Lang-Vojta's conjecture for the moduli of Fano threefolds of Picard rank 1, index 1 and degree 4

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Abstract

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We verify Lang-Vojta's conjecture for the moduli of Fano threefolds of Picard rank 1, index 1 and degree 4. This leads us to studying the infinitesimal Torelli problem for quasi-smooth weighted complete intersections. We give a proof for the fact that the infinitesimal Torelli map can be described as a multiplication in the associated Jacobi ring. We also study the geometry of the moduli stack and show that it is stratified via the two types of such Fano threefolds given by Iskovskikh's classification. Furthermore, we work on the persistence conjecture. We generalize a criterion that says that geometric hyperbolicity implies the persistence of arithmetic hyperbolicity to the case of algebraic stacks.

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Contents

Chapter 1

Introduction

Lang-Vojta's conjectures suggest that there are equivalences of certain phenomena in complex and arithmetic geometry. In particular, it is conjectured that Brody hyperbolicity (the non-existence of non-constant entire maps) is equivalent to arithmetic hyperbolicity (finiteness of integral points, i.e., points valued in a finitely generated domain of characteristic zero); see [36, 50].

Before we formulate the conjecture, we compare the notions for the following examples of varieties. We start with the affine line $\mathbb{A}^1$. Of course, there are infinitely many integral points on $\mathbb{A}^1$, and there are non-constant holomorphic maps $\mathbb{C} \to \mathbb{C}$.

Now we remove a point. The map $x \mapsto (x, 1/x)$ defines an isomorphism of the variety $\mathbb{A}^1 \setminus \{0\}$ with the vanishing set $V(xy-1) \subseteq \mathbb{A}^2$. Therefore, the integral points of $\mathbb{A}^1 \setminus \{0\}$ valued in a finitely generated domain R are in 1-1 correspondence with the units $R^\times$. Hence, for example, the variety $\mathbb{A}^1 \setminus \{0\}$ has infinitely many integral points valued in $\mathbb{Z}[1/2]$. On the other hand, there are non-constant entire maps. For example, the exponential map $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is non-constant and holomorphic.

When we remove two points, the situation changes. The variety $\mathbb{A}^1 \setminus \{0,1\}$ is isomorphic to the vanishing set $V(x(x-1)y-1) \subseteq \mathbb{A}^2$. Hence, the integral points of $\mathbb{A}^1 \setminus \{0,1\}$ valued in a finitely generated domain R are in 1-1 correspondence with elements $x \in R$ such that $x(= x-0)$ and $x-1$ are units. Hence, we get a correspondence with the solutions of the unit equation

$$x + y = 1, \quad x, y \in R^\times.$$

By a famous theorem of Siegel [70], given any finitely generated domain $R \subseteq \overline{\mathbb{Q}}$, this equation has only finitely many solutions in R. On the other hand, all holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ are constant. This is known as Picard's Little Theorem.

Next, we study the conjecture for a smooth projective curve X of genus g. The curve X is Brody hyperbolic if and only if its universal cover is Brody hyperbolic. If $g = 0$, then X is $\mathbb{P}^1$. If $g = 1$, then X is a torus, and the universal cover is $\mathbb{C}$. Hence, if $g \leq 1$, then there are non-constant entire maps. If $g \geq 2$, then by the uniformization theorem, the universal cover is the upper half plane, which is biholomorphic to the unit disc. Hence, all entire maps are constant by Liouville's theorem. On the other hand, if $g = 0$, then there are infinitely many points with integral coordinates. If $g = 1$, then X can be given the structure of an elliptic curve, and we can use the group structure to produce infinitely many integral points from just one integral non-torsion point. If $g \geq 2$, then there are only finitely many integral points. This is a theorem by Faltings [19], formerly known as the Mordell conjecture.

The $g = 1$ case has a natural generalization to higher dimensions. Namely, if A is an abelian variety of dimension $d > 0$, then the analytification $A_{\mathbb{C}}^{\mathrm{an}}$ is a complex torus of dimension d. Hence, it is uniformized by $\mathbb{C}^d$. So it is not Brody hyperbolic. On the other hand, with a similar argument as in the case of elliptic curves, we see that A is not arithmetically hyperbolic.

We collect the evidence in the following table.

Variety	Brody hyperbolic	Arithmetically hyperbolic
$\mathbb{A}^1$	no	no
$\mathbb{A}^1 \setminus \{0\}$	no	no
$\mathbb{A}^1 \setminus \{0,1\}$	yes	yes
curve of genus $g \in \{0,1\}$	no	no
curve of genus $g \geq 2$	yes	yes
abelian variety	no	no

One way of producing examples for Lang-Vojta's conjectures in higher dimensions is to consider closed subvarieties of abelian varieties. Let A be an abelian variety, and let $X \subseteq A$ be a closed subvariety. Clearly if X contains the translate of a positive dimensional abelian subvariety of A, then X is neither arithmetically hyperbolic nor Brody hyperbolic. Interestingly, this is an equivalence not just an implication. For Brody hyperbolicity, this is a theorem of Bloch–Ochiai–Kawamata [43]. For Arithmetic hyperbolicity, this is a theorem of Faltings [21]. Hence, the conjecture is verified for closed subvarieties of abelian varieties.

For more evidence on Lang-Vojta's conjectures, see [5, 6, 14, 22, 34, 54, 76]. In this work, we study Lang-Vojta's conjecture for the moduli of (smooth) Fano threefolds of Picard rank 1, index 1 and degree 4, or shortly type (1,1,4); see Definition 4.1.1 and Definition 4.1.4. To formulate the conjecture, we first specify what the notions of hyperbolicity mean for stacks.

Definition (Arithmetic hyperbolicity)**.** If X is a stack and S is a scheme, then by $\pi_0(X(S))$ we denote the set of isomorphism classes of objects of X over S. Let k be an algebraically closed field, and let X be a finitely presented algebraic stack over k. If $A \subseteq k$ is a subring, then a *model for X over A* is a stack $\mathcal{X}$ over A such that $\mathcal{X}_k \cong X$. Following [39], we say that X is *arithmetically hyperbolic over k* if there is a $\mathbb{Z}$-finitely generated subring $A \subseteq k$ and a (finitely presented) model $\mathcal{X}$ for X over A such that, for any $\mathbb{Z}$-finitely generated subring $A' \subseteq k$ containing A, the set

$$\operatorname{im}\left(\pi_0(\mathcal{X}(A')) \to \pi_0(\mathcal{X}(k))\right)$$

is finite. Following [37], we say that X is *absolutely arithmetically hyperbolic* if X_L is arithmetically hyperbolic over L for every algebraically closed field extension $L \supseteq k$.

Definition (Brody hyperbolicity)**.** Let X be a finite type integral separated Deligne-Mumford stack with quasi-projective coarse space over $\mathbb{C}$. We say X is *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X^{\text{an}}$ is *isotrivial*, i.e., the composition

$$\mathbb{C} \to X^{\text{an}} \to X^{\text{an,coarse}}$$

is constant.

Lang-Vojta's conjecture says that a Brody hyperbolic quasi-projective variety should be arithmetically hyperbolic. The following conjecture is a stacky generalization of this.

Conjecture (Stacky Lang-Vojta's conjecture)**.** Let X be a finite type integral separated Deligne-Mumford stack with quasi-projective coarse space over $\mathbb{C}$. The stack X is Brody hyperbolic if and only if X is arithmetically hyperbolic over $\mathbb{C}$.

Let us test this conjecture for the moduli stack $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g over $\mathbb{C}$; see Definition 5.2.2. Every element of the Siegel upper-half space

$$\Omega \in \mathbb{H}_g := \left\{ \mathbb{C}^{g\times g} \middle| \Omega^T = \Omega, \, \mathrm{Im}(\Omega) > 0 \right\}$$

gives a complex torus $X_\Omega := \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ with an associated principal polarization, and any principally polarized complex torus is obtained this way. The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbb{H}_g$. Given $\Omega \in \mathbb{H}_g$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, we define

$$\mu_M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}.$$

Two matrices $\Omega_1, \Omega_2 \in \mathbb{H}_g$ define isomorphic principally polarized complex tori if and only if they lie in the same $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit, and the moduli stack is given as the stack quotient

$$\mathcal{A}_g^{\mathrm{an}} = [\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g];$$

see [4]. Hence, $\mathcal{A}_g^{\mathrm{an}}$ is uniformized by $\mathbb{H}_g$, which is biholomorphic to a bounded domain; see [71, Page 8-9]. Therefore, the stack $\mathcal{A}_g$ is Brody hyperbolic by Liouville's theorem; see [42, Remark 2.9]. On the other hand, the stack $\mathcal{A}_g$ is absolutely arithmetically hyperbolic. This is a famous theorem of Faltings [20] conjectured by Shafarevich at the 1962 ICM. Considering this example, we note the following. For a finite type integral separated Delinge-Mumford stack X with quasi-projective coarse space over $\mathbb{C}$, being Brody hyperbolic is not synonymous with its coarse space being Brody hyperbolic. For example, the coarse space of $\mathcal{A}_1$ is the j-line $\mathbb{A}^1$, which is not Brody hyperbolic.

One way of approaching Lang-Vojta's conjecture for a certain moduli stack is by relating the problem to $\mathcal{A}_g$ via a period map. Let $\mathcal{F}$ be the moduli stack of Fano threefolds of type $(1,1,4)$; see Section 4.1 for definition. For $\mathcal{F}$, the period map is quasi-finite; see Theorem 5.2.3. Any stack with a quasi-finite morphism to $\mathcal{A}_g$ is Brody hyperbolic, because $\mathcal{A}_g$ is; see [1, § 0.3] or [36, 50]. Therefore (by Lang-Vojta's conjecture), $\mathcal{F}$ should be arithmetically hyperbolic. Our first main result is the verification of Lang-Vojta's conjecture for this moduli stack.

Theorem I (Lang-Vojta's conjecture for moduli of Fano threefolds of type (1,1,4))**.** *The moduli stack $\mathcal{F}$ of Fano threefolds of type $(1,1,4)$ is absolutely arithmetically hyperbolic.*

We will now give an overview of the ingredients of the proof.

Quasi-finiteness of the period map and infinitesimal Torelli for weighted complete intersections

A crucial input is the fact that the period map for Fano threefolds of type (1,1,4) is quasi-finite. Our proof of this fact is inspired by [40]. There, the quasi-finiteness of the period map is deduced from the injectivity of its differential. The question that asks whether the period map has an injective differential is called the infinitesimal Torelli problem. The problem can be formulated very concretely for a smooth projective variety X over $\mathbb{C}$ of dimension n. Namely, we say that X

satisfies infinitesimal Torelli if the map

$$\mathrm{H}^1(X,\Theta_X^1) \to \bigoplus_{p+q=n} \mathrm{Hom}_{\mathbb{C}}\left(\mathrm{H}^p(X,\Omega_X^q),\mathrm{H}^{p+1}(X,\Omega_X^{q-1})\right)$$

induced by the contraction map is injective. Here, the tangent sheaf is denoted by Θ_X^1. Among others, there are the following works on this problem which have been influential for our work: curves [2], hypersurfaces in projective space [12, 18], hypersurfaces in weighted projective space [66], complete intersections in projective space [62, 75, 77], zerosets of sections of vector bundles [25], certain cyclic covers of a Hirzebruch surface [47], complete intersections in certain homogeneous Kähler manifolds [46], some weighted complete intersections [78], certain Fano quasi-smooth weighted hypersurfaces [23] and some elliptic surfaces [44, 45, 65].

The methods used in many of these studies have in common that they describe the cohomology groups relevant for the infinitesimal Torelli map as components of a so-called Jacobi ring and argue that the map can be interpreted as multiplication by some element in this ring. By Iskovskikh's classification [30], Fano threefolds of type (1,1,4) are certain quasi-smooth weighted complete intersections. Hence, it is helpful to generalize this Jacobi ring method to the case of such complete intersections. Following [17], we introduce the terminology:

Definition. Let B be a ring and let $W = (W_0,\dots,W_n) \in \mathbb{Z}_{>1}^{n+1}$ be a tuple of positive integers. Let $S_{B,W} = B[x_0,\dots,x_n]$ be the graded polynomial algebra with $\deg(x_i) = W_i$. The *weighted projective space with weights W over B* is the scheme

$$\mathbb{P}_B(W) = \operatorname{Proj} S_{B,W}.$$

If the base ring is specified in the discussion or needs no further specification, then we will simply write S_W and $\mathbb{P}(W)$.

Let $c \le n$, and let $d = (d_1,\dots,d_c) \in \mathbb{Z}_{>0}^c$ be an ordered tuple of positive integers. A closed subvariety $X \subseteq \mathbb{P}(W)$ is a *weighted complete intersection of degree d* if X has codimension c, and if there are homogeneous polynomials $f_i \in (S_W)_{d_i}$ such that X is the vanishing locus $V_+(f_1,\dots,f_c)$ of these polynomials. A weighted complete intersection $X = V_+(f_1,\dots,f_c) \subseteq \mathbb{P}(W)$ is *quasi-smooth* if its *affine cone* $A(X) := \operatorname{Spec} S_W/(f_1,\dots,f_c) \setminus \{0\}$ is smooth.

Given a quasi-smooth weighted complete intersection X over $\mathbb{C}$ as above, we can define generalized sheaves of differentials $\tilde{\Omega}_X^q$ (see Section 3.4) allowing a decomposition

$$\mathrm{H}^{n-c}(X,\mathbb{C}) = \bigoplus_{p+q=n-c} \mathrm{H}^p(X,\tilde{\Omega}_X^q)$$

that coincides with the usual Hodge decomposition in case X is smooth; see Theorem 3.4.3. Consider the polynomial $F = y_1 f_1 + \dots + y_c f_c \in \mathbb{C}[x_0,\dots,x_n,y_1,\dots,y_c]$, which is homogeneous with respect to the bigrading given by $\deg(x_i) = (0,w_i)$ and $\deg(y_j) = (1,-d_j)$. The *Jacobi ring associated to the complete intersection X* is the bigraded ring

$$R = \mathbb{C}[x_0,\dots,x_n,y_1,\dots,y_c]/(\partial_{x_0}F,\dots,\partial_{x_n}F,\partial_{y_1}F,\dots,\partial_{y_c}F).$$

Our second main result can be interpreted as giving an explicit description of the differential of the period map associated to a quasi-smooth weighted complete intersection in terms of its Jacobi ring.

Theorem II (Infinitesimal Torelli map for weighted complete intersections)**.** *Let*

$$X = V_+(f_1, \ldots, f_c) \subseteq \mathbb{P}_{\mathbb{C}}(W_0, \ldots, W_n)$$

be a quasi-smooth weighted complete intersection of degree $(d_1, \ldots, d_c)$ with tangent sheaf Θ^1_X of dimension $\dim(X) = n - c > 2$. Let R be the associated Jacobi ring. Let $\nu = \sum W_i - \sum d_j$. For all integers $p \in \mathbb{Z}$ with $0 < p < n - c$ and $p \neq n - c - p$, there are isomorphisms

$$\mathrm{H}^{n-c-p}(X, \tilde{\Omega}^p_X) \cong \mathrm{Hom}_k(R_{p,-\nu}, \mathbb{C})$$

and

$$\mathrm{H}^1(X, \Theta^1_X) \cong R_{1,0}$$

such that the diagram

$$\begin{array}{ccc}
\mathrm{H}^1(X,\Theta^1_X) & \longrightarrow & \mathrm{Hom}(\mathrm{H}^{n-c-p}(X,\tilde{\Omega}^p_X), \mathrm{H}^{n-c-p+1}(X,\tilde{\Omega}^{p-1}_X)) \\
\downarrow \cong & & \downarrow \cong \\
 & & \mathrm{Hom}(\mathrm{Hom}_{\mathbb{C}}(R_{p,-\nu},\mathbb{C}), \mathrm{Hom}_{\mathbb{C}}(R_{p-1,-\nu},\mathbb{C})) \\
 & & \downarrow \cong \\
R_{1,0} & \xrightarrow{\mu} & \mathrm{Hom}(R_{p-1,-\nu}, R_{p,-\nu})
\end{array}$$

where μ is the map that sends $\alpha \in R_{1,0}$ to the multiplication-by-α map commutes.

As an application of this theorem, we investigate the infinitesimal Torelli problem for Fano threefolds of type (1,1,4). By Iskovskikh's classification, there are two types of such varieties; see [30, Table 3.5]. The varieties of the first type are smooth quartics in $\mathbb{P}^4$. For smooth hypersurfaces in projective space, the infinitesimal Torelli problem is completely understood. In particular, smooth quartic threefolds satisfy infinitesimal Torelli; see [12]. The second type of Fano threefolds with Picard rank 1, index 1 and degree 4 are called *hyperelliptic*; each such Fano threefold X is a double cover of a smooth quadric $Q \subseteq \mathbb{P}^4$ ramified along a smooth divisor of degree 8 in Q. Such a double cover comes naturally with an involution ι associated to the double cover; see Section 4.3. It turns out that such hyperelliptic Fano threefolds do **not** satisfy infinitesimal Torelli, i.e., the period map on the moduli of Fano threefolds of type (1,1,4) does not have an injective differential. However, the following result says that the "restricted" period map on the locus of hyperelliptic Fano threefolds does have an injective differential.

Theorem III (Infinitesimal Torelli for hyperelliptic Fano threefolds)**.** *Let X be a hyperelliptic Fano threefold of Picard rank 1, index 1 and degree 4 over $\mathbb{C}$. Let $\iota \in \mathrm{Aut}(X)$ be the involution associated to the double cover. Then the ι-invariant part of the infinitesimal Torelli map*

$$\mathrm{H}^1(X, \Theta_X)^\iota \to \bigoplus_{p+q=3} \mathrm{Hom}_{\mathbb{C}}\left(\mathrm{H}^p(X, \Omega^q_X), \mathrm{H}^{p+1}(X, \Omega^{q-1}_X)\right)$$

is injective.

As explained in [40, Section 3.5], among the Fano threefolds of Picard rank 1 and index 1, infinitesimal Torelli is satisfied if the degree is 2, 6 or 8, and it is known to fail for degrees 10 and 14. Our work deals with one of the remaining cases, namely that of degree 4.

Note that the failure of infinitesimal Torelli for Fano threefolds of type (1,1,4) is analogous to the failure of infinitesimal Torelli for curves of genus $g \geq 2$. Such a curve satisfies infinitesimal Torelli if and only if it is non-hyperelliptic [13], but the period map restricted to the hyperelliptic locus is unramified [49].

Stratification of the moduli of Fano threefolds of type (1,1,4)

As explained above, hyperelliptic Fano threefolds of type (1,1,4) do not strictly satisfy infinitesimal Torelli. However, we can work around this by using the following result on the structure of the moduli stack $\mathcal{F}$. Let $\mathcal{Q} \subseteq \mathcal{F}$ be the locus of smooth quartics and let $\mathcal{H} \subseteq \mathcal{F}$ be the locus of hyperelliptic Fano threefolds; see Section 4.4 for definitions.

Theorem IV (Stratification of the moduli of Fano threefolds of type (1,1,4))**.** *The inclusion $\mathcal{H} \to \mathcal{F}$ is a closed immersion, the inclusion $\mathcal{Q} \to \mathcal{F}$ is an open immersion, and for any field k with $2 \in k^\times$, we have*

$$\mathcal{F}(k) = \mathcal{Q}(k) \sqcup \mathcal{H}(k).$$

With this result in mind, the quasi-finiteness of the period map can be checked after restricting it to each stratum separately. The map described in Theorem III is the differential of the period map restricted to the hyperelliptic locus; see Theorem 4.5.1. Hence, it follows that the period map is quasi-finite; see Theorem 5.2.3.

Arithmetic hyperbolicity and persistence conjecture

The proof of Theorem I works in two steps. The first step is to prove the arithmetic hyperbolicity over some $\mathbb{C}$. The arithmetic hyperbolicity over $\overline{\mathbb{Q}}$ is implied immediately. As explained in [39, Section 4], this can be formulated as follows.

Theorem V (Shafarevich conjecture for Fano threefolds of type (1,1,4))**.** *Let K be a number field and let S be a finite set of places on K. Then the set of K-isomorphism classes of Fano threefolds of Picard rank 1, index 1 and degree 4 over K with good reduction outside S (see Definition 4.1.2) is finite.*

In [19] Faltings proved that given a number field K, a finite set of places S on K and a positive integer g, there are only finitely many isomorphism classes of abelian varieties of dimension g over K with good reduction outside S. Faltings's finiteness result illustrates a more general phenomenon, commonly referred to as the Shafarevich conjecture, that the set of objects of fixed type over a number field K with good reduction outside S should be finite. This far-reaching conjecture has been verified for K3 surfaces (and hyperkähler varieties) [3, 58, 68, 73], cyclic covers [41], ample hypersurfaces in abelian varieties [51] (building on [52]), polycurves [35, 59], flag varieties [32], complete intersections of Hodge level at most one [31], certain types of Fano threefolds [40], del Pezzo surfaces [67], and Enriques surfaces [74]. Interestingly, the Shafarevich conjecture for Fano threefolds can fail; see [40, Theorem 1.4] for a precise statement. However, Theorem V verifies the Shafarevich conjecture for Fano threefolds of type (1,1,4). Note that this is one of the cases for which the Shafarevich conjecture was not handled by Loughran and Javanpeykar [40].

The second step is to show that the arithmetic hyperbolicity persists for field extensions. For an arbitrary stack, there should be no difference between being arithmetically hyperbolic over $\overline{\mathbb{Q}}$ and being absolutely arithmetically hyperbolic, i.e., arithmetic hyperbolicity should persist over all field extensions. This is formalized by the following conjecture for stacks alluded to in [39, Remark 4.13].

Conjecture (Persistence conjecture)**.** Let k be an algebraically closed field of characteristic zero, and let X be a finitely presented algebraic stack over k. If X is arithmetically hyperbolic over k, then X is absolutely arithmetically hyperbolic.

This is a stacky version of [34, Conjecture 1.1] (see also [36, Conjecture 17.5]). The conjecture says, in particular, that the finiteness of rational points over number fields on a projective variety over $\overline{\mathbb{Q}}$ should imply the finiteness over all finitely generated fields; this was formulated as a precise question by Lang in [50, p. 202]. However, known proofs for persistence usually require another geometric input. For example, in [56], we showed that arithmetic hyperbolicity combined with a function field analogue of artihmetic hyperbolicity implies absolute arithmetic hyperbolicity for affine varieties. However, the argument given in this work is more in line with [37]. We prove stacky generalizations of results found there and show that geometric hyperbolicity (see Section 5.1 for Definition) implies persistence; see Theorem 5.1.3.

Most of the results presented in this work can also be found in our papers [55] (Chapter 2 + 3) and [57] (Chapter 4 + 5).

