Long-range voter model on the real line

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Abstract

In the classical voter model on the integer lattice, a voter with one of two opinions, say 0 or 1, is placed at each location on the lattice. Each voter has an alarm clock set to an exponentially distributed random time. When the clock rings, the voter adopts the opinion of a randomly chosen neighbour and the clock is set at a new random time. This process satisfies a moment duality with a system of coalescing random walks.

Here we are interested in the situation with an uncountable number of voters, placed at each point of the real line. We allow them to adopt the opinions of other voters that are far away. Specifically, we describe a measure valued process satisfying a moment duality relation with a coalescing system of symmetric α-stable processes, where $\alpha \in (1, 2)$. Such a process was constructed by Steven N. Evans in 1997.

The purpose of this thesis is to analyze this so called long-range voter model on the real line. We discuss the Hausdorff dimension of the interface between opinions and examine the survival probability of colonies of opinion 1 at a given time point. In addition, for a simplified model, we calculate the dimension of the exceptional time points where the support of opinion 1 is unbounded. Furthermore, we show that the process arises as limit of solutions of a stochastic partial differential equation with accelerated noise.

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Contents

1. Introduction

The voter model is an interacting particle system that was introduced independently by Clifford and Sudbury [CS73] and Holley and Liggett [HL75] in the 1970s. At each site of some discrete geographic space S sits a person who has one of several possible opinions. These persons are called voters. Each voter has a clock which rings after some standard exponentially distributed random time, independently of the other clocks. When a voter's clock rings, the voter adopts the opinion of a person who is chosen according to a given probability distribution on the site space and the clock is set to a new random time. In addition to the interpretation with the voters, the model can also be used in other application contexts, for example as a model for a territorial conflict between different populations or a random coloring of some graph. In the case of two different opinions or types or colors, each site of the space can be occupied or vacant, for example by one party or a particle.

Considering as the geographical space the lattice $S = \mathbb{Z}^d$ for some dimension $d \in \mathbb{N}$ and as possible opinions 0 and 1, one can formalize the voter model according to [Li85, Chapter V] as follows: It is a Markov process $(\eta_t)_{t\geq 0}$ taking values in $E = \{0,1\}^S$ whose generator $\mathcal{A}$ acts on functions $f\colon E \to \mathbb{R}$ which depend on at most a finite number of coordinates like

$$(\mathcal{A}f)(\eta) = \sum_{x\in\mathbb{Z}^d} c(x,\eta)(f(\eta^{(x)}) - f(\eta)), \qquad \eta \in E.$$

Here $\eta^{(x)}\colon \mathbb{Z}^d \to \{0,1\}$ is defined via

$$\eta^{(x)}(y) := \begin{cases} 1-\eta(x), & \text{if } y = x, \\ \eta(y), & \text{if } y \neq x \end{cases}$$

and the so called jump rate $c(x,\eta)$ from η to $\eta^{(x)}$ for some $x \in \mathbb{Z}^d$ is given by

$$c(x,\eta) := \sum_{\substack{y\in\mathbb{Z}^d,\\ \eta(y)\neq\eta(x)}} p(x,y)$$

where $p(\cdot,\cdot)$ denotes the transition probability of some irreducible Markov chain on $\mathbb{Z}^d$. Then $\eta_t(x) \in \{0,1\}$ indicates the opinion of the voter at site $x \in \mathbb{Z}^d$ at time $t \geq 0$. For $x, y \in \mathbb{Z}^d$ the value $p(x,y)$ can be interpreted as the probability that the voter at site x adopts the opinion of the voter at site y given the voter at x is chosen to change the opinion at this time. For simplicity, we will henceforth assume that p is symmetric and translation-invariant, i.e. $p(x,y) = p(y,x) = p(0, y-x)$ for each $x, y \in \mathbb{Z}^d$.

It is also possible to construct the voter model in a rigorous way based on the intuition with the exponentially distributed times at which the changes of opinion occur. This leads to a graphical representation which is described in detail for example in [Du88, Chapter 2]. Using this one can compute the distribution of $\eta_t(x)$ for some fixed $x \in \mathbb{Z}^d$ and $t \geq 0$ given some initial condition $\eta_0 \in E$: One starts a continuous-time random walk with increment distribution $p(0,\cdot)$ in x and evaluates η_0 at the position of this random walk at time t. This gives a realisation of $\eta_t(x)$. To compute the opinions of several voters at some time, one can proceed in the same way, but the different random walks are independent only until two walkers meet. When two walkers meet, they coalesce and continue following together a random walk until they meet another walker. This is the well known duality between the voter model and coalescing random walks: Consider a finite set $A \subset \mathbb{Z}^d$. Start an independent random walk at each point of A. Each time two walkers meet they coalesce and follow together a random walk until they meet another walker. Thus let Y_t^A be the set of positions of this coalescing random walk at time $t \geq 0$. Then one has

$$\mathbf{P}\left[\eta_t(x) = 1 \text{ for all } x \in A\right] = \mathbf{P}\left[\eta_0(y) = 1 \text{ for all } y \in Y_t^A\right].$$

Since $\eta_t(x) \in \{0,1\}$ for each $x \in \mathbb{Z}^d$, this can also be written in terms of moments:

$$\mathbf{E}\left[\prod_{x\in A} \eta_t(x)\right] = \mathbf{E}\left[\prod_{y \in Y_t^A} \eta_0(y)\right].$$

The duality allows to obtain results on the voter model using facts about random walks. Assume that the random walk is simple, i.e.

$$p(x,y) := \begin{cases} \frac{1}{2d}, & \text{if } \sum_{i=1}^d |y_i - x_i|^2 = 1, \\ 0, & \text{else.} \end{cases}$$

For the situation where the initial condition is a Bernoulli measure with intensity $\vartheta \in (0,1)$, that is

$$\mathbf{P} \circ \eta_0^{-1} = (\vartheta\delta_1 + (1-\vartheta)\delta_0)^{\otimes \mathbb{Z}^d},$$

[CS73] and [HL75] obtained that the process η_t converges in distribution to a random variable η_∞ as $t \longrightarrow \infty$ satisfying

$$\mathbf{P}[\eta_\infty \in \cdot] = \begin{cases} \vartheta\delta_{\mathbf{1}} + (1-\vartheta)\delta_{\mathbf{0}}, & \text{if } d \in \{1,2\}, \\ \nu_\vartheta, & \text{if } d \geq 3, \end{cases}$$

where $\delta_{\mathbf{1}}$ or $\delta_{\mathbf{0}}$ denotes the dirac measure on all 1's or 0's and ν_ϑ is an ergodic nontrivial equilibrium measure with intensity ϑ. More generally, the same dichotomy can be observed depending on the reccurence or transience of the associated random walk. In the recurrent case, i.e. $d \leq 2$ for simple random walks, one can show for an arbitrary initial condition η_0 that for $x, y \in \mathbb{Z}^d$

$$\lim_{t\to\infty} \mathbf{P}\left[\eta_t(x) \neq \eta_t(y)\right] = 0.$$

This means in the long term there is local consensus between the two opinions 0 and 1. If one of the two opinions initially has a bounded support, it can be proven that this opinion will eventually die out. In dimension $d = 1$, the so-called interface at a certain time $t \geq 0$ is defined as $\{x \in \mathbb{Z} : \eta_t(x) \neq \eta_t(x-1)\}$. It can be interpreted as the transition area between the two opinions. [Sc78, Section 2] described the interface through a system of annihilating random walks. These are random walks in which both walkers die when they meet each other.

While the model described so far assumes a countable number of voters on the discrete set $\mathbb{Z}^d$, it is also possible to construct a voter model in a situation with continuous space. Steven N. Evans introduced in [Ev97] a process which fulfills the moment duality of the voter model with a coalescing system of Markov processes taking values in a Polish space. Important examples of the dual process are coalescing Brownian motions or coalescing α-stable processes with $\alpha \in (1, 2)$ on the real line $\mathbb{R}$. Thus in these cases we refer to the process that [Ev97] constructed as a nearest-neighbour or long-range voter model on the real line. We only consider the one-dimensional case, since two Brownian motions or two stable processes in higher dimensions with probability 1 do not meet. Further we continue to look at the two-type situation, while [Ev97] even allows infinitely many types. Formally, the process is a measure-valued Feller process $(u_t)_{t\geq 0}$ where u_t is a measure on $\mathbb{R}$ with some density $u_t : \mathbb{R} \to [0, 1]$ that fulfills an analogous moment duality as the classical voter model: For each $t \geq 0$, $n \in \mathbb{N}$ and for Lebesgue-almost all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, it holds

$$\mathbf{E}\left[\prod_{i=1}^{n} u_t(x_i)\right] = \mathbf{E}\left[\prod_{y\in\Xi_t^{\{x_1,\ldots,x_n\}}} u_0(y)\right] \tag{1.1}$$

where Ξ_t^A denotes the set of positions of particles performing a coalescing Brownian motion or coalescing α-stable process at time t starting in a finite set $A \subset \mathbb{R}$. Thus $u_t(x) \in [0, 1]$ can be first of all interpreted as the proportion of opinion 1 of the voter at site $x \in \mathbb{R}$ at time $t \geq 0$. But [Ev97] showed that, for each fixed $t > 0$, one has almost surely $u_t(x) \in \{0, 1\}$ for Lebesgue-almost all $x \in \mathbb{R}$. Therefore, we can really interpret $u_t(x)$ as the opinion of the voter at site x at time t. It should also be noticed that, in contrast to the discrete voter model, the moment duality is used here to show existence of the process. The discrete model was introduced via the generator or graphical construction and duality is an application of it.

