Semilinear parabolic systems and the Stokes equations in non-local function spaces

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CHAPTER 1

Introduction

In the 19th century Navier and Stokes developed an equation for studying hydrodynamics which is today known as the *Navier-Stokes equations*:

$$\begin{aligned}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla\phi &= 0 && \text{in } \mathbb{R}^3 \times (0,T) \\
\operatorname{div}(u) &= 0 && \text{in } \mathbb{R}^3 \times (0,T) \qquad \text{(NSE)} \\
u(0) &= u_0 && \text{in } \mathbb{R}^3.
\end{aligned}$$

Here $u\colon [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ describes the velocity of the fluid and $\phi\colon [0,T) \times \mathbb{R}^3 \to \mathbb{R}$ its pressure. The initial velocity of the fluid is given by u_0.

While the Navier-Stokes equations might be best known for their appearance as one of the Millennium problems [41], another interesting question is about the function space of the initial value.

Note that the Navier-Stokes equations have a natural scaling, meaning that if $u(x,t)$ is a solution of NSE, then also $u_\lambda(x,t) := \lambda u(\lambda^2 t, \lambda x)$ is a solution of the same equation. A Banach space X is called *critical* for the Navier-Stokes equations if the norm in $\mathrm{L}^\infty(0,T;X)$ is invariant under the natural scaling. Existence of solutions in such critical spaces is a particularly interesting topic.

For the Navier-Stokes equation the first result of this form is due to Fujita and Kato [42] from 1964 where they proved global well-posedness for initial data $u_0 \in \dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{R}^3)$ whenever $\|u_0\|_{\dot{\mathrm{H}}^{\frac{1}{2}}}$ is sufficiently small. The smallness condition is important for the global existence, otherwise only local well-posedness, i.e., local-in-time existence and uniqueness is obtained.

For the Lebesgue space $\mathrm{L}^3(\mathbb{R}^3)$ global-in-time well-posedness, again under a smallness condition for the initial value, was proven independently by Kato [58], Giga and Miyakawa [51], and Giga [50] in 1984, 1985 and 1986 respectively. Let us mention here that the approach of Giga consists of an abstract result which is not restricted to the Navier-Stokes equations and also includes the case of smoothly bounded domains.

In 1989, Giga and Miyakawa [52] considered the setting where the vorticity of the initial condition, i.e., $\operatorname{curl} u_0$, is a measure belonging to some Morrey space. Moreover, Kato [59] in 1992 showed that for small initial data there exists a unique global-in-time solution in the Morrey space $\mathrm{L}^{p,3-p}(\mathbb{R}^3)$ without considering the initial vorticity. Note that for $p = 3$ the Lebesgue space result is recovered.

Later in 1995, Cannone generalized Kato's theorem in his PhD thesis [16] as he first considered data $u_0 \in \mathrm{L}^3(\mathbb{R}^3)$ but with the smallness condition in the homogeneous Besov space $\dot{\mathrm{B}}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for $3 < p \leq 6$. In the subsequent paper [17] from 1997, he lifts the assumption $u_0 \in \mathrm{L}^3(\mathbb{R}^3)$ to $u_0 \in \dot{\mathrm{B}}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ which further allowed him to prove an existence and uniqueness result for self-similar solutions in that Besov space.

In 1996, Planchon [78] extended Cannone's result to $u_0 \in \mathrm{L}^p(\mathbb{R}^3) \cap \dot{\mathrm{B}}^{-1+\frac{3}{2p}}_{2p,\infty}(\mathbb{R}^3)$ for all $p > 3/2$. Moreover, he conjectured that $p > 3/2$ might only be a technical hurdle and that the lower bound could be decreased to 1. For a proof of this fact we refer to [12, Theorem 5.40].

In their seminal paper [62] from 2001, Koch and Tataru showed global-in-time well-posedness under a smallness condition in the space BMO^{-1}. Note that this space contains the critical Lebesgue, Morrey and Besov spaces mentioned before. Thus, their result is indeed an extension of previous results.

In a recent preprint from 2025, Coiculescu and Palasek [21] proved that the smallness condition in [62] is sharp in the sense that there exists large data admitting two distinct smooth global solutions.

In the light of the so far mentioned results it is a natural question which is the largest critical space where the Navier-Stokes equations are locally well-posed for small data. In fact any critical space is continuously embedded in the Besov space $\dot{\mathrm{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ see, e.g., [18, Proposition 7]. In 2001, Montgomery-Smith [72] considered an equation of Navier-Stokes type, which he called *cheap Navier-Stokes equations* and showed blow-up the in the endpoint Besov space. However, the case of the full Navier-Stokes equations was open until 2008 when Bourgain and Pavlović [15] showed ill-posedness in $\dot{\mathrm{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. To be precise, they proved that there exist arbitrarily small initial data in $\dot{\mathrm{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ producing solutions which are arbitrarily large in $\dot{\mathrm{B}}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ after an arbitrarily short time.

More results on ill-posedness include the work of Yoneda [99] from 2010, in which the author proves ill-posedness in the Besov space $\dot{\mathrm{B}}^{-1}_{\infty,q}(\mathbb{R}^3)$ for $q > 2$ and a generalized Besov space in between $\dot{\mathrm{B}}^{-1}_{\infty,2}(\mathbb{R}^3)$ and $\dot{\mathrm{B}}^{-1}_{\infty,q}(\mathbb{R}^3)$ for $q > 2$. Moreover, using different techniques Wang [95] extended this results to the case $1 \leq q \leq 2$.

Turning to the half-space, literature is much more sparse. Difficulties are generated for example by the fact that Helmholtz projection and Laplacian do not commute. Moreover, having a boundary requires to choose boundary conditions where Dirichlet boundary conditions are the most popular ones, i.e.,

$$\text{(HNSE)} \qquad \begin{aligned} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\phi &= 0 && \text{in } \mathbb{R}^3_+ \times \mathbb{R}_+ \\ \operatorname{div} u &= 0 && \text{in } \mathbb{R}^3_+ \times \mathbb{R}_+ \\ u(x,t) &= 0 && \text{on } \partial\mathbb{R}^3_+ \times \mathbb{R}_+ \\ u(x,0) &= u_0 && \text{in } \mathbb{R}^3_+. \end{aligned}$$

In 1987 Ukai [93] deduced a solution formula for the Stokes system which implied L^p-L^q-estimates for the Stokes semigroup. These estimates allow to construct a global strong solution to (HNSE) under the condition that u is small in $\mathrm{L}^3(\mathbb{R}^3_+)$. Later, Cannone, Planchon and Schonbek [19] improved on this result by imposing the smallness condition in the homogeneous Besov space $\dot{\mathrm{B}}^{-1+3/p}_{p,\infty}(\mathbb{R}^3_+)$ with $3 < p \leq 6$. Recently, Watanabe [96] proved the existence of a unique strong solution whenever $u_0 \in \dot{\mathrm{B}}^{-1+3/p}_{p,1}(\mathbb{R}^3_+)$ with $2 < p < \infty$. In 2024, Diebou [28] examined steady-state solutions of the Navier-Stokes equations using tent spaces.

However, as mentioned by Watanabe, the approach of Koch and Tataru for initial values in BMO^{-1} is not applicable on the half-space. In 2017, Auscher and Frey [8] provided an alternative proof for the Koch-Tataru theorem which allows them to treat Navier-Stokes type equations where the Laplacian is replaced by a general divergence form operator which is not necessarily self-adjoint. The question whether their approach can be applied to the half-space case sets the starting point for the second part of this thesis.

A different direction of study starts with the works of Giga and Miyakawa [52] and Kato [59] on Morrey spaces. Also in 1992, Taylor [88] considered general parabolic equations as well as the Navier-Stokes equations on compact Riemannian manifolds, thus complementing the aforementioned results. For another result on the full space but on the related Morrey-Campanato space we refer to the work of Lemarié-Rieusset [64]. Further results on $\mathbb{R}^d$ but not for the Navier-Stokes equations include the work of Wu [98] from 1997 on equations of quasi-geostrophic type and the work of Wakabayashi [94] from 2018 on parabolic-parabolic Keller-Segel systems. An overview of the importance of Morrey spaces for the Navier-Stokes and Euler equations can be found in [65]. Turning to the elliptic setting we refer to the work of Lieberman [66] from 2003 and the references therein.

Going away from the full space, the literature again is much more sparse. In 2013, de Almeida and Ferreira considered the Navier-Stokes equations on the half-space equipped with initial and boundary data in suitable Morrey-(type) spaces, see [25]. They prove existence and uniqueness of mild solutions in said spaces.

Maekawa, Miura and Prange [68] considered the Navier-Stokes equations in locally uniform Lebesgue spaces on $\mathbb{R}^d_+$ which are somewhat similar to Morrey spaces. They showed analyticity of the Stokes semigroup as well as short time existence to the Navier-Stokes equations. For domains, Shen [83] in 2003 considered boundary value problems for elliptic systems on Lipschitz domains with boundary data in L^2-based Morrey spaces. Other than that we are not aware of works on parabolic equations or related ones on Lipschitz domains.

In 1987, Chiarenza and Frasca [20] considered the Hardy-Littlewood maximal function as well as Riesz potentials in Morrey spaces. There is quite some literature on extrapolation to weighted Morrey spaces with weights in the Muckenhoupt class. In 2013, Shi, Fu and Zhao [86] consider an operator bounded on a weighted L^p-space that additionally satisfies point-wise estimates under suitable vanishing conditions on the argument. Then, they deduce boundedness of this operator on certain Morrey spaces with the same weight.

Further results on weighted Morrey spaces are due to Rosenthal and Schmeisser [80] as well as Duoandikoetxea and Rosenthal [30]. In difference to the aforementioned result they assume boundedness in weighted L^p-spaces for all Muckenhoupt weights (for some p_0) and obtain Morrey space boundedness for a much larger class of spaces, in particular for all integrability exponents $1 < p < \infty$. In our context, however, p-sensitivity is crucial and so these results do not seem to be appropriate here. This sets the need for a different extrapolation technique from Lebesgue to Morrey spaces.

Overview

This thesis is structured as follows. In Chapter 2 we introduce the concept of *sectorial operators* and the function spaces we will be working in. Moreover, we define *Lipschitz domains* and collect important results on this topic.

The main part of this thesis is split into two parts. In Chapters 3–5 we consider elliptic operators on Morrey spaces on Lipschitz domains. We first establish the notion of *weak reverse Hölder estimates*. We then examine under which circumstances a bounded linear operator has an extension from the scale of Lebesgue spaces to the one of Morrey spaces. As for the techniques employed in that part of this thesis, Shen's L^p-extrapolation theorem [84] serves as an important tool to extrapolate operators from Lebesgue to Morrey spaces. Thus, in our results we will complement L^p-boundedness with suitable weak reverse Hölder estimates to obtain estimates in Morrey spaces. In Chapter 4 we apply the extrapolation results from Chapter 3 first to elliptic operators and then, to some extent, also to the Stokes operator. In the case of elliptic operators we consider Dirichlet as well as Neumann boundary conditions. As both operators emerge through form theory the proofs are mostly similar, however, at times suitable restrictions are necessary in the case of Neumann boundary conditions. As an example we mention $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates for the corresponding semigroup which for Neumann boundary conditions are uniform only in a finite time interval $[0, T]$, even for mean value free functions. Turning to the Stokes operator our results are much more sparse. This is primarily due to the lack of L^p-L^∞-estimates which, at least to our understanding, are crucial for $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates. In Chapter 5, we provide the Morrey space version of an iteration scheme for semilinear problems of the form

$$\begin{aligned} \partial_t u(t) + Au(t) &= Fu(t), \\ u(0) &= a, \end{aligned}$$

that were first established by Giga [50] in the L^p-setting and later refined by Tolksdorf [90]. Here, L^p-L^q-estimates are replaced by $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates so that the Stokes operator is currently not treatable in this set-up. Subsequently, we apply this iteration scheme to solve semilinear parabolic problems where the elliptic part is an elliptic operator in divergence form with Neumann or Dirichlet boundary conditions. Moreover, we assume a certain form of the non-linearity. The original goal of this first part was to transfer the well-posedness result of Kato on the Navier-Stokes equations in Morrey spaces, see [59], to Lipschitz domains. However, at least to our knowledge, this is currently still out of reach as $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates for the Stokes semigroup do not seem to be available.

The starting point for the second part of this thesis was the aforementioned question of whether the approach of Auscher and Frey for the well-posedness of the Navier-Stokes equations is also applicable on the half-space. In their approach they write

$$\begin{aligned} \int_0^t \mathrm{e}^{(t-s)\Delta}\, \mathbb{P} \operatorname{div} \alpha(s,\cdot)\,\mathrm{d}s &= \int_0^t \mathrm{e}^{(t-s)\Delta}\, \Delta(s\Delta)^{-1}(\mathrm{Id} - \mathrm{e}^{2s\Delta}) s^{\frac{1}{2}} \mathbb{P} \operatorname{div} s^{\frac{1}{2}} \alpha(s,\cdot)\,\mathrm{d}s \\ &\quad + \int_0^\infty \mathrm{e}^{(t+s)\Delta}\, \mathbb{P} \operatorname{div} \alpha(s,\cdot)\,\mathrm{d}s \\ &\quad - \int_t^\infty \mathrm{e}^{(t+s)\Delta}\, \mathbb{P} s^{-\frac{1}{2}} \operatorname{div} s^{\frac{1}{2}} \alpha(s,\cdot)\,\mathrm{d}s \\ &=: \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \end{aligned}$$

The term $\mathcal{A}_3$ represents an error term and usually is well-behaved. Whilst the Hardy estimates appearing in estimating the operator $\mathcal{A}_2$ remain open, we were able to prove boundedness of the maximal regularity operator on the tent space $\mathrm{T}^{\infty,2}$ which plays a crucial role in estimating $\mathcal{A}_1$. To this end, in Chapter 6 we investigate how to extend the Stokes semigroup to non divergence-free functions.

The natural way via the Helmholtz projection is not suitable as it does not respect so-called *off-diagonal estimates*. However, this type of estimate is crucially used for extending the maximal regularity operator from L^2 to the Tent space $\mathrm{T}^{\infty,2}$. It is for this reason that we take a detour via the Stokes resolvent system. The solution formula for the Stokes resolvent problem on the half-space allows to extend the Stokes resolvent to non divergence-free functions. Nonetheless, we need to spend some time to show that this extended resolvent, while possibly not being an actual resolvent, still enjoys important properties of resolvents.

In Chapter 7 we use this extended resolvent to define the maximal regularity operator for the Stokes operator on non divergence-free functions. One obstacle in this procedure stems from the fact that in the non divergence-free setting the pressure can not be disregarded. This motivates the definition of an extended Stokes operator defined on a tuple of functions. In this setting we can then show L^2-boundedness of the extended maximal regularity operator as well as the desired off-diagonal estimates. Finally, in Chapter 8 we take advantage of a general procedure for lifting operators from Lebesgue spaces to Tent spaces and show that the maximal regularity operator on the Tent space $\mathrm{T}^{\infty,2}$ is bounded. Moreover, in the spirit of [10] also time weights and Tent spaces $\mathrm{T}^{p,2}$ with finite p are considered. Our work can be seen as a complement of the recent work [9] which considers parabolic problems and addresses the difficulty of initial values in tent spaces.

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For reasons of privacy protection, the acknowledgements are left blank in the electronic version of this thesis.

CHAPTER 2

Fundamentals in the theory of function spaces

2.1 Sectorial operators and semigroup theory

The aim of this section is to introduce sectorial operators and give a short overview on basics of semigroup theory needed in this thesis. The goal of semigroup theory is to extend the exponential functions from real numbers, matrices or bounded operator, all cases in which the exponential series converges, to unbounded operators. In this setting, an approach via the exponential series is clearly not available. However, there are several alternatives, for example the Hille approximation using Euler's approximation $\mathrm{e}^t = \lim_{k\to\infty}\left(1+\frac{t}{k}\right)^k$. Moreover, Yosida's approximation which is based on approximating the unbounded operator in question by bounded operators has to be mentioned. For more details we refer to [36, Chapter II, 3.3]. In this thesis we use the approach via the Cauchy integral formula. For $t > 0$, a complex number $a \in \mathbb{C}$ and a suitable curve γ we have

$$\mathrm{e}^{-ta} = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{e}^{tz}}{z+a}\,\mathrm{d}z.$$

If we want an analogue of this formula but for an operator A, resolvent estimates in a suitable sector of the complex plane are required. This leads to the notion of *sectorial operators* we introduce now. Before that let us make precise what we mean by a sector.

Definition 2.1.1. For $\theta \in (0,\pi)$ define the sector in the complex plane

$$\mathrm{S}_\theta := \{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \theta\}$$

and for $\theta = 0$ define $\mathrm{S}_\theta := (0,\infty)$.

Definition 2.1.2. A linear operator A in a Banach space X with domain $\mathcal{D}(A) \subseteq X$ is called *sectorial of angle* ω if there exists $\omega \in [0, \pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\mathrm{S}_\omega}$ and if for every $\theta \in (\omega, \pi]$ there exists a constant $C > 0$ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C$$

for all $\lambda \in \mathrm{S}_{\pi-\theta}$.

Remark 2.1.3. A sectorial operator is closed as closedness is preserved under taking inverses.

As an example we note that the operator $A = -\Delta$ on $\mathrm{L}^2(\mathbb{R}^d)$ (in fact on $\mathrm{L}^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$) is sectorial. For a proof we refer to [54, Chap. 8]. It is a classical result, see, e.g., [36, Sect. II.4], that if A is sectorial of angle $\omega \in [0, \pi/2)$, then $-A$ generates an analytic semigroup via the Cauchy integral of the resolvent. For $t > 0$ define

$$\mathrm{e}^{-tA} := \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{tz}(z + A)^{-1} \, \mathrm{d}z,$$

where γ is any piecewise smooth curve in $\mathrm{S}_{\pi-\omega}$ going from $\infty \, \mathrm{e}^{-\mathrm{i}(\pi-\theta)}$ to $\infty \, \mathrm{e}^{\mathrm{i}(\pi-\theta)}$ for some $\theta \in (\omega, \pi/2)$. The semigroup can be defined for complex numbers in a suitable sector, for the scope of this thesis, however, positive real arguments are sufficient.

Note that we do not require A to be densely defined. Consequently, $(\mathrm{e}^{-tA})_{t>0}$ might not be strongly continuous. To be precise, [67, Proposition 2.1.4.] states that $\lim_{t\to 0^+} \mathrm{e}^{-tA} x = x$ if and only if $x \in \overline{\mathcal{D}(A)}$.

Recall that reflexivity of the Banach space implies that a sectorial operator is densely defined, see [54, Proposition 2.1.1]. We will later use this fact to show that Morrey spaces are not reflexive.

2.2 Function spaces

In this section we introduce the underlying spaces for our work, namely tent spaces as well as Morrey spaces.

2.2.1 Tent spaces

Tent spaces, with elliptic scaling, were first introduced by Coifman, Meyer and Stein in 1985, see [22]. For $1 \leq p, q < \infty$ the authors defined the space $\mathrm{T}^{p,q}_{\mathrm{ell}}((0,\infty) \times \mathbb{R}^d)$ as the equivalence classes of those functions $f \colon (0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\left(\int_{\mathbb{R}^d} \left(\int_0^\infty \int_{B(x,t)} |f(t,y)|^q \frac{\mathrm{d}y \, \mathrm{d}t}{t^{d+1}} \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Here, the elliptic scaling is manifested by the use of balls $B(x,t)$, meaning center x and radius t, and normalization parameter $t^{-(d+1)}$. In this thesis we are concerned with the (Navier)-Stokes equations which are a system of parabolic equations so that a parabolic scaling seems more appropriate. In our setting we restrict ourselves to $q = 2$, however, we need to include the case $p = \infty$ in which the definition is slightly different. We now consider the half-space case, i.e., functions defined on $(0,\infty) \times \mathbb{R}^d_+$. By $\mathbb{R}^d_+$ we mean $\mathbb{R}^{d-1} \times \mathbb{R}_+$ and elements $x \in \mathbb{R}^d_+$ will sometimes be denoted as $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$ and $x_d > 0$.

Definition 2.2.1 (Parabolic tent spaces)**.** For $p \in [1,\infty)$ we define the tent space $\mathrm{T}^{p,2}((0,\infty) \times \mathbb{R}^d_+)$ (with parabolic scaling) as the space of (equivalence classes of) measurable functions f on $(0,\infty) \times \mathbb{R}^d_+$ with values in $\mathbb{R}^d$ such that

$$\|f\|_{\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+)} \coloneqq \left(\int_{\mathbb{R}^d_+} \left(\int_0^\infty \int_{B(x,\sqrt{t})\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} |f(t,y)|^2 \,\mathrm{d}y\,\mathrm{d}t \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we define $\mathrm{T}^{\infty,2}((0,\infty) \times \mathbb{R}^d_+)$ as the space of (equivalence classes of) measurable functions f on $(0,\infty) \times \mathbb{R}^d_+$ with values in $\mathbb{R}^d$ such that

$$\|f\|_{\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)} \coloneqq \sup_{x\in\mathbb{R}^d_+} \sup_{t>0} \left(\int_0^t \int_{B(x,\sqrt{t})\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} |f(s,y)|^2 \,\mathrm{d}y\,\mathrm{d}s \right)^{\frac{1}{2}} < \infty.$$

Remark 2.2.2.

- The tent space $\mathrm{T}^{p,2}((0,\infty) \times \mathbb{R}^d)$ is defined analogously after replacing $\mathbb{R}^d_+$ by $\mathbb{R}^d$.
- Note that one can switch between the two scalings as explained in [8]. To be precise for $1 \leq p < \infty$ we have $f \in \mathrm{T}^{p,2}((0,\infty) \times \mathbb{R}^d)$ if and only if $tf(t^2,\cdot) \in \mathrm{T}^{p,2}_{\mathrm{ell}}((0,\infty) \times \mathbb{R}^d)$. Indeed, the substitution $s = t^2$ shows

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \left(\int_0^\infty \int_{B(x,t)} |tf(t^2,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{d+1}} \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} \\
&\quad = \left(\int_{\mathbb{R}^d} \left(\int_0^\infty \int_{B(x,\sqrt{s})} s|f(s,y)|^2 \frac{\mathrm{d}y}{t^{d/2+1/2}} \frac{\mathrm{d}s}{s^{\frac{1}{2}}} \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} \\
&\quad = \left(\int_{\mathbb{R}^d} \left(\int_0^\infty \int_{B(x,\sqrt{s})} |f(s,y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{s^{\frac{d}{2}}} \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}$$

  The same proof applies on the half-space and for general $1 \leq q < \infty$ with the adaptation that $tf(t^2,\cdot)$ is replaced by $t^{\frac{2}{q}} f(t^2,\cdot)$. For $p = \infty$ the situation is more involved as the definition of $\mathrm{T}^{\infty,2}_{\mathrm{ell}}((0,\infty) \times \mathbb{R}^d)$ varies in this case. However, the same correspondence as for $p < \infty$ holds.

- As long as there is no confusion about the set the functions are defined on, we do not mention the set in the norm.

For $p < \infty$ the inner integral in the definition of the tent space norm is taken over a cone, i.e., over a set of the form $\{(y,t) \in \mathbb{R}^{d+} \times \mathbb{R}_+ \colon |x-y| < \sqrt{t}\}$ for some $x \in \mathbb{R}^d_+$. It is natural to ask whether the corresponding function space changes when replacing $\sqrt{t}$ by $\alpha\sqrt{t}$ for some $\alpha > 0$, denoting the angle of aperture of the cone. Additionally, we consider time weights of the form t^β with $\beta \in \mathbb{R}$.

Definition 2.2.3. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. For $p \in [1,\infty)$ define the tent space $\mathrm{T}^{p,2}_\alpha((0,\infty)\times\mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x)$ as the space of measurable functions $f\colon (0,\infty)\times\mathbb{R}^d_+ \to \mathbb{R}^d$ such that

$$\|f\|_{\mathrm{T}^{p,2}_\alpha(t^\beta \, \mathrm{d}t \, \mathrm{d}x)} \coloneqq \left(\int_{\mathbb{R}^d_+} \left(\int_0^\infty \int_{B(x,\alpha\sqrt{t})\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} |f(t,y)|^2 \, \mathrm{d}y \, t^\beta \mathrm{d}t \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Moreover, define the tent space $\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+, s^\beta \, \mathrm{d}s \, \mathrm{d}x)$ as the space of those measurable functions $f\colon (0,\infty)\times\mathbb{R}^d_+ \to \mathbb{R}^d$ such that

$$\|f\|_{\mathrm{T}^{\infty,2}(s^\beta \, \mathrm{d}s \, \mathrm{d}x)} \coloneqq \sup_{x\in\overline{\mathbb{R}^d_+}} \sup_{t>0} \left(\int_0^t \int_{B(x,\sqrt{t})\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} |f(s,y)|^2 \, \mathrm{d}y \, s^\beta \mathrm{d}s \right)^{\frac{1}{2}} < \infty.$$

As in the whole space setting, the definition of $\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x)$ simplifies, with the same proof, for $p = 2$ as follows:

Proposition 2.2.4. *Let $\beta \in \mathbb{R}$. Then it holds $\mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x) = \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x)$ with equivalence of norms.*

Proof. Writing χ_A for the characteristic function of a set A and applying Fubini's theorem, for $f \in \mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x)$ we have

$$\begin{aligned}
\|f\|^2_{\mathrm{T}^{2,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x)} &= \int_{\mathbb{R}^d_+} \int_0^\infty \int_{B(x,\sqrt{t})\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} |f(t,y)|^2 \, \mathrm{d}y \, t^\beta \mathrm{d}t \, \mathrm{d}x \\
&= \int_0^\infty \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \chi_{B(x,\sqrt{t})}(y) \, \mathrm{d}x \, t^{-\frac{d}{2}} |f(t,y)|^2 \, \mathrm{d}y \, t^\beta \mathrm{d}t \\
&\simeq \int_0^\infty \int_{\mathbb{R}^d_+} |f(t,y)|^2 \, \mathrm{d}y \, t^\beta \mathrm{d}t.
\end{aligned}$$

In the last step we used that due to $x \in \mathbb{R}^d_+$ the intersection $B(x,\sqrt{t}) \cap \mathbb{R}^d_+$ is of size $t^{\frac{d}{2}}$. □

Next, we establish boundedness properties of the extension-by-zero operator. These will be useful to transfer results from the full-space case to the half-space one.

Proposition 2.2.5. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. For $p \in [1, \infty)$ and $f \in \mathrm{T}^{p,2}_{\alpha}((0,\infty) \times \mathbb{R}^d_+, t^\beta \,\mathrm{d}t \,\mathrm{d}x)$ define*

$$\begin{aligned} E_0 f \colon (0,\infty) \times \mathbb{R}^d &\to \mathbb{R}^d \\ (t,x) &\mapsto \begin{cases} f(t,x), & x \in \mathbb{R}^d_+, \\ 0, & \text{else}, \end{cases} \end{aligned}$$

and do so analogously for $f \in \mathrm{T}^{\infty,2}((0,\infty) \times \mathbb{R}^d_+, s^\beta \,\mathrm{d}s \,\mathrm{d}x)$. For $p \in [1,\infty)$ we then have $E_0 f \in \mathrm{T}^{p,2}_\alpha((0,\infty) \times \mathbb{R}^d, t^\beta \,\mathrm{d}t \,\mathrm{d}x)$ with

$$\|f\|_{\mathrm{T}^{p,2}_\alpha((0,\infty)\times\mathbb{R}^d_+, t^\beta \,\mathrm{d}t\,\mathrm{d}x)} \le \|E_0 f\|_{\mathrm{T}^{p,2}_\alpha((0,\infty)\times\mathbb{R}^d, t^\beta \,\mathrm{d}t\,\mathrm{d}x)} \le 2\|f\|_{\mathrm{T}^{p,2}_\alpha((0,\infty)\times\mathbb{R}^d_+, t^\beta \,\mathrm{d}t\,\mathrm{d}x)}.$$

For $p = \infty$ the same statement holds true.

Proof. For simplicity, we only prove the statement for $\alpha = 1$ and $\beta = 0$. For general parameters the same proof applies.

Note that first inequality is trivially satisfied. For the second inequality let $f \in \mathrm{T}^{p,2}((0,\infty) \times \mathbb{R}^d_+)$ and $E_0 f$ as defined above. We first consider the case $p = \infty$. By definition, we need to estimate

$$\left(\int_0^t \int_{B(x,\sqrt{t}) \cap \mathbb{R}^d} t^{-\frac{d}{2}} |E_0 f(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \right)^{\frac{1}{2}} \tag{2.1}$$

uniformly in $x \in \mathbb{R}^d$ and $t > 0$. For $x \in \overline{\mathbb{R}^d_+}$ and $t > 0$, we can estimate (2.1) by $\|f\|_{\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)}$ as $E_0 f \equiv 0$ on $\mathbb{R}^d_-$. Let us now consider $x \in \mathbb{R}^d_-$ and $t > 0$. Then we have $B(x,\sqrt{t}) \cap \mathbb{R}^d_+ \subseteq B(x^*, \sqrt{t})$ where x^* is the reflection of x at the half-space. As $E_0 f = 0$ in $\mathbb{R}^d_-$, we obtain

$$\left(\int_0^t \int_{B(x,\sqrt{t}) \cap \mathbb{R}^d} t^{-\frac{d}{2}} |E_0 f(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \right)^{\frac{1}{2}} \le \left(\int_0^t \int_{B(x^*,\sqrt{t}) \cap \mathbb{R}^d_+} t^{-\frac{d}{2}} |f(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \right)^{\frac{1}{2}}$$

so that also in this case (2.1) is controlled by $\|f\|_{\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)}$. Hence, we obtain $E_0 f \in \mathrm{T}^{\infty,2}((0,\infty) \times \mathbb{R}^d)$ with $\|f\|_{\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)} \ge \|E_0 f\|_{\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d)}$.

Now, we consider the case $p < \infty$. To show $E_0 f \in \mathrm{T}^{p,2}((0,\infty) \times \mathbb{R}^d)$, writing $B := B(x, \sqrt{t})$ for $x \in \mathbb{R}^d$ and $t > 0$, we split the norm defining integral as follows

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} \left(\int_0^\infty \int_B t^{-\frac{d}{2}} |E_0 f(t,y)|^2 \,\mathrm{d}y \,\mathrm{d}t \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} \le \\ &\qquad \left(\int_{\mathbb{R}^d_+} \left(\int_0^\infty \int_B t^{-\frac{d}{2}} |E_0 f(t,y)|^2 \,\mathrm{d}y \,\mathrm{d}t \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} \\ &\qquad + \left(\int_{\mathbb{R}^d_-} \left(\int_0^\infty \int_B t^{-\frac{d}{2}} |E_0 f(t,y)|^2 \,\mathrm{d}y \,\mathrm{d}t \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

As before, the first integral can be bounded by $\|f\|_{\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+)}$. For the second we use again that $B(x,\sqrt{t})\cap\mathbb{R}^d_+ \subseteq B(x^*,\sqrt{t})$ holds for $x\in\mathbb{R}^d_-$ and $t>0$ and conclude as for $p=\infty$. □

Note that for investigating the effect of changing the aperture it is sufficient to understand the unweighted case as explained in [10, Corollary 2.2]. It was already shown by Coifman, Meyer and Stein in [22, Proposition 4] that the function space stays the same and $\|\cdot\|_{\mathrm{T}^{p,2}_\alpha}$ defines an equivalent norm on $\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d)$. In 2008, the equivalence constants were improved by Hytönen, van Neerven and Portal in [57, Theorem 4.3], to be precise the power of α was lowered from d to $d/\min(p,2)$ which later will be crucial. Moreover, they proved sharpness of this estimate up to the logarithmic factor which was removed by Auscher [5] in 2011. Our result then is the half-space version in the weighted case, however, sharpness remains open. Here, and in the remainder of this thesis we use the notation $A\lesssim B$ as an abbreviation for the existence of an implicit constant $C>0$ such that $A\le CB$ holds. If both $A\lesssim B$ and $B\lesssim A$ hold, we write $A\simeq B$.

Proposition 2.2.6. *Let $1<p<\infty$, $\alpha\ge 1$ and $\beta\in\mathbb{R}$. Then there exists a constant $C>0$ such that, for all $f\in\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+,t^\beta\,\mathrm{d}t\,\mathrm{d}x)$ we have*

$$\|f\|_{\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}\le\|f\|_{\mathrm{T}^{p,2}_\alpha(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}\lesssim(1+\log\alpha)\alpha^{\frac{d}{\tau}}\|f\|_{\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}$$

where $\tau=\min(p,2)$ and the implicit only depends on d and p.

Proof. Let $f\in\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+,t^\beta\,\mathrm{d}t\,\mathrm{d}x)$. Using its extension by zero E_0f and Proposition 2.2.5 followed by Corollary 2.2 in [10], we obtain

$$\begin{aligned}\|f\|_{\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+,t^\beta\,\mathrm{d}t\,\mathrm{d}x)}&\le\|E_0f\|_{\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d,t^\beta\,\mathrm{d}t\,\mathrm{d}x)}\\&\le\|E_0f\|_{\mathrm{T}^{p,2}_\alpha((0,\infty)\times\mathbb{R}^d,t^\beta\,\mathrm{d}t\,\mathrm{d}x)}\\&\le C(1+\log\alpha)\alpha^{\frac{d}{\tau}}\|E_0f\|_{\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d,t^\beta\,\mathrm{d}t\,\mathrm{d}x)}\\&\le 2C(1+\log\alpha)\alpha^{\frac{d}{\tau}}\|f\|_{\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+,t^\beta\,\mathrm{d}t\,\mathrm{d}x)},\end{aligned}$$

where C is the constant from Corollary 2.2 in [10]. Using Proposition 2.2.5 for $\mathrm{T}^{p,2}_\alpha((0,\infty)\times\mathbb{R}^d,t^\beta\,\mathrm{d}t\,\mathrm{d}x)$, we conclude the proof. □

We end this subsection with a remark on a slightly different definition for tent spaces and a notational convention.

Remark 2.2.7. Sometimes it is more convenient to work with cubes instead of balls in the inner integral of the tent space norm. To be precise, instead of a ball $B(x,\sqrt{t})$ we can also choose a cube $Q(x,\sqrt{t})$ with center x and side length $2\sqrt{t}$.

This defines the same space of functions with an equivalent norm. Indeed, for $x \in \overline{\mathbb{R}^d_+}$ and $t > 0$ we have

$$B(x, \sqrt{t}) \subseteq Q(x, \sqrt{t}) \subseteq B(x, \sqrt{d}\sqrt{t}).$$

For $p = \infty$ this concludes the proof as the supremum is taken over all $t > 0$. For $p \in [1, \infty)$ the claim follows from Proposition 2.2.6 on changing the aperture.

Convention 2.2.8. In the remainder of this thesis we will only work with tent spaces on $(0, \infty) \times \mathbb{R}^d_+$ so this part of the notation of the norm will be dropped from now on. Moreover, if $\alpha = 1$ or $\beta = 0$ we also drop the respective parts, i.e., for $1 \leq p < \infty$ we write $\mathrm{T}^{p,2} := \mathrm{T}^{p,2}_1(t^0 \, \mathrm{d}t \, \mathrm{d}x)$.

2.2.2 Morrey spaces

Morrey spaces were introduced by Morrey in 1938, see [73]. For further works on Morrey spaces we refer to the overviews [76] and [1] as well as the references therein.

Here, we want to work on $\mathbb{R}^d$ as well as on bounded domains. As the definition of Morrey spaces on bounded domains is at first sight slightly different, we have two different definitions.

Definition 2.2.9. For $0 \leq \lambda < d$ and $1 \leq p < \infty$ define the Morrey space $\mathrm{L}^{p,\lambda}(\mathbb{R}^d)$ as the space of all $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f\|^p_{\mathrm{L}^{p,\lambda}(\mathbb{R}^d)} := \sup_{\substack{x_0 \in \mathbb{R}^d \\ r > 0}} \frac{1}{r^\lambda} \int_{B(x_0, r)} |f(x)|^p \, \mathrm{d}x < \infty.$$

In the following, let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set.

Definition 2.2.10. For $0 \leq \lambda < d$ and $1 \leq p < \infty$ the Morrey space $\mathrm{L}^{p,\lambda}(\Omega)$ is defined as the space of all $f \in \mathrm{L}^p(\Omega)$ that satisfy

$$\|f\|^p_{\mathrm{L}^{p,\lambda}(\Omega)} := \sup_{\substack{x_0 \in \Omega \\ 0 < r \leq 1}} \frac{1}{r^\lambda} \int_{B(x_0, r) \cap \Omega} |f(x)|^p \, \mathrm{d}x < \infty.$$

Notice that Hölder's inequality together with $\lambda < d$ and $r \leq 1$ implies that the Morrey spaces are nested for increasing p. Next, we show that it is not necessary to restrict to small radii in the previous definition, which will be useful later on. Moreover, we consider E_0, the extension by 0 from a bounded domain to $\mathbb{R}^d$ similarly to Proposition 2.2.5 and prove that this is a bounded operator.

Lemma 2.2.11. *Let $\Omega \subseteq \mathbb{R}^d$ be bounded and measurable. On $\mathrm{L}^{p,\lambda}(\Omega)$ we consider the norm $\widetilde{\|\cdot\|}_{\mathrm{L}^{p,\lambda}(\Omega)}$ defined by*

$$\widetilde{\|f\|}^p_{\mathrm{L}^{p,\lambda}(\Omega)} \coloneqq \sup_{\substack{x_0 \in \Omega, \\ 0<r<\infty}} \frac{1}{r^\lambda} \int_{B(x_0,r)\cap\Omega} |f(x)|^p \,\mathrm{d}x$$

and constitutes an equivalent norm on $\mathrm{L}^{p,\lambda}(\Omega)$.

Proof. Note that $\|f\|_{\mathrm{L}^{p,\lambda}(\Omega)} \leq \widetilde{\|f\|}_{\mathrm{L}^{p,\lambda}(\Omega)}$ is trivially true for $f \in \mathrm{L}^{p,\lambda}(\Omega)$.

For the reverse inequality we start by noting that for $x_0 \in \Omega$ and $r \geq \operatorname{diam}(\Omega)$ the expression

$$\frac{1}{r^\lambda} \int_{B(x_0,r)\cap\Omega} |f(x)|^p \,\mathrm{d}x$$

is decreasing in r as for such r we have $B(x_0,r)\cap\Omega = \Omega$. Hence, it suffices to treat the case $r \leq \operatorname{diam}(\Omega)$. It is clear that for $\operatorname{diam}(\Omega) \leq 1$, there is nothing to show. Otherwise, let $x_0 \in \Omega$ and $r \in (1, \operatorname{diam}(\Omega))$ be arbitrary. We then want to cover $B(x_0,r)$ with balls $B(x_i,\tilde{r})$ where $x_i \in \Omega$ for $i = 1,\dots,N$ for some $N \in \mathbb{N}$. Those balls shall all have the same radius $\tilde{r} \leq 1$, furthermore the radius $\tilde{r}$ needs to be comparable to r. One way to achieve that is to choose $\tilde{r} \coloneqq \frac{r}{\operatorname{diam}(\Omega)}$. Then, the number of such balls necessary to cover $B(x_0,r)$ depends only on $\operatorname{diam}(\Omega)$ and the dimension d.

In fact, $N \coloneqq \lceil \operatorname{diam}(\Omega)\sqrt{d}\rceil^d$ balls suffice. To see this, note that $B(x_0,r)$ is contained in $Q(x_0,r)$, the cube centred in x_0 with side length $2r$. Moreover, the largest cube in a ball with radius $\tilde{r}$ has side length $2\tilde{r}/\sqrt{d}$. Then, $\lceil \operatorname{diam}(\Omega)\sqrt{d}\rceil^d$ of such cubes cover the cube $Q(x_0,r)$. In particular, the corresponding balls cover the ball $B(x_0,r)$.

Hence, there are N balls with centres x_i, $i = 1,\dots,N$ and radius $\frac{r}{\operatorname{diam}(\Omega)}$ such that $B(x_0,r) \subset \bigcup_{i=1}^N B(x_i, \frac{r}{\operatorname{diam}(\Omega)})$. Using the triangle inequality we have

$$\begin{aligned}\left(\frac{1}{r^\lambda}\int_{B(x,r)\cap\Omega} |f(y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} &\leq \sum_{i=1}^N \left(\frac{\operatorname{diam}(\Omega)^\lambda}{\operatorname{diam}(\Omega)^\lambda r^\lambda}\int_{B(x_i,\frac{r}{\operatorname{diam}(\Omega)})\cap\Omega} |f(y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} \\ &\leq \operatorname{diam}(\Omega)^{-\frac{\lambda}{p}} N \|f\|_{\mathrm{L}^{p,\lambda}(\Omega)}\end{aligned}$$

which shows the desired estimate. Now that $\widetilde{\|\cdot\|}_{\mathrm{L}^{p,\lambda}(\Omega)}$ is well-defined, it is clear that it also defines a norm on $\mathrm{L}^{p,\lambda}(\Omega)$. □

From now on, we do not distinguish betweens these two norms and use $\|\cdot\|_{\mathrm{L}^{p,\lambda}(\Omega)}$ for both. Equipped with the previous result, we now compare Morrey spaces with Lebesgue spaces.

Proposition 2.2.12. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set, $1 \leq p < \infty$ and $0 \leq \lambda < d$. Then $\mathrm{L}^{p,\lambda}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ with*

$$\|f\|_{\mathrm{L}^p(\Omega)} \leq (\operatorname{diam}(\Omega))^{\frac{\lambda}{p}} \|f\|_{\mathrm{L}^{p,\lambda}(\Omega)}$$

for all $f \in \mathrm{L}^{p,\lambda}(\Omega)$.

Proof. Let $f \in \mathrm{L}^{p,\lambda}(\Omega)$ and $x_0 \in \Omega$ be arbitrary. Then $B(x_0, \operatorname{diam}(\Omega)) \cap \Omega = \Omega$ holds true. Consequently, we have

$$\begin{aligned}\int_\Omega |f(x)|^p \,\mathrm{d}x &= \frac{(\operatorname{diam}(\Omega))^\lambda}{(\operatorname{diam}(\Omega))^\lambda} \int_{B(x_0,\operatorname{diam}(\Omega))\cap\Omega} |f(y)|^p \,\mathrm{d}y \\ &\leq (\operatorname{diam}(\Omega))^\lambda \|f\|^p_{\mathrm{L}^{p,\lambda}(\Omega)}\end{aligned}$$

which after taking the p-th root concludes the proof. □

The inclusion of $\mathrm{L}^q(\Omega)$ into $\mathrm{L}^{p,\lambda}(\Omega)$ (for suitable $q \geq p$) can be proven without any boundedness assumption.

Proposition 2.2.13. *Let $\Omega \subseteq \mathbb{R}^d$ be either a bounded measurable set or $\mathbb{R}^d$ and let $0 \leq \lambda < d$ and $1 \leq p < \infty$. Then, for $q = \frac{d}{d-\lambda} p \geq p$ we have*

$$\|f\|_{\mathrm{L}^{p,\lambda}(\Omega)} \lesssim \|f\|_{\mathrm{L}^q(\Omega)}$$

for all $f \in \mathrm{L}^q(\Omega)$. Moreover, if Ω is bounded, the result is true even for all $q \geq \frac{d}{d-\lambda} p$.

Proof. Let $f \in \mathrm{L}^q(\Omega)$ be arbitrary. In order to estimate the $\mathrm{L}^{p,\lambda}(\Omega)$-norm, fix $x_0 \in \Omega$ and $r > 0$. By Jensen's inequality we then have

$$\begin{aligned}\left(\frac{1}{r^\lambda} \int_{B(x_0,r)\cap\Omega} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} &= |B(x_0,r)\cap\Omega|^{\frac{1}{p}} r^{-\frac{\lambda}{p}} \left(\fint_{B(x_0,r)\cap\Omega} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq |B(x_0,r)\cap\Omega|^{\frac{1}{p}-\frac{1}{q}} r^{-\frac{\lambda}{p}} \left(\int_{B(x_0,r)\cap\Omega} |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\lesssim r^{\frac{d-\lambda}{p}-\frac{d}{q}} \|f\|_{\mathrm{L}^q(\Omega)},\end{aligned}$$

as $\frac{1}{p} - \frac{1}{q} \geq 0$. Note that $q = \frac{d}{d-\lambda} p$ is equivalent to $\frac{d-\lambda}{p} = \frac{d}{q}$ and this concludes the proof of the first part. The second statement is a direct consequence of the inclusion of Lebesgue spaces on bounded sets (sets of finite measure to be precise) due to Hölder's inequality. □

To conclude our introductory part on Morrey spaces, we show that functions in $\mathrm{L}^{p,\lambda}(\Omega)$ can continuously be extended to $\mathbb{R}^d$.

Lemma 2.2.14. *Let $1 \leq p < \infty$ and $0 \leq \lambda < d$. For a bounded and measurable set $\Omega \subseteq \mathbb{R}^d$ the extension operator $E_0 \colon \mathrm{L}^{p,\lambda}(\Omega) \to \mathrm{L}^{p,\lambda}(\mathbb{R}^d)$ is bounded.*

Proof. Let $x_0 \in \mathbb{R}^d$ and $r > 0$ be arbitrary. If $B(x_0, r) \cap \Omega = \emptyset$, then there is nothing to show. So, assume $B(x_0, r) \cap \Omega \neq \emptyset$, in other words, there exists $y \in B(x_0, r) \cap \Omega$. Then we have $B(y, 2r) \supset B(x_0, r)$ and thus

$$\begin{aligned}\left(\frac{1}{r^\lambda}\int_{B(x_0,r)} |E_0 f(y)|^p \, \mathrm{d}y\right)^{\frac{1}{p}} &= \left(\frac{1}{r^\lambda}\int_{B(x_0,r)\cap\Omega} |Ef(y)|^p \, \mathrm{d}y\right)^{\frac{1}{p}} \\ &\leq 2^{\lambda p}\left(\frac{1}{(2r)^\lambda}\int_{B(y,2r)\cap\Omega} |f(y)|^p \, \mathrm{d}y\right)^{\frac{1}{p}} \\ &\leq 2^{\lambda p}\|f\|_{\mathrm{L}^{p,\lambda}(\Omega)}.\end{aligned}$$

We then conclude that E_0 is bounded with $\|E_0\| \leq 2^{\lambda p}$. $\square$

2.3 Lipschitz domains

In this section we introduce Lipschitz domains. In the literature on partial differential equations on a domain $\Omega \subseteq \mathbb{R}^d$ it is often assumed that Ω has smooth enough boundary, usually meaning at least $\mathcal{C}^1$-boundary. This, however, is not suitable for applications as in real life, the domains in question may have edges or corners. In this thesis, we will consider elliptic systems as well as the Stokes equation on domains which are not $\mathcal{C}^1$ but merely Lipschitz. We will make this definition precise in a second but for now we think of a domain lying above the graph of a Lipschitz continuous function (a so-called special Lipschitz domain). In view of Rademacher's theorem (a Lipschitz continuous function is differentiable almost everywhere) this is a natural extension of $\mathcal{C}^1$-domains.

As for notation, for $x \in \mathbb{R}^d$, we write $x' = (x_1, \dots, x_{d-1})$ and for sets $A, B \subseteq \mathbb{R}^d$, we define $A \pm B \coloneqq \{a \pm b \colon a \in A \text{ and } b \in B\}$. Let us now rigorously define Lipschitz domains following [85].

Definition 2.3.1. A bounded, open and connected set $\Omega \subseteq \mathbb{R}^d$ is called a *bounded strong Lipschitz domain*, or short, *bounded Lipschitz domain*, if there exist $r_0, M > 0$ such that for each point $y \in \partial\Omega$ there exists a Lipschitz continuous function $\eta_y \colon \mathbb{R}^{d-1} \to \mathbb{R}$ with $\eta_y(0) = 0$ and $\|\nabla \eta_y\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and a rotation $R_y \colon \mathbb{R}^d \to \mathbb{R}^d$ such that for all $0 < r \leq r_0$ we have

$$\begin{aligned} R_y(\Omega - \{y\}) \cap D(r) &= D_{\eta_y}(r), \\ R_y(\partial\Omega - \{y\}) \cap D(r) &= I_{\eta_y}(r), \end{aligned}$$

where

$$\begin{aligned} D(r) &:= \{(x', x_d)\colon |x'| < r, |x_d| < 10d(M+1)r\}, \\ D_{\eta_y}(r) &:= \{(x', x_d)\colon |x'| < r, \eta_x(x') < x_d < 10d(M+1)r\}, \\ I_{\eta_y}(r) &:= \{(x', x_d)\colon |x'| < r, \eta_x(x') = x_d\}. \end{aligned}$$

Sets of the form $D_{\eta_y}(r)$ with η, y and r as above will be called *Lipschitz cylinders.* Lipschitz cylinders are indeed Lipschitz domains on their own. However, the proof is technical and we refer to [90, Lemma 1.3.25].

Another useful notion is the one of special Lipschitz domains.

Definition 2.3.2. Let $\varphi\colon \mathbb{R}^{d-1} \to \mathbb{R}$ satisfy

$$|\varphi(x') - \varphi(y')| \leq M|x' - y'|$$

for all $x', y' \in \mathbb{R}^{d-1}$, i.e., φ is Lipschitz continuous with Lipschitz constant at most M. Then, a set D is called *special Lipschitz domain* if

$$D = R\{(x', x_d) \in \mathbb{R}^d\colon x_d > \varphi(x')\}$$

for some rotation R. The smallest M for which the above holds is called *bound* of the special Lipschitz domain D.

In order to define surface measures on the boundary of Lipschitz domains, we first need to introduce some notation. The next part is taken from [90, Section 1.3].

For a bounded Lipschitz domain Ω, $y \in \partial\Omega$ and $0 < r \leq r_0$, let R_x denote the rotation according to Definition 2.3.1. Define $U_{y,r} := \{y\} + R_y^{-1}D(r)$. Moreover, we call the function

$$\begin{aligned} \Phi_{y,r}\colon B'(0,r) &\to \partial\Omega \cap U_{y,r} \\ x' &\mapsto y + R_y^{-1}\begin{pmatrix} x' \\ \eta_y(x') \end{pmatrix}. \end{aligned}$$

coordinate function, where $B'(0, r)$ denotes the ball in $\mathbb{R}^{d-1}$ centred in 0 and with radius r.

An important notion concerning Lipschitz domains is the so-called Lipschitz character. We follow the discussion in [77, Section 5].

Definition 2.3.3. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain and $x_1, \dots, x_n \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^n$ covers $\partial\Omega$. We say that a constant $C > 0$ depends on the *Lipschitz character* of Ω if it depends on n and M, where M is constant from the definition of Lipschitz domains in Definition 2.3.1.

Now, we are able to introduce surface measures. To this end let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, fix finitely many points $x_1, \ldots, x_n \in \partial\Omega$ and a radius $0 < r \leq r_0$ such that the family $(U_{x_i,r})_{i=1}^n$ covers $\partial\Omega$. Such points exist by compactness of $\partial\Omega$.

Definition 2.3.4. A set $A \subseteq \partial\Omega$ is *measurable* if for all $1 \leq i \leq n$ the set $\Phi_{x_i,r}^{-1}(A \cap U_{x_i,r})$ is measurable with respect to the $(d-1)$-dimensional Lebesgue measure. Define the sets

$$A_1 \coloneqq A \cap U_{x_1,r}, \qquad A_{k+1} \coloneqq (A \cap U_{x_{k+1},r}) \setminus \bigcup_{i=1}^k A_i.$$

Then we define the surface measure $\sigma(A)$ as follows

$$\sigma(A) \coloneqq \sum_{i=1}^n \int_{\Phi_{x_i,r}^{-1}(A_i)} (1 + |\nabla_{y'} \eta_{x_i}(y')|^2)^{\frac{1}{2}} \, \mathrm{d}y'. \tag{2.2}$$

Recall that due to polar coordinates for $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and any ball $B(x,r)$ we have

$$\int_{B(x,r)} f(y) \, \mathrm{d}y = \int_0^r \int_{\partial B(x,s)} f(z) \, \mathrm{d}\sigma(z) \, \mathrm{d}s$$

where σ is the surface measure on $\partial B(x,s)$. We aim to use a similar argument later on in the derivation of reverse Hölder estimates, but with $B(x,r)$ replaced by a Lipschitz cylinder. In order to make this argument precise we need the so-called co-area formula. For a proof we refer to [40, Theorem 3.2.12].

Theorem 2.3.5 (Co-area formula). *Let $d \geq 2$. If $f \colon \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous and $\mathfrak{g}$ is the representative of a function $g \in \mathrm{L}^1(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \mathfrak{g}(x) \left[\sum_{i=1}^d |\partial_i f(x)|^2 \right]^{\frac{1}{2}} \mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(y)} \mathfrak{g}(x) \, \mathrm{d}m_{d-1}(x) \, \mathrm{d}y,$$

where m_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$.

Recall that the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^{d-1}$ coincides up to a constant with the $(d-1)$-dimensional Lebesgue measure, see, e.g., [35, Satz III.2.9]. We apply the co-area formula to prove the following lemma. We will see in the proof that it is crucial to exclude the set $I_\eta(2)$ on the right-hand side of the assumption.

Lemma 2.3.6. *Let $1 \leq p < \infty$ and $u \in \mathrm{L}^2(\mathbb{R}^d)$ satisfy*

$$\left(\int_{D_\eta(1)} |u|^p \, \mathrm{d}x \right)^{\frac{2}{p}} \leq C \int_{\partial D_\eta(t) \setminus I_\eta(2)} |u|^2 \, \mathrm{d}\sigma$$

for some $C > 0$ and all $t \in (1,2)$, then also

$$\left(\int_{D_\eta(1)} |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq C' \left(\int_{D_\eta(2)} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

holds true for some $C' > 0$ depending on C and d.

Proof. We integrate the inequality for t between 1 and 2 to obtain

$$\begin{aligned}\left(\int_{D_\eta(1)} |u|^p \,\mathrm{d}x\right)^{\frac{2}{p}} &\leq C \int_1^2 \int_{\partial D_\eta(t)\setminus I_\eta(2)} |u|^2 \,\mathrm{d}\sigma \,\mathrm{d}t \\ &\leq C \int_1^2 \int_{\partial D_\eta(t)\setminus I_\eta(2)} |u|^2 \,\mathrm{d}m_{d-1} \,\mathrm{d}t\end{aligned}$$

where we used that the surface measure is comparable to the $(d-1)$-dimensional Lebesgue measure and the previous note about comparability of Hausdorff and Lebesgue measure. It remains to apply the co-area formula in the following way. Define a function f via

$$\begin{aligned} f\colon \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto t \;\Leftrightarrow x \in \partial D(t) \\ 0 &\mapsto 0. \end{aligned}$$

For $x \in \partial D(t)$ and $y \in \partial D(s)$, by definition of the cylinders, we have

$$|f(x) - f(y)| = |t - s| \leq |x - y|$$

so that f is Lipschitz continuous with Lipschitz constant 1. In particular, the partial derivatives appearing in the co-area formula are bounded by 1 in light of the mean value theorem. Moreover, note that $D_\eta(2) \cap f^{-1}(t) = \partial D_\eta(t) \setminus I(2)$. Hence, an application of the co-area formula with $g = |u|^2 \chi_{D_\eta(2)}$ concludes the proof. $\square$

Later on we need to extend functions from a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ to the full space. If a function satisfies Dirichlet boundary conditions, the extension by zero works as expected. However, dealing with Neumann boundary conditions, things are more delicate. In general, the question whether there is a bounded extension operator from $\mathrm{W}^{k,p}(\Omega)$ to $\mathrm{W}^{k,p}(\mathbb{R}^d)$ is its own field of study and leads to the notion of $\mathrm{W}^{k,p}$-extension domains or universal Sobolev extension domains. Using this theory, one can show that Lipschitz domains are universal Sobolev extension domains but a lot of definitions would be required. For the sake of completeness we remark that every Lipschitz domain is an (ε, δ)-domain, see [32, Lemma 2.2.20], and every (ε, δ)-domain is a universal Sobolev extension domain, see [79, Theorem 8].

However, such a sophisticated method is not necessary in our setting. For our purpose the extension result by Stein [87, Section VI Theorem 3.5] suffices. We now introduce the notion of a minimally smooth boundary.

Definition 2.3.7. Let D be an open set in $\mathbb{R}^d$ and let ∂D denote its boundary. We say that ∂D is minimally smooth if there exists an $\varepsilon > 0$, an integer N, a constant $M > 0$ and a sequence $U_1, U_2, \ldots,$ of open sets so that:

(i) If $x \in \partial D$, then $B(x, \varepsilon) \subseteq U_i$ for some i.

(ii) No point of $\mathbb{R}^d$ is contained in more than N of the U_i.

(iii) For each i there exists a special Lipschitz domain D_i, whose bound does not exceed M so that
$$U_i \cap D = U_i \cap D_i.$$

Theorem 2.3.8. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then, $\partial\Omega$ is minimally smooth.*

Proof. For $x \in \partial\Omega$ let R_x denote the rotation from Definition 2.3.1. Then the family $(R_x^{-1}(R_x x + D(r_0/2)))_{x\in\partial\Omega}$ is an open cover of $\partial\Omega$. As rotations are linear maps, these sets in fact coincide with the sets $U_{x,r_0/2}$ introduced earlier. As Ω is bounded, its boundary $\partial\Omega$ is compact and thus there are finitely many points $x_1, \ldots, x_N \in \partial\Omega$ such that the family $(U_{x_i,r_0/2})_{i=1,\ldots,N}$ covers $\partial\Omega$.

We then claim that with the choice of $U_i = U_{x_i,r_0}$ and $\varepsilon = r_0/4$, the conditions of Definition 2.3.7 are satisfied.

The second condition is trivially satisfied, as we only use finitely many U_i. The first condition is a consequence of the fact that $U_{x_i,r_0/2}$ covers $\partial\Omega$ and for $y \in U_{x_i,r_0/2}$ also $B(y, r_0/4) \subseteq U_{x_i,r_0}$ is true. As Ω is a Lipschitz domain, we have $R_{x_i}(\Omega \cap U_i) = D_{\eta_{x_i}}(r_0)$. Hence, choosing $D_i := \{(x', x_d) \in \mathbb{R}^d \colon x_d > \eta_{x_i}(x')\}$ we have $\Omega \cap U_i = U_i \cap R_{x_i}^{-1} D_i$. Then, the claim follows because the property of special Lipschitz domains, by definition, is preserved under rotations. □

Let us now state the announced extension theorem by Stein.

Theorem 2.3.9. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists an operator $\mathfrak{E}$ mapping functions on Ω to functions on $\mathbb{R}^d$ with the properties*

(i) $\mathfrak{E}(f)|_\Omega = f$*, that is, $\mathfrak{E}$ is an extension operator.*

(ii) *For $1 \leq p \leq \infty$, $\mathfrak{E}$ maps $\mathrm{W}^{1,p}(\Omega)$ continuously into $\mathrm{W}^{1,p}(\mathbb{R}^d)$.*

A consequence of the extension theorem is that a bounded Lipschitz domain is a so called d-set in the following sense.

Theorem 2.3.10. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $r_0 > 0$. Then there exists a constant $C > 0$ depending on r_0 such that for all $x \in \overline{\Omega}$ and $0 < r \leq r_0$ it holds*

$$|B(x, r) \cap \Omega| \geq Cr^d.$$

Proof. For $x \in \Omega$ and $r_0 = 1$ this is the statement of [55, Theorem 2]. General radii $r_0 > 0$ are a consequence of [32, Lemma 1.2.23]. General centres follow from the fact that for $x \in \overline{\Omega}$ and $r > 0$ there exists $y \in \Omega$ satisfying $B(y, r/2) \subseteq B(x, r)$. □

To conclude this introductory chapter, we establish a Poincaré inequality on Lipschitz cylinders. Again, this can be proved in more generality, see [34, Section 7], but requires more technical tools. As the general case is not necessary in this thesis, we restrict ourselves to Lipschitz cylinders. Moreover, in contrast to the classical Poincaré inequality where the function is required to vanish on the whole boundary, we only assume this on a part of the boundary. More precisely, we later consider solutions to a resolvent problem on Ω with Dirichlet boundary values. If, for a second, we assume the solution u to be smooth, then our boundary condition on the Lipschitz cylinder would read $u = 0$ on $I(r)$. Hence, we need to define Sobolev spaces with partially vanishing trace.

Definition 2.3.11. Let $\Omega \subseteq \mathbb{R}^d$ a domain and $E \subseteq \overline{\Omega}$. Define

$$\begin{aligned} \mathcal{C}_E^\infty(\mathbb{R}^d) &\coloneqq \{u \in \mathcal{C}_c^\infty(\mathbb{R}^d) \colon \operatorname{dist}(\operatorname{supp}(u), E) > 0\} \\ \mathcal{C}_E^\infty(\Omega) &\coloneqq \{u_{|\Omega} \colon u \in \mathcal{C}_E^\infty(\mathbb{R}^d)\} \end{aligned}$$

and finally the Sobolev space with partially vanishing trace as the completion of $\mathcal{C}_E^\infty(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$, i.e.,

$$\mathrm{W}_E^{k,p}(\Omega) \coloneqq \overline{\mathcal{C}_E^\infty(\Omega)}^{\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}}.$$

This indeed is an extension of the classical Sobolev spaces as $\mathrm{W}_{\partial\Omega}^{k,p}(\Omega) = \mathrm{W}_0^{k,p}(\Omega)$ holds true. For a proof of this fact and further details on Sobolev spaces with partially vanishing trace we refer to [32, Section 1.2.5]. Coming back to the Poincaré inequality, the general version mentioned before require technical tools which go beyond the scope of this thesis. However, in our situation of a Lipschitz cylinder $D_\eta(r)$ and $\mathrm{W}_E^{k,p}(\Omega)$ with $E = I_\eta(r)$ an elementary proof is available which we present next.

