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Two proofs of a Jantzen Conjecture for Whittaker modules

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Abstract

We define a filtration of a standard Whittaker module over a complex semisimple Lie algebra and and establish its fundamental properties. Our filtration specializes to the Jantzen filtration of a Verma module for a certain choice of parameter. We prove that embeddings of standard Whittaker modules are strict with respect to our filtration, and that the filtration layers are semisimple. This provides a generalization of the Jantzen conjectures to Whittaker modules. We prove these statements in two ways. First, we give an algebraic proof which compares Whittaker modules to Verma modules using a functor introduced by Backelin. Second, we give a geometric proof using mixed twistor $\mathcal{D}$-modules.

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Contents

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1 Introduction

1.1 Overview

In this paper, we state and prove a Jantzen conjecture for Whittaker modules, generalizing a famous theorem of Beilinson–Bernstein. We provide two proofs — one algebraic (relating Whittaker modules to Verma modules in category $\mathcal{O}$), and one geometric (using mixed twistor $\mathcal{D}$-modules).

The Jantzen conjectures for Verma modules are a classical result in the representation theory of semisimple Lie algebras. They establish that a certain algorithmically-defined filtration of a Verma module satisfies remarkable functoriality and semisimplicity properties. The conjectures were proven by Beilinson–Bernstein in [4] using mixed structures on categories of holonomic $\mathcal{D}$-modules.

We define an analogous filtration on a standard Whittaker module. Our main result (Theorem 2.5.8) is that our filtration satisfies the same functoriality and semisimplicity properties as the Jantzen filtration of a Verma module. Our theorem contains the Jantzen conjectures for Verma modules as a special case.

This paper completes a program set out in [7, 9, 27] to establish which results from category $\mathcal{O}$ can be lifted to the Whittaker category $\mathcal{N}$. Previous work on Whittaker modules is based on two main approaches. The first approach is algebraic, relating Whittaker modules to category $\mathcal{O}$, then using known results about highest weight modules to deduce properties of Whittaker modules. The second approach is geometric, using Beilinson–Bernstein localization to relate Whittaker modules to twisted equivariant $\mathcal{D}$-modules on the associated flag variety. In this paper we use both approaches, developing significant extensions of existing techniques in each setting. One notable aspect of our geometric proof is that it relies on the machinery of mixed twistor $\mathcal{D}$-modules developed in [22, 23, 28, 29]. Mixed twistor $\mathcal{D}$-modules can be seen as an instance of Simpson's meta theorem [33] applied to Saito's mixed Hodge modules [30, 31]. Our work provides one of the first applications of this technology to geometric representation theory.

In the remainder of the introduction, we explain our results in more detail.

1.2 Whittaker modules

Our theorem deals with a class of representations called standard Whittaker modules. In this section, we describe their construction and basic properties.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ fixed Cartan and Borel subalgebras. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent radical of $\mathfrak{b}$. An *η-Whittaker module*[1] is a $\mathfrak{g}$-module which is cyclically generated by a vector on which $\mathfrak{n}$ acts by a character $\eta : \mathfrak{n} \to \mathbb{C}$.

When η is the zero character, Verma modules, highest weight modules, and simple finite-dimensional $\mathfrak{g}$-modules satisfy this condition, providing examples of 0-Whittaker modules. At the other extreme, when η is nondegenerate (meaning that it does not vanish on any simple root space in $\mathfrak{n}$) Kostant showed that η-Whittaker modules have an elementary description: simple nondegenerate Whittaker modules are uniquely determined by their infinitesimal character, and equivalence classes of nondegenerate Whittaker modules are in bijection with ideals in the center of the universal enveloping algebra of $\mathfrak{g}$ [16].

For partially degenerate η, [19] provided a classification of simple η-Whittaker modules as unique irreducible quotients of *standard Whittaker modules*, which are defined as follows. Let Π_η be the subset of simple roots for which η does not vanish on the corresponding root space. The set Π_η determines a Levi subalgebra and parabolic subalgebra[2] $\mathfrak{l}_\eta \subset \mathfrak{p}_\eta \subset \mathfrak{g}$. Denote by $Z(\mathfrak{l}_\eta) \subset U(\mathfrak{l}_\eta)$ the center of the universal enveloping algebra of $\mathfrak{l}_\eta$ and $\mathfrak{n}_\eta = \mathfrak{n} \cap \mathfrak{l}_\eta$. Let $\rho, \rho_\eta \in \mathfrak{h}^*$ be the half-sum of positive roots of $\mathfrak{g}$ and $\mathfrak{l}_\eta$, respectively. For $\lambda \in \mathfrak{h}^*$, the corresponding standard Whittaker module is

$$M(\lambda, \eta) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} U(\mathfrak{l}_\eta) \otimes_{Z(\mathfrak{l}_\eta)\otimes U(\mathfrak{n}_\eta)} \mathbb{C}_{\lambda-\rho+\rho_\eta,\eta}. \tag{1.1}$$

Here $\mathbb{C}_{\lambda-\rho+\rho_\eta,\eta}$ is the one-dimensional $Z(\mathfrak{l}_\eta) \otimes U(\mathfrak{n}_\eta)$-module on which $Z(\mathfrak{l}_\eta)$ acts by the infinitesimal character corresponding to $\lambda - \rho + \rho_\eta$, and $U(\mathfrak{n}_\eta)$ acts by η.

When $\eta = 0$, the standard Whittaker module $M(\lambda, 0)$ in (1.1) is the Verma module of highest weight $\lambda - \rho$, and the corresponding simple Whittaker module is its unique irreducible quotient. When η is non-degenerate, the standard Whittaker module in (1.1) is irreducible for all choices of λ. For partially degenerate η, the standard Whittaker module interpolates between these two extremes.

McDowell's classification scheme bears a strong resemblance to the classification of simple highest weight modules in category $\mathcal{O}$, and even includes it as a special case.

[1] We drop the η and use the term *Whittaker module* when the character η is clear from context.

[2] See Sect. 2.1-2.2 for precise definitions and ρ-shift conventions.

Motivated by this analogy, Miličić–Soergel initiated a categorical approach to Whittaker modules in [25], defining a category $\mathcal{N}$ whose simple objects are precisely the simple Whittaker modules. Working within this category, Miličić–Soergel and Backelin [2] provided a description of composition series multiplicities of standard Whittaker modules in terms of parabolic Kazhdan–Lusztig polynomials, further demonstrating the similarities between the categories $\mathcal{O}$ and $\mathcal{N}$.

In [26], Miličić–Soergel proposed a geometric approach to the study of category $\mathcal{N}$, using Beilinson–Bernstein localization to relate Whittaker modules to twisted equivariant $\mathcal{D}$-modules on the flag variety. This approach allowed results about $\mathcal{N}$ to be proven directly, instead of by reduction to known results in $\mathcal{O}$, which was the proof strategy in [2, 25]. The geometric approach was further developed in [27] and [36] where a geometric algorithm for computing composition multiplicities was established that provides a direct $\mathcal{D}$-module proof of the results in [2, 25].

The parallel algebraic and geometric approaches to studying category $\mathcal{N}$ have demonstrated many similarities with category $\mathcal{O}$. However, there are two major departures from highest weight theory which provide obstacles to furthering the study of Whittaker modules:

(1) Unlike Verma modules, standard Whittaker modules admit many distinct contravariant forms [7]. This makes it difficult to generalize arguments about Verma modules which rely on the uniqueness of the Shapovalov form [32].
(2) The $\mathcal{D}$-modules corresponding to Whittaker modules have irregular singularities [26]. This prevents the use of the Riemann–Hilbert correspondence to compare Whittaker modules to categories of perverse sheaves, and, in particular, it removes the possibility of studying them using the mixed ℓ-adic sheaves of [12] or the mixed Hodge modules of [30, 31].

The obstacles above made one fundamental result in category $\mathcal{O}$ seem especially out of reach - the Jantzen conjectures. The Jantzen conjectures describe the properties of the Jantzen filtration of a Verma module, which is defined using the unique Shapovalov form. By obstacle (1) above, there is no clear generalization of this filtration to standard Whittaker modules since there is no unique contravariant form. Moreover, the proof of the Jantzen conjectures by Beilinson–Bernstein in [4] relies on passage to categories of mixed ℓ-adic sheaves or mixed Hodge modules, so obstacle (2) eliminates the possibility of a direct generalization.

In this paper, we explain how one can circumvent obstacles (1) and (2) to both state and prove a Jantzen conjecture for Whittaker modules. We use two key tools. The first is the unique contravariant pairing between a standard Whittaker module and a Verma module introduced in [9], which gives us an appropriate substitute for the Shapovalov form. This allows us to overcome obstacle (1). The second is the theory of mixed twistor $\mathcal{D}$-modules [23], which encapsulates holonomic $\mathcal{D}$-modules with irregular singularities and serves as a replacement for mixed Hodge modules. This allows us to overcome obstacle (2).

We describe our generalization of the Jantzen conjectures in more detail in the following section.

1.3 A Jantzen conjecture

A key result in [9] is that blocks of category $\mathcal{N}$ have the structure of highest weight categories. In particular, for each $\lambda \in \mathfrak{h}^*$ and $\mathfrak{n}$-character η, there exists a standard Whittaker module $M(\lambda, \eta)$, a costandard module $M^\vee(\lambda, \eta)$, and a canonical map

$$\psi : M(\lambda, \eta) \to M^\vee(\lambda, \eta)$$

in category $\mathcal{N}$. The canonical map has an explicit realization in terms of the unique contravariant pairing between $M(\lambda, \eta)$ and the Verma module $M(\lambda)$ (Proposition 2.2.6).

To define the Jantzen filtration, we need a deformation of the above set-up in a specified direction $\gamma \in \mathfrak{h}^*$. Let $T = \mathcal{O}(\mathbb{C}\gamma)$ be the ring of regular functions on the line $\mathbb{C}\gamma \subset \mathfrak{h}^*$. Identify T with the polynomial ring $\mathbb{C}[s]$, and set $A = T_{(s)}$ to be the localization of T with respect to the prime ideal (s). In Section 2.4, we define deformed standard and costandard Whittaker modules, which are $(\mathfrak{g}, A)$-bimodules admitting a canonical $(\mathfrak{g}, A)$-bimodule homomorphism

$$\psi_A : M_A(\lambda, \eta) \to M_A^\vee(\lambda, \eta).$$

Setting $s = 0$ recovers the non-deformed set-up above, and setting $\eta = 0$ recovers the canonical map from deformed Verma modules to deformed dual Verma modules studied in [34].

The deformed module $M_A^\vee(\lambda, \eta)$ has a $(\mathfrak{g}, A)$-module filtration given by powers of s, and the *Jantzen filtration* of $M_A(\lambda, \eta)$ is defined to be the filtration obtained by pulling back this "powers of s" filtration along the canonical map ψ_A:

$$M_A(\lambda, \eta)_{-i} = \{m \in M_A(\lambda, \eta) \mid \psi_A(m) \in M_A^\vee(\lambda, \eta)s^i\}.$$

Setting $s = 0$, we recover a filtration of $M(\lambda, \eta)$ which we call the Jantzen filtration. When $\eta = 0$, this specializes to the classical Jantzen filtration of a Verma module defined in [15] using the Shapovalov form.

Remark 1.3.1 (Computability of Jantzen filtration) Because the Jantzen filtration is defined in terms of an explicitly defined contravariant pairing (see (5.1) in [9]), the Jantzen filtration is directly computable. In contrast, other representation-theoretic filtrations such as the socle filtration are known to exist, but are difficult to compute algorithmically.

Despite being computable, it is unclear from the definition what properties the Jantzen filtration should satisfy. Jantzen conjectured that for Verma modules, his filtration satisfies strong functoriality and semisimplicity properties. In particular, the *Jantzen conjectures* are the statements (posed as questions in [15, Section 5.17]) that for $\gamma = \rho$, embeddings of Verma modules are strict[3] for Jantzen filtrations, and the

[3] By *strict* we mean that if $\iota : M(\mu) \hookrightarrow M(\lambda)$ is an embedding, the filtration on $\iota(M(\mu))$ induced from the Jantzen filtration on $M(\lambda)$ agrees with the Jantzen filtration on $M(\mu)$ up to a shift.

Jantzen filtration coincides with the socle filtration. In particular, the filtration layers of the Jantzen filtration of a Verma module are semisimple.

In the years following Jantzen's conjectures, it became clear that the Jantzen filtration carries fundamental information about Verma modules and category $\mathcal{O}$. Work of Barbasch [3], Gabber-Joseph [14], and others showed that the Jantzen conjectures imply the Kazhdan–Lusztig conjectures for Verma modules. In fact, they imply a stronger statement, relating multiplicities of simple modules in layers of the Jantzen filtration to coefficients of Kazhdan–Lusztig polynomials [14].

The Jantzen conjectures for Verma modules were proven by Beilinson–Bernstein in [4] using weight theory and mixed ℓ-adic sheaves. Twenty years later, Williamson provided an algebraic proof using Soergel bimodules [35], building on earlier work of Kübel and Soergel [17, 18, 34]. Williamson's proof also demonstrated that the Jantzen filtration depends on the deformation parameter $\gamma \in \mathfrak{h}^*$, answering a longstanding question of Deodhar.

The main theorem of this paper (which appears as Theorem 2.5.8 in the body of the text) is a generalization of Jantzen's conjectures to standard Whittaker modules, using the Jantzen filtration of $M(\lambda, \eta)$ defined above.

Theorem 1.3.2 (The Jantzen conjectures for Whittaker modules)

(i) (Hereditary property) Embeddings of standard Whittaker modules are strict for Jantzen filtrations.

(ii) (Semisimplicity property) The filtration layers of the Jantzen filtration of $M(\lambda, \eta)$ are semisimple.

When $\eta = 0$, Theorem 1.3.2 specializes to the hereditary and semisimplicity property for the Jantzen filtration in Verma modules. We provide two proofs of Theorem 1.3.2. In the following sections, we outline the strategies of each proof. We note that we do not claim that the Jantzen filtration coincides with the socle filtration and leave this as a conjecture.

1.4 An algebraic approach

Our first approach to Theorem 1.3.2 uses a functor defined by Backelin to relate Whittaker modules to category $\mathcal{O}$. The functor is constructed as follows. For any $\mathfrak{g}$-module M, denote by

$$\Gamma_\eta(M) = \{m \in M \mid (X - \eta(X))^k m = 0 \text{ for all } X \in \mathfrak{n} \text{ and } k \gg 0\}$$

the set of generalized η-Whittaker vectors in M. This set is $\mathfrak{g}$-stable [9, Lem. 6.1]. If $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$ is a weight module, then define the completion of M to be the direct product of weight spaces:

$$\overline{M} = \prod_{\mu \in \mathfrak{h}^*} M_\mu.$$

In [2], Backelin defines a functor

$$\overline{\Gamma}_\eta : \mathcal{O} \to \mathcal{N};\, M \mapsto \Gamma_\eta(\overline{M}).$$

Backelin's functor is exact and sends Verma modules to standard Whittaker modules [2, Prop. 6.9], so it is a useful tool in using relating properties of standard Whittaker modules to properties of Verma modules.

The approach of our proof is to show that $\overline{\Gamma}_\eta$ sends the Jantzen filtration of a Verma module to the Jantzen filtration of a standard Whittaker module. This allows us to use the Jantzen conjecture for Verma modules to deduce the Jantzen conjecture for Whittaker modules. In particular, the semisimplicity in Theorem 1.3.2.(ii) follows immediately from the semisimplicity of filtration layers in category $\mathcal{O}$, and the hereditary property in Theorem 1.3.2.(i) follows from the fact that $\overline{\Gamma}_\eta$ is a quotient functor when restricted to blocks of $\mathcal{O}$ [1, Cor. 2].

To show that $\overline{\Gamma}_\eta$ matches Jantzen filtrations, we need to define a deformed version of $\overline{\Gamma}_\eta$ and establish its properties. We do this in Section 2.6. The key result is Theorem 2.6.4, which demonstrates that the deformed Backelin functor sends certain deformed Verma modules (those for which the corresponding $\mathfrak{l}_\eta$-Verma module is simple) to deformed Whittaker modules. Our proof relies on a factorization of the functor $\overline{\Gamma}_\eta$ into a non-degenerate piece and a zero piece. Combining this with a careful analysis of partial weight spaces in deformed Verma modules, we are able to deduce the theorem.

It seems likely that one can adapt the algebraic arguments from [14] and [3] to deduce from the hereditary property that the Jantzen filtration equals the socle filtration. For this, one would need to get a precise understanding of deformed translation functors on deformed standard Whittaker modules.

1.5 A geometric approach

Our second proof of Theorem 1.3.2 is a direct proof using $\mathcal{D}$-modules. This proof generalizes Beilinson–Bernstein's original approach in [4] by replacing mixed ℓ-adic sheaves with mixed twistor $\mathcal{D}$-modules. We explain the strategy below.

An essential ingredient in the construction of Whittaker $\mathcal{D}$-modules is the exponential $\mathcal{D}$-module (see Remark 3.2.1), which is holonomic but not regular. This prevents us from using the Riemann–Hilbert correspondence to compare Whittaker $\mathcal{D}$-modules to constructible sheaves and their mixed upgrades. Instead, we use the mixed twistor $\mathcal{D}$-modules introduced by Sabbah and Mochizuki, which encapsulate $\mathcal{D}$-modules with irregular singularities.

Like their mixed Hodge module counterparts, mixed twistor $\mathcal{D}$-modules carry weight filtrations with strong functoriality and semisimplicity properties, similar to those in the Jantzen conjectures. Moreover, for mixed twistor $\mathcal{D}$-modules constructed using maximal extension functors, the weight filtration agrees with the monodromy filtration, which is explicitly computable. The strategy of our proof is to show that the $\mathcal{D}$-modules corresponding to standard Whittaker modules fit into this framework, and thus the Jantzen filtration inherits the properties of weight filtrations.

Our first step is to upgrade the geometric results of [26, 27], which use twisted sheaves of differential operators on the flag variety, to H-monodromic $\mathcal{D}$-modules on base affine space. We do this in §3.2. Once in the setting of base affine space, we are able to realize the $\mathcal{D}$-modules corresponding to standard Whittaker modules as subsheaves of maximal extension functors. This endows them with a monodromy filtration.

Next, we argue that the global sections of the monodromy filtration align with the Jantzen filtration. A key component is recognising the geometric incarnation of the deformed standard Whittaker module which arises in the construction of the maximal extension functor. This is Lemma 3.4.2. Because this critical step is given minimal justification in [4], we have included a careful argument that reduces to Verma modules when $\eta = 0$.

Finally, we show that Whittaker $\mathcal{D}$-modules admit a mixed twistor structure, and hence carry a weight filtration. This allows us to use results in [23] to establish that the weight filtration agrees with the monodromy filtration on Whittaker $\mathcal{D}$-modules. This completes the proof: the functoriality of the weight filtration implies the hereditary property of the Jantzen filtration (Theorem 1.3.2.(i)) and the semisimplicity of filtration layers of the weight filtration implies Theorem 1.3.2.(ii).

