Einstein Proves the Mohr–Mascheroni Theorem

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On March 5, 1930, Julian Liban, a high-school student from Hirschberg im Riesengebirge, present-day Jelenia Góra, in Poland, wrote a letter to the famous professor Albert Einstein (1879–1955) in Berlin.[1] On the occasion of their final high-school exam (*Abiturium*), he was asking Einstein for help in finding "the simplest solution" to two problems: given two points A and B, how could one (a) construct two points C and D that would complete a square over AB, and (b) how could one construct the midpoint between A and B? The condition, however, in both cases was to use only a compass, not making use of a straightedge.

What Liban was seeking Einstein's help with was part of a famous more general problem, that of proving that one can do any Euclidean construction using only a compass. In 1672, the Danish geometer Georg Mohr (1640–1697) had published a little book entitled *Euclides Danicus*, in which he elaborated on a Euclidean geometry using only a compass. The book had been published in Amsterdam in Dutch [11], and another version had been printed at the time in Danish. It had received little or no attention at the time. When Liban wrote his letter to Einstein, however, it had just been republished in a new edition in Copenhagen [12]. The theorem that one can do any Euclidean construction using only a compass had been popularized by the Italian geometer Lorenzo Mascheroni (1750–1800), who in 1797 had published a little treatise, *La geometria del compasso* [9], proving essentially the same theorem that Georg Mohr had proved before him. The Einstein Archives contain notes for a response by Einstein to Liban; see Figure 1.

Both tasks, he observes, can be carried out using 3, 4, 5 Pythagorean triangles, but his manuscript gives little indication how he had envisaged the construction. We can find a valid idea for a solution to Liban's first question, but apparently not an explicit solution to his second. Indeed, internal notes at the time by Einstein's secretary, Helen Dukas, who after 1955 continued to work with his papers and created the Einstein Archives, indicate that a response to Liban may never have been sent.

A year and a half later, in late November 1931, Einstein visited his good friend Paul Ehrenfest (1880–1933) and his family in Leiden before embarking on a transatlantic trip to Pasadena, California. Paul was a theoretical physicist, his wife, Tatiana Ehrenfest-Afanassieva (1876–1964), a mathematician, both of them still known today, among other things, for a jointly authored review of the foundations of statistical mechanics [6]. Their eldest daughter, born in 1905, who had been given the same first name as her mother, was also a mathematician and was fondly referred to in mock mathematical jargon as her mother's derivative, Tania′ (Tania prime). Apparently, the problem of Euclidean constructions with compass alone had been raised during

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[1]Albert Einstein Archives (AEA) 25 156, [7, Abs. 531].