Proposition 2.3.12 (Poincaré inequality on Lipschitz cylinders)**.** *Let Ω be a bounded Lipschitz domain and $D_\eta(r)$ with $0 < r \leq r_0$ a corresponding Lipschitz cylinder. Define $E \coloneqq I_\eta(r)$. Then, there exist a constant C depending only on d and the Lipschitz character of Ω such that for all $u \in \mathrm{W}_E^{1,2}(\Omega)$ we have*

$$\int_{D_\eta(r)} |u|^2 \, \mathrm{d}x \leq C \int_{D_\eta(r)} |\nabla u|^2 \, \mathrm{d}x.$$

Proof. The strategy of the proof is essentially the one for $\mathrm{W}_0^{1,2}(\Omega)$ where Ω is a bounded domain. By density it suffices to consider $u \in \mathcal{C}_E^\infty(D_\eta(r))$. Denote by $\tilde{u}$ its extension by 0 to $D(r)$. It is clear that $\tilde{u} \in \mathcal{C}^\infty(D(r))$. Writing $x' = (x_1, \dots, x_{d-1})$, for all $x \in D_\eta(r)$ we have

$$\begin{aligned} u(x) &= \tilde{u}(x) \\ &= \tilde{u}(x_1, \dots, x_{d-1}, \eta(x')) + \int_{\eta(x')}^{x_d} \partial_d \tilde{u}(x_1, \dots, x_{d-1}, t) \, \mathrm{d}t. \end{aligned}$$

Hence by Hölder's inequality,

$$\begin{aligned} |u(x)|^2 &\leq \left(\int_{\eta(x')}^{x_d} |\partial_d \tilde{u}(x_1, \dots, x_{d-1}, t)| \, \mathrm{d}t \right)^2 \\ &\leq \left(\int_{\eta(x')}^{10d(M+1)r_0} 1 \cdot |\partial_d \tilde{u}(x_1, \dots, x_{d-1}, t)| \, \mathrm{d}t \right)^2 \\ &\leq (2 \cdot 10d(M+1)r_0) \int_{\eta(x')}^{x_d} |\partial_d \tilde{u}(x_1, \dots, x_{d-1}, t)|^2 \, \mathrm{d}t. \end{aligned}$$

Integrating this point-wise estimate with respect to $|x_d| \leq 10d(M+1)r_0$ implies

$$\int_{-10d(M+1)r_0}^{10d(M+1)r_0} |\tilde{u}(x)|^2 \, \mathrm{d}x_d \leq (2 \cdot 10d(M+1)r_0)^2 \int_{-10d(M+1)r_0}^{10d(M+1)r_0} |\partial_d \tilde{u}(x_1, \dots, x_{d-1}, t)|^2 \, \mathrm{d}t.$$

Integrating over $D(r)$ with respect to the first $(d-1)$ components yields the desired estimate for all $u \in \mathcal{C}_E^\infty(D_\eta(r))$ due to $|\partial_d \tilde{u}| \leq |\nabla u|$. We conclude the proof by density. $\square$

CHAPTER 3

Extrapolation on Morrey spaces: abstract results

In this chapter we are concerned with the question of when and how boundedness of operators transfers from Lebesgue to Morrey spaces. The crucial ingredient in this matter are so-called *weak reverse Hölder estimates*. They were first studied in [48] to obtain regularity results for certain higher order elliptic partial differential equations. For a relation of weak reverse Hölder estimates to generalized Muckenhoupt weights we refer to [61].

Starting with a (sub)linear bounded operator on an L^2-space satisfying weak reverse Hölder estimates, it was shown by Shen in [84] that this operator is bounded on certain L^p-spaces. Here, we want to establish similar results in Morrey spaces. First, we introduce weak reverse Hölder estimates and study some of their important properties. Next, we turn to extrapolation results for Morrey spaces, first on the full space and then on bounded sets. The general strategy for bounded sets will be to extend the operator in question to the full space and then use the corresponding results on the full space.

3.1 Weak reverse Hölder estimates

In this section we introduce the notion of weak reverse and uniform weak reverse Hölder estimates and prove several results which will be useful later on.

Definition 3.1.1 ((uniform) weak reverse Hölder estimates)**.** Let $\Omega \subseteq \mathbb{R}^d$ be measurable and T be a bounded linear operator on $\mathrm{L}^2(\Omega)$. Assume that there exist constants $q > 2$, $R_0 > 0$ and $C > 0$ such that the following holds.

- For all $x_0 \in \overline{\Omega}$, all $0 < r < R_0$ and all $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ the estimate

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,2r)\cap\Omega} |Tf|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{3.1}$$

  holds. Then we say that T satisfies *weak reverse Hölder estimates*. For $q = \infty$ replace the left-hand side by the $\mathrm{L}^\infty(B(x_0, r) \cap \Omega)$-norm of Tf.

- Let moreover $\nu > 0$. For all $0 < 2r \leq R < R_0$, $x_0 \in \overline{\Omega}$ and all $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 2R) \cap \Omega$ the estimate

$$\left(\int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C r^{d\left(\frac{1}{q}-\frac{1}{2}\right)}\left(\left(\frac{r}{R}\right)^{\nu}\int_{B(x_0,R)\cap\Omega} |Tf|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

  holds. Then we say that T satisfies *uniform weak reverse Hölder estimates with parameter* ν. Again, for $q = \infty$ replace the left-hand side by the $\mathrm{L}^\infty(B(x_0, r) \cap \Omega)$-norm of Tf.

A few remarks are in order.

Remark 3.1.2.

- In the original paper [84] by Shen, the author assumes the weak reverse Hölder estimates for bounded measurable functions (with compact support in case of $\Omega = \mathbb{R}^d$). Working in an L^p-setting this is sufficient as one has several density arguments at their disposal. Unfortunately, such arguments are not available for Morrey spaces which is why we work with $f \in \mathrm{L}^2(\Omega)$. We will later return to the topic of dense subspaces in Morrey spaces.

- By a result by Giaquinta and Modica [48, Proposition 5.1], weak reverse Hölder estimates are self-improving in the following sense. Let $q > 1$ and $f \in \mathrm{L}^q_{\mathrm{loc}}(\Omega)$ be a non negative function. Suppose that for some constants $C > 0, R_0 > 0$

$$\left(\fint_{B(x_0,r)} f^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\fint_{B(x_0,2r)} f \,\mathrm{d}x$$

  holds for all $x_0 \in \Omega$, $0 < r < \min(R_0, \frac{\operatorname{dist}(x_0,\partial\Omega)}{2})$. Then $f \in \mathrm{L}^{q+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon = \varepsilon(d, q, C) > 0$ and there exists a constant $c = c(d, q, \varepsilon, C)$ such that

$$\left(\fint_{B(x_0,r)} f^{q+\varepsilon} \,\mathrm{d}x\right)^{\frac{1}{q+\varepsilon}} \leq c\left(\fint_{B(x_0,2r)} f^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

  holds for all $x_0 \in \Omega$ and $0 < r < \min(R_0, \frac{\operatorname{dist}(x_0,\partial\Omega)}{2})$. We will see later in this section that a similar result is also true for uniform weak reverse Hölder estimates.

- If T satisfies uniform weak reverse Hölder estimates for some $\mu > d$, then T is local, i.e., if $f \equiv 0$ in $B(x_0, R) \cap \Omega$ for some $0 < R < R_0$, then $Tf \equiv 0$ in $B(x_0, R) \cap \Omega$. Indeed, let $y \in B(x_0, R) \cap \Omega$ and set $\delta := \operatorname{dist}(y, \partial B(x_0, R))$. For $0 < 2r < \delta$ we then find
$$\left(\fint_{B(y,r)} |\chi_\Omega T f|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\left(\frac{r}{\delta}\right)^{\mu} \frac{1}{r^d} \int_{B(y,\delta)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$
Since $\mu > d$, the right-hand side tends to 0 as $r \to 0$. The Lebesgue differentiation theorem then implies that $Tf \equiv 0$ in $B(x_0, R) \cap \Omega$.

- The definition of (uniform) weak reverse Hölder estimates is tailored to what is necessary to prove extrapolation results. However, in applications the situation is more delicate. Usually, one can show weak reverse Hölder estimates for balls centred on the boundary or balls completely contained in Ω, see e.g., [97, Corollary 3.1]. Nonetheless, for our desired extrapolation results on bounded domains, this is too weak. Fortunately, it turns out that this is not a problem due a covering argument presented in [90, Lemma 3.2.2]. To be precise, the following result is true.

Lemma 3.1.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a measurable set. Moreover, let T be a bounded linear operator on $\mathrm{L}^2(\Omega)$. Assume that there exist constants $q > 2, R_0 > 0$ and $C > 0$ such that for all balls $B(x_0, r)$ with $0 < r < R_0$ that satisfy either $x_0 \in \partial\Omega$ or $B(x_0, 3r) \subseteq \Omega$ and all $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ we have*
$$\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$
Then there exists a constant $C' > 0$ depending on d, q and C such that T satisfies weak reverse Hölder estimates with constant C'.

Proof. Let $x_0 \in \overline{\Omega}$ and $0 < r < R_0$ be arbitrary. Moreover, let $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. As announced, we want to apply [90, Lemma 3.2.2]. To this end we need to show the existence of a constant $C > 0$ independent of x_0, r and f (so that the same is true for C') such that
$$\left(\frac{1}{\tilde{r}^d} \int_{\tilde{B}\cap\Omega} |Tf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\frac{1}{\tilde{r}^d} \int_{2\tilde{B}\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}$$
holds for all balls $\tilde{B} = B(\tilde{x}, \tilde{r})$ satisfying $B(\tilde{x}, 3\tilde{r}) \subseteq B(x_0, 3r)$ which are either centred on $\partial\Omega$ or satisfy $B(\tilde{x}, 3\tilde{r}) \subseteq \Omega$. We write $2\tilde{B} := B(\tilde{x}, 2\tilde{r}$. So let $B(\tilde{x}, \tilde{r})$ be such a ball. Then, $f \equiv 0$ on $B(\tilde{x}, 3\tilde{r}) \cap \Omega \subseteq B(x_0, 3r) \cap \Omega$ is true and the desired estimate is an immediate consequence of the assumptions on T. Finally, an application of [90, Lemma 3.2.2] with $\alpha = 2$ yields the claim. □

One can generalize the notion of weak reverse Hölder estimates by considering an L^p-norm instead of instead of the L^2-norm in the right-hand side of (3.1). This leads to the following definition.

Definition 3.1.4. Let $1 \leq p < \infty$. Moreover, let $\Omega \subseteq \mathbb{R}^d$ be measurable and T be a bounded linear operator on $\mathrm{L}^p(\Omega)$. Assume that there exist constants $q > p$, $R_0 > 0$ and $C > 0$ such that the following holds. For all $x_0 \in \overline{\Omega}$, all $0 < r < R_0$ and all $f \in \mathrm{L}^p(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ the estimate, with the suitable adaptation for $q = \infty$,

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega} |Tf|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \tag{3.2}$$

holds. Then we say that T satisfies L^p*-weak reverse Hölder estimates.*

Note that if $\Omega \subseteq \mathbb{R}^d$ is a bounded d-set, e.g., a bounded Lipschitz domain, then due to Hölder's inequality L^2-weak reverse Hölder estimates imply L^p-weak reverse Hölder estimates for $p > 2$. For an extension of this implication for $p < 2$ the following result by Barton and Mayboroda [13, Lemma 4.38] is of great use.

Lemma 3.1.5. *Let $0 < p_0 < q \leq \infty$. Suppose that $U \subseteq \mathbb{R}^d$ is a domain and that u is a function defined on U with the property that, whenever $B(x_0, 2r) \subseteq U$, we have the bound*

$$\left(\frac{1}{r^d}\int_{B(x_0,r)} |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\frac{1}{r^d}\int_{B(x_0,2r)} |u|^{p_0} \,\mathrm{d}x\right)^{\frac{1}{p_0}}$$

for some constant $C > 0$ depending only on p_0, q, d and u. Then for every $0 < p < \infty$, there is some constant $C(p) > 0$ depending only on p, p_0, q and C such that

$$\left(\frac{1}{r^d}\int_{B(x_0,r)} |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C(p) \left(\frac{1}{r^d}\int_{B(x_0,2r)} |u|^{p} \,\mathrm{d}x\right)^{\frac{1}{p}}$$

holds.

Let us now prove that, under suitable assumptions, L^2-weak reverse Hölder estimates imply L^p-weak reverse Hölder estimates.

Theorem 3.1.6. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set. Moreover, let $1 \leq p < \infty$, $\max\{2, p\} < q < \infty$ and $T \in \mathcal{L}(\mathrm{L}^p(\Omega)) \cap \mathcal{L}(\mathrm{L}^2(\Omega))$ be a bounded operator on $\mathrm{L}^p(\Omega)$ and on $\mathrm{L}^2(\Omega)$. If T satisfies L^2-weak reverse Hölder estimates with constants q and $R_0 > 0$, then T also satisfies L^p-weak reverse Hölder estimates with the same constants.*

Proof. The case $p > 2$ is an immediate consequence of Hölder's inequality. Let now $p < 2$. We only consider $q < \infty$, but the same proof applies for $q = \infty$. Fix $x_0 \in \overline{\Omega}$ and $0 < r < R_0$. By Lemma 3.1.5 applied to the extension by zero of Tf to $B(x_0, 2r)$ we get the desired inequality for all $f \in \mathrm{L}^p(\Omega) \cap \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ (even for all $p > 0$). It remains to upgrade this to all $f \in \mathrm{L}^p(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ via a density argument.

So let $f \in \mathrm{L}^p(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ be arbitrary and choose a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}_\mathrm{c}^\infty(\Omega)$ with $f_n \to f$ in $\mathrm{L}^p(\Omega)$. Then,

$$\tilde{f}_n := f_n(1 - \chi_{B(x_0,3r)}) \in \mathrm{L}^p(\Omega) \cap \mathrm{L}^2(\Omega)$$

and $\tilde{f}_n \equiv 0$ on $B(x_0, 3r) \cap \Omega$ for all $n \in \mathbb{N}$. Moreover, $\tilde{f}_n \to f$ in $\mathrm{L}^p(\Omega)$ for $n \to \infty$ as $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. By means of the L^p-weak reverse Hölder estimates for $\tilde{f}_n$ we have

$$\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |T\tilde{f}_n - T\tilde{f}_m|^q \, \mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |T(\tilde{f}_n - \tilde{f}_m)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \to 0$$

for $n, m \to \infty$ as T is a bounded operator on $\mathrm{L}^p(\Omega)$. By completeness of the space $\mathrm{L}^q(B(x_0, r) \cap \Omega)$ we deduce that $(T\tilde{f}_n)_{n\in\mathbb{N}}$ is convergent in said space. As the boundedness of T on $\mathrm{L}^p(\Omega)$ implies that $(T\tilde{f}_n)_{n\in\mathbb{N}}$ converges point-wise almost everywhere (at least along a subsequence) to Tf we obtain $T\tilde{f}_n \to Tf$ in $\mathrm{L}^q(B(x_0, r) \cap \Omega)$ as $n \to \infty$. Combining the previous arguments yields

$$\begin{aligned}
\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |Tf|^q \, \mathrm{d}x\right)^{\frac{1}{q}} &= \lim_{n\to\infty} \left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |T\tilde{f}_n|^q \, \mathrm{d}x\right)^{\frac{1}{q}} \\
&\lesssim \lim_{n\to\infty} \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |T\tilde{f}_n|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |Tf|^p \, \mathrm{d}x\right)^{\frac{1}{p}}
\end{aligned}$$

and this concludes the proof. □

Sometimes weak reverse Hölder estimates might not be sufficient and uniform weak reverse Hölder estimates are required instead. However, this does not complicate matters as weak reverse Hölder estimates imply uniform weak reverse Hölder estimates, at least for certain parameters ν.

Proposition 3.1.7. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable, $R_0 > 0$, $q > 2$. Assume that T satisfies weak reverse Hölder estimates with constants R_0 and q. Then T also satisfies uniform weak reverse Hölder estimates with constants R_0, q and $\nu = 2d\left(\frac{1}{2} - \frac{1}{q}\right)$.*

Proof. Let $0 < 2r \leq R < R_0$ and $x_0 \in \overline{\Omega}$. For $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 2R) \cap \Omega$ we have

$$\begin{aligned}
\left(\int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} &\leq \left(\frac{R^d}{R^d} \int_{B(x_0,R/2)\cap\Omega} |Tf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\leq C R^{\frac{d}{q}} \left(\frac{1}{R^d} \int_{B(x_0,R)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&= C \left(\frac{R}{r} \right)^{d\left(\frac{1}{q}-\frac{1}{2}\right)} r^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\int_{B(x_0,R)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&= C r^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{r}{R} \right)^{\nu} \int_{B(x_0,R)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}
\end{aligned}$$

with $\nu = 2d\left(\frac{1}{2} - \frac{1}{q}\right)$. For the second inequality we note that we have both $2\frac{R}{2} = R < R_0$ and $f \equiv 0$ on $B(x_0, \frac{3}{2}R) \cap \Omega \subseteq B(x_0, 2R) \cap \Omega$ so that the weak reverse Hölder estimates for T apply. □

The proof of the next result follows the lines of [90, Lemma 3.2.3] but for uniform weak reverse Hölder estimates. The philosophy of the result is the same, namely that the constant R_0 in Definition 3.1.1 is not of great importance. This will be particularly useful in the case of bounded domains as we will see in the proof of Theorem 3.2.3.

Lemma 3.1.8. *Let $f \in \mathrm{L}^2(\mathbb{R}^d)$, let $q > 2$, $\mu > 0$ and $c, c' > 1$. Moreover, let $x_0 \in \mathbb{R}^d$ and $0 < cr' \leq R' < R_0'$ for some $R_0' > 0$. Assume that for some $R_0 \leq R_0'$ there exists a constant $C > 0$ such that for all $y \in B(x_0, r')$ and $0 < cr \leq R < R_0$ with $B(y, 2R) \subseteq B(x_0, 2R')$, the function f satisfies*

$$\left(\int_{B(y,r)} |f|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C r^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{r}{R} \right)^{\nu} \int_{B(y,R)} |f|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Then, there exists a constant $C' > 0$ depending only on c, c', R_0, R_0', C and d such that

$$\left(\int_{B(x_0,r')} |f|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C'(r')^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{r'}{R'} \right)^{\nu} \int_{B(x_0,R')} |f|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Proof. We denote $B := B(x_0, r')$ and define $\overline{r} := \varepsilon r' \frac{R_0}{R_0'}$ where $\varepsilon := \min\{\frac{c'-1}{5c}, \frac{1}{2}\frac{R_0'}{R_0}\}$. Then, $\overline{r} \leq r'$ and the family $\{B(y, \overline{r})\}_{y \in B}$ is a covering of B. Thus Vitali's covering lemma, see [37, Theorem 1.5.1], states that there exists an at most countable index-family $\mathcal{F}$ such that the balls $\{B(y, \overline{r})\}_{y \in \mathcal{F}}$ are pairwise disjoint and additionally satisfy $B \subset \bigcup_{y\in\mathcal{F}} B(y, 5\overline{r})$. Let $\#\mathcal{F}$ denote the number of elements in $\mathcal{F}$. Then, due to the choice of ε and $\overline{r} \leq r'$ we have

$$|B(0,1)|\overline{r}^d \#\mathcal{F} = \sum_{y\in\mathcal{F}} |B(y,\overline{r})| \leq |B(x, 2r')| = (2r')^d |B(0,1)|$$

and hence $\#\mathcal{F}$ is finite with an upper bound which depends on ε, R_0, R_0' and d only. Then we have

$$\begin{aligned}\left(\int_{B(x_0,r')} |f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} &\leq \sum_{y\in\mathcal{F}} \left(\int_{B(y,5\overline{r})} |f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\leq \sum_{y\in\mathcal{F}} C\overline{r}^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{5c' R_0' \overline{r}}{(c'-1)R' R_0}\right)^{\mu} \int_{B(y, \frac{c'-1}{c'} R' \frac{R_0}{R_0'})} |f|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

In the second step we applied the uniform weak reverse Hölder estimates with $r = 5\overline{r}$ and $R = \frac{c'-1}{c'} R' \frac{R_0}{R_0'}$. A simple estimates show

$$5c\overline{r} = 5c\varepsilon r' \frac{R_0}{R_0'} \leq 5c\varepsilon \frac{R'}{c'} \frac{R_0}{R_0'} \leq \frac{c'-1}{c'} \frac{R'}{R_0'} R_0 < R_0$$

by definition of ε. Moreover, we obtain $B(y, 2R) \subseteq B(x_0, r' + 2R)$ for $y \in \mathcal{F}$. Note that

$$r' + 2R \leq \frac{R'}{c'} + 2\frac{c'-1}{c'} R' \frac{R_0}{R_0'} \leq 2R' \tag{3.3}$$

is true, so that $B(y, 2R) \subseteq B(x_0, 2R')$ holds for all $y \in \mathcal{F}$. This validates the use of the uniform weak reverse Hölder estimates. To complete the proof, note that $B(y, R) \subseteq B(x_0, R')$ follows by adapting the calculation in (3.3). Indeed, this yields

$$\left(\int_{B(x_0,r')} |f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_{R_0,R_0',\varepsilon,d,c,c'} (r')^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{r'}{R'}\right)^{\nu} \int_{B(x_0,R')} |f|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

what was to be shown. □

Equipped with this result we shall now show that uniform weak reverse Hölder estimates indeed imply weak reverse Hölder estimates, as the name suggests.

Lemma 3.1.9. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable and assume that T satisfies uniform weak reverse Hölder estimates with constants $q > 2$, $R_0 > 0$ and parameter $\nu > 0$. Then, T also satisfies weak reverse Hölder estimates with the same constants q and R_0.*

Proof. By an application of the previous lemma with $c = 2$ and $c' = 3/2$, we can assume that T satisfies uniform weak reverse Hölder estimates but for $0 < \frac{3}{2}r \leq R < R_0$. Let now $x_0 \in \overline{\Omega}$, $0 < r < R_0$ and $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. With $R \coloneqq \frac{3}{2}r$ we have $B(x_0, 2R) = B(x_0, 3r)$ and the uniform weak reverse Hölder estimates apply:

$$\left(\int_{B(x_0,r)\cap\Omega} |Tf|^q \mathrm{d}x\right)^{\frac{1}{q}} \leq C r^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\int_{B(x_0,\frac{3}{2}r)\cap\Omega} |Tf|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

In view of $B(x_0, 3/2r) \subseteq B(x_0, 2r)$ the claim follows. □

Taking into account that weak reverse Hölder estimates imply uniform weak reverse Hölder estimates and that the former are self-improving, the same results also follows for the latter. However, there might be a price to pay with respect to the parameter ν as in the following result ν' is independent of ν

Proposition 3.1.10. *Let $\Omega \subset \mathbb{R}^d$ be measurable, $R_0 > 0$, $q > 2$ and $\nu > 0$. Assume that T satisfies uniform weak reverse Hölder estimates with constants q, R_0 and parameter ν. Then there exists $p > q$ such that T satisfies uniform weak reverse Hölder estimates with constants p, R_0 and parameter $\nu' = 2d\left(\frac{1}{2} - \frac{1}{p}\right)$.*

Proof. This is a direct consequence of the previous lemma combined with the self-improving property of weak reverse Hölder estimates mentioned in Remark 3.1.2 and Proposition 3.1.7. □

To conclude this section on weak reverse Hölder estimates let us prove result in the spirit of [90, Lemma 3.2.2] but not on the radius R_0 but on the size of the ball where f is supposed to vanish.

Lemma 3.1.11. *Let Ω be bounded and measurable, $1 \leq p < \infty$, T be a bounded operator on $\mathrm{L}^p(\Omega)$, $q > p$ and $\alpha > 3$. Assume that there exists $R_0 > 0$ and $C > 0$ such that for all balls $B(x_0, r)$ with $x_0 \in \overline{\Omega}$ and $0 < r < R_0$ that satisfy either $B(x_0, \alpha r) \subseteq \Omega$ or $x_0 \in \partial\Omega$ and all $f \in \mathrm{L}^p(\Omega)$ with $f \equiv 0$ on $B(x_0, \alpha r) \cap \Omega$ we have*

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega} |Tf|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Then, there exists a constant $C' > 0$ depending on p, q, α, d and C such that T satisfies L^p-weak reverse Hölder estimates with constants q and R_0.

Proof. Let $x_0 \in \overline{\Omega}$ and $0 < r < R_0$ with $B(x_0, 3r) \subseteq \Omega$ if $x_0 \notin \partial\Omega$ be arbitrary. Moreover, let $f \in \mathrm{L}^p(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. In the spirit of the proof of [90, Lemma 3.2.2] set

$$c \coloneqq \max\left\{\frac{2}{5\alpha + 1}, \frac{1}{11}\right\}.$$

Moreover, introduce the sets

$$\mathcal{I}_1 \coloneqq \{y \in B(x_0, r) \cap \Omega \colon B(y, cr) \subseteq \Omega\},$$
$$\mathcal{I}_2 \coloneqq \{y' \in \partial\Omega \colon \exists y \in B(x_0, r) \cap \Omega \text{ s.t. } y' \in B(y, cr)\}.$$

Then, for all $y \in \mathcal{I} \coloneqq \mathcal{I}_1 \cup \mathcal{I}_2$ it holds $B(y, 10cr) \subseteq B(x_0, 2r)$ as well as $B(y, 5\alpha cr) \subseteq B(x_0, 3r)$. Moreover,

$$B(x_0, r) \cap \Omega \subseteq \Omega \cap \bigcup_{y \in \mathcal{I}} B(y, cr)$$

is satisfied. Then, Vitali's covering lemma, see [37, Theorem 1.5.1], states that there exist a at most countable index set $\mathcal{F} \subseteq \mathcal{I}$ such that the balls $\{B(y, cr)\}_{y \in \mathcal{F}}$ are pairwise disjoint and

$$\bigcup_{y \in \mathcal{I}} B(y, cr) \subseteq \bigcup_{y \in \mathcal{F}} B(y, 5cr).$$

As in the proof of Lemma 3.1.8, for the number of elements in $\mathcal{F}$, $\#\mathcal{F}$, we have

$$|B(0,1)| c^d r^d \#\mathcal{F} = \sum_{y \in \mathcal{F}} |B(y, cr)| \leq ||B(x_0, 2r)| = |B(0,1)| 2^d r^d$$

so that $\#\mathcal{F} \leq \frac{2^d}{c^d}$ follows. Then,

$$\begin{aligned}
\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |Tf|^q \, \mathrm{d}x \right)^{\frac{1}{q}} &\leq \left(\sum_{y \in \mathcal{F}} \frac{1}{r^d} \frac{(5c)^d}{(5c)^d} \int_{B(y,5cr)\cap\Omega} |Tf|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\leq C (5c)^{\frac{d}{q} - \frac{d}{p}} \left(\sum_{y \in \mathcal{F}} \frac{1}{r^d} \int_{B(y,10cr)\cap\Omega} |Tf|^p \, \mathrm{d}x \right)^{\frac{1}{p}}
\end{aligned}$$

as $B(y, \alpha 5cr) \subseteq B(x_0, 3r)$. Moreover, due to $B(y, 10cr) \subseteq B(x_0, 2r)$ we can further estimate

$$\begin{aligned}
&\leq C (5c)^{\frac{d}{q} - \frac{d}{p}} \left(\sum_{y \in \mathcal{F}} \frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |Tf|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\leq C (5c)^{\frac{d}{q} - \frac{d}{p}} \left(\frac{2}{c} \right)^{\frac{d}{p}} \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |Tf|^p \, \mathrm{d}x \right)^{\frac{1}{p}}
\end{aligned}$$

and this is the desired L^p-weak reverse Hölder estimate. □

3.2 Morrey extrapolation on $\mathbb{R}^d$ and bounded sets

In this section we first prove extrapolation results on the full space and then use those to obtain similar results on bounded measurable sets. Our results will be similar to those in the L^p-extrapolation theorem by Shen [84, Theorem 3.1]. A natural starting point is to complement Shen's assumption of L^2-boundedness and weak reverse Hölder estimates with $\mathrm{L}^{2,\lambda}$-boundedness for some $0 < \lambda < d$. This yields extrapolation to $\mathrm{L}^{q,\mu}$-spaces for suitable $q > 2$ and $0 < \mu < d$. Note that in the Morrey space setting the usual strategy of dealing with $q < 2$, namely duality is not available, see e.g., [1, Chapter 5]. In order to extrapolate $\mathrm{L}^{2,\lambda}$-boundedness to $\mathrm{L}^{q,\lambda}$-boundedness uniform weak reverse Hölder estimates are required. However, we show that in certain situations the assumption of $\mathrm{L}^{2,\lambda}$-boundedness can be waived. Moreover, we will see that for suitable ν we can obtain $\mathrm{L}^{p,\nu}$-boundedness from L^p-boundedness as well as weak reverse Hölder estimates. This strategy then allows to also consider $q < 2$.

Before we present our first extrapolation result, we first deduce a condition on p, q for the space $\mathrm{L}^{q,\mu}(\mathbb{R}^d)$ to be included in $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$. Notice that Hölder's inequality implies for $q \geq 2$

$$\begin{aligned}\left(\frac{1}{r^\lambda}\int_{B(x_0,r)}|f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} &\leq C(d) r^{-\frac{\lambda}{2}+d(\frac{1}{2}-\frac{1}{q})}\left(\int_{B(x_0,r)}|f(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\simeq \left(\frac{1}{r^\mu}\int_{B(x_0,r)}|f(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}},\end{aligned}$$

where $C(d) > 0$ depends only on d and where

$$\mu := d - \frac{q}{2}(d-\lambda).$$

Next, the condition

$$\frac{1}{2} - \frac{\lambda}{2d} \leq \frac{1}{q} \leq \frac{1}{2}$$

implies that $0 \leq \mu < d$. We deduce that for such numbers q the inclusion

$$\mathrm{L}^{q,\mu}(\mathbb{R}^d) \subset \mathrm{L}^{2,\lambda}(\mathbb{R}^d)$$

is valid. Our first extrapolation result then reads as follows.

Theorem 3.2.1. *Let* $0 \leq \lambda < d$ *and* $q > 2$ *satisfy*

$$\frac{1}{2} - \frac{\lambda}{2d} \leq \frac{1}{q} \leq \frac{1}{2},$$

and define

$$\mu := d - \frac{q}{2}(d-\lambda).$$

Let T be a linear operator mapping $\mathrm{L}^2(\mathbb{R}^d)$ boundedly into $\mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$ boundedly into $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$. Assume furthermore, that T satisfies weak reverse Hölder estimates with exponent q and $R_0 = \infty$. Then, the restriction of T to $\mathrm{L}^{q,\mu}(\mathbb{R}^d)$ is a bounded operator from $\mathrm{L}^{q,\mu}(\mathbb{R}^d)$ into $\mathrm{L}^{q,\mu}(\mathbb{R}^d)$. The operator norm of T on $\mathrm{L}^{q,\mu}(\mathbb{R}^d)$ depends on d, q, λ, the constant in the weak reverse Hölder estimates as well as the operator norm of T on $\mathrm{L}^2(\mathbb{R}^d)$ and on $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$.

Proof. First of all, let $x_0 \in \mathbb{R}^d$ and $r > 0$, and abbreviate $B(x_0, r) =: B$. Fix $f \in \mathrm{L}^{q,\mu}(\mathbb{R}^d) \subset \mathrm{L}^{2,\lambda}(\mathbb{R}^d)$.

We divide the proof into two parts, by treating each of the terms of the splitting

$$Tf = T(f\chi_{3B}) + T(f\chi_{\mathbb{R}^d\setminus 3B}) \tag{3.4}$$

separately. Notice that this is well-defined, as $f \in \mathrm{L}^{q,\mu}(\mathbb{R}^d)$ implies $f\chi_A \in \mathrm{L}^{q,\mu}(\mathbb{R}^d)$ for any measurable set A.

For the estimate of the first term recall that $f \in \mathrm{L}^{q,\mu}(\mathbb{R}^d)$ implies $f\chi_{3B} \in \mathrm{L}^q(3B)$ (see Proposition 2.2.12). This indicates that an L^q-theory of the operator might suffice in order to estimate the first term. The classical L^q-extrapolation theorem of Shen [84, Theorem 3.3] together with the self-improving property of weak reverse Hölder estimates mentioned in Remark 3.1.2 implies that the operator T extrapolates to a bounded operator on $\mathrm{L}^q(\mathbb{R}^d)$ with operator norm bounded by constant depending on d, q, the operator norm of T on $\mathrm{L}^2(\mathbb{R}^d)$ and the constant in the weak reverse Hölder estimates. Then,

$$\|Tf\chi_{3B}\|_{\mathrm{L}^q(3B)} \leq \|Tf\chi_{3B}\|_{\mathrm{L}^q(\mathbb{R}^d)} \leq \|T\|_{\mathcal{L}(\mathrm{L}^q(\mathbb{R}^d))}\|f\|_{\mathrm{L}^q(3B)}.$$

A multiplication by $r^{-\mu/q}$ implies

$$\left(\frac{1}{r^\mu}\int_B |T(f\chi_{3B})|^q \,\mathrm{d}y\right)^{\frac{1}{q}} \leq C\left(\frac{1}{r^\mu}\int_{3B}|f|^q \,\mathrm{d}y\right)^{\frac{1}{q}} \leq C\|f\|_{\mathrm{L}^{q,\mu}(\mathbb{R}^d)}. \tag{3.5}$$

It remains to estimate the second term in (3.4). Since the argument of the operator is vanishing on $3B$, a further application of the weak reverse Hölder estimate might help. Indeed, an elementary estimate involving only the weak reverse Hölder estimate, the $\mathrm{L}^{2,\lambda}$-boundedness of T, and (3.2) implies

$$
\begin{aligned}
\Big(\frac{1}{r^\mu}\int_B |T(f\chi_{\mathbb{R}^d\setminus 3B})|^q \,\mathrm{d}y\Big)^{\frac{1}{q}} &\leq C r^{\frac{1}{q}(d-\mu)}\Big(\frac{1}{r^d}\int_{2B}|T(f\chi_{\mathbb{R}^d\setminus 3B})|^2\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&= C 2^{\frac{\lambda}{2}} r^{\frac{1}{q}(d-\mu)-\frac{d}{2}+\frac{\lambda}{2}}\Big(\frac{1}{(2r)^\lambda}\int_{2B}|T(f\chi_{\mathbb{R}^d\setminus 3B})|^2\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\leq C r^{\frac{1}{q}(d-\mu)-\frac{d}{2}+\frac{\lambda}{2}}\|f\|_{\mathrm{L}^{2,\lambda}(\mathbb{R}^d)} \\
&\leq C r^{\frac{1}{q}(d-\mu)-\frac{d}{2}+\frac{\lambda}{2}}\|f\|_{\mathrm{L}^{q,\mu}(\mathbb{R}^d)}.
\end{aligned}
$$

Notice that the definition of μ implies

$$\frac{1}{q}(d-\mu)-\frac{d}{2}+\frac{\lambda}{2} = \frac{1}{q}\Big(d-\Big[d-\frac{q}{2}(d-\lambda)\Big]\Big)-\frac{d}{2}+\frac{\lambda}{2}=0.$$

Consequently,

$$\Big(\frac{1}{r^\mu}\int_B |T(f\chi_{\mathbb{R}^d\setminus 3B})|^q\,\mathrm{d}y\Big)^{\frac{1}{q}} \leq C\|f\|_{\mathrm{L}^{q,\mu}(\mathbb{R}^d)} \tag{3.6}$$

with C being independent of x and r. Combining (3.4), (3.5), and (3.6) delivers

$$\Big(\frac{1}{r^\mu}\int_B |Tf|^q\,\mathrm{d}y\Big)^{\frac{1}{q}} \leq C\|f\|_{\mathrm{L}^{q,\mu}(\mathbb{R}^d)}$$

with C being independent of x and r. Taking the supremum over x and r concludes the proof. □

Next, we present an extrapolation theorem where the parameter λ does not change. In contrast to the previous theorem we now assume uniform weak reverse Hölder estimates. However, in this situation the inclusion $\mathrm{L}^{q,\lambda}(\mathbb{R}^d) \subseteq \mathrm{L}^{2,\lambda}(\mathbb{R}^d)$ is no longer valid. Hence, our operator is only defined on the intersection $\mathrm{L}^{q,\lambda}(\mathbb{R}^d) \cap \mathrm{L}^{2,\lambda}(\mathbb{R}^d)$.

Theorem 3.2.2 (Whole space case)**.** *Let $\nu > 0$, $0 \leq \lambda < \min\{\nu, d\}$, and $q > 2$ satisfy the relation*

$$\frac{\lambda}{q} + d\Big(\frac{1}{2}-\frac{1}{q}\Big) < \frac{\nu}{2}. \tag{3.7}$$

Let T be a linear operator mapping $\mathrm{L}^2(\mathbb{R}^d)$ boundedly into $\mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$ boundedly into $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$. Assume furthermore, that T satisfies uniform weak reverse Hölder estimates with exponent q, parameter ν and $R_0 = \infty$. Then the restriction of T to $\mathrm{L}^{2,\lambda}(\mathbb{R}^d) \cap \mathrm{L}^{q,\lambda}(\mathbb{R}^d)$ is bounded with respect to the $\mathrm{L}^{q,\lambda}(\mathbb{R}^d)$-norm with a constant depending only on d, q, λ, ν, the constant in the weak reverse Hölder estimates and the operator norm of T on $\mathrm{L}^2(\mathbb{R}^d)$ and on $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$.

Proof. Let $k \in \mathbb{N}$ and $f \in \mathrm{L}^{2,\lambda}(\mathbb{R}^d) \cap \mathrm{L}^{q,\lambda}(\mathbb{R}^d)$. Denote by $B \coloneqq B(x_0, r)$ the ball with center $x_0 \in \mathbb{R}^d$ and radius $r > 0$. By linearity of T, we can write

$$Tf = T(f\chi_{4B}) + \sum_{n=1}^{k} T(f\chi_{2^{n+2}B\setminus 2^{n+1}B}) + T(f\chi_{\mathbb{R}^d\setminus 2^{k+2}B}).$$

Step 1: Estimate of the first term.

Recall that we showed in Lemma 3.1.9 that uniform weak reverse Hölder estimates imply weak reverse Hölder estimates. Hence, this term can be handled analogously to the first step in the proof of Theorem 3.2.1, i.e., there exists a constant $C > 0$ independent of x_0 and r such that

$$\left(\frac{1}{r^\lambda}\int_B |T(f\chi_{4B})|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\|f\|_{\mathrm{L}^{q,\lambda}(\mathbb{R}^d)}. \tag{3.8}$$

Step 2: Estimate of the second term.

To estimate the sum, we employ the uniform weak reverse Hölder estimates. More precisely, for $1 \leq n \leq k$ we have

$$\left(\int_B |T(f\chi_{2^{n+2}B\setminus 2^{n+1}B})|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq Cr^{d(\frac{1}{q}-\frac{1}{2})}\left(2^{-n\mu}\int_{2^nB} |T(f\chi_{2^{n+2}B\setminus 2^{n+1}B})|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

An application of Jensen's inequality delivers

$$\leq C2^{dn(\frac{1}{2}-\frac{1}{q})-\frac{n\mu}{2}}\left(\int_{2^nB} |T(f\chi_{2^{n+2}B\setminus 2^{n+1}B})|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Finally, as in Step 1, we note that since T is $\mathrm{L}^2(\mathbb{R}^d)$-bounded and satisfies weak reverse Hölder estimates, it is also $\mathrm{L}^q(\mathbb{R}^d)$-bounded by [84, Theorem 3.1] together with the self-improving property of weak reverse Hölder estimates. Consequently,

$$\left(\int_B |T(f\chi_{2^{n+2}B\setminus 2^{n+1}B})|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C2^{dn(\frac{1}{2}-\frac{1}{q})-\frac{n\mu}{2}}\left(\int_{2^{n+2}B} |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Dividing by $r^{\frac{\lambda}{q}}$ delivers

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_B |T(f\chi_{2^{n+2}B\setminus 2^{n+1}B})|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\qquad\leq C2^{dn(\frac{1}{2}-\frac{1}{q})-\frac{n\mu}{2}}\left(\frac{1}{r^\lambda}\int_{2^{n+2}B} |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\qquad= C2^{dn(\frac{1}{2}-\frac{1}{q})-\frac{n\mu}{2}+\frac{\lambda}{q}(n+2)}\left(\frac{1}{(2^{n+2}r)^\lambda}\int_{2^{n+2}B} |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.
\end{aligned}$$

Altogether, there exists a constant $C > 0$ not depending on x_0, r, and n such that

$$\sum_{n=1}^{k} \left(\frac{1}{r^\lambda} \int_B |T(f\chi_{2^{n+2}B \setminus 2^{n+1}B})|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \sum_{n=1}^{\infty} 2^{dn(\frac{1}{2}-\frac{1}{q}) - \frac{n\mu}{2} + \frac{\lambda n}{q}} \|f\|_{\mathrm{L}^{q,\lambda}(\mathbb{R}^d)}. \tag{3.9}$$

By virtue of (3.7) the sum is finite.

Step 3: Estimate of the third term.

Finally, we check that the third term vanishes in the limit $k \to \infty$. This follows by a simple computation that combines the uniform weak reverse Hölder estimates with the $\mathrm{L}^{2,\lambda}$-boundedness of T. More precisely,

$$\begin{aligned} \left(\int_B |T(f\chi_{\mathbb{R}^d \setminus 2^{k+2}B})|^q \mathrm{d}x \right)^{\frac{1}{q}} &\leq C r^{d(\frac{1}{q}-\frac{1}{2})} \left(2^{-\mu(k+1)} \int_{2^{k+1}B} |T(f\chi_{\mathbb{R}^d \setminus 2^{k+2}B})|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq C \|T\|_{\mathcal{L}(\mathrm{L}^{2,\lambda}(\mathbb{R}^d))} r^{d(\frac{1}{q}-\frac{1}{2})+\frac{\lambda}{2}} 2^{\frac{1}{2}(\lambda-\mu)(k+1)} \|f\|_{\mathrm{L}^{2,\lambda}(\mathbb{R}^d)} \end{aligned}$$

with the right-hand side going to zero as $k \to \infty$ since $\lambda < \mu$.

Summarizing (3.8), (3.9), and the third step, we deduce

$$\begin{aligned} \left(\frac{1}{r^\lambda} \int_B |Tf|^q \, \mathrm{d}x \right)^{\frac{1}{q}} &\leq \liminf_{k \to \infty} \Bigg\{ \left(\frac{1}{r^\lambda} \int_B |T(f\chi_{4B})|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\quad + \sum_{n=1}^{k} \left(\frac{1}{r^\lambda} \int_B |T(f\chi_{2^{n+2}B \setminus 2^{n+1}B})|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\quad + \left(\frac{1}{r^\lambda} \int_B |T(f\chi_{\mathbb{R}^d \setminus 2^{k+2}B})|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \Bigg\} \\ &\leq C \|f\|_{\mathrm{L}^{q,\lambda}(\mathbb{R}^d)}. \end{aligned}$$

By taking the supremum over all $x_0 \in \mathbb{R}^d$ and $r > 0$, we have

$$\|Tf\|_{\mathrm{L}^{q,\lambda}(\mathbb{R}^d)} \leq C \|f\|_{\mathrm{L}^{q,\lambda}(\mathbb{R}^d)}$$

and this shows the desired estimate as $f \in \mathrm{L}^{2,\lambda} \cap \mathrm{L}^{q,\lambda}$ was arbitrary. □

In the previous theorems we proved two different extrapolations on the full space. Next, we consider the case of bounded measurable sets. Note that no assumptions on the connectedness of Ω or the regularity of $\partial\Omega$ are made.

Theorem 3.2.3. *Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable. Let $\nu > 0$, $0 \leq \lambda < \min\{\nu, d\}$ and $q > 2$ satisfy*

$$\frac{\lambda}{q} + d\left(\frac{1}{2} - \frac{1}{q}\right) < \frac{\nu}{2}.$$

Let T be a linear operator mapping $\mathrm{L}^2(\Omega)$ boundedly into $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^{2,\lambda}(\Omega)$ boundedly into $\mathrm{L}^{2,\lambda}(\Omega)$ satisfying uniform weak reverse Hölder estimates with exponent q, parameter ν and radius $R_0 > 0$. Then T restricts to a bounded operator from $\mathrm{L}^{q,\lambda}(\Omega)$ to $\mathrm{L}^{q,\lambda}(\Omega)$ with operator norm depending on d, q, λ, ν, R_0, Ω, the constant in the weak reverse Hölder estimates and the operator norm of T on $\mathrm{L}^2(\Omega)$ and on $\mathrm{L}^{2,\lambda}(\Omega)$.

Proof. The strategy of the proof is to define an operator acting on the full space which satisfies the assumptions of Theorem 3.2.2. The natural way is to proceed as follows: We define $\mathcal{T}$ on $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$ by

$$\mathcal{T} f \coloneqq E_0(Tf|_\Omega),$$

where E_0 denotes extension by zero as in Lemma 2.2.14. First, we need to check that $\mathcal{T}$ is in fact well-defined. Clearly, $f|_\Omega \in \mathrm{L}^2(\Omega)$ holds true for $f \in \mathrm{L}^2(\mathbb{R}^d)$. Taking a closer look at the definition of the Morrey-norm, also $f|_\Omega \in \mathrm{L}^{2,\lambda}(\Omega)$ with $\|f\|_{\mathrm{L}^{2,\lambda}(\Omega)} \leq \|f|_\Omega\|_{\mathrm{L}^{2,\lambda}(\mathbb{R}^d)} \leq \|f\|_{\mathrm{L}^{2,\lambda}(\mathbb{R}^d)}$ is clear. By assumption on T we then know $T(f|_\Omega) \in \mathrm{L}^2(\Omega) \cap \mathrm{L}^{2,\lambda}(\Omega)$. As the extension by zero does not change the L^2-norm, we conclude $\mathcal{T} f \in \mathrm{L}^2(\mathbb{R}^d)$. Using Lemma 2.2.14 we infer $\mathcal{T} f \in \mathrm{L}^{2,\lambda}(\mathbb{R}^d)$. Moreover, the previous arguments imply that $\mathcal{T}$ is bounded on $\mathrm{L}^2(\mathbb{R}^d)$ and on $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$. Hence, it remains to verify the uniform weak reverse Hölder inequality.

First of all, we note that the inequality is fulfilled for R large enough due to the condition $f \equiv 0$ in $B(x_0, 2R) \cap \Omega$ and this is where the boundedness of Ω is crucial. To be precise, assume that $R > \operatorname{diam}(\Omega)$ and $B(x_0, R) \cap \Omega \neq \emptyset$ (otherwise the inequality is trivially true). By choice of R we then have $B(x_0, 2R) \supset \Omega$ and thus $f \equiv 0$ in Ω. By definition of $\mathcal{T}$ we deduce $\mathcal{T} f = 0$ and the inequality holds. Hence, we already showed uniform weak reverse Hölder estimates for $x_0 \in \mathbb{R}^d$ and $0 < 2r \leq R$ whenever $R > \operatorname{diam}(\Omega)$.

Next, we show that the inequality is satisfied for all $x_0 \in \mathbb{R}^d$ and $0 < 8r \leq R < \tilde{R}_0$ for some $\tilde{R}_0 > 0$. In case of $B(x_0, r) \cap \Omega = \emptyset$ there is nothing to show. Otherwise, if $B(x_0, r) \cap \partial\Omega \neq \emptyset$ there exists $y \in B(x_0, r) \cap \Omega$ and as $B(y, R) \subseteq B(x_0, 2R)$ the uniform weak reverse Hölder estimates from the assumption apply. Hence, we have

$$\begin{aligned}
\left(\int_{B(x_0,r)\cap\Omega} |Tf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} &\leq \left(\int_{B(y,2r)\cap\Omega} |Tf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\leq C(2r)^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{4r}{R}\right)^{\mu} \int_{B(y,R/2)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq C(2r)^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{4r}{R}\right)^{\mu} \int_{B(x_0,R)\cap\Omega} |Tf|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned}$$

For the second step to be valid the following inequalities need to hold

$$4r \leq \frac{R}{2} \leq \frac{\tilde{R}_0}{2} \leq R_0$$

The first and second one are true by choice of r and R. The last one is true if we choose for example $\tilde{R}_0 := 2R_0$. Hence, we established the uniform weak reverse Hölder inequality for all $x_0 \in \mathbb{R}^d$ and all $0 < 8r \leq R < 2R_0$. By Lemma 3.1.8 this is sufficient to conclude the proof. □

Next, under slightly weaker conditions we show local $\mathrm{L}^{p,\lambda}$-regularity. Note that the weakening consists of not having uniform weak reverse Hölder estimates for balls centred on $\partial\Omega$.

Theorem 3.2.4. *Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable and $K \subset \Omega$ compact. Let $\mu > 0$, $0 \leq \lambda < \min\{\mu, d\}$ and $q > 2$ satisfy*

$$\frac{\lambda}{q} + d\left(\frac{1}{2} - \frac{1}{q}\right) < \frac{\mu}{2}.$$

Let T be a linear operator mapping $\mathrm{L}^2(\Omega)$ boundedly into $\mathrm{L}^2(K)$ and $\mathrm{L}^{2,\lambda}(\Omega)$ boundedly into $\mathrm{L}^{2,\lambda}(K)$ such that there exists a constant $C > 0$ such that for all $x \in \Omega$ and all $0 < 2r \leq R$ with $B(x, R) \subset \Omega$ and all $f \in \mathrm{L}^{2,\lambda}(\Omega)$ with $f \equiv 0$ on $B(x, 2R) \cap \Omega$ it holds

$$\left(\int_{B(x,r)\cap\Omega} |Tf|^q \,\mathrm{d}y\right)^{\frac{1}{q}} \leq C r^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left(\left(\frac{r}{R}\right)^{\mu} \int_{B(x,R)\cap\Omega} |Tf|^2 \,\mathrm{d}y\right)^{\frac{1}{2}}.$$

Then T restricts to a bounded operator from $\mathrm{L}^{q,\lambda}(\Omega)$ to $\mathrm{L}^{q,\lambda}(K)$ with operator norm depending on d, q, λ, ν, K, Ω, C and the operator norm of T on $\mathrm{L}^2(\Omega)$ and on $\mathrm{L}^{2,\lambda}(\Omega)$.

Proof. We proceed as in the proof of the previous theorem. Define on $\mathrm{L}^2(\mathbb{R}^d)$ the operator $\mathcal{T}f := E_0(Tf|_\Omega)$. Again it remains to verify the uniform weak reverse Hölder inequality for $x \in \mathbb{R}^d$ and $0 < 2r \leq R < R_0$ for some chosen R_0. We first remark that for $R_0 := \frac{1}{3}\operatorname{dist}(K, \partial\Omega)$ and $x \in \mathbb{R}^d$ one of following two scenarios occurs. We either have $B(x, R_0) \cap K = \emptyset$ or $B(x, R_0) \subset \Omega$. In the first case there is nothing to show as the inequality is trivially true. In the second case the inequality holds for $0 < 2r \leq R < R_0$ by assumption. One can then proceed as in the proof of Theorem 3.2.3 to conclude the proof. □

The next result was at first only considered for $p = 2$ in order to show that the assumption of $\mathrm{L}^{2,\lambda}(\mathbb{R}^d)$-boundedness is superfluous (at least for a certain λ). However, it turns out that the results remains true for general p which is particular useful to deal with Morrey spaces $\mathrm{L}^{p,\mu}$ where $p < 2$.

Theorem 3.2.5. *Let $p \in [1,\infty)$, $q > p$ and $0 < \lambda < d - \frac{dp}{q}$. Let $T\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d)$ be a linear and bounded operator satisfying L^p-weak reverse Hölder estimates with exponent q and $R_0 = \infty$. Then the restriction $T\colon \mathrm{L}^p(\mathbb{R}^d) \cap \mathrm{L}^{p,\lambda}(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d) \cap \mathrm{L}^{p,\lambda}(\mathbb{R}^d)$ is bounded with respect to the $\mathrm{L}^{p,\lambda}(\mathbb{R}^d)$-norm. The operator norm of T depends on p, q, λ, the constant in the L^p-weak reverse Hölder estimate and the L^p-norm of T.*

Proof. Let $f \in \mathrm{L}^p(\mathbb{R}^d) \cap \mathrm{L}^{p,\lambda}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ and $r > 0$ be arbitrary but fixed. For $R > r$ we have

$$\int_{B(x_0,r)} |Tf|^p \,\mathrm{d}x \lesssim \int_{B(x_0,r)} |T(f\chi_{B(x_0,3R)})|^p \,\mathrm{d}x + \int_{B(x_0,r)} |T(f(1-\chi_{B(x_0,3R)}))|^p \,\mathrm{d}x.$$

Again, we estimate both terms separately. By virtue of the L^p-boundedness of T we can estimate the first one as

$$\int_{B(x_0,r)} |T(f\chi_{B(x_0,3R)})|^p \,\mathrm{d}x \lesssim R^\lambda \|f\|^p_{\mathrm{L}^{p,\lambda}(\mathbb{R}^d)}.$$

For the second one, we use the condition $r < R$ together with Hölder's inequality and the L^p-weak reverse Hölder estimates (3.2) to obtain

$$\begin{aligned}
&\left(\int_{B(x_0,r)} |T(f(1-\chi_{B(x_0,3R)}))|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\
&\qquad \lesssim r^{\frac{d}{p}-\frac{d}{q}} R^{\frac{d}{q}} \left(\frac{1}{R^d} \int_{B(x_0,R)} |T(f(1-\chi_{B(x_0,3R)}))|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\qquad \lesssim r^{\frac{d}{p}-\frac{d}{q}} R^{\frac{d}{q}} \left(\frac{1}{R^d} \int_{B(x_0,2R)} |T(f(1-\chi_{B(x_0,3R)}))|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the p-th power and using the linearity of T we get

$$\begin{aligned}
&\int_{B(x_0,r)} |T(f(1-\chi_{B(x_0,3R)}))|^p \,\mathrm{d}x \\
&\qquad \lesssim r^{d-\frac{dp}{q}} R^{\frac{dp}{q}} \left(\frac{1}{R^d} \int_{B(x_0,2R)} |T(f(1-\chi_{B(x_0,3R)}))|^p \,\mathrm{d}x \right) \\
&\qquad \lesssim \left(\frac{r}{R}\right)^{d-\frac{dp}{q}} \int_{B(x_0,2R)} |Tf|^p \,\mathrm{d}x + \int_{B(x_0,2R)} |T(f\chi_{B(x_0,3R)})|^p \,\mathrm{d}x.
\end{aligned}$$

Note that $r < R$ and $d - \frac{dp}{q} > 0$, so we were able estimate $\left(\frac{r}{R}\right)^{d-\frac{dp}{q}} \leq 1$ in the second summand. Combining this with the estimate for the first term yields

$$\int_{B(x_0,r)} |Tf|^p \,\mathrm{d}x \lesssim \left(\frac{r}{R}\right)^{d-\frac{dp}{q}} \int_{B(x_0,R)} |Tf|^p \,\mathrm{d}x + R^\lambda \|f\|^p_{\mathrm{L}^{p,\lambda}(\mathbb{R}^d)}.$$

Defining $\phi(s) := \int_{B(x_0,s)} |Tf|^p \, \mathrm{d}x$, we are in the situation of [47, Lemma 5.13], provided that $d - \frac{dp}{q} > \lambda$. Applying the lemma we deduce

$$\frac{1}{r^\lambda} \int_{B(x_0,r)} |Tf|^p \, \mathrm{d}x \lesssim \frac{1}{R^\lambda} \int_{B(x_0,2R)} |Tf|^p \, \mathrm{d}x + \|f\|^p_{\mathrm{L}^{p,\lambda}(\mathbb{R}^d)}.$$

Next, use the L^p-boundedness of T and $R \to \infty$ to obtain

$$\frac{1}{r^\lambda} \int_{B(x_0,r)} |Tf|^p \, \mathrm{d}x \lesssim \|f\|^p_{\mathrm{L}^{p,\lambda}(\mathbb{R}^d)}.$$

Taking the p-th root and the supremum with respect to r and x_0 completes the proof. □

A result similar to Theorem 3.2.5 also holds for bounded sets.

Corollary 3.2.6. *Let $\Omega \subseteq \mathbb{R}^d$ be bounded and measurable. Let $p \in [1, \infty)$, $q > p$ and $0 < \lambda < d - \frac{dp}{q}$ and let $T \colon \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega)$ be a linear bounded operator satisfying L^p-weak reverse Hölder estimates with exponent q and radius $R_0 > 0$. Then the restriction of $T \colon \mathrm{L}^{p,\lambda}(\Omega) \to \mathrm{L}^{p,\lambda}(\Omega)$ is bounded with respect to the $\mathrm{L}^{p,\lambda}(\Omega)$ norm with operator norm depending on d, p, q, λ, R_0, Ω, the constant in the L^p-weak reverse Hölder estimate and the operator norm of T on $\mathrm{L}^p(\Omega)$.*

Proof. We show that the operator $\mathcal{T}$ defined on $\mathrm{L}^p(\mathbb{R}^d)$ via $\mathcal{T}f := E_0(T(f_{|\Omega}))$ satisfies the weak reverse Hölder estimates for all radii. Here E_0 denotes as before the extension by zero outside of Ω. For large R this follows from the boundedness of Ω as explained in the proof of Theorem 3.2.3. For the intermediate radii we appeal to [90, Lemma 3.2.3]. Note that the lemma remains true, with an analogous proof, for L^p-weak reverse Hölder estimates. Now, the claim follows from Theorem 3.2.5. □

To complete this chapter we formulate two results on how we will later use the extrapolation results presented here.

Theorem 3.2.7. *Let $\Omega \subseteq \mathbb{R}^d$ be bounded and open. Let $T \colon \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ be bounded and linear. Assume further that T satisfies weak reverse Hölder estimates for all exponents $2 < q < \infty$ and radius $R_0 > 0$. Then for all $p \geq 2$ and all $0 \leq \lambda < d$, T restricts to a bounded operator from $\mathrm{L}^{p,\lambda}(\Omega) \to \mathrm{L}^{p,\lambda}(\Omega)$.*

Proof. We plan on using the previous corollary. Let $p \geq 2$ and $0 \leq \lambda < d$ be arbitrary. Then, the L^p-boundedness is a consequence of the L^2-boundedness and the weak reverse Hölder estimates as Tolksdorf [90, Theorem 3.1.2] generalized Shen's extrapolation theorem to hold on arbitrary open sets Ω instead only the full space or bounded Lipschitz domains. As the weak reverse Hölder estimates hold by assumption for all exponents $q > 2$, they particularly hold for q large enough so that $\lambda < d - \frac{dp}{q}$ is satisfied. Note that we assumed only L^2-weak reverse Hölder estimates but the previous corollary requires L^p-weak reverse Hölder estimates. However, these follow due to Theorem 3.1.6. Hence, we can apply the previous corollary which immediately yields the desired result. $\square$

In applications, our operators may not satisfy reverse Hölder inequalities for all $2 < q < \infty$ but only for $2 < q < q_0$ for some $q_0 > 2$. Under these assumptions we have the following result.

Theorem 3.2.8. *Let $\Omega \subseteq \mathbb{R}^d$ be bounded and open. Let $T \colon \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ be bounded and linear. Assume further that for some $q_0 > 0$ the operator T satisfies weak reverse Hölder estimates for all exponents $2 < q < q_0 < \infty$ and radius $R_0 > 0$. Then for all $2 \leq p < q_0$ and all $0 \leq \lambda < d - \frac{dp}{q_0}$, T restricts to a bounded operator from $\mathrm{L}^{p,\lambda}(\Omega) \to \mathrm{L}^{p,\lambda}(\Omega)$.*

Proof. The proof essentially is the same as the one of the previous theorem. However, in the light of the previous proof, the restriction on the exponents in weak reverse Hölder estimates directly corresponds to the range of p and λ stated in this theorem. $\square$

CHAPTER 4

Applications

4.1 Elliptic operators on Morrey spaces

In this section we examine elliptic operators in divergence form on Morrey spaces. As we work on Lipschitz domains, boundary conditions are required. In this thesis Dirichlet as well as Neumann boundary conditions are considered. Our results are similar in both cases even though for Neumann boundary conditions some results require the restriction to mean-value free functions. Our goal is to obtain a solution theory for semilinear parabolic problems in Morrey spaces similar to the original result by Giga [50] and the refinement by Tolksdorf [90]. In this context an $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimate is necessary. With the methods presented here, those are available only in dimension 3. We later discuss this issue in more depth. From now on, unless stated otherwise, let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain.

This section is structured as follows. On $\mathrm{L}^2(\Omega)$ we introduce elliptic operators in divergence form subject to Dirichlet or Neumann boundary conditions via form theory. For an introduction to form theory we refer to the book by Kato [60]. This approach has the advantage that both boundary conditions can be considered simultaneously. Building on an L^2-theory, weak reverse Hölder estimates provide an L^p-theory in the light of Shen's extrapolation theorem [84, Theorem 3.3]. For Neumann boundary conditions those were proved by Wei and Zhang in [97] while for Dirichlet boundary conditions they are consequences of results by Shen [82]. As we were not able to find the latter results in the literature, we provide a proof in this thesis.