It seems likely that one can adapt the point-wise purity arguments from [4] to deduce that the Jantzen filtration equal the socle filtration. However, it is not clear to us if all their arguments involving mixed ℓ-adic sheaves translate to mixed twistor modules.

1.6 Structure of the paper

This paper is structured as follows.

Section 2.1: We define Whittaker modules and Miličić–Soergel's category $\mathcal{N}$.

Section 2.2: We define standard and costandard Whittaker modules and record their basic properties.

Section 2.3: We recall Backelin's functor from category $\mathcal{O}$ to category $\mathcal{N}$ and its important properties.

Section 2.4: We define deformed category $\mathcal{O}$, deformed Verma/dual Verma modules, and deformed standard/costandard Whittaker modules.

Section 2.5: We define the Jantzen filtration of Verma modules and standard Whittaker modules. First, it is necessary to extend the results in [9] about contravariant pairings to the deformed setting. We state the Jantzen conjecture for Whittaker modules.

Section 2.6: We define a deformed version of Backelin's functor and establish its basic properties. This involves a careful analysis of the set of Whittaker vectors in completed deformed Whittaker modules. The main result is Theorem 2.6.4, which establishes that the deformed Backelin functor sends deformed Verma modules to deformed standard Whittaker modules.

Section 2.7: We prove that the Backelin functor sends the Jantzen filtration of a Verma module to the Jantzen filtration of the corresponding standard Whittaker module. This allows us to prove the Jantzen conjecture algebraically.

Section 3.1: We set our $\mathcal{D}$-module conventions and record basic facts about functors. We define a map from the extended universal enveloping algebra $\widetilde{U}$ to global differential operators on base affine space $\widetilde{X}$ which gives the global sections of $\mathcal{D}_{\widetilde{X}}$-modules the structure of $\widetilde{U}$-modules. We introduce monodromic $\mathcal{D}$-modules, and explain their role in our story.

Section 3.2: We briefly recall the geometric approach to Whittaker modules set out in [26, 27], then explain how this can be formulated in terms of monodromic $\mathcal{D}$-modules.

Section 3.3: We define the monodromy filtration and explain how it can be used to provide a filtration on $\mathcal{D}$-modules obtained via maximal extension functors. We explain how Whittaker $\mathcal{D}$-modules can be made to fit into this framework.

Section 3.4: We show that the global sections functor sends the monodromy filtration on a Whittaker $\mathcal{D}$-module to the Jantzen filtration on the standard Whittaker module.

Section 4.1: We establish background and conventions for mixed twistor $\mathcal{D}$-modules.

Section 4.2: We show that standard Whittaker modules (and their embeddings) lift to (maps of) mixed twistor modules.

Section 4.3: We provide our geometric proof of the Jantzen conjecture.

2 Whittaker modules

2.1 Definitions

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ the nilpotent radical of $\mathfrak{b}$, and $\mathfrak{h}$ a Cartan subalgebra such that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Set $\bar{\mathfrak{n}}$ to be the nilpotent radical of the opposite Borel subalgebra $\bar{\mathfrak{b}}$.

Let $\Pi \subseteq \Sigma^+ \subseteq \Sigma \subseteq \mathfrak{h}^*$ be the sets of simple and positive roots in the root system of $\mathfrak{g}$ determined by our choice of $\mathfrak{b}$, and let (W, S) be the corresponding Coxeter system. Denote by $\mathfrak{g}_\alpha$ the root space corresponding to $\alpha \in \Sigma$, so $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.[4] Set $\rho \in \mathfrak{h}^*$ to be the half-sum of positive roots.

For $\alpha \in \Sigma$, denote by $\alpha^\vee \in \Sigma^\vee$ the corresponding coroot in the dual root system. We say that a weight $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha^\vee(\lambda) \neq 0$ for any $\alpha \in \Sigma$, *integral* if $\alpha^\vee(\lambda) \in \mathbb{Z}$ for all $\alpha \in \Sigma$, and *antidominant* if $\alpha^\vee(\lambda) \notin \mathbb{Z}_{>0}$.

Let $\operatorname{ch} \mathfrak{n}$ be the set of Lie algebra homomorphisms $\eta : \mathfrak{n} \to \mathbb{C}$. For any Lie algebra $\mathfrak{a}$, denote by $U(\mathfrak{a})$ its universal enveloping algebra and $Z(\mathfrak{a})$ the center of $U(\mathfrak{a})$. Any character $\eta \in \operatorname{ch} \mathfrak{n}$ can be extended uniquely to an algebra homomorphism $\eta : U(\mathfrak{n}) \to \mathbb{C}$ which we call by the same name.

Definition 2.1.1 Let $\eta \in \operatorname{ch} \mathfrak{n}$. An *$\eta$-Whittaker vector* in a $\mathfrak{g}$-module M is a vector $m \in M$ such that $X \cdot m = \eta(X)m$ for all $X \in \mathfrak{n}$. A *Whittaker module* is a $\mathfrak{g}$-module which is cyclically generated by an η-Whittaker vector for some $\eta \in \operatorname{ch} \mathfrak{n}$.

In [25], Miličić–Soergel introduced a category whose simple objects are the simple Whittaker modules.

Definition 2.1.2 The *Whittaker category* $\mathcal{N}$ is the category of finitely generated $U(\mathfrak{g})$-modules which are $U(\mathfrak{n})$-finite and $Z(\mathfrak{g})$-finite.

[4] Our convention that root spaces in $\mathfrak{n}$ correspond to positive roots is the opposite to that in [4], where $\mathfrak{n}$ is comprised of negative root spaces. Due to this choice, results in [4] which are stated for dominant $\lambda \in \mathfrak{h}^*$ hold for antidominant $\lambda \in \mathfrak{h}^*$ in this paper. This becomes relevant in Section 3.

The Whittaker category has a block decomposition given by the finiteness conditions in Definition 2.1.2. For $\lambda \in \mathfrak{h}^*$, set

$$\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}; z \mapsto (\lambda - \rho)p(z), \tag{2.1}$$

where

$$p : Z(\mathfrak{g}) \to U(\mathfrak{h}) \tag{2.2}$$

is the Harish-Chandra homomorphism, defined as projection onto the first coordinate in the direct sum decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}). \tag{2.3}$$

Under these conventions, $\chi_\lambda = \chi_\mu$ if and only if $\mu = w\lambda$ for some $w \in W$, where $w\lambda$ denotes the usual (not dot) action of W on $\mathfrak{h}^*$. Set $\theta = W\lambda$ to be the Weyl group orbit of λ in $\mathfrak{h}^*$. Then $J_\theta := \ker \chi_\lambda$ is a maximal ideal in $Z(\mathfrak{g})$. Set $\mathcal{N}(\theta)$ and $\mathcal{N}(\hat{\theta})$ to be the full subcategories of $\mathcal{N}$ consisting of modules annihilated by J_θ and some power of J_θ, respectively. The objects in $\mathcal{N}(\theta)$ (resp. $\mathcal{N}(\hat{\theta})$) are the modules in $\mathcal{N}$ with infinitesimal character (resp. generalized infinitesimal character) χ_λ.

For a character $\eta \in \operatorname{ch}\mathfrak{n}$, set $\mathcal{N}(\eta)$ to be the full subcategory of $\mathcal{N}$ consisting of modules V satisfying

$$V = \{v \in V \mid (X - \eta(X))^k v = 0, X \in \mathfrak{n}, \text{ for some } k \in \mathbb{N}\}.$$

Set $\mathcal{N}(\hat{\theta}, \eta) = \mathcal{N}(\hat{\theta}) \cap \mathcal{N}(\eta)$ and $\mathcal{N}(\theta, \eta) = \mathcal{N}(\theta) \cap \mathcal{N}(\eta)$. Clearly every simple Whittaker module is in some $\mathcal{N}(\theta, \eta)$.

Proposition 2.1.3 ([25, 26]) *The category $\mathcal{N}$ decomposes*

$$\mathcal{N} = \bigoplus_{\theta \in W\backslash\mathfrak{h}^*} \bigoplus_{\eta \in \operatorname{ch}\mathfrak{n}} \mathcal{N}(\hat{\theta}, \eta).$$

Let N be the unipotent subgroup of $\operatorname{Int}\mathfrak{g}$ such that $\operatorname{Lie} N = \mathfrak{n}$. Given a module $V \in \mathcal{N}(\eta)$, we can twist by $-\eta$ to obtain a module $V \otimes \mathbb{C}_{-\eta} \in \mathcal{N}(0)$ on which the action of $\mathfrak{n}$ is locally nilpotent, and hence integrates to an algebraic action of N. The natural isomorphism $V \xrightarrow{\sim} V \otimes \mathbb{C}_{-\eta}, v \mapsto v \otimes 1$ induces an algebraic action of N on V whose differential differs from the $\mathfrak{n}$-action by η. This establishes that the categories $\mathcal{N}(\theta, \eta)$ can be viewed as "twisted Harish-Chandra modules".

Proposition 2.1.4 ([26, Lem. 2.3]) *There is an equivalence of categories*

$$\mathcal{N}(\theta, \eta) = \mathcal{M}_{f.g.}(U_\theta, N, \eta),$$

where U_θ is the quotient of $U(\mathfrak{g})$ by the ideal generated by J_θ and $\mathcal{M}_{f.g.}(U_\theta, N, \eta)$ is the category of finitely generated U_θ-modules admitting an algebraic action of N whose differential differs from the $\mathfrak{n}$-action obtained by restriction exactly by η.

2.2 Standard/costandard modules and contravariant pairings

The categories $\mathcal{N}(\theta, \eta)$ have a natural set of standard, costandard, and simple objects, and with respect to these objects, they are highest weight categories [9, Cor. 7.4]. In this section, we review the construction of standard, costandard, and simple objects, and recall the classification of contravariant pairings between standard Whittaker modules and Verma modules introduced in [9].

A character $\eta \in \operatorname{ch}\mathfrak{n}$ determines a subset of simple roots

$$\Pi_\eta := \{\alpha \in \Pi \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}. \tag{2.4}$$

Let Σ_η be the root system generated by Π_η and W_η the Weyl group of Σ_η. Denote by

$$\mathfrak{l}_\eta = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Sigma_\eta} \mathfrak{g}_\alpha, \quad \mathfrak{n}_\eta = \mathfrak{l}_\eta \cap \mathfrak{n}, \quad \bar{\mathfrak{n}}_\eta = \mathfrak{l}_\eta \cap \bar{\mathfrak{n}}, \quad \mathfrak{n}^\eta = \bigoplus_{\alpha\in\Sigma^+\setminus\Sigma_\eta^+} \mathfrak{g}_\alpha, \quad \mathfrak{p}_\eta = \mathfrak{l}_\eta \oplus \mathfrak{n}^\eta. \tag{2.5}$$

The Levi subalgebra $\mathfrak{l}_\eta$ is reductive with center

$$\mathfrak{h}^\eta = \{H \in \mathfrak{h} \mid \alpha(H) = 0, \alpha \in \Pi_\eta\}. \tag{2.6}$$

Let

$$p_\eta : Z(\mathfrak{l}_\eta) \to U(\mathfrak{h}) \tag{2.7}$$

be the Harish-Chandra homomorphism of $U(\mathfrak{l}_\eta)$, defined as projection onto the first coordinate in the direct sum decomposition

$$U(\mathfrak{l}_\eta) = U(\mathfrak{h}) \oplus (\bar{\mathfrak{n}}_\eta U(\mathfrak{l}_\eta) + U(\mathfrak{l}_\eta)\mathfrak{n}_\eta). \tag{2.8}$$

For $\lambda \in \mathfrak{h}^*$, let

$$\chi_\eta^\lambda : Z(\mathfrak{l}_\eta) \to \mathbb{C}; z \mapsto (\lambda - \rho_\eta)p_\eta(z) \tag{2.9}$$

be the corresponding infinitesimal character. Here $\rho_\eta = \frac{1}{2}\sum_{\alpha\in\Sigma_\eta^+} \alpha$ is the half-sum of positive roots in Σ_η. We have $\chi_\eta^\lambda = \chi_\eta^\mu$ if and only if $\mu = w\lambda$ for some $w \in W_\eta$.

From the data $(\lambda, \eta) \in \mathfrak{h}^* \times \operatorname{ch}\mathfrak{n}$, we construct a $U(\mathfrak{l}_\eta)$-module

$$Y(\lambda, \eta) := U(\mathfrak{l}_\eta) \otimes_{Z(\mathfrak{l}_\eta)\otimes U(\mathfrak{n}_\eta)} \mathbb{C}_{\chi_\eta^\lambda, \eta}. \tag{2.10}$$

Here $\mathbb{C}_{\chi_\eta^\lambda}$ is the one-dimensional $Z(\mathfrak{l}_\eta) \otimes U(\mathfrak{n}_\eta)$-module with action

$$z \otimes u \cdot v = \chi_\eta^\lambda(z)\eta(u)v \tag{2.11}$$

for $z \in Z(\mathfrak{l}_\eta)$, $u \in U(\mathfrak{n}_\eta)$, $v \in \mathbb{C}$. By construction, $Y(\lambda, \eta) \simeq Y(\mu, \eta)$ if and only if $\mu = w\lambda$ for some $w \in W_\eta$. For any $\lambda \in \mathfrak{h}^*$, $Y(\lambda, \eta)$ is an irreducible $U(\mathfrak{l}_\eta)$-module [19, Prop. 2.3].

Definition 2.2.1 The *standard Whittaker module* associated to $(\lambda, \eta) \in \mathfrak{h}^* \times \operatorname{ch}\mathfrak{n}$ is the $U(\mathfrak{g})$-module

$$M(\lambda, \eta) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} Y(\lambda - \rho + \rho_\eta, \eta).$$

Here we view $Y(\lambda - \rho + \rho_\eta)$ as a $U(\mathfrak{p}_\eta)$-module by letting $\mathfrak{n}^\eta$ act trivially.

The properties of standard Whittaker modules were established in [19]. We collect them in the following proposition for future reference.

Proposition 2.2.2 *(i) If $\theta = W\lambda$, then $M(\lambda, \eta) \in \mathcal{N}(\theta, \eta)$.*

(ii) Two standard modules $M(\lambda, \eta)$ and $M(\mu, \eta)$ are isomorphic if and only if $\mu \in W_\eta \lambda$.

(iii) The standard module $M(\lambda, \eta)$ is a Whittaker module generated by the η-Whittaker vector $1 \otimes 1 \otimes 1$.

(iv) For $\lambda \in \mathfrak{h}^*$, denote by λ_η the restriction of λ to $\mathfrak{h}^\eta$ (2.6). There is a natural partial order on $\mathfrak{h}^{\eta *} = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{h}^\eta, \mathbb{C})$ obtained from the partial order on $\mathfrak{h}^*$. The Lie algebra $\mathfrak{h}^\eta$ acts semisimply on $M(\lambda, \eta)$, and with respect to this action, $M(\lambda, \eta)$ decomposes into $\mathfrak{h}^\eta$-weight spaces:

$$M(\lambda, \eta) = \bigoplus_{\nu_\eta \leqslant \lambda_\eta - \rho_\eta} M(\lambda, \eta)_{\nu_\eta}.$$

Each $\mathfrak{h}^\eta$-weight space $M(\lambda, \eta)_{\nu_\eta}$ is $U(\mathfrak{l}_\eta)$-stable, and as $U(\mathfrak{l}_\eta)$-modules,

$$M(\lambda, \eta)_{\nu_\eta} = U(\bar{\mathfrak{n}}^\eta)_{\mu_\eta} \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_\eta, \eta)$$

for some $\mu_\eta \leqslant 0$ in $\mathfrak{h}^{\eta *}$. Here $\bar{\mathfrak{n}}^\eta = \bigoplus_{\alpha \in -(\Sigma^+ \setminus \Sigma_\eta^+)} \mathfrak{g}_\alpha$.

(v) The standard module $M(\lambda, \eta)$ has a unique irreducible quotient $L(\lambda, \eta)$, which is also an η-Whittaker module. All simple Whittaker modules are isomorphic to $L(\lambda, \eta)$ for some $(\lambda, \eta) \in \mathfrak{h}^* \times \operatorname{ch}\mathfrak{n}$, and the collection of $L(\lambda, \eta)$ forms a complete set of simple modules in $\mathcal{N}$.

(vi) If $\eta = 0$, then $M(\lambda, 0)$ is a Verma module of highest weight $\lambda - \rho$.

In [9], costandard modules in category $\mathcal{N}$ were constructed. For a $U(\mathfrak{g})$-module V, denote by

$$(V)_\eta := \{v \in V \mid (X - \eta(X))^k v = 0,\ X \in \mathfrak{n} \text{ for some } k \in \mathbb{N}\} \tag{2.12}$$

the set of η-twisted $\mathfrak{n}$-finite vectors in V. For any $U(\mathfrak{g})$-module, $(V)_\eta$ is $U(\mathfrak{g})$-stable [9, Lem. 6.1]. For a $U(\mathfrak{g})$-module V, we denote by $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ the full linear dual of V, which we consider as a $U(\mathfrak{g})$-module via the action $u \cdot f(-) = f(\tau(u) \cdot -)$, where τ is the transpose antiautomorphism of $\mathfrak{g}$.

Definition 2.2.3 The *costandard module* in $\mathcal{N}$ associated to $(\lambda, \eta) \in \mathfrak{h}^* \times \operatorname{ch}\mathfrak{n}$ is

$$M^\vee(\lambda, \eta) := (M(\lambda)^*)_\eta,$$

where $M(\lambda)$ is the Verma module of highest weight $\lambda - \rho$.

Unlike standard modules, costandard modules are not Whittaker modules, though they are in category $\mathcal{N}$. Standard and costandard modules in $\mathcal{N}$ are related via contravariant pairings.

Definition 2.2.4 A *contravariant pairing* between the standard Whittaker module $M(\lambda, \eta)$ and the Verma module $M(\mu)$ is a bilinear pairing

$$(\cdot, \cdot) : M(\lambda, \eta) \times M(\mu) \to \mathbb{C}$$

such that $(ux, y) = (x, \tau(u)y)$ for all $u \in U(\mathfrak{g})$ and $y \in M(\mu)$.

Theorem 2.2.5 [9, Thm. 5.2] *If $\lambda \in \mathfrak{h}^*$ is regular, there exists a contravariant pairing between $M(\lambda, \eta)$ and $M(\mu)$ if and only if $\mu \in W_\eta\lambda$. If a contravariant pairing between $M(\lambda, \eta)$ and $M(\mu)$ exists, it is unique up to scaling.*

Contravariant pairings on standard Whittaker modules are a generalization of contravariant forms on Verma modules, and they provide the canonical map between standard and costandard objects in category $\mathcal{N}$. The properties of costandard Whittaker modules were developed in [9], and we list them in the following proposition for future reference.