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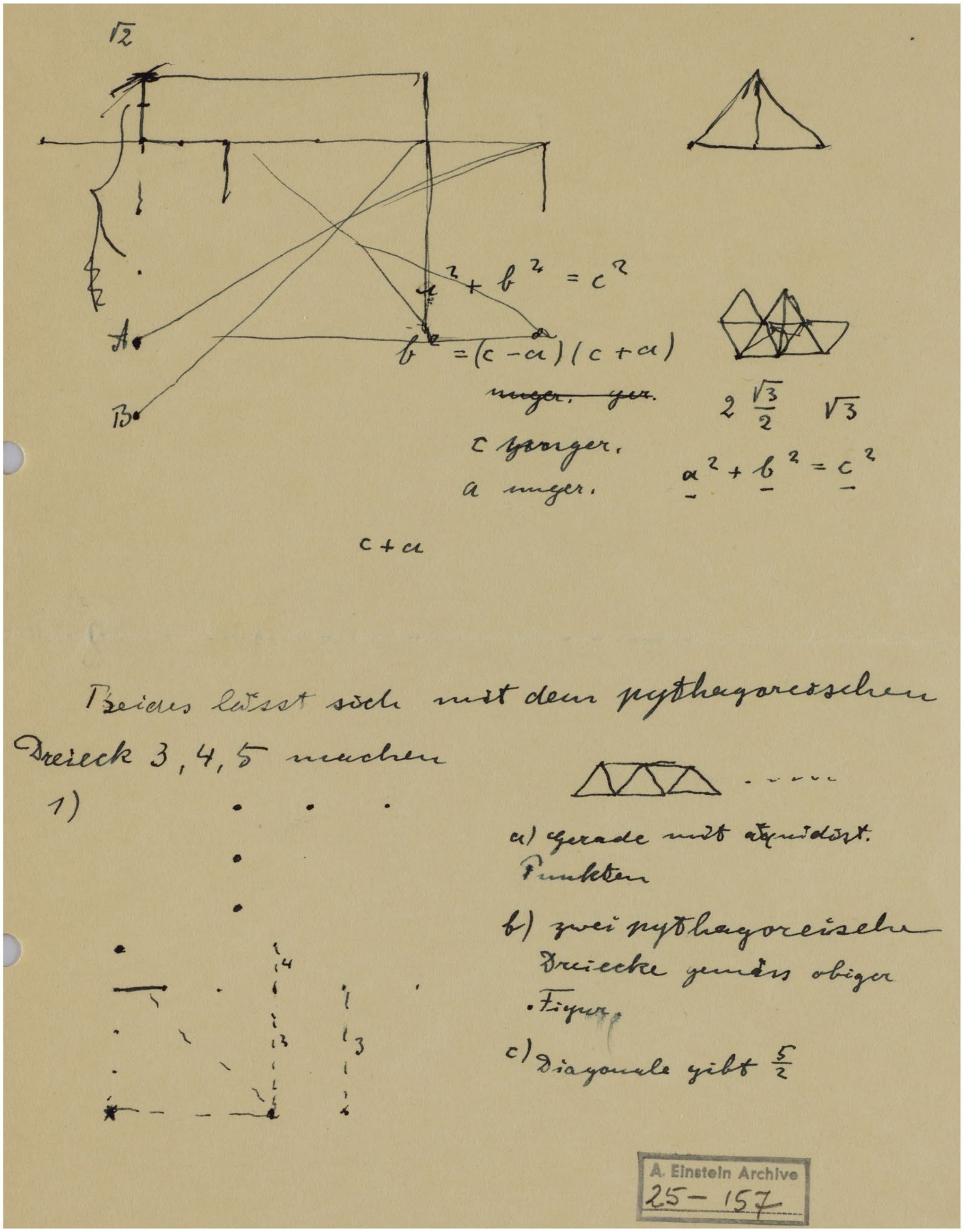

Figure 1. Einstein's draft notes for a response to Julian Liban, AEA 25-157, [7, Doc. 279]. (© The Hebrew University of Jerusalem.).

Einstein's visit with the Ehrenfests. A few days later, when Einstein had embarked on his transatlantic steamer, he noted for December 3 in his travel diary, "Yesterday and today I crafted together the proof that all Euclidean constructions can be carried out with the compass alone. Tania Ehrenfest didn't want to believe it."[2]

Indeed, there is a letter to Tania (prime) Ehrenfest, sent on December 19 from the Panama Canal (AEA 10 288) with accompanying notes (ASEA 10 287), in which Einstein fully lays out a proof of the Mohr–Mascheroni theorem; see Figure 2.

In his letter, he wrote, "On the boat, I worked on this pretty triviality that we talked about. All constructions

[2]"Gestern und heute konstruierte ich mir den Beweis dafür zusammen, dass alle Euklidischen Konstruktionen mit dem Zirkel allein ausgeführt werden können. Tania Ehrenfest hatte es nicht glauben wollen" [AEA 29 136].

