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Some remarks on spin-orbits of unit vectors



Tariq Syed

Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55128 Mainz, Germany

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ABSTRACT

For $n \in \mathbb{N}$ and a commutative ring R with $2 \in R^\times$, the group $SL_n(R)$ acts on the set $Um_n(R)$ of unimodular vectors of length n and $Spin_{2n}(R)$ acts on the set of unit vectors $U_{2n-1}(R)$. We give an example of a ring for which the comparison map $Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R)$ fails to be bijective.

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1. Introduction

Let R be a commutative ring with $2 \in R^\times$ and $n \in \mathbb{N}$. We denote by $Um_n(R)$ the set of unimodular vectors of length n , i.e., the set of vectors $a = (a_1, \dots, a_n)^t \in R^n$ with $a_i \in R$, $1 \leq i \leq n$, such that

E-mail address: tariq.syed@gmx.de.

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$\langle a_1, \dots, a_n \rangle = R$; furthermore, we denote by $U_{2n-1}(R)$ the corresponding set of unit vectors, i.e., the set of elements $(a, b) \in R^n \oplus R^n$ with $\sum_{i=1}^n a_i b_i = 1$, where $a = (a_1, \dots, a_n)^t$ and $b = (b_1, \dots, b_n)^t$. The group $SL_n(R)$ and hence any of its subgroups act on $Um_n(R)$, while the group $Spin_{2n}(R)$ acts on $U_{2n-1}(R)$. The orbit spaces $Um_n(R)/SL_n(R)$ play a central role in the study of stably free modules over commutative rings (e.g., see [8], [14] or [18]). There are well-defined comparison maps between orbit spaces

$$Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R),$$

which are automatically surjective. Vineeth Chintala has proven in [6] that the analogous comparison maps

$$Um_n(R)/E_n(R) \rightarrow U_{2n-1}(R)/Epin_{2n}(R)$$

between the corresponding elementary orbit spaces are bijective for $n \geq 3$; this laid the foundation for the study of stably free R -modules via Spin-orbits of unit vectors. It is therefore natural to ask whether the comparison maps between $SL_n(R)$ -orbits and $Spin_{2n}(R)$ -orbits are also bijective for commutative rings when $n \geq 3$.

While we observe that the comparison map is bijective for some specific classes of rings (cf. Theorem 3.1), we give a negative answer to this question in this paper. For any field k with $\text{char}(k) \neq 2$ and $n \in \mathbb{N}$, we let $S_{2n-1} = k[x_1, \dots, x_n, y_1, \dots, y_n]/\langle \sum_{i=1}^n x_i y_i - 1 \rangle$. While the comparison map for unimodular vectors of length 3 is bijective for S_5 (cf. Corollary 3.5), we prove the following result for S_7 :

Theorem. *If $R = S_7$ and k is infinite perfect with $\text{char}(k) \neq 2$, then the comparison map $Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R)$ is not injective.*

In particular, the ring $R = S_7$ gives an example of a ring for which the comparison map between $SL_n(R)$ -orbits and $Spin_{2n}(R)$ -orbits fails to be bijective.

The paper is structured as follows: In Section 2.1 we introduce the comparison maps between the orbit space of unimodular vectors under the action of $SL_n(R)$ and the orbit space of unit vectors under the action of $Spin_{2n}(R)$; in Section 2.2 we introduce the degree maps and discuss the connection between the degree maps and Spin-orbits of unit vectors. Section 2.3 serves as a brief introduction to \mathbb{A}^1 -homotopy theory as needed for this paper. Finally, in Section 3 we will prove some bijectivity results for the comparison maps above and eventually prove the main result of this paper.

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2. Preliminaries

Throughout this paper, R will always denote a commutative ring with $2 \in R^\times$.

2.1. The comparison maps

At first, we introduce the set of unimodular vectors:

Definition 2.1. For $n \in \mathbb{N}$, we denote by $Um_n(R)$ the set of unimodular vectors of length n over R , i.e., the set of vectors $a = (a_1, \dots, a_n)^t \in R^n$ with $a_i \in R$, $1 \leq i \leq n$, such that $\langle a_1, \dots, a_n \rangle = R$.

The group $GL_n(R)$ of invertible matrices of rank n over R acts on the left on $Um_n(R)$ by matrix multiplication. In particular, all subgroups of $GL_n(R)$ and hence $SL_n(R)$ and $E_n(R)$ act on $Um_n(R)$. In this paper unimodular vectors are usually thought of as column vectors, but in the literature unimodular vectors are often defined as unimodular rows or unimodular row vectors; in the latter case, $GL_n(R)$ and its subgroups obviously act on the right on the set of unimodular rows of length n over R by matrix multiplication. The two definitions make no essential difference: As a matter of fact, transposition of vectors and matrices induces a bijection between the orbit space of unimodular column vectors of length n over R under the left action of $GL_n(R)$ and the orbit space of unimodular row vectors of length n over R under the right action of $GL_n(R)$; analogous statements hold for the subgroups $E_n(R)$ and $SL_n(R)$ of $GL_n(R)$. Therefore we will just speak of unimodular vectors of length n over R and usually think of them as unimodular column vectors and elements of R^n in this paper. Now consider the space $H(R^n) = R^n \oplus R^n$ equipped with the quadratic form

$$q(a, b) = \sum_{i=1}^n a_i b_i,$$

where $a = (a_1, \dots, a_n)^t, b = (b_1, \dots, b_n)^t \in R^n$.

Definition 2.2. The set $U_{2n-1}(R)$ of unit vectors is the set of elements (a, b) in $H(R^n)$ with $q(a, b) = 1$.

Recall now from [11, Chapter IV, §1] that the Clifford algebra $Cl(V, q)$ of $V = H(R^n)$ is the quotient of the tensor algebra

$$T(V) = R \oplus V \oplus V^{\otimes 2} \oplus \dots$$

by the two-sided ideal generated by the elements of the form $x \otimes x - q(x)$ for $x \in V$. Grading $T(V)$ by even and odd degrees, $Cl(V, q)$ inherits a $\mathbb{Z}/2\mathbb{Z}$ -grading. Furthermore, there is an inclusion $V \hookrightarrow Cl(V, q), (v, w) \mapsto (v, w)$.

Now let $a = (a_1, \dots, a_n)^t, b = (b_1, \dots, b_n)^t \in R^n$. Following the notation in [2] (and not the notation of [5] or [6]), the Suslin matrix can be defined inductively by $\alpha_n(a, b) = (a_1)$ if $n = 1$ and

$$\alpha_n(a, b) = \begin{pmatrix} a_1 id_{2^{n-2}} & \alpha_{n-1}(a', b') \\ -\alpha_{n-1}(b', a')^t & b_1 id_{2^{n-2}} \end{pmatrix}$$

if $n \geq 2$, where $a' = (a_2, \dots, a_n)^t, b' = (b_2, \dots, b_n)^t \in R^{n-1}$. In [14, Lemma 5.1] Suslin proved that $\det(\alpha_n(a, b)) = (a^t b)^{2^{n-2}}$ if $n \geq 2$; in particular, if $a = (a_1, \dots, a_n)^t$ is a unimodular vector of length n and $b = (b_1, \dots, b_n)^t$ defines a section of a , i.e., $q(a, b) = \sum_{i=1}^n a_i b_i = 1$, then $\alpha_n(a, b) \in SL_{2^{n-1}}(R)$.

Similarly, one defines $\overline{\alpha_n(a, b)} = (b_1)$ if $n = 1$ and

$$\overline{\alpha_n(a, b)} = \begin{pmatrix} b_1 id_{2^{n-2}} & -\alpha_{n-1}(a', b') \\ \alpha_{n-1}(b', a')^t & a_1 id_{2^{n-2}} \end{pmatrix}$$

if $n \geq 2$, where again $a' = (a_2, \dots, a_n)^t, b' = (b_2, \dots, b_n)^t \in R^{n-1}$. It was proven in [14, Lemma 5.1] that one has $\alpha_n(b, a)^t = \overline{\alpha_n(a, b)}$ and $\alpha_n(a, b)\overline{\alpha_n(a, b)} = \alpha_n(a, b)\alpha_n(a, b) = (a \cdot b^t)id_{2^{n-1}}$.