Outline

The technical heart of this work is Chapter 3. There, we prove our results on infinitesimal Torelli: Theorem II and Theorem III. The description in terms of the Jacobi ring is the consequence of a comparison of pairings of certain spectral sequences. In Chapter 2, we develop the necessary theory for describing such pairings. The introduction of Chapter 3 contains an overview of the proof of Theorem II. The reader is advised to have a brief look at this overview first, since it motivates the concepts developed in Chapter 2. In Chapter 4, we study the structure of the moduli of Fano threefolds of type (1,1,4) and prove Theorem IV. In Chapter 5, we prove our results on the persistence conjecture, Theorem V and Theorem I.

Chapter 2

Pairings of spectral sequences

In this chapter, we introduce pairings of filtered complexes. We will see that such pairings induce a pairing of the associated spectral sequences.

2.1 Multigraded differential modules

In this section, we introduce multi-graded differential modules, which is a notion used for example in [9]. In particular, this notion describes single and double complexes and pages of spectral sequences.

Let R be a (commutative) ring or, more generally, the structure sheaf $\mathcal{O}_T$ of a scheme T. A differential d on an n-graded R-module $E = \bigoplus_{p \in \mathbb{Z}^n} E^p$ for us is always considered to be an R-linear self map that is homogeneous of a certain degree with $d \circ d = 0$. Whenever we consider a module together with multiple differentials defined on it, we require the differentials to commute pairwise.

Let (E_1, d_1) and (E_2, d_2) be differential n-graded R-modules with homogeneous differentials of the same degree $a \in \mathbb{Z}^n$. Then the tensor product $E_1 \otimes E_2$ comes with an induced $(2n)$-grading

$$E_1 \otimes E_2 = \bigoplus_{(p,q) \in \mathbb{Z}^n \times \mathbb{Z}^n} E_1^p \otimes E_2^q$$

and the two homogeneous differentials $d_1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes d_2$, giving us a bidifferential $2n$-graded R-module.

Definition 2.1.1. Let $n \in \mathbb{Z}_{>0}$ be a positive integer and let (E, d_1, d_2) be a bidifferential $2n$-graded R-module. Write the degree of d_i as (a_i, b_i) where $a_i, b_i \in \mathbb{Z}^n$. Suppose $a_1 + b_1 = a_2 + b_2$, then we define the *associated total differential n-graded R-module of* (E, d_1, d_2) to be the n-graded module

$$\mathrm{Tot}(E) = \bigoplus_{p \in \mathbb{Z}^n} \mathrm{Tot}^p(E)$$

where

$$\mathrm{Tot}^p(E) = \bigoplus_{\substack{s,t \in \mathbb{Z}^n \\ s+t=p}} E^{s,t}$$

with homogeneous differential $d \in \mathrm{End}(\mathrm{Tot}(E))$ of degree $a_1 + b_1 = a_2 + b_2$ defined by $d|_{E^{s,t}} = d_1 + (-1)^{s_1} d_2$.

Example 2.1.2. Let $(K^{\bullet,\bullet}, d_1, d_2)$ be a double complex. Then $K = \bigoplus_{p,q\in\mathbb{Z}} K^{p,q}$ is a bigraded module and d_1, d_2 define differentials of degree $(1,0)$, $(0,1)$ on K, thus giving K the structure of a bidifferential bigraded module. In fact, giving the data of a double complex is equivalent to defining a bigraded module with differentials of degree $(1,0)$ and $(0,1)$. Similarly, a complex $(L^\bullet, d)$ can be identified with the differential graded module $(L = \bigoplus_{p\in\mathbb{Z}} L^p, d)$. Under these identifications, the total single complex associated to $K^{\bullet,\bullet}$ and the total differential graded module associated to $K^{\bullet,\bullet}$ are the same.

Example 2.1.3. For us, the total differential bigraded module associated to a tensor product of bigraded differential modules with differentials of the same degree $a \in \mathbb{Z}^2$ is of particular interest. So let (E_1, d_1) and (E_2, d_2) be differential bigraded modules. Then the differentials $d_1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes d_2$ on the quadgraded module $E_1 \otimes E_2$ have degrees $(a_1, a_2, 0, 0)$ and $(0, 0, a_1, a_2)$. For $p, q \in \mathbb{Z}$, we have

$$\mathrm{Tot}^{p,q}(E_1 \otimes E_2) = \bigoplus_{\substack{s+t=p \\ u+v=q}} E_1^{s,u} \otimes E_2^{t,v}.$$

On $E^{s,u} \otimes E^{t,v}$ the differential is given as

$$d_{Tot}|_{E^{s,u}\otimes E^{t,v}} = d_1|_{E^{s,u}} \otimes \mathrm{id}_{E^{t,v}} + (-1)^s \,\mathrm{id}\,|_{E^{s,u}} \otimes d_2|_{E^{t,v}}.$$

2.2 Pairings of filtered complexes

In this section, we explain how a pairing of filtered complexes induces a pairing of the associated homology complexes that respects the induced filtration. Let R be a ring or, more generally, the structure sheaf $R = \mathcal{O}_T$ of a scheme T. All modules are considered to be R-modules and all single (resp. double) complexes are considered to be single (resp. double) complexes of R-modules.

Let (K, d), $(K_1^\bullet, d_1)$ and $(K_2^\bullet, d_2)$ be complexes. The total complex of the tensor product of $K_1^\bullet$ and $K_2^\bullet$, as introduced in Section 2.1, is given by

$$\mathrm{Tot}^n(K_1^\bullet \otimes K_2^\bullet) = \bigoplus_{p+q=n} K_1^p \otimes K_2^q$$

with the differential given by

$$d_{Tot}|_{K_1^p \otimes K_2^q} = d_1 \otimes \mathrm{id}\,|_{K_2^q} + (-1)^p \,\mathrm{id}\,|_{K_1^p} \otimes d_2.$$

Definition 2.2.1. A *pairing of complexes from* $(K_1^\bullet, d_1)$ *and* $(K_2^\bullet, d_2)$ *to* (K, d) is a morphism of complexes

$$\phi : (\mathrm{Tot}^\bullet(K_1^\bullet \otimes K_2^\bullet)), d_{Tot}) \to (K^\bullet, d).$$

For $p, q \in \mathbb{Z}$, we let $\phi^{p,q}$ denote the map $\phi^{p,q} \colon K_1^p \otimes K_2^q \to K^{p+q}$ induced by ϕ.

From now on let R be a ring. A pairing of complexes induces a pairing of the associated homology complexes.

Lemma 2.2.2. *Let* $(K^\bullet, d)$*,* $(K_1^\bullet, d_1)$ *and* $(K_2^\bullet, d_2)$ *be complexes and let*

$$\phi \colon \mathrm{Tot}^\bullet(K_1^\bullet \otimes K_2^\bullet) \to K^\bullet$$

be a pairing of complexes. Then ϕ induces a pairing of the associated homology complexes

$$\overline{\phi}\colon\ \mathrm{Tot}^\bullet(\mathrm{H}^\bullet(K_1^\bullet, d_1) \otimes \mathrm{H}^\bullet(K_2, d_2)) \to \mathrm{H}^\bullet(K^\bullet, d).$$

Proof. If $\alpha \in \ker(K_1^p \to K_1^{p+1})$, $\beta \in \ker(K_2^q \to K_2^{q+1})$, then

$$d(\phi(\alpha \otimes \beta)) = \phi(d_{Tot}(\alpha \otimes \beta)) = \phi(d_1(\alpha) \otimes \beta + (-1)^p \alpha \otimes d_2(\beta)) = 0.$$

Hence $\phi(\alpha \otimes \beta) \in \ker(K^{p+q} \to K^{p+q+1})$. If furthermore $\alpha = d_1(\gamma)$ for some $\gamma \in K_1^{p-1}$, then

$$d(\phi(\gamma \otimes \beta)) = \phi(d_{Tot}(\gamma \otimes \beta)) = \phi(\alpha \otimes \beta)$$

lies in the image $\mathrm{im}(K^{p+q-1} \to K^{p+q})$. Similarly, we see that this also holds if $\beta \in \mathrm{im}(d_2)$. Hence, $\phi^{p,q}$ induces a map

$$\mathrm{H}^p(K_1^\bullet, d_1) \otimes \mathrm{H}^q(K_2^\bullet, d_2) \to \mathrm{H}^{p+q}(K^\bullet, d).$$

These maps give a pairing of the homology complexes since the differentials of homology complexes are all zero. *q.e.d.*

A *filtered complex* is a triple $(K^\bullet, d, F)$, where $K^\bullet$ is a complex with differential d, and F is a decreasing filtration on $K^\bullet$ compatible with the differential, i.e., for each $n \in \mathbb{Z}$, we have a decreasing filtration

$$K^n \supseteq \ldots \supseteq F^pK^n \supseteq F^{p+1}K^n \supseteq \ldots$$

such that $d(F^pK^n) \subseteq F^pK^{n+1}$ for all $n, p \in \mathbb{Z}$.

Given a filtered complex $(K^\bullet, d, F)$, there is an *induced filtration* on the homology complex $\mathrm{H}^\bullet(K^\bullet, d)$ given by

$$F^p\mathrm{H}^n(K^\bullet, d)) := \mathrm{im}(\mathrm{H}^n(F^pK^\bullet) \to \mathrm{H}^n(K^\bullet)) = \frac{\ker(d) \cap F^pK^n + \mathrm{im}(d) \cap K^n}{\mathrm{im}(d) \cap K^n}.$$

For the associated graded, we have

$$\mathrm{gr}^p\mathrm{H}^n(K^\bullet) := \frac{F^p\mathrm{H}^n(K^\bullet, d))}{F^{p+1}\mathrm{H}^n(K^\bullet, d)} = \frac{\ker(d) \cap F^pK^n}{\ker(d) \cap F^{p+1}K^n + \mathrm{im}(d) \cap F^pK^n}; \tag{2.2.1}$$

see [72, Tag 0BDT].

Definition 2.2.3. Let $(K^\bullet, d, F)$,$(K_1^\bullet, d_1, F_1)$ and $(K_2^\bullet, d_2, F_2)$ be filtered complexes. A pairing of complexes

$$\phi\colon\ \mathrm{Tot}^\bullet(K_1^\bullet \otimes K_2^\bullet) \to K^\bullet$$

is a *pairing of filtered complexes* if it is *compatible with the filtrations*, that is

$$\phi^{p,q}(\alpha \otimes \beta) \in F^{i+j}K^{p+q}$$

for all $\alpha \in F_1^iK_1^p$ and $\beta \in F_2^jK_2^q$.

From the definition, it is evident that for each $p, q \in \mathbb{Z}$, such a pairing of filtered complexes induces a pairing of complexes

$$\phi\colon\ \mathrm{Tot}^\bullet(F_1^pK_1^\bullet \otimes F_2^qK_2^\bullet) \to F^{p+q}K^\bullet.$$

Hence the induced pairing of the homology complexes

$$\overline{\phi}\colon \operatorname{Tot}^\bullet(\mathrm{H}^\bullet(K_1^\bullet, d_1) \otimes \mathrm{H}^\bullet(K_2^\bullet, d_2)) \to \mathrm{H}^\bullet(K^\bullet, d)$$

from Lemma 2.2.2 is compatible with the induced filtrations on the homology complexes. Therefore, for each $p, q, i, j \in \mathbb{Z}$, we get induced maps $\overline{\phi}^{p,q,i,j}$ and $\operatorname{gr}^{p,q,i,j}(\phi)$ making the diagram

$$\begin{CD}
\mathrm{H}^p(K_1^\bullet) \otimes \mathrm{H}^q(K_2^\bullet) @<{\alpha^{p,i}\otimes\alpha^{q,j}}<< F^i\mathrm{H}^p(K_1^\bullet) \otimes F^j\mathrm{H}^q(K_2^\bullet) @>{\beta^{p,i}\otimes\beta^{q,j}}>> \operatorname{gr}^i \mathrm{H}^p(K_1^\bullet) \otimes \operatorname{gr}^j \mathrm{H}^q(K_2^\bullet) \\
@V{\overline{\phi}^{p,q}}VV @V{\overline{\phi}^{p,q,i,j}}VV @V{\operatorname{gr}^{p,q,i,j}(\phi)}VV \\
\mathrm{H}^{p+q}(K^\bullet) @<{\alpha^{p+q,i+j}}<< F^{i+j}\mathrm{H}^{p+q}(K^\bullet) @>{\beta^{p+q,i+j}}>> \operatorname{gr}^{i+j} \mathrm{H}^{p+q}(K^\bullet)
\end{CD} \tag{2.2.2}$$

commute, where the maps $\alpha^{a,b}$ denote the natural injections and the maps $\beta^{a,b}$ denote the natural surjections.

2.3 Spectral pairing

In this section, we follow [72, Tag 012K] and explain how to construct the spectral sequence associated to a filtered complex. Building on this, we show that a pairing of filtered complexes induces a pairing of the associated spectral sequences.

Let R be a ring. All complexes are complexes of R-modules. A *spectral sequence* is given by the data

$$E = (E_r, d_r)_{r \in \mathbb{Z}_{\geq 0}}$$

where E_r is an R-module and $d_r \in \operatorname{End}(E_r)$ is a differential such that

$$E_{r+1} = \mathrm{H}(E_r, d_r) := \ker(d_r) / \operatorname{im}(d_r).$$

We call E_r the *r-th page of E*. A *bigrading on the spectral sequence E* is given by a direct sum decomposition for each page $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$ such that the differential d_r decomposes into a direct sum of maps

$$d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and we have

$$E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \operatorname{im} d_r^{p-r,q+r-1}.$$

We can associate a bigraded spectral sequence to a filtered complex $(K^\bullet, d, F)$ in the following way. We define

$$Z_r^{p,q} = \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_r^{p,q} = \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and $E_r^{p,q} = Z_r^{p,q} / B_r^{p,q}$. Now set $B_r = \bigoplus_{p,q} B_r^{p,q}$, $Z_r = \bigoplus_{p,q} Z_r^{p,q}$ and $E_r = \bigoplus_{p,q} E_r^{p,q}$. Define the map $d_r\colon E_r \to E_r$ as the direct sum of the maps

$$d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}\colon \quad z + F^{p+1}K^{p+q} \mapsto d(z) + F^{p+r+1}K^{p+q+1}$$

where $z \in F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1})$. This defines the *bigraded spectral sequence* (E_r, d_r) *associated to the filtered complex* $(K^\bullet, d, F)$.

Definition 2.3.1. Let $(E_r, d_r), ({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ be bigraded spectral sequences, and let $\phi = (\phi_r)_{r\in\mathbb{Z}_{\geq 0}}$ be a collection of morphisms of bigraded differential modules

$$\phi_r\colon \operatorname{Tot}^{\bullet,\bullet}({}'E_r^{\bullet,\bullet} \otimes {}''E_r^{\bullet,\bullet}) \to E_r^{\bullet,\bullet}$$

that are homogeneous of degree 0. We denote by

$$\phi_r^{s,t,u,v}\colon {}'E_r^{s,u} \otimes {}''E_r^{t,v} \to E_r^{s+t,u+v}$$

the map induced by ϕ. The collection ϕ is called a *pairing of bigraded spectral sequences* if $\phi_{r+1}^{s,t,u,v}$ is induced by $\phi_r^{s,t,u,v}$ for all $r, s, t, u, v \in \mathbb{Z}$, $r \geq 0$.

Lemma 2.3.2. *Let* $((K^\bullet, d, F), (E_r, d_r))$, $(({}'K^\bullet, {}'d, {}'F), ({}'E_r, {}'d_r))$ *and* $(({}''K^\bullet, {}''d, {}''F), ({}''E_r, {}''d_r))$ *be pairs of filtered complexes with their associated bigraded spectral sequences. Any pairing of filtered complexes*

$$\phi : \operatorname{Tot}^\bullet(({}'K^\bullet, {}'d, {}'F) \otimes ({}''K^\bullet, {}''d, {}''F)) \to (K^\bullet, d, F)$$

induces a pairing of the associated spectral sequences $\tilde{\phi} = (\tilde{\phi}_r)_{r\in\mathbb{Z}_{\geq 0}}$,

$$\tilde{\phi}_r\colon \operatorname{Tot}^{\bullet,\bullet}(({}'E_r^{\bullet,\bullet}, {}'d_r) \otimes ({}''E_r^{\bullet,\bullet}, {}''d_r)) \to (E_r^{\bullet,\bullet}, d_r).$$

Proof. First, we show that ϕ induces a map ${}'Z_r^{s,u} \otimes {}''Z_r^{t,v} \to Z_r^{s+t,u+v}$. Since ϕ respects the filtration, if

$$\alpha \in {}'F^s\,{}'K^{s+u} \cap {}'d^{-1}({}'F^{r+s}\,{}'K^{s+u+1})$$

and

$$\beta \in {}''F^t\,{}''K^{t+v} \cap {}''d^{-1}({}''F^{r+t}\,{}''K^{t+v+1}),$$

then

$$\phi^{s+u,t+v}(\alpha \otimes \beta) \in F^{s+t}K^{s+t+u+v}.$$

Furthermore,

$$\begin{aligned} &d(\phi^{s+u,t+v}(\alpha \otimes \beta)) \\ &= \phi \circ d_{Tot}(\alpha \otimes \beta) \\ &= \phi({}'d(\alpha) \otimes \beta) + (-1)^{s+u}\phi(\alpha \otimes {}''d(\beta)). \end{aligned}$$

Since ${}'d(\alpha) \in {}'F^{r+s}\,{}'K^{s+u+1}$ and ${}''d(\beta) \in {}''F^{r+t}\,{}''K^{t+v+1}$, we conclude

$$\phi^{s+u,t+v}(\alpha \otimes \beta) \in d^{-1}(F^{r+s+t}K^{1+s+t+u+v}).$$

Similarly, it is seen that under this map, ${}'B_r^{s,u} \otimes {}''Z_r^{t,v}$ and ${}'Z_r^{s,u} \otimes {}''B_r^{t,v}$ are both mapped to $B_r^{s+t,u+v}$. Hence, the map ϕ induces maps

$$\tilde{\phi}_r^{s,t,u,v} : {}'E_r^{s,u} \otimes {}''E_r^{t,v} \to E_r^{s+t,u+v}.$$

That $\tilde{\phi}_{r+1}^{s,t,u,v}$ is induced by $\tilde{\phi}_r^{s,t,u,v}$ follows from the fact that both maps are induced by ϕ. *q.e.d.*

For a filtered complex $(K^\bullet, d, F)$, we define

$$Z_\infty^{p,q} = \bigcap_r Z_r^{p,q} = \bigcap_r \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_\infty^{p,q} = \bigcup_r B_r^{p,q} = \bigcup_r \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$. If we now suppose that the filtration is finite, i.e., for all $n \in \mathbb{Z}$, there are $l, m \in \mathbb{Z}$ such that $F^lK^n = K^n$ and $F^mK^n = 0$, then the chains

$$Z_0^{p,q} \supseteq \dots Z_r^{p,q} \supseteq Z_{r+1}^{p,q} \supseteq \dots$$

and

$$B_0^{p,q} \subseteq \dots B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \dots$$

become stationary and assume $Z_\infty^{p,q}$ and $B_\infty^{p,q}$ after finitely many steps. We have

$$Z_\infty^{p,q} = \frac{F^pK^{p+q} \cap \ker(d) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_\infty^{p,q} = \frac{F^pK^{p+q} \cap \operatorname{im}(d) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}.$$

If we now put $n = p + q$ and compare with Equation (2.2.1), we get an identity

$$\operatorname{gr}^p \mathrm{H}^n(K^\bullet) = \frac{\ker(d) \cap F^pK^n + F^{p+1}K^n}{\operatorname{im}(d) \cap F^pK^n + F^{p+1}K^n} = E_\infty^{p,q}. \tag{2.3.3}$$

Theorem 2.3.3. *Let* $((K^\bullet, d, F), (E_r, d_r))$, $(('K^\bullet, 'd, 'F), ('E_r, 'd_r))$ *and* $((''K^\bullet, ''d, ''F), (''E_r, ''d_r))$ *be pairs of filtered complexes with their associated bigraded spectral sequences such that all the filtrations are finite, and let*

$$\phi\colon \operatorname{Tot}^\bullet('K^\bullet \otimes ''K^\bullet) \to K^\bullet$$

be a pairing of filtered complexes. The induced pairing of the associated spectral sequences induces a pairing of bigraded modules

$$\tilde{\phi}_\infty\colon \operatorname{Tot}^\bullet('E_\infty^{\bullet,\bullet} \otimes ''E_\infty^{\bullet,\bullet}) \to E_\infty^{\bullet,\bullet}$$

such that for all $i, j, p, q \in \mathbb{Z}$, *the diagram*

$$\begin{array}{ccc} 'E_\infty^{i,p} \otimes ''E_\infty^{j,q} & \xrightarrow{=} & \operatorname{gr}^i \mathrm{H}^{p+i}('K^\bullet) \otimes \operatorname{gr}^j \mathrm{H}^{q+j}(''K^\bullet) \\ \downarrow \tilde{\phi}_\infty^{i,j,p,q} & & \downarrow \operatorname{gr}^{p+i,q+j,i,j}(\phi) \\ E_\infty^{i+j,p+q} & \xrightarrow{=} & \operatorname{gr}^{i+j} \mathrm{H}^{p+q+i+j}(K^\bullet) \end{array}$$

commutes.

Proof. By Lemma 2.3.2, the pairing of filtered complexes ϕ induces a pairing of spectral sequences

$$\tilde{\phi}_r \colon \operatorname{Tot}^{\bullet,\bullet}(({}'E_r^{\bullet,\bullet}, {}'d_r) \otimes ({}''E_r^{\bullet,\bullet}, {}''d_r)) \to (E_r^{\bullet,\bullet}, d_r).$$

For each $i, j, p, q \in \mathbb{Z}$ the modules $E_r^{i,p}$, ${}'E_r^{i,p}$ and ${}''E_r^{i,p}$ assume $E_\infty^{i,p}$, ${}'E_\infty^{i,p}$ and ${}''E_\infty^{i,p}$ after finitely many pages. Hence the maps $\tilde{\phi}_r^{i,j,p+i,q+j}$ converge to a map

$$\tilde{\phi}_\infty^{i,j,p,q} \colon {}'E_\infty^{i,p} \otimes {}''E_\infty^{j,q} \to E_\infty^{i+j,p+q}.$$

It coincides with $\operatorname{gr}^{p+i,q+j,i,j}(\phi)$ as both maps are induced by ϕ. *q.e.d.*

Chapter 3

Jacobi ring of weighted complete intersections

In this chapter, we will prove Theorem II and Theorem III. The proof of Theorem II is quite involved. We give a brief overview of the argument. Details will be discussed in the following sections.

