Periodic Higgs-de Rham flows and representations of algebraic fundamental groups

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Abstract

Let $k := \bar{\mathbb{F}}_p$ for $p > 2$, $W_n(k) := W(k)/p^n$ and X_n be a projective smooth $W_n(k)$-scheme that is $W_{n+1}(k)$-liftable. For all $n \geq 1$, we construct explicitly a functor, which we call the inverse Cartier functor, from a subcategory of Higgs bundles over X_n to a subcategory of flat Bundles over X_n. Then we introduce the notion of periodic Higgs-de Rham flows and show that a periodic Higgs-de Rham flow is equivalent to a Fontaine-Faltings module. Together with a p-adic analogue of Riemann-Hilbert correspondence established by Faltings, we obtain a coarse p-adic Simpson correspondence.

Zusammenfassung

Sei $k := \bar{\mathbb{F}}_p$, $p > 2$, $W_n(k) := W(k)/p^n$, und X_n ein projektives glattes $W_n(k)$-Schema, das $W_{n+1}(k)$-aufhebbar ist. Für alle $n \geq 1$, konstruieren wir ausdrücklich einen Funktor, den wir inversen Cartier-Funktor nennen, von einer Unterkategorie der Higgs-Bündel über X_n, nach einer Unterkategorie der glatten Bündel über X_n. Dann führen wir den Begriff der periodischen Higgs-de Rham Flüsse ein und zeigen, dass ein periodischer Higgs-de Rham Flüss einem Fontaine-Faltings-Modul entspricht. Zusammen mit einem p-adischen Analogon der Riemann-Hilbert-Korrespondenz, wie Faltings vorgeschlagen hat, erhalten wir eine grobe p-adische Simpson-Korrespondenz.

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1 Introduction

N. Hitchin [Hi] introduced rank two stable Higgs bundles over a compact Riemann surface X and showed that they correspond naturally to irreducible representations of the fundamental group $\pi_1(X)$ by solving Yang-Mills equations, which has generalized the earlier works of Donaldson, Unblenbeck-Yau for polystable vector bundles. Later, C. Simpson[Si] obtained the full correspondence for polystable Higgs bundles over complex projective manifolds of any dimension.

As a matter of fact, Simpson correspondence consists of two steps: the first step is the Riemann-Hilbert correspondence from complex polarized variations of Hodge structures ($\mathbb{C}$-PVHS) on a complex manifold X to monodromy representations of $\pi_1(X)$; the second step, ascribed to Simpson, is a correspondence from systems of Hodge bundles, which are polystable Higgs bundles with trivial Chern classes, to $\mathbb{C}$-PVHS.

Recall that a $\mathbb{C}$-PVHS on a complex manifold X is a quadruple $(H, \nabla, Fil^{\cdot}, \varphi)$ that consists of a vector bundle H over X with an integrable connection ∇, a decreasing filtration $Fil^{\cdot}$ of H by holomorphic subbundles that satisfies the Griffiths transversality with respect to ∇, and a horizontal bilinear form φ satisfying the Hodge-Riemann bilinear relation. We call (H, ∇) flat bundle and $Fil^{\cdot}$ Hodge filtration. Systems of Hodge bundles are a special type of Higgs bundles that can be decomposed as a direct sum as follows:

$$(E, \theta) \quad = \quad (\oplus_{i+j=n} E^{i,j}, \oplus_{i+j=n} \theta^{i,j}),$$

where for $1 \leq i \leq n$, $\theta^{i,j}$ is an $\mathcal{O}_X$-linear map:

$$\theta^{i,j} : E^{i,j} \to E^{i-1,j+1} \otimes \Omega_X,$$

and $\theta^{0,n}$ is the zero map.

Starting with a polystable system of Hodge bundles (E, θ) with trivial Chern classes, by solving the Yang-Mills-Higgs equation on (E, θ), Simpson obtained a solution guaranteed by the stability condition, which are called Yang-Mills-Higgs metric h on E; the metric connection of h together with θ equips E with an integrable connection ∇ and a filtration $Fil^{\cdot}$, while the alternate sum of h with respect to the Hodge decomposition of E gives a polarization φ. Therefore, a $\mathbb{C}$-PVHS is associated to the system of Hodge bundles (E,θ).

It has been attempted by many to establish a p-adic Simpson correspondence over a smooth proper $W(k)$-scheme $\mathbf{X}$, where $k=\bar{\mathbb{F}}_p$ for some odd prime number p. In particular, G. Faltings has accomplished the first step, i.e., a p-adic analogue of the Riemann-Hilbert correspondence. The p-adic analogue of $\mathbb{C}$-PVHS introduced by Faltings in [Fa] is a quadruple $(H,\nabla,Fil^{\cdot},\varphi)$, which we call Fontaine-Faltings module, consisting of a vector bundle H over $\mathbf{X}$ with an integrable connection ∇, a decreasing filtration $Fil^{\cdot}$ of H with length n that satisfies the Griffiths transversality, and the relative Frobenius map φ which is horizontal and satisfies p-divisible property.

Moreover, when $n\leq p-2$, Faltings has shown that the category of Fontaine-Faltings modules $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$ is equivalent to the category of dual-crystalline sheaves, which is a full subcategory of the category of étale local systems over the generic fiber $\mathbf{X}^0$. These results are geometric analogues of the arithmetic ones obtained earlier by Fontaine-Laffaille [FL] on the category of integral p-adic representations of a local Galois group, and can be regarded as a p-adic analogue of the classical Riemann-Hilbert correspondence. We have been motivated to build a p-adic Simpson correspondence by studying relations between Fontaine-Faltings modules and systems of Hodge bundles. In the thesis, we are aimed to establish a correspondence from systems of Hodge bundles over $\mathbf{X}$ to Fontaine-Faltings modules.

We work inductively on $W_n(k):=W(k)/p^n$ instead of on $W(k)$. Let X_1 be a smooth projective scheme over k with $W_2(k)$-liftings. Our work begins with an explicit construction of the inverse Cartier functor from the category of nilpotent Higgs bundles of order less than p over X_1 to the category of flat bundles with nilpotent p-curvatures of order less than p over X_1. The inverse Cartier gives an equivalence between the two categories. The same equivalence has also been obtained by Ogus-Vologodsky [OV] abstractly by using Azumaya algebras; the inverse Cartier in [OV] is different from ours only by a minus sign. We will construct the inverse Cartier in two steps: first, locally define the inverse Cartier functor; second, define transition morphisms to glue the local inverse Cartier functor into a global one. The definition of the transition morphisms makes use of an important lemma of Deligne-Illusie, which roughly says that the obstruction to the lifting of the absolute Frobenius morphism of X_1/k to a $W_2(k)$-lifting X_2 lies in $H^1(X_1,F_0^*T_{X_1})$. An advantage of our method is that one can see immediately that the inverse Cartier functor has a close relation with the Fontaine-Faltings modules, since the transition morphism in some sense can be viewed as a characteristic-p version of the Taylor formula appeared in the definition of Fontaine-Faltings modules. In fact, our idea arises from the interpretation of Fontaine-Faltings modules and our construction is designed to be related to Fontaine-Faltings modules.

Furthermore, we construct explicitly the inverse Cartier over $W_n(k)$, whose existence seems unclear from the method of Ogus-Vologodsky. A significant observation in our construction is that to a de Rham bundle $(H, \nabla, Fil^{\cdot})$ over X_n, we can associate a twisted flat bundle $(\tilde{H}, \tilde{\nabla})$ which has p-connection and can be viewed as an intermediate object between $(H, \nabla, Fil^{\cdot})$ and the associated graded Higgs bundle $\mathrm{Gr}_{Fil^{\cdot}}(H, \nabla)$. Actually, the key point is that the twisted flat bundle $(\tilde{H}, \tilde{\nabla})$ can be reconstructed from $\mathrm{Gr}_{Fil^{\cdot}}(H, \nabla)$ and the reduction modulo p^{n-1} of $(H, \nabla, Fil^{\cdot})$. Such a type of twisted flat bundles, due to the p-divisible property, has already appeared in the formulation of Fontaine-Faltings module. To obtain the inverse Cartier functor, we fix the Frobenius pullback of the twisted flat bundles as local models and glue them by Taylor formula.

Once the inverse Cartier functor over $W_n(k)$ becomes available, the notion of periodic Higgs-de Rham flow comes up naturally, and we prove that the category of periodic Higgs-de Rham flows is equivalent to that of Fontaine-Faltings modules. Therefore, we obtain a p-adic Simpson correspondence in some sense. In the characteristic-p case, beyond the scope of this thesis, a correspondence involving stability has been further developed in [LSZ], where the notion of strongly semistable Higgs bundle is introduced and it is shown that a Higgs bundle is preperiodic (become periodic after some steps) if and only if it is strongly semistable with trivial Chern classes; moreover, it is proved that for a system of Hodge bundles of rank $\leq p$, semistability is equivalent to strong semistability. The latter has also been proved by Adrian Langer independently with a slightly different approach (see [AL1]).

In the end, we would like to mention that the concept of Higgs-de Rham flows in the characteristic-p case has played an important role in Adrian Langer's proof of Bogomolov-Giesecker inequality and Miyaoka-Yau inequality in characteristic p (see [AL2]).

The thesis is organized as follows. In Chapter 2, we briefly review Faltings' theory [Fa]. Section 2.1 is on the category $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$; Section 2.2 is on the periodic ring $B^+(R)$; Section 2.3 is on the functor from $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$ to the category of representations of $\pi_1(\mathbf{X^0})$, in particular, it is shown to be fully faithful.

In Chapter 3, we discuss the inverse Cartier functor and Cartier transform functors in characteristic p. The construction of the inverse Cartier functor and Cartier transform functor are given respectively in Sections 3.1 and 3.3. A discussion on the relation of the inverse Cartier and Fontaine-Faltings modules is made in Section 3.2. In Section 3.4, we show that our construction is equivalent to the one of Ougs-Vologodsky up to a minus sign.

In Chapter 4, we discuss periodic Higgs-de Rham flows in characteristic p. In Section 4.1, we introduce the notion of periodic Higgs-de Rham flows and show that a 1-periodic Higgs-de Rham flow is equivalent to a strict p-torsion Fontaine-Faltings modules. In Section 4.2, we generalize this equivalence to the case when the period-1 condition is dropped.

In Chapter 5, we generalize the results from Chapters 3 and 4 to the W_n-level. In Section 5.1, we construct the inverse Cartier functor in the W_n-case. In Section 5.2, we discuss periodic Higgs-de Rham flows in the W_n-case.

In Chapter 6, we discuss rudimentary results of our theory. What is worth mentioning is the full faithfulness of the grading functor in Theorem 6.0.7.

This thesis is a part of the joint work with my supervisor Prof. Kang Zuo and Prof. Mao Sheng.

2 Falting's theory

2.1 $\mathfrak{MF}$ categories

Let $k := \overline{\mathbb{F}}_p$ for some prime $p > 2$, $V = W(k)$, and K be its fraction field. We call a V- algebra R to be small if there exists an étale map $V[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \to R$. Let R be a small V- algebra and $\overline{R}$ be the normalization of R over the maximal étale extension of $R[\frac{1}{p}]$. Denote by $\widehat{R}$ (resp. $\widehat{\overline{R}}$) the p-adic completion of R (resp. $\overline{R}$).

First recall the categories $\mathfrak{MF}_{big}(R)$, $\mathfrak{MF}(R)$, $\mathfrak{MF}^{\nabla}(R)$ from [Fa]. Since R is a smooth V-algebra, we can choose a semilinear endomorphism $\Phi : \widehat{R} \to \widehat{R}$ such that it is a lift of the absolute Frobenius of R/pR. Note that Φ induces a map $d\Phi_* : \Omega_{R/V} \otimes_{R,\Phi} \widehat{R} \to \Omega_{R/V} \otimes_R \widehat{R}$, which is divisible by p. An object of $\mathfrak{MF}_{big}(R)$ consists of a p-torsion R-module M (here p-torsion means every element is annihilate by p^n for some $n > 0$), a sequence of p-torsion R-module $F^i(M)$, and sequences of R-linear maps $F^i(M) \to F^{i-1}(M)$, $F^i(M) \to M$, and $\varphi^i : F^i(M) \otimes_{R,\Phi} R \to M$, subject to the following conditions:

i) The composition $F^i(M) \to F^{i-1}(M) \to M$ is the map $F^i(M) \to M$;

ii) The map $F^i(M) \to M$ is an isomorphism for $i \ll 0$;

iii) The composition of φ^{i-1} with $F^i(M) \to F^{i-1}(M)$ is $p\varphi^i$.

Morphisms between objects are morphisms of R-modules satisfying obvious conditions, and this makes $\mathfrak{MF}_{big}(R)$ an abelian category. We define $\tilde{M}$ as $\oplus F^i(M)$ modulo an equivalent relation. The equivalent relation is as follows: for $m \in F^i M$, the image of m in $F^i(M) \to F^{i-1}(M)$ is equivalent to $p \cdot m \in F^i(M)$. Now axiom iii) is equivalent to the fact that φ induces a Φ-linear map $\tilde{M} \to M$, or an R-linear map

$$\varphi : \tilde{M} \otimes_{R,\Phi} R \to M.$$

Remark 2.1.1. The image of R under Φ may not be contained in R. However, for p-torsion module M, it makes sense to say $M \otimes_{R,\Phi} R$. In fact, if M is annihilated by p^n, then M is a R/p^nR-module. As $R/p^nR = \widehat{R}/p^n\widehat{R}$, Φ induces a map $\Phi_n : R/p^nR \to R/p^nR$. So $M \otimes_{R,\Phi} R$ is defined to be $M \otimes_{R/p^nR,\ \Phi_n} R/p^nR$.

We define $\mathfrak{MF}(R)$ as the full subcategory of $\mathfrak{MF}_{big}(R)$ whose objects are tuples

$$\{M, F^i(M), \varphi\}$$

such that M and all $F^i(M)$ are finitely generated p-torsion R-modules; $F^i(M) = 0$ for $i \ll 0$; $F^i(M) \to F^{i-1}(M) \to M$ are injections; and φ induces an isomorphism $\tilde{M} \otimes_{R,\Phi} R \to M$. $\mathfrak{MF}(R)$ is an abelian subcategory of $\mathfrak{MF}_{big}(R)$. Note that $\mathfrak{MF}(R)$ depends on the choice of the Frobenius-lift Φ.

Next we introduce integrable connections: an object of $\mathfrak{MF}^{\nabla}(R)$ is an object $\{M, F^i(M), \varphi\}$ of $\mathfrak{MF}(R)$ together with an integrable connection $\nabla : M \to M \otimes_R \Omega_{R/V}$ such that it satisfies Griffiths-transversality :

$$\nabla(F^i(M)) \subset F^{i-1}(M) \otimes_R \Omega_{R/V},$$

and the maps $\varphi^i : F^i(M) \otimes_{R,\Phi} R \to M$ are parallel. Being parallel means that

$$\nabla \circ \varphi^i = (\varphi^{i-1} \otimes_R \frac{d\Phi_*}{p}) \circ \nabla : F^i(M) \otimes_{R,\Phi} R \to F^{i-1}(M) \otimes_R \Omega_{R/V}.$$

Another way to look at this is that ∇ on M induces an integrable connection $\tilde{\nabla}$ on $\tilde{M} \otimes_{R,\Phi} R$, as follows: For $m \in F^i(M)$, $m \otimes 1$ is mapped to $(1 \otimes \frac{d\Phi_*}{p})(\nabla(m)) \in F^{i-1}(M) \otimes_{R,\Phi} \Omega_{R/V}$.

Then the parallel condition is equivalent to that $\varphi : \tilde{M} \otimes_{R,\Phi} R \to M$ are horizontal. We can easily see that the connection on $\tilde{M} \otimes_{R,\Phi} R$ is nilpotent, so it follows that ∇ is nilpotent.

For integers $a \le b$, we define $\mathfrak{MF}^{\nabla}_{[a,b]}(R)$ as the full subcategory of $\mathfrak{MF}^{\nabla}(R)$ consisting of objects with $F^a(M) = M$, $F^{b+1} = (0)$. We claim that for $b - a < p$, the category $\mathfrak{MF}^{\nabla}_{[a,b]}(R)$ is independent of the choice of Φ.

Theorem 2.1.2. *Assume $0 \le b - a \le p - 1$, and $p > 2$. Then for any two choices of Φ there is an equivalence between the corresponding categories $\mathfrak{MF}^{\nabla}_{[a,b]}(R)$. These equivalences satisfy the obvious cocycle condition, so that up to a canonical isomorphism $\mathfrak{MF}^{\nabla}_{[a,b]}(R)$ is independent of the choice of Φ.*

Proof. We may assume that $a = 0$, $b = p - 1$. Suppose we have two Frobenius lifts Φ and Ψ, we want to define an isomorphism

$$\alpha : \tilde{M} \otimes_{R,\Phi} R \to \tilde{M} \otimes_{R,\Psi} R.$$

Choose local coordinates $\{t_1, \dots t_d\}$, i.e., an étale morphism $V[t_1, \dots, t_d] \to R$. Let $\partial_i = \frac{\partial}{\partial t_i}$ denote the dual base of R-derivations. Via ∇ these operate on M, and for any multi-index $J = (j_1, \dots, j_d)$ we get an endomorphism $\nabla(\partial)^J \triangleq (\nabla_{\partial_1})^{j_1} \cdots (\nabla_{\partial_d})^{j_d}$ of M. Also $(\Phi(t) - \Psi(t))^J$ denotes the monomial $\prod_{i=1}^d (\Phi(t_i) - \Psi(t_i))^{j_i}$, it is divisible by $p^{|J|}$, $|J| = j_1 + \cdots + j_d$ is the order of J. Finally $J! = j_1! \cdots j_d!$.

Now for $m \in F^i(M)$ viewed as an element of $\tilde{M}$, its image under α is given by the formula

$$\alpha(m \otimes 1) = \sum_J \nabla(\partial)^J(m) \otimes (\Phi(t) - \Psi(t))^J / (J! \cdot p^{min\{|J|,i\}}).$$

Here $\nabla(\partial)^J(m)$ is considered as an element of $F^{max\{0,i-|J|\}}(M)$, and the sum is over all multi-indices J. Note that the terms $(\Phi(t) - \Psi(t))^J / (J! \cdot p^{min\{|J|,i\}})$ are elements of R and converge to zero in the p-adic topology, so that the sum is finite.

Finally we can verify the following properties:

- Using the Taylor's formula : $\Phi(r) = \sum_J \Psi(\partial^J(r)) \otimes (\Phi(t) - \Psi(t))^J / J!$ for $r \in R$, we see that α defines an R-linear isomorphism from $\tilde{M} \otimes_{R,\Phi} R$ to $\tilde{M} \otimes_{R,\Psi} R$;
- For three different Frobenius-lifts Φ_1, Φ_2, Φ_3 the α's satisfy transitivity;
- α is independent of the choice of local coordinates $t_1 \dots, t_d$;
- α is parallel with respect to the connections.

□

As an application of the above theorem, we can define $\mathfrak{MF}^{\nabla}_{[a,b]}(X)$ for any smooth V-scheme X, by gluing the data obtained from local Frobenius lifts Φ. So an object of $\mathfrak{MF}^{\nabla}_{[a,b]}(X)$ is a globally defined de Rham bundle $(H, \nabla, Fil^{\cdot})$ such that for each affine open sets U with a choice of Frobenius lifting $F_{\hat{U}} : \hat{U} \to \hat{U}$, there is a morphism φ_U; and for difference choice of open subschemes and Frobenius liftings, the φ_U's are related by the Taylor formula. We call objects of $\mathfrak{MF}^{\nabla}_{[a,b]}(X)$ Fontaine-Faltings modules.

2.2 The period ring $B^+(R)$

We have the following lemma ([Lo] Corollary 2.3).

Lemma 2.2.1. *The absolute Frobenius map $\overline{R}/p\overline{R} \to \overline{R}/p\overline{R}$ is surjective.*

Proof. Let $\overline{K}$ denote an algebraic closure of K, and $\overline{V}$ be the integral closure of V. Let R_∞ be the subring of $\overline{R}$, generated by p-power roots $\{T_i^{\pm\frac{1}{p^n}} | n \in \mathbb{N}, 1 \leq i \leq d\}$ over $R \otimes_V \overline{V}$. We see that the absolute Frobenius map on R_∞/pR_∞ is surjective. By Faltings' almost purity theorem, $\overline{R}$ is almost étale over R_∞. It implies that the absolute Frobenius map on $\overline{R}/p\overline{R}$ is an almost isomorphism (see[GR] Theorem 3.5.13). That is for $x \in \overline{R}$, there exists $y, z \in \overline{R}$, such that

$$p^{\frac{1}{2}} \cdot x = y^p + p \cdot z.$$

Set

$$w := y \cdot p^{-\frac{1}{2p}} \in \overline{R}[\frac{1}{p}].$$

Then we have

$$w^p = y^p \cdot p^{-\frac{1}{2}} = x - p^{\frac{1}{2}} \cdot z \in \overline{R}.$$

As $\overline{R}$ is integrally closed, $w \in \overline{R}$, then

$$x = w^p + p^{\frac{1}{2}} \cdot z.$$

Apply the same trick to z, we obtain $u, v \in \overline{R}$, such that

$$z = u^p + p^{\frac{1}{2}} \cdot v.$$

So we have

$$x \equiv (w + p^{\frac{1}{2p}} u)^p \ mod\ p.$$

$\square$

We will construct $B^+(R)$ following Fontaine's method. Consider the ring

$$S = \lim_{\leftarrow}(\overline{R}/p\overline{R}).$$

Where the limit is taken over a projective system of rings indexed by $\mathbb{N}$, with all transition-maps $\overline{R}/p\overline{R} \to \overline{R}/p\overline{R}$ given by absolute Frobenius maps. So elements of S are sequences $\{r_n \in \overline{R}/p\overline{R} | n \in$

$\mathbb{N}\}$ such that $r_n = r_{n+1}^p$. Since S is a ring of characteristic p, the absolute Frobenius map **Fr** is bijective on S.