4.1.1 L^2-theory and weak reverse Hölder estimates

As mentioned in the introduction of this section in the L^2-setting we use form theory. Here, we consider systems of m equations. Consequently, we write $\mathrm{L}^p(\Omega)$ for the space of L^p-functions with values in $\mathbb{R}^m$ and do so analogously for other function spaces. Let $\mathrm{W}^{1,2}_0(\Omega) \subseteq V \subseteq \mathrm{W}^{1,2}(\Omega)$. For later purposes we only consider the cases $V = \mathrm{W}^{1,2}_0(\Omega)$ which corresponds to Dirichlet boundary conditions and $V = \mathrm{W}^{1,2}(\Omega)$ which corresponds to Neumann boundary conditions. Other boundary conditions can be encoded via other choices of V. Define the form

$$
\begin{aligned}
a \colon V \times V &\to \mathbb{C} \\
(u, v) &\mapsto \int_\Omega \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{3} a_{ij}^{\alpha\beta} \partial_j u^\beta \overline{\partial_i v^\alpha} \, \mathrm{d}x
\end{aligned}
$$

where the coefficients $a_{ij}^{\alpha\beta}$ are constant, real, symmetric and strongly elliptic. By the latter we mean that there exists $\Lambda > 0$ such that

$$
\Lambda |\xi|^2 \leq \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{3} a_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \leq \Lambda^{-1} |\xi|^2
$$

holds for all $\xi = (\xi_i^\alpha) \in \mathbb{R}^{3m}$. Note that this assumption is as in [97] and is stronger than the Legendre-Hadamard condition in [82]. For further discussion on this topic we refer to [97, Remark 1.4]. Then, by classical form theory, see [60, Chapter 6], this gives rise to a sectorial operator A_2 on $\mathrm{L}^2(\Omega)$ defined via

$$
\begin{aligned}
\mathcal{D}(A_2) &\coloneqq \Big\{ u \in V \colon \exists! f \in \mathrm{L}^2(\Omega) \colon a(u, v) = \int_\Omega f \cdot \overline{v} \, \mathrm{d}x \ \forall v \in V \Big\} \\
A_2 u &\coloneqq f.
\end{aligned}
$$

For $2 < p < \infty$, we define the operator A_p as the part of A_2 in $\mathrm{L}^p(\Omega)$:

$$
\begin{aligned}
\mathcal{D}(A_p) &\coloneqq \{ u \in \mathcal{D}(A_2) \cap \mathrm{L}^p(\Omega) \colon A_2 u \in \mathrm{L}^p(\Omega) \} \\
A_p u &\coloneqq A_2 u.
\end{aligned}
$$

For $1 < p < 2$ we use duality to define A_p. It is now natural to ask whether A_p is sectorial, too. Answering this questions requires knowledge of the behaviour of the resolvent of A_p on $\mathrm{L}^p(\Omega)$ which hints that Shen's L^p-extrapolation theorem might be of use.

Next, we introduce the relevant resolvent problems. Let $\theta \in (\pi/2, \pi)$ and for $\lambda \in \mathrm{S}_\theta = \{ z \in \mathbb{C} \setminus \{0\} \colon |\arg(z)| < \theta \}$ consider the problem

$$
\lambda u + A_2 u = f \tag{4.1}
$$

for $f \in \mathrm{L}^2(\Omega)$. For $V = \mathrm{W}^{1,2}(\Omega)$ this is the Neumann resolvent problem which then reads

$$\begin{aligned} \lambda u - \operatorname{div}(L\nabla u) &= f \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4.2}$$

where

$$\left(\frac{\partial u}{\partial \nu}\right)^{\alpha} := \sum_{\beta=1}^{m} \sum_{i,j=1}^{3} a_{ij}^{\alpha\beta} \partial_j u^{\beta} N_i, \quad \alpha = 1, 2, \dots, m. \tag{4.3}$$

Here $N = (N_1, N_2, N_3)$ is the outward unit normal to $\partial\Omega$. For $V = \mathrm{W}_0^{1,2}(\Omega)$, (4.1) is the Dirichlet resolvent problem:

$$\begin{aligned} \lambda u - \operatorname{div}(L\nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.4}$$

In the Dirichlet case also $\lambda = 0$ is allowed. In both cases L represents the coefficients of the corresponding operator.

Returning to the question of sectoriality of A_p, recall that sectoriality of A_2 includes L^2-boundedness of the resolvents. Hence for an application of Shen's extrapolation theorem, it remains to show weak reverse Hölder estimates. In [97], Wei and Zhang considered the case of Neumann boundary conditions and obtained weak reverse Hölder estimates for the resolvent and its gradient. Note that in [82], Shen uses Green functions to obtain L^p-boundedness of the resolvent. However, his paper contains all necessary ingredients to prove weak reverse Hölder estimates which will be useful for Morrey space theory, too. Before we provide a proof of those, let us now precisely state the aforementioned result by Wei and Zhang [97, Proposition 3.1].

Theorem 4.1.1. *Let $V = \mathrm{W}^{1,2}(\Omega)$, $\theta \in (\pi/2, \pi)$ and $r_0 > 0$ the radius from the definition of Lipschitz domains, see Definition 2.3.1. Then there exists $\varepsilon > 0$ depending only on θ, Ω and Λ such that for all $\lambda \in \mathrm{S}_\theta$ the following holds. For all $x_0 \in \overline{\Omega}$ and $0 < r \le r_0$ satisfying either $x_0 \in \partial\Omega$ or $B(x_0, 3r) \subseteq \Omega$ and all $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ the solution u to (4.2) satisfies*

$$\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \quad \forall\, 2 < q < 3 + \varepsilon, \tag{4.5}$$

$$\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \quad \forall\, 2 < q < \infty. \tag{4.6}$$

The implicit constants only depend on q, θ, Ω and Λ.

For the Dirichlet resolvent problem the general idea is to follow the strategy of [97] but using the results from [82]. Here, we note that instead of the sets used by Wei and Zhang we work with Lipschitz cylinders. The interior estimate is the same as the first part of the proof of [97, Proposition 3.1] and also the boundary case follows the idea presented there.

Theorem 4.1.2. *Let $V = \mathrm{W}^{1,2}_0(\Omega)$, $\theta \in (\pi/2, \pi)$ and $r_0 > 0$ the radius from the definition of Lipschitz domains, see Definition 2.3.1. Then there exists $\varepsilon > 0$ depending only on θ, Ω and Λ such that the following holds for all $\lambda \in \mathrm{S}_\theta \cup \{0\}$. For all $x_0 \in \overline{\Omega}$ and $0 < r \leq r_0$ satisfying either $x_0 \in \partial\Omega$ or $B(x_0, 3r) \subseteq \Omega$ and all $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ the solution u to (4.4) satisfies*

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \quad \forall\, 2 < q < 3 + \varepsilon, \tag{4.7}$$

$$\left(\frac{1}{r^d}\int_{B(x_0,r)\cap\Omega} |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \quad \forall\, 2 < q < \infty. \tag{4.8}$$

The implicit constants only depend on q, θ, Ω and Λ.

Proof. Interior balls are taken care of by the point-wise estimate in [97, Lemma 3.2] and the argumentation in [97, Proposition 3.1]. For $x_0 \in \partial\Omega$, after a rotation and translation, we may assume $x_0 = 0$. As Ω is a Lipschitz domain, there exists a Lipschitz function $\eta\colon \mathbb{R}^2 \to \mathbb{R}$ with $\eta(0) = 0$ such that for all $0 < r \leq r_0$ we have

$$D(r) \cap (\Omega \setminus \{0\}) = \{(x', x_3) \in \mathbb{R}^3 \colon |x'| < r, \eta(x') < x_3 < 30(M+1)r\} = D_\eta(r)$$

where $M \geq \|\nabla\eta\|_{\mathrm{L}^\infty(\mathbb{R}^2)}$ and $D_\eta(r)$ denotes the Lipschitz cylinder introduced in Definition 2.3.1. Moreover, recall

$$I_\eta(r) \coloneqq \{(x', x_3) \in \mathbb{R}^3 \colon |x'| < r, \eta(x') = x_3\}$$

and that a Lipschitz cylinder is a Lipschitz domain. We will first prove the desired estimate on the level of Lipschitz cylinders and then argue why this is sufficient. By scaling it suffices to consider $r = 1$ as the scaled function $\eta_r(x) \coloneqq r^{-1}\eta(rx)$ also satisfies $\|\nabla\eta_r\|_{\mathrm{L}^\infty(\mathbb{R}^2)} \leq M$ and so the Lipschitz constant is invariant under scaling. Let now the function $f \in \mathrm{L}^2(\Omega)$ satisfy $f \equiv 0$ on $D_\eta(6)$. For $x \in \partial\Omega$ introduce the non-tangential maximal function $(\nabla u)^*$ defined by

$$(\nabla u)^*(x) \coloneqq \sup\{|\nabla u(y)| \colon y \in \Omega \text{ s.t. } |y - x| < 2\,\mathrm{dist}(y, \partial\Omega)\}.$$

For $s \in (1,2)$ by [85, Equation (5.20)] and [82, Theorem 0.12] we have

$$\begin{aligned}\left(\int_{D_\eta(1)} |\nabla u|^3 \,\mathrm{d}x\right)^{\frac{2}{3}} &\lesssim \int_{\partial D_\eta(s)} |(\nabla u)^*|^2 \,\mathrm{d}\sigma(x) \\ &\lesssim \int_{\partial D_\eta(s)} |\nabla_{\tan} u|^2 + ||\lambda|^{\frac{1}{2}} u|^2 + |u|^2 \,\mathrm{d}\sigma(x) \\ &= \int_{\partial D_\eta(s)\setminus I_\eta(2)} |\nabla_{\tan} u|^2 + |\lambda||u|^2 + |u|^2 \,\mathrm{d}\sigma(x).\end{aligned}$$

Here $\nabla_{\tan}$ denotes the tangential derivative. Both this term and u vanish on $\partial\Omega$ and consequently on $I_\eta(r)$ by the boundary condition so that the last equality is justified. Next, apply Lemma 2.3.6 and use $|\nabla_{\tan} u| \leq |\nabla u|$ to obtain

$$\left(\int_{D_\eta(1)} |\nabla u|^3 \,\mathrm{d}x\right)^{\frac{2}{3}} \lesssim \int_{D_\eta(2)} |\nabla u|^2 \,\mathrm{d}x + \int_{D_\eta(2)} |\lambda||u|^2 \,\mathrm{d}x + \int_{D_\eta(2)} |u|^2 \,\mathrm{d}x.$$

In order to get rid of the first two terms we apply the Caccioppoli inequality [82, Lemma 2.1] for $k = 0$ (which is not stated there but is proved first) to obtain

$$\left(\int_{D_\eta(1)} |\nabla u|^3 \,\mathrm{d}x\right)^{\frac{1}{3}} \lesssim \left(\int_{D_\eta(4)} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.9}$$

In light of working in dimension 3, an application of Sobolev's inequality implies that for all $2 < q < \infty$ we have

$$\left(\int_{D_\eta(1)} |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\int_{D_\eta(4)} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.10}$$

Moreover, for the gradient note that $u = 0$ on $\partial D(4) \cap \partial\Omega$ by the boundary condition on u, so an application of Poincaré's inequality (see Proposition 2.3.12) together with Jensen's inequality to (4.9) yields

$$\left(\int_{D_\eta(1)} |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\int_{D_\eta(4)} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{4.11}$$

for all $2 < q \leq 3$. This concludes the proof on the level of Lipschitz cylinders for $r = 1$. Let us now make the scaling argument precise. To this end let $0 < r \leq r_0/6$ and $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $D_\eta(6r)$. Then, $u_r(y) = u(ry)$ is the solution of (4.4) with right-hand side $f_r(y) = f(ry)$ and f_r satisfies $f_r \equiv 0$ on $D_\eta(6)$. Thus, the transformation rule implies

$$\left(\frac{1}{r^d} \int_{D_\eta(r)} |u(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} = \left(\int_{D_{\eta_r}(1)} |u(ry)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Applying the previously proven estimate and again the transformation rule implies weak reverse Hölder estimates on Lipschitz cylinders.

Now, let us argue that it suffices to consider Lipschitz cylinders. Note that both

$$\begin{aligned} B(x_0, r) \cap \Omega &\subseteq D_\eta(r) \\ D_\eta(4r) &\subseteq B(x_0, \alpha r) \cap \Omega \end{aligned}$$

hold where $\alpha \coloneqq \sqrt{3^2 \cdot 10^2 (M+1)^2 + 4^2} > 3$. Then if $f \equiv 0$ on $B(x_0, (\alpha+4)r) \cap \Omega$, also $f \equiv 0$ on $D_\eta(6r)$ follows. Hence, for $2 < q < \infty$ we obtain weak reverse Hölder estimates of the form

$$\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |u|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d} \int_{B(x_0,\alpha r)\cap\Omega} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

under the condition $f \equiv 0$ on $B(x_0, (\alpha+4)r) \cap \Omega$. Applying [90, Lemma 3.2.2] allows to change the factor α on the right-hand side of the integral to 2, while Lemma 3.1.11 changes the vanishing condition to $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. The condition $0 < r \leq r_0$ instead of $0 < r \leq r_0/6$ is obtained by [90, Lemma 3.2.3]. This proves the desired result in the range $2 < q < \infty$ for u and in the range $2 < q \leq 3$ for ∇u.

Finally, the self-improving property of weak reverse Hölder estimates, see Remark 3.1.2, improves the range for exponents for the gradient to $2 < q < 3 + \varepsilon$ for some $\varepsilon > 0$. □

Remark 4.1.3.

- Note that the above estimates are not the weak reverse Hölder estimates we introduced in Section 3.1. However, recall that Lemma 3.1.3 shows that the above estimates indeed imply weak reverse Hölder estimates. In particular, in view of [90, Lemma 3.2.3] the constant R_0 is not of great importance and for any $\tilde{R}_0 > 0$ there exists an implicit constant such that weak reverse Hölder estimates hold for any $0 < r < \tilde{R}_0$.

- Higher dimensions, i.e. $d \geq 4$, can also be considered. Then, the weak reverse Hölder estimates for the gradient hold for $2 < q < \frac{2d}{d-1} + \varepsilon$ and for the resolvent for $2 < q < \frac{2d}{d-3} + \varepsilon$. For the Neumann problem this was proved by Wei and Zhang and for the Dirichlet problem it is contained in our proof analogously to the Neumann problem.

Working in dimension 3 allows to obtain L^2-L^∞-weak reverse Hölder estimates by combining weak reverse Hölder estimates for $(\lambda + A_2)^{-1}$ and $\nabla(\lambda + A_2)^{-1}$ with the Gagliardo-Nirenberg inequality. Note that Neumann boundary conditions require more attention as, unlike for Dirichlet boundary conditions, an extension by zero operator is not available.

Lemma 4.1.4. *Let $\theta \in (\pi/2, \pi)$ and $V = \mathrm{W}^{1,2}(\Omega)$ or $V = \mathrm{W}^{1,2}_0(\Omega)$. Then for the radius r_0 from the definition of a Lipschitz domain the following holds for all $\lambda \in \mathrm{S}_\theta$ if $V = \mathrm{W}^{1,2}(\Omega)$ or $\lambda \in \mathrm{S}_\theta \cup \{0\}$ if $V = \mathrm{W}^{1,2}_0(\Omega)$. For all $x_0 \in \overline{\Omega}$ and $0 < r \leq r_0$ and $f \in \mathrm{L}^2(\Omega)$ satisfying $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ the solution u to (4.1) satisfies*

$$\|u\|_{\mathrm{L}^\infty(B(x_0,r)\cap\Omega)} \lesssim \left(\frac{1}{r^d}\int_{B(x_0,2r)\cap\Omega} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The implicit constant only depends on θ, Ω and Λ.

Proof. By a covering argument in the spirit of Lemma 3.1.3 and [90, Lemma 3.2.2] it suffices to consider the cases $x_0 \in \partial\Omega$ or $B(x_0, 3r) \subseteq \Omega$. The interior estimate is a consequence of [97, Lemma 3.2].

For balls centred on the boundary, the idea is to estimate the L^∞-norm by the Gagliardo-Nirenberg inequality on $\mathbb{R}^3$. Note that for $x_0 \in \partial\Omega$ and $r > 0$ the intersection $B(x_0, r) \cap \Omega$ is not necessarily a domain so that Stein's extension theorem is of no use here. For Dirichlet boundary conditions extension by zero is possible. Recall that we denote this operator by E_0. For Neumann boundary conditions however, as already mentioned, a different approach is necessary. Let us first consider the case of Dirichlet boundary conditions as this serves as a blueprint for how to proceed for Neumann boundary conditions.

Let $x_0 \in \partial\Omega$, $0 < r \leq r_0$ where $r_0 > 0$ is the constant from Definition 2.3.1 belonging to Ω and $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, r) \cap \Omega$. Moreover, denote by u the solution to (4.1). By a scaling argument as in the proof of Theorem 4.1.2 it suffices to consider $r = 1$. Let $p = 3 + \varepsilon/2$ with $\varepsilon > 0$ as in Theorem 4.1.2 and set $\vartheta \coloneqq 3/p < 1$. Note that then p depends on θ, Ω and Λ. Then, the extension by zero and the Gagliardo-Nirenberg inequality [74] deliver

$$\begin{aligned}\|u\|_{\mathrm{L}^\infty(B(x_0,1)\cap\Omega)} &\lesssim \|E_0 u\|^{\vartheta}_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \|E_0 u\|^{1-\vartheta}_{\mathrm{L}^p(\mathbb{R}^d)} \\ &\lesssim \left(\int_{B(x_0,1)\cap\Omega} |u|^p + |\nabla u|^p \,\mathrm{d}x\right)^{\frac{\vartheta}{p}} \left(\int_{B(x_0,1)\cap\Omega} |u|^p \,\mathrm{d}x\right)^{\frac{1-\vartheta}{p}},\end{aligned}$$

where the implicit constant depends on θ, Ω and Λ. Now apply the weak reverse Hölder estimates from Theorem 4.1.2. In the first factor use [90, Lemma 3.2.2] to obtain a ball with radius $3/2$ instead of radius 2 as the domain of integration on the right-hand side. This yields

$$\|u\|_{\mathrm{L}^\infty(B(x_0,1)\cap\Omega)} \lesssim \left(\int_{B(x_0,3/2)\cap\Omega} |u|^2 + |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{\vartheta}{2}} \left(\int_{B(x_0,2)\cap\Omega} |u|^2 \,\mathrm{d}x\right)^{\frac{1-\vartheta}{2}}, \tag{4.12}$$

where the implicit constant depends on p, θ, Ω and Λ. Finally, appeal to Caccioppoli's inequality [82, Lemma 2.1] to estimate

$$\left(\int_{B(x_0,3/2)\cap\Omega} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim \left(\int_{B(x_0,2)\cap\Omega} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \tag{4.13}$$

where the implicit constant depends on Λ. Combining (4.12) and (4.13) then yields the desired weak reverse Hölder estimates for $r = 1$.

For Neumann boundary conditions we employ an extension procedure described by Tolksdorf in [90, Proposition 7.1.3]. Note that in that chapter of his thesis a more general definition of Lipschitz domains was used, see [90, Assumption 7.1.1 and Remark 7.1.2 (1)]. It is explained in [53, p. 7f.] that the there used definition is indeed more general than ours. Moreover, Tolksdorf introduces a constant $\tilde{M} \geq 1$ which in his setting bounds the Lipschitz constant of the coordinate charts. In particular, this constant $\tilde{M}$ corresponds to the Lipschitz constant M in our setting.

His extension operator, which essentially is an even reflection, operates on functions defined on a set $U_{x_0,r}$, for $x_0 \in \partial\Omega$ and $0 < r \leq 1/4$, instead of a ball $B(x_0, r)$. For a precise definition we refer to [90, Remark 7.1.2 (2)]. Important for our work is that these sets satisfy the following relation, see [90, Remark 7.1.2 (3)]:

$$B(x_0, r) \subseteq U_{x_0, M\sqrt{3}r} \subseteq B(x_0, M^2\sqrt{3}r).$$

This then serves to extend functions, defined on Ω, from $B(x_0, r) \cap \Omega$ to $B(x_0, r)$. From this ball we then can extend to $\mathbb{R}^d$ via Stein's extension theorem and use the Gagliardo-Nirenberg inequality.

Again by the aforementioned scaling argument it suffices to cover $r = 1$. Let $f \in \mathrm{L}^2(\Omega)$ satisfy $f \equiv 0$ on $B(x_0, 3\alpha r) \cap \Omega$ with $\alpha = M^2\sqrt{3} > 1$. Denote by u the solution to (4.2) with right-hand side f. As for Dirichlet boundary conditions, set $p \coloneqq 3 + \varepsilon/2$ with $\varepsilon > 0$ as in Theorem 4.1.1 so that weak reverse Hölder estimates for the gradient hold. With $\vartheta = 3/p < 1$ we obtain by Gagliardo-Nirenberg and the $\mathrm{W}^{1,p}$-boundedness of Tolksdorf's extension operator that

$$\|u\|_{\mathrm{L}^\infty(B(x_0,1)\cap\Omega)} \lesssim \|u\|^{\vartheta}_{\mathrm{W}^{1,p}(B(x_0,M^2\sqrt{3})\cap\Omega)} \|u\|^{1-\vartheta}_{\mathrm{L}^p(B(x_0,M^2\sqrt{3})\cap\Omega)}$$

holds true with implicit constant depending on θ, Ω and Λ. Now proceed as in the Dirichlet case, i.e., use the weak reverse Hölder estimates from Theorem 4.1.1 as well as Caccioppoli's inequality for Neumann boundary conditions [97, Corollary 3.1]. This yields weak reverse Hölder estimates of the form

$$\|u\|_{\mathrm{L}^\infty(B(x_0,1)\cap\Omega)} \lesssim \left(\int_{B(x_0,2\alpha)\cap\Omega} |u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \tag{4.14}$$

for $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ on $B(x_0, 3\alpha)$. The implicit constant depends on θ, Ω and Λ.

Finally, apply [90, Lemma 3.2.2] to change the radius on the right-hand side of (4.14) to 2 and Lemma 3.1.11 for (4.14) to hold under the condition $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. Note that while in both results an L^q-norm with finite q is considered, the same proofs apply for the L^∞-norm as well. $\square$

4.1.2 L^p-theory

In view of the weak reverse Hölder estimates (Theorems 4.1.1 and 4.1.2) together with Shen's extrapolation theorem the following result can be obtained.

Theorem 4.1.5. *Let $V = \mathrm{W}^{1,2}_0(\Omega)$ or $V = \mathrm{W}^{1,2}(\Omega)$ and $1 < p < \infty$. Then, the operator A_p is sectorial of angle 0 on $\mathrm{L}^p(\Omega)$ and hence $-A_p$ generates a strongly continuous analytic semigroup on $\mathrm{L}^p(\Omega)$.*

Proof. For the Neumann problem this was proved by Wei and Zhang, see [97, Theorem 1.1 and Corollary 1.1]. For Dirichlet boundary data the same proof applies in light of Theorem 4.1.2. However, Shen used Green functions, see [82, Theorem 0.3], to prove the result as he proved his extrapolation results 10 years later. □

Remark 4.1.6. Recall that sectoriality requires a resolvent estimate of the form $\|(\lambda + A_p)^{-1}\| \lesssim \frac{1}{|\lambda|}$ for $\lambda \in \mathrm{S}_\theta$ with $\theta \in (\pi/2, \pi)$. However, for Dirichlet boundary conditions the stronger estimate

$$\|(\lambda + A_p)^{-1} f\|_{\mathrm{L}^p(\Omega)} \lesssim \frac{1}{1 + |\lambda|} \|f\|_{\mathrm{L}^p(\Omega)} \tag{4.15}$$

is valid for $f \in \mathrm{L}^p(\Omega)$ and $\lambda \in \mathrm{S}_\theta$, see [82, Theorem 0.3]. In particular, by analyticity of the resolvent map, this implies $0 \in \rho(A_p)$ with the same estimate so that (4.15) holds for all $\lambda \in \mathrm{S}_\theta \cup \{0\}$. For the Neumann resolvent problem such an estimate can not hold for all $f \in \mathrm{L}^p(\Omega)$ as invertibility of A_p fails. However, restricting to mean value free functions, estimate (4.15) holds true for $\lambda \in \mathrm{S}_\theta \cup \{0\}$, see [97, Theorem 1.1].

Having an L^p-theory for the resolvent we can use results from Chapter 3 to obtain L^p-L^∞-weak reverse Hölder inequalities for any $p > 1$.

Corollary 4.1.7. *Let $\theta \in (\pi/2, \pi)$, $2 < q \leq \infty$ and $1 < p < q$. Moreover, let $V = \mathrm{W}^{1,2}(\Omega)$ or $V = \mathrm{W}^{1,2}_0(\Omega)$. Then for the radius r_0 from the definition of a Lipschitz domain the following holds for all $\lambda \in \mathrm{S}_\theta$ if $V = \mathrm{W}^{1,2}(\Omega)$ or for all $\lambda \in \mathrm{S}_\theta \cup \{0\}$ if $V = \mathrm{W}^{1,2}_0(\Omega)$. For all $x_0 \in \overline{\Omega}$ and $0 < r \leq r_0$ and $f \in \mathrm{L}^p(\Omega)$ satisfying $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$ the function $u := (\lambda + A_p)^{-1} f$ satisfies*

$$\|u\|_{\mathrm{L}^\infty(B(x_0,r)\cap\Omega)} \lesssim \Big(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |u|^p \, \mathrm{d}x \Big)^{\frac{1}{p}}$$

if $q = \infty$ or

$$\Big(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |u|^q \, \mathrm{d}x \Big)^{\frac{1}{q}} \lesssim \Big(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |u|^p \, \mathrm{d}x \Big)^{\frac{1}{p}}$$

if $q < \infty$. The implicit constant only depends on q, p, θ, Ω and Λ.

Proof. We want to apply Theorem 3.1.6 to $T \coloneqq (\lambda + A_p)^{-1}$. By Theorem 4.1.1, Theorem 4.1.2 and Lemma 4.1.4, T satisfies the necessary L^2-weak reverse Hölder estimates. Hence, the claim follows from Theorem 3.1.6 as T is continuous on $\mathrm{L}^p(\Omega)$ for $1 < p < \infty$ by Shen's L^p-extrapolation theorem. □

From now on we will drop the index p and only write A instead of A_p. As mentioned in the introduction, an important step for a Morrey space theory for semilinear parabolic problems are $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates for the semigroup. Examining the proof of [59, Lemma 2.1] suggests that uniform estimates play a crucial role in establishing such $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates. Thus, we first aim for an L^p-L^∞-estimate for the resolvent. The strategy of the proof is to proceed similarly to the proof of weak L^2-L^∞-reverse Hölder estimates so that gradient estimates for the resolvent are necessary. Wei and Zhang proved gradient estimates for $2 \leq p < 3 + \varepsilon$, see [97, Proposition 4.1], for the Neumann resolvent problem by means of Shen's extrapolation theorem. For the Dirichlet resolvent problem Shen in [82, Theorem 0.5] obtained gradient estimate for $\frac{3}{2} - \delta < p < 3 + \delta$ for some $\delta > 0$. Here, we prove that in both cases those gradient estimates remain true for $1 < p < 2$.

Theorem 4.1.8. *Let $\theta \in (\pi/2, \pi)$ and $V \in \{\mathrm{W}^{1,2}(\Omega), \mathrm{W}^{1,2}_0(\Omega)\}$. Then there exists $\varepsilon > 0$ depending on θ, Ω and Λ such that for all $\lambda \in \mathrm{S}_\theta$ the operator $|\lambda|^{\frac{1}{2}} \nabla(\lambda + A)^{-1}$ is bounded on $\mathrm{L}^p(\Omega)$ for $1 < p < 3 + \varepsilon$. The operator norm depends on p, θ, Ω and Λ.*

Proof. Let $\varepsilon > 0$ as in Theorem 4.1.1 or Theorem 4.1.2 depending on V. Then, for $2 \leq p < 3 + \varepsilon$ the claim follows from Shen's extrapolation theorem. Thus, let now $p < 2$. For $f \in \mathrm{L}^p(\Omega)$ we estimate $\|\nabla(\lambda + A)^{-1} f\|_{\mathrm{L}^p(\Omega)}$ via duality. For $g \in \mathrm{C}^\infty_c(\Omega)$ with $\|g\|_{\mathrm{L}^q(\Omega)} \leq 1$ where $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$
\begin{aligned}
\langle \nabla(\lambda + A)^{-1} f, g \rangle_{p,q} &\leq \langle (\lambda + A)^{-1} f, \operatorname{div} g \rangle_{p,q} \leq \langle f, (\overline{\lambda} + A^*)^{-1} \operatorname{div} g \rangle_{p,q} \\
&\leq \|f\|_{\mathrm{L}^p} \|(\overline{\lambda} + A)^{-1} \operatorname{div} g\|_{\mathrm{L}^q}
\end{aligned}
$$

by Hölder's inequality. Here, $\langle \cdot, \cdot \rangle_{p,q}$ denotes the dual pairing between $\mathrm{L}^p(\Omega)$ and $\mathrm{L}^q(\Omega)$. Hence, it suffices to understand $(\lambda + A)^{-1} \operatorname{div}$ on $\mathrm{L}^q(\Omega)$. Note that A has real, symmetric coefficients and we can write λ instead of $\overline{\lambda}$ as S_θ is stable under taking conjugate. Moreover $p < 2$ implies $q > 2$ so that we are in the realm of Shen's L^q-extrapolation theorem.

In order to apply this theorem we need to define our operator on $\mathrm{L}^2(\Omega)$ which is not immediate as we are dealing with $T = (\lambda + A)^{-1} \operatorname{div}$. We proceed by the form method and define Tf for $f \in \mathrm{L}^2(\Omega)$ as the unique solution of

$$
\lambda \int_\Omega u \cdot \overline{v} \, \mathrm{d}x + \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{3} \int_\Omega a^{\alpha\beta}_{ij} \partial_j u^\beta \overline{\partial_i v^\alpha} \, \mathrm{d}x = - \int_\Omega f \overline{\nabla v} \, \mathrm{d}x
$$

for all $v \in V$. Choosing $v = u$ and using Hölder's and Young's inequality together with [90, Lemma 5.2.4] we recover that $|\lambda|^{\frac{1}{2}}(\lambda + A)^{-1} \operatorname{div}$ is uniformly bounded on $\mathrm{L}^2(\Omega)$ with respect to λ.

In order to apply Shen's extrapolation theorem it remains to show weak reverse Hölder estimates for $u = (\lambda+A)^{-1} \operatorname{div} f$. To this end consider some ball $B(x,r)$ with $x \in \Omega$ and $0 < r < r_0$ with r_0 as in Theorem 4.1.1 or Theorem 4.1.2, respectively, and $f \in \mathrm{L}^2(\Omega)$ with $f \equiv 0$ in $B(x,3r) \cap \Omega$. Then u by definition solves $\lambda u + Au = 0$ in $B(x,3r) \cap \Omega$. By Theorem 4.1.1 or Theorem 4.1.2, u satisfies weak reverse Hölder estimates for any exponent $q \in (2,\infty)$ and by Shen's extrapolation theorem T is bounded on $\mathrm{L}^q(\Omega)$ for $q \in (2,\infty)$ and this concludes the proof. □

Remark 4.1.9. For $V = \mathrm{W}^{1,2}_0(\Omega)$ (i.e. Dirichlet boundary conditions) a stronger estimate in the spirit of Remark 4.1.6 remains true also for the gradient. To be precise, in this case for $\lambda \in \mathrm{S}_\theta$, with θ as in the previous theorem, the operator $(1 + |\lambda|)^{\frac{1}{2}} \nabla(\lambda + A)^{-1}$ is bounded on $\mathrm{L}^p(\Omega)$ uniformly in λ so the boundedness extends to $\lambda = 0$. For a proof we refer to [82, Theorem 0.5].

In [84] the author shows that L^2-reverse Hölder estimates with exponent p imply L^p-boundedness of the Riesz transform if $V = \mathrm{W}^{1,2}_0(\Omega)$ in the case of scalar equations. However, this results remains true in the context of systems and even for Neumann boundary conditions. For systems under Dirichlet boundary conditions Shen's proof essentially stays the same, only noting that [11, Theorem 4] needs to be replaced by the corresponding results for systems in [33].

Theorem 4.1.10. *Let $V = \mathrm{W}^{1,2}(\Omega)$, $\theta \in (\pi/2, \pi)$, $\varepsilon > 0$ as in Theorem 4.1.1 and $2 \le p < 3 + \varepsilon$. Then, for all $\lambda \in \mathrm{S}_\theta$ and all $f \in \mathrm{L}^p(\Omega)$ the function u given by $u := (\lambda + A)^{-1} f$ satisfies*

$$\|\nabla u\|_{\mathrm{L}^p(\Omega)} \lesssim \|(1 + A)^{\frac{1}{2}} u\|_{\mathrm{L}^p(\Omega)}. \tag{4.16}$$

The implicit constant only depends on p, θ, Ω and Λ.

Proof. The case $p = 2$ is classical in form theory as the form in question is real and symmetric, see [60, Theorem 2.23]. Let now $2 < p < 3+\varepsilon$. Note that it suffices to prove

$$\|\nabla(1 + A)^{-\frac{1}{2}} h\|_{\mathrm{L}^p(\Omega)} \lesssim \|h\|_{\mathrm{L}^p(\Omega)}$$

for all $h \in \mathcal{R}((1 + A)^{\frac{1}{2}})$. Again we proceed by testing with $g \in \mathrm{C}^\infty_c(\Omega)$ satisfying $\|g\|_{\mathrm{L}^{p'}(\Omega)} \le 1$ where $\frac{1}{p} + \frac{1}{p'} = 1$. By applying Hölder's inequality it remains to estimate

$$\sup \|(1 + A^*)^{-\frac{1}{2}} \operatorname{div} g\|_{\mathrm{L}^{p'}(\Omega)}$$

where the supremum is taken over all $g \in \mathrm{C}^\infty_c(\Omega)$ with $\|g\|_{\mathrm{L}^{p'}(\Omega)} \le 1$. As in the previous proof we omit the adjoint. Next, we write

$$(1 + A)^{-\frac{1}{2}} \operatorname{div} g = (1 + A)^{\frac{1}{2}} A^{-\frac{1}{2}} A^{\frac{1}{2}} (1 + A)^{-\frac{1}{2}} (1 + A)^{-\frac{1}{2}} \operatorname{div} g. \tag{4.17}$$

Here $L^{\frac{1}{2}}(1+A)^{-\frac{1}{2}}$ is well-defined as $\mathcal{R}((1+A)^{-\frac{1}{2}}) = \mathcal{D}((1+A)^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{2}})$ where the last equality is due to [54, Proposition 3.1.9 a)]. Note that $A^{-\frac{1}{2}}$ well-defined, as, in the above formula, it acts on divergence-free functions. Indeed, $(1+A)^{-1}\operatorname{div} g$ is mean value free by the equation as can be seen by an integration by parts. As $A^{\frac{1}{2}}$ preserves this property by [54, Theorem 2.3.3.d)], we can restrict ourselves to mean value free functions and there A is bijective, so negative fractional powers are well-defined.

The composition $S \coloneqq (1+A)^{\frac{1}{2}}A^{-\frac{1}{2}}$ is well-defined with the same argumentation as for $A^{\frac{1}{2}}(1+A)^{-\frac{1}{2}}$. Moreover the composition is closed. Indeed, $(1+A)^{\frac{1}{2}}$ is closed and the continuity of $A^{-\frac{1}{2}}$ is due to the continuity of A^{-1} on mean value free functions (see the second part of Theorem 1.1 in [97]) and [54, Proposition 3.1.1.a)], recall that for an injective sectorial operator also its inverse is sectorial). Hence, S is a closed operator defined on the Banach space of mean-value free L^p-functions and therefore, by the closed graph theorem, continuous. Thus, in order to control (4.17) in the $\mathrm{L}^{p'}$-norm it suffices to estimate the $\mathrm{L}^{p'}$-norm of $A^{\frac{1}{2}}(1+A)^{-1}\operatorname{div} g$. Note that $p > 2$ implies $p' < 2$. First, we employ [33, Theorem 1.2] which after an application of Poincaré's inequality states

$$
\begin{aligned}
\|A^{\frac{1}{2}}(1+A)^{-1}\operatorname{div} g\|_{\mathrm{L}^q(\Omega)} &\lesssim \|\nabla(1+A)^{-1}\operatorname{div} g\|_{\mathrm{W}^{1,q}(\Omega)} \\
&\lesssim \|\nabla(1+A)^{-1}\operatorname{div} g\|_{\mathrm{L}^q(\Omega)}
\end{aligned}
$$

for all $1 < q < 2$. Note that $\fint_\Omega \operatorname{div} g = 0$ due to integration by parts and therefore also $(1+A)^{-1}\operatorname{div} g$ has mean 0. Hence, Poincaré's inequality indeed can be applied.

It remains to prove $\mathrm{L}^{p'}$-boundedness for the operator $T = \nabla(1+A)^{-1}\operatorname{div}$ where $(1+A)^{-1}\operatorname{div}$ is defined as in the previous proof. Note that due to a duality argument it suffices to show L^p-boundedness of the same operator which allows us to employ Shen's extrapolation theorem. Recall that the associated form in question is real and symmetric. The L^2-estimate is again a consequence of [90, Lemma 5.2.4] and testing. With the same argument as in the previous result we can show that $\nabla(1+A)^{-1}\operatorname{div}$ satisfies the same weak reverse Hölder estimates as $\nabla(1+A)^{-1}$. Thus, by Theorem 4.1.1 and Shen's extrapolation theorem we deduce L^q-boundedness for $2 < q < 3+\varepsilon$ for some $\varepsilon > 0$ as in Theorem 4.1.1. In view of the aforementioned duality argument this concludes the proof. □

Now, we are in the position to prove L^p-L^∞-estimates for the resolvent. However, in the case of Neumann boundary conditions we need to assume f to be mean value free, as ensures that Poincaré's inequality is applicable. For Dirichlet boundary conditions no such assumption is necessary.

Proposition 4.1.11. *Let $\theta \in (\pi/2, \pi)$, $V \in \{\mathrm{W}^{1,2}(\Omega), \mathrm{W}^{1,2}_0(\Omega)\}$ and $\varepsilon > 0$ as in Theorem 4.1.1 or Theorem 4.1.2, respectively. Moreover, let $3/2 < p < 3+\varepsilon$.*

Then, for all $\lambda \in \mathrm{S}_\theta$ and $f \in \mathrm{L}^p(\Omega)$, with the additional assumption $\int_\Omega f \,\mathrm{d}x = 0$ if $V = \mathrm{W}^{1,2}(\Omega)$, the function $u := (\lambda + A)^{-1} f$ satisfies

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \lesssim |\lambda|^{-\frac{1}{2}(2-\frac{3}{p})} \|f\|_{\mathrm{L}^p(\Omega)}.$$

Here, the implicit constant depends only on p, θ, Ω and Λ.

Proof. Our strategy is to prove L^p-L^∞-estimates in two different ranges of p and then conclude by Riesz-Thorin interpolation, see, e.g., [14]. We present the proof in the case of Neumann boundary conditions. In case of Dirichlet boundary conditions no sophisticated tool as Stein's extension operator is necessary as extension by zero is possible.

We use the extension operator $\mathfrak{E}$ given by Stein in [87, VI, Section 3.1, Theorem 5] which is applicable as we showed in Theorem 2.3.8 that the boundary of a bounded Lipschitz domain is minimally smooth. For $3 < p < 3 + \varepsilon$ we then apply the Gagliardo-Nirenberg inequality on $\mathbb{R}^d$, see [74] with $\vartheta = 3/p < 1$ to the extended function and use the boundedness of the extension operator to obtain

$$\begin{aligned}
\|u\|_{\mathrm{L}^\infty(\Omega)} &\lesssim \|\mathfrak{E}u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \\
&\lesssim \|\mathfrak{E}u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{1-\vartheta} \|\nabla \mathfrak{E}u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\vartheta} \\
&\lesssim \|u\|_{\mathrm{L}^p(\Omega)}^{1-\vartheta} \|u\|_{\mathrm{W}^{1,p}(\Omega)}^{\vartheta}.
\end{aligned}$$

Note that due to the use of the extension operator a full Sobolev norm appears. Therefore, we can not immediately use the boundedness estimates for u and ∇u as this would yield an incorrect behaviour in λ. However, note that for mean value free functions f also the solution u to the resolvent problem (4.2) is mean value free. Then Poincaré's inequality on Ω delivers

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)}^{1-\vartheta} \|\nabla u\|_{\mathrm{L}^p(\Omega)}^{\vartheta} \tag{4.18}$$

Next, we use the boundedness of the Riesz transform, then Sobolev's inequality on both factors and finally again boundedness of the Riesz transform on the second one to get

$$\begin{aligned}
\|u\|_{\mathrm{L}^\infty(\Omega)} &\lesssim \|u\|_{\mathrm{L}^p(\Omega)}^{1-\vartheta} \|(1+A)^{\frac{1}{2}} u\|_{\mathrm{L}^p(\Omega)}^{\vartheta} \\
&\lesssim \|\nabla u\|_{\mathrm{L}^{p_*}(\Omega)}^{1-\vartheta} \|(1+A) u\|_{\mathrm{L}^{p_*}(\Omega)}^{\vartheta}
\end{aligned}$$

where $p_* := \frac{3p}{3+p}$ is the lower Sobolev conjugate of p. Recall that in Theorem 4.1.8 we showed $\|\nabla u\|_{\mathrm{L}^{p_*}(\Omega)} \lesssim |\lambda|^{-\frac{1}{2}} \|f\|_{\mathrm{L}^{p_*}(\Omega)}$ even for $p_* < 2$ which was not included in [97, Proposition 4.1]. For the second factor note that $(1 + A)u = f + (1 - \lambda)u$

and thus, in light of Remark 4.1.6 as f is mean value free in the case of Neumann boundary conditions, we obtain

$$\|(1+A)u\|_{\mathrm{L}^{p_*}(\Omega)} \leq \|f\|_{\mathrm{L}^{p_*}(\Omega)} + (1+|\lambda|)\|u\|_{\mathrm{L}^{p_*}(\Omega)} \lesssim \|f\|_{\mathrm{L}^{p_*}(\Omega)}.$$

Combining the previous calculations yields

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \lesssim |\lambda|^{-\frac{1}{2}(1-\theta)}\|f\|_{\mathrm{L}^{p_*}(\Omega)} = |\lambda|^{\frac{3}{2}\frac{1}{p_*}-1}\|f\|_{\mathrm{L}^{p_*}(\Omega)},$$

where we made use of $\frac{1}{p} = \frac{1}{p_*} - \frac{1}{3}$. Thus we showed an L^{p_*}-L^∞-estimate whenever $\frac{3}{2} < p_* < \frac{3}{2} + \varepsilon$.

On the other hand note that for $3 < p < 3 + \varepsilon$ we can estimate $\|u\|_{\mathrm{L}^p(\Omega)}$ and $\|\nabla u\|_{\mathrm{L}^p(\Omega)}$ in (4.18) directly by means of Theorem 4.1.5 and Theorem 4.1.8. This then yields

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \lesssim |\lambda|^{\frac{3}{2}\frac{1}{p}-1}\|f\|_{\mathrm{L}^p(\Omega)}.$$

By Riesz-Thorin interpolation we then obtain the desired estimate for all $\frac{3}{2} < p < 3 + \varepsilon$. □

The previous result contains two restrictions, namely an upper bound on p and in the case of Neumann boundary conditions the condition that f is mean value free. Both these restrictions can be lifted using a decomposition argument which in return comes with the restriction that the resolvent parameter must be bounded away from 0. For simplicity we choose $|\lambda| \geq 1$ but of course any other positive lower bound is equally possible.

Theorem 4.1.12. *Let $\theta \in (\pi/2, \pi)$, $V \in \{\mathrm{W}^{1,2}_0(\Omega), \mathrm{W}^{1,2}(\Omega)\}$ and $\frac{3}{2} < p < \infty$. Then, for all $\lambda \in \mathrm{S}_\theta$ satisfying $|\lambda| \geq 1$ the operator $(\lambda + A)^{-1}$ is bounded from $\mathrm{L}^p(\Omega)$ to $\mathrm{L}^\infty(\Omega)$ with*

$$\|(\lambda + A)^{-1}\|_{\mathrm{L}^\infty(\Omega)} \lesssim |\lambda|^{-\frac{1}{2}(2-\frac{3}{p})}\|f\|_{\mathrm{L}^p(\Omega)}$$

for all $f \in \mathrm{L}^p(\Omega)$. Here, the implicit constant depends only on p, θ, Ω and Λ.

Proof. We only prove the statement for Neumann boundary conditions as for Dirichlet boundary conditions the same steps apply but not mean value needs to be introduced.

Due to density it suffices to consider $f \in \mathrm{L}^2(\Omega) \cap \mathrm{L}^p(\Omega)$. By the dominated convergence theorem, for $x_0 \in \Omega$ we have

$$f = f\chi_{B(x_0, 2|\lambda|^{-\frac{1}{2}}) \cap \Omega} + \sum_{\ell=1}^\infty f\chi_{C_\ell \cap \Omega}$$

in $\mathrm{L}^p(\Omega)$ where $C_\ell = B(x_0, 2^{\ell+1}|\lambda|^{-\frac{1}{2}}) \setminus B(x_0, 2^\ell|\lambda|^{-\frac{1}{2}})$. As $T := (\lambda + A)^{-1}$ is continuous on $\mathrm{L}^p(\Omega)$, we also obtain

$$Tf = T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega}) + \sum_{\ell=1}^{\infty} T(f\chi_{C_\ell\cap\Omega})$$

in $\mathrm{L}^p(\Omega)$ and thus point-wise almost everywhere. For the first term we want to apply the previous L^p-L^∞-estimate, however, the argument does not need be mean value free. To overcome this issue, add and subtract

$$f_B := \frac{1}{|\Omega|}\int_\Omega f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})}\,\mathrm{d}x.$$

Note that $f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} - f_B$ is mean value free over Ω. By linearity of T we find

$$T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega}) = T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} - f_B) + T(f_B)$$

and thus the following equality holds almost everywhere

$$Tf(x_0) = T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} - f_B)(x_0) + \sum_{\ell=1}^{\infty} T(f\chi_{C_\ell\cap\Omega})(x_0) + T(f_B)(x_0).$$

As $\mathrm{L}^\infty(\Omega)$ is a Banach space, it suffices to show that every term on the right-hand side is in $\mathrm{L}^\infty(\Omega)$ and the series converges absolutely. We start by considering the first summand. By means of the L^p-L^∞-estimates we have for $\frac{3}{2} < q \leq 3 + \varepsilon$ with $q \leq p$

$$\begin{aligned}|T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} - f_B)(x_0)| &\lesssim |\lambda|^{\frac{1}{2}(\frac{3}{q}-\frac{1}{\infty})-1}\left(\int_\Omega |f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})} - f_B|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\lesssim |\lambda|^{\frac{1}{2}(\frac{3}{q}-\frac{1}{\infty})-1}\left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.\end{aligned}$$

Here, the last inequality holds due to

$$\left(\int_\Omega |f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})} - f_B|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^q\,\mathrm{d}x\right)^{\frac{1}{q}} + |f_B||\Omega|^{\frac{1}{q}}$$

and

$$|f_B| \leq \fint_\Omega |f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega}|\,\mathrm{d}x \leq |\Omega|^{-\frac{1}{q}}\left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Lastly, we apply Jensen's inequality on $B(x_0, 2|\lambda|^{-\frac{1}{2}}) \cap \Omega$ to obtain

$$
\begin{aligned}
&|\lambda|^{\frac{1}{2}\cdot\frac{3}{q}-1} \left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\qquad \lesssim |\lambda|^{\frac{1}{2}\frac{3}{q}-1} |B(x_0, 2|\lambda|^{-\frac{1}{2}}) \cap \Omega|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}
$$

As $\frac{1}{q} - \frac{1}{p} \geq 0$, we deduce

$$
|B(x_0, 2|\lambda|^{-\frac{1}{2}}) \cap \Omega|^{\frac{1}{q}-\frac{1}{p}} \lesssim |\lambda|^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}.
$$

Combining the previous calculations we conclude

$$
\begin{aligned}
|T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} - f_B)(x_0)| &\lesssim |\lambda|^{\frac{3}{2}\cdot\frac{1}{p}-1} \left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\lesssim |\lambda|^{\frac{3}{2}\cdot\frac{1}{p}-1} \|f\|_{\mathrm{L}^p(\Omega)}.
\end{aligned}
$$

For the second summand we have by the L^∞-weak reverse Hölder estimates with exponent p from Corollary 4.1.7, recall that by [90, Lemma 3.2.3] and the argument in Theorem 3.2.3 we have those for all radii, and the L^p-boundedness of the resolvent

$$
\begin{aligned}
|T(f\chi_{C_\ell\cap\Omega})(x_0)| &\lesssim \left((2^\ell|\lambda|^{-\frac{1}{2}})^{-3} \int_{B(x_0,\frac{2}{3}2^\ell|\lambda|^{-\frac{1}{2}})\cap\Omega} |T(f\chi_{C_\ell\cap\Omega})|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\lesssim \left(2^{3\ell}|\lambda|^{-\frac{3}{2}}\right)^{-\frac{1}{p}} |\lambda|^{-1} \left(\int_{C_\ell\cap\Omega} |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\lesssim \left(2^{3\ell}|\lambda|^{-\frac{3}{2}}\right)^{-\frac{1}{p}} |\lambda|^{-1} \left(\int_{\Omega} |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&= 2^{-\frac{3\ell}{p}} |\lambda|^{\frac{3}{2p}-1} \|f\|_{\mathrm{L}^p(\Omega)}.
\end{aligned}
$$

This is summable in ℓ and it remains to deal with $T(f_B)$. By definition, $T(f_B)$ is the solution v of $\lambda v + Av = f_B$ with Neumann boundary conditions. As used before, we know $\lambda \fint_\Omega v \, \mathrm{d}x = \fint_\Omega f_B \, \mathrm{d}x = f_B$. This implies

$$
\lambda(v - \fint_\Omega f \, \mathrm{d}x) + A(v - \fint_\Omega v \, \mathrm{d}x) = 0
$$

and as $\lambda \in \rho(-A)$ we deduce $v = ⨍_\Omega v \, \mathrm{d}x$. Consequently, we obtain $v = \frac{f_B}{\lambda}$. Applying Jensen's inequality we have

$$\begin{aligned}
\|T(f_B)\|_{\mathrm{L}^\infty(\Omega)} &= \frac{|f_B|}{|\lambda|} = \frac{1}{|\lambda|}\frac{1}{|\Omega|}\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f| \, \mathrm{d}x \\
&\lesssim \frac{1}{|\lambda|}\frac{|B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega|}{|\Omega|}\left(⨍_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega} |f|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \\
&\lesssim |\lambda|^{-1}|B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega|^{-\frac{1}{p}}\|f\|_{\mathrm{L}^p(\Omega)} \\
&\lesssim |\lambda|^{-1}|\lambda|^{\frac{3}{2p}}\|f\|_{\mathrm{L}^p(\Omega)}
\end{aligned}$$

and in light of Theorem 2.3.10. It is here that $|\lambda| \geq 1$ is crucial as this condition guarantees $2|\lambda|^{-\frac{1}{2}} \leq 2$. For each term we showed an estimate of the same type and this concludes the proof. □

To conclude this section we collect estimates for the semigroup. Note that due to Cauchy's integral formula the lower bound $|\lambda| \geq 1$ corresponds to an upper bound for t.

Theorem 4.1.13. (i) *Let $V = \mathrm{W}^{1,2}(\Omega)$. Then for all $1 < p < \infty$ the estimate*

$$\|T(t)f\|_{\mathrm{L}^p(\Omega)} \lesssim \|f\|_{\mathrm{L}^p(\Omega)} \tag{4.19}$$

holds for all $t > 0$ and all $f \in \mathrm{L}^p(\Omega)$. Assume now $\frac{3}{2} < p < 3+\varepsilon$ and that $f \in \mathrm{L}^p(\Omega)$ satisfies $⨍_\Omega f \, \mathrm{d}x = 0$ or $\frac{3}{2} < p < \infty$ and $f \in \mathrm{L}^p(\Omega)$. Then

$$\|T(t)f\|_{\mathrm{L}^\infty(\Omega)} \lesssim t^{-\frac{3}{2}\frac{1}{p}}\|f\|_{\mathrm{L}^p(\Omega)} \tag{4.20}$$

for all $0 < t < \infty$ in the first case and all $0 < t \leq 1$ in the second case. The implicit constants depend on p, Ω and Λ.

(ii) *Let now $V = \mathrm{W}^{1,2}_0(\Omega)$. Then, for $1 < p < \infty$, there exists $\omega_p > 0$ such that*

$$\|T(t)f\|_{\mathrm{L}^p(\Omega)} \lesssim \mathrm{e}^{-\omega_p t}\|f\|_{\mathrm{L}^p(\Omega)} \tag{4.21}$$

holds for all $0 < t < \infty$ and all $f \in \mathrm{L}^p(\Omega)$. Let $\frac{3}{2} < p < \infty$. Then

$$\|T(t)f\|_{\mathrm{L}^\infty(\Omega)} \lesssim t^{-\frac{3}{2}\frac{1}{p}}\|f\|_{\mathrm{L}^p(\Omega)} \tag{4.22}$$

holds for all $0 < t < \infty$ and all $f \in \mathrm{L}^p(\Omega)$. The implicit constants depend on p, Ω and Λ.

(iii) *Let $\frac{3}{2} < p < \infty$ and $q \geq p$. Then*

$$\|T(t)f\|_{\mathrm{L}^q(\Omega)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{\mathrm{L}^p(\Omega)}$$

holds for all $f \in \mathrm{L}^p(\Omega)$ and all $0 < t \leq 1$ if $V = \mathrm{W}^{1,2}(\Omega)$ and all $0 < t < \infty$ if $V = \mathrm{W}_0^{1,2}(\Omega)$. The implicit constants depend on p, q, Ω and Λ.

Proof. The result for Neumann boundary conditions, i.e., the first part, is an immediate consequence of the resolvent estimates in Theorem 4.1.5, Proposition 4.1.11 and Theorem 4.1.12 in light of the transference principle [91, Proposition 3.7]. Recall that the semigroup is defined via the Cauchy integral so that small resolvent parameters λ correspond to large times t.

For the second part, i.e., Dirichlet boundary conditions, recall that in Remark 4.1.6 we deduced $0 \in \rho(A_p)$. It is classical, via a Neumann series argument, that the resolvent set is open, see, e.g., [54, Proposition A.2.3]. Combining this with the sectoriality of A_p implies that the spectral bound of $-A_p$ is negative. Then, [67, Corollary 2.3.2] states the growth bound (Lunardi calls it the *type of the semigroup*) of the semigroup coincides with the spectral bound. This implies the claimed exponential decay. The L^p-L^∞-estimate results of a combination of the exponential decay and the L^p-L^∞-estimate for small times which on their own follow from Theorem 4.1.12 and [91, Proposition 3.7]. Indeed, it holds

$$\begin{aligned}\|T(t)f\|_{\mathrm{L}^\infty(\Omega)} = \|T(1)T(t-1)f\|_{\mathrm{L}^\infty(\Omega)} &\lesssim \|T(t-1)f\|_{\mathrm{L}^p(\Omega)} \\ &\lesssim \mathrm{e}^{-\omega_p t} \|f\|_{\mathrm{L}^p(\Omega)} \\ &\lesssim t^{-\frac{3}{2}\frac{1}{p}} \|f\|_{\mathrm{L}^p(\Omega)}.\end{aligned}$$

Finally, the L^p-L^q-estimate follows by interpolating the L^p-L^∞-estimate with the L^p-boundedness. □

4.1.3 Morrey space theory

Next, we turn to Morrey spaces and define $A_{p,\mu}$ as the part of A_p in $\mathrm{L}^{p,\mu}(\Omega)$. To be precise, for $1 < p < \infty$ and $0 < \mu < d$ we define $A_{p,\mu}$ on $\mathrm{L}^{p,\mu}(\Omega)$ by

$$\begin{aligned}\mathcal{D}(A_{p,\mu}) &\coloneqq \{u \in \mathcal{D}(A_p) \cap \mathrm{L}^{p,\mu}(\Omega) \colon A_p u \in \mathrm{L}^{p,\mu}(\Omega)\} \\ A_{p,\mu} u &\coloneqq A_p u.\end{aligned}$$

Note that this definition makes sense as for bounded domains we have $\mathrm{L}^{p,\mu}(\Omega) \subseteq \mathrm{L}^p(\Omega)$. Here, we do not consider the case $\mu = 0$ as $\mathrm{L}^{p,0}(\Omega) = \mathrm{L}^p(\Omega)$ holds.

Next, we want to show that $A_{p,\mu}$ is a sectorial operator. To this end, let $f \in \mathrm{L}^{p,\mu}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ and $\lambda \in \mathrm{S}_\theta \setminus \{0\}$ be arbitrary. Then, $u \coloneqq (\lambda + A_p)^{-1} f \in \mathcal{D}(A_p)$. It remains to show $u \in \mathrm{L}^{p,\mu}(\Omega)$ with $\|u\|_{\mathrm{L}^{p,\mu}(\Omega)} \lesssim \frac{1}{|\lambda|} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}$. This is the result of the following theorem.

Theorem 4.1.14. *Let* $\theta \in (\pi/2, \pi)$, $V \in \{\mathrm{W}_0^{1,2}(\Omega), \mathrm{W}^{1,2}(\Omega)\}$, $1 < p < \infty$ *and* $0 < \mu < d$. *Then for all* $\lambda \in \mathrm{S}_\theta$ *the estimate*

$$\|(\lambda + A_p)^{-1} f\|_{\mathrm{L}^{p,\mu}(\Omega)} \lesssim \frac{1}{|\lambda|} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}$$

holds for all $f \in \mathrm{L}^{p,\mu}(\Omega)$ *and the implicit constant only depends on* p, θ, Ω *and* Λ. *In particular,* $-A_{p,\mu}$ *generates an analytic semigroup on* $\mathrm{L}^{p,\mu}(\Omega)$.

Proof. We apply Corollary 3.2.6. The L^p-boundedness of $\lambda(\lambda + A_p)^{-1}$ was proven in Theorem 4.1.5. By Corollary 4.1.7 and Lemma 3.1.3 weak reverse Hölder estimates of the form

$$\left(\frac{1}{r^d} \int_{B(x_0,r)\cap\Omega} |(\lambda + A_p)^{-1} f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \lesssim \left(\frac{1}{r^d} \int_{B(x_0,2r)\cap\Omega} |(\lambda + A_p)^{-1} f|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

hold true for any $x_0 \in \Omega$, $r > 0$, $q > p$ and $f \in \mathrm{L}^2(\Omega)$ satisfying $f \equiv 0$ on $B(x_0, 3r) \cap \Omega$. This allows us to apply Corollary 3.2.6 and this concludes the proof. $\square$

Remark 4.1.15. For Dirichlet boundary conditions, similarly to the L^p-setting, we obtain stronger resolvent estimates in Morrey spaces, too. Indeed, extrapolating the L^p-estimate in Remark 4.1.6 to Morrey spaces by Corollary 3.2.6 yields that for all $1 < p < \infty$ and $0 < \mu < 3$ the estimate

$$\|(\lambda + A_{p,\mu})^{-1} f\|_{\mathrm{L}^{p,\mu}(\Omega)} \lesssim \frac{1}{1 + |\lambda|} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)} \tag{4.23}$$

is valid for all $\lambda \in \mathrm{S}_\theta$ and $f \in \mathrm{L}^{p,\mu}(\Omega)$. Again, this implies $0 \in \rho(A_{p,\mu})$. Using the same argumentation as in the proof of Theorem 4.1.13, it follows that in case of Dirichlet boundary conditions, the corresponding semigroup has exponential decay on $\mathrm{L}^{p,\mu}(\Omega)$.

For Neumann boundary conditions such estimates remain open. Note that in L^p-setting it was necessary to restrict to mean value free functions. However, our extrapolation results require decomposition arguments which do not respect this condition.

We now collect further results on the level of resolvents before turning to semigroup theory. Again, we drop the indices and write A instead of $A_{p,\mu}$ if there is no confusion about the corresponding space. As a consequence of the last argument of the proof of Theorem 4.1.8 we also obtain $\mathrm{L}^{p,\mu}$-boundedness for the operator $|\lambda|^{\frac{1}{2}}(\lambda + A)^{-1} \operatorname{div}$. Those are particularly interesting as the corresponding estimates for the semigroup will be crucial to proving existence of solutions to a semilinear partial differential equation.

Proposition 4.1.16. *Let $\theta \in (\pi/2, \pi)$ and $V \in \{\mathrm{W}_0^{1,2}(\Omega), \mathrm{W}^{1,2}(\Omega)\}$. Moreover, let $\varepsilon > 0$ be as in Theorem 4.1.1 or Theorem 4.1.2, respectively. Then, there exists $\delta > 0$ depending only on ε such that for $\frac{3}{2} - \delta < p < \infty$ and $0 < \mu < 3$ the operator $T := |\lambda|^{\frac{1}{2}}(\lambda + A)^{-1} \operatorname{div}$ is bounded on $\mathrm{L}^{p,\mu}(\Omega)$ for all $\lambda \in \mathrm{S}_\theta$.*

Proof. As deduced in the proof of Theorem 4.1.8, $|\lambda|^{\frac{1}{2}}(\lambda + A)^{-1} \operatorname{div}$ satisfies the same weak reverse Hölder estimates as $(\lambda + A)^{-1}$ which implies that T is bounded on L^p for $p > 2$. For $p < 2$ note that the adjoint is $|\lambda|^{\frac{1}{2}}\nabla(\overline{\lambda} + A)^{-1}$ and by the reverse Hölder estimates (4.5) this operator is bounded on $\mathrm{L}^{p'}$ for $2 < p' < 3+\varepsilon$, i.e., we obtain L^p-boundedness for $\frac{3}{2} - \delta < p < \infty$ for some $\delta > 0$. Finally, we proceed as in Theorem 4.1.14. Then, the weak reverse Hölder estimates from Corollary 4.1.7 allow us to apply the extrapolation result from Corollary 3.2.6 to conclude the proof. ☐

Next, we can combine weak reverse Hölder inequalities with L^p-L^∞-estimates for the resolvent in order to obtain $\mathrm{L}^{p,\mu}$-L^∞-estimates. These estimates are the next intermediate step to prove the desired $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates. Similar to the proof of L^p-L^∞-estimates for the Neumann resolvent problem, a further restriction on λ in the form of a lower bound on the absolute value is necessary. Again, we choose $|\lambda| \geq 1$ but of course any other positive lower bound is equally possible. However, note that here this restriction is necessary independent of choice of boundary conditions.

