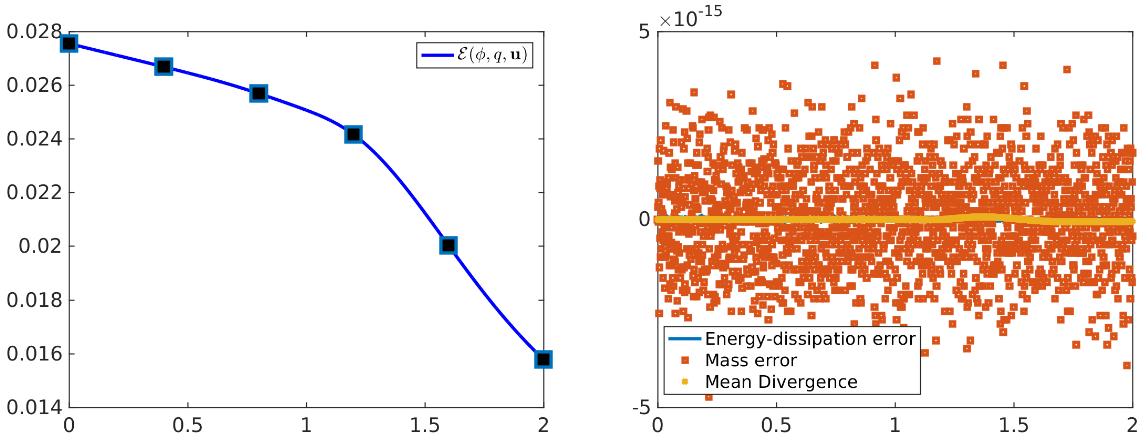

Figure 8.4.: (Left): Evolution of the energy $\mathcal{E}(\phi, \mathbf{u})$ and (Right): Energy-dissipation, mass conservation and mean divergence error for Experiment 8.2.1.

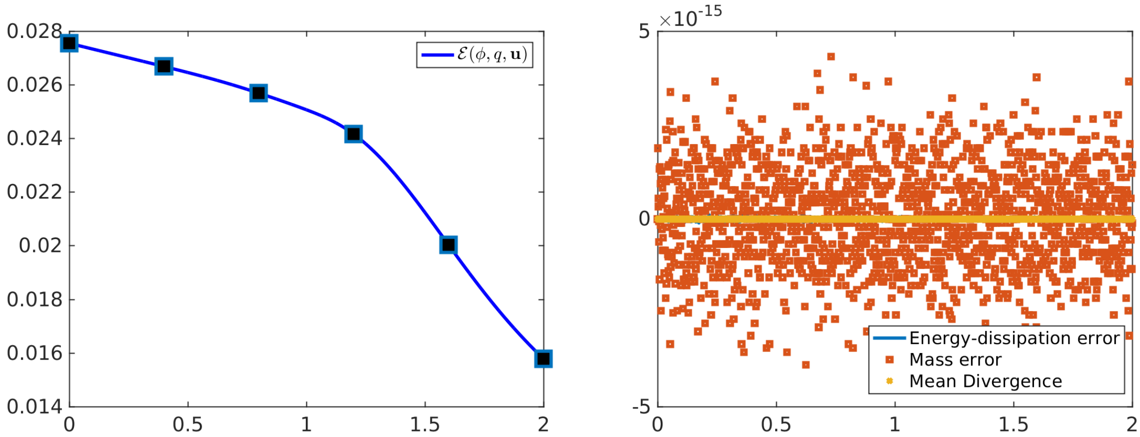

Figure 8.5.: (Left): Evolution of the energy $\mathcal{E}(\phi, \mathbf{u})$ and (Right): Energy-dissipation, mass conservation and mean divergence error for Experiment 8.2.1.

8.2.1. Experimental convergence error

We will consider the above tests to compute the experimental order of convergence (eoc). Since no analytic solution is available, the discretisation error will be estimated by comparing discrete solutions on uniformly refined meshes in space and time. In order to measure the eoc we introduce the discrete error by

$$e_{h,\tau} := \max_{t^n\in\mathcal{I}_\tau}\left(\left\|\phi_{h,\tau}(t^n)-\phi_{\frac{h}{2},\frac{\tau}{2}}(t^n)\right\|_1^2+\left\|q_{h,\tau}(t^n)-q_{\frac{h}{2},\frac{\tau}{2}}(t^n)\right\|_0^2+\left\|\mathbf{u}_{h,\tau}(t^n)-\mathbf{u}_{\frac{h}{2},\frac{\tau}{2}}(t^n)\right\|_0^2\right) + \left\|\bar{\mu}_{h,\tau}-\bar{\mu}_{\frac{h}{2},\frac{\tau}{2}}\right\|_{L^2(H^1)}^2+\left\|\bar{q}_{h,\tau}-\bar{q}_{\frac{h}{2},\frac{\tau}{2}}\right\|_{L^2(H^1)}^2+\left\|\bar{\mathbf{u}}_{h,\tau}-\bar{\mathbf{u}}_{\frac{h}{2},\frac{\tau}{2}}\right\|_{L^2(H^1)}^2.$$

Furthermore, we introduce the following additional errors by

$$e_{p,h,\tau} := \left\|\bar{p}_{h,\tau}-\bar{p}_{\frac{h}{2},\frac{\tau}{2}}\right\|_{L^2(L^2)}^2, \qquad e_{\mathbf{u},h,\tau} := \max_{t^n\in\mathcal{I}_\tau}\left\|\mathbf{u}_{h,\tau}(t^n)-\mathbf{u}_{\frac{h}{2},\frac{\tau}{2}}(t^n)\right\|_0^2$$
$$e_{q,h,\tau} := \max_{t^n\in\mathcal{I}_\tau}\left\|q_{h,\tau}(t^n)-q_{\frac{h}{2},\frac{\tau}{2}}(t^n)\right\|_0^2.$$

For the semi-discrete case, we consider a fixed time step τ^* and define the error quantities by

$$e_{h} := \max_{t^n\in\mathcal{I}_\tau}\left(\left\|\phi_{h,\tau^*}(t^n)-\phi_{\frac{h}{2},\tau^*}(t^n)\right\|_1^2+\left\|q_{h,\tau^*}(t^n)-q_{\frac{h}{2},\tau^*}(t^n)\right\|_0^2+\left\|\mathbf{u}_{h,\tau^*}(t^n)-\mathbf{u}_{\frac{h}{2},\tau^*}(t^n)\right\|_0^2\right) + \left\|\bar{\mu}_{h,\tau^*}-\bar{\mu}_{\frac{h}{2},\tau^*}\right\|_{L^2(H^1)}^2+\left\|\bar{q}_{h,\tau^*}-\bar{q}_{\frac{h}{2},\tau^*}\right\|_{L^2(H^1)}^2+\left\|\bar{\mathbf{u}}_{h,\tau^*}-\bar{\mathbf{u}}_{\frac{h}{2},\tau^*}\right\|_{L^2(H^1)}^2.$$

Similarly, we introduce the following additional errors by

$$e_{p,h} := \left\|\bar{p}_{h,\tau^*}-\bar{p}_{\frac{h}{2},\tau^*}\right\|_{L^2(L^2)}^2, \qquad e_{\mathbf{u},h} := \max_{t^n\in\mathcal{I}_\tau}\left\|\mathbf{u}_{h,\tau^*}(t^n)-\mathbf{u}_{\frac{h}{2},\tau^*}(t^n)\right\|_0^2$$
$$e_{q,h} := \max_{t^n\in\mathcal{I}_\tau}\left\|q_{h,\tau^*}(t^n)-q_{\frac{h}{2},\tau^*}(t^n)\right\|_0^2.$$

We note that again $\bar{g}$ denotes the L^2-projection onto the time mesh $\mathcal{I}_\tau$. The experimental order of convergence can be then computed via

$$\mathrm{eoc} := \log_2\left(\frac{e_{h,\tau}}{e_{h/2,\tau/2}}\right), \text{ or } \mathrm{eoc} := \log_2\left(\frac{e_{h,\tau^*}}{e_{h/2,\tau^*}}\right),$$

both for the fully discrete or semi-discrete error. Of course, similar formulas are valid for the additional errors in the velocity $\mathbf{u}$, pressure p and bulk stress q.

The results for the three test cases can be found in the Tables 8.1–8.10. In what follows we will discuss the results of our computations obtained on a sequence of uniformly refined meshes. In the case of model H, we consider the mesh size $h_k = 2^{-(3+k)}$, $k = 0, \ldots, 3$ and time steps $\tau_k = 0.025h_k$. For the results concerning the semi-discretization, the time step is chosen $\tau^* = 1.28 \cdot 2^{-9}$.
In the case of the CHNSQ model, we use the mesh size $h_k = 2^{-(2+k)}$, $k = 0, \ldots, 3$ and time steps $\tau_k = 0.01h_k$. For the results concerning the semi-discretization, the time step is chosen $\tau^* = 1.024 \cdot 2^{-10}$. Since nested grids are used in all our computations, the discrete errors can be computed exactly, i.e., we do not introduce an additional interpolation error by projection onto finer or coarser meshes.

In the case of model H we observe the eoc of 4th order, hence due to squared norms, we have the second order experimental convergence in space and time. Similarly, the pressure converges with the second order in space and time. Furthermore, we observe the third order superconvergence of the velocity in the L^2-norm in space. We have no proof for this superconvergence. Note that, this fact is known for smooth solutions of incompressible Navier-Stokes equations, see [68] for superconvergence in space. Such a property is typically related to the smoothness of the dual problem.

Due to the complexity for the test cases involving non-zero A, we present the full convergence test until $T = 2$ for four different meshes and until $T = 0.3$ for five different meshes. We observe again that the error $e_{h,\tau}$ converge almost with second order in space and time, as well as the pressure. Again we observe the third order superconvergence for the velocity. In the bulk stress, the superconvergence of order three is not observed. This could be due to large time steps or simply that is a property that does not hold for the bulk stress. We also observe that the error and the eoc of the semi-discrete and the fully discrete case almost coincide. This is due to the choice of a small time step to speed up the iteration of the nonlinear system. This implies that the error in this regime is dominated by the space discretisation error.

Let us shortly summarize the results of the section.

1. The numerical results match the theoretical results in terms of the error estimates. Furthermore, we observe that the energy-dissipation identity (8.1) holds up to machine precision.

2. We observe a superconvergence property in the L^2-norm for the velocity, which is not theoretically proven, but at least to some degree expected.

3. The experimental convergence for the test cases with constant and nonlinear A does not really deviate. This can be understood as a first hint that the error analysis can indeed be extended to the nonlinear case, i.e. beyond assumption (A9).

Table 8.1.: **Model H:** Errors and experimental convergence rates for the semi-discrete approximation for Experiment 8.2.1, final time $T = 2$.

k	e_h	eoc	$e_{\mathbf{u},h}$	eoc	$e_{p,h}$	eoc
0	$5.743 \cdot 10^{-0}$	—	$3.669 \cdot 10^{-3}$	—	$4.866 \cdot 10^{-4}$	—
1	$1.521 \cdot 10^{-0}$	1.916	$4.101 \cdot 10^{-4}$	3.084	$1.442 \cdot 10^{-4}$	1.754
2	$1.634 \cdot 10^{-1}$	3.217	$1.156 \cdot 10^{-5}$	5.122	$1.586 \cdot 10^{-5}$	3.184
3	$1.757 \cdot 10^{-2}$	3.928	$1.757 \cdot 10^{-7}$	6.038	$5.544 \cdot 10^{-7}$	4.839

Table 8.2.: **Model H:** Errors and experimental convergence rates for the fully discrete approximation for Experiment 8.2.1, final time $T = 2$.

k	$e_{h,\tau}$	eoc	$e_{\mathbf{u},h,\tau}$	eoc	$e_{p,h,\tau}$	eoc
0	$5.748 \cdot 10^{-0}$	—	$3.468 \cdot 10^{-3}$	—	$4.863 \cdot 10^{-4}$	—
1	$1.523 \cdot 10^{-0}$	1.916	$4.011 \cdot 10^{-4}$	3.084	$1.442 \cdot 10^{-4}$	1.753
2	$1.637 \cdot 10^{-1}$	3.217	$1.156 \cdot 10^{-5}$	5.123	$1.586 \cdot 10^{-5}$	3.184
3	$1.074 \cdot 10^{-2}$	3.929	$1.757 \cdot 10^{-7}$	6.084	$5.544 \cdot 10^{-7}$	4.839

Table 8.3.: **Nonlinear A:** Errors and experimental convergence rates for the semi-discrete approximation for Experiment 8.2.1, final time $T = 2$.

k	e_h	eoc	$e_{\mathbf{u},h}$	eoc	$e_{p,h}$	eoc	$e_{q,h}$	eoc
0	$4.573 \cdot 10^{-1}$	—	$2.121 \cdot 10^{-4}$	—	$3.019 \cdot 10^{-5}$	—	$1.749 \cdot 10^{-7}$	—
1	$7.392 \cdot 10^{-2}$	2.629	$2.154 \cdot 10^{-5}$	3.265	$2.173 \cdot 10^{-6}$	3.796	$1.270 \cdot 10^{-8}$	3.782
2	$5.074 \cdot 10^{-3}$	3.864	$2.550 \cdot 10^{-7}$	6.412	$8.890 \cdot 10^{-8}$	4.611	$4.551 \cdot 10^{-10}$	4.803

Table 8.4.: **Nonlinear A:** Errors and experimental convergence rates for the fully discrete approximation for Experiment 8.2.1, final time $T = 2$.

k	$e_{h,\tau}$	eoc	$e_{\mathbf{u},h,\tau}$	eoc	$e_{p,h,\tau}$	eoc	$e_{q,h,\tau}$	eoc
0	$4.578 \cdot 10^{-1}$	—	$2.121 \cdot 10^{-4}$	—	$3.018 \cdot 10^{-5}$	—	$1.749 \cdot 10^{-7}$	—
1	$7.412 \cdot 10^{-2}$	2.626	$2.154 \cdot 10^{-5}$	3.265	$2.172 \cdot 10^{-6}$	3.796	$1.270 \cdot 10^{-8}$	3.782
2	$5.086 \cdot 10^{-3}$	3.865	$2.550 \cdot 10^{-7}$	6.412	$8.889 \cdot 10^{-8}$	4.611	$4.551 \cdot 10^{-10}$	4.803

Table 8.5.: **Nonlinear A:** Errors and experimental convergence rates for the semi-discrete approximation for Experiment 8.2.1, final time $T = 0.3$.

k	e_h	eoc	$e_{\mathbf{u},h}$	eoc	$e_{p,h}$	eoc	$e_{q,h}$	eoc
0	$7.121 \cdot 10^{-2}$	—	$4.563 \cdot 10^{-5}$	—	$5.029 \cdot 10^{-6}$	—	$9.795 \cdot 10^{-8}$	—
1	$5.681 \cdot 10^{-3}$	3.647	$2.715 \cdot 10^{-6}$	4.047	$3.648 \cdot 10^{-7}$	3.782	$1.911 \cdot 10^{-9}$	5.679
2	$4.914 \cdot 10^{-4}$	3.531	$1.273 \cdot 10^{-7}$	4.415	$1.392 \cdot 10^{-8}$	4.711	$4.709 \cdot 10^{-11}$	5.343
3	$3.257 \cdot 10^{-5}$	3.915	$1.203 \cdot 10^{-9}$	6.724	$6.486 \cdot 10^{-10}$	4.423	$1.599 \cdot 10^{-12}$	4.880