We now form the ring of Witt-vectors $W(S)$. And we can define a homomorphism $\theta : W(S) \rightarrow \widehat{\overline{R}}$ as follows: if $[s_0, s_1, \ldots]$ is an element of $W(S)$, with each s_n as a sequence $\{s_{nm} \in \overline{R}/p\overline{R} | m \geq 0\}$. Lift the s_{nm} to elements r_{nm} of $\overline{R}$, and consider

$$r_{0m}^{p^m} + p \cdot r_{1m}^{p^{m-1}} + p^2 \cdot r_{2m}^{p^{m-2}} + \cdots p^m \cdot r_{mm} \in \overline{R}/p^{m+1}\overline{R}.$$

when varying m these form a compatible projective system, hence we obtain the desired ring-homomorphism $W(S) \rightarrow \widehat{\overline{R}}$. Since the Frobenius is surjective on $\overline{R}/p\overline{R}$, then θ is surjective. We denote by I the kernel of θ. Note that I is a principal ideal. If we choose a sequence of p-power roots of p in $\overline{V} \subset \overline{R}$, that is $\{\nu_n, n \geq 0\}$, with $\nu_0 = p$, and $\nu_n = \nu_{n+1}^p$. Then reducing the ν_n modulo p, we define an element $\underline{p} \in S$. Similarly we define an element $\underline{-1} \in S$, by taking p-power roots of -1. Let $\xi = [\underline{p}, (\underline{-1})^p, 0, 0, \ldots]$. Then I is generated by ξ.

Now we construct the divided power hull $D_I(W(S))$. Then $B^+(R)$ is defined as the completion of $D_I(W(S))$ with respect to the topology defined by the divided power ideals. So it is a $B^+(V)$ algebra, which in turn is an algebra over V. And the Galois-group $Gal(\overline{R}/R)$ operates continuously on $B^+(R)$. Note that the Galois invariant of $B^+(R)$ in general is not equal to R. And by the same calculation as in [FL], the Frobenius **Fr** can be extended from $W(S)$ to $B^+(R)$. We define a decreasing filtration $F^{\cdot}$ on $B^+(R)$ by defining F^n to be the n-th divided power ideal of I. Then $B^+(R) = \varprojlim B^+(R)/F^n(B^+(R))$. And $gr_F^0(B^+(R)) = \widehat{\overline{R}}$.

There exists a canonical map $\alpha : \mathbb{Z}_p(1) \rightarrow B^+(R)^*$ defined as follows : $\mathbb{Z}_p(1)$ can be described as the multiplicative groups of sequences of p-powers roots $\{\zeta_n, n \geq 0\}$, with $\zeta_n = \zeta_{n+1}^p$. The reduction modulo p of such a sequence defines an element $\underline{\zeta}$ of S, and by Teichmüller lifting we can associate $[\underline{\zeta}, 0 \ldots] \in W(S)$ to it. Note that α is Galois equivariant, and $\varphi(\alpha(x)) = \alpha(x)^p$. Also $\alpha(\mathbb{Z}_p(1)) \subset 1 + I$, so taking logarithmic defines an additive map $\beta : \mathbb{Z}_p(1) \rightarrow F^1$. Again β is Galois equivariant and $\varphi(\beta(x)) = p\beta(x)$. If x is a generator of $\mathbb{Z}_p(1)$,then $\frac{\beta(x)}{x} \in B^+(R)$, and θ maps it to an element of the form $u \cdot p^{\frac{1}{p-1}}$, where u is an unit .

2.3 Functors between Fontaine-Faltings modules and Galois representations

First we are going to define a functor from $\mathfrak{MF}^{\nabla}(R)$ to the category of Galois-representations as in [FL] does. Let $D = B^+(R)[\frac{1}{p}]/B^+(R)$, it has a filtration $F^i(D)$ and the maps $\varphi^i = \mathbf{Fr}/p^i$ are from $F^i(D)$ to D. But it is not in $\mathfrak{MF}_{big}(R)$, because there is no natural R-module structure. However, as R is naturally contained in $B^+(R)/F^1(B^+(R))$, and as R is smooth over V, we can lift the inclusion above to a V-linear map $R \to B^+(R)$. Furthermore, as $F^1(B^+(R))$ admits divided powers, for any R-module M with integrable connection, the tensor product $M \otimes_R B^+(R)$ is up to canonical isomorphism independent of the choice of the lift, by applying Taylor formula similarly as Section 2.1. To any object $M \in \mathfrak{MF}^{\nabla}(R)$, we can associate an object $M \otimes_R B^+(R)$ with a filtration. As $(\tilde{M} \otimes_{R,\Phi} R) \otimes_R B^+(R)$ is canonically isomorphic to $(M \otimes_R B^+(R))\tilde{} \otimes_{B^+(R),\mathbf{Fr}} B^+(R)$, it follows that the map $\varphi : \tilde{M} \otimes_{R,\Phi} R \to M$ induces a map $(M \otimes_R B^+(R))\tilde{} \to M \otimes_R B^+(R)$. So we can define $\mathbf{D}(M) = Hom(M \otimes_R B^+(R), D)$, where the homomorphisms are $B^+(R)$-linear, preserve filtrations and commute with the φ's.

We want to define an action of the Galois group $\Gamma \triangleq \mathrm{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}])$ on $M \otimes_R B^+(R)$. If we choose local parameters $\{t_1, \ldots, t_d\}$, the action on the p-power roots of the $\{t_i\}$, defines a homomorphism $\Gamma \to \mathbb{Z}_p(1)^d$. For $\sigma \in \Gamma$, we denote the components of its image by $\sigma_i \in \mathbb{Z}_p(1)$, we may apply the homomorphism β defined in section 2. If we let ∂_i denote the derivation dual to $d\log(t_i) = \frac{dt_i}{t_i}$. Then we associate σ a vector fields valued in $B^+(R)$, by $\tau(\sigma) \triangleq \sum \partial_i \beta(\sigma_i)$. Then $\sigma \in \Gamma$ defines a map $\exp(\nabla_{\tau(\sigma)})$ from M to $M \otimes_R B^+(R)$. Since Γ acts on $B^+(R)$, we extend the above to a Γ action on $M \otimes_R B^+(R)$ compatible with the action on $B^+(R)$. This also defines a Galois-action on $\mathbf{D}(M)$.

In summary, we define a functor $\mathbf{D}$ from the category $\mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$ to the category of $\mathbf{Z}_p$-Γ modules.

Lemma 2.3.1. *suppose $M \in \mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$ then:*

1. *if $pM = (0)$, L is an $\mathbb{F}_p$-Γ module, and we have a map $L \to \mathbf{D}(M)$ which does not factor over $\mathbf{D}(M/N)$ for any nontrivial quotient M/N of M in $M \in \mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$. Then $rank_{R/pR}(M) \leq dim_{\mathbb{F}_p}(L)$, and equality implies $L \cong \mathbf{D}(M)$.*

2. *The same holds for arbitrary $M \in \mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$ and finite $\mathbb{Z}_p$-Γ modules L, provided we*

replace $\dim_{\mathbb{F}_p}$ *by length* $_{\mathbb{Z}_p}$ *and rank by length after localization in the prime-ideal* pR.

We define an adjoint functor $\mathbf{E}$ of $\mathbf{D}$ by the rule $\mathbf{E}(L) = \varinjlim\{M | L \to \mathbf{D}(M)\}$, where the direct limit is taken over all pairs consisting of $M \in \mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$ and a map $L \to \mathbf{D}(M)$, and the transition maps are the obvious ones. It is enough to consider pairs such that the map does not factor $\mathbf{D}(M/N)$ for a nontrivial quotient M/N of M, and these form an ordered set. By lemma 2.3.1 this ordered set has a maximal element which we call $\mathbf{E}(L)$, and we know that its length is less than or equal to the length of L, with equality holding only if $L \cong \mathbf{D}(M)$. Quite formally it follows that $Hom(M, \mathbf{E}(L)) \cong Hom(L, \mathbf{D}(M))$, where the left side denotes a homomorphism in $\mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$, and on the right side we have the Galois linear maps. As $\mathbf{E}$ is left exact, and $\mathbf{D}$ is faithful, it implies that the map η from M into $\mathbf{E}(\mathbf{D}(M))$ is always injective. By the length inequality, we derive that η is an isomorphism, and so $\mathbf{D}$ is fully faithful. Finally the image of $\mathbf{D}$ consists of those L such that the length of $\mathbf{E}(L)$ is equal the length of L. As we have always the length inequality, and as $\mathbf{E}$ is left exact, it follows that the image of $\mathbf{D}$ is closed under taking subobjects and quotients, which we call dual-crystalline representations.

Theorem 2.3.2. $\mathbf{D}$ *induces an equivalence between* $\mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$ *and the full subcategory of finite* $\mathbb{Z}_p$-Γ *modules whose objects are dual-crystalline representations. This subcategory is closed under taking subobjects and quotients.*

Assume that X is a proper smooth V-scheme. Cover X by open affine subschemes Spec(R) with R small. The categories $\mathfrak{MF}^{\nabla}_{[0,p-2]}(R)$ are glued to a category $\mathfrak{MF}^{\nabla}_{[0,p-2]}(X)$, and the functors $\mathbf{D}$ associate to each object of this category a compatible system of étale sheaves on $\mathrm{spec}(R[\frac{1}{p}])$. These can be glued into a locally constant sheaf on $X \otimes_V K$. It follows that

Theorem 2.3.3. *Suppose* X *is a proper smooth* V*-scheme. There exists a fully faithful contravariant functor* $\mathbf{D}$ *from* $\mathfrak{MF}^{\nabla}_{[0,p-2]}(X)$ *to the category of locally étale* p*-torsion sheaves on* $X \otimes_V K$*. The image is closed under taking subobjects and quotients. Locally* $\mathbf{D}$ *is given by the procedure above.*

3 Inverse Cartier and Cartier transform in char p

Let X_1 be a smooth connected algebraic variety over k of dimension d. Let $W_2 := W_2(k)$. Assume that X_1 has a W_2-lifting, i.e., a smooth W_2-scheme X_2 with closed fiber X_1.

A Higgs bundle (E, θ) over X_1 is a vector bundle E with an integrable Higgs field θ, which is an $\mathcal{O}_{X_1}$-linear morphism

$$\theta : E \to E \otimes \Omega_{X_1}.$$

Locally over an open affine subscheme $U \subset X_1$ with local coordinates $\{t_1, \cdots, t_d\}$, the Higgs field θ can be written as $\theta = \sum \theta_i dt_i$, where $\theta_i = \partial_{t_i} \lrcorner \theta$ is an endomorphism of E. The integrability of the Higgs field is then equivalent to the commuting relations $[\theta_i, \theta_j] = 0$ for $1 \leq i < j \leq d$. And We say that the Higgs bundle is nilpotent of order less than p, if $\prod_{j=1}^d \theta_j^{i_j} = 0$ once $\sum_{j=1}^d i_j \geq p$. Let $HIG_{p-1}(X_1)$ denote the category of Higgs bundles of order less than p over X_1.

A flat bundle (H, ∇) over X_1 is a vector bundle H with a integrable connection ∇, which is a k-derivation

$$\nabla : H \to H \otimes \Omega_{X_1}.$$

We recall some basics on connection and curvature. Take a local basis e_U of sections of H over U. Then over U, $\nabla = d + A_U$, where A_U is a matrix-valued one-form given by $\nabla(e_U) = A_U e_U$. More precisely, the formula means that for a local section $s = s_U e_U$ of H over U, where s_U is a row vector with elements in $\mathcal{O}_{X_1}(U)$, one has $\nabla(s) = ds_U e_U + s_U A_U e_U$. Let A'_U be the connection one-form of ∇ under another local basis $e'_U = M e_U$ of H over U, where M is an invertible matrix with entries in $\mathcal{O}_{X_1}(U)$. Then one has the transformation formula:

$$A'_U = dMM^{-1} + MA_U M^{-1}.$$

Finally, the curvature K of ∇ is defined to the composite

$$H \xrightarrow{\nabla} H \otimes \Omega_{X_1} \xrightarrow{\nabla^1} H \otimes \Omega^2_{X_1},$$

where $\nabla^1(h \otimes \omega) = \nabla(h) \wedge \omega + h \otimes d\omega$. Note that K is $\mathcal{O}_{X_1}$-linear and under the local basis e_U of H over U is expressed by the formula $K_U = dA_U + A_U \wedge A_U$. We say that the connection is integrable if its curvature is zero. Let $MIC(X_1)$ denote the category of flat bundles over X_1.

3.1 Inverse Cartier

Now we proceed to the inverse Cartier functor C_1^{-1} from the category $HIG_{p-1}(X_1)$ to the category $MIC(X_1)$.

Let $(E, \theta) \in HIG_{p-1}(X_1)$, we construct $(H, \nabla) := C_1^{-1}(E, \theta)$ as follows: take an affine covering $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in I}$ of X_2, whose closed fibers $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ form an affine covering of X_1. Note that over each U'_α, we can take a lifting $F_\alpha : U'_\alpha \to U'_\alpha$ of the absolute Frobenius $F_0 : U_\alpha \to U_\alpha$. The composite of $\mathcal{O}_{U'_\alpha}$-morphisms

$$F_\alpha^* \Omega_{U'_\alpha} \overset{dF_\alpha}{\to} p\Omega_{U'_\alpha} \overset{\frac{1}{[p]}}{\cong} \Omega_{U_\alpha}$$

descends to an $\mathcal{O}_{U_\alpha}$-morphism

$$\frac{dF_\alpha}{p} : F_0^* \Omega_{U_\alpha} \to \Omega_{U_\alpha}.$$

Consider the vector bundle $H_\alpha := F_0^*(E|_{U_\alpha})$ over U_α, where $E|_{U_\alpha}$ denotes the restriction of E over U_α. Define a connection ∇_α on H_α by the formula

$$\nabla_\alpha = \nabla_{can} + \frac{dF_\alpha}{p}(F_0^* \theta),$$

where ∇_{can} is the canonical connection on H_α, i.e., the sections of $F_0^{-1}(E|_{U_\alpha})$ are horizontal under ∇_{can}.

Lemma 3.1.1. *The connection ∇_α on H_α is integrable.*

Proof. As θ is integrable, i.e., $\theta \wedge \theta = 0$, it follows that $F_0^*(\theta) \wedge F_0^*(\theta) = F_0^*(\theta \wedge \theta) = 0$. Furthermore,

$$\frac{dF_\alpha}{p}(F_0^* \theta) \wedge \frac{dF_\alpha}{p}(F_0^* \theta) = (\bigwedge^2 \frac{dF_\alpha}{p})(F_0^* \theta \wedge F_0^* \theta) = 0.$$

It is left to show that $d(\frac{dF_\alpha}{p}(F_0^* \theta)) = 0$. This is done by local computation: by definition, for $\omega \in \Omega_{U_\alpha}$,

$$\frac{dF_\alpha}{p}(F_0^* \omega) = \frac{1}{[p]}(dF_\alpha(F_\alpha^* \omega')),$$

where $\omega' \in \Omega_{U'_\alpha}$ is any lifting of ω. Then

$$d \circ \frac{dF_\alpha}{p}(F_0^*\omega) = d \circ \frac{1}{[p]}(dF_\alpha(F_\alpha^*\omega')) = \frac{1}{[p]}(d \circ dF_\alpha(F_\alpha^*\omega')).$$

We may write $\omega' = \sum_i f_i dg_i$ for $f_i, g_i \in \mathcal{O}_{U'_\alpha}$. Then

$$d(dF_\alpha(F_\alpha^*\omega')) = \sum_i d(F_\alpha^* f_i) \wedge d(F_\alpha^* g_i) \in p^2\Omega^2_{U'_\alpha} = 0.$$

Thus $d(\frac{dF_\alpha}{p}(F_0^*\omega)) = 0$. Clearly it follows that $d(\frac{dF_\alpha}{p}(F_0^*\theta)) = 0$. The lemma is proved. $\square$

Thus we have defined a local flat bundle $(H_\alpha, \nabla_\alpha)$ over each $U_\alpha \in \mathcal{U}$. In order to glue them into a global one, we need to provide a set of isomorphisms

$$\{G_{\alpha\beta} \in Iso\left(H_\alpha|_{U_{\alpha\beta}}, H_\beta|_{U_{\alpha\beta}}\right) = Aut(F_0^*E|_{U_{\alpha\beta}})\},$$

where $U_{\alpha\beta} := U_\alpha \cap U_\beta$, satisfying

(i) the bundle gluing condition over $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$:

$$G_{\beta\gamma} \circ G_{\alpha\beta} = G_{\alpha\gamma};$$

(ii) the connection gluing condition over $U_{\alpha\beta}$:

$$\nabla_\beta \circ G_{\alpha\beta} = (G_{\alpha\beta} \otimes id) \circ \nabla_\alpha.$$

We shall need a lemma of Deligne-Illusie,

Lemma 3.1.2. *There are homomorphisms $h_{\alpha\beta} : F_0^*\Omega_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}$ satisfying the following two properties:*

(i) over $F_0^{-1}\Omega_{U_{\alpha\beta}}$, we have $\frac{dF_\alpha}{p} - \frac{dF_\beta}{p} = dh_{\alpha\beta}$;

(ii) the cocycle condition over $U_{\alpha\beta\gamma}$: $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$.

Proof. Consider the W_2-morphism $G_\alpha : Z'_\alpha \to U'_{\alpha\beta} := U'_\alpha \cap U'_\beta$ sitting in the following Cartesian diagram:

$$\begin{array}{ccc} Z'_\alpha & \xrightarrow{G_\alpha} & U'_{\alpha\beta} \\ {\scriptstyle j'_\alpha}\downarrow & & \downarrow{\scriptstyle i'_\alpha} \\ U'_\alpha & \xrightarrow{F_\alpha} & U'_\alpha, \end{array}$$

where i'_α is the natural inclusion. By reduction modulo p, we obtain the following Cartesian square

$$\begin{array}{ccc} Z_\alpha & \xrightarrow{G_0} & U_{\alpha\beta} \\ {\scriptstyle j_\alpha}\downarrow & & \downarrow{\scriptstyle i_\alpha} \\ U_\alpha & \xrightarrow{F_0} & U_\alpha. \end{array}$$

Thus we see that Z_α is $U_{\alpha\beta}$ and $G_\alpha : Z'_\alpha \to U'_\alpha$ is a lifting of the absolute Frobenius F_0 over $U_{\alpha\beta}$. Similarly for (U'_β, F_β), we have $G_\beta : Z'_\beta \to U'_{\alpha\beta}$ which is also a lifting of $F_0 : U_{\alpha\beta} \to U_{\alpha\beta}$. Now we apply [Il, Lemma 5.4] to the pair $(G_\alpha : Z'_\alpha \to U'_{\alpha\beta},\ G_\beta : Z'_\beta \to U'_{\alpha\beta})$ of Frobenis liftings of the absolute Frobenius F_0 on $U_{\alpha\beta}$, we get the homomorphisms $h_{\alpha\beta} : F_0^*\Omega_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}$ such that $\frac{dF_\alpha}{p} - \frac{dF_\beta}{p} = dh_{\alpha\beta}$ and $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$. □

By the lemma, we have an $\mathcal{O}_{U_{\alpha\beta}}$-linear morphism $h_{\alpha\beta}(F_0^*\theta) : F_0^*E|_{U_{\alpha\beta}} \to F_0^*E|_{U_{\alpha\beta}}$. By the assumption on the order of the nilpotent Higgs field θ, the morphism

$$G_{\alpha\beta} := \exp[h_{\alpha\beta}(F_0^*\theta)]$$

is well-defined and in fact equal to the finite sum $\sum_{i=0}^{p-1} \frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!}$. clearly $G_{\alpha\beta}$ is an isomorphism with $\exp[-h_{\alpha\beta}(F_0^*\theta)]$ to be its inverse. Now we can use the $\{G_{\alpha\beta} : H_\alpha|_{U_{\alpha\beta}} \cong H_\beta|_{U_{\alpha\beta}}\}$ as the gluing isomorphisms for $\{H_\alpha\}_{\alpha\in\mathcal{U}}$. Then we have the following:

Theorem 3.1.3. *The local flat bundles $\{(H_\alpha, \nabla_\alpha)\}_{\alpha\in\mathcal{U}}$ are glued into a global flat bundle (H, ∇) via the isomorphism $\{G_{\alpha\beta}\}$.*

Proof. The proof is divided into two steps.