Theorem 4.1.17. *Let $\theta \in (\pi/2, \pi)$, $V \in \{\mathrm{W}_0^{1,2}(\Omega), \mathrm{W}^{1,2}(\Omega)\}$, $\frac{3}{2} < p < \infty$ and $0 < \mu < 3$. Then for $\lambda \in \Sigma_\theta \setminus \{0\}$ satisfying $|\lambda| \geq 1$ the operator $Tf := (\lambda + A)^{-1} f$ is bounded from $\mathrm{L}^{p,\mu}(\Omega)$ to $\mathrm{L}^\infty(\Omega)$ with*

$$\|(\lambda + A)^{-1} f\|_{\mathrm{L}^\infty(\Omega)} \lesssim |\lambda|^{\frac{1}{2}\left(\frac{3-\mu}{p}\right)-1} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}$$

for all $f \in \mathrm{L}^{p,\mu}(\Omega)$. The implicit constant depends p, μ, θ, Ω and Λ.

Proof. Let $f \in \mathrm{L}^{p,\mu}(\Omega)$ and $x_0 \in \Omega$ be arbitrary. We proceed as in the proof of Theorem 4.1.12. Note that due that theorem neither a distinction between the boundary conditions nor adding of the average f_B is necessary. With the same arguments as in the proof of Theorem 4.1.12 we obtain that for almost every $x_0 \in \Omega$ the representation

$$Tf(x_0) = T\left(f\chi_{B(x_0, 2|\lambda|^{-\frac{1}{2}}) \cap \Omega}\right)(x_0) + \sum_{\ell=1}^{\infty} T(f\chi_{C_\ell \cap \Omega})(x_0)$$

is valid. The sets $C_\ell = B(x_0, 2^{\ell+1}|\lambda|^{-\frac{1}{2}}) \setminus B(x_0, 2^\ell|\lambda|^{-\frac{1}{2}})$ are as defined before. Again it suffices to show that every term on the right-hand side is in $\mathrm{L}^\infty(\Omega)$ and the series converges absolutely. We start by considering the first summand. Applying Theorem 4.1.12 yields

$$\begin{aligned}|T(f\chi_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega})(x_0)| &\lesssim |\lambda|^{\frac{1}{2}\frac{3}{p}-1}\left(\int_{B(x_0,2|\lambda|^{-\frac{1}{2}})\cap\Omega}|f|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\lesssim |\lambda|^{\frac{1}{2}\left(\frac{3-\mu}{p}\right)-1}\|f\|_{\mathrm{L}^{p,\mu}(\Omega)}.\end{aligned}$$

For the second summand we essentially repeat the calculation from the proof of Theorem 4.1.12: By the weak reverse Hölder estimates and the L^p-boundedness of the resolvent we have

$$\begin{aligned}|T(f\chi_{C_\ell\cap\Omega})(x_0)| &\lesssim \left(\left(2^\ell|\lambda|^{-\frac{1}{2}}\right)^{-3}\int_{B(x_0,\frac{2}{3}2^\ell|\lambda|^{-\frac{1}{2}})\cap\Omega}|T(f\chi_{C_\ell\cap\Omega})|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\lesssim \left(2^{3\ell}|\lambda|^{-\frac{3}{2}}\right)^{-\frac{1}{p}}|\lambda|^{-1}\left(\int_{C_\ell\cap\Omega}|f|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\lesssim \left(2^{3\ell}|\lambda|^{-\frac{3}{2}}\right)^{-\frac{1}{p}}|\lambda|^{-1}\left(\left(\frac{2^\ell|\lambda|^{-\frac{1}{2}}}{2^\ell|\lambda|^{-\frac{1}{2}}}\right)^{\mu}\int_{B(x_0,2^\ell|\lambda|^{-\frac{1}{2}})\cap\Omega}|f|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\lesssim 2^{-\frac{3\ell}{p}}|\lambda|^{\frac{3}{2p}-1}\left(2^\ell|\lambda|^{-\frac{1}{2}}\right)^{\frac{\mu}{p}}\|f\|_{\mathrm{L}^{p,\mu}(\Omega)}\\ &= 2^{\frac{\mu-3}{p}\ell}|\lambda|^{\frac{1}{2}\frac{3-\mu}{p}-1}\|f\|_{\mathrm{L}^{p,\mu}(\Omega)}.\end{aligned}$$

Due to $\mu < 3$ this is summable in l and this completes the proof. □

Remark 4.1.18. Regarding the semigroup $T(t)$, recall that the lower bound on resolvent parameters corresponds to an upper bound of t. This means that $\mathrm{L}^{p,\mu}$-L^∞-estimates will not be uniform in $0 < t < \infty$ but only in $0 < t < T_0$ with $T_0 < \infty$. We make this more precise later. However, we note at this point that the behaviour in λ proved here implies the correct decay for the semigroup.

Whilst gradient estimates are not crucial for our application, we nonetheless state them for the sake of completeness. No new argument is needed here, only a combination of previous results.

Theorem 4.1.19. *Let $\theta \in (\pi/2, \pi)$, $V \in \{\mathrm{W}^{1,2}_0(\Omega), \mathrm{W}^{1,2}(\Omega)\}$, $2 < q < 3 + \varepsilon$ and $0 < \mu < 3\left(1 - \frac{q}{3+\varepsilon}\right)$ where $\varepsilon > 0$ is as in Theorem 4.1.2 or Theorem 4.1.1, respectively. Then, for all $\lambda \in \mathrm{S}_\theta$ the operator $|\lambda|^{\frac{1}{2}}\nabla(\lambda+A)^{-1}$ is bounded on $\mathrm{L}^{q,\mu}(\Omega)$.*

Proof. The L^2-boundedness was already proven by Wei and Zhang for the Neumann problem and by Shen for the Dirichlet problem. Weak reverse Hölder estimates for the gradient were obtained in Theorem 4.1.1 and Theorem 4.1.2 respectively. Then, an application of Theorem 3.2.8 concludes the proof. □

From now on we write $(T(t))_{t\geq 0}$ for the semigroup generated by $-A$. As mentioned earlier, in our setting analytic semigroups do not need to be strongly continuous on $\mathrm{L}^{p,\mu}(\Omega)$. However, [67, Proposition 2.1.4 (i)] states that the family $(T(t))_{t>0}$ is strongly continuous precisely on $\overline{\mathcal{D}(A_{p,\mu})}$. Hence, now aim to find a suitable description of $\overline{\mathcal{D}(A_{p,\mu})}$. Recall that by Proposition 2.2.13 we have $\mathrm{L}^q(\Omega) \hookrightarrow \mathrm{L}^{p,\mu}(\Omega)$ for $q \geq \frac{3}{3-\mu}p$. Using L^p-theory, we obtain the following result:

Proposition 4.1.20. *Let $\frac{3}{2} < p < \infty$, $0 < \mu < 3$ and $\frac{3}{3-\mu}p \leq q < \infty$. Then, $\overline{\mathcal{D}(A_{p,\mu})} = \overline{\mathrm{L}^q(\Omega)}$ where both closures are to be understood with respect to the $\mathrm{L}^{p,\mu}$-norm.*

Proof. To show that the semigroup is continuous on $\overline{\mathrm{L}^q(\Omega)}^{\|\cdot\|_{\mathrm{L}^{p,\mu}(\Omega)}}$ with respect to the $\mathrm{L}^{p,\mu}(\Omega)$-norm is an $\frac{\varepsilon}{3}$ argument. To be precise, let $f \in \overline{\mathrm{L}^q(\Omega)}^{\|\cdot\|_{\mathrm{L}^{p,\mu}(\Omega)}}$ and $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^q(\Omega)$ a sequence with $f_n \to f$ in $\mathrm{L}^{p,\mu}(\Omega)$. Then, by the triangle inequality, the boundedness of the semigroup together with the embedding $\mathrm{L}^q(\Omega) \hookrightarrow \mathrm{L}^{p,\mu}(\Omega)$ we have

$$\begin{aligned}\|T(t)f - f\|_{\mathrm{L}^{p,\mu}(\Omega)} &= \|T(t)f - T(t)f_n + T(t)f_n - f_n + f_n - f\|_{\mathrm{L}^{p,\mu}(\Omega)} \\ &\lesssim 2\|f_n - f\|_{\mathrm{L}^{p,\mu}(\Omega)} + \|T(t)f_n - f_n\|_{\mathrm{L}^q(\Omega)}.\end{aligned}$$

Let $\varepsilon > 0$. The first term is independent of t and can be bounded by $\frac{\varepsilon}{2}$ for n large by the choice of the sequence $(f_n)_{n\in\mathbb{N}}$. For such n large enough we can find $T_0 > 0$ small enough, such that the second term can also be bounded by $\frac{\varepsilon}{2}$ for $t < T_0$ due to the strong continuity of the semigroup on $\mathrm{L}^q(\Omega)$ with respect to the L^q-norm. In light of [67, Proposition 2.1.4 (i)] the inclusion $\overline{\mathcal{D}(A_{p,\mu})} \supseteq \overline{\mathrm{L}^q(\Omega)}$ follows.

For the other inclusion note that $\mathrm{L}^{p,\mu}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ and the semigroup on Lebesgue spaces has, at least for small times, L^p-L^q smoothing by Theorem 4.1.13. More precisely, for $f \in \overline{\mathcal{D}(A_{p,\mu})} \subseteq \mathrm{L}^p(\Omega)$ and $0 < t \leq 1$ we have $T(t)f \in \mathrm{L}^q(\Omega)$. As $T(t)f \to f$ for $t \to 0$ with respect to $\|\cdot\|_{\mathrm{L}^{p,\mu}(\Omega)}$, this shows that f can be approximated in $\mathrm{L}^{p,\mu}(\Omega)$ by functions in $\mathrm{L}^q(\Omega)$. $\square$

Nonetheless, for solving the abstract Cauchy problem mentioned in the introduction we need a dense subspace contained in $\mathrm{L}^{r,\mu}(\Omega)$ for some $r > p$. One possible candidate for this is $\mathrm{C}_c^\infty(\Omega)$, the smooth, compactly supported functions on Ω.

Theorem 4.1.21. *Let $\frac{3}{2} < p < \infty$, $0 < \mu < 3$. Then, $\overline{\mathcal{D}(A_{p,\mu})} = \overline{\mathrm{C}_c^\infty(\Omega)}$ where both closures are to be understood with respect to the $\mathrm{L}^{p,\mu}$-norm.*

Proof. Note that the inclusion $\overline{\mathrm{C}_c^\infty(\Omega)} \subseteq \overline{\mathcal{D}(A_{p,\mu})}$ is an immediate consequence of the previous result as $\mathrm{C}_c^\infty(\Omega) \subseteq \mathrm{L}^q(\Omega)$ holds true for all $1 \leq q < \infty$.

For the other inclusion let $f \in \overline{\mathcal{D}(A_{p,\mu})}$ be arbitrary. Then, by [67, Proposition 2.1.4 (i)] it holds $T(t)f \to f$ in $\mathrm{L}^{p,\mu}(\Omega)$ as $t > 0$. Regarding the regularity of $T(t)f$, we note that for fixed $t > 0$ the function $T(t)f$ is the weak solution of an elliptic system with constant coefficients and right-hand side 0. Then, [47, Corollary 5.15] implies that for fixed $t > 0$ the function $T(t)f$ is smooth on any compactly contained subset of Ω. In order to obtain an approximation that is smooth and compactly supported in Ω multiply these functions with a suitable cut-off. However, it then remains to show that this does not change anything in regard to approximation with respect to the $\mathrm{L}^{p,\mu}(\Omega)$-norm. To this end, for $n \in \mathbb{N}$, let φ_n be a smooth cut-off with compact support in Ω, satisfying $0 \leq \varphi_n \leq 1$, $\varphi_n = 0$ on $\{x \in \Omega : \mathrm{d}(x, \partial\Omega) < \frac{1}{n}\}$ and $\varphi_n = 1$ on $\{x \in \Omega : \mathrm{d}(x, \partial\Omega) > \frac{2}{n}\}$. Then $\varphi_n T(t)f$ is smooth and compactly supported in Ω for any $t > 0$ and $n \in \mathbb{N}$. Now, let $x_0 \in \Omega$ and $0 < r \leq 1$ be arbitrary but fixed. With $B_n \coloneqq \{x \in \Omega \colon \mathrm{d}(x, \partial\Omega) \leq \frac{2}{n}\} \cap B(x_0, r)$ we then have for $q > p$ to be specified later that

$$\begin{aligned}
\left(\frac{1}{r^\mu} \int_{B(x_0,r)\cap\Omega} |T(t)f - \varphi_n T(t)f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} &= \left(\frac{1}{r^\mu} \int_{B_n} |(1-\varphi_n)T(t)f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{r^\mu} \int_{B_n} |T(t)f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&= r^{-\frac{\mu}{p}} |B_n|^{\frac{1}{p}} \left(\fint_{B_n\cap\Omega} |T(t)f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq r^{-\frac{\mu}{p}} |B_n|^{\frac{1}{p}} \left(\fint_{B_n\cap\Omega} |T(t)f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= r^{-\frac{\mu}{p}} |B_n|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B_n\cap\Omega} |T(t)f|^q \,\mathrm{d}x\right)^{\frac{1}{q}}
\end{aligned}$$

holds true. Note that the last integral is well-defined for any $q > p$ as $f \in \mathrm{L}^{p,\mu}(\Omega)$ implies $f \in \mathrm{L}^p(\Omega)$ and thus, by the L^p-L^q-smoothing, $T(t)f \in \mathrm{L}^q(\Omega)$. Moreover, as $q > p$ implies $\frac{1}{p} - \frac{1}{q} > 0$ we can estimate $|B_n|^{\frac{1}{p}-\frac{1}{q}} \leq r^{d(\frac{1}{p}-\frac{1}{q})}$. Then, choosing $q = \frac{3}{3-\mu}p$ guarantees that the powers in r cancel out, and we find

$$\left(\frac{1}{r^\mu} \int_{B(x_0,r)\cap\Omega} |T(t)f - \varphi_n T(t)f|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_{B_n} |T(t)f|^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

It remains to note $B_n \subseteq \{x \in \Omega \colon \mathrm{d}(x, \partial\Omega) \leq \frac{2}{n}\}$ by definition. Then, the right hand side is independent of x_0 and r. Thus, an application of the dominated convergence theorem shows that the right-hand side tends to 0 as $n \to \infty$ and this implies that $\varphi_n T(t)f \to T(t)f$ in $\mathrm{L}^{p,\mu}(\Omega)$ as $n \to \infty$ for all $0 < t < 1$. To conclude estimate

$$\|\varphi_n T(t)f - f\|_{\mathrm{L}^{p,\mu}(\Omega)} \leq \|\varphi_n T(t)f - T(t)f\|_{\mathrm{L}^{p,\mu}(\Omega)} + \|T(t)f - f\|_{\mathrm{L}^{p,\mu}(\Omega)}$$

by the triangle inequality. By the previous arguments the right-hand side tends to 0 as $n \to \infty$ and $t \to 0$. Hence, the set where the semigroup is strongly continuous coincides with the closure of $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ in $\mathrm{L}^{p,\mu}(\Omega)$. □

Recall that in Section 2.1 we noted that in a reflexive Banach space a sectorial operator is densely defined. In the light of the aforementioned result by Zorko we deduce that Morrey spaces $\mathrm{L}^{p,\mu}(\Omega)$ with $1 < p < \infty$ and $0 < \mu < d$ are not reflexive. It remains to answer the question whether the closure of $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ in $\mathrm{L}^{p,\mu}(\Omega)$ coincides with $\mathrm{L}^{p,\mu}(\Omega)$. It turns out that this question is independent of the geometry of Ω and was answered negatively by Zorko in 1986 in [100, Example after Proposition 2]. Hence, the following definition is justified.

Definition 4.1.22. Let $\Omega \subseteq \mathbb{R}^d$ be bounded and measurable. For $0 < \mu < d$ and $1 \leq p < \infty$ we define $\mathring{\mathrm{L}}^{p,\mu}(\Omega)$ as the closure of $\mathrm{C}_c^\infty(\Omega)$ in $\mathrm{L}^{p,\mu}(\Omega)$.

This notation is inspired by [81]. Moreover, following a result by Zorko [100, Proposition 3], for $\Omega = \mathbb{R}^d$, this space is characterized as the space of those functions in $\mathrm{L}^{p,\mu}(\mathbb{R}^d)$ such that the translation is a continuous operator with respect to the $\mathrm{L}^{p,\mu}(\mathbb{R}^d)$-norm. Recall that this space also appears in the work of Kato and in the case $\Omega = \mathbb{R}^d$ it follows from the heat kernel representation that $\mathring{\mathrm{L}}^{p,\mu}(\mathbb{R}^d)$ is the largest subspace where the heat semigroup is continuous (see [59, Corollary 3.2]). This space will be important later on when solving semilinear parabolic partial differential equations. Let us now return to Morrey spaces and collect estimates for the semigroup.

Theorem 4.1.23. *Let $V \in \{\mathrm{W}_0^{1,2}(\Omega), \mathrm{W}^{1,2}(\Omega)\}$ and $0 < \mu < 3$. Then there exists $\delta > 0$ such that the estimates*

$$\begin{aligned} \|T(t)f\|_{\mathrm{L}^{p,\mu}(\Omega)} &\lesssim \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}, \quad 1 < p < \infty, \\ \|T(t) \operatorname{div} f\|_{\mathrm{L}^{p,\mu}(\Omega)} &\lesssim t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}, \quad \frac{3}{2} - \delta < p < \infty \end{aligned}$$

hold for all $f \in \mathrm{L}^{p,\mu}(\Omega)$ and $t > 0$. The implicit constants depend on p, μ, Ω and Λ. Moreover, for $T > 0$ there exists a constant $\widetilde{N_T} > 0$ depending on p, μ, Ω, Λ and T such that

$$\|T(t)f\|_{\mathrm{L}^\infty(\Omega)} \leq \widetilde{N_T} t^{-\frac{1}{2}\left(\frac{3-\mu}{p}\right)} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}, \quad \frac{3}{2} < p < \infty \tag{4.24}$$

holds for all $f \in \mathrm{L}^{p,\mu}(\Omega)$ and $0 < t < T$. In the case $V = \mathrm{W}_0^{1,2}(\Omega)$ the constant $\widetilde{N_T}$ is uniformly bounded in $0 < T < \infty$ so that (4.24) holds uniformly for $0 < t < \infty$.

Proof. This is an immediate consequence of Theorem 4.1.14, Theorem 4.1.17 and Proposition 4.1.16 together with the transference principle for the correspondence between semigroup and resolvent estimates (see [91, Proposition 3.7]). The stronger estimate for Dirichlet boundary conditions is a consequence of the exponential decay of the semigroup obtained in Remark 4.1.15 and the argument in the proof of Theorem 4.1.13. □

Finally, we are in the position to prove $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$-estimates for the semigroup.

Theorem 4.1.24. *Let $V \in \{\mathrm{W}_0^{1,2}(\Omega), \mathrm{W}^{1,2}(\Omega)\}$ and $0 < \mu < 3$. Let moreover $\frac{3}{2} < p < \infty$ and $p \le q < \infty$. Then for all $T > 0$ there exists a constant $M_T > 0$ depending on p, q, μ, Ω, Λ and T such that*

$$\|T(t)f\|_{\mathrm{L}^{q,\mu}(\Omega)} \le M_T t^{-\frac{1}{2}\left(\frac{3-\mu}{p} - \frac{3-\mu}{q}\right)} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)} \tag{4.25}$$

holds for all $f \in \mathrm{L}^{p,\mu}(\Omega)$ and all $0 < t < T$. In the case $V = \mathrm{W}_0^{1,2}(\Omega)$ the constant M_T is uniformly bounded in $0 < T < \infty$ so that (4.25) *holds uniformly for $0 < t < \infty$.*

Proof. We calculate for $x_0 \in \Omega$ and $r > 0$ for $B \coloneqq B(x_0, r)$ by virtue of the two previous estimates

$$\begin{aligned}\left(\frac{1}{r^\mu}\int_{B\cap\Omega} |T(t)f|^{q-p+p}\,\mathrm{d}x\right)^{\frac{1}{q}} &\le \|T(t)f\|_{\mathrm{L}^\infty(\Omega)}^{1-\frac{p}{q}} \left(\frac{1}{r^\mu}\int_{B\cap\Omega} |Tf|^p\,\mathrm{d}x\right)^{\frac{1}{p}\cdot\frac{p}{q}} \\ &\lesssim M_T t^{-\frac{1}{2}\frac{3-\mu}{p}(1-\frac{p}{q})} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}^{1-\frac{p}{q}} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}^{\frac{p}{q}} \\ &= M_T t^{-\frac{1}{2}\left(\frac{3-\mu}{p} - \frac{3-\mu}{q}\right)} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)}\end{aligned}$$

and this proves the claim. □

Having those estimates allows us to prove $\mathrm{L}^{p,\mu}$-$\mathrm{L}^{q,\mu}$ estimates for $T(t)\operatorname{div}$:

Theorem 4.1.25. *Let $V \in \{\mathrm{W}_0^{1,2}(\Omega), \mathrm{W}^{1,2}(\Omega)\}$ and $0 < \mu < 3$. Let moreover $\frac{3}{2} < p < \infty$ and $p \le q < \infty$. Then for all $T > 0$ there exists a constant $N_T > 0$ such that*

$$\|T(t)\operatorname{div} f\|_{\mathrm{L}^{q,\mu}(\Omega)} \le N_T t^{-\frac{1}{2}\left(\frac{3-\mu}{p} - \frac{3-\mu}{q}\right) - \frac{1}{2}} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)} \tag{4.26}$$

holds for all $f \in \mathrm{L}^{p,\mu}(\Omega)$ and all $0 < t < T$. In the case $V = \mathrm{W}_0^{1,2}(\Omega)$ the constant N_T is uniformly bounded in $0 < T < \infty$ so that (4.26) *holds uniformly for $0 < t < \infty$.*

Proof. Note that $T(t)\operatorname{div} = T(t/2)T(t/2)\operatorname{div}$ by the semigroup law. Then, a combination of (4.24) and Theorem 4.1.24 concludes the proof. □

4.2 Stokes on Lipschitz domains

Let $\Omega \subseteq \mathbb{R}^d$, $d \ge 2$, be a bounded Lipschitz domain. In this section we consider the Dirichlet problem for the Stokes system

$$\begin{aligned} -\Delta u + \nabla\phi &= f &&\text{in } \Omega \\ \operatorname{div}(u) &= 0 &&\text{in } \Omega \\ u &= 0 &&\text{on } \partial\Omega. \end{aligned}$$

Similarly to elliptic systems we realize the Stokes operator on $\mathrm{L}^2(\Omega)$ by the form method. Then we move on the L^p-scale and finally to Morrey spaces. To encode the divergence-free condition we need the following function spaces. Here, $\mathrm{C}^\infty_{c,\sigma}(\Omega)$ denotes those smooth functions with compact support and values in $\mathbb{C}^d$ whose divergence vanishes. The appropriate spaces for the Stokes operator are then defined as closure of $\mathrm{C}^\infty_{c,\sigma}(\Omega)$ with respect to distinct norms. Define $\mathrm{L}^p_\sigma(\Omega) \coloneqq \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{\mathrm{L}^p(\Omega)}}$ and similarly $\mathrm{W}^{1,p}_{0,\sigma}(\Omega) \coloneqq \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{\mathrm{W}^{1,p}(\Omega)}}$. On $\mathrm{H}^1_{0,\sigma}(\Omega) \coloneqq \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ introduce the form

$$\begin{aligned} a\colon \mathrm{H}^1_{0,\sigma}(\Omega) \times \mathrm{H}^1_{0,\sigma}(\Omega) &\to \mathbb{C} \\ (u,v) &\mapsto \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x. \end{aligned}$$

The Stokes operator on $\mathrm{L}^2(\Omega)$ with Dirichlet boundary conditions, denoted by A_2, is defined as follows

$$\begin{aligned} \mathcal{D}(A_2) &\coloneqq \Big\{u \in \mathrm{H}^1_{0,\sigma}(\Omega)\colon \exists! f \in \mathrm{L}^2_\sigma(\Omega) \text{ s.t. } \forall v \in \mathrm{H}^1_{0,\sigma}(\Omega) : a(u,v) = \int_\Omega f\overline{v}\,\mathrm{d}x\Big\} \\ A_2 u &\coloneqq f. \end{aligned}$$

The following result is well-known and we only state it for the sake of completeness. For a proof we refer to [90, Remark 5.2.2 and Proposition 5.2.5].

Proposition 4.2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and A_2 the Stokes operator. Then we have*

(i) *A_2 is closed with dense domain. Furthermore, $0 \in \rho(A_2)$.*

(ii) *$\sigma(A) \subset (0,\infty)$ and for all $\theta \in (0,\pi)$ there exists $C > 0$ such that*

$$\|\lambda(\lambda + A_2)^{-1}\|_{\mathcal{L}(\mathrm{L}^2_\sigma(\Omega))} \leq C, \text{ for all } \lambda \in \mathrm{S}_\theta. \tag{4.27}$$

In particular, $-A_2$ generates a bounded analytic semigroup on $\mathrm{L}^2_\sigma(\Omega)$.

Mitrea and Monniaux proved, see [70, Theorem 4.7], the following characterisation of the Stokes operator.

Theorem 4.2.2. *If $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain and A_2 is the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$, then*

$$\mathcal{D}(A_2) = \{u \in \mathrm{H}^1_{0,\sigma}(\Omega)\colon \exists \phi \in \mathrm{L}^2(\Omega) \textit{ s.t. } -\Delta u + \nabla\phi \in \mathrm{L}^2_\sigma(\Omega)\}.$$

Here, $-\Delta u + \nabla\phi$ is to be understood in the sense of distributions. Moreover, for $u \in \mathcal{D}(A_2)$ and corresponding $\phi \in \mathrm{L}^2(\Omega)$ we have

$$A_2 u = -\Delta u + \nabla\phi.$$

Next, we consider the case $p > 2$ and define A_p as the part of A_2 in $\mathrm{L}^p_\sigma(\Omega)$, i.e.,

$$\begin{aligned}\mathcal{D}(A_p) &\coloneqq \{u \in \mathcal{D}(A_2) \cap \mathrm{L}^p_\sigma(\Omega) \colon A_2 u \in \mathrm{L}^p_\sigma(\Omega)\}\\ A_p u &\coloneqq A_2 u.\end{aligned}$$

The question of an L^p-theory for the Stokes operator is complicated by the roughness of the underlying domain. Indeed, for bounded domains with smooth boundary it was shown by Giga in [49] that for $p \in (1,\infty)$ the Stokes operator generates an analytic semigroup on $\mathrm{L}^p(\Omega)$. On bounded C^1-domains this result was proved by Geng and Shen [45] in 2025. In the recently published paper [46] the same authors managed to show that Stokes operator generates an analytic semigroup on $\mathrm{L}^\infty_\sigma(\Omega)$ whenever either $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, is a bounded C^1-domain or $\Omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain. Even though this result is a breakthrough in the theory of the Stokes operator in nonsmooth domains, in the following we restrict ourselves to the results obtained via weak reverse Hölder estimates as this is the appropriate setting for Morrey spaces. On Lipschitz domains there seems to be a connection to the boundedness of the Helmholtz projection on divergence-free functions. For $d = 3$, Taylor conjectured in [89] that for a given bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that for $3/2 - \varepsilon < p < 3 + \varepsilon$ the Stokes operator generates a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$. This is indeed the range of p in which Fabes, Mendez and Mitrea in [39] showed boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega)$. To be precise, they assumed the boundary of Ω to be connected. This condition was later removed by Mitrea and Taylor, see [71, Proposition 8.3]. Shortly after Taylor's conjecture, Deuring constructed in [27] bounded Lipschitz domains such that for p large enough the Stokes operator does not even generate a strongly continuous semigroup on $\mathrm{L}^p_\sigma(\Omega)$. In 2012, Shen proved Taylor's conjecture for $d \geq 3$ in his seminal paper [85]. He showed weak reverse Hölder estimates for the Stokes resolvent problem which allowed him to extrapolate the boundedness estimate of the resolvent to $\mathrm{L}^p(\Omega)$ with p in the conjectured range. For an extension of Shen's result in weighted L^p-spaces on a three-dimensional bounded Lipschitz domain we refer to [29]. Here, we are only concerned with $p > 2$. Let us now present the weak reverse Hölder estimates proved by Shen.

Theorem 4.2.3. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain and let $\theta \in (0,\pi)$. Then, there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that for any $\lambda \in \mathrm{S}_\theta$ the Stokes resolvent satisfies weak reverse Hölder estimates for all $2 < p < \frac{2d}{d-1} + \varepsilon$.*

In view of Lemma 3.1.3, this is the covering argument mentioned by Shen in [85, Lemma 6.2], it suffices to consider interior balls, i.e., balls completely contained in Ω and balls centred on the boundary of Ω. Moreover, due to the self-improving property of weak reverse Hölder estimates (see the third part in Remark 3.1.2) it suffices to show weak reverse Hölder estimates for $p = \frac{2d}{d-1}$.

To provide a complete picture and add some details, we present Shen's proof of the weak reverse Hölder estimates in the following. The case of interior balls is easier and we start with those balls. By [85, Remark 5.7] we know that if (u, ϕ) is a solution to

$$\begin{aligned} \lambda u - \Delta u + \nabla \phi &= 0 \\ \operatorname{div}(u) &= 0 \end{aligned} \tag{4.28}$$

in $B(x_0, r)$, then for all $l \in \mathbb{N}$ the following estimate holds true

$$|\nabla^l u(x_0)| \leq \frac{C_l}{r^l} \left(⨍_{B(x_0,r)} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \tag{4.29}$$

where $C_l > 0$ depends only on d, l and θ. We claim that this estimate already implies interior reverse Hölder estimates.

Lemma 4.2.4. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain. Moreover, let $x_0 \in \Omega$ and $r > 0$ be such that $B(x_0, 3r) \subseteq \Omega$ is satisfied. If (u, ϕ) solves (4.28) in $B(x_0, 3r)$, then for any $2 < p < \infty$ there exists a constant C, depending only on p, d and θ such that the weak reverse Hölder estimate*

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^d} \int_{B(x_0,2r)} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

holds.

Proof. Let $y \in B(x_0, r)$ be arbitrary. Note that $B(y, r) \subseteq B(x_0, 3r)$, thus (u, ϕ) solves (4.28) on $B(y, r)$ and consequently (4.29) is satisfied. To be precise, we have

$$|u(y)| \leq C_0 \left(\frac{1}{r^d} \int_{B(y,r)} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \leq C_0 \left(\frac{1}{r^d} \int_{B(x_0,2r)} |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

As the constant is independent of y, we now take the p-th power of the previous estimate and integrate this over $B(x_0, r)$ to obtain

$$\left(\int_{B(x_0,r)} |u|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \leq C_0 \left(\int_{B(x_0,r)} \left(\frac{1}{r^d} \int_{B(x_0,2r)} |u|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \mathrm{d}y \right)^{\frac{1}{p}}.$$

Note that the outer integral does not depend on y so that dividing by $|B(x_0, r)|$ concludes the proof. □

Next, we consider weak reverse Hölder estimates for balls centred on $\partial\Omega$. For the sake of completeness we first state [85, Theorem 5.6]. Recall that the Lipschitz character of a bounded Lipschitz domain Ω was introduced in Definition 2.3.3.

Theorem 4.2.5. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, be a bounded Lipschitz domain with connected boundary. Let $u \in \mathrm{H}^1(\Omega)$ and $\phi \in \mathrm{L}^2(\Omega)$. Suppose that (u, ϕ) satisfy the Stokes resolvent system* (4.28) *in Ω for some $\lambda \in \mathrm{S}_\theta$. Then*

$$\left(\int_\Omega |u|^{p_d} \,\mathrm{d}x\right)^{\frac{1}{p_d}} \leq C \left(\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma\right)^{\frac{1}{2}} \tag{4.30}$$

where $p_d = \frac{2d}{d-1}$ and C depends on d, θ and the Lipschitz character of Ω.

Now we are in the position to prove weak reverse Hölder estimates also for balls centred on $\partial\Omega$ (see [85, Lemma 6.1]).

Lemma 4.2.6. *Let η be a Lipschitz function with $\|\eta\|_\infty \leq M$ and $u \in \mathrm{H}^1(D_\eta(2r))$ and $\phi \in \mathrm{L}^2(D_\eta(2r))$. Suppose that (u, ϕ) satisfies the Stokes resolvent system* (4.28) *in $D_\eta(2r)$ and $u = 0$ on $I_\eta(2r)$ for some $0 < r < \infty$ and $\lambda \in \mathrm{S}_\theta$. Then u satisfies weak reverse Hölder estimates for the form*

$$\left(\frac{1}{r^d}\int_{D_\eta(r)} |u|^{p_d} \,\mathrm{d}x\right)^{\frac{1}{p_d}} \lesssim \left(\frac{1}{r^d}\int_{D_\eta(2r)} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{4.31}$$

where $p_d = \frac{2d}{d-1}$.

Proof. By rescaling we may assume that $r = 1$. Let $t \in (1, 2)$ and apply Theorem 4.2.5 to u in the Lipschitz domain $D_\eta(t)$ to obtain

$$\left(\int_{D_\eta(t)} |u|^{p_d} \,\mathrm{d}x\right)^{\frac{2}{p_d}} \leq C \int_{\partial D_\eta(t)} |u|^2 \,\mathrm{d}\sigma,$$

where C depends only on d, θ and M. Note that by [90, Remark 1.3.27], the Lipschitz character of $D_\eta(t)$, is uniform in t for $t \in (1, 2)$. Since $u = 0$ on $I_\eta(2)$ by assumption, it follows that

$$\left(\int_{D_\eta(t)} |u|^{p_d} \,\mathrm{d}x\right)^{\frac{2}{p_d}} \leq C \int_{\partial D_\eta(t)\setminus I_\eta(2)} |u|^2 \,\mathrm{d}\sigma. \tag{4.32}$$

Finally, we appeal to Lemma 2.3.6 to conclude the proof. □

Note that weak reverse Hölder estimates are again proved only on Lipschitz cylinders and not on balls, analogously to the elliptic setting, see Theorem 4.1.2. However, the argumentation presented in the proof of Theorem 4.1.2 is independent of the operator in question so that it applies here as well. This covers the case $d \geq 3$.

Regarding the case $d = 2$ note that Shen states in the introduction of [85] that his approach should also work. This would answer the conjecture made by Mitrea, see [69, Conjecture 1.2], who in 2002 showed boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega)$ for $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$. Indeed in [44], Gabel and Tolksdorf examined the method developed by Shen and successfully adapted it to the case $d = 2$. To be precise, they show weak reverse Hölder estimates for $2 < p < 4+\varepsilon$ which generalizes the results by Shen to the two dimensional setting. For details we refer to their paper or the Masters thesis of Gabel [43]. Combining the results by Gabel and Tolksdorf as well as Shen, the following result is obtained.

Theorem 4.2.7. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain. For $2 < p < \frac{2d}{d-1} + \varepsilon$, the operator A_p is sectorial of angle 0 and $-A_p$ generates an analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$.*

For the L^p-setting there also is a characterisation of the domain of the Stokes operator, see [90, Theorem 5.2.11]:

Theorem 4.2.8. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator A_p is given by

$$\mathcal{D}(A_p) = \{u \in \mathrm{W}^{1,p}_{0,\sigma}(\Omega) \colon \exists \phi \in \mathrm{L}^p(\Omega) \text{ s.t. } -\Delta u + \nabla \phi \in \mathrm{L}^p_\sigma(\Omega)\}$$

where $-\Delta u + \nabla \pi$ is be understood in the sense of distributions. If ϕ is the pressure belonging to $u \in \mathcal{D}(A_p)$, then $A_p u$ is given by

$$A_p u = -\Delta u + \nabla \phi.$$

Now, we want to consider the Stokes operator on Morrey spaces. As ground space, we define $\mathrm{L}^{p,\mu}_\sigma(\Omega) \coloneqq \mathrm{L}^{p,\mu}(\Omega) \cap \mathrm{L}^p_\sigma(\Omega)$. Define $A_{p,\mu}$ as the part of A_p in $\mathrm{L}^{p,\mu}_\sigma(\Omega)$, i.e.,

$$\begin{aligned} \mathcal{D}(A_{p,\mu}) &\coloneqq \{u \in \mathrm{L}^{p,\mu}_\sigma(\Omega) \cap \mathcal{D}(A_p) \colon A_p u \in \mathrm{L}^{p,\mu}_\sigma(\Omega)\} \\ A_{p,\mu} u &\coloneqq A_p u. \end{aligned}$$

For $2 \le p < \frac{2d}{d-1} + \varepsilon$ (with ε as in the previous theorem), we have

$$\begin{aligned}\mathcal{D}(A_{p,\mu}) &= \{u \in \mathrm{L}^{p,\mu}_\sigma(\Omega) \cap \mathrm{W}^{1,p}_{0,\sigma}(\Omega) \colon \exists \phi \in \mathrm{L}^p(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^{p,\mu}_\sigma(\Omega)\} \\ A_{p,\mu} u &= -\Delta u + \nabla \phi.\end{aligned}$$

Then we obtain the following result for the Stokes operator on Morrey spaces. The proof is the adaptation to Morrey spaces of the proof of [43, Theorem 1.17]. For further details on the Helmholtz projection we refer to [90, Section 1.1.4].

Theorem 4.2.9. *Let $\Omega \subseteq \mathbb{R}^d$, $d \ge 2$, be a bounded Lipschitz domain. For $2 \le p < \frac{2d}{d-1} + \varepsilon$ and $0 \le \mu < d(1 - \frac{p}{pd+\varepsilon})$ the operator $A_{p,\mu}$ is sectorial of angle 0 and $-A_{p,\mu}$ generates an analytic semigroup on $\mathrm{L}^{p,\mu}_\sigma(\Omega)$.*

Proof. For $\lambda \in \mathrm{S}_\theta$ consider the operator

$$\begin{aligned}T_\lambda \colon \mathrm{L}^2(\Omega) &\to \mathrm{L}^2_\sigma(\Omega) \\ f &\mapsto (|\lambda| + 1)(\lambda + A_2)^{-1} \mathbb{P}_2 f\end{aligned}$$

where $\mathbb{P}_2$ is the Helmholtz projection on $\mathrm{L}^2(\Omega)$. It is classical that the operator family T_λ is uniformly bounded on $\mathrm{L}^2(\Omega)$, see, e.g., [43, Theorem 1.17]. Gabel and Tolksdorf then showed that this operator family additionally satisfies weak reverse Hölder estimates for $2 < p < 4 + \varepsilon$ for some $\varepsilon > 0$. For $d \ge 3$, Shen proved the same weak reverse Hölder estimates in the range $2 < p < \frac{2d}{d-1} + \varepsilon$. Applying Theorem 3.2.8, we obtain uniform boundedness of T_λ on $\mathrm{L}^{p,\mu}(\Omega)$ for p and μ as in the statement of the theorem. Finally let us consider $f \in \mathrm{L}^{p,\mu}_\sigma(\Omega)$. Then, $u := (\lambda + A_p)^{-1} f \in \mathcal{D}(A_p)$. Moreover, $\mathbb{P}_2 f = f$ holds, so that $u = (1 + |\lambda|)^{-1} T_\lambda f$ is true. Next, by the boundedness of T_λ on $\mathrm{L}^{p,\mu}(\Omega)$ we deduce $u \in \mathrm{L}^{p,\mu}(\Omega)$ which in turn implies $u \in \mathrm{L}^{p,\mu}_\sigma(\Omega)$. Moreover,

$$A_p u = f - \lambda u \in \mathrm{L}^{p,\mu}_\sigma(\Omega) \tag{4.33}$$

holds and thus $u \in \mathcal{D}(A_{p,\mu})$ follows. Additionally, we verify $(\lambda + A_{p,\mu})u = f$ and therefore $u = (\lambda + A_{p,\mu})^{-1} f$. Finally, the necessary decay in λ is imminent by definition of T_λ. □

Transferring further results from the case of elliptic systems appears to be out of reach due to the lack of L^p-L^∞-estimates. We close this section with a remark regarding the strong continuity of the generated the generated semigroup.

Remark 4.2.10. In the spirit of elliptic problems we may define

$$\mathring{\mathrm{L}}^{p,\mu}_\sigma(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{\mathrm{L}^{p,\mu}(\Omega)}}. \tag{4.34}$$

By definition, this is a subspace of $\mathrm{L}^{p,\mu}(\Omega)$ and also of $\mathrm{L}^p_\sigma(\Omega)$ and hence of $\overline{\mathrm{L}^{p,\mu}_\sigma(\Omega)}$. It remains open whether this space, as in the elliptic setting, coincides with $\overline{\mathcal{D}(A_{p,\mu})}$. Note that the proof of the elliptic case does not apply here as multiplying with a cut-off destroys the divergence-free condition.

CHAPTER 5

Solving an abstract semilinear Cauchy problem à la Giga

We consider general problems of the form

$$
\text{(ACP)} \qquad \begin{aligned} \partial_t u(t) + Au(t) &= Fu(t), \\ u(0) &= a, \end{aligned}
$$

where F describes the nonlinear part of the equation and $-A$ generates an analytic semigroup on, possibly a subspace of, certain Morrey space $\mathrm{L}^{p,\mu}$. Note that necessary boundary conditions are encoded in the operator A. Such problems are called *abstract Cauchy problems*.

Often in the theory of partial differential equations the notion of classical solutions is not suitable and other concepts of solutions are required. Here focus on *mild solutions*. Before giving a precise definition in a moment let us mention that u is a mild solution to (ACP) if it satisfies the variation of constants formula, also called Duhamel formula. Further concepts include so-called *strong solutions* or *weak solutions*. Broadly speaking, strong solutions require less spatial regularity and satisfy in the question only almost everywhere. For a more detailed discussion on strong solutions and their relation to mild solutions we refer to [75, Chapter 4.2]. Weak solutions are defined via testing the equation with suitable test functions. For an introduction on this matter we refer to [38, Chapter 7.1].

In this chapter we aim to prove a Morrey space version of [90, Theorem 6.3.5] where the author proves a general theorem on Giga's iteration scheme for semilinear parabolic problems. In the second part we then apply this abstract theorem to elliptic systems with Dirichlet or Neumann boundary conditions. More precisely, we assume a certain structure on the nonlinearity F and then show well-posedness in critical Morrey spaces for small initial value. For Dirichlet boundary conditions we obtain global-in-time solutions, for Neumann boundary conditions however, only local-in-time existence is proved.

Coming back to (ACP), in the L^p-setting a first solution theory was developed by Giga in [50]. We refer to that paper and the references therein for an historic overview. In the context of the Navier-Stokes equations we further mention the results by Fujita and Kato [42] as well as Kato [58]. In the latter paper a result similar to the one in [50] is proved but focused on the Navier-Stokes equations instead of an abstract Cauchy problem. For an L^p-based solution theory of the Navier-Stokes equations on bounded Lipschitz domains the assumptions by Giga are too strong as he required an analytic semigroup for all $1 < p < \infty$. In his PhD-thesis [90], Tolksdorf proved a version of Giga's theorem which takes care of this issue. For a Morrey space version further adjustments are necessary, the general structure of the proof, however, stays the same. One major difference between Lebesgue and Morrey spaces is that smooth compactly supported functions are not dense in the latter one. Note that a density argument is crucial as Giga's proof requires higher integrability for the treatise of the initial value.

To close the introductory part of this chapter let us define mild solutions of (ACP) in Morrey spaces.

Definition 5.0.1. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set. Assume that $a \in \mathrm{L}^{p,\mu}(\Omega)$ for some $1 < p < \infty$ and $0 < \mu < d$. Then, a continuous function $u \colon [0,T) \to \mathrm{L}^{p,\mu}(\Omega)$ is a mild solution of (ACP) on $[0,T)$ if it satisfies

$$u(t) = \mathrm{e}^{-tA}\, a + \int_0^t \mathrm{e}^{-(t-s)A}\, Fu(s)\,\mathrm{d}s$$

for all $0 < t < T$.

5.1 Giga's iteration scheme in Morrey spaces

While for applications we are interested in three-dimensional bounded Lipschitz domains no such restriction is necessary for the general iteration scheme. So let, $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$ be a bounded measurable set.

Moreover, let $0 < \mu < d$ and $1 < p_- < p^\circ < p_+ < \infty$ be real numbers. For $p_- < p < p_+$ let $\mathrm{U}^{p,\mu}$ be a closed subspace of $\mathring{\mathrm{L}}^{p,\mu}(\Omega)$, see Definition 4.1.22, and let P_p be a continuous projector from $\mathring{\mathrm{L}}^{p,\mu}(\Omega)$ onto $\mathrm{U}^{p,\mu}$ such that the restriction of P_p to $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ is independent of p.

Choosing $\mathring{\mathrm{L}}^{p,\mu}(\Omega)$ allows for the desired density argument, by definition of this space. Moreover, this choice guarantees the strong continuity of the semigroups in our application to elliptic systems as in that setting this space is the closure of the domain the operators in question, see Theorem 4.1.21.

In the spirit of [90, Assumption 6.3.2] we assume the following properties. Note that, in difference to the work of Tolksdorf, we use $d-\mu$ instead of n and only require $d-\mu>0$ instead of $n\geq 1$.

Assumption 5.1.1. (i) *For all $p_- < p < p_+$ the operators $-A_{p,\mu}\colon \mathcal{D}(A_{p,\mu}) \subset \mathrm{U}^{p,\mu} \to \mathrm{U}^{p,\mu}$ generate strongly continuous analytic semigroups of operators $(\mathrm{e}^{-tA_{p,\mu}})_{t\geq 0}$ on $\mathrm{U}^{p,\mu}$ that are consistent on the $\mathrm{U}^{p,\mu}$ scale. Moreover, there is a constant $m\geq 1$ such that for every fixed $0<T<\infty$ there exists $M_T>0$ such that for all $f\in \mathrm{U}^{p,\mu}$ and $0<t<T$ the estimate*

$$\|\,\mathrm{e}^{-tA_{p,\mu}} f\|_{\mathrm{U}^{q,\mu}} \leq M_T t^{-\sigma} \|f\|_{\mathrm{U}^{p,\mu}}$$

holds with $\sigma := \frac{d-\mu}{m}\left(\frac{1}{p}-\frac{1}{q}\right)$ and $p_- < p \leq q < p_+$.

(ii) *There exist constants $\alpha>0$ and $0\leq\gamma<m$ such that for each $p_- < p \leq p^\circ$ the nonlinear term Fu can be decomposed into the composition $Fu = \Gamma G u$ with Γ and G having the following properties. The operator $\Gamma\colon \mathcal{D}(\Gamma)\subset \mathrm{L}^{p,\mu}(\Omega) \to \mathrm{U}^{p,\mu}$ is linear and densely defined. Moreover, for each $t>0$ and $p^\circ \leq q < p_+$ the operator $\mathrm{e}^{-tA_{p,\mu}}\Gamma$ extends to a bounded operator from $\mathrm{L}^{p,\mu}(\Omega)$ into $\mathrm{L}^{q,\mu}(\Omega)$ such that for each given $0<T<\infty$ there exists a constant $N_{1,T}>0$ such that for all $f\in \mathrm{L}^{p,\mu}(\Omega)$ and all $0<t<T$*

$$\|\,\mathrm{e}^{-tA_p}\,\Gamma f\|_{\mathrm{U}^{q,\mu}} \leq N_{1,T} t^{-\sigma-\frac{\gamma}{m}} \|f\|_{\mathrm{L}^{p,\mu}(\Omega)},$$

where m and σ are the same numbers as in the first part.

The operator G is supposed to reflect the nonlinear part and satisfies $G0=0$. Moreover, there exists a constant $N_2>0$ such that

$$\|Gv-Gw\|_{\mathrm{L}^{p,\mu}(\Omega)} \leq N_2 \|v-w\|_{\mathrm{U}^{p(1+\alpha),\mu}} (\|v\|^{\alpha}_{\mathrm{U}^{p(1+\alpha),\mu}} + \|w\|^{\alpha}_{\mathrm{U}^{p(1+\alpha),\mu}})$$

for all $v,w\in \mathrm{U}^{p(1+\alpha),\mu}$.

Assumption 5.1.2. *With p_-, p_+ and α as above, assume that*

$$\max\{p_-(1+\alpha), p^\circ\} < \min\{p_+, p^\circ(1+\alpha)\}$$

holds true.

For a in some $\mathrm{U}^{r,\mu}$-space and some $p_- < p \leq p_+$ introduce the iteration scheme

$$\begin{aligned} u_0(t) &:= \mathrm{e}^{-tA_{r,\mu}}\, a, \\ u_{j+1}(t) &:= u_0(t) + Su_j(t) := u_0(t) + \int_0^t \mathrm{e}^{-(t-\tau)A_{p,\mu}}\, Fu_j(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Note that a fixed point of this iteration is, under sufficient regularity in time, a mild solution of (ACP). In the following we denote by $\mathrm{BC}([0,T_0);\mathrm{U}^{p,\mu})$ the space of bounded continuous functions on $[0,T_0)$ with values in $\mathrm{U}^{p,\mu}$. Then, the Morrey space version of [90, Theorem 6.3.5] reads as follows:

Theorem 5.1.3. *Under the last two assumptions the following holds true for every $T > 0$.*

(i) *Let p_0, p_0' denote*

$$p_0 := \frac{(d-\mu)\alpha}{m-\gamma} \text{ and } p_0' := \max\left\{p_0, p^\circ, \frac{(d-\mu)p^\circ(1+\alpha)}{d-\mu+p^\circ m}\right\}.$$

Then, $p_0 = p_0'$ holds if $p_0 \geq p^\circ$. Suppose further, that $a \in \mathrm{U}^{r,\mu}$ for a fixed $p_0' < r < \min\{p_+, p^\circ(1+\alpha)\}$ or if $p_0 > p^\circ$, r could even be equal to p_0. Then there is $0 < T_0 \leq T$ and a function u on $[0, T_0)$ that satisfies

$$\begin{aligned} &t \mapsto t^\sigma u(t) \in \mathrm{BC}([0,T_0); \mathrm{U}^{p,\mu}) && \text{for } r \leq p < \min\{p_+, p^\circ(1+\alpha)\}, \\ &t^\sigma \|u(t)\|_{\mathrm{U}^{p,\mu}} \to 0 \text{ as } t \to 0 && \text{for } r < p < \min\{p_+, p^\circ(1+\alpha)\}, \end{aligned}$$

whenever $\sigma := \frac{d-\mu}{m}\left(\frac{1}{r} - \frac{1}{p}\right)$ satisfies additionally $0 \leq \sigma < \frac{1}{1+\alpha}$. Moreover, for every $\max\{r, p_-(1+\alpha)\} < p < \min\{p_+, p^\circ(1+\alpha)\}$ and $h := \frac{p}{1+\alpha}$, u solves the equation

$$u(t) = \mathrm{e}^{-tA_{r,\mu}}\, a + \int_0^t \mathrm{e}^{-(t-\tau)A_{h,\mu}}\, \Gamma G u(\tau)\, \mathrm{d}\tau \qquad (t \in (0, T_0)). \tag{5.1}$$

(ii) *If $r > p_0'$, the lifespan has the lower bound*

$$T_0 \geq C\|a\|_{\mathrm{U}^{r,\mu}}^{-\frac{\alpha}{1-\beta(r)}} \tag{5.2}$$

with $\beta(r) := \frac{1}{m}(\gamma + \frac{(d-\mu)\alpha}{r})$ and $C > 0$ depending only on α, M_T, $N_{1,T}$, N_2, γ, n, m, r, μ and p.

(iii) *If $p_0 > p^\circ$ and*

$$p_0 \leq p < \min\{p_+, p^\circ(1+\alpha)\} \text{ with } \frac{d-\mu}{m}\left(\frac{1}{p_0} - \frac{1}{p}\right) < \frac{1}{1+\alpha},$$

there is a positive constant $\mathcal{C}$ such that if $\|a\|_{\mathrm{U}^{p_0,\mu}} < \mathcal{C}$, then T_0 equals T. Furthermore, if M_T and $N_{1,T}$ are uniformly bounded in $0 < T < \infty$, we can take $T = \infty$ and get

$$\|u\|_{\mathrm{U}^{p,\mu}} \leq Ct^{-\sigma}, \quad (0 < t < \infty), \tag{5.3}$$

with C and $\mathcal{C}$ depending only on α, M_T, $N_{1,T}$, N_2, γ, n, m, r, μ and p (if $T = \infty$, these constants depend on the uniform bounds of M_T and $N_{1,T}$).

Proof. The first calculation of verifying $p_0 = p_0'$ if $p_0 \geq p^\circ$ is independent of the spaces we work on. Note that compared to [90, Theorem 6.3.5] we need to replace n by $d-\mu$ but this does not affect the proof regardless of the fact that $d-\mu \geq 1$ might not be true.

Let us continue with the next step. Assume that $p_-(1+\alpha) < p \leq p^\circ(1+\alpha)$ and $p^\circ \leq s < p_+$. Define $\delta := \frac{1}{m}(\frac{d-\mu}{s} - \frac{d-\mu}{p})$, $h := \frac{p}{1+\alpha}$ and $\beta(p) := \frac{1}{m}(\gamma + \frac{(d-\mu)\alpha}{p})$. Then, we claim that for all $0 < t < T$ and all $v, w \in \mathrm{U}^{p,\mu}$ we have

$$\| \mathrm{e}^{-tA_h}(Fv - Fw)\|_{\mathrm{U}^{s,\mu}} \leq N_{1,T} N_2 t^{\delta-\beta(p)} \|v-w\|_{\mathrm{U}^{p,\mu}} (\|v\|^\alpha_{\mathrm{U}^{p,\mu}} + \|w\|^\alpha_{\mathrm{U}^{p,\mu}}). \tag{5.4}$$

In order to prove this, we use the second part of Assumption 5.1.1. By the first estimate therein we have

$$\begin{aligned} \| \mathrm{e}^{-tA_h}(Fv - Fw)\|_{\mathrm{U}^{s,\mu}} &= \| \mathrm{e}^{-tA_h} \Gamma(Gv - Gw)\|_{\mathrm{U}^{s,\mu}} \\ &\leq N_{1,T} t^{\delta-\beta(p)} \|Gv - Gw\|_{\mathrm{L}^{h,\mu}(\Omega)}. \end{aligned}$$

Applying the second estimate therein concludes the proof of (5.4).

Next, we fix a number $0 < T_0 \leq T$ and continue by establishing an a priori estimate for the quantity

$$K_j := K_j(T_0) := \sup_{0<t<T_0} t^\sigma \|u_j(t)\|_{\mathrm{U}^{p,\mu}}$$

where σ and p are chosen such that $\sigma = \frac{1}{m}(\frac{d-\mu}{r} - \frac{d-\mu}{p})$ and

$$\begin{aligned} &\max\left\{0, \frac{d-\mu}{m}\left(\frac{1}{r} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)}\right)\right\} < \sigma \quad \text{and} \\ &\sigma < \min\left\{\frac{1}{1+\alpha}, \frac{d-\mu}{m}\left(\frac{1}{r} - \frac{1}{\min\{p_+, p^\circ(1+\alpha)\}}\right)\right\} \end{aligned} \tag{5.5}$$

hold. If this is possible, this implies

$$p < \min\{p_+, p^\circ(1+\alpha)\}, \quad p > \max\{p_-(1+\alpha), p^\circ\}, \quad \text{and } p > r,$$

so that particularly $p > p_0$. With the same argumentation as in [90], in order to prove that the choice of p for (5.5) to hold is possible, it is sufficient to prove that

$$\frac{d-\mu}{m}\left(\frac{1}{r} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)}\right) < \frac{1}{1+\alpha}.$$

We have $\mu < d$ so that $d - \mu > 0$ holds. Hence, we can use the same argumentation (after replacing n by $d - \mu$) as in [90] to show that the choice in (5.5) is possible. Recall that our iteration scheme is defined via

$$u_{j+1} = u_0 + Su_j.$$

In order to estimate u_{j+1} use the auxiliary estimate (5.4) with $v = u_j$, $w = 0$ and $s = p$ to obtain

$$t^\sigma \|Su_j(t)\|_{\mathrm{U}^{p,\mu}} \leq t^\sigma \int_0^t \frac{N_{1,T} N_2}{(t-\tau)^{\beta(p)}} \|u_j(\tau)\|_{\mathrm{U}^{p,\mu}}^{1+\alpha} \,\mathrm{d}\tau. \tag{5.6}$$

Here, the condition $\max\{p_-(1+\alpha), p^\circ\} < p < \min\{p_+, p^\circ(1+\alpha)\}$ is crucial as it allows us to apply (5.4). Next, estimating $t^\sigma \|u_{j+1}(t)\|_{\mathrm{U}^{p,\mu}}$ by means of the triangle inequality and the previous calculation yields

$$\begin{aligned} t^\sigma \|u_{j+1}(t)\|_{U^{p,\mu}} &\leq K_0 + t^\sigma \int_0^t \frac{N_{1,T} N_2}{(t-\tau)^{\beta(p)}} \|u_j(\tau)\|_{\mathrm{U}^{p,\mu}}^{1+\alpha} \,\mathrm{d}\tau \\ &\leq K_0 + t^{\sigma - \beta(p)} N_{1,T} N_2 K_j^{1+\alpha} \int_0^t \frac{1}{(1-\frac{\tau}{t})^{\beta(p)}} \frac{1}{\tau^{\sigma(1+\alpha)}} \,\mathrm{d}\tau \\ &= K_0 + t^{\sigma - \beta(p)} N_{1,T} N_2 K_j^{1+\alpha} t^{-\sigma(1+\alpha)} t \int_0^1 \frac{1}{(1-s)^{\beta(p)}} \frac{1}{s^{\sigma(1+\alpha)}} \,\mathrm{d}s, \end{aligned}$$

where in the last step we performed the substitution $s = \frac{\tau}{t}$. Note that this integral, denoted by B, is finite due to $\sigma < \frac{1}{1+\alpha}$ which deals with the singularity in 0 and the singularity in 1 is controlled because of $p > p_0$, $\beta(p_0) = 1$ and the fact that the function β is strictly decreasing. Ordering all powers of t in combination with $\beta(p) + \alpha\sigma = \beta(r)$ and taking the supremum with respect to t yields the iterative estimate

$$K_{j+1} \leq K_0 + N_{1,T} N_2 B K_j^{1+\alpha} T_0^{1-\beta(r)}.$$

As Giga and Tolksdorf we use, for a technical reason, the less sharp estimate

$$K_{j+1} \leq K_0 + 2N_{1,T} N_2 B K_j^{1+\alpha} T_0^{1-\beta(r)}.$$

Assume now that the inequality

$$T_0^{1-\beta(r)} K_0^\alpha < \left(\frac{\alpha}{1+\alpha}\right)^\alpha \frac{1}{2(1+\alpha) N_{1,T} N_2 B} \tag{5.7}$$

is valid. Define $K \coloneqq K(T_0) \coloneqq \frac{1+\alpha}{\alpha} K_0(T_0)$. Then, by an elementary calculation, we find that

$$K_j < K \quad (j \geq 0), \tag{5.8}$$

$$2N_{1,T} N_2 B T_0^{1-\beta(r)} K^\alpha < \frac{1}{1+\alpha}, \tag{5.9}$$

and

$$K \to 0 \text{ as } K_0 \to 0. \tag{5.10}$$

Let us first prove (5.9). Plugging $K = \frac{1+\alpha}{\alpha} K_0$ into (5.7) readily yields (5.9). Then, (5.8) is proved via induction. For $j = 0$, the statement is trivially satisfied as $\alpha > 0$. For the inductive step let $j \geq 0$ and suppose $K_j < K$. We want to show $K_{j+1} < K$. Using the bound on K_j and the definition of K we find

$$\begin{aligned} K_{j+1} &\leq K_0 + 2N_{1,T}N_2 B K_j^{1+\alpha} T_0^{1-\beta(r)} \\ &< \frac{\alpha}{1+\alpha} K + 2N_{1,T}N_2 B K^{1+\alpha} T_0^{1-\beta(r)} \\ &< \frac{\alpha}{1+\alpha} K + \frac{1}{1+\alpha} K = K \end{aligned}$$

and this proves (5.8). Finally, (5.10) is an immediate consequence of the definition of K. This means that under the condition (5.7) we have an a priori bound for K_j.

Next, we deduce conditions for T_0 and a guaranteeing (5.7). We start by proving that for $p > r$ we have

$$t^\sigma \| \mathrm{e}^{-tA_{r,\mu}} a \|_{\mathrm{U}^{p,\mu}} \to 0 \quad \text{as} \quad t \to 0. \tag{5.11}$$

Here it is crucial that the initial value a is taken from only $\mathring{\mathrm{L}}^{p,\mu}(\Omega)$ and not from $\mathrm{L}^{p,\mu}(\Omega)$ as smooth compactly supported functions are not dense in the latter space and even more, suitable (with more integrability) dense subspaces are not available. Recall that $\mathrm{C}_\mathrm{c}^\infty(\Omega) \subseteq \mathrm{L}^{p,\mu}(\Omega)$ for all $1 \leq p < \infty$.