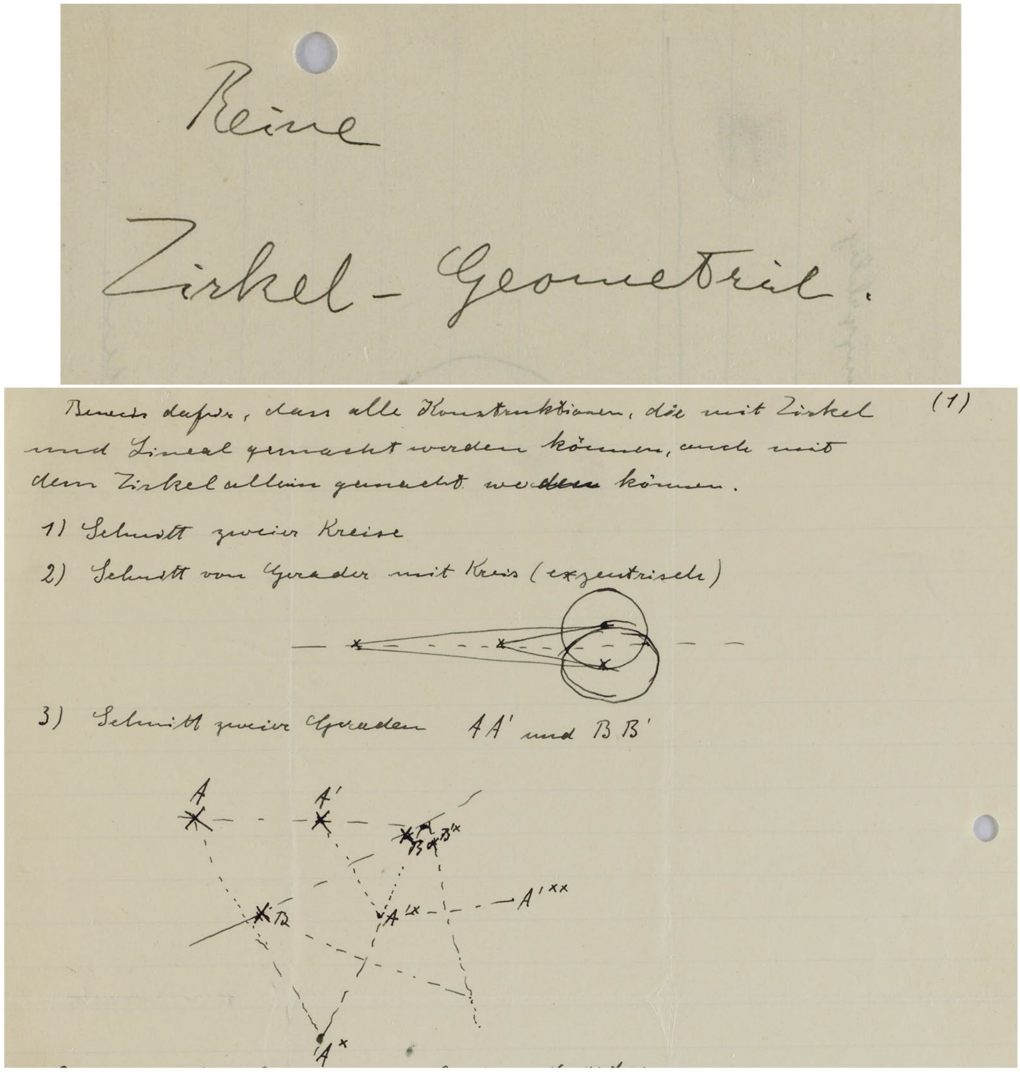

Figure 2. Title line and beginning of Einstein's notes for a proof that all constructions that can be done with compass and straightedge can also be done with compass alone, AEA 10-287. (© The Hebrew University of Jerusalem.).

can indeed be done with the compass alone, even if it's not exactly convenient."[3]

Einstein's Proof

In the following, I explicate Einstein's proof as it was sketched out in his notes for Tania Ehrenfest (AEA 10 287), closely following Einstein's notes (see, for example, Figures 2 and 3 and the left-hand side of Figure 4), adding only occasionally a bit of clarification, notation, or explanation.

Einstein begins by stating that in order to prove the Mohr–Mascheroni theorem, we need to show that three fundamental constructions can be carried out with the compass alone:

1. the intersection points of two circles with given radii and midpoints;
2. the intersection points of a straight line given by two points with a circle;
3. the intersection point of two straight lines each given by two distinct points.

These are the three fundamental constructions of plane Euclidean geometry, and if one can show that each of them can be done by compass alone, then so can any Euclidean construction. The various steps of the proof that Einstein lays out are the following.

