

A VERY SHORT PROOF OF SIDORENKO'S INEQUALITY FOR COUNTS OF HOMOMORPHISMS BETWEEN GRAPHS

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Abstract

A fundamental extremality result due to Sidorenko [‘A partially ordered set of functionals corresponding to graphs’, *Discrete Math.* **131**(1–3) (1994), 263–277] states that among all connected graphs G on k vertices, the k -vertex star maximises the number of graph homomorphisms of G into any graph H . We provide a new short proof of this result using only a simple recursive counting argument for trees and Hölder’s inequality.

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Let $\# \text{hom}(G, H)$ denote the number of graph homomorphisms from a graph $G = (V(G), E(G))$ into an image graph $H = (V(H), E(H))$, that is,

$$\text{hom}(G, H) = \{f : V(G) \rightarrow V(H) \mid (u, v) \in E(G) \Rightarrow (f(u), f(v)) \in E(H)\}.$$

The following general inequality was proven by Sidorenko [6].

THEOREM 1. *Let G denote any connected graph on $k + 1$ vertices and S_k the star graph with k edges. Then, $\# \text{hom}(G, H) \leq \# \text{hom}(S_k, H)$ for any graph H .*

Define a partial order \leq on the set \mathcal{G}_k of connected graphs on $k + 1$ vertices by declaring $G' \geq G$ if $\# \text{hom}(G', H) \geq \# \text{hom}(G, H)$ for every graph H .

Aside from explicitly classifying the maximal elements of \leq and its usefulness for counting problems in graphs, Sidorenko’s inequality is a valuable tool in various applications. For instance, if A is any positive symmetric $n \times n$ matrix, Theorem 1 implies the inequality

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$$\sum_{i,j \leq n} A_{i,j}^k \leq \sum_{i=1}^n \left(\sum_{j=1}^n A_{i,j} \right)^k,$$

which is due to Hoffman [3]. Further applications can be found in the context of noise sensitivity of Boolean functions [2] and in the theory of complex networks, where Theorem 1 is central in deriving moment criteria for the continuity of clustering coefficients under local weak convergence [4].

Three proofs of Theorem 1 can be found in the literature. Sidorenko's original proof relies on the remarkable fact that the relation $G \leq G'$ is in one-to-one correspondence to an ordering of integral functionals on measure spaces which can be associated with G and G' . Certain combinatorial operations on graphs map to log-convex combinations of these functionals for which Hölder-type inequalities hold. The corresponding inequalities for the homomorphism counts are then used to establish the extremality of star graphs and further relations between the homomorphism counts of concrete examples of graphs.

Csikvári and Lin [1] provide another proof of Theorem 1 that is close in spirit to Sidorenko's work, but uses the Wiener index (the sum of all distances between pairs of vertices in a graph) and more elementary combinatorial operations on graphs to conclude the proof.

Finally, Levin and Peres [5] prove Theorem 1 by a brief and elegant probabilistic argument that connects the homomorphism count to the stationary distribution of the simple random walk on the target graph.

The aim of this note is to present a new and remarkably elementary proof of Sidorenko's bound that relies solely on a short recursive enumeration argument and Hölder's inequality on finite probability spaces.

PROOF OF THEOREM 1. Fix an arbitrary image graph $H = (V(H), E(H))$. Observe that removing edges from G can only increase $\# \text{hom}(G, H)$; hence, it suffices to show

$$\# \text{hom}(T, H) \leq \# \text{hom}(S_k, H) \quad (1)$$

whenever T is any k -edge tree. Let $\mathcal{T}(k, \ell)$ denote the set of k -edge trees with precisely $\ell \leq k$ leaves. In particular, $\mathcal{T}(k, k) = \{S_k\}$. The bound (1) follows if we can show that

$$\max_{T \in \mathcal{T}(k, \ell)} \# \text{hom}(T, H) \leq \max_{T \in \mathcal{T}(k, \ell+1)} \# \text{hom}(T, H) \quad \text{for any } \ell < k.$$

To this end, we demonstrate that for every nonstar $T \in \mathcal{T}(k, \ell)$, there exists some k -edge tree T' with one more leaf that admits at least the same number of homomorphisms into H as T . Denote by $\text{sk}(T)$ the *skeleton tree* of T , obtained by removing all leaves from T . Since $\ell < k$, $\text{sk}(T)$ has at least two leaves. We designate the two leaves b_1, b_2 of $\text{sk}(T)$, and denote by $T(b_1, b_2)$ the graph obtained from T by removing all leaves adjacent to b_1 and b_2 . We write \vec{d}_1, \vec{d}_2 for the number of leaves removed at b_1 and b_2 , respectively. Calculating $\# \text{hom}(T, H)$ by first counting all maps of $T(b_1, b_2)$ and then the possible choices for the images of the remaining leaves yields

$$\begin{aligned} \# \text{hom}(T, H) &= \sum_{u,v \in V(H)} \#\{f \in \text{hom}(T(b_1, b_2), H) : f(b_1) = u, f(b_2) = v\} \text{deg}(u)^{\vec{d}_1} \text{deg}(v)^{\vec{d}_2} \\ &= \# \text{hom}(T(b_1, b_2), H) \sum_{u,v \in V(H)} p(u, v) \text{deg}(u)^{\vec{d}_1} \text{deg}(v)^{\vec{d}_2}, \end{aligned} \tag{2}$$

where $p(u, v)$ denotes the probability that a uniformly chosen map in $\text{hom}(T(b_1, b_2), H)$ maps b_1 to u and b_2 to v . Denoting the marginals of $p(\cdot, \cdot)$ by $p_1(\cdot)$ and $p_2(\cdot)$, we conclude with the help of Hölder’s inequality,

$$\begin{aligned} \# \text{hom}(T, H) &\leq \# \text{hom}(T(b_1, b_2), H) \\ &\cdot \left(\sum_{u \in V(H)} p_1(u) \text{deg}(u)^{\vec{d}_1 + \vec{d}_2} \right)^{\vec{d}_1 / (\vec{d}_1 + \vec{d}_2)} \left(\sum_{u \in V(H)} p_2(u) \text{deg}(u)^{\vec{d}_1 + \vec{d}_2} \right)^{\vec{d}_2 / (\vec{d}_1 + \vec{d}_2)}. \end{aligned} \tag{3}$$