Ingredients of the proof of Theorem II

For smooth complete intersections $X = V(f_1, \ldots, f_c) \subseteq \mathbb{P}^n$ in usual projective space, similar results to Theorem II have been achieved by relating the Infinitesimal variations of Hodge structure (IVHS) of X to the IVHS of the hypersurface $V(F) \subseteq \mathbb{P}(E)$, where $E = \bigoplus \mathcal{O}_{\mathbb{P}^n}(d_i)$ and $F = y_1 f_1 + \cdots + f_c y_c$; see [75]. To avoid problems of this geometric approach arising from the singular nature of the surrounding weighted projective space in our case, we will use another purely algebraic approach inspired by the calculations of Flenner; see [24, Section 8].

Let $X \subseteq \mathbb{P}(W)$ be a quasi-smooth complete intersection of codimension c in an n-dimensional weighted projective space. Let U be its affine cone. The first step is to express the contraction pairing in cohomology

$$\alpha \colon \mathrm{H}^{n-c-p}(U, \Omega^p_U) \otimes \mathrm{H}^1(U, \Theta^1_U) \to \mathrm{H}^{n-c-p+1}(U, \Omega^{p-1}_U)$$

in terms of multiplication in the Jacobi ring; see Lemma 3.3.2. To do this, in Section 3.1, we construct exact sequences

$$0 \to K_p^{-p} \to \cdots \to K_p^0 \to \Omega^p_U \to 0$$

and

$$0 \to \Theta^1_U \to K^0_{-1} \to K^1_{-1} \to 0,$$

where $K_p^\bullet$ and $K_{-1}^\bullet$ are complexes of free $\mathcal{O}_U$-modules; see Theorem 3.1.1. We then show that the contraction pairing

$$\Omega^p_U \otimes \Theta^1_U \xrightarrow{\gamma} \Omega^{p-1}_U$$

extends to a pairing of these "resolutions"

$$\tilde{\gamma} \colon \operatorname{Tot}^\bullet\left(K_p^\bullet \otimes K_{-1}^\bullet\right) \to K_{p-1}^\bullet;$$

see Lemma 3.1.2. The homology of the total Čech complex computes the hypercohomology of these "resolutions", i.e.,

$$\mathbb{H}^q(U, K_p^\bullet) = \mathrm{H}^q(\operatorname{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, K_p^\bullet))),$$

where $\mathcal{U}$ is an open affine covering of U. The pairing $\tilde{\gamma}$ induces a pairing

$$\gamma' : \mathrm{Tot}^\bullet\left(\mathbb{H}^\bullet(U, K_p^\bullet) \otimes \mathbb{H}^\bullet(U, K_{-1}^\bullet)\right) \to \mathbb{H}^\bullet(U, K_{p-1}^\bullet).$$

For our argument, we do not need to calculate this pairing of hypercohomology complexes explicitly. The key point is that we can construct two different pairings of spectral sequences that converge towards this pairing. The pairing γ' is just an intermediate step to identify the pairings of spectral sequences. The total complex

$$L_p^\bullet := \mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, K_p^\bullet))$$

comes with two filtrations F_1 and F_2 given by

$$F_1^r(L_p^q) = \bigoplus_{i+j=q, i\geq r} \check{C}^i(\mathcal{U}, (K_p^j))$$

and

$$F_2^r(L_p^q) = \bigoplus_{i+j=q, j\geq r} \check{C}^i(\mathcal{U}, (K_p^j)).$$

For each filtration F_i of the complex $L_p^\bullet$, there is an associated spectral sequence $(E_{i,p,r}^{\bullet,\bullet})_{r\in\mathbb{Z}_{\geq 0}}$. And the pairing $\tilde{\gamma}$ induces pairings of these spectral sequences, one for each filtration. For the filtration F_1, the entries of page 0 of the spectral sequences are given by the entries of the Čech double complex. We get a pairing

$$\begin{array}{ccc}
\begin{pmatrix} \check{C}^{n-c}(\mathcal{U}, (K_p^{-p})) & \cdots & \check{C}^{n-c}(\mathcal{U}, (K_p^0)) \\ \vdots & & \vdots \\ \check{C}^0(\mathcal{U}, (K_p^{-p})) & \cdots & \check{C}^0(\mathcal{U}, (K_p^0)) \end{pmatrix} & & E_{i,p,0}^{\bullet,\bullet} \\
\otimes & = & \otimes \\
\begin{pmatrix} \check{C}^{n-c}(\mathcal{U}, (K_{-1}^0)) & \check{C}^{n-c}(\mathcal{U}, (K_{-1}^1)) \\ \vdots & \vdots \\ \check{C}^0(\mathcal{U}, (K_{-1}^0)) & \check{C}^0(\mathcal{U}, (K_{-1}^1)) \end{pmatrix} & & E_{i,-1,0}^{\bullet,\bullet} \\
\downarrow & & \downarrow \\
\begin{pmatrix} \check{C}^{n-c}(\mathcal{U}, (K_{p-1}^{-p+1})) & \cdots & \check{C}^{n-c}(\mathcal{U}, (K_{p-1}^0)) \\ \vdots & & \vdots \\ \check{C}^0(\mathcal{U}, (K_{p-1}^{-p+1})) & \cdots & \check{C}^0(\mathcal{U}, (K_{p-1}^0)) \end{pmatrix} & = & E_{i,p-1,0}^{\bullet,\bullet}.
\end{array} \tag{3.0.1}$$

In the diagram above, we omit the rows above row $n-c$ since these vanish on later pages. For the filtration F_2, page 0 is the transposed of Picture 3.0.1. Visually, the second page of the spectral sequences are calculated as follows. For the filtration F_1, we first compute homology in horizontally and then vertically. The paring on the second page

$$\begin{pmatrix} E^{n-c,0}_{1,p,2} = \mathrm{H}^{n-c}(U,\Omega^p_U) \\ \vdots \\ E^{0,0}_{1,p,2} = \mathrm{H}^0(U,\Omega^p_U) \end{pmatrix} \otimes \begin{pmatrix} E^{n-c,0}_{1,-1,2} = \mathrm{H}^{n-c}(U,\Theta^1_U) \\ \vdots \\ E^{0,0}_{1,-1,2} = \mathrm{H}^0(U,\Theta^1_U) \end{pmatrix} \to \begin{pmatrix} E^{n-c,0}_{1,p-1,2} = \mathrm{H}^{n-c}(U,\Omega^{p-1}_U) \\ \vdots \\ E^{0,0}_{1,p-1,2} = \mathrm{H}^0(U,\Omega^{p-1}_U) \end{pmatrix}$$

is the pairing induced by the contraction map. For the filtration F_2, we start with Picture 3.0.1, compute homology vertically, then horizontally and then transpose. The homology of the Čech complex calculates the cohomology. As we explain in Section 3.2, for a free $\mathcal{O}_U$-module M the cohomology group $\mathrm{H}^q(U,M)$ vanishes for $q \neq 0, n-c$. Hence, on the first and second page all rows except row 0 and $n-c$ vanish. The pairing on the second page becomes

$$\begin{pmatrix} E^{0,0}_{2,p,2} & 0\cdots 0 & E^{0,n-c}_{2,p,2} \\ \vdots & \vdots\quad\vdots & \vdots \\ E^{-p,0}_{2,p,2} & 0\cdots 0 & E^{-p,n-c}_{2,p,2} \end{pmatrix} \otimes \begin{pmatrix} E^{1,0}_{2,-1,2} & 0\cdots 0 & E^{1,n-c}_{2,-1,2} \\ E^{0,0}_{2,-1,2} & 0\cdots 0 & E^{0,n-c}_{2,-1,2} \end{pmatrix} \to \begin{pmatrix} E^{0,0}_{2,p-1,2} & 0\cdots 0 & E^{0,n-c}_{2,p-1,2} \\ \vdots & \vdots\quad\vdots & \vdots \\ E^{-p+1,0}_{2,p-1,2} & 0\cdots 0 & E^{-p+1,n-c}_{2,p-1,2} \end{pmatrix}.$$

For both filtrations, the spectral sequences converge on the second page. Given any $u \in \mathbb{Z}$, there is at most one combination of integers $s,t \in \mathbb{Z}$ with $s+t = u$ and $E^{s,t}_{i,p,2} \neq 0$. Note by the assumptions of Theorem II, we have $p < n-c$. Hence, for such s,t, we have $\mathrm{H}^u(L^\bullet_p) = E^{s,t}_{i,p,2}$. Hence, for both filtrations, the pairings of spectral sequences are identified with the pairing of complexes induced by $\tilde{\gamma}$

$$\gamma' \colon \operatorname{Tot}^\bullet\left(\mathrm{H}^\bullet(L^\bullet_p) \otimes \mathrm{H}^\bullet(L^\bullet_{-1})\right) \to \mathrm{H}^\bullet(L^\bullet_{p-1}).$$

In particular, if we consider the filtration F_1, then the pairing

$$\mathrm{H}^{n-c-p}(L^\bullet_p) \otimes \mathrm{H}^1(L^\bullet_{-1}) \to \mathrm{H}^{n-c-p+1}(L^\bullet_{p-1})$$

is identified with α. And for F_2, we get an identification with the pairing

$$E^{-p,n-c}_{2,p,2} \otimes E^{1,0}_{2,-1,2} \to E^{-p+1,n-c}_{2,p-1,2}.$$

In Section 3.3, we show that this map can be expressed in terms of multiplication in the Jacobi ring. In Section 3.5, the map γ is then related to the actual infinitesimal Torelli map.

Further result on automorphisms

In Section 3.6, we use Theorem II to explicitly calculate the infinitesimal Torelli map for smooth Fano threefolds of type $(1,1,4)$ and prove Theorem III. These calculations also allow us to study the action of the automorphism group on the cohomology groups. This is a very natural thing to do; see for example [11, 33, 48]. We have the following result.

Theorem 3.0.1. *Let X be a (smooth) Fano threefold of Picard rank 1, index 1 and degree 4 over $\mathbb{C}$. Then the following statements hold.*

(1) The automorphism group $\operatorname{Aut}(X)$ acts faithfully on $\mathrm{H}^1(X,\Theta^1_X)$.

(2) If X is a smooth quartic, then $\mathrm{Aut}(X)$ *acts faithfully on* $\mathrm{H}^3(X,\mathbb{C})$.

(3) If X is hyperelliptic, then the kernel $\ker\left(\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{H}^3(X,\mathbb{C}))\right)$ *is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by the involution ι.*

3.1 The contraction pairing on the affine cone

Let k be a field of characteristic zero and let $X = V_+(f_1,\ldots,f_c) \subseteq \mathbb{P}_k(W_0,\ldots,W_n)$ be a quasi-smooth weighted complete intersection of degree $(d_1,\ldots,d_c)$ with coordinate ring

$$A = S_W/(f_1,\ldots,f_c)$$

and affine cone $U = Y \setminus \{0\}$, where $Y = \operatorname{Spec} A$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}$ be the ideal sheaf of U in $\mathbb{A}^{n+1} \setminus \{0\}$. Let Ω^1_U be the sheaf of k-differentials on U and let Θ^1_U be its dual, namely the tangent sheaf. Let p be an integer satisfying $1 \leq p \leq n-c$. Building on Flenner's calculations [24, Section 8], in this section we will construct free resolutions of the sheaves Ω^p_U and extend the contraction pairing

$$\Omega^p_U \otimes \Theta^1_U \xrightarrow{\gamma} \Omega^{p-1}_U$$

to these resolutions and their associated total Čech cohomology complexes.

The resolutions

The conormal sequence associated to the closed immersion of the smooth complete intersection U into $\mathbb{A}^{n+1} \setminus \{0\}$, namely

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{\mathbb{A}^{n+1}\setminus\{0\}} \otimes \mathcal{O}_U \to \Omega^1_U \to 0,$$

is exact and Ω^1_U is locally free; see [28, Theorem II.8.17]. This uses the smoothness of U. The $\mathcal{O}_U$-module $\Omega^1_{\mathbb{A}^{n+1}\setminus\{0\}} \otimes \mathcal{O}_U$ is free of rank $n+1$ and spanned by the elements $dx_0,\ldots,dx_n$. The conormal sheaf $\mathcal{I}/\mathcal{I}^2$ of the complete intersection U is free of rank c and is generated by the elements $f_1,\ldots,f_c$. Hence the conormal sequence is described by the exact sequence

$$0 \to F = \bigoplus_{i\in\{1,\ldots,c\}} \mathcal{O}_U \cdot y_i \xrightarrow{\phi} G = \bigoplus_{j\in\{0,\ldots n\}} \mathcal{O}_U \cdot dx_j \xrightarrow{\pi} \Omega^1_U \to 0 \tag{3.1.2}$$

of $\mathcal{O}_U$-modules, where the y_i are basis elements, the morphism ϕ is the $\mathcal{O}_U$-linear map with

$$\phi(y_i) = d(f_i) = \sum_{j=0}^{n} \partial_j(f_i) \cdot dx_j$$

and π is the natural surjection. Note, if we set $\deg(y_i) = d_i$ and $\deg(dx_i) = w_i$, then the induces morphisms $\phi_U : \Gamma(U,F) \to \Gamma(U,G)$ and $\pi_U : \Gamma(U,G) \to \Gamma(U,\Omega^1_U)$ are homogeneous of degree 0.

For any quasi-coherent $\mathcal{O}_U$-module N and $r \in \mathbb{Z}_{\geq 0}$, let $S^r(N)$ denote the r-th symmetric power of N. As the $\mathcal{O}_U$-module F is free with a basis $y_1,\ldots,y_c$, the symmetric power $S^r(F)$ is free with a basis formed by the elements

$$y^\lambda := y_1^{\lambda_1} \cdot \ldots \cdot y_c^{\lambda_c}$$

where $\lambda \in \mathbb{Z}^c_{\geq 0}$ with $\sum \lambda_i = r$. For the notation y^λ, we will allow $\lambda \in \mathbb{Z}^c$. Namely, if $\lambda_i < 0$ for some i, then we set $y^\lambda = 0$. Similarly to [53, example (ii)], we define the complex $(K_p^\bullet, d_{K_p}^\bullet)$ of $\mathcal{O}_U$-modules with components

$$K_p^q = S^{-q}(F) \otimes \bigwedge^{p+q}(G)$$

for $-p \leq q \leq 0$ and $K_p^q = 0$ otherwise and differential

$$K_p^q = S^{-q}(F) \otimes \bigwedge^{p+q}(G) \to K_p^{q+1} = S^{-q-1}(F) \otimes \bigwedge^{p+q+1}(G)$$

given as the $\mathcal{O}_U$-linear map that sends $y^\lambda \otimes \omega$, where $\lambda \in \mathbb{Z}^c_{\geq 0}$ with $\sum_{i=c} \lambda_i = -q$ and $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_{p+q}}$, to

$$\sum_{i=1}^{c} y^{\lambda - e_i} \otimes d(f_i) \wedge \omega,$$

where $e_i \in \mathbb{Z}^c$ denotes the i-th standard basis vector.

By composing it with the natural surjection $K_p^0 = \bigwedge^p(G) \to \Omega_U^p$, we get a complex

$$0 \to K_p^{-p} \to \cdots \to K_p^0 \to \Omega_U^p \to 0.$$

Note for $p = 1$ this is Sequence (3.1.2). By dualizing the exact sequence (3.1.2) of locally free sheaves, we get an exact sequence

$$0 \to \Theta_U^1 \xrightarrow{\pi^*} G^* = \bigoplus_{i=0}^{n} \mathcal{O}_U \cdot \delta_i \xrightarrow{\phi^*} F^* = \bigoplus_{j=1}^{c} \mathcal{O}_U \cdot y_j^* \to 0,$$

where the elements $\delta_0, \ldots, \delta_n$ are the dual basis for $dx_0, \ldots, dx_n$ and the elements $y_1^*, \ldots, y_c^*$ are the dual basis for $y_1, \ldots, y_c$. The differential ϕ^* maps δ_i to $\sum_{j=1}^{c} \partial_{x_i}(f_j) \cdot y_j^*$. Again, we set $\deg(\delta_i) = -w_i$ and $\deg(y_i^*) = -d_i$, so that the sequence becomes homogeneous of degree 0 on global sections. We define the complex $(K_{-1}^\bullet, d_{K_{-1}}^\bullet)$ with components $K_{-1}^0 = G^*$, $K_{-1}^1 = F^*$ and $K_{-1}^q = 0$ if $q \notin \{0, 1\}$ and differential ϕ^*. These complexes give the desired resolutions.

Theorem 3.1.1. *In the situation above, for every $p \in \{1, \ldots, n-c\}$, the complex of $\mathcal{O}_U$-modules*

$$0 \to K_p^{-p} \to \cdots \to K_p^0 \to \Omega_U^p \to 0$$

is exact. Furthermore the complex of $\mathcal{O}_U$-modules

$$0 \to \Theta_U^1 \to K_{-1}^0 \to K_{-1}^1 \to 0$$

is exact.

Proof. We have already proven the second statement and the first statement for $p = 1$. Let $p > 1$ and let $V = \operatorname{Spec} B \subseteq U$ be any affine open such that the sheaf Ω_U^1 restricted to V is free. Let $M =$

$\Gamma(V, \Omega^1_U)$ be a free B-module. Hence it is m-torsion-free (see [53, Introduction] for definition) for any positive integer $m \in \mathbb{Z}_{>0}$. Hence by applying [53, Theorem 3.1] (note, since $\mathrm{char}(k) = 0$, the ring B is a $\mathbb{Q}$-algebra and hence the divided powers used in that reference are isomorphic to symmetric powers) to M with the free resolution

$$0 \to \Gamma(V, K_1^{-1}) \to \Gamma(V, K_1^0) \to M \to 0,$$

we see that the complex

$$0 \to \Gamma(V, K_p^{-p}) \to \cdots \to \Gamma(V, K_p^0) \to \Gamma(V, \Omega^p_U) \to 0$$

is exact. Since Ω^1_U is locally free, we can cover U with affine opens V such that the restriction is free. So we are done. *q.e.d.*

The pairing of resolutions

There are $\mathcal{O}_U$-bilinear contraction maps

$$\begin{aligned} \tilde{\gamma}_G \colon \quad & \bigwedge^q (G) \times G^* \to \bigwedge^{q-1} (G) \\ (dx_{i_1} \wedge \cdots \wedge dx_{i_q}, \theta) & \mapsto \sum_{j=1}^q (-1)^j \theta(dx_{i_j}) dx_{i_1} \wedge \ldots d\hat{x}_{i_j} \ldots \wedge dx_{i_q} \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}_F \colon S^q(F) \times F^* & \to S^{q-1}(F) \\ (y^\lambda, \mu) & \mapsto \sum_{i=1}^r y^{\lambda - e_i} \mu(y_i). \end{aligned}$$

These contraction maps induce morphisms

$$\mathrm{id}_F \otimes \tilde{\gamma}_G \colon K_p^q \otimes K_{-1}^0 \to K_{p-1}^q$$

and

$$\mathrm{id}_G \otimes \tilde{\gamma}_F \colon K_p^q \otimes K_{-1}^1 \to K_{p-1}^{q+1}.$$

We define

$$\tilde{\gamma}^q \colon \ \mathrm{Tot}^q \left(K_p^\bullet \otimes K_{-1}^\bullet \right) = K_p^{q-1} \otimes K_{-1}^1 \oplus K_p^q \otimes K_{-1}^0 \to K_{p-1}^q$$

as

$$\tilde{\gamma}^q = \mathrm{id}_G \otimes \tilde{\gamma}_F \oplus (-1)^q \, \mathrm{id}_F \otimes \tilde{\gamma}_G.$$

These maps are compatible with the differentials and extend γ.

Lemma 3.1.2. *The maps above define a pairing of complexes*

$$\tilde{\gamma} \colon \ \mathrm{Tot}^\bullet \left(K_p^\bullet \otimes K_{-1}^\bullet \right) \to K_{p-1}^\bullet$$

and induce the contraction pairing, i.e., given sections θ of Θ^1_U and ω' of $\bigwedge^p G$ with $\omega := (\bigwedge^p \pi)(\omega')$, we have

$$\gamma(\omega, \theta) = \left(\bigwedge^{p-1} \pi\right) \circ \tilde{\gamma}(\omega', \pi^*(\theta)).$$

Proof. First, we show that the maps define a morphism of $\mathcal{O}_U$-modules

$$\mathrm{Tot}^q\left(K_p^\bullet \otimes K_{-1}^\bullet\right) = K_p^{q-1} \otimes K_{-1}^1 \oplus K_p^q \otimes K_{-1}^0 \to K_{p-1}^q.$$

As all maps involved are $\mathcal{O}_U$-linear, we may verify $d_{K_{p-1}} \circ \tilde{\gamma} = \tilde{\gamma} \circ d_{Tot}$ on generating elments. So let

$$\xi = (y^\lambda \otimes \omega) \otimes \mu \in K_p^{q-1}(U) \otimes K_{-1}^1(U).$$

Then

$$\begin{aligned} d_{Tot}(\xi) &= d_{K_p}(y^\lambda \otimes \omega) \otimes \mu + (-1)^{q-1} y^\lambda \otimes \omega \otimes d_{K_{-1}}(\mu) \\ &= \sum_{j=1}^{c} y^{\lambda - e_j} \otimes d(f_j) \wedge \omega \otimes \mu. \end{aligned}$$

Hence,

$$\tilde{\gamma} \circ d_{Tot}(\xi) = \sum_{l=1}^{c} \sum_{j=1}^{c} \mu(y_l) \cdot y^{\lambda - e_j - e_l} \otimes d(f_j) \wedge \omega.$$

On the other hand,

$$\tilde{\gamma}(\xi) = \sum_{l=1}^{c} \mu(y_l) \cdot y^{\lambda - e_l} \otimes \beta_j \wedge \omega.$$

Hence,

$$d_{K_{p-1}} \circ \tilde{\gamma}(\xi) = \sum_{j=1}^{c} \sum_{l=1}^{c} \mu(y_l) \cdot y^{\lambda - e_j - e_l} \otimes d(f_j) \wedge \omega.$$

Now let

$$\eta = \left(y^\lambda \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p+q}}\right) \otimes \theta \in K_p^q(U) \otimes K_{-1}^0(U).$$

Then

$$\begin{aligned} d_{Tot}(\eta) &= d_{K_p}(y^\lambda \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p+q}}) \otimes \theta \\ &\quad + (-1)^q y^\lambda \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p+q}} \otimes d_{K_{-1}}(\theta) \\ &= \sum_{j=1}^{c} y^{\lambda - e_j} \otimes d(f_j) \wedge b_{i_1} \wedge \cdots \wedge b_{i_{p+q}} \otimes \theta \\ &\quad + (-1)^q y^\lambda \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p+q}} \otimes \phi^*(\theta). \end{aligned}$$

Hence,

$$\begin{aligned}\tilde{\gamma} \circ d_{Tot}(\eta) &= (-1)^{q+1} \sum_{j=1}^{c} \theta(d(f_j)) \cdot y^{\lambda - e_j} \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p+q}} \\ &+ (-1)^{q+1} \sum_{j=1}^{c} \sum_{l=1}^{p+q} (-1)^{l+1} \theta(b_{i_l}) \cdot y^{\lambda - e_j} \otimes d(f_j) \wedge b_{i_1} \wedge \dots \hat{b_{i_l}} \cdots \wedge b_{i_{p+q}} \\ &+ (-1)^{q} \sum_{j=1}^{c} \theta(d(f_j)) \cdot y^{\lambda - e_j} \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p+q}} \\ &= (-1)^{q+1} \sum_{j=1}^{c} \sum_{l=1}^{p+q} (-1)^{l+1} \theta(b_{i_l}) \cdot y^{\lambda - e_j} \otimes d(f_j) \wedge b_{i_1} \wedge \dots \hat{b_{i_l}} \cdots \wedge b_{i_{p+q}}.\end{aligned}$$

On the other hand,

$$\tilde{\gamma}(\eta) = \sum_{l=1}^{p+q} (-1)^{l} \theta(b_{i_l}) y^{\lambda} \otimes b_{i_1} \wedge \dots \hat{b_{i_l}} \cdots \wedge b_{i_{p+q}}.$$

Hence,

$$d_{K_{p-1}} \circ \tilde{\gamma}(\eta) = (-1)^{q} \sum_{j=1}^{c} \sum_{l=1}^{p+q} (-1)^{l} \theta(b_{i_l}) y^{\lambda - e_j} \otimes d(f_j) \wedge b_{i_1} \wedge \dots \hat{b_{i_l}} \cdots \wedge b_{i_{p+q}}.$$

q.e.d.