Altogether, one obtains maps

$$\alpha_n : U_{2n-1}(R) \rightarrow GL_{2^{n-1}}(R)$$

and

$$\overline{\alpha}_n : U_{2n-1}(R) \rightarrow GL_{2n-1}(R).$$

One can use Suslin matrices in order to obtain a description of the Clifford algebra $Cl(V, q)$ (cf. [5, Section 2] and [5, Section 2.3]): Let us denote by $M_{2^n}(R)$ the set of $2^n \times 2^n$ -matrices over R . The assignment

$$\phi(a, b) = \begin{pmatrix} 0 & \alpha_n(a, b) \\ \alpha_n(a, b) & 0 \end{pmatrix}$$

descends to a map $Cl(V, q) \rightarrow M_{2^n}(R)$ and is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras, where matrices of the form

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

are of degree 0 and matrices of the form

$$\begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix}$$

are of degree 1 (cf. [5, Theorem 2.3 and remarks thereafter]). Furthermore, one can inductively define $2^{n-1} \times 2^{n-1}$ -matrices J_n by $J_1 = 1$ if $n = 1$ and

$$J_n = \begin{pmatrix} 0 & J_{n-1} \\ -J_{n-1} & 0 \end{pmatrix}$$

if $n \geq 2$ is even or

$$J_n = \begin{pmatrix} J_{n-1} & 0 \\ 0 & -J_{n-1} \end{pmatrix}$$

if $n \geq 2$ is odd. Note again that we use the notation from [2]. By induction, one sees that $J_n J_n^t = J_n^t J_n = id_{2^{n-1}}$ and $J_n^{-1} = (-1)^{\frac{n(n-1)}{2}} J_n$. It follows that $M^* = J_{n+1} M^t J_{n+1}^t$, where $M \in M_{2^n}(R)$, defines an involution on $M_{2^n}(R)$. As one can check, this involution corresponds to the so-called standard involution of the Clifford algebra $Cl(V, q)$ (cf. [5, Theorem 3.1]).

Now let us denote by $Cl = Cl(V, q) = Cl_0 \oplus Cl_1$ the $\mathbb{Z}/2\mathbb{Z}$ -grading of $Cl(V, q)$.

Definition 2.3. For any $n \in \mathbb{N}$, the Spin group is defined as $Spin_{2n}(R) := \{g \in Cl_0 \mid gg^* = 1, gH(R^n)g^{-1} = H(R^n)\}$.

Following [11, Chapter IV, §6], there is a map

$$\pi : Spin_{2n}(R) \rightarrow SO_{2n}(R)$$

defined by $\pi(g) : R^{2n} \rightarrow R^{2n}, (a, b) \mapsto g \cdot (a, b) \cdot g^{-1}$. If we let $U_{2n-1}(R) := \{(a, b) \in R^n \oplus R^n \mid q(a, b) = 1\}$ as above, then it is clear that $O_{2n}(R)$ and hence $SO_{2n}(R)$ act on the left on $U_{2n-1}(R)$ by matrix multiplication. Consequently, one obtains a left action of $Spin_{2n}(R)$ on $U_{2n-1}(R)$ via π . If $n \geq 3$, one has a map

$$Um_n(R) \rightarrow U_{2n-1}(R)/SO_{2n}(R)$$

which sends any $v \in Um_n(R)$ to the orbit of (v, w) , where w is any section of v . It follows for example from [6, Theorem 4.3] that this map is well-defined for $n \geq 3$.

Thus, from now on, let us assume $n \geq 3$. Let $v_1, v_2 \in Um_n(R)$ with sections w_1 and w_2 and assume $\sigma v_1 = v_2$ for some $\sigma \in SL_n(R)$. Then

$$H(\sigma) = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{t-1} \end{pmatrix} \in SO_{2n}(R)$$

and hence $(v_1, w_1) \sim_{SO_{2n}(R)} H(\sigma)(v_1, w_1) = (\sigma v_1, \sigma^{t-1} w_1) \sim_{SO_{2n}(R)} (v_2, w_2)$, where the latter equivalence holds again because of [6, Theorem 4.3]. In particular, it follows that the map $Um_n(R) \rightarrow U_{2n-1}(R)/SO_{2n}(R)$ descends to a map

$$Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/SO_{2n}(R).$$

Furthermore, there is an injective homomorphism $\wedge : SL_n(R) \rightarrow Spin_{2n}(R)$ which lifts the homomorphism $H : SL_n(R) \rightarrow SO_{2n}(R)$ along π (cf. [11, Chapter IV, §6]), so by using again [6, Theorem 4.3] one obtains a sequence of surjections between orbit spaces

$$Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R) \rightarrow U_{2n-1}(R)/SO_{2n}(R).$$

We now try to compare $SL_n(R)$ -orbits of $Um_n(R)$ with $Spin_{2n}(R)$ -orbits of $U_{2n-1}(R)$. So let us investigate the injectivity of the comparison map

$$Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R)$$

above. First of all, let us denote by $St(x, y)$ the subgroup of $Spin_{2n}(R)$ which stabilizes $(x, y) \in U_{2n-1}(R)$. We now prove the following criterion for the injectivity of the comparison map:

Theorem 2.4. *The natural map $Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R)$ above is injective (and hence bijective) if and only if, for any $(x, y) \in U_{2n-1}(R)$, one has $Spin_{2n}(R) = \wedge(SL_n(R))Epin_{2n}(R)St(x, y)$. Furthermore, the orbit space $Um_n(R)/SL_n(R)$ is trivial if and only if the orbit space $U_{2n-1}(R)/Spin_{2n}(R)$ is trivial and $Spin_{2n}(R) = \wedge(SL_n(R))Epin_{2n}(R)St(u_n)$, where $u_n = (e_n, e_n)$ and $e_n = (1, 0, \dots, 0)^t \in R^n$.*

Remark 2.5. Let G be a group. For two subgroups $M, N \subset G$, let $MN = \{mn | m \in M, n \in N\}$ and $NM = \{nm | n \in N, m \in M\}$. Recall that M and N are called permuting if $MN = NM$; this is the case if and only if the set MN is a subgroup of G . Now recall that $Epin_{2n}(R)$ is a normal subgroup of $Spin_{2n}(R)$ for $n \geq 3$: Indeed, $EO_{2n}(R)$ is a normal subgroup of $SO_{2n}(R)$ (cf. [15, Theorem 2.12]) and $Epin_{2n}(R)$ is the preimage of $EO_{2n}(R)$ under the homomorphism π above (cf. [3, Remarks after Proposition 4.3.3]). It follows that $Epin_{2n}(R)$ permutes with any subgroup of $Spin_{2n}(R)$ and, moreover, $Spin_{2n}(R)/Epin_{2n}(R)$ is a group. We denote by $\overline{\wedge SL_n(R)}$ and $\overline{St(x, y)}$ the images of $\wedge(SL_n(R))$ and $St(x, y)$ under the canonical projection $Spin_{2n}(R) \rightarrow Spin_{2n}(R)/Epin_{2n}(R)$ respectively. The criterion in Theorem 2.4 above essentially means that $Spin_{2n}(R)/Epin_{2n}(R) = \overline{\wedge SL_n(R)} \overline{St(x, y)}$ for any $(x, y) \in U_{2n-1}(R)$. In particular, it follows that the criterion in the theorem is equivalent to the equality $Spin_{2n}(R) = St(x, y)Epin_{2n}(R) \wedge(SL_n(R))$ for any $(x, y) \in U_{2n-1}(R)$. Furthermore, it follows completely analogously that the equality $SO_{2n}(R) = H(SL_n(R))EO_{2n}(R)St(x, y)$ in Theorem 2.6 below is equivalent to the equality $SO_{2n}(R) = St(x, y)EO_{2n}(R)H(SL_n(R))$.