Proposition 2.2.6 *(i) Two costandard modules $M^\vee(\lambda, \eta)$ and $M^\vee(\mu, \eta)$ are isomorphic if and only if $\mu \in W_\eta\lambda$.*
(ii) The costandard module $M^\vee(\lambda, \eta)$ is an object in the category $\mathcal{N}(\theta, \eta)$, where $\theta = W\lambda$.
(iii) The unique contravariant pairing $\langle\cdot, \cdot\rangle : M(\lambda, \eta) \times M(\lambda) \to \mathbb{C}$ (Theorem 2.2.5, (2.25)) induces a canonical $\mathfrak{g}$-module homomorphism $M(\lambda, \eta) \to M^\vee(\lambda, \eta)$.
(iv) In the Grothendieck group $K\mathcal{N}(\theta, \eta)$, $[M(\lambda, \eta)] = [M^\vee(\lambda, \eta)]$, so the standard and costandard module corresponding to a given parameter have the same composition factors.
(v) The costandard module $M^\vee(\lambda, \eta)$ has a unique irreducible submodule which is isomorphic to $L(\lambda, \eta)$.

2.3 Backelin's functor

An important tool for in our algebraic proof of the Jantzen conjecture is Backelin's functor comparing category $\mathcal{N}$ and $\mathcal{O}$.

For a $\mathfrak{g}$-modules that are semisimple as $\mathfrak{h}$-modules, *Backelin's functor*, introduced in [2], is defined as

$$\overline{\Gamma}_\eta(M) = \Gamma_\eta(\overline{M}).$$

Here, $\overline{M} = \prod_{\mu\in\mathfrak{h}} M_\mu$ is the completion (which is again an $\mathfrak{g}$-module) and for an $\mathfrak{n}$-module N,

$$\Gamma_\eta(N) = \{n \in N \mid \exists k \geqslant 0 \text{ with } U_\eta(\mathfrak{n})^k n = 0\}$$

denotes the subset of generalized Whittaker vectors.

We list some important properties of $\overline{\Gamma}_\eta$ shown in [2] and [1] for future reference.

Proposition 2.3.1 *(i) The functor $\overline{\Gamma}_\eta$ restricts to an exact functor*

$$\overline{\Gamma}_\eta : \mathcal{O} \to \mathcal{N}$$

between the BGG category $\mathcal{O}$ (which consists of finitely generated $\mathfrak{g}$-modules which are $\mathfrak{h}$-semisimple and $\mathfrak{n}$-locally finite) and the Whittaker category. The functor is a quotient functor onto its essential image.

(ii) For $\lambda^* \in \mathfrak{h}$, $\overline{\Gamma}_\eta(M(\lambda)) = M(\lambda, \eta)$.

(iii) Let λ be anti-dominant. Then $\overline{\Gamma}_\eta(L(w\lambda)) = L(w\lambda, \eta)$ if w is a longest coset representative in $W_\eta \backslash W$. Else $\overline{\Gamma}_\eta(L(w\lambda)) = 0$.

(iv) For $M \in \mathcal{O}$ and E finite-dimensional, there is a natural isomorphism $\overline{\Gamma}_\eta(M \otimes E) \cong \overline{\Gamma}_\eta(M) \otimes E$.

In Sect. 2.6, we will discuss a deformed version of the Backelin's functor.

2.4 Deformed modules

To define the Jantzen filtrations algebraically, we need to introduce deformed versions of the objects above. To begin, we recall the construction of deformed category $\mathcal{O}$, following [34].

2.4.1 Deformed category $\mathcal{O}$

Let T be a commutative unital $\mathbb{C}$-algebra with a distinguished algebra homomorphism

$$U(\mathfrak{h}) = S(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*) \xrightarrow{\varphi} T. \tag{2.13}$$

Let $\mathfrak{g}$-mod-T denote the category of bimodules over $\mathfrak{g}$ and T. For $\lambda \in \mathfrak{h}^*$ and $M \in \mathfrak{g}$-mod-T, define the λ-*deformed weight space* to be

$$M^\lambda := \{m \in M \mid (H - \lambda(H))m = m\varphi(H) \text{ for all } H \in \mathfrak{h}\}. \tag{2.14}$$

Definition 2.4.1 *Deformed category* $\mathcal{O}$ is the full subcategory $\mathcal{O}_T \subseteq \mathfrak{g}$-mod-$T$ of locally $\mathfrak{n}$-finite modules in $\mathfrak{g}$-mod-T with the property that $M = \bigoplus_\lambda M^\lambda$.

We are especially interested in two classes of objects in $\mathcal{O}_T$: deformed Verma modules and deformed dual Verma modules.

Definition 2.4.2 For $\lambda \in \mathfrak{h}^*$, the corresponding *deformed Verma module* is the $(\mathfrak{g}, T)$-bimodule

$$M_T(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} T_{\lambda-\rho},$$

where $\mathfrak{g}$ acts by left multiplication on the first tensor factor and T acts on the second tensor factor (which is isomorphic to T) by right multiplication. The $U(\mathfrak{b})$-module structure on $T_{\lambda-\rho}$ is given by extending the $\mathfrak{h}$-action

$$H \cdot t = (\lambda - \rho + \varphi)(H)t$$

for $H \in \mathfrak{h}$ and $t \in T$ trivially to $\mathfrak{b}$.

Definition 2.4.3 The *deformed dual Verma module* $M_T^\vee(\lambda)$ corresponding to $\lambda \in \mathfrak{h}^*$ is the sum of the deformed weight spaces in

$$\mathrm{ind}_{\bar{\mathfrak{b}}}^{\mathfrak{g}}(T_{\lambda-\rho}) = \mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), T_{\lambda-\rho}).$$

Here $\mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), T_{\lambda-\rho})$ refers to homomorphisms of left $U(\bar{\mathfrak{b}})$-modules. It is given the structure of a left $U(\mathfrak{g})$-module via the right action of $U(\mathfrak{g})$ on itself.

Remark 2.4.4 As $(\mathfrak{g}, T)$-bimodules,

$$\mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), T_{\lambda-\rho}) \simeq \mathrm{Hom}_T(M_T(\lambda), T) \tag{2.15}$$

where $\mathrm{Hom}_T(M_T(\lambda), T)$ denotes the set of T-module homomorphisms from $M_T(\lambda)$ to T. The set $\mathrm{Hom}_T(M_T(\lambda), T)$ is given the structure of a left $U(\mathfrak{g})$-module via the action

$$u \cdot f(v \otimes t) = f(\tau(u)v \otimes t),$$

for $u, v \in U(\mathfrak{g})$, $f \in \mathrm{Hom}_T(M_T(\lambda), T)$, and $t \in T_{\lambda-\rho}$. The isomorphism (2.15) comes from the left $U(\mathfrak{g})$-module isomorphism [5]

$$\mathrm{Hom}^l_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), T_{\lambda-\rho}) \simeq \mathrm{Hom}^r_{U(\mathfrak{b})}(U(\mathfrak{g}), T_{\lambda-\rho})$$

and an application of tensor-hom adjunction for right modules.

The bimodules $M_T(\lambda)$ and $M_T^\vee(\lambda)$ are objects in $\mathcal{O}_T$. The deformed weight spaces of $M_T(\lambda)$ and $M_T^\vee(\lambda)$ are free finite-rank T-modules, and in both modules, for $T \neq 0$, the rank of the deformed $(\lambda - \rho - \nu)$-weight space is $\dim_{\mathbb{C}} U(\mathfrak{n})^\nu$.

Remark 2.4.5 If we specialize to the case $T = \mathbb{C}$ and let φ be evaluation at zero, then $\mathcal{O}_T$ is a version of Bernstein–Gelfand–Gelfand's category $\mathcal{O}$ which is missing some of the usual finiteness conditions. In this case, the $M_T(\lambda)$ and $M_T^\vee(\lambda)$ are the Verma module and dual Verma module of highest weight $\lambda - \rho$.

5 Here superscripts on Hom indicate whether we are considering left or right modules, and the left $U(\mathfrak{g})$-module structure on $\mathrm{Hom}^r_{U(\mathfrak{b})}(U(\mathfrak{g}), T_{\lambda-\rho})$ comes from the right action of $U(\mathfrak{g})$ on itself via left multiplication by the transpose.

2.4.2 Deformed Whittaker modules

Within the category of $(\mathfrak{g}, T)$-bimodules, we can also define deformed standard and costandard Whittaker modules.

Definition 2.4.6 For a pair $(\lambda, \eta) \in \mathfrak{h}^* \times \mathrm{ch}\, \mathfrak{n}$, we define a *deformed standard Whittaker module* to be the $(\mathfrak{g}, T)$-bimodule

$$M_T(\lambda, \eta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} U(\mathfrak{l}_\eta) \otimes_{Z(\mathfrak{l}_\eta)\otimes U(\mathfrak{n}_\eta)} T_{\lambda-\rho+\rho_\eta,\eta}$$

where $\mathfrak{g}$ acts by left multiplication on the first tensor factor and T acts by right multiplication on the final tensor factor. The $Z(\mathfrak{l}_\eta) \otimes U(\mathfrak{n}_\eta)$-module structure on $T_{\lambda-\rho+\rho_\eta}$ is given as follows: for $z \otimes u \in Z(\mathfrak{l}_\eta) \otimes U(\mathfrak{n}_\eta)$ and $t \in T_{\lambda-\rho+\rho_\eta}$,

$$z \otimes u \cdot t = (\chi_\eta^{\lambda-\rho+\rho_\eta} + \varphi \circ p_\eta)(z)\eta(u)t,$$

where p_η is the Harish-Chandra homomorphism of $\mathfrak{l}_\eta$ (2.7), $\chi_\eta^{\lambda-\rho+\rho_\eta}$ is the infinitesimal character of $Z(\mathfrak{l}_\eta)$ corresponding to $\lambda - \rho + \rho_\eta$ (2.9), and φ is as in (2.13).

Because $\chi_\eta^{\lambda+\rho-\rho_\eta} = \chi_\eta^{w\lambda+\rho-\rho_\eta}$ for all $w \in W_\eta$, we see immediately from the definition that

$$M_T(\lambda, \eta) \simeq M_T(w\lambda, \eta) \tag{2.16}$$

for all $w \in W_\eta$.

Recall that for a $U(\mathfrak{g})$-module V, $(V)_\eta$ denotes the $U(\mathfrak{g})$-submodule of η-twisted $\mathfrak{n}$-finite vectors in V (2.12).

Definition 2.4.7 For a pair $(\lambda, \eta) \in \mathfrak{h}^* \times \mathrm{ch}\, \mathfrak{n}$, the corresponding *deformed costandard Whittaker module* is the $(\mathfrak{g}, T)$-bimodule

$$M_T^\vee(\lambda, \eta) = \left(\mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), T_{\lambda-\rho})\right)_\eta.$$

Here the $U(\mathfrak{g})$-module structure on $\mathrm{Hom}_{U(\bar{\mathfrak{b}})}(U(\mathfrak{g}), T_{\lambda-\rho})$ is given by right multiplication of $U(\mathfrak{g})$ on itself.

2.5 Two algebraic Jantzen filtrations

2.5.1 The Jantzen filtration of a Verma module

There is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_T}(M_T(\lambda), M_T^\vee(\lambda)) \xrightarrow{\sim} T \tag{2.17}$$

induced from restriction to the deformed $(\lambda-\rho)$-weight space together with the canonical identifications $M_T(\lambda)^{\lambda-\rho} \xrightarrow{\sim} T$ and $M_T^\vee(\lambda)^{\lambda-\rho} \xrightarrow{\sim} T$ [34, Prop. 2.12]. Under this isomorphism, $1 \in T$ distinguishes a canonical $(\mathfrak{g}, T)$-bimodule homomorphism

$$\psi : M_T(\lambda) \to M_T^\vee(\lambda). \tag{2.18}$$

To define the Jantzen filtration of a deformed Verma module, we must specialize slightly. Let A be a local $\mathbb{C}$-algebra and $\varphi : U(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*) \to A$ a distinguished ring homomorphism such that $\varphi(\mathfrak{h}) \subseteq \mathfrak{m}$, where $\mathfrak{m}$ is the unique maximal ideal of A.

For any right A-module M, there is a filtration by A-submodules $M\mathfrak{m}^i \subset M$ and a reduction map

$$\text{red} : M \to M/M\mathfrak{m}, m \mapsto \overline{m}. \tag{2.19}$$

Applying this to the A-module $M_A(\lambda)$, we see that the filtration layers are $\mathfrak{g}$-stable, so we obtain a surjective $\mathfrak{g}$-module homomorphism

$$\text{red} : M_A(\lambda) \to M_A(\lambda)/M_A(\lambda)\mathfrak{m} = M(\lambda). \tag{2.20}$$

Similarly, the reduction map for $M_A^\vee(\lambda)$ gives a surjective $\mathfrak{g}$-module homomorphism from $M_A^\vee(\lambda)$ to the dual Verma module $M^\vee(\lambda)$. Pulling back the filtration above along the canonical map ψ (2.18), we obtain a $(\mathfrak{g}, A)$-bimodule filtration of $M_A(\lambda)$.

Definition 2.5.1 The *Jantzen filtration of $M_A(\lambda)$* is the $(\mathfrak{g}, A)$-bimodule filtration

$$M_A(\lambda)_{-i} := \{m \in M_A(\lambda) \mid \psi(m) \in M_A^\vee(\lambda)\mathfrak{m}^i\},$$

where ψ is the canonical map in (2.18). By applying the map red (2.20) to the filtration layers, we obtain in this way a filtration $M(\lambda)_\bullet$ of $M(\lambda)$, which we refer to as the *Jantzen filtration of $M(\lambda)$*.

Remark 2.5.2 It is convenient for us to view the Jantzen filtration as an ascending filtration, as this aligns more naturally with the geometric filtrations occurring in Sect. 3. Because of this, our definition of the Jantzen filtration differs slightly from that in [15], where it is defined as a descending filtration.

This filtration was introduced using contravariant forms in work of Jantzen [15]. In the case where the deformation direction is ρ, the filtration satisfies surprising functoriality and semisimplicity properties. More precisely, we further specialize to the following setting.

Set $T = \mathcal{O}(\mathbb{C}\rho)$ to be the ring of regular functions on the line $\mathbb{C}\rho \subset \mathfrak{h}^*$. A choice of a linear functional $v : \mathbb{C}\rho \to \mathbb{C}$ gives an isomorphism $T \cong \mathbb{C}[v]$. We fix such an identification. Set

$$A = T_{(v)} \tag{2.21}$$

to be the local $\mathbb{C}$-algebra obtained from T by inverting all polynomials not divisible by v, and

$$\varphi : \mathcal{O}(\mathfrak{h}^*) \to A \tag{2.22}$$

to be the composition of the restriction map $\mathcal{O}(\mathfrak{h}^*) \to T$ with inclusion $T \hookrightarrow A$.

Theorem 2.5.3 [4, Cor. 5.3.4, Cor. 5.3.1] (The Jantzen Conjectures for Verma modules) *Let A and T be as in* (2.21).

(1) If $\iota : M(\mu) \hookrightarrow M(\lambda)$ is an embedding of Verma modules, then the filtration on $\iota(M(\mu))$ induced from the Jantzen filtration on $M(\lambda)$ agrees with the Jantzen filtration on $M(\mu)$, up to a shift.
(2) The filtration layers $M(\lambda)_i/M(\lambda)_{i-1}$ of the Jantzen filtration of a Verma module $M(\lambda)$ are semisimple for all i.

The strategy of Beilinson–Bernstein's proof of Theorem 2.5.3 is to relate the Jantzen filtration to the weight filtration of the corresponding $\mathcal{D}$-module. We will follow a similar course for Whittaker modules in Sect. 4.

2.5.2 The Jantzen filtration of a standard Whittaker module

We can generalize the classical construction above to Whittaker modules using the canonical map between standard and costandard Whittaker modules (Proposition 2.2.6(iii)). Our first step is to establish the existence of a canonical map between standard and costandard Whittaker modules on the deformed level. To do so, we use A-valued contravariant pairings.

We return to the setting where A is a local $\mathbb{C}$-algebra equipped with a distinguished algebra homomorphism $\varphi : U(\mathfrak{h}) \to A$ which maps $\mathfrak{h}$ into the maximal ideal $\mathfrak{m}$.

Definition 2.5.4 An *A-contravariant pairing* between $(\mathfrak{g}, A)$-bimodules V and W is an A-valued A-bilinear pairing

$$(\cdot, \cdot)_A : V \times W \to A$$

such that for all $u \in U(\mathfrak{g})$, $v \in V$ and $w \in W$, $(uv, w)_A = (v, \tau(u)w)_A$, where $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the transpose antiautomorphism.

Clearly the set $\Psi_A(V, W)$ of A-contravariant pairings between V and W is a right A-module. We are interested in the A-module

$$\Psi_A := \Psi_A(M_A(\lambda, \eta), M_A(\lambda)) \tag{2.23}$$

of A-contravariant pairings between deformed standard Whittaker modules and deformed Verma modules, which is closely related to the $\mathbb{C}$-vector space

$$\Psi := \{\text{contravariant pairings between } M(\lambda, \eta) \text{ and } M(\lambda)\} \tag{2.24}$$

studied in [9]. By Theorem 2.2.5, Ψ is 1-dimensional.

Let $w = 1 \otimes 1 \otimes 1 \in M(\lambda, \eta)$ be the generating Whittaker vector and $v = 1 \otimes 1 \in M(\lambda)$ be the generating highest weight vector. Denote by

$$\langle \cdot, \cdot \rangle : M(\lambda, \eta) \otimes M(\lambda) \to \mathbb{C} \tag{2.25}$$

the unique contravariant pairing such that $\langle w, v\rangle = 1$ (Theorem 2.2.5). Set $\underline{w} = 1 \otimes 1 \otimes 1 \in M_A(\lambda, \eta)$ and $\underline{v} = 1 \otimes 1 \in M_A(\lambda)$. Clearly $\underline{w}$ (resp. $\underline{v}$) generates $M_A(\lambda, \eta)$ (resp. $M_A(\lambda)$) as a $(\mathfrak{g}, A)$-bimodule. We define a A-bilinear pairing

$$\langle \cdot, \cdot \rangle_A : M_A(\lambda, \eta) \otimes M_A(\lambda) \to A \tag{2.26}$$

by $\langle u\underline{w}a, u'\underline{v}b\rangle_A := \langle uw, u'v\rangle ab$ for $u, u' \in U(\mathfrak{g}), a, b \in A$.

Proposition 2.5.5 *The map $\langle \cdot, \cdot \rangle_A$ in (2.26) is an A-contravariant pairing of $M_A(\lambda, \eta)$ and $M_A(\lambda)$.*

Proof By construction, $\langle \cdot, \cdot \rangle_A$ is A-bilinear and contravariant. It remains to show that it is well-defined.

A vector $x \in M_A(\lambda, \eta)$ can be represented as $u\underline{w}a$ for $u \in U(\mathfrak{g}), a \in A$ in many different ways, and similarly for $y = u'\underline{v}b \in M_A(\lambda)$. We will show that our definition is independent of such choices. Because $\langle \cdot, \cdot \rangle_A$ is defined to be A-bilinear and $M_A(\lambda, \eta)$ and $M_A(\lambda)$ are free as A-modules, it suffices to consider the case when $a = b = 1$.