The pairing of the total Čech complexes

Let $\mathcal{U}$ be an open affine covering of U. For $p \in \{-1, 1, \dots, n-c\}$, let $\check{C}^\bullet(\mathcal{U}, K_p^\bullet)$ be the Čech double complex (as defined in [72, Tag 01FP]) and let

$$L_p^\bullet = \mathrm{Tot}^\bullet\left(\check{C}^\bullet(\mathcal{U}, K_p^\bullet)\right)$$

be the associated total complex. We consider the cup product map of complexes

$$\cup\colon\ \mathrm{Tot}^\bullet(L_p^\bullet \otimes L_{-1}^\bullet) \to \mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \mathrm{Tot}^\bullet(K_p^\bullet \otimes K_{-1}^\bullet)))$$

as defined in [72, Tag 07MB] and compose it with the map

$$\mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \mathrm{Tot}^\bullet(K_p^\bullet \otimes K_{-1}^\bullet))) \to \mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, K_{p-1}^\bullet))$$

induced by the pairing of complexes from Lemma 3.1.2 to obtain a pairing of complexes

$$\bar{\gamma}\colon\ \mathrm{Tot}^\bullet(L_p^\bullet \otimes L_{-1}^\bullet) \to L_{p-1}^\bullet.$$

3.2 Cohomology for weighted complete intersections

In this section, we explain how to calculate the cohomology of certain coherent sheaves on weighted complete intersections. We give an overview of results on that matter found in [17] and [24, Section 8].

Definition 3.2.1. For any integer $l \in \mathbb{Z}$, let $S_W(l)$ be the graded S_W module with

$$S_W(l)_m = (S_W)_{l+m}$$

for all $m \in \mathbb{Z}$. The sheaf $\mathcal{O}_{\mathbb{P}(W)}(l)$ is the $\mathcal{O}_{\mathbb{P}(W)}$-module associated to $S_W(l)$. For any closed subvariety $Y \xrightarrow{\alpha} \mathbb{P}(W)$, we define the sheaf

$$\mathcal{O}_Y(l) := \alpha^* \mathcal{O}_{\mathbb{P}(W)}(l).$$

We start with weighted projective space, where a similar statement can be found in [17]. The proof for the case of usual projective space, found in [28, Theorem III.5.1], also works in the general case.

Lemma 3.2.2. *Let k be a field, let $W \in \mathbb{N}^{n+1}$ be weights, let $S_W = k[x_0, \dots, x_n]$ be the weighted polynomial algebra and let $\mathbb{P} = \mathbb{P}_k(W) = \operatorname{Proj} S_W$ be the weighted projective space. Then the following statements hold.*

(1) *The natural map $S_W \to \bigoplus_{l \in \mathbb{Z}} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ of graded S_W-modules is an isomorphism.*

(2) *We have $\mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l)) = 0$ for $q \neq 0, n$ and $l \in \mathbb{Z}$.*

(3) *In Čech cohomology with respect to the covering $\mathcal{U} = \{D_+(x_i)\}$, we have*

$$\begin{aligned}&\bigoplus_{l \in \mathbb{Z}} \check{\mathrm{H}}^n(\mathcal{U}, \mathbb{P}, \mathcal{O}_{\mathbb{P}}(l)) \\ &= \operatorname{coker}\left(k\langle x_0^{\alpha_0} \dots x_n^{\alpha_n} \mid \text{there exists } i \text{ with } \alpha_i \geq 0\rangle \to S_W[1/x_0, \dots, 1/x_n]\right).\end{aligned}$$

Proof. We compute the Čech cohomology with repect to the covering $\mathcal{U} = \{D_+(x_i)\}$. We have

$$\check{C}^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}}(l)) = \bigoplus_{i_0, \dots, i_q} S_W[1/x_{i_0}, \dots, 1/x_{i_q}]_l.$$

This immediately proves (3). And (1) follows since

$$\bigcap_{i=0}^{n} S_W[1/x_i]_l = (S_W)_l.$$

It remains to prove statement (2). We proceed by induction on n. In the case $n = 0$, there is nothing to show. So we assume $n > 0$. If we localize the Čech complex at x_n, then the resulting complex describes the Čech complex of the $\mathcal{O}_{D_+(x_n)}$-module $S_W[1/x_n](l)$ on the affine scheme $D_+(x_n) = \operatorname{Spec} S_W[1/x_n]_0$ with respect to to the covering $\{D_+(x_i) \cap D_+(x_n)\}$. As $D_+(x_n)$ is affine, we see that $\mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))[1/x_n] = 0$ for $q > 0$. Hence, every element of $\mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ is annihilated by some power of x_n. To complete the proof, we will show that multiplication by x_n induces a bijective endomorphism of $\bigoplus_l \mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$. Let $W' = (W_0, \ldots, W_{n-1})$ and $\mathbb{P}' = \mathbb{P}(W')$. We consider the exact sequence of graded S_W-modules

$$0 \to S_W(l - W_n) \xrightarrow{\cdot x_n} S_W(l) \to S_{W'}(l) \to 0.$$

We get an associated long exact cohomology sequence

$$\cdots \to \mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l - W_n)) \xrightarrow{\cdot x_n} \mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l)) \to \mathrm{H}^q(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(l)) \to \ldots.$$

The sequence

$$0 \to \mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l - W_n)) \xrightarrow{\cdot x_n} \mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l)) \to \mathrm{H}^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(l)) \to 0$$

is exact by (1), and $\mathrm{H}^q(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(l)) = 0$ for $0 < q < n-1$ by induction. By (3), the k-vector space $\mathrm{H}^{n-1}(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(l))$ is generated by the monomial $x_0^{\alpha_0} \ldots x_{n-1}^{\alpha_{n-1}}$ where $\alpha_i < 0$ for all $0 \leq i < n$. Furthermore, the kernel of the map

$$\mathrm{H}^n(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l - W_n)) \xrightarrow{\cdot x_n} \mathrm{H}^n(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$$

is generated by the monomials $x_0^{\alpha_0} \ldots x_n^{\alpha_n}$ where $\alpha_i < 0$ for all $0 \leq i < n$ and $\alpha_n = -1$. The border map

$$\mathrm{H}^{n-1}(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(l)) \to \mathrm{H}^n(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l - W_n))$$

is given as multiplication with $1/x_n$. This map is clearly injective. Hence, the long exact sequence shows that the multiplication by x_n is a bijective endomorphism of $\mathrm{H}^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ for $0 < q < n$. *q.e.d.*

To better handle the top cohomology, we introduce the k-dual module.

Definition 3.2.3. Let k be a field, let A be a k-algebra and let M be a graded A-module. We define the *k-dual module of M* to be the graded A-module $\mathrm{D}(M) = \bigoplus_{l \in \mathbb{Z}} \mathrm{D}(M)_l$ with $\mathrm{D}(M)_l = \mathrm{Hom}_k(M_{-l}, k)$.

Let $\alpha : M \to N$ be a morphism of graded A-modules that is homogeneous of degree l. Then the map

$$\mathrm{D}(\alpha) \colon \mathrm{D}(N) \to \mathrm{D}(M) \colon f \to f \circ \alpha$$

defines a morphism of the same degree l.

Example 3.2.4. If $A = S_W$ is a weighted polynomial algebra, then

$$\mathrm{D}(S_W)_l = \mathrm{Hom}_k((S_W)_{-l}, k).$$

Here $(S_W)_{-l}$ is spanned by the monomials $x_0^{\alpha_0} \ldots x_n^{\alpha_n}$ with $\sum \alpha_i W_i = -l$. We denote the corresponding dual basis elements by $\phi_{\alpha_0, \ldots, \alpha_n} \in \mathrm{D}(S_W)_l$.

We list some important properties of the functor D.

Lemma 3.2.5. *Let k be a field, let A be a finitely generated k-algebra and let $\alpha : M \to N$ be a morphism of finitely generated graded A-modules that is homogeneous of degree l. Then*

(1) $\mathrm{D}(\ker(\alpha)) = \mathrm{coker}(\mathrm{D}(\alpha))$,

(2) $\mathrm{D}(\mathrm{coker}(\alpha)) = \ker(\mathrm{D}(\alpha))$,

(3) *α is surjective if and only if $\mathrm{D}(\alpha)$ is injective and*

(4) *α is injective if and only if $\mathrm{D}(\alpha)$ is surjective.*

Proof. By assumption, every fixed degree part of M and N is a finite-dimensional k-vector space. The assertions now follow as dualizing is exact for such vector spaces. *q.e.d.*

Remark 3.2.6. Let $|W| = \sum W_i$. The k-vector space $\check{\mathrm{H}}^n(\mathcal{U}, \mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(l))$ vanishes if $l > -|W|$. If $l \leq -|W|$, then the vector space is spanned by the elements $x_0^{-1-\alpha_0} \dots x_n^{-1-\alpha_n}$ where $\alpha \in (\mathbb{Z}_{\geq 0})^{n+1}$ and $-|W| - \sum \alpha_i W_i = l$. The k linear map

$$\bigoplus_{l \in \mathbb{Z}} \check{\mathrm{H}}^n(\mathcal{U}, \mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(l)) \to \mathrm{D}(S)(|W|)$$

that maps $x_0^{-1-\alpha_0} \dots x_n^{-1-\alpha_n}$ to $\phi_{\alpha_0,\dots,\alpha_n}$ defines an isomorphism of graded S-modules.

Consider a complete intersection $X = V(f_1, \dots, f_c) \subseteq \mathbb{P}(W)$ of codimension c and degree $(d_1, \dots, d_c)$ in $\mathbb{P}(W)$. The surjection of coordinate rings

$$S_W \to A := S_W/(f_1, \dots, f_c)$$

naturally induces an embedding $\mathrm{D}(A) \subseteq \mathrm{D}(S)$; see Lemma 3.2.5. For $r \in \{1, \dots, c\}$, the scheme

$$X_r = \mathrm{Proj}\, A_r, \quad A_r = S_W/(f_1, \dots, f_r)$$

is a weighted complete intersection of codimension r. We have a chain of closed immersions

$$X = X_c \subseteq \dots \subseteq X_0 := \mathbb{P}(W).$$

For every $l \in \mathbb{Z}$ and $1 \leq r \leq c-1$, the ideal sheaf sequence

$$0 \to \mathcal{O}_{X_r}(l - d_{r+1}) \xrightarrow{\cdot f_{r+1}} \mathcal{O}_{X_r}(l) \to \mathcal{O}_{X_{r+1}}(l) \to 0$$

is exact, as $f_1, \dots, f_{r+1}$ is a regular sequence. Considering the associated long exact cohomology sequence, we get the following result.

Lemma 3.2.7. *Let $l \in \mathbb{Z}$ and let X be a quasi-smooth weighted complete intersection of codimension c as above. Let $|W| = \sum_{i=0}^n W_i$, and let $\nu = |W| - \sum_{i=1}^c d_i$. Suppose $\dim(X) = n - c \geq 1$. Then*

(1) *the natural map $A \to \bigoplus_{l \in \mathbb{Z}} \mathrm{H}^0(X, \mathcal{O}_X(l))$ is an isomorphism of graded A-modules,*

(2) $\mathrm{H}^q(X, \mathcal{O}_X(l)) = 0$ *for $q \neq 0, n-c$ and $l \in \mathbb{Z}$, and*

(3) $\bigoplus_{l \in \mathbb{Z}} \mathrm{H}^{n-c}(X, \mathcal{O}_X(l)) \cong \mathrm{D}(A)\,(\nu)$.

Proof. We argue by induction on the codimension. Lemma 3.2.2 and Remark 3.2.6 prove the statements for $c = 0$. So assume the lemma holds true for X_r and consider the long exact cohomology sequence associated to the ideal sheaf sequence

$$\begin{array}{l}
0 \longrightarrow A_r(-d_{r+1}) \longrightarrow A_r \longrightarrow \bigoplus_{l\in\mathbb{Z}} \mathrm{H}^0(X_{r+1}, \mathcal{O}_{X_{r+1}}(l)) \\
\longrightarrow 0 \longrightarrow 0 \longrightarrow \bigoplus_{l\in\mathbb{Z}} \mathrm{H}^1(X_{r+1}, \mathcal{O}_{X_{r+1}}(l)) \\
\dashrightarrow 0 \longrightarrow 0 \longrightarrow \bigoplus_{l\in\mathbb{Z}} \mathrm{H}^{n-r-1}(X_{r+1}, \mathcal{O}_{X_{r+1}}(l)) \\
\longrightarrow \mathrm{D}(A_r)(\nu_r - d_{r+1}) \longrightarrow \mathrm{D}(A_r)(\nu_r) \longrightarrow 0,
\end{array}$$

where $\nu_r = |W| - \sum_{i=1}^r d_i$. We see

$$\bigoplus_{l\in\mathbb{Z}} \mathrm{H}^0(X_{r+1}, \mathcal{O}_{X_{r+1}}(l)) = \frac{A_r}{f_{r+1}\cdot A_r(-d_{r+1})} = A_{r+1}.$$

By Lemma 3.2.5,

$$\begin{aligned}
\bigoplus_{l\in\mathbb{Z}} \mathrm{H}^{n-r-1}(X_{r+1}, \mathcal{O}_{X_{r+1}}(l)) &= \ker(\mathrm{D}(A_r)(\nu_r - d_{r+1}) \xrightarrow{\mathrm{D}(\cdot f_{r+1})} D(A_r)(\nu_r)) \\
&= \mathrm{D}(\operatorname{coker}(A_r(-\nu_r) \xrightarrow{\cdot f_{r+1}} A_r(-\nu_r + d_{r+1}))) \\
&= \mathrm{D}(A_{r+1})(\nu_{r+1}).
\end{aligned}$$

q.e.d.

Remark 3.2.8. If A is the coordinate ring of a weighted complete intersection X with affine cone $U = Y \setminus \{0\}$, where $Y = \operatorname{Spec} A$ and M is a graded A-module, then there is a natural isomorphism of graded A-modules

$$\mathrm{H}^q(U, M^\sim|_U) \cong \bigoplus_{l\in\mathbb{Z}} \mathrm{H}^q(X, (M(l))^\sim),$$

where $M(l)$ denotes the module M with grading shifted by l, and $(\_)^\sim$ denotes the functor that associates to an A-module its associated $\mathcal{O}_Y$-module (respectively its associated graded $\mathcal{O}_X$-module). This isomorphism can be established by compairing the Čech cohomology with respect to the coverings $\{D(x_i)\}$ for U and $\{D_+(x_i)\}$ for X or with methods of local cohomology (see [24, Section 8]). We can use this identification to bring the results above in a more compact form. In particular, we have

$$\mathrm{H}^q(U, \mathcal{O}_U) \cong \bigoplus_{l\in\mathbb{Z}} \mathrm{H}^q(X, \mathcal{O}_X(l)).$$

3.3 The Jacobi ring of a weighted complete intersection

In this section, we will introduce the Jacobi ring of a weighted complete intersection and explain how cohomology can be expressed in terms of it. Our methods build on Flenner's calculation in [24, Section 8]. We continue with notations and conventions from Section 3.1. As all components of the complexes $K_p^\bullet$ are free, the homology of the associated total complex of the Čech double complexes with respect to the affine covering $\mathcal{U}$ calculates the hypercohomology of these complexes, i.e.,

$$\mathbb{H}^q(U, K_p^\bullet) = \mathrm{H}^q(\mathrm{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, K_p^\bullet))) = \mathrm{H}^q(L_p^\bullet);$$

see [72, Tag 0FLH]. The total complex associated to a double complex comes with two filtrations F_1 and F_2 given by

$$F_1^r(L_p^q) = \bigoplus_{i+j=q, i\geq r} \check{C}^i(\mathcal{U}, K_p^j)$$

and

$$F_2^r(L_p^q) = \bigoplus_{i+j=q, j\geq r} \check{C}^i(\mathcal{U}, K_p^j);$$

see [72, Tag 012X]. The pairing $\bar{\gamma}$ is compatible with these filtrations. Hence, by Theorem 2.3.3, we get pairings of the associated spectral sequences, one for each filtration. We denote the spectral sequences associated to the filtered complex $(L_p^\bullet, F_i)$ by $(E_{i,p,r}^{\bullet,\bullet})_{r\in\mathbb{Z}_{\geq 0}}$. See Section 2.3 or [72, Tag 0130] for formulas for the computation of the pages of these spectral sequences. We first compute the pairing of spectral sequences associated to the filtration F_1. By Theorem 3.1.1, on the first page, we see

$$E_{1,p,1}^{s,t} = \mathrm{H}^t(\check{C}^s(\mathcal{U}, K_p^\bullet)) = \begin{cases} \check{C}^s(\mathcal{U}, \Omega_U^p) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

if $p > 0$ and

$$E_{1,-1,1}^{s,t} = \mathrm{H}^t(\check{C}^s(\mathcal{U}, K_{-1}^\bullet) = \begin{cases} \check{C}^s(\mathcal{U}, \Theta_U^1) & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, all spectral sequences converge on the second page with

$$E_{1,p,\infty}^{s,t} = E_{1,p,2}^{s,t} = \begin{cases} \mathrm{H}^s(U, \Omega_U^p)) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E_{1,-1,\infty}^{s,t} = E_{1,-1,2}^{s,t} = \begin{cases} \mathrm{H}^s(U, \Theta_U^1) & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 2.3.3, there is a pairing induced by $\bar{\gamma}$

$$\mathrm{Tot}^\bullet(E_{1,p,\infty}^{\bullet,\bullet} \otimes E_{1,-1,\infty}^{\bullet,\bullet}) \to E_{1,p-1,\infty}^{\bullet,\bullet}.$$

In particular, we obtain a pairing

$$\mathrm{H}^{s_1}(U, \Omega_U^p) \otimes \mathrm{H}^{s_2}(U, \Theta_U^1) \to \mathrm{H}^{s_1+s_2}(U, \Omega_U^{p-1}).$$

We note that it is the pairing induced by the contraction map $\gamma\colon \Omega_U^p \otimes \Theta_U^1 \to \Omega_U^{p-1}$ on cohomology, see Lemma 3.1.2. Now, we compute the pairing of spectral sequences associated to the filtration F_2. On the first page, we see

$$E_{2,p,1}^{s,t} = \mathrm{H}^t(\check{C}^\bullet(\mathcal{U}, K_p^s)) = \mathrm{H}^t(U, K_p^s).$$

All modules involved in the complex $K_p^\bullet$ are free. So by Lemma 3.2.7, we see that the spectral sequence satisfies $E_{2,p,1}^{s,t} = 0$ if $t \neq 0, n-c$. We made the assumption that $p < n-c$. Hence, we see that the spectral sequences converge on page 2 since the differential never connects non-vanishing parts on later pages. We have

$$E_{2,p,\infty}^{s,t} = E_{2,p,2}^{s,t} = \begin{cases} \mathrm{H}^s(\mathrm{H}^t(U, K_p^\bullet)) & \text{if } t \in \{0, n-c\} \\ 0 & \text{otherwise} \end{cases}.$$

The pairing

$$\mathrm{Tot}^\bullet(E_{2,p,\infty}^{\bullet,\bullet} \otimes E_{2,-1,\infty}^{\bullet,\bullet}) \to E_{2,p-1,\infty}^{\bullet,\bullet}$$

is the one induced by the pairing

$$\mathrm{Tot}^\bullet(K_p^\bullet \otimes K_{-1}^\bullet) \to K_{p-1}^\bullet$$

on cohomology. Note that for both filtrations, all spectral sequences converge in such a way that for each integer m there is only one combination of (s,t) depending on m such that $s+t=m$ and

$$\mathrm{gr}^s \mathrm{H}^m(L_p^\bullet) = E_{i,p,\infty}^{s,t} \neq 0;$$

see Equation (2.3.3)). That means

$$F^q \mathrm{H}^m(L_p^\bullet) = \begin{cases} 0 & \text{if } q > s \\ \mathrm{H}^m(L_p^\bullet) & \text{if } q \leq s \end{cases}$$

and therefore in the diagram

$$\mathrm{H}^m(L_p^\bullet) \xleftarrow{\alpha^{m,s}} F^s\mathrm{H}^m(L_p^\bullet) \xrightarrow{\beta^{m,s}} \mathrm{gr}^s \mathrm{H}^m(L_p^\bullet),$$

the maps $\alpha^{m,s}$ and $\beta^{m,s}$ are both isomorphisms. We combine Diagram (2.2.2) for suitable choices of i and j with the diagram from Theorem 2.3.3 to get a commutative diagram

$$\begin{array}{ccccc} E_{1,p,\infty}^{n-c-p,0} \otimes E_{1,-1,\infty}^{1,0} & \longrightarrow & \mathrm{H}^{n-c-p}(L_p^\bullet) \otimes \mathrm{H}^1(L_{-1}^\bullet) & \longleftarrow & E_{2,p,\infty}^{-p,n-c} \otimes E_{2,-1,\infty}^{1,0} \\ \downarrow & & \downarrow & & \downarrow \\ E_{1,p-1,\infty}^{n-c-p+1,0} & \longrightarrow & \mathrm{H}^{n-c-p+1}(L_{p-1}^\bullet) & \longleftarrow & E_{2,p-1,\infty}^{1-p,n-c} \end{array}$$

where the horizontal morphisms are isomorphisms. Thus, we have identified the pairings of spectral sequences for the filtrations F_1 and F_2 with each other. As shown above, the pairing

$$E_{1,p,\infty}^{n-c-p,0} \otimes E_{1,-1,\infty}^{1,0} \to E_{1,p-1,\infty}^{n-c-p+1,0}$$

is identified with the contraction map

$$\mathrm{H}^{n-c-p}(U, \Omega_U^p) \otimes \mathrm{H}^1(U, \Theta_U^1) \to \mathrm{H}^{n-c-p+1}(U, \Omega_U^{p-1}).$$

On the other hand, the pairing

$$E_{2,p,\infty}^{-p,n-c} \otimes E_{2,-1,\infty}^{1,0} \to E_{1,p-1,\infty}^{1-p,n-c}$$

is the pairing

$$\mathrm{H}^{-p}(\mathrm{H}^{n-c}(U, K_p^\bullet)) \otimes \mathrm{H}^1(\mathrm{H}^0(U, K_{-1}^\bullet)) \to \mathrm{H}^{1-p}(\mathrm{H}^{n-c}(U, K_{p-1}^\bullet))$$

induced by $\tilde{\gamma}$. We now explicitly calculate all the cohomology groups involved in this pairing. The group $\mathrm{H}^{-p}(\mathrm{H}^{n-c}(U, K_p^\bullet)$ is the kernel of the map

$$\mathrm{H}^{n-c}(U, K_p^{-p}) \to \mathrm{H}^{n-c}(U, K_p^{1-p}).$$

We compute:

$$\mathrm{H}^{n-c}(U, K_p^{-p}) = \mathrm{H}^{n-c}(U, (S^p(F))) = \bigoplus_{\sum \beta_i = p} \mathrm{H}^{n-c}(U, \mathcal{O}_U) \cdot y^\beta,$$

$$\mathrm{H}^{n-c}(U, K_p^{1-p}) = \bigoplus_{i=0}^{n} \bigoplus_{\sum \beta_i = p-1} \mathrm{H}^{n-c}(U, \mathcal{O}_U) \cdot y^\beta dx_i.$$

We note that $\deg(y^\beta) = \sum \beta_i d_i$, and $\deg(dx_i) = W_i$. Hence, by Lemma 3.2.7 and Remark 3.2.8, we see that

$$\mathrm{H}^{n-c}(U, K_p^{-p}) = \bigoplus_{\sum \beta_i = p} \mathrm{D}\left(A\left(-\nu + \sum \beta_i d_i\right)\right) \cdot y^\beta,$$

and that

$$\mathrm{H}^{n-c}(U, K_p^{-p}) = \bigoplus_{i=0}^{n} \bigoplus_{\sum \beta_i = p-1} \mathrm{D}\left(A\left(-\nu + W_i + \sum \beta_i d_i\right)\right) \cdot y^\beta dx_i.$$

By Lemma 3.2.5, the kernel of the map $\mathrm{H}^{n-c}(U, K_p^{-p}) \to \mathrm{H}^{n-c}(U, K_p^{1-p})$ is the k-dual of the cokernel of the map

$$\alpha \colon \bigoplus_{i=0}^{n} \bigoplus_{\sum \beta_i = p-1} A\left(-\nu + W_i + \sum \beta_i d_i\right) \cdot y^\beta dx_i \to \bigoplus_{\sum \beta_i = p} A\left(-\nu + \sum \beta_i d_i\right) \cdot y^\beta$$

that maps $y^\beta dx_i$ to $\sum_j \partial_{x_i}(f_j) y^{\beta + e_j}$. To describe the cokernel of this map, we introduce the Jacobi ring.