Step 1: Bundle gluing. We show the cocycle condition holds, i.e.,

$$G_{\beta\gamma} \circ G_{\alpha\beta} = G_{\alpha\gamma}, \quad \forall \alpha, \beta, \gamma.$$

By direct computation,

$$G_{\beta\gamma} \circ G_{\alpha\beta} \quad = \quad \exp[h_{\beta\gamma}(F_0^*\theta)] \exp[h_{\alpha\beta}(F_0^*\theta)].$$

It follows from the integrability of the Higgs field that the two morphisms $h_{\alpha\beta}(F_0^*\theta)$ and $h_{\beta\gamma}(F_0^*\theta)$ commute with each other. Thus

$$G_{\beta\gamma} \circ G_{\alpha\beta} = \exp[(h_{\beta\gamma} + h_{\alpha\beta})(F_0^*\theta)] = \exp[h_{\alpha\gamma}(F_0^*\theta)] = G_{\alpha\gamma},$$

where the second equality follows from Lemma 3.1.2 (ii).

Step 2: Connection gluing. We show that the local connections $\{\nabla_\alpha\}$ coincide on the overlaps, that is

$$(G_{\alpha\beta}\otimes id)\circ\nabla_\alpha=\nabla_\beta\circ G_{\alpha\beta},\quad \forall\alpha,\beta.$$

It suffices to show

$$\frac{dF_\alpha}{p}(F_0^*\theta)=G_{\alpha\beta}^{-1}\circ dG_{\alpha\beta}+G_{\alpha\beta}^{-1}\circ\frac{dF_\beta}{p}(F_0^*\theta)\circ G_{\alpha\beta}.$$

We see that

$$G_{\alpha\beta}^{-1}\circ dG_{\alpha\beta}=-dG_{\alpha\beta}^{-1}\circ G_{\alpha\beta}=dh_{\alpha\beta}(F_0^*\theta);$$

And as $G_{\alpha\beta}$ commutes with $\frac{dF_\beta}{p}(F_0^*\theta)$ due to the integrability of the Higgs field,

$$G_{\alpha\beta}^{-1}\circ\frac{dF_\beta}{p}(F_0^*\theta)\circ G_{\alpha\beta}=\frac{dF_\beta}{p}(F_0^*\theta).$$

So

$$G_{\alpha\beta}^{-1}\circ dG_{\alpha\beta}+G_{\alpha\beta}^{-1}\circ\frac{dF_\beta}{p}(F_0^*\theta)\circ G_{\alpha\beta}=dh_{\alpha\beta}(F_0^*\theta)+\frac{dF_\beta}{p}(F_0^*\theta)=\frac{dF_\alpha}{p}(F_0^*\theta),$$

where the last equality uses Lemma 3.1.2 (i). □

Definition 3.1.4. By the above construction we define the inverse functor $C_1^{-1}: HIG_{p-1}(X_1)\to MIC(X_1)$ which maps (E,θ) to (H,∇).

Remark 3.1.5. The isomorphism class of $C_1^{-1}(E,\theta)$ depends on neither the choice of an affine covering $\mathcal{U}'$ of X_2 nor the choice of Frobenius liftings. However, it does depend on the choice of W_2-lifting of X_1.

3.2 Relation with strict p-torsion Fontaine-Faltings modules

Now suppose that $\mathbf{X}$ is a smooth proper scheme defined over $W(k)$. Denote by X_1 (resp. X_2) the reduction modulo p (resp. p^2) of $\mathbf{X}$.

Let $(H,\nabla,Fil^\cdot,\varphi)$ be a strict p-torsion object annihilated by p of $\mathfrak{MF}^\nabla_{[0,n]}(\mathbf{X})$, where $n\le p-1$. As a strict p-torsion object, (H,∇) descends to a flat bundle over X_1, still denoted by (H,∇). Let $(E,\theta):=Gr_{Fil^\cdot}(H,\nabla)$ be the associated graded Higgs bundle over X_1. Let $(H_{exp},\nabla_{exp}):=C_1^{-1}(E,\theta)$, where the inverse Cartier C_1^{-1} is defined with respect to X_2. We have the following:

Proposition 3.2.1. *φ induces an isomorphism*

$$\tilde{\varphi} : (H_{exp}, \nabla_{exp}) \cong (H, \nabla).$$

Proof. We choose a small affine covering $\{\mathbf{U}_\alpha\}$ of $\mathbf{X}$ together with Frobenius lifting $F_{\hat{\mathbf{U}}_\alpha}$ over each $\hat{\mathbf{U}}_\alpha$, where $\hat{\mathbf{U}}_\alpha$ denotes the p-adic completion of $\mathbf{U}_\alpha$. Denote by U'_α and F_α (resp. U_α and F_0) the reduction modulo p^2 (resp. p) of $\hat{\mathbf{U}}_\alpha$ and $F_{\hat{\mathbf{U}}_\alpha}$.

Recall that for each pair (U'_α, F_α), φ induces a local isomorphism:

$$\tilde{\varphi}_{F_\alpha} : H_{exp}|_{U_\alpha} = F_0^*(E|_{U_\alpha}) \cong H|_{U_\alpha}.$$

Here note that $E = \tilde{H}$, which has already appeared in the definition of Fontaine-Faltings module and this identification is only true for strict p-torsion object. Then the isomorphism $\tilde{\varphi}$ is obtained by gluing the set of local isomorphisms $\{\tilde{\varphi}_{F_\alpha}\}$.

Step 1: Bundle isomorphism. By choosing a local basis e of sections of E over $U_{\alpha\beta}$, we are going to show that over $U_{\alpha\beta}$,

$$\tilde{\varphi}_{F_\alpha}(F_0^* e) = \tilde{\varphi}_{F_\beta} \circ G_{\alpha\beta}(F_0^* e).$$

Choose a coordinate $\{t_1, \cdots t_d\}$ of $U_{\alpha\beta}$. For a multi-index $\underline{j} = (j_1, \cdots, j_d)$, we put

$$\theta_\partial^{\underline{j}} = (\partial_{t_1} \lrcorner \theta)^{j_1} \cdots (\partial_{t_d} \lrcorner \theta)^{j_d},$$

$$z^{\underline{j}} = \prod_{l=1}^{d} z_l^{j_l}, \qquad \text{where } z_l = \left(\frac{F_\alpha - F_\beta}{[p]}\right)(F_0^* t_l).$$

As Φ is horizontal under ∇, according to the Taylor formula, we have

$$\tilde{\Phi}_{\tilde{F}_\alpha}(F_0^* e) = \tilde{\Phi}_{\tilde{F}_\beta} \circ \left(1 + \sum_{|\underline{j}|=1}^{n} F_0^*(\theta_\partial^{\underline{j}}) \cdot \frac{z^{\underline{j}}}{\underline{j}\,!}\right)(F_0^* e).$$

So it suffices to show

$$G_{\alpha\beta} = 1 + \sum_{|\underline{j}|=1}^{n} F_0^*(\theta_\partial^{\underline{j}}) \cdot \frac{z^{\underline{j}}}{\underline{j}\,!}. \tag{3.2.1.1}$$

As

$$h_{\alpha\beta}(F_0^* \theta) = \sum_{l=1}^{d} F_0^*(\partial_{t_l} \lrcorner \theta) h_{\alpha\beta}(F_0^* dt_l),$$

and

$$h_{\alpha\beta}(F_0^* dt_l) = \left(\frac{F_\alpha - F_\beta}{p}\right)(F_0^* t_l) = z_l,$$

it follows that

$$\frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!} = \frac{(\sum_{l=1}^d F_0^*(\partial_{t_l} \lrcorner \theta) z_l)^i}{i!} = \sum_{|\underline{j}|=i} \frac{F_0^*(\theta_\partial^{\underline{j}}) z^{\underline{j}}}{\underline{j}\,!}.$$

Recall that

$$G_{\alpha\beta} = \sum_{i=0}^n \frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!},$$

then (3.2.1.1) follows.

Step 2: Connection isomorphism. We need to show that under the above isomorphism, the connection ∇_{exp} on H_{exp} is equal to the connection ∇ on H. Take any section e of E over U_α. And set $\theta_l := \theta(\partial t_l)$. By the horizontal property of φ, we have

$$\nabla[\tilde{\varphi}_{F_\alpha}(F_0^* e)] = \tilde{\varphi}_{F_\alpha} \circ \left[\sum_{l=1}^d F_0^*\theta_l \frac{F_\alpha}{[p]}(F_0^* dt_l)\right](F_0^* e).$$

As

$$\nabla_{exp}(F_0^* e) = \left[\sum_{l=1}^d F_0^*\theta_l \cdot \frac{F_\alpha}{[p]}(F_0^* dt_l)\right](F_0^* e),$$

it follows that

$$\tilde{\varphi}_{F_\alpha}(\nabla_{exp}(F_0^* e)) = \left[\sum_{l=1}^d F_0^*\theta_l \frac{F_\alpha}{[p]}(F_0^* dt_l)\right] \tilde{\varphi}_{F_\alpha}(F_0^* e) = \nabla[\tilde{\varphi}_{F_\alpha}(F_0^* e)].$$

□

3.3 Cartier transform

In this section we proceed to another direction, the Cartier transform functor C_1 from the category of flat bundles with nilpotent p-curvatures of order less than p to the category of Higgs bundles.

Let (H, ∇_H) be a flat bundle on X_1 whose p-curvature map

$$\psi_H : F_0^* T_{X_1} \to \mathrm{End}(H)$$

is nilpotent of order less than p. We construct $(E, \theta) := C_1(H, \nabla_H)$ as follows:

First, choose an affine covering $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in I}$ of X_2 and Frobenius liftings $F_\alpha : U'_\alpha \to U'_\alpha$ for all $\alpha \in I$ as before. Also recall the map

$$\zeta_\alpha := \frac{dF_\alpha}{p} : F_0^*\Omega_{U_\alpha} \to \Omega_{U_\alpha}.$$

By lemma (3.1.2), we have a section $h_{\alpha\beta}$ of $F_0^*(T_{X_1})$ over $U_{\alpha\beta}$ such that

$$dh_{\alpha\beta} = \zeta_\alpha - \zeta_\beta.$$

Let $H_\alpha := H|_{U_\alpha}$, $\psi_\alpha := \psi_H|_{U_\alpha}$ and $\nabla_\alpha := \nabla_H|_{U_\alpha}$. We define a new connection ∇'_α on H_α as follows:

$$\nabla'_\alpha = \nabla_\alpha + \zeta_\alpha(\psi_\alpha).$$

Set

$$J_{\alpha\beta} := \exp(\psi_H(h_{\alpha\beta})) \in Aut(H|_{U_{\alpha\beta}}).$$

Obviously, it satisfies the cocycle condition

$$J_{\beta\gamma} \circ J_{\alpha\beta} = J_{\alpha\gamma}.$$

So we can use $\{J_{\alpha\beta} : H_\alpha \to H_\beta\}$ to glue $\{H_\alpha\}$ into a new vector bundle H'.

Lemma 3.3.1. *$\{\nabla'_\alpha\}$ are glued into a connection on H', denoted by ∇'; The p-curvature map $\{\psi_\alpha\}$ is also glued into a morphism $F_0^* T_{X_1} \to \mathrm{End}(H')$, still denoted by ψ_H.*

Proof. We first prove the gluing condition for ∇'. It suffices to show

$$\zeta_\alpha(\psi_H) = J_{\alpha\beta}^{-1} \circ dJ_{\alpha\beta} + J_{\alpha\beta}^{-1} \circ \zeta_\beta(\psi_H) \circ J_{\alpha\beta}.$$

However we see that

$$J_{\alpha\beta}^{-1} \circ dJ_{\alpha\beta} = -dJ_{\alpha\beta}^{-1} \circ J_{\alpha\beta} = d(\psi_H(h_{\alpha\beta})) = \zeta_\alpha(\psi_H) - \zeta_\beta(\psi_H),$$

and

$$J_{\alpha\beta}^{-1} \zeta_\beta(\psi_H) J_{\alpha\beta} = \zeta_\beta(\psi_H).$$

So

$$dJ_{\alpha\beta} J_{\alpha\beta}^{-1} + J_{\alpha\beta}\zeta_\beta(\psi_H) J_{\alpha\beta}^{-1} = \zeta_\alpha(\psi_H) - \zeta_\beta(\psi_H) + \zeta_\beta(\psi_H) = \zeta_\alpha(\psi_H).$$

Next, we show the gluing condition for ψ_H. It suffices to show that

$$\psi_H J_{\alpha\beta} = J_{\alpha\beta} \psi_H.$$

This is obvious. $\square$

Lemma 3.3.2. *The p-curvature map ψ'_H of ∇' is a zero map.*

Proof. It suffice to check locally over each U_α. Set $\mathcal{E}_\alpha := \mathcal{O}_{U_\alpha} \oplus F_0^* \Omega_{U_\alpha}$. We define a connection on $\mathcal{E}_\alpha$:

$$\begin{aligned} \nabla_{\mathcal{E}_\alpha} : \mathcal{E}_\alpha &\longrightarrow \mathcal{E}_\alpha \otimes \Omega_{U_\alpha} \\ (f, g \otimes \omega) &\longrightarrow (df + g \otimes \zeta_\alpha(1 \otimes \omega), (1 \otimes \omega) \otimes dg), \end{aligned}$$

for $f, g \in \mathcal{O}_{U_\alpha}$, and $\omega \in \Omega_{U_\alpha}$.

To calculate its p-curvature map $\psi_{\mathcal{E}_\alpha} : F_0^* T_{U_\alpha} \to \mathrm{End}(\mathcal{E}_\alpha)$, we choose coordinates $\{t_1 \cdots t_d\}$ for U_α. When restricted to the first component $\mathcal{O}_{U_\alpha}$, $\psi_{\mathcal{E}_\alpha}$ is a zero map, because $\nabla_{\mathcal{E}_\alpha}$ is just the canonical connection.

For $\omega = dt_j \in \Omega_{U_\alpha}$,

$$\psi_{\mathcal{E}_\alpha}(1 \otimes \partial t_i)(1 \otimes dt_j) = (\nabla_{\mathcal{E}_\alpha}(\partial t_i))^p (1 \otimes dt_j) = (\frac{\partial}{\partial t_i})^{p-1}(\partial t_i \lrcorner \zeta_\alpha (1 \otimes dt_j)) = -\delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$. So we have

$$\psi_{\mathcal{E}_\alpha}(f, g \otimes \omega) = -(g, 0) \otimes \omega.$$

Let $S^{<p}(\Omega_{U_\alpha}) := \bigoplus_{i<p} S^i \Omega_{U_\alpha}$. Extend the connection $\nabla_{\mathcal{E}_\alpha}$ to a connection ∇_{Ω_α} on $F_0^*(S^{<p}(\Omega_{U_\alpha}))$ by Leibniz rule. It is easy to see that the p-curvature map ψ_{Ω_α} of ∇_{Ω_α} is as follows: for $\omega \in S^{<p}(\Omega_{U_\alpha})$,

$$\psi_{\Omega_\alpha}(1 \otimes \partial t_k)(1 \otimes \omega) = -1 \otimes \partial t_k \lrcorner \omega.$$

∇_{Ω_α} induces a dual connection ∇_{T_α} on $\mathcal{T}_\alpha := F_0^*(S^{<p} T_{U_\alpha}) := \bigoplus_{i<p} F_0^*(S^i T_{U_\alpha})$.

To calculate the p-curvature of ∇_{T_α}, we need some calculation about ∇_{T_α}. Let $f_{ik} = \partial t_k \lrcorner \zeta_\alpha (1 \otimes dt_i)$, 1_T denote the unit in $\mathcal{T}_\alpha$, and 1_Ω denote the unit in $F_0^*(S^{<p}\Omega_{U_\alpha})$. Then

$$< \nabla_{T_\alpha}(\partial t_k)(1_T), 1_\Omega >= - < 1_T, \nabla_{\Omega_\alpha}(\partial t_k)(1_\Omega) >= 0,$$

$$< \nabla_{T_\alpha}(\partial t_k)(1_T), 1 \otimes dt_i >= - < 1_T, \nabla_{\Omega_\alpha}(\partial t_k)(1 \otimes dt_i) >= -f_{ik},$$

and for $m > 1$

$$< \nabla_{T_\alpha}(\partial t_k), 1 \otimes dt_{i_1} \cdots dt_{i_m} >= 0.$$

So

$$\nabla_{T_\alpha}(\partial t_k)(1_T) = \sum_{i=1}^{d} -f_{ik} \otimes \partial t_i. \tag{3.3.2.1}$$

Now we calculate the p-curvature map ψ_{T_α} of ∇_{T_α}:

$$\begin{aligned}
& < \psi_{T_\alpha}(1 \otimes \partial t_k)(1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}), 1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d} > \\
= \ & < [\nabla_{T_\alpha}(\partial t_k)]^p (1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}), 1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d} > \\
= & - < 1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, [\nabla_{\Omega_\alpha}(\partial t_k)]^p (1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d}) > \\
= & - < 1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, \psi_{\Omega_\alpha}(1 \otimes \partial t_k)(1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d}) > \\
= & \qquad < 1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, 1 \otimes \partial t_k \lrcorner ((dt_1)^{j_1} \cdots (dt_d)^{j_d}) > \\
= & \qquad < 1 \otimes \partial t_k \cdot (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, 1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d} > .
\end{aligned}$$

So for $\tau \in \mathcal{T}_\alpha$, $\upsilon \in F_0^*(T_{U_\alpha})$,

$$\psi_T(\upsilon)(\tau) = \upsilon \cdot \tau. \tag{3.3.2.2}$$

Let $F_0^* T_{U_\alpha}$ act on $\mathcal{T}_\alpha$ via the map ψ_{T_α}, then the action can be naturally extended to an action of $\mathcal{L}_\alpha := F_0^*(S^{<p}(T_{U_\alpha}))$ on $\mathcal{T}_\alpha$. $\mathcal{L}_\alpha$ has a natural ring structure induced by the following isomorphism,

$$\mathcal{L}_\alpha \cong F_0^*(S^{\cdot} T_{U_\alpha})/I,$$

with I the ideal $F_0^*(S^{\geq p} T_{U_\alpha}) := \bigoplus_{i \geq p} F_0^* S^i T_{U_\alpha}$.

By (3.3.2.2), this action is just the multiplication map. So $\mathcal{T}_\alpha$ is a rank 1 free module over $\mathcal{L}_\alpha$ with basis 1_T. On the other hand, $\mathcal{L}_\alpha$ acts on H_α via the p-curvature map ψ_α. Now we consider the following $\mathcal{O}_{U_\alpha}$-module:

$$\mathscr{H}om_{\mathcal{L}_\alpha}(\mathcal{T}_\alpha, H_\alpha).$$

There is a natural isomorphism:

$$\begin{aligned} \lambda_\alpha : \mathscr{H}om_{\mathcal{L}_\alpha}(\mathcal{T}_\alpha, H_\alpha) \cong \quad & H_\alpha, \\ \phi \ \to \ & \phi(1_T). \end{aligned}$$

Note that $\mathscr{H}om_{\mathcal{L}_\alpha}(\mathcal{T}_\alpha, H_\alpha)$ has a natural connection $\tilde{\nabla}_\alpha$ induced by ∇_{T_α} on $\mathcal{T}_\alpha$ and ∇_α on H_α. Via the isomorphism λ_α, it induces a connection ∇'_α on H_α. As $\mathcal{L}_\alpha$ acts as zero map on $\mathscr{H}om_{\mathcal{L}_\alpha}(\mathcal{T}_\alpha, H_\alpha)$, so the p-curvature map $\psi'_\alpha$of ∇'_α is a zero map.

Now we calculate the ∇'_α. For $\phi \in \mathscr{H}om_{\mathcal{L}_\alpha}(\mathcal{T}_\alpha, H_\alpha)$, set $e = \phi(1_T)$. Then

$$\begin{aligned} \nabla'_\alpha(\partial t_k)(e) &= (\tilde{\nabla}_\alpha(\partial t_k)\phi)(1_T) = \nabla_\alpha(\partial t_k)(\phi(1_T)) - \phi(\nabla_{T_\alpha}(\partial t_k)(1_T)) \\ &= \nabla_\alpha(\partial t_k)(e) + \textstyle\sum_{i=1}^d \phi(f_{ik}\partial t_i) = \nabla_\alpha(\partial t_k)(e) + \sum_{i=1}^d \phi(\psi_{T_\alpha}(f_{ik}\partial t_i)(1_T)) \\ &= \nabla_\alpha(\partial t_k) + \textstyle\sum_{i=1}^d \psi_\alpha(f_{ik}\partial t_i)(\phi(1_T)) = \nabla_\alpha(\partial t_k) + \partial t_k \lrcorner \zeta_\alpha(\psi_\alpha(e)). \end{aligned}$$

So $\nabla'_\alpha = \nabla_\alpha + \zeta_\alpha(\psi_\alpha)$. ☐

Now as the p-curvature map ψ'_H of ∇' is a zero map, we can use the Cartier descend. Set

$$(E, \theta) := (H', \psi_H)^{\nabla'}.$$

This is the Higgs bundle as desired.