Let $u, u', z, z' \in U(\mathfrak{g})$ be such that

$$x = u\underline{w} = z\underline{w} \text{ and } y = u'\underline{v} = z'\underline{v}.$$

Then $u - z \in \mathrm{Ann}_{U(\mathfrak{g})}\, \underline{w}$ and $u' - z' \in \mathrm{Ann}_{U(\mathfrak{g})}\, \underline{v}$. In fact, $u - z \in \mathrm{Ann}_{U(\mathfrak{g})}\, w$ and $u' - z' \in \mathrm{Ann}_{U(\mathfrak{g})}\, v$, because the reduction map $\mathrm{red} : M_A \to M$ (2.19) for $M = M(\lambda, \eta)$ or $M = M(\lambda)$ is a $U(\mathfrak{g})$-module homomorphism. This implies that

$$\langle uw, u'v\rangle = \langle zw, z'v\rangle,$$

because $\langle \cdot, \cdot \rangle$ is well defined, and hence

$$\langle u\underline{w}, u'\underline{v}\rangle_A = \langle z\underline{w}, z'\underline{v}\rangle_A.$$

□

We have constructed one element $\langle \cdot, \cdot \rangle_A \in \Psi_A$. For any $a \neq 1 \in A$, $\langle \cdot, \cdot \rangle_A a$ is another (distinct) element of Ψ_A, so $A \subseteq \Psi_A$. We claim that there are no other A-contravariant pairings.

Proposition 2.5.6 *As a right A-module, $\Psi_A \simeq A$.*

Proof Let $(\cdot, \cdot)_A$ be an A-contravariant pairing. From $(\cdot, \cdot)_A$, we can define a contravariant pairing

$$(\cdot, \cdot) : M(\lambda, \eta) \times M(\lambda) \to \mathbb{C}$$

by $(uw, u'v) := \mathrm{red}(u\overline{w}, u'\overline{v})_A$, where $\mathrm{red} : A \to \mathbb{C}$ is the reduction map (2.19) for the left A-module A. This defines a map

$$g : \Psi_A \to \Psi; (\cdot, \cdot)_A \mapsto (\cdot, \cdot). \tag{2.27}$$

By the construction above Proposition 2.5.5, we also have a map

$$f : \Psi \to \Psi_A, \tag{2.28}$$

with $f(\langle\cdot,\cdot\rangle) = \langle\cdot,\cdot\rangle_A$. (Here $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_A$ are the specific pairings defined in (2.25) and (2.26).)

The maps f and g are not inverse to one another. But, since both $M_A(\lambda,\eta)$ and $M_A(\lambda)$ are cyclically generated, for each A-contravariant pairing $(\cdot,\cdot)_A$, there exists $a \in A$, given by the pairing of the generators under $(\cdot,\cdot)_A$, such that

$$f \circ g((\cdot,\cdot)_A)a = (\cdot,\cdot)_A. \tag{2.29}$$

Now, by Theorem 2.2.5,

$$g((\cdot,\cdot)_A) = (\cdot,\cdot) = c\langle\cdot,\cdot\rangle$$

for some $c \in \mathbb{C}$, where $\langle\cdot,\cdot\rangle$ is the unique pairing in (2.25). From this we see that for an A-contravariant pairing $(\cdot,\cdot)_A$, we have on one hand,

$$f \circ g((\cdot,\cdot)_A)a = (\cdot,\cdot)_A$$

for some $a \in A$, and on the other hand,

$$f \circ g((\cdot,\cdot)_A) = f(c\langle\cdot,\cdot\rangle) = c\langle\cdot,\cdot\rangle_A.$$

Hence $(\cdot,\cdot)_A = c\langle\cdot,\cdot\rangle_A a$ is an A-multiple of $\langle\cdot,\cdot\rangle_A$. □

We can use Proposition 2.5.6 to construct a canonical map between deformed standard and costandard Whittaker modules. Recall that by Remark 2.4.4, we can view a deformed costandard Whittaker module (Definition 2.4.7) as a submodule of the full A-linear dual of a deformed Verma module:

$$M_A^\vee(\lambda,\eta) \simeq (\mathrm{Hom}_A(M_A(\lambda),A))_\eta \subseteq \mathrm{Hom}_A(M_A(\lambda),A).$$

There is a canonical A-contravariant form $\langle\cdot,\cdot\rangle_A \in \Psi_A$ corresponding to $1 \in A$ with $\langle \underline{w}, \underline{v}\rangle_A = 1$ (Proposition 2.5.6). This induces a $(\mathfrak{g}, A)$-bimodule morphism

$$\psi_\eta : M_A(\lambda,\eta) \to \mathrm{Hom}_A(M_A(\lambda),A); m \mapsto \langle m,\cdot\rangle_A. \tag{2.30}$$

Because ψ_η is a $\mathfrak{g}$-module homomorphism and $(M_A(\lambda,\eta))_\eta = M_A(\lambda,\eta)$, the image of ψ_η lands in $M_A^\vee(\lambda,\eta) = (\mathrm{Hom}_A(M_A(\lambda),A))_\eta$.

Now we can define a Jantzen filtration of $M_A(\lambda,\eta)$ analogously to Definition 2.5.1 by pulling back the natural filtration of $M_A^\vee(\lambda,\eta)$ along ψ_η.

Definition 2.5.7 The *Jantzen filtration of* $M_A(\lambda,\eta)$ is the $(\mathfrak{g}, A)$-bimodule filtration

$$M_A(\lambda,\eta)_{-i} := \{m \in M_A(\lambda,\eta) \mid \psi_\eta(m) \in M_A^\vee(\lambda,\eta)\mathfrak{m}^i\},$$

where $\mathfrak{m}$ is the unique maximal ideal of A. By applying the reduction map 2.19 to the filtration layers, we obtain a filtration $M(\lambda, \eta)_\bullet$ of $M(\lambda, \eta)$ which we refer to as the *Jantzen filtration of* $M(\lambda, \eta)$.

The Jantzen filtration of a standard Whittaker module satisfies the same hereditary and semisimplicity properties as the Jantzen filtration of a Verma module. This is our main result.

Theorem 2.5.8 (The Jantzen conjectures for Whittaker modules) *Let A be the local $\mathbb{C}$-algebra in* (2.21) *determined by ρ.*

(1) If $\iota : M(\mu, \eta) \hookrightarrow M(\lambda, \eta)$ is an embedding of standard Whittaker modules, then the filtration on $\iota(M(\mu, \eta))$ induced from the Jantzen filtration on $M(\lambda, \eta)$ agrees with the Jantzen filtration on $M(\mu, \eta)$, up to a shift.

(2) Then filtration layers $M(\lambda, \eta)_i/M(\lambda, \eta)_{i-1}$ of the Jantzen filtration of a standard Whittaker module $M(\lambda, \eta)$ are semisimple for all i.

We will prove this theorem in two ways: first, by comparing it to the Jantzen filtration of a Verma module using a functor introduced by Backelin, and second, by comparing it to the weight filtration on the corresponding mixed twistor $\mathcal{D}$-module.

2.6 Whittaker vectors in completed deformed Verma modules

In this section we lift results of Kostant [16] on Whittaker vectors in completed Verma modules to the deformed case.

We return to the setting where A is a local $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}$ and fix a map $\varphi : U(\mathfrak{h}) \to A$ with $\varphi(\mathfrak{h}) \subseteq \mathfrak{m}$. Let $\lambda \in \mathfrak{h}^*$. The deformed Verma module $M_A(\lambda)$ has deformed weight spaces of the form

$$M_A(\lambda)^{\lambda-\rho+\mu} \cong U(\bar{\mathfrak{n}})_\mu \otimes A$$

for a weight μ of $U(\bar{\mathfrak{n}})$. Denote by $U(\mathfrak{n})^\dagger = \bigoplus_{\mu \in \mathfrak{h}} (U(\mathfrak{n})_\mu)^*$ the graded dual of $U(\mathfrak{n})$ with the contragredient action of $\mathfrak{n}$. We consider the $(\mathfrak{n}, A)$-bimodule $U(\mathfrak{n})^\dagger \otimes A$ where $\mathfrak{n}$ acts by left and A by right multiplication. Then we get the following deformed version of [16, Prop. 3.8].

Proposition 2.6.1 *Let $\lambda \in \mathfrak{h}^*$. If $M(\lambda)$ is simple, then there is an isomorphism of $(\mathfrak{n}, A)$-bimodules*

$$\beta : M_A(\lambda) \to U(\mathfrak{n})^\dagger \otimes A.$$

Proof Denote by $v = 1 \otimes 1 \in M_A(\lambda)$ the highest weight generator and by $l : M_A(\lambda) \to A$ the A-linear map determined by $l(v) = 1$ and $l(n) = 0$ for all $n \in M_A(\lambda)^\mu$ for $\mu \neq \lambda - \rho$.

Let $m \in M_A(\lambda)$. Define a $\mathbb{C}$-linear map $\beta_m \in \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), A)$ by $\beta_m(u) = l(\check{u}m)$ for $u \in U(\mathfrak{n})$. Here $\widecheck{uw} = \check{w}\check{u}$ for $u, w \in U(\mathfrak{n})$ and $\check{x} = -x$ for $x \in \mathfrak{n}$.

For $x \in \mathfrak{n}$, $u \in U(\mathfrak{n})$ and $m \in M_A(\lambda)$, we have

$$\beta_{xm}(u) = l(\check{u}xm) = -l(\check{u}\check{x}m) = -l(\widecheck{xu}m) = \beta_m(-xu) = x \cdot \beta_m(u)$$

where $x \cdot \beta_m$ denotes the contragredient action of x on β_m. Moreover, let $m \in M_A(\lambda)^{\lambda-\rho-\mu}$ and $u \in U(\mathfrak{n})_{\mu'}$ for $\mu' \neq \mu$. Then $\beta_m(u) = l(\check{u}m) = 0$ since $\check{u}m \in M_A(\lambda)^{\neq\lambda-\rho}$. It follows that $\beta : m \mapsto \beta_m$ is a map of $(\mathfrak{n}, A)$-bimodules with image in $U(\mathfrak{n})^\dagger \otimes A \subset \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), \mathbb{C}) \otimes A \subset \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), A)$ which maps $M_A(\lambda)^{\lambda-\rho-\mu}$ to $(U(\mathfrak{n})_\mu)^* \otimes A$.

By reducing mod $\mathfrak{m}$ (2.19), we obtain a commutative diagram

$$\begin{tikzcd} M_A(\lambda)^{\lambda-\rho-\mu} \arrow[r, "\beta"] \arrow[d, "\mathrm{red}"] & (U(\mathfrak{n})_\mu)^* \otimes A \arrow[d, "\mathrm{red}"] \\ M(\lambda)_{\lambda-\rho-\mu} \arrow[r, "\overline{\beta}"] & (U(\mathfrak{n})_\mu)^* \end{tikzcd}$$

where $\overline{\beta}$ is an isomorphism by [16, Prop. 3.8] using that $M(\lambda)$ is simple. Since A is local and the weight spaces are finitely generated free A-modules, Nakayama's lemma implies that β is an isomorphism. □

We now consider a deformed version of Backelin's functor $\overline{\Gamma}_\eta$, see Section 2.3. For $M \in \mathfrak{h}$-mod-A we define the *completion* $\overline{M}$ by

$$\overline{M} = \prod_{\mu\in\mathfrak{h}^*} M^\mu. \tag{2.31}$$

Let $\eta \in \mathfrak{n}^*$ and

$$U_\eta(\mathfrak{n}) := \ker(\eta : U(\mathfrak{n}) \to \mathbb{C}). \tag{2.32}$$

Moreover, we define

$$\mathrm{Wh}_\eta(M) = \{m \in M \mid (X - \eta(X))m = 0 \text{ for all } X \in \mathfrak{n}\}, \tag{2.33}$$
$$\Gamma_\eta(M) = \{m \in M \mid \exists k \geqslant 0 \text{ with } U_\eta(\mathfrak{n})^k m = 0\}, \text{ and} \tag{2.34}$$
$$\Omega_\eta(M) = U(\mathfrak{g})\,\mathrm{Wh}_\eta(M). \tag{2.35}$$

We have $\Omega_\eta(M) \subset \Gamma_\eta(M)$ by [16, Prop. 4.2.1]. Γ_η admits a filtration by

$$\Gamma_\eta(M)_n = \{m \in M \mid U_\eta(\mathfrak{n})^n m = 0\} = \mathrm{Hom}_{U(\mathfrak{n})}(U(\mathfrak{n})/U_\eta(\mathfrak{n})^n, M). \tag{2.36}$$

Moreover, Ω_η admits a filtration by $\Omega_\eta(M)_n = U(\mathfrak{g})_{\leqslant n}\,\mathrm{Wh}_\eta(M)$ where $U(\mathfrak{g})_{\leqslant n} \subset U(\mathfrak{g})$ is the subspace of all terms of order up to $n \geqslant 0$. We abbreviate $\overline{\mathrm{Wh}}_\eta(M) = \mathrm{Wh}_\eta(\overline{M})$, $\overline{\Gamma}_\eta(M) = \Gamma_\eta(\overline{M})$ and so on.

Theorem 2.6.2 *Let $\lambda \in \mathfrak{h}^*$ and $\eta \in \mathfrak{n}^*$ be non-degenerate. Assume that $M(\lambda)$ is simple.*

(1) There is an isomorphism of $(Z(\mathfrak{g}) \otimes U(\mathfrak{n}), A)$-bimodules

$$A_{\lambda-\rho} \cong \overline{\mathrm{Wh}}_\eta(M_A(\lambda)).$$

Where $Z(\mathfrak{g}) \otimes U(\mathfrak{n})$ acts on the left hand side via

$$z \otimes u \cdot a = (\chi_\lambda + \varphi \circ p)(z)\eta(u)a,$$

where χ_λ and p are as in (2.1) and (2.2). (See also Definition 2.4.6).

(2) Assume that A is a PID. Then there is an isomorphism of $(\mathfrak{g}, A)$-bimodules

$$M_A(\lambda, \eta) = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})\otimes U(\mathfrak{n})} A_{\lambda-\rho} \cong \overline{\Omega}_\eta(M_A(\lambda)) = \overline{\Gamma}_\eta(M_A(\lambda)).$$

Proof (1) By Proposition 2.6.1 there is an isomorphism of $(\mathfrak{n}, A)$-bimodules

$$\overline{M_A(\lambda)} \cong \overline{U(\mathfrak{n})^\dagger \otimes A} = \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), A).$$

Now $U(\mathfrak{n}) = \mathbb{C} \cdot 1 \oplus U_\eta(\mathfrak{n})$ and $U(\mathfrak{n})/U_\eta(\mathfrak{n}) \cong \mathbb{C}$. We obtain

$$\overline{\mathrm{Wh}}_\eta(M_A(\lambda)) = \mathrm{Wh}_\eta(\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), A)) = \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n})/U_\eta(\mathfrak{n}), A) \cong A_\eta$$

as a $(\mathfrak{n}, A)$-bimodule. (Here A_η is the free rank 1 A-module with $\mathfrak{n}$-action given by η.) Now $Z(\mathfrak{g})$ acts on $M_A(\lambda)$ and $\overline{\mathrm{Wh}}_\eta(M_A(\lambda))$ in the same way. The first statement follows.

(2) We first make the following abbreviations

$$\begin{aligned} W &= \overline{\mathrm{Wh}}_\eta(M_A(\lambda)) \cong A_{\lambda-\rho}, \\ M &= U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})\otimes U(\mathfrak{n})} W \cong M_A(\lambda, \eta) \text{ and} \\ N &= \overline{\Gamma}_\eta(M_A(\lambda)). \end{aligned}$$

Then, $\mathrm{Wh}_\eta\, M = 1 \otimes W$ and $M = \Gamma_\eta(M) = \Omega_\eta(M)$. Since $W \subset N$, multiplication yields a map of $(\mathfrak{g}, A)$-bimodules

$$\phi : M \to N, \ X \otimes w \mapsto Xw$$

and $\mathrm{im}(\phi) \subset \Omega_\eta(N)$.

Our goal is to show that the map ϕ is actually an isomorphism. The map ϕ becomes an isomorphism mod $\mathfrak{m}$ by [16, Thm. 3.8, Lem. 3.9, Thm. 4.4]. As an A-module, M is torsion free since $U(\mathfrak{g})$ is free as an $Z(\mathfrak{g}) \otimes U(\mathfrak{n})$-module [9, Lem. 4.1] and N is a torsion free A-module because $M_A(\lambda)$ is torsion free. However, to conclude that ϕ is also an isomorphism, we cannot apply Nakayama's lemma directly, since M and N are not finitely generated as A-modules. We will instead pass to certain finitely generated torsion free submodules of M and N.

We first show that ϕ is injective with image $\Omega_\eta(N)$. For each $n \geqslant 0$, the map ϕ restricts to

$$\phi_n : \Omega_\eta(M)_n \to \Omega_\eta(N)_n$$

and both sides of the equation are finitely generated free A-modules. Here we use that a finitely generated torsion free module over the local PID A is free. Since ϕ is an isomorphism mod $\mathfrak{m}$, so is ϕ_n. Hence, by Nakayama's lemma ϕ_n is an isomorphism.

Since $M = \Omega_\eta(M) = \bigcup_n \Omega_\eta(M)_n$ and each ϕ_n injective, so is ϕ and the image of ϕ is $\Omega_\eta(N) = \bigcup_n \Omega_\eta(N)_n$.

We now show that ϕ is an isomorphism by a similar argument. The isomorphism $\overline{M_A(\lambda)} \cong \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), A)$ shows that

$$\begin{aligned}\Gamma_\eta(N)_n &\cong \mathrm{Hom}_{U(\mathfrak{n})}(U(\mathfrak{n})/U_\eta(\mathfrak{n})^n, \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}), A)) \\ &\cong \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n})/U_\eta(\mathfrak{n})^n, A)\end{aligned}$$

is finitely generated as an A-module. Since ϕ is injective and $\phi(\Gamma_\eta(M)_n) \subset \Gamma_\eta(N)_n$, it follows that $\Gamma_\eta(M)_n$ is also finitely generated. We can hence apply Nakayama's lemma again to see that ϕ induces an isomorphism $\Gamma_\eta(M)_n \cong \Gamma_\eta(N)_n$. Since $M = \Gamma_\eta(M) = \bigcup_n \Gamma_\eta(M)_n$ and $N = \Gamma_\eta(N) = \bigcup_n \Gamma_\eta(N)_n$ it follows that ϕ is an isomorphism. □

Remark 2.6.3 The statements probably hold true without assuming that $M(\lambda)$ is simple or that A is a PID. However, the condition that A is local is essential.

Theorem 2.6.4 *Let $\lambda \in \mathfrak{h}^*$ such that the $\mathfrak{l}_\eta$-Verma module of highest weight $\lambda - \rho_\eta$, $M^{\mathfrak{l}_\eta}(\lambda)$, is simple. Assume that A is a PID. Then there is an isomorphism*

$$M_A(\lambda, \eta) \cong \overline{\Gamma}_\eta(M_A(\lambda)).$$

Proof Our proof uses techniques similar to the proof of [2, Lem. 6.5].