The nearest-neighbour voter model on the real line is a well studied process. [HOV18, Theorem 2.8 (a)] proved that it can be approximated by solutions of the stochastic partial differential equation

$$\partial_t u_t^{[\gamma]}(x) = \frac{1}{2}\Delta u_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x)(1-u_t^{[\gamma]}(x))}\dot{W}_{t,x}, \qquad t \geq 0,\ x \in \mathbb{R}, \tag{1.2}$$

as $\gamma \to \infty$. Here $\dot{W}$ is a space-time white noise. For $\gamma \in (0,\infty)$, this model is called continuous-space stepping stone model with parameter γ. Here both opinions can coexist at the same site and the type is changed at a rate proportional to the proportions of the respective types at a site. [Tr95] showed that the interface of a rescaling of (1.2) converges in distribution to annihilating Brownian motion. [HOV18] identified the interface of the voter model with annihilating Brownian motion. In particular, it was shown in [HOV18, Theorem 2.12] that the voter model can also be constructed from the annihilating system, as is also possible in the discrete case. Further due to [AS11] and [GSW16] there is also a graphical representation using the so called dual Brownian web. Roughly speaking the Brownian web is a system of coalescing Brownian motion where at each space-time-point one starts a new Brownian motion. For more details on the nearest-neighbour voter model see [HOV21, Section 3] and [BO21].

The aim of this work is to study the long-range voter model on the real line whose dual process is a system of coalescing α-stable processes with $\alpha \in (1,2)$. One difficulty is that the paths of stable processes are not continuous. This complicates the examination of coalescing systems of such processes. If one starts with a compact interval of opinion 1, in the Brownian case this opinion survives as long as it takes for two independent Brownian motions that start at the two boundary points of the compact set to meet. This follows from the characterization of the interface mentioned above. In the stable case it is not clear what the interface looks like. Intuitively it should be more chaotic like a fractal. A main goal of this work is to get a better understanding of this. Therefore, we prove an upper bound on the Hausdorff dimension of the interface and give heuristics on a lower bound. Further we compute the survival probability of a small colony of opinion 1 at some fixed time. We also show that the support remains compact at fixed times. This opens the question of whether there may be random time points when the support is unbounded. We will consider some toy model and compute the Hausdorff dimension of these exceptional time points in that simpler model. Moreover, we will prove that the long-range voter-model on the real line arises as the limit of the finite rate model (1.2) as $\gamma \to \infty$ where the Laplace operator is replaced by the fractional Laplace operator.

To achieve these goals, duality is often the main tool that will be used in proofs. In fact, the moment duality formula (1.1) is a priori the only characterization of the process available. Therefore it is important to understand coalescing stable processes. In [EMS13] it is shown that the system "comes down from infinity" if one starts the particles in a compact subset of the real line. We will use various interim results from [EMS13] for studying the voter model.

Here is the outline: In Chapter 2, we will discuss basic properties of stable processes and coalescing systems. When defining the coalescing system, we will follow [EMS13] and cite various results. We then introduce the long-range voter model on the real line in Chapter 3 with the result of [Ev97] and prove basic properties that follow from the moment duality. Further we show that the support stays bounded at fixed times. These properties are required in particular in the following two chapters. In Chapter 4, we prove an upper bound on the Hausdorff dimension of the interface. In Chapter 5, we give bounds on the survival probability of small colonies at fixed times and derive that they die out almost surely. In Chapter 6, we introduce the toy model and compute the Hausdorff dimension of the exceptional time points for it. In Chapter 7, we show that the finite rate model converges to the voter model. Finally, in Chapter 8, we discuss some open questions and conjectures including a lower bound on the Hausdorff dimension of the interface.

2. Preliminaries

2.1. Stable processes

In this section, we introduce stable processes from which we build the dual process of the voter model, the coalescing stable process. Since stable processes are Lévy processes we start defining this class of processes (see [Be96] and [Kyp14]). Here we always restrict ourselves to the one-dimensional case. After presenting basic distribution properties of stable processes, we later cite results about time reversal, hitting times and local times. In the following sections of that chapter we introduce the coalescing stable process (Section 2.2) and present stable limit theorems (Section 2.3).

Definition 2.1. *A Lévy process is a real-valued stochastic process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ and càdlàg paths on some filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ such that*

(i) $X_t - X_s$ *is independent of* $\sigma(X_u\colon u \leq s)$ *for all* $s \in [0,t]$,

(ii) $X_t - X_s \overset{d}{=} X_{t-s}$ *for all* $s \in [0,t]$.

The characteristic function of a Lévy processes X can be written in the form

$$\mathbf{E}\left[e^{i\vartheta X_t}\right] = e^{-t\Psi(\vartheta)}, \qquad \vartheta \in \mathbb{R},$$

where $\Psi\colon \mathbb{R} \to \mathbb{C}$ is called the *characteristic exponent* of X. It determines the law of X as a random variable with values in $D([0,\infty),\mathbb{R})$, the space of real-valued càdlàg-paths. In general, if E is some metric space, we denote by $D([0,\infty),E)$ the space of E-valued càdlàg-paths which we endow with the Skorokhod topology (see [EK86, Section 3.5] or [JS03, Chapter VI]).

Definition 2.2. *A strictly stable process with index $\alpha \in (0,2]$ is a Lévy process X such that for each $t > 0$*

$$X_t \overset{d}{=} t^{\frac{1}{\alpha}} X_1.$$

According to [Be96, Section VIII.1, p. 217] the characteristic exponent $\Psi(\vartheta)$ of a strictly stable process with index $\alpha \in (0,2]$ is for $\vartheta \in \mathbb{R}$ given by

$$\Psi(\vartheta) = \begin{cases} c\,|\vartheta|^{\alpha}\left(1 - i\beta \operatorname{sgn}(\vartheta)\tan\left(\frac{\pi\alpha}{2}\right)\right), & \text{if } \alpha \in (0,1)\cup(1,2), \\ c\,|\vartheta| + di\vartheta, & \text{if } \alpha = 1, \\ c\vartheta^2, & \text{if } \alpha = 2. \end{cases} \tag{2.1}$$

where $c > 0$, $d \in \mathbb{R}$ and $\beta \in [-1,1]$.

Definition 2.3. *We call the distribution of a real-valued random variable Z strictly stable with index $\alpha \in (0,2]$ if for $\vartheta \in \mathbb{R}$, its characteristic function is given by*

$$\mathbf{E}\left[e^{i\vartheta Z}\right] = e^{-\Psi(\vartheta)}$$

with Ψ from (2.1).

Let X be a strictly stable process with index $\alpha \in (0,2]$. We assume from now on $\beta = d = 0$, that is, the process is symmetric and its characteristic exponent is given by

$$\Psi(\vartheta) = c\,|\vartheta|^{\alpha}, \qquad \vartheta \in \mathbb{R}.$$

We call the symmetric strictly stable process simply an *α-stable process* or *stable process* (with rate c) and analogously we call the distribution of X_1 an *α-stable distribution* or *stable distribution* (with rate c). In the case $\alpha = 1$, X is the so called Cauchy process, in the case $\alpha = 2$, the process has continuous sample paths and is Brownian motion. Furthermore, we refer to the stable process with $c = 1$ as the *standard stable process* and to the stable distribution with $c = 1$ as the *standard stable distribution*.