Definition 3.3.1. Let $X = V(f_1, \ldots, f_c) \subseteq \mathbb{P}(W)$ be a weighted complete intersection of multidegree $(d_1, \ldots, d_c)$. Let $k[x_0, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring with bigrading $\deg(x_i) = (0, w_i)$, $\deg(y_j) = (1, -d_j)$. The polynomial $F = y_1 f_1 + \cdots + y_c f_c$ is bihomogeneous of degree $(1, 0)$. We define the *Jacobi ring of Y* to be the bigraded ring

$$R = k[x_0, \ldots, x_n, y_1, \ldots, y_c] / (\partial_{x_0}(F), \ldots, \partial_{x_n}(F), \partial_{y_0}(F), \ldots, \partial_{y_c}(F)).$$

We see that $\operatorname{coker}(\alpha)$ is the part of R in which we fix the first degree to be p. In fact, if we view this part $R_{p,*}$ as a graded module via $\deg_2$, we get an isomorphism

$$\operatorname{coker}(\alpha) \cong R_{p,*}(-\nu)$$

of graded modules. This shows

$$\begin{aligned} \mathrm{H}^{-p}(\mathrm{H}^{n-c}(U, K_p^\bullet)) &= \ker(\mathrm{H}^{n-c}(U, K_p^{-p}) \to \mathrm{H}^{n-c}(U, K_p^{1-p})) \\ &= \mathrm{D}(\operatorname{coker}(\alpha)) \\ &= \mathrm{D}(R_{p,*})(\nu). \end{aligned}$$

Next we calculate $\mathrm{H}^1(\mathrm{H}^0(U, K_{-1}^\bullet))$. It is the cokernel of the map

$$\mathrm{H}^0(G^*) = \bigoplus_{i=0}^{n} A(W_i) \cdot \delta_i \xrightarrow{\phi^*} \mathrm{H}^0(F^*) = \bigoplus_{j=1}^{c} A(d_j) \cdot y_j^*,$$

where the differential maps δ_i to $\sum_{j=1}^{c} \partial_{x_i}(f_j) \cdot y_j^*$. By Lemma 3.2.7 and Remark 3.2.8, we can identify F^* with $\mathrm{H}^0(U, F^*)$ and G^* with $\mathrm{H}^0(U, G^*)$. Hence it can be identified with the $\deg_1 = 1$ part of the Jacobi ring, namely

$$\mathrm{H}^1(\mathrm{H}^0(U, K_{-1}^\bullet)) = R_{1,*}.$$

For $x \in R_{1,*}$, let $m_x \colon R_{p-1,*} \to R_{p,*}$ be the multiplication by x. Now under these identifications, the pairing is explicitly given as

$$\mathrm{D}(R_{p,*})(\nu) \otimes R_{1,*} \to \mathrm{D}(R_{p-1,*})(\nu) \colon \quad \varphi \otimes x \mapsto \varphi \circ m_x.$$

We have proven the following.

Lemma 3.3.2. *Let A be the coordinate ring of a quasi-smooth weighted projective complete intersection $X = V(f_1, \ldots, f_c) \subseteq \mathbb{P}(W_0, \ldots, W_n)$ of degree $(d_1, \ldots, d_c)$ with affine cone U. Let $\nu = \sum W_i - \sum d_j$. There are isomorphisms $\mathrm{H}^{n-c-p}(U, \Omega_U^p) \cong \mathrm{D}(R_{p,*})(\nu)$ and $\mathrm{H}^1(U, \Theta_U^1) \cong R_{1,*}$. Under theses isomorphisms the contraction pairing*

$$\mathrm{H}^{n-c-p}(U, \Omega_U^p) \otimes \mathrm{H}^1(U, \Theta_U^1) \to \mathrm{H}^{n-c-p+1}(U, \Omega_U^{p-1})$$

is the pairing

$$\mathrm{D}(R_{p,*})(\nu) \otimes R_{1,*} \to \mathrm{D}(R_{p-1,*})(\nu) \colon \quad \varphi \otimes x \mapsto \varphi \circ m_x$$

Remark 3.3.3. Giving the pairing

$$\mathrm{H}^{n-c-p}(U, \Omega_U^p) \otimes \mathrm{H}^1(U, \Theta_U^1) \to \mathrm{H}^{n-c-p+1}(U, \Omega_U^{p-1}).$$

is equivalent to giving a map

$$\mathrm{H}^1(U, \Theta_U^1) \to \operatorname{Hom}(\mathrm{H}^{n-c-p}(U, \Omega_U^p), \mathrm{H}^{n-c-p+1}(U, \Omega_U^{p-1})).$$

Under the identifications given in Lemma 3.3.2, this is the map

$$R_{1,*} \to \operatorname{Hom}(\mathrm{D}(R_{p,*})(\nu), \mathrm{D}(R_{p-1,*})(\nu)) = \operatorname{Hom}(R_{p-1,*}(-\nu), R_{p,*}(-\nu))$$

that sends a homogeneous element $x \in R_{1,*}$ to m_x.

3.4 Hodge structure on V-varieties

Following [63, Section 2.5], we recall some facts about almost Kähler V-varieties (e.g., quasi-smooth weighted complete intersections).

Definition 3.4.1. A complex analytic space X is an *n-dimensional V-manifold* if there is an open cover $X = \bigcup X_i$ such that $X_i = U_i/G_i$ is the quotient of an open subset $U_i \subseteq \mathbb{C}^n$ by a finite group G_i acting holomorphically on X_i. A V-manifold X is *almost Kähler* if there exists a manifold Y that is bimeromorphic to a Kähler manifold and a *proper modification* $f : Y \to X$, i.e., a proper holomorphic map which is biholomorphic over the complement of a nowhere dense analytic subset.

There are generalized sheaves of differentials on almost Kähler V-manifolds.

Definition 3.4.2. Let X be a V-manifold. Let $i\colon X_{sm} \to X$ be the inclusion map of the smooth locus. Define

$$\tilde{\Omega}^p_X = i_*\Omega^p_{X_{sm}}.$$

The cohomology of these sheaves determines a Hodge structure, which coincides with the usual Hodge decomposition in the compact Kähler case; see [63, Theorem 2.43] and its proof.

Theorem 3.4.3. *Let X be an almost Kähler V-manifold. Then, the complex $\tilde{\Omega}^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}_X$. Furthermore the spectral sequence in hypercohomology*

$$E_1^{p,q} = \mathrm{H}^q(X, \tilde{\Omega}^q_X) \Rightarrow \mathrm{H}^{p+q}(X, \mathbb{C})$$

degenerates on page 1, and $\mathrm{H}^l(X, \mathbb{Q})$ admits a Hodge structure of weight l given by

$$\mathrm{H}^l(X, \mathbb{Q}) \otimes \mathbb{C} = \mathrm{H}^l(X, \mathbb{C}) = \bigoplus_{p+q=l} \mathrm{H}^q(X, \tilde{\Omega}^q_X).$$

As remarked in [24, Section 7] there are multiple equivalent ways of defining the $\tilde{\Omega}^p_X$. For us, the identification with the reflexive hull of the usual sheaf of differentials is relevant.

Lemma 3.4.4. *Let k be an algebraically closed field and let X be a normal integral scheme of finite type over k and let $i\colon X_{sm} \to X$ denote the inclusion of the smooth locus. Then there is a canonical isomorphism*

$$(\Omega^p_X)^{**} \to i_*\Omega^p_{X_{sm}}.$$

Proof. The restriction map $\Omega^p_X \to i_*\Omega^p_{X_{sm}}$ induces a map of the corresponding reflexive hulls $(\Omega^p_X)^{**} \to (i_*\Omega^p_{X_{sm}})^{**}$. As $\Omega^p_{X_{sm}}$ is reflexive, there is a canonical isomorphism $(i_*\Omega^p_{X_{sm}})^{**} \cong i_*\Omega^p_{X_{sm}}$. The induced map

$$(\Omega^p_X)^{**} \to i_*\Omega^p_{X_{sm}}$$

is a map of reflexive sheaves that restricted to X_{sm} is an isomorphism. Note since X is normal, the complement of the smooth locus $X \setminus X_{sm}$ has a codimension of at least 2. Hence it is an isomorphism by [29, Proposition 1.6]. *q.e.d.*

Remark 3.4.5. If $X \subseteq \mathbb{P}_{\mathbb{C}}(W)$ is a quasi-smooth weighted projective variety, then X is normal; see [17, Proposition 1.3.3] for the case $X = \mathbb{P}_{\mathbb{C}}(W)$, the argument given there, namely that X is the quotient of its smooth (and hence normal) affine cone by a finite group, also applies in the general case. Hence by Lemma 3.4.4 the generalized sheaf of differentials $\tilde{\Omega}^p_X$ is canonically isomorphic to the reflexive hull $(\Omega^p_X)^{**}$. In particular, the tangent sheaf Θ^1_X is therefore canonically isomorphic to the dual $\tilde{\Theta}^1_X := (\tilde{\Omega}^1_X)^*$ of $\tilde{\Omega}^1_X$.

3.5 Infinitesimal Torelli for weighted complete intersections

In this section, we proof Theorem II. We continue with notations from Section 3.3. From now on we choose the base field $k = \mathbb{C}$.

Let $X = V(f_1, \dots, f_c) \subseteq \mathbb{P}_{\mathbb{C}}(W_0, \dots, W_n)$ be a weighted complete intersection of degree $(d_1, \dots, d_c)$ with affine cone U. Let $A = S_W/(f_1, \dots, f_c)$ be its coordinate ring. Let $Y = \operatorname{Spec} A$, and let $U = Y \setminus \{0\}$ be the affine cone. Let Ω^1_A be the sheaf of $\mathbb{C}$-differentials on A, and let $\Omega^p_A = \bigwedge^p \Omega^1_A$. We define the *Euler map* as the A-linear morphism

$$\xi \colon \Omega^p_A \to \Omega^{p-1}_A$$

that sends $dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ to $\sum_{j=1}^p (-1)^j W_j x_j \cdot dx_{i_1} \wedge \ldots d\hat{x}_{i_j} \ldots \wedge dx_{i_p}$. The associated $\mathcal{O}_Y$-module $(\Omega^p_A)^\sim$ is the sheaf of p-Forms on Y. Hence, we see that

$$(\Omega^p_A)^\sim|_U = \Omega^p_Y|_U = \Omega^p_U.$$

Therefore by Remark 3.2.6, there is a natural isomorphism

$$\mathrm{H}^q(U, \Omega^p_U) \cong \bigoplus_{l \in \mathbb{Z}} \mathrm{H}^q(X, (\Omega^p_A(l)^\sim). \tag{3.5.3}$$

Lemma 3.5.1 ([24, Lemma 8.9]). *For all integers $l \in \mathbb{Z}$, the complex $((\Omega^\bullet_A(l))^\sim, \xi)$ of sheaves on X is exact and the kernel of $(\Omega^p_A(0))^\sim \xrightarrow{\xi} (\Omega^{p-1}_A(0))^\sim$ is canonically isomorphic to $\tilde{\Omega}^p_X$.*

Lemma 3.5.1 gives us short exact sequences

$$0 \to \tilde{\Omega}^p_X \to (\Omega^p_A(0))^\sim \xrightarrow{\xi} \tilde{\Omega}^{p-1}_X \to 0. \tag{3.5.4}$$

There is the following vanishing result.

Lemma 3.5.2 ([24, Lemma 8.10]). *In the situation above, the following statements hold.*

(1) We have $\mathrm{H}^q(U, ((\Omega^p_A(l))^\sim) = 0$, if $p + q \neq n - c, n - c + 1$ and $0 < q < n - c$.

(2) The map $\mathrm{H}^0(X, (\Omega^p_A(0))^\sim) \xrightarrow{\xi} \mathrm{H}^0(X, \tilde{\Omega}^{p-1}_X)$ is surjective if $p \geq 2$ and has cokernel isomorphic to $\mathbb{C}$ if $p = 1$.

These results allow us to calculate the relevant cohomology groups.

Lemma 3.5.3. *In the situation above, the following identities hold.*

(1) For $0 < p < n - c$*:*

$$\mathrm{H}^q(X, \tilde{\Omega}_X^p) = \begin{cases} 0 & \text{if } 0 < q < n - c - p, q \neq p \\ \mathbb{C} & \text{if } 0 < q < n - c - p, q = p \\ \mathrm{Hom}_{\mathbb{C}}(R_{p,-\nu}, \mathbb{C}) & \text{if } q = n - c - p, q \neq p \\ \mathbb{C} \oplus \mathrm{Hom}_{\mathbb{C}}(R_{p,-\nu}, \mathbb{C}) & \text{if } q = p = n - c - p. \end{cases}$$

(2)

$$\mathrm{H}^1(X, \Theta_X^1) = R_{1,0}.$$

Proof. We first prove (1). We argue by induction on p. In each step we consider the long exact cohomology sequences associated to the short exact Sequence (3.5.4) and use Lemmata 3.2.7, 3.3.2, 3.5.2 and Isomorphism 3.5.3 to compute certain cohomology groups. Let $p = 1$. We know $\tilde{\Omega}_X^0 = A^\sim$. Hence, it follows Lemma 3.2.7 that $\mathrm{H}^q(X, \tilde{\Omega}_X^0) = 0$ for $0 < q < n - c$. The long exact sequence is

$$\begin{array}{l} 0 \to \mathrm{H}^0(X, \tilde{\Omega}_X^1) \to \mathrm{H}^0(X, (\Omega_A^1(0))^\sim) \to \mathrm{H}^0(X, \tilde{\Omega}_X^0) \to \\ \to \mathrm{H}^1(X, \tilde{\Omega}_X^1) \to 0 \to 0 \dashrightarrow \\ \dashrightarrow \mathrm{H}^{n-c-2}(X, \tilde{\Omega}_X^1) \to 0 \to 0 \to \\ \to \mathrm{H}^{n-c-1}(X, \tilde{\Omega}_X^1) \to \mathrm{Hom}_{\mathbb{C}}(R_{1,-\nu}, \mathbb{C}) \to 0. \end{array}$$

The assertion for $p = 1$ immediately follows. Now assume that $2 \leq p < n - c - p - 1$ and that the result holds for $p - 1$. We see the assertion is also true for p by considering the long exact sequence

$$\begin{array}{l} 0 \to \mathrm{H}^0(X, \tilde{\Omega}_X^p) \to \mathrm{H}^0(X, (\Omega_A^p(0))^\sim) \to \mathrm{H}^0(X, \tilde{\Omega}_X^{p-1}) \to \\ \to \mathrm{H}^1(X, \tilde{\Omega}_X^p) \to 0 \to 0 \dashrightarrow \\ \dashrightarrow \mathrm{H}^{p-1}(X, \tilde{\Omega}_X^p) \to 0 \to \mathbb{C} \to \\ \to \mathrm{H}^p(X, \tilde{\Omega}_X^p) \to 0 \to 0 \dashrightarrow \\ \dashrightarrow \mathrm{H}^{n-c-p-1}(X, \tilde{\Omega}_X^p) \to 0 \to 0 \to \\ \to \mathrm{H}^{n-c-p}(X, \tilde{\Omega}_X^p) \to \mathrm{Hom}_{\mathbb{C}}(R_{p,-\nu}, \mathbb{C}) \to 0. \end{array}$$

Similarly the result is verified in case $p \geq n - c - p - 1$.

Now we prove (2). If we dualize Sequence 3.5.4 for $p = 1$ and consider the associated long exact sequence, we get

$$\mathrm{H}^1(X, \Theta_X^0) \to \mathrm{H}^1(X, \Theta_U^1)_0 \to \mathrm{H}^1(X, \Theta_X^1) \to \mathrm{H}^2(X, \Theta_X^0);$$

see Remark 3.4.5. Under the assumption that $2 < n - c$, we have $\mathrm{H}^1(X, \Theta_X^0) = \mathrm{H}^2(X, \Theta_X^0) = 0$ and therefore

$$\mathrm{H}^1(X, \Theta_X^1) = \mathrm{H}^1(X, \Theta_U^1))_0 = R_{1,0}.$$

This proves the lemma. *q.e.d.*

Proof of Theorem II. The statement is a combination of Lemma 3.5.3, Lemma 3.3.2 and Remark 3.3.3. *q.e.d.*

3.6 Infinitesimal Torelli for hyperelliptic Fano threefolds of type (1,1,4)

In this section, we will prove Theorem III and Theorem 3.0.1. Any hyperelliptic Fano threefold of Picard rank 1, index 1 and degree 4 over $\mathbb{C}$ is a weighted complete intersection

$$X = V_+(z^2 - f, g) \subset \mathbb{P}_{\mathbb{C}}(1,1,1,1,1,2) = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_4, z]$$

with $f, g \in \mathbb{C}[x_0, \dots, x_4]$, $\deg(g) = 2$, $\deg(f) = 4$; see [30, Theorem II.2.2.ii)]. It is a double cover of the smooth quadric $V(g) \subseteq \mathbb{P}^4$ with ramification along the smooth surface $V(f, g) \subseteq \mathbb{P}^4$. Since $V(g)$ is a smooth quadric, after a change of coordinates, we may assume $g = x_0^2 + \cdots + x_4^2$. Write $h_i = \partial_{x_i}(f)/2$. Then the Jacobi ring of X is given by

$$R = \mathbb{C}[x_0, \dots, x_4, z, y_2, y_4]/(f - z^2, g, y_2 x_0 - y_4 h_0, \dots, y_2 x_4 - y_4 h_4, y_4 \cdot z).$$

We apply Theorem II to a complete intersection of this type. We calculate $\nu = 7 - 6 = 1$ and therefore

$$\mathrm{H}^1(X, \Theta_X) \cong R_{1,0},\ \mathrm{H}^1(X, \tilde{\Omega}_X^2) \cong \operatorname{Hom}_{\mathbb{C}}(R_{1,-1}, \mathbb{C}),\ \mathrm{H}^2(X, \tilde{\Omega}_X^1) \cong \operatorname{Hom}_{\mathbb{C}}(R_{2,-1}, \mathbb{C}).$$

There are surjections

$$y_2 \cdot \mathbb{C}[x_0, \dots, x_4, z]_2 \oplus y_4 \cdot \mathbb{C}[x_0, \dots, x_4, z]_4 \to R_{1,0},$$

$$y_2 \cdot \mathbb{C}[x_0, \dots, x_4, z]_1 \oplus y_4 \cdot \mathbb{C}[x_0, \dots, x_4, z]_3 \to R_{1,-1},$$

$$y_2^2 \cdot \mathbb{C}[x_0, \dots, x_4, z]_3 \oplus y_2 y_4 \cdot \mathbb{C}[x_0, \dots, x_4, z]_5 \oplus y_4^2 \cdot \mathbb{C}[x_0, \dots, x_4, z]_7 \to R_{2,-1}.$$

Let $B = \mathbb{C}[x_0, \dots, x_4]/(f, g)$. Using the relations $y_2 x_i = y_4 h_i$ and $y_4 z = 0$, we see

$$R_{1,0} \cong (y_2 B_2 \oplus \mathbb{C} \cdot y_2 z \oplus y_4 B_4)/(y_2 x_i - y_4 h_i) \cong \mathbb{C} \cdot y_2 z \oplus y_4 B_4,$$

$$R_{1,-1} \cong (y_2 B_1 \oplus y_4 B_3)/(y_2 x_i - y_4 h_i) \cong y_4 B_3,$$

$$R_{2,-1} \cong \left(y_2^2 \cdot B_3 \oplus y_2 y_4 B_5 \oplus y_4^2 B_7\right)/(y_2 x_i - y_4 h_i) \cong y_4^2 B_7.$$

Note that there are injections

$$T_1 := y_2 B_2 \oplus \mathbb{C} \cdot y_2 z \to R_{1,0}$$

and

$$T_2 := y_2 B_1 \to R_{1,-1}.$$

We will need the following Lemma to prove Theorem 3.0.1.