We first fix some notation. For $i = 1, 2$ denote by $\mathfrak{l}_i \subset \mathfrak{p}_i \subset \mathfrak{g}$ the Levi and standard parabolic corresponding to the simple roots α with $\eta(X_\alpha) \neq 0$ and $\eta(X_\alpha) = 0$, respectively. In particular, $\mathfrak{l}_1 = \mathfrak{l}_\eta$ in the notation of (2.5). Let ρ_i be the half-sum of positive roots in the corresponding root systems. Denote the opposite parabolic by $\overline{\mathfrak{p}}_i$. Let $\mathfrak{n}_i = \mathfrak{n} \cap \mathfrak{l}_i$ and $\overline{\mathfrak{n}}_i = \overline{\mathfrak{n}} \cap \mathfrak{l}_i$. Denote by $\overline{\Gamma}^i_\eta$ the functor (2.34) for $\eta|_{\mathfrak{n}_i}$. Denote by $\mathfrak{z}_i = \mathfrak{z}(\mathfrak{l}_i)$ the respective centers.

It follows that $\mathfrak{h} = \mathfrak{z}_1 + \mathfrak{z}_2$. Moreover, the nil radical of $\overline{\mathfrak{p}}_1$ is $\overline{\mathfrak{u}}_1 = \overline{\mathfrak{n}}_2 + [\overline{\mathfrak{n}}_1, \overline{\mathfrak{n}}_2]$ and $\mathfrak{l}_1$ acts on it with the adjoint action. Similarly, the nil radical of $\overline{\mathfrak{p}}_2$ is $\overline{\mathfrak{u}}_2 = \overline{\mathfrak{n}}_1 + [\overline{\mathfrak{n}}_1, \overline{\mathfrak{n}}_2]$ with an adjoint action of $\mathfrak{l}_2$.

Abbreviate $M = M_A(\lambda)$. For $\mu_i \in \mathfrak{z}_i^*$ denote by M^{μ_i} the corresponding deformed weight spaces. Hence, we obtain the deformed $\mathfrak{h}$-weight spaces of M as $M^{\mu_1,\mu_2} = M^{\mu_1} \cap M^{\mu_2}$.

The deformed weight spaces with respect to $\mathfrak{z}_i$ admit the following explicit description. We have

$$\begin{aligned}M &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_{\lambda-\rho} \\ &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_i)} U(\mathfrak{l}_i) \otimes_{U(\mathfrak{l}_i \cap \mathfrak{b})} A_{\lambda-\rho} \\ &= U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_i)} M_A^{\mathfrak{l}_i}(\lambda) \\ &= U(\overline{\mathfrak{u}}_i) \otimes_{\mathbb{C}} M_A^{\mathfrak{l}_i}(\lambda)\end{aligned}$$

where the last isomorphism holds on the level of $(\mathfrak{l}_i, A)$-bimodules and $M_A^{\mathfrak{l}_i}(\lambda)$ refers to the deformed Verma module for $\mathfrak{l}_i$ of highest weight $\lambda - \rho_i$. We hence obtain for

$\mu_i \in \mathfrak{z}_i^*$ the explicit description $M^{\mu_i} = U(\overline{\mathfrak{u}}_i)_{\mu_i-(\lambda-\rho)|_{\mathfrak{z}_i}} \otimes M_A^{\mathfrak{l}_i}(\lambda)$. (This follows from the description of $\mathfrak{z}_1$ in (2.6) and the analogous statement for $\mathfrak{z}_2$.)

In particular, for the $\mathfrak{l}_1$-module M^{μ_1}, we have

$$\begin{aligned}\overline{\Gamma}_\eta^1(M^{\mu_1}) &= \overline{\Gamma}_\eta^1(U(\overline{\mathfrak{u}}_1)_{\mu_1-(\lambda-\rho)|_{\mathfrak{z}_1}} \otimes M_A^{\mathfrak{l}_1}(\lambda)) \\ &= U(\overline{\mathfrak{u}}_1)_{\mu_1-(\lambda-\rho)|_{\mathfrak{z}_1}} \otimes \overline{\Gamma}_\eta^1(M_A^{\mathfrak{l}_1}(\lambda)) \\ &= U(\overline{\mathfrak{u}}_1)_{\mu_1-(\lambda-\rho)|_{\mathfrak{z}_1}} \otimes M_A^{\mathfrak{l}_1}(\lambda,\eta)\end{aligned}$$

where the second equality follows since $U(\overline{\mathfrak{u}}_1)_{\mu_1-(\lambda-\rho)|_{\mathfrak{z}_1}}$ is finite-dimensional and $\overline{\Gamma}_\eta^1$ commutes with tensor products with finite-dimensional modules by [1, Prop. 3] (there the statement is proven in the undeformed case, but translates unchanged), and the third equality follows from Theorem 2.6.2, using that η is non-degenerate for $\mathfrak{l}_1$ and $M^{\mathfrak{l}_1}(\lambda)$ is assumed to be simple.

Moreover, for a $\mathfrak{z}_2$-weight $\mu_2 \in \mathfrak{z}_2^*$, the deformed weight space M^{μ_2} is $\mathfrak{l}_2$-stable, and we have

$$\begin{aligned}\overline{\Gamma}_\eta^2(M^{\mu_2}) &= \overline{\Gamma}_\eta^2(U(\overline{\mathfrak{u}}_2)_{\mu_2-(\lambda-\rho)|_{\mathfrak{z}_2}} \otimes \overline{M}_A^{\mathfrak{l}_2}(\lambda)) \\ &= U(\overline{\mathfrak{u}}_2)_{\mu_2-(\lambda-\rho)|_{\mathfrak{z}_2}} \otimes \overline{\Gamma}_\eta^2(M_A^{\mathfrak{l}_2}(\lambda)) \\ &= U(\overline{\mathfrak{u}}_2)_{\mu_2-(\lambda-\rho)|_{\mathfrak{z}_2}} \otimes \overline{\Gamma}_0^2(M_A^{\mathfrak{l}_2}(\lambda)) \\ &= U(\overline{\mathfrak{u}}_2)_{\mu_2-(\lambda-\rho)|_{\mathfrak{z}_2}} \otimes M_A^{\mathfrak{l}_2}(\lambda) \\ &= M^{\mu_2}.\end{aligned}$$

Here, the second equality again holds by [1, Prop. 3]. The third equality follows from the fact that η vanishes on $\mathfrak{n}_2$. The fourth equality holds because there can only be finitely many deformed weight spaces containing highest weight vectors, so $\mathfrak{n}_2$-finite vectors can only be supported in finitely many weight spaces, see [2, Rmk. 3.3].

Hence, in total we get an inclusion

$$\begin{aligned}\overline{\Gamma}_\eta(M) = \Gamma_\eta(\overline{M}) &= \Gamma_\eta^1\Gamma_0^2\left(\prod_{\mu_2}\prod_{\mu_1} M^{\mu_1,\mu_2}\right) \\ &\subseteq \Gamma_\eta^1\left(\prod_{\mu_2}\overline{\Gamma}_0^2(M^{\mu_2})\right) = \Gamma_\eta^1\left(\prod_{\mu_2} M^{\mu_2}\right) \\ &= \Gamma_\eta^1\left(\prod_{\mu_2}\bigoplus_{\mu_1} M^{\mu_1,\mu_2}\right) = \bigoplus_{\mu_1}\overline{\Gamma}_\eta^1\left(M^{\mu_1}\right) \\ &= \bigoplus_{\mu_1} U(\overline{\mathfrak{u}}_1)_{\mu_1-(\lambda-\rho)|_{\mathfrak{z}_1}} \otimes_{\mathbb{C}} M_A^{\mathfrak{l}_1}(\lambda,\eta).\end{aligned}$$

In particular, by considering the summand $U(\overline{\mathfrak{u}}_1)_0 \otimes_{\mathbb{C}} M_A^{\mathfrak{l}_1}(\lambda, \eta) = M_A^{\mathfrak{l}_1}(\lambda, \eta)$, we obtain a natural multiplication map

$$\phi : M_A(\lambda, \eta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M_A^{\mathfrak{l}_1}(\lambda, \eta) \to \overline{\Gamma}_\eta(M)$$

By writing $M_A(\lambda, \eta) = U(\overline{\mathfrak{u}}_1) \otimes M_A^{\mathfrak{l}_1}(\lambda, \eta)$ we see that $M_A(\lambda, \eta)$ has the same decomposition as the space containing $\overline{\Gamma}_\eta(M)$ considered above. It follows that ϕ is an isomorphism. □

Remark 2.6.5 Actually, the same proof shows that

$$\overline{\Gamma}_\eta(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} M) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} \overline{\Gamma}^1_\eta(M)$$

for every module M in deformed category $\mathcal{O}_A$ of $\mathfrak{l}_1$. For this one uses that for each module N in category $\mathcal{O}_A(\mathfrak{g})$ we have N^{μ_2} is in category $\mathcal{O}_A(\mathfrak{l}_2)$ for each $\mu_2 \in \mathfrak{z}_2^*$, see [13, Thm. 2.3.2].

2.7 The Jantzen conjectures via Category $\mathcal{O}$

Using the results of Sect. 2.6, we can prove the Jantzen conjectures for Whittaker modules (Theorem 2.5.8) using the Jantzen conjectures for Verma modules (Theorem 2.5.3). We dedicate this section to doing so.

We have established that for a local PID A with distinguished homomorphism $\varphi : U(\mathfrak{h}) \to A$ sending $\mathfrak{h}$ into the maximal ideal $\mathfrak{m}$, the functor

$$\overline{\Gamma}_\eta : \mathcal{O}_A \to \mathfrak{g}\text{-mod-}A$$

of (2.34) sends $M_A(\lambda)$ to $M_A(\lambda, \eta)$, where $\lambda \in \mathfrak{h}^*$ is chosen such that the $\mathfrak{l}_\eta$-Verma module $M^{\mathfrak{l}_\eta}(\lambda)$ is simple (Theorem 2.6.4). For every $M_A(\lambda, \eta)$, such a λ exists by (2.16).

Moreover, $\overline{\Gamma}_\eta$ also sends $M_A^\vee(\lambda)$ to $M_A^\vee(\lambda, \eta)$. This follows from the fact that the completion (2.31) of a deformed dual Verma module is canonically isomorphic to the A-linear dual of a deformed Verma module; i.e.

$$\overline{M_A^\vee(\lambda)} = \mathrm{Hom}_A(M_A(\lambda), A).$$

Hence the functor $\overline{\Gamma}_\eta$ sends the canonical $(\mathfrak{g}, A)$-module homomorphism

$$\psi : M_A(\lambda) \to M_A^\vee(\lambda)$$

of (2.18) to the canonical $(\mathfrak{g}, A)$-module homomorphism

$$\psi_\eta : M_A(\lambda, \eta) \to M_A^\vee(\lambda, \eta)$$

of (2.30).

By construction, the functor $\overline{\Gamma}_\eta$ has the property that for any $a \in A$ and $(\mathfrak{g}, A)$-bimodule M,

$$\overline{\Gamma}_\eta(Ma) = \overline{\Gamma}_\eta(M)a,$$

so $\overline{\Gamma}_\eta$ maps the Jantzen filtration of $M_A(\lambda)$ (Definition 2.5.1) to the Jantzen filtration of $M_A(\lambda, \eta)$ (Definition 2.5.7).

Since $\overline{\Gamma}_\eta$ is a quotient functor onto its essential image, see Proposition 2.3.1, all morphisms in $\mathcal{N}$ between standard Whittaker modules are obtained from morphisms between Verma modules. Moreover, because $\overline{\Gamma}_\eta$ is exact, it sends injective morphisms to injective morphisms. Hence all embeddings of standard Whittaker modules are of the form

$$M(\mu, \eta) \xhookrightarrow{\overline{\Gamma}_\eta(\iota)} M(\lambda, \eta),$$

where $\iota : M(\mu) \hookrightarrow M(\lambda)$ is an embedding in category $\mathcal{O}$. Hence Theorem 2.5.8(i) follows from Theorem 2.5.3(i). Moreover, as a quotient functor $\overline{\Gamma}_\eta$ preserves semisimple objects, see Proposition 2.3.1. Hence Theorem 2.5.8(ii) follows from Theorem 2.5.3(ii).

3 $\mathcal{D}$-Modules

3.1 Preliminaries

We start by listing our geometric conventions and recording the necessary background results.

3.1.1 $\mathcal{D}$-module notation and conventions

We adopt the following notational conventions for $\mathcal{D}$-modules. Our conventions mostly[6] follow [20, 21].

- If Y is a smooth algebraic variety, $\mathcal{O}_Y$ is the sheaf of regular functions on Y and $\mathcal{D}_Y$ is the sheaf of differential operators on Y.
- We denote by $\mathcal{M}_{qc}(\mathcal{D}_Y)$, $\mathcal{M}_{coh}(\mathcal{D}_Y)$, $\mathcal{M}_{hol}(\mathcal{D}_Y)$ the categories of quasi-coherent, coherent, and holonomic $\mathcal{D}_Y$-modules on Y, respectively. We denote by $D^b_{qc}(\mathcal{D}_Y)$, $D^b_{coh}(\mathcal{D}_Y)$, $D^b_{hol}(\mathcal{D}_Y)$ the corresponding bounded derived categories.
- For a morphism $f : Y \to Z$ of smooth algebraic varieties, we have functors

$$f^+ : \mathcal{M}_{qc}(\mathcal{D}_Z) \to \mathcal{M}_{qc}(\mathcal{D}_Y)$$
$$f^! : D^b_{qc}(\mathcal{D}_Z) \to D^b_{qc}(\mathcal{D}_Y)$$
$$f_+ : D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_Z)$$
$$f_! : D^b_{coh}(\mathcal{D}_Y) \to D^b_{coh}(\mathcal{D}_Z)$$

[6] In places where they differ, we explicitly state definitions. If definitions are not explicitly stated, they are the same as the sources above.

The functors f^+, $f^!$, f_+ are defined in the standard way (see [20]), and the !-pushforward is the twist of the +-pushforward by duality:

$$f_! := \mathbb{D}_Z \circ f_+ \circ \mathbb{D}_Y,$$

for the duality functor $\mathbb{D}_\square : D^b_{coh}(\mathcal{D}_\square) \to D^b_{coh}(\mathcal{D}_\square)$, $\square = Y, Z$, given by

$$\mathbb{D}_\square(\mathcal{V}^\bullet) := RHom_{\mathcal{D}_\square}(\mathcal{V}^\bullet, \mathcal{D}_\square)[\dim \square].$$

- If $i : Y \to Z$ is an immersion, then i_+ is the right derived functor of the left exact functor
$$H^0 \circ i_+ \circ [0] : \mathcal{M}_{qc}(\mathcal{D}_Y) \to \mathcal{M}_{qc}(\mathcal{D}_Z),$$
where $[0] : \mathcal{M}_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_Y)$ sends a $\mathcal{D}_Y$-module $\mathcal{V}$ to the complex $0 \to \cdots \to 0 \to \mathcal{V} \to 0 \to \cdots 0$ concentrated in degree 0. In this setting, we refer to $H^0 \circ i_+ \circ [0]$ as i_+.
- If $\mathcal{V} \in \mathcal{M}_{hol}(\mathcal{D}_Y)$, then $\mathbb{D}_Y(\mathcal{V}[0]) \in D^b_{hol}(\mathcal{D}_Y)$ has holonomic cohomology, and $H^p(\mathbb{D}_Y(\mathcal{V}[0])) = 0$ for all $p \neq 0$. In this case, $\mathbb{D}_Y$ descends to a functor on holonomic $\mathcal{D}_Y$-modules, which we refer to by the same symbol:
$$\begin{aligned}\mathbb{D}_Y : \mathcal{M}_{hol}(\mathcal{D}_Y) &\to \mathcal{M}_{hol}(\mathcal{D}_Y)\\ \mathcal{V} &\mapsto H^0(RHom_{\mathcal{D}_Y}(\mathcal{V}[0], \mathcal{D}_Y)[\dim Y])\end{aligned}$$
- Hence if $i : Y \to Z$ is an immersion, we can consider $i_!$ as a functor on holonomic modules:
$$i_! : \mathcal{M}_{hol}(\mathcal{D}_Y) \to \mathcal{M}_{hol}(\mathcal{D}_Z).$$
- If $f : Y \to Z$ is proper, then f_+ preserves the derived category of coherent $\mathcal{D}$-modules, and on this category $f_! = f_+$. In particular, if $i : Y \to Z$ is a closed immersion, then the functors $i_!$ and i_+ agree on holonomic $\mathcal{D}_Y$-modules:
$$i_+ = i_! : \mathcal{M}_{hol}(\mathcal{D}_Y) \to \mathcal{M}_{hol}(\mathcal{D}_Z).$$
- If Y admits an action by an algebraic group K with finitely many orbits, then we denote by $\mathcal{M}_{qc}(\mathcal{D}_Y, K)$ the category of strongly K-equivariant quasicoherent $\mathcal{D}_Y$-modules, and by $\mathcal{M}_{qc}(\mathcal{D}_Y, K)_{\text{weak}}$ the category of weakly K-equivariant quasicoherent $\mathcal{D}_Y$-modules. (See [24] for a reference on strong and weak equivariance.)

3.1.2 The flag variety and base affine space

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and denote by G the simply connected semisimple Lie group corresponding to $\mathfrak{g}$. Fix a Borel subgroup $B \subset G$, and denote by $N \subset B$ the unipotent radical of B. The abstract torus

$$H := B/N \tag{3.1}$$

does not depend on choice of B [11, Lem. 6.1.1]. Set

$$X := G/B \text{ and } \widetilde{X} := G/N. \tag{3.2}$$

The variety X is the flag variety of $\mathfrak{g}$, and we refer to the variety $\widetilde{X}$ as *base affine space*. Both X and $\widetilde{X}$ have left actions of G coming from left multiplication. There is also a natural right action of H on $\widetilde{X}$ by right multiplication, and with respect to this action, the quotient map

$$\pi : \widetilde{X} \to X \tag{3.3}$$

is a principal G-equivariant H-bundle.

3.1.3 Differential operators on base affine space and the universal enveloping algebra

Global differential operators on $\widetilde{X}$ are related to the extended universal enveloping algebra. More specifically, by differentiating the left G-action and right H-action described above (which clearly commute), one obtains an algebra homomorphism

$$U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{h}) \to \Gamma(\widetilde{X}, \mathcal{D}_{\widetilde{X}}).$$

This homomorphism factors through the quotient [4, §2]

$$\widetilde{U} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h}). \tag{3.4}$$

Here $Z(\mathfrak{g})$ acts on $U(\mathfrak{h})$ via the composition of the Harish-Chandra homomorphism (2.2) with the ρ-shift endomorphism

$$\tau : U(\mathfrak{h}) \to U(\mathfrak{h}) \tag{3.5}$$

which sends $h \in \mathfrak{h}$ to $\tau(h) = h - \rho(h)$. We refer to the algebra $\widetilde{U}$ as the *extended universal enveloping algebra*. By construction, $\widetilde{U}$ contains both $U(\mathfrak{g})$ and $U(\mathfrak{h})$ as subalgebras, and the subalgebra $U(\mathfrak{h}) \subset \widetilde{U}$ is the center. Moreover, there is a natural action of W on $\widetilde{U}$ coming from the W-action[7] on $U(\mathfrak{h})$, and the Harish-Chandra isomorphism[8] $\tau \circ p : Z(\mathfrak{g}) \xrightarrow{\simeq} U(\mathfrak{h})^W$ guarantees that $\widetilde{U}^W \simeq U(\mathfrak{g})$.