By Fourier inversion, X_t has a density $p_t^{(c,\alpha)}: \mathbb{R} \to (0,\infty)$, i.e., for $t > 0$ and $x \in \mathbb{R}$,

$$p_t^{(c,\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\vartheta} e^{-tc|\vartheta|^{\alpha}}\, dx.$$

From the scaling property of the stable process it follows that the density fulfills for $t > 0$ and $x \in \mathbb{R}$

$$p_t^{(c,\alpha)}(x) = t^{-\frac{1}{\alpha}} p_1^{(c,\alpha)}(t^{-\frac{1}{\alpha}} x).$$

We simply write $p_t^{(\alpha)} := p_t^{(1,\alpha)}$ for $t > 0$. The density $p_t^{(\alpha)}$ is continuous and symmetric and according to [BG60, Theorem 2.1] (see also [BJ07, Lemma 3]) in the case $\alpha \in (0,2)$ there exist constants $c_1(\alpha), c_2(\alpha) > 0$ such that for $t > 0$ and $x \in \mathbb{R}$

$$c_1(\alpha)\min\left\{t^{-\frac{1}{\alpha}}, t\,|x|^{-\alpha-1}\right\} \le p_t^{(\alpha)}(x) \le c_2(\alpha)\min\left\{t^{-\frac{1}{\alpha}}, t\,|x|^{-\alpha-1}\right\}. \tag{2.2}$$

For each $x \in \mathbb{R}$ denote by $\mathbf{P}_x$ the law of $x + X$ under $\mathbf{P}$. We refer to $x + X$ as a stable process starting in x. According to [Be96, Propositions I.2.5, I.2.6], X is a strong Markov process and the corresponding semigroup has the Feller property. We can write down the scaling property in the following way. For $x \in \mathbb{R}$ and $c > 0$, we have

$$\mathbf{P}_x\left[(c^{-\frac{1}{\alpha}} X_{ct})_{t\ge 0} \in \cdot\right] = \mathbf{P}_{c^{-1/\alpha}x}\left[(X_t)_{t\ge 0} \in \cdot\right]. \tag{2.3}$$

The generator of a standard stable process is called fractional Laplacian operator and is denoted by $\mathcal{L}_\alpha = -(-\Delta)^{\alpha/2}$, that is

$$\left(\mathcal{L}_\alpha f\right)(x) = \lim_{t\searrow 0}\frac{1}{t}\left(\int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x) f(y)\,dy \; - f(x)\right), \qquad x \in \mathbb{R}, \tag{2.4}$$

for each $f\colon \mathbb{R}\to\mathbb{R}$ such that the limit on the right-hand side in (2.4) exists. For further equivalent definitions of the fractional Laplacian, see for example [Kw17].

Now we quote from [Be96] a time-reversal property that applies to symmetric Lévy processes. We specifically consider two independent stable processes $X=(X_t)_{t\geq 0}$ and $Y=(Y_t)_{t\geq 0}$. Assume that they are defined on the same filtered probability space $(\Omega,\mathcal{A},(\mathcal{F}_t)_{t\geq 0},\mathbf{P})$. For $(x,y)\in\mathbb{R}^2$, we denote by $\mathbf{P}_{(x,y)}$ the law of (X,Y) under $\mathbf{P}$. For fixed $t>0$, define

$$\widetilde{X}_s := \begin{cases} X_{(t-s)-}, & \text{if } s\in[0,t),\\ X_0, & \text{if } s\in[t,\infty),\end{cases} \qquad \widetilde{Y}_s := \begin{cases} Y_{(t-s)-}, & \text{if } s\in[0,t),\\ Y_0, & \text{if } s\in[t,\infty),\end{cases}$$

where for $s>0$,

$$X_{s-} := \lim_{u\nearrow s} X_u.$$

When we write $\mathbf{P}_{(x,y)}\left[(\widetilde{X},\widetilde{Y})\in\cdot\right]$ we mean that $(X_0,Y_0)=(x,y)$.

Proposition 2.4 ([Be96])**.** *Let X and Y be two independent stable processes of index $\alpha\in(0,2]$ and $t>0$. For an $\mathcal{F}_t$-measurable functional $F\colon D([0,\infty),\mathbb{R})^2\to[0,\infty)$ we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{E}_{(x,y)}\left[F(X,Y)\right]\,dy\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{E}_{(x,y)}\left[F(\widetilde{X},\widetilde{Y})\right]\,dy\,dx. \tag{2.5}$$

Proof. This is proved in [Be96, Section II.1, p. 46 above]. □

Next we sum up results on hitting times of stable processes (see [GR17] for general results on hitting times of symmetric Lévy processes). A stable process hits points almost surely if and only if $\alpha\in(1,2]$.

Proposition 2.5 ([YYY09],[Po67],[GR17])**.** *Let $(X_t)_{t\geq 0}$ be a standard stable process of index $\alpha\in(1,2]$ and let*

$$\tau := \inf\left\{t>0\,:\,X_t=0\right\}.$$

(i) For $x\in\mathbb{R}$ and $t>0$, we have

$$\mathbf{P}_x\left[\tau\leq t\right] = \int_0^1 \frac{(1-s)^{-\left(1-\frac{1}{\alpha}\right)}s^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)}\cdot\frac{p_1^{(\alpha)}\left(x(ts)^{-\frac{1}{\alpha}}\right)}{p_1^{(\alpha)}(0)}\,ds,$$

where Γ denotes the gamma function, i.e. $\Gamma(x)=\int_0^\infty u^{x-1}e^{-u}\,du$ for $x>0$.

(ii) There are constants $C_{2.5.1}=C_{2.5.1}(\alpha)>0$, $C_{2.5.2}=C_{2.5.2}(\alpha)>0$ and $C_{2.5.3}=C_{2.5.3}(\alpha)>0$ such that for $x\in\mathbb{R}$ and $t>0$, we have

$$C_{2.5.1}\min\left\{C_{2.5.2}t^{-\left(1-\frac{1}{\alpha}\right)}\left|x\right|^{\alpha-1},1\right\}\leq\mathbf{P}_x\left[\tau>t\right]\leq\min\left\{C_{2.5.3}t^{-\left(1-\frac{1}{\alpha}\right)}\left|x\right|^{\alpha-1},1\right\}.$$

Proof. The formula in (i) is given in the proof of [YYY09, Theorem 5.4]. The asymptotics of the non-hitting-probability in (ii) was proved in [Po67, Theorem 2]. The given bounds follow directly from more general results for symmetric Lévy processes in [GR17]: Let

$$K(x) := \int_0^\infty \left(p_s^{(\alpha)}(0) - p_s^{(\alpha)}(x)\right) ds = \left(2\Gamma(\alpha)\sin\left(\frac{(\alpha-1)\pi}{2}\right)\right)^{-1} |x|^{\alpha-1}$$

for $x \in \mathbb{R}$ (for the second equality see for example [YYY09, Lemma 4.1 (ii)]). According to [GR17, Proposition 3.1] for $x \in \mathbb{R}$ and $t > 0$, we have

$$\mathbf{P}_x[\tau > t] \leq 51\pi^3 \frac{K(x)}{t\Psi^{-1}(1/t)} = C_{2.5.3} t^{-\left(1-\frac{1}{\alpha}\right)} |x|^{\alpha-1}$$

for some constant $C_{2.5.3} = C_{2.5.3}(\alpha) > 0$. In the last equality we plugged in $\Psi^{-1}(y) = y^{\frac{1}{\alpha}}$ due to $\Psi(y) = y^\alpha$ for $y > 0$. Using [GR17, Lemma 3.3] we get the existence of constants $C_{2.5.1} = C_{2.5.1}(\alpha) > 0$ and $C_{2.5.2} = C_{2.5.2}(\alpha) > 0$ such that for $x \in \mathbb{R}$ and $t > 0$

$$\mathbf{P}_x[\tau > t] \geq C_{2.5.1} \min\left\{\frac{K(x)}{K(1/\Psi^{-1}(1/t))}, 1\right\} \geq C_{2.5.1} \min\left\{C_{2.5.2} t^{-\left(1-\frac{1}{\alpha}\right)} |x|^{\alpha-1}, 1\right\}.$$

□

We close this section with introducing the notion of local times of stable processes where we cite the occupation time formula and give an upper bound on the Laplace transform of the local time.

Proposition 2.6 ([Bo64],[Ba88])**.** *Let $(X_t)_{t\geq 0}$ be a stable process of index $\alpha \in (1,2]$. There exists a process $(L_t^{(z)})_{t\geq 0, z\in\mathbb{R}}$, which is jointly continuous in both variables, such that for each bounded and measurable $f\colon \mathbb{R} \to \mathbb{R}$ and $t \geq 0$, the occupation time formula holds, that is,* $\mathbf{P}$*-almost surely*

$$\int_0^t f(X_s)\, ds = \int_{-\infty}^\infty f(z) L_t^{(z)}\, dz. \tag{2.6}$$

$(L_t^{(z)})_{t\geq 0}$ is called local time process of $(X_t)_{t\geq 0}$ in $z \in \mathbb{R}$.