Proof. Let $\varphi \in Spin_{2n}(R)$ such that $(v, w) = \varphi \cdot (x, y)$. Note that elements of $St(x, y)$ do not change the right-hand vector (x, y) by definition. Elements of $Epin_{2n}(R)$ will not change the $E_n(R)$ -orbits

of the unimodular vector x in question because of [6, Theorem 4.4]. Hence if one has $Spin_{2n}(R) = \wedge(SL_n(R))Ep_{in_{2n}}(R)St(x, y)$, then v will be in the same $SL_n(R)$ -orbit as x .

Conversely, let us fix $(x, y) \in U_{2n-1}(R)$. Assume that $(v, w) \sim_{Spin_{2n}(R)} (x, y)$ always implies $v \sim_{SL_n(R)} x$ and let $\varphi \in Spin_{2n}(R)$. Let $(v, w) = \varphi \cdot (x, y)$. Then there is $\varphi' \in SL_n(R)$ such that $\wedge(\varphi') \cdot (v, w) = (\varphi'v, \varphi'^{t-1}w) = (x, y')$ for some y' . Furthermore, there is $\varphi'' \in Ep_{in_{2n}}(R)$ with $\varphi'' \cdot (x, y') = (x, y)$ by [6, Theorem 4.3]. But then $\varphi'' \wedge (\varphi')\varphi \in St(x, y)$. So $\varphi \in \wedge(SL_n(R))Ep_{in_{2n}}(R)St(x, y)$ and, for any $(x, y) \in U_{2n-1}(R)$, $Spin_{2n}(R) = \wedge(SL_n(R))Ep_{in_{2n}}(R)St(x, y)$. This proves the first statement in the theorem.

The second statement in the theorem follows easily from the first paragraph of this proof by realizing that one only has to consider the vector u_n in order to show that $Um_n(R)/SL_n(R)$ is trivial if $U_{2n-1}(R)/Spin_{2n}(R)$ is trivial. \square

Note that $St(u_n)$ is just $Spin_{2n-1}(R)$. Of course, the criterion in the theorem is automatically satisfied whenever the orbit space $Um_n(R)/SL_n(R)$ is trivial. Later we will be able to give more examples of rings for which the criterion in Theorem 2.4 holds.

Analogously, we now investigate the injectivity of the comparison map

$$Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/SO_{2n}(R).$$

By abuse of notation, let us also denote by $St(x, y)$ the subgroup of $SO_{2n}(R)$ which stabilizes $(x, y) \in U_{2n-1}(R)$; note that $St(u_n)$ is just $SO_{2n-1}(R)$. We prove the following criterion for the equality of $SL_n(R)$ -orbits of unimodular vectors and $SO_{2n}(R)$ -orbits of unit vectors:

Theorem 2.6. *The natural map $Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/SO_{2n}(R)$ is injective (and hence bijective) if and only if, for any $(x, y) \in U_{2n-1}(R)$, one has $SO_{2n}(R) = H(SL_n(R))EO_{2n}(R)St(x, y)$. Furthermore, $Um_n(R)/SL_n(R)$ is trivial if and only if the orbit space $U_{2n-1}(R)/SO_{2n}(R)$ is trivial and $SO_{2n}(R) = H(SL_n(R))EO_{2n}(R)St(u_n)$, where $u_n = (e_n, e_n)$ and $e_n = (1, 0, \dots, 0)^t \in R^n$.*

Proof. The proof is very similar to the proof of Theorem 2.4 above, but for the reader's convenience we still give the full argument: Let $\varphi \in SO_{2n}(R)$ such that $(v, w) = \varphi \cdot (x, y)$. Again, note that elements of $St(x, y)$ do not change the right-hand vector (x, y) by definition. Furthermore, elements of $EO_{2n}(R)$ will not change the $E_n(R)$ -orbits of the corresponding unimodular vector x because of [6, Theorem 4.4] again. Thus, if $SO_{2n}(R) = H(SL_n(R))EO_{2n}(R)St(x, y)$, then v lies in the same $SL_n(R)$ -orbit as x .

Conversely, we fix $(x, y) \in U_{2n-1}(R)$. Assume that $(v, w) \sim_{SO_{2n}(R)} (x, y)$ always implies $v \sim_{SL_n(R)} x$ and let $\varphi \in SO_{2n}(R)$. Let $(v, w) = \varphi \cdot (x, y)$. By assumption, there is $\varphi' \in H(SL_n(R))$ such that $H(\varphi') \cdot (v, w) = (\varphi'v, \varphi'^{t-1}w) = (x, y')$ for some y' . By [6, Theorem 4.3], there is $\varphi'' \in EO_{2n}(R)$ with $\varphi'' \cdot (x, y') = (x, y)$. But then $\varphi''H(\varphi')\varphi \in St(x, y)$. So $\varphi \in H(SL_n(R))EO_{2n}(R)St(x, y)$ and, for any $(x, y) \in U_{2n-1}(R)$, $SO_{2n}(R) = H(SL_n(R))EO_{2n}(R)St(x, y)$. This proves the first statement in the theorem.

Again, the second statement in the theorem follows from the first paragraph by realizing that one only has to consider the vector u_n in order to show that $Um_n(R)/SL_n(R)$ is trivial in case $U_{2n-1}(R)/SO_{2n}(R)$ is trivial. \square

2.2. Degree maps

In this section, we introduce the degree maps already studied in [14] and [2, Section 3] and discuss their relationship with Spin-orbits of unit vectors.

For all $n \in \mathbb{N}$, we denote by $Sp_{2n}(R)$ the group of symplectic matrices of rank $2n$, by $O_{2n}(R)$ the group of orthogonal matrices of rank $2n$, by $S_{2n}(R)$ the set of invertible symmetric matrices of rank $2n$, by $A_{2n}(R)$ the

set of invertible alternating matrices of rank $2n$ and by $\tilde{A}_{2n}(R)$ its subset of invertible alternating matrices of rank $2n$ with Pfaffian 1. The group $GL_{2n}(R)$ acts on the set $S_{2n}(R)$ by $M \mapsto \varphi^t M \varphi$ for $\varphi \in GL_{2n}(R)$ and $M \in S_{2n}(R)$. Similarly, $GL_{2n}(R)$ acts on the set $A_{2n}(R)$ by $M \mapsto \varphi^t M \varphi$ for $\varphi \in GL_{2n}(R)$ and $M \in A_{2n}(R)$; this action also induces an action of $SL_{2n}(R)$ on $\tilde{A}_{2n}(R)$. We denote by $S_{2n}(R)/GL_{2n}(R)$, $A_{2n}(R)/GL_{2n}(R)$ and $\tilde{A}_{2n}(R)/SL_{2n}(R)$ the corresponding orbit spaces.

For $m, n \geq 1$, there are embeddings $S_{2n}(R) \rightarrow S_{2n+2m}(R), M \mapsto M \perp \sigma_{2m}$, and $A_{2n}(R) \rightarrow A_{2n+2m}(R), M \mapsto M \perp \psi_{2m}$, where the matrices σ_{2m} and ψ_{2m} are defined inductively by

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\sigma_{2m} = \sigma_{2m-2} \perp \sigma_2$ and $\psi_{2m} = \psi_{2m-2} \perp \psi_2$ for $m \geq 2$. We denote by $S(R) = \text{colim}_n S_{2n}(R)$ and $A(R) = \text{colim}_n A_{2n}(R)$ the direct limits under the respective embeddings.