Remark 3.3.3. The isomorphism class of (E, θ) depends neither on the the choice of an affine covering $\mathcal{U}'$ of X_2 nor on the choice of Frobenius liftings. However, it does depend on the choice of W_2-lifting of X_1.

3.4 Comparison with the construction of Ogus-Vologodsky

In this section, we want to show that our construction of the inverse Cartier functor coincides with Ougs-Vologodsky's abstract construction up to a minus sign. In this section we will adopt the notation in [OV] without explanation.

Consider the following commutative diagram :

$$\begin{tikzcd} X_1 \arrow[r, "F"] \arrow[dr] & X_1^{(1)} \arrow[r, "\pi"] \arrow[d] & X_1 \arrow[d] \\ & S \arrow[r, "\sigma"] & S, \end{tikzcd}$$

where σ is the absolute Frobenius map of S, $X_1^{(1)}$ is the base change of X_1, the composite of F and π is just the absolutely Frobenius map F_0 of X_1. Similarly denote by $U_\alpha^{(1)}$ and $U_{\alpha\beta}^{(1)}$ respective the base change of U_α and $U_{\alpha\beta}$ with respect to σ.

To compare the inverse Cartier construction, we first recall the inverse Cartier from [OV, Theorem 2.8]. Given a Higgs bundle (E, θ) over X_1 of nilpotent order less than p. Let $(E^{(1)}, \theta^{(1)}) := \pi^*(E, \theta)$. Set $\tilde{S} := Spec(W_2(k))$. By abusing notation, we also use σ to denote the absolute Frobenius on $\tilde{S}$. Set $\tilde{\pi} : \tilde{X}_2 := X_2 \otimes_\sigma \tilde{S} \to X_2$, and $\mathscr{X}/\mathscr{S} := (X_1/S, \tilde{X}_2/\tilde{S})$.

Then $C^{-1}_{\mathscr{X}/\mathscr{S}}(E^{(1)}, \theta^{(1)})$ is defined to be

$$(M, \nabla_M) := \mathscr{B}_{\mathscr{X}/\mathscr{S}} \otimes_{\hat{\Gamma}.T_{X_1''}} \iota^*(E^{(1)}).$$

Now for (U'_α, F_α), let $\tilde{U}_\alpha := U'_\alpha \otimes_\sigma \tilde{S}$ and $\tilde{F}_\alpha$ be the composite of F_α and $\tilde{\pi}^{-1}$. There is an isomorphism

$$\sigma_\alpha : \mathscr{B}_{\mathscr{X}/\mathscr{S}}|_{U_\alpha} \cong F^*\hat{\Gamma}.T_{U_\alpha^{(1)}},$$

which induces more isomorphisms

$$\xi_\alpha : M|_{U_\alpha} \cong F^*\hat{\Gamma}.T_{U_\alpha^{(1)}} \otimes_{\hat{\Gamma}.T_{U_\alpha^{(1)}}} \iota^* E^{(1)} \cong F_0^*\hat{\Gamma}.T_{U_\alpha} \otimes_{\hat{\Gamma}.T_{U_\alpha}} \iota^* E,$$

and

$$\varpi_\alpha : F_0^*\hat{\Gamma}.T_{U_\alpha} \otimes_{\hat{\Gamma}.T_{U_\alpha}} \iota^* E \cong F_0^*\iota^* E \cong F_0^* E.$$

Let $\eta_\alpha := \varpi_\alpha \circ \xi_\alpha$. Under the isomorphism η_α, ∇_M induces a connection ∇_α on $F_0^* E$. For any section e of E over U_α,

$$\nabla_\alpha(\partial t_k)(1 \otimes e) = \varpi_\alpha(\nabla_{\partial t_k}(1_T) \otimes e) = \varpi_\alpha(\textstyle\sum_{i=1}^d (f_{ik} \otimes \partial t_i) \otimes e)$$
$$= -\textstyle\sum_{i=1}^d \varpi_\alpha(f_{ik} \otimes \theta_{\partial t_i}(e)) = -\textstyle\sum_{i=1}^d f_{ik}\varpi_\alpha(1 \otimes \theta_{\partial t_i}(e)) = -\partial t_k \lrcorner \zeta_\alpha(\theta(e)).$$

Note that the connection on $F_0^*\hat{\Gamma}.T_{U_\alpha}$ in [OV] is different from ours in Section 3.3 by a minus sign. So

$$\nabla_\alpha = \nabla_{can} - \zeta(\theta),$$

where ∇_{can} is the canonical connection.

On the overlap $U_{\alpha\beta}$, $\sigma_\beta \circ \sigma_\alpha^{-1}$ is just the multiplication map by $\exp(h_{\alpha\beta})$, here $h_{\alpha\beta}$ is considered as a section of $F^*T_{U^{(1)}_{\alpha\beta}}$. Set

$$\bar{J}_{\alpha\beta} := \eta_\beta \circ \eta_\alpha^{-1},$$

then

$$\bar{J}_{\alpha\beta}(1 \otimes e) = \exp(h_{\alpha\beta}) \otimes \pi^*(e) = \exp(-h_{\alpha\beta}(F_0^*\theta(e))).$$

Thus the inverse Cartier in [OV] is equivalent to using the transition isomorphisms

$$\{\bar{J}_{\alpha\beta} = \exp(-h_{\alpha\beta}(\theta(e)))\}$$

to glue the local models

$$\{(M_\alpha = F_0^*E|_{U_\alpha}, \nabla_\alpha = \nabla_{can} - \zeta_\alpha(\theta))\}.$$

Clearly this is exactly our inverse Cartier defined in Section 3.1 for $(E, -\theta)$.

Next we compare the Cartier transform. Also recall the Cartier transform in [OV, Theorem 2.8]. Given a flat bundle (H, ∇_H) , whose p-curvature map ψ_H is nilpotent of order less than p, $C_{\mathscr{X}/\mathscr{S}}(H, \nabla)$ is defined to be

$$(E^{(1)}, \theta^{(1)}) = \iota^* \mathscr{H}om_{D^\gamma_{X_1/S}}(\mathscr{B}_{\mathscr{X}/\mathscr{S}}, H).$$

Let $D_{X_1/S}$ be the sheaf of algebra of PD-differential operators on X_1. Set $D_\alpha := D_{X_1/S}|_{U_\alpha}$, and $(E, \theta) := (\pi^{-1})^*(E^{(1)}, \theta^{(1)})$. As the p-curvature map ψ_H is nilpotent of order less than p, the σ_α induces isomorphisms

$$\mu_\alpha : E|_{U_\alpha} \cong \iota^* \mathscr{H}om_{F_0^*T_{U_\alpha}}(F_0^*T_{U_\alpha}, H|_{U_\alpha})^{D_\alpha},$$

and

$$\kappa_\alpha : \iota^* \mathscr{H}om_{F_0^*T_{U_\alpha}}(F_0^*T_{U_\alpha}, H|_{U_\alpha})^{D_\alpha} \cong H|_{U_\alpha}^{\nabla'_\alpha},$$

where ∇'_α is the connection on $H_\alpha := H|_{U_\alpha}$ induced via the isomorphism κ_α from the connection on $\mathscr{H}om_{F_0^*T_{U_\alpha}}(F_0^*T_{U_\alpha}, H_\alpha)$ which is induced by the connection ∇_{T_α} on $F_0^*T_{U_\alpha}$ and ∇_H on H.

Now the calculation in the proof of Lemma (3.3.2) implies that

$$\nabla'_\alpha = \nabla_H + \zeta_\alpha(\psi_H).$$

Set $\rho_\alpha := \kappa_\alpha \circ \mu_\alpha$. Via the isomorphism ρ_α, the Higgs field θ induces a Higgs field θ_α on $H_\alpha^{\nabla'_\alpha}$. For any section e of H over U_α annihilated by ∇'_α,and of the form $e = \phi(1_T)$ for some section $\phi \in \mathscr{H}om_{F_0^* T_{U_\alpha}}(F_0^* T_{U_\alpha}, H_\alpha)(U_\alpha)$,

$$\theta_\alpha(\partial t_k)(e) = \kappa_\alpha(\psi_H(\partial t_k) \circ \phi) = -\psi_H(\partial t_k)(\phi(1_T)) = -\psi_H(\partial t_k)(e).$$

Therefore, $\theta_\alpha := -\psi_H$.

On the overlap $U_{\alpha\beta}$, set $J_{\alpha\beta} := \rho_\beta \circ \rho_\alpha^{-1}$. Then for $e \in H_\alpha^{\nabla'_\alpha}$ with $e = \phi(1_T)$ for $\phi \in \mathscr{H}om_{F_0^* T_{U_\alpha}}(F_0^* T_{U_\alpha}, H_\alpha)(U_\alpha)$,

$$\begin{aligned} J_{\alpha\beta}(e) &= \phi(\exp(h_{\alpha\beta})) = \phi(\psi_T(\exp(h_{\alpha\beta}))(1_T)) \\ &= \exp(\psi_H(h_{\alpha\beta}))(\phi(1_T)) = \exp(\psi_H(h_{\alpha\beta}))(e). \end{aligned}$$

Thus

$$J_{\alpha\beta} = \exp(\psi_H(h_{\alpha\beta})).$$

So the pullback of Cartier transform construction in [OV], $(\pi^{-1})^* C_{\mathscr{X}/\mathscr{S}}(H, \nabla)$ is equivalent to using

$$\{J_{\alpha\beta} = \exp(\psi_H(h_{\alpha\beta}))\}$$

to glue local models

$$\{(E_\alpha, \theta_\alpha) := (H_\alpha, -\psi_H)^{\nabla'_\alpha}\}.$$

It is just our Cartier transform of (H, ∇) if the sign of the Higgs field is changed.

4 Periodic flows in char p and strict p-torsion Fontaine-Faltings modules

In this chapter, We fix a smooth projective variety $\mathbf{X}$ defined over $W(k)$, with generic fiber $\mathbf{X}^0 = \mathbf{X} \otimes Spec(K)$ and closed fiber X_1 over k. We also denote by X_2 the reduction modulo p^2 of $\mathbf{X}$. For convenience, we fix a small affine open covering $\{\mathbf{U}_\alpha\}$ of $\mathbf{X}$, together with an absolute Frobenius lifting $F_{\hat{\mathbf{U}}_\alpha}$ over each $\hat{\mathbf{U}}_\alpha$ (the p-adic completion of $\mathbf{U}_\alpha$). Denote by U_α and F_0 (resp. U'_α and F_α) the reduction modulo p (resp.p^2) of $\hat{\mathbf{U}}_\alpha$ and $F_{\hat{\mathbf{U}}_\alpha}$.

From now on, a Higgs bundle always means a system of Hodge bundles:

$$(E = \oplus_{i+j=n} E^{i,j}, \theta = \oplus_{i+j=n} \theta^{i,j}), \qquad \theta^{i,j} : E^{i,j} \to E^{i-1,j+1} \otimes \Omega.$$

4.1 1-periodic flows and equivalence of categories

First, we define the category $\mathcal{HB}_{n,f}(X_2)$, abbreviated as $\mathcal{HB}_f$, of periodic Higgs-de Rham flows of length f.

Its object is a tuple $(E, \theta, Fil_0^\cdot, \cdots, Fil_{f-1}^\cdot, \phi)$ with the following properties:

(1) (E, θ) is a Higgs bundle on X_1;

(2) Set $(E_0, \theta_0) = (E, \theta)$. For $0 \leq i \leq f-1$, $Fil_i^\cdot$ is a decreasing filtration on $(H_i, \nabla_i) := C_1^{-1}(E_i, \theta_i)$ such that $(E_{i+1}, \theta_{i+1}) := Gr_{Fil_i^\cdot}(H_i, \nabla_i)$ is a well-defined Higgs bundle. To be precise, $Fil_i^\cdot$ satisfies that $Fil_i^0 = H_i$, $Fil_i^{n+1} = 0$, $Gr_{Fil_i^\cdot}(H_i)$ is locally free, and Griffiths transversality holds. Here C_1^{-1} is defined with respect to X_2;

(3) ϕ is an isomorphism of Higgs bundles

$$\phi : Gr_{Fil_{f-1}^\cdot} \circ C_1^{-1}(E_{f-1}, \theta_{f-1}) \cong (E, \theta).$$

A periodic Higgs-de Rham flow can be represented as the following diagram:

$$\begin{tikzcd}
 & (H_0,\nabla_0) \arrow[dr, "Gr_{Fil_0^{\cdot}}"] & & (H_{f-1},\nabla_{f-1}) \arrow[dr, "Gr_{Fil_{f-1}^{\cdot}}"] & \\
(E_0,\theta_0) \arrow[ur, "C_1^{-1}"] & & \cdots \arrow[ur, "C_1^{-1}"] & & (E_f,\theta_f). \arrow[llll, bend left, "\overset{\phi}{\cong}"]
\end{tikzcd}$$

A morphism between two objects is a morphism of Higgs bundles preserving the additional structures. As an illustration, we explain a morphism in the category $\mathcal{HB}_1$ in detail: Let $(E_i,\theta_i,Fil_i^{\cdot},\phi_i), i = 1,2$ be two objects and

$$f:(E_1,\theta_1)\to(E_2,\theta_2)$$

a morphism of Higgs bundle. For f be become a morphism in $\mathcal{HB}_1$, it is required that :

(1) the induced morphism

$$C_1^{-1}(f):C_1^{-1}(E_1,\theta_1)\to C_1^{-1}(E_2,\theta_2) \tag{4.1.0.1}$$

preserves the filtrations;

(2) the morphism $GrC_1^{-1}(f):Gr_{Fil_1^{\cdot}}C_1^{-1}(E_1,\theta_1)\to Gr_{Fil_2^{\cdot}}C_1^{-1}(E_2,\theta_2)$, induced from applying the grading functor Gr_{Fil} to (4.1.0.1), fits into the following commutative diagram:

$$\begin{array}{ccc} Gr_{Fil_1^{\cdot}}C_1^{-1}(E_1,\theta_1) & \xrightarrow{\phi_1} & (E_1,\theta_1) \\ {\scriptstyle GrC_1^{-1}(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Gr_{Fil_2^{\cdot}}C_1^{-1}(E_2,\theta_2) & \xrightarrow{\phi_2} & (E_2,\theta_2). \end{array} \tag{4.1.0.2}$$

We introduce another category $\mathfrak{MF}^{\nabla}_{[0,n],f}(X_2)$, a modification of Faltings' category $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$. For each $f\in\mathbb{N}$, let $\mathbb{F}_{p^f}$ be the unique extension of $\mathbb{F}_p$ inside k of degree f. An object in $\mathfrak{MF}^{\nabla}_{[0,n],f}(X_2)$ is a tuple $(H,\nabla,Fil^{\cdot},\varphi,\iota)$, where $(H,\nabla,Fil^{\cdot},\varphi)$ is a strict p-torsion object in $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$ and

$$\iota:\mathbb{F}_{p^f}\hookrightarrow \mathrm{End}_{\mathfrak{MF}}(H,\nabla,Fil^{\cdot},\varphi)$$

is an embedding of $\mathbb{F}_p$-algebras. A morphism in $\mathfrak{MF}^{\nabla}_{[0,n],f}(X_2)$ is a morphism in $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$ compatible with the endomorphism ι . Clearly, the category $\mathfrak{MF}^{\nabla}_{[0,n],f}(X_2)$ for $f=1$ is just the strict p-torsion subcategory of $\mathfrak{MF}^{\nabla}_{[0,n]}(\mathbf{X})$.

Proposition 4.1.1. *There is a natural one-to-one correspondence between the Faltings' category $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$ and the category $\mathcal{HB}_{n,1}(X_2)$.*

To prove the above proposition, we shall need the following lemma which shows a functor $\mathcal{GR}$ from the category $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$ to the category $\mathcal{HB}_{n,1}(X_2)$.

Lemma 4.1.2. *Let $(H, \nabla, Fil^{\cdot}, \varphi)$ be an object in $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$. Set $(E, \theta) := Gr_{Fil^{\cdot}}(H, \nabla)$. Then there exists a filtration $Fil^{\cdot}_{\exp}$ on $C_1^{-1}(E, \theta)$, which together with $Fil^{\cdot}$ and φ induces an isomorphism of Higgs bundles*

$$\phi_{\exp} : Gr_{Fil^{\cdot}_{exp}}(C_1^{-1}(E, \theta)) \cong (E, \theta).$$

Proof. In Proposition 3.2.1, we have shown that φ induces a global isomorphism of flat bundles

$$\tilde{\varphi} : C_1^{-1}(E, \theta) \cong (H, \nabla).$$

So we can define $Fil^{\cdot}_{exp}$ on $C_1^{-1}(E, \theta)$ to be the inverse image of $Fil^{\cdot}$ on H by $\tilde{\varphi}$ and deduce tautologically an isomorphism of Higgs bundles

$$\phi_{\exp} = Gr(\tilde{\varphi}) : Gr_{Fil^{\cdot}_{exp}}(C_1^{-1}(E, \theta)) \cong (E, \theta).$$

$\square$

Therefore, we can define a functor $\mathcal{GR} : \mathfrak{MF}^{\nabla}_{[0,n],1}(X_2) \to \mathcal{HB}_{n,1}(X_2)$ that maps $(H, \nabla, Fil^{\cdot}, \varphi)$ to $(E, \theta, Fil^{\cdot}_{exp}, \phi_{\exp})$.

Next, we show that the functor C_1^{-1} induces a functor in the opposite direction. Given an object $(E, \theta, Fil^{\cdot}, \phi) \in \mathcal{HB}_{n,1}(X_2)$, we define a de Rham bundle

$$(H, \nabla, Fil^{\cdot}) = (C_1^{-1}(E, \theta), Fil^{\cdot}).$$

To produce a relative Frobenius φ from ϕ, by definition it suffices to associate to each pair (U'_α, F_α) an $\mathcal{O}_{U_\alpha}$-morphism

$$\varphi_{F_\alpha} : F_0^* Gr_{Fil^{\cdot}} H|_{U_\alpha} \to H|_{U_\alpha},$$

satisfying the following conditions:

(1) Strong p-divisibility, that is, φ_{F_α} is an isomorphism;

(2) φ_{F_α} is horizontal;

(3) Over each $U_\alpha \cap U_\beta$, φ_{F_α} and φ_{F_β} are related via the Taylor formula. More precisely, if the gluing map for $H_\alpha := H|_{U_\alpha}$ and $H_\beta := H|_{U_\beta}$ is $G_{\alpha\beta}$, then the following diagram is commutative:

$$\begin{array}{ccc} F_0^*(Gr_{Fil^\cdot} H_\alpha|_{U_\alpha \cap U_\beta}) & \xrightarrow{\varphi_{F_\alpha}|_{U_\alpha\cap U_\beta}} & H_\alpha|_{U_\alpha \cap U_{\alpha\beta}} \\ \widetilde{G}_{\alpha\beta}\Big\downarrow & & \Big\downarrow G_{\alpha\beta} \\ F_0^*(Gr_{Fil^\cdot} H_\beta|_{U_\alpha \cap U_\beta}) & \longrightarrow & H_\beta|_{U_\alpha \cap U_\beta} \\ \varepsilon_{\alpha\beta}\Big\downarrow & & \Big\downarrow Id \\ F_0^*(Gr_{Fil^\cdot} H_\beta|_{U_\alpha \cap U_\beta}) & \xrightarrow{\varphi_{F_\beta}|_{U_\alpha\cap U_\beta}} & H_\beta|_{U_\alpha \cap U_\beta}, \end{array}$$

where $\widetilde{G}_{\alpha\beta}$ is induced by $G_{\alpha\beta}$, and $\varepsilon_{\alpha\beta}$ is an $\mathcal{O}_{U_\alpha\cap U_\beta}$-linear map defined by the Taylor formula as follows: for e any sections of $Gr_{Fil^\cdot} H_\beta|_{U_\alpha\cap U_\beta}$,

$$\varepsilon_{\alpha\beta}(e\otimes 1) = e\otimes 1 + \sum_{|\underline{k}|=1}^{n} \theta_\partial^{\underline{k}}(e) \otimes \frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!},$$

where $\underline{k} = (k_1, \cdots, k_d)$ is a multi-index, $z^{\underline{k}} = \prod_{i=1}^d z_i^{k_i}$ with $z_k = F_\alpha(t'_k) - F_\beta(t'_k)$, $\theta_\partial^{\underline{k}} = (\partial_{t_1} \lrcorner \theta)^{k_1} \cdots (\partial_{t_d} \lrcorner \theta)^{k_d}$, and $\{t'_1, \cdots, t'_d\}$ is a system of étale local coordinates of $U'_{\alpha\beta}$ and a lifting of the system of étale local coordinates $\{t_1, \cdots, t_d\}$ of $U_{\alpha\beta}$. Note that z_k is independent of the choice of the lifting of $\{t_1, \cdots, t_d\}$.