Lemma 3.6.1. *If $\varphi \in \mathrm{Aut}(X)$, then there are linear polynomials $\lambda_i \in k[x_0, \ldots, x_4]_1$ and $b \in \mathbb{C}^*$ such that, for all $(x_0 : \cdots : x_4 : z) \in X(\mathbb{C})$, we have*

$$\varphi(x_0 : \cdots : x_4 : z) = (\lambda_0 : \cdots : \lambda_4 : bz).$$

Proof. The anticanonical bundle of X is isomorphic to $\mathcal{O}_X(1)$; see [17, Theorem 3.3.4]. The cohomology group $\mathrm{H}^0(X, \mathcal{O}_X(1))$ is a 5-dimensional vector space generated by $x_0, \ldots, x_4$, and $\mathrm{H}^0(X, \mathcal{O}_X(2))$ is a 15-dimensional vector space generated by $x_0^2, x_0x_1, \ldots, x_4^2, z$. Any automorphism $\varphi \in \mathrm{Aut}(X)$ induces an automorphism of these cohomology groups. Hence φ is of the form

$$\varphi(x_0, \ldots, x_4, z) = (\lambda_0, \ldots, \lambda_4, bz + q),$$

where $\lambda_i \in k[x_0, \ldots, x_4]_1$, $b \in \mathbb{C}^*$ and $q \in \mathbb{C}[x_0, \ldots, x_4]_2$. Note if $g(x_0, \ldots, x_4) = 0$, then there is a $z \in \mathbb{C}$ such that $(x_0, \ldots, x_4, z) \in X$. This shows $g(\lambda_0, \ldots, \lambda_4)$ vanishes on $V_+(g) \subseteq \mathbb{P}^4$. By Hilbert's Nullstellensatz, $g(\lambda_0, \ldots, \lambda_4) = \nu g$ for some $\nu \in \mathbb{C}^*$. Furthermore, again by Hilbert's Nullstellensatz, we see

$$(bz + q)^2 - f(\lambda_0, \ldots, \lambda_4) \in (z^2 - f, g).$$

It follows that g divides q. As q vanishes on $V_+(g)$, we can put $q = 0$. *q.e.d.*

In particular, the involution coming from the double cover is given by

$$\iota \colon X \to X \colon \quad (x_0, \ldots, x_4, z) \mapsto (x_0, \ldots, x_4, -z).$$

Proof of Theorem 3.0.1. (1): Consider an automorphism $\varphi \in \mathrm{Aut}(X)$ as in Lemma 3.6.1. If φ operates trivially on $\mathrm{H}^1(X, \Theta_X^1)$, then it operates trivially on T_1. Therefore, we have $b = 1$ $\varphi(x_ix_j) = x_ix_j$ for all i, j. This shows either $\lambda_i = x_i$ for all i or $\lambda_i = -x_i$ for all i. Note in $\mathbb{P}(1,1,1,1,1,2)_{\mathbb{C}}$, the coordinates $(x_0 : \ldots : x_4, z)$ and $(-x_0 : \ldots : -x_4, z)$ define the same point. Hence $\varphi = \mathrm{id}$. (2): As mentioned in the introduction, this is already proven; see [31, Proposition 2.12].

(3): If φ acts trivially on $\mathrm{H}^3(X, \mathbb{C})$, then it acts trivially on $\mathrm{H}^{2,1}$. In particular, such a φ then operates trivially on T_2. Hence, we have $\lambda_i = x_i$ for $i \in \{0, \ldots, 4\}$. As φ has to preserve the equation $z^2 - f$, we see $b \in \{1, -1\}$. This implies $\varphi \in \{\mathrm{id}, \iota\}$. *q.e.d.*

Proof of Theorem III. From the explicit descriptions above, we calculate that the involution invariant part $H^1(X, \Theta_X)^\iota$ is

$$(R_{1,0})^\iota \cong y_4 B_2.$$

Hence by Theorem II, the involution invariant infinitesimal Torelli map can be identified with the map

$$B_4 \to \mathrm{Hom}\,(B_3, B_7)\,.$$

The sequence f, g is regular as these polynomials define a complete intersection in $\mathbb{P}^4$. We can find polynomials h_1, h_2, h_3 such that f, g, h_1, h_2, h_3 is regular. Note that we can choose these polynomials of arbitrarily large degrees. Now, by Macaulay's theorem [79, Corollary 6.20], the map

$$\left(\frac{\mathbb{C}[x_0, \ldots, x_4]}{(f, g, h_1, h_2, h_3)}\right)_4 \to \mathrm{Hom}\left(\left(\frac{\mathbb{C}[x_0, \ldots, x_4]}{(f, g, h_1, h_2, h_3)}\right)_3, \left(\frac{\mathbb{C}[x_0, \ldots, x_4]}{(f, g, h_1, h_2, h_3)}\right)_7\right)$$

is injective. *q.e.d.*

Chapter 4

The moduli stack of Fano threefolds of Picard rank 1, index 1 and degree 4

In this chapter, we study the moduli stack of Fano threefolds of Picard rank 1, index 1 and degree 4 (in short type (1,1,4)). Our main goal is to prove Theorem IV. After Iskovskikh's classification [30, table 3.5], the Fano threefolds of type (1,1,4) come in two types:

a) smooth quartics, and

b) double covers of smooth quadrics in $\mathbb{P}^4$ ramified along smooth divisors of degree 8.

The latter type is called hyperelliptic. As explained in Section 4.2, both types are weighted complete intersections. We first prove the fact that this holds more generally. Namely, families of Fano threefolds of type $(1,1,4)$ are weighted complete intersections, locally for the Zariski topology.

Theorem 4.0.1 (Zariski local presentation)**.** *Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a Fano threefold of type (1,1,4); see Section 4.1 for Definition. Let $b \in B$ be a point. There is an affine open neighbourhood $U = \operatorname{Spec} R \subseteq B$ of b and polynomials $q_2, q_4 \in R[x_0, \dots, x_4, z]$ of degree $\deg(q_2) = 2$ and $\deg(q_4) = 4$ with respect to the grading given by $\deg(x_i) = 1$ and $\deg(z) = 2$ such that the Fano threefold $f|_U\colon f^{-1}(U) \to U$ is isomorphic to $V_+(q_2, q_4) \subseteq \mathbb{P}_R(1,1,1,1,1,2)$, and $\omega^{-1}_{f^{-1}(U)/U}$ is identified with $\mathcal{O}(1)$.*

Not that the statement of Theorem 4.0.1 can also be formulated as there being an isomorphism of polarized schemes

$$(f|_U\colon f^{-1}(U) \to U, \omega^{-1}_{f^{-1}(U)/U}) \cong (V_+(q_2, q_4), \mathcal{O}(1)).$$

This will be our main tool for the proof of Theorem IV, as it allows us to study the properties of Fano threefolds as properties of the local equations.

4.1 The stack of Fano threefolds

Following [40, Section 2-3], we introduce the stack of Fano threefolds.

Definition 4.1.1. Let k be a field. A *Fano variety over k* is a smooth proper geometrically integral variety S over k such that its anticanonical bundle is ample.

Let B be a scheme. A *Fano scheme over B* (or *family of Fano varieties over B*) is a smooth proper morphism $X \to B$ of schemes whose fibres are Fano varieties. A *Fano n-fold over B* is a Fano scheme of relative dimension n.

Definition 4.1.2. Let B be a Dedekind scheme with function field K. A Fano variety X over K has *good reduction over* B if there is a Fano scheme $\mathcal{X} \to B$ and an isomorphism $\mathcal{X}_K \cong X$.

Remark 4.1.3. Let $f\colon X \to B$ be a smooth proper morphism with geometrically connected fibres of relative dimension n. Then $\omega_{X/B} = \wedge^n \Omega_{X/B}$ is a line bundle. Since f is proper, the line bundle $\omega_{X/B}^{-1}$ is relatively ample if and only if it is fibre-wise ample; see [26, Theorem 4.7.1]. Hence $X \to B$ is a Fano scheme if and only if $\omega_{X/B}^{-1}$ is relatively ample.

Definition 4.1.4. Let k be an algebraically closed field and let X be a Fano threefold over k. Then we define:

(1) The *Picard rank of* X is $\rho(X) = \mathrm{rank}_{\mathbb{Z}} \operatorname{Pic} X$.

(2) The *index of* X is $r(X) = \max\{m \in \mathbb{N} \mid \omega_X^{-1}/m \in \operatorname{Pic}(X)\}$.

(3) The *degree of* X is the triple intersection number $d(X) = (\omega_X^{-1}/r(X))^3$.

(4) The *type of* X is the triple $(\rho(X), r(X), d(X))$.

Definition 4.1.5. We define a fibred category $p\colon \mathrm{Fano} \to \mathrm{Sch}$, where for a scheme B, the objects of $\mathrm{Fano}(B)$ are Fano threefolds over B. A morphism $(f\colon X \to B) \to (f'\colon X' \to B')$ of two Fano three in Fano is given by a pair (g, h), where $g\colon B \to B'$ and $h\colon X \to X'$ are morphisms of schemes such that the square

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{g} & B' \end{array}$$

is cartesian. The functor p is the forgetful functor, that remembers only the base scheme.

Given a triple of positive integers $(\rho, r, d) \in \mathbb{N}^3$, we define $\mathrm{Fano}_{\rho,r,d} \to \mathrm{Sch}$ to be the full fibred subcategory of Fano threefolds $f\colon X \to B$ such that all geometric fibres of f are Fano varieties of Picard rank ρ, index r and degree d. We define $\mathcal{F}$ to be the fibred category $(\mathrm{Fano}_{1,1,4})_{\mathbb{Z}[1/2]}$.

Proposition 4.1.6.

(1) The fibred category $\mathcal{F}$ is a finite type algebraic stack with an affine diagonal over $\mathbb{Z}[1/2]$.

(2) The stack $\mathcal{F}_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$.

(3) There is an $N \in \mathbb{N}$ such that $\mathcal{F}_{\mathbb{Z}[1/N]}$ is separated over $\mathbb{Z}[1/N]$.

(4) There is an $N \in \mathbb{N}$ such that for all schemes B over $\mathbb{Z}[1/N]$ and $X, Y \in \mathcal{F}(B)$, the morphism $\mathrm{Isom}_B(X, Y) \to B$ is finite.

(5) The stack $\mathcal{F}_{\mathbb{Q}}$ is a Deligne-Mumford stack.

Proof. Note that (1), (2), (3) and (4) follows from [40, Lemma 3.5, 3.6, 3.7, 3.8]. Finally, (5) follows from the fact that a finite type separated algebraic stack with an affine diagonal over $\mathbb{Q}$ is Deligne-Mumford. *q.e.d.*

4.2 Fano threefolds of Picard rank 1, index 1, degree 4 over fields

Fano threefolds over an algebraically closed field have been classified by Iskovskikh [30, Table 3.5] in characteristic 0. His result was later generalized by Shepherd-Barron [69] to positive characteristic in the case where the Picard rank is 1. In this section, we will generalize the characterization for Fano threefolds of type $(1,1,4)$ to the case of non algebraically closed fields of characteristic not equal to 2. In Section 4.3, we will then generalize it to families of Fano threefolds.

Theorem 4.2.1 (Iskovskikh-Shepherd-Barron [69, Propositions 4.1+4.3])**.** *Let k be an algebraically closed field. Let X be a Fano threefold of type $(1,1,4)$ over k.*

a) Then X is a smooth quartic in $\mathbb{P}^4_k$, or

b) X is a double cover of a smooth quadric in $\mathbb{P}^4_k$ ramified along a smooth surface of degree 8.

Let k be a field with $2 \in k^\times$. We consider the weighted projective space

$$\mathbb{P}_k(1,1,1,1,1,2) = \operatorname{Proj} k[x_0,\dots,x_4,z]$$

with weights $\deg(x_i) = 1$ for $i \in \{0,\dots,4\}$ and $\deg(z) = 2$; see [17] for an introduction on weighted projective varieties. Varieties of both types a) and b) represent special cases of smooth weighted complete intersections of degree $(2,4)$. Considering type a), let

$$X = V_+(q_4) \subseteq \mathbb{P}^4_k = \operatorname{Proj} k[x_0,\dots,x_4]$$

be a smooth quartic. Then X is isomorphic to the complete intersection

$$X' = V_+(z,q_4) \subseteq \mathbb{P}_k(1,1,1,1,1,2).$$

Considering type b), let X be a double cover of a smooth quadric $V_+(q_2) \subseteq \mathbb{P}^4$ which is ramified along a smooth surface $V_+(q_2,q_4)$ of degree 8. Then X is isomorphic to

$$X' = V_+(q_2, q_4 - z^2) \subseteq \mathbb{P}_k(1,1,1,1,1,2).$$

The double cover map is given via projection onto the first 5 homogeneous coordinates. Note in both cases X' does not contain the point $Q = (0:0:0:0:0:1)$. In fact, the following more general statement holds.

Lemma 4.2.2. *If $Y \subseteq \mathbb{P}_k(1,1,1,1,1,2)$ be a positive-dimensional smooth complete intersection, then Y does not contain Q.*

Proof. The point Q is contained in the open affine neighbourhood

$$U = D_+(z) = \operatorname{Spec} k\left[\frac{x_i x_j}{z}\middle| 0 \le i \le j \le 4\right] \subseteq \mathbb{P}_k(1,1,1,1,1,2).$$

The generators $w_{i,j} = \frac{x_i x_j}{z}$ satisfy relations

$$q_{\alpha,\sigma} = w_{\alpha_1,\alpha_2} w_{\alpha_3,\alpha_4} - w_{\alpha_{\sigma 1},\alpha_{\sigma 2}} w_{\alpha_{\sigma 3},\alpha_{\sigma 4}},$$

where $\alpha \in \{0,\ldots,4\}^4$ with $\alpha_1 \leq \alpha_2$ and $\alpha_3 \leq \alpha_4$ and σ is a permutation of $\{1,2,3,4\}$ with $\alpha_{\sigma 1} \leq \alpha_{\sigma 2}$ and $\alpha_{\sigma 3} \leq \alpha_{\sigma 4}$. We see $U \cong \operatorname{Spec} k[w_{i,j}]/(q_{\alpha,\sigma})$. The point Q corresponds to the point with coordinates $w_{i,j} = 0$. Note that $\frac{\partial q_{\alpha,\sigma}}{\partial w_{i,j}}(Q) = 0$ for all i, j, α, σ. Therefore it is not possible for $U \cap Y$ to satisfy the Jacobi criterion if Y does contain Q. *q.e.d.*

In the following, we will find an explicit description for the anti-canonical bundle of the weighted complete intersection X'. By [17, Theorem 3.3.4], there is an isomorphism

$$\omega_{X'/k} \cong \mathcal{O}_{X'}(-1). \tag{4.2.1}$$

Note that for a closed subvariety $Y \subseteq \mathbb{P}_k(W_1,\ldots,W_r)$ with weighted coordinate ring A, the sheaf $\mathcal{O}_Y(m)$ is the graded $\mathcal{O}_Y$-module associated to the degree shifted graded module $A(m)$. In general, it is not true that $\mathcal{O}_Y(m)$ is a line bundle or that $\mathcal{O}_Y(m) \otimes \mathcal{O}_Y(l) \cong \mathcal{O}_Y(m+l)$; see [17, Section 1.5]. For example, if $P = \mathbb{P}_k(1,1,1,1,1,2)$ as above, then $\mathcal{O}_P(1) \otimes \mathcal{O}_P(1) \ncong \mathcal{O}_P(2)$. This is because in any neighbourhood of $Q = (0:0:0:0:0:1)$, the section z of $\mathcal{O}_P(2)$ is not a product of sections of $\mathcal{O}_P(1)$. However, this is the only problematic point.

Lemma 4.2.3. *Let k be a field, let $X \subseteq \mathbb{P}_k(1,1,1,1,1,2)$ be a closed subvariety that does not contain the point $Q = (0:0:0:0:0:1)$. Then for any $m, l \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(m)$ is a line bundle and the multiplication map induces an isomorphism*

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(l) \cong \mathcal{O}_X(m+l).$$

Proof. We have $X \subseteq \mathbb{P}_k(1,1,1,1,1,2) \setminus \{Q\} = \bigcup D_+(x_i)$, where

$$D_+(x_i) = \operatorname{Spec} k\left[\frac{x_0}{x_i},\ldots,\frac{x_4}{x_i},\frac{z}{x_i^2}\right].$$

Let $U_i = X \cap D_+(x_i)$. The assertion follows, since the multiplication map

$$\mathcal{O}_X|_{U_i} \xrightarrow{\cdot x_i^m} \mathcal{O}_X(m)|_{U_i}$$

is an isomorphism. *q.e.d.*

By Lemma 4.2.2 and Lemma 4.2.3, the Isomorphism (4.2.1) induces an isomorphism

$$\omega_{X'/k}^{-i} \cong \mathcal{O}_{X'}(i) \tag{4.2.2}$$

for all $i \in \mathbb{Z}$. We need the following technical Lemma.

Lemma 4.2.4. *Let k be a field with $2 \in k^\times$. Let $S = k[x_0,\ldots,x_4,z]$ be the weighted polynomial algebra with $\deg(x_i) = 1$ and $\deg(z) = 2$. Let $q_2 \in S_2$ and $q_4 \in S_4$ such that $q_2 \neq 0$ and q_2 does not divide q_4. Set $T = S/(q_2,q_4)$. Then $\phi(i) = \dim_k T_i$ satisfies $\phi(1) = 5$, $\phi(2) = 15$, $\phi(3) = 35$ and $\phi(4) = 69$. In case $q_2 = z$, we have $T \cong k[x_0,\ldots,x_4]/(q_4)$.*

Proof. In the case that q_2 depends on z, after a change of coordinates, we may assume that $q_2 = z$. Hence, $T \cong k[x_0,\ldots,x_4]/(q_4)$, and the dimensions are easily calculated. In case that q_2 does not depend on z, after a change of coordinates, we may assume $q_2 = x_0^2 + q_2'$, where $q_2' \in k[x_0,\ldots,x_4]_2$ has no x_0^2-term. Hence the 14 monomials x_ix_j where $0 \leq i \leq j \leq 4$ and $(i,j) \neq (0,0)$ together with z form a basis for T_2. Hence $\phi(2) = 14 + 1 = 15$. Similarly, we see the monomial $x_0 x_i x_j$ where $1 \leq i \leq j \leq 4$ together with $x_i x_j x_l$ where $1 \leq i \leq j \leq l \leq 4$ and $zx_0, \dots, zx_4$ form a basis for the degree 3 part. Hence $\phi(3) = 10 + 20 + 5$. Similarly, we see that the degree 4 part of $k[x_0, \dots, x_4, z]/(q_2)$ has dimension 70. As we have to mod out one more relation in degree 4, we get $\phi(4) = 69$. *q.e.d.*

The proof of the following proposition will utilize the fact that our Fano threefolds come with a canonical embedding into weighted projective space associated to the anticanonical bundle.

Proposition 4.2.5. *Let k be a field with $2 \in k^\times$, let X be a Fano threefold of type $(1,1,4)$ over k. Then the function $\phi(i) = \dim_k \mathrm{H}^0(X, \omega_{X/k}^{-i})$ satisfies $\phi(1) = 5, \phi(2) = 15, \phi(3) = 35$ and $\phi(4) = 69$. Let $\xi_0, \dots, \xi_4$ be a basis for $\mathrm{H}^0(X, \omega_{X/k}^{-1})$. Then the following statements hold.*

(1) *If $X_{\bar{k}}$ is of type a), then the variety X is a smooth quartic in $\mathbb{P}^4_k$. The monomials of degree 4, $\xi_0^4, \xi_0^3\xi_1, \dots, \xi_4^4$ satisfy a relation q_4. The map $\xi_i \mapsto x_i$ induces an isomorphism of graded k-algebras*

$$\bigoplus_{i \geq 0} \mathrm{H}^0(X, \omega_{X/k}^{-i}) \cong k[x_0, \dots, x_4]/(q_4)$$

and an isomorphism $X \cong V_+(q_4) \subseteq \mathbb{P}^4_k$ of varieties.

(2) *If $X_{\bar{k}}$ is of type b), then the variety X is a double cover of a smooth quadric in $\mathbb{P}^4_k$ ramified along a smooth surface of degree 8. The monomials of degree 2, $\xi_0^2, \xi_0\xi_1, \dots, \xi_4^2$ satisfy a relation q_2 and span a 14-dimensional subspace of $\mathrm{H}^0(X, \omega_{X/k}^{-2})$. Let $\zeta \in \mathrm{H}^0(X, \omega_{X/k}^{-2})$ be an element completing those monomials to a generating set. There are polynomials $q_2 \in k[x_0, \dots, x_4]_2$ and $q_4 \in k[x_0, \dots, x_4, z]_4$ such that the map $\xi_i \mapsto x_i, \zeta \mapsto z$ induces an isomorphism if graded k-algebras*

$$\bigoplus_{i \geq 0} \mathrm{H}^0(X, \omega_{X/k}^{-i}) \cong k[x_0, \dots, x_4, z]/(q_2, q_4 - z^2)$$

and an isomorphism $X \cong V_+(q_2, q_4 - z^2) \subseteq \mathbb{P}_k(1,1,1,1,1,2)$ of varieties.

Proof. We will prove (2). The proof of (1) is similar. Let X be a Fano threefold over k of type (1,1,4) such that there is an isomorphism

$$X_{\bar{k}} \cong V_+(q_2', q_4' - z^2) \subseteq \mathbb{P}_{\bar{k}}(1,1,1,1,1,2),$$

where $q_j' \in k[x_0, \dots, x_4]_j$. Isomorphism (4.2.2) induces an isomorphism

$$\bigoplus_{i \geq 0} \mathrm{H}^0\left(X_{\bar{k}}, \omega_{X_{\bar{k}}/\bar{k}}^{-i}\right) \cong k[x_0, \dots, x_4, z]/(q_2', q_4' - z^2) =: R$$

of graded k-algebras. Furthermore for each $i \geq 0$, there is an isomorphism

$$\mathrm{H}^0(X, \omega_{X/k}^{-i}) \otimes \bar{k} \cong \mathrm{H}^0\left(X_{\bar{k}}, \omega_{X_{\bar{k}}/\bar{k}}^{-i}\right).$$

Hence, the value $\phi(i)$ can be determined by counting the elements of a $\bar{k}$-vector space basis for the degree i part R_i; see Lemma 4.2.4 for the computations. The basis $\xi_0, \dots, \xi_4$ of $\mathrm{H}^0(X, \omega_{X/k}^{-1})$ is also a basis for $\mathrm{H}^0(X_{\bar{k}}, \omega^{-1}_{X_{\bar{k}}/\bar{k}})$. The image of the multiplication pairing $R_1 \otimes R_1 \to R_2$ is generated by the 15 monomials $x_0^2, x_0x_1, \ldots, x_4^2$. These monomials satisfy the relation q_2'. Therefore, the image has dimension 14. Hence, also the monomials $\xi_0^2, \xi_0\xi_1, \ldots, \xi_4^2$ generate a 14-dimensional subspace of $\mathrm{H}^0(X, \omega^{-2}_{X/k})$, and therefore satisfy a relation.