Now we discuss an equivalence relation on the set $S(R)$: Two matrices $M \in S_{2m}(R)$ and $N \in S_{2n}(R)$ are said to be equivalent, $M \sim N$, if there exists $i \geq 1$ and $E \in E_{2n+2m+2i}(R)$ with

$$M \perp \sigma_{2n+2i} = E^t(N \perp \sigma_{2m+2i})E.$$

It follows from [4, Lemme 4.5.1.9] that the block sum of matrices equips the set of equivalence classes $S(R)/\sim$ with the structure of an abelian group. We also refer the reader to [2, Section 2] for a discussion of the group $S(R)/\sim$.

Analogously, we have an equivalence relation on the set $A(R)$: Two matrices $M \in A_{2m}(R)$ and $N \in A_{2n}(R)$ are said to be equivalent, $M \sim N$, if there exists $i \geq 1$ and $E \in E_{2n+2m+2i}(R)$ with

$$M \perp \psi_{2n+2i} = E^t(N \perp \psi_{2m+2i})E.$$

Then it follows from [16, §3] that the block sum of matrices equips the set of equivalence classes $W'_E(R) := A(R)/\sim$ with the structure of an abelian group. One can also consider the direct limit $\tilde{A}(R) = \text{colim}_n \tilde{A}_{2n}(R)$ with respect to the induced embeddings $\tilde{A}_{2n}(R) \rightarrow \tilde{A}_{2n+2m}(R), M \mapsto M \perp \psi_{2m}$; then it follows again from [16, §3] that the set of equivalence classes $W_E(R) := \tilde{A}(R)/\sim$ is an abelian group. This group is also called the elementary symplectic Witt group and clearly is a subgroup of $W'_E(R)$; we refer the reader to [17, Section 3.A] for a discussion of the groups $W'_E(R)$ and $W_E(R)$.

One can also define a quotient of $W_E(R)$ which is usually denoted by $W_{SL}(R)$ as follows: In this case two matrices $M \in \tilde{A}_{2m}(R)$ and $N \in \tilde{A}_{2n}(R)$ are said to be equivalent, $M \sim_{SL} N$, if there exists $i \geq 1$ and $\varphi \in SL_{2n+2m+2i}(R)$ with

$$M \perp \psi_{2n+2i} = \varphi^t(N \perp \psi_{2m+2i})\varphi.$$

It follows again from [16, §3] that the block sum of matrices equips the set of equivalence classes $W_{SL}(R) := \tilde{A}(R)/\sim_{SL}$ with the structure of an abelian group. We also refer the reader to [18, Section 2.B] for a detailed discussion of this group.

If R is a smooth affine algebra over a field k with $\text{char}(k) \neq 2$, then it is argued in [2, Section 2.3] that the abelian group $S(R)/\sim$ can be identified with the higher Grothendieck-Witt group $GW_1^1(R)$; furthermore, the group $W'_E(R)$ can be identified with the higher Grothendieck-Witt group $GW_1^3(R)$. We refer the reader to [2, Section 2] for a detailed discussion of the higher Grothendieck-Witt groups $GW_i^j(R)$ for $i, j \in \mathbb{Z}$. For the purpose of this paper, we simply mention that there exist exact Karoubi periodicity sequences

$$K_1(R) \xrightarrow{H_{1,1}} GW_1^1(R) \xrightarrow{\eta} GW_0^0(R) \xrightarrow{f_{0,0}} K_0(R)$$

and

$$K_1(R) \xrightarrow{H_{1,3}} GW_1^3(R) \xrightarrow{\eta} GW_0^2(R) \xrightarrow{f_{0,2}} K_0(R).$$

The groups $K_i(R)$, $i = 0, 1$, are the usual algebraic K -theory groups; the group $GW_0^0(R) = GW(R)$ is the Grothendieck-Witt group of non-degenerate symmetric bilinear forms over R , while the group $GW_0^2(R) = KSp_0(R)$ is the 0th symplectic K -theory group (of non-degenerate alternating bilinear forms). The homomorphisms $f_{0,0}$ and $f_{0,2}$ are the usual forgetful homomorphisms, while the homomorphisms $H_{1,1}$ and $H_{1,3}$ are called hyperbolic homomorphisms. The hyperbolic homomorphism $H_{1,1}$ is defined by $M \mapsto M^t \sigma_{2n} M \in S_{2n}(R)$ for $M \in GL_{2n}(R)$, while $H_{1,3}$ is defined by $M \mapsto M^t \psi_{2n} M \in A_{2n}(R)$ for $M \in GL_{2n}(R)$. Following [2, Section 2.4], the homomorphism η in each sequence is defined by $M \mapsto [M] - [\sigma_{2n}] \in GW(R)$ for $M \in S_{2n}(R)$ and respectively by $M \mapsto [M] - [\psi_{2n}] \in KSp_0(R)$ for $M \in A_{2n}(R)$; as the definitions of the forgetful homomorphisms and the homomorphisms $H_{1,1}$, $H_{1,3}$ and η make sense over any commutative ring R with $2 \in R^\times$, we will use the notation for these homomorphisms in this general case as well. Finally, the group $W_{SL}(R)$ can be identified with the cokernel of $H_{1,3}$ and the homomorphism η identifies $W_{SL}(R)$ with the kernel of the forgetful homomorphism $KSp_0(R) \rightarrow K_0(R)$.

Now let R be any commutative ring with $2 \in R^\times$ again. The $2n \times 2n$ -matrices τ_{2n} over R are defined inductively by

$$\tau_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\tau_{2n} = \tau_{2n-2} \perp \tau_2$ for $n \geq 2$. One also defines matrices $E_n \in GL_{2^{n-1}}(R)$ inductively by $E_n = id_{2^{n-1}}$ for $n = 1, 2$ and by

$$E_n := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & J_{n-1}^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau_{2^{n-2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sigma_{2^{n-2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \psi_{2^{n-2}} \end{pmatrix} & \text{if } n \equiv 0 \pmod 4 \\ \begin{pmatrix} E_{n-1} & 0 \\ 0 & E_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \psi_{2^{n-2}} \end{pmatrix} & \text{if } n \equiv 1 \pmod 4 \\ \begin{pmatrix} 1 & 0 \\ 0 & J_{n-1}^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau_{2^{n-2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & \psi_{2^{n-2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_{2^{n-2}} \end{pmatrix} & \text{if } n \equiv 2 \pmod 4 \\ \begin{pmatrix} E_{n-1} & 0 \\ 0 & E_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_{2^{n-2}} \end{pmatrix} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

for $n \geq 3$. For any $(v, w) \in U_{2^{n-1}}(R)$, it follows from [14, Lemma 5.3] and [2, Lemma 3.3.1] that the matrices $E_n^{-1} \alpha_n^t(v, w) E_n$ are orthogonal if $n \equiv 0 \pmod 4$ and symplectic if $n \equiv 2 \pmod 4$ and that the matrices $E_n^t \alpha_n(v, w) J_n E_n$ are invertible symmetric if $n \equiv 1 \pmod 4$ and invertible alternating if $n \equiv 3 \pmod 4$.

Hence setting

$$\Psi_n(v, w) := \begin{cases} E_n^{-1} \alpha_n^t(v, w) E_n & \text{if } n \text{ is even} \\ E_n^t \alpha_n(v, w) J_n E_n & \text{if } n \text{ is odd} \end{cases}$$

for any $(v, w) \in U_{2^{n-1}}(R)$ defines a map

$$\Psi_n : U_{2^{n-1}}(R) \rightarrow \begin{cases} O_{2^{n-1}}(R) & \text{if } n \equiv 0 \pmod 4 \\ S_{2^{n-1}}(R) & \text{if } n \equiv 1 \pmod 4 \\ Sp_{2^{n-1}}(R) & \text{if } n \equiv 2 \pmod 4 \\ A_{2^{n-1}}(R) & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

which we will call the degree map (see also [2, Section 3.3]).