Proof. This is proved in [Bo64, Theorem 1 & Section 4]. See also [Ba88, Theorem 2]. □

Lemma 2.7 ([AT00],[BHO16])**.** *Let $(X_t)_{t\geq 0}$ be a stable process of index $\alpha \in (1,2]$ and let $(L_t^{(z)})_{t\geq 0}$ its local time process in $z \in \mathbb{R}$. For each bounded and measurable function $f\colon \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}$, we have* $\mathbf{P}$*-almost surely*

$$\int_0^t f(X_s, s, \cdot)\, ds = \int_{-\infty}^\infty \int_0^t f(z, s, \cdot)\, dL_s^{(z)}\, dz.$$

Proof. This follows from the occupation time formula (2.6). See [BHO16, Lemma A.10] or [AT00, Lemma 2] for a proof in the Brownian case $\alpha = 2$, which works exactly the same here. □

Lemma 2.8. *Let $(L_t^{(0)})_{t\geq 0}$ be the local time process of a standard stable process of index $\alpha \in (1,2]$ in 0. There exists a constant $C_{2.8} = C_{2.8}(\alpha) > 0$ such that for $\lambda > 0$ and $t > 0$ we have*

$$\mathbf{E}\left[e^{-\lambda L_t^{(0)}}\right] \leq \frac{C_{2.8}}{\lambda} t^{-\left(1-\frac{1}{\alpha}\right)}.$$

Proof. For $u \geq 0$ let

$$\tau_u := \inf\left\{t \geq 0 \,:\, L_t^{(0)} > u\right\}.$$

$(\tau_u)_{u\geq 0}$ is the inverse local time process. According to [Be99, Section 2.2 & Proposition 8.1] the process $(\tau_u)_{u\geq 0}$ is a stable subordinator with index $\beta = 1 - \frac{1}{\alpha} \in (0,1)$, that is, $\mathbf{E}[e^{-\lambda \tau_u}] = e^{-uc\lambda^\beta}$ for $\lambda > 0$, $u \geq 0$ and some constant $c = c(\beta) = c(\alpha) > 0$. By [Kh11, Lecture 8, Theorem 10], there exists a constant $C_{2.8} = C_{2.8}(\beta) = C_{2.8}(\alpha) > 0$ such that for $z > 0$, we have

$$\mathbf{P}[\tau_1 > z] \leq C_{2.8} z^{-\beta}.$$

Using the scaling property $\tau_u \overset{d}{=} u^{\frac{1}{\beta}} \tau_1$ this yields

$$\mathbf{P}[\tau_u > z] \leq C_{2.8} u z^{-\beta} = C_{2.8} u z^{-\left(1-\frac{1}{\alpha}\right)}.$$

With this tail bound we are now ready to bound the Laplace transform of $L_t^{(0)}$ for $t \geq 0$.

$$\begin{aligned}
\mathbf{E}\left[e^{-\lambda L_t^{(0)}}\right] &= \int_0^1 \mathbf{P}\left[e^{-\lambda L_t^{(0)}} \geq x\right] dx \\
&= \int_0^1 \mathbf{P}\left[L_t^{(0)} \leq -\frac{1}{\lambda}\ln(x)\right] dx \\
&= \int_0^1 \mathbf{P}\left[\tau_{-\frac{1}{\lambda}\ln(x)} > t\right] dx \\
&\leq -\frac{C_{2.8}}{\lambda} t^{-\left(1-\frac{1}{\alpha}\right)} \int_0^1 \ln(x)\, dx \\
&= \frac{C_{2.8}}{\lambda} t^{-\left(1-\frac{1}{\alpha}\right)}.
\end{aligned}$$

□

2.2. Coalescing stable process

Having established basic definitions and properties of stable processes in the last section we introduce now the *coalescing stable process*. Informally speaking, one starts several independent stable processes of index $\alpha \in (1,2]$ somewhere on the real line and if two processes meet, they coalesce. This process was examined in detail in [EMS13] and we follow the notation there and present here the constructions and most important results that are relevant to us. Note that in the Brownian case $\alpha = 2$, such processes have already been introduced in [Ar79]. We start giving a formal construction of a *vector-valued coalescing system* of a finite or countable number of stable processes and afterwards we introduce the *set-valued coalescing system* which allows to start a stable process in each point of a closed subset of the real line (for example the real line itself). The constructions follows [EMS13, Section 2].

Vector-valued coalescing system
Let $N \in \mathbb{N} \cup \{\infty\}$ and

$$[N] = \begin{cases} \{1, 2, \ldots, N\}, & \text{if } N < \infty, \\ \mathbb{N}, & \text{if } N = \infty. \end{cases}$$

Let $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(N)}$ be independent α-stable processes of index $\alpha \in (1, 2]$ starting in some points $\xi_0^{(1)}, \xi_0^{(2)}, \ldots, \xi_0^{(N)} \in \mathbb{R}$ (if $N = \infty$ we take a family of countable many processes). Fix a bijection $\sigma \colon [N] \to [N]$. The map can be interpreted as follows: If particle $\sigma(i)$ and $\sigma(j)$ with $i < j$ hit each other, then particle $\sigma(j)$ dies and follows the path of particle $\sigma(i)$. In the construction of the coalescing system, σ describes the order in which we "lay down" the stable processes (first we lay down process number $\sigma(1)$, then $\sigma(2)$ and so on). Now we give a formal inductive construction of the vector-valued coalescing process, which we denote by $\zeta^\sigma = (\zeta^{\sigma,1}, \zeta^{\sigma,2}, \ldots)$. It takes values in the space $D([0,\infty), \mathbb{R})^N$. Let $\zeta^{\sigma,\sigma(1)} := \xi^{(\sigma(1))}$. Assume that $\zeta^{\sigma,\sigma(1)}, \zeta^{\sigma,\sigma(2)}, \ldots, \zeta^{\sigma,\sigma(i-1)}$ are already constructed for some $i \geq 2$. Set

$$\tau_i := \inf\left\{t \geq 0 \,:\, \xi_t^{(\sigma(i))} \in \left\{\zeta_t^{\sigma,\sigma(1)}, \zeta_t^{\sigma,\sigma(2)}, \ldots, \zeta_t^{\sigma,\sigma(i-1)}\right\}\right\}.$$

This is the first coalescence time of particle number $\sigma(i)$. Note that τ_i is almost surely finite due to the assumption $\alpha \in (1, 2]$. Further let

$$J_i := \min\left\{j \in \{1, 2, \ldots, i-1\} \,:\, \xi_{\tau_i}^{(\sigma(i))} = \zeta_{\tau_i}^{\sigma,\sigma(j)}\right\}.$$

This means particle number $\sigma(J_i)$ is the coalescence partner of particle $\sigma(i)$. For $t \geq 0$ let

$$\zeta_t^{\sigma,\sigma(i)} := \begin{cases} \xi_t^{(\sigma(i))}, & \text{if } t < \tau_i, \\ \zeta_t^{\sigma,\sigma(J_i)}, & \text{if } t \geq \tau_i. \end{cases}$$

Lemma 2.9 ([EMS13])**.** *The distribution of ζ^σ does not depend on the choice of the bijection $\sigma \colon [N] \to [N]$.*

Proof. This is [EMS13, Lemma 2.1]. □

Let $\mathrm{id} \colon [N] \to [N]$ be the identical mapping. The previous lemma motivates to define $\zeta := \zeta^{\mathrm{id}}$. We write $\zeta = (\zeta^{(1)}, \zeta^{(2)}, \ldots)$. Define the right-continuous filtration $(\mathcal{F}_t^{\mathrm{coal}})_{t \geq 0}$ via

$$\mathcal{F}_t^{\mathrm{coal}} := \bigcap_{\varepsilon > 0} \sigma(\xi_s^{(i)} \colon s \leq t + \varepsilon, i \in [N]).$$

Lemma 2.10 ([EMS13]). *The stochastic process ζ is strong Markov with respect to the filtration $(\mathcal{F}_t^{\text{coal}})_{t\geq 0}$.*

Proof. This is [EMS13, Lemma 2.2]. $\square$

Set-valued coalescing system
For $t \geq 0$, define

$$\Xi_t := \overline{\left\{\zeta_t^{(i)} \,:\, i \in [N]\right\}},$$

where $\overline{B}$ denotes the topological closure in $\mathbb{R}$ of a set $B \subset \mathbb{R}$. We refer to $(\Xi_t)_{t\geq 0}$ as the set-valued coalescing system. The process $(\Xi_t)_{t\geq 0}$ takes values in the set $\mathcal{C}_\mathbb{R}$ of nonempty closed subsets of $\mathbb{R}$. One can topologize that state space as follows (see [EMS13, Section 2.2]): Consider the one-point compactification $\mathbb{R}^*$ of $\mathbb{R}$ with some metric $d_{\mathbb{R}^*}$. Let $\mathcal{K}_{\mathbb{R}^*}$ be the set of nonempty compact subsets of $\mathbb{R}^*$ with the Hausdorff metric $d_{\mathcal{K}_{\mathbb{R}^*}}$, i.e.

$$d_{\mathcal{K}_{\mathbb{R}^*}}(K_1, K_2) = \inf\left\{\varepsilon > 0 \,:\, K_1^\varepsilon \supset K_2 \text{ and } K_2^\varepsilon \supset K_1\right\}$$

where $K_1, K_2 \in \mathcal{K}_{\mathbb{R}^*}$ and with

$$K^\varepsilon := \{y \in \mathbb{R}^* \,:\, d_{\mathbb{R}^*}(x, y) < \varepsilon \text{ for some } x \in \mathbb{R}^*\}$$

for $K \in \mathcal{K}_{\mathbb{R}^*}$ and $\varepsilon > 0$. Let $C \in \mathcal{C}_\mathbb{R}$ be closed in $\mathbb{R}$. Since the closure $\overline{C}^{\mathbb{R}^*}$ of C in $\mathbb{R}^*$ is compact in $\mathbb{R}^*$, i.e. $\overline{C}^{\mathbb{R}^*} \in \mathcal{K}_{\mathbb{R}^*}$, the Hausdorff metric $d_{\mathcal{K}_{\mathbb{R}^*}}$ induces a metric $d_{\mathcal{C}_\mathbb{R}}$ on $\mathcal{C}_\mathbb{R}$ via

$$d_{\mathcal{C}_\mathbb{R}}(C_1, C_2) := d_{\mathcal{K}_{\mathbb{R}^*}}(\overline{C}_1^{\mathbb{R}^*}, \overline{C}_2^{\mathbb{R}^*})$$

for $C_1, C_2 \in \mathcal{C}_\mathbb{R}$.