Table 8.6.: **Nonlinear A:** Errors and experimental convergence rates for the fully discrete approximation for Experiment 8.2.1, final time $T = 0.3$.

k	$e_{h,\tau}$	eoc	$e_{\mathbf{u},h,\tau}$	eoc	$e_{p,h,\tau}$	eoc	$e_{q,h,\tau}$	eoc
0	$7.138 \cdot 10^{-2}$	—	$4.863 \cdot 10^{-4}$	—	$5.019 \cdot 10^{-6}$	—	$1.749 \cdot 10^{-7}$	—
1	$5.710 \cdot 10^{-3}$	3.643	$1.442 \cdot 10^{-4}$	4.016	$2.648 \cdot 10^{-7}$	3.782	$1.270 \cdot 10^{-8}$	5.679
2	$4.977 \cdot 10^{-4}$	3.520	$1.586 \cdot 10^{-5}$	4.415	$1.392 \cdot 10^{-8}$	4.711	$4.551 \cdot 10^{-10}$	5.343
3	$3.287 \cdot 10^{-5}$	3.920	$1.203 \cdot 10^{-9}$	6.724	$6.486 \cdot 10^{-10}$	4.423	$1.599 \cdot 10^{-12}$	4.880

Table 8.7.: **Constant A:** Errors and experimental convergence rates for the semi-discrete approximation for Experiment 8.2.1, final time $T = 2$.

k	e_h	eoc	$e_{\mathbf{u},h}$	eoc	$e_{p,h}$	eoc	$e_{q,h}$	eoc
0	$4.572 \cdot 10^{-1}$	—	$2.121 \cdot 10^{-4}$	—	$3.018 \cdot 10^{-5}$	—	$1.669 \cdot 10^{-7}$	—
1	$7.390 \cdot 10^{-2}$	2.629	$2.154 \cdot 10^{-5}$	3.265	$2.172 \cdot 10^{-6}$	3.796	$1.109 \cdot 10^{-8}$	3.911
2	$5.073 \cdot 10^{-3}$	3.864	$2.548 \cdot 10^{-7}$	6.412	$8.888 \cdot 10^{-8}$	4.611	$5.092 \cdot 10^{-10}$	4.445

Table 8.8.: **Constant A:** Errors and experimental convergence rates for the fully discrete approximation for Experiment 8.2.1, final time $T = 2$.

k	e_h	eoc	$e_{\mathbf{u},h}$	eoc	$e_{p,h}$	eoc	$e_{q,h}$	eoc
0	$4.578 \cdot 10^{-1}$	—	$2.121 \cdot 10^{-4}$	—	$3.018 \cdot 10^{-5}$	—	$1.669 \cdot 10^{-7}$	—
1	$7.411 \cdot 10^{-2}$	2.626	$2.154 \cdot 10^{-5}$	3.265	$2.172 \cdot 10^{-6}$	3.796	$1.109 \cdot 10^{-8}$	3.911
2	$5.085 \cdot 10^{-3}$	3.865	$2.548 \cdot 10^{-7}$	6.412	$8.888 \cdot 10^{-8}$	4.611	$5.092 \cdot 10^{-10}$	4.444

Table 8.9.: **Constant A:** Errors and experimental convergence rates for the semi-discrete approximation for Experiment 8.2.1, final time $T = 0.3$.

k	e_h	eoc	$e_{\mathbf{u},h}$	eoc	$e_{p,h}$	eoc	$e_{q,h}$	eoc
0	$2.044 \cdot 10^{-1}$	—	$1.618 \cdot 10^{-4}$	—	$4.736 \cdot 10^{-6}$	—	$1.311 \cdot 10^{-7}$	—
1	$2.105 \cdot 10^{-2}$	3.279	$1.756 \cdot 10^{-5}$	3.233	$3.209 \cdot 10^{-7}$	3.883	$6.385 \cdot 10^{-9}$	4.360
2	$1.303 \cdot 10^{-3}$	4.013	$2.111 \cdot 10^{-7}$	6.386	$1.288 \cdot 10^{-8}$	4.637	$3.732 \cdot 10^{-10}$	4.096
3	$6.675 \cdot 10^{-5}$	4.287	$1.352 \cdot 10^{-9}$	7.250	$6.229 \cdot 10^{-10}$	4.371	$1.151 \cdot 10^{-11}$	5.018

Table 8.10.: **Constant A:** Errors and experimental convergence rates for the fully discrete approximation for Experiment 8.2.1, final time $T = 0.3$.

k	$e_{h,\tau}$	eoc	$e_{\mathbf{u},h,\tau}$	eoc	$e_{p,h,\tau}$	eoc	$e_{q,h,\tau}$	eoc
0	$2.045 \cdot 10^{-1}$	—	$1.618 \cdot 10^{-4}$	—	$4.735 \cdot 10^{-6}$	—	$1.311 \cdot 10^{-7}$	—
1	$2.108 \cdot 10^{-2}$	3.277	$1.756 \cdot 10^{-4}$	3.239	$3.209 \cdot 10^{-7}$	3.883	$6.385 \cdot 10^{-9}$	4.360
2	$1.305 \cdot 10^{-3}$	4.013	$2.111 \cdot 10^{-5}$	6.386	$1.288 \cdot 10^{-8}$	4.637	$3.731 \cdot 10^{-10}$	4.096
3	$6.681 \cdot 10^{-5}$	4.288	$1.352 \cdot 10^{-9}$	7.251	$6.229 \cdot 10^{-10}$	4.371	$1.151 \cdot 10^{-11}$	5.018

8.3. Viscoelastic phase separation

Here, we study the evolutionary behaviour of the CHNSQ model, i.e., System S.4 in the context of viscoelastic phase separation. We again consider the three cases $A = 0$, $A = 1$ and a nonlinear relation for A. We consider the same initial data for all simulations.

Experiment 8.3.1.

$$\begin{aligned} \phi_0 &= 0.4 + \xi(x, y), \qquad q_0 = 0, \\ \mathbf{u}_0 &= 10^{-3}(-\sin(\pi x)^2 \sin(2\pi y), \sin(\pi y)^2 \sin(2\pi x))^\top. \end{aligned}$$

with $\xi(x, y)$ is a uniformly distributed random variable with range $[-0.0025, 0.0025]$ and the parameters $\gamma = 0.001, \varepsilon_0 = \varepsilon_1 = 0$. For the nonlinear parametric functions we chose $b(\phi) = 1.6 \cdot \phi^2(1-\phi)^2 + 10^{-3}$, $f(\phi) = 16(\phi - 0.95)^2(\phi - 0.05)^2$, $\kappa(\phi) = 10^{-3}(10\phi^2 + 10^{-4})^{-1}$, $\eta(\phi) = 10^{-3}(1 + \phi^2)$ and

$$A = \begin{cases} 0, & \text{for model H} \\ \frac{1}{2}\Big[1 + \tanh(10[\cot(\pi\phi^*) - \cot(\pi\phi)])\Big] & \text{for CHNSQ} \\ 1, & \text{for CHNSQ.} \end{cases}$$

Note that the nonlinear variant of A can be understood as the interpolation between the limiting cases $A = 0$ and $A = 1$. We fix $h = 2^{-5}$ and $\tau = 5 \cdot 10^{-3}$. We refer again to the case $A = 0$ as model H, to $A = 1$ as the constant case, and the remaining case is called the nonlinear case. In principle, the configuration is chosen to mimic the experimental setup shown in Figure 1.2, i.e., we start from an almost constant state with small perturbations.

In the Figures 8.1-8.8, 8.10–8.12 and 8.13–8.16 we show the evolution of the volume fraction ϕ, the velocity $\mathbf{u}$, and the energy for model H, the CHNSQ model with nonlinear A and constant A, respectively. We observe that for model H the separation into small droplets is almost instantaneous, while for the CHNSQ model we observe a slow network-like pattern evolution. For the CHNSQ model, we show the volume fraction with a fixed colour scale ranging from $[0, 1]$ and colour scale which adjusts to the minimal and maximal values of ϕ for the given time step. We observe that in the case of nonlinear A, the dynamics evolve much faster than for the constant A, and we observe a lengthy network trident at the final time.

In order to emphasise that the model is capable of simulating viscoelastic phase separation, we compare the results either to the real experiment, i.e., Figure 1.2 or to the scaling regimes, i.e., Figure 2.1. We observe that in principle, the simulations show the same effects of viscoelastic phase separation. In details, by recalling Figure 1.2 and 2.1 we observe that until $t = 0.4$ we exhibit the evolution from frozen state to the solvent-rich droplet phase. Afterwards, the volume of the polymer richer phase starts to shrink and developed network-like patterns until $t = 3.2$. Here the phase inversion happens, i.e., change of the visible dominant phase. After $t = 3.2$ we enter the relaxational regime, i.e., the network starts to break and relax towards smaller structures. Note that the final regime, i.e., the hydrodynamic regime, is not visible here, since it requires much larger times. Furthermore, the instantaneous droplet phase in model H corresponds to the droplet regime, i.e., Figure 2.1 (f).

Let us shortly summarize the results of this section. The appearance of the bulk stress, i.e. for constant or nonlinear A, in the CHNSQ model allows us to reproduce experimental observation of viscoelastic phase separation to some degree. While, for model H, we do not observe the more complex dynamics. This again underlines that model H is not capable to describe the complex dynamics of viscoelastic phase separation, while the more complex CHNSQ model seems to be capable. The main dynamics in the case of constant and nonlinear A is very similar, however, it seems that the evolution of the dynamics for nonlinear A is faster. Again, conservation of mass and the energy identity (8.1) holds up to machine precision.

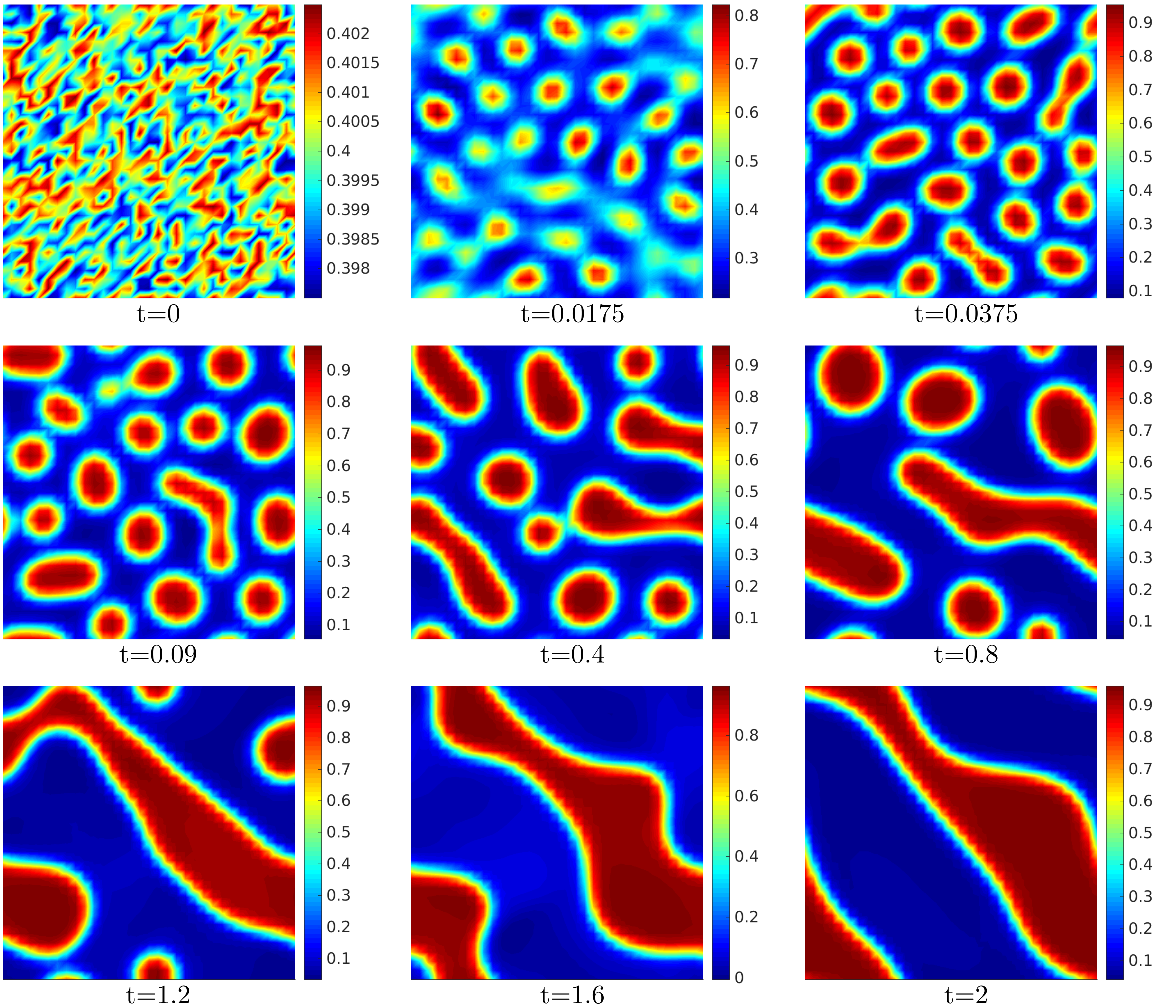

Figure 8.6.: **Model H:** Snapshots of the volume fraction ϕ for Experiment 8.3.1.