By density, there is a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ with $a_i \to a$ in $\mathrm{L}^{r,\mu}(\Omega)$. Moreover, we assumed the projection P_q to be independent of q on $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ so that $P_q a_i \in \mathrm{U}^{q,\mu}$ for all $p_- < q < p_+$. Applying the estimate on the semigroup, we obtain

$$\begin{aligned} t^\sigma \| \mathrm{e}^{-tA_{r,\mu}} a \|_{\mathrm{U}^{p,\mu}} &\leq t^\sigma \| \mathrm{e}^{-tA_{r,\mu}} (a - P_r a_i) \|_{\mathrm{U}^{p,\mu}} + t^\sigma \| \mathrm{e}^{-tA_{r,\mu}} P_r a_i \|_{\mathrm{U}^{p,\mu}} \\ &\leq M_T \| P_r a - P_r a_i \|_{\mathrm{U}^{r,\mu}} + t^{\frac{d-\mu}{2m}\left(\frac{1}{r} - \frac{1}{p}\right)} \| P_r a_i \|_{\mathrm{U}^{\frac{2rp}{r+p},\mu}} \end{aligned}$$

for all $i \in \mathbb{N}$. For given $\varepsilon > 0$, due to the boundedness of P_r and $a_i \to a$ in $\mathrm{L}^{r,\mu}$, we can find $i_0 \in \mathbb{N}$ such that $M_T \| P_r a - P_r a_{i_0} \|_{\mathrm{U}^{r,\mu}} < \frac{\varepsilon}{2}$. Additionally, as $r > p$ and $\frac{2rp}{r+p} < p_+$ we can choose $t > 0$ small enough such that the second term is bounded by $\frac{\varepsilon}{2}$ as well. This proves the desired convergence. Particularly, this proves that

$$K_0(T_0) \to 0 \quad \text{as} \quad T_0 \to 0. \tag{5.12}$$

Next, we verify that the assumptions of the theorem imply the validity of (5.7). If $r > p_0'$, so in particular $r > p_0$, then $\beta(r) < 1$ follows due to $\beta(p_0) = 1$. Now, the condition $T_0 < C\|a\|_{\mathrm{U}^{r,\mu}}^{-\frac{\alpha}{1-\beta(r)}}$ with

$$C = \left[\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} \frac{1}{2(1+\alpha)BN_{1,T}N_2M_T^{\alpha}}\right]^{\frac{1}{1-\beta(r)}}$$

ensures the validity of (5.7) since the boundedness of the semigroup implies that $K_0/\|a\|_{\mathrm{U}^{r,\mu}}$ is bounded by M_T. Indeed,

$$\begin{aligned} T_0^{1-\beta(r)}K_0^{\alpha} &\leq C^{1-\beta(r)}\|a\|_{U^{r,\mu}}^{-\alpha}K_0^{\alpha} \\ &\leq M_T^{\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\frac{1}{2(1+\alpha)BN_{1,T}N_2M_T^{\alpha}} \\ &= \left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\frac{1}{2(1+\alpha)BN_{1,T}N_2}. \end{aligned}$$

Hence, the estimate (5.2) on the lifespan follows.

In the case $p_0 > p^{\circ}$ and $r = p_0$, the convergence (5.12) shows that (5.7) is valid for small T_0. Moreover, since $\beta(p_0) = 1$, (5.7) does not include T_0 explicitly, so that it holds with $T_0 = T$ whenever

$$\|a\|_{\mathrm{U}^{p_0,\mu}} < \frac{\alpha}{1+\alpha}\left(\frac{1}{2(1+\alpha)BN_{1,T}N_2M_T^{\alpha}}\right)^{\frac{1}{\alpha}}.$$

If M_T and $N_{1,T}$ are uniformly bounded in $0 < T < \infty$ this implies that K is finite for $T_0 = \infty$ and (5.7) is finite for small $\|a\|_{\mathrm{U}^{p_0,\mu}}$ as well.

It remains to show convergence of $(u_j)_{j\in\mathbb{N}}$ and then check that the limit function is the desired function u. Moreover, (5.3) needs to be shown. Finally, (5.5) has to be relaxed so that all $0 \leq \sigma < \frac{1}{1+\alpha}$ considered. However, this is standard and completely analogous to the proof of Tolksdorf so that we do not include this here. □

Remark 5.1.4. Depending on the equation, stronger assumptions as in 5.1.1, but not as strong as in [50], might be satisfied. In particular, we consider the case where $p_+ = \infty$ and the decomposition in the second part is valid for all $p_- < p \leq q < \infty$.

5.2 Iteration for elliptic systems with Neumann or Dirichlet boundary condition

In this section we consider semilinear parabolic problems on bounded Lipschitz domains with a particular nonlinearity as well as suitable boundary conditions. To be precise, for a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ and $T > 0$, we are interested in solving

$$\text{(SLP)} \qquad \begin{aligned} \partial u_t - \operatorname{div}(L\nabla u) &= \operatorname{div}(Gu) \quad \text{in } (0,T)\times\Omega \\ u(0) &= a \quad \text{in } \Omega \end{aligned}$$

complemented with either Dirichlet boundary conditions, i.e.,

$$u = 0 \quad \text{on } (0,T)\times\partial\Omega$$

or Neumann boundary conditions, i.e.,

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega$$

where $\frac{\partial u}{\partial \nu}$ is defined as in (4.3). Here, $G\colon \mathbb{C}^m \to \mathbb{C}^{d\times m}$ describes the nonlinearity to be specified later and the divergence of a matrix is defined column-wise. We assume the coefficients L to be constant, real, symmetric, strongly elliptic in the sense of Subsection 4.1.1. Then, we are in the setting of Section 4.1. Recall that in contrast to Neumann boundary conditions, for Dirichlet boundary conditions we were able to prove uniform-in-time estimates. This difference will also be reflected in this section's results. Our goal is obtain a well-posedness result in the critical space $\mathrm{L}^{3-\mu,\mu}(\Omega)$ in the spirit of the full space results on the Navier-Stokes equations in [59]. While for Dirichlet boundary conditions and suitable μ, global-in-time mild solutions are available under a smallness condition for the initial value, for Neumann boundary conditions this question remains open.

In order to achieve this, we work in the setting of Section 5.1 and write (SLP) as an abstract Cauchy problem of the form

$$\text{(5.13)} \qquad \begin{aligned} \partial_t u(t) + Au(t) &= Fu \\ u(0) &= a \end{aligned}$$

where $-A$ the operator corresponding to an elliptic system with the same assumptions on the coefficients and boundary conditions as above.

Let us now argue that (SLP) indeed fits into this setting. For $1 < p < \infty$ and $0 < \mu < 3$ we choose $\mathrm{U}^{p,\mu} = \mathring{\mathrm{L}}^{p,\mu}(\Omega)$. This then fixes the projection P_p to be the identity. Let us now argue that Assumption 5.1.1 is satisfied.

Lemma 5.2.1. *In the above situation Assumption 5.1.1 is satisfied up to the estimate on the nonlinearity G. For the appearing constants we find $m = 2$, $\gamma = 1$, $p_- = \frac{3}{2}$, $p_+ = \infty$ and $p^\circ \in (3/2, \infty)$ can be chosen arbitrarily.*

Proof. Theorem 4.1.14 and Theorem 4.1.21 state that $-A_{p,\mu}$ generates a strongly continuous analytic semigroup on $\mathrm{U}^{p,\mu}$. Furthermore, recall that in Theorem 4.1.24 we have shown that for any $\frac{3}{2} < p < \infty$ and $p \leq q < \infty$ and $0 < T < \infty$ there exists a constant M_T such that

$$\| \mathrm{e}^{-tA_p} f\|_{\mathrm{U}^{q,\mu}} \leq M_T t^{-\frac{3-\mu}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\mathrm{U}^{p,\mu}}$$

holds for all $0 < t < T$. Moreover, in the case of Dirichlet boundary conditions the constant M_T is uniformly bounded in $0 < T < \infty$. Regarding the estimate on the nonlinear term we have shown in Theorem 4.1.25 that for any $\frac{3}{2} < p < \infty$ and $p \leq q < \infty$ as well as $0 < T < \infty$ there exists a constant $N_T > 0$ such

$$\| \mathrm{e}^{-tA_p} \operatorname{div} f\|_{\mathrm{U}^{q,\mu}} \lesssim N_T t^{-\frac{3-\mu}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{\mathrm{L}^{p,\mu}}$$

is satisfied for all $0 < t < T$. Again, in the case of Dirichlet boundary conditions the constant N_T is uniformly bounded in $0 < T < \infty$. This fixes $\gamma = 1$. ☐

In order to completely satisfy Assumption 5.1.1 we finally assume that the nonlinearity G satisfies the estimate

$$\|Gv - Gw\|_{\mathrm{L}^{p,\mu}(\Omega)} \lesssim \|v-w\|_{U^{2p,\mu}} \left(\|v\|_{U^{2p,\mu}} + \|w\|_{U^{2p,\mu}}\right), \tag{5.14}$$

for all $v, w \in \mathrm{U}^{2p,\mu}$. This corresponds to $\alpha = 1$.

Remark 5.2.2.
- Note that in contrast to the situation in [90] we do not require two separate ranges for p and q and no additional variable p° is needed here. However, we want to apply Theorem 5.1.3 to our setting and for this application a careful choice of p° will be necessary later on.
- Revising these assumptions we notice that the assumptions here are indeed stronger than those in Theorem 5.1.3 but still weaker (due to $p_- = \frac{3}{2}$) than the result by Giga (his result is on the L^p-scale but holds with the modifications presented in the previous section also on the Morrey-space scale).
- If in the case of Neumann boundary conditions one could prove global-in-time estimates on mean value free functions, one could further restrict $\mathrm{U}^{p,\mu}$ to mean value free functions as well. Similarly for the Stokes operator $\mathrm{U}^{p,\mu}$ would need to be adapted to only contain divergence free functions.

Before looking into the proof let us turn to Assumption 5.1.2 and the remaining constants in Theorem 5.1.3.

Remark 5.2.3.
- As mentioned before, no p° was needed so far in our setting. However, for fixed $p^\circ \in (3/2, \infty)$ we can choose p_+ large enough so that Assumption 5.1.2 is satisfied.

- Concerning the remaining constants in Theorem 5.1.3, in our setting we have

$$p_0 := \frac{(d-\mu)\alpha}{m-\gamma} = 3-\mu.$$

In Theorem 5.1.3 the lower bound for the integrability exponent r of the initial value is provided by the parameter p_0' which depends on p°. As mentioned before, our setting does not require the constant p°, but $p^\circ \in (3/2,\infty)$ can be chosen arbitrarily. Examining the definition of p_0' suggests that the following modification is sensible. We define

$$\tilde{p_0}' := \max\left\{3-\mu, \frac{3}{2}, \frac{(3-\mu)\frac{3}{2}\cdot 2}{3-\mu+\frac{3}{2}\cdot 2}\right\}.$$

We claim that $\tilde{p_0}' = \max\{3-\mu, \frac{3}{2}\}$. Indeed, we have

$$\frac{3}{2} > \frac{3(3-\mu)}{3-\mu+3} \Leftrightarrow 3-\frac{1}{2}\mu > 3-\mu$$

and this is a true statement due to $\mu > 0$.

Note that in the following we restrict to $\mu > 0$ as $\mathrm{L}^{p,0}(\Omega) = \mathrm{L}^p(\Omega)$ and this is already understood in the light of the abstract result [90, Theorem 6.3.5]. In the introduction we mentioned the concept of critical spaces, i.e., a space X is critical for an equation with a certain scaling if the norm in $\mathrm{L}^\infty(0,T;X)$ is invariant under said scaling. For parabolic problems of the above form the scaling is, as for the Navier-Stokes equations, given by $u_\lambda(x,t) := \lambda u(\lambda^2 t, \lambda x)$. In the scale of Lebesgue spaces the space $\mathrm{L}^3(\Omega)$ is critical, for Morrey spaces the spaces $\mathrm{L}^{3-\mu,\mu}(\Omega)$ with $0 \leq \mu \leq 2$ are critical. The condition $\mu \leq 2$ is due to the fact, that Morrey spaces are only defined for $p \geq 1$. Let us now prove the existence of mild solutions to certain semilinear parabolic equations in critical Morrey spaces.

Theorem 5.2.4. *Assume that G is subject to* (5.14). *Let $0 < \mu < 3$. Let $a \in \mathrm{U}^{r,\mu}$ with $r \geq 3-\mu$ if $0 < \mu < 3/2$ and $r > 3/2$ if $\frac{3}{2} < \mu < 3$. Then, for any $T > 0$ there exists $0 < T_0 \leq T$ and a function u on $[0,T_0)$ that satisfies*

$$t \mapsto t^\sigma u(t) \in \mathrm{BC}([0,T_0);\mathrm{U}^{p,\mu}) \qquad \text{for } r \leq p < \infty, \tag{5.15}$$

$$t^\sigma \|u(t)\|_{\mathrm{U}^{p,\mu}} \to 0 \text{ as } t \to 0 \qquad \text{for } r < p < \infty, \tag{5.16}$$

where $\sigma = \frac{d-\mu}{m}\left(\frac{1}{r}-\frac{1}{p}\right)$. Moreover, for every $\max\{3,r\} < p < \infty$ and $h = \frac{p}{2}$, u solves the equation

$$u(t) = \mathrm{e}^{-tA_{r,\mu}}\, a + \int_0^t \mathrm{e}^{-(t-\tau)A_{h,\mu}} \operatorname{div} Gu(\tau)\,\mathrm{d}\tau \tag{5.17}$$

that is, u is a mild solution of (SLP). *If $r > \max\{3-\mu, \frac{3}{2}\}$, the lifespan has the lower bound*

$$T_0 \geq C\|a\|_{\mathrm{L}^{r,\mu}(\Omega)}^{-\frac{1}{1-\beta(r)}}$$

with $\beta(r) := \frac{1}{2}\left(1 + \frac{3-\mu}{r}\right)$ and $C > 0$ depending on r.

If $3-\mu > \frac{3}{2}$, there is a positive constant $\mathcal{C} > 0$ such that if $\|a\|_{\mathrm{L}^{3-\mu,\mu}(\Omega)} < \mathcal{C}$, then $T_0 = T$. Furthermore, in the case of Dirichlet boundary conditions, we can take $T = \infty$ and for all $3-\mu < p < \infty$ there exists a constant C such that

$$\|u(t)\|_{\mathrm{L}^{p,\mu}(\Omega)} \leq Ct^{-\sigma}.$$

holds for all $0 < t < \infty$.

Proof. As already announced we want to apply the previous result with a suitable choice of p° and p_+. We first delay choosing p_+ but in generally it will be chosen sufficiently large.

Consider first the case of $0 < \mu < \frac{3}{2}$. Then we have $p_- = \frac{3}{2} < 3-\mu = p_0$. Consequently, we can choose $p^\circ \in (p_-, p_0)$ so that in particular $p^\circ < p_0$ holds true. By the previous theorem we conclude $p_0' = p_0$ so that $r = p_0$ is an admissible choice. For $r > p_0$ note that we can choose any $p^\circ > p_-$ and choosing such p° leads to the following restriction on r:

$$p_0' < r < \min\{p_+, 2p^\circ\}.$$

This is the first appearance of p_+. As discussed, we can choose p_+ as large as necessary which implies the upper bound above is $2p^\circ$. Recall that $p^\circ > \frac{1}{2}(3-\mu)$ implies $p_0' = p^\circ$. Therefore, choosing $\frac{1}{2}(3-\mu) < p^\circ < 3-\mu$ allows us to deal with $p_0 \leq r < 2(3-\mu)$. For larger r choose $p^\circ > 3-\mu$ so that we can deal with $p^\circ < r < 2p^\circ$. As p° was arbitrary, we obtain $p_0 \leq r < \infty$ as desired.

Next, consider the case $\frac{3}{2} < \mu < 3$. Then, $3-\mu < \frac{3}{2}$ and hence, there does not exist p° with $p^\circ < p_0 = 3-\mu$. Thus, we need to proceed differently. To this end, let $\varepsilon > 0$ and set $p^\circ = \frac{3}{2} + \varepsilon$. Then, $p_0 < p^\circ$ and moreover

$$\frac{3}{2} + \varepsilon > \frac{(3-\mu)(\frac{3}{2}+\varepsilon)2}{3-\mu+2(\frac{3}{2}+\varepsilon)}$$

is equivalent to

$$3 - \frac{\mu}{2} + \varepsilon > 3 - \mu$$

which is true as $\mu, \varepsilon > 0$. Hence, $p_0' = \frac{3}{2} + \varepsilon$ and the previous theorem allows us to deal with $\frac{3}{2} + \varepsilon < r < 2(\frac{3}{2} + \varepsilon)$. As $\varepsilon > 0$ was arbitrary, we can treat $\frac{3}{2} < r < \infty$.

Applying Theorem 5.1.3 then yields the existence of a function u on $[0, T)$ that for fixed $r > \max\{3-\mu, \frac{3}{2}\}$ (and suitably chosen p°) that (5.15) and (5.16) hold for $r \leq p < 2p^\circ$ and $r < p < 2p^\circ$ respectively whenever $\sigma = \frac{3-\mu}{2}(\frac{1}{r} - \frac{1}{p}) < \frac{1}{2}$ is satisfied. However, due to $r \geq 3-\mu$ and $p < \infty$ we have

$$\frac{3-\mu}{2}\left(\frac{1}{r} - \frac{1}{p}\right) < \frac{3-\mu}{2}\frac{1}{r} \leq \frac{3-\mu}{2}\frac{1}{3-\mu} = \frac{1}{2}$$

so that this condition is automatically satisfied. Additionally, the theorem also states that u solves (5.17) for $\max\{r, 3\} < p < 2p^\circ$ and $h = p/2$.

Hence, it remains to show that (5.15)–(5.17) also hold for $p > 2p^\circ$. For this, we return to the proof of Theorem 5.1.3. Due to $p > 2p^\circ$, also $p > 2p_-$ and thus $\frac{p}{2} > p_-$ is true. Consequently, (5.4) holds for $p_- < \frac{p}{2} < s < \infty$, so in particular $s = p$ is included. Hence, (5.4) holds for all p that we are interested in and it is therefore not necessary to introduce an additional condition as in the proof of the previous theorem.

Finally, we can prove the continuity as before and also boundedness, behaviour in 0 and convergence of the sequence follows analogously. Note that the final part on relaxing conditions on s and p is not necessary here because we are only concerned with large p.

We conclude the proof by recalling that in case of Dirichlet boundary conditions the constants M_T and N_T are uniformly bounded in $0 < T < \infty$. Thus, in this case we may choose $T = \infty$. $\square$

CHAPTER 6

Extension of the Stokes semigroup to non divergence-free functions

In the second part of this thesis we aim for generalizations of boundedness results for the maximal regularity operator on Morrey spaces. First, we mention the result by Auscher and Frey, see [8, Lemma 3.3]. As explained therein their estimates is based on the work by Auscher, Monniaux and Portal [10]. Here, we follow, to a certain degree, the same strategy. However, considering the Stokes operator and its semigroup on the half-space is more delicate.

The aim of this chapter is to extend the Stokes semigroup in a suitable manner to non divergence-free functions. Note that the extension via the Helmholtz projection is not appropriate for our purposes as the resulting semigroup does not satisfy any off-diagonal estimates, see for example the introduction of [92].

A natural way to circumvent this issue is to use the solution formula for the resolvent problem to extend the resolvent to non divergence-free functions. Then, an extension of the semigroup can be defined by means of the Cauchy integral formula analogously to the setting of sectorial operators as explained in Section 2.1. However, one has to make sure that important properties of the semigroup are not lost by this procedure and that the resulting resolvent and semigroup behave as they should. Nonetheless, this approach also has its drawback as it is no longer possible to remove the pressure. This leads to some technical difficulties explained in the subsequent chapters.

6.1 The Stokes resolvent problem on $\mathrm{L}^2_\sigma(\mathbb{R}^d_+)$

In this section we review the Stokes resolvent problem on the half-space $\mathbb{R}^d_+$ and use the solution formula as well as point-wise estimates deduced by Maekawa, Miura and Prange in [68] to obtain L^1-estimates on the corresponding kernels. These estimates will play a crucial role in our extension procedure.

For the rest of this thesis fix $\varepsilon \in (0, \pi)$ and let $d \geq 3$. Then, for $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ we consider the Stokes resolvent problem on the half-space, i.e., the problem

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = f, & \text{in } \mathbb{R}^d_+, \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^d_+, \\ u = 0, & \text{on } \partial\mathbb{R}^d_+ \end{cases}$$

for $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d_+)$. It is known, see, e.g., [26], that after a Fourier transform in the tangential variables the solution $\hat{u} = \hat{v} + \hat{w}$ is for all $\xi \in \mathbb{R}^{d-1}$ and $y_d > 0$ given by

$$\hat{v}(\xi, y_d) = \frac{1}{2\omega_\lambda(\xi)} \int_0^\infty \left(\mathrm{e}^{-\omega_\lambda(\xi)|y_d - z_d|} - \mathrm{e}^{-\omega_\lambda(\xi)(y_d+z_d)} \right) \hat{f}(\xi, z_d) \, \mathrm{d}z_d, \tag{6.1}$$

$$\widehat{w}'(\xi, y_d) = \int_0^\infty \frac{\xi}{|\xi|} \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi)z_d} \, \xi \cdot \hat{f}'(\xi, z_d) \, \mathrm{d}z_d \tag{6.2}$$

and

$$\widehat{w}_d(\xi, y_d) = \mathrm{i} \int_0^\infty \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi)z_d} \, \xi \cdot \hat{f}'(\xi, z_d) \, \mathrm{d}z_d, \tag{6.3}$$

where $\omega_\lambda(\xi) \coloneqq \sqrt{\lambda + |\xi|^2}$. Here and in the following, $|\cdot|$ denotes the 2-norm, but due to norm equivalence in $\mathbb{R}^{d-1}$, this choice is not of great importance. The associated pressure ϕ in Fourier variables is given for all $\xi \in \mathbb{R}^{d-1}$ and $y_d > 0$ by

$$\hat{\phi}(\xi, y_d) = \int_0^\infty \mathrm{e}^{-|\xi|y_d} \, \mathrm{e}^{-\omega_\lambda(\xi)z_d} \left(\frac{1}{|\xi|} + \frac{1}{\omega_\lambda(\xi)} \right) \mathrm{i}\xi \cdot \hat{f}'(\xi, z_d) \, \mathrm{d}z_d. \tag{6.4}$$

Taking the inverse Fourier transform, we obtain a kernel representation for the solution $u = v + w$. For this, we distinguish between several kernels. We first define the ones connected with the Dirichlet-Laplace part v. These are two kernels $k_{1,\lambda} \colon \mathbb{R}^d \to \mathbb{C}$ and $k_{2,\lambda} \colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}$ that are given for all $y' \in \mathbb{R}^{d-1}$ and $y_d \in \mathbb{R}$, by

$$k_{1,\lambda}(y', y_d) \coloneqq \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y' \cdot \xi} \frac{1}{2\omega_\lambda(\xi)} \mathrm{e}^{-\omega_\lambda(\xi)|y_d|} \, \mathrm{d}\xi, \tag{6.5}$$

and for all $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$, by

$$k_{2,\lambda}(y', y_d, z_d) := \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{1}{2\omega_\lambda(\xi)} \mathrm{e}^{-\omega_\lambda(\xi)(y_d+z_d)} \,\mathrm{d}\xi. \tag{6.6}$$

For the nonlocal part w, the kernels $r'_\lambda \colon \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{C}^{(d-1)\times(d-1)}$ and $r_{d,\lambda} \colon \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{C}^{d-1}$ are defined as follows: For all $y' \in \mathbb{R}^{d-1}$ and y_d, $z_d > 0$,

$$r'_\lambda(y', y_d, z_d) := \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi)z_d} \frac{\xi \otimes \xi}{|\xi|} \,\mathrm{d}\xi \tag{6.7}$$

$$r_{d,\lambda}(y', y_d, z_d) := \int_{\mathbb{R}^{d-1}} \mathrm{i}\,\mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi)z_d} \,\xi \,\mathrm{d}\xi. \tag{6.8}$$

The kernel $q_\lambda \colon \mathbb{R}^{d-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}^{d-1}$ associated to the pressure is defined in the following way: For all $y' \in \mathbb{R}^{d-1}$ and y_d, $z_d > 0$,

$$q_\lambda(y', y_d, z_d) := \mathrm{i} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \,\mathrm{e}^{-|\xi|y_d} \,\mathrm{e}^{-\omega_\lambda(\xi)z_d} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\omega_\lambda(\xi)} \right) \mathrm{d}\xi. \tag{6.9}$$

Then, we obtain the following representation formulae: For all $y' \in \mathbb{R}^{d-1}$ and all $y_d > 0$ we have

$$\begin{aligned} v(y', y_d) &= \int_{\mathbb{R}^{d-1}} \int_0^\infty k_{1,\lambda}(y' - z', y_d - z_d) f(z', z_d) \,\mathrm{d}z_d \,\mathrm{d}z' \\ &\quad + \int_{\mathbb{R}^{d-1}} \int_0^\infty k_{2,\lambda}(y' - z', y_d, z_d) f(z', z_d) \,\mathrm{d}z_d \,\mathrm{d}z', \end{aligned} \tag{6.10}$$

$$w'(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty r'_\lambda(y' - z', y_d, z_d) f'(z', z_d) \,\mathrm{d}z_d \,\mathrm{d}z', \tag{6.11}$$

$$w_d(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty r_{d,\lambda}(y' - z', y_d, z_d) \cdot f'(z', z_d) \,\mathrm{d}z_d \,\mathrm{d}z' \tag{6.12}$$

and

$$\phi(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty q_\lambda(y' - z', y_d, z_d) \cdot f'(z', z_d) \,\mathrm{d}z_d \,\mathrm{d}z'. \tag{6.13}$$

Later in this thesis we will need several estimates on the kernel or its gradient. Some estimates on the kernel were already proved in [26] but particularly for the gradient estimates we rely on the point-wise kernel bounds derived in [68, Propositions 3.2, 3.5 and 3.7]. To be precise Maekawa, Miura and Prange proved the following estimates:

Proposition 6.1.1. *Let $\lambda \in \mathrm{S}_{\pi-\varepsilon}$. Then there exists $c(d,\varepsilon) > 0$ such that for $y' \in \mathbb{R}^{d-1}$, $y_d \in \mathbb{R}$ we have*

$$|k_{1,\lambda}(y', y_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}|y_d|}}{(|y_d| + |y'|)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(|y_d| + |y'|))^2},$$
$$|\nabla k_{1,\lambda}(y', y_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}|y_d|}}{(|y_d| + |y'|)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(|y_d| + |y'|))^2}.$$

Moreover, for $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$ we have

$$|k_{2,\lambda}(y', y_d, z_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(y_d+z_d)}}{(y_d + z_d + |y'|)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d + |y'|))^2},$$
$$|\nabla k_{2,\lambda}(y', y_d, z_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(y_d+z_d)}}{(y_d + z_d + |y'|)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d + |y'|))^2}.$$

Finally, for $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$ we have

$$\begin{aligned}
&|r'_\lambda(y', y_d, z_d)| + |r_{d,\lambda}(y', y_d, z_d)| \\
&\qquad \lesssim \frac{y_d}{(y_d + z_d + |y'|)^{d-1}} \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} z_d}}{(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d + |y'|))(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d))}, \\
&|\nabla_y r'_\lambda(y', y_d, z_d)| + |\nabla_y r_{d,\lambda}(y', y_d, z_d)| \\
&\qquad \lesssim \frac{1}{(y_d + z_d + |y'|)^{d-1}} \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} z_d}}{(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d + |y'|))(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d))}.
\end{aligned}$$

In all cases the implicit constant depends only on d and ε.

As a first step, we present elementary L^1-estimates of these kernels that will be used frequently in the course of this thesis.

Lemma 6.1.2. *For all $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ we have the following kernel estimates:*

(i) $\int_{\mathbb{R}^d} |k_{1,\lambda}(x)| \,\mathrm{d}x \lesssim \frac{1}{|\lambda|}$.

(ii) *For all $x_d, y_d > 0$ we have*

$$\int_{\mathbb{R}^{d-1}} |k_{2,\lambda}(x', x_d, y_d)| \,\mathrm{d}x' \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{|\lambda|^{\frac{1}{2}}}.$$

(iii) *For all $x_d, y_d > 0$ we have*

$$\int_{\mathbb{R}^{d-1}} |r'_\lambda(x', x_d, y_d)| + |r_{d,\lambda}(x', x_d, y_d)| \,\mathrm{d}x' \lesssim \frac{x_d \,\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))|\lambda|^{\frac{1}{2}}(x_d + y_d)}.$$

In all of the above statements the implicit constants only depend on d and ε.

Proof. For the L^1-bound on $k_{1,\lambda}$ introduce polar coordinates in $\mathbb{R}^{d-1}$ to obtain

$$\int_{\mathbb{R}^d} |k_{1,\lambda}(x)| \, \mathrm{d}x \lesssim \int_{\mathbb{R}} \int_0^\infty \frac{r^{d-2} \, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} |x_d|}}{(|x_d| + r)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(|x_d| + r))^2} \, \mathrm{d}r \, \mathrm{d}x_d.$$

Perform in the inner integral the substitution $s = r/|x_d|$, so that

$$\begin{aligned} \int_{\mathbb{R}^d} |k_{1,\lambda}(x)| \, \mathrm{d}x &\lesssim \int_{\mathbb{R}} \int_0^\infty \frac{(s|x_d|)^{d-2} \, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} |x_d|} |x_d|}{(|x_d| + s|x_d|)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(|x_d| + s|x_d|))^2} \, \mathrm{d}s \, \mathrm{d}x_d \\ &= \int_{\mathbb{R}} |x_d| \, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} |x_d|} \int_0^\infty \frac{s^{d-2}}{(1+s)^{d-2}(1 + |\lambda|^{\frac{1}{2}} |x_d|(1+s))^2} \, \mathrm{d}s \, \mathrm{d}x_d. \end{aligned}$$

Substitute $t = |\lambda|^{\frac{1}{2}} x_d$ to obtain

$$\int_{\mathbb{R}^d} |k_{1,\lambda}(x)| \, \mathrm{d}x \lesssim \frac{1}{|\lambda|} \int_{\mathbb{R}} |t| \, \mathrm{e}^{-c|t|} \int_0^\infty \frac{s^{d-2}}{(1+s)^{d-2}(1 + |t|(1+s))^2} \, \mathrm{d}s \, \mathrm{d}t.$$

To show that this double integral is finite, split the inner integral into an integral from 0 to $\frac{1}{|t|}$ and one from $\frac{1}{|t|}$ to ∞, yielding

$$\begin{aligned} \int_0^{\frac{1}{|t|}} \frac{s^{d-2}}{(1+s)^{d-2}(1 + |t|(1+s))^2} \, \mathrm{d}s + \int_{\frac{1}{|t|}}^\infty \frac{s^{d-2}}{(1+s)^{d-2}(1 + |t|(1+s))^2} \, \mathrm{d}s \\ \leq \int_0^{\frac{1}{|t|}} 1 \, \mathrm{d}s + \int_{\frac{1}{|t|}}^\infty \frac{1}{(1 + |t|s)^2} \, \mathrm{d}s \\ = \frac{1}{|t|} + \frac{1}{|t|} \int_1^\infty \frac{1}{(1+x)^2} \, \mathrm{d}x. \end{aligned}$$

Plugging this into the double integral shows the claim for $k_{1,\lambda}$. We proceed with the other kernels. Using the transformation $z' = |\lambda|^{\frac{1}{2}} x'$ and polar coordinates in $\mathbb{R}^{d-1}$, we have

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |k_{2,\lambda}(x', x_d, y_d)| \, \mathrm{d}x' &\lesssim \int_{\mathbb{R}^{d-1}} \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d + y_d)}}{(x_d + y_d + |x'|)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d + |x'|))^2} \, \mathrm{d}x' \\ &= \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d + y_d)}}{|\lambda|^{\frac{1}{2}}} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + |z'|)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + |z'|)^2} \, \mathrm{d}z' \\ &\lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d + y_d)}}{|\lambda|^{\frac{1}{2}}} \int_0^\infty \frac{r^{d-2}}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + r)^{d-2}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + r)^2} \, \mathrm{d}r \\ &\leq \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d + y_d)}}{|\lambda|^{\frac{1}{2}}} \int_0^\infty \frac{1}{(1+r)^2} \, \mathrm{d}r. \end{aligned}$$

For the kernels belonging to the non-local part we similarly have

$$
\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |r'_\lambda(x', x_d, y_d)| \, \mathrm{d}x' + \int_{\mathbb{R}^{d-1}} |r_{d,\lambda}(x', x_d, y_d)| \, \mathrm{d}x' \\
&\lesssim \int_{\mathbb{R}^{d-1}} \frac{x_d \, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{(x_d + y_d + |x'|)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d + |x'|))(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \, \mathrm{d}x' \\
&= \frac{x_d \, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{1 + |\lambda|^{\frac{1}{2}}(x_d + y_d)} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + |z'|)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + |z'|)} \, \mathrm{d}z' \\
&\lesssim \frac{x_d \, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{1 + |\lambda|^{\frac{1}{2}}(x_d + y_d)} \int_0^\infty \frac{r^{d-2}}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + r)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + r)} \, \mathrm{d}r.
\end{aligned}
$$

In the following, we concentrate on the integral term. Here, we split the integral at $r = |\lambda|^{\frac{1}{2}}(x_d + y_d)$ to get

$$
\begin{aligned}
&\int_0^\infty \frac{r^{d-2}}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + r)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + r)} \, \mathrm{d}r \\
&= \int_0^{|\lambda|^{\frac{1}{2}}(x_d + y_d)} \frac{r^{d-2}}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + r)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + r)} \, \mathrm{d}r \\
&\qquad + \int_{|\lambda|^{\frac{1}{2}}(x_d + y_d)}^\infty \frac{r^{d-2}}{(|\lambda|^{\frac{1}{2}}(x_d + y_d) + r)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d) + r)} \, \mathrm{d}r \\
&\le \frac{1}{(|\lambda|^{\frac{1}{2}}(x_d + y_d))^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \int_0^{|\lambda|^{\frac{1}{2}}(x_d + y_d)} r^{d-2} \, \mathrm{d}r \\
&\qquad + \int_{|\lambda|^{\frac{1}{2}}(x_d + y_d)}^\infty \frac{1}{r(1 + r)} \, \mathrm{d}r \\
&= \frac{1}{(d-1)(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} + \int_{|\lambda|^{\frac{1}{2}}(x_d + y_d)}^\infty \frac{1}{r(1 + r)} \, \mathrm{d}r \\
&\le \frac{1}{(d-1)(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} + \frac{1}{|\lambda|^{\frac{1}{2}}(x_d + y_d)} \\
&\lesssim \frac{1}{|\lambda|^{\frac{1}{2}}(x_d + y_d)},
\end{aligned}
$$

where in the second to last step we used L'Hôpital's rule to deduce that both $\lim_{r \to 0} \log\left(\frac{r+1}{r}\right) r = 0$ and $\lim_{r \to \infty} \log\left(\frac{r+1}{r}\right) r = 1$ hold. To be precise, for $a > 0$ it holds

$$
\begin{aligned}
\int_a^\infty \frac{1}{r(1 + r)} \, \mathrm{d}r &= \log\left(\frac{a+1}{a}\right) \\
&= \log\left(\frac{a+1}{a}\right) \frac{a}{a} \le \frac{1}{a}
\end{aligned}
$$

and this concludes the proof. □

Next, we prove similar L^1-estimates for the gradients. Here, ∇_x denotes the gradient with respect to $x = (x', x_d)$.

Lemma 6.1.3. *For all $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ the following estimates hold true:*

(i) $\int_{\mathbb{R}^d} |\nabla k_{1,\lambda}(x)| \,\mathrm{d}x \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}}$.

(ii) *For all $x_d, y_d > 0$ we have*

$$\int_{\mathbb{R}^{d-1}} |\nabla k_{2,\lambda}(x', x_d, y_d)| \,\mathrm{d}x' \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \cdot \bigg(\frac{1}{(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} + \log\bigg(1 + \frac{1}{|\lambda|^{\frac{1}{2}}(x_d + y_d)}\bigg)\bigg).$$

(iii) *For all $x_d, y_d > 0$ we have*

$$\int_{\mathbb{R}^{d-1}} |\nabla_x r'_\lambda(x', x_d, y_d)| + |\nabla_x r_{d,\lambda}(x', x_d, y_d)| \,\mathrm{d}y' \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \cdot \bigg(\frac{1}{(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} + \log\bigg(1 + \frac{1}{|\lambda|^{\frac{1}{2}}(x_d + y_d)}\bigg)\bigg).$$

In all of the above statements the implicit constants only depend on d and ε.

Proof. The point-wise bounds deduced in [68], see Proposition 6.1.1, together with polar coordinates in $\mathbb{R}^{d-1}$ as well as the substitutions $s = r/|x_d|$ and $t = |\lambda|^{\frac{1}{2}} x_d$ imply

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla k_{1,\lambda}(x)| \,\mathrm{d}x &\lesssim \int_{\mathbb{R}} \int_0^\infty \frac{r^{d-2}\, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}}|x_d|}}{(|x_d| + r)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(r + |x_d|))^2} \,\mathrm{d}r \,\mathrm{d}x_d \\ &= \int_{\mathbb{R}} \int_0^\infty \frac{s^{d-2}\, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}}|x_d|}}{(1 + s)^{d-1}(1 + |\lambda|^{\frac{1}{2}}|x_d|(1 + s))^2} \,\mathrm{d}s \,\mathrm{d}x_d \\ &= \frac{1}{|\lambda|^{\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{e}^{-c|t|} \int_0^\infty \frac{s^{d-2}}{(1 + s)^{d-1}(1 + |t|(1 + s))^2} \,\mathrm{d}s \,\mathrm{d}t. \end{aligned}$$

To conclude the estimate for $\nabla k_{1,\lambda}$, it remains to show that the double integral converges. To this end, we bound the exponential by 1 and cancel out the factor s^{d-2}. This yields

$$\int_{\mathbb{R}^d} |\nabla k_{1,\lambda}(y)| \,\mathrm{d}y \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}} \int_{\mathbb{R}} \int_0^\infty \frac{1}{(1 + s)(1 + |t|(1 + s))^2} \,\mathrm{d}s \,\mathrm{d}t.$$

After an application of Fubini's theorem the resulting integral can be computed explicitly. To be precise, by means of the transformation $r = t(1+s)$ we have

$$\int_{\mathbb{R}} \int_0^\infty \frac{1}{(1+s)(1+|t|(1+s))^2} \, \mathrm{d}s \, \mathrm{d}t = 2 \int_0^\infty \int_0^\infty \frac{1}{(1+r)^2} \, \mathrm{d}r \, \frac{1}{(1+s)^2} \, \mathrm{d}s = 2.$$

Next, we consider the second part of the statement. Again using the point-wise estimates from [68], see Proposition 6.1.1, and the substitution $z' = |\lambda|^{\frac{1}{2}} x'$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |\nabla k_{2,\lambda}(x', x_d, y_d)| \, \mathrm{d}x' \\
&\qquad \lesssim \int_{\mathbb{R}^{d-1}} \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{(|x'| + x_d + y_d)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(|x'| + x_d + y_d))^2} \, \mathrm{d}x' \\
&\qquad = \mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)} \int_{\mathbb{R}^{d-1}} \frac{1}{(|z'| + |\lambda|^{\frac{1}{2}}(x_d + y_d))^{d-1}(1 + |z'| + |\lambda|^{\frac{1}{2}}(x_d + y_d))^2} \, \mathrm{d}z' \\
&\qquad \lesssim B \int_{\mathbb{R}^{d-1}} \frac{1}{(|z'| + |\lambda|^{\frac{1}{2}}(x_d + y_d))^{d-1}(1 + |z'| + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \, \mathrm{d}z'
\end{aligned}$$

where

$$B := \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))}.$$

Using polar coordinates in $\mathbb{R}^{d-1}$ and splitting the integral at $r = |\lambda|^{\frac{1}{2}}(x_d + y_d)$ yields

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \frac{1}{(|z'| + |\lambda|^{\frac{1}{2}}(x_d + y_d))^{d-1}(1 + |z'| + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \, \mathrm{d}z' \\
&\qquad \lesssim \int_0^\infty \frac{r^{d-2}}{(r + |\lambda|^{\frac{1}{2}}(x_d + y_d))^{d-1}(1 + r + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \, \mathrm{d}r \\
&\qquad \lesssim \int_0^{|\lambda|^{\frac{1}{2}}(x_d+_d)} \frac{1}{|\lambda|^{\frac{1}{2}}(x_d + y_d)} \frac{1}{1 + |\lambda|^{\frac{1}{2}}(x_d + y_d)} \, \mathrm{d}r \\
&\qquad\qquad + \int_{|\lambda|^{\frac{1}{2}}(x_d+y_d)}^\infty \frac{1}{(r + |\lambda|^{\frac{1}{2}}(x_d + y_d))(1 + r + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \, \mathrm{d}r \\
&\qquad \lesssim \frac{1}{1 + |\lambda|^{\frac{1}{2}}(x_d + y_d)} + \log\left(1 + \frac{1}{|\lambda|^{\frac{1}{2}}(x_d + y_d)}\right)
\end{aligned}$$

which proves the desired estimate for the second kernel. It remains to examine the third kernel. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |\nabla_x r'_\lambda(x', x_d, y_d)| \, \mathrm{d}x' + \int_{\mathbb{R}^{d-1}} |\nabla_x r_{\lambda,d}(x', x_d, y_d)| \, \mathrm{d}x' \\
&\qquad \lesssim \int_{\mathbb{R}^{d-1}} \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{(x_d + y_d + |x'|)^{d-1}(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d + |x'|))(1 + |\lambda|^{\frac{1}{2}}(x_d + y_d))} \, \mathrm{d}x'
\end{aligned}$$

$$= \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} y_d}}{(1+|\lambda|^{\frac{1}{2}}(x_d+y_d))} \int_{\mathbb{R}^{d-1}} \frac{1}{(x_d+y_d+|x'|)^{d-1}(1+|\lambda|^{\frac{1}{2}}(x_d+y_d+|x'|))} \,\mathrm{d}x'.$$

After the substitution $z' = |\lambda|^{\frac{1}{2}} x'$ this is the same integral as for $\nabla k_{2,\lambda}$ and this completes the proof. $\square$

For the pressure ϕ given by (6.13) no such L^1-estimates are needed and point-wise estimates are sufficient. Hence, we postpone estimates for the pressure to the next section.

6.2 Stokes resolvent problem on $\mathrm{L}^p(\mathbb{R}^d_+)$

Recall that our goal is to find an extension of the Stokes semigroup to $\mathrm{L}^p(\mathbb{R}^d_+)$ that satisfies off-diagonal estimates. To this end, we first extend the resolvent by means of the formulae from the last section. We then show that this extended resolvent is well-defined on $\mathrm{L}^p(\mathbb{R}^d_+)$ for $1 < p < \infty$, satisfies the first resolvent identity and deserves at least partly its name as it provides a solution to $\lambda u - \Delta u + \nabla \phi = f$. This part, albeit not difficult, takes up a substantial amount of this section as it requires careful calculations in the Fourier space. Moreover, we show L^p-L^q-estimates for the extended resolvent belonging to u. Have in mind that working with non divergence-free functions forces us to also consider the pressure. This creates some technical difficulties, for example it remains open whether the extended resolvent for u is again a resolvent in its own right.

We then use the resolvent estimates to define, by means of the Cauchy integral formula, operator families $E^u(t)$ and $E^\phi(t)$ for $t > 0$ which extend the Stokes semigroup to $\mathrm{L}^p(\mathbb{R}^d_+)$. Moreover, we prove that the family $(E^u(t))_{t>0}$ satisfies the semigroup law, is bounded on $\mathrm{L}^p(\mathbb{R}^d_+)$ and also satisfies L^p-L^q-estimates.

Let us now make our understanding of extended resolvents more precise.

Definition 6.2.1. Let $\lambda \in \mathrm{S}_{\pi-\varepsilon}$. Then we define

$$(6.14) \qquad \begin{aligned} R^u(\lambda)\colon \mathrm{L}^2(\mathbb{R}^d_+) &\to \mathrm{L}^2(\mathbb{R}^d_+) \\ f &\mapsto v + w \end{aligned}$$

where v and w are defined in (6.10) – (6.12). Moreover, define

$$(6.15) \qquad \begin{aligned} R^\phi(\lambda)\colon \mathrm{L}^2(\mathbb{R}^d_+) &\to \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+) \\ f &\mapsto \phi \end{aligned}$$

where ϕ is defined in (6.13). We call the tupel $R(\lambda) \coloneqq (R^u(\lambda), R^\phi(\lambda))$ *extended resolvent.*

Note that for $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ we also have the representation via the inverse Fourier transform in the tangential variables, namely

$$R^u(\lambda)f = \mathcal{F}^{-1}(\hat{v} + \hat{w})$$

where $\hat{v}$ and $\hat{w}$ are defined in (6.1) – (6.3). Similarly, we have $R^\phi(\lambda)f = \mathcal{F}^{-1}(\hat{\phi})$ as defined in (6.4). The former definition will be more helpful to show estimates, while the latter turns out to be more suitable to show the first resolvent identity. As a matter of fact, R^u can be defined even on $\mathrm{L}^p(\mathbb{R}^d_+)$ for $1 < p < \infty$. To be precise, R^u and ∇R^u satisfy the following estimate extending the well-known results from $\mathrm{L}^p_\sigma(\mathbb{R}^d_+)$.

Proposition 6.2.2. *Let $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ and $1 < p < \infty$. Then the extended resolvent $R^u(\lambda)\colon \mathrm{L}^p(\mathbb{R}^d_+) \to \mathrm{L}^p(\mathbb{R}^d_+)$ as well as its gradient are well-defined and satisfy*

$$\|R^u(\lambda)f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \lesssim \frac{1}{|\lambda|}\|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \tag{6.16}$$

$$\|\nabla R^u(\lambda)f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}}\|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \tag{6.17}$$

for all $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ and the implicit constants only depend on d, p and ε.

Proof. The first inequality is proven in [26, Theorem 3.5]. Note that they assume f to be divergence-free but the proof carries over to non divergence-free functions verbatim.

For the second part note that the term involving $\nabla k_{1,\lambda}$ is of convolution type. Thus, Young's convolution inequality together with Lemma 6.1.3 yields the desired estimate. For the other kernels we proceed as in the proof of the first part given in [26]: Specifically, we use [26, Lemma 3.3 a)] with $n = d-1$ by which it suffices to show uniform bounds for the kernels in suitable Lebesgue spaces. To be more precise, we have to show

$$\Big(\int_0^\infty \Big(\int_0^\infty \|k(\cdot, x_d, y_d)\|_{\mathrm{L}^1(\mathbb{R}^{d-1})}^{p'} \,\mathrm{d}y_d\Big)^{\frac{p}{p'}} \,\mathrm{d}x_d\Big)^{\frac{1}{p}} \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}} \tag{6.18}$$

where p' as usual is the Hölder conjugate of p and $k \in \{\nabla k_{2,\lambda}, \nabla_x r'_\lambda + \nabla_x r_{d,\lambda}\}$. We consider first $k = \nabla k_{2,\lambda}$ and use the estimate deduced in Lemma 6.1.3:

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} |\nabla k_{2,\lambda}(x', x_d, y_d)| \,\mathrm{d}x' \lesssim{}& \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{(1+|\lambda|^{\frac{1}{2}}(x_d+y_d))^2} \\ &+ \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{(1+|\lambda|^{\frac{1}{2}}(x_d+y_d))} \log\Big(1 + \frac{1}{|\lambda|^{\frac{1}{2}}(x_d+y_d)}\Big).\end{aligned}$$

Using the triangle inequality, we can plug both summands into the left-hand side of (6.18) separately. In case of the first summand we use the substitutions $\tilde{y}_d = |\lambda|^{\frac{1}{2}} y_d$ and $\tilde{x}_d = |\lambda|^{\frac{1}{2}} x_d$ to obtain

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-cp'|\lambda|^{\frac{1}{2}}(y_d+x_d)}}{(1+|\lambda|^{\frac{1}{2}}(x_d+y_d))^{2p'}} \,\mathrm{d}y_d\right)^{\frac{p}{p'}} \mathrm{d}x_d\right)^{\frac{1}{p}} \\
&= \frac{1}{|\lambda|^{\frac{1}{2}(\frac{1}{p}+\frac{1}{p'})}} \left(\int_0^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-cp'(\tilde{x}_d+\tilde{y}_d)}}{(1+\tilde{x}_d+\tilde{y}_d)^{2p'}} \,\mathrm{d}\tilde{y}_d\right)^{\frac{p}{p'}} \mathrm{d}\tilde{x}_d\right)^{\frac{1}{p}}.
\end{aligned}$$

An easy way to see that the remaining integral is finite is to estimate the denominator by 1 and then use the decay of the exponential. Due to $\frac{1}{p}+\frac{1}{p'}=1$ the desired estimate follows. For the second summand we need to estimate

$$\left(\int_0^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-cp'|\lambda|^{\frac{1}{2}}(x_d+y_d)}}{(1+|\lambda|^{\frac{1}{2}}(x_d+y_d))^{p'}} \log\left(1+\frac{1}{|\lambda|^{\frac{1}{2}}(x_d+y_d)}\right)^{p'} \mathrm{d}y_d\right)^{\frac{p}{p'}} \mathrm{d}x_d\right)^{\frac{1}{p}}.$$

The decay in λ, i.e., the factor $|\lambda|^{-\frac{1}{2}}$, is obtained by virtue of the same substitutions as in the first case. Estimating the integral reveals

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-cp'(\tilde{x}_d+\tilde{y}_d)}}{(1+(\tilde{x}_d+\tilde{y}_d))^{p'}} \log\left(1+\frac{1}{\tilde{x}_d+\tilde{y}_d}\right)^{p'} \mathrm{d}\tilde{y}_d\right)^{\frac{p}{p'}} \mathrm{d}\tilde{x}_d\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^\infty \left(\int_0^\infty \mathrm{e}^{-cp'\tilde{y}_d} \log\left(1+\frac{1}{\tilde{y}_d}\right)^{p'} \mathrm{d}\tilde{y}_d\right)^{\frac{p}{p'}} \mathrm{e}^{-cp\tilde{x}_d}\,\mathrm{d}\tilde{x}_d\right)^{\frac{1}{p}}.
\end{aligned}$$

Note that the singularity in zero of the logarithmic term is an integrable one. For large $\tilde{y}_d$, the logarithmic term is bounded so that the exponential decay guarantees the convergence of the $\tilde{y}_d$-integral. Convergence of the $\tilde{x}_d$-integral is clear.

Hence, it remains to prove (6.18) for $k = \nabla_x r'_\lambda + \nabla_x r_{d,\lambda}$. Even though the L^1-estimate is almost the same as for $\nabla k_{2,\lambda}$, we must be slightly more careful, as there is no exponential decay in x_d. The factor $|\lambda|^{-\frac{1}{2}}$ is obtained as before and after the substitutions it remains to show that the following integral converges:

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-cp'y_d}}{(1+x_d+y_d)^{p'}} \left(\frac{1}{1+x_d+y_d} + \log\left(1+\frac{1}{x_d+y_d}\right)\right)^{p'} \mathrm{d}y_d\right)^{\frac{p}{p'}} \mathrm{d}x_d\right)^{\frac{1}{p}} \lesssim \\
&\left(\int_0^\infty \left(\int_0^\infty \frac{\mathrm{e}^{-cp'y_d}}{(1+x_d+y_d)^{p'}} \left(\frac{1}{(1+x_d+y_d)^{p'}} + \log\left(1+\frac{1}{x_d+y_d}\right)^{p'}\right) \mathrm{d}y_d\right)^{\frac{p}{p'}} \mathrm{d}x_d\right)^{\frac{1}{p}}
\end{aligned}$$

where we used norm equivalence in $\mathbb{R}^2$. Next, except from appearance in the exponential, use $y_d > 0$ so that convergence of the remaining integral with respect to y_d is clear. The integral with respect to x_d then reads

$$\int_0^\infty \frac{1}{(1+x_d)^{2p}} + \frac{1}{(1+x_d)^p} \log\left(1+\frac{1}{x_d}\right)^p \mathrm{d}x_d.$$

Clearly, the first summand is integrable. For the second summand notice that the logarithmic term is bounded for large x_d, for example for $x_d \geq 1$. Due to $p > 1$ this shows integrability of the second summand for large x_d as well. Regarding small x_d note the first factor is bounded and so the logarithmic term is important. Hence, it remains to check whether

$$\int_0^1 \log\left(1+\frac{1}{x_d}\right)^p \mathrm{d}x_d$$

is finite for $p > 1$. This can for example be verified by noting that the integrand is positive and

$$\log\left(1+\frac{1}{x_d}\right) < \log\left(\frac{2}{x}\right)$$

holds true for $x_d \in (0,1]$ as well as due to the fact that $\log(x_d)^p$ is integrable on $(0,1]$ for $p > 1$. Thus we have shown (6.18) for all kernels and conclude that (6.17) holds true. $\square$

By interpolation together with the previous proposition we obtain, for suitable exponents p and q, L^p-L^q-estimates for $R^u(\lambda)$.

Proposition 6.2.3. *Let $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ and let $1 < p \leq q < \infty$ with $\frac{1}{d} \geq \frac{1}{p} - \frac{1}{q} \geq 0$. For all $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ we have*

$$\|R^u(\lambda)f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} \lesssim \frac{1}{|\lambda|^{1-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}} \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)}$$

where the implicit constant only depends on d, p, q and ε.

Proof. Let $1 < p \leq q < \infty$ with $\frac{1}{d} \geq \frac{1}{p} - \frac{1}{q} \geq 0$, set $\theta := d\left(\frac{1}{p} - \frac{1}{q}\right) \in [0,1]$ and let $f \in \mathrm{L}^p(\mathbb{R}^d_+)$. We then apply the Gagliardo-Nirenberg inequality on the half-space and Proposition 6.2.2 to obtain

$$\begin{aligned} \|R^u(\lambda)f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} &\lesssim \|\nabla R^u(\lambda)f\|^\theta_{\mathrm{L}^p(\mathbb{R}^d_+)} \|R^u(\lambda)f\|^{1-\theta}_{\mathrm{L}^p} \\ &\lesssim \frac{1}{|\lambda|^{\frac{1}{2}\theta}} \frac{1}{|\lambda|^{1-\theta}} \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)}. \end{aligned}$$

Note that the original proof by Nirenberg [74] already contains the half-space case. Due to $\frac{1}{2}\theta + 1 - \theta = 1 - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$ this completes the proof. $\square$

Turning to the pressure term note that no global L^p-estimate is available. However, we can show that the pressure is uniformly bounded away from the half-space boundary. To be precise, our result reads as follows:

Proposition 6.2.4. *Let $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ and $\lambda \in \mathrm{S}_{\pi-\varepsilon}$. For $\delta > 0$ there exists a constant $C = C(\delta, d, \varepsilon)$ such that for all $y = (y', y_d) \in \mathbb{R}^d_+$ with $y_d > \delta$ we have*

$$|R^\phi(\lambda) f(y', y_d)| \leq C(\delta, d, \varepsilon) \frac{1}{|\lambda|^{\frac{1}{4}}} \|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}.$$

Proof. The point-wise estimate for the pressure provided by [68, Proposition 3.7] reads

$$|q_\lambda(y', y_d, z_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} z_d}}{(y_d + z_d + |y'|)^{d-1}} \tag{6.19}$$

for all $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$. For $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ and $y = (y', y_d) \in \mathbb{R}^d_+$ with $y_d > \delta$ we then have

$$\begin{aligned}
|R^\phi(\lambda) f(y', y_d)| &\lesssim \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} z_d}}{(y_d + z_d + |y' - z'|)^{d-1}} |f'(z', z_d)| \,\mathrm{d}z_d \,\mathrm{d}z' \\
&\lesssim \left(\int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{\mathrm{e}^{-2c|\lambda|^{\frac{1}{2}} z_d}}{(y_d + z_d + |y' - z'|)^{2(d-1)}} \,\mathrm{d}z_d \,\mathrm{d}z' \right)^{\frac{1}{2}} \|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}.
\end{aligned}$$

Now, we focus on the integral term and use $y_d > \delta$, $z_d > 0$ as well as translation in $\mathbb{R}^{d-1}$ to obtain

$$\int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{\mathrm{e}^{-2c|\lambda|^{\frac{1}{2}} z_d}}{(y_d + z_d + |y' - z'|)^{2(d-1)}} \,\mathrm{d}z_d \,\mathrm{d}z' \lesssim \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{\mathrm{e}^{-2c|\lambda|^{\frac{1}{2}} z_d}}{(\delta + |z'|)^{2(d-1)}} \,\mathrm{d}z_d \,\mathrm{d}z'.$$

Considering the inner integral delivers

$$\int_0^\infty \mathrm{e}^{-2c|\lambda|^{\frac{1}{2}} z_d} \,\mathrm{d}z_d \lesssim \frac{1}{|\lambda|^{\frac{1}{2}}}.$$

Here, the implicit constant depends on ε, as c depends on ε. For the remaining integral use polar coordinates in $\mathbb{R}^{d-1}$ to infer

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \frac{1}{(\delta + |z'|)^{2d-2}} \,\mathrm{d}z' &= \int_0^\infty \frac{r^{d-2}}{(\delta + r)^{d-2+d}} \,\mathrm{d}r \\
&\leq \int_0^\infty \frac{1}{(\delta + r)^d} =: C(\delta, d) < \infty.
\end{aligned}$$

Combining the previous estimates concludes the proof. □

Next, we prove that the generalised resolvents R^u and R^ϕ indeed yield a solution to the Stokes resolvent problem but without the divergence-freeness. Here, further regularity of $R^u(\lambda)f$ and $R^\phi(\lambda)f$ is required. For the pressure note that the kernel corresponding to $\nabla R^\phi(\lambda)f$ satisfies (6.19) but with exponent d in the denominator, see [68, Proposition 3.7]. Then, the proof of the previous proposition applies as well. For the second derivatives of the kernels belonging to $R^u(\lambda)f$ the estimates derived by Maekawa, Miura and Prange in [68, Proposition 3.2 and Proposition 3.5] are stronger than those for first order derivatives of the pressure. For the precise point-wise estimates we refer to Proposition 7.2.1 In particular, estimates of the form (6.19) are satisfied in this setting as well and this guarantees the existence of $\Delta R^u(\lambda)f$.

Proposition 6.2.5. *For $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ and $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ we have $\lambda R^u(\lambda)f - \Delta R^u(\lambda)f + \nabla R^\phi(\lambda)f = f$ in the sense of distributions.*

Proof. Let $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ and $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ be arbitrary and v, w and p as defined in (6.10)–(6.13). We have to prove

$$\lambda(v + w) - \Delta(v + w) + \nabla\phi = f.$$

As v is the solution to the Dirichlet-Laplace resolvent problem, it follows that $\lambda v - \Delta v = f$ is true. Thus, it remains to show $\lambda w - \Delta w + \nabla\phi = 0$. To this end we use the Fourier transform on $\mathbb{R}^{d-1}$, hence it suffices to show the equivalent statement

$$\lambda\mathcal{F}^{-1}\hat{w} - \Delta\mathcal{F}^{-1}\hat{w} + \nabla\mathcal{F}^{-1}\hat{\phi} = 0. \tag{6.20}$$

We calculate $\Delta\mathcal{F}^{-1}\hat{w}$ and distinguish between the first $d-1$ components and the last one. As the Fourier transform is taken in $d-1$ components only, the first $d-1$ summands of the Laplacian translate to a factor $-|\xi|^2$ inside the inverse Fourier transform. Using this and the representation (6.2) of $\hat{w}'$ we have

$$\begin{aligned}-\Delta\mathcal{F}^{-1}\hat{w}' = \mathcal{F}^{-1}&\left(\int_0^\infty \xi|\xi|\frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\xi\cdot\hat{f}'(\xi, z_d)\,\mathrm{d}z_d\right)\\ &- \mathcal{F}^{-1}\left(\int_0^\infty \frac{\xi}{|\xi|}\frac{|\xi|^2\,\mathrm{e}^{-|\xi|y_d} - \omega_\lambda(\xi)^2\,\mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\xi\cdot\hat{f}'(\xi, z_d)\,\mathrm{d}z_d\right).\end{aligned}$$

By definition, $\omega_\lambda(\xi)^2 = \lambda + |\xi|^2$ holds, so by (6.2) in the first $d-1$ components we have

$$\lambda\mathcal{F}^{-1}\hat{w}' - \Delta\mathcal{F}^{-1}\hat{w}' = \mathcal{F}^{-1}\left(\int_0^\infty \frac{\xi}{|\xi|}\frac{\lambda\,\mathrm{e}^{-|\xi|y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\xi\cdot\hat{f}'(\xi, z_d)\,\mathrm{d}z_d\right).$$

It remains to calculate the gradient of the pressure with respect to the first $d-1$ coordinates. As the Fourier transform is also taken in the first $d-1$ coordinates only, the gradient translates to a factor of $\mathrm{i}\xi$ and we obtain

$$\nabla_{x'}\mathcal{F}^{-1}\hat{\phi} = -\mathcal{F}^{-1}\int_0^\infty \xi\,\mathrm{e}^{-|\xi|y_d}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\left(\frac{1}{|\xi|}+\frac{1}{\omega_\lambda(\xi)}\right)\xi\cdot\hat{f}'(\xi,z_d)\,\mathrm{d}z_d.$$

Next, we add both terms and use the identity $\frac{\lambda}{\omega_\lambda(\xi)-|\xi|} = \omega_\lambda(\xi)+|\xi|$ to obtain in the first $d-1$ components

$$\begin{aligned}
&\lambda\mathcal{F}^{-1}\hat{w}' - \Delta\mathcal{F}^{-1}\hat{w}' + \nabla_{x'}\mathcal{F}^{-1}\hat{\phi}\\
&= \mathcal{F}^{-1}\int_0^\infty \xi\,\mathrm{e}^{-|\xi|y_d}\,\frac{\omega_\lambda(\xi)+|\xi|}{\omega_\lambda(\xi)|\xi|}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\xi\cdot\hat{f}'(\xi,z_d)\,\mathrm{d}z_d\\
&\quad - \mathcal{F}^{-1}\int_0^\infty \xi\,\mathrm{e}^{-|\xi|y_d}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\left(\frac{1}{|\xi|}+\frac{1}{\omega_\lambda(\xi)}\right)\xi\cdot\hat{f}'(\xi,z_d)\,\mathrm{d}z_d\\
&= 0.
\end{aligned}$$

Hence, it remains to check the last component. With a similar argument, we have

$$\lambda\mathcal{F}^{-1}\hat{w}_d - \Delta\mathcal{F}^{-1}\hat{w}_d = \mathcal{F}^{-1}\int_0^\infty \mathrm{i}\frac{\lambda\,\mathrm{e}^{-|\xi|y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi)-|\xi|)}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\xi\cdot\hat{f}'(\xi,z_d)\,\mathrm{d}z_d.$$

For the pressure it holds

$$\partial_d\mathcal{F}^{-1}\hat{\phi} = -\mathcal{F}^{-1}\int_0^\infty \mathrm{i}|\xi|\,\mathrm{e}^{-|\xi|y_d}\,\mathrm{e}^{-\omega_\lambda(\xi)z_d}\left(\frac{1}{|\xi|}+\frac{1}{\omega_\lambda(\xi)}\right)\xi\cdot\hat{f}'(\xi,z_d)\,\mathrm{d}z_d.$$

Adding both terms and again using $\frac{\lambda}{\omega_\lambda(\xi)-|\xi|} = \omega_\lambda(\xi)+|\xi|$ yields

$$\lambda\mathcal{F}^{-1}\hat{w}_d - \Delta\mathcal{F}^{-1}\hat{w}_d + \partial_d\mathcal{F}^{-1}\hat{\phi} = 0$$

which implies (6.20) and thus concludes the proof. □

It remains open whether $R^u(\lambda)$ is injective for some $\lambda \in \mathrm{S}_{\pi-\varepsilon}$. If this was the case, we would obtain that the family of extended resolvents $(R^u(\lambda))_{\lambda\in\mathrm{S}_{\pi-\varepsilon}}$ is indeed a resolvent (for a suitable operator). However, we are able to prove that the family $(R^u(\lambda))_{\lambda\in\mathrm{S}_{\pi-\varepsilon}}$ satisfies the first resolvent identity on $\mathrm{L}^2(\mathbb{R}^d_+)$. Note that this result is crucial as it guarantees that the Cauchy integral of the extended resolvent $R^u(\lambda)$ yields a family obeying the semigroup law. Moreover, it remains open whether a similar result is true for the family $(R^\phi(\lambda))_{\lambda\in\mathrm{S}_{\pi-\varepsilon}}$.