For $H = \{H_\alpha := F_0^*E|_{U_\alpha}, G_{\alpha\beta}\}$, $G_{\alpha\beta} = \varepsilon_{\alpha\beta}$. We define φ_{F_α} to be

$$F_0^* Gr_{Fil^\cdot} H|_{U_\alpha} \xrightarrow{F_0^*\phi} F_0^*E|_{U_\alpha}.$$

By construction, φ_{F_α} is strongly p-divisible, and satisfies condition (1). And as ϕ is globally defined, we have the following commutative diagram:

$$\begin{array}{ccc} Gr_{Fil^\cdot} H_\alpha|_{U_\alpha\cap U_\beta} & \xrightarrow{\phi|_{U_\alpha\cap U_\beta}} & E|_{U_\alpha\cap U_\beta} \\ Gr(G_{\alpha\beta})\Big\downarrow & & \Big\downarrow Id \\ Gr_{Fil^\cdot} H_\beta|_{U_\alpha\cap U_\beta} & \xrightarrow{\phi|_{U_\alpha\cap U_\beta}} & E|_{U_\alpha\cap U_\alpha}. \end{array}$$

Pulling back the above diagram by F_0^* , we get the following commutative diagram:

$$\begin{array}{ccc} F_0^* Gr_{Fil^\cdot} H_\alpha|_{U_\alpha\cap U_\beta} & \xrightarrow{F_0^*(\phi)|_{U_\alpha\cap U_\alpha}} & F_0^*E|_{U_\alpha\cap U_\alpha} \\ \widetilde{G_{\alpha\beta}}\Big\downarrow & & \Big\downarrow Id \\ F_0^* Gr_{Fil^\cdot} H_\beta|_{U_\alpha\cap U_\beta} & \xrightarrow{F_0^*(\phi)|_{U_\alpha\cap U_\beta}} & F_0^*E|_{U_\alpha\cap U_\beta}, \end{array}$$

Then we extend it to the following diagram

$$\begin{array}{ccc} F_0^*Gr_{Fil^\cdot}H_\alpha|_{U_\alpha\cap U_\beta} & \xrightarrow{F_0^*(\phi)|_{U_\alpha\cap U_\beta}} & F_0^*E|_{U_\alpha\cap U_\beta} \\ \widetilde{G_{\alpha\beta}}\big\downarrow & & \big\downarrow Id \\ F_0^*Gr_{Fil^\cdot}H_\beta|_{U_\alpha\cap U_\beta} & \xrightarrow{F_0^*(\phi)|_{U_\alpha\cap U_\beta}} & F_0^*E|_{U_\alpha\cap U_\beta} \\ \varepsilon_{\alpha\beta}\big\downarrow & & \big\downarrow G_{\alpha\beta} \\ F_0^*Gr_{Fil^\cdot}H_\beta|_{U_\alpha\cap U_\beta} & \xrightarrow{F_0^*(\phi)|_{U_\alpha\cap U_\beta}} & F_0^*E|_{U_\alpha\cap U_\beta} \end{array} \tag{4.1.2.1}$$

As ϕ commutes with θ, we have

$$\begin{aligned} G_{\alpha\beta}\circ F_0^*(\phi)(e\otimes 1) =& \phi(e)\otimes 1+\textstyle\sum_{|\underline{k}|=1}^n \theta_{\partial}^{\underline{k}}(\phi(e))\otimes \frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!} \\ =& \phi(e)\otimes 1+\textstyle\sum_{|\underline{k}|=1}^n \phi(\theta_{\partial}^{\underline{k}}(e))\otimes \frac{z^{\underline{k}}}{p^{|\underline{k}|}\underline{k}!} \\ =& F_0^*(\phi)\circ\varepsilon_{\alpha\beta}. \end{aligned}$$

So diagram (4.1.2.1) is commutative. Thus φ_{F_α} satisfies condition (3).

Lemma 4.1.3. *φ_{F_α} is horizontal with respect to ∇.*

Proof. Put $\tilde{H}=Gr_{Fil^\cdot}H$, $\theta'=Gr_{Fil^\cdot}\nabla$, $\varphi_\alpha=\varphi_{F_\alpha}$. By definition, it is equivalent to that the following diagram is commutative:

$$\begin{array}{ccc} F_0^*\tilde{H}|_{U_\alpha} & \xrightarrow{\varphi_\alpha} & H|_{U_\alpha} \\ F_\alpha^*\tilde{\nabla}\big\downarrow & & \nabla\big\downarrow \\ F_0^*\tilde{H}|_{U_\alpha}\otimes\Omega_{U_\alpha} & \xrightarrow{\varphi_\alpha\otimes id} & H|_{U_\alpha}\otimes\Omega_{U_\alpha}, \end{array}$$

where $F_\alpha^*\tilde{\nabla}$ is induced by $\frac{dF_\alpha}{p}(F_0^*\theta')$ via Leibniz rule, i.e., it is the composite map

$$F_0^*\tilde{H}|_{U_\alpha}\xrightarrow{F_0^*\theta'}F_0^*\tilde{H}|_{U_\alpha}\otimes F_0^*\Omega_{U_\alpha}\xrightarrow{id\otimes\frac{dF_\alpha}{p}}F_0^*\tilde{H}|_{U_\alpha}\otimes\Omega_{U_\alpha}.$$

It is reduced to show the following diagram commutes:

$$\begin{array}{ccc} F_0^*\tilde{H}|_{U_\alpha} & \xrightarrow{F_0^*\phi} & F_0^*E|_{U_\alpha} \\ \frac{dF_\alpha}{p}(F_0^*\theta')\big\downarrow & & \frac{dF_\alpha}{p}(F_0^*\theta)\big\downarrow \\ F_0^*\tilde{H}|_{U_\alpha}\otimes\Omega_{U_\alpha} & \xrightarrow{F_0^*\phi\otimes id} & F_0^*E|_{U_\alpha}\otimes\Omega_{U_\alpha}. \end{array} \tag{4.1.3.1}$$

As ϕ is a morphism of Higgs bundles, the following diagram is commutative:

$$\begin{array}{ccc} \tilde{H}|_{U_\alpha} & \xrightarrow{\phi} & E|_{U_\alpha} \\ \theta'\big\downarrow & & \big\downarrow\theta \\ \tilde{H}|_{U_\alpha}\otimes\Omega_{U_\alpha} & \xrightarrow{\phi\otimes id} & E|_{U_\alpha}\otimes\Omega_{U_\alpha}. \end{array}$$

The pull-back via F_0^* of the above diagram yields the following commutative diagram:

$$\begin{tikzcd}
F_0^*\tilde{H}|_{U_\alpha} \arrow[r, "F_0^*\phi"] \arrow[d, "F_0^*\theta'"'] & F_0^*E|_{U_\alpha} \arrow[d, "F_0^*\theta"'] \arrow[ddr, bend left, "\frac{dF_\alpha}{p}F_0^*\theta"] & \\
F_0^*\tilde{H}|_{U_\alpha}\otimes F_0^*\Omega_{U_\alpha} \arrow[r, "F_0^*\phi\otimes id"] \arrow[drr, "F_0^*\phi\otimes\frac{dF_\alpha}{p}"'] & F_0^*E|_{U_\alpha}\otimes F_0^*\Omega_{U_\alpha} \arrow[dr, "id\otimes\frac{dF_\alpha}{p}"] & \\
 & & F_0^*E|_{U_\alpha}\otimes\Omega_{U_\alpha}.
\end{tikzcd} \tag{4.1.3.2}$$

The commutativity of diagram (4.1.3.1) follows now from that of the diagram (4.1.3.2). $\square$

The above lemma provides us with the functor $\mathcal{C}_1^{-1}: \mathcal{HB}_{n,1}(X_2) \to \mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$ in the opposite direction, which maps $(E,\theta,Fil^{\cdot},\phi)$ to $(H,\nabla,Fil^{\cdot},\varphi)$. Now we can prove Proposition 4.1.1.

Proof of Proposition 4.1. It suffices to check the following relations:

$$\mathcal{GR}\circ\mathcal{C}_1^{-1}\cong Id,\quad \mathcal{C}_1^{-1}\circ\mathcal{GR}\cong Id.$$

First define a natural isomorphism $\mathcal{A}$ from $\mathcal{C}_1^{-1}\circ\mathcal{GR}$ to Id as follows: for $(H,\nabla,Fil^{\cdot},\varphi)\in\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$, put

$$(E,\theta,Fil^{\cdot},\phi)=\mathcal{GR}(H,\nabla,Fil^{\cdot},\varphi),\quad (H',\nabla',Fil'^{\cdot},\varphi')=\mathcal{C}_1^{-1}(E,\theta,Fil^{\cdot},\phi).$$

Then we define $\mathcal{A}(H,\nabla,Fil^{\cdot},\varphi)$ to be

$$\tilde{\varphi}:(H',\nabla')=C_1^{-1}\circ Gr_{Fil^{\cdot}}(H,\nabla)\cong(H,\nabla).$$

It induces an isomorphism

$$F_0^*Gr(\tilde{\varphi}):F_0^*GrH'\cong F_0^*GrH.$$

Over each U_α, we have the following commutative diagram:

$$\begin{tikzcd}
F_0^*GrH' \arrow[r, "\varphi'_{F_\alpha}"] \arrow[d, "F_0^*Gr(\tilde{\varphi}|_{U_\alpha})"'] & H' \arrow[d, "\tilde{\varphi}|_{U_\alpha}"] \\
F_0^*GrH \arrow[r, "\varphi_{F_\alpha}"] & H.
\end{tikzcd}$$

It is obvious that $\tilde{\varphi}$ preserves filtrations, and is compatible with connections and relative Frobenius maps. So $\tilde{\varphi}$ is an isomorphism from $(H',\nabla',Fil'^{\cdot},\varphi')$ to $(H,\nabla,Fil^{\cdot},\varphi)$ in the category $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$.

Given a morphism in $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$

$$f : (H_1, \nabla_1, F_1^{\cdot}, \varphi_1) \to (H_2, \nabla_2, F_2^{\cdot}, \varphi_2),$$

we obtain

$$\mathcal{C}_1^{-1} \circ \mathcal{GR}(f) = C_1^{-1} Gr(f) : \mathcal{C}_1^{-1} \circ \mathcal{GR}(H_1) = H_1' \to H_2' = \mathcal{C}_1^{-1} \circ \mathcal{GR}(H_2),$$

fitting into the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}_1^{-1} \circ \mathcal{GR}(H_1) = H_1' \arrow[r, "\mathcal{C}_1^{-1}\circ\mathcal{GR}(f)"] \arrow[d, "\mathcal{A}_{H_1}"'] & H_2' = \mathcal{C}_1^{-1} \circ \mathcal{GR}(H_2) \arrow[d, "\mathcal{A}_{H_2}"] \\ H_1 \arrow[r, "f"] & H_2. \end{tikzcd} \tag{4.1.3.3}$$

The commutativity of the diagram (4.1.3.3) can be checked locally; Over each U_α, it is the consequence of the following diagram:

$$\begin{tikzcd} F_0^* Gr(H_1|_{U_\alpha}) \arrow[r, "F_0^* Gr(f)"] \arrow[d, "\varphi_{F_\alpha,1}"'] & F_0^* Gr(H_2|_{U_\alpha}) \arrow[d, "\varphi_{F_\alpha,2}"] \\ H_1 \arrow[r, "f"] & H_2, \end{tikzcd}$$

which follows from the definition of morphisms in $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$. The diagram (4.1.3.3) implies that $\mathcal{A}$ is a natural equivalence between $\mathcal{C}_1^{-1} \circ \mathcal{GR}$ and Id.

Second, define a natural isomorphism $\mathcal{B}$ from $\mathcal{GR} \circ \mathcal{C}_1^{-1}$ to Id as follows: for $(E, \theta, Fil^{\cdot}, \phi) \in \mathcal{HB}_{n,1}(X_2)$, put

$$(H, \nabla, Fil, \varphi) = \mathcal{C}_1^{-1}(E, \theta, Fil^{\cdot}, \phi) \quad (E', \theta', Fil'^{\cdot}, \phi') = \mathcal{GR}(H, \nabla, Fil^{\cdot}, \varphi).$$

Note that $E' = GrC_1^{-1}(E)$,

$$\tilde{\varphi} = C_1^{-1}(\phi) : C_1^{-1}(E') \cong C_1^{-1}(E).$$

$\phi' = Gr(\tilde{\varphi})$, and $Fil'^{\cdot}$ is the pullback of $Fil^{\cdot}$ via $\tilde{\varphi}$.

Now we define $\mathcal{B}(E, \theta, Fil^{\cdot}, \phi)$ to be

$$\phi : E' = GrC_1^{-1}(E) \cong E.$$

We see that $C_1^{-1}(\phi) = \tilde{\varphi}$ is compatible with filtrations, and the following diagram is commutative:

$$\begin{tikzcd} GrC_1^{-1}(E') \arrow[r, "\phi' = Gr(\tilde{\varphi})"] \arrow[d, "GrC_1^{-1}(\phi) = Gr(\tilde{\varphi})"'] & E' \arrow[d, "\phi"] \\ GrC_1^{-1}(E) \arrow[r, "\phi"] & E. \end{tikzcd}$$

Therefore $\mathcal{B}(E,\theta,Fil^{\cdot},\phi) = \phi$ is an isomorphism from $(E',\theta',Fil'^{\cdot},\phi')$ to $(E,\theta,Fil^{\cdot},\phi)$ in the category $\mathcal{HB}_{n,1}(X_2)$.

Given a morphism in $\mathcal{HB}_{n,1}(X_2)$

$$g : (E_1,\theta_1,Fil_1^{\cdot},\phi_1) \to (E_2,\theta_2,Fil_2^{\cdot},\phi_2),$$

we obtain $\mathcal{GR}\circ\mathcal{C}_1^{-1}(g) = GrC_1^{-1}(g) : E_1' \to E_2'$. By definition of morphisms in $\mathcal{HB}_{n,1}(X_2)$, g satisfies the commutative diagram (4.1.0.2), which implies that the following diagram is commutative:

$$\begin{tikzcd} E_1' \arrow[r, "GrC_1^{-1}(g)"] \arrow[d, "\mathcal{B}(E_1,\theta_1,Fil_1,\phi_1)=\phi_1"'] & E_2' \arrow[d, "\mathcal{B}(E_2,\theta_2',Fil_2,\phi_2)=\phi_2"] \\ E_1 \arrow[r, "g"] & E_2. \end{tikzcd}$$

So $\mathcal{B}$ is a natural equivalence between $\mathcal{GR}\circ\mathcal{C}_1^{-1}$ and Id. □

4.2 General equivalence

To deal with periodic Higgs-de Rham flows of length greater than 1, we shall introduce an intermediate category i.e., the category of periodic Higgs-de Rham flows of length 1 with endomorphism structure $\mathbb{F}_{p^f}$ denoted by $\mathcal{HB}^f$. An object of $\mathcal{HB}^f$ is a quintuple $(E,\theta,Fil,\phi,\iota)$, where (E,θ,Fil,ϕ) is an object in $\mathcal{HB}_{n,1}(X_2)$ and $\iota : \mathbb{F}_{p^f} \hookrightarrow \mathrm{End}_{\mathcal{HB}_{n,1}(X_2)}(E,\theta,Fil,\phi)$ is an embedding of $\mathbb{F}_p$-algebras. As a direct consequence of Proposition 4.1.1, we have

Corollary 4.2.1. *The category $\mathfrak{MF}^{\nabla}_{[0,n],f}(X_2)$ is equivalent to the category $\mathcal{HB}^f$ of periodic Higgs-de Rham flows of length 1 with endomorphism structure $\mathbb{F}_{p^f}$.*

Moreover, we have :

Proposition 4.2.2. *There is a one-to-one correspondence between the category $\mathcal{HB}_{n,f}(X_2)$ of periodic Higgs-de Rham flows of length f and the category $\mathcal{HB}^f$ of periodic Higgs-de Rham flows of length 1 with endomorphism structure $\mathbb{F}_{p^f}$.*

We start with an object $(E,\theta,Fil_0^{\cdot},\cdots,Fil_{f-1}^{\cdot},\phi)$ in $\mathcal{HB}_{n,f}(X_2)$. Set

$$(G,\eta) := \bigoplus_{i=0}^{f-1}(E_i,\theta_i),$$

where $(E_0,\theta_0) = (E,\theta)$. As the functor C_1^{-1} is compatible with direct sum, one has the identification

$$C_1^{-1}(G,\eta) = \bigoplus_{i=0}^{f-1} C_1^{-1}(E_i,\theta_i).$$

We equip a filtration $Fil^{\cdot}$ on $C_1^{-1}(G,\eta)$ by $\bigoplus_{i=0}^{f-1} Fil_i$ via the above identification. Also ϕ induces a natural isomorphism of Higgs bundles

$$\tilde{\phi}: Gr_{Fil}C_1^{-1}(G,\eta) \cong (G,\eta).$$

$\tilde{\phi}$ is defined as follows: because

$$Gr_{Fil}C_1^{-1}(G,\eta) = \bigoplus_{i=0}^{f-1} Gr_{Fil_i}C_1^{-1}(E_i,\theta_i),$$

for $0 \leq i \leq f-2$, $\tilde{\phi}$ maps the factor $Gr_{Fil_i}(E_i,\theta_i)$ identically to the factor (E_{i+1},θ_{i+1}) and maps the last factor $Gr_{Fil_{f-1}}(E_{f-1},\theta_{f-1})$ isomorphically to (E_0,θ_0) via ϕ. Thus the so-constructed quadruple $(G,\eta,Fil,\tilde{\phi})$ is an object in $\mathcal{HB}_{n,1}(X_2)$.

Lemma 4.2.3. *For an object $(E,\theta,Fil_0,\cdots,Fil_{f-1},\phi)$ in $\mathcal{HB}_{n,f}(X_2)$, there is a natural embedding of $\mathbb{F}_p$-algebras*

$$\iota: \mathbb{F}_{p^f} \to \mathrm{End}_{\mathcal{HB}_{n,f}(X_2)}(G,\eta,Fil,\tilde{\phi}).$$

Thus the extended tuple $(G,\eta,Fil,\tilde{\phi},\iota)$ is an object in $\mathcal{HB}^f$.

Proof. For simplicity we only prove the case of $f=2$. The general case can be proved similarly. Choose a primitive element ξ in the field extension $\mathbb{F}_{p^2}/\mathbb{F}_p$ once and for all. To define the embedding ι, it suffices to specify the image $s := \iota(\xi)$. Write $(G,\eta) = (E_0,\theta_0)\oplus(E_1,\theta_1)$. Then define $s := m_\xi \oplus m_{\xi^p}$, where $m_{\xi^{p^i}}$ is the multiplication maps by ξ^{p^i} for $i=0,1$. Clearly s is defines an endomorphism of (G,η) and preserves $Fil^{\cdot}$ on $C_1^{-1}(G,\eta)$. Write $(Gr_{Fil^{\cdot}}C_1^{-1})(s)$ to be the induced endomorphism of $Gr_{Fil}C_1^{-1}(G,\eta)$. It remains to verify the commutativity

$$\tilde{\phi}\circ s = (Gr_{Fil}\circ C_1^{-1})(s)\circ\tilde{\phi}.$$

In terms of a local basis, it boils down to the equation

$$\begin{pmatrix} 0 & 1 \\ \phi & 0 \end{pmatrix}\begin{pmatrix} \xi & 0 \\ 0 & \xi^p \end{pmatrix} = \begin{pmatrix} \xi^p & 0 \\ 0 & \xi \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \phi & 0 \end{pmatrix},$$

which is obvious. □

Conversely, given an object (G,η,Fil,ϕ,ι) in the category $\mathcal{HB}^f$, we can associate to it an object in $\mathcal{HB}_{n,f}(X_2)$ as follows: (G,η) is decomposed to a direct sum of eigenspace of $\iota(\xi)$

$$(G,\eta) = \bigoplus_{i=0}^{f-1}(G_i,\eta_i),$$

where (G_i, η_i) is the eigenspace with respect to the eigenvalue ξ^{p^i}. The isomorphism $C_1^{-1}(\iota(\xi))$ induces a decomposition of the de Rham bundle as well:

$$(C_1^{-1}(G, \eta), Fil^{\cdot}) = \bigoplus_{i=0}^{f-1} (C_1^{-1}(G_i, \eta_i), Fil_i^{\cdot}).$$

Under the decomposition, the isomorphism $\phi : Gr_{Fil^{\cdot}} C_1^{-1}(G, \eta) \cong (G, \eta)$ decomposes into $\oplus_{i=0}^{f-1} \phi_i$ such that

$$\phi_i : Gr_{Fil_i^{\cdot}} C_1^{-1}(G_i, \eta_i) \cong (G_{\overline{i+1}}, \theta_{\overline{i+1}}),$$

where $\overline{i+1}$ means the representative of $i+1$ mod f between 0 and $f-1$.