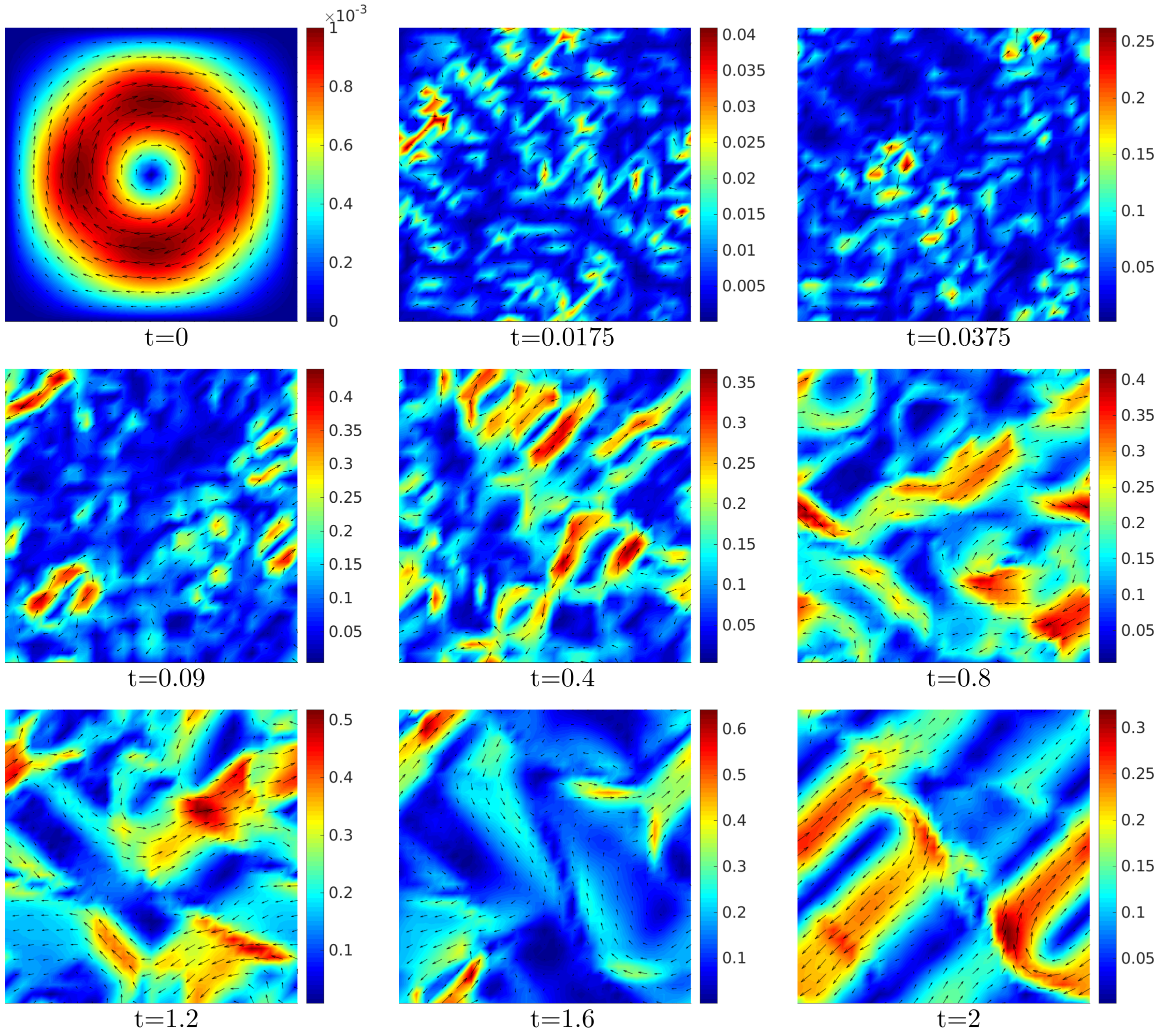

Figure 8.7.: **Model H:** Snapshots of the velocity $\mathbf{u}$ for Experiment 8.3.1.

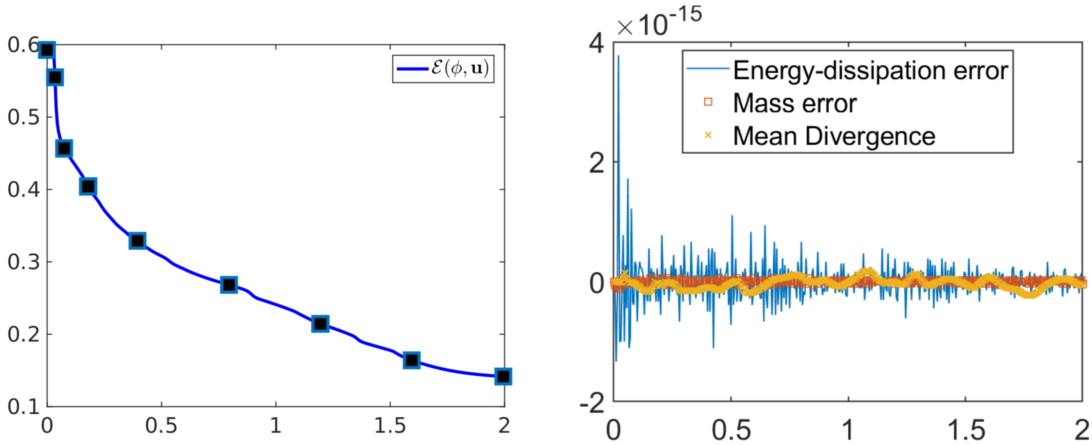

Figure 8.8.: **Model H:** (Left): Evolution of the energy $\mathcal{E}(\phi, \mathbf{u})$ and (Right): Energy-dissipation, mass conservation and mean divergence error for Experiment 8.3.1.

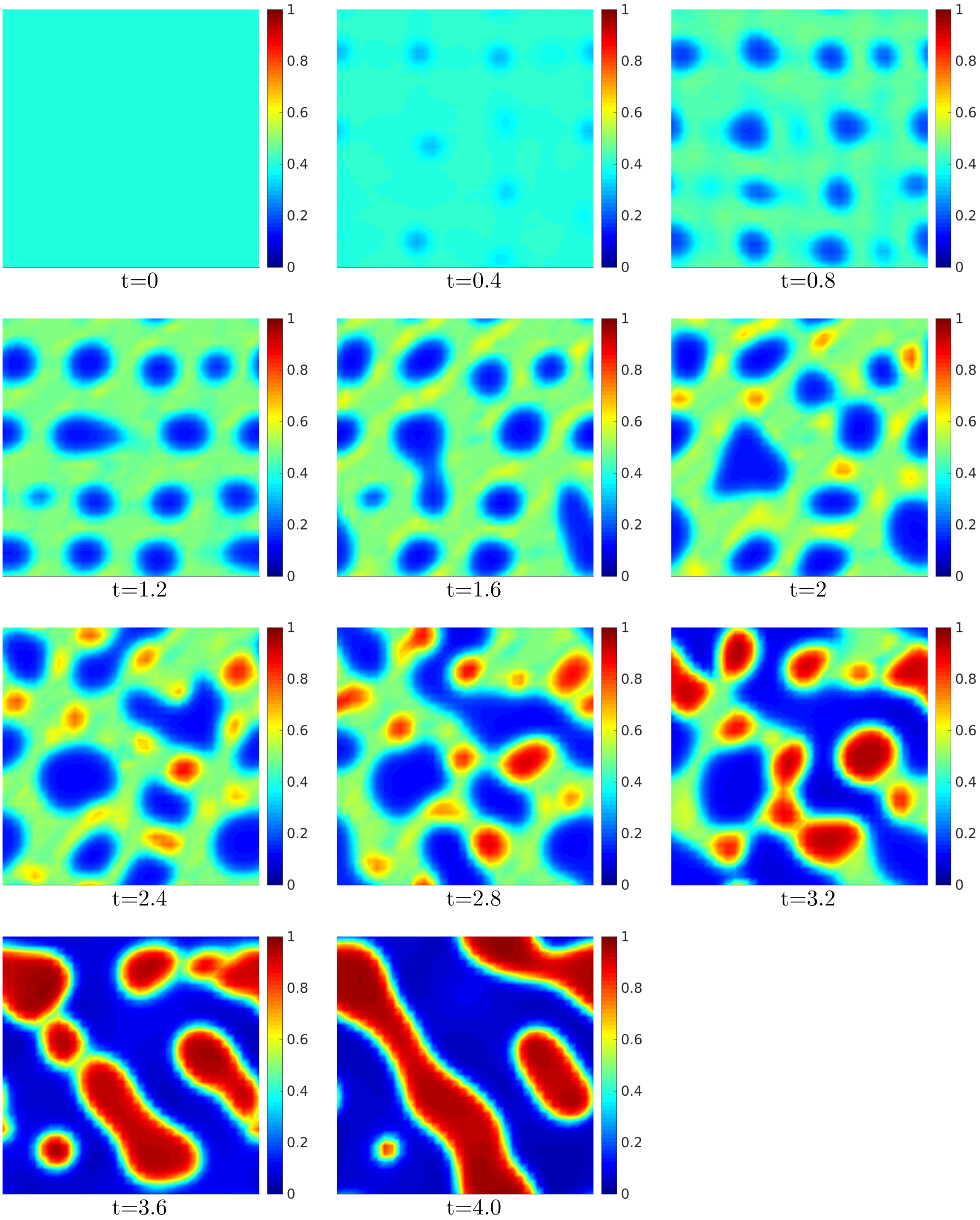

Figure 8.9.: **Nonlinear A:** Snapshots of the volume fraction ϕ with the fixed colour map $[0, 1]$ for Experiment 8.3.1.

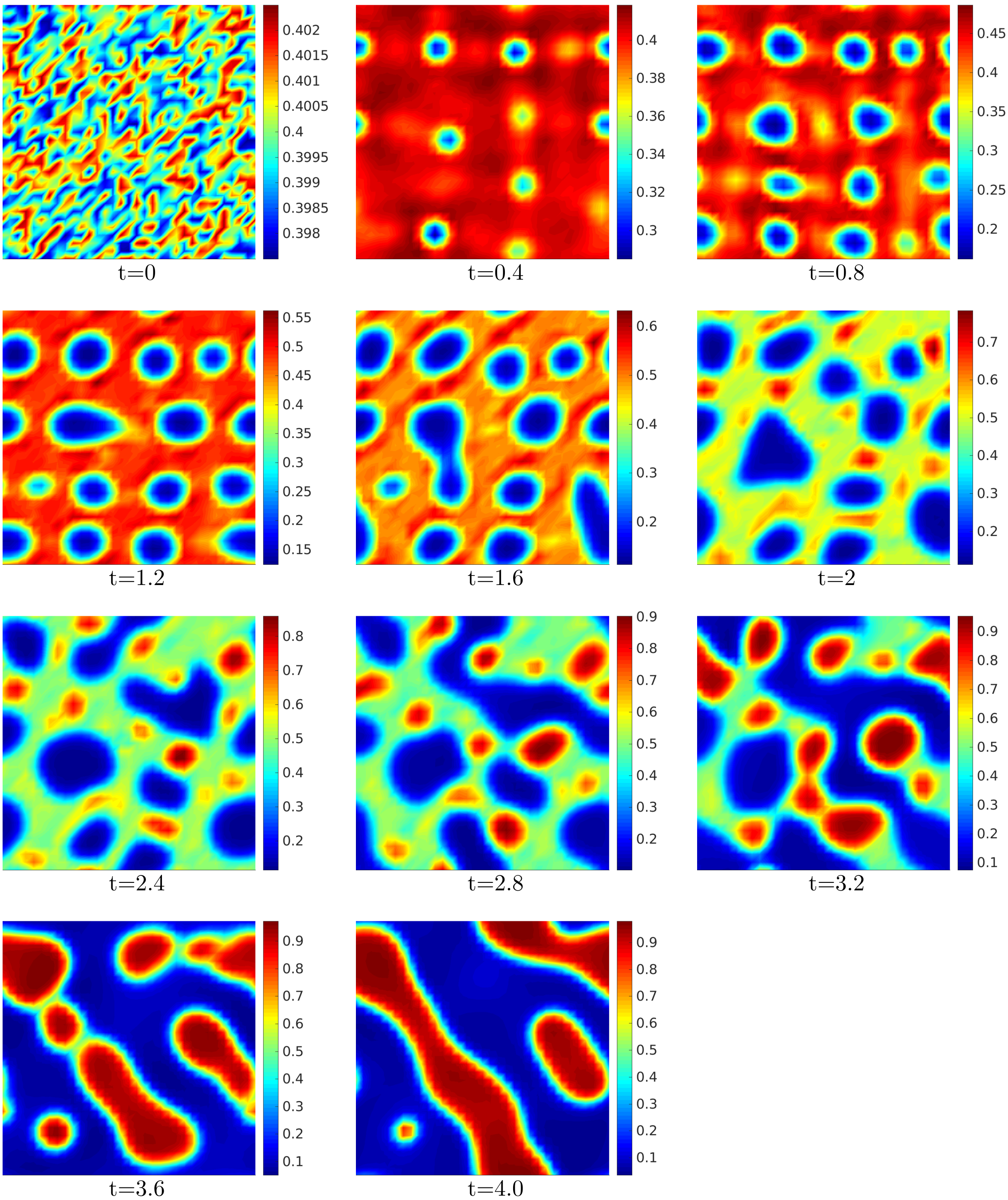

Figure 8.10.: **Nonlinear A:** Snapshots of the volume fraction ϕ for Experiment 8.3.1.

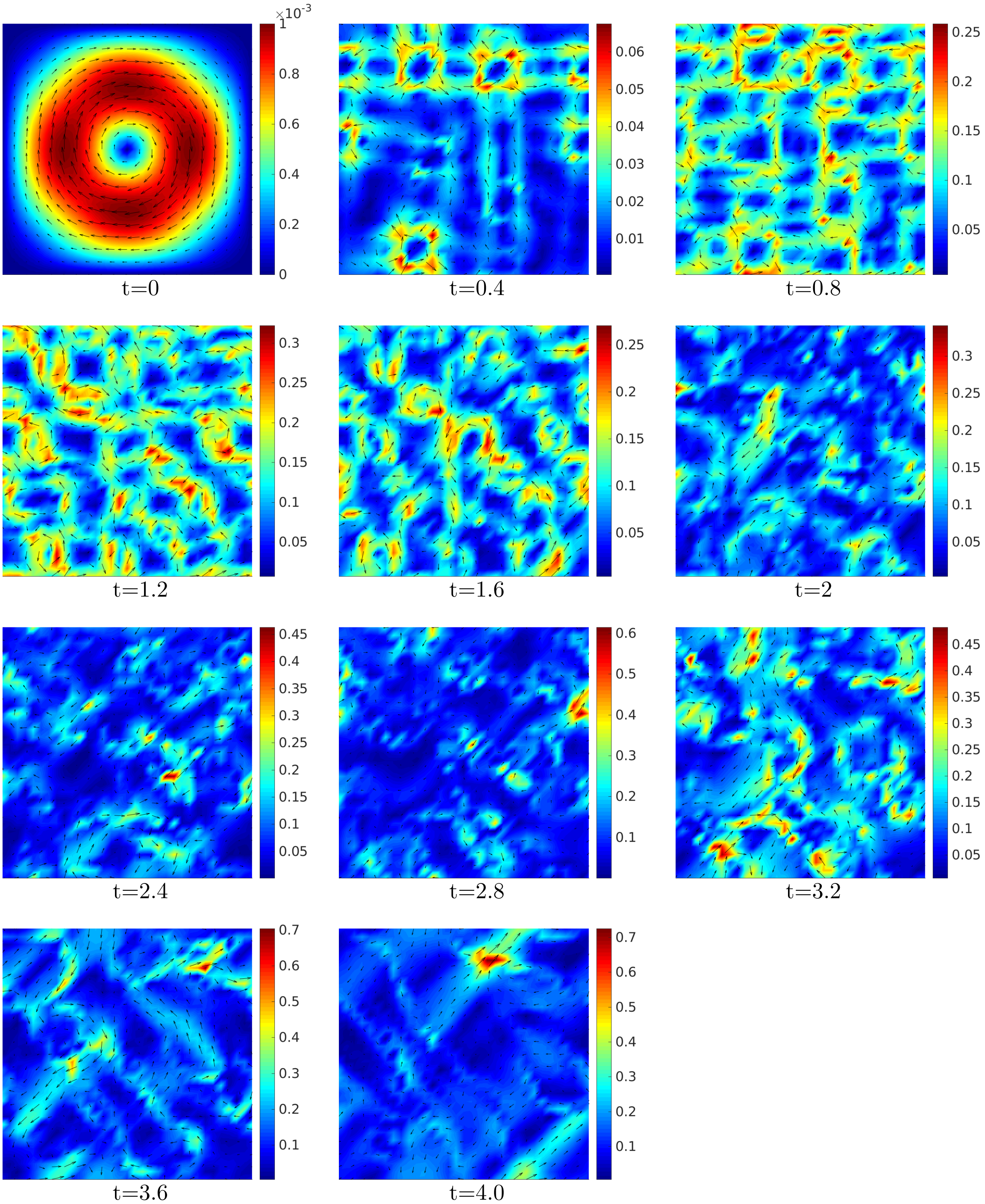

Figure 8.11.: **Nonlinear A:** Snapshots of the velocity $\mathbf{u}$ for Experiment 8.3.1.