1. Since we are using a compass, nothing needs to be shown for task 1.
2. The second fundamental construction task splits into two cases, depending on whether the straight line passes through the center of the circle. For the off-centric case, the standard solution is readily given by

[3]"Ich habe mich auf dem Schiff mit dieser hübschen Spielerei beschäftigt, über die wir uns unterhalten haben. Es lassen sich wirklich alle Konstruktionen mit dem Zirkel allein ausführen, wenn auch nicht gerade bequem" (AEA 10-288).

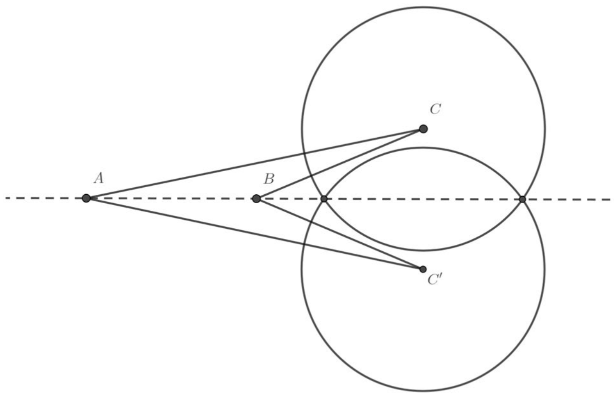

Figure 3. Constructing the intersection points between a line and a circle: Point C is mirrored along the line AB by drawing circles around A and B of radii AC and BC, respectively, AEA 10-287.

mirroring the midpoint of the circle at the straight line, a construction indicated by Einstein with a little sketch like that in Figure 3.

Here the circle's midpoint C is mirrored to C' by drawing circles around A and B of radii AC and BC

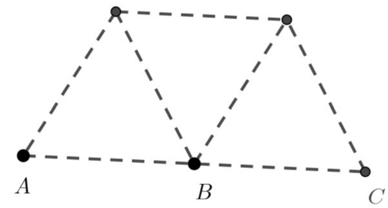

Figure 5. A given line AB can be repeated along itself by constructing equilateral triangles.

respectively. They intersect at C and C'. Then a circle is drawn around C' with the same radius as for the original circle, and the intersection points of these two circles are also the intersection points of the straight line through A and B with the circle around C.

The centric case was left unresolved for the moment by Einstein.

3. Instead, he begins to discuss the third fundamental task. He first observes that repeated mirroring of one line in the other as in the previous construction reduces the problem to that of constructing the center of a circle given three equidistant points on it (see Figure 4).