To emphasize the initial condition of the process, we also write $\Xi_t^x := \Xi_t$ if

$$x = (\zeta_0^{(1)}, \zeta_0^{(2)}, \dots).$$

Proposition 2.11 ([EMS13]). *Let $x, y \in \mathbb{R}^N$ such that*

$$\overline{\{x_i \,:\, i \in [N]\}} = \overline{\{y_i \,:\, i \in [N]\}},$$

then $(\Xi_t^x)_{t>0}$ and $(\Xi_t^y)_{t>0}$ have the same distribution.

Proof. This is [EMS13, Proposition 2.5]. $\square$

The last proposition motivates to introduce the notation $\Xi_t^A := \Xi_t$ if

$$A = \overline{\left\{\zeta_0^{(i)} \,:\, i \in [N]\right\}}.$$

So we can talk about the distribution of the set-valued coalescing system given an initial condition $A \subset \mathbb{R}$ where A is nonempty and closed. In fact the distribution is the same as the distribution of the coalescing system starting in an arbitrary countable dense subset of A. Therefore we also write $\Xi_t^A := \Xi_t$ in the case when

$$A = \left\{ \zeta_0^{(i)} \, : \, i \in [N] \right\}.$$

Remark 2.12. Let $\emptyset \neq A_1 \subset A_2 \subset \mathbb{R}$ be two finite sets. By constructing the coalescing system via laying down first the stable processes starting in A_1 and then in $A_2 \setminus A_1$ we can assume for $t \geq 0$

$$\Xi_t^{A_1} \subset \Xi_t^{A_2}.$$

Now we argue why a similar inclusion holds in the case where $\emptyset \neq A_1 \subset A_2 \subset \mathbb{R}$ are closed sets: The coalescing system Ξ^{A_2} is built from some vector-valued coalescing system $\zeta = (\zeta^{(1)}, \zeta^{(2)}, \dots)$ based on independent stable processes $\xi^{(1)}, \xi^{(2)}, \dots$, such that $\{\xi_0^{(i)} \, : \, i \in \mathbb{N}\}$ is a countable dense subset of A_2. Let

$$I := \left\{ i \in \mathbb{N} \, : \, \xi_0^{(i)} \in A_1 \right\}$$

and write $I = \{i_1, i_2, \dots\}$ with $1 \leq i_1 < i_2 < \dots$. Define

$$\widetilde{\zeta} := (\widetilde{\zeta}^{(1)}, \widetilde{\zeta}^{(2)}, \dots) := (\zeta^{(i_1)}, \zeta^{(i_2)}, \dots)$$

and let for $t \geq 0$

$$\widetilde{\Xi}_t^{A_1} := \overline{\left\{ \widetilde{\zeta}_t^{(i)} \, : \, i \in \mathbb{N} \right\}} = \overline{\left\{ \zeta_t^{(i_k)} \, : \, k \in \mathbb{N} \right\}},$$

be the corresponding set-valued coalescing system. By construction we have $\widetilde{\Xi}_t^{A_1} \subset \Xi_t^{A_2}$ for each $t \geq 0$ and according to Lemma 2.9 $\widetilde{\Xi}^{A_1}$ has the same distribution as Ξ^{A_1}. The discussion above shows for nonempty closed sets $A_1, A_2, \dots \subset \mathbb{R}$ such that $A_1 \subset A_2 \subset \dots$ we have for each measurable set $B \subset \mathbb{R}$ and $t \geq 0$

$$\mathbf{P}\left[\Xi_t^{A_1} \cap B \neq \emptyset\right] \leq \mathbf{P}\left[\Xi_t^{A_2} \cap B \neq \emptyset\right]$$

and

$$\lim_{n \to \infty} \mathbf{P}\left[\Xi_t^{A_n} \cap B \neq \emptyset\right] = \mathbf{P}\left[\Xi_t^{\overline{\bigcup_{n=1}^{\infty} A_n}} \cap B \neq \emptyset\right].$$

Corollary 2.13 ([EMS13])**.** *The stochastic process Ξ is strong Markov with respect to the filtration $(\mathcal{F}_t^{\mathrm{coal}})_{t \geq 0}$.*

Proof. This is [EMS13, Corollary 2.6]. □

One main result of [EMS13] is that almost surely $|\Xi_t^A| < \infty$ for each $t > 0$ if A is a compact set (see [EMS13, Theorem 6.1 (a)]). An important step to that is an error bound for the probability that the reduction to a fixed finite number of particles doesn't happen quickly or that the particles move too far around up to that time. We use this result later in several ways, therefore we give its precise formulation which needs some notation: For a nonempty closed $A \subset \mathbb{R}$ and $\delta > 0$, define the δ-fattening of A to be

$$A^\delta := \{y \in \mathbb{R} \,:\, |y - x| < \delta \text{ for some } x \in A\}$$

and define the range of $(\Xi_t^A)_{t\ge 0}$ until time $t_0 > 0$ as

$$\mathcal{R}(A; [0, t_0]) := \bigcup_{s \le t_0} \Xi_s^A.$$

Further, for $m \in \mathbb{N}$, let

$$\tau_m^A := \inf \left\{ s \ge 0 \,:\, \left|\Xi_s^A\right| \le m \right\}.$$

Let $(X_t)_{t\ge 0}$ be a standard stable process of index $\alpha \in (1, 2)$. Fix $\beta > 0$ and let

$$\underline{p} := \underline{p}(\beta) := \mathbf{P}_1[X_s = 0 \text{ for some } s \in (0, \beta)] > 0.$$

Set $\gamma := 1/(1 - \underline{p}/5) > 1$, let $\eta \in \left(\frac{\alpha-1}{2}, \alpha - 1\right)$, $h := 1 - (1 + \eta)/\alpha > 0$, $0 < \varepsilon \le \frac{1}{2}$, $\nu_i := \varepsilon\gamma^{-hi}$ and $\eta_i := \beta 2^\alpha \gamma^{-\alpha i}$ for $i \in \mathbb{N}_0$.

Lemma 2.14 ([EMS13])**.** *There is a constant $C_{2.14} = C_{2.14}(\varepsilon) > 0$ such that for each $m \in \mathbb{N}_0$, $\ell > 0$, $A \subset \mathbb{R}$ with $\operatorname{diam}(A) \le \frac{\ell}{2}$ and $|A| = \lceil \gamma^m \rceil$ and each $k \in \mathbb{N}_0$ with $k \le m$, one has the bound*

$$\mathbf{P}\left[\left\{\tau^A_{\lceil\gamma^k\rceil} > \ell^\alpha \sum_{i=k+1}^m \eta_i\right\} \cup \left\{\mathcal{R}(A; [0, \tau^A_{\lceil\gamma^k\rceil}]) \not\subseteq (A)^{\ell \sum_{i=k+1}^m \nu_i}\right\}\right] \le C_{2.14}\gamma^{-\eta k}.$$

Proof. This is [EMS13, Lemma 6.10 (a)]. □

We will use this result in the following way.

Corollary 2.15. *There are $t_0 > 0$ and $\gamma > 1$ and for each $\eta \in \left(\frac{\alpha-1}{2}, \alpha - 1\right)$, there is a constant $C_{2.15} = C_{2.15}(\eta, \alpha) > 0$ such that for each $m \in \mathbb{N}$, $\ell > 0$, $A \subset \mathbb{R}$ with $\operatorname{diam}(A) \le \ell$ and $|A| = \lceil\gamma^m\rceil$ and each $k \in \mathbb{N}_0$ with $k \le m$, one has*

$$\mathbf{P}\left[\left\{\tau^A_{\lceil\gamma^k\rceil} > \ell^\alpha t_0\right\} \cup \left\{\mathcal{R}(A; [0, \tau^A_{\lceil\gamma^k\rceil}]) \not\subseteq (A)^\ell\right\}\right] \le C_{2.15}\gamma^{-\eta k}.$$

Proof. Let $\eta \in \left(\frac{\alpha-1}{2}, \alpha - 1\right)$, $h := 1 - (1 + \eta)/\alpha > 0$ and $\gamma > 1$ as in Lemma 2.14. First fix $\ell > 0$, $A \subset \mathbb{R}$ with $\operatorname{diam}(A) \le \frac{\ell}{2}$ and $|A| = \lceil\gamma^m\rceil$ for some $m \in \mathbb{N}_0$. Further let $\varepsilon < \frac{1}{2}(1 - \gamma^{-h}) < \frac{1}{2}$ which implies

$$\sum_{i=k+1}^\infty \nu_i = \varepsilon \sum_{i=k+1}^\infty \gamma^{-hi} = \varepsilon \frac{\gamma^{-h(k+1)}}{1 - \gamma^{-h}} < \frac{1}{2}\gamma^{-h(k+1)} < \frac{1}{2}.$$