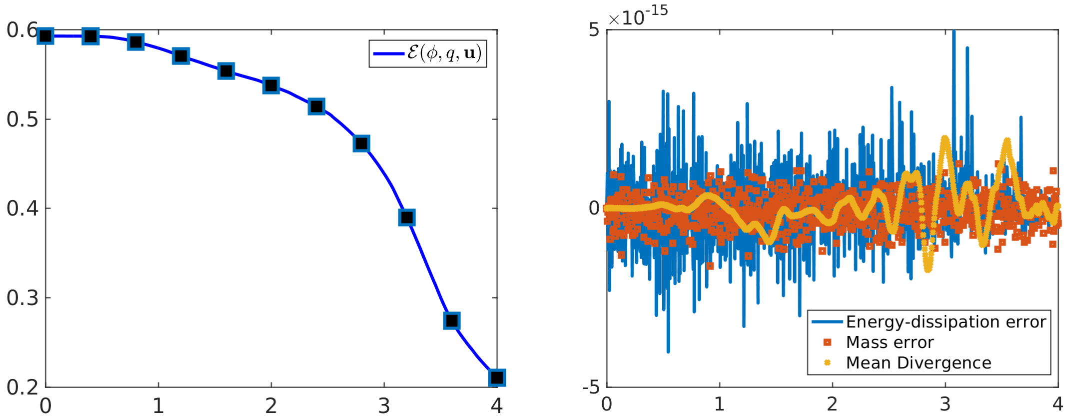

Figure 8.12.: **Nonlinear A:** (Left): Evolution of the energy $\mathcal{E}(\phi, q, \mathbf{u})$ and (Right): Energy-dissipation, mass conservation and mean divergence error for Experiment 8.3.1.

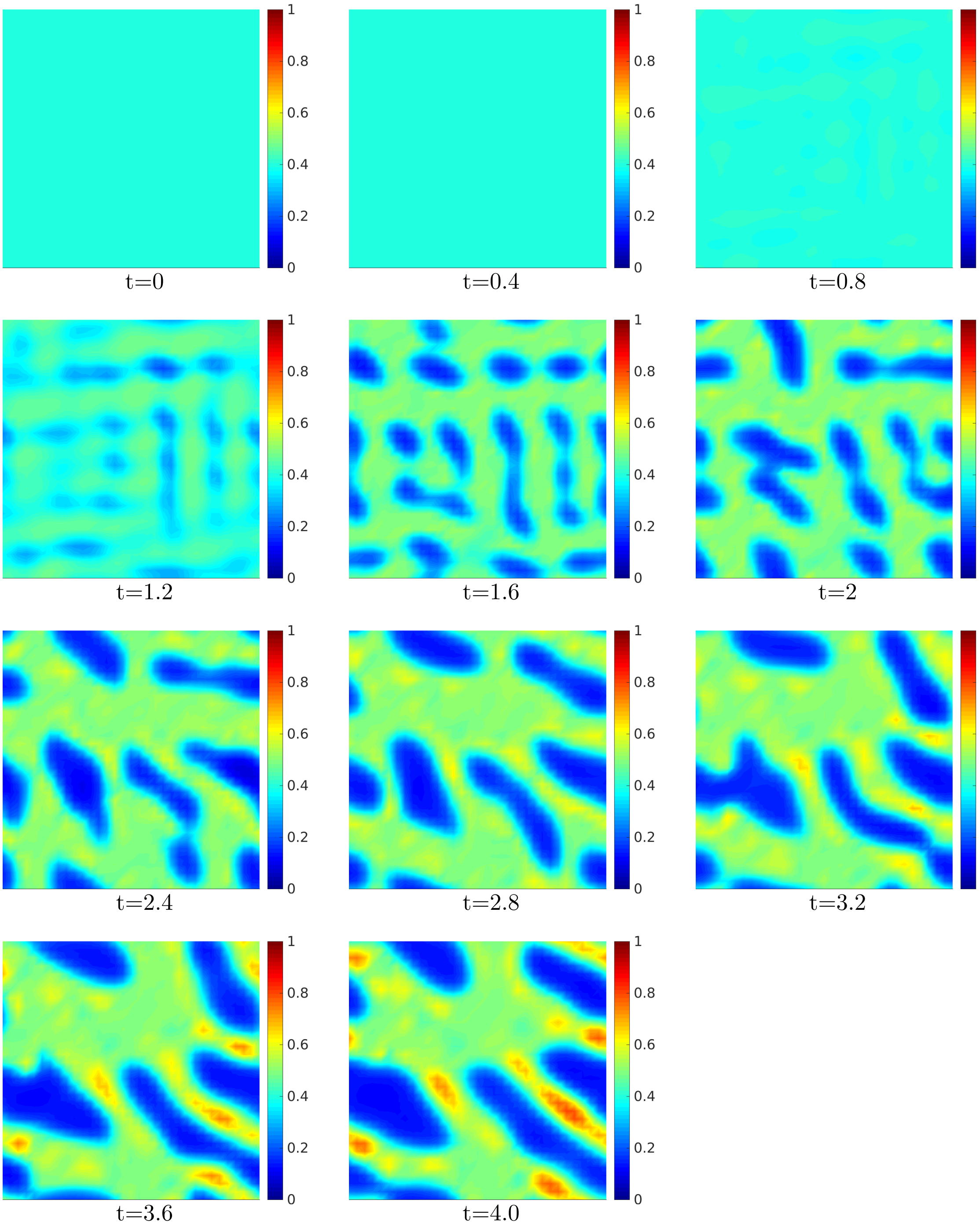

Figure 8.13.: **Constant A:** Snapshots of the volume fraction ϕ with the fixed colour map $[0, 1]$ for Experiment 8.3.1.

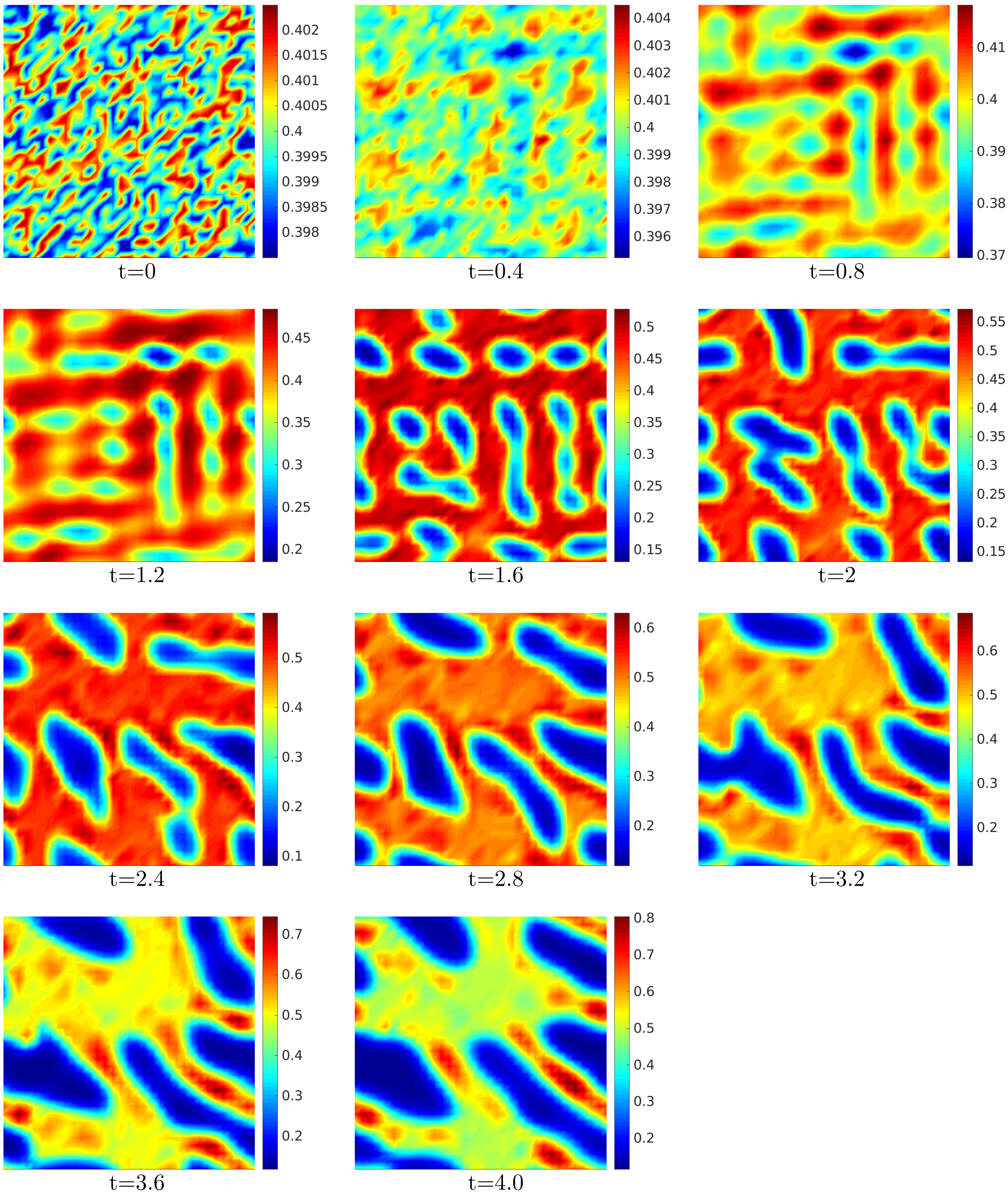

Figure 8.14.: **Constant A:** Snapshots of the volume fraction ϕ for Experiment 8.3.1.

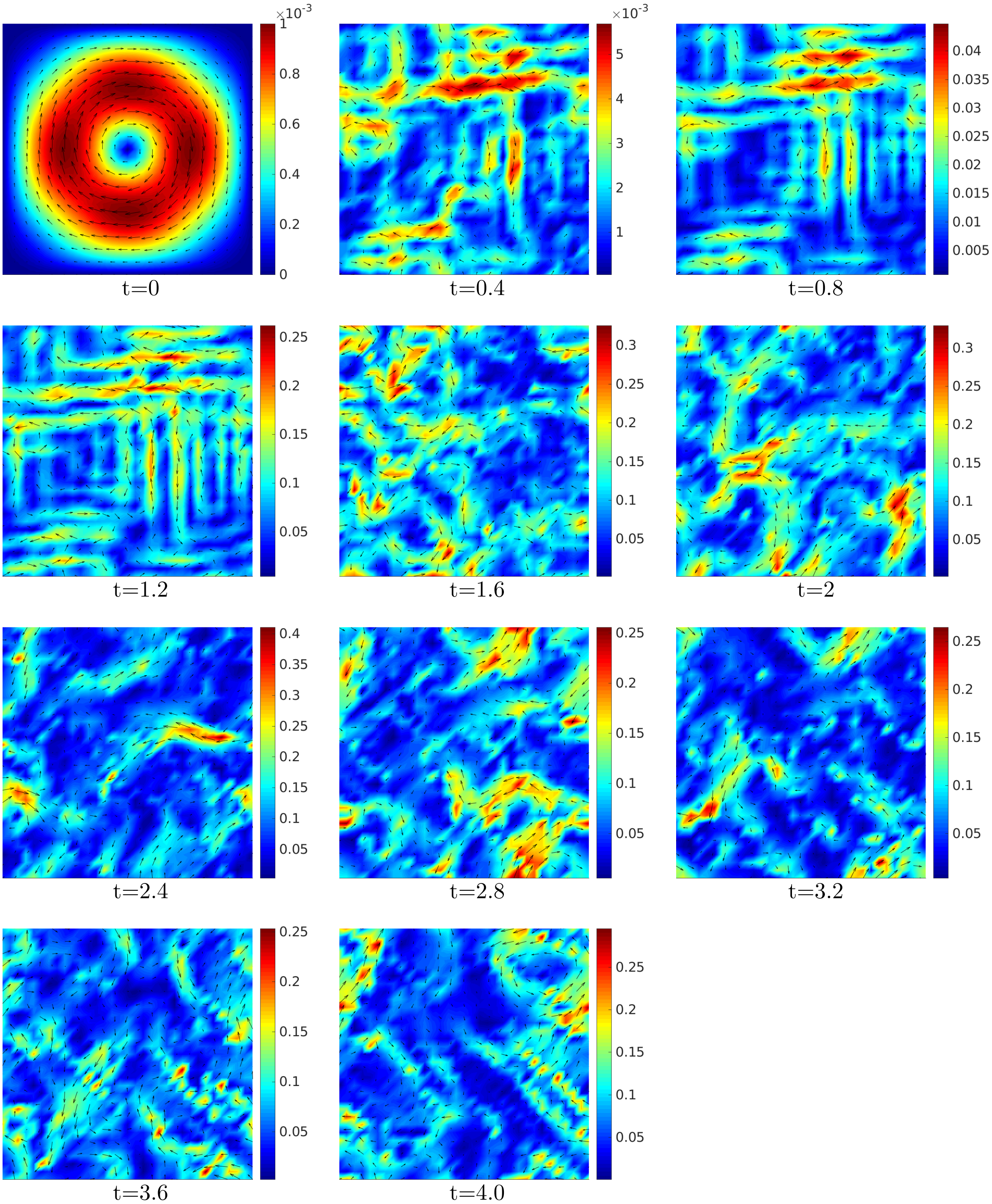

Figure 8.15.: **Constant A:** Snapshots of the velocity **u** for Experiment 8.3.1.

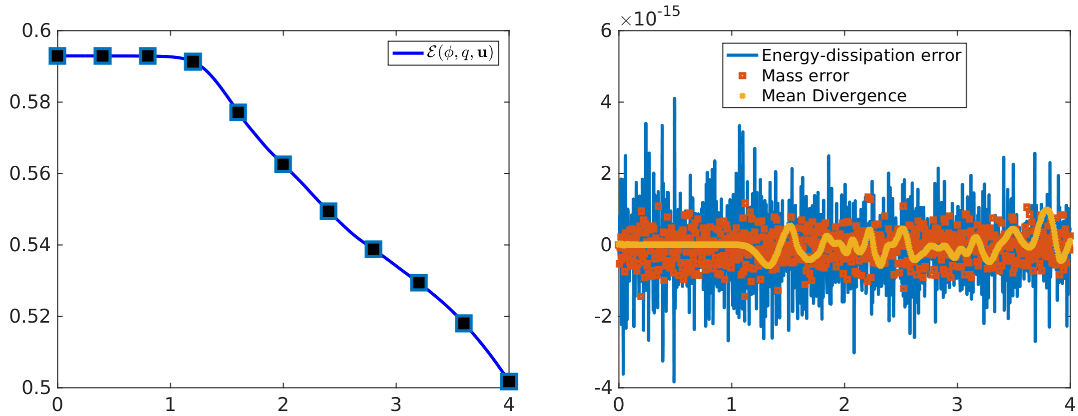

Figure 8.16.: **Constant A:** (Left): Evolution of the energy $\mathcal{E}(\phi, q, \mathbf{u})$ and (Right): Energy-dissipation, mass conservation and mean divergence error for Experiment 8.3.1.