Let $q_2 \in k[x_0, \ldots, x_4]_2 \setminus \{0\}$ such that $q_2(\xi_0, \ldots, \xi_4) = 0$, and let $\zeta \in \mathrm{H}^0(X, \omega^{-2}_{X/k})$ be an element completing those monomials to a generating set. The vector space $(k[x_0, \ldots, x_4, z]/(q_2))_4$ has dimension 70. Hence the map

$$(k[x_0, \ldots, x_4, z]/(q_2))_4 \to \mathrm{H}^0(X, \omega^{-4}_{X/k})$$

given by $x_i \mapsto \xi_i, z \mapsto \zeta$ has a non-zero kernel. Let $q_4'' \neq 0$ be an element of the kernel. Then the maps

$$\bigoplus_{i \geq 0} \mathrm{H}^0(X, \omega^{-i}_{X/k}) \to k[x_0, \ldots, x_4, z]/(q_2, q_4'')$$

and

$$X \to V_+(q_2, q_4'') \subseteq \mathbb{P}_k(1,1,1,1,1,2)$$

are isomorphisms after the flat base change to the algebraic closure. Therefore, they were isomorphisms, to begin with. By Lemma 4.2.2, the variety $V_+(q_2, q_4'')$ does not contain Q. Hence, q_4'' has a z^2-term and after a suitable change of coordinates, we get $q_4'' = q_4 - z^2$. *q.e.d.*

4.3 Families of Fano threefolds of Picard rank 1, index 1, degree 4

As seen in Section 4.2, over fields of characteristic unequal to 2, there are two mutually exclusive types of Fano threefolds of Picard rank 1, index 1 and degree 4, namely smooth quartics and hyperelliptic ones. In this section, we will study how these types behave in families of Fano threefolds over schemes B over $\mathbb{Z}[1/2]$. The main tool in this study will be Theorem 4.0.1, which says that families of such Fano threefolds Zariski locally are weighted complete intersections.

Local presentation as weighted complete intersection

We need the following technical lemma, which is a combination of Proposition 4.2.5 with a "cohomology and basechange" argument.

Lemma 4.3.1. *Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a Fano threefold of type (1,1,4) and let $l \in \mathbb{Z}_{>0}$. Then the following statements hold.*

(1) The formation of $f_*\omega^{-l}_{X/B}$ commutes with arbitrary base change.

(2) The sheaf $f_*\omega^{-l}_{X/B}$ is locally free.

(3) Let $r(l) = \operatorname{rank} f_*\omega^{-l}_{X/B}$. We have $r(1) = 5$, $r(2) = 5$, $r(3) = 35$ and $r(4) = 69$.

(4) Assume additionally that B is affine. Let $b \in B$ be a point. Then the natural map

$$\mathrm{H}^0(X, \omega^{-l}_{X/B}) \otimes k(b) \to \mathrm{H}^0\left(X_b, \omega^{-l}_{X_b/k(b)}\right): \quad \xi \otimes \lambda \mapsto \lambda \cdot \xi|_b$$

is an isomorphism.

Proof. The morphism f is proper and smooth of relative dimension 3. Hence f is locally of finite presentation and $\omega_{X/B}^{-l}$ is a sheaf of finite presentation on X and flat over B. By Proposition 4.2.5, for any point $b \in B$, the fibre X_b is a weighted complete intersection of degree $(2,4)$ in $\mathbb{P}_{k(b)}(1,1,1,1,1,2)$, and $\omega_{X/B}^{-l}|_{X_b}$ is isomorphic to $\mathcal{O}_{X_b}(l)$; see Isomorphism (4.2.2). Hence $h^1\left(X_b, \omega_{X_b/k(b)}^{-l}\right) = 0$ and the function

$$b \to h^0\left(X_b, \omega_{X_b/k(b)}^{-l}\right)$$

is constant; see Proposition 4.2.5 and Lemma 3.2.2. By applying [7, Lemma 1.1.5], we get (1) and (2). We get (4) from (1) by considering the base change along $\operatorname{Spec} k(b) \to B$. Assertion (3) is a consequence of (4) and Proposition 4.2.5. *q.e.d.*

Proof of Theorem 4.0.1. We will successively shrink B and thereby also X to find a suitable U. By Lemma 4.3.1.(2), we can shrink B such that $B = \operatorname{Spec} R$ is affine and $f_*\omega_{X/B}^{-l}$ is free for $l \in \{1,2,3,4\}$. We choose a basis $\xi_0, \ldots, \xi_4$ for the free R-module $\mathrm{H}^0(X, \omega_{X/B}^{-1}) = \mathrm{H}^0(B, f_*\omega_{X/B}^{-1})$. Then the restrictions $\xi_0|_b, \ldots, \xi_4|_b$ form a basis for $\mathrm{H}^0\left(X_b, \omega_{X_b}^{-1}\right)$ by Lemma 4.3.1.(4). By Proposition 4.2.5, we know that the 15 elements $\xi_0|_b^2, \xi_0\xi_1|_b, \ldots, \xi_4|_b^2$ generate a subspace of dimension at least 14 in the 15-dimensional vector space $\mathrm{H}^0\left(X_b, \omega_{X_b/k(b)}^{-2}\right)$. We choose $\tilde{z} \in \mathrm{H}^0\left(X_b, \omega_{X_b/k(b)}^{-2}\right) \setminus \{0\}$ such that $\xi_0|_b^2, \xi_0\xi_1|_b, \ldots, \xi_4|_b^2, \tilde{z}$ is a set of generators for $\mathrm{H}^0\left(X_b, \omega_{X_b/k(b)}^{-2}\right)$. After shrinking B, we may find a $\zeta \in \mathrm{H}^0(X, \omega_{X/B}^{-2})$ such that $\zeta|_b = \tilde{z}$. Shrinking B again, we may assume that $\xi_0^2, \xi_0\xi_1, \ldots, \xi_4^2, \zeta$ generates $\mathrm{H}^0(X, \omega_{X/B}^{-2})$. Hence these elements define a surjection $\phi\colon R^{16} \to \mathrm{H}^0(X, \omega_{X/B}^{-2})$. Let $K = \ker(\phi)$. As $\mathrm{H}^0(X, \omega_{X/B}^{-2})$ is a free R-module of rank 15, we get an exact sequence

$$0 \to K \otimes k(b) \to k(b)^{16} \to \mathrm{H}^0\left(X_b, \omega_{X_b/k(b)}^{-2}\right) \to 0,$$

where $K \otimes k(b)$ is a 1-dimensional $k(b)$ vector space. After shrinking B, we can find $\lambda \in K$ sucht that $\lambda|_b$ generates $K \otimes k(b)$. In particular $\lambda|_b \neq 0$. We shrink B such that $\lambda|_p \neq 0$ for all $p \in B$. Then $\lambda|_p$ generates $K \otimes k(p)$ for all $p \in P$. The element λ corresponds to a polynomial $q_2 \in R[x_0, \ldots, x_4, z]_2$ such that $q_2(\xi_0, \ldots, \xi_4, \zeta) = 0$.

By Proposition 4.2.5, the restrictions

$$\xi_0^4|_p, \xi_0^3|_p\xi_1|_p, \ldots, \xi_4^4|_p, \xi_i|_p\xi_j|_p\zeta|_p, \zeta|_p^2$$

generate $\mathrm{H}^0\left(X_p, \omega_{X_p/k(p)}^{-4}\right)$. The map

$$\left(\frac{k(p)[x_0, \ldots, x_4, z]}{q_2|_p}\right)_4 \to \mathrm{H}^0\left(X_p, \omega_{X_p/k(p)}^{-4}\right)$$

that maps $x_i \mapsto \xi_i|_p$ and $z \mapsto \zeta|_p$ has a 1 dimensional kernel. As above, after shrinking B, we find a global generator ν for this kernel. The section ν corresponds to a polynomial $q_4 \in R[x_0, \ldots, x_4, z]_4$ such that $q_4(\xi_0, \ldots, \xi_4, \zeta) = 0$. The map $x_i \mapsto \xi_i$, $z \mapsto \zeta$ induces a morphism

$$X \to V_+(q_2, q_4) \subseteq \mathbb{P}_R(1,1,1,1,1,2).$$

Again by Proposition 4.2.5, this is an isomorphism fibre-wise. Hence it is an isomorphism. *q.e.d.*

The hyperelliptic and smooth quartic locus

Definition 4.3.2. Let B be a scheme over $\mathbb{Z}[1/2]$. A Fano scheme $f\colon X \to B$ of type (1,1,4) is called *hyperelliptic* if there is an open affine cover $B = \bigcup_{i\in I} U_i$, $U_i = \operatorname{Spec} R_i$ such that the polarized scheme

$$(f|_{f^{-1}(U_i)}\colon f^{-1}(U_i) \to U_i, \omega^{-1}_{f^{-1}(U_i)/U_i})$$

is isomorphic to a complete intersection $(V_+(q_2, q_4), \mathcal{O}(1))$ in $\mathbb{P}_{R_i}(1,1,1,1,1,2)$ with $q_2 \in R_i[x_0, \dots, x_4]$.

Remark 4.3.3. Theorem 4.0.1 guarantees that there is an open affine cover on which a Fano scheme is a weighted complete intersection. The additional condition imposed by being hyperelliptic is that q_2 does not depend on z. Note that every geometric fibre of a hyperelliptic Fano threefold is hyperelliptic in the sense of Iskovskik's classification. More generally, the following statement holds.

Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a hyperelliptic Fano threefold of type (1,1,4). Then for every field k and every point $b \in B(k)$, the fibre X_b is a double cover of a smooth quadric in $\mathbb{P}^4_k$ ramified along a smooth surface of degree 8. Indeed, since $f\colon X \to B$ is hyperelliptic, each fibre is a weighted complete intersection $V_+(q_2, q_4') \subseteq \mathbb{P}_k(1,1,1,1,1,2)$ with q_2 not depending on z. By Lemma 4.2.2, the polynomial q_4' has a z^2-term and we may assume that $q_4' = q_4 - z^2$ for some $q_4 \in k[x_0, \dots, x_4]$ after a suitable change of coordinates. The variety $V_+(q_2, q_4 - z^2)$ is a double cover of $V_+(q_2) \subseteq \mathbb{P}^4_k$ ramified along $V_+(q_2, q_4)$.

The condition on q_2 in the definition of hyperelliptic does not depend on the choice of the open covering.

Lemma 4.3.4. *Let B be a scheme over $\mathbb{Z}[1/2]$. Let $f\colon X \to B$ be a hyperelliptic Fano scheme of type (1,1,4). Let $U = \operatorname{Spec} R \subset B$ be an open affine subscheme such that there is an isomorphism of polarized schemes*

$$(f|_U\colon f^{-1}(U), \omega_{f^{-1}(U_i)/U}) \cong (V_+(q_2, q_4), \mathcal{O}(1))$$

over U. Write $q_2 = q_2' + cz$ with $q_2' \in R[x_0, \dots, x_4]$. Then we have $c = 0$.

Proof. Note that $c = 0$ if and only if there is an open affine covering $U = \bigcup V_j$ such that $c|_{V_j} = 0$. As $f\colon X \to B$ is hyperelliptic, after shrinking U if necessary, we may assume that $f^{-1}(U)$ has a presentation $V_+(q_2'', q_4'') \subseteq \mathbb{P}_R(1,1,1,1,1,2)$ with $q_j'' \in R[x_0, \dots, x_4]_j$. Thus, there is an isomorphism over U of polarized schemes

$$(V_+(q_2' + c, q_4), \mathcal{O}(1)) \cong (V_+(q_2'', q_4''), \mathcal{O}(1)).$$

The isomorphism of polarized schemes induces an isomorphism

$$\alpha\colon R[x_0, \dots, x_4, z]/(q_2' + cz, q_4) \to R[x_0, \dots, x_4, z]/(q_2'', q_4'')$$

of graded R-algebras. Since z is of degree 2, the image $\alpha(z)$ has to be of degree 2. We write $\alpha(z) = az + p$ with $a \in R$ and $p \in R[x_0, \dots, x_4]_2$. As the degree 2 relation q_2'' does not depend on z, it follows from the subjectivity of α that a is invertible. Since $a \cdot c = 0$, it follows that $c = 0$, as required. *q.e.d.*

Definition 4.3.5. Let B be a scheme over $\mathbb{Z}[1/2]$. A Fano scheme $f\colon X \to B$ of type (1,1,4) is a *smooth quartic* if every geometric fibre is a smooth quartic.

One might ask why we did not define hyperelliptic Fano threefolds via a fibre-wise criterion as well. The problem with such a definition is that over non-reduced bases, it would allow for "hyperelliptic" Fano threefolds that are not double covers, as the following example shows.

Example 4.3.6. Let k be a field with $2 \in k^\times$. Consider the family

$$X = V_+(x_0^2 + \cdots + x_4^2 + \epsilon z, x_0^4 + \cdots + x_4^4 - z^2) \subseteq \mathbb{P}_{k[\epsilon]}(1,1,1,1,1,2)$$

over $k[\epsilon] := k[t]/(t^2)$. This family has just one geometric fibre, which is hyperelliptic. However, it is neither hyperelliptic nor a smooth quartic.

At the same time, this is an example of a first-order deformation of a hyperelliptic Fano threefold that is not hyperelliptic. This is in accordance with the fact that the locus of hyperelliptic Fanos is a closed substack of $\mathcal{F}$.

Proposition/Definition 4.3.7 (hyperelliptic locus)**.** *Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a Fano threefold of type (1,1,4). Then there is a closed subscheme $c\colon B_{hyp} \to B$ such that $X \times_B B_{hyp} \to B_{hyp}$ is hyperelliptic with the following universal property. If $g\colon T \to B$ is a morphism of schemes such that $X \times_B T$ is hyperelliptic, then there is a unique morphism $g'\colon T \to B_{hyp}$ such that $g = c \circ g'$. Moreover, B_{hyp} is the set of points $b \in B$ such that the geometric fibre $X_{\bar{b}}$ over $k(\bar{b})$ is hyperelliptic. We call B_{hyp} the* hyperelliptic locus of $f\colon X \to B$.

Proof. The universal property shows that if it exists, the hyperelliptic locus is unique up to a unique isomorphism. Therefore, the assertion is local on B, because the universality of the hyperelliptic locus allows us to construct the locus locally and glue it together afterwards. By Theorem 4.0.1, we may therefore assume that $B = \operatorname{Spec} R$ is affine and $X = V_+(q_2, q_4) \subseteq \mathbb{P}_R(1,1,1,1,1,2)$ is a weighted complete intersection. We write $q_2 = q_2' + cz$, where $q_2 \in R[x_0, \ldots, x_4]_2$ and $c \in R$. We set $B_{hyp} = V(c) \subseteq B$.

The assertion that a morphism $g\colon T \to B$ is a morphism of schemes such that $X \times_B T$ is hyperelliptic factors uniquely through B_{hyp} is local on T. We may therefore assume that $T = \operatorname{Spec} S$ is affine. Let $\alpha\colon R \to S$ be the morphism of rings corresponding to g. Now

$$X \times_B T \cong V_+(\alpha(q_4), \alpha(q_2') + \alpha(c)z) \subseteq \mathbb{P}_S(1,1,1,1,1,2)$$

is a presentation as weighted complete intersection. Therefore $\alpha(c) = 0$ by Lemma 4.3.4. Hence g factors uniquely though $B_{hyp} = V(c) = \operatorname{Spec} R/(c)$. *q.e.d.*

There is a similar statement for the smooth quartic locus that is proven completely analogous.

Proposition/Definition 4.3.8. *(smooth quartic locus) Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a Fano threefold of type (1,1,4). There is an open subscheme $o\colon B_{sq} \to B$ such that $f\colon f^{-1}(B_{sq}) \to B_{sq}$ is a smooth quartic with the following universal property. If $g\colon T \to B$ is a morphism of schemes such that $X \times_B T$ is a smooth quartic, then there is a unique morphism $g'\colon T \to B_{sq}$ such that $g = c \circ g'$. Furthermore B_{sq} is the complement of B_{hyp} in B.*

The involution on a hyperelliptic Fano scheme

Next, we want to construct an involution on hyperelliptic Fano threefolds. Note that local presentations can be chosen in a certain form.

Lemma 4.3.9. *Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a hyperelliptic Fano threefold of type (1,1,4). Let $b \in B$ be a point. There is an affine open neighbourhood $U = \operatorname{Spec} R \subseteq B$ of b and polynomials $q_2, q_4 \in R[x_0, \ldots, x_4]$ of degree $\deg(q_2) = 2$ and $\deg(q_4) = 4$ such that there is a isomorphism*

$$(f^{-1}(U), \omega^{-1}_{f^{-1}(U)/U}) \cong (V_+(q_2, q_4 - z^2), \mathcal{O}(1))$$

of polarized schemes over U.

Proof. By Theorem 4.0.1, we can find a local presentation $V_+(q_2, q_4)$ with $q_2, q_4 \in R[x_0, \ldots, x_4]$. Write $q_4 = q_4' + p_2 z + dz^2$ with $q_4', p_2 \in R[x_0, \ldots, x_4]$, $d \in R$. The coefficient d is invertible. Otherwise there would be a fibre with $d \otimes k(b) = 0$. This is impossible as this fibre then would be singular by Lemma 4.2.2. After the change of coordinates $z \mapsto z + \frac{p_2}{2d}$, we get the desired equations. *q.e.d.*

Definition 4.3.10. Let B be a scheme over $\mathbb{Z}[1/2]$ and let $f\colon X \to B$ be a hyperelliptic Fano threefold of type (1,1,4). The *involution induced by the double cover* is the map $\iota \in \operatorname{Aut}_B(X)$ that over any open affine $U \subseteq B$ with $(f^{-1}(U), \omega^{-1}_{f^{-1}(U)/U}) \cong (V_+(q_2, q_4 - z^2), \mathcal{O}(1))$ as in Lemma 4.3.9 is given by $z \mapsto -z$.

To see that the involution is well defined, we have to show that the defined maps agree on the intersection of two open affines over which X has a presentation of the given form. Since we can cover the intersection with smaller open affines, it suffices to show the following statement. If $U = \operatorname{Spec} R \subseteq B$ is open affine and $q_2, q_4, q_2', q_4' \in R[x_0, \ldots, x_4]$ are polynomials such that

$$(V_+(q_2, q_4 - z^2), \mathcal{O}(1)) \cong (f^{-1}(U), \omega^{-1}_{f^{-1}(U)/U}) \cong (V_+(q_2', q_4' - z^2), \mathcal{O}(1)),$$

then the maps that are given by $z \mapsto -z$ are identified under the isomorphisms. The isomorphism of polarized schemes induces an isomorphism of graded R-algebras

$$\alpha\colon R[x_0, \ldots, x_4, z]/(q_2, q_4 - z^2) \to R[x_0, \ldots, x_4, z]/(q_2', q_4' - z^2).$$

To preserve the equations, the image $\alpha(z) = az + p$ now has to satisfy $p = 0$. Hence α commutes with the map $z \mapsto -z$.

4.4 The stacks of hyperelliptic and smooth quartic Fanos

In this section, we define the stack of hyperelliptic and smooth quartic Fanos and prove Theorem IV.

Definition 4.4.1. Let $\mathcal{F}$ over $\mathbb{Z}[1/2]$ be the stack of Fano threefolds of type (1,1,4) as defined in Section 4.1.

(1) We define $\mathcal{H}$ to be the full fibred subcategory of $\mathcal{F}$ of those Fano threefolds $f\colon X \to B$ that are hyperelliptic.

(2) We define $\mathcal{Q}$ to be the full fibred subcategory of $\mathcal{F}$ of those Fano threefolds $f\colon X \to B$ that are smooth quartic.

Lemma 4.4.2. *The categories fibred in groupoids*

$$\mathcal{H} \to \operatorname{Spec} \mathbb{Z}[1/2]$$

and

$$\mathcal{Q} \to \operatorname{Spec} \mathbb{Z}[1/2]$$

are stacks.

Proof. For the statement on smooth quartics, we refer to [8]. We prove the statement about $\mathcal{H}$. Since $\mathcal{H}$ is a full subcategory of the stack $\mathcal{F}$, it is a prestack. It remains to show that any descent datum for any given fppf cover is effective. To do so, we consider a cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

where $f\colon X \to B$ and $f'\colon X' \to B'$ are Fano threefolds with the latter being hyperelliptic and g is an fppf morphism. Let $B = \bigcup_{i \in I} U_i$, $U_i = \operatorname{Spec} R_i$ be an open affine cover such that for each $i \in I$ there are $q_4 \in R_i[x_0, \dots, x_4, z]_4$, $q_2 \in R_i[x_0, \dots, x_4]_2$ and $c \in R_i$ such that

$$(V_+(q_2 + cz, q_4), \mathcal{O}(1)) \cong (f^{-1}(U_i), \omega^{-1}_{f^{-1}(U_i)/U_i}).$$

Now we can find a cover $B' = \bigcup_{i \in I, j \in J_i} V_{i,j}$, where $V_{i,j}$ is quasi compact and $g(V_{i,j}) = U_i$. Fix some i, j. Let $V_{i,j} = \bigcup_{m=1}^{n} W_m$ be a finite open affine cover; see [60, Corollary 1.1.6]. Set $W = \coprod_{m=1}^{n} W_m$. Then W is affine, $X' \times_{B'} W$ is hyperelliptic and the map $W \to U_i$ induced by g is faithfully flat. The corresponding ring map

$$\alpha\colon R_i \to S := \Gamma(W, \mathcal{O}_W)$$

is faithfully flat, hence injective. As

$$X' \times_{B'} W \cong V_+(\alpha(q_2) + \alpha(c)z, \alpha(q_4)) \subseteq \mathbb{P}_S(1,1,1,1,1,2)$$

is hyperelliptic we see that $\alpha(c) = 0$ by Lemma 4.3.4. Hence $c = 0$. As i, j were chosen arbitrarily, this proves that $f\colon X \to B$ is hyperelliptic. *q.e.d.*

There is the following description for the 2-fibre product of fibred categories, which will be helpful in the proof of Theorem IV; see [60, Proposition 3.4.13].