Put $(E, \theta) = (G_0, \theta_0)$.

Lemma 4.2.4. *The filtrations $\{Fil_i^{\cdot}\}$ and isomorphisms of Higgs bundles $\{\phi_i\}$ induce inductively filtrations $\widetilde{Fil}_i^{\cdot}$ on $C_1^{-1}(E_i, \theta_i), i = 0, \cdots, f-1$ and an isomorphism of Higgs bundles*

$$\tilde{\phi} : Gr_{\widetilde{Fil}_{f-1}^{\cdot}}(E_{f-1}, \theta_{f-1}) \cong (E, \theta).$$

Thus the extended tuple $(E, \theta, \widetilde{Fil}_0^{\cdot}, \cdots, \widetilde{Fil}_{f-1}^{\cdot}, \tilde{\phi})$ is an object of $\mathcal{HB}_{n,f}(X_2)$.

Proof. Again we shall assume $f = 2$. The filtration $\widetilde{Fil}_0^{\cdot}$ on $C_1^{-1}(E_0, \theta_0)$ is just $Fil_0^{\cdot}$. Via the isomorphism

$$C_1^{-1}(\phi_0) : C_1^{-1} Gr_{Fil_0^{\cdot}} C_1^{-1}(G_0, \eta_0) \cong C_1^{-1}(G_1, \eta_1),$$

we obtain a filtration $\widetilde{Fil}_1^{\cdot}$ on $C_1^{-1}(E_1, \theta_1)$ from $Fil_1^{\cdot}$. Finally we define $\tilde{\phi}$ to be the composite:

$$Gr_{\widetilde{Fil}_1^{\cdot}}(E_1, \theta_1) = Gr_{\widetilde{Fil}_1^{\cdot}} C_1^{-1} Gr_{\widetilde{Fil}_0^{\cdot}} C_1^{-1}(E, \theta) \xrightarrow{Gr_{\widetilde{Fil}_1^{\cdot}} C_1^{-1}(\phi_0)} Gr_{\widetilde{Fil}_1^{\cdot}} C_1^{-1}(G_1, \eta_1) \xrightarrow{\phi_1} (E, \theta).$$

□

We come to the proof of Proposition 4.2.2.

Proof of Proposition 4.2.2 . Note that Lemma 4.2.3 gives us a functor $\mathcal{E}$ from $\mathcal{HB}_{n,f}(X_2)$ to $\mathcal{HB}^f$, while Lemma 4.2.4 provides a functor $\mathcal{F}$ in the opposite direction. We show that they give an equivalence of categories. Obviously,

$$\mathcal{F} \circ \mathcal{E} = Id.$$

So it remains to give a natural isomorphism τ between $\mathcal{E} \circ \mathcal{F}$ and Id. Again we assume that $f = 2$ in the following argument. For $(E, \theta, Fil^{\cdot}, \phi, \iota)$, put

$$\mathcal{F}\{(E, \theta, Fil^{\cdot}, \phi, \iota)\} = (G, \eta, Fil_0^{\cdot}, Fil_1^{\cdot}, \tilde{\phi}),$$

$$\mathcal{E}(G,\eta,Fil_0^{\cdot},Fil_1^{\cdot},\tilde{\phi}) = (E',\theta',Fil'^{\cdot},\phi',\iota').$$

Note that $(E',\theta') = (G,\eta) \oplus Gr_{Fil_0^{\cdot}} C_1^{-1}(G,\eta)$, we define an isomorphism of Higgs bundles by

$$Id \oplus \phi_0 : (E',\theta') = (G,\eta) \oplus Gr_{Fil_0^{\cdot}} C_1^{-1}(G,\eta) \cong (E_0,\theta_0) \oplus (E_1,\theta_1) = (E,\theta).$$

It is easy to check that the above isomorphism gives an isomorphism $\tau(E,\theta,Fil,\phi,\iota)$ in the category $\mathcal{HB}^f$. The functorial property of τ can be easily verified. $\square$

Summarizing corollary 4.2.1 and proposition 4.2.2, we arrive the following theorem:

Theorem 4.2.5. *There is a one-to-one correspondence between the category $\mathfrak{MF}^{\nabla}_{[0,n],f}(X_2)$ and the category $\mathcal{HB}_{n,f}(X_2)$.*

According to Faltings's theory in Chapter 2, we have

Corollary 4.2.6. *There is an equivalence of categories between the category of dual crystalline $\mathbb{F}_{p^f}$-representations of $\pi_1(\mathbf{X}^0)$ and the category of periodic Higgs-de Rham flows of length f.*

Proof. Under the functor $\mathbf{D}$, an $\mathbb{F}_{p^f}$-endomorphism structure on an object of $\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$ is mapped to an $\mathbb{F}_{p^f}$-endomorphism structure on the corresponding $\mathbb{F}_p$-representation, and vice versa. The result is then a direct consequence of Theorem 4.2.5. $\square$

5 Inverse Cartier and periodic flows over W_n

In this chapter, we construct the inverse Cartier functor and define periodic flows in the W_n-level by induction on n. Assume that $n \geq 2$, let X_n be a smooth proper W_n-scheme of dimension d and X_{n-1} be its reduction modulo p^{n-1} .

5.1 Inverse Cartier over W_n

Let $(H, \nabla, Fil^{\cdot})$ be a de Rham bundle over X_n. Define $\tilde{H} := \oplus Fil^i / \sim$, where the equivalent relation $\sim$ is as follows: for $s \in Fil^i$, the image of s in $Fil^i \to Fil^{i-1}$ is equivalent to $p \cdot s \in Fil^i$. ∇ on H induces a p-connection $\tilde{\nabla}$ on $\tilde{H}$ as follows: for $s \in Fil^i$, denote by $\tilde{s}$ its image in $\tilde{H}$; Regard $\nabla(s)$ as a section of $Fil^{i-1} \otimes \Omega_{X_n}$, and denote by $\widetilde{\nabla(s)}$ its image in $\tilde{H} \otimes \Omega_{X_n}$, define

$$\tilde{\nabla}(\tilde{s}) := \widetilde{\nabla(s)}.$$

We called $(\tilde{H}, \tilde{\nabla})$ the twisted flat bundle (vector bundle with p-connection) associated to the de Rham bundle $(H, \nabla, Fil^{\cdot})$.

Lemma 5.1.1. *(i) For $i = 1, 2$, let $(H_i, \nabla_i, Fil_i^{\cdot})$ be two de Rham bundles over X_n and denote by $(\bar{H}_i, \bar{\nabla}_i, \bar{Fil}_i^{\cdot})$ the corresponding reduction modulo p^{n-1} objects over X_{n-1}. Suppose we have two isomorphisms :*

$$\bar{f}_{12} : (\bar{H}_1, \bar{\nabla}_1, \bar{Fil}_1^{\cdot}) \cong (\bar{H}_2, \bar{\nabla}_2, \bar{Fil}_2^{\cdot})$$

$$f_{12}^G : Gr_{Fil_1}(H_1, \nabla_1) \cong Gr_{Fil_2}(H_2, \nabla_2).$$

such that

$$Gr(\bar{f}_{12}) = f_{12}^G, \ (mod\ p^{n-1}).$$

Then there exists an isomorphism between the associated twisted flat bundles:

$$\tilde{f}_{12} : (\tilde{H}_1, \tilde{\nabla}_1) \cong (\tilde{H}_2, \tilde{\nabla}_2).$$

(ii) *If we have a third object* $(H_3, \nabla_3, Fil_3^{\cdot})$*, and two pairs of morphisms* $(\bar{f}_{23}, f_{23}^G)$*,* $(\bar{f}_{13}, f_{13}^G)$*, such that*

$$\bar{f}_{13} = \bar{f}_{23} \circ \bar{f}_{12},$$

$$f_{13}^G = f_{23}^G \circ f_{12}^G.$$

Then

$$\tilde{f}_{13} = \tilde{f}_{23} \circ \tilde{f}_{12}.$$

Proof. (i) Without loss of generality, we assume that the weight of the filtration is 1, i.e., $Fil_i^0 = H_i$, $Fil_i^2 = 0$. And set $Fil_i := Fil_i^1$. Now f_{12}^G induces an isomorphism $Fil_1 \cong Fil_2$, which is exactly $\tilde{f}_{12}$ restricted to Fil_1. For any section s of H_1 over any small open subscheme of X_n, denote by $\tilde{s}$ (resp. $\hat{s}$, $\bar{s}$) its image in $\tilde{H}_1$ (resp. H_1/Fil_1, $\bar{H}_1$). Consider the set of sections of H_2 whose image in H_2 mod p^{n-1} (resp. H_2/Fil_2) is $\bar{f}_{12}(\bar{s})$ (resp. $f_{12}^G(\hat{s})$). Note that the difference of any two such sections lies in $p^{n-1} \cdot Fil_2$. Thus it determines a unique section $\tilde{t}$ of $\tilde{H}_2$. Then define $\tilde{f}_{12} : \tilde{s} \to \tilde{t}$. We see easily that $\tilde{f}_{12}$ is a well-defined isomorphism.

(ii) This is obvious by the above construction.

□

Consider the following category $\mathcal{D}^\flat\mathcal{H}_n$: its object is a set of datum $(E, \theta, \bar{H}, \bar{\nabla}, \bar{Fil}^{\cdot}, \bar{\psi})$, where (E, θ) is a Higgs bundles nilpotent of order less than $p-1$ over X_n; $(\bar{H}, \bar{\nabla}, \bar{Fil}^{\cdot})$ is a filtered de Rham bundle over X_{n-1}; and $\bar{\psi}$ is an isomorphism

$$\bar{\psi} : Gr_{\bar{Fil}^{\cdot}}(\bar{H}, \bar{\nabla}) \cong (\bar{E}, \bar{\theta}),$$

where $(\bar{E}, \bar{\theta})$ is the reduction modulo p^{n-1} of (E, θ).

Corollary 5.1.2. *There exists a functor* $\mathcal{T}_n$ *from the category* $\mathcal{D}^\flat\mathcal{H}_n$ *to the category of twisted flat bundle over* X_n.

Proof. We first show that for a small enough affine open subscheme U of X_n, there exists a W_n-lifting $(H_U, \nabla_U, Fil^{\cdot}_U, \psi_U)$ of $(\bar{H}, \bar{\nabla}, \bar{Fil}^{\cdot}, \bar{\psi})$ over U, where

$$\psi_U : Gr_{Fil^{\cdot}_U}(H_U, \nabla_U) \cong (E|_U, \theta|_U).$$

Assume that $E = \oplus_{k=0}^m E^{m-k,k}$, $\theta = \oplus_{k=0}^{m-1} \theta_k$, with $\theta_k : E^{m-k,k} \to E^{m-k-1,k+1}$. For $0 \leq k \leq m$, we take a basis $\{e_k\}$ of $E^{m-k,k}$ over U. Under this basis, θ_k can be represented by a matrix of differential 1-forms, still denote by θ_k. Set $\bar{\theta}_k = \theta_k \mod p^{n-1}$, and $\bar{e}_k = e_k \, mod \, p^{n-1}$, for $0 \leq k \leq m$. Take a local basis $\{f'_k\}_{0 \leq k \leq m}$ of $Gr_{\overline{Fil}}(\bar{H})$ such that

$$\bar{\psi}(f'_k) = \bar{e}_k.$$

Also choose a local basis $\{\bar{f}_k\}_{0 \leq k \leq m}$ of $\bar{H}$ such that its image in $Gr_{\overline{Fil}}(\bar{H})$ is $\{f'_k\}_{0 \leq k \leq m}$. Under the basis $\{\bar{f}_k\}$, $\bar{\nabla}$ can be represented by a matrix of differential 1-forms $\bar{a}_{ij}$, i.e,

$$\bar{\nabla}(\bar{f}_i) = \sum \bar{a}_{ij} \bar{f}_j.$$

By Griffiths transversality, one can see that $(\bar{a}_{ij}) = 0$ for $j > i+1$. For $i \geq j$, take a matrix of differential 1-forms a_{ij} over U which lift $\bar{a}_{ij}$. For $j = i+1$, set $a_{ij} = \theta_i$. For $j > i+1$, set $a_{ij} = 0$.

Let H_U be a W_n-lifting of $\bar{H}|_U$ with a local basis $\{f_k\}_{0 \leq k \leq m}$, of which reduction modulo p^{n-1} basis is $\{\bar{f}_k\}_{0 \leq k \leq m}$. Let Fil^{m-k}_U be the subbundle of H_U generated by the base $\{f_j\}_{0 \leq j \leq k}$, which defines a filtration $Fil^{\cdot}_U$ on H_U. Then define a connection ∇_U on H_U as follows:

$$\nabla_U(f_i) = \sum_{i=0}^m a_{ij} f_j, \quad \forall \ i.$$

Denote by $\{\hat{f}_k\}_{0 \leq k \leq m}$ the corresponding basis of $Gr_{Fil^{\cdot}_U}(H_U, \nabla_U)$. Therefore, an isomorphism define $\psi_U : Gr_{Fil^{\cdot}_U}(H_U, \nabla_U) \cong (E|_U, \theta|_U)$ can be given by

$$\psi_U(\hat{f}_k) = e_k. \quad \forall \ k.$$

Next, let $(\tilde{H}, \tilde{\nabla})$ denote the twisted flat bundle associated to $(H_U, \nabla_U, Fil^{\cdot}_U)$, we want to show that $\tilde{\nabla}$ is integrable, i.e, $\tilde{\nabla}(\partial t_i)$ commute with $\tilde{\nabla}(\partial t_j)$ for a system of local coordinates $\{t_1, \cdots t_d\}$.

Let $\tilde{f}_k$ denote the image of $f_k \in Fil^{m-k}_U)$ in $\tilde{H}$. Then $\{\tilde{f}_k\}_{1 \leq k \leq m}$ form a basis of $\tilde{H}$ over U. As the matrix of ∇_U under the basis $\{f_k\}_{1 \leq k \leq m}$ is $A = (a_{ij})$ with $a_{ij} = 0$ for $j > i+1$, then the matrix of $\tilde{\nabla}$ under the basis $\{\tilde{f}_k\}_{1 \leq k \leq m}$ is $\tilde{A} = (\tilde{a}_{ij})$ with $\tilde{a}_{ij} = p^{i+1-j} a_{ij}$ for $j \leq i+1$, and the rest are zeros. Then $\tilde{\nabla}(\partial t_i) \circ \tilde{\nabla}(\partial t_j)$ under this basis is represented by the matrix

$$p \partial t_i (\partial t_j \lrcorner \tilde{A}) + (\partial t_j \tilde{A}) \cdot (\partial t_i \tilde{A}).$$

It suffices to show that

$$p\partial t_i(\partial t_j \lrcorner \tilde{A}) + (\partial t_j \tilde{A}) \cdot (\partial t_i \tilde{A}) - p\partial t_j(\partial t_i \lrcorner \tilde{A}) - (\partial t_i \tilde{A}) \cdot (\partial t_j \tilde{A}) = 0.$$

Or equivalently: for $0 \leq r \leq m-2$ with $t = r+1$ and $s = r+2$,

$$(\partial t_i \lrcorner a_{rt})(\partial t_j \lrcorner a_{ts}) - (\partial t_j \lrcorner a_{rt})(\partial t_i \lrcorner a_{ts}) = 0,$$

which is just the integrability of the Higgs field θ;

And for $0 \leq r, s \leq m$ with $s \leq r+1$,

$$p^{r+2-s}(\partial t_i \wedge \partial t_j) \lrcorner (da_{rs}) = p^{r+2-s} \sum_{t=0}^{m} \left[(\partial t_i \lrcorner a_{rt})(\partial t_j \lrcorner a_{ts}) - (\partial t_j \lrcorner a_{rt})(\partial t_i \lrcorner a_{ts})\right],$$

which follows from the integrability of $\bar{\nabla}$, i.e.,

$$(\partial \bar{t}_i \wedge \partial \bar{t}_j) \lrcorner (d\bar{a}_{rs}) = \sum_{t=0}^{m} \left[(\partial \bar{t}_i \lrcorner \bar{a}_{rt})(\partial \bar{t}_j \lrcorner \bar{a}_{ts}) - (\partial \bar{t}_j \lrcorner \bar{a}_{rt})(\partial \bar{t}_i \lrcorner \bar{a}_{ts})\right],$$

where $\{\bar{t}_1, \cdots, \bar{t}_d\}$ is the reduction of $\{t_1, \cdots, t_d\}$, and hence a system of local coordinate over $\bar{U}$.

Finally we take a small affine open covering $X_n = \cup_{i \in I} U_i$ (resp. $X_{n-1} = \cup_{i \in I} U_i^\flat$) such that over each U_i, there exists a lifting $(H_i, \nabla_i, Fil_i^\cdot, \psi_i)$ of $(\bar{H}, \bar{\nabla}, \bar{Fil}^\cdot, \bar{\psi})$ over U_i. We fix one such lifting over each U_i.

By the Lemma 5.1.1, there exist isomorphisms

$$\tilde{\psi}_{ij} : (\tilde{H}_i, \tilde{\nabla}_i)|_{U_{ij}} \cong (\tilde{H}_j, \tilde{\nabla}_j)|_{U_{ij}}, \quad \forall\ i, j,$$

such that $\tilde{\psi}_{jk} \circ \tilde{\psi}_{ij} = \tilde{\psi}_{ik}$ over U_{ijk}, $\forall\ i, j, k$. So we can glue them into a global twisted de Rham bundle $(\tilde{H}_{-1}, \tilde{\nabla}_{-1})$. Still by the Lemma 5.1.1, we see that $(\tilde{H}_{-1}, \tilde{\nabla}_{-1})$ is independent of the choice of the lifting $(H_i, \nabla_i, Fil_i^\cdot, \psi_i)$ of $(\bar{H}, \bar{\nabla}, \bar{Fil}^\cdot, \bar{\psi})$. Therefore, we obtain a well-defined functor

$$\mathcal{T}_n : (E, \theta, \bar{H}, \bar{\nabla}, \bar{Fil}^\cdot, \bar{\psi}) \to (\tilde{H}_{-1}, \tilde{\nabla}_{-1}).$$

□

Fixing a W_{n+1}-lifting X_{n+1} of X_n, we are going to define a functor C_n^{-1} from the category $\mathcal{D}^\flat\mathcal{H}_n$ to the category of flat bundles over X_n as follows: Given an object $(E, \theta, \bar{H}, \bar{\nabla}, \bar{Fil}^\cdot, \bar{\psi})$ of $\mathcal{D}^\flat\mathcal{H}_n$, by applying the functor $\mathcal{T}_n$ defined in Corollary 5.1.2, we obtain a twisted flat bundle $(\tilde{H}_{-1}, \tilde{\nabla}_{-1})$ over X_n.

Choose an affine covering $X_{n+1} = \cup_{i\in I} U_i^\sharp$ (resp. $X_n = \cup_{i\in I} U_i$). and Frobenius liftings $\{F_i^\sharp : U_i^\sharp \to U_i^\sharp\}_{i\in I}$ (resp. $\{F_i : U_i \to U_i\}_{i\in I}$). We first locally associated a flat bundle (H_i, ∇_i) to $(E, \theta, \bar{H}, \bar{\nabla}, \bar{Fil}^{\cdot}, \bar{\psi})$ over each U_i. Note that there is a map

$$\frac{dF_i^\sharp}{p} : F_i^*\Omega_{U_i} \to \Omega_{U_i}, \text{ over } U_i,$$

we can define a local flat bundle (H_i, ∇_i) as follow:

$$H_i := F_i^*(\tilde{H}_{-1}|_{U_i}),$$

and $\nabla_i(f \otimes e) = df \otimes e + f \cdot (\frac{dF_i^\sharp}{p} \otimes 1)(1 \otimes \tilde{\nabla}_{-1}(e))$, for $e \in \tilde{H}_{-1}|_{U_i}$ and $f \in \mathcal{O}_{U_i}$.

Lemma 5.1.3. *∇_i is a well-defined integrable connection on H_i.*

Proof. First, we show that ∇_i is well-defined, i.e., $\nabla_i(1 \otimes fe) = \nabla_i(F_i^*(f) \otimes e)$. This can be verified by direct calculation as follows:

$$\begin{aligned}\nabla_i(1 \otimes fe) &= (\tfrac{dF_i^\sharp}{p} \otimes 1)(1 \otimes \tilde{\nabla}_{-1}(fe)) = (\tfrac{dF_i^\sharp}{p} \otimes 1)(1 \otimes pdf \cdot e + 1 \otimes f \cdot \tilde{\nabla}_{-1}(e)) \\ &= d(F_i^*(f)) \otimes e + F_i^*(f) \cdot (\tfrac{dF_i^\sharp}{p} \otimes 1)(1 \otimes \tilde{\nabla}_{-1}(e)) = \nabla_i(F_i^*(f) \otimes e).\end{aligned}$$

where the second equality follows from the fact that $\tilde{\nabla}_{-1}$ is a p-connection, i.e., $\tilde{\nabla}_{-1}(fe) = pdf \cdot e + f\tilde{\nabla}_{-1}(e)$.