8.3.1. Structure Factor

In this subsection, we will compare to so-called structure factors for the above three experiments. We introduce the averaged structure factor $S(q,t) : [0,2\pi) \times [0,T] \to \mathbb{R}_+$ for a given wave number in two steps. First we introduce the full structure factor $S(\mathbf{q},t) : [0,2\pi)^d \times [0,T] \to \mathbb{R}_+$ via the Fourier transformation as

$$S(\mathbf{q},t) = \left|\langle \phi(t), e^{i\mathbf{q}\cdot x}\rangle\right|^2. \tag{8.2}$$

The above expression is now averaged over spherical shells of width $\omega > 0$ as follows

$$S(q,t) = \int_{Z(q,\omega)} S(\mathbf{q},t)\,\mathrm{d}\mathbf{q}, \qquad Z(q,\omega) := \{\mathbf{q} \in \mathbb{R}^d : q - \omega < \|\mathbf{q}\|_2 \leq q\}.$$

For the Cahn-Hilliard type models we immediately obtain for wavenumber $q = 0$ that $S(0,t) = \langle \phi(t), 1\rangle^2 = \langle \phi(0), 1\rangle^2$, by the conservation of mass. The results for Experiment 8.3.1 are given in Figure 8.17. In principle, we observe that in the case of model H, the structure factor is almost the same for all snapshots. While we observe that for constant and nonlinear A that the height of $S(q,t)$ grows in time. In comparison of both viscoelastic models, we observe that in the nonlinear case the structure factors grow faster in time than in the constant case.

According to [10, 28] one expects that the wavenumber $q_{\max}$ corresponding to the maximal peak of $S(q,t)$, i.e. $\max_q S(q,t)$, is inversely proportional to the characteristic length scale, which is associated to the growth of domains. This means that $S(q,t)$ is a measure of how fast the domains evolve under the dynamics of the system. Hence, we can observe that this length scale for the viscoelastic phase separation problems grows much slower than in the case of model H. From an experimental point of view, this is expected, since the slow domain growth described by the dynamics of the viscoelastic phase separation is the reason why the experiment, i.e., the process of phase separation, visualised in Figure 1.2 is observable. In the context of standard phase separation processes, the length scale growth is so fast, that it is almost impossible to observe the process of phase separation experimentally. For more simulations and more details on the physical background, we refer to our latest article [28]. There we have compared similar simulations with a mesoscopic model based on an algorithm that combines molecular dynamics with Lattice-Boltzmann techniques, see [125].

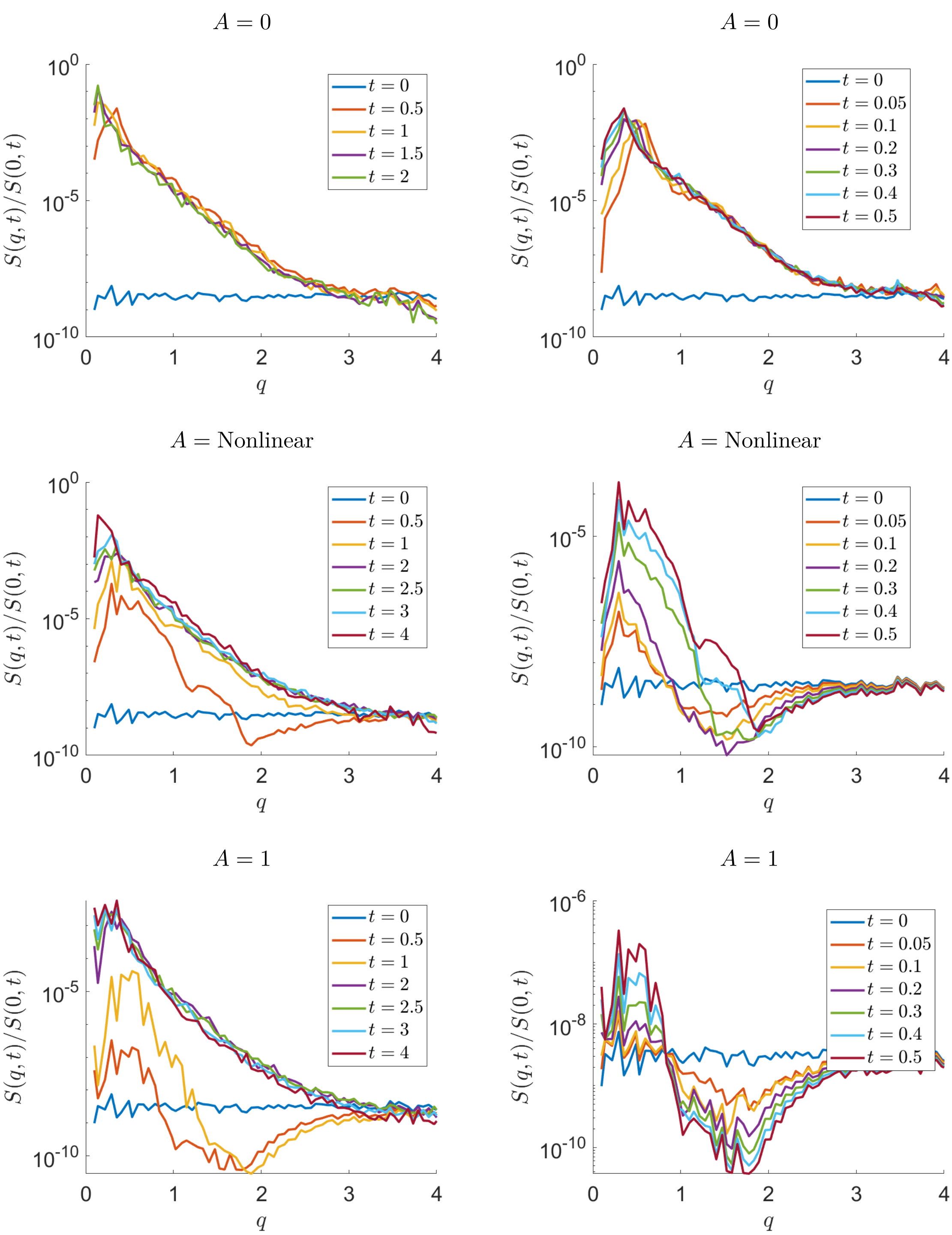

Figure 8.17.: Comparison of the structure factors for Experiment 8.3.1. (Left:) Overview over full timescale. (Right:) Comparison for small timescales.

9 Summary and outlook

In this chapter, we will first summarize the results of the second part of the thesis on numerical analysis. Afterwards, we will comment on several extensions and related topics to give a brief outlook of what is possible or still open. To this end, we recall the summary of the first part on well-posedness in Section 4.13.

9.1. Summary of Part II: Numerical analysis

In the Chapters 5-8 we considered the numerical analysis for the CHNSQ model. In Chapter 5 we gave an overview of the relevant methods known in the literature and stated the main problems which arise in the discretisation of the CHNSQ model. Furthermore, we recalled the relevant tools for the numerical approximation used in the chapters afterwards. In Chapter 6 we analysed a semi-discretisation in space using conforming finite elements with quadratic/linear inf-sup stable elements. We have proven order-optimal second order convergence in space with sharp regularity assumptions. Under the Galerkin projection the inherent energy-dissipative structure is preserved. Using a discrete version of the stability result, derived via relative energy methods, the error analysis can be performed by choosing the discrete perturbed solution by suitable projections. In order to simplify the final estimates, we restricted this part of the proof using the simplification $A = 1$. In principle, the result can be extended to suitable nonlinear A in a straightforward but tedious fashion.

In Chapter 7 we employed a Petrov-Galerkin discretisation in time on the semi-discrete problem and obtained a fully discrete scheme. We have proven optimal second order error estimates in space and time using realistic regularity of the true solution via a discrete relative energy method. The energy-dissipative structure is preserved even at the fully discrete level. In principle, the proof structure follows the semi-discrete case. However, it has to be adapted to capture the discrete nature of the time approximation. Again, only the error estimate is subjected to the simplification, $A = 1$, and this can be extended to the nonlinear case.

We emphasize that the techniques and results immediately translate to suitable reductions of the CHNSQ model, i.e., the Cahn-Hilliard equation, the Cahn-Hilliard-Bulk-Stress model, the Cahn-Hilliard-Navier-Stokes model and also the Navier-Stokes equations. Furthermore, the techniques can be extended to more complicated models, for instance, the Cahn-Hilliard-Navier-Stokes system, with non-constant density and interface width.

In Chapter 8 we present numerical experiments which illustrate theoretical convergence result in various cases. Finally, we applied the method for viscoelastic phase separation and demonstrated that the model is indeed able to reproduce the network structure, the phase inversion observed in numerical experiments, cf. Figure 1.2.

9.2. Outlook

In this section, we would like to discuss future directions and possible generalizations of the simplifications we considered here. Let us also recall the outlook of the first part of the thesis in Section 4.12.

Error analysis for nonlinear A
First, the rigorous error analysis results can be generalized for non-constant A. In principle, one can start by estimating the residuals in Lemma 6.4.1 or 7.4.1. Comparison with the continuous case, i.e., Section 4.6 implies that the uniform bounds $\Delta_h \phi_h, \Delta_h \bar{\phi}_{h,\tau} \in L^2(0,T;L^2(\Omega))$ will be crucial. Here Δ_h denotes the discrete Laplacian whose control can be attained in standard manner, see for instance [47, 46].

Higher-order discretisation
In principle, the same proof can be applied for higher-order finite element approximations in space without any difficulties. High order discretisation in time is possible by considering a modified Petrov-Galerkin formulation with P_k^c/P_{k-1} as ansatz and test functions. Rigorous error analysis of such a scheme is an ongoing work.

A posteriori estimates
In the a posteriori analysis one reverses the idea of the a priori analysis. Instead of using the discrete stability estimate we employ the continuous stability estimate and choose $\hat{z}$ as a suitable reconstruction of the discrete solution $z_{h,\tau}$. Since we already have the stability estimate on the continuous level, this would be very interesting to study, especially in the context of adaptive mesh refinement in τ, h.

Large time behaviour
Another very interesting question is the large time behaviour of our system. We have given an outlook on this for the weak solutions in Section 4.12. A natural question is whether the numerical method we have proposed preserves this convergence property. In the case of the Cahn-Hilliard equation this is proven for several methods, see, [8, 101]. In the Navier-Stokes case, this in principle depends only on the correct discretisation of the convective term $\mathbf{c}(\mathbf{u}, \mathbf{u}, \mathbf{v})$ and the associated boundary conditions. To our knowledge there is no work considering large time behaviour for discretisation for the model H and hence not for the CHNSQ model. We believe that our numerical method for the model H and the CHNSQ model can be proven to preserve the long time behaviour, at least with Neumann-Dirichlet boundary conditions. The periodic case seems to be much more involved.

BDF-type extrapolations
Following [124] one can study suitable linearisation or extrapolations backwards in time in the nonlinear terms. We note that in principle the parts of the equation, where we linearised the perturbed system around the discrete solution, can be used to apply such simplifications. Exactly these terms do not contribute to the energy or appear in positive coefficient functions. Hence, this does not harm the energy dissipative structure and therefore the stability estimate translates verbatim. We note that so far our study does not allow linearising the potential. We mention, for instance, [116], for a second order linear scheme and observe except the potential this can be obtained by second order extrapolation of Problem P.3.

Numerical methods for the Peterlin model
To derive schemes for the full viscoelastic phase separation model, i.e., System S.3 we also need to consider discretisation strategies for the Peterlin model, i.e., System S.5. In two space dimensions methods with rigorous convergence proofs, using the special energy estimate (2.13), are already available [98, 99, 109]. Applying our strategy, i.e., the Petrov-Galerkin scheme in time, and conforming inf-sup stable finite elements in space, we can preserve the energy estimates (2.13) and (2.11). Computing time averages yields the time-stepping scheme proposed in [109].

However, the development of a scheme that preserves the total energy structure (2.16) seems to be much more involved. We discuss the main problems. To obtain the total energy, we have to test with $\mathbf{C}^{-1}$. On the discrete level, this is not allowed, since the inverse matrix does not belong to the discrete subspace in general. The second problem is to preserve the positive definiteness on the discrete level, which immediately limits one to use linear finite elements. The only methods available in the literature which rigorously preserves the total energy can be found in [12] and [16] for the Oldroyd-B and FENE-P model, respectively. The rigorous analysis is restricted to two space dimensions and the proof is rather difficult and involves several regularisations and limiting processes. However, the main core of their work is a combination of mass-lumping and convex regularisation of the logarithmic terms. It would be interesting to study the Peterlin model with reformulations and the discrete variational concepts as proposed by Egger [49].

Parameter calibration
From an application point of view, a more involved parameter study for the model is necessary to identify the relevant parameters and calibrate the model. Therefore, identification of these objects from experimental data is necessary.

Time and length scale quantification
The structure factors are only a first tool to quantify the dynamic properties of the evolution. More detailed quantification of the network structure is possible, for instance, via the so-called Minkowski functionals [10], with an application to standard phase separation see, [100]. In principle, these functionals are related to many topological properties of the networks, such as the number of holes and curvature of the network.

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A Theoretical framework

For completeness, we present here several technical and mathematical results and relevant notations, which are used throughout the whole thesis, which can also be found in [104].