Global sections of $\mathcal{D}_{\widetilde{X}}$-modules have the structure of $\widetilde{U}$-modules via the map

$$\alpha : \widetilde{U} \to \Gamma(\widetilde{X}, \mathcal{D}_{\widetilde{X}}). \tag{3.6}$$

3.1.4 Monodromic $\mathcal{D}$-Modules

The $\mathcal{D}$-modules which arise in the geometric construction of the Jantzen filtration have an additional structure—they are H-monodromic. Using H-monodromic $\mathcal{D}$-modules on base affine space, one can study $\mathfrak{g}$-modules of generalized infinitesimal

[7] Recall that it is our convention to work with the usual action of W on $\mathfrak{h}^*$, not the dot action.

[8] Here p is the projection in (2.2).

character. (In contrast, using the more classical approach of modules over sheaves of twisted differential operators on the flag variety, one can only study $\mathfrak{g}$-modules of strict infinitesimal character.)

Recall the principal H-bundle $\pi : \widetilde{X} \to X$ from (3.3). An *H-monodromic $\mathcal{D}$-module on X* is a weakly H-equivariant $\mathcal{D}_{\widetilde{X}}$-module. Such $\mathcal{D}$-modules have an alternate characterization in terms of H-invariant sections of $\pi_\bullet \mathcal{D}_{\widetilde{X}}$. Specifically, the right H-action on $\widetilde{X}$ induces an H-action on $\pi_\bullet \mathcal{D}_{\widetilde{X}}$ by algebra automorphisms, where $\pi_\bullet$ denotes the sheaf-theoretic direct image functor. We define

$$\widetilde{\mathcal{D}} := [\pi_\bullet \mathcal{D}_{\widetilde{X}}]^H \subset \pi_\bullet \mathcal{D}_{\widetilde{X}} \tag{3.7}$$

to be the sheaf of H-invariant sections of $\pi_\bullet \mathcal{D}_{\widetilde{X}}$. This is a sheaf of algebras. There is an equivalence of categories[9] [4, §1.8.9, 2.5]

$$\mathcal{M}_{qc}(\mathcal{D}_{\widetilde{X}}, H)_{\text{weak}} \simeq \mathcal{M}_{qc}(\widetilde{\mathcal{D}}). \tag{3.8}$$

Moreover, because the left G-action and right H-action commute, the image of the homomorphism α in (3.6) is contained in $\Gamma(\widetilde{X}, \mathcal{D}_{\widetilde{X}})^H = \Gamma(X, \widetilde{\mathcal{D}})$. The resulting map is an algebra isomorphism [8]

$$\alpha : \widetilde{U} \xrightarrow{\simeq} \Gamma(X, \widetilde{\mathcal{D}}). \tag{3.9}$$

Hence the global sections of H-monodromic $\mathcal{D}$-modules on X have the structure of $\widetilde{U}$-modules.

Remark 3.1.1 (Relationship to twisted differential operators) There is a natural embedding $S(\mathfrak{h}) \hookrightarrow \widetilde{\mathcal{D}}$ coming from the action of the Lie algebra of H, whose image lies in (and, in fact, is equal to) the center of $\widetilde{\mathcal{D}}$ [4, §2.5]. For $\lambda \in \mathfrak{h}^*$, denote by $\mathfrak{m}_\lambda \subset S(\mathfrak{h})$ the maximal ideal corresponding to $\lambda + \rho$[10]. The sheaf $\mathcal{D}_\lambda := \widetilde{D}/\mathfrak{m}_\lambda \widetilde{\mathcal{D}}$ is a sheaf of twisted differential operators (TDOs) on the flag varity X. Therefore, $\widetilde{\mathcal{D}}$-modules on which $\mathfrak{m}_\lambda$ acts trivially can be identified with $\mathcal{D}_\lambda$-modules.

The relationships described above can be summarized in the following commutative diagram.

$$\begin{tikzcd}
\mathcal{M}_{qc}(\mathcal{D}_{\widetilde{X}}, H)_{\text{weak}} \arrow[r, "\text{forget equiv.}"] \arrow[d, "\Gamma"'] & \mathcal{M}_{qc}(\mathcal{D}_{\widetilde{X}}) \arrow[r, "\pi_\bullet"] & \mathcal{M}_{qc}(\pi_\bullet \mathcal{D}_{\widetilde{X}}) \arrow[d, "\text{restrict}"] \\
\mathcal{M}(\widetilde{U}) & & \mathcal{M}_{qc}(\widetilde{\mathcal{D}}) \arrow[ll, "\Gamma"']
\end{tikzcd} \tag{3.10}$$

The composition of the top two horizontal arrows and the right-most vertical arrow is the equivalence in (3.8).

[9] Here, similar to the definition for $\mathcal{D}$-modules in Section 3.1.1, $\mathcal{M}_{qc}(\widetilde{\mathcal{D}})$ denotes the category of $\widetilde{\mathcal{D}}$-modules which are quasicoherent as $\mathcal{O}_X$-modules.

[10] Our ρ-shift conventions are chosen so that the definition of $\mathcal{D}_\lambda$ aligns with that in [7, 21, 26, 27]. These conventions differ from those in [4] by a negative ρ-shift.

3.2 Whittaker $\mathcal{D}$-modules

In this section, we describe the $\mathcal{D}$-modules which can be used to study the standard and costandard Whittaker modules of Z Sect. 2.2.

Let $\mathfrak{b} = \operatorname{Lie} B$ be the Lie algebra of B and $\mathfrak{n} = \operatorname{Lie} N$ the Lie algebra of N. Choose a character $\eta \in \operatorname{ch} \mathfrak{n}$. In [26, 27], the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ of *η-twisted Harish-Chandra sheaves* on X is studied. An object in this category is a coherent $\mathcal{D}_\lambda$-module[11] $\mathcal{V}$ which is N-equivariant as an $\mathcal{O}_X$-module and satisfies the additional properties that the morphism $\mathcal{D}_\lambda \otimes \mathcal{V} \to \mathcal{V}$ is N-equivariant, and if $\mu : \mathfrak{n} \to \operatorname{End} \mathcal{V}$ denotes the differential of the N-action and $\pi : \mathcal{D}_\lambda \to \operatorname{End} \mathcal{V}$ the $\mathcal{D}_\lambda$-action, then for any $\xi \in \mathfrak{n}$,

$$\pi(\xi) = \mu(\xi) + \eta(\xi).$$

Twisted Harish-Chandra sheaves are holonomic $\mathcal{D}_\lambda$-modules, and hence have finite length [26, Lem. 1.1].

In [26] simple objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are classified. They are parametrized by pairs (Q, τ), where Q is an N-orbit in X and τ is an η-twisted connection on Q (i.e., an irreducible $(\mathcal{D}_Q, N, \eta)$-module). In particular, an N-orbit (Bruhat cell) $C(w)$ admits an η-twisted connection exactly when $w = w^C$ is the unique longest coset representative in a coset $C \in W_\eta \backslash W$, where $W_\eta \subset W$ is the subgroup generated by the simple roots with the property that η does not vanish on the corresponding root subspace, as in Sect. 2.2 [26, Lem. 4.1, Thm. 4.2]. On such a Bruhat cell $C(w^C)$, the unique η-twisted connection is the exponential $\mathcal{D}_{C(w^C)}$-module $\mathcal{O}_{C(w^C),\eta}$. As an N-equivariant $\mathcal{O}_{C(w^C)}$-module, $\mathcal{O}_{C(w^C),\eta}$ is isomorphic to $\mathcal{O}_{C(w^C)}$, but the $\mathcal{D}_\lambda$-module structure is twisted by η.

Remark 3.2.1 (The exponential $\mathcal{D}$-module) The η-twisted connection $\mathcal{O}_{C(w^C),\eta}$ is "exponential" in the following sense. The exponential function $f(x) = e^x$ on $\mathbb{A}^1$ is the solution of the ordinary differential equation $\partial f = f$, or $(\partial - 1)f = 0$. The corresponding $D_{\mathbb{A}^1}$-module is the exponential $\mathcal{D}_{\mathbb{A}^1}$-module

$$\exp := D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\partial - 1).$$

Here $D_{\mathbb{A}^1}$ denotes the global differential operators on $\mathbb{A}^1$; i.e., the Weyl algebra. Corresponding to the Lie algebra character $\eta \in \operatorname{ch} \mathfrak{n}$ is an additive character $\eta : N \to \mathbb{A}^1$, which we call by the same name. We can use the character η to construct exponential $\mathcal{D}$-modules on certain Bruhat cells. In particular, if for all $x \in C(w)$, $\eta|_{\operatorname{stab}_N x} = 1$, then η factors through $N/\operatorname{stab}_N x \simeq C(w)$:

$$\begin{tikzcd} N \arrow[r] \arrow[rr, bend left, "\eta"] & N/\operatorname{stab}_N x \arrow[r, "\overline{\eta}"] & \mathbb{A}^1. \end{tikzcd}$$

For such a Bruhat cell $C(w)$, we can pull back the exponential $D_{\mathbb{A}^1}$-module exp to an exponential $\mathcal{D}$-module $\overline{\eta}^+\exp$ on $C(w)$. Lemma 4.1 in [26] establishes that

[11] Here $\lambda \in \mathfrak{h}^*$ is fixed and $\mathcal{D}_\lambda$ is the TDO in Remark 3.1.1.

$\eta|_{\mathrm{stab}_N x} = 1$ precisely when $w = w^C$ is a longest coset representative for some $C \in W_\eta \backslash W$. The $\mathcal{D}$-modules constructed in this way are the η-twisted connections $\mathcal{O}_{C(w^C),\eta}$ described above.

The category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ is a highest weight category [9, Thm. 7.2]. By the discussion above, it has finitely many simple objects, parameterized by $W_\eta \backslash W$. Standard and costandard objects are given by !- and +-pushforwards (respectively) of the exponential $\mathcal{D}_\lambda$-modules $\mathcal{O}_{C(w^C),\eta}$ for $C \in W_\eta \backslash W$. When $\lambda \in \mathfrak{h}^*$ is regular and antidominant[12], the global sections of standard and costandard $(\mathcal{D}_\lambda, N, \eta)$-modules are the standard and costandard Whittaker modules defined in Sect. 2.2. Specifically, we have the following relation.

Theorem 3.2.2 [27, Thm. 9], [9, Lem. 7.3] *Let $\lambda \in \mathfrak{h}^*$ be antidominant, $\eta \in \mathrm{ch}\,\mathfrak{n}$, and w^C the longest coset representative in $C \in W_\eta \backslash W$. The standard and costandard $(\mathcal{D}_\lambda, N, \eta)$-modules $i_{C(w^C)!}\mathcal{O}_{C(w^C),\eta}$ and $i_{C(w^C)+}\mathcal{O}_{C(w^C),\eta}$ have global sections*

$$\Gamma(X, i_{C(w^C)!}\mathcal{O}_{C(w^C),\eta}) = M(w^C\lambda, \eta) \tag{3.11}$$

$$\Gamma(X, i_{C(w^C)+}\mathcal{O}_{C(w^C),\eta}) = M^\vee(w^C\lambda, \eta). \tag{3.12}$$

Using the set-up in Sect. 3.1, we can also find standard and costandard Whittaker modules within the global sections of $\mathcal{D}_{\widetilde{X}}$-modules. This perspective will be necessary in the construction of the Jantzen filtration. Let $Q = C(w^C)$ be a Bruhat cell corresponding to a longest coset representative $w^C \in C \in W_\eta \backslash W$, and denote by $\widetilde{Q} = \pi^{-1}(Q)$ its preimage in $\widetilde{X}$. Because the map $\pi : \widetilde{X} \to X$ is a G-equivariant principal H-bundle, it preserves categories of η-twisted Harish-Chandra sheaves under $\mathcal{D}$-module functors. Hence the $\mathcal{D}_{\widetilde{Q}}$-module $\pi^+\mathcal{O}_{Q,\eta} = \mathcal{O}_{\widetilde{Q},\eta}$ obtained by pulling back the exponential $\mathcal{D}_Q$-module $\mathcal{O}_{Q,\eta}$ to $\widetilde{Q}$ is a $(\mathcal{D}_{\widetilde{Q}}, N, \eta)$-module.

Let $i_{\widetilde{Q}} : \widetilde{Q} \hookrightarrow \widetilde{X}$ be inclusion, and consider the $\mathcal{D}_{\widetilde{X}}$-modules

$$i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta} \text{ and } i_{\widetilde{Q}+}\mathcal{O}_{\widetilde{Q},\eta}. \tag{3.13}$$

These are H-monodromic $\mathcal{D}$-modules on X, so we can consider the corresponding $\widetilde{\mathcal{D}}$-modules under the correspondence (3.8). Specifically, these are the $\pi_\bullet\mathcal{D}_{\widetilde{X}}$-modules $\pi_\bullet i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta}$ and $\pi_\bullet i_{\widetilde{Q}+}\mathcal{O}_{\widetilde{Q},\eta}$, with the restricted action of $\widetilde{\mathcal{D}} \subset \pi_\bullet\mathcal{D}_{\widetilde{X}}$. For $\lambda \in \mathfrak{h}^*$, consider the subsheaves

$$\left(\pi_\bullet i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta}\right)_\lambda \text{ and } \left(\pi_\bullet i_{\widetilde{Q}+}\mathcal{O}_{\widetilde{Q},\eta}\right)_\lambda \tag{3.14}$$

on which $\mathfrak{m}_\lambda$ acts trivially. By Remark 3.1.1, these can be considered as $\mathcal{D}_\lambda$-modules on X. For integral λ, (3.14) are isomorphic to the standard and costandard (respectively) η-twisted Harish-Chandra sheaves $i_{Q!}\mathcal{O}_{Q,\eta}$ and $i_{Q+}\mathcal{O}_{Q,\eta}$ [4, Lem. 2.5.4].

This allows us to study standard and costandard η-twisted Harish-Chandra sheaves for all integral λ simultaneously by working with the $\mathcal{D}_{\widetilde{X}}$-modules (3.13). By taking

[12] See Sect. 2.1 for our conventions with these definitions.

global sections on which the ideal $\widetilde{U}\mathfrak{m}_\lambda \subset \widetilde{U}$ acts trivially, we recover the standard and costandard Whittaker modules of Sect. 2.2. In particular, if for a $\widetilde{U}$-module M, we denote by $(M)_\lambda$ the sub-$U(\mathfrak{g})$-module on which $\widetilde{U}\mathfrak{m}_\lambda$ acts trivially, by diagram (3.10) and Theorem 3.2.2, we have the following relationship.

Lemma 3.2.3 *Let $\lambda \in \mathfrak{h}^*$ be antidominant, $\eta \in \operatorname{ch}\mathfrak{n}$, and w^C the longest coset representative in $C \in W_\eta \backslash W$. Denote by $\widetilde{C}(w^C) = \pi^{-1}(C(w^C))$ the preimage of the Bruhat cell $C(w^C)$ under $\pi : \widetilde{X} \to X$. We have the following isomorphism of $U(\mathfrak{g})$-modules.*

$$\left(\Gamma(\widetilde{X}, i_{\widetilde{C}(w^C)!}\mathcal{O}_{\widetilde{C}(w^C),\eta})\right)_\lambda = M(w^C\lambda, \eta)$$

3.3 A geometric Jantzen filtration via monodromy

In [4], Beilinson–Bernstein construct "geometric Jantzen filtrations" of certain $\mathcal{D}$-modules using monodromy filtrations. In this section, we describe their construction in detail, and explain how the Whittaker $\mathcal{D}$-modules introduced in Sect. 3.2 can be made to fit into this framework.

3.3.1 Monodromy filtrations in abelian categories

Let $\mathcal{A}$ be an abelian category, A an object in $\mathcal{A}$, and $s \in \operatorname{End}_{\mathcal{A}}(A)$ a nilpotent endomorphism. The Jacobson–Morozov theorem [12, Prop. 1.6.1] guarantees the existence of a unique increasing exhaustive filtration $\mu_\bullet$ of A satisfying two conditions:

(i) $s\mu_i \subset \mu_{i-2}$ for all i, and
(ii) the endomorphism s^k induces an isomorphism $\operatorname{gr}_k^\mu A \simeq \operatorname{gr}_{-k}^\mu A$ for all k.

This is the *monodromy filtration* of A. The monodromy filtration naturally induces a filtration $J_{!,\bullet}$ on the sub-object $\ker s$ and a filtration $J_{+,\bullet}$ on the quotient $\operatorname{coker} s$. The filtrations $J_{!,\bullet}$ and $J_{+,\bullet}$ admit a description in terms of powers of s. Explicitly, if we denote by

$$\mathcal{K}^p := \begin{cases} \ker s^{p+1} & \text{for } p \geqslant 0; \\ 0 & \text{for } p < 0, \end{cases} \text{ and } \mathcal{I}^p := \begin{cases} \operatorname{im} s^p & \text{for } p > 0; \\ A & \text{for } p \leqslant 0 \end{cases} \tag{3.15}$$

the kernel and image filtrations of A, then

$$J_{!,i} = \ker s \cap \mathcal{I}^{-i} \text{ and } J_{+,i} = (\mathcal{K}^i + \operatorname{im} s)/\operatorname{im} s. \tag{3.16}$$

3.3.2 The maximal extension functor

To utilize the monodromy filtration in the setting of holonomic $\mathcal{D}$-modules, we need a procedure for producing $\mathcal{D}$-modules with nilpotent endomorphisms. The maximal extension functor accomplishes this. In this section, we recall the maximal extension for $\mathcal{D}$-modules, which is a special case of the construction in [5].

Let $f : Y \to \mathbb{A}^1$ be a regular function on a smooth algebraic variety Y. This gives rise to an open-closed decomposition of Y:

$$U := f^{-1}(\mathbb{A}^1 - \{0\}) \overset{j}{\hookrightarrow} Y \overset{i}{\hookleftarrow} f^{-1}(0). \tag{3.17}$$

For $n \in \mathbb{N}$, denote by

$$I^{(n)} := (\mathcal{O}_{\mathbb{A}^1-\{0\}} \otimes \mathbb{C}[s]/s^n)t^s$$

the free rank one $\mathcal{O}_{\mathbb{A}^1-\{0\}} \otimes \mathbb{C}[s]/s^n$-module generated by the symbol t^s. The action

$$\partial_t \cdot t^s = st^{-1}t^s \tag{3.18}$$

gives $I^{(n)}$ the structure of a $\mathcal{D}_{\mathbb{A}^1-\{0\}}$-module. Any $\mathcal{D}_U$-module $\mathcal{M}_U$ (where $U = f^{-1}(\mathbb{A}^1 - \{0\})$ as in (3.17)) can be deformed using $I^{(n)}$ in the following way. For $n \in \mathbb{N}$, set

$$f^s\mathcal{M}_U^{(n)} := f^+I^{(n)} \otimes_{\mathcal{O}_U} \mathcal{M}_U. \tag{3.19}$$

We consider this a "deformation" because $f^s\mathcal{M}_U^{(1)} = \mathcal{M}_U$. Note that multiplication by $s \in \mathbb{C}[s]/s^n$ acts nilpotently on $f^s\mathcal{M}_U^{(n)}$ by construction.