Finally, let us now return to the study of Spin-orbits of unit vectors: If $n \geq 3$ is odd, the action of $Spin_{2n}(R)$ on $U_{2n-1}(R)$ can be described as follows (cf. [5, Section 4]): Any $\varphi \in Spin_{2n}(R)$ has the form $\varphi = \text{diag}(g, g^{*-1})$ for $g \in GL_{2n-1}(R)$ under the identification $Cl(V, q) \cong M_{2n}(R)$ (cf. [5, Theorem 4.3]); here g^* denotes the standard involution of $g \in M_{2n-1}(R)$. If $(v, w) \in U_{2n-1}(R)$, then the matrix $g\alpha_n(v, w)g^*$ is again a Suslin matrix of some unit vectors, i.e., $g\alpha_n(v, w)g^* = \alpha_n(v', w')$ for some $(v', w') \in U_{2n-1}(R)$, and this is the action of φ on (v, w) (cf. [5, Section 4.1]).

Moreover, if we associate to any Suslin matrix $\alpha_n(v, w)$ the matrix

$$\Psi_n(v, w) = E_n^t \alpha_n(v, w) J_n E_n$$

from the definition of the degree map, this gives a bijection from Suslin matrices to matrices of the form $\Psi_n(v, w)$; in this sense, no information is lost from the Suslin matrix construction. Indeed, the map Ψ_n is obtained from the Suslin matrix map $\alpha_n : U_{2n-1}(R) \rightarrow GL_{2n-1}(R)$ by considering the composite of α_n and the bijection $GL_{2n-1}(R) \rightarrow GL_{2n-1}(R), M \mapsto E_n^t M J_n E_n$ and realizing that the image of this composite lies in fact in $S_{2n-1}(R)$ if $n \equiv 1 \pmod 4$ or in $A_{2n-1}(R)$ if $n \equiv 3 \pmod 4$. Furthermore, note that $\Psi_n(e_n, e_n) = \sigma_{2n-1}$ if $n \equiv 1 \pmod 4$ and that $\Psi_n(e_n, e_n) = \psi_{2n-1}$ if $n \equiv 3 \pmod 4$ (cf. [2, Lemma 3.3.1]).

The action of $Spin_{2n}(R)$ translates to

$$\begin{aligned} \Psi_n(v', w') &= E_n^t g \alpha_n(v, w) g^* J_n E_n = E_n^t g \alpha_n(v, w) J_n g^t J_n^t J_n E_n = E_n^t g \alpha_n(v, w) J_n g^t E_n \\ &= (E_n^t g E_n^{t-1})(E_n^t \alpha_n(v, w) J_n E_n)(E_n^{-1} g^t E_n) = g' \Psi_n(v, w) g'^t, \end{aligned}$$

where (v', w') is again the action of $\varphi = \text{diag}(g, g^{*-1})$ on (v, w) and $g' = E_n^t g E_n^{t-1}$. This description has some consequences that we now explain:

Let us first explain why the determinant of $\Psi_n(v, w)$ is 1 if $n \equiv 1 \pmod 4$ and that the Pfaffian of $\Psi_n(v, w)$ is 1 if $n \equiv 3 \pmod 4$; note that it suffices to check this locally, i.e., after localization at any prime ideal. But over a local ring every given unimodular vector (v, w) is completable to a matrix in $E_n(R)$ and the homomorphism $\wedge : SL_n(R) \rightarrow Spin_{2n}(R)$ always maps $E_n(R)$ into $Ep_{in_{2n}}(R)$, so we can assume that some $\varphi = (g, g^*) \in Ep_{in_{2n}}(R)$ transforms (v, w) into $u_n = (e_n, e_n)$; one can then check easily via the explicit generators of $Ep_{in_{2n}}(R)$ given at the end of [6, Section 3.3] that the determinant of g has to be 1.

In particular, using the computation of $\Psi_n(v', w')$ above for $(v', w') = u_n$, it follows locally and then also over any commutative ring R that the determinant of $\Psi_n(v, w)$ is indeed 1 if $n \equiv 1 \pmod 4$ and that the Pfaffian of $\Psi_n(v, w)$ is 1 if $n \equiv 3 \pmod 4$, as claimed. For any commutative ring R and $\varphi = \text{diag}(g, g^{*-1}) \in Spin_{2n}(R)$, it then follows again from the computation above that the determinant of g' and hence of g is a square root of 1 if $n \equiv 1 \pmod 4$ and equal to 1 if $n \equiv 3 \pmod 4$.

By composition we obtain induced maps

$$Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R) \rightarrow S_{2n-1}(R)/GL_{2n-1}(R) \rightarrow \text{coker}(K_1(R)) \xrightarrow{H_{1,1}} S(R)/\sim$$

for $n \equiv 1 \pmod 4$ and

$$\begin{aligned} Um_n(R)/SL_n(R) &\rightarrow U_{2n-1}(R)/Spin_{2n}(R) \rightarrow \tilde{A}_{2n-1}(R)/SL_{2n-1}(R) \\ &\rightarrow \text{coker}(K_1(R)) \xrightarrow{H_{1,3}} W'_E(R) = W_{SL}(R) \end{aligned}$$

for $n \equiv 3 \pmod 4$. By abuse of notation, we call these composites again Ψ_n or Ψ_n modulo SL.

It also follows from the discussion above that the injective homomorphism $St(x, y) \rightarrow GL_{2n-1}(R) \xrightarrow{\cong} GL_{2n-1}(R), (g, g^{*-1}) \mapsto g \mapsto g' = E_n^t g E_n^{t-1}$ factors through $O(\Psi_n(x, y))$ if $n \equiv 1 \pmod 4$ and $Sp(\Psi_n(x, y))$ if $n \equiv 3 \pmod 4$.

It is a natural question whether the map Ψ_n is injective or, in other words, detects the non-triviality of $Um_n(R)/SL_n(R)$, i.e., (v, w) and (v', w') represent isomorphic stably free modules if and only if their images under Ψ_n coincide.

Proposition 2.7. *Let R be a ring and $n \geq 3$ odd. If the degree map Ψ_n modulo SL is injective, then the natural map $Um_n(R)/SL_n(R) \rightarrow U_{2n-1}(R)/Spin_{2n}(R)$ above is bijective.*

Proof. Follows directly from the factorizations above. \square

Remark 2.8. Recall from [17] and [18] that there is a generalized Vaserstein symbol $V_\theta : Um_3(R)/SL_3(R) \rightarrow W_{SL}(R)$ modulo SL associated to any isomorphism $\theta : R \xrightarrow{\cong} \det(R^2)$ via the canonical identification $\tilde{V}_{SL}(R) \cong W_{SL}(R)$ from [18, Section 2.C]. If we let $e_1 = (1, 0)^t, e_2 = (0, 1)^t \in R^2$, then there is a canonical isomorphism $\theta : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$. The map Ψ_3 modulo SL coincides with the Vaserstein symbol modulo SL associated to θ . However, the map Ψ_3 modulo SL does not coincide with the original Vaserstein symbol considered in [16, §5]; as a matter of fact, the original Vaserstein symbol coincides with the generalized Vaserstein symbol associated to $-\theta$ via the canonical identification $\tilde{V}_{SL}(R) \cong W_{SL}(R)$ (cf. [17, Section 4.B]). Altogether, it follows from Proposition 2.7 that the comparison map $Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R)$ is bijective whenever the generalized Vaserstein symbol modulo SL associated to the isomorphism θ above is injective.