We assume, without loss of generality, that

$$\sum_{u \in V(H)} p_1(u) \text{deg}(u)^{\vec{d}_1 + \vec{d}_2} \geq \sum_{u \in V(H)} p_2(u) \text{deg}(u)^{\vec{d}_1 + \vec{d}_2},$$

since if the opposite inequality holds, we may reverse the choice of b_1, b_2 at the beginning. Thus, (3) leads to

$$\begin{aligned} \# \text{hom}(T, H) &\leq \# \text{hom}(T(b_1, b_2), H) \sum_{u \in V(H)} p_1(u) \text{deg}(u)^{\vec{d}_1 + \vec{d}_2} \\ &= \sum_{u \in V(H)} \#\{f \in \text{hom}(T(b_1, b_2), H) : f(b_1) = u\} \text{deg}(u)^{\vec{d}_1 + \vec{d}_2}. \end{aligned} \tag{4}$$

The last expression equals $\# \text{hom}(T', H)$, where $T' \in \mathcal{T}(k, \ell + 1)$ is obtained from $T(b_1, b_2)$ by attaching $\vec{d}_1 + \vec{d}_2$ leaves to b_1 and consequently has precisely one more leaf than T . □

In fact, the above line of reasoning also establishes a slightly stronger form of the statement that coincides with Sidorenko’s original formulation of the result with only a minor refinement.

COROLLARY 2 [6, Theorem 1.2]. *Let T denote any k -edge tree, S_k the star graph with k edges and $T_{k-1,1}$ any tree obtained by attaching a single leaf to a leaf of S_{k-1} . Then, $T \leq T_{k-1,1} \leq S_k$.*

PROOF. All graphs in $\mathcal{T}(k, k - 1)$ have K_2 as their skeleton tree. Since the latter graph is symmetric under swapping b_1 and b_2 , it follows that the distribution $p(\cdot, \cdot)$ used in (2) is symmetric. Let (U, V) denote a pair of random variables with distribution $p(\cdot, \cdot)$. Then,

$$\begin{aligned} \sum_{u,v \in V(H)} p(u,v) \deg(u)^{\vec{d}_1} \deg(v)^{\vec{d}_2} &= \mathbb{E}[(\deg(U)^{\vec{d}_1 + \vec{d}_2})^{\vec{d}_1 / (\vec{d}_1 + \vec{d}_2)} (\deg(V)^{\vec{d}_1 + \vec{d}_2})^{\vec{d}_2 / (\vec{d}_1 + \vec{d}_2)}] \\ &= \mathbb{E}[e^{p_1 g(U) + (1-p_1)g(V)}] \end{aligned}$$

for $g(v) = (\vec{d}_1 + \vec{d}_2) \log(\deg(v))$ and $p_1 = \vec{d}_1 / (\vec{d}_1 + \vec{d}_2)$. Since $p(u, v) = p(v, u)$, the map $\phi : [0, 1] \rightarrow [0, \infty)$ given by

$$\phi(p) = \mathbb{E}[e^{pg(U) + (1-p)g(V)}]$$

is symmetric. The proof of Theorem 1 shows that ϕ attains its maxima at 0 and 1. Furthermore,

$$\phi'(p) = \mathbb{E}[(g(U) - g(V))e^{pg(U) + (1-p)g(V)}], \quad \phi''(p) = \mathbb{E}[(g(U) - g(V))^2 e^{pg(U) + (1-p)g(V)}],$$

and hence ϕ is convex and attains its minimum at $p = 1/2$. Consequently, ϕ is nondecreasing on $[1/2, 1]$, which implies that $\mathcal{T}(k, k-1)$ is totally ordered with respect to \leq , where the minimum is attained at $\vec{d}_1 = \vec{d}_2 = (k-1)/2$ if k is odd and at $\vec{d}_1 = k/2, \vec{d}_2 = k/2 - 1$ if k is even, and the maximum is attained at $\vec{d}_1 = k-2, \vec{d}_2 = 1$. \square

We conclude this note by remarking that one can replace the use of Hölder's inequality in the proof of Theorem 1 by an equally short argument using the weighted AM–GM inequality. It is an elementary analytic exercise to see that the two inequalities are in fact equivalent, so both variants of our argument are equally fundamental.

ALTERNATIVE PROOF OF THEOREM 1 WITHOUT HÖLDER'S INEQUALITY. Employing the notation from the proof of Corollary 2, we see that

$$\phi(p) \leq p\mathbb{E}[e^{g(U)}] + (1-p)\mathbb{E}[e^{g(V)}] \leq \max\{\mathbb{E}[e^{g(U)}], \mathbb{E}[e^{g(V)}]\}, \quad p \in [0, 1],$$

where the first inequality follows from applying the weighted AM–GM inequality. The term on the right-hand side is precisely the term appearing on the right-hand side of (4).

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