Assume that $\mathcal{M}_U$ is holonomic. There is a canonical map between the ! and + pushforward of $f^s\mathcal{M}_U^{(n)}$ to Y:

$$\text{can} : j_! f^s\mathcal{M}_U^{(n)} \to j_+ f^s\mathcal{M}_U^{(n)}. \tag{3.20}$$

We denote the composition of the canonical map with multiplication by s by

$$s^1(n) := s \circ \text{can} : j_! f^s\mathcal{M}_U^{(n)} \to j_+ f^s\mathcal{M}_U^{(n)}. \tag{3.21}$$

For large enough n, the cokernel of $s^1(n)$ stabilizes [5, Lem. 2.1]. Define the **maximal extension of $\boldsymbol{\mathcal{M}}_U$** to be the $\mathcal{D}_Y$-module

$$\Xi_f\mathcal{M}_U := \operatorname{coker} s^1(n) \tag{3.22}$$

for n large enough that $\operatorname{coker} s^1(n)$ has stabilized. By construction, $\Xi_f\mathcal{M}_U$ comes equipped with a nilpotent endomorphism, and $(\Xi_f\mathcal{M}_U)|_U = \mathcal{M}_U$.

This construction defines an exact functor [4, Lem. 4.2.1(i)]

$$\Xi_f : \mathcal{M}_{hol}(\mathcal{D}_U) \to \mathcal{M}_{hol}(\mathcal{D}_Y). \tag{3.23}$$

Moreover, there are canonical short exact sequences [4, Lem. 4.2.1(ii)']

$$0 \to j_!\mathcal{M}_U \to \Xi_f\mathcal{M}_U \to \text{coker}(\text{can}) \to 0 \tag{3.24}$$
$$0 \to \text{coker}(\text{can}) \to \Xi_f\mathcal{M}_U \to j_+\mathcal{M}_U \to 0 \tag{3.25}$$

with $j_! = \ker(s : \Xi_f \to \Xi_f)$ and $j_+ = \operatorname{coker}(s : \Xi_f \to \Xi_f)$.

3.3.3 Geometric Jantzen filtrations

We continue working in the setting of Sect. 3.3.2. For any regular function $f : Y \to \mathbb{A}^1$ and holonomic $\mathcal{D}_U$-module $\mathcal{M}_U$, the $\mathcal{D}_Y$-module $\Xi_f \mathcal{M}_U$ has a monodromy filtration given by the nilpotent endomorphism s. Moreover, the sub- and quotient $\mathcal{D}_Y$-modules

$$j_! \mathcal{M}_U = \ker(s : \Xi_f \mathcal{M}_U \to \Xi_f \mathcal{M}_U) \text{ and } j_+ \mathcal{M}_U = \operatorname{coker}(s : \Xi_f \mathcal{M}_U \to \Xi_f \mathcal{M}_U) \tag{3.26}$$

obtain the filtrations $J_{!,\bullet}$ and $J_{+,\bullet}$ of (3.16). We call the filtrations $J_{!,\bullet}$ and $J_{+,\bullet}$ the **geometric Jantzen filtrations** of $j_! \mathcal{M}_U$ and $j_+ \mathcal{M}_U$, respectively.

3.3.4 Geometric Jantzen filtrations on Whittaker $\mathcal{D}$-modules

Now we specialize to the setting of base affine space $\widetilde{X}$. Let $Q \subset X$ be an N-orbit admitting an η-twisted connection; i.e., a Bruhat cell $C(w^C)$ corresponding to a longest coset representative $w^C \in C \in W_\eta \backslash W$, as in Sect. 3.2. Let $\widetilde{Q} = \pi^{-1}(Q)$ be the corresponding union of N-orbits in $\widetilde{X}$. Fix a positive[13] regular integral weight $\varphi \in \mathfrak{h}^*$. It is explained in [4, §3.4, Lem. 3.5.1] that corresponding to φ is a unique (up to multiplication by a constant) non-zero N-invariant function $f_\varphi : \widetilde{Q} \to \mathbb{A}^1$ satisfying

$$f_\varphi(h\widetilde{x}) = (\exp \varphi)(h) f_\varphi(\widetilde{x}) \tag{3.27}$$

for all $h \in H, \widetilde{x} \in \widetilde{X}$. Moreover, f_φ is a regular function on the closure $\overline{\widetilde{Q}}$ of $\widetilde{Q}$ with the property that $f_\varphi^{-1}(0) = \overline{\widetilde{Q}} \backslash \widetilde{Q}$, and f_φ extends to a regular function on $\widetilde{X}$ which we call by the same name[14].

Let $U := f_\varphi^{-1}(\mathbb{A}^1 - \{0\}) \subset \widetilde{X}$. Because $f_\varphi^{-1}(0) = \overline{\widetilde{Q}} \backslash \widetilde{Q}$, $\widetilde{Q}$ is closed in U. The regular function $f_\varphi : \widetilde{X} \to \mathbb{A}^1$ determines a maximal extension functor

$$\Xi_\varphi := \Xi_{f_\varphi} : \mathcal{M}_{hol}(\mathcal{D}_U) \to \mathcal{M}_{hol}(\mathcal{D}_{\widetilde{X}}) \tag{3.28}$$

by the process laid out in Sect. 3.3.2. Denote by $\mathcal{M}_{hol}^{\overline{\widetilde{Q}}}(\mathcal{D}_{\widetilde{X}}) \subset \mathcal{M}_{hol}(\mathcal{D}_{\widetilde{X}})$ (resp. $\mathcal{M}_{hol}^{\widetilde{Q}}(\mathcal{D}_U) \subset \mathcal{M}_{hol}(\mathcal{D}_U)$) the category of holonomic $\mathcal{D}_{\widetilde{X}}$-modules (resp. $\mathcal{D}_U$-modules) supported on $\overline{\widetilde{Q}}$ (resp. $\widetilde{Q}$). By Kashiwara's theorem [20, Thm. 12.6], there are equivalences of categories

$$\mathcal{M}_{hol}^{\overline{\widetilde{Q}}}(\mathcal{D}_{\widetilde{X}}) \simeq \mathcal{M}_{hol}(\mathcal{D}_{\overline{\widetilde{Q}}}) \text{ and } \mathcal{M}_{hol}^{\widetilde{Q}}(\mathcal{D}_U) \simeq \mathcal{M}_{hol}(\mathcal{D}_{\widetilde{Q}}). \tag{3.29}$$

[13] By a *positive* weight, we mean a weight $\lambda \in \mathfrak{h}^*$ such that $\alpha^\vee(\lambda) > 0$ for all positive roots $\lambda \in \Sigma^+$.

[14] See proof of Lemma 3.5.1 in [4] for an explicit construction of this function. Note that this construction works for any N-orbit in X, not just those admitting η-twisted connections.

The functor Ξ_φ (3.28) transforms the subcategory $\mathcal{M}^{\widetilde{Q}}_{hol}(\mathcal{D}_U)$ into $\mathcal{M}^{\overline{\widetilde{Q}}}_{hol}(\mathcal{D}_{\widetilde{X}})$ [4, Rmk. 4.2.2 (ii)], so using (3.29), we have

$$\mathcal{M}_{hol}(\mathcal{D}_{\widetilde{Q}}) \simeq \mathcal{M}^{\widetilde{Q}}_{hol}(\mathcal{D}_U) \xrightarrow{\Xi_\varphi} \mathcal{M}^{\overline{\widetilde{Q}}}_{hol}(\mathcal{D}_{\widetilde{X}}). \tag{3.30}$$

In this way, the $\mathcal{D}_{\widetilde{X}}$-modules $i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta}$ and $i_{\widetilde{Q}+}\mathcal{O}_{\widetilde{Q},\eta}$ of (3.13) obtain geometric Jantzen filtrations from the monodromy filtration on $\Xi_\varphi\mathcal{O}_{\widetilde{Q},\eta}$. A priori, these filtrations depend on choice of $\varphi \in \mathfrak{h}^*$.

3.4 Relationship between algebraic and geometric Jantzen filtration

The geometric Jantzen filtration on $i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta}$ (3.13) described in Sect. 3.3.4 aligns with the algebraic Jantzen filtration on the corresponding standard Whittaker modules (Definition 2.2.1) under the global sections functor. This relationship is not immediately obvious from the definitions, so we dedicate this section to explaining the connection.

We return to the setting of §3.3.2. Let $f : Y \to \mathbb{A}^1$ be a regular function on a smooth algebraic variety and $U = f^{-1}(\mathbb{A}^1 - \{0\})$. As in (3.21), for any holonomic $\mathcal{D}_U$-module $\mathcal{M}_U$ and $n \in \mathbb{N}$, we have a map

$$s^1(n) := s \circ \mathrm{can} : j_! f^s \mathcal{M}_U^{(n)} \to j_+ f^s \mathcal{M}_U^{(n)}. \tag{3.31}$$

Assume n is large enough so that $\mathrm{coker}\, s^1(n)$ has stabilized. There are two natural copies of the $\mathcal{D}_Y$-module $j_!\mathcal{M}_U$ appearing in (3.31):

(1) as a quotient of the domain,

$$j_! f^s \mathcal{M}_U^{(n)} / s j_! f^s \mathcal{M}_U^{(n)} \simeq j_! \mathcal{M}_U,$$

(2) as a submodule of the cokernel,

$$\ker(s : \Xi_f \mathcal{M}_U \to \Xi_f \mathcal{M}_U) \simeq j_! \mathcal{M}_U.$$

Each of these copies of $j_!\mathcal{M}_U$ obtains a natural filtration from (3.31) in the following way.

Filtration 1: There is a filtration $S_\bullet$ on $j_+ f^s \mathcal{M}_U^{(n)}$ given by powers of s:

$$S_i := s^{-i} j_+ f^s \mathcal{M}_U^{(n)}. \tag{3.32}$$

where we set $s^i = \mathrm{id}$ for $i \leqslant 0$. Pulling back the filtration $S_\bullet$ along the canonical map can (3.20), we obtain a filtration $P_\bullet$ of $j_! f^s \mathcal{M}_U^{(n)}$:

$$P_i := \mathrm{can}^{-1}(s^{-i} j_+ f^s \mathcal{M}_U^{(n)}).$$

The filtration $P_\bullet$ on $j_! f^s \mathcal{M}_U^{(n)}$ induces a filtration on the quotient $j_! f^s \mathcal{M}_U^{(n)} / s j_! f^s \mathcal{M}_U^{(n)}$. Call this filtration $A_\bullet$:

$$A_\bullet := \text{ filtration on } j_! \mathcal{M}_U \text{ induced from } P_\bullet. \tag{3.33}$$

Filtration 2: As explained in Sect. 3.3.2–Sect. 3.3.3, the maximal extension $\Xi_f \mathcal{M}_U = \operatorname{coker} s^1(n)$ has a monodromy filtration, which induces a geometric Jantzen filtration $J_{!,\bullet}$ on $\ker(s : \Xi_f \mathcal{M}_U \to \Xi_f \mathcal{M}_U)$. This filtration can be described in terms of powers of s acting on $\Xi_f \mathcal{M}_U$ (3.16):

$$J_{!,i} := \ker s \cap \operatorname{im} s^{-i}, \tag{3.34}$$

where it is taken that $\operatorname{im} s^i = \Xi_f \mathcal{M}_U$ for $i \leqslant 0$

The two copies of $j_! \mathcal{M}_U$ in (3.31) can be identified as follows. First we have the following commutative diagram where each square is Cartesian

$$\begin{tikzcd}
j_! f^s \mathcal{M}_U^{(n)} \arrow[r, "s"] \arrow[d] & j_! f^s \mathcal{M}_U^{(n)} \arrow[r, "\mathrm{can}"] \arrow[d] & j_+ f^s \mathcal{M}_U^{(n)} \arrow[d] \\
0 \arrow[r] & j_! \mathcal{M}_U \arrow[r, "\overline{\mathrm{can}}"] & \Xi_f \mathcal{M}_U
\end{tikzcd}$$

where we use the definition of the maximal extension in (3.22), that the canonical map can, see (3.20), commutes with s and that $j_! f^s \mathcal{M}_U^{(n)} / s j_! f^s \mathcal{M}_U^{(n)} \simeq j_! \mathcal{M}_U$.

By construction, $\operatorname{im} \overline{\mathrm{can}} = \ker(s : \Xi_f \mathcal{M}_U \to \Xi_f \mathcal{M}_U)$. In this way, $\overline{\mathrm{can}}$ provides an isomorphism between the two copies of $j_! \mathcal{M}_U$ in (3.31). Moreover, because the geometric Jantzen filtration on $\ker(s : \Xi_f \mathcal{M}_U \to \Xi_f \mathcal{M}_U)$ can be realized in terms of the image filtration (3.34), it is clear that under $\overline{\mathrm{can}}$, Filtration (3.33) and Filtration (3.34) align.

Remark 3.4.1 The alignment of the two filtrations above is most easily seen through an example. In [10], a detailed $\mathfrak{sl}_2(\mathbb{C})$ example is given. We encourage the interested reader to view Figure 8 in that paper.

Now we specialize to the setting of Sect. 3.3.4. We will see that in these circumstances, Filtration 1 above aligns with the algebraic Jantzen filtration under the global sections functor, hence the global sections of the geometric Jantzen filtration (Filtration 2 above) also align with the algebraic Jantzen filtration.

Set $Y = \widetilde{X}$ to be base affine space, $\widetilde{Q}$ to be the preimage under $\pi : \widetilde{X} \to X$ of an N-orbit admitting an η-twisted connection, and let f_φ be the regular function on $\widetilde{X}$ given by (3.27) corresponding to a positive regular integral weight $\varphi \in \mathfrak{h}^*$. As above, set $U := f_\varphi^{-1}(\mathbb{A}^1 - \{0\})$. We have the following commuting inclusions of subvarieties:

$$\begin{tikzcd} U \arrow[r, hook, "j_U"] & \widetilde{X} \\ \widetilde{Q} \arrow[u, hook, "k"] \arrow[ur, hook, "i_{\widetilde{Q}}"] \arrow[r, hook, "j_{\widetilde{Q}}"'] & \overline{\widetilde{Q}} \arrow[u, hook, "\overline{k}"'] \end{tikzcd} \tag{3.35}$$

The maps k and $\overline{k}$ are closed immersions, and the maps j_U and $j_{\widetilde{Q}}$ are open immersions. Applying the filtration construction above to the $\mathcal{D}_U$-module $k_!\mathcal{O}_{\widetilde{Q},\eta} = k_+\mathcal{O}_{\widetilde{Q},\eta}$, we obtain from Filtration 1 a filtration of the $\mathcal{D}_{\widetilde{X}}$-module $j_{U!}f_\varphi^s k_!\mathcal{O}_{\widetilde{Q},\eta}^{(n)}$ by pulling back the "powers of s" filtration along the canonical map. By the projection formula [20, §4],

$$j_{U!}f_\varphi^s k_!\mathcal{O}_{\widetilde{Q},\eta}^{(n)} = i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}. \tag{3.36}$$

Hence under the equivalence (3.8), we obtain a filtration of the corresponding $\widetilde{\mathcal{D}}$-module $\pi_\bullet i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}$.

For $\lambda \in \mathfrak{h}^*$, denote by

$$\left(\pi_\bullet i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}\right)_{\widehat{\lambda}} \tag{3.37}$$

the subsheaf of $\pi_\bullet i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}$ annihilated by a power of $\mathfrak{m}_\lambda\widetilde{\mathcal{D}} \subset \widetilde{\mathcal{D}}$, where $\mathfrak{m}_\lambda \subset U(\mathfrak{h})$ is the ideal defined in Remark 3.1.1. The subsheaf (3.37) is $\widetilde{\mathcal{D}}$-stable, and its global sections have the structure of a $\widetilde{U}$-module which is annihilated by the same power of $\mathfrak{m}_\lambda \subset \widetilde{U}$ [4, §3.3]. In particular, for regular integral antidominant $\lambda \in \mathfrak{h}^*$, the global sections of (3.37) can be considered as a $U(\mathfrak{g})$-module with generalized infinitesimal character χ_λ [4, §3.1], where χ_λ is as in (2.1). In fact, the global sections of the $\widetilde{\mathcal{D}}$-module (3.37) are a deformed Whittaker module.

Lemma 3.4.2 *Let $\lambda \in \mathfrak{h}^*$ be a regular integral antidominant weight, $Q = C(w^C) \subset X$ a Bruhat cell corresponding to a longest coset representative in $C \in W_\eta\backslash W$, and $\widetilde{Q} = \pi^{-1}(Q) \subset \widetilde{X}$ the corresponding union of N-orbits in base affine space. For any $n \in \mathbb{Z}_{>0}$,*

$$\Gamma\left(X, \left(\pi_\bullet i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}\right)_{\widehat{\lambda}}\right) \cong M_{A^{(n)}}(w^C\lambda, \eta)$$

as $(U(\mathfrak{g}), A^{(n)})$-modules, where $A = \mathbb{C}[[s]]$, $A^{(n)} = A/s^n = \mathbb{C}[s]/s^n$, and $M_{A^{(n)}}(w^C\lambda, \eta)$ is the deformed Whittaker module (Definition 2.4.6) with respect to the algebra homomorphism $\psi : U(\mathfrak{h}) \to A^{(n)}$ mapping $h \in \mathfrak{h}$ to $\varphi(h)s$.

Proof The case $n = 1$ is Theorem 3.2.2. An adaptation of the proof in [27, §4] for $\widetilde{\mathcal{D}}$-modules can be used to prove the lemma for $n > 1$. Alternatively, the statement follows from the fact that both objects are uniquely determined as iterated extensions of the $n = 1$ case. We expand on this perspective below.

To ease notation, set

$$\mathcal{M}_\lambda^{(n)} := \left(\pi_\bullet i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}\right)_{\widehat{\lambda}} \text{ and } M_\lambda^{(n)} := M_{A^{(n)}}(w^C\lambda, \eta).$$

Using adjunctions of $\mathcal{D}$-module functors and an η-twisted adaptation of the Langland's classification in [4, Section 2.6], one can show that

$$\operatorname{Ext}^1_{\widetilde{\mathcal{D}}}\left(\mathcal{M}_\lambda^{(1)}, \mathcal{M}_\lambda^{(n)}\right) = \operatorname{Ext}^1_{U(\mathfrak{h})}\left(A_{\lambda+\rho}^{(1)}, A_{\lambda+\rho}^{(n)}\right). \tag{3.38}$$

On the other hand, it follows from tensor-hom adjunction and the Harish-Chandra isomorphism for $\mathcal{Z}(\mathfrak{l}_\eta)$ that

$$\operatorname{Ext}^1_{U(\mathfrak{g})}\left(M_\lambda^{(1)}, M_\lambda^{(n)}\right) = \operatorname{Ext}^1_{U(\mathfrak{h})}\left(A_\mu^{(1)}, A_\mu^{(n)}\right) \tag{3.39}$$

for $\mu = w^C\lambda - \rho + \rho_\eta$. Since both μ and $\lambda + \rho$ are regular, the second Ext groups in each equality (3.38) and (3.39) can be canonically identified.