Furthermore, the assignment $(g, g^{*-1}) \mapsto g$ induces isomorphisms $Spin_6(R) \cong SL_4(R)$ and $Epin_6(R) \cong E_4(R)$ (cf. [5, Theorems 6.1 and 7.5]); in particular, it follows that the assignment $(g, g^{*-1}) \mapsto E_3^t g E_3^{t-1}$ also induces an isomorphism $Spin_6(R) \cong SL_4(R)$. Furthermore, one can check easily that any matrix in $\tilde{A}_4(R)$ is of the form $\Psi_3(v, w)$ for some $(v, w) \in U_5(R)$ and therefore the assignment $(v, w) \mapsto \Psi_3(v, w)$ induces a bijection $U_5(R) \cong \tilde{A}_4(R)$. Hence it follows that the map $U_5(R)/Spin_6(R) \rightarrow \tilde{A}_4(R)/SL_4(R)$ is bijective and the discussion above shows that $St(x, y)$ corresponds to $Sp(\chi)$ with $\chi = \Psi_3(x, y)$ under the isomorphism $Spin_6(R) \rightarrow SL_4(R), (g, g^{*-1}) \mapsto E_3^t g E_3^{t-1}$.

If n is even, the action of $Spin_{2n}(R)$ on $U_{2n-1}(R)$ can be described as follows (cf. [5, Section 5]): Any $\varphi \in Spin_{2n}(R)$ has the form $\varphi = diag(\varphi_1, \varphi_2)$ for $\varphi_1, \varphi_2 \in GL_{2n-1}(R)$ under the identification $Cl(V, q) \cong M_{2n}(R)$. Furthermore, both φ_1 and φ_2 satisfy $\varphi_i^* = \varphi_i^{-1}$ for $i = 1, 2$; it follows from this that $E_n^{-1} \varphi_i E_n \in O_{2n-1}(R)$ if $n \equiv 0 \pmod 4$ and $E_n^{-1} \varphi_i E_n \in Sp_{2n-1}(R)$ if $n \equiv 2 \pmod 4$, because the equality $\varphi_i^* = J_n \varphi_i^t J_n^t$ implies

$$J_n^t = \varphi_i^t J_n^t \varphi_i.$$

Using $E_n^t J_n^t E_n = \sigma_{2n-1}$ if $n \equiv 0 \pmod 4$ and $E_n^t J_n^t E_n = -\psi_{2n-1}$ if $n \equiv 2 \pmod 4$ (cf. [2, Lemma 3.3.1]), one obtains the statement.

If $(v, w) \in U_{2n-1}(R)$, then the matrix $\varphi_1 \alpha_n(v, w) \varphi_2^{-1}$ is again a Suslin matrix of some unit vectors, i.e., $\varphi_1 \alpha_n(v, w) \varphi_2^{-1} = \alpha_n(v', w')$ for some $(v', w') \in U_{2n-1}(R)$, and this is essentially the action of φ on (v, w) .

Moreover, if we associate to any Suslin matrix $\alpha_n(v, w)$ the matrix

$$\Psi_n(v, w) = E_n^{-1} \alpha_n^t(v, w) E_n,$$

from the definition of the degree map, this gives a bijection from Suslin matrices to matrices of the form $\Psi_n(v, w)$. The action of $Spin_{2n}(R)$ translates to

$$\begin{aligned} \Psi_n(v', w') &= E_n^{-1} \varphi_2^{-t} \alpha_n^t(v, w) \varphi_1^t E_n = (E_n^{-1} \varphi_2^{-t} E_n) (E_n^{-1} \alpha_n^t(v, w) E_n) (E_n^{-1} \varphi_1^t E_n) \\ &= (E_n^{-1} \varphi_2^{-t} E_n) \Psi_n(v, w) (E_n^{-1} \varphi_1^t E_n). \end{aligned}$$

It follows from the equality $\varphi_1\alpha_n(v, w)\varphi_2^{-1} = \alpha_n(v', w')$ that always $\det(\varphi_1) = \det(\varphi_2)$ and $\det(\varphi)$ is a square; for $(v, w) = (e_n, e_n)$, it follows that $\varphi_1\varphi_2^{-1}$ is a Suslin matrix of some unit vectors, which we denote by $\alpha(\varphi)$.

Altogether, we have an embedding $Spin_{2n}(R) \rightarrow O_{2^{n-1}}(R) \times O_{2^{n-1}}(R)$ if $n \equiv 0 \pmod 4$ and $Spin_{2n}(R) \rightarrow Sp_{2^{n-1}}(R) \times Sp_{2^{n-1}}(R)$ if $n \equiv 2 \pmod 4$. Furthermore, if $\varphi \in St(e_n, e_n)$ it follows that $\varphi_1 = \varphi_2$ and hence we have an embedding $Spin_{2n-1}(R) \rightarrow O_{2^{n-1}}(R)$ if $n \equiv 0 \pmod 4$ and $Spin_{2n-1}(R) \rightarrow Sp_{2^{n-1}}(R)$ if $n \equiv 2 \pmod 4$. Unfortunately, if n is even, it is not clear at all whether there exist analogues of the Ψ_n modulo SL maps as in the odd case.

2.3. Motivic homotopy theory

In this section we outline the construction of the unstable \mathbb{A}^1 -homotopy category over a field given in [13]; for an introduction to model categories, we refer the reader to [10]. The underlying idea of \mathbb{A}^1 -homotopy theory is to develop a homotopy theory of schemes in which the affine line \mathbb{A}^1 plays the role of the unit interval $[0, 1]$ in topology. So let k be a fixed base field.

We denote by Sm_k the category of smooth separated schemes of finite type over k and then we consider the category $Spc_k = \Delta^{op}Shv_{Nis}(Sm_k)$ of simplicial Nisnevich sheaves over Sm_k ; we will also refer to simplicial Nisnevich sheaves over Sm_k as spaces. Both the category Sm_k and the category $\Delta^{op}Sets$ of simplicial sets can be embedded into Spc_k . Following [13], there is a model structure on this category in which cofibrations are simply given by monomorphisms and weak equivalences are given by morphisms which induce weak equivalences of simplicial sets on stalks; this model structure is usually called the simplicial model structure and weak equivalences with respect to this model structure are also called simplicial weak equivalences. The \mathbb{A}^1 -model structure is then obtained as a left Bousfield localization of the simplicial model structure with respect to the projection morphisms $\mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{X}$; its weak equivalences are called \mathbb{A}^1 -weak equivalences. The associated homotopy category (which is obtained from Spc_k by inverting \mathbb{A}^1 -weak equivalences) is usually denoted $\mathcal{H}(k)$ and called the unstable \mathbb{A}^1 -homotopy category over k . There is a pointed version of this category which is constructed completely analogously by considering the category $Spc_{k,\bullet}$ of pointed simplicial Nisnevich sheaves over Sm_k (which are usually referred to as pointed spaces); it is denoted $\mathcal{H}_\bullet(k)$ and called the pointed unstable \mathbb{A}^1 -homotopy category. The category $Spc_{k,\bullet}$ is a pointed model category and hence features the formalism of fiber and cofiber sequences (cf. [10, Chapter 6]).

As in topology, there is a smash product $(\mathcal{X}, x) \wedge (\mathcal{Y}, y)$ for two pointed spaces (\mathcal{X}, x) and (\mathcal{Y}, y) . The simplicial suspension functor is defined as the functor $\Sigma_s = (S^1, *) \wedge - : Spc_{k,\bullet} \rightarrow Spc_{k,\bullet}$; it has a right adjoint $\Omega_s : Spc_{k,\bullet} \rightarrow Spc_{k,\bullet}$ called the loop space functor. This adjoint pair of functors forms a Quillen adjunction.