A.1. Notation and functional spaces

Let $d \in \{2, 3\}$ be the space dimension and let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ be the vector of space coordinates. Further, let u, v be scalar function, $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_d), \mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_d)$ be vector-valued functions and $\mathbf{C} = \{\mathbf{C}_{ij}\}, \mathbf{D} = \{\mathbf{D}_{ij}\}, i, j, = 1, \dots d$, be $d \times d$ matrix-valued functions. We introduce the following notation

$$
\begin{aligned}
(\nabla u)_i &= \partial_{x_i} u, & (\nabla \mathbf{u})_{ij} &= \partial_{x_j} \mathbf{u}_i, & (\nabla \mathbf{C})_{ijk} &= \partial_{x_k} \mathbf{C}_{ij}, \\
\Delta u &= \sum_{i=1}^{d} \partial_{x_i}^2 u, & (\Delta \mathbf{u})_i &= \sum_{j=1}^{d} \partial_{x_j}^2 \mathbf{u}_i, & (\Delta \mathbf{C})_{ij} &= \sum_{k=1}^{d} \partial_{x_k}^2 \mathbf{C}_{ij}, \\
\mathbf{u} \cdot \mathbf{v} &= \sum_{i=1}^{d} u_i v_i, & \mathbf{C} : \mathbf{D} &= \sum_{i,j=1}^{d} \mathbf{C}_{ij} \mathbf{D}_{i,j}, & \nabla \mathbf{C} : \nabla \mathbf{D} &= \sum_{i,j,k=1}^{d} \partial_{x_k} \mathbf{C}_{ij} \partial_{x_k} \mathbf{D}_{ij}, \\
\mathbf{v} \cdot \nabla u &= \sum_{i=1}^{d} \mathbf{v}_i \partial_{x_i} u, & ((\mathbf{v} \cdot \nabla) \mathbf{u})_i &= \sum_{j=1}^{d} \mathbf{v}_j \partial_{x_j} \mathbf{u}_i, & ((\mathbf{v} \cdot \nabla) \mathbf{C})_{ij} &= \sum_{k=1}^{d} \mathbf{v}_k \partial_{x_k} \mathbf{C}_{ij}, \\
\operatorname{div}(\mathbf{u}) &= \sum_{i=1}^{d} \partial_{x_i} \mathbf{u}_i, & (\operatorname{div}(\mathbf{C}))_i &= \sum_{j=1}^{d} \partial_{x_j} \mathbf{C}_{ji}.
\end{aligned}
$$

We denote the deformation tensor or the symmetric part of the velocity gradient by

$$
\mathrm{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top). \tag{A.1}
$$

Let $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be the multi index and $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Then we denote the α-th partial derivative by D^α and mean

$$
D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d} \text{ and } D^0 u := u.
$$

Further, we denote by $\mathrm{tr}(\mathbf{C}), \det(\mathbf{C}), \mathbf{C}^\top$ the trace, determinant and transposed respectively and introduce the notation

$$\mathbf{C} : \mathbf{D} = \sum_{i,j} \mathbf{C}_{ij}\mathbf{D}_{ij} = \mathrm{tr}(\mathbf{C}\mathbf{D}^\top) = \mathrm{tr}(\mathbf{C}^\top\mathbf{D}) \text{ for } \mathbf{C}, \mathbf{D} \in \mathbb{R}^{m\times n}. \tag{A.2}$$

While the inner product for vectors is the standard euclidean scalar product, the inner product for the real $n \times n$ matrices will be induced by the Frobenius norm, i.e.

$$\|\mathbf{D}\|_F = \left(\sum_{i,j} |\mathbf{D}_{i,j}|^2\right)^{1/2} = \mathrm{tr}(\mathbf{D}\mathbf{D}^\top).$$

Functional spaces

From now on the space domain $\Omega \subset \mathbb{R}^d$ is assumed to be bounded with at least a Lipschitz-continuous boundary $\partial\Omega$.

The Hölder spaces $C^k(\bar{\Omega}), C_0^\infty(\bar{\Omega}), C_{\mathrm{div}}^\infty(\bar{\Omega})$ [4]

For $k \geq 0$ we introduce the Hölder spaces $C^k(\bar{\Omega})$ as the space of all k-times differentiable function in Ω and by $C^\infty(\Omega)$ the intersection of all $C^k(\bar{\Omega})$. This is the space of all infinitely many times continuously differentiable function in Ω. The spaces are equipped with the norm

$$\|u\|_{C^k} := \max_{0\leq|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha u| \text{ if } k < \infty.$$

We introduce the abbreviations $C(\bar{\Omega}) := C^0(\bar{\Omega})$ and introduce the spaces,

$$C_0^\infty(\bar{\Omega}) := \left\{u : u \in C^\infty(\bar{\Omega}),\ \mathrm{supp}(u) \text{ compact in } \Omega\right\},$$
$$C_{0,\mathrm{div}}^\infty(\bar{\Omega}) := \left\{\mathbf{u} : \mathbf{u} \in C_0^\infty(\bar{\Omega}), \mathrm{div}\,(\mathbf{u}) = 0\right\}.$$

The Lebesque space $L^p(\Omega), L_0^2(\Omega), L_{\mathrm{div}}^2(\Omega)^d, L_S^2(\Omega)^{d\times d}, L_{SPD}^2(\Omega)^{d\times d}$ [4, 59, 108, 122]

We denote by $L^p(\Omega)$ for $1 \leq p < \infty$ the Lebesque space of all measurable function where the p-th power is Lebesque-integrable in Ω. Furthermore, the space $L^\infty(\Omega)$ is the space of essentially bounded function in Ω. The norms of the above space are given by

$$\|u\|_{0,p} := \left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{1/p}, \quad \|u\|_{0,\infty} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|,$$

with the abbreviation $\|\cdot\|_{0,2} = \|\cdot\|_0$. The special case of $L^2(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{L^2} := \int_\Omega u(x)v(x)dx.$$

The typical spaces in fluid dynamics are defined via

$$L_0^2(\Omega) := \left\{u : u \in L^2(\Omega), \int_\Omega u \, dx = 0\right\}, \quad L_{\mathrm{div}}^2(\Omega)^d := \overline{C_{0,\mathrm{div}}^\infty(\Omega)}^{\|\cdot\|_0}$$

and denote the space of mean-free L^2-functions and the space of divergence-free L^2-functions. Here divergence-free is understood in the sense of distributions. we denote the space of square-integrable symmetric and symmetric positive definite $d \times d$ matrices by $L^2_S(\Omega)^{d\times d}, L^2_{SPD}(\Omega)^{d\times d}$, respectively.

The Sobolev spaces $H^k(\Omega), W^{k,p}(\Omega), H^1_{\text{div}}(\Omega)^d, H^1_S(\Omega)^{d\times d}$ [4, 56, 122]

We denote the Sobolev space

$$W^{k,p}(\Omega) := \left\{u : D^\alpha u \in L^p(\Omega), \text{ for } \alpha \in \mathbb{N}^d, |\alpha| \le k\right\}, \tag{A.3}$$

where the derivative $D^\alpha u$ are defined in the sense of distributions. Moreover, in the case $p = 2$ we set $W^{k,2}(\Omega) = H^k(\Omega)$. In this case $H^k(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{H^k} := \int_\Omega \sum_{|\alpha|\le k} D^\alpha u(x) D^\alpha v(x)\, \mathrm{d}x.$$

We denote the norm of the Sobolev space by

$$\|u\|_{k,p} := \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{0,p}\right)^{1/p}$$

and in the case of $p = 2$ we abbreviate $\|\cdot\|_{k,2} = \|\cdot\|_k$. The relevant space for fluid dynamics is given by

$$H^1_{\text{div}}(\Omega)^d := \overline{C^\infty_{0,\text{div}}(\Omega)}^{\|\cdot\|_1}.$$

The Sobolev space of symmetric $d \times d$ matrices is denoted by $H^1_S(\Omega)^{d\times d}$.

The Bochner space $L^p(0, T; X(\Omega))$ [56, 108]

Let $X(\Omega)$ be a Banach space and given $1 \le p \le \infty$. For a function $u(t, x)$ defined on $(0, T) \times \Omega$ for every $t \in (0, T)$ the function $u(t)(x) := u(t, x)$ is measurable, and it is an element of the space $X(\Omega)$. We therefore denote the Bochner spaces by

$$L^p(0, T; X(\Omega)) := \left\{u : u(t) : (0, T) \to X(\Omega), \left(\int_0^t \|u(t)\|_X dt\right)^{1/p} < \infty\right\},$$
$$L^\infty(0, T; X(\Omega)) := \left\{u : u(t) : (0, T) \to X(\Omega), \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X < \infty\right\}$$

with the corresponding norms

$$\|u\|_{L^p(X)} := \left(\int_0^T \|u(t)\|_X \,\mathrm{d}t\right)^{1/p}, \quad \|u\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X.$$

Similarly, we introduce the Bochner-Sobolev spaces $W^{k,p}(0, T; X(\Omega))$ by

$$W^{k,p}(0, T; X(\Omega)) := \left\{u \in L^p(0, T; X(\Omega)), \frac{\partial^j u}{\partial t^j} \in L^p(0, T; X(\Omega)), j = 1, \dots, k\right\}.$$

The dual spaces $X', W^{-k,p}(\Omega), L^{p'}(0,T;X(\Omega)')$

Let X be a normed, linear space. Then the dual space X' is the space of all bounded linear functionals $g : X \to \mathbb{R}$ which together with the norm

$$\|g\|_{X'} := \sup_{\|u\|_X \neq 0} \frac{\langle g, u\rangle_{X' \times X}}{\|u\|_X}, \tag{A.4}$$

makes it a Banach space. Here the inner product is replaced by the so-called *dual paring* between X and X'. In this context for $1 < p < \infty$ we introduce the dual space to the Sobolev space $W^{k,p}(\Omega)$ as $W^{k,p}(\Omega)' := W^{-k,p'}(\Omega)$ where p' denotes the conjugate exponent, i.e. $1/p + 1/p' = 1$. In fact, we will also abbreviate

$$\langle u, v\rangle_{H^k \times H^{-k}} = \langle u, v\rangle \text{ and } \langle v, u\rangle_{H^{-k} \times H^k} = \langle v, u\rangle. \tag{A.5}$$

It will be clear from the context which dual paring or inner product is used. Similarly, the dual space of the Bochner space $L^p(0,T;X(\Omega))$ is defined by the dual space of both components $L^p(0,T;X(\Omega))' = L^{p'}(0,T;X(\Omega)')$ if X is reflexive.

A.2. Symmetric positive definite matrices

Here we will give a short review about symmetric and positive definite matrices and list relevant properties, see [1, 78, 108] for details and proofs. Furthermore, we will introduce the concept of matrix functions.

Symmetry and definiteness for matrices

A matrix $\mathbf{D} \in \mathbb{R}^{d\times d}$ is symmetric if $\mathbf{D} = \mathbf{D}^\top$, where $\mathbf{D}^\top$ denotes the transposed of $\mathbf{D}$. The following properties hold:

- The matrix has d real eigenvalues.
- There exists an orthogonal decomposition (the eigendecomposition) $\mathbf{D} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^\top$, with a real orthogonal matrix $\mathbf{Q}$, where the columns are given by the eigenvectors of $\mathbf{D}$. Moreover, $\boldsymbol{\Lambda}$ is a real diagonal matrix with the eigenvalues on the diagonal.
- A matrix $\mathbf{D}$ is positive definite if and only if all eigenvalues are positive.
- A matrix $\mathbf{D}$ is positive semi-definite if and only if all eigenvalues are non-negative.
- A matrix $\mathbf{D}$ is indefinite if there is at least one positive and one negative eigenvalue.

We denote the space of all symmetric positive definite matrices in $\mathbb{R}^{d\times d}$ by $\mathcal{S}^d_+$.

Matrix norm

In what follows we provide some norms and equivalence of norms for matrices. First we denote by $\|\mathbf{D}\|_{0,p} := \int_\Omega \sum_{i,j=1}^d |\mathbf{D}_{i,j}|^p$, which for $p = 2$ is the Lebesque-Frobenius norm.

Lemma A.2.1 ([96]). *Let* $\mathbf{D} \in \mathcal{S}^d_+$. *Then the following holds*

$$\|\mathbf{D}\|^p_{0,p} \leq \|\mathrm{tr}(\mathbf{D})\|^p_{0,p} \leq d^{p-1}\|\mathbf{D}\|^p_{0,p}, \qquad p \geq 2. \tag{A.6}$$

Let $\mathbf{D} \in \mathbb{R}^{d\times d}$ *then it holds*

$$\|\mathrm{tr}(\mathbf{D})\|^p_{0,p} \leq d^{p-1}\|\mathbf{D}\|^p_{0,p}, \qquad p \geq 2.$$

Matrix function

Let $\mathbf{D} \in \mathcal{S}^d_+$. Then we define the matrix function $f(\mathbf{D})$ via the orthogonal decomposition as

$$f(\mathbf{D}) := \mathbf{Q} f(\mathbf{\Lambda})\mathbf{Q},$$

where $f(\mathbf{\Lambda})$ is the application of f on each eigenvalue λ_i. Although the orthogonal decomposition is not unique, the matrix function is uniquely defined. In the following we present the main matrix function we need. First we consider the logarithm by $f(x) = \ln(x)$, i.e. we need $\lambda_i > 0$ to define the real logarithm of a matrix by

$$\ln(\mathbf{D}) := \mathbf{Q}\ln(\mathbf{\Lambda})\mathbf{Q}.$$

The second application is the inverse matrix, i.e. $f(x) = 1/x$, and we obtain

$$\mathbf{D}^{-1} := \mathbf{Q}\frac{1}{\mathbf{\Lambda}}\mathbf{Q}.$$

Consider a matrix $\mathbf{D} \in \mathcal{S}^d_+$. Then the following holds

$$\mathrm{tr}(\ln \mathbf{D}) = \ln \det \mathbf{D}, \tag{A.7}$$

$$\mathrm{tr}(\mathbf{D})\mathbf{D} - 2\ln\mathbf{D} - \mathbf{I} \text{ is symmetric and } \mathrm{tr}(\mathrm{tr}(\mathbf{D})\mathbf{D} - 2\ln\mathbf{D} - \mathbf{I}) \geq 0, \tag{A.8}$$

$$\mathbf{D} + \mathbf{D}^{-1} - 2\mathbf{I} \text{ is symmetric and } \mathrm{tr}(\mathbf{D} + \mathbf{D}^{-1} - 2\mathbf{I}) \geq 0. \tag{A.9}$$

The proof can be found in [104] in the case of $d = 2$. However, since the proof is based on the eigenvalue representation of the trace this extends verbatim to all space dimensions.

Cayley-Hamilton [62]:

Furthermore, we can deduce from the Cayley-Hamilton theorem in two and space dimensions

$$\mathbf{C}^2 - \mathrm{tr}(\mathbf{C})\mathbf{C} + \det(\mathbf{C})\mathbf{I} = 0, \tag{A.10}$$

$$\mathbf{C}^3 - \mathrm{tr}(\mathbf{C})\mathbf{C}^2 + \frac{1}{2}[\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)]\mathbf{C} - \det(\mathbf{C})\mathbf{I} = 0. \tag{A.11}$$

A direct consequence of divergence-freedom of the velocity $\mathbf{u}$, see [96], yields the following result.

Let $\mathbf{C} \in \mathbb{R}^{2\times 2}$ be a symmetric tensor and let $\mathbf{u} \in \mathbb{R}^2$ be a divergence-free vector field. Then the following identity holds true

$$\mathrm{tr}(\mathbf{C})\mathbf{C} : \nabla\mathbf{u} = \frac{1}{2}\Big[(\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T\Big] : \mathbf{C}. \tag{A.12}$$

Jacobi's formula [23]:

Furthermore, for a positive definite matrix function $\mathbf{C} \in C^1([0,T))$ the following Jacobi formula holds

$$\partial_t \mathbf{C} : \mathbf{C}^{-1} = \operatorname{tr}(\mathbf{C}^{-1}\partial_t \mathbf{C}) = \partial_t \operatorname{tr}(\ln \mathbf{C}). \tag{A.13}$$

Lemma A.2.2 (Matrix laplacian, [17])**.** *Let $\mathbf{D} \in H^2(\Omega)^{m\times m} \cap C^1(\overline{\Omega})^{m\times m}, m \in \mathbb{N}$, be a symmetric matrix function, which is uniformly positive definite on $\overline{\Omega}$ and satisfies homogeneous Neumann boundary conditions, then*

$$\int_\Omega \Delta \mathbf{D} : \mathbf{D}^{-1}\, \mathrm{d}x = -\int_\Omega \nabla \mathbf{D} : \nabla \mathbf{D}^{-1}\, \mathrm{d}x \geq \frac{1}{m}\int_\Omega |\nabla \operatorname{tr}(\log \mathbf{D})|^2\, \mathrm{d}x \geq 0. \tag{A.14}$$

A.3. Inequalities and useful lemmas

In what follows, we state several inequalities that are needed in the thesis.