Pushforward along the natural $U(\mathfrak{h})$-module homomorphism $\iota : A_\mu^{(1)} \to A_\mu^{(n)}$ given by multiplication with s^{n-1} yields a map

$$\iota_* : \operatorname{Ext}^1_{U(\mathfrak{h})}\left(A_\mu^{(1)}, A_\mu^{(1)}\right) \to \operatorname{Ext}^1_{U(\mathfrak{h})}\left(A_\mu^{(1)}, A_\mu^{(n)}\right).$$

Using the Koszul resolution of $A_\mu^{(1)}$, the domain of ι_* may be identified with $\mathfrak{h}^*$. Now, both $\mathcal{M}_\lambda^{(n+1)}$ and $M_\lambda^{(n+1)}$ arise in (3.38) and (3.39) as the extensions corresponding to the element $\iota_*(\varphi)$. □

A similar statement holds for deformed costandard objects, and these are compatible with filtrations.

From Lemma 3.4.2, we can see that the geometric Jantzen filtration of Sect. 3.3.3 aligns with the algebraic Jantzen filtration of Definition 2.5.7. Specifically, we have shown that we can realize the geometric Jantzen filtration on the standard $(\mathcal{D}_{\widetilde{X}}, N, \eta)$-module $i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta}$ as the filtration obtained by pulling back the "powers of s" filtration (3.32) along the canonical map between deformed standard and costandard $(\mathcal{D}_{\widetilde{X}}, N, \eta)$-modules and then descending to the quotient

$$i_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)}/si_{\widetilde{Q}!}f_\varphi^s\mathcal{O}_{\widetilde{Q},\eta}^{(n)} \simeq i_{\widetilde{Q}!}\mathcal{O}_{\widetilde{Q},\eta}.$$

By specifying a regular, antidominant and integral weight $\lambda \in \mathfrak{h}^*$ and taking global sections, we obtain a filtration on the corresponding standard Whittaker module $M(w^C\lambda, \eta)$, which is given by pulling back the "powers of s" filtration along the canonical map between standard and costandard deformed Whittaker modules. This is exactly how the algebraic Jantzen filtration is defined. By varying the N-orbit $Q = C(w^C)$, we obtain the result for all standard Whittaker modules $M(\mu, \lambda)$ corresponding to regular integral weights $\mu \in \mathfrak{h}^*$.

4 Geometric proof of the Jantzen conjecture

We now show how the Jantzen conjecture for Whittaker modules follows from the theory of mixed twistor $\mathcal{D}$-modules.

4.1 Background on mixed twistor modules

We will make use of the following notations, definitions and properties of mixed twistor modules, following [23].

4.1.1 Basic notations

For a smooth complex variety Y, we denote by $\mathrm{MTM}(Y)$ the abelian category of *algebraic mixed twistor $\mathcal{D}$-modules on Y*. We denote the constant twistor module on Y by $\underline{\mathcal{O}}_Y$, to distinguish it from the $\mathcal{D}$-module $\mathcal{O}_Y$.

For $w \in \mathbb{Z}$, we denote by $\mathrm{MT}(Y, w) \subset \mathrm{MTM}(Y)$ the subcategory of *polarizable pure twistor modules* of weight w. This is an abelian semisimple category.

4.1.2 Weights and Tate twist

A mixed twistor module $M \in \mathrm{MTM}(Y)$ is equipped with an increasing filtration $W_\bullet$, called the *weight filtration*, such that the associated graded $\mathrm{Gr}^W_w(M) \in \mathrm{MT}(Y, w)$ is pure of weight w and hence semisimple.

For $k \in \frac{1}{2}\mathbb{Z}$, the Tate twist (k) on $\mathrm{MTM}(Y)$ is an automorphism which shifts the weight filtration $W_w M(k) = W_{w+2k} M$ and maps $\mathrm{MT}(Y, w)$ to $\mathrm{MT}(Y, w - 2k)$. In the notation of [23, Section 2.1.8],

$$M(\tfrac{q-p}{2}) = M \otimes \mathcal{U}(p, q).$$

A very important result is that any morphism of mixed twistor modules is strict with respect to the weight filtration, see [23, Lemma 7.1.2].

4.1.3 Realization functor

There is an exact realization functor to the category of holonomic $\mathcal{D}$-modules

$$\Upsilon = \Upsilon^{\lambda_0} : \mathrm{MTM}(Y) \to \mathcal{M}_{hol}(\mathcal{D}_Y).$$

The definition involves the choice of a generic $\lambda_0 \in \mathbb{C}$ which we fix here. Hence, all holonomic $\mathcal{D}$-modules that lift to MTM are equipped with a weight filtration. Moreover, all morphisms that lift to MTM are strict for these weight filtrations.

Denote by $\mathcal{M}_{hol}(\mathcal{D}_Y)_{\mathrm{ss}} \subset \mathcal{M}_{hol}(\mathcal{D}_Y)$ the subcategory of semisimple holonomic $\mathcal{D}$-modules. Then, for $w \in \mathbb{Z}$, the realization functor restricts to a essentially surjective functor

$$\Upsilon : \mathrm{MT}(Y, w) \to \mathcal{M}_{hol}(\mathcal{D}_Y)_{\mathrm{ss}}.$$

In particular, each semisimple holonomic $\mathcal{D}$-module on Y admits a lift to a polarizable pure twistor $\mathcal{D}$-module, see [22, Theorem 19.4.1 and 19.4.2].

4.1.4 Six operations

The six operations for $\mathcal{D}$-modules lift to MTM and commute with the realization functor Υ, see [23, Section 14.1.5.3, Proposition 14.3.28, Proposition 14.3.19, Section 14.3.3 and 14.3.5].

In comparison with $\mathcal{D}$-modules, the pushforward/pullback functors for mixed twistor modules introduce Tate twists. By [23, Section 14.3.3.3], if $f : Y \times Z \to Y$ is the projection map and $d = \dim Z$, then

$$f^*M = M \boxtimes \underline{\mathcal{O}}_Z(-\tfrac{1}{2}d)[-d] \text{ and } f^!M = M \boxtimes \underline{\mathcal{O}}_Z(\tfrac{1}{2}d)[d].$$

By [23, Section 14.3.3.2], if $i : Y \hookrightarrow Z$ is a closed immersion and $d = \dim Z - \dim Y$, then

$$i^*\underline{\mathcal{O}}_Z = \underline{\mathcal{O}}_Y(\tfrac{1}{2}d)[d] \text{ and } i^!\underline{\mathcal{O}}_Z = \underline{\mathcal{O}}_Y(-\tfrac{1}{2}d)[-d].$$

4.1.5 Beilinson gluing

Let $f : Y \to \mathbb{A}^1$ be a function on a smooth variety Y and $j : U = f^{-1}(\mathbb{G}_m) \to Y$. Beilinson's maximal extension functor Ξ_f for $\mathcal{D}$-modules, see Section 3.3.2, admits a lift

$$\Xi_f = \Xi_f^{(0)} : \mathrm{MTM}(U) \to \mathrm{MTM}(Y).$$

The monodromy filtrations on $\Xi_f(M)$, $j_!(M)$ and $j_+(M)$ for $M \in \mathrm{MTM}(U)$ coincide with the respective weight filtrations, see [23, Proposition 11.3.6]. The associated graded of the monodromy filtration is hence a sum of pure twistor modules and, after applying Υ, a semisimple holonomic $\mathcal{D}$-module.

4.2 Lifting Whittaker $\mathcal{D}$-modules to mixed twistor modules

We now explain how the Whittaker $\mathcal{D}$-modules considered in Sect. 3.2 can be equipped with a mixed twistor structure. That is, we will lift $\mathcal{D}$-modules M to mixed twistor modules $\underline{M}$ such that $\Upsilon(\underline{M}) = M$, where Υ is the realization functor. This will equip the $\mathcal{D}$-modules M with a weight filtration which we will use in the proof of the Jantzen conjecture. To construct these lifts, we will use that each semisimple holonomic $\mathcal{D}$-module admits such a lift and that the six functors for holonomic $\mathcal{D}$-modules lift to mixed twistor modules.

Following Remark 3.2.1, denote by $\exp \in \mathcal{M}_{hol}(\mathcal{D}_{\mathbb{A}^1})$ the exponential $\mathcal{D}$-module on $\mathbb{A}^1$. Since exp is simple holonomic, it admits a lift $\underline{\exp} \in \mathrm{MT}(\mathbb{A}^1, 0)$ to a weight zero pure twistor module, see Sect. 4.1.3.

For longest coset representative w^C for $C \in W_\eta \backslash W$ there is a map $\overline{\eta} : C(w^C) \to \mathbb{A}^1$ and we denote by

$$\underline{\mathcal{O}}_{C(w^C),\eta} = \overline{\eta}^! \underline{\exp}(-\tfrac{1}{2}(\ell(w^C) - 1))[-(\ell(w^C) - 1)]$$

the pullback. The map $\overline{\eta}$ can be identified with a projection of the form $\mathbb{A}^{\ell(w^C)} = \mathbb{A}^{\ell(w^C)-1} \times \mathbb{A}^1 \to \mathbb{A}^1$. Under this identification $\underline{\mathcal{O}}_{C(w^C),\eta}$ corresponds to the external tensor product

$$\underline{\mathcal{O}}_{\mathbb{A}^{\ell(w^C)-1}} \boxtimes \underline{\exp} \in \mathrm{MT}(\mathbb{A}^{\ell(w^C)}, 0),$$

see Sect. 4.1.4. Hence, we obtain that $\underline{\mathcal{O}}_{C(w^C),\eta} \in \mathrm{MT}(C(w^C), 0)$ is a weight zero pure twistor module.

Similarly, by pulling back along the (trivializable) principal H-bundle $\pi : \widetilde{C}(w^C) \to C(w^C)$ we obtain the weight zero pure twistor module

$$\underline{\mathcal{O}}_{\widetilde{C}(w^C),\eta} = \pi^! \underline{\mathcal{O}}_{C(w^C),\eta}(-\tfrac{1}{2}\dim H)[-\dim H] \in \mathrm{MT}(\widetilde{C}(w^C), 0).$$

Next, we consider the direct image with compact support

$$\underline{\widetilde{\mathcal{M}}}_{C,\eta} = i_{\widetilde{C}(w^C)!}\underline{\mathcal{O}}_{\widetilde{C}(w^C),\eta} \in \mathrm{MTM}(\widetilde{X})$$

in base affine space.

As explained in Sect. 3.3.4, $\widetilde{C}(w^C)$ arises as the non-zero locus of a regular function on $\widetilde{X}$ intersected with the closure $\overline{\widetilde{C}(w^C)}$. Hence the object $\underline{\widetilde{\mathcal{M}}}_{C,\eta} = i_{\widetilde{C}(w^C)!}$ is equipped with a monodromy filtration $J_{!,\bullet}$ which agrees with the weight filtration $W_\bullet$, see Section 4.1.5. By applying the realization functor we obtain the holonomic $\mathcal{D}$-module

$$\Upsilon(\underline{\widetilde{\mathcal{M}}}_{C,\eta}) = \widetilde{\mathcal{M}}_{C,\eta} = i_{\widetilde{C}(w^C)!}\underline{\mathcal{O}}_{\widetilde{C}(w^C),\eta} \in \mathcal{M}_{hol}(\mathcal{D}_{\widetilde{X}})$$

with filtration $J_{!,\bullet} = W_\bullet$.

Now let $\lambda \in \mathfrak{h}^*$ be anti-dominant and regular. By Lemma 3.2.3 passing to global sections and λ-invariants we obtain the standard Whittaker module

$$M(w^C\lambda, \eta) = \Gamma(\widetilde{X}, \widetilde{\mathcal{M}}_{C,\eta})_\lambda$$

and by Sect. 3.4, the Jantzen filtration $M(w^C\lambda, \eta)_\bullet$, is induced by the monodromy filtration $J_{!,\bullet}$. Hence, we obtain the following result.

Theorem 4.2.1 *The Jantzen filtration of the standard Whittaker module $M(w^C\lambda, \eta)$ is induced by the weight filtration on the mixed twistor module $\underline{\widetilde{\mathcal{M}}}_{C,\eta}$.*

We will now show that the inclusions of standard Whittaker modules lift to maps of the corresponding mixed twistor modules.

Theorem 4.2.2 *Let w^{C_1}, w^{C_2} be longest coset representatives for $C_1, C_2 \in W_\eta\backslash W$ with $w^{C_1} \leqslant w^{C_2}$ and let $d = \ell(w^{C_2}) - \ell(w^{C_1})$. Then the embedding $M(w^{C_1}\lambda, \eta) \hookrightarrow M(w^{C_2}\lambda, \eta)$ lifts to an element in*

$$\mathrm{Hom}_{\mathrm{MTM}(\widetilde{X})}(\underline{\widetilde{\mathcal{M}}}_{C_1,\eta}, \underline{\widetilde{\mathcal{M}}}_{C_2,\eta}(-\tfrac{1}{2}d)).$$

Proof Since the modules $\widetilde{\underline{\mathcal{M}}}_{C,\eta}$ arise from pullback under the map $G/N \to G/B$, we may instead work with the standard modules

$$\underline{\mathcal{M}}_{C,\eta} = \iota_{C,!}\underline{\mathcal{O}}_{C(w^C),\eta} \in \mathrm{MTM}(X)$$

and obtain the desired map by pullback. By induction, we may assume that $w^{C_1} = w$ and $w^{C_2} = ws$ for a simple reflection s.

We first consider the case $w = e$, $ws = s$ and $\eta = 0$. In this case, we may assume that $G = P_s = B \cup BsB$ and identify $G/B = \mathbb{P}^1 = \mathrm{pt} \cup \mathbb{A}^1$ so that

$$\underline{\mathcal{M}}_{C(e),0} = i_!\underline{\mathcal{O}}_{\mathrm{pt}} \text{ and } \underline{\mathcal{M}}_{C(s),0} = j_!\underline{\mathcal{O}}_{\mathbb{A}^1}.$$

In this case, the morphism, say ι, arises from the boundary map

$$i_!\underline{\mathcal{O}}_{\mathrm{pt}}(\tfrac{1}{2})[1] = i_!i^*\underline{\mathcal{O}}_{\mathbb{P}^1} \to j_!j^*\underline{\mathcal{O}}_{\mathbb{P}^1}[1] = j_!\underline{\mathcal{O}}_{\mathbb{A}^1}[1]$$

in the localization long exact sequence for $\mathbb{P}^1 = \mathrm{pt} \cup \mathbb{A}^1$ applied to $\underline{\mathcal{O}}_{\mathbb{P}^1}$. Since all underlying $\mathcal{D}$-modules are regular here, this also follows from [4, Corollary 5.3.4].

We now treat the case of a general w and η. For this, we first recall some standard properties on the Bruhat decomposition and root subgroups, see for example [6]. For $x \in W$, we denote $N'_x = N \cap xN^-x^{-1}$ and $N_x = N \cap xNx^{-1}$. As varieties, N'_x and N_x are direct products of the root subgroups they contain, and multiplication yields an isomorphism $N'_x \times N_x \xrightarrow{\sim} N$. In particular, N'_s is the root subgroup for the simple root α_s associated to s. Using Remark 3.2.1, we obtain $N_x = \mathrm{Stab}_N(xB/B) \subset \ker(\eta)$ if and only if x is a longest coset representative in $W_\eta \backslash W$.

Since $ws > w$, multiplication yields an isomorphism $N'_w \times wN'_sw^{-1} \to N'_{ws}$ and $wN'_sw^{-1} \subset N_w$. Hence, $wN'_sw^{-1} \subset \ker(\eta)$.

Denote by $P_s = B \cup BsB$ the standard parabolic subgroup associated to s. Multiplication yields and isomorphism

$$\psi : N'_w \times \{w\} \times P_s/B \xrightarrow{\sim} C(w) \cup C(ws) \subset G/B$$

compatible with the decomposition $P_s/B = C(e) \cup C(s)$. Moreover, ψ is compatible with the parametrizations of the cells in G/B and P_s/B. We have the following commutative diagram

$$\begin{array}{ccc} N'_w \times \{w\} \times N'_s \times \{s\} & \longrightarrow & C(ws) \\ \downarrow & & \uparrow \\ N'_w \times wN'_sw^{-1} \times \{ws\} & \longrightarrow & N'_{ws} \times \{ws\} \end{array}$$

and a similar, simpler diagram for $\widetilde{C}(w)$. Hence, under ψ we see that η acts trivially on both cells in P_s/N and we can write the standard twistor modules as external tensor products,

$$\underline{\mathcal{M}}_{C(w),\eta} = \underline{\mathcal{O}}_{N'_w,\eta} \boxtimes \underline{\mathcal{M}}_{C(e),0} \text{ and } \underline{\mathcal{M}}_{C(w),\eta} = \underline{\mathcal{O}}_{N'_w,\eta} \boxtimes \underline{\mathcal{M}}_{C(s),0},$$

were $\underline{\mathcal{O}}_{N'_w,\eta} = \eta^! \underline{\exp}(-\frac{1}{2}(\ell(w)-1))[-(\ell(w)-1)]$. We obtain our desired morphism as id $\boxtimes\iota$, where ι is the map $\underline{\mathcal{M}}_{C(e),0} \to \underline{\mathcal{M}}_{C(s),0}(-\frac{1}{2})$ constructed above. □

4.3 Geometric proof of the Jantzen conjectures

We can now give a geometric proof of the Jantzen conjecture. By Theorem 4.2.1, the Jantzen filtration on $M(w^C\lambda, \eta)$ is induced from the weight filtration of the mixed twistor module $\widetilde{\underline{\mathcal{M}}}_{C,\eta}$. By Section 4.1.2, the associated graded of the weight filtration is semisimple, hence the same is true for the Jantzen filtatrion. This proves the second part of Theorem 2.5.8.

By Theorem 4.2.2, the embedding of standard Whittaker modules lift to maps of the associated mixed twistor modules (up to a Tate twist which is equal to half the difference in length). Hence, these maps are strict with respect to the weight (=Jantzen) filtration, see Sect. 4.1.2, and we obtain with respect to the canonical inclusion

$$M(w^{C_1}\lambda, \eta)_i = M(w^{C_1}\lambda, \eta) \cap M(w^{C_2}\lambda, \eta)_{i-\ell(w^{C_2})+\ell(w^{C_1})}$$

for $w^{C_1} \leqslant w^{C_2}$ longest coset representatives in $W_\eta\backslash W$. This is the first part of Theorem 2.5.8.

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