For two spaces \mathcal{X}, \mathcal{Y} , we denote by $[\mathcal{X}, \mathcal{Y}]_{\mathcal{H}(k)}$ the set of morphisms from \mathcal{X} to \mathcal{Y} in the category $\mathcal{H}(k)$; similarly, for two pointed spaces $(\mathcal{X}, x), (\mathcal{Y}, y)$, we denote by $[(\mathcal{X}, x), (\mathcal{Y}, y)]_{\mathcal{H}_\bullet(k)}$ the set of morphisms from (\mathcal{X}, x) to (\mathcal{Y}, y) in the pointed category $\mathcal{H}_\bullet(k)$.

Now let $n \geq 1$. Then we define $S_{2n-1} = k[x_1, \dots, x_n, y_1, \dots, y_n] / \langle \sum_{i=1}^n x_i y_i - 1 \rangle$ and let $Q_{2n-1} = Spec(S_{2n-1})$ be the smooth affine hypersurface in \mathbb{A}^{2n} . The morphism $p_{2n-1} : Q_{2n-1} \rightarrow \mathbb{A}^n \setminus 0$ induced by projection on the coefficients x_1, \dots, x_n is locally trivial with fibers isomorphic to \mathbb{A}^{n-1} and therefore an \mathbb{A}^1 -weak equivalence (cf. [1], [2]). In particular, we conclude that there is a pointed \mathbb{A}^1 -weak equivalence

$$\mathbb{A}^n \setminus 0 \simeq_{\mathbb{A}^1} Q_{2n-1}$$

if we let $\mathbb{A}^n \setminus 0$ have the basepoint $(1, 0, \dots, 0)$ and if we let Q_{2n-1} have the basepoint $(1, 0, \dots, 0, 1, 0, \dots, 0)$. Furthermore, if R is an affine k -algebra and $X = Spec(R)$, it is easy to see that

$$Um_n(R) \cong Hom_{Sch_k}(X, \mathbb{A}^n \setminus 0)$$

and

$$U_{2n-1}(R) = \{(a, b) | a, b \in R^n, q(a, b) = 1\} = Hom_{Sch_k}(X, Q_{2n-1}),$$

where Sch_k is the category of Noetherian k -schemes of finite Krull dimension. If R is furthermore smooth over k , $char(k) \neq 2$ and $n \geq 3$, then [12, Remark 7.10] and [7, Theorem 2.1] imply that in fact

$$Um_n(R)/E_n(R) \cong [X, \mathbb{A}^n \setminus 0]_{\mathcal{H}(k)}.$$

It was proven in [12, Corollary 5.43] that for a perfect field k and $n \geq 3$ there is a canonical bijection

$$[\mathbb{A}^n \setminus 0, \mathbb{A}^n \setminus 0]_{\mathcal{H}(k)} \cong GW(k)$$

called motivic Brouwer degree by analogy with the classical Brouwer degree in algebraic topology; here $GW(k)$ denotes the Grothendieck-Witt ring of non-degenerate symmetric bilinear forms over k .

Finally, we quickly introduce some notions of contractibility in motivic homotopy theory:

Definition 2.9. A space $\mathcal{X} \in Spc_k$ is called \mathbb{A}^1 -contractible if the unique morphism $\mathcal{X} \rightarrow Spec(k)$ is an \mathbb{A}^1 -weak equivalence.

Definition 2.10. Let $X \in Sm_k$ and x be a closed point. Then the pointed space (X, x) is stably \mathbb{A}^1 -contractible if $\mathbb{P}_k^{1 \wedge n} \wedge (X, x)$ is an \mathbb{A}^1 -contractible space for some $n \geq 0$.

Now let us work over a base field k which admits an embedding $\iota : k \hookrightarrow \mathbb{C}$. By means of such an embedding, one may associate a complex manifold X_ι^{an} to any smooth variety X over k .

Definition 2.11. Let k be a field which admits an embedding into \mathbb{C} . A smooth affine k -variety X is called topologically contractible if the manifold X_ι^{an} is a contractible topological space for any embedding $\iota : k \hookrightarrow \mathbb{C}$.

3. Results

We can now prove the main results in this paper. As usual, R always denotes a commutative ring with $2 \in R^\times$.

Theorem 3.1. *The map $Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R)$ is a bijection if*

- R is a Noetherian ring of dimension ≤ 2 (with $2 \in R^\times$)
- R is a smooth affine algebra of dimension 3 over a perfect field k such that $6 \in k^\times$ and $c.d.(k) \leq 1$
- R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $6 \in k^\times$

In particular, $SL_4(R) = \wedge(SL_3(R))Epin_4(R)Sp(\chi)$ in these cases, where χ is any non-degenerate alternating form with Pfaffian 1.

Proof. The bijectivity of the map $Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R)$ follows from Remark 2.8 and the bijectivity of the Vaserstein symbol modulo SL: The case of a Noetherian ring of dimension ≤ 2 follows easily from the criteria for the injectivity and surjectivity of the Vaserstein symbol modulo SL proven in [18, Theorems 3.2 and 3.6], the other two cases are covered by [19, Theorems 3.9 and 3.11]. The second statement follows directly from Theorem 2.4 and Remark 2.8. \square

There is a sufficient criterion for the equality of $Spin_{2n}(R)$ -orbits and $SO_{2n}(R)$ -orbits:

Proposition 3.2. *Assume R is a smooth affine algebra over a field k with $char(k) \neq 2$. If $R^\times = k^\times$, the 2-torsion of $Pic(R)$ is trivial and k is algebraically closed, then $Spin_{2n}(R)$ -orbits and $SO_{2n}(R)$ -orbits of $U_{2n-1}(R)$ coincide.*

Proof. Following [11, Chapter III, §3] there is an exact sequence

$$0 \rightarrow R^\times / (R^\times)^2 \rightarrow Disc(R) \rightarrow Pic(R)\{2\} \rightarrow 0.$$

Under the assumptions it follows that $Disc(R)$ is trivial and hence $Spin_{2n}(R) \rightarrow SO_{2n}(R)$ is surjective by [11, Chapter IV, Theorem 6.2.6], which implies the equality of the orbits of $U_{2n-1}(R)$. \square

Corollary 3.3. *Let $X = Spec(R)$ be smooth affine complex variety of dimension 3 or 4. Assume that one of the following conditions is satisfied:*

- a) X is topologically contractible and $R^\times = \mathbb{C} \setminus 0$
- b) X is stably \mathbb{A}^1 -contractible

Then there are bijections $Um_3(R)/SL_3(R) \xrightarrow{\cong} U_5(R)/Spin_6(R) \xrightarrow{\cong} U_5(R)/SO_6(R)$.

Proof. It was famously proven in [9] that $Pic(R)$ is trivial whenever $X = Spec(R)$ is a topologically contractible smooth affine complex variety (of any dimension); a well-known representability result (cf. [20, Section 6.1]) for motivic cohomology groups (and, in particular, for Picard groups) in stable motivic homotopy theory directly implies that the Picard group of a stably \mathbb{A}^1 -contractible smooth affine complex variety (of any dimension) is also trivial. Hence the statement follows from Theorem 3.1 and Proposition 3.2. \square

Recall that for any field k with characteristic $\neq 2$, we denote by S_{2n-1} the smooth affine k -algebra $k[x_1, \dots, x_n, y_1, \dots, y_n] / \langle \sum_{i=1}^n x_i y_i - 1 \rangle$. Furthermore, we let $Q_{2n-1} = Spec(S_{2n-1})$.