Lemma 2.14 yields for $k \in \mathbb{N}_0$ with $k \leq m$

$$\begin{aligned}
&\mathbf{P}\left[\left\{\tau^A_{\lceil\gamma^k\rceil} > \ell^\alpha \sum_{i=k+1}^{\infty} \eta_i\right\} \cup \left\{\mathcal{R}(A;[0,\tau^A_{\lceil\gamma^k\rceil}]) \nsubseteq (A)^{\frac{\ell}{2}}\right\}\right] \\
&\qquad \leq \mathbf{P}\left[\left\{\tau^A_{\lceil\gamma^k\rceil} > \ell^\alpha \sum_{i=k+1}^{m} \eta_i\right\} \cup \left\{\mathcal{R}(A;[0,\tau^A_{\lceil\gamma^k\rceil}]) \nsubseteq (A)^{\ell\sum_{i=k+1}^{m}\nu_i}\right\}\right] \\
&\qquad \leq C_{2.15}\gamma^{-\eta k}
\end{aligned}$$

with $C_{2.15} = C_{2.14}$. Since $C_{2.14}$ depends of ε and we chose ε dependent of h which depends on α and η, the constant $C_{2.15}$ depends on η and α.
Now consider A with $\operatorname{diam}(A) \leq \ell$ instead of $\operatorname{diam}(A) \leq \frac{\ell}{2}$ and choose $t_0 := 2^\alpha \sum_{i=1}^{\infty} \eta_i$ to receive the desired result. □

Next we give a bound on the probability that the coalescing system starting in initial sets far away from the origin hits a fixed interval. Let $\frac{\alpha-1}{2} < \eta < \alpha - 1$, $d := \frac{2}{\eta} > 1$. For $r \in \mathbb{N}$ define

$$J_{r,1} := \left[-\sum_{j=1}^{r} j^d, -\sum_{j=1}^{r-1} j^d\right) \qquad \text{and} \qquad J_{r,2} := \left[\sum_{j=1}^{r-1} j^d, \sum_{j=1}^{r} j^d\right).$$

Then $(J_{r,i})_{r\in\mathbb{N}, i\in\{1,2\}}$ is a partition of $\mathbb{R}$ into bounded intervals, where $J_{r,i}$ has length r^d for $r \in \mathbb{N}$, $i \in \{1,2\}$. The following result is a part of the proof of [EMS13, Theorem 6.3] where it is proven for $a = 1$. We repeat the proof here to emphasize the dependence on a.

Lemma 2.16 ([EMS13])**.** *Let $a > 0$. There exist $C_{2.16.1}, C_{2.16.2} > 0$ (which may depend on α and d) and $r_0 = r_0(\alpha, a, d) \in \mathbb{N}$ such that, for each $t > 0$, $r \geq r_0$, $i \in \{1,2\}$ and each nonempty closed set $A \subset J_{r,i}$ we have*

$$\mathbf{P}\left[\Xi^A_t \cap [-a,a] \neq \emptyset\right] \leq C_{2.16.1} r^{-2} + C_{2.16.2} t r^{-\alpha}.$$

Proof. Let $A \subset J_{r,i}$ for some $i \in \{1,2\}$ and $r \in \mathbb{N}$. Consider $\gamma > 1$ from Corollary 2.15. According to Remark 2.12 it is enough to prove the result in the case $|A| = \lceil\gamma^m\rceil$ for some $m \in \mathbb{N}_0$. The rough idea is to divide the hitting probability according to whether the particles move quickly away from $J_{r,i}$ or not. In the first case we can use Corollary 2.15, in the latter case we can use the tails of stable process.
According to Corollary 2.15 there are $t_0 > 0$ and a constant $C_{2.16.1} := C_{2.16.1}(\alpha, d) := C_{2.15}(\eta, \alpha) > 0$ such that (note $\operatorname{diam}(A) \leq r^d$) for $k \leq m$

$$\mathbf{P}\left[\mathcal{R}(A;[0,\tau^A_{\lceil\gamma^k\rceil}]) \nsubseteq (J_{r,i})^{r^d}\right] \leq C_{2.16.1}\gamma^{-\eta k}.$$

For $r \in \mathbb{N}$ define $b := b(r) := (2/\eta)\lceil\log_\gamma(r)\rceil = d\lceil\log_\gamma(r)\rceil$. We will use the last bound later with $k = b$ which leads to

$$\mathbf{P}\left[\mathcal{R}(A;[0,\tau^A_{\lceil\gamma^b\rceil}]) \not\subseteq (J_{r,i})^{r^d}\right] \leq C_{2.16.1} r^{-2}. \tag{2.7}$$

Next we need a lower bound on the distance of $(J_{r,i})^{r^d}$ and $[-a,a]$ for sufficient large r, $i \in \{1,2\}$. Choose $r_1 = r_1(a,d) \in \mathbb{N}$ with

$$\left(\sum_{j=1}^{r_1-1} j^d\right) > a.$$

For $r \geq r_1$ we have

$$\inf_{x\in[-a,a],y\in J_{r,i}} |x-y| \geq \left(\sum_{j=1}^{r-1} j^d\right) - a > \left(\int_0^{r-1} x^d\,dx\right) - a = \frac{1}{d+1}(r-1)^{d+1} - a.$$

There exists $r_2 = r_2(a,d) \geq r_1$ with

$$\frac{1}{d+1}(r-1)^{d+1} - \frac{1}{4(d+1)} r^{d+1} \geq r^d + a$$

for all $r \geq r_2$. This implies

$$\inf_{x\in[-a,a],y\in J_{r,i}} |x-y| \geq r^d + \frac{1}{4(d+1)} r^{d+1}$$

respectively

$$\inf_{x\in[-a,a],y\in (J_{r,i})^{r^d}} |x-y| \geq \frac{1}{4(d+1)} r^{d+1}$$

for $r \geq r_2$ and $i \in \{1,2\}$.

Let $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ be independent standard stable processes of index $\alpha \in (1,2]$ with $(X_0^{(1)}, X_0^{(2)}, \ldots, X_0^{(n)}) = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$. Due to [EMS13, Corollary 6.8 (a)] there is a constant $C_{2.16.3} = C_{2.16.3}(\alpha) > 0$ such that for $t > 0$ and $u > 0$ we have

$$\mathbf{P}\left[\sup_{0\leq s\leq t}\left|X_s^{(i)} - x^{(i)}\right| > u \text{ for some } i \in \{1,2,\ldots,n\}\right] \leq C_{2.16.3} n t u^{-\alpha}. \tag{2.8}$$

Let $n = \lceil\gamma^b\rceil$. Since $\lceil x\rceil \leq \frac{3}{2}x$ for $x \geq 2$ there is $r_3 = r_3(\gamma,d) \in \mathbb{N}$ such that for $r \geq r_3$

$$\lceil\gamma^b\rceil = \lceil\gamma^{d\lceil\log_\gamma(r)\rceil}\rceil \leq \frac{3}{2}\gamma^{d\lceil\log_\gamma(r)\rceil} \leq \frac{3}{2}\gamma^d r^d$$

where we used $\lceil x\rceil \leq x+1$ for $x \in \mathbb{R}$ in the last inequality. For $r \geq r_3$, $i \in \{1,2\}$ and

$(X_0^{(1)}, \dots, X_0^{(\lceil \gamma^b \rceil)}) = (x^{(1)}, \dots, x^{(\lceil \gamma^b \rceil)})$ with $x^{(1)}, \dots, x^{(\lceil \gamma^b \rceil)} \in (J_{r,i})^{r^d}$ we have with (2.8)

$$\begin{aligned}
&\mathbf{P}\left[\sup_{0\le s\le t}\left|X_s^{(i)} - x^{(i)}\right| > \frac{1}{4(d+1)} r^{d+1} \text{ for some } i \in \left\{1, \dots, \lceil \gamma^b \rceil\right\}\right] \\
&\le C_{2.16.3}\lceil \gamma^b \rceil t \left(\frac{1}{4(d+1)} r^{d+1}\right)^{-\alpha} \\
&\le C_{2.16.2} t r^{-\alpha(d+1)+d} \\
&= C_{2.16.2} t r^{-\alpha - d(\alpha-1)} \\
&\le C_{2.16.2} t r^{-\alpha}
\end{aligned} \tag{2.9}$$

for some constant $C_{2.16.2} = C_{2.16.2}(\alpha, \gamma, d) > 0$ where we used $d(\alpha - 1) > 0$ in the last inequality. Since the distance of $(J_{r,i})^{r^d}$ and $[-a, a]$ is at least $\frac{1}{4(d+1)} r^{d+1}$ for $r \ge r_2$ and $i \in \{1, 2\}$, we get from (2.9) with $r_0 := \max\{r_2, r_3\}$