Proposition 6.2.6. *For $\lambda,\mu \in \mathrm{S}_{\pi-\varepsilon}$ we have $R^u(\lambda) - R^u(\mu) = (\mu-\lambda)R^u(\lambda)R^u(\mu)$ as operators on $\mathrm{L}^2(\mathbb{R}^d_+)$.*

Proof. For the proof to be easier to follow we introduce the following notation: Given $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ and $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ we denote by $\hat{v}_{\lambda,f}$ the function $\hat{v}$ as defined in (6.1) and do so analogously for $\hat{w}'$ and $\hat{w}_d$. Moreover, we define $g(\xi,\cdot) := \xi \cdot \widehat{f}'(\xi,\cdot)$ as well as $a := \omega_\lambda(\xi)$ and $b := \omega_\mu(\xi)$.

With this notation and the fact that the Fourier transform in the tangential variables is an isomorphism on $\mathrm{L}^2(\mathbb{R}^d_+)$ it suffices to prove

$$
\begin{aligned}
&\hat{v}_{\lambda,f} + \hat{w}_{\lambda,f} - \hat{v}_{\mu,f} - \hat{w}_{\mu,f} \\
&\quad = (\mu-\lambda)(\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}} + \hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}} + \hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}} + \hat{w}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}).
\end{aligned}
\tag{6.21}
$$

Here, we note that as v is the solution to the Dirichlet-Laplace resolvent problem, it satisfies the resolvent identity, i.e.,

$$
\hat{v}_{\lambda,f} - \hat{v}_{\mu,f} = (\mu-\lambda)\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}}
$$

holds. Hence, it remains to show

$$
\hat{w}_{\lambda,f} - \hat{w}_{\mu,f} = (\mu-\lambda)(\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}} + \hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}} + \hat{w}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}).
\tag{6.22}
$$

To deal with the remaining terms on the right-hand side of (6.22) we calculate them one by one and start with $\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}$. Note that the formulae for $\hat{w}'$ and $\hat{w}_d$ differ only in one factor. Hence, choosing $\ell = (\frac{\xi}{|\xi|}, \mathrm{i})^t$ we can deal with all components simultaneously. Plugging in the definition of $\hat{v}$ and $\hat{w}$ (see (6.1)–(6.3)) together with Fubini's theorem we get for $\xi \in \mathbb{R}^{d-1}$ and $y_d > 0$ that

$$
\begin{aligned}
&\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}(\xi,y_d) \\
&\quad = \frac{1}{2a}\int_0^\infty \Big(\mathrm{e}^{-a|y_d-z_d|} - \mathrm{e}^{-a(y_d+z_d)}\Big)\, \hat{w}_{\mu,f}(\xi,z_d)\,\mathrm{d}z_d \\
&\quad = \frac{\ell}{2a}\int_0^\infty \Big(\mathrm{e}^{-a|y_d-z_d|} - \mathrm{e}^{-a(y_d+z_d)}\Big) \int_0^\infty \frac{\mathrm{e}^{-|\xi|z_d} - \mathrm{e}^{-bz_d}}{b(b-|\xi|)}\,\mathrm{e}^{-bx_d}\, g(\xi,x_d)\,\mathrm{d}x_d\,\mathrm{d}z_d \\
&\quad = \frac{\ell}{2a}\int_0^\infty\int_0^\infty \Big(\mathrm{e}^{-a|y_d-z_d|} - \mathrm{e}^{-a(y_d+z_d)}\Big)\frac{\mathrm{e}^{-|\xi|z_d} - \mathrm{e}^{-bz_d}}{b(b-|\xi|)}\,\mathrm{e}^{-bx_d}\, g(\xi,x_d)\,\mathrm{d}z_d\,\mathrm{d}x_d.
\end{aligned}
$$

Next, we pull the terms which are independent of z_d out of the inner integral. The remainder of this inner integral requires a not difficult but lengthy calculation. Indeed, we have

$$
\begin{aligned}
&\int_0^\infty \Big(\mathrm{e}^{-a|y_d-z_d|} - \mathrm{e}^{-a(y_d+z_d)}\Big)\Big(\mathrm{e}^{-|\xi|z_d} - \mathrm{e}^{-bz_d}\Big)\,\mathrm{d}z_d \\
&\quad = \int_0^\infty (\mathrm{e}^{-a|y_d-z_d|-|\xi|z_d} - \mathrm{e}^{-a|y_d-z_d|-bz_d} - \mathrm{e}^{-a(y_d+z_d)-|\xi|z_d} + \mathrm{e}^{-a(y_d+z_d)-bz_d})\,\mathrm{d}z_d \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}
\end{aligned}
$$

and calculate each of these integrals separately. Clearly, III and IV are easier to deal with as the domain of integration does not have to be split in those cases. We start with I and obtain

$$\begin{aligned}
\mathrm{I} &= \int_0^{y_d} \mathrm{e}^{-a(y_d-z_d)-|\xi|z_d}\,\mathrm{d}z_d + \int_{y_d}^{\infty} \mathrm{e}^{-a(z_d-y_d)-|\xi|z_d}\,\mathrm{d}z_d \\
&= \mathrm{e}^{-ay_d}\int_0^{y_d} \mathrm{e}^{z_d(a-|\xi|)}\,\mathrm{d}z_d + \mathrm{e}^{ay_d}\int_{y_d}^{\infty} \mathrm{e}^{-z_d(a+|\xi|)}\,\mathrm{d}z_d \\
&= \frac{\mathrm{e}^{-ay_d}}{a-|\xi|}\left(\mathrm{e}^{y_d(a-|\xi|)} - 1\right) - \frac{\mathrm{e}^{ay_d}}{a+|\xi|}\left(0-\mathrm{e}^{-y_d(a+|\xi|)}\right) = \frac{\mathrm{e}^{-|\xi|y_d}-\mathrm{e}^{-ay_d}}{a-|\xi|} + \frac{\mathrm{e}^{-|\xi|y_d}}{a+|\xi|}.
\end{aligned}$$

Next, a similar calculation for II yields

$$\begin{aligned}
\mathrm{II} &= -\int_0^{y_d} \mathrm{e}^{-a(y_d-z_d)-bz_d}\,\mathrm{d}z_d - \int_{y_d}^{\infty} \mathrm{e}^{-a(z_d-y_d)-bz_d}\,\mathrm{d}z_d \\
&= -\mathrm{e}^{-ay_d}\int_0^{y_d} \mathrm{e}^{z_d(a-b)}\,\mathrm{d}z_d - \mathrm{e}^{ay_d}\int_{y_d}^{\infty} \mathrm{e}^{-z_d(a+b)}\,\mathrm{d}z_d \\
&= -\frac{\mathrm{e}^{-ay_d}}{a-b}\left(\mathrm{e}^{y_d(a-b)} - 1\right) + \frac{\mathrm{e}^{ay_d}}{a+b}\left(0-\mathrm{e}^{-y_d(a+b)}\right) \\
&= \frac{-\mathrm{e}^{-by_d}+\mathrm{e}^{-ay_d}}{a-b} - \frac{\mathrm{e}^{-by_d}}{a+b}.
\end{aligned}$$

Now, it remains to cover the last two terms. We have

$$\mathrm{III} = -\int_0^{\infty} \mathrm{e}^{-a(y_d+z_d)-|\xi|z_d}\,\mathrm{d}z_d = -\mathrm{e}^{-ay_d}\int_0^{\infty} \mathrm{e}^{-z_d(a+|\xi|)}\,\mathrm{d}z_d = -\frac{\mathrm{e}^{-ay_d}}{a+|\xi|}$$

and

$$\mathrm{IV} = \int_0^{\infty} \mathrm{e}^{-a(y_d+z_d)-bz_d}\,\mathrm{d}z_d = \mathrm{e}^{-ay_d}\int_0^{\infty} \mathrm{e}^{-z_d(a+b)}\,\mathrm{d}z_d = \frac{\mathrm{e}^{-ay_d}}{a+b}.$$

Combining these results yields

$$\begin{aligned}
\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV} &= \frac{\mathrm{e}^{-|\xi|y_d}-\mathrm{e}^{-ay_d}}{a-|\xi|} + \frac{\mathrm{e}^{-|\xi|y_d}}{a+|\xi|} + \frac{-\mathrm{e}^{-by_d}+\mathrm{e}^{-ay_d}}{a-b} \\
&\quad - \frac{\mathrm{e}^{-by_d}}{a+b} - \frac{\mathrm{e}^{-ay_d}}{a+|\xi|} + \frac{\mathrm{e}^{-ay_d}}{a+b} \\
&= \frac{\left(\mathrm{e}^{-|\xi|y_d}-\mathrm{e}^{-ay_d}\right)(a+|\xi|) + \left(\mathrm{e}^{-|\xi|y_d}-\mathrm{e}^{-ay_d}\right)(a-|\xi|)}{a^2-|\xi|^2} \\
&\quad + \frac{\left(\mathrm{e}^{-ay_d}-\mathrm{e}^{-by_d}\right)(a+b) + \left(\mathrm{e}^{-ay_d}-\mathrm{e}^{-by_d}\right)(a-b)}{a^2-b^2} \\
&= 2a\left(\frac{\mathrm{e}^{-|\xi|y_d}-\mathrm{e}^{-ay_d}}{\lambda} + \frac{\mathrm{e}^{-ay_d}-\mathrm{e}^{-by_d}}{\lambda-\mu}\right)
\end{aligned}$$

by definition of a and b. Next, we plug this into the original calculation of $\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}$ and obtain

$$\begin{aligned}(6.23)\qquad &\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}(\xi, y_d)\\ &= \ell \int_0^\infty \frac{\mathrm{e}^{-bx_d}\, g(\xi, x_d)}{b(b-|\xi|)} \left(\frac{\mathrm{e}^{-ay_d} - \mathrm{e}^{-by_d}}{\lambda - \mu} + \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}}{\lambda} \right) \mathrm{d}x_d.\end{aligned}$$

Having dealt with the first term on the right-hand side of (6.22), we now turn our eyes to the second one. We proceed similarly and then use Fubini's theorem to obtain

$$\begin{aligned}&\hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}}(\xi, y_d)\\ &= \ell \int_0^\infty \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}}{a(a-|\xi|)}\, \mathrm{e}^{-az_d}\, \xi \cdot \hat{v}'_{\mu,f}(\xi, z_d)\, \mathrm{d}z_d\\ &= \ell \int_0^\infty \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}}{a(a-|\xi|)}\, \mathrm{e}^{-az_d}\, \xi \cdot \frac{1}{2b} \int_0^\infty \left(\mathrm{e}^{-b|z_d - x_d|} - \mathrm{e}^{-b(z_d+x_d)} \right) \hat{f}'(\xi, x_d)\, \mathrm{d}x_d\, \mathrm{d}z_d\\ &= \frac{\ell}{2b} \int_0^\infty \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}}{a(a-|\xi|)} g(\xi, x_d) \int_0^\infty \mathrm{e}^{-az_d} \left(\mathrm{e}^{-b|z_d - x_d|} - \mathrm{e}^{-b(z_d+x_d)} \right) \mathrm{d}z_d\, \mathrm{d}x_d.\end{aligned}$$

Calculating the z_d-integral yields

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-az_d} \left(\mathrm{e}^{-b|z_d - x_d|} - \mathrm{e}^{-b(z_d+x_d)} \right) \mathrm{d}z_d\\ &\quad= \int_0^{x_d} \mathrm{e}^{-az_d}\, \mathrm{e}^{-b(x_d - z_d)}\, \mathrm{d}z_d + \int_{x_d}^\infty \mathrm{e}^{-az_d}\, \mathrm{e}^{-b(z_d - x_d)}\, \mathrm{d}z_d - \mathrm{e}^{-bx_d} \int_0^\infty \mathrm{e}^{-z_d(a+b)}\, \mathrm{d}z_d\\ &\quad= \mathrm{e}^{-bx_d} \int_0^{x_d} \mathrm{e}^{-z_d(a-b)}\, \mathrm{d}z_d + \mathrm{e}^{bx_d} \int_{x_d}^\infty \mathrm{e}^{-z_d(a+b)}\, \mathrm{d}z_d - \mathrm{e}^{-bx_d} \int_0^\infty \mathrm{e}^{-z_d(a+b)}\, \mathrm{d}z_d\\ &\quad= -\frac{\mathrm{e}^{-bx_d}}{a-b} \left(\mathrm{e}^{-x_d(a-b)} - 1 \right) - \frac{\mathrm{e}^{bx_d}}{a+b} \left(0 - \mathrm{e}^{-x_d(a+b)} \right) + \frac{\mathrm{e}^{-bx_d}}{a+b} (0-1)\\ &\quad= \frac{-\mathrm{e}^{-ax_d} + \mathrm{e}^{-bx_d}}{a-b} + \frac{\mathrm{e}^{-ax_d} - \mathrm{e}^{-bx_d}}{a+b}\\ &\quad= \frac{1}{a^2 - b^2} \left((a+b)(\mathrm{e}^{-bx_d} - \mathrm{e}^{-ax_d}) + (a-b)(\mathrm{e}^{-ax_d} - \mathrm{e}^{-bx_d}) \right)\\ &\quad= \frac{2b}{\lambda - \mu} \left(\mathrm{e}^{-bx_d} - \mathrm{e}^{-ax_d} \right).\end{aligned}$$

We plug this into the representation above and obtain

$$(6.24)\qquad \hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}}(\xi, y_d) = \ell \int_0^\infty \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}}{a(a-|\xi|)} g(\xi, x_d) \frac{1}{\lambda - \mu} \left(\mathrm{e}^{-bx_d} - \mathrm{e}^{-ax_d} \right) \mathrm{d}x_d.$$

Before considering the third term on the right-hand side of (6.22) let us recall how the left-hand side of (6.22) looks like. Having in mind the Fourier representation of $\hat{w}$, see (6.2) and (6.3), we find

$$\hat{w}_{\lambda,f}(\xi, y_d) = \ell \int_0^\infty \frac{\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}}{a(a-|\xi|)}\, \mathrm{e}^{-az_d}\, g(\xi, z_d)\, \mathrm{d}z_d,$$

$$-\hat{w}_{\mu,f}(\xi, y_d) = -\ell \int_0^\infty \frac{\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-b y_d}}{b(b-|\xi|)} \, \mathrm{e}^{-b z_d} \, g(\xi, z_d) \, \mathrm{d} z_d.$$

Then, we see that $\hat{w}_{\lambda,f}$ is (almost) appearing in (6.24). To be precise, we have

$$\begin{aligned}(\mu-\lambda)\hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}}(\xi, y_d) &= \ell \int_0^\infty \frac{\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d}}{a(a-|\xi|)} g(\xi, x_d) \left(\mathrm{e}^{-a x_d} - \mathrm{e}^{-b x_d}\right) \mathrm{d} x_d \\ &= \hat{w}_{\lambda,f}(\xi, y_d) - \int_0^\infty \frac{\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d}}{a(a-|\xi|)} g(\xi, x_d) \, \mathrm{e}^{-b x_d} \, \mathrm{d} x_d.\end{aligned}$$

Similarly, examining the representation (6.23) of $\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}$ we see that $-\hat{w}_{\mu,f}$ (almost) appears as well. Indeed, adding $\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-|\xi| y_d}$ we obtain

$$\begin{aligned}&(\mu-\lambda)\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}(\xi, y_d) \\ &= (\mu-\lambda)\ell \int_0^\infty \frac{g(\xi, x_d)\, \mathrm{e}^{-b x_d}}{b(b-|\xi|)} \left(\frac{\mathrm{e}^{-a y_d} - \mathrm{e}^{-|\xi| y_d} + \mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-b y_d}}{\lambda - \mu} + \frac{\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d}}{\lambda} \right) \mathrm{d} x_d \\ &= -\hat{w}_{\mu,f}(\xi, y_d) + \ell \int_0^\infty \frac{\mathrm{e}^{-b x_d}\, g(\xi, x_d)}{b(b-|\xi|)} \Big(\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d} + \frac{\mu - \lambda}{\lambda} \Big(\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d} \Big)\Big) \, \mathrm{d} x_d \\ &= -\hat{w}_{\mu,f}(\xi, y_d) + \ell \int_0^\infty \frac{\mathrm{e}^{-b x_d}\, g(\xi, x_d)}{b(b-|\xi|)} \frac{\mu}{\lambda} (\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d}) \, \mathrm{d} x_d.\end{aligned}$$

Combining this calculation with the one for $(\mu-\lambda)\hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}}(\xi, y_d)$ yields

$$\begin{aligned}&(\mu-\lambda)(\hat{v}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}} + \hat{w}_{\lambda,\mathcal{F}^{-1}\hat{v}_{\mu,f}})(\xi, y_d) \\ &\quad = \hat{w}_{\lambda,f}(\xi, y_d) - \hat{w}_{\mu,f}(\xi, y_d) \\ &\qquad + \ell \int_0^\infty (\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d})\, \mathrm{e}^{-b x_d}\, g(\xi, x_d) \Big(\frac{\mu}{\lambda b(b-|\xi|)} - \frac{1}{a(a-|\xi|)} \Big) \, \mathrm{d} x_d\end{aligned} \tag{6.25}$$

Now, it remains to deal with the third term on the right-hand side of (6.22). Using the introduced notation we can again cover all components with one calculation. We have

$$\begin{aligned}&\hat{w}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}(\xi, y_d) \\ &= \ell \int_0^\infty \int_0^\infty \frac{\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d}}{a(a-|\xi|)} \, \mathrm{e}^{-a z_d}\, \xi \cdot \frac{\xi}{|\xi|} \frac{\mathrm{e}^{-|\xi| z_d} - \mathrm{e}^{-b z_d}}{b(b-|\xi|)} \, \mathrm{e}^{-b x_d}\, g(\xi, x_d) \, \mathrm{d} x_d \, \mathrm{d} z_d \\ &= \ell \int_0^\infty \frac{\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-a y_d}}{a(a-|\xi|) b(b-|\xi|)} \, \mathrm{e}^{-b x_d} \, |\xi| g(\xi, x_d) \int_0^\infty \mathrm{e}^{-a z_d} \left(\mathrm{e}^{-|\xi| z_d} - \mathrm{e}^{-b z_d} \right) \mathrm{d} z_d \, \mathrm{d} x_d.\end{aligned}$$

Again, we compute the inner integral separately. It holds

$$\int_0^\infty \mathrm{e}^{-a z_d} \left(\mathrm{e}^{-|\xi| z_d} - \mathrm{e}^{-b z_d} \right) \mathrm{d} z_d = \int_0^\infty \mathrm{e}^{-z_d(a+|\xi|)} \, \mathrm{d} z_d - \int_0^\infty \mathrm{e}^{-z_d(a+b)} \, \mathrm{d} z_d$$

$$= \frac{1}{a+|\xi|} - \frac{1}{a+b}.$$

Hence, we have

$$\begin{aligned}(6.26)\qquad &(\mu-\lambda)\hat{w}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}(\xi,y_d) \\ &= \ell \int_0^\infty \frac{\left(\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-ay_d}\right)\mathrm{e}^{-bx_d}\, g(\xi,x_d)}{a(a-|\xi|)b(b-|\xi|)} |\xi|(\mu-\lambda)\left(\frac{1}{a+|\xi|} - \frac{1}{a+b}\right) \mathrm{d}x_d.\end{aligned}$$

Comparing (6.25) and (6.26) reveals that (6.22), and thus the resolvent identity, is proven if $-\hat{w}_{\lambda,\mathcal{F}^{-1}\hat{w}_{\mu,f}}(\xi,y_d)$ coincides with the last summand in (6.25). To this end, use $\lambda = (a-|\xi)(a+|\xi|)$ as well as $\mu = (b-|\xi|)(b+|\xi)$ for $\xi \in \mathbb{R}^{d-1}$ in the last factor of (6.25). We thus have

$$\begin{aligned}\frac{\mu}{\lambda b(b-|\xi|)} - \frac{1}{a(a-|\xi|)} &= \frac{(b-|\xi|)(b+|\xi|)}{(a-|\xi|)(a+|\xi|)b(b-|\xi|)} - \frac{1}{a(a-|\xi|)} \\ &= \frac{a(b+|\xi|) - b(a+|\xi|)}{(a-|\xi|)(a+|\xi|)ab} = \frac{(a-b)|\xi|}{(a-|\xi|)(a+|\xi|)ab}.\end{aligned}$$

As the integrals in (6.26) and (6.25) have the same structure and $(b–a)(b+a) = \mu–\lambda$ holds, it suffices to examine

$$\frac{|\xi|(b-a)(b+a)}{a(a-|\xi|)b(b-|\xi|)}\left(\frac{1}{a+|\xi|} - \frac{1}{a+b}\right) + \frac{(a-b)|\xi|}{(a-|\xi|)(a+|\xi|)ab}.$$

Taking a closer look reveals that there are some more common factors so that it is sufficient to compute

$$\begin{aligned}&\frac{(b-a)(b+a)}{(b-|\xi|)}\left(\frac{1}{a+|\xi|} - \frac{1}{a+b}\right) + \frac{(a-b)}{(a+|\xi|)} \\ &\qquad = \frac{(b-a)(b+a)}{(b-|\xi|)}\frac{(a+b-a-|\xi|)}{(a+|\xi|)(a+b)} + \frac{(a-b)}{(a+|\xi|)} \\ &\qquad = \frac{b-a}{a+|\xi|} + \frac{a-b}{a+|\xi|} = 0.\end{aligned}$$

Hence, (6.22) and consequently the resolvent identity for R^u is proved. □

As a corollary, we deduce that the resolvent identity also holds when we consider the operators on $\mathrm{L}^p(\mathbb{R}^d_+)$.

Corollary 6.2.7. *Let $1 < p < \infty$. For $\lambda,\mu \in \mathrm{S}_{\pi-\varepsilon}$ we have $R^u(\lambda) - R^u(\mu) = (\mu-\lambda)R^u(\lambda)R^u(\mu)$ as operators on $\mathrm{L}^p(\mathbb{R}^d_+)$.*

Proof. By the previous proposition the resolvent identity is true on $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d_+) \subset \mathrm{L}^p(\mathbb{R}^d_+)$. As by Proposition 6.2.2 the resolvent is a continuous operator on $\mathrm{L}^p(\mathbb{R}^d_+)$ and $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d_+)$ is dense in $\mathrm{L}^p(\mathbb{R}^d_+)$, we conclude the proof. □

6.3 Stokes semigroup on $\mathrm{L}^p(\mathbb{R}^d_+)$

In this section we define the extended Stokes semigroup on $\mathrm{L}^p(\mathbb{R}^d_+)$ using the resolvent estimates from the previous section. The results are as expected, however, due to the extension to non divergence-free functions, some work is required.

Definition 6.3.1. Let $1 < p < \infty$. Then we define the extended Stokes semigroup $E^u(t)$ for $t > 0$ by

$$E^u(t)f \coloneqq \frac{1}{2\pi \mathrm{i}} \int_{\gamma_t} \mathrm{e}^{\lambda t} R^u(\lambda) f \,\mathrm{d}\lambda$$

for $f \in \mathrm{L}^p(\mathbb{R}^d_+)$. Here γ_t is the path parametrizing the boundary of $\mathrm{S}_\varphi \setminus B(0, t^{-1})$ for some $\varphi \in (\pi/2, \pi)$ counter-clockwise.

The following theorem now states that these operators are indeed well-defined, define a semigroup and satisfy L^p-L^q-estimates.

Theorem 6.3.2. *Let $1 < p \leq q < \infty$. For the operator $E^u(t)$ defined as above and $f \in \mathrm{L}^p(\mathbb{R}^d_+)$, the following estimates hold true for all $t > 0$:*

$$\|E^u(t)f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \lesssim \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \tag{6.27}$$

$$\|E^u(t)f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)}. \tag{6.28}$$

Moreover, the family $(E^u(t))_{t>0}$ satisfies the semigroup law. Here, the implicit constants only depend on d, p and q.

Proof. The first estimate is an immediate consequence of the first part of Proposition 6.2.2 and [91, Proposition 3.7]. Note that their proof is valid even though $R^u(\lambda)$ is not a resolvent family in the classical sense. Knowing that $E^u(t)$ is well-defined, it follows from Corollary 6.2.7 that the family $(E^u(t))_{t>0}$ is a semigroup on $\mathrm{L}^p(\mathbb{R}^d_+)$. For a detailed proof of this fact see [36, 4.3 Proposition]. For $1 < p \leq q < \infty$ with $\frac{1}{d} \geq \frac{1}{p} - \frac{1}{q}$ the second part directly follows from Proposition 6.2.3 together with the transference principle [91, Proposition 3.7]. For arbitrary $1 < p \leq q < \infty$ there exists $k \in \mathbb{N}$ such that $\frac{k}{d} \geq \frac{1}{p} - \frac{1}{q}$ holds. We then use the semigroup law to write $E^u(t) = E^u\left(\frac{t}{k}\right) \cdots E^u\left(\frac{t}{k}\right)$. In each time step we can go from $\mathrm{L}^r(\mathbb{R}^d_+)$ to $\mathrm{L}^s(\mathbb{R}^d_+)$ with the desired estimate where $1 < s \leq r < \infty$ satisfy $\frac{1}{d} \geq \frac{1}{s} - \frac{1}{r}$. This means that after k steps we arrive at the statement of the theorem. □

Remark 6.3.3. Note, that $R^u(\lambda)$ is holomorphic in λ so that by Cauchy's integral formula the path γ_t can be replaced by any path in $\mathrm{S}_{\pi-\varepsilon}$ going from $\mathrm{e}^{-i\theta\infty}$ to $\mathrm{e}^{i\theta\infty}$ for $\theta \in (\pi/2, \pi)$. However, for the L^p-estimate it is crucial to take the path γ_t as otherwise the behaviour in t will not be the desired one.

Moreover, $E^u(t)$ satisfies the following gradient estimates:

Theorem 6.3.4. *Let $1 < p \leq q < \infty$. For $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ and all $t > 0$ the estimates*

$$\|\nabla E^u(t) f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \lesssim t^{-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \tag{6.29}$$

$$\|\nabla E^u(t) f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)} \tag{6.30}$$

hold. The implicit constants only depend on d.

Proof. Again, the first part is an immediate consequence of Proposition 6.2.2. For the second part note that $E^u(t)$ satisfies the semigroup law and so we can combine the first estimate with L^p-L^q-estimates for the semigroup to obtain

$$\begin{aligned}\|\nabla E^u(t) f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} &= \|\nabla E^u(t/2) E^u(t/2) f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} \\ &\lesssim t^{-\frac{1}{2}} \|E^u(t/2) f\|_{\mathrm{L}^q} (\mathbb{R}^d_+) \\ &\lesssim t^{-\frac{1}{2}} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{\mathrm{L}^p(\mathbb{R}^d_+)}\end{aligned}$$

and this concludes the proof. □

As previously discussed, in our setting it is crucial to also understand the behaviour of the pressure. Hence, it is natural to define an object analogous to $E^u(t)$ but now for the pressure.

Definition 6.3.5. For $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ and $t > 0$ we define

$$E^\phi(t) f := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_t} \mathrm{e}^{\lambda t} R^\phi(\lambda) f \, \mathrm{d}\lambda.$$

Here, γ_t is as in Definition 6.3.1.

Remark 6.3.6. Note that we are not able to prove global L^p-estimates for $E^\phi(t)$. This is because the point-wise estimates of the kernel of the pressure are not suitably behaved. Recall that in Proposition 6.2.4 we showed a point-wise estimate for $R^\phi(\lambda)$ which, away from the half-space boundary, is uniform. Plugging this estimate into the Cauchy integral formula shows that $E^\phi(t)f$ is well-defined for $f \in \mathrm{L}^2(\mathbb{R}^d_+)$. For further estimates on $E^\phi(t)$ we refer to Proposition 7.1.9 in the next chapter. Finally, note that without a resolvent identity for R^ϕ one does not expect a semigroup law to hold.

CHAPTER 7

Off-diagonal and L^2-estimates

The goal of this chapter is twofold. In the first part we aim for an L^2-estimate for the maximal regularity operator. It turns out that for such an estimate additional assumptions are necessary. To be precise, we need to restrict to functions with compact support with respect to the spatial variables. However, this is not really a restriction as a density argument allows to waive the assumption on compact support. Recalling $\mathrm{T}^{2,2} = \mathrm{L}^2$ we obtain a first boundedness result on a tent space. To this end it will be also necessary to extend the Stokes operator to tupels of functions as working on non divergence-free functions restrains us from recovering the pressure through the velocity.

Recall that we were unable to use the Helmholtz projection to extend the Stokes semigroup to non divergence-free functions as then off-diagonal estimates are no longer satisfied. In the second part of this chapter we now verify that our extension procedure respects off-diagonal estimates. The proof requires a careful decomposition of the domain of integration as well as the point-wise estimates proven by [68].

7.1 The extended maximal regularity operator

In this section we want to extend the maximal regularity operator to $\mathrm{L}^2(\mathbb{R}^d_+)$. Let us first assume that $F\colon [0,\infty) \to \mathrm{L}^2(\mathbb{R}^d_+)$ takes values in $\mathrm{L}^2_\sigma(\mathbb{R}^d_+)$. Then, taking Fourier transform in time and using that the Laplace transform of the semigroup is the resolvent, it is known, see, e.g., [63, 1.5 Discussion], that the maximal regularity operator can be rewritten as

$$\tilde{A}\int_0^t \mathrm{e}^{-(t-s)\tilde{A}}\, F(s)\,\mathrm{d}s = [\mathcal{F}^{-1}[\mathrm{i}\xi(\mathrm{i}\xi-\tilde{A})^{-1}-\mathrm{Id}]\mathcal{F}F\chi_{(0,\infty)}](t). \tag{7.1}$$

Here, $\tilde{A}$ is the classical Stokes operator on $\mathbb{R}^d_+$ and $\mathrm{e}^{-t\tilde{A}}$ the generated semigroup. Our aim is to prove a similar formula but without the assumption that F takes values in $\mathrm{L}^2_\sigma(\mathbb{R}^d_+)$. It will turn out that both representations have their merits and our proof of the main result heavily relies on the possibility to choose the more suitable representation in every situation.

The crucial ingredient in proving (7.1) is the fact that the Laplace transform of the Stokes semigroup on the imaginary axis is the corresponding resolvent. Note that on the right-half plane this is a general result in semigroup theory, see e.g., [3, the proof of Theorem 3.7.11], and does not depend on the concrete operator.

Thus, our aim is to prove that the Laplace transform of the extended semigroup $(E^u(t))_{t\geq 0}$ is in fact the extended resolvent R^u. In this context we need the decay of $E^u(t)$ stated in Theorem 6.3.2.

To this end we will proceed as follows: we will show that the Laplace transform of the extended semigroup is well-defined for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and that it there coincides with the extended resolvent. Next, we define the Laplace transform for $z \in \mathrm{i}\mathbb{R}$ as a distribution. Here, the L^p-L^q-estimates will play a crucial role. Finally, we will then be able to show that also for those z the Laplace transform coincides with the extended resolvent in the sense of distributions. Unfortunately, the same strategy, and related ones, do not work for the extended pressure $(E^\phi(t))_{t>0}$ as the kernel estimates are not sufficiently well-behaved. However, it is possible to prove (7.1) via a different strategy.

First we consider the Laplace transform on the right-half plane. Recall that the path γ_k was introduced in Definition 6.3.1.

Lemma 7.1.1. *Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Then the Laplace transform of $E^u(t)$ at z defined by*

$$\mathcal{L}(E^u)(z) := \int_0^\infty \mathrm{e}^{-zt}\, E^u(t)\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-zt}\int_{\gamma_k} \mathrm{e}^{\lambda t}\, R^u(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}t,$$

for k sufficiently large, exists as an operator on $\mathrm{L}^p(\mathbb{R}^d_+)$ and coincides there with $R^u(z)$. The same result is true with E^u replaced by E^ϕ and R^u replaced by R^ϕ.

Proof. We only prove the statement for E^u and R^u as for E^ϕ and R^ϕ the same proof applies. Recall that by Remark 6.3.3 the definition of $E^u(t)$ is independent of the particular path, so that we can indeed use the paths γ_k. Let $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ and

$k \in \mathbb{N}$ large enough that $\operatorname{Re} z > \frac{1}{k}$. Then $\operatorname{Re} z > \operatorname{Re} \lambda$ for all $\lambda \in \gamma_k$. By Fubini's theorem we then have

$$\begin{aligned}\mathcal{L}(E^u)(z)f &= \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-zt} \int_{\gamma_k} \mathrm{e}^{\lambda t} R^u(\lambda) f \, \mathrm{d}\lambda \, \mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_k} \int_0^\infty \mathrm{e}^{t(\lambda - z)} R^u(\lambda) f \, \mathrm{d}t \, \mathrm{d}\lambda \\ &= -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_k} \frac{R^u(\lambda) f}{\lambda - z} \, \mathrm{d}\lambda = R^u(z) f\end{aligned}$$

where in the last step we used the Cauchy integral formula. Here, the minus sign cancels with the fact that the curves passes clockwise by z. □

Note that this exact argument does not work for $z \in \mathrm{i}\mathbb{R}$ as it is not possible to find a path γ on which $\operatorname{Re}(\lambda - z) > 0$ holds true. Hence, we need to make a small detour in this case and define $\mathcal{L}(E^u)(z)f$ as a distribution.

Lemma 7.1.2. *Let $z \in \mathrm{i}\mathbb{R}$ and $d \geq 3$. For $f \in \mathrm{L}^2(\mathbb{R}^d_+)$, $\mathcal{L}(E^u)(z)f$ induces a distribution.*

Proof. Let $r > \frac{2d}{d-2}$ and $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d_+)$. Splitting the integral at $t = 1$ and using Hölder's inequality as well as Hardy's inequality in dimension 1, see, e.g., [56], for $t > 1$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d_+} \int_0^\infty \mathrm{e}^{-zt} E^u(t) f \, \mathrm{d}t \, \varphi \, \mathrm{d}x \leq{}& (\|\varphi\|_{\mathrm{L}^2(\mathbb{R}^d_+)} + \|x_d \varphi\|_{\mathrm{L}^{r'}(\mathbb{R}^d_+)}) \\ &\cdot \Big(\int_0^1 \|E^u(t) f\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \, \mathrm{d}t + \int_1^\infty \|\nabla E^u(t) f\|_{\mathrm{L}^r(\mathbb{R}^d_+)} \, \mathrm{d}t \Big)\end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. The convergence of the first integral is clear by the uniform L^2-boundedness of the family $(E^u(t))$. Hence, it remains to show convergence of the second integral. Using the L^p-L^q-estimates for the gradient of the semigroup proven in Theorem 6.3.4 it holds

$$\int_1^\infty \|\nabla E^u(t) f\|_{\mathrm{L}^r(\mathbb{R}^d_+)} \, \mathrm{d}t \lesssim \int_1^\infty t^{-\frac{d}{2}(\frac{1}{2} - \frac{1}{r}) - \frac{1}{2}} \, \mathrm{d}t \|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}$$

and the integral is finite by the choice of r. □

Now, we are able to prove that the Laplace transform of the semigroup $(E^u(t))_{t>0}$ coincides with the resolvent R^u on the imaginary axis.

Proposition 7.1.3. *Let $d \geq 3$, $z \in \mathrm{i}\mathbb{R} \setminus \{0\}$ and $(z_n)_{n \in \mathbb{N}}$ with $\operatorname{Re} z_n > 0$ and $z_n \to z$. Then we have $\mathcal{L}(E^u)(z_n) f \to \mathcal{L}(E^u)(z) f$ in the sense of distributions for all $f \in \mathrm{L}^2(\mathbb{R}^d_+)$. In particular, $\mathcal{L}(E^u)(z) f = R^u(z) f$ is true for those f.*

Proof. Convergence of the integral can be shown as in the previous lemma. Then, an application of the dominated convergence theorem proves the first part. Note that the resolvent identity proved in Proposition 6.2.6 implies the continuity of the map $\lambda \mapsto R^u(\lambda)$ in $\mathrm{S}_{\pi-\varepsilon}$ in view of the L^2-boundedness of the resolvent. $\square$

Although the case $p = 2$ is sufficient for the scope of this thesis, we next prove that the case $1 < p < \infty$ can be considered. However, we need to restrict ourselves to functions with compact support.

Lemma 7.1.4. *Let $1 < p < \infty$ and $z \in \mathrm{i}\mathbb{R}$. For $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ with compact support $\mathcal{L}(E^u)(z)f$ induces a distribution.*

Proof. As f has compact support, there exists $1 < q < \frac{d}{2}$ such that $f \in \mathrm{L}^q(\mathbb{R}^d_+)$. Let $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d_+)$ be a test function and $r > q$ such that $\frac{1}{q} - \frac{1}{r} > \frac{2}{d}$. Then by Hölder's inequality for $\frac{1}{r} + \frac{1}{r'}$ we have

$$\int_{\mathbb{R}^d_+} \int_0^\infty \mathrm{e}^{-zt}\, E^u(t)f \,\mathrm{d}t \,\varphi \,\mathrm{d}x \leq \|\varphi\|_{\mathrm{L}^{r'}(\mathbb{R}^d_+)} \int_0^\infty \|E^u(t)f\|_{\mathrm{L}^r(\mathbb{R}^d_+)} \,\mathrm{d}t.$$

To check that the integral converges, we split the integral at some fixed time $t < \infty$, say $t = 1$, and obtain with Theorem 6.3.2 that

$$\int_0^\infty \|E^u(t)f\|_{\mathrm{L}^r(\mathbb{R}^d_+)} \,\mathrm{d}t \lesssim \|f\|_{\mathrm{L}^r} + \|f\|_{\mathrm{L}^q(\mathbb{R}^d_+)} \int_1^\infty t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \,\mathrm{d}t < \infty$$

holds by choice of r and q. $\square$

Proposition 7.1.5. *Let $1 < p < \infty$, $z \in \mathrm{i}\mathbb{R} \setminus \{0\}$ and $(z_n)_{n\in\mathbb{N}}$ with $\operatorname{Re} z_n > 0$ and $z_n \to z$. Then we have $\mathcal{L}(E^u)(z_n)f \to \mathcal{L}(E^u)(z)f$ in the sense of distributions for all $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ with compact support. In particular, $\mathcal{L}(E^u)(z)f = R^u(z)f$ is true for those f.*

Proof. Let $z \in \mathrm{i}\mathbb{R}$ and $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ with $\operatorname{Re} z_n > 0$ and $z_n \to z$. For $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d_+)$ and r as in the previous proof we then have by Hölder's inequality

$$\left| \int_{\mathbb{R}^d_+} \int_0^\infty \left(\mathrm{e}^{-zt} - \mathrm{e}^{-z_n t}\right) E^u(t)f \,\mathrm{d}t \,\varphi \,\mathrm{d}x \right| \leq \|\varphi\|_{\mathrm{L}^{r'}} \int_0^\infty \left|\mathrm{e}^{-zt} - \mathrm{e}^{-z_n t}\right| \|E^u(t)f\|_{\mathrm{L}^r} \,\mathrm{d}t.$$

The convergence of the last integral can be shown as in the previous lemma. Then, an application of the dominated convergence theorem concludes the proof. For the continuity of the extended resolvent on $\mathrm{L}^p(\mathbb{R}^d_+)$ in λ argue as in Proposition 7.1.3. Thus, we conclude that $\mathcal{L}(E^u)(z)f = R^u(z)f$ holds for all $f \in \mathrm{L}^p(\mathbb{R}^d_+)$ with compact support. $\square$

Now we are at the delicate point where we would like to show an analogue of Lemma 7.1.2 but for the pressure. Unfortunately, due to the weaker decay estimates this does not seem to be possible. Thus, a different strategy for establishing (7.1) will be needed. We present this strategy in next subsection.

7.1.1 Extended Stokes operator

Before we can extend the Fourier representation of the maximal regularity operator (7.1) to non divergence-free functions, it is first necessary to extend the Stokes operator. As the divergence-free condition is crucial to obtain the pressure from the velocity, our operator will be defined on a tupel (u, ϕ) consisting of velocity and pressure.

Definition 7.1.6. Define an operator $A\colon \mathrm{L}^2(\mathbb{R}^d_+) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+) \to \mathrm{L}^2(\mathbb{R}^d_+)$ via

$$\mathcal{D}(A) \coloneqq \{(u, \phi) \in \mathrm{H}^1(\mathbb{R}^d_+) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+) : -\Delta u + \nabla \phi \in \mathrm{L}^2(\mathbb{R}^d_+)\}$$

and $A(u, \phi) \coloneqq -\Delta u + \nabla \phi$ for $(u, \phi) \in \mathcal{D}(A)$ where derivatives are meant in the sense of distributions.

Remark 7.1.7. Before working towards an analogue of (7.1) let us mention that A with domain $\mathcal{D}(A)$ is a closed operator. Note that $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$ is not a Banach space, but merely a Fréchet space. However, this does not pose any problems.

Our goal is to show $\int_0^t E(t-s)f(s)\,\mathrm{d}s \in \mathcal{D}(A)$ for all $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ where $E(t) = (E^u(t), E^\phi(t))$. Recall that the extended semigroup was defined via the Cauchy integral of the extended resolvent. Hence, we start with a similar result on the level of resolvents:

Proposition 7.1.8. *For $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ and $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ we have $(R^u(\lambda)f, R^\phi(\lambda)f) \in \mathcal{D}(A)$.*

Proof. Let $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ and $f \in \mathrm{L}^2(\mathbb{R}^d_+)$. We know by Proposition 6.2.2 that $R^u(\lambda)f \in \mathrm{H}^1(\mathbb{R}^d_+)$ and by Proposition 6.2.4 that $R^\phi(\lambda)f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$. Moreover, the resolvent identity stated in Proposition 6.2.5 implies

$$-\Delta R^u(\lambda)f + \nabla R^\phi(\lambda)f = f - \lambda R^u(\lambda)f \in \mathrm{L}^2(\mathbb{R}^d_+)$$

so that $(R^u(\lambda)f, R^\phi(\lambda)f) \in \mathcal{D}(A)$ with $A((R^u(\lambda)f, R^\phi(\lambda)f)) = f - \lambda R^u(\lambda)f$. $\square$

In a next step we examine the extended semigroup.

Proposition 7.1.9. *Let $f \in \mathrm{L}^2(\mathbb{R}^d_+)$. Then $(E^u(t)f, E^\phi(t)f) \in \mathcal{D}(A)$ for all $t > 0$. In particular,*

$$\|tA(E^u(t)f, E^\phi(t)f)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \lesssim \|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}$$

holds where the implicit constant depends on d.

Proof. Let $f \in \mathrm{L}^2(\mathbb{R}^d_+)$ and $t > 0$. The regularity for $E^u(t)f$ was proven in Theorem 6.3.2 and Theorem 6.3.4. For $E^\phi(t)f$ we obtain with Proposition 6.2.4:

$$\begin{aligned}\|E^\phi(t)f\|_{\mathrm{L}^2(K)} &\lesssim \int_{\gamma_t} \mathrm{e}^{-\lambda t} \, \|R^\phi(\lambda)f\|_{\mathrm{L}^2(K)} \, |\mathrm{d}\lambda| \\ &\lesssim t^{-3/4}\|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}\end{aligned}$$

for any compact set $K \subset \mathbb{R}^d_+$. Here, the implicit constant depends on K. This shows the desired regularity.

For the distributional derivative let us argue that we can change derivative and Cauchy integral. Indeed, by the distributional derivative for the resolvents we have

$$\int_{\gamma_t} \mathrm{e}^{-\lambda t}(-\Delta R^u(t)f + \nabla R^\phi(t)f) \, \mathrm{d}\lambda = \int_{\gamma_t} \mathrm{e}^{-\lambda t}(f - \lambda R^u(\lambda)f \, \mathrm{d}\lambda.$$

Further note that the Cauchy integral of the first term vanishes as we integrate an holomorphic function. By the L^2-boundedness of $\lambda R^u(\lambda)$ deduced in Proposition 6.2.2, the integral on the right-hand side converges. Thus, we can change the order of distributional derivative and integral and infer

$$\| - \Delta E^u(t)f + \nabla E^\phi(t)f\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \leq t^{-1}\|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}.$$

This completes the proof. $\square$

Finally, we consider the time integral $(\int_0^t E^u(t-s)f(s)\,\mathrm{d}s, \int_0^t E^\phi(t-s)f(s)\,\mathrm{d}s)$ for $f \in \mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))$ and first show the desired regularities.

Proposition 7.1.10. *Let $t > 0$. The map*

$$f \mapsto \int_0^t E^u(t-s)f(s,\cdot)\,\mathrm{d}s$$

is bounded from $\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))$ to $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{W}^{1,2}(\mathbb{R}^d_+))$. Moreover for every compact set $K \subset \mathbb{R}^d_+$ there exists a constant $C(K) > 0$ such that

$$\left\|\int_0^t E^\phi(t-s)f(s,\cdot)\,\mathrm{d}s\right\|_{\mathrm{L}^\infty(K)} \leq C(K)t^{\frac{1}{4}}\|f\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))}$$

for all $t > 0$ and $f \in \mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))$.

Proof. The L^2-estimate is an immediate consequence of the uniform L^2-boundedness of $E(t)$ and the inclusion of L^p-spaces on finite measure spaces (here finite time interval). For the gradient, slightly more caution is required. Taking the L^2-norm inside the integral yields

$$\int_0^t \|\nabla E^u(t-s)f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \,\mathrm{d}s \lesssim \int_0^t (t-s)^{-\frac{1}{2}} \|f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \,\mathrm{d}s.$$

We want to write this integral as the convolution of two suitable functions. To this end note that $s^{-\frac{1}{2}} \in \mathrm{L}^1([0,T])$ for any $T>0$ and $\|f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \in \mathrm{L}^2((0,\infty))$. Thus, for each $T>0$ the convolution $t^{-\frac{1}{2}}\chi_{[0,T]} * \|f(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)}\chi_{[0,T]}$ exists for almost all t by Young's convolution inequality. For $t \leq T$ we have

$$\begin{aligned} t^{-\frac{1}{2}}\chi_{[0,T]} * \|f(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)}\chi_{[0,T]}(t) &= \int_0^T (t-s)^{-\frac{1}{2}}\chi_{[0,T]}(t-s)\|f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \,\mathrm{d}s \\ &= \int_0^t (t-s)^{-\frac{1}{2}} \|f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \,\mathrm{d}s. \end{aligned}$$

Hence, the right-hand side converges for almost all $t \leq T$ and as T was arbitrary for almost all $t>0$. The L^2-regularity in time is again a consequence of Young's convolution inequality.

For the second estimate let $y \in K$ be arbitrary and note $y_d > c(K)$ for some $c(K) > 0$ independent of y. Applying Proposition 6.2.4, we obtain

$$\begin{aligned} &\left| \int_0^t \int_{\gamma_{t-s}} e^{\lambda(t-s)} \int_{\mathbb{R}^d_+} R^\phi(\lambda) f(s,\cdot) \,\mathrm{d}\lambda \,\mathrm{d}s \right| \\ &\lesssim C(K) \int_0^t \int_{\gamma_{t-s}} \mathrm{e}^{\operatorname{Re}\lambda(t-s)} \|f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} |\lambda|^{-\frac{1}{4}} \,|\mathrm{d}\lambda| \,\mathrm{d}s \\ &\lesssim C(K) \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \,\mathrm{d}s. \end{aligned}$$

Note that the last integral is again of convolution type so Young's convolution inequality yields the desired estimate. $\square$

For the estimate of the distributional derivative we must proceed differently since in the proof of Proposition 7.1.9 we showed

$$\| -\Delta E^u(t) f + \nabla E^\phi(t) f\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \leq t^{-1}\|f\|_{\mathrm{L}^2(\mathbb{R}^d_+)}$$

and the right-side is not integrable with respect to t.

As mentioned earlier in this section the classical argument via the Laplace transform is not available due to the presence of the pressure. This manifests the need of a different approach that we present now. Instead of $f \in \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$ we first consider $g \in \mathrm{C}_\mathrm{c}^\infty((0,\infty);\mathrm{L}^2(\mathbb{R}^d_+))$ and show $\int_0^t E(t-s)g(s)\,\mathrm{d}s \in \mathcal{D}(A)$ for almost all $t>0$. The closedness of A will us then allow to conclude the same for general $f \in \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$. The differentiability of g is here to be understood in the Fréchet-sense.

Theorem 7.1.11. *Let $g \in \mathrm{C}_c^\infty((0,\infty);\mathrm{L}^2(\mathbb{R}_+^d))$. Then for almost every $t > 0$ we find*

$$\int_0^t E(t-s)g(s)\,\mathrm{d}s \in \mathcal{D}(A)$$

with

$$A\int_0^t E(t-s)g(s)\,\mathrm{d}s = [\mathcal{F}^{-1}([\mathrm{Id} - \mathrm{i}\xi R^u(\mathrm{i}\xi)]\mathcal{F}g)(t).$$

Proof. By definition of A we have to show two things, namely

$$\int_0^t E(t-s)g(s)\,\mathrm{d}s \in \mathrm{H}^1(\mathbb{R}_+^d) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+^d) \tag{7.2}$$

and

$$-\Delta\int_0^t E^u(t-s)g(s)\,\mathrm{d}s + \nabla\int_0^t E^\phi(t-s)g(s)\,\mathrm{d}s \in \mathrm{L}^2(\mathbb{R}_+^d) \tag{7.3}$$

for almost all $t > 0$ where the derivatives are meant in the distributional sense. Note that (7.2) was established in Proposition 7.1.10. The distributional derivatives are defined via an integration by parts. For $h \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}_+^d)$ we thus consider

$$\int_{\mathbb{R}_+^d}\Big(\int_0^t E(t-s)g(s)\,\mathrm{d}s\Big)(x)\cdot(-\Delta h(x), \operatorname{div} h(x))\,\mathrm{d}x =: \Big\langle\int_0^t E(t-s)g(s)\,\mathrm{d}s, A'h\Big\rangle$$

where $\cdot$ is the scalar product in $\mathbb{R}^2$, $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathrm{L}^2(\mathbb{R}_+^d)$ and $A'h \coloneqq (-\Delta h, \operatorname{div} h)$. For further examinations we test with $\varphi \in \mathcal{S}(\mathbb{R};\mathbb{R})$ satisfying $\mathcal{F}^{-1}\varphi(0) = 0$. Note that these functions are dense in $\mathrm{L}^2(\mathbb{R}_+^d)$, by, e.g., [23, Lemma 3.6]. Here $\mathcal{F}$ is the one dimensional Fourier transform and $\mathcal{S}$ denotes the Schwartz class. We use the convention

$$\mathcal{F}\varphi(t) \coloneqq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi t}\,\varphi(\xi)\,\mathrm{d}\xi.$$

Note that due to the compact support of h, the space integral is taken only over a compact subset of $\mathbb{R}_+^d$. On such a set the term $E^\phi(t-s)g$ is well-behaved for $s \to t$ by Proposition 6.2.4 in the sense that the L^2-norm behaves like $(t-s)^{-\frac{3}{4}}$. Together with the condition $\varphi \in \mathcal{S}(\mathbb{R};\mathbb{R})$, this allows us to employ Fubini's theorem to rewrite

$$\int_{\mathbb{R}}\Big\langle\int_0^t E(t-s)g(s)\,\mathrm{d}s, A'h\Big\rangle\varphi(t)\,\mathrm{d}t = \Big\langle\int_{\mathbb{R}}\int_0^t E(t-s)g(s)\,\mathrm{d}s\,\varphi(t)\,\mathrm{d}t, A'h\Big\rangle. \tag{7.4}$$

The next goal is to show

$$\int_{\mathbb{R}}\int_0^t E(t-s)g(s)\,\mathrm{d}s\,\varphi(t)\,\mathrm{d}t \in \mathcal{D}(A). \tag{7.5}$$

Here, due to the L^2-scalar product with a compactly supported function, the spatial variables are from now on restricted to said compact set. By virtue of the transformation $\tilde{s} = t - s$, $\operatorname{supp}(g) \subseteq (0, \infty)$ and renaming the variables we find

$$\begin{aligned}\int_{\mathbb{R}} \int_0^t E(t-s) g(s) \,\mathrm{d}s \, \varphi(t) \,\mathrm{d}t &= \int_{\mathbb{R}} \int_0^t E(\tilde{s}) g(t-\tilde{s}) \,\mathrm{d}\tilde{s} \, \varphi(t) \,\mathrm{d}t \\ &= \int_{\mathbb{R}} \int_0^\infty E(\tilde{s}) g(t-\tilde{s}) \,\mathrm{d}\tilde{s} \, \varphi(t) \,\mathrm{d}t \\ &= \int_{\mathbb{R}} \int_0^\infty E(t) g(s-t) \,\mathrm{d}t \, \varphi(s) \,\mathrm{d}s.\end{aligned}$$

Next, we apply Fubini's theorem to change the order of integration to obtain

$$\begin{aligned}\int_{\mathbb{R}} \int_0^t E(t-s) g(s) \,\mathrm{d}s \, \varphi(t) \,\mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}} E(t) \varphi(s) g(s-t) \,\mathrm{d}s \,\mathrm{d}t \\ &= \int_0^\infty E(t) \int_{\mathbb{R}} \varphi(s) g(s-t) \,\mathrm{d}s \,\mathrm{d}t \\ &= \int_0^\infty E(t) (\varphi * g(-\cdot))(t) \,\mathrm{d}t.\end{aligned}$$

Introducing Fourier and inverse Fourier transform with respect to time yields

$$\begin{aligned}\int_{\mathbb{R}} \int_0^t E(t-s) g(s) \,\mathrm{d}s \, \varphi(t) \,\mathrm{d}t &= \int_0^\infty E(t) \mathcal{F}\mathcal{F}^{-1} (\varphi(-\cdot) * g)(t) \,\mathrm{d}t. \\ &= \sqrt{2\pi} \int_0^\infty E(t) \mathcal{F}(\mathcal{F}^{-1}\varphi \mathcal{F}^{-1}(g(-\cdot)))(t) \,\mathrm{d}t \\ &= \sqrt{2\pi} \int_0^\infty E(t) \mathcal{F}(\mathcal{F}^{-1}\varphi \mathcal{F} g)(t) \,\mathrm{d}t.\end{aligned}$$

For the following argument write $\psi := \mathcal{F}^{-1}\varphi\mathcal{F}g$ and note that $\psi(0) = 0$ holds by assumption on φ. Now, for $\varepsilon > 0$ by Fubini's theorem we have

$$\begin{aligned}\sqrt{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\varepsilon t} E(t) \chi_{[0,\infty)}(t) \mathcal{F}\psi(t) \,\mathrm{d}t &= \sqrt{2\pi} \int_0^\infty \mathrm{e}^{-\varepsilon t} E(t) \mathcal{F}\psi(t) \,\mathrm{d}t \\ &= \int_0^\infty \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi t - \varepsilon t} E(t) \psi(\xi) \,\mathrm{d}\xi \,\mathrm{d}t \\ &= \int_{\mathbb{R}} \int_0^\infty \mathrm{e}^{(-\mathrm{i}\xi - \varepsilon)t} E(t) \,\mathrm{d}t \, \psi(\xi) \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \int_0^\infty \frac{1}{2\pi\mathrm{i}} \int_{\gamma_{2/\varepsilon}} \mathrm{e}^{(-\mathrm{i}\xi - \varepsilon + \lambda)t} R(\lambda) \,\mathrm{d}\lambda \,\mathrm{d}t \, \psi(\xi) \,\mathrm{d}\xi. \qquad (7.6)\end{aligned}$$

Recall that $\gamma_{2/\varepsilon}$ parametrizes the boundary of $\mathrm{S}_\varphi \setminus B(0, \varepsilon/2)$ for some $\varphi \in (\pi/2, \pi)$ counter-clockwise. Next, we want to apply Fubini's theorem to the two innermost integrals. Here, it is now crucial that the spatial test function h in (7.4) has compact support so that the resulting function is only evaluated at a compact subset of $\mathbb{R}^d_+$ and Proposition 6.2.4 applies. We find

$$\begin{aligned}\int_0^\infty \int_{\gamma_{2/\varepsilon}} \| \mathrm{e}^{(-\mathrm{i}\xi-\varepsilon+\lambda)t} R(\lambda)\| \, |\mathrm{d}\lambda| \, \mathrm{d}t &\lesssim \int_0^\infty \int_{\gamma_{2/\varepsilon}} \mathrm{e}^{(\operatorname{Re}\lambda-\varepsilon)t} \Big(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^{\frac{1}{4}}}\Big) |\mathrm{d}\lambda| \, \mathrm{d}t \\ &= \int_{\gamma_{2/\varepsilon}} \Big(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^{\frac{1}{4}}}\Big) \int_0^\infty \mathrm{e}^{(\operatorname{Re}\lambda-\varepsilon)t} \, \mathrm{d}t \, |\mathrm{d}\lambda| \\ &= \int_{\gamma_{2/\varepsilon}} \Big(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^{\frac{1}{4}}}\Big) \frac{1}{\varepsilon - \operatorname{Re}\lambda} |\mathrm{d}\lambda|.\end{aligned}$$

Here the choice of the path in the Cauchy integral is crucial as for $\lambda \in \gamma_{2/\varepsilon}$ we have $\operatorname{Re}\lambda < \varepsilon/2$ and so the final integral converges. Thus, we can apply Fubini's theorem in (7.6) to obtain

$$\begin{aligned}\sqrt{2\pi} \int_0^\infty \mathrm{e}^{-\varepsilon t} \, E(t)\mathcal{F}\psi(t) \, \mathrm{d}t &= \int_{\mathbb{R}} \frac{1}{2\pi\mathrm{i}} \int_{\gamma_{2/\varepsilon}} \int_0^\infty \mathrm{e}^{(-\mathrm{i}\xi-\varepsilon+\lambda)t} \, \mathrm{d}t \, R(\lambda) \, \mathrm{d}\lambda \, \psi(\xi) \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \frac{1}{2\pi\mathrm{i}} \int_{\gamma_{2/\varepsilon}} -\frac{R(\lambda)}{\lambda - (\varepsilon + \mathrm{i}\xi)} \, \mathrm{d}\lambda \, \psi(\xi) \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}} R(\varepsilon + \mathrm{i}\xi)\psi(\xi) \, \mathrm{d}\xi\end{aligned} \tag{7.7}$$

by the Cauchy integral formula. Again, it is crucial that we evaluate only at spatial variables in a compact subset of $\mathbb{R}^d_+$ so that $\int_{\mathbb{R}} R(\varepsilon+\mathrm{i}\xi)\psi(\xi) \, \mathrm{d}\xi \in \mathrm{H}^1(\mathbb{R}^d_+) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$ for each $\varepsilon > 0$. Moreover, by Fubini's theorem we find

$$A \int_{\mathbb{R}} R(\varepsilon + \mathrm{i}\xi)\psi(\xi) \, \mathrm{d}\xi = \int_{\mathbb{R}} AR(\varepsilon + \mathrm{i}\xi)\psi(\xi) \, \mathrm{d}\xi.$$

Next, we want to consider the limit $\varepsilon \to 0$. For the left-hand side of (7.7), Proposition 6.2.4 together with the dominated convergence theorem implies

$$\sqrt{2\pi} \int_0^\infty \mathrm{e}^{-\varepsilon t} \, E(t)\mathcal{F}\psi(t) \, \mathrm{d}t \to \sqrt{2\pi} \int_0^\infty E(t)\mathcal{F}\psi(t) \, \mathrm{d}t \quad \text{in } \mathrm{H}^1(\mathbb{R}^d_+) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$$

as $\varepsilon \to 0$. For the right-hand side of (7.7) it is now crucial that $\psi(0) = 0$ holds. First note

$$\int_{\mathbb{R}} R(\mathrm{i}\xi)\psi(\xi) \, \mathrm{d}\xi \in \mathrm{H}^1(\mathbb{R}^d_+) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+).$$

For the second component, i.e., the pressure term, this again is a consequence of Proposition 6.2.4. For L^2-convergence in the first component recall the estimate $\|R^u(\mathrm{i}\xi)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \lesssim \frac{1}{|\xi|}$. Finally, as ψ is a Schwartz function, continuously differentiable with bounded derivative would in fact suffice, and satisfies $\psi(0) = 0$, for small ξ the function $\psi(\xi)$ behaves like ξ by the mean value theorem. This ensures convergence of the integral in $\mathrm{L}^2(\mathbb{R}^d_+)$ as ψ decays sufficiently fast at ∞. For convergence of the gradient the classical estimate from (6.17) is sufficient. Then, the dominated convergence theorem implies

$$\int_{\mathbb{R}} R(\varepsilon + \mathrm{i}\xi)\psi(\xi)\,\mathrm{d}\xi \to \int_{\mathbb{R}} R(\mathrm{i}\xi)\psi(\xi)\,\mathrm{d}\xi \quad \text{in } \mathrm{H}^1(\mathbb{R}^d_+) \times \mathrm{L}_{\mathrm{loc}}(\mathbb{R}^d_+)$$

as $\varepsilon \to 0$. Moreover, on $\mathrm{L}^2(\mathbb{R}^d_+)$ the family $AR(\mathrm{i}\xi + \varepsilon)$ is uniformly bounded with respect to ε and we have $AR(\mathrm{i}\xi + \varepsilon) \to AR(\mathrm{i}\xi)$ as operators on $\mathrm{L}^2(\mathbb{R}^d_+)$ for all $\xi \neq 0$. Now, the closedness of A implies

$$\sqrt{2\pi}\int_0^\infty E(t)\mathcal{F}\psi(t)\,\mathrm{d}t = \int_{\mathbb{R}} R(\mathrm{i}\xi)\psi(\xi)\,\mathrm{d}\xi \in \mathcal{D}(A)$$

and

$$A\int_0^\infty E(t)\mathcal{F}\psi(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} AR(\mathrm{i}\xi)\psi(\xi)\,\mathrm{d}\xi.$$

Hence, (7.5) is proved. Moreover, we obtain

$$A\int_{\mathbb{R}}\int_0^t E(t-s)g(s)\,\mathrm{d}s\,\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}} AR(\mathrm{i}\xi)\psi(\xi)\,\mathrm{d}\xi. \tag{7.8}$$

It remains to rewrite the right-hand side of (7.8). In the following we abbreviate $X \coloneqq \mathrm{L}^2(\mathbb{R}^d_+)$. Then $g \in \mathrm{C}^\infty((0,\infty);\mathrm{L}^2(\mathbb{R}^d_+))$ implies $\mathcal{F}g \in \mathcal{S}(\mathbb{R};X)$. As the map $\xi \mapsto AR(\mathrm{i}\xi)$ is bounded from $\mathbb{R}\setminus\{0\}$ with values in $\mathcal{L}(X)$ by Proposition 7.1.8 we conclude that the map

$$\xi \mapsto AR(\mathrm{i}\xi)\mathcal{F}g(\xi)$$

is rapidly decreasing and so particularly in $\mathrm{L}^1(\mathbb{R};X)\cap\mathrm{L}^2(\mathbb{R};X)$ where the integrability is to be understood in the sense of Bochner. Thus, taking Fourier inverse with respect to ξ yields

$$\mathcal{F}^{-1}(AR(\mathrm{i}\xi)^{-1}\mathcal{F}g) \in \mathrm{L}^\infty(\mathbb{R};X)\cap\mathrm{L}^1(\mathbb{R};X).$$

Let now $\varphi \in \mathcal{S}(\mathbb{R};\mathbb{R})$ be arbitrary. Then, by Fubini's theorem we find

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}^{-1}(AR(\mathrm{i}\xi)\mathcal{F}g)(s)\varphi(s)\,\mathrm{d}s &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}s\xi}\,AR(\mathrm{i}\xi)\mathcal{F}g(\xi)\varphi(s)\,\mathrm{d}\xi\,\mathrm{d}s\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}s\xi}\,\varphi(s)\,\mathrm{d}s AR(\mathrm{i}\xi)\mathcal{F}g(\xi)\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}} AR(\mathrm{i}\xi)(\mathcal{F}^{-1}\varphi)(\xi)\mathcal{F}g(\xi)\,\mathrm{d}\xi.\\ &= \int_{\mathbb{R}} AR(\mathrm{i}\xi)\psi(\xi)\,\mathrm{d}\xi.\end{aligned}$$

Plugging this into (7.8) and going back to the beginning of the proof in (7.4) we obtain

$$\begin{aligned}\int_{\mathbb{R}} \Big\langle \int_0^t E(t-s)g(s)\,\mathrm{d}s, A'h\Big\rangle \varphi(t)\,\mathrm{d}t &= \Big\langle \int_{\mathbb{R}} \int_0^t E(t-s)g(s)\,\mathrm{d}s\,\varphi(t)\,\mathrm{d}t, A'h\Big\rangle \\ &= \Big\langle \int_{\mathbb{R}} \mathcal{F}^{-1}(AR(\mathrm{i}\xi)\mathcal{F}g)(s)\varphi(s)\,\mathrm{d}s, h\Big\rangle \\ &= \int_{\mathbb{R}} \Big\langle \mathcal{F}^{-1}(AR(\mathrm{i}\xi)\mathcal{F}g)(t), h\Big\rangle \varphi(t)\,\mathrm{d}s\end{aligned}$$

where in the last step we used Fubini's theorem as in the beginning of the proof. As $\varphi \in \mathcal{S}(\mathbb{R};\mathbb{R})$ is arbitrary up to $\mathcal{F}^{-1}\varphi(0) = 0$ and this class of functions is dense in $\mathrm{L}^2(\mathbb{R})$ it follows that

$$\Big\langle \int_0^t E(t-s)g(s)\,\mathrm{d}s, A'h\Big\rangle = \Big\langle \mathcal{F}^{-1}(AR(\mathrm{i}\xi)\mathcal{F}g)(t), h\Big\rangle$$

holds for almost every $t \geq 0$. Thus, we proved

$$\int_0^t E(t-s)g(s)\,\mathrm{d}s \in \mathcal{D}(A)$$

for almost every $t \geq 0$ with

$$A\int_0^t E(t-s)g(s)\,\mathrm{d}s = \mathcal{F}^{-1}(AR(\mathrm{i}\xi)\mathcal{F}g)(t).$$

In the light of the first resolvent identity, see Proposition 6.2.5, this concludes the proof. $\square$

As an easy corollary we obtain an L^2-estimate on the maximal regularity operator.