If $n \geq 3$ is odd and k is infinite perfect with $char(k) \neq 2$, we will see below that the orbit space $Um_n(S_{2n-1})/SL_n(S_{2n-1})$ as well as the abelian group $\ker(GW_0^{n-1}(S_{2n-1}) \rightarrow K_0(S_{2n-1}))$ can be computed explicitly in terms of the Witt group $W(k)$ of non-degenerate symmetric bilinear forms by using results proven in [1] and [2]; as a consequence, this enables us to understand the map Ψ_n modulo SL as follows:

Theorem 3.4. *If $n \geq 3$ is odd, k is infinite perfect with $char(k) \neq 2$, there is a canonical bijection $Um_n(S_{2n-1})/SL_n(S_{2n-1}) \cong W(k) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/(n-1)!\mathbb{Z}$, where we consider the fiber product with respect to the rank modulo 2 map $W(k) \rightarrow \mathbb{Z}/2\mathbb{Z}$ and the canonical surjection $\mathbb{Z}/(n-1)!\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$, while there is a canonical isomorphism $\ker(GW_0^{n-1}(S_{2n-1}) \rightarrow K_0(S_{2n-1})) \cong W(k)$. Under these identifications the map Ψ_n modulo SL for $R = S_{2n-1}$ then corresponds to the projection $W(k) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/(n-1)!\mathbb{Z} \rightarrow W(k)$.*

Proof. By the proof of [1, Theorem 4.10], the Witt-valued Brouwer degree map

$$Um_n(S_{2n-1})/E_n(S_{2n-1}) = [Q_{2n-1}, \mathbb{A}^n \setminus 0]_{\mathcal{H}_\bullet(k)} = GW(k) \rightarrow W(k)$$

and the rank map

$$Um_n(S_{2n-1})/E_n(S_{2n-1}) = [Q_{2n-1}, \mathbb{A}^n \setminus 0]_{\mathcal{H}_\bullet(k)} = GW(k) \rightarrow \mathbb{Z}$$

induce the identification $Um_n(S_{2n-1})/SL_n(S_{2n-1}) \cong W(k) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/(n-1)!\mathbb{Z}$.

By [2, Proposition 3.4.3], the group $\ker(GW_0^{n-1}(S_{2n-1}) \rightarrow K_0(S_{2n-1}))$ is a free $W(k)$ -module of rank 1 generated by $\eta\Psi_n(x, y)$, where $x = (x_1, \dots, x_n)^t$ and $y = (y_1, \dots, y_n)^t$. Now consider the composite

$$\begin{aligned} Um_n(S_{2n-1})/E_n(S_{2n-1}) &= [Q_{2n-1}, \mathbb{A}^n \setminus 0]_{\mathcal{H}_\bullet(k)} = GW(k) \xrightarrow{\eta\Psi_n} \ker(GW_0^{n-1}(S_{2n-1}) \rightarrow K_0(S_{2n-1})) \\ &= W(k) \cdot \eta\Psi_n(x, y). \end{aligned}$$

Since the Witt-valued Brouwer degree of (x, y) is $1 \in W(k)$ and $\eta\Psi_n$ is $GW(k)$ -linear, we can identify $\eta\Psi_n$ with the projection $GW(k) \rightarrow W(k)$ above and the theorem follows. \square

Corollary 3.5. *If k is infinite perfect with $\text{char}(k) \neq 2$, then Ψ_3 modulo SL is a bijection for $R = S_5$. In particular, $Um_3(S_5)/SL_3(S_5) \xrightarrow{\cong} U_5(S_5)/Spin_6(S_5)$ is a bijection.*

Proof. This follows immediately from Proposition 2.7 and Theorem 3.4. \square

Theorem 3.6. *If $R = S_7$ and k is infinite perfect with $\text{char}(k) \neq 2$, then $E_n(R)$ acts transitively on $Um_n(R)$ for $n \geq 5$; furthermore, $SL_4(R)e_4 \neq Sp_4(R)e_4$, where $e_4 = (1, 0, 0, 0)^t \in R^4$.*

Proof. The first statement is clear as $Um_n(R)/E_n(R) \cong [Q_7, Q_{2n-1}]_{\mathcal{H}(k)} = *$ for $n \geq 5$ by [12, Corollary 5.43].

For the second statement, recall that $Um_4(R)/SL_4(R) \cong \mathcal{V}_3^o(R) \cong \mathbb{Z}/3!\mathbb{Z}$ by [1, Theorem 4.8], where the representatives for the $3! = 6$ distinct orbits are exactly given by the vectors $(x_1^m, x_2, x_3, x_4)^t$ for $1 \leq m \leq 6$. By Suslin’s famous theorem on completion of unimodular vectors, the vectors $(x_1^6, x_2, x_3, x_4)^t$ and $(x_1^{12}, x_2, x_3, x_4)^t$ are completable to an invertible matrix of determinant 1.

In particular, the vectors $(x_1^6, x_2, x_3, x_4)^t$ and $(x_1^{12}, x_2, x_3, x_4)^t$ are in the $SL_4(R)$ -orbit of $e_4 = (1, 0, 0, 0)^t \in R^4$. We show that at least one of these vectors is not in the $Sp_4(R)$ -orbit of e_4 : Assume that both of them are in fact in the $Sp_4(R)$ -orbit of e_4 . Then there would be a symplectic matrix transforming the unimodular vector $(x_1^6, x_2, x_3, x_4)^t$ into $(x_1^{12}, x_2, x_3, x_4)^t$. But by [7, Proposition 3.5] this would mean that the associated symplectic modules of rank 2 are isomorphic, which is not true by [1, Remark 4.15]. This proves the second statement. \square

Theorem 3.7. *If $R = S_7$ and k is infinite perfect with $\text{char}(k) \neq 2$, then the comparison map $Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R)$ is not injective (and not bijective) and Ψ_3 modulo SL induces a bijection $U_5(S_7)/Spin_6(R) \xrightarrow{\cong} W_{SL}(R)$.*

Proof. Consider the factorization

$$\Psi_3 : Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R) \rightarrow \tilde{A}_4(R)/SL_4(R) \rightarrow W_{SL}(R).$$

By Remark 2.8, the map $U_5(R)/Spin_6(R) \rightarrow \tilde{A}_4(R)/SL_4(R)$ is bijective and Ψ_3 modulo SL coincides with the Vaserstein symbol modulo SL associated to $\theta : R \xrightarrow{\cong} \det(R^2), 1 \mapsto e_1 \wedge e_2$, via the canonical identification $\tilde{V}_{SL}(R) \cong W_{SL}(R)$ from [18, Section 2.C]. As $E_n(R)$ acts transitively on $Um_n(R)$ for $n \geq 5$ by Theorem 3.6, the Vaserstein symbol modulo SL is surjective (cf. [17, Theorem 4.5]); hence the same holds for the map $\tilde{A}_4(R)/SL_4(R) \rightarrow W_{SL}(R)$. As $E_{2n}(R)$ acts transitively on $Um_{2n}(R)$ for $n \geq 3$, also the group $Sp(\chi)$ acts transitively on $Um_{2n}(R)$ for $n \geq 3$ and for any invertible alternating matrix χ of rank $2n$ (cf. [16, Lemma 5.5]). Therefore the injectivity of the map $\tilde{A}_4(R)/SL_4(R) \rightarrow W_{SL}(R)$ follows from [18, Lemma 3.4]; note that [18, Lemma 3.4] holds for any commutative ring and does not use the dimension assumption at the beginning of [18, Section 3.B]. So the map $\tilde{A}_4(R)/SL_4(R) \rightarrow W_{SL}(R)$ is bijective.

Similarly, as $E_n(R)$ acts transitively on $Um_n(R)$ for $n \geq 5$, the statement of [18, Theorem 3.6] still holds for R even though $\dim(R) > 4$. In particular, it follows easily that the map Ψ_3 modulo SL can

only be injective if $SL_4(R)e_4 = Sp_4(R)e_4$, where $e_4 = (1, 0, 0, 0)^t \in R^4$. But, by Theorem 3.6 above, we already know that $SL_4(R)e_4 \neq Sp_4(R)e_4$. So Ψ_3 modulo SL is not injective and hence $Um_3(R)/SL_3(R) \rightarrow U_5(R)/Spin_6(R)$ cannot be injective either. This finishes the proof. \square

CRedit authorship contribution statement

Tariq Syed: Writing – review & editing, Writing – original draft, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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