$$\mathbf{P}\left[\Xi_s^{A_b} \cap [-a, a] \neq \emptyset \text{ for some } s \in [0, t]\right] \le C_{2.16.2} t r^{-\alpha} \tag{2.10}$$

for all $r \ge r_0$, $i \in \{1, 2\}$ and $A_b \subset (J_{r,i})^{r^d}$ with $|A_b| = \lceil \gamma^b \rceil$. Now let $r \ge r_0$. For $A \subset J_{r,i}$, $i \in \{1, 2\}$, $|A| = \lceil \gamma^m \rceil$, $m \in \mathbb{N}_0$ we have

$$\begin{aligned}
&\mathbf{P}\Big[\Xi_s^A \cap [-a, a] \neq \emptyset \text{ for some } s \in [0, t]\Big] \\
&= \mathbf{P}\Big[\Big\{\Xi_s^A \cap [-a, a] \neq \emptyset \text{ for some } s \in [0, t]\Big\} \cap \Big\{\mathcal{R}(A; [0, \tau^A_{\lceil \gamma^b \rceil}]) \nsubseteq (J_{r,i})^{r^d}\Big\}\Big] \\
&\quad + \mathbf{P}\Big[\Big\{\Xi_s^A \cap [-a, a] \neq \emptyset \text{ for some } s \in [0, t]\Big\} \cap \Big\{\mathcal{R}(A; [0, \tau^A_{\lceil \gamma^b \rceil}]) \subseteq (J_{r,i})^{r^d},\ \tau^A_{\lceil \gamma^b \rceil} < t\Big\}\Big] \\
&\quad + \mathbf{P}\Big[\Big\{\Xi_s^A \cap [-a, a] \neq \emptyset \text{ for some } s \in [0, t]\Big\} \cap \Big\{\mathcal{R}(A; [0, \tau^A_{\lceil \gamma^b \rceil}]) \subseteq (J_{r,i})^{r^d},\ \tau^A_{\lceil \gamma^b \rceil} \ge t\Big\}\Big]
\end{aligned}$$

The first summand is bounded by $C_{2.16.1} r^{-2}$ (see (2.7)) and using the strong Markov property of the coalescing system (see Corollary 2.13) the second summand is bounded by $C_{2.16.2} t r^{-\alpha}$ (see (2.10)). The third summand is equal to zero since $[-a, a] \cap (J_{r,i})^{r^d} = \emptyset$ for $r \ge r_0$. We conclude

$$\mathbf{P}\Big[\Xi_t^A \cap [-a, a] \neq \emptyset\Big] \le C_{2.16.1} r^{-2} + C_{2.16.2} t r^{-\alpha}.$$

□

2.3. Stable limit theorems and convergence of coalescing systems

Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed real valued random variables. It is well known due to the central limit theorem that rescaled sums of such random variables converge in distribution to the normal distribution if $\mathbf{E}[Z_1^2] < \infty$. If

$\mathbf{E}[Z_1^2] = \infty$ there exist analogs to the central limit theorem where stable distributions occur in the limit. One can make use of this to show that a rescaled coalescing system of certain random walks converges to the coalescing stable process of index $\alpha \in (1,2)$. In this section, we present the necessary steps to obtain this result (formulated in Corollary 2.23) since we need it later. For (local) central limit theorems for stable distributions we refer to [GK54] and for functional limit theorems in $D([0,\infty),\mathbb{R})$ to [Sk57]. Furthermore, [Wh02, Section 4.5] gives an overview of both. The convergence of the coalescing system is sketched in [MRV19, Section 6], but we give here some more details. We may remark that in [NRS05, Lemma 5.1] it is shown that coalescing random walks whose increment distribution has second moments converge to coalescing Brownian motion. Therefore, we exclude in this section the case $\alpha = 2$. In the following, we write $S_n := s_0 + \sum_{i=1}^n Z_i$ for $n \in \mathbb{N}$ with $s_0 \in \mathbb{R}$ and refer to $(S_n)_{n\in\mathbb{N}_0}$ as a (discrete-time) random walk and to $Z_1, Z_2, \ldots$ as the increments of the random walk. If $X, X_1, X_2, \ldots$ are random variables with values in a metric space E, we write $X_n \xRightarrow{n\to\infty} X$ if $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X.

Definition 2.17. *Let $s_0 = 0$. The distribution of Z_1 belongs to the domain of attraction of an α-stable distribution μ, $\alpha \in (0,2]$, if there exist $(c_n)_{n\in\mathbb{N}} \subset (0,\infty)$ and $(m_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that*

$$c_n^{-1}(S_n - m_n) \xRightarrow{n\to\infty} Z$$

with an α-stable distributed random variable $Z \sim \mu$. The distribution of Z_1 belongs to the normal domain of attraction an α-stable distribution, if we can choose $c_n = an^{\frac{1}{\alpha}}$ for some constant $a > 0$.

We may remark that the notion of (normal) domain of attraction is not restricted to symmetric stable processes, but in that what follows we will assume as before that stable processes are symmetric. Therefore we further focus on distributions of Z_1 that are symmetric. The following characterization result is due to [GK54, Section 35], our formulation of it is based on [Wh02, Theorem 4.5.2].

Proposition 2.18 ([GK54],[Wh02])**.** *Let $\alpha \in (0,2)$ and let Z_1 be a real-valued random variable with a symmetric distribution, i.e. $Z_1 \overset{d}{=} -Z_1$. The distribution of Z_1 belongs to the normal domain of attraction with $a = 1$, i.e. $c_n = n^{\frac{1}{\alpha}}$, of an α-stable distribution with rate $c > 0$, if and only if*

$$\lim_{x\to\infty} c^{-1} x^{\alpha} \mathbf{P}[Z_1 > x] = C_\alpha^{\mathrm{dom}}$$

with

$$C_\alpha^{\mathrm{dom}} = \frac{1}{2}\left(\int_0^\infty x^{-\alpha}\sin(x)\,dx\right).$$

Proof. This is [Wh02, Theorem 4.5.2]. There the case $c = 1$ was proved. The general case $c > 0$ follows from the scaling property of the stable distribution. □

Now we can formulate the functional limit theorem that goes back to [Sk57, Theorem 2.7].

Proposition 2.19 ([Sk57])**.** *Let* $\alpha \in (0,2)$*,* $s_0 = 0$ *and assume that* Z_1 *satisfies the requirements of Proposition 2.18 for some* $c > 0$*. Then*

$$\left(\frac{S_{\lfloor nt\rfloor}}{n^{\frac{1}{\alpha}}}\right)_{t\geq 0} \xrightarrow{n\to\infty} (X_t)_{t\geq 0} \qquad \text{in } D([0,\infty),\mathbb{R}),$$

where $X = (X_t)_{t\geq 0}$ *is an* α*-stable process with rate* $c > 0$*.*

Proof. This is [Sk57, Theorem 2.7]. □

To prove convergence of coalescing systems of certain random walks it is enough to show convergence of the hitting times of the random walks. Here we follow the proof of [MRV19, Lemma 6.1]. We need the following local limit theorem (see [GK54, Chapter 9, Section 50] or [IL71, Theorem 4.2.1]) for random walks on a lattice. Since the formulation of the theorem in [GK54] and [IL71] needs a kind of maximality property of the lattice, we assume $\mathbf{P}[Z_1 = k] > 0$ for all $k \in \mathbb{Z}$ to fulfill this property.

Proposition 2.20 ([GK54])**.** *Let* $\alpha \in (0,2)$*,* $s_0 = 0$ *and assume that* Z_1 *satisfies the requirements of Proposition 2.18 for some* $c > 0$ *as well as* $\mathbf{P}[Z_1 = k] > 0$ *for all* $k \in \mathbb{Z}$*. Consider some increasing sequence* $(t_n)_{n\in\mathbb{N}} \subset [1,\infty)$ *with* $\lim_{n\to\infty} t_n = \infty$*. It holds*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\lfloor t_n\rfloor^{\frac{1}{\alpha}}\mathbf{P}[S_{\lfloor t_n\rfloor} = \lfloor t_n^{\frac{1}{\alpha}}x\rfloor] - p_1^{(c,\alpha)}(x)\right| = 0.$$