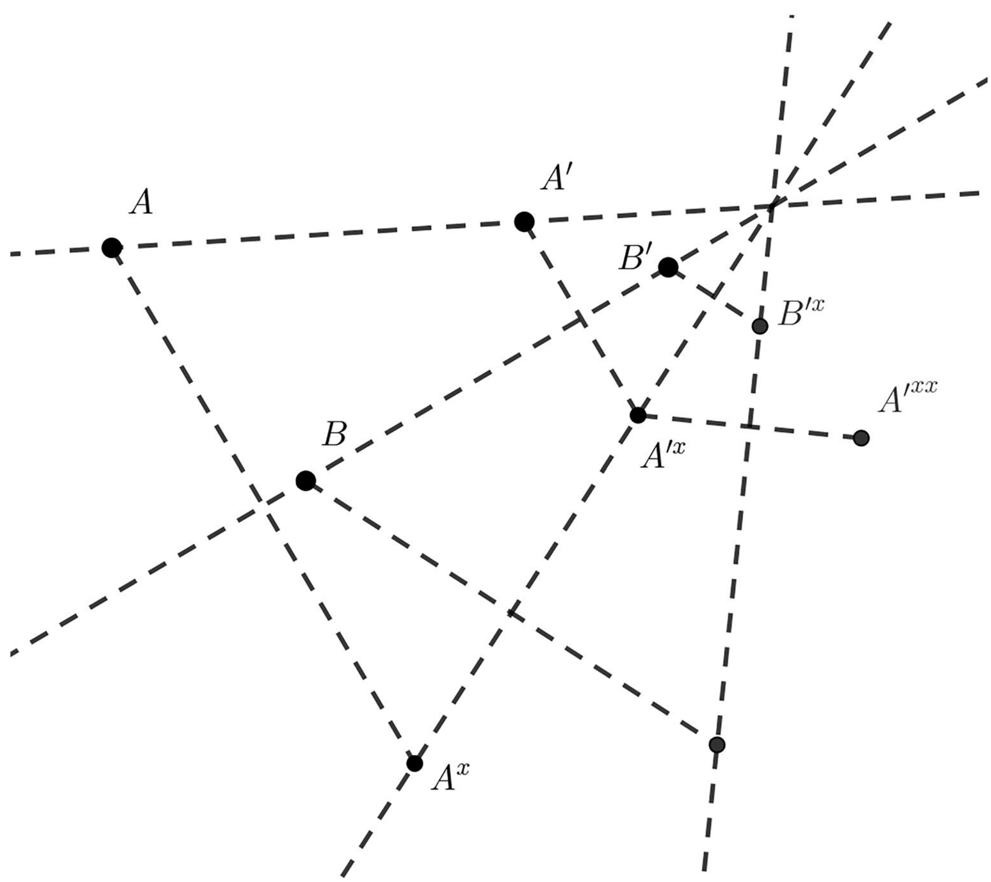

Figure 4. Repeated mirroring of points A, A' along the straight line BB' and B, B' along $A^x A'^x$ reduces the task of constructing the intersection point of AA' and BB' to that of finding the center of a circle through three equidistant points A, A', A'' (e.g., A', A'^x, A'^{xx} from the left-hand image) on it.

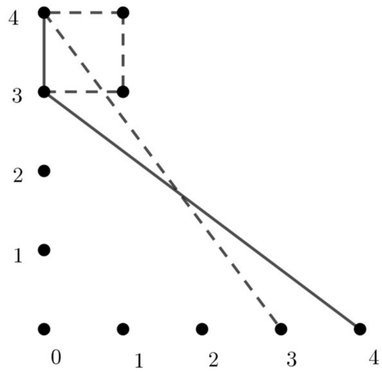

Figure 6. A unit square can be constructed using 3, 4, 5 right triangles.

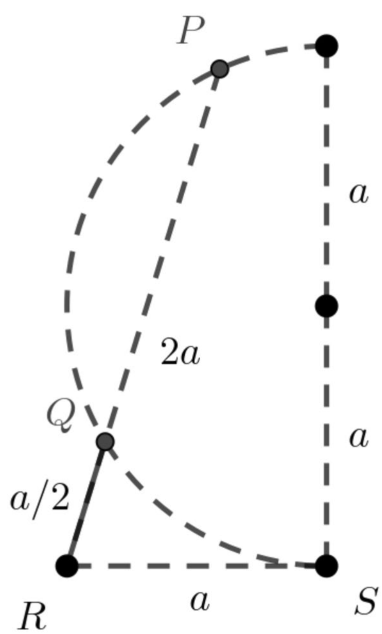

Figure 7. A line of length $a/2$ can be constructed using the tangent–secant theorem.

Einstein proceeds by focusing first on the auxiliary task of constructing a quadratic point net of unit squares starting from a unit length. He breaks this task up into several subtasks:

(a) A given line AB can be extended by constructing a sequence of equilateral triangles, as in Figure 5. Here A, B, and C are on a straight line such that $AB = AC$.
(b) This process can be repeated to create a sequence of equidistant points along a line.
(c) Using right triangles with sides of length 3, 4, and 5, one can construct an orthogonal unit distance as in Figure 6: Given points $0, 1, \ldots, 4, \ldots$ of unit distance apart on a line, one can construct, for example, the distance from 3 to 4 on the perpendicular line through the point 0 by drawing circles of radius 5 around 3 and 4 and circles around 0 of radii 4 and 3, respectively. This process can be repeated to create a unit square. Clearly, this is also the solution to Liban's first question as indicated in his draft letter.
(d) In order to construct the half-distance $a/2$, Einstein invokes the tangent–secant theorem (readily identified, though not by Einstein, as Book III, Theorem 36, of Euclid's *Elements*). To apply the theorem, he writes $x = a/2$ as $a^2 = x \cdot 2a$ and constructs x as in Figure 7. Begin by constructing perpendicular line segments of length a and $2a$ and drawing a circle of radius a as shown in the figure. Then the segment RS is tangent to the circle. Drawing a circle of radius $2a$ around R then intersects the first circle at the point P, and we have then $|RP| = 2a$. The point Q can then be constructed by the second fundamental task. The tangent–secant theorem gives $|RS|^2 = |RQ| \cdot |RP|$, or $a^2 = x \cdot (2a)$, yielding $x = a/2$.
(e) Given $a > b > 0$, the distance $\sqrt{a^2 - b^2}$ can be constructed using Thales's theorem (as before, we can identify this as Book III, Theorem 31 of the *Elements*), as in Figure 8.This latter construction now allows Einstein to construct any length $\sqrt{na^2}$ for $n \in \mathbb{N}$. He wrote