Next, we show that ∇_i is integrable. Fix a system of local coordinates $\{t_1, \cdots t_d\}$ of U_i. By definition, $\tilde{\nabla}_{-1}$ is integrable, i.e., $\tilde{\nabla}_{-1}(\partial t_\alpha)$ commutes with $\tilde{\nabla}_{-1}(\partial t_\beta)$ for $1 \leq \alpha, \beta \leq d$. One cansee that for $1 \leq j, k \leq d$ and $e \in \tilde{H}_{-1}|_{U_i}$,

$$\nabla_i(\partial t_j)(e) = \sum_{\alpha=1}^{d} \left(\partial t_j \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\alpha) \right) \otimes \tilde{\nabla}_{-1}(\partial t_\alpha)(e).$$

By calculation, we obtain

$$\begin{aligned}\nabla_i(\partial t_k) \circ \nabla_i(\partial t_j)(1 \otimes e) &= \textstyle\sum_{\alpha=1}^d \partial_{t_k} \left(\partial t_j \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\alpha) \right) \otimes \tilde{\nabla}_{-1}(\partial t_\alpha)(e) \\ &+ \textstyle\sum_{1\leq \alpha,\beta \leq d} \left(\partial t_j \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\alpha) \right) \cdot \left(\partial t_k \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\beta) \right) \otimes \tilde{\nabla}_{-1}(\partial t_\beta) \circ \tilde{\nabla}_{-1}(\partial t_\alpha)(e) \\ &= \textstyle\sum_{\alpha=1}^d \partial_{t_j} \left(\partial t_k \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\alpha) \right) \otimes \tilde{\nabla}_{-1}(\partial t_\alpha)(e) \\ &+ \textstyle\sum_{1\leq \alpha,\beta \leq d} \left(\partial t_j \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\alpha) \right) \cdot \left(\partial t_k \lrcorner \frac{dF_i^\sharp}{p}(1 \otimes dt_\beta) \right) \otimes \tilde{\nabla}_{-1}(\partial t_\alpha) \circ \tilde{\nabla}_{-1}(\partial t_\beta)(e) \\ &= \nabla_i(\partial t_j) \circ \nabla_i(\partial t_k)(1 \otimes e),\end{aligned}$$

where the second equality is due to the fact that $\tilde{\nabla}_{-1}(\partial t_\alpha)$ commutes with $\tilde{\nabla}_{-1}(\partial t_\beta)$.

Therefore, for all $f \in \mathcal{O}_{U_i}$, $e \in \tilde{H}_{-1}|_{U_i}$, $1 \leq k, j \leq d$, we have

$$\begin{aligned}
&\nabla_i(\partial t_k) \circ \nabla_i(\partial t_j)(f \otimes e) \\
&= \tfrac{\partial^2 f}{\partial t_k \partial t_j} \otimes e + \tfrac{\partial f}{\partial t_j} \cdot \nabla_i(\partial t_k)(1 \otimes e) + \tfrac{\partial f}{\partial t_k} \cdot \nabla_i(\partial t_j)(1 \otimes e) + f \cdot \nabla_i(\partial t_k)\nabla_i(\partial t_j)(1 \otimes e) \\
&= \tfrac{\partial^2 f}{\partial t_j \partial t_k} \otimes e + \tfrac{\partial f}{\partial t_j} \cdot \nabla_i(\partial t_k)(1 \otimes e) + \tfrac{\partial f}{\partial t_k} \cdot \nabla_i(\partial t_j)(1 \otimes e) + f \cdot \nabla_i(\partial t_j)\nabla_i(\partial t_k)(1 \otimes e) \\
&= \nabla_i(\partial t_i) \circ \nabla_i(\partial t_k)(f \otimes e),
\end{aligned}$$

which means that ∇_i is integrable. □

Next we glue locally defined flat bundles (H_i, ∇_i) to a global one by the Taylor formula as follows: choose a system of local coordinates $\{t_1^\sharp, \cdots, t_d^\sharp\}$ over $U_{ij}^\sharp$ (resp. $\{t_1, \cdots, t_d\}$ over U_{ij}).

Define an isomorphism

$$G_{ij} : H_i|_{U_{ij}} \cong H_j|_{U_{ij}}$$

as follows: for any section $\tilde{s}$ of $\tilde{H}_{-1}$ over U_{ij},

$$G_{ij}(\tilde{s} \otimes 1) = \sum_J \tilde{\nabla}(\partial)^J(\tilde{s}) \otimes \frac{z^J}{J!}, \tag{5.1.3.1}$$

where the sum is over all multi-indices $J := (j_1, \cdots, j_d)$, $J! = \prod_{k=1}^d j_k!$,

$$\tilde{\nabla}(\partial)^J := (\tilde{\nabla}_{\partial t_1})^{j_1} \circ \cdots \circ (\tilde{\nabla}_{\partial t_d})^{j_d},$$

and $z^j = \prod_{k=1}^d z_k^{j_k}$, with

$$z_k = \frac{(F_i^\sharp)^*(t_k^\sharp) - (F_j^\sharp)^*(t_k^\sharp)}{p}. \tag{5.1.3.2}$$

Lemma 5.1.4. *The above $\{G_{ij}\}_{i,j\in I}$ satisfies the cocycle condition and is compatible with connection $\{\nabla_i\}_{i\in I}$, thus it defines a global de Rham bundle $(H, \nabla) = C_n^{-1}(E, \theta)$.*

Proof. First we check the cocycle condition by direct calculation.

Similarly as in (5.1.3.2), let $z_l := \frac{(F_i^\sharp)^*(t_l^\sharp) - (F_j^\sharp)^*(t_l^\sharp)}{p}$ over U_{ij}, $\hat{z}_l = \frac{(F_j^\sharp)^*(t_l^\sharp) - (F_k^\sharp)^*(t_l^\sharp)}{p}$ over U_{jk}, $\tilde{z}_l = \frac{(F_i^\sharp)^*(t_l^\sharp) - (F_k^\sharp)^*(t_l^\sharp)}{p}$ over U_{ik}. Then $\tilde{z}_l = z_l + \hat{z}_l$ over U_{ijk}. For any section $\tilde{s} \in \tilde{H}_{-1}$ over U_{ijk},

$$\begin{aligned}
G_{jk} \circ G_{ij}(\tilde{s} \otimes 1) &= \textstyle\sum_I \sum_J \tilde{\nabla}(\partial)^{I+J}(\tilde{s}) \otimes \frac{z^J}{J!} \cdot \frac{\hat{z}^I}{I!} \\
&= \textstyle\sum_K \tilde{\nabla}(\partial)^K(\tilde{s}) \otimes \left(\sum_{I+J=K} \frac{z^J}{J!} \cdot \frac{\hat{z}^I}{I!}\right) \\
&= \textstyle\sum_K \tilde{\nabla}(\partial)^K(\tilde{s}) \otimes \frac{(z+\hat{z})^K}{K!} \\
&= \textstyle\sum_K \tilde{\nabla}(\partial)^K(\tilde{s}) \otimes \frac{\tilde{z}^K}{K!} \\
&= G_{ik}(\tilde{s} \otimes 1).
\end{aligned}$$

Thus $G_{jk} \circ G_{ij} = G_{ik}$.

Next we prove that G_{ij} is horizontal. For $1 \leq k \leq d$, we identity k with the multiple index $(0, \cdots, 1, \cdots, 0)$, which vanishes at all positions except the k-th. Let $\tilde{s}$ be a section of $\tilde{H}_{-1}$ over U_{ij}, then

$$(id \otimes G_{ij})(\nabla_i(\tilde{s} \otimes 1)) = \sum_J \sum_{k=1}^d \tilde{\nabla}(\partial)^{J+k}(\tilde{s}) \otimes \frac{z^J}{J!} \cdot \frac{dF_i^\sharp}{p}(dt_k),$$

and

$$\nabla_j(G_{ij}(\tilde{s} \otimes 1)) = \sum_J \sum_{k=1}^d \tilde{\nabla}(\partial)^{J+k}(\tilde{s}) \otimes \frac{z^J}{J!} \cdot \left(\frac{dF_j^\sharp}{p}(dt_k) + dz_k\right).$$

As

$$dz_k = \frac{dF_i^\sharp}{p}(dt_k) - \frac{dF_j^\sharp}{p}(dt_k),$$

Clearly

$$(id \otimes G_{ij})(\nabla_i(\tilde{s}) \otimes 1) = \nabla_j(G_{ij}(\tilde{s} \otimes 1)).$$

We complete the proof. □

Definition 5.1.5. By the above construction, we define the W_n-level inverse Cartier functor $C_n^{-1} : \mathcal{D}^\flat\mathcal{H}_n \to MIC(X_n)$.

Remark 5.1.6. The construction of C_n^{-1} depends on the W_{n+1}-lifting X_{n+1} of X_n.

We have the following obvious lemma.

Lemma 5.1.7. *Suppose that $(H, \nabla, Fil^\cdot, \varphi)$ is a strict p^n-torsion Fontaine-Faltings module over X_n. Let $(\bar{H}, \bar{\nabla}, \bar{Fil}^\cdot)$ denote its reduction modulo p^{n-1} object over X_{n-1}. Let $(E, \theta) = Gr_{Fil^\cdot}(H, \nabla)$. Using these datum and we can construct $C_n^{-1}(E, \theta)$. Then φ induces an isomorphism:*

$$\tilde{\varphi} : C_n^{-1}(E, \theta) \cong (H, \nabla).$$

5.2 Equivalent of categories over W_n

Definition 5.2.1. A 1-periodic Higgs- de Rham flow (call 1-periodic flow for short) over X_n consists of the following data:

(1) A 1-periodic flow over X_{n-1} :

$$\begin{tikzcd}
& (C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0), \bar{Fil}^{\cdot}_0) \arrow[dr, "Gr_{\bar{Fil}^{\cdot}_0}"] & \\
(\bar{E}_0,\bar{\theta}_0) \arrow[ur, "C_{n-1}^{-1}"] & & Gr_{\bar{Fil}^{\cdot}_0} \circ C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0) \arrow[ll, bend left, "\bar{\psi}", "\cong"']
\end{tikzcd}$$

(2) A Higgs bundle (E,θ) nilpotent of order less than $p-1$ over X_n, whose reduction modulo p^{n-1} is exactly $(\bar{E}_0,\bar{\theta}_0)$.

(3) A lifting of $\bar{Fil}^{\cdot}_0$, i.e., a filtration $Fil^{\cdot}$ on $C_n^{-1}(E,\theta)$ which is defined w.r.t. the data $(C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0), \bar{Fil}^{\cdot}_0, \bar{\psi})$ and (E,θ).

(4) A lifting of $\bar{\psi}$, i.e., an isomorphism

$$\psi : Gr_{Fil^{\cdot}}(C_n^{-1}(E,\theta)) \cong (E,\theta).$$

Theorem 5.2.2. *The category of1-periodic flows over X_n is equivalent to the category of strict p^n-torsion Fontaine-Faltings modules over X_n.*

Proof. The proof is similar to the case of $n=1$. First we define a functor $\mathcal{GR}_n$ from the category of strict p^n-torsion Fonatine-Faltings modules to the category of 1-periodic flows over X_n.

Given a strict p^n-torsion Fontaine-Faltings module $(H,\nabla,Fil^{\cdot},\varphi)$, we know that its reduction modulo p^{n-1} induces a periodic flow over X_{n-1}:

$$\begin{tikzcd}
& (C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0), \bar{Fil}^{\cdot}_0) \arrow[dr, "Gr_{\bar{Fil}^{\cdot}_0}"] & \\
(\bar{E}_0,\bar{\theta}_0) \arrow[ur, "C_{n-1}^{-1}"] & & Gr_{\bar{Fil}^{\cdot}_0} \circ C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0). \arrow[ll, bend left, "\bar{\psi}", "\cong"']
\end{tikzcd}$$

Using the data $(C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0), \bar{Fil}^{\cdot}_0, \bar{\psi})$ and $(E,\theta) := Gr_{Fil^{\cdot}}(H,\nabla)$, we obtain a de Rham bundle $C_n^{-1}(E,\theta)$ over X_n.

By Lemma 5.1.7, we see that $\tilde{\varphi} : C_n^{-1}(E,\theta) \cong (H,\nabla)$; The pullback of the filtration induces a filtration $Fil^{\cdot}_{exp}$ on $C_n^{-1}(E,\theta)$. By taking grading, we get an isomorphism

$$\psi := Gr(\tilde{\varphi}) : Gr_{Fil^{\cdot}_{\mathrm{exp}}}(C_n^{-1}(E,\theta)) \cong Gr_{Fil^{\cdot}}(H,\nabla) = (E,\theta).$$

Thus we obtain a 1-periodic flow over X_n, and define $\mathcal{GR}_n(H,\nabla,Fil^{\cdot},\varphi)=(E,\theta,Fil^{\cdot}_{exp},\psi)$.

Next, we show that the functor C_n^{-1} induces a functor $\mathcal{C}_n^{-1}$ in the opposite direction. Given a 1-periodic flow over X_n

$$\begin{tikzcd}
 & (C_n^{-1}(E,\theta), Fil^{\cdot}) \arrow[dr, "Gr_{Fil^{\cdot}}"] & \\
(E,\theta) \arrow[ur, "C_n^{-1}"] & & Gr_{Fil^{\cdot}} \circ C_n^{-1}(E,\theta). \arrow[ll, bend left, "\psi"', "\cong"]
\end{tikzcd}$$

We obtain a filtered de Rham bundle $(H,\nabla,Fil^{\cdot}):=(C_n^{-1}(E,\theta),Fil^{\cdot})$. It suffices to define an isomorphism φ. Consider the construction of C_n^{-1}, using the mod p^{n-1} periodic flow

$$\begin{tikzcd}
 & (C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0), \bar{Fil}^{\cdot}_0) \arrow[dr, "Gr_{\bar{Fil}^{\cdot}_0}"] & \\
(\bar{E}_0,\bar{\theta}_0) \arrow[ur, "C_{n-1}^{-1}"] & & Gr_{\bar{Fil}^{\cdot}_0} \circ C_{n-1}^{-1}(\bar{E}_0,\bar{\theta}_0). \arrow[ll, bend left, "\bar{\psi}"', "\cong"]
\end{tikzcd}$$

By Corollary 5.1.2, ψ induces an isomorphism :

$$\mathcal{T}(\psi):(\tilde{H},\tilde{\nabla})\cong(\tilde{H}_{-1},\tilde{\nabla}_{-1}),$$

where the right hand side is the twisted flat bundle appeared in the construction of C_n^{-1}. For $i\in I$, we define an isomorphism

$$\varphi_i=F_i^*(\mathcal{T}(\psi)):F_i^*\tilde{H}|_{U_i}\cong F_i^*(\tilde{H}_{-1})|_{U_i}=H|_{U_i},$$

which fits into the following commutative diagram:

$$\begin{tikzcd}
\left(F_i^*\tilde{H},\frac{dF_i^\sharp}{p}(F_i^*(\tilde{\nabla}))\right) \arrow[r, "\varphi_i"] \arrow[d, "G_{ij}"'] & (F_i^*\tilde{H}_{-1},\nabla_i) \arrow[d, "G_{ij}"] \\
\left(F_j^*\tilde{H},\frac{dF_j^\sharp}{p}(F_j^*(\tilde{\nabla}))\right) \arrow[r, "\varphi_j"] & (F_j^*\tilde{H}_{-1},\nabla_j).
\end{tikzcd}$$

where "$\frac{dF_i^\sharp}{p}(F_i^*(\tilde{\nabla}))$" is viewed as a connection induced by the map $\frac{dF_i^\sharp}{p}(F_i^*(\tilde{\nabla}))$ via the Leibniz rule, G_{ij} on the left side is the given by (5.1.3.1) with respect to $\tilde{\nabla}$, and G_{ij} on the right side is with respect to $\tilde{\nabla}_{-1}$.

This diagram implies that the $\{\varphi_i\}$'s are related by the Taylor formula and hence are horizontal. So we obtain a strict p^n-torsion Fontaine-Faltings module $(H, \nabla, Fil^\cdot, \varphi)$ over X_n. And the functor $\mathcal{C}_n^{-1}$ is defined as

$$\mathcal{C}_n^{-1}(E, \theta, Fil^\cdot, \psi) = (H, \nabla, Fil^\cdot, \varphi).$$

Finally, we prove the following isomorphisms of functors:

$$\mathcal{GR}_n \circ \mathcal{C}_n^{-1} \cong Id, \qquad \mathcal{C}_n^{-1} \circ \mathcal{GR}_n \cong Id.$$

First define a natural isomorphism $\mathcal{A}_n$ from $\mathcal{C}_n^{-1} \circ \mathcal{GR}_n$ to Id. Given a strict p^n-torsion Fontaine Faltings module $(H, \nabla, Fil^\cdot, \varphi)$ on X_n, set

$$(E, \theta, Fil^\cdot, \psi) := \mathcal{GR}_n(H, \nabla, Fil^\cdot, \varphi), \qquad (H', \nabla'.Fil'^\cdot, \varphi') := \mathcal{C}_n^{-1}(E, \theta, Fil^\cdot, \psi).$$

Then we define $\mathcal{A}_n(H, \nabla, Fil^\cdot, \varphi)$ to be

$$\tilde{\varphi} : (H', \nabla', Fil'^\cdot) = (C_n^{-1} \circ Gr_{Fil^\cdot}(H, \nabla), Fil^\cdot) \cong (H, \nabla, Fil^\cdot).$$

To check that $\tilde{\varphi}$ is an isomorphism in the category of strict p^n-torsion Fontaine-Faltings modules, we consider the induced isomorphism $\tilde{\tilde{\varphi}}$

$$\tilde{\tilde{\varphi}} : (\tilde{H}', \tilde{\nabla}') \cong (\tilde{H}, \tilde{\nabla}).$$

and the following commutative diagram over over each U_i:

$$\begin{tikzcd} F_i^*\tilde{H}' \arrow[r, "\varphi_i'"] \arrow[d, "F_i^*\tilde{\tilde{\varphi}}"'] & H' \arrow[d, "\tilde{\varphi}"] \\ F_i^*\tilde{H} \arrow[r, "\varphi_i"] & H. \end{tikzcd}$$

from which it is clear that $\tilde{\varphi}$ has the required property.

Now suppose that we have a morphism of Fontaine-Faltings modules

$$f : (H_1, \nabla_1, Fil_1^\cdot, \varphi_1) \to (H_2, \nabla_2, Fil_2^\cdot, \varphi_2),$$

then we obtain

$$\mathcal{C}_n^{-1} \circ \mathcal{GR}_n(f) = C_n^{-1}Gr(f) : \mathcal{C}_n^{-1} \circ \mathcal{GR}_n(H_1) = H_1' \to H_2' = \mathcal{C}_n^{-1} \circ \mathcal{GR}_n(H_2).$$

To see that $\mathcal{A}_n$ is a natural equivalence, it is equivalent to show the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C}_n^{-1}\circ\mathcal{GR}_n(H_1)=H_1' & \xrightarrow{\mathcal{C}_n^{-1}\circ\mathcal{GR}_n(f)} & H_2'=\mathcal{C}_n^{-1}\circ\mathcal{GR}_n(H_2) \\ \Big\downarrow{\scriptstyle\mathcal{A}_{H_1}} & & \Big\downarrow{\scriptstyle\mathcal{A}_{H_2}} \\ H_1 & \xrightarrow{f} & H_2. \end{array} \tag{5.2.2.1}$$

The commutativity of (5.2.2.1) can be checked locally. Over each U_i, by definition of morphism in the category of Fontaine-Faltings modules, we have the following diagram:

$$\begin{array}{ccc} F_i^*\tilde{H}_1|_{U_i} & \xrightarrow{F_i^*Gr(f)} & F_i^*\tilde{H}_2|_{U_i} \\ \Big\downarrow{\scriptstyle\varphi_{1,i}} & & \Big\downarrow{\scriptstyle\varphi_{2,i}} \\ H_1 & \xrightarrow{f} & H_2. \end{array}$$

which implies that 5.2.2.1 is commutative over U_i.