Corollary 7.1.12. *There exists $C > 0$ such that for all $g \in \mathrm{C}_c^\infty((0,\infty);\mathrm{L}^2(\mathbb{R}^d_+))$ and all $\tau \in (0,\infty]$ the estimate*

$$\Big\| A\int_0^t [E(t-s)g(s,\cdot)](x)\,\mathrm{d}s\Big\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\mathbb{R}^d_+))} \leq C\|g\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\mathbb{R}^d_+))}$$

holds true.

Proof. Consider first the case $\tau = \infty$. By the previous theorem and Plancherel's theorem it suffices to show that the operators

$$\mathrm{i}\xi R^u(\mathrm{i}\xi)\colon \mathrm{L}^2(\mathbb{R}^d_+) \to \mathrm{L}^2(\mathbb{R}^d_+)$$

are uniformly bounded with respect to ξ. Recalling that this is precisely the statement of Proposition 6.2.2 concludes the proof for $\tau = \infty$. For $\tau < \infty$ note that the maximal regularity operator only captures times $s \leq t \leq \tau$ so that the left hand-side does not change when replacing g by $\chi_{(0,\tau)}g$. Then, the result is an immediate consequence of the case $\tau = \infty$. $\square$

Next, we upgrade the previous result to general L^2-functions via a density argument and the closedness of A.

Proposition 7.1.13. *Let $f \in \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$. Then for almost all $t>0$ we have*

$$\int_0^t E(t-s)f(s)\,\mathrm{d}s \in \mathcal{D}(A)$$

with

$$A\int_0^t E(t-s)f(s)\,\mathrm{d}s = [\mathcal{F}^{-1}[\mathrm{Id}-\mathrm{i}\xi R^u(\mathrm{i}\xi)]\mathcal{F}f\chi_{(0,\infty)}](t).$$

Moreover, there exists $C>0$ such that for all $\tau\in(0,\infty]$ and all $f\in\mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$ the estimate

$$\Big\| A\int_0^t [E(t-s)f(s,\cdot)](x)\,\mathrm{d}s\Big\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\mathbb{R}^d_+))} \leq C\|f\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\mathbb{R}^d_+))}$$

holds true.

Proof. Let $f\in\mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$ be arbitrary. By density, there exists a sequence $(g_n)_{n\in\mathbb{N}}\subseteq \mathrm{C}_\mathrm{c}^\infty((0,\infty);\mathrm{L}^2(\mathbb{R}^d_+))$ with $g_n\to f$ for $n\to\infty$ in $\mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$. Then, by Proposition 7.1.10 we find $\int_0^t E(t-s)g_n(s)\,\mathrm{d}s \to \int_0^t E(t-s)f(s)\,\mathrm{d}s$ for $n\to\infty$ in $\mathrm{H}^1(\mathbb{R}^d_+)\times\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$. Moreover, by argument in the previous corollary we obtain

$$A\int_0^t E(t-s)g_n(s)\,\mathrm{d}s \to [\mathcal{F}^{-1}[\mathrm{Id}-\mathrm{i}\xi R^u(\mathrm{i}\xi)]\mathcal{F}f\chi_{(0,\infty)}](t) \quad \text{in } \mathrm{L}^2(\mathbb{R}^d_+).$$

Thus, the closedness of A shows

$$\int_0^t E(t-s)f(s)\,\mathrm{d}s \in \mathcal{D}(A)$$

with

$$A\int_0^t E(t-s)f(s)\,\mathrm{d}s = [\mathcal{F}^{-1}[\mathrm{Id}-\mathrm{i}\xi R^u(\mathrm{i}\xi)]\mathcal{F}f\chi_{(0,\infty)}](t).$$

The desired estimate follows as well from the previous corollary. □

In order to deal with weighted tent spaces, we need an analogue of Proposition 7.1.13 for weighted L^2-spaces. Here, we follow the proof from [6, Theorem 1.3]. Recall from Proposition 7.1.9 that the family $tAE(t)$ is uniformly bounded on $\mathrm{L}^2(\mathbb{R}^d_+)$. As a preliminary result we prove a version of Schur's lemma.

Lemma 7.1.14. *If $U(t,s)$, for $s,t>0$ are bounded linear operators on $\mathrm{L}^2(\mathbb{R}^d_+)$ with bounds $\|U(t,s)\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d_+))} \le h(t/s)$ and $C=\int_0^\infty h(u)/u\,\mathrm{d}u<\infty$, we then have the boundedness estimate*

$$\left|\!\left|\!\left| \int_0^\infty U(t,s)f(s)\frac{\mathrm{d}s}{s} \right|\!\right|\!\right| \le C |\!|\!| f(s) |\!|\!|$$

for all $f\in \mathrm{L}^2((0,\infty);\mathrm{L}^2(\mathbb{R}^d_+))$ with compact support. Here,

$$|\!|\!| f(t) |\!|\!| := \left(\int_0^\infty \|f(t)\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}. \tag{7.9}$$

Proof. Let $f\in \mathrm{L}^2((0,\infty);\mathrm{L}^2(\mathbb{R}^d_+))$ with compact support be arbitrary. Then,

$$\left|\!\left|\!\left| \int_0^\infty U(t,s)f(s)\frac{\mathrm{d}s}{s} \right|\!\right|\!\right| \le \left(\int_0^\infty \left| \int_0^\infty \|U(t,s)\| \|f(s)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \frac{\mathrm{d}s}{s} \right|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \tag{7.10}$$

holds true. We first consider only the inner integral. Using the bound on $\|U(t,s)\|$ as well as the Cauchy-Schwarz inequality, we find

$$\begin{aligned} \left| \int_0^\infty \|U(t,s)\| \|f(s)\|_{\mathrm{L}^2(\mathbb{R}^d_+} \frac{\mathrm{d}s}{s} \right|^2 &\le \left| \int_0^\infty h\Big(\frac{t}{s}\Big) \|f(s)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \frac{\mathrm{d}s}{s} \right|^2 \\ &\le \left(\int_0^\infty h\Big(\frac{t}{s}\Big)\frac{t}{s}\,\mathrm{d}s \right) \left(\int_0^\infty h\Big(\frac{t}{s}\Big) \|f(s)\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)} \frac{s}{t}\frac{\mathrm{d}s}{s^2} \right). \end{aligned}$$

In the first factor use the substitution $u=t/s$ to obtain

$$\int_0^\infty h\left(\frac{t}{s}\right)\frac{t}{s}\,\mathrm{d}s = t\int_0^\infty h(u)\,\frac{\mathrm{d}u}{u} \le Ct$$

Plugging this into (7.10) yields

$$\left|\!\left|\!\left| \int_0^\infty U(t,s)f(s)\frac{\mathrm{d}s}{s} \right|\!\right|\!\right| \le \sqrt{C} \left(\int_0^\infty \int_0^\infty h\left(\frac{t}{s}\right) \|f(s)\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}. \tag{7.11}$$

After an application of Fubini's theorem on the right-hand side, the substitution $u=t/s$ in the integral with respect to t delivers

$$\begin{aligned} \int_0^\infty h\left(\frac{t}{s}\right)\frac{\mathrm{d}t}{t} &= \int_0^\infty h(u)\frac{1}{su}s\,\mathrm{d}u \\ &= \int_0^\infty h(u)\,\frac{\mathrm{d}u}{u} = C. \end{aligned}$$

Combining this with (7.11) yields

$$\left\|\left\| \int_0^\infty U(t,s)f(s)\frac{\mathrm{d}s}{s} \right\|\right\| \le C\left(\int_0^\infty \|f(s)\|_{\mathrm{L}^2(\mathbb{R}^d_+)}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{2}}$$

what we wanted to show. □

Now, we have all the necessary ingredients to prove boundedness of the maximal regularity operator on weighted L^2-spaces.

Proposition 7.1.15. *For all $\beta \in (-\infty, 1)$ there exists a constant $C > 0$ depending on β and d such that for all $\tau \in (0,\infty]$ and all $f \in \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+, t^\beta\,\mathrm{d}t)$ we have*

$$\left\| A\int_0^t [E(t-s)f(s,\cdot)](x)\,\mathrm{d}s \right\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\mathbb{R}^d_+),t^\beta\,\mathrm{d}t)} \le C\|f\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\mathbb{R}^d_+),t^\beta\,\mathrm{d}t))}.$$

Proof. By density it suffices to consider functions $f \in \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$ with compact support in time. Let now $\beta < 1$ and $\tau = \infty$. In view of Proposition 7.1.13 we can assume $\beta \neq 0$. Define $\alpha \coloneqq \beta/2$. Then α satisfies $0 \neq \alpha < 1/2$. Moreover, for $f \in \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$ with compact support in time define

$$\mathcal{M}_+f(t) \coloneqq \int_0^t AE(t-s)f(s,\cdot)\,\mathrm{d}s.$$

With this notation we have

$$\|\mathcal{M}_+f(t)\|_{\mathrm{L}^2(t^\beta\,\mathrm{d}t\,\mathrm{d}x)} = \|t^\alpha\mathcal{M}_+f(t)\|_{\mathrm{L}^2(\mathrm{d}t,\mathrm{d}x)}.$$

With $f_\alpha(s) \coloneqq s^\alpha f(s)$, we obtain

$$t^\alpha\mathcal{M}_+f(t) = \mathcal{M}_+(f\alpha)(t) + \int_0^t AE(t-s)(t^\alpha - s^\alpha)f(s)\,\mathrm{d}s.$$

The first term is controlled by Proposition 7.1.13. For the second term use the notation introduced in (7.9) to obtain

$$\left\| \int_0^t AE(t-s)(t^\alpha - s^\alpha)f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^2(0,\infty)} = \left\|\left\| \int_0^\infty U(t,s)g(s)\frac{\mathrm{d}s}{s} \right\|\right\|$$

with $g(s) = s^{1/2+\alpha}f(s)$ and $U(t,s) = AE(t-s)(t^\alpha - s^\alpha)s^{1/2-\alpha}t^{1/2}$ for $s < t$ and 0 otherwise. We now want to apply Lemma 7.1.14. By definition of g we have $|||g(t)||| = \|f\|_{\mathrm{L}^2(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}$ so that it remains to estimate $\|U(t,s)\|$. As the family $tAE(t)$ is bounded, we obtain

$$\|U(t,s)\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d_+))} \lesssim \frac{|t^\alpha - s^\alpha|}{|t-s|}s^{\frac{1}{2}-\alpha}t^\alpha, \quad s < t. \tag{7.12}$$

The mean value theorem implies that the first factor is of order $t^{\alpha-1}$ as $s < t$. Then, the right-hand side of (7.12) is of order $(s/t)^{1/2-\max(\alpha,0)}$ so that $\|U(t,s)\| \leq h(t/s)$ holds with $h(u) := u^{\max(\alpha,0)-1/2}$ for $u > 1$ and 0 otherwise. As $\alpha < \frac{1}{2}$ we obtain

$$\int_0^\infty h(u)\frac{\mathrm{d}u}{u} < \infty$$

and so an application of Lemma 7.1.14 concludes the proof for $\tau = \infty$. For $\tau < \infty$ the same argumentation as in Corollary 7.1.12 applies. □

7.2 Off-diagonal estimates

In this subsection we prove the announced off-diagonal estimate for $AE(t-s)$. Recall that A is the extended Stokes operator and $E(t-s) = (E^u(t-s), E^\pi(t-s))$.

To this end, we use the Cauchy integral representation for the semigroup together with the kernel representation for the resolvent. Here, the following kernel estimates from [68, Propositions 3.2, 3.5 and 3.7] mentioned earlier are crucial.

Proposition 7.2.1. *There exists $c = c(d,\varepsilon)$ such that for all $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ the following kernel estimates hold. For all $y = (y', y_d) \in \mathbb{R}^d$ we have*

$$|\nabla^2 k_{1,\lambda}(y', y_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}|y_d|}}{(|y_d| + |y'|)^d(1 + |\lambda|^{\frac{1}{2}}(|y_d| + |y'|))}.$$

Moreover, for $y' \in \mathbb{R}^{d-1}$, $y_d, t_d > 0$ we have

$$|\nabla_y^2 k_{2,\lambda}(y', y_d, z_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}(y_d+z_d)}}{(y_d + z_d + |y'|)^d(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d + |y'|))}.$$

For $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$ we have

$$|\nabla_y^2 r'_\lambda(y', y_d, z_d)| + |\nabla_y^2 r_{d,\lambda}(y', y_d, z_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}z_d}}{(y_d + z_d + |y'|)^d(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d))}.$$

Finally, for $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$ we have

$$|\nabla_y q_\lambda(y', y_d, z_d)| \lesssim \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}}z_d}}{(y_d + z_d + |y'|)^d}.$$

Note that the estimate for the gradient of the pressure and in fact the second order derivatives of the other kernels satisfy the same estimate as the gradient of the pressure. Let us now derive the off-diagonal estimates. Recall that $Q(x_0, \sqrt{\tau})$ denotes the cube with center x_0 and side length $2\sqrt{\tau}$.

Proposition 7.2.2. *Let $x_0 \in \mathbb{R}^d_+$, $s, t, \tau > 0$, $\ell \in \mathbb{N}_{>0}$ and set $C_l \coloneqq Q(x_0, 2^{l+1}\sqrt{\tau}) \setminus Q(x_0, 2^l\sqrt{\tau})$. Then for all $f \in \mathrm{L}^2_{\mathrm{loc}}(\overline{\mathbb{R}^d_+})$ it holds*

$$\|x \mapsto [AE(t-s)\chi_{C_l \cap \mathbb{R}^d_+} f](x)\|_{\mathrm{L}^2(Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+)} \lesssim 2^{-l\frac{d+1}{2}} \tau^{-\frac{1}{4}} (t-s)^{-\frac{3}{4}} \|f\|_{\mathrm{L}^2(C_l \cap \mathbb{R}^d_+)}$$

and the constant only depends on d.

Proof. Let $x_0 \in \mathbb{R}^d_+$ $s, t, \tau > 0$ and $\ell \geq 1$ be arbitrary. For $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$ we want to argue that the operator A can be taken inside the Cauchy integral so that

$$\begin{aligned} AE(t-s)\chi_{C_\ell \cap \mathbb{R}^d_+} f = &- \int_{\gamma_{t-s}} \mathrm{e}^{(t-s)\lambda}\, \Delta R^u(\lambda) \chi_{C_\ell \cap \mathbb{R}^d_+} f \,\mathrm{d}\lambda \\ &+ \int_{\gamma_{t-s}} \mathrm{e}^{(t-s)\lambda}\, \nabla R^\phi(\lambda) \chi_{C_\ell \cap \mathbb{R}^d_+} f \,\mathrm{d}\lambda \end{aligned}$$

holds true. As both the Laplacian and the gradient are closed operators it suffices to show that the right-hand side is well defined. As already mentioned before this proposition, all terms in fact satisfy the same estimates as the gradient of the pressure. The point-wise bounds then deliver for $x \in \mathbb{R}^d_+ \cap Q(x_0, \sqrt{\tau})$ that

$$\begin{aligned} |\nabla R^\phi(\lambda) \chi_{C_\ell \cap \mathbb{R}^d_+} f(x)| &\leq \int_{C_\ell \cap \mathbb{R}^d_+} |\nabla_x q_\lambda(x' - y', x_d, y_d) f(y)| \,\mathrm{d}y \\ &\lesssim \int_{C_\ell \cap \mathbb{R}^d_+} \frac{\mathrm{e}^{-c|\lambda|^{1/2} y_d} |f(y)|}{(x_d + y_d + |x' - y'|)^d} \,\mathrm{d}y. \end{aligned}$$

To proceed, we distinguish the two cases $Q(x_0, \sqrt{\tau}) \cap \partial\mathbb{R}^d_+ \neq \emptyset$ and $Q(x_0, \sqrt{\tau}) \cap \partial\mathbb{R}^d_+ = \emptyset$.

In case of $Q(x_0, \sqrt{\tau}) \cap \partial\mathbb{R}^d_+ \neq \emptyset$, the "lower" part of C_ℓ is completely contained in the lower half-space. This means, that if $y \in C_\ell \cap \mathbb{R}^d_+$ satisfies $y_d < (x_0)_d + 2^\ell\sqrt{\tau}$ then also $\|y' - x'\|_\infty \geq (2^\ell - 1)\sqrt{\tau}$ holds for $x \in Q(x_0, \sqrt{\tau})$. Due to norm equivalence in $\mathbb{R}^{d-1}$ this implies $|y' - x'| \gtrsim 2^\ell\sqrt{\tau}$ for all $x \in Q(x_0, \sqrt{\tau})$ with a constant depending only on d. Introduce the sets

$$\begin{aligned} S_\ell &\coloneqq \{y \in C_\ell \cap \mathbb{R}^d_+ \colon y_d < (x_0)_d + 2^\ell\sqrt{\tau}\}, \\ T_\ell &\coloneqq \{y \in C_\ell \cap \mathbb{R}^d_+ \colon y_d \geq (x_0)_d + 2^\ell\sqrt{\tau}\}. \end{aligned} \tag{7.13}$$

Hölder's inequality delivers

$$\begin{aligned} &\int_{C_\ell \cap \mathbb{R}^d_+} \frac{\mathrm{e}^{-c|\lambda|^{1/2} y_d} |f(y)|}{(x_d + y_d + |x' - y'|)^d} \,\mathrm{d}y \\ &\qquad \leq \bigg(\int_{C_\ell \cap \mathbb{R}^d_+} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \,\mathrm{d}y \bigg)^{\frac{1}{2}} \|f\|_{\mathrm{L}^2(C_\ell \cap \mathbb{R}^d_+)}. \end{aligned}$$

By virtue of the sets S_ℓ and T_ℓ we can decompose the integral on the right-hand side by

$$\Big(\int_{C_\ell \cap \mathbb{R}^d_+} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \, \mathrm{d}y\Big)^{\frac{1}{2}} \leq \Big(\int_{S_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \, \mathrm{d}y\Big)^{\frac{1}{2}} + \Big(\int_{T_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \, \mathrm{d}y\Big)^{\frac{1}{2}}.$$

Taking the $\mathrm{L}^2(Q(x_0, \sqrt{\tau}) \cap \mathbb{R}^d_+)$-norm of each of these norms delivers

$$\begin{aligned} &\Big(\int_{Q(x_0,\sqrt{\tau})} \int_{S_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \, \mathrm{d}y \, \mathrm{d}x\Big)^{\frac{1}{2}} \\ &\qquad \lesssim \tau^{\frac{d}{4}} (2^\ell \sqrt{\tau})^{\frac{d-1}{2}} \frac{\tau^{-\frac{d}{2}}}{(2^\ell)^d} \Big(\int_0^{(x_0)_d + 2^\ell \sqrt{\tau}} \mathrm{e}^{-2c|\lambda|^{1/2} y_d} \, \mathrm{d}y_d\Big)^{\frac{1}{2}} \\ &\qquad \lesssim \tau^{\frac{d}{4}} (2^\ell \sqrt{\tau})^{\frac{d-1}{2}} \frac{\tau^{-\frac{d}{2}}}{2^{\ell d}} |\lambda|^{-\frac{1}{4}} \\ &\qquad = 2^{-\ell \frac{d+1}{2}} |\lambda|^{-\frac{1}{4}} \tau^{-\frac{1}{4}} \end{aligned}$$

and

$$\begin{aligned} \Big(\int_{Q(x_0,\sqrt{\tau})} \int_{T_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \, \mathrm{d}y \, \mathrm{d}x\Big)^{\frac{1}{2}} &\lesssim 2^{-\ell d} \tau^{-\frac{d}{2}} \, \mathrm{e}^{-c|\lambda|^{1/2} 2^\ell \sqrt{\tau}} \, \tau^{\frac{d}{4}} 2^{\ell \frac{d}{2}} \tau^{\frac{d}{4}} \\ &= 2^{-\ell \frac{d}{2}} \, \mathrm{e}^{-c|\lambda|^{1/2} 2^\ell \sqrt{\tau}} \\ &\lesssim 2^{-\ell \frac{d+1}{2}} |\lambda|^{-\frac{1}{4}} \tau^{-\frac{1}{4}}. \end{aligned}$$

Consider now the case $Q(x_0, \sqrt{\tau}) \cap \partial \mathbb{R}^d_+ = \emptyset$. In this case, there exists a smallest $N \in \mathbb{N}_0$ such that $Q(x_0, 2^{N+1} \sqrt{\tau}) \cap \partial \mathbb{R}^d_+ \neq \emptyset$. Then, for each $\ell \in \mathbb{N}$ with $\ell \geq N + 1$ the "bottom" part of the set $Q(x_0, 2^{\ell+1} \sqrt{\tau}) \setminus Q(x_0, 2^\ell \sqrt{\tau})$ lies in the lower half-space. Moreover, for $x \in Q(x_0, \sqrt{\tau})$ it holds

$$x_d \geq (2^N - 1) \sqrt{\tau}. \tag{7.14}$$

Here, we distinguish again two cases, namely, the case that $\ell \geq N + 1$ and the case that $\ell \leq N$. Notice that since $\ell \geq 1$, the latter case is only of interest for $N \geq 1$.

Let us first consider $\ell \geq N + 1$. This situation is similar to the first case. Indeed, decompose C_ℓ into a union of two sets S_ℓ and T_ℓ as in (7.13). Then, we find

$$\Big(\int_{Q(x_0,\sqrt{\tau})} \int_{S_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \, \mathrm{d}y \, \mathrm{d}x\Big)^{\frac{1}{2}} \lesssim 2^{-\ell \frac{d+1}{2}} |\lambda|^{-\frac{1}{4}} \tau^{-\frac{1}{4}}$$

and

$$\left(\int_{Q(x_0,\sqrt{\tau})} \int_{T_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{2}} \lesssim 2^{-\ell \frac{d+1}{2}} |\lambda|^{-\frac{1}{4}} \tau^{-\frac{1}{4}}.$$

Let us now consider $\ell \leq N$. In this case, the set C_ℓ is completely contained inside the half-space $\mathbb{R}^d_+$. Decompose C_ℓ into three parts S_ℓ, B_ℓ, and T_ℓ. The set S_ℓ describes the "side" part of C_ℓ, the set B_ℓ the "bottom" part of C_ℓ, and T_ℓ the "top" part of C_ℓ. To be precise, in the spirit of (7.13) but slightly different we define

$$\begin{aligned} T_\ell &\coloneqq \{y \in C_\ell \cap \mathbb{R}^d_+ \colon y_d \geq (x_0)_d + 2^\ell \sqrt{\tau}\} \\ B_\ell &\coloneqq \{y \in C_\ell \cap \mathbb{R}^d_+ \colon 0 < y_d \leq (x_0)_d - 2^\ell \sqrt{\tau}\} \\ S_\ell &\coloneqq C_\ell \cap \mathbb{R}^d_+ \setminus (T_\ell \cup B_\ell). \end{aligned} \tag{7.15}$$

Note that as before for $y \in S_\ell$ we have $|y' - x_0'| \gtrsim 2^\ell \sqrt{\tau}$. The $\mathrm{L}^2(Q(x_0, \sqrt{\tau}))$-norm is estimated by

$$\begin{aligned} &\left(\int_{Q(x_0,\sqrt{\tau})} \int_{C_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \left(\int_{Q(x_0,\sqrt{\tau})} \int_{S_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{Q(x_0,\sqrt{\tau})} \int_{B_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{Q(x_0,\sqrt{\tau})} \int_{T_\ell} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + |x' - y'|)^{2d}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &=: \aleph_\ell + \beth_\ell + \tilde{\beth}_\ell. \end{aligned}$$

Let us start by estimating the side part. Using the lower bound $|y' - x'| \gtrsim (2^l - 1)\sqrt{\tau}$ for $x \in Q(x_0, \sqrt{\tau})$ we obtain

$$\begin{aligned} \aleph_\ell &\lesssim (2^\ell \sqrt{\tau})^{\frac{d-1}{2}} \tau^{\frac{d-1}{4}} \left(\int_{(x_0)_d - \sqrt{\tau}}^{(x_0)_d + \sqrt{\tau}} \int_{(x_0)_d - 2^\ell \sqrt{\tau}}^{(x_0)_d + 2^\ell \sqrt{\tau}} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d + (2^\ell - 1)\sqrt{\tau})^{2d}} \,\mathrm{d}y_d \,\mathrm{d}x_d \right)^{\frac{1}{2}} \\ &\lesssim (2^\ell \sqrt{\tau})^{\frac{d-1}{2}} \tau^{\frac{d}{4}} \left(\int_0^\infty \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{((2^\ell - 1)\sqrt{\tau}))^{2d}} \,\mathrm{d}y_d \right)^{\frac{1}{2}} \\ &\lesssim 2^{-\ell d} \tau^{-\frac{d}{2}} (2^\ell \sqrt{\tau})^{\frac{d-1}{2}} \tau^{\frac{d}{4}} |\lambda|^{-\frac{1}{4}} \\ &\leq 2^{-\ell \frac{d+1}{2}} \tau^{-\frac{1}{4}} |\lambda|^{-\frac{1}{4}}. \end{aligned}$$

Recall that $N \geq 1$ and the lower bound (7.14) on $x_d \in Q(x_0, \sqrt{\tau})$ so that the bottom term is estimated by

$$\begin{aligned}
\beth_\ell &\leq 2^{\ell\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \Bigg(\int_{(x_0)_d - \sqrt{\tau}}^{(x_0)_d + \sqrt{\tau}} \int_{\max\{(x_0)_d - 2^{\ell+1}\sqrt{\tau}, 0\}}^{(x_0)_d - 2^\ell \sqrt{\tau}} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{(x_d + y_d)^{2d}} \, \mathrm{d}y_d \, \mathrm{d}x_d \Bigg)^{\frac{1}{2}} \\
&\leq 2^{\ell\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \Bigg(\int_{(x_0)_d - \sqrt{\tau}}^{(x_0)_d + \sqrt{\tau}} \int_{\max\{(x_0)_d - 2^{\ell+1}\sqrt{\tau}, 0\}}^{(x_0)_d - 2^\ell \sqrt{\tau}} \frac{\mathrm{e}^{-2c|\lambda|^{1/2} y_d}}{((2^N - 1)\sqrt{\tau})^{2d}} \, \mathrm{d}y_d \, \mathrm{d}x_d \Bigg)^{\frac{1}{2}} \\
&\leq 2^{\ell\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \tau^{\frac{1}{4}} 2^{-Nd} \tau^{-\frac{d}{2}} \Bigg(\int_0^\infty \mathrm{e}^{-2c|\lambda|^{1/2} y_d} \, \mathrm{d}y_d \Bigg)^{\frac{1}{2}} \\
&\leq 2^{\ell\frac{d-1}{2}} \tau^{\frac{d-1}{2}} \tau^{\frac{1}{4}} 2^{-Nd} \tau^{-\frac{d}{2}} |\lambda|^{-\frac{1}{4}} \\
&= 2^{\ell\frac{d-1}{2}} 2^{-Nd} \tau^{-\frac{1}{4}} |\lambda|^{-\frac{1}{4}}.
\end{aligned}$$

Notice that $2^{\ell d} 2^{-Nd} \leq 1$ so that

$$\beth_\ell \leq 2^{\ell\frac{d-1}{2} - \ell d} \tau^{-\frac{1}{4}} |\lambda|^{-\frac{1}{4}} = 2^{-\ell\frac{d+1}{2}} \tau^{-\frac{1}{4}} |\lambda|^{-\frac{1}{4}}.$$

For the top term $\widetilde{\beth}_\ell$ the same estimate is true. Indeed, in the previous calculation the fact we integrated over B_ℓ is only visible in the domain of integration in the y_d-integral. In the final step, however, the y_d-integral is taken over $(0, \infty)$ and this estimate is also valid when integrating over the top T_ℓ.

Concluding, we proved that in each possible case an estimate of the same type holds true. Thus, plugging this estimate inside the Cauchy integral delivers

$$\|x \mapsto [AE(t-s)\chi_{C_\ell \cap \mathbb{R}^d_+} f](x)\|_{\mathrm{L}^2(Q(x_0, \sqrt{\tau}) \cap \mathbb{R}^d_+)} \lesssim 2^{-\ell\frac{d+1}{2}} \tau^{-\frac{1}{4}} (t-s)^{-\frac{3}{4}} \|f\|_{\mathrm{L}^2(C_\ell \cap \mathbb{R}^d_+)}$$

what was to be shown. □

CHAPTER 8

Maximal regularity operator on tent spaces

In the previous chapter we defined the maximal regularity operator on the tent space $\mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+)$ and showed boundedness in this space. In the first part of this chapter we describe a general method to extend an operator from $\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))$ to $\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)$. In this procedure, off-diagonal estimates will be crucial. The approach presented here can be understood as the tent space version of the one used by Auscher and Egert in [7, Proposition 5.1] where the authors extend an operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^\infty(\mathbb{R}^d)$.

Moreover, we show that the maximal regularity operator is covered by this framework and that this extension coincides (whenever this is meaningful) with the already defined maximal regularity operator on $\mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+)$. This is necessary for two reasons. First of all, in the proof of $\mathrm{T}^{\infty,2}$-boundedness we want to be able to switch between these two extensions. Showing boundedness of the maximal regularity operator on $\mathrm{T}^{\infty,2}$ requires a dyadic decomposition of $\mathbb{R}^d_+$. For the local estimate, meaning that the argument is supported on a ball, the representation via the Fourier transform is useful while for the non-local part off-diagonal estimates come into play. The second reason is that we want to employ an interpolation argument to cover the case $2<p<\infty$.

For $p<2$ such an approach is not readily available. Additionally, the approach of Auscher, Monniaux and Portal in [10] for elliptic operators cannot be immediately transferred as there it is decisive to work with a semigroup. Recall that in our setting the pressure can not be omitted and the resulting operator does not necessarily satisfy the semigroup law. Thus, an alternative way has to be found. Working with smooth functions, an additional integration by parts is possible in the solution formulae for the Stokes resolvent problem. Note that the resulting kernels are different from the ones in [68]. However, they are similar enough to exhibit the same point-wise estimates. For this we use a space of smooth functions whose derivatives satisfy suitable polynomial estimates. Finally, the estimates on the new kernel allow us to obtain the same result as Auscher, Monniaux and Portal in the elliptic setting.

8.1 Defining operators on tent spaces

In this section we discuss how to define the maximal regularity operator on tent spaces. For $\mathrm{T}^{2,2}$, which by Proposition 2.2.4 coincides with $\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))$, we use the fact that for $f \in \mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathbb{R}^d_+))$ we expressed the maximal regularity operator in terms of the Fourier transform in time. This allowed us to show in Proposition 7.1.13 that on such functions the maximal regularity operator is bounded in the L^2-sense.

Now, we want to define the maximal regularity operator on $\mathrm{T}^{p,2}$ for $p \neq 2$ and start with $p = \infty$. Note that in this case, no density argument is available. The general idea is to lift an operator T from $\mathrm{T}^{2,2}$ to $\mathrm{T}^{\infty,2}$ in the following way:

$$Tf \coloneqq \lim_{n\to\infty} T(\chi_{B_n} f),$$

where the limit is understood in the $\mathrm{L}^2_{\mathrm{loc}}$-sense and the sets $B_n \subseteq (0,\infty)\times\mathbb{R}^d_+$ are bounded and satisfy $\bigcup_{n\in\mathbb{N}} B_n = (0,\infty)\times\mathbb{R}^d_+$. Note that $T(\chi_{B_n} f)$ is well-defined as for a function $f \in \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)$ the multiplication $\chi_{B_n} f$ is in $\mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+)$. Convergence is usually guaranteed by off-diagonal estimates (see [7, Proposition 5.1] for the case of Lebesgue spaces). Note that this strategy does not take advantage of special form of the maximal regularity operator but can be applied in a broader setting as well.

Applying this lifting procedure to the maximal regularity operator thus requires off-diagonal estimates which we show in a moment. Before let us prove a technical lemma that will be needed here and in the proving boundedness of the maximal regularity on tent spaces.

Lemma 8.1.1. *Let $\tau > 0$. Then there exists a constant $C > 0$ such that for all $f \in \mathrm{T}^{\infty,2}(t^\beta \,\mathrm{d}t\,\mathrm{d}x)$, all $\beta \in (-\infty,0]$ and all compact sets $A \subseteq \mathbb{R}^d$ the estimate*

$$\left\| t \mapsto \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \right\|_{\mathrm{L}^2(0,\tau,t^\beta)} \le C\tau^{\frac{1}{4}} \left(\int_0^\tau \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \right)^{\frac{1}{2}}$$

holds true. Moreover, for $k \geq 0$ and $\beta \in (-\infty, 1)$ we find

$$\Big\| t \mapsto \int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big\|_{\mathrm{L}^2(0,\tau,t^\beta)} \leq C\tau^{\frac{1}{4}} 2^{-\frac{k}{2}(1-\beta)} \Big(\int_0^{2^{-k}\tau} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \Big)^{\frac{1}{2}}.$$

Proof. Let us with the first statement. For $t \in (0,\tau)$ estimate

$$\Big(\int_0^t (t-s)^{-\frac{3}{4}} \,\mathrm{d}s \Big)^{\frac{1}{2}} \leq Ct^{\frac{1}{8}} \leq C\tau^{\frac{1}{8}}.$$

By Hölder's inequality we find

$$\begin{aligned}
&\Big\| t \mapsto \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big\|_{\mathrm{L}^2(0,\tau,t^\beta)} \\
&\leq \Big\| t \mapsto \Big(\int_0^t (t-s)^{-\frac{3}{4}} \,\mathrm{d}s \Big)^{\frac{1}{2}} \Big(\int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&\lesssim \tau^{\frac{1}{8}} \Big\| t \mapsto \Big(\int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&= \tau^{\frac{1}{8}} \Big(\int_0^\tau \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \, t^\beta \,\mathrm{d}t \Big)^{\frac{1}{2}}.
\end{aligned}$$

As we assumed $\beta \leq 0$, we can estimate $t^\beta \leq s^\beta$. Together with Fubini's theorem we find

$$\begin{aligned}
&\tau^{\frac{1}{8}} \Big(\int_0^\tau \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \, t^\beta \,\mathrm{d}t \Big)^{\frac{1}{2}} \\
&\qquad \leq \tau^{\frac{1}{8}} \Big(\int_0^\tau \int_s^\tau (t-s)^{-\frac{3}{4}} \,\mathrm{d}t \, \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \Big)^{\frac{1}{2}} \\
&\qquad \lesssim \tau^{\frac{1}{4}} \Big(\int_0^\tau \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \Big)^{\frac{1}{2}}.
\end{aligned}$$

The second statement is shown by similar means. For $t \in (0,\tau)$ estimate

$$\int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \,\mathrm{d}s \lesssim \int_{2^{-k-1}t}^{2^{-k}t} t^{-\frac{3}{4}} \,\mathrm{d}s \lesssim 2^{-k} t^{\frac{1}{4}} \lesssim 2^{-k}\tau^{\frac{1}{4}}. \tag{8.1}$$

Then, again by Hölder's inequality, we find

$$
\begin{aligned}
&\Big\| t \mapsto \int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&\lesssim \Big\| t \mapsto \Big(\int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \,\mathrm{d}s \Big)^{\frac{1}{2}} \Big(\int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&\lesssim \tau^{\frac{1}{8}} 2^{-\frac{k}{2}} \Big(\int_0^\tau \int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \, t^\beta \,\mathrm{d}t \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Now, focus on the double integral and use Fubini's theorem

$$
\begin{aligned}
&\Big(\int_0^\tau \int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \, t^\beta \,\mathrm{d}t \Big)^{\frac{1}{2}} \\
&\qquad \le \Big(\int_0^{2^{-k}\tau} \int_{2^k s}^{2^{k+1}s} t^\beta (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}t \,\mathrm{d}s \Big)^{\frac{1}{2}} \\
&\qquad \le \Big(\int_0^{2^{-k}\tau} \int_{2^k s}^{2^{k+1}s} (2^k s)^\beta (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}t \,\mathrm{d}s \Big)^{\frac{1}{2}} \\
&\qquad = 2^{\frac{k}{2}\beta} \Big(\int_0^{2^{-k}\tau} \int_{2^k s}^{2^{k+1}s} (t-s)^{-\frac{3}{4}} \,\mathrm{d}t \, \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \, s^\beta \,\mathrm{d}s \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Next estimate

$$
\int_{2^k s}^{2^{k+1}s} (t-s)^{-\frac{3}{4}} \,\mathrm{d}t \lesssim (2^{k+1}s)^{\frac{1}{4}} \lesssim (2^k 2^{-k}\tau)^{\frac{1}{4}} = \tau^{\frac{1}{4}}
$$

as $0 < s < 2^{-k}\tau$. Putting everything together delivers

$$
\begin{aligned}
&\Big\| t \mapsto \int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} \,\mathrm{d}s \Big\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&\qquad\qquad \lesssim \tau^{\frac{1}{4}} 2^{-\frac{k}{2}(1-\beta)} \Big(\int_0^{2^{-k}\tau} \|f(s,\cdot)\|^2_{\mathrm{L}^2(A\cap\mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \Big)^{\frac{1}{2}}
\end{aligned}
$$

what we wanted to show. □

Note that the following result is similar to the already proven off-diagonal estimates in Proposition 7.2.2 but as we work with different sets, we provide the proof here. Recall that $Q(0, t_0)$ is the cube with center 0 and side length $2t_0$.

Proposition 8.1.2. *Let $\beta \in (-\infty, 1)$. Moreover, let $K \subset (0,\infty) \times \mathbb{R}^d_+$ be compact, and choose $t_0 > 0$ such that $K \subset (0, t_0) \times (Q(0, t_0) \cap \mathbb{R}^d_+)$. Moreover, let $R > 0$*

satisfy $t_0 < R$. *Then, for* $k \geq 1$ *and* $f \in \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+, s^\beta\,\mathrm{d}t\,\mathrm{d}x)$ *we have the following estimate*

$$\begin{aligned}\Big\|(t,x)\mapsto\int_0^t AE(t-s)\chi_{C_\ell\cap\mathbb{R}^d_+}f(s,\cdot)\,\mathrm{d}s[x]\Big\|_{\mathrm{L}^2(K,t^\beta\,\mathrm{d}t\,\mathrm{d}x)}&\\ \lesssim (t_0)^{\frac{d}{2}+\frac{1}{4}}2^{-\frac{\ell}{2}}R^{-\frac{1}{2}}\|f\|_{\mathrm{T}^{\infty,2}(s^\beta\,\mathrm{d}t\,\mathrm{d}x)}&\end{aligned}\tag{8.2}$$

where $C_\ell \coloneqq Q(0,2^{\ell+1}R)\setminus Q(0,2^\ell R)$ *and the implicit constant depends on* d.

Proof. As mentioned, the proof is similar to the one of Proposition 7.2.2.

In this spirit introduce the sets $S_\ell \coloneqq \{y\in C_\ell\cap\mathbb{R}^d_+ : y_d < 2^\ell R\}$ and $T_\ell \coloneqq \{y\in C_\ell\cap\mathbb{R}^d_+ : y_d \geq 2^\ell R\}$. Note that the lower part of C_ℓ is completely contained in the lower half-space, so no bottom part of C_ℓ is present. Hölder's inequality delivers

$$\begin{aligned}&\int_{C_\ell\cap\mathbb{R}^d_+}\frac{\mathrm{e}^{-c|\lambda|^{\frac12}y_d}|f(s,y)|}{(x_d+y_d+|x'-y'|)^d}\,\mathrm{d}y\\ &\qquad\lesssim\Big(\int_{C_\ell\cap\mathbb{R}^d_+}\frac{\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}}{(x_d+y_d+|x'-y'|)^{2d}}\,\mathrm{d}y\Big)^{\frac12}\|f(s,\cdot)\|_{\mathrm{L}^2(C_k\cap\mathbb{R}^d_+)}.\end{aligned}$$

By virtue of the sets S_ℓ and T_ℓ we can decompose the integral on the right-hand side by

$$\begin{aligned}\Big(\int_{C_\ell\cap\mathbb{R}^d_+}\frac{\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}}{(x_d+y_d+|x'-y'|)^{2d}}\,\mathrm{d}y\Big)^{\frac12}&\leq\Big(\int_{S_\ell}\frac{\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}}{(x_d+y_d+|x'-y'|)^{2d}}\,\mathrm{d}y\Big)^{\frac12}\\ &\quad+\Big(\int_{T_\ell}\frac{\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}}{(x_d+y_d+|x'-y'|)^{2d}}\,\mathrm{d}y\Big)^{\frac12}.\end{aligned}$$

Taking the $\mathrm{L}^2(Q(0,t_0)\cap\mathbb{R}^d_+)$-norm of these terms delivers

$$\begin{aligned}&\Big(\int_{Q(0,t_0)\cap\mathbb{R}^d_+}\int_{S_\ell}\frac{\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}}{(x_d+y_d+|x'-y'|)^{2d}}\,\mathrm{d}y\,\mathrm{d}x\Big)^{\frac12}\\ &\qquad\lesssim (t_0)^{\frac{d}{2}}(2^\ell R)^{\frac{d-1}{2}}(2^\ell R-t_0)^{-d}\Big(\int_0^{2^\ell R}\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}\,\mathrm{d}y_d\Big)^{\frac12}\\ &\qquad\lesssim (t_0)^{\frac{d}{2}}(2^\ell R)^{\frac{d-1}{2}}(2^\ell R)^{-d}|\lambda|^{-\frac14}\\ &\qquad= (t_0)^{\frac{d}{2}}(2^\ell R)^{-\frac{d+1}{2}}|\lambda|^{-\frac14}\end{aligned}$$

and

$$\begin{aligned}\Big(\int_{Q(0,t_0)\cap\mathbb{R}^d_+}\int_{T_\ell}\frac{\mathrm{e}^{-2c|\lambda|^{\frac12}y_d}}{(x_d+y_d+|x'-y'|)^{2d}}\,\mathrm{d}y\,\mathrm{d}x\Big)^{\frac12}&\lesssim (t_0)^{\frac{d}{2}}(2^\ell R)^{-d}(2^\ell R)^{\frac{d}{2}}\,\mathrm{e}^{-c|\lambda|^{\frac12}2^\ell R}\\ &\lesssim (t_0)^{\frac{d}{2}}(2^\ell R)^{-\frac{d}{2}}\Big(|\lambda|^{\frac12}2^\ell R\Big)^{-\frac12}\\ &= (t_0)^{\frac{d}{2}}(2^\ell R)^{-\frac{d+1}{2}}|\lambda|^{-\frac14}.\end{aligned}$$

Plugging this estimate into the Cauchy integral yields

$$\begin{aligned}\Big\| x \mapsto \int_0^t AE(t-s)\chi_{C_\ell \cap \mathbb{R}^d_+} f(s,\cdot)\,\mathrm{d}s[x] \Big\|_{\mathrm{L}^2(Q(0,t_0)\cap\mathbb{R}^d_+)} \\ \lesssim (t_0)^{\frac{d}{2}} (2^\ell R)^{-\frac{d+1}{2}} \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(C_\ell\cap\mathbb{R}^d_+)}\,\mathrm{d}s.\end{aligned}$$

Finally, we integrate over $(0,t)$ and take the $\mathrm{L}^2(0,t_0,t^\beta)$-norm. For $\beta \in (-\infty,0]$, by means of Lemma 8.1.1 and $t_0 < R$ we find

$$\begin{aligned}&(t_0)^{\frac{d}{2}} (2^\ell R)^{-\frac{d+1}{2}} \Big\| t \mapsto \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(Q(0,2^{\ell+1}R)\cap\mathbb{R}^d_+))}\,\mathrm{d}s \Big\|_{\mathrm{L}^2(0,t_0,t^\beta)} \\ &\lesssim (t_0)^{\frac{d}{2}+\frac{1}{4}} (2^\ell R)^{-\frac{d+1}{2}} \Big(\int_0^{t_0} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(0,2^{\ell+1}R)\cap\mathbb{R}^d_+)} s^\beta\,\mathrm{d}s \Big)^{\frac{1}{2}} \\ &\lesssim (t_0)^{\frac{d}{2}+\frac{1}{4}} (2^\ell R)^{-\frac{1}{2}} \Big((2^{\ell+1}R)^{-d} \int_0^{2^{2\ell+2}R^2} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(0,2^{\ell+1}R)\cap\mathbb{R}^d_+)} s^\beta\,\mathrm{d}s \Big)^{\frac{1}{2}} \\ &\lesssim (t_0)^{\frac{d}{2}+\frac{1}{4}} (2^\ell R)^{-\frac{1}{2}} \|f\|_{\mathrm{T}^{\infty,2}(s^\beta\,\mathrm{d}s\,\mathrm{d}x)}.\end{aligned}$$

This completes the proof for $\beta \in (-\infty,0]$. For $\beta \in (0,1)$ we use the further decomposition $\int_0^t \mathrm{d}s = \sum_{k=0}^\infty \int_{2^{-k-1}t}^{2^{-k}t} \mathrm{d}s$. For $k \geq 0$ again by Lemma 8.1.1 and $t_0 < R$ we find

$$\begin{aligned}&(t_0)^{\frac{d}{2}} (2^\ell R)^{-\frac{d+1}{2}} \Big\| t \mapsto \int_{2^{-(k+1)}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(Q(0,2^{\ell+1}R)\cap\mathbb{R}^d_+))}\,\mathrm{d}s \Big\|_{\mathrm{L}^2(0,t_0,t^\beta)} \\ &\lesssim (t_0)^{\frac{d}{2}} (2^\ell R)^{-\frac{d+1}{2}} (t_0)^{\frac{1}{4}} 2^{-\frac{k}{2}(1-\beta)} \Big(\int_0^{2^{-k}t_0} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(0,2^{\ell+1}R)\cap\mathbb{R}^d_+)} s^\beta\,\mathrm{d}s \Big)^{\frac{1}{2}} \\ &\lesssim (t_0)^{\frac{d}{2}+\frac{1}{4}} (2^\ell R)^{-\frac{1}{2}} 2^{-\frac{k}{2}(1-\beta)} \Big((2^{\ell+1}R)^{-d} \int_0^{2^{2\ell+2}R^2} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(0,2^{\ell+1}R)\cap\mathbb{R}^d_+)} s^\beta\,\mathrm{d}s \Big)^{\frac{1}{2}} \\ &\lesssim (t_0)^{\frac{d}{2}+\frac{1}{4}} (2^\ell R)^{-\frac{1}{2}} 2^{-\frac{k}{2}(1-\beta)} \|f\|_{\mathrm{T}^{\infty,2}(s^\beta\,\mathrm{d}s\,\mathrm{d}x)}.\end{aligned}$$

Due to $\beta < 1$ this is summable in k and proves the desired estimate. □

With this estimate at hand, we can now show $\mathrm{L}^2_{\mathrm{loc}}$-convergence.

Proposition 8.1.3. *Let $\beta \in (-\infty,2)$. For $f \in \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+, s^\beta\,\mathrm{d}s\mathrm{d}x)$ the limit*

$$T_{\infty,2,\beta} f \coloneqq \lim_{R\to\infty} A \int_0^t E(t-s)\chi_{Q(0,R)\cap\mathbb{R}^d_+} f(s,\cdot)\,\mathrm{d}s$$

exists in $\mathrm{L}^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d_+, s^\beta\,\mathrm{d}s\,\mathrm{d}x)$.

Proof. We show that this sequence is Cauchy in $\mathrm{L}^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d_+, s^\beta \,\mathrm{d}s\,\mathrm{d}x)$. Let $K \subseteq (0,\infty)\times\mathbb{R}^d_+$ be compact and choose t_0 such that $K \subset (0,t_0)\times Q(0,t_0)$. Moreover, let $R > t_0$ and $\tilde{R} > R$. We then consider

$$\begin{aligned}
&A\int_0^t E(t-s)(\chi_{Q(0,2\tilde{R})} - \chi_{Q(0,2R)})f(s,\cdot)\,\mathrm{d}s\\
&\quad = A\int_0^t E(t-s)\left(\chi_{Q(0,2\tilde{R})\setminus Q(0,2R)}\right)f(s,\cdot)\,\mathrm{d}s\\
&\quad = \int_0^t AE(t-s)\left(\chi_{Q(0,2\tilde{R})\setminus Q(0,2R)}\sum_{k=1}^\infty \chi_{C_k}\right)f(s,\cdot)\,\mathrm{d}s.
\end{aligned}$$

Note that the second step is true due to $\sum_{k=1}^\infty \chi_{C_k\cap\mathbb{R}^d_+} = 1$ on $\mathbb{R}^d_+ \setminus Q(0,2R)$ as the characteristic function vanishes on $Q(0,2R)$. The sets C_k are defined as in the previous proposition. Moreover, by the previous proposition A can be taken inside the integral. Finally, an application of the previous proposition together with the fact that the right-hand side of (8.2) is summable in k and vanishes as $R\to\infty$. $\square$

Remark 8.1.4.

- Recall that in the $\mathrm{T}^{\infty,2}$-norm the supremum is taken over all cubes $Q(x_0,r)$ with center $x_0\in\mathbb{R}^d_+$ and side length $2r$. Hence, it stands to reason that for $f\in\mathrm{T}^{\infty,2}$ the definition of $T_{\infty,2}$ does not depend on the center of the cube, i.e., yields the same result if $\chi_{Q(x_0,R)\cap\mathbb{R}^d_+}$ is used as a cut-off. To this end, we remark that for $R>0$ large enough, e.g., $R > 2\|x_0\|_\infty$, every point $x\in Q(0,R)\triangle Q(x_0,R)$ satisfies $x_d > \frac{R}{2}$ or $|x'| > \frac{R}{2}$. Here $\triangle$ denotes the symmetric difference. This together with Proposition 8.1.2 implies

$$\left\|\int_0^t AE(t-s)(\chi_{Q(0,R)} - \chi_{Q(x_0,R)})f(s)\,\mathrm{d}s\right\|_{\mathrm{L}^2(K)} \to 0$$

  for $R\to\infty$ for all $x_0\in\mathbb{R}^d_+$ and any compact set $K\subseteq(0,\infty)\times\mathbb{R}^d_+$.

- Let $x_0\in\mathbb{R}^d_+$ and $R>0$ be fixed. Then, $(2^kR)_{(k\in\mathbb{N})}$ converges to ∞. Using again dyadic decomposition and the linearity of our operator implies

$$T_{\infty,2}f = \sum_{k=0}^\infty T_{\infty,2}\chi_{C_k\cap\mathbb{R}^d_+}f$$

  where the limit is to be understood in $\mathrm{L}^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d_+)$ and $C_0 = Q(x_0,2R)$ and $C_k = Q(x_0,2^{k+1}R)\setminus Q(x_0,2^kR)$.

Finally, we show that our extension $T_{\infty,2}$ coincides with $T_{2,2}$ on $\mathrm{T}^{\infty,2}\cap\mathrm{T}^{2,2}$. This result is crucial as it will allow us to obtain a boundedness result for $2<p<\infty$ by interpolation.

Theorem 8.1.5. *Let $f \in \mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+) \cap \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)$. Then $T_{2,2}f = T_{\infty,2,0}f$.*

Proof. An application of the dominated convergence theorem together with

$$\mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+) = \mathrm{L}^2((0,\infty)\times\mathbb{R}^d_+),$$

see Proposition 2.2.4, shows that for $f \in \mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+) \cap \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)$ the cut-off $f_R \coloneqq \chi_{(0,R)\times Q(0,R)}f$ converges to f in the $\mathrm{T}^{2,2}$-norm as $R\to\infty$. By continuity of the maximal regularity operator on $\mathrm{T}^{2,2}((0,\infty)\times\mathbb{R}^d_+)$ we obtain $T_{2,2}f = \lim_{R\to\infty} T_{2,2}f_R$ where the limit is to be understood in the L^2-sense. Recall that on the tent space $\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)$ we defined our extension via off-diagonal estimates as $T_{\infty,2}f = \lim_{R\to\infty} T_{2,2}f_R$ where the limit is be understood in the $\mathrm{L}^2_{\mathrm{loc}}$-sense. Hence, the limits coincide point-wise almost everywhere and thus our extensions are compatible in the desired sense. □

Remark 8.1.6. The above theorem, although formulated only for the unweighted case, holds in the weighted case as well. In particular, if $f \in \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+, s^\beta\,\mathrm{d}s\,\mathrm{d}x) \cap \mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+, s^{\tilde\beta}\,\mathrm{d}s\,\mathrm{d}x)$ for $\beta,\tilde\beta \in (-\infty,1)$ then $T_{\infty,2,\beta}f = T_{\infty,2,\tilde\beta}f$ holds true. The argument is the same as in the previous proof as $\mathrm{L}^2_{\mathrm{loc}}$-convergence, also in weighted spaces, implies convergence point-wise almost everywhere.

Next, consider $p<\infty$. In this case, an application of the dominated convergence theorem shows that $\mathrm{L}^2_{\mathrm{c}}((0,\infty)\times\mathbb{R}^d_+)$ is dense in $\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+, t^\beta\,\mathrm{d}t\,\mathrm{d}x)$ (with respect to the $\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$-norm). Assuming boundedness of $T_{2,2}$ on $\mathrm{L}^2_{\mathrm{c}}((0,\infty)\times\mathbb{R}^d_+)$ with respect to the $\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$-norm, the natural way of defining $T_{p,2,\beta}$ is the following: For $f\in\mathrm{T}^{p,2}((0,\infty)\times\mathbb{R}^d_+, t^\beta\,\mathrm{d}t\,\mathrm{d}x)$ define

$$T_{p,2,\beta}f \coloneqq \lim_{r\to\infty} T_{2,2}\chi_{Q(0,r)\cap\mathbb{R}^d_+}f \tag{8.3}$$

where the limit is taken with respect to the $\mathrm{T}^{p,2}$-norm.

Consequently, it is sufficient to estimate the maximal regularity operator on this subspace. By linearity, for all $f\in\mathrm{L}^2_{\mathrm{c}}((0,\infty)\times\mathbb{R}^d_+)$ and all $x_0\in\mathbb{R}^d_+$ and $r>0$ we have

$$T_{2,2}f = \sum_{\ell=0}^\infty T_{2,2}\chi_{C_\ell}f$$

point-wise almost everywhere, where the sets C_ℓ are taken with respect to the cube $Q(x_0,r)$. Note that due to the compact support of f the series is in fact a finite sum. However, for the boundedness estimate in $\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$ we aim for a uniform estimate in f so that the above notation is justified. We postpone the proof thereof for later.

8.2 The main theorem

The goal of this section is prove the following theorem.

Theorem 8.2.1. *Let $\beta \in (-\infty, 1)$ and $\frac{2d}{d+1} < p \leq \infty$ satisfy $p > \frac{2d}{d+2-2\beta}$. Then, there exists a bounded linear operator*

$$T_{p,2,\beta} \colon \mathrm{T}^{p,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x) \to \mathrm{T}^{p,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x)$$

extending the operator $T_{2,2}$ from $\mathrm{L}^2_c((0,\infty) \times \mathbb{R}^d_+)$ to $\mathrm{T}^{p,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x)$. Moreover, $T_{p,2,\beta}$ is defined via (8.3)*.*

First of all, note that the case $p = 2$ is covered in light of Proposition 7.1.13 and $\mathrm{T}^{2,2}((0,\infty) \times \mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x) = \mathrm{L}^2((0,\infty) \times \mathbb{R}^d_+, t^\beta \, \mathrm{d}t \, \mathrm{d}x)$ due to Proposition 2.2.4.

Remark 8.2.2. It is natural question to ask if the range of β is sharp. In the case of a bounded analytic semigroup this was answered negatively for any $\beta \geq 1$ in [6, Theorem 1.5]. We did not consider this topic in this thesis but conjecture that $\beta \geq 1$ fails as well in our setting.

8.2.1 The case $p > 2$

In this subsection we consider the case $p > 2$. To this end we show the boundedness for $p = \infty$. The case $2 < p < \infty$ then follows by interpolation.