$$(za)^2 - a^2 = \left(z^2 - 1\right)a^2\,,$$
$$[(za)^2 - a^2] - a^2 = \left(z^2 - 2\right)a^2\,,$$
$$\ldots$$

Beginning with an integer z large enough, $z^2 > n$, one can repeatedly construct sides of length $z^2 - k$, $k = 1, 2, \ldots$ until $z^2 - k = n$.
(f) Given a and b with $b > a$, one can construct a^2/b making use again of the tangent–secant theorem, as in Figure 9.

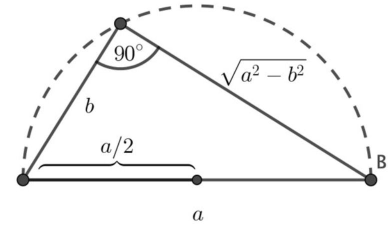

Figure 8. The length $\sqrt{a^2 - b^2}$ can be constructed using Thales's theorem.

In order to make sure that a circle of radius b intersects a circle of radius na, we need an integer $n \geq 1$ that also satisfies $b < 2na$.

With these auxiliary construction solutions at hand, Einstein now returns to the second fundamental problem, which, as we have seen, is equivalent to constructing the center of a circle through three equidistant points on it.

With the notation of Figure 10, he can write

$$\left(\frac{S}{2}\right)^2 + x^2 = s^2\,,$$
$$\left(\frac{S}{2}\right)^2 + (r - x)^2 = r^2\,,$$

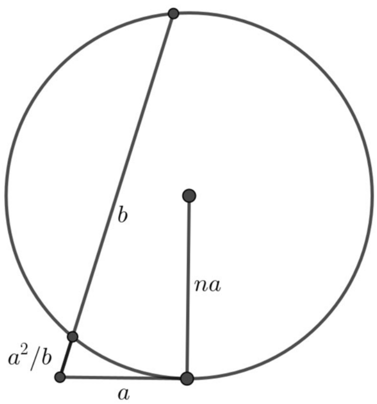

Figure 9. Constructing a^2/b using the tangent–secant theorem for integer $n \geq 1$ with $a < b < 2na$.

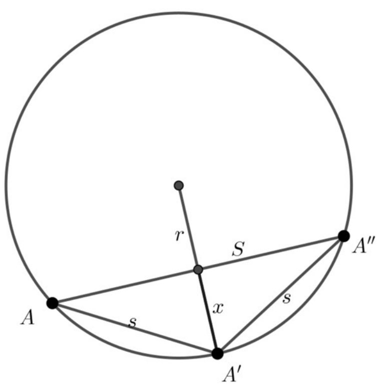

Figure 10. Constructing the radius r.

to obtain

$$x = \sqrt{s^2 - \left(\frac{S}{2}\right)^2} \tag{1}$$

and

$$r = \frac{s^2}{2\sqrt{s^2 - \left(\frac{S}{2}\right)^2}} . \tag{2}$$

Since we always have $4s^2 > S^2$, we can construct x according to (e).

If we want to construct r, we have to make sure that $x > s$, which requires $s > S/\sqrt{3}$. This condition is not satisfied if the three points A', A', A'' are too closely aligned. If that is the case, however, we can return to the original fundamental task and repeat the mirroring of the points as many times as required to decrease the angle between AA' and $A'A''$ to less than 120°.