Lemma 4.4.3. *Let C be a category and let $p_i\colon F_i \to C$ be fibred categories (i=1,2,3). Let $c\colon F_1 \to F_3$ and $d\colon F_2 \to F_3$ be morphisms of fibred categories. The 2-fibre product $F_1 \times_{F_3} F_2$ has the following description. The objects are triples (x_1, x_2, σ), where $x_i \in F_i$ with $p_1(x_1) = p_2(x_2)$, and $\sigma : c(x_1) \to d(x_2)$ is an isomorphism in the fibre category $F_3(p_1(x_1)) = F_3(p_2(x_2))$. A morphism*

$$(x_1', x_2', \sigma') \to (x_1, x_2, \sigma)$$

is a pair of morphisms $f_i\colon x_i' \to x_i$ in F_i ($i = 1, 2$) such that $p_1(f_1) = p_2(f_2)$ and such that the diagram

$$\begin{array}{ccc} c(x_1') & \xrightarrow{c(f_1)} & c(x_1) \\ \downarrow{\sigma'} & & \downarrow{\sigma} \\ d(x_2') & \xrightarrow{d(f_2)} & d(x_2) \end{array}$$

commutes.

Proof of Theorem IV. By Proposition 4.2.5, for any field k with $2 \in k^\times$, we have $\mathcal{F}(k) = \mathcal{Q}(k) \sqcup \mathcal{H}(k)$. That $\mathcal{H} \to \mathcal{F}$ is a representable closed immersion and $\mathcal{Q} \to \mathcal{F}$ is a representable open immersion is a reformulation of Proposition 4.3.7 and Proposition 4.3.8, respectively. We will explain this for the statement about $\mathcal{H}$. The other one is similar.

Let B be schemes over $\mathbb{Z}[1/2]$, and let $B \to \mathcal{F}$ be a morphism of stacks corresponding to a Fano threefold $(f\colon X \to B) \in \mathcal{F}(B)$. Let $\gamma : B_{hyp} \to B$ be the hyperelliptic locus as in Proposition 4.3.7. We claim that B_{hyp} represents the 2-fibre product $B \times_{\mathcal{F}} \mathcal{H}$. More specifically, we claim that the functor

$$\lambda\colon \mathrm{Sch}/B_{hyp} \to (\mathrm{Sch}/B) \times_{\mathcal{F}} \mathcal{H}$$

that maps a morphism of schemes $h : T \to B_{hyp}$ to the triple

$$(\gamma \circ h\colon T \to B, X \times_{f,B,\gamma\circ h} T \to T, \mathrm{id}\colon X \times_B T \to X \times_B T)$$

is an equivalence of categories (compare with Lemma 4.4.3 for a description of the 2-fibre product). We first show the essential surjectivity. Let

$$\xi = (g\colon T \to B, \phi\colon Y \to T, \sigma\colon X \times_B T \to Y)$$

be a triple, where g is a morphism of schemes, Y is an hyperelliptic Fano threefold over T and σ is an isomorphism. Then in particular, $X \times_B T$ is hyperelliptic. Hence by Proposition 4.3.7, there is a morphism $h : T \to B_{hyp}$ such that $g = \gamma \circ h$. We have

$$\lambda(h) = (g, \pi_2\colon X \times_B T \to T, \mathrm{id}).$$

The pair of maps (id_T, σ) defines an isomorphism between $\lambda(h)$ and ξ.

In order to show that λ is fully faithful, we have to show that for two morphisms $h' : T' \to B_{hyp}$ and $h : T \to B_{hyp}$, giving a morphism $T' \to T$ over B_{hyp} is the same as giving a morphism

$$(c \circ h', X \times_B T', \mathrm{id}) \to (c \circ h, X \times_B T, \mathrm{id})$$

in $(\mathrm{Sch}/B) \times_{\mathcal{F}} \mathcal{H}$. By Lemma 4.4.3, such a morphism is given by a pair of morphisms $f_1\colon T' \to T$ and $f_2\colon X \times_B T' \to X \times_B T$ such that the diagrams

$$\begin{array}{ccc} X \times_B T' & \xrightarrow{f_2} & X \times_B T \\ \downarrow & & \downarrow \\ T' & \xrightarrow{f_1} & T \end{array}$$

and

$$\begin{array}{ccc} X \times_B T' & \xrightarrow{id \times f_1} & X \times_B T \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ X \times_B T' & \xrightarrow{f_2} & X \times_B T \end{array}$$

commute. Therefore, the morphism is completely determined by the morphism f_1. By definition, the map f_1 is a morphism over B and by Proposition 4.3.7, it is a morphism over B_{hyp}. *q.e.d.*

4.5 First-order deformations of hyperelliptic Fano threefolds of type (1,1,4)

In this section, we determine the space of first-order deformations of hyperelliptic Fano threefolds of type (1,1,4), i.e., the "tangent space" of the moduli of hyperelliptic Fano threefolds $\mathcal{H}$. Let k be a field with $2 \in k^\times$, let $k[\epsilon] := k[t]/(t^2)$ be the ring of dual numbers over k and let $f\colon X \to \operatorname{Spec} k$ be a hyperelliptic Fano threefold of type (1,1,4). A *first-order lift* of X is given by a pair of morphisms

$$(\tilde{f}\colon \tilde{X} \to \operatorname{Spec} k[\epsilon], i\colon X \to \tilde{X})$$

such that the square

$$\begin{array}{ccc} X & \xrightarrow{i} & \tilde{X} \\ \downarrow f & & \downarrow \tilde{f} \\ \operatorname{Spec} k & \xrightarrow{g} & \operatorname{Spec} k[\epsilon] \end{array}$$

is cartesian, where g is the closed immersion given by $\epsilon \mapsto 0$. Two lifts

$$(\tilde{f}'\colon \tilde{X}' \to \operatorname{Spec} k[\epsilon], i'\colon X \to \tilde{X}) \text{ and } (\tilde{f}\colon \tilde{X} \to \operatorname{Spec} k[\epsilon], i\colon X \to \tilde{X})$$

are considered to be equivalent if there is an isomorphism $\phi\colon \tilde{X}' \to \tilde{X}$ over $k[\epsilon]$ such that $\phi \circ i' = i$. An equivalence class of first-order lifts is called a *first-order deformation*.

We are interested in characterizing first-order deformations of X which are again hyperelliptic. For this note that X comes with an involution associated to the double cover $\iota\colon X \to X$; see Definition 4.3.10. If $(\tilde{f}\colon \tilde{X} \to \operatorname{Spec} k[\epsilon], i\colon X \to \tilde{X})$ is a first-order lift such that $\tilde{f}\colon \tilde{X} \to \operatorname{Spec} k[\epsilon]$ is hyperelliptic, then there is an involution $\tilde{\iota}\colon \tilde{X} \to \tilde{X}$ which extends ι. From the explicit description of X as a weighted complete intersection it follows that a lift is hyperelliptic if and only if the involution of X extends to the lift. This observation allows us to determine the space of deformations that are hyperelliptic.

Theorem 4.5.1. *If X is a hyperelliptic Fano threefold of type $(1,1,4)$ over a field k with $2 \in k^\times$ with involution ι, then the space of first-order deformations of X that are hyperelliptic is the subspace of ι-invariants*

$$\mathrm{H}^1(X, \Theta_X)^\iota \subseteq \mathrm{H}^1(X, \Theta_X)$$

of the space of all first-order deformations.

Proof. Let $G = \mathbb{Z}/2\mathbb{Z}$. Then G acts on X via ι. By the observation above and [10, Proposition 3.2.7], the space of first-order deformations of X that are hyperelliptic is given by the equivariant sheaf cohomology group $\mathrm{H}^1(X; G, T_X)$. By Theorem 3.0.1, the action of G on Θ_X is faithful.

Therefore we can apply [27, Proposition 5.2.3] to see that

$$\mathrm{H}^1(X; G, \Theta_X) = \mathrm{H}^1(X, \Theta_X)^G = \mathrm{H}^1(X, \Theta_X)^\iota.$$

q.e.d.

Chapter 5

Persistence of arithmetic hyperbolicity

In this chapter, we formulate and prove our results on the persistence conjecture. They allow us to finish the proof of Theorem V and Theorem I.

5.1 Geometric hyperbolicity and persistence

Let k be an algebraically closed field of characteristic zero. We follow [38, §2] and say that a finitely presented algebraic stack X over k is *geometrically hyperbolic over* k if, for every smooth integral curve C over k, every point $c \in C(k)$ and object $x \in X(k)$, the set of isomorphism classes of morphisms $f : C \to X$ with $f(c) \cong x$ is finite. We will show that geometric hyperbolicity implies the persistence of arithmetic hyperbolicity. The proof of this fact uses an inductive argument. The following lemma gives us the induction step.

Lemma 5.1.1. *Let $k \subseteq L$ be an extension of algebraically closed fields of characteristic zero such that L is of transcendence degree* 1 *over k. Let X be a finite type separated arithmetically hyperbolic Deligne-Mumford stack over k. If X is geometrically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*

Proof. (If X is a variety, then this is [34, Lemma 4.2]. We adapt the proof of *loc. cit.* to the setting of stacks.)

The notion of arithmetic hyperbolicity is independent of the chosen model; check for [39, Lemma 4.8]. We choose a $\mathbb{Z}$-finitely generated subring $A \subseteq k$ and a model $\mathcal{X}$ for X over A. Since X is separated and Deligne-Mumford, it has a finite diagonal. The property of having a finite diagonal spreads out; see [64, B.3]. Hence after possibly replacing A with a finitely generated extension contained in K, we may and do assume that $\mathcal{X}$ has a finite diagonal. Note that $\mathcal{X}$ is also a model for X_L.

Let $B \subseteq L$ be a $\mathbb{Z}$-finitely generated subring containing A. We will find a $\mathbb{Z}$-finitely generated subring $B' \subseteq L$ containing B such that $\pi_0(\mathcal{X}(B'))$ is finite. We may assume B is not contained in k. Otherwise, we are done since X is arithmetically hyperbolic over k.

The morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$ and $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ are generically smooth and finitely presented. As smoothness spreads out, we can find finitely generated extensions $A \subseteq A' \subseteq k$ and $B \subseteq B' \subseteq L$ such that $A' \subseteq B'$ and both $\operatorname{Spec} B' \to \operatorname{Spec} A'$ and $\operatorname{Spec} A' \to \operatorname{Spec} \mathbb{Z}$ are smooth. The scheme $\mathcal{C} = \operatorname{Spec} B'$ is integral and smooth of relative dimension 1 over $\operatorname{Spec} A'$. As k is algebraically closed, the affine curve $C = \mathcal{C}_k$ has k-sections. Hence, after possibly replacing A' and B' by finitely generated extensions still satisfying $A' \subseteq k$ and $B' \subseteq L$ and the smoothness properties above, there is a section $c \in \mathcal{C}(A')$. Since A' is smooth over $\mathbb{Z}$, it is in particular integrally closed. Therefore, as X is arithmetically hyperbolic over k, by applying [39, Theorem 4.23], we see that $\pi_0(\mathcal{X}(A'))$ is finite. For any morphism $f : \mathcal{C} \to \mathcal{X}$, we have $f(c) \in \mathcal{X}(A')$. Therefore, we get an inclusion

$$\pi_0(\mathcal{X}(B')) = \pi_0(\mathcal{X}(\mathcal{C})) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\mathcal{X}) \subseteq \bigcup_{x\in\pi_0(\mathcal{X}(A'))} \mathrm{Hom}_{\mathcal{C}}((\mathcal{C},c),(\mathcal{X},x)).$$

Furthermore, consider the diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}((\mathcal{C},c),(\mathcal{X},x)) & \xrightarrow{\alpha} & \mathrm{Hom}_k((C,c_k),(X,x_k)) \\
\downarrow{\scriptstyle \iota} & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\mathcal{X}) = \pi_0(\mathcal{X}(B')) & \xrightarrow{\beta} & \pi_0(\mathcal{X}(L)) = \mathrm{Hom}(L,X)
\end{array}$$

The natural inclusion map ι is injective and the pullback map β has finite fibres by [39, Proposition 4.19]. Hence α has finite fibres. Since X is geometrically hyperbolic over k, we know that the set $\mathrm{Hom}_k((C,c_k),(X,x_k))$ is finite. From this, we deduce that $\pi_0(\mathcal{X}(B'))$ is finite. *q.e.d.*

Lemma 5.1.2. *Let $k \subseteq L$ be an extension of algebraically closed fields of characteristic zero and let X be a finite type separated arithmetically hyperbolic Deligne-Mumford stack over k such that, for every algebraically closed subfield $K \subseteq L$ containing k, the stack X_K is geometrically hyperbolic over K. Then X_L is arithmetically hyperbolic over L.*

Proof. As arithmetic hyperbolicity expresses the finiteness of points valued in $\mathbb{Z}$-finitely generated integral domains of characteristic zero, the stack X_L is arithmetically hyperbolic if and only if X_K is arithmetically hyperbolic for all algebraically closed subfields $K \subseteq L$ that have a finite transcendence degree over k. Therefore, we may and do assume that L/k has a finite transcendence degree. Let $k = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = L$ be a chain of extensions of algebraically closed fields each having transcendence degree one. We apply Lemma 5.1.1 inductively to see that X_{K_i} is arithmetically hyperbolic for all i. *q.e.d.*

Theorem 5.1.3. *Let k be an uncountable algebraically closed field of characteristic zero. Let X be a finite type separated Deligne-Mumford stack over k. If X is arithmetically hyperbolic over k and geometrically hyperbolic over k, then X_L/L is arithmetically hyperbolic over L and geometrically hyperbolic over L for any algebraically closed field extension $k \subseteq L$.*

Proof. This is a combination of Lemma 5.1.2 with the fact that geometric hyperbolicity over uncountable fields persists [38, Lemma 2.4]. *q.e.d.*

The following two lemmata will be used to prove that $\mathcal{F}$ is geometrically hyperbolic.

Lemma 5.1.4. *Let X be a finite type separated Deligne-Mumford stack over a field k and let C be a smooth integral curve over k with function field K. Then the map $\pi_0(X(C)) \to \pi_0(X(K))$ is injective.*

Proof. Let $g_1, g_2 \in X(C)$ be objects. Since the diagonal $\Delta\colon X \to X \times_k X$ is finite, the Isom-sheaf $\mathrm{Isom}_{X(C)}(g_1,g_2)$ is a finite scheme over C. In particular, every generic section of

$$\mathrm{Isom}_{X(C)}(g_1,g_2) \to C$$

extends by the valuative criterion of properness (uniquely) to a section. I.e. if g_1, g_2 are isomorphic over K, then they are isomorphic over C. *q.e.d.*

Lemma 5.1.5. *Let $f\colon X \to Y$ be a quasi-finite representable morphism of separated finite type Deligne-Mumford stacks over a field k. If Y is geometrically hyperbolic, then X is geometrically hyperbolic.*

Proof. Let C be a smooth integral curve over k and $c \in C(k)$, $x \in X(k)$ points. Let $y = f(x) \in Y(k)$. The set $\mathrm{Hom}_k((C,c),(Y,y))$ is finite. Therefore it suffices to show that for any fixed $g \in \mathrm{Hom}_k((C,c),(Y,y))$, there are only finitely many isomorphism classes of morphisms $\phi\colon (C,c) \to (X,x)$ such that $f \circ \phi \cong g$. Consider the generic point $\eta\colon \operatorname{Spec} k(C) \to C$. As f is quasi-finite, there are only finitely many possibilities for the image of η in X up to isomorphism. Hence by Lemma 5.1.4 there are only finitely many possibilities for the isomorphism class of ϕ. *q.e.d.*

5.2 Period map and arithmetic hyperbolicity

In this section, we will proof Theorem I and Theorem V. The following well-known lemma will be used to prove the arithmetic hyperbolicity of the stack $\mathcal{F}$.

Lemma 5.2.1. *Let $f\colon X \to Y$ be a morphism of finite type separated Deligne-Mumford stacks. If f is injective on tangent spaces, then f is quasi-finite.*

Proof. Let $Y' \to Y$ be an étale surjective morphism with Y' a scheme and let $X' \to X \times_Y Y'$ be an an étale surjective morphism with X' a scheme. Then the morphism $X' \to Y'$ induced by f is injective on tangent spaces. Hence it is unramified and therefore quasi-finite; see [72, Tag 02V5, Tag 02BG]. This shows that f is quasi-finite. *q.e.d.*

Following [61], we define the stack of principally polarized abelian varieties.

Definition 5.2.2. Let $g \in \mathbb{Z}_{\geq 0}$. We define $\mathcal{A}_g$ category of principally polarized abelian varieties as follows.

The objects objects are triples (S, A, λ), where A is an abelian scheme over S of relative dimension g, and $\lambda\colon A \to A^\vee$ is a polarization of degree 1 ($A^\vee$ denotes the dual abelian scheme of A).

A morphism $(S', A', \lambda') \to (S, A, \lambda)$ is a cartesian diagram

$$\begin{array}{ccc} A' & \xrightarrow{a} & A \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

such that the induced diagram

$$\begin{array}{ccc} A' & \xrightarrow{a} & S' \times_S A \\ \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \mathrm{id} \times \lambda} \\ (A')^\vee & \xleftarrow[a^*]{} & S' \times_S A^\vee \end{array}$$

commutes. The category $\mathcal{A}_g$ becomes a category fibered in groupoids over S via the projection functor

$$p\colon \mathcal{A}_g \to (\mathrm{Sch}/S)\colon (S, A, \lambda) \mapsto S.$$

Recall that $\mathcal{A}_g$ is a finite type separated Deligne-Mumford stack over $\mathbb{Z}$; see [61]. The analytification $\mathcal{A}_{g,\mathbb{C}}^{\mathrm{an}}$ is the moduli-stack of principally polarized complex tori. If X is a Fano threefold of type $(1,1,4)$ over $\mathbb{C}$, then $\mathrm{H}^{3,0}(X)$ and $\mathrm{H}^{0,3}(X)$vanish. We obtain a Hodge decomposition

$$\mathrm{H}^3(X, \mathbb{C}) = \mathrm{H}^{1,2}(X) \oplus \mathrm{H}^{2,1}(X),$$

i.e., it is of Hodge level one. In Section 3.6, we used Theorem II to find an explicit description for $\mathrm{H}^{1,2}(X)$ and $\mathrm{H}^{2,1}(X)$ in the case that X is hyperelliptic. Given a presentation

$$X \cong V_+(z^2 - f, g) \subseteq \mathbb{P}_{\mathbb{C}}(1,1,1,1,1,2)$$

with $f, g \in \mathbb{C}[x_0, \ldots, x_4]$, $\deg(g) = 2$ and $\deg(f) = 4$, the dual of $\mathrm{H}^{2,1}(X)$ is isomorphic to

$$\left(\frac{\mathbb{C}[x_0, \ldots, x_4]}{(f,g)}\right)_3.$$

From this, we see $\dim(\mathrm{H}^{2,1}(X)) = 30$. On the other hand if $X \cong V_+(h) \subseteq \mathbb{P}^4_{\mathbb{C}}$ is a smooth quartic, then by Theorem II, the dimension of $\mathrm{H}^{2,1}(X)$ equals the dimension of

$$\left(\frac{\mathbb{C}[x_0, \ldots, x_4, y]}{(h, y\partial_{x_0}h, \ldots, y\partial_{x_4}h)}\right)_{(1,-1)} \cong \left(\frac{\mathbb{C}[x_0, \ldots, x_4]}{(\partial_{x_0}h, \ldots, \partial_{x_4}h)}\right)_3,$$

which is also 30. Hence, the *intermediate Jacobian* of X

$$J(X) := \mathrm{H}^{1,2}(X)/\mathrm{H}^3(X, \mathbb{Z})$$

is a principally polarized abelian variety of dimension 30. There is a complex analytic period map

$$p : \mathcal{F}_{\mathbb{C}}{}^{\mathrm{an}} \to \mathcal{A}^{\mathrm{an}}_{30,\mathbb{C}},$$

which point-wise associates to a Fano threefold its intermediate Jacobian; see [15], where the period map is constructed for threefolds of Hodge level 1, also see [16, 40]. Our main result is that this period map has finite fibres in the setting of Fano threefolds of type $(1,1,4)$.

Theorem 5.2.3. *The period map*

$$p : \mathcal{F}_{\mathbb{C}}{}^{an} \to \mathcal{A}^{an}_{30,\mathbb{C}}$$

is quasi-finite.

Proof. First, we consider the period map restricted to the smooth quartic locus. The infinitesimal Torelli problem for hypersurfaces is completely understood; see [12]. In particular, if X is a smooth quartic, the differential of the period map

$$(dp)_X \colon \mathrm{H}^1(X, \Theta_X) \to \bigoplus_{p+q=n} \mathrm{Hom}_{\mathbb{C}}\left(H^p(X, \Omega^q_X), H^{p+1}(X, \Omega^{q-1}_X)\right)$$

is injective. Therefore, the period map restricted to $\mathcal{Q}$ is quasi-finite by Lemma 5.2.1.

Furthermore, by Theorem 4.5.1, if X is a hyperelliptic Fano threefold over $\mathbb{C}$ with involution ι asscociated to the double cover, then the tangent space of $\mathcal{H}$ at the object X is $\mathrm{H}^1(X, \Theta_X)^\iota$. Therefore, by Theorem III, the differential of the period map restricted to $\mathcal{H}$ is injective. Again by Lemma 5.2.1, we conclude that $p|_{\mathcal{H}}$ is quasi-finite.

Since $\mathcal{F}$ is the union of $\mathcal{Q}$ and $\mathcal{H}$ (Theorem IV), the period map p is quasi-finite. *q.e.d.*

Proof of Theorem I. By [20], the stack $(\mathcal{A}_{30})_{\overline{\mathbb{Q}}}$ is absolutely arithmetically hyperbolic. Moreover, by [38, Theorem 1.7], the stack $(\mathcal{A}_{30})_{\mathbb{C}}$ is geometrically hyperbolic. Note by Proposition 4.1.6. the stack $\mathcal{F}_{\mathbb{C}}$ is Deligne-Mumford. Hence by [39, Theorem 6.4], Lemma 5.1.5 and Theorem 5.2.3, the stack $\mathcal{F}_{\mathbb{C}}$ is arithmetically hyperbolic and geometrically hyperbolic. Hence by Theorem 5.1.3, the stack X_L is arithmetically and geometrically hyperbolic for any algebraically closed field extension $L \supseteq \mathbb{C}$. *q.e.d.*

Proof of Theorem V. Let K be a number field and let S be a finite set of places on K. We may assume that all places lying over 2 are contained in S. As seen in [40, Proposition 2.10], if B is a connected scheme and $f : X \to B$ is a Fano threefold, then the type is constant in the fibres of f. Hence, the set of isomorphism classes of Fano threefolds X of type $(1, 1, 4)$ over K with good reduction outside S is the image

$$\mathrm{im}\left(\pi_0(\mathcal{F}(O_{K,S})) \to \pi_0(\mathcal{F}(K))\right).$$

Since $\mathcal{F}_{\overline{\mathbb{Q}}}$ is arithmetically hyperbolic by Theorem I, this set is finite by [39, Theorem 4.22]. *q.e.d.*

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