Continuous Gronwall lemma [128]:

Lemma A.3.1. *Let $T > 0$, $v, g \in C[0,T]$ and $\lambda \in L^1(0,T)$ be given. Further, assume that*

$$v(t) \leq g(t) + \int_0^t \lambda(s)v(s)ds, \quad 0 \leq t \leq T,$$

and that $\lambda(t) \geq 0$ for a.a. $0 \leq t \leq T$. Then

$$v(t) \leq g(t) + \int_0^t g(s)\lambda(s)e^{\int_s^t \lambda(r)dr}ds, \qquad 0 \leq t \leq T. \tag{A.15}$$

A similar result also holds on the discrete level.

Discrete Gronwall lemma [108]:

Lemma A.3.2. *Let $(a^n)_n$, $(b_n)_n$, and $(c_n)_n$ be given positive sequences, satisfying*

$$u_n + b_n \leq e^{\lambda}u_{n-1} + c_n, \qquad n \geq 0$$

with some $\lambda \in \mathbb{R}$. Then

$$u_n + \sum_{k=0}^n e^{\sum_{j=k}^n \lambda_j} b_k \leq e^{\sum_{j=0}^n \lambda_j} u_0 + \sum_{k=0}^n e^{\sum_{j=k}^n \lambda_j} g_k, \quad n \geq 0. \tag{A.16}$$

Hölder inequality [56, 59]:

Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ with $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following holds

$$\int_\Omega |u||v|\, \mathrm{d}x \leq \|u\|_{0,p}\|v\|_{0,q}.$$

Furthermore, let $u_i \in L^{p_i}(\Omega), i = 1, \ldots, m$, with $1 \leq p_i \leq \infty$ and $\sum_{i=1}^m p_i^{-1} = r^{-1}$ then the following inequality holds

$$\left\|\prod_{i=1}^m u_i\right\|_{0,r} \leq \prod_{i=1}^m \|u_i\|_{0,p_i}.$$

Young inequality [56]:

Let $q < p, q < \infty$ conjugated exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then the inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \text{ and } ab \leq \delta a^p + c(\delta) b^q$$

hold for $a, b, \delta > 0$ and $c(\delta) = (\delta p)^{-q/p} q^{-1}$. In fact the constant $c(\delta)$ depends inversely on δ and we will genuinely denote them using the pair $(\delta, c(\delta))$ for the Yong inequality.

Sobolev embeddings [4, 56, 122]:

The following embedding holds

$$W^{1,p}(\Omega) \subset L^q(\Omega), \frac{1}{q} = \frac{1}{p} - \frac{1}{d}, \quad W^{k,p}(\Omega) \subset C(\Omega), pk > d. \tag{A.17}$$

Furthermore, the following compact embedding holds

$$W^{1,p}(\Omega) \Subset L^q(\Omega), \frac{1}{q} > \frac{1}{p} - \frac{1}{d}. \tag{A.18}$$

Poincaré inequality [123]:

Let $\mathbf{u} \in H^1(\Omega)$ then the following inequality holds

$$\|u\|_1 \leq C_p \left(\|\nabla u\|_0 + \left| \int_\Omega u \, \mathrm{d}x \right| \right) \tag{A.19}$$

where the constant C_p is the Poincaré constant which is known to be related to the diameter of Ω.

Interpolation inequalities [85], [67]:

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain. Then the following inequalities hold true

$$\|u\|_{0,3} \leq c\|u\|_{0,2}^{1/2} \|\nabla u\|_{0,2}^{1/2} + c_1 \|u\|_{0,2}^2, \qquad u \in H^1(\Omega), \text{ for } d \in \{2,3\} \tag{A.20}$$

$$\|u\|_{0,4} \leq c\|u\|_{0,2}^{1/2} \|\nabla u\|_{0,2}^{1/2} + c_1 \|u\|_{0,2}^2, \qquad u \in H^1(\Omega), \text{ for } d = 2 \tag{A.21}$$

$$\|u\|_{0,4} \leq c\|u\|_{0,2}^{1/4} \|\nabla u\|_{0,2}^{3/4} + c_1 \|u\|_{0,2}^2, \qquad u \in H^1(\Omega), \text{ for } d = 3. \tag{A.22}$$

Let $v \in L^\infty(0,T; L^p(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$. Then $v \in L^r(0,T; L^q(\Omega))$ and the estimate

$$\|v\|_{L^r(L^q)} \leq c(d,p,q,|\Omega|,T)(\|v\|_{L^\infty(L^p)} + \|v\|_{L^p(W^{1,p})}), \text{ with } \frac{1}{r} = \frac{d}{p^2} - \frac{d}{pq} \tag{A.23}$$

holds, if the following constraints are satisfied

$$q \in \left[p, \frac{dp}{d-p}\right], r \in [p, \infty) \text{ if } 1 < p < d, \qquad q \in [p, \infty), r \in \left(\frac{p^2}{d}, \infty\right], \text{ if } 1 < d < p.$$

In fact for $p = 2$ the symmetric space, i.e. $r = q$ is given by

$$v \in L^4(0,T; L^4(\Omega)) \text{ for } d = 2 \qquad v \in L^{10/3}(0,T; L^{10/3}(\Omega)) \text{ for } d = 3. \tag{A.24}$$

Moreover, for $d = 3$ and $p = 2$,

$$v \in L^4(0,T; L^3(\Omega)) \cap L^3(0,T; L^4(\Omega)). \tag{A.25}$$

Weak and strong convergence [55]:

Given a sequence $\{u_n\}_{n=1}^{\infty} \subset L^p(\Omega)$ for $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. A sequence is called weakly convergent to its limit $u \in L^p(\Omega)$, denoted by $u_n \rightharpoonup u$ in $L^p(\Omega)$ if

$$\int_{\Omega} u_n(x)v(x)\,\mathrm{d}x \longrightarrow \int_{\Omega} u(x)v(x)\,\mathrm{d}x, \forall v \in L^{p'}(\Omega). \tag{A.26}$$

In the case $p = \infty$ we say that a sequence $\{u\}_{n=1}^{\infty} \subset L^{\infty}(\Omega)$ converge weakly-*, denoted by $u_i \rightharpoonup^* u$ in $L^{\infty}(\Omega)$ if

$$\int_{\Omega} u_i(x)v(x)\,\mathrm{d}x \to \int_{\Omega} u(x)v(x)\,\mathrm{d}x, \forall v \in L^1(\Omega). \tag{A.27}$$

If $1 \leq p < \infty$ then weak/weak-* convergence to $u \in L^p(\Omega)$ implies that every u_n is bounded in $L^p(\Omega)$ and $\|u\|_{0,p} \leq \liminf_{n\to\infty} \|u_n\|_{0,p}$.

Lemma A.3.3 (Consequences of Banch-Alaoglu)**.** *Suppose $1 < p < \infty$ and the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^p(\Omega)$. Then there is a subsequence, still denoted by $\{u_n\}_{n=1}^{\infty}$, and a function $u \in L^p(\Omega)$ such that*

$$u_i \rightharpoonup u \text{ in } L^p(\Omega).$$

Note that the result also holds for $p = \infty$, by replacing weak with weak- convergence and remember that this is false for $p = 1$.*

We call a sequence $\{u_n\}_{n=1}^{\infty} \subset L^p(\Omega)$ for $1 \leq p < \infty$ strongly convergent to its limit $u \in L^p(\Omega)$, denoted by $u_n \to u$ in $L^p(\Omega)$, if

$$\lim_{n\to\infty} \|u_i - u\|_{0,p} = 0.$$

Finally, we remark that for strongly convergent sequences in $L^p(\Omega), p > 1$ one can extract a subsequence which converges almost everywhere in Ω to its limit.

Lemma A.3.4. *Let $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ be two sequences and D be a bounded domain with the following properties*

1. *$u_n \to u$ a.e in D and $\|u_n\|_{0,\infty} \leq c < \infty$ for all n.*
2. *$v_n \rightharpoonup v$ in $L^2(D)$.*

Then the product $u_n v_n$ converges weakly to uv in $L^2(D)$.

Lemma A.3.5 (Vitali lemma [63])**.** *Let $M \subset \mathbb{R}^d$ be a measurable and bounded set. Let the sequence $\{f_m\}_{m\in\mathbb{N}}$ be uniformly bounded in $L^q(M)$ for $q > 1$. Finally, let $f_m \to f$ a.e. in M for some $f \in L^q(M)$. Then*

$$\int_M f_m \to \int_M f.$$

Lemma A.3.6 (Aubin-Lions lemma [24])**.** *Let $X \subset Y \subset Z$ be three Hilbert spaces, and suppose that the embedding of X in Y is compact and Y in Z is continuous. Then* *i) For any $p, q \in (1, \infty)$ the embedding*

$$\left\{ f \in L^p(0,T;X), \frac{df}{dt} \in L^q(0,T;Z) \right\} \subset L^p(0,T;Y) \tag{A.28}$$

is compact.

ii) For any $p > 1$ the embedding

$$\left\{ f \in L^\infty(0,T;X), \frac{df}{dt} \in L^q(0,T;Z) \right\} \subset C([0,T];Y) \tag{A.29}$$

is compact.

iii) The following continuous embeddings holds

$$\left\{ f \in L^2(0,T;X), \frac{df}{dt} \in L^2(0,T;Z) \right\} \subset C(0,T;[X,Z]_{\frac{1}{2}}). \tag{A.30}$$

The space $[X,Y]_{\frac{1}{2}}$ is an interpolation space, see [24]. In the thesis we only use $[H^{-1}(\Omega), H^3(\Omega)]_{\frac{1}{2}} = H^1(\Omega)$, see again [24].

Convective trilinear forms [123, 96]:

For any open set $\Omega \subset \mathbb{R}^d$ it holds that the forms

$$\begin{aligned} \mathbf{c}(\mathbf{u};\phi,\psi) \equiv \langle \mathbf{u}\cdot\nabla\phi,\psi\rangle, \qquad \mathbf{c}(\mathbf{u};\mathbf{v},\mathbf{w}) \equiv \langle (\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}\rangle,& \\ \mathbf{c}(\mathbf{u};\mathbf{C},\mathbf{D}) \equiv \langle (\mathbf{u}\cdot\nabla)\mathbf{C},\mathbf{D}\rangle& \end{aligned} \tag{A.31}$$

are continuous and trilinear on $H^1_{\text{div}}(\Omega)^d \times H^1(\Omega) \times H^1(\Omega)$, $H^1_{\text{div}}(\Omega)^d \times H^1_{\text{div}}(\Omega)^d \times H^1_{\text{div}}(\Omega)^d$ and $H^1_{\text{div}}(\Omega)^d \times H^1(\Omega)^{d\times d} \times H^1(\Omega)^{d\times d}$, respectively. Furthermore, the forms are skew-symmetric, i.e.,

$$\mathbf{c}(\mathbf{u};\phi,\psi) = -\mathbf{c}(\mathbf{u};\psi,\phi), \qquad \mathbf{u} \in H^1_{\text{div}}(\Omega)^d, \phi,\psi \in H^1(\Omega), \tag{A.32}$$

$$\mathbf{c}(\mathbf{u};\mathbf{v},\mathbf{w}) = -\mathbf{c}(\mathbf{u};\mathbf{w},\mathbf{v}), \qquad \mathbf{u} \in H^1_{\text{div}}(\Omega)^d, \mathbf{v},\mathbf{w} \in H^1(\Omega)^d, \tag{A.33}$$

$$\mathbf{c}(\mathbf{u};\mathbf{C},\mathbf{D}) = -\mathbf{c}(\mathbf{u};\mathbf{D},\mathbf{C}), \qquad \mathbf{u} \in H^1_{\text{div}}(\Omega)^d, \mathbf{C},\mathbf{D} \in H^1(\Omega)^{d\times d}. \tag{A.34}$$

Proof of Lemma 5.3.2:

We can now return to the proof of Lemma 5.3.2. We start with considering a single time element $J = (t^{n-1}, t^n)$ and show that

$$\|\bar{u}\bar{v} - \overline{uv}\|_{0,p} \leq C\tau^2(\|u\|_{2,p}\|v\|_{1,\infty} + \|u\|_{1,\infty}\|v\|_{2,p}), \tag{A.35}$$

where $\|\cdot\|_{k,p} = \|\cdot\|_{W^{k,p}(J)}$ and $\bar{a} = \bar{\pi}^0_\tau a$ denotes the time average of a over J. In addition, we denote by $\tilde{a} := a(t^{n-1/2})$ the constant interpolant of a at the midpoint $t^{n-1/2} := (t^n + t^{n-1})/2$. Then we estimate via triangle inequality

$$\begin{aligned} \|\bar{u}\bar{v} - \overline{uv}\|_{0,p} &\leq \|\bar{u}\bar{v} - \tilde{u}\bar{v}\|_{0,p} + \|\tilde{u}\bar{v} - \tilde{u}\tilde{v}\|_{0,p} + \|\tilde{u}\tilde{v} - \widetilde{uv}\|_{0,p} + \|\widetilde{uv} - \overline{uv}\|_{0,p} \\ &= (i) + (ii) + (iii) + (iv). \end{aligned}$$

In order to bound the individual terms, we use the superconvergence estimate

$$\|\bar{a} - \tilde{a}\|_{0,p} \leq C\tau^2 \|a\|_{2,p}, \tag{A.36}$$

which follows by observing that $\bar{a} - \tilde{a} = 0$ for $a \in P_1(J)$ and using the Bramble-Hilbert lemma and a scaling argument; see [26] for details. We can then estimate the first term in the above error expansion by

$$(i) \leq \|\bar{u} - \tilde{u}\|_{0,p} \|\bar{v}\|_{0,\infty} \leq C\tau^2 \|u\|_{2,p} \|v\|_{0,\infty}.$$

Following similar lines, we observe that $(ii) \leq C\tau^2 \|u\|_{0,\infty} \|v\|_{k+2,p}$. The third term vanishes identically, i.e., $(iii) = 0$. Applying (A.36), the last term can be bounded by,

$$(iv) \leq C\tau^2 \|uv\|_{2,p} \leq C'\tau^2 (\|u\|_{2,p} \|v\|_{1,\infty)} + \|u\|_{1,\infty} \|v\|_{2,p}).$$

This proves the desired estimate (A.35) for one single time element $J = (t^{n-1}, t^n)$. The global estimate (5.16) then follows by summation over the elements and using the the continuous embedding $\|a\|_{L^\infty(0,T)} \leq \|a\|_{W^{1,\infty}(0,T)} \leq C\|a\|_{W^{2,p}(0,T)}$.