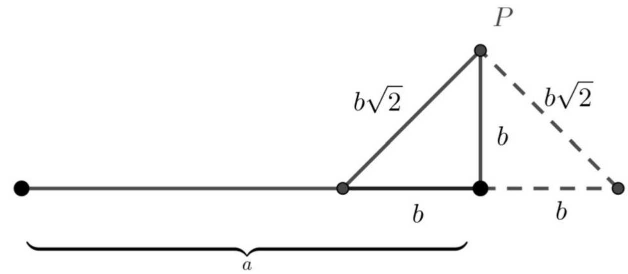

Figure 11. Constructing the sum and difference $a \pm b$.

We now complete the second fundamental task by considering the centric case, in which the straight line passes through the center of the circle. Einstein finally shows that intersection points of a line and a circle can be constructed if the line passes through the center of the circle. He observes that this can be done if the sum and difference of two distances can be constructed. He also remarks that this can be achieved if the distance $\sqrt{a^2 + b^2}$ can be constructed; see Figure 11.

Given a and b, one can construct a right angle by finding P as the intersection of circles around the endpoints of the segment of length a shown in Figure 11 of radii $\sqrt{a^2 + b^2}$ and b, respectively. The sum and difference, $a \pm b$, are then found by drawing a circle around P of radius $b\sqrt{2}$, constructed as in item 2(e) above.

Einstein comments that if one takes, without loss of generality, $a > b$, then one can construct

$$(n+1)a^2 - nb^2 = x^2 ,$$
$$na^2 - (n+1)b^2 = y^2 ,$$

where one can always take n large enough to make x real. The construction of $x^2 - y^2$ is then equivalent to constructing $a^2 + b^2$.

Some Remarks

Einstein's elementary proof, explicated here from his notes, is not a polished proof. Nor is it entirely original, although it does give an individual solution.[4] Mohr and Mascheroni had long before shown how Euclidean constructions could be done with compass only. A proof of the Mohr–Mascheroni theorem had been given by August Adler (1863–1923) in 1890 by reducing the problem to that of constructing inversion at a circle [1, 2]. It is not unlikely that Einstein had heard about the Mascheroni theorem as a young student at the Zurich Polytechnic.[5] Historically, it is noteworthy that Einstein carried out the proof at all and at that time. We can be rather certain that he "crafted together" the proof by himself while aboard the ship without recourse to existing literature, both because he says so in his travel diary and because it is very unlikely that he took much printed matter with him on the long sea voyage.

[4] For another short elementary proof and further references, see, e.g., [8].

[5] For a discussion of Einstein's mathematical training as a student, see [13, Section 2].

But he had heard about the theorem before and had tried to convince the young Tania Ehrenfest of the viability of a "pure compass geometry." Einstein's pondering on the Mohr–Mascheroni problem reflects a more generally shared interest in more abstract mathematical questions. Shortly before, in 1928, Mohr's book had been translated into German and reprinted with a preface by Johannes Hjelmslev, who had sketched Mohr's principal ideas. This publication had been reviewed in various places, among others in the German *Jahresbericht der Deutschen Mathematiker-Vereinigung* and in the *Monatshefte für Mathematik und Physik* [10, 15] and in the *Bulletin of the American Mathematical Society* [4]. The historian of mathematics Florian Cajori had also reviewed the Mohr reprint in the *American Mathematical Monthly* [3]. He added a few remarks on the prehistory of the Mascheroni problem and pointed out parallel ideas on the solution of Euclidean constructions using only a straightedge and a fixed circle by Jacob Steiner going back to 1833 [14]. And a little later, Heinrich Dörrie published a widely read book on famous problems in mathematics that included a chapter on the Mohr–Mascheroni theorem [5]. Clearly, the interest in the possibilities of a "pure compass geometry" here was not so much in the construction as in a reflection on the inherent possibilities of positing systematically and axiomatically different constructive tools.

As for Einstein, we know that he cherished the peace and quiet of long trips aboard transoceanic steamers to think about scientific problems of unified field theory. The fact that he ruminated on elementary Euclidean geometry after discussing the constructibility problem testifies to his turning more and more to mathematics in his later years. As he said to his secretary, Helen Dukas, on an outing in his own sailboat on a lake near Berlin in September 1930, "This [sailing] and mathematics are the most beautiful things in the world" (AEA 38 592.1).

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