Second define a natural isomorphism $\mathcal{B}_n$ from $\mathcal{GR}_n\circ\mathcal{C}_n^{-1}$ to Id. Given a 1-periodic flow $(E,\theta,Fil^{\cdot},\psi)$ over X_n, set

$$(H,\nabla,Fil,\varphi)=\mathcal{C}_n^{-1}(E,\theta,Fil^{\cdot},\psi)\quad (E',\theta',Fil'^{\cdot},\psi')=\mathcal{GR}_n(H,\nabla,Fil^{\cdot},\varphi).$$

Note that $E'=GrC_n^{-1}(E)$,

$$\tilde{\varphi}=C_n^{-1}(\psi):C_n^{-1}(E')\cong C_n^{-1}(E).$$

$\psi'=Gr(\tilde{\varphi})$, and $Fil'^{\cdot}$ is just the pullback via $\tilde{\varphi}$ of the filtration $Fil^{\cdot}$.

Now we define $\mathcal{B}(E,\theta,Fil^{\cdot},\psi)$ to be

$$\psi:E'=GrC_n^{-1}(E)\cong E.$$

Because $C_n^{-1}(\psi)=\tilde{\varphi}$ is compatible with filtration, and the following diagram is commutative:

$$\begin{array}{ccc} GrC_n^{-1}(E') & \xrightarrow{\psi'=Gr(\tilde{\varphi})} & E' \\ \Big\downarrow{\scriptstyle GrC_n^{-1}(\psi)=Gr(\tilde{\varphi})} & & \Big\downarrow{\scriptstyle\psi} \\ GrC_n^{-1}(E) & \xrightarrow{\psi} & E, \end{array}$$

thus $\mathcal{B}(E,\theta,Fil^{\cdot},\psi)=\psi$ is an isomorphism from $(E',\theta',Fil'^{\cdot},\psi')$ to $(E,\theta,Fil^{\cdot},\psi)$ in the category of periodic flows .

Now suppose that we have a morphism in the category of periodic flows,

$$g:(E_1,\theta_1,Fil_1^{\cdot},\psi_1)\to(E_2,\theta_2,Fil_2^{\cdot},\psi_2).$$

then we obtain $\mathcal{GR}_n\circ\mathcal{C}_n^{-1}(g)=GrC_n^{-1}(g):E_1'\to E_2'$.

By definition of the morphism in the category of periodic flows , g satisfies the following commutative diagram:

$$\begin{array}{ccc} E_1' & \xrightarrow{GrC_n^{-1}(g)} & E_2' \\ {\scriptstyle \mathcal{B}(E_1,\theta_1,Fil_1,\psi_1)=\psi_1}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(E_2,\theta_2',Fil_2,\psi_2)=\psi_2} \\ E_1 & \xrightarrow{g} & E_2. \end{array}$$

Therefore, $\mathcal{B}$ is a natural equivalence between $\mathcal{GR}_n \circ \mathcal{C}_n^{-1}$ and Id.

□

Finally for natural number $f > 1$, we can define the category of periodic length f Higgs-de Rham flows in a similar way as characteristic p case in Section 4.1. and Using the same algebraic trick as in Section 4.2, we have the following:

Theorem 5.2.3. *There is a one to one correspondence between the category of dual crystalline $W_n(\mathbb{F}_{p^f})$-representation of $\pi(\mathbf{X}^0)$ and the category of periodic length f Higgs-de Rham flows over X_n.*

We conclude this chapter with an example.

Example 5.2.4. Let A_1 be an ordinary abelian variety defined over k of dimension g. By Serre-Tate theory, it has the canonical lifting A over $W(k)$ with the Frobenius lifting $F : A \to A$. Consider the following Higgs bundle (E, θ) over A/W:

$$E^{1,0} \oplus E^{0,1} = \Omega_A \oplus \mathcal{O}_A, \qquad \theta^{1,0} = id : \Omega_A \to \mathcal{O}_A \otimes \Omega_A.$$

We claim that the above Higgs bundle is 1-periodic. Let $(E_n, \theta_n) \equiv (E, \theta)\ mod\ p^n, \forall n$. First we construct a 1-periodic Higgs-de Rham flow on A_1. Since A_2 has the global Frobenius lifting F_2, it follows that

$$C_1^{-1}(E_1, \theta_1) := (H_1, \nabla_1),$$

with

$$H_1 := F_1^* E_1 \qquad \text{and} \qquad \nabla_1 = \nabla_{can} + \frac{dF_2}{p}(F_1^*\theta_1).$$

Define a weight-1 Hodge filtration on (H_1, ∇_1) is defined by $Fil_1 = F_1^*\Omega_{A_1} = \Omega_{A_1}$. Set

$$(E_1', \theta_1') := Gr_{Fil_1}(H_1, \nabla_1).$$

Then

$$E_1' = \Omega_{A_1} \oplus \mathcal{O}_{A_1}, \qquad \theta_1'^{1,0} = \frac{dF_2}{p}(F_1^*\theta_1).$$

Because A_1 is ordinary, the Hasse-Witt map

$$\frac{dF_2}{p} : H^0(A_1, \Omega_{A_1}) \to H^0(A_1, \Omega_{A_1})$$

is an isomorphism, and thus $\theta_1'^{1,0}$ is an isomorphism. We proceed to show that (E_1', θ_1') is isomorphic to (E_1, θ_1). Indeed, there is one natural choice of an isomorphism $\psi_1 : E_1' \to E_1$ of isomorphisms described as follows: its $(0,1)$-component mapping $\mathcal{O}_{A_1}$ to itself is the identity, and its $(1,0)$-component mapping Ω_{A_1} to itself is the unique isomorphism commuting with the Higgs fields. Therefore, we have obtained a 1-periodic flow over A_1 as claimed:

$$\begin{tikzcd}
 & ((H_1, \nabla_1), Fil_1) \arrow[dr, "Gr_{Fil_1}"] & \\
(E_1, \theta_1) \arrow[ur, "C_1^{-1}"] & & (E_1', \theta_1'). \arrow[ll, bend left, "\overset{\psi_1}{\cong}"]
\end{tikzcd}$$

Next we proceed to the W_2-level. Using the fact that A_3 has the Frobenius lifting F_3, one computes that $\tilde{H}_{-1,2} = \Omega_{A_2} \oplus \mathcal{O}_{A_2}$, and $\tilde{\nabla}_{-1,2} = p\nabla_{can} + \theta_2$. So

$$C_2^{-1}(E_2, \theta_2) = (H_2, \nabla_2),$$

where

$$H_2 = F_2^* E_2 = \Omega_{A_2} \oplus \mathcal{O}_{A_2} \qquad \text{and} \qquad \nabla_2 = \nabla_{can} + \frac{dF_3}{p}(F_2^* \tilde{\nabla}_{-1,2}).$$

Now we take the filtration $Fil_2 = \Omega_{A_2}$, which lifts Fil_1. Then the associated graded Higgs bundle is

$$E_2' = \Omega_{A_2} \oplus \mathcal{O}_{A_2},\ \theta_2'^{1,0} = \frac{dF_3}{p}(F_2^* \theta_2)$$

which lifts (E_1', θ_1'). Again there is an obvious isomorphism

$$\psi_2 : (E_2', \theta_2') \to (E_2, \theta_2)$$

which lifts ψ_1 and whose $(0,1)$-component is the identity map. So we obtain a 1-periodic flow over A_2:

$$\begin{tikzcd}
 & ((H_2, \nabla_2), Fil_2) \arrow[dr, "Gr_{Fil_2}"] & \\
(E_2, \theta_2) \arrow[ur, "C_2^{-1}"] & & (E_2', \theta_2') \arrow[ll, bend left, "\overset{\psi_2}{\cong}"]
\end{tikzcd}$$

Then one continues and constructs inductively a 1-periodic flow over A_n, for $n \geq 1$:

$$\begin{tikzcd}
& ((H_n, \nabla_n), Fil_n) \arrow[dr, "Gr_{Fil_n}"] & \\
(E_n, \theta_n) \arrow[ur, "C_n^{-1}"] & & (E'_n, \theta'_n). \arrow[ll, bend left, "\psi_n"', "\cong"]
\end{tikzcd}$$

By taking the inverse limit, we obtain a 1-periodic Higgs-de Rham flow over A/W, and hence a rank-$g+1$ crystalline $\mathbb{Z}_p$-representation of Hodge-Tate weight one of the generic fiber A^0 of A/W, which corresponds to a p-divisible group over A/W by [Fa][Theorem 7.1].

6 Rudimentary results

Let ρ be a crystalline $\mathbb{F}_{p^f}$-representation of $\pi_1(\mathbf{X}^0)$, and $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi)$ the corresponding periodic Higgs-de Rham flow of length f. For

$$(E_f, \theta_f) = Gr_{Fil_{f-1}}(H_{f-1}, \nabla_{f-1}),$$

by pull back $C_1^{-1}(\phi)$ induces a filtration $C_1^{-1}(\phi)^* Fil^{\cdot}_0$ on $C_1^{-1}(E_f, \theta_f)$ and an isomorphism $GrC_1^{-1}(\phi)$ of Higgs bundles by taking the grading. It is easy to check that

$$(E_1, \theta_1, Fil^{\cdot}_1, \cdots, Fil^{\cdot}_{f-1}, C_1^{-1}(\phi)^* Fil^{\cdot}_0, GrC_1^{-1}(\phi))$$

is an object in $\mathcal{HB}_{n,f}(X_2)$, which is called the *shift* of $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{f-1}, \phi)$. For any multiple $lf, l \geq 1$, we can lengthen $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{f-1}, \phi)$ to an object $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{lf-1}, \phi_l)$ of $\mathcal{HB}_{n,lf}(X_2)$ similarly as above: define inductively the induced filtration on $(H_j, \nabla_j), f \leq j \leq lf-1$ from $Fil^{\cdot}_i$s via ϕ; Define the isomorphism $\phi_l : (E_{lf}, \theta_{lf}) \cong (E_0, \theta_0)$ to be the composite of the isomorphisms of Higgs bundles $(GrC_1^{-1})^{l'f}(\phi) : (E_{(l'+1)f}, \theta_{(l'+1)f}) \cong (E_{l'f}, \theta_{l'f}), 0 \leq l' \leq l-1$. $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{lf-1}, \phi_l)$ is called the l-th *lengthening* of $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{f-1}, \phi)$. By the above construction, we obtain the following result.

Proposition 6.0.5. *Let ρ and $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{f-1}, \phi)$ be as above. Then:*

(i) The shift of $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{f-1}, \phi)$ corresponds to $\rho^\sigma = \rho \otimes_{\mathbb{F}_{p^f}, \sigma} \mathbb{F}_{p^f}$, the σ-conjugation of ρ. Here $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^f} | \mathbb{F}_p)$ is the Frobenius element.

(ii) For $l \in \mathbb{N}$, the l-th lengthening of $(E, \theta, Fil^{\cdot}_0, \cdots, Fil^{\cdot}_{f-1}, \phi)$ corresponds to the base extension $\rho \otimes_{\mathbb{F}_{p^f}} \mathbb{F}_{p^{lf}}$.

Given a periodic Higgs-de Rham flow

$$\begin{tikzcd}
& (H_0, \nabla_0) \arrow[dr, "Gr_{Fil_0}"] & & (H_1, \nabla_1) \arrow[dr, "Gr_{Fil_1}"] & \\
(E_0, \theta_0) \arrow[ur, "C_1^{-1}"] & & (E_1, \theta_1) \arrow[ur, "C_1^{-1}"] & & \ldots,
\end{tikzcd}$$

we make the following observation:

Lemma 6.0.6. *If $(E,\theta)=(E_0,\theta_0)$ is Higgs stable, then there is a unique periodic Higgs-de Rham flow with leading term (E,θ) up to isomorphism and l-lengthening.*

Proof. For $(E,\theta,Fil_0^{\cdot},\cdots,Fil_{f-1}^{\cdot},\phi)\in\mathcal{HB}_{n,f}(X_2)$, with (E,θ) Higgs stable, we show that the data $Fil_i^{\cdot}$, for $0\leq i\leq f-1$ and ϕ are uniquely determined up to isomorphism. By Theorem 4.2.5, it corresponds to

$$(H,Fil^{\cdot},\nabla,\varphi,\iota)\in\mathfrak{MF}^{\nabla}_{[0,n],1}(X_2)$$

satisfying $Gr_{Fil}(H,\nabla)=\bigoplus_{i=1}^{f}(E_i,\theta_i)$. Since

$$(Gr_{Fil}\circ C_1^{-1})^i(E_f,\theta_f)=(E_i,\theta_i),\quad 1\leq i\leq f-1,$$

each (E_i,θ_i) is also Higgs stable by Corollary 4.4 [SZ]. Because the filtration on a flat bundle which satisfies the Griffiths transversality and whose grading is Higgs stable, we see inductively that $Fil_i^{\cdot}$ is unique. Now we consider ϕ. For another choice φ, one notes that $\varphi\circ\phi^{-1}$ is an automorphism of (E,θ). As it is stable, one must have $\varphi=\lambda\phi$ for a nonzero λ in k. It is easy to see there is an isomorphism in $\mathcal{HB}_{n,f}(X_2)$:

$$(E,\theta,Fil_0,\cdots,Fil_{f-1}^{\cdot},\phi)\cong(E^{\cdot},\theta,Fil_0^{\cdot},\cdots,Fil_{f-1}^{\cdot},\lambda\phi).$$

□

Theorem 6.0.7. *Let $\mathbf{X}$ be a smooth proper scheme over $W(k)$. For $i=1,2$, let $(H_i,\nabla_i,Fil_i,\varphi_i)$ be a torsion free Fontaine-Faltings modules over $\mathbf{X}$ and (E_i,θ_i) be the associated Higgs bundles of the respective filtered de Rham bundles. If (E_1,θ_1) is isomorphic to (E_2,θ_2) and E_i,θ_i mod p is Higgs stable for $i=1,2$. Then $(H_i,\nabla_i,Fil_i^{\cdot},\varphi_i)$ are isomorphic.*

Proof. We prove that $(H_1,\nabla_1,Fil_1,\varphi_1)\cong(H_2,\nabla_2,Fil_2,\varphi_2)$ *modulo* p^n by induction on n. the mod p^n objects are isomorphic.

Let $(H_{i,n},\nabla_{i,n}Fil_{i,n}^{\cdot})$ and $(E_{i,n},\theta_{i,n})$ be the reduction modulo p^n of $(H_i,\nabla_i,Fil_i^{\cdot})$ and (E_i,θ_i). For $n=1$, it is already proved by Lemma6.0.6. Assume that $(H_1,\nabla_1,Fil_1,\varphi_1)\cong(H_2,\nabla_2,Fil_2,\varphi_2)$ *modulo* p^{n-1} Apply the W_n-level inverse Cartier to $(E_{i,n},\theta_{i,n})$, we obtain

$$(H_{1,n},\nabla_{1,n})\cong(H_{2,n},\nabla_{2,n}).\tag{6.0.7.1}$$

We claim that the filtrations on $(H_{i,n}, \nabla_{i,n})$ are preserved under the above isomorphism. If not, let a be the largest number such that Fil_1^a not equal to Fil_2^a, and b be the largest number such that $Fil_1^{a-j} \subseteq Fil_2^{b-j}$ for $j \geq 0$. Define

$$f_n : (E_{1,n}, \theta_{1,n}) \to (E_{2,n}, \theta_{2,n}),$$

by setting $f_n|_{Fil_1^{a+j}/Fil_1^{a+j+1}} = 0$ for $j \geq 1$ and for $j \leq 0$ f_n is the following projection:

$$Fil_1^{a+j}/Fil^{a+j+1} \to Fil_2^{b+j}/Fil_2^{b+j+1}.$$

Clearly f_n is a nonzero morphism of Higgs bundles. Note that $f_n \equiv 0 \, mod \, p^{n-1}$ zero. The composite map of f_n with the map $\frac{1}{[p^{n-1}]}$ induces a nonzero morphism of Higgs bundles:

$$(E_{1,1}, \theta_{1,1}) \to (E_{2,1}, \theta_{2,1}).$$

Which is neither zero nor an isomorphism. This contradicts with the fact that a morphism between stable Higgs bundles is either zero map or isomorphism. Therefore, the isomorphism (6.0.7.1) preserves the filtrations.

Moreover, as the automorphism of $(E_{i,n}, \theta_{i,n})$ are only scalars, so φ_1 is equal to φ_2 up to a scalar. Thus $(H_1, \nabla_1, Fil_1, \varphi_1) \cong (H_2, \nabla_2, Fil_2, \varphi_2)$ *modulo* p^n. $\square$

When the leading term (E, θ) of a periodic Higgs-de Rham flow is only Higgs semistable, the above uniqueness statement is no longer true. We shall make the following

Assumption 6.0.8. Given an object $(E, \theta, Fil_0, \cdots, Fil_{f-1}, \phi) \in \mathcal{HB}_{n,f}(X_2)$. for each $0 \leq i \leq f-1$, the filtration $Fil_i^{\cdot}$ on H_i is preserved by all automorphisms of (H_i, ∇_i).

The isomorphism $\varphi : (E_f, \theta_f) \cong (E_0, \theta_0)$ induces

$$(GrC_1^{-1})^{nf}(\varphi) : (E_{(n+1)f}, \theta_{(n+1)f}) \cong (E_{nf}, \theta_{nf}).$$

For $-1 \leq i < j$, we define $\varphi_{j,i}$ to be the isomorphism

$$(GrC_1^{-1})^{(i+1)f}(\varphi) \circ \cdots \circ (GrC_1^{-1})^{jf}(\varphi) : (E_{(j+1)f}, \theta_{(j+1)f}) \cong (E_{(i+1)f}, \theta_{(i+1)f}).$$

For $i = -1$, put $\varphi_j := \varphi_{j,-1}$.

Lemma 6.0.9. *For any two isomorphisms $\varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0)$, there exists a pair (i, j) with $0 \leq i < j$ such that $\phi_{j,i} \circ \varphi_{j,i}^{-1} = id$.*

Proof. Let $\tau_s = \phi_s \circ \varphi_s^{-1}$. Then τ_s is an automorphism of (E_0, θ_0). Moreover, each element in the set $\{\tau_s\}_{s\in\mathbb{N}}$ is defined over the same finite subfield of k. As there are only finite many automorphisms of (E_0, θ_0) defined over the same finite subfield, there exists $j > i \geq 0$ such that $\tau_j = \tau_i$. So the lemma follows. $\square$

Proposition 6.0.10. *Under Assumption 6.0.8. Let (i, j) be a pair given by Lemma 6.0.9 for two given isomorphisms $\varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0)$. Then there is an isomorphism in $\mathcal{HB}_{n,(j-i)f}(X_2)$:*

$$(E, \theta, Fil_0, \cdots, Fil_{(j-i)f-1}, \varphi_{j-i-1}) \cong (E, \theta, Fil_0, \cdots, Fil_{(j-i)f-1}, \phi_{j-i-1}).$$

Proof. Put $\beta = \phi_i \circ \varphi_i^{-1} : (E_0, \theta_0) \cong (E_0, \theta_0)$. We shall check that it induces an isomorphism in $\mathcal{HB}_{n,(j-i)f}(X_2)$. By Assumption 6.0.8, $C_1^{-1}(GrC_1^{-1})^m(\beta)$ for $m \geq 0$ always preserves the filtration. We need only to check that β is compatible with ϕ_{j-i-1} as well as φ_{j-i-1}. So it suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
E_{(j-i)f} \arrow[r, "\varphi_{j-i-1}"] \arrow[d, "\varphi_{j,j-i-1}^{-1}"'] & E_0 \arrow[d, "\varphi_i^{-1}"'] \\
E_{(j+1)f} \arrow[d, "\phi_{j,j-i-1}"'] & E_{(i+1)f} \arrow[d, "\phi_i"'] \\
E_{(j-i)f} \arrow[r, "\phi_{j-i-1}"] & E_0,
\end{tikzcd}$$

which in turn follows from that the following diagram is commutative:

$$\begin{tikzcd}
E_{(j-i)f} \arrow[d, "\varphi_{j,j-i-1}^{-1}"'] & E_0 \arrow[l, "\varphi_{j-i-1}^{-1}"'] \arrow[d, "\varphi_i^{-1}"'] \\
E_{(j+1)f} \arrow[d, "\phi_{j,j-i-1}"'] & E_{(i+1)f} \arrow[d, "\phi_i"'] \\
E_{(j-i)f} \arrow[r, "\phi_{j-i-1}"] & E_0.
\end{tikzcd}$$

In the above diagram, the anti-clockwise direction is

$$\phi_{j-i-1} \circ \phi_{j,j-i-1} \circ \varphi_{j,j-i-1}^{-1} \circ \varphi_{j-i-1}^{-1} = \phi_j \circ \varphi_j^{-1} = \phi_i \circ (\phi_{j,i} \circ \varphi_{j,i}^{-1}) \circ \varphi_i.$$

By the assumptions on for (i, j), we have $\phi_{j,i} \circ \varphi_{j,i}^{-1} = id$, so the anti-clockwise direction is $\phi_i \circ \varphi_i$, which is exactly the clockwise direction. Therefore, β is compatible with ϕ_{j-i-1} and φ_{j-i-1}, and induces an isomorphism in $\mathcal{HB}_{n,(j-i)f}(X_2)$, as required. $\square$

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