Proof of Lemma 5.3.3

Let us now turn to the proof of Lemma 5.3.3. We will consider the single time interval $J = (t^{n-1}, t^n)$ and show that

$$\|\overline{g(\bar{\phi})} - \overline{g(\phi)}\|_{0,p} \leq C\tau^2 \|g(\phi)\|_{2,p}$$

We denote again by $\tilde{a} := a(t^{n-1/2})$ the constant interpolant of a at $t^{n-1/2}$ and estimate

$$\|\overline{g(\bar{\phi})} - \overline{g(\phi)}\|_{0,p} \leq \|\overline{g(\bar{\phi})} - \widetilde{g(\phi)}\|_{0,p} + \|\widetilde{g(\phi)} - \overline{g(\phi)}\|_{0,p}.$$

The second term is already of second order using (A.36). The first term can be written as

$$\|\overline{g(\bar{\phi})} - \widetilde{g(\phi)}\|_{0,p} = \|\overline{g(\tilde{\phi})} - \widetilde{g(\phi)}\|_{0,p} = \|g(\phi(t^{n-1/2})) - g(\phi(t^{n-1/2}))\|_{0,p} = 0.$$

The full estimate (5.17) follows from summation over the elements.

B Simulation appendix

B.1. Numerical results for the CHNSQ model for Experiment 8.2.1

For completeness, we present the simulations results of Experiment 8.2.1 for the CHNSQ model. We observe that the dynamics match of the model H quite well, cf. Figures 8.1 and 8.2.

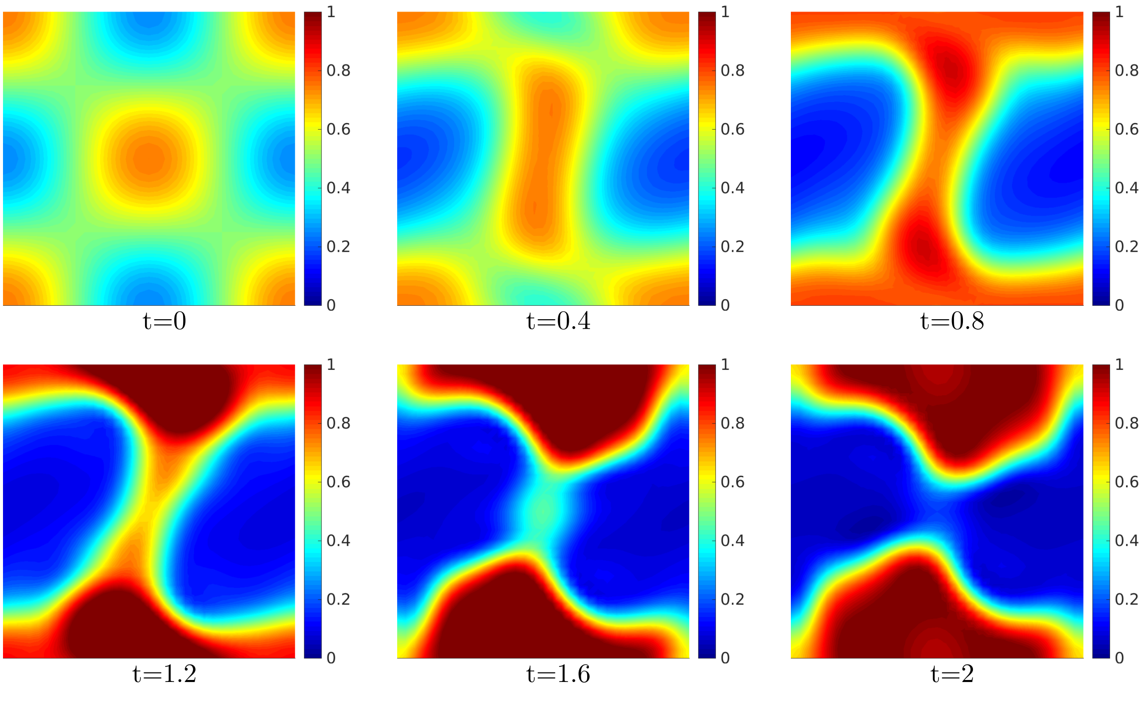

Figure B.1.: **Nonlinear A:** Snapshots of the volume fraction ϕ for Experiment 8.2.1.

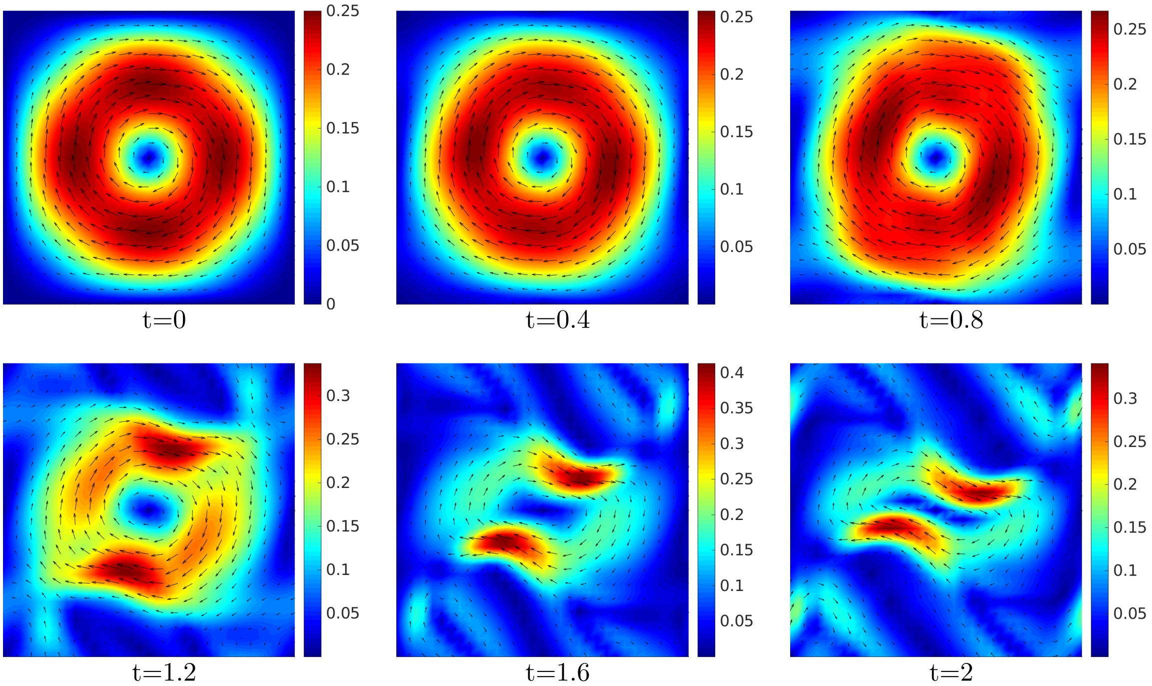

Figure B.2.: **Nonlinear A:** Snapshots of the velocity **u** for Experiment 8.2.1.

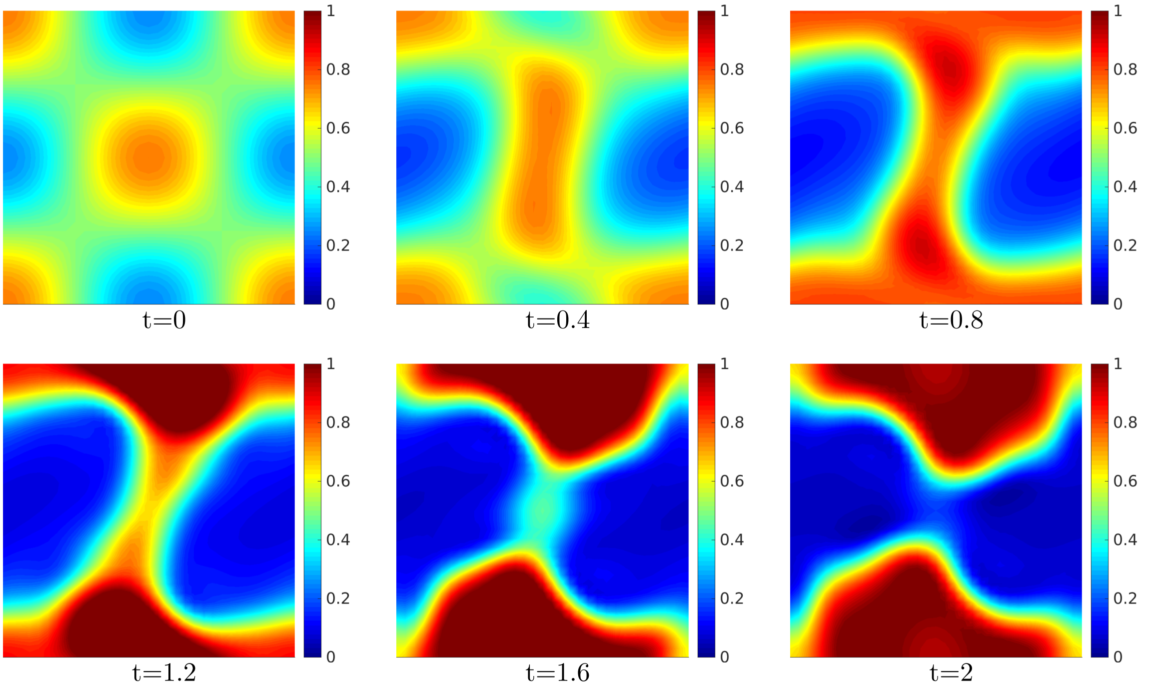

Figure B.3.: **Constant A:** Snapshots of the volume fraction ϕ for Experiment 8.2.1.

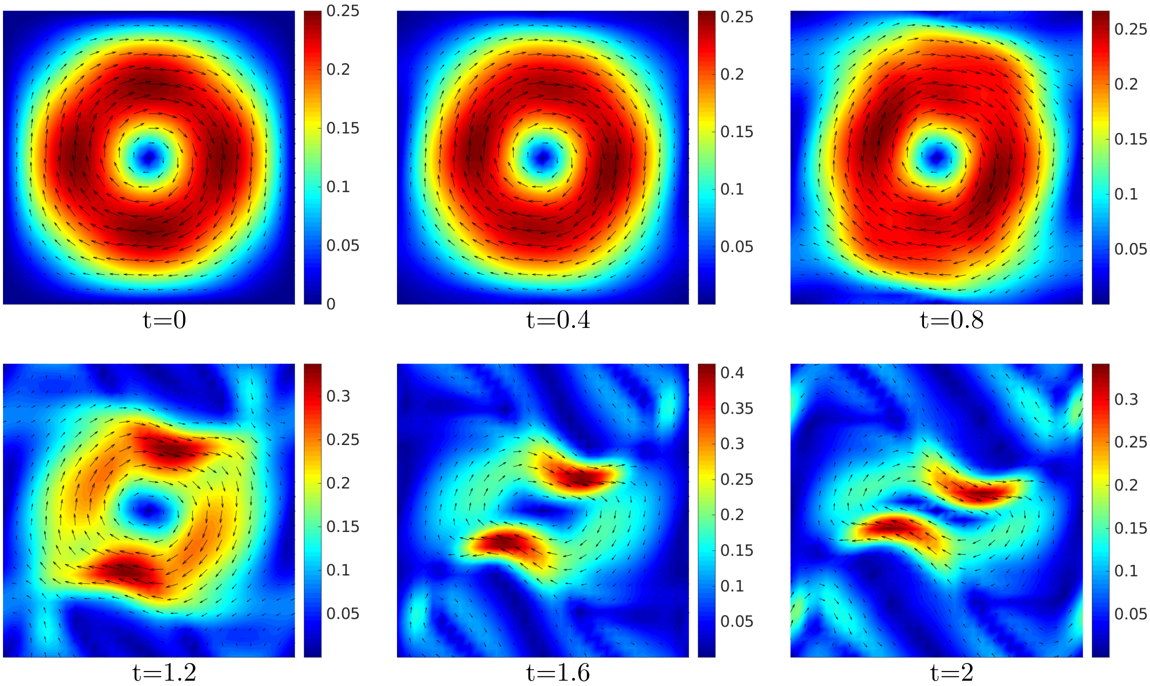

Figure B.4.: **Constant A:** Snapshots of the velocity **u** for Experiment 8.2.1.

Within the research project, the following publications or preprints were written. In the following, we will state the works, my main contributions to the publications and where parts of the publications can be found in this thesis.

1. A. Brunk, B. Dünweg, H. Egger, O. Habrich, M. Lukáčová-Medviďová, and D. Spiller, Analysis of a viscoelastic phase separation model. Condens. Matter Phys., 33 (2021), p. 234002.

   The main contribution in this work is the development of the model reduction, the review of existing results and the macroscopic simulations. We use this work as a main reference for the model derivation.

2. D. Spiller, A. Brunk, O. Habrich, H. Egger, M. Lukáčová-Medviďová, and B. Dünweg, Systematic derivation of hydrodynamic equations for viscoelastic phase separation, J. Condens. Matter Phys., 33 (2021), p. 364001.

   The contribution in this work is careful calculations to obtain a physically relevant model, which satisfies the Second Law of Thermodynamics.

3. A. Brunk, Y. Lu, and M. Lukáčová-Medviďová, Existence, regularity and weak-strong uniqueness for the three-dimensional Peterlin viscoelastic model, 2021. Accepted in Commun. Math. Sci.

   The main contribution is the existence result and weak-strong uniqueness principle as well as the illustrating simulations. The existence result and its proof can be found in the thesis in Chapter 3.

4. A. Brunk, H. Egger, O. Habrich, and M. Lukáčová-Medviďová, Relative energy estimates for the Cahn-Hilliard equation with concentration dependent mobility, 2021. Submitted to SIAM J. Numer. Anal.

   The main contributions is the relative energy estimate and the error analysis for the Cahn-Hilliard equation. The techniques in this work are extended to a more complex model in the theoretical part in Chapter 4 and the numerical part in the Chapters 6 and 7.

5. A. Brunk and M. Lukáčová-Medviďová, Global existence of weak solutions to viscoelastic phase separation: Part I Regular Case, 2019. Submitted to Nonlinearity.

   The contribution in this work is the development of a suitable concept of weak solutions, its existence proof and the illustrating simulations. The results of this paper servers as a basis for the existence proofs in Chapter 3, however extending the results to three space dimensions.

6. A. Brunk and M. Lukáčová-Medviďová, Global existence of weak solutions to viscoelastic phase separation: Part II Degenerate Case, 2020. Submitted to Nonlinearity.

   The main contribution in this work is the development of a suitable concept of weak solutions, the associated existence proof and the illustrating simulations. The results are only used as a reference in the outlook, cf. Section 4.12.

7. A. Brunk and M. Lukáčová-Medviďová, Relative energy and weak-strong uniqueness of the two-phase viscoelastic phase separation model, 2021. Submitted to Appl. Math. Mech.

   The main contribution is the development of a suitable relative energy concept, the relative energy estimates and the resulting weak-strong uniqueness principle. These results inspired the relative energy method in Chapter 4. However, we adopted a more general method for the proofs.