Theorem 8.2.3. *Let $\beta \in (-\infty, 1)$. Then, the maximal regularity operator is bounded on $\mathrm{T}^{\infty,2}_\beta((0,\infty) \times \mathbb{R}^d_+)$ with operator norm depending on d and β.*

Proof. Let $f \in \mathrm{T}^{\infty,2}(t^\beta \mathrm{d}t \, \mathrm{d}x)$ be arbitrary. Again it will be more convenient to work with cubes instead of balls in the definition of the tent space norm. We fix $x_0 \in \mathbb{R}^d_+$ and $\tau > 0$. The sets C_ℓ appearing in the following are taken with respect to x_0 and $\sqrt{\tau}$, i.e., $C_\ell = Q(x_0, 2^{\ell+1}\sqrt{\tau}) \setminus Q(x_0, 2^\ell \sqrt{\tau})$. Estimate by means of Remark 8.1.4 and the triangle inequality

(8.4)

$$\begin{aligned}
&\Big(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+} \Big| A \int_0^t [E(t-s) f(s,\cdot)](x) \, \mathrm{d}s \Big|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \Big)^{\frac{1}{2}} \\
&\leq \Big(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+} \Big| A \int_0^t [E(t-s) \chi_{Q(x_0, 2\sqrt{\tau}) \cap \mathbb{R}^d_+} f(s,\cdot)](x) \, \mathrm{d}s \Big|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \Big)^{\frac{1}{2}} \\
&+ \sum_{\ell=1}^\infty \Big(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+} \Big| A \int_0^t [E(t-s) \chi_{C_\ell \cap \mathbb{R}^d_+} f(s,\cdot)](x) \, \mathrm{d}s \Big|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \Big)^{\frac{1}{2}} \\
&=: \mathrm{I} + \sum_{\ell=1}^\infty \mathrm{II}_\ell.
\end{aligned}$$

The first term can be estimated by Proposition 7.1.15. For the second term we first consider $\beta \leq 0$ as it is the easier case. The off-diagonal estimates proven in Proposition 7.2.2 together with Lemma 8.1.1 with $A = C_\ell$ imply

$$\begin{aligned}
&\left(\tau^{-\frac{d}{2}} \int_0^\tau \int_{\mathbb{R}^d_+ \cap Q(x_0,\sqrt{\tau})} \left| A \int_0^t [E(t-s)\chi_{C_\ell \cap \mathbb{R}^d_+} f(s,\cdot)](x) \,\mathrm{d}s \right|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \right)^{\frac{1}{2}} \\
&\qquad \lesssim \tau^{-\frac{d}{4}-\frac{1}{4}} 2^{-\ell \frac{d+1}{2}} \left\| t \mapsto \int_0^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(C_\ell \cap \mathbb{R}^d_+)} \,\mathrm{d}s \right\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&\qquad \lesssim \tau^{-\frac{d}{4}} 2^{-\ell \frac{d+1}{2}} \left(\int_0^\tau \|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell \cap \mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \right)^{\frac{1}{2}}.
\end{aligned}$$

Now, estimate $\tau \leq 2^{2l+2}\tau$ and use $C_\ell \subseteq Q(x_0, 2^{\ell+1}\sqrt{\tau})$ to obtain

$$\begin{aligned}
&\tau^{-\frac{d}{4}} 2^{-\ell \frac{d+1}{2}} \left(\int_0^\tau \|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell \cap \mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \right)^{\frac{1}{2}} \\
&\qquad \lesssim 2^{-\ell \frac{d+1}{2}} 2^{\ell \frac{d}{2}} \left((2^{2\ell}\tau)^{-\frac{d}{2}} \int_0^{2^{2\ell+2}\tau} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(x_0,2^{\ell+1}\sqrt{\tau}) \cap \mathbb{R}^d_+)} s^\beta \,\mathrm{d}s \right)^{\frac{1}{2}} \\
&\qquad \lesssim 2^{-\frac{\ell}{2}} \|f\|_{\mathrm{T}^{\infty,2}(t^\beta \,\mathrm{d}t \,\mathrm{d}y)}.
\end{aligned}$$

Plugging everything into (8.4) finally yields

$$\begin{aligned}
&\left(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+} \left| A \int_0^t [E(t-s) f(s,\cdot)](x) \,\mathrm{d}s \right|^2 \mathrm{d}x \, t^\beta \,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq \mathrm{I} + \sum_{\ell=1}^\infty \mathrm{II}_\ell \\
&\leq C \|f\|_{\mathrm{T}^{\infty,2}(t^\beta \,\mathrm{d}t \,\mathrm{d}y)} + C \sum_{\ell=1}^\infty 2^{-\frac{\ell}{2}} \|f\|_{\mathrm{T}^{\infty,2}(t^\beta \,\mathrm{d}t \,\mathrm{d}y)}
\end{aligned}$$

for $\beta \leq 0$. This completes the proof for non-positive time-weights.

For $\beta \in (0,1)$ we need to be slightly more careful and further decompose $\int_0^t \mathrm{d}s = \sum_{k=0}^\infty \int_{2^{-k-1}t}^{2^{-k}t} \mathrm{d}s$ in the terms II_ℓ as done in [10, Theorem 3.2]. For fixed $\ell \geq 1$ our estimate then reads

$$\begin{aligned}
&\left(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+} \left| A \int_0^t [E(t-s)\chi_{C_\ell \cap \mathbb{R}^d_+} f(s,\cdot)](x) \,\mathrm{d}s \right|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq \sum_{k=0}^\infty \left(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau}) \cap \mathbb{R}^d_+} \left| A \int_{2^{-k-1}t}^{2^{-k}t} [E(t-s)\chi_{C_\ell \cap \mathbb{R}^d_+} f(s,\cdot)](x) \,\mathrm{d}s \right|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \right)^{\frac{1}{2}} \\
&=: \sum_{k=0}^\infty \mathrm{II}_{k,\ell}
\end{aligned}$$

For $k \geq 0$ use Minkowski's integral inequality and Hölder's inequality as for $\beta \leq 0$ and Lemma 8.1.1 with $A = C_\ell$ to obtain

$$\begin{aligned}
&\left(\tau^{-\frac{d}{2}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau})\cap\mathbb{R}^d_+} \left| A \int_{2^{-k-1}t}^{2^{-k}t} [E(t-s)\chi_{C_\ell\cap\mathbb{R}^d_+} f(s,\cdot)](x) \, \mathrm{d}s \right|^2 \mathrm{d}x \, t^\beta \mathrm{d}t \right)^{\frac{1}{2}} \\
&\qquad \lesssim \tau^{-\frac{d}{4}-\frac{1}{4}} 2^{-\ell\frac{d+1}{2}} \left\| t \mapsto \int_{2^{-k-1}t}^{2^{-k}t} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(C_\ell\cap\mathbb{R}^d_+)} \, \mathrm{d}s \right\|_{\mathrm{L}^2(0,\tau;t^\beta)} \\
&\qquad \lesssim \tau^{-\frac{d}{4}} 2^{-\ell\frac{d+1}{2}} 2^{-\frac{k}{2}(1-\beta)} \left(\int_0^{2^{-k}\tau} \|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell\cap\mathbb{R}^d_+)} s^\beta \, \mathrm{d}s \right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, using $2^{-k}\tau \leq 2^{2\ell+2}\tau$ and $C_\ell \subseteq Q(x_0, 2^{\ell+1}\sqrt{\tau})$ yields

$$\begin{aligned}
\mathrm{II}_{k,l} &\lesssim \tau^{-\frac{d}{4}} 2^{-\ell\frac{d+1}{2}} 2^{-\frac{k}{2}(1-\beta)} \left(\int_0^{2^{2\ell+2}\tau} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(x_0,2^{\ell+1}\sqrt{\tau})\cap\mathbb{R}^d_+)} s^\beta \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\lesssim 2^{-\ell\frac{d+1}{2}} 2^{l\frac{d}{2}} 2^{-\frac{k}{2}(1-\beta)} \left((2^{2l}\tau)^{-\frac{d}{2}} \int_0^{2^{2l+2}\tau} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(x_0,2^{\ell+1}\sqrt{\tau})\cap\mathbb{R}^d_+)} s^\beta \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\lesssim 2^{-\frac{\ell}{2}} 2^{-\frac{k}{2}(1-\beta)} \|f\|_{\mathrm{T}^{\infty,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}y)}.
\end{aligned}$$

Note that $\beta < 1$ implies that the right-hand side is summable with respect to k and ℓ and this concludes the case of positive time-weights. □

Now, we can cover the case $2 < p < \infty$ by interpolation.

Theorem 8.2.4. *Let $2 < p < \infty$. Then for all $\beta \in (-\infty, 1)$ the restriction of $T_{2,2}$ to $\mathrm{L}^2_c((0,\infty)\times\mathbb{R}^d_+)$ is bounded with respect to the $\mathrm{T}^{p,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x)$-norm and thus has a bounded extension to $\mathrm{T}^{p,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x)$.*

Proof. We aim to interpolate the operators $T_{2,2}$ and $T_{\infty,2}$. Recall that we showed $T_{2,2} = T_{\infty,2}$ on $\mathrm{T}^{2,2} \cap \mathrm{T}^{\infty,2}$, so particularly on $\mathrm{L}^2_c((0,\infty)\times\mathbb{R}^d_+)$. The boundedness of $T_{2,2}$ on $\mathrm{L}^2_c((0,\infty)\times\mathbb{R}^d_+)$ with respect to the $\mathrm{T}^{p,2}(t^\beta \, \mathrm{d}t \, \mathrm{d}x)$-norm is then a consequence of the interpolation result for Tent spaces by Coifman, Meyer and Stein (see [22, Theorem 4]). Note that their statement is only for the unweighted case. For the weighted case we refer to [2, Theorem 2.1] and the references therein. □

8.2.2 The case $p < 2$

For $p < 2$ one has to be more careful. Examining the proof of Auscher, Monniaux and Portal in [10] reveals that they crucially use the semigroup law. Unfortunately, in our setting no such law is available for the pressure. Hence, we need to come up with a different technique.

The classes M_k

Here, we introduce a notion measuring the growth of a smooth function at infinity and 0. It turns out that this notion is particularly useful in our setting.

Definition 8.2.5. For $k \in \mathbb{Z}$ we define M_k as the set of all functions $f \in \mathrm{C}^\infty(\mathbb{R}^{d-1} \setminus \{0\})$ that satisfy

$$|\nabla^\alpha f(\xi)| \lesssim |\xi|^{k-|\alpha|} \tag{8.5}$$

for all multi-indices α and all $\xi \neq 0$.

The following two results will be vastly used in proving kernel estimates. Recall that for $\lambda \in \mathrm{S}_{\pi-\varepsilon}$ for some $\varepsilon > 0$ we defined $\omega_\lambda(\xi) = \sqrt{\lambda + |\xi|^2}$.

Lemma 8.2.6. *Let $k, l \in \mathbb{Z}$. If $f \in M_k$ and $g \in M_l$, then $fg \in M_{k+l}$.*

Proof. This is an immediate consequence of the higher order product rule. □

Lemma 8.2.7. *Let $\varepsilon > 0$ and $\lambda \in \mathrm{S}_{\pi-\varepsilon}$. Then $\frac{|\xi|}{\omega_\lambda(\xi)} \in M_0$.*

Proof. We prove this by providing a representation for arbitrary derivatives of $\frac{|\xi|}{\omega_\lambda(\xi)}$ which will imply the desired estimates.

We claim that

$$\nabla^\gamma \frac{|\xi|}{\omega_\lambda(\xi)} = \sum_{i=1}^{n_\gamma} p_{k_i}(\xi) |\xi|^{\alpha_i} \omega_\lambda(\xi)^{\beta_i}$$

where p_{k_i} is a polynomial of degree k_i and $k_i + \alpha_i + \beta_i = -|\gamma|$ holds for all $i \leq n_\gamma$ which is a natural number dependent on γ.

We prove this via induction over $|\gamma|$. For $\gamma = 0$ is this clear. As it is sufficient to consider only one summand we compute for $j \in \{1, \dots, d\}$

$$\begin{aligned} \partial_j (p_k(\xi)|\xi|^\alpha \omega_\lambda(\xi)^\beta) &= \partial_j p_k(\xi)|\xi|^\alpha \omega_\lambda(\xi)^\beta + \alpha p_k(\xi)|\xi|^{\alpha-1} \frac{\xi_j}{|\xi|} \omega_\lambda(\xi)^\beta \\ &\quad + p_k(\xi)|\xi|^\alpha \beta \omega_\lambda(\xi)^{\beta-1} \frac{\xi_j}{\omega_\lambda(\xi)} \end{aligned}$$

which proves the desired representation as $\partial_j p_k$ is either a polynomial of degree $k-1$ or 0. □

Reformulation of a bad term

In [10, Theorem 3.1] the authors deal with the last term J_0 in by means of the semigroup property. To be precise, they write

$$(8.6)\quad \int_{\frac{t}{2}}^{t} L\,\mathrm{e}^{-(t-s)L} f(s,\cdot)\,\mathrm{d}s = \int_0^t L\,\mathrm{e}^{-(t-s)L} f(s,\cdot)\,\mathrm{d}s - \mathrm{e}^{-\frac{t}{2}L}\int_0^{\frac{t}{2}} \mathrm{e}^{-(t/2-s)L} f(s,\cdot)\,\mathrm{d}s$$

where $-L$ is the generator of a bounded analytic semigroup. Then, they employ L^2-boundedness of the maximal regularity operator as well as the uniform boundedness of the semigroup to conclude their proof. Unfortunately, this approach does not work in our setting, as the family $(E^{\phi}(t))_{t>0}$ does not satisfy a semigroup law.

For $f \in \mathrm{C}_c^\infty((0,\infty)\times\mathbb{R}^d_+)$ however, we can prove a relation similar to (8.6) in our extended setting as well. To this end we proceed in three steps. First we derive a new kernel representation of $A(R^u(\lambda)f, R^{\phi}(\lambda)f)$ by an integration by parts argument. This new kernel will be integrated against ∇f instead of f. Hence, it is reasonable to expect that these kernels behave like first order derivatives of the kernels belonging to u, see (6.5)–(6.8), or like the kernel belonging to the pressure ϕ, see (6.9).

In the second step we examine the proof for the kernel estimates given by Maekawa, Miura and Prange in [68] to show that this expectation is indeed fulfilled. Finally, in the third step we use the newly derived kernel estimates to show that the left-hand side of (8.6) converges and then use the Cauchy integral representation of our extended semigroup to obtain a relation similar to (8.6). This allows to then proceed similarly to [10] and conclude the desired boundedness estimate by already proven estimates for the maximal regularity operator as well as for the family $(E^u(t))_{t>0}$.

So, let us establish a representation for $A((R^u(\lambda)f, R^{\phi}(\lambda)f)$ where $f\in\mathrm{C}_c^\infty(\mathbb{R}^d_+)$. Recall that $u = v + w$ and ϕ were defined in (6.10)–(6.13).

Lemma 8.2.8. *Let $f \in \mathrm{C}_c^\infty(\mathbb{R}^d_+)$. For $y=(y',y_d)\in\mathbb{R}^d_+$ we have*

$$\begin{aligned}\nabla_{y'}^2 v(y',y_d) &= \int_{\mathbb{R}^{d-1}}\int_0^\infty \nabla_{y'} k_{1,\lambda}(y'-z', y_d - z_d)\nabla_{z'} f(z',z_d)\,\mathrm{d}z_d\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^{d-1}}\int_0^\infty \nabla_{y'} k_{2,\lambda}(y'-z', y_d, z_d)\nabla_{z'} f(z',z_d)\,\mathrm{d}z_d\,\mathrm{d}z.\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_{y_d}^2 v(y',y_d) &= \int_{\mathbb{R}^{d-1}}\int_0^\infty \partial_{y_d} k_{1,\lambda}(y'-z', y_d - z_d)\partial_{z_d} f(z',z_d)\,\mathrm{d}z_d\,\mathrm{d}z\\ &\quad - \int_{\mathbb{R}^{d-1}}\int_0^\infty \partial_{y_d} k_{2,\lambda}(y'-z', y_d, z_d)\partial_{z_d} f(z',z_d)\,\mathrm{d}z_d\,\mathrm{d}z.\end{aligned}$$

For the non-local part w and $y = (y', y_d) \in \mathbb{R}^d_+$ we have with $r_\lambda(y', y_d, z_d) = (r'_\lambda(y', y_d, z_d), r_{d,\lambda}(y', y_d, z_d))$

$$\nabla^2_{y'} w(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty \nabla_{y'} r_\lambda(y' - z', y_d, z_d) \nabla_{z'} f'(z', z_d) \, \mathrm{d}z_d \, \mathrm{d}z'$$

and

$$\partial^2_{y_d} w'(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty \tilde{r}'_\lambda(y' - z', y_d, z_d) \partial_{z_d} f'(z', z_d) \, \mathrm{d}z_d \, \mathrm{d}z'$$

where

$$\tilde{r}'_\lambda(y', y_d, z_d) = \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{\frac{|\xi|^2}{\omega_\lambda(\xi)} \mathrm{e}^{-|\xi| y_d} - \omega_\lambda(\xi) \, \mathrm{e}^{-\omega_\lambda(\xi) y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi) z_d} \frac{\xi \otimes \xi}{|\xi|} \, \mathrm{d}\xi$$

for $y = (y', y_d) \in \mathbb{R}^d_+$ and $z_d > 0$. For the remaining derivative of the non-local part w we have

$$\partial^2_{y_d} w_d(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty \tilde{r}_{d,\lambda}(y' - z', y_d, z_d) \cdot \partial z_d f'(z', z_d) \, \mathrm{d}z_d \, \mathrm{d}z'$$

where

$$\tilde{r}_{d,\lambda}(y', y_d, z_d) = \mathrm{i} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{\frac{|\xi|^2}{\omega_\lambda(\xi)} \mathrm{e}^{-|\xi| y_d} - \omega_\lambda(\xi) \, \mathrm{e}^{-\omega_\lambda(\xi) y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi) z_d} \, \xi \, \mathrm{d}\xi.$$

For the pressure we have

$$\nabla_{y'} \phi(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty q_\lambda(y' - z', y_d, z_d) \nabla_{z'} f'(z', z_d) \, \mathrm{d}z_d \, \mathrm{d}z'$$

with q_λ as in (6.9)*. Moreover, we have*

$$\partial_{y_d} \phi(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty \tilde{q}_\lambda(y' - z', y_d, z_d) \cdot \partial_{z_d} f'(z', z_d) \, \mathrm{d}z_d \, \mathrm{d}z'$$

where

$$\tilde{q}_\lambda(y', y_d, z_d) = -\mathrm{i} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{|\xi|}{\omega_\lambda(\xi)} \mathrm{e}^{-|\xi| y_d} \mathrm{e}^{-\omega_\lambda(\xi) z_d} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\omega_\lambda(\xi)} \right) \mathrm{d}\xi.$$

Proof. The statement for all derivatives with respect to the first $d - 1$ variables is an immediate consequence of the formulas (6.10)–(6.13) and the definition of the respective kernels as

$$\nabla_{y'} \mathrm{e}^{\mathrm{i}(y' - x')\cdot\xi} = -\nabla_{z'} \mathrm{e}^{\mathrm{i}(y' - z')\cdot\xi}$$

holds true and we can integrate by parts by assumption on f. For $\partial_{y_d}^2 v(y', y_d)$ note that $k_{1,\lambda}$ is again of convolution type and in $k_{2,\lambda}$ the relevant factor is $\mathrm{e}^{-\omega_\lambda(\xi)(y_d+z_d)}$ which leads to the minus sign.

Turning to the non-local part w it suffices to consider $\partial_{y_d}^2 r_\lambda'(y', y_d, z_d)$. The representation of $\partial_{y_d}^2 r_{d,\lambda}(y', y_d, z_d)$ is proved analogously. A straightforward calculation shows

$$\begin{aligned}\partial_{y_d}^2 r_\lambda'(y', y_d, z_d) &= \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{|\xi|^2 \mathrm{e}^{-|\xi|y_d} - \omega_\lambda(\xi)^2 \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} \mathrm{e}^{-\omega_\lambda(\xi)z_d} \frac{\xi\otimes\xi}{|\xi|} \,\mathrm{d}\xi \\ &= -\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{|\xi|^2 \mathrm{e}^{-|\xi|y_d} - \omega_\lambda(\xi)^2 \mathrm{e}^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)^2(\omega_\lambda(\xi) - |\xi|)} \partial_{z_d} \mathrm{e}^{-\omega_\lambda(\xi)z_d} \frac{\xi\otimes\xi}{|\xi|} \,\mathrm{d}\xi\end{aligned}$$

Finally, for the pressure we find

$$\begin{aligned}\partial_{y_d} q_\lambda(y', y_d, z_d) &= -\mathrm{i}\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} |\xi| \,\mathrm{e}^{-|\xi|y_d} \mathrm{e}^{-\omega_\lambda(\xi)y_d} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\omega_\lambda(\xi)}\right) \mathrm{d}\xi \\ &= \mathrm{i}\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} \frac{|\xi|}{\omega_\lambda(\xi)} \mathrm{e}^{-|\xi|y_d} \mathrm{e}^{-\omega_\lambda(\xi)y_d} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\omega_\lambda(\xi)}\right) \mathrm{d}\xi\end{aligned}$$

and in light of integration by parts this concludes the proof. □

Recall that for $t > 0$ and $f \in \mathrm{C}_\mathrm{c}^\infty((0,\infty)\times\mathbb{R}_+^d)$ we are interested in convergence of the integral

$$\int_{\frac{t}{2}}^t AE(t-s)f(s,\cdot)\,\mathrm{d}s.$$

In the light of Lemma 8.2.8 convergence of the integral over ∇_v as well as all derivatives with respect to y' is a consequence of Proposition 7.1.10. For the remaining terms further arguments are necessary. Next, we will show that the new kernels derived in Lemma 8.2.8 satisfy the same point-wise estimates as $\partial_{y_d} r_\lambda'$, $\partial_{y_d} r_{d,\lambda}$ and q_λ respectively. To this end we examine the proofs by Maekawa, Miura and Prange, see [68, Proposition 3.5, Proposition 3.7]. A crucial tool is [68, Lemma 3.1] which we state for the sake of completeness. For a proof we refer to the paper by Maekawa, Miura and Prange.

Lemma 8.2.9. *Let $m \in \mathrm{C}^\infty(\mathbb{R}^{d-1}\setminus\{0\})$ be a smooth Fourier multiplier. Let K be the kernel associated with m, i.e., for all $y' \in \mathbb{R}^{d-1}$ define*

$$K(y') := \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi} m(\xi)\,\mathrm{d}\xi,$$

Assume that there exists $n > 1-d$ and positive constants $c_0(d,n,m)$, $C_0(d,n,m) < \infty$ such that for all $\alpha \in \mathbb{N}$ satisfying $0 \le \alpha \le n+d$ and for all $\xi \in \mathbb{R}^{d-1}\setminus\{0\}$ it holds

$$|\nabla^\alpha m(\xi)| \le C_0|\xi|^{n-\alpha} \mathrm{e}^{-c_0|\xi|}.$$

Then there exists a constant $C(d,n,c_0,C_0) < \infty$ such that for all $y' \in \mathbb{R}^{d-1} \setminus \{0\}$ the estimate

$$|K(y')| \leq \frac{C}{|y'|^{n+d-1}}$$

is satisfied.

In deriving estimates on the kernels we will, as Maekawa, Miura and Prange in [68], consider $|\lambda| = 1$ and obtain general λ by scaling. To this end we note that the new kernels satisfy the following scaling relations:

$$\begin{aligned}
\tilde{r}'_\lambda(y', y_d, z_d) &= |\lambda|^{\frac{d-1}{2}} \tilde{r}'_{\frac{\lambda}{|\lambda|}}(|\lambda|^{\frac{1}{2}} y', |\lambda|^{\frac{1}{2}} y_d, |\lambda|^{\frac{1}{2}} z_d)\\
\tilde{r}_{d,\lambda}(y', y_d, z_d) &= |\lambda|^{\frac{d-1}{2}} \tilde{r}_{d,\frac{\lambda}{|\lambda|}}(|\lambda|^{\frac{1}{2}} y', |\lambda|^{\frac{1}{2}} y_d, |\lambda|^{\frac{1}{2}} z_d)\\
\tilde{q}_\lambda(y', y_d, z_d) &= |\lambda|^{\frac{d-1}{2}} \tilde{q}_{\frac{\lambda}{|\lambda|}}(|\lambda|^{\frac{1}{2}} y', |\lambda|^{\frac{1}{2}} y_d, |\lambda|^{\frac{1}{2}} z_d).
\end{aligned}$$

The above scaling allows to consider $|\lambda| = 1$ in the proof of the following kernel estimates which read as follows.

Lemma 8.2.10. *Let $\varepsilon > 0$ and $\lambda \in \mathrm{S}_{\pi-\varepsilon}$. Then there exists constants $c = c(d,\varepsilon) > 0$ and $C = C(d,\varepsilon) > 0$ depending only on d and ε such that for all $y' \in \mathbb{R}^{d-1}$, $y_d, z_d > 0$*

$$\begin{aligned}
&|\tilde{r}'_\lambda(y', y_d, z_d)| + |\tilde{r}_{d,\lambda}(y', y_d, z_d)|\\
&\quad \leq \frac{C}{(y_d + z_d + |y'|)^{d-1}} \frac{\mathrm{e}^{-c|\lambda|^{\frac{1}{2}} z_d}}{(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d + |y'|))(1 + |\lambda|^{\frac{1}{2}}(y_d + z_d))}
\end{aligned}$$

and

$$|\tilde{q}_\lambda(y', y_d, z_d)| \leq \frac{C\, \mathrm{e}^{-c|\lambda|^{\frac{1}{2}} z_d}}{(y_d + z_d + |y'|)^{d-1}}$$

hold.

Proof. We will only prove the estimate for $\tilde{r}'_\lambda$ as the proof for the two other terms follows the same strategy. Note that the kernels derived in Lemma 8.2.8 coincide with the ones of $\partial_{y_d} r'_\lambda$, $\partial_{y_d} r_{d,\lambda}$ and q_λ up to a factor lying in M_0, respectively. As mentioned before, here we will examine the proofs of [68, Proposition 3.5, Proposition 3.7] to show that the same estimates hold. First, recall the we defined $\omega_\lambda(\xi) := \sqrt{\lambda + |\xi|^2}$ and thus for all $\lambda \in \Sigma_{\pi-\varepsilon}$ we have the estimate

$$|\omega_\lambda(\xi)| \gtrsim |\lambda|^{\frac{1}{2}} + |\xi| \tag{8.7}$$

where the implicit constant depends only on ε. Moreover, the identity

$$\frac{1}{\omega_\lambda(\xi) - \xi} = \frac{\omega_\lambda(\xi) + \xi}{\lambda} \tag{8.8}$$

holds true. Hence, we obtain

$$\begin{aligned}\tilde{r}'_\lambda(y', y_d, z_d) = &\frac{1}{\lambda}\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi}\left(\frac{|\xi|^2}{\omega_\lambda(\xi)}\,\mathrm{e}^{-|\xi|y_d} - \omega_\lambda(\xi)\,\mathrm{e}^{-\omega_\lambda(\xi)y_d}\right)\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\frac{\xi\otimes\xi}{|\xi|}\,\mathrm{d}\xi\\ &+\frac{1}{\lambda}\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi}\left(\frac{|\xi|^2}{\omega_\lambda(\xi)}\,\mathrm{e}^{-|\xi|y_d} - \omega_\lambda(\xi)\,\mathrm{e}^{-\omega_\lambda(\xi)y_d}\right)\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\frac{\xi\otimes\xi}{\omega_\lambda(\xi)}\,\mathrm{d}\xi\end{aligned}$$

By scaling to $|\lambda| = 1$ and $\frac{1}{\omega_\lambda(\xi)} \lesssim \frac{1}{|\xi|}$ it is enough to estimate

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi}\left(\frac{|\xi|^2}{\omega_\lambda(\xi)}\,\mathrm{e}^{-|\xi|y_d} - \omega_\lambda(\xi)\,\mathrm{e}^{-\omega_\lambda(\xi)y_d}\right)\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\frac{\xi\otimes\xi}{|\xi|}\,\mathrm{d}\xi\\ &\quad=\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi}\,\frac{|\xi|}{\omega_\lambda(\xi)}|\xi|\left(\mathrm{e}^{-|\xi|y_d} - \mathrm{e}^{-\omega_\lambda(\xi)y_d}\right)\mathrm{e}^{-\omega_\lambda(\xi)z_d}\,\frac{\xi\otimes\xi}{|\xi|}\,\mathrm{d}\xi\\ &\quad\quad+\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}y'\cdot\xi}(|\xi| - \omega_\lambda(\xi))\left(1+\frac{|\xi|}{\omega_\lambda(\xi)}\right)\mathrm{e}^{-\omega_\lambda(\xi)(y_d+z_d)}\,\frac{\xi\otimes\xi}{|\xi|}\,\mathrm{d}\xi\\ &\quad\coloneqq \mathrm{I} + \mathrm{II}.\end{aligned} \tag{8.9}$$

We start with the second term which is slightly easier to estimate. Here, (8.7) will be crucial. Introduce a smooth cut-off $\varphi_2 \in \mathrm{C}^\infty(\mathbb{R}^{d-1})$ satisfying $\varphi_2 \equiv 1$ on $B(0,2)$ and $\varphi_2 \equiv 0$ on $\mathbb{R}^{d-1} \setminus B(0,3)$ to split II into $\mathrm{II}_{\mathrm{low}}$ and $\mathrm{II}_{\mathrm{high}}$ by multiplying the integrand by $\varphi_2 + (1-\varphi_2)$. Then, we have

$$\begin{aligned}|\mathrm{II}_{\mathrm{low}}| &\lesssim \int_{|\xi|\leq 3} \big||\xi| - \omega_\lambda(\xi)\big|\left|1+\frac{|\xi|}{\omega_\lambda(\xi)}\right|\mathrm{e}^{-\omega_\lambda(\xi)(y_d+z_d)}\,\mathrm{d}\xi\\ &\lesssim \mathrm{e}^{-2c(y_d+z_d)}\end{aligned} \tag{8.10}$$

for some c depending only on d and ε due to (8.7). For $|y'| \leq 1$ we then conclude

$$\begin{aligned}|\mathrm{II}_{\mathrm{low}}| &\lesssim \mathrm{e}^{-2c(y_d+z_d)}\,\frac{(y_d+z_d+|y'|)^{d-1}}{(y_d+z_d+|y'|)^{d-1}}\,\frac{(1+y_d+z_d+|y'|)(1+y_d+z_d)}{(1+y_d+z_d+|y'|)(1+y_d+z_d)}\\ &\leq \mathrm{e}^{-2c(y_d+z_d)}\,\frac{(y_d+z_d+1)^{d-1}}{(y_d+z_d+|y'|)^{d-1}}\,\frac{(1+y_d+z_d+1)(1+y_d+z_d)}{(1+y_d+z_d+|y'|)(1+y_d+z_d)}\\ &\lesssim \frac{1}{(y_d+z_d+|y'|)^{d-1}}\,\frac{\mathrm{e}^{-cz_d}}{(1+y_d+z_d+|y'|)(1+y_d+z_d)}\end{aligned}$$

which is the desired estimate. For $|y'| \geq 1$ we employ Lemma 8.2.9. For $\alpha \in \mathbb{N}$ satisfying $0 \leq \alpha \leq d+1$ and all $\xi \in \mathbb{R}^{d-1}$ we find

$$\begin{aligned}&\left|\nabla_\xi^\alpha\left\{(|\xi| - \omega_\lambda(\xi))\left(1+\frac{|\xi|}{\omega_\lambda(\xi)}\right)\varphi_2(\xi)\,\mathrm{e}^{-\omega_\lambda(\xi)(y_d+z_d)}\,\frac{\xi\otimes\xi}{|\xi|}\right\}\right|\\ &\qquad\qquad\lesssim \mathrm{e}^{-2c(y_d+z_d)}\,|\xi|^{-\alpha+1}\varphi_2(\xi)\end{aligned}$$

for some $c = c(d, \varepsilon)$. Then Lemma 8.2.9 implies

$$|\mathrm{II}_{\mathrm{low}}| \lesssim \frac{\mathrm{e}^{-c(y_d+z_d)}}{|y'|^d}.$$

Then the desired estimate for $\mathrm{II}_{\mathrm{low}}$ follows as before by expanding and using the decay of the exponential. For technical reasons we postpone the treatise of $\mathrm{II}_{\mathrm{high}}$ to the end of the proof. Coming back to (8.9) it remains to estimate I. Note that up to the factor $\frac{|\xi|^2}{\omega_\lambda(\xi)} \in M_1$, this coincides with the term on the right-hand side of (3.25) in [68]. Note that this term behaves slightly worse than II as there is no exponential decay in y_d.

First consider the case $y_d \geq 1$. Then, use the substitution $\eta = \xi y_d$ to obtain

$$\mathrm{I} = y_d^{-(d+1)} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}\tilde{y}'\cdot\eta} \frac{|\eta|^2}{\sqrt{\lambda y_d^2 + |\eta|^2}} \left(\mathrm{e}^{-|\eta|} - \mathrm{e}^{-\sqrt{\lambda y_d^2+|\eta|^2}}\right) \mathrm{e}^{-\omega_\lambda(\eta/y_d)z_d} \frac{\eta\otimes\eta}{|\eta|} \,\mathrm{d}\eta$$

where $\tilde{y}' = \frac{y'}{y_d}$. We claim that

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}\tilde{y}'\cdot\eta} \frac{|\eta|^2}{\sqrt{\lambda y_d^2 + |\eta|^2}} \left(\mathrm{e}^{-|\eta|} - \mathrm{e}^{-\sqrt{\lambda y_d^2+|\eta|^2}}\right) \mathrm{e}^{-\omega_\lambda(\eta/y_d)z_d} \frac{\eta\otimes\eta}{|\eta|} \,\mathrm{d}\eta \\ \lesssim \frac{\mathrm{e}^{-cz_d}}{(1+|\tilde{y}'|)^{d+1}} \end{aligned} \tag{8.11}$$

holds true. If this is the case, the desired estimate follows in light of

$$\begin{aligned} |\mathrm{I}| &\lesssim \frac{\mathrm{e}^{-cz_d}}{(y_d+|y'|)^d y_d} \frac{(y_d+z_d+|y'|)^{d-1}}{(y_d+z_d+|y'|)^{d-1}} \frac{(1+y_d+z_d+|y'|)(1+y_d+z_d)}{(1+y_d+z_d+|y'|)(1+y_d+z_d)} \\ &\lesssim \frac{1}{(y_d+z_d+|y'|)^{d-1}} \frac{\mathrm{e}^{-cz_d}}{(1+y_d+z_d+|y'|)(1+y_d+z_d)} \end{aligned}$$

as $y_d \geq 1$. To establish (8.11) we distinguish $|\tilde{y}'| \geq 1$ and $|\tilde{y}'| \leq 1$. With a calculation similar to (8.10) we find

$$\begin{aligned} \Big| \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}\tilde{y}'\cdot\eta} \frac{|\eta|^2}{\sqrt{\lambda y_d^2 + |\eta|^2}} \left(\mathrm{e}^{-|\eta|} - \mathrm{e}^{-\sqrt{\lambda y_d^2+|\eta|^2}}\right) \mathrm{e}^{-\omega_\lambda(\eta/y_d)z_d} \frac{\eta\otimes\eta}{|\eta|} \,\mathrm{d}\eta \Big| \\ \lesssim \int_{\mathbb{R}^{d-1}} |\eta| \left(\mathrm{e}^{-|\eta|} - \mathrm{e}^{-c|\eta|}\right) \mathrm{e}^{-cz_d} |\eta| \mathrm{d}\eta \\ \lesssim \mathrm{e}^{-cz_d} \end{aligned}$$

due to (8.7) and $y_d \geq 1$. This concludes the proof of (8.11) for $|\tilde{y}|' \leq 1$. For $|\tilde{y}'| \geq 1$ we again aim to apply Lemma 8.2.9. To this end, we note

$$\left| \nabla_\eta^\alpha \left\{ \frac{|\eta|^2}{\sqrt{\lambda y_d^2 + |\eta|^2}} (\mathrm{e}^{-|\eta|} - \mathrm{e}^{-\sqrt{\lambda y_d^2+|\eta|^2}})\, \mathrm{e}^{-\omega_\lambda(\frac{\eta}{y_d})z_d} \frac{\eta\otimes\eta}{|\eta|} \right\} \right| \lesssim \mathrm{e}^{-cz_d}\, \mathrm{e}^{-c_0|\eta|}\, |\eta|^{-\alpha+2}$$

for all $0 \leq \alpha \leq d+2$. Here, (8.7) and $y_d \geq 1$ are crucial to deal with (derivatives of) $\sqrt{\lambda y_d^2 + |\eta|^2}$. This concludes the proof of (8.11) for $|\tilde{y}'| \geq 1$ and thus in case of $y_d \geq 1$.

For $0 \leq y_d \leq 1$ a further distinction is needed. As in [68] introduce a smooth cut-off with respect to ξ. As the case of low frequencies is dealt with by adapting the proof of Maekawa, Miura and Prange in the same manner as before we do not present it here. For high frequencies however, a new strategy involving integration by parts is used in [68]. Hence, we present this argument once. The remainder of the proof follows the same spirit and therefore will be omitted.

For working with high frequencies, we assume $|\xi| \geq 2$ from now on and denote the term to be estimated by I_{high}. Note that it holds

$$\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-\omega_\lambda(\xi) y_d} = \Big(1 - \mathrm{e}^{(|\xi| - \omega_\lambda(\xi))\, y_d)}\Big)\, \mathrm{e}^{-|\xi| y_d}\,.$$

Moreover, as derived in [68], we find

$$|\xi| - \omega_\lambda(\xi) = -\frac{\lambda}{2|\xi|} + \mathcal{O}\left(\frac{\lambda^2}{|\xi|^3}\right)$$

for $|\xi| \geq 2$ and $|\lambda| = 1$. Consequently, we obtain

$$\left|\nabla_\xi^\alpha \left\{ \frac{|\xi|}{\omega_\lambda(\xi)} |\xi| \left(\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-\omega_\lambda(\xi) y_d}\right) \mathrm{e}^{-\omega_\lambda(\xi) z_d} \, \frac{\xi \otimes \xi}{|\xi|} \right\}\right| \lesssim \mathrm{e}^{-c z_d}\, \mathrm{e}^{-c|\xi|(y_d + z_d)}\, |\xi|^{-\alpha+1}$$

for all $|\xi| \geq 2$, $0 \leq y_d \leq 1$ and $0 \leq \alpha \leq d+1$. For the final part a further case distinction between $|y'| \geq \frac{1}{4}$ and $|y'| \leq \frac{1}{4}$ is needed. For $|y'| \geq \frac{1}{4}$ an integration by parts argument together with the estimates on the derivatives of order $d+1$ implies for $j = 1, \ldots, d-1$

$$\begin{aligned}\left|(-\mathrm{i} y_j)^{d+1} \mathrm{I}_{\text{high}}\right| &= \left|\int_{|\xi| \geq 2} \mathrm{e}^{\mathrm{i} y' \cdot \xi}\, \partial_\xi^{d+2} \left(\frac{|\xi|}{\omega_\lambda(\xi)} |\xi| \left(\mathrm{e}^{-|\xi| y_d} - \mathrm{e}^{-\omega_\lambda(\xi) y_d}\right) \mathrm{e}^{-\omega_\lambda(\xi) z_d}\, \frac{\xi \otimes \xi}{|\xi|} \right) \mathrm{d}\xi \right| \\ &\lesssim \mathrm{e}^{-c z_d} \int_{|\xi| \geq 2} |\xi|^{-d}\, \mathrm{d}\xi \lesssim \mathrm{e}^{-c z_d}\,.\end{aligned}$$

With this at hand the desired estimate follows promptly in light of

$$\begin{aligned}\frac{\mathrm{e}^{-c z_d}}{|y'|^{d+2}} &\lesssim \frac{\mathrm{e}^{-c z_d}}{|y'|^d} \frac{(y_d + z_d + |y'|)^{d-1}}{(y_d + z_d + |y'|)^{d-1}} \frac{(1 + y_d + z_d + |y'|)(1 + y_d + z_d)}{(1 + y_d + z_d + |y'|)(1 + y_d + z_d)} \\ &\lesssim \frac{1}{(y_d + z_d + |y'|)^{d-1}} \frac{\mathrm{e}^{-c z_d}}{(1 + y_d + z_d + |y'|)(1 + y_d + z_d)}.\end{aligned}$$

Note that here both $|y'| \geq \frac{1}{4}$ and $0 \leq y_d \leq 1$ are crucial. In order to cover the case $|y'| \leq \frac{1}{4}$ a final distinction is necessary. All the necessary ingredients were already covered and we refer to the proof of [68, Proposition 3.5] together with suitable adaptations that we presented in this proof.

It remains to cover the term $\mathrm{II}_{\mathrm{high}}$. In fact, the argumentation for $\mathrm{I}_{\mathrm{high}}$ applies in this case as well. Note that due to the additional exponential decay in y_d these estimates work uniformly in $y_d \in \mathbb{R}_+$ and so no distinction between the cases $y_d \geq 1$ and $0 < y_d < 1$ is necessary. □

With these kernel estimates at hand we can finally prove an analogous relation of (8.6) in our setting. As an immediate consequence we also obtain a (weighted) L^2-estimate that we present in the following result as well.

Theorem 8.2.11. *Let $\beta \in (-\infty, 1)$. For $f \in \mathrm{C}_c^\infty((0,\infty) \times \mathbb{R}^d_+)$, the representation*

$$A \int_{t/2}^t E(t-s) f(s) \,\mathrm{d}s = A \int_0^t E(t-s) f(s) \,\mathrm{d}s - E^u(t/2) A \int_0^{t/2} E(t/2 - s) f(s) \,\mathrm{d}s$$

is valid. Moreover, for $f \in \mathrm{L}^2((0,\infty) \times \mathbb{R}^d_+, t^{\beta - \frac{d}{2}} \,\mathrm{d}t \,\mathrm{d}x)$ we have

$$\Big\| A \int_{t/2}^t E(t-s) f(s, \cdot) \,\mathrm{d}s \Big\|_{\mathrm{L}^2(t^{\beta - \frac{d}{2}} \,\mathrm{d}t \,\mathrm{d}x)} \lesssim \|f\|_{\mathrm{L}^2(t^{\beta - \frac{d}{2}} \,\mathrm{d}t \,\mathrm{d}x)} \tag{8.12}$$

and the implicit constant depends only on β and d.

Proof. Due to density it suffices to consider $f \in \mathrm{C}_\mathrm{c}^\infty((0,\infty) \times \mathbb{R}^d_+)$. In view of Lemma 8.2.10 and Proposition 7.1.10 we deduce that the integral

$$\int_{t/2}^t A E(t-s) f(s, \cdot) \,\mathrm{d}s$$

is convergent for all $f \in \mathrm{C}_\mathrm{c}^\infty((0,\infty) \times \mathbb{R}^d_+)$. We then claim that

$$A E(t-s) f(s, \cdot) = E^u \left(\frac{t}{2} \right) A E(t/2 - s) f(s, \cdot) \tag{8.13}$$

is true for all $0 < s < t/2$ and prove this via the Cauchy integral. Define $\gamma \coloneqq \gamma_1$ as introduced in Definition 6.3.1. Using Proposition 6.2.5 we have

$$\begin{aligned} A(E^u(t-s)f(s), E^\phi(t-s)f(s)) &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{-(t-s)\lambda} \Big(\nabla R^\phi(\lambda) f(s) - \Delta R^u(\lambda) f(s) \Big) \,\mathrm{d}\lambda \\ &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{-(t-s)\lambda} \left(f(s) - \lambda R^u(\lambda) f(s) \right) \,\mathrm{d}\lambda \\ &= -\frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{-(t-s)\lambda} \lambda R^u(\lambda) f(s) \,\mathrm{d}\lambda \end{aligned}$$

as the first integral vanishes due to holomorphicity. Now, we consider the right-hand side of (8.13). Further set $\gamma' \coloneqq \gamma + c$ for some $c > 0$, i.e., γ' is a right-shift of γ. With this notation use the previous calculation together with Proposition 6.2.6 to obtain

$$\begin{aligned}
E^u\left(\frac{t}{2}\right) AE(t/2-s)f(s) &= -\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{1}{2\pi\mathrm{i}}\int_{\gamma'} \mathrm{e}^{-\frac{t}{2}\lambda-(\frac{t}{2}-s)\mu}\, \mu R^u(\lambda)R^u(\mu)f(s)\,\mathrm{d}\mu\,\mathrm{d}\lambda \\
&= \frac{1}{2\pi\mathrm{i}}\int_{\gamma'} \frac{1}{2\pi\mathrm{i}}\int_{\gamma} \mathrm{e}^{-\frac{t}{2}\lambda-(\frac{t}{2}-s)\mu}\, \frac{\mu}{\lambda-\mu}R^u(\lambda)f(s)\,\mathrm{d}\lambda\,\mathrm{d}\mu \\
&\quad - \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{1}{2\pi\mathrm{i}}\int_{\gamma'} \mathrm{e}^{-\frac{t}{2}\lambda-(\frac{t}{2}-s)\mu}\, \frac{\mu}{\lambda-\mu}R^u(\mu)f(s)\,\mathrm{d}\mu\,\mathrm{d}\lambda. \\
&= -\frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{e}^{-(t-s)\lambda}\,\lambda R^u(\lambda)f(s)\,\mathrm{d}\lambda.
\end{aligned}$$

In the last step, the first summand vanished by holomorphicity of the integrand while for the second we applied Cauchy's integral formula. Here it is crucial that γ is on the left of γ'. Finally, (8.13) implies

$$A\int_{t/2}^t E(t-s)f(s)\,\mathrm{d}s = A\int_0^t E(t-s)f(s)\,\mathrm{d}s - E^u(t/2)A\int_0^{t/2} E(t/2-s)f(s)\,\mathrm{d}s$$

and this is the desired representation formula. For the L^2-estimate note that the right-hand side is bounded on $\mathrm{L}^2(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$ by the boundedness of $T_{2,2}$ on $\mathrm{L}^2(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$ (see Proposition 7.1.15) and the uniform boundedness of $(E^u(t))_{t>0}$ on $\mathrm{L}^2(\mathbb{R}^d_+)$ (see Theorem 6.3.2). □

Proof of Theorem 8.2.1 for $p < 2$

Finally, we are in the position the prove the remaining part of Theorem 8.2.1, i.e., the case $p < 2$.

Theorem 8.2.12. *Let $\beta \in (-\infty, 1)$ and $\frac{2d}{d+1} < p < 2$ satisfy $p > \frac{2d}{d+2(1-\beta)}$. Then, the restriction of $T_{2,2}$ to $\mathrm{C}_c^\infty((0,\infty)\times\mathbb{R}^d_+)$ is bounded with respect to the $\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$-norm and thus extends to a bounded operator on $\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$. The operator norm depends on the operator of $T_{2,2}$, β, p and d.*

Proof. Let $\beta \in (-\infty, 1)$ and $\frac{2d}{d+1} < p < 2$ satisfy $p > \frac{2d}{d+2(1-\beta)}$. Moreover, let $f \in \mathrm{C}_c^\infty((0,\infty)\times\mathbb{R}^d_+)$. Again, it will be more convenient to work with cubes instead of balls in the definition of the tent space norm. For $x \in \mathbb{R}^d_+$ and $t > 0$ let $Q \coloneqq Q(x,\sqrt{t})$ denote the cube with center x and side length $2\sqrt{t}$. Moreover, denote $2Q \coloneqq Q(x, 2\sqrt{t})$. Additionally, for $\ell \geq 1$ set $C_\ell(x,t) \coloneqq (Q(x, 2^{\ell+1}\sqrt{t}) \setminus$

$Q(x, 2^\ell\sqrt{t})) \cap \mathbb{R}^d_+$. In the spirit of the decomposition (8.4) in the proof of Theorem 8.2.3 decompose

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d_+}\left(\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}|A\int_0^t [E(t-s)f(s,\cdot)](y)\,\mathrm{d}s|^2\,\mathrm{d}y\ t^\beta \mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}}\\
&\leq \sum_{k=1}^\infty \mathrm{I}_k + \sum_{\ell=1}^\infty\sum_{k=1}^\infty \mathrm{II}_{k,\ell} + \sum_{\ell=0}^\infty \mathrm{III}_\ell,
\end{aligned}$$

where for $k \geq 1$

$$\mathrm{I}_k := \left(\int_{\mathbb{R}^d_+}\left(\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{2^{-k-1}t}^{2^{-k}t} [E(t-s)\chi_{2Q\cap\mathbb{R}^d_+}f(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\, t^\beta \mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}},$$

for $k, \ell \geq 1$

$$\mathrm{II}_{k,\ell} := \left(\int_{\mathbb{R}^d_+}\left(\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{2^{-k-1}t}^{2^{-k}t} [E(t-s)\chi_{C_\ell}f(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\, t^\beta \mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}},$$

and finally for $l \geq 0$

$$\mathrm{III}_\ell := \left(\int_{\mathbb{R}^d_+}\left(\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{\frac{t}{2}}^{t} [E(t-s)\chi_{C_\ell}f(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\, t^\beta \mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}}$$

with the convention $C_0 := 2Q\cap\mathbb{R}^d_+$. We start with the second sum and first estimate the inner integral. For each $k, \ell \in \mathbb{N}$ by means of Minkowski's integral inequality and the off-diagonal estimates from Proposition 7.2.2 we find

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{2^{-k-1}t}^{2^{-k}t} [E(t-s)\chi_{C_\ell}f(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\, t^\beta \mathrm{d}t\\
&\leq \int_0^\infty t^{-\frac{d}{2}}\left(\int_{2^{-k-1}t}^{2^{-k}t}\left(\int_{\mathbb{R}^d_+\cap Q}|[AE(t-s)\chi_{C_\ell}f(s,\cdot)](y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}s\right)^2 t^\beta\mathrm{d}t\\
&\lesssim \int_0^\infty t^{-\frac{d}{2}}\left(\int_{2^{-k-1}t}^{2^{-k}t} 2^{-\ell(d+1)}t^{-\frac{1}{4}}(t-s)^{-\frac{3}{4}}\|f(s,\cdot)\|_{\mathrm{L}^2(C_\ell)}\,\mathrm{d}s\right)^2 t^\beta\mathrm{d}t\\
&\leq 2^{-\ell(d+1)}\int_0^\infty t^{-\frac{d}{2}+\beta-\frac{1}{2}}\int_{2^{-k-1}t}^{2^{-k}t}(t-s)^{-\frac{3}{4}}\|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell)}\,\mathrm{d}s\left(\int_{2^{-k-1}t}^{2^{-k}t}(t-s)^{-\frac{3}{4}}\,\mathrm{d}s\right)\mathrm{d}t
\end{aligned}$$

where in the last step we used Hölder's inequality in the s-variable. Estimate, as in (8.1),

$$\int_{2^{-k-1}t}^{2^{-k}t}(t-s)^{-\frac{3}{4}}\,\mathrm{d}s \lesssim 2^{-k}t^{\frac{1}{4}}$$

to obtain

$$\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{2^{-k-1}t}^{2^{-k}t}[E(t-s)\chi_{C_\ell}f(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\,t^\beta\mathrm{d}t$$
$$\lesssim 2^{-k}2^{-\ell(d+1)}\int_0^\infty t^{-\frac{d}{2}-\frac{1}{4}+\beta}\int_{2^{-k-1}t}^{2^{-k}t}(t-s)^{-\frac{3}{4}}\|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell)}\,\mathrm{d}s\,\mathrm{d}t.$$

Next, $s\in(2^{-k-1}t,2^{-k}t)$ implies $t\leq 2^{k+1}s$ so that $C_\ell\subseteq Q(x,2^{l+1}2^{\frac{k+1}{2}}\sqrt{s})\cap\mathbb{R}^d_+$ follows. Hence, applying Fubini's theorem we find

$$\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{2^{-k-1}t}^{2^{-k}t}[E(t-s)\chi_{C_\ell}F(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\,t^\beta\mathrm{d}t$$
$$\lesssim 2^{-k}2^{-\ell(d+1)}\int_0^\infty t^{-\frac{d}{2}-\frac{1}{4}+\beta}\int_{2^{-k-1}t}^{2^{-k}t}(t-s)^{-\frac{3}{4}}\|f(s,\cdot)\chi_{Q(x,2^{l+1}2^{\frac{k+1}{2}}\sqrt{s})\cap\mathbb{R}^d_+}\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)}\,\mathrm{d}s\,\mathrm{d}t$$
$$= 2^{-k}2^{-\ell(d+1)}\int_0^\infty \int_{2^k s}^{2^{k+1}s} t^{-\frac{d}{2}-\frac{1}{4}+\beta}(t-s)^{-\frac{3}{4}}\,\mathrm{d}t\|f(s,\cdot)\chi_{Q(x,2^{l+1}2^{\frac{k+1}{2}}\sqrt{s})\cap\mathbb{R}^d_+}\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)}\,\mathrm{d}s.$$

Finally, note that $-\frac{d}{2}-\frac{1}{4}+\beta<0$ as $d\geq 3$ and $\beta<1$ so that the estimate

$$\int_{2^k s}^{2^{k+1}s} t^{-\frac{d}{2}-\frac{1}{4}+\beta}(t-s)^{-\frac{3}{4}}\,\mathrm{d}t\lesssim 2^k s(2^k s)^{-\frac{d}{2}-\frac{1}{4}+\beta}(2^k s)^{-\frac{3}{4}}=(2^k s)^{-\frac{d}{2}+\beta}$$

holds. Thus, we obtain

$$\int_0^\infty \int_{\mathbb{R}^d_+\cap Q} t^{-\frac{d}{2}}\Big|A\int_{2^{-k-1}t}^{2^{-k}t}[E(t-s)\chi_{C_\ell}f(s,\cdot)](y)\,\mathrm{d}s\Big|^2\,\mathrm{d}y\,t^\beta\mathrm{d}t$$
$$\lesssim 2^{-k}2^{-\ell(d+1)}2^{-k(\frac{d}{2}-\beta)}\int_0^\infty s^{-\frac{d}{2}+\beta}\|f(s,\cdot)\chi_{Q(x,2^{l+1}2^{\frac{k+1}{2}}\sqrt{s})\cap\mathbb{R}^d_+}\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)}\,\mathrm{d}s.$$

In order to recover $\mathrm{II}_{k,\ell}$ it remains to take the square root and the $\mathrm{L}^p(\mathbb{R}^d_+)$-norm with respect to x. Then, the previous estimates together with the change of aperture formula from Theorem 2.2.6 with $\tau=p$ as $p<2$, imply

$$|\mathrm{II}_{k,\ell}|\lesssim 2^{-\ell\frac{d+1}{2}}2^{-k(\frac{1}{2}+\frac{d}{4}-\frac{\beta}{2})}\Big(\int_{\mathbb{R}^d_+}\Big(\int_0^\infty s^{-\frac{d}{2}+\beta}\|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(x,2^{l+1}2^{\frac{k+1}{2}}\sqrt{s})\cap\mathbb{R}^d_+)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\lesssim 2^{-\ell\frac{d+1}{2}}2^{-k(\frac{1}{2}+\frac{d}{4}-\frac{\beta}{2})}(\ell+1)\frac{k+1}{2}(2^{\ell+1}2^{\frac{k+1}{2}})^{\frac{d}{p}}\|f\|_{\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}.$$

Here, we used $\log(x)\leq x$ for $x>0$. Summing over k and ℓ, the resulting series converges if the conditions $\frac{1}{2}+\frac{d}{4}-\frac{\beta}{2}-\frac{d}{2p}>0$ and $\frac{d+1}{2}-\frac{d}{p}>0$ are satisfied. The first condition is equivalent to $p>\frac{2d}{d+2-2\beta}$ and the second condition holds if $p>\frac{2d}{d+1}$. Note that both are satisfied by assumption on p.

It remains to check the other summands. For I_k we need the L^2-boundedness of the family $tAE(t)$ which was obtained in Proposition 7.1.9. Indeed, for fixed $k \in \mathbb{N}$ and $x \in \mathbb{R}^d_+$ we have

$$
\begin{aligned}
&\int_0^\infty \int_{Q\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} |A \int_{2^{-k-1}t}^{2^{-k}t} [E(t-s)\chi_{Q\cap\mathbb{R}^d_+} f(s,\cdot)](y)\,\mathrm{d}s|^2 \,\mathrm{d}y\, t^\beta \,\mathrm{d}t \\
&\leq \int_0^\infty \int_{Q\cap\mathbb{R}^d_+} t^{-\frac{d}{2}} \left(\int_{2^{-k-1}t}^{2^{-k}t} |[(t-s)AE(t-s)\chi_{Q\cap\mathbb{R}^d_+} f(s,\cdot)](y)| \frac{\mathrm{d}s}{t-s} \right)^2 \mathrm{d}y\, t^\beta \,\mathrm{d}t \\
&\leq \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} t \left(\int_{Q\cap\mathbb{R}^d_+} |(t-s)AE(t-s)\chi_{Q\cap\mathbb{R}^d_+} f(s,\cdot)(y)|^2 \,\mathrm{d}y \right) t^{\beta-\frac{d}{2}-2} \,\mathrm{d}s \,\mathrm{d}t \\
&\leq \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} \|\chi_{Q(x,\sqrt{t})\cap\mathbb{R}^d_+} F(s,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)} t^{\beta-\frac{d}{2}-1} \,\mathrm{d}s \,\mathrm{d}t \\
&\leq 2^{-k} \int_0^\infty \int_{2^k s}^{2^{k+1}s} t^{\beta-\frac{d}{2}-1} \,\mathrm{d}t \|\chi_{Q(x,2^{\frac{k}{2}}\sqrt{s})\cap\mathbb{R}^d_+} f(s,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)} \,\mathrm{d}s \\
&\leq 2^{-k(\frac{d}{2}+1-\beta)} \int_0^\infty \|\chi_{Q(x,2^{\frac{k}{2}}\sqrt{s})\cap\mathbb{R}^d_+} f(s,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d_+)} s^{\beta-\frac{d}{2}} \,\mathrm{d}s
\end{aligned}
$$

where we used $t-s \sim t$ and Fubini's theorem. After an integration with respect to x we apply the change of aperture formula as for $\mathrm{II}_{k,\ell}$ and obtain

$$
|\mathrm{I}_k| \lesssim k 2^{-k(\frac{1}{2}(\frac{d}{2}+1-\beta)-\frac{d}{2p})} \|f\|_{\mathrm{T}^{p,2}(t^\beta \,\mathrm{d}t \,\mathrm{d}x)}.
$$

Summing over k requires the same condition as for $\mathrm{II}_{k,\ell}$. Next, we consider III_ℓ for $\ell \geq 1$. The case $\ell = 0$ will be dealt with afterwards. After applying the off-diagonal estimates we have

$$
\begin{aligned}
&\int_0^\infty t^{-\frac{d}{2}} \left(\int_{t/2}^t 2^{-\ell\frac{d+1}{2}} t^{-\frac{1}{4}} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|_{\mathrm{L}^2(C_\ell(t))} \,\mathrm{d}s \right)^2 t^\beta \,\mathrm{d}t \\
&\lesssim 2^{-\ell(d+1)} \int_0^\infty t^{-\frac{d}{2}-\frac{1}{2}+\beta} \int_{t/2}^t (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell(t))} \,\mathrm{d}s \left(\int_{t/2}^t (t-s)^{-\frac{3}{4}} \,\mathrm{d}s \right) \mathrm{d}t \\
&= 2^{-\ell(d+1)} \int_0^\infty \int_s^{2s} t^{-\frac{d}{2}-\frac{1}{4}+\beta} (t-s)^{-\frac{3}{4}} \|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell(t))} \,\mathrm{d}t \,\mathrm{d}s \\
&\lesssim 2^{-\ell(d+1)} \int_0^\infty s^{-\frac{d}{2}-\frac{1}{4}+\beta} \|f(s,\cdot)\|^2_{\mathrm{L}^2(C_\ell(2s))} \int_s^{2s} (t-s)^{-\frac{3}{4}} \,\mathrm{d}t \,\mathrm{d}s \\
&\lesssim 2^{-\ell(d+1)} \int_0^\infty s^{-\frac{d}{2}+\beta} \|f(s,\cdot)\|^2_{\mathrm{L}^2(Q(x,2^{\ell+1}\sqrt{2}\sqrt{s})\cap\mathbb{R}^d_+)} \,\mathrm{d}s.
\end{aligned}
$$

Integrating this over x and using the change of aperture formula from Theorem 2.2.6 yields

$$\begin{aligned}|\mathrm{III}_\ell| &\lesssim 2^{-\ell\frac{(d+1)}{2}}\left(\int_{\mathbb{R}^d_+}\left(\int_0^\infty s^{-\frac{d}{2}+\beta}\|f(s,\cdot)\|^2_{\mathrm{L}^2(B(x,2^{\ell+1}\sqrt{2}\sqrt{s}))}\,\mathrm{d}s\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{2}}\\ &\lesssim 2^{-\ell\frac{(d+1)}{2}}\|f\|_{\mathrm{T}^{p,2}_{2^{\ell+1}\sqrt{2}}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}\\ &\lesssim 2^{-\ell\frac{(d+1)}{2}}(l+1)(2^{\ell+1})^{\frac{d}{p}}\|f\|_{\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)}.\end{aligned}$$

Again, this is summable provided that $p > \frac{2d}{d+1}$. Finally, we consider the case $\ell = 0$. Enlarging the domain of integration, we find

$$\mathrm{III}_0 \leq \left(\int_{\mathbb{R}^d_+}\mathrm{III}_0(x)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}} \tag{8.14}$$

where

$$\mathrm{III}_0(x) \coloneqq \int_0^\infty\int_{\mathbb{R}^d_+} t^{-\frac{d}{2}}\left|A\int_{t/2}^t[E(t-s)\chi_{Q(x,2\sqrt{t})\cap\mathbb{R}^d_+}f(s,\cdot)](y)\,\mathrm{d}s\right|^2\mathrm{d}y\,t^\beta\,\mathrm{d}t.$$

Estimate $\mathrm{III}_0(x)$ by Theorem 8.2.11 and plug the result into (8.14) to obtain

$$\mathrm{III}_0 \lesssim \left(\int_{\mathbb{R}^d_+}\left(\int_0^\infty\int_{Q(x,2\sqrt{t})\cap\mathbb{R}^d_+} t^{-\frac{d}{2}}|f(t,y)|^2\mathrm{d}y\,t^\beta\,\mathrm{d}t\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{1}{p}}.$$

To conclude the proof, recover the $\mathrm{T}^{p,2}(t^\beta\,\mathrm{d}t\,\mathrm{d}x)$-norm by changing the aperture by Theorem 2.2.6. □

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Curriculum Vitæ

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