Proof. Denote by $\varphi_X : \mathbb{R} \to \mathbb{C}$ the characteristic function of a real-valued random variable X. By Fourier inversion (see [Kl13, Theorem 15.10, Exercise 15.1.6]) we have for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\mathbf{P}[S_{\lfloor t_n\rfloor} = \lfloor t_n^{\frac{1}{\alpha}}x\rfloor] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(-i\xi\lfloor t_n^{\frac{1}{\alpha}}x\rfloor\right)\varphi_{S_{\lfloor t_n\rfloor}}(\xi)\,d\xi$$

and

$$p_1^{(c,\alpha)}(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\exp(-i\xi x)\varphi_{X_1}(\xi)\,d\xi,$$

where $(X_t)_{t\geq 0}$ is an α-stable process with rate $c > 0$. Thus

$$\begin{aligned}&\left|\lfloor t_n\rfloor^{\frac{1}{\alpha}}\mathbf{P}[S_{\lfloor t_n\rfloor} = \lfloor t_n^{\frac{1}{\alpha}}x\rfloor] - p_1^{(c,\alpha)}(x)\right|\\ &\leq \int_{\mathbb{R}}\left|\exp\left(-i\xi\frac{\lfloor t_n^{\frac{1}{\alpha}}x\rfloor}{\lfloor t_n\rfloor^{\frac{1}{\alpha}}}\right)\varphi_{Z_1}\left(\frac{\xi}{\lfloor t_n\rfloor^{\frac{1}{\alpha}}}\right)^{\lfloor t_n\rfloor}\mathbf{1}_{\left[-\pi\lfloor t_n\rfloor^{\frac{1}{\alpha}},\pi\lfloor t_n\rfloor^{\frac{1}{\alpha}}\right]}(\xi) - \exp(-i\xi x)\varphi_{X_1}(\xi)\right|d\xi,\end{aligned}$$

Now with the same proof as in [GK54, Chapter 9, Section 50] or [IL71, Theorem 4.2.1], where only the case $t_n = n$ is considered, one can show that the right-hand side of the last equation converges to zero (note that $(\lfloor t_n\rfloor)_{n\in\mathbb{N}} \subset \mathbb{N}$). □

Lemma 2.21. *Let $\alpha \in (1,2)$, $s_0 = 0$ and assume that Z_1 satisfies the requirements of Proposition 2.18 for some $c > 0$ as well as $\mathbf{P}[Z_1 = k] > 0$ for all $k \in \mathbb{Z}$. Let $a \in \mathbb{R}$ and*

$$T^{(n)} := \inf\left\{ j \in \mathbb{N}_0 \,:\, S_j = \lfloor a n^{\frac{1}{\alpha}} \rfloor \right\}, \qquad T := \inf\left\{ t \geq 0 \,:\, X_t = a \right\}.$$

Then we have

$$\frac{1}{n} T^{(n)} \xRightarrow{n\to\infty} T.$$

Proof. We show convergence of the Laplace transforms (this implies convergence in distribution, see [Bi99, Example 5.5]). For $\lambda > 0$ let

$$u^{(\lambda)}(x) = \int_0^\infty e^{-\lambda t} p_t^{(c,\alpha)}(x)\, dt, \qquad x \in \mathbb{R}$$

and

$$v^{(\lambda)}(k) = \sum_{j=0}^{\infty} e^{-\lambda j} \mathbf{P}[S_j = k] = \int_0^\infty e^{-\lambda \lfloor t \rfloor} \mathbf{P}[S_{\lfloor t \rfloor} = k]\, dt, \qquad k \in \mathbb{Z}.$$

One has (due to [Be96, Corollary II.5.18] for the first equality and using the Markov property at time $T^{(n)}$ in the second equality)

$$u^\lambda(a) = u^\lambda(0) \mathbf{E}[e^{-\lambda T}], \qquad v^{\frac{\lambda}{n}}(\lfloor a n^{\frac{1}{\alpha}} \rfloor) = v^{\frac{\lambda}{n}}(0) \mathbf{E}[e^{-\frac{\lambda}{n} T^{(n)}}],$$

so it is enough to show that

$$\lim_{n\to\infty} n^{\frac{1}{\alpha}-1} v^{\frac{\lambda}{n}}(\lfloor x n^{\frac{1}{\alpha}} \rfloor) = u^\lambda(x) \tag{2.11}$$

for $x \in \mathbb{R}$ (we need this for $x \in \{0, a\}$). We compute for $n \in \mathbb{N}$

$$\begin{aligned} n^{\frac{1}{\alpha}-1} v^{\frac{\lambda}{n}}(\lfloor x n^{\frac{1}{\alpha}} \rfloor) &= n^{\frac{1}{\alpha}-1} \int_0^\infty e^{-\lambda \frac{\lfloor t \rfloor}{n}} \mathbf{P}[S_{\lfloor t \rfloor} = \lfloor n^{\frac{1}{\alpha}} x \rfloor]\, dt \\ &= \int_0^\infty e^{-\lambda \frac{\lfloor nt \rfloor}{n}} n^{\frac{1}{\alpha}} \mathbf{P}[S_{\lfloor nt \rfloor} = \lfloor n^{\frac{1}{\alpha}} x \rfloor]\, dt \\ &= \int_0^\infty e^{-\lambda \frac{\lfloor nt \rfloor}{n}} t^{-\frac{1}{\alpha}} (nt)^{\frac{1}{\alpha}} \mathbf{P}[S_{\lfloor nt \rfloor} = \lfloor (nt)^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}} x \rfloor]\, dt. \end{aligned}$$

If we call the intergrand in the last integral $f^{(n)}(t)$, we have using Proposition 2.20 for each $t > 0$

$$\lim_{n\to\infty} f^{(n)}(t) = e^{-\lambda t} t^{-\frac{1}{\alpha}} p_1^{(c,\alpha)}(t^{-\frac{1}{\alpha}} x) = e^{-\lambda t} p_t^{(c,\alpha)}(x).$$

Now we want to use dominated convergence: Due to Proposition 2.20 there exists a constant $C_{2.21.1} > 0$ such that

$$\sup_{t\geq 1} \sup_{x\in\mathbb{R}} \left| \lfloor t \rfloor^{\frac{1}{\alpha}} \mathbf{P}[S_{\lfloor t \rfloor} = \lfloor t^{\frac{1}{\alpha}} x \rfloor] - p_1^{(c,\alpha)}(x) \right| \leq C_{2.21.1}. \tag{2.12}$$

Further there is a constant $C_{2.21.2} > 0$ such that $\sup_{y\geq 2} \frac{y}{y-1} \leq C_{2.21.2}$. Using $\lfloor y \rfloor \geq y - 1$

for $y \geq 0$ we get for $nt \leq 2$

$$\left|f^{(n)}(t)\right| \leq 2^{\frac{1}{\alpha}} e^{-\lambda \frac{\lfloor nt \rfloor}{n}} t^{-\frac{1}{\alpha}} \leq 2^{\frac{1}{\alpha}} e^{\lambda} e^{-\lambda t} t^{-\frac{1}{\alpha}}$$

and for $nt \geq 2$

$$\begin{aligned}\left|f^{(n)}(t)\right| &= e^{-\lambda \frac{\lfloor nt \rfloor}{n}} t^{-\frac{1}{\alpha}} \left(\frac{nt}{\lfloor nt \rfloor}\right)^{\frac{1}{\alpha}} \lfloor nt \rfloor^{\frac{1}{\alpha}} \mathbf{P}[S_{\lfloor nt \rfloor} = \lfloor (nt)^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}} x \rfloor] \\ &\leq e^{\lambda} \left(\frac{nt}{nt-1}\right)^{\frac{1}{\alpha}} C_{2.21.3} e^{-\lambda t} t^{-\frac{1}{\alpha}} \\ &\leq e^{\lambda} C_{2.21.2}^{\frac{1}{\alpha}} C_{2.21.3} e^{-\lambda t} t^{-\frac{1}{\alpha}}\end{aligned}$$

for some constant $C_{2.21.3} > 0$, where we used (2.12) in the first inequality. So there exists a constant $C_{2.21.4} > 0$ such that for each $t > 0$

$$\sup_{n \in \mathbb{N}} \left|f^{(n)}(t)\right| \leq C_{2.21.4} e^{-\lambda t} t^{-\frac{1}{\alpha}}.$$

Thus dominated convergence gives (2.11). □

Corollary 2.22. *Let $\alpha \in (1, 2)$, $x_1 \in \mathbb{R}$ and $(X_t^{(1)})_{t \geq 0}$ and $(X_t^{(2)})_{t \geq 0}$ be two independent standard stable processes with $X_0^{(1)} = x_1$ and $X_0^{(2)} = 0$. Further let $Z_1^{(1)}, Z_2^{(1)}, Z_3^{(1)}, \ldots,$ $Z_1^{(2)}, Z_2^{(2)}, Z_3^{(2)}, \ldots$ be independent copies of a random variable satisfying the requirements of Proposition 2.18 with $c = 1$ as well as $\mathbf{P}[Z_1^{(1)} = k] > 0$ for each $k \in \mathbb{Z}$. Now let*

$$S_j^{(n,1)} := \lfloor x_1 n^{\frac{1}{\alpha}} \rfloor + \sum_{\ell=1}^{j} Z_\ell^{(1)}, \qquad S_j^{(n,2)} := 0 + \sum_{\ell=1}^{j} Z_\ell^{(2)}$$

and

$$T^{(n)} := \inf\left\{j \in \mathbb{N}_0 \,:\, S_j^{(n,1)} = S_j^{(n,2)}\right\}, \qquad T := \inf\left\{t \geq 0 \,:\, X_t^{(1)} = X_t^{(2)}\right\}.$$

Then we have

$$\frac{1}{n} T^{(n)} \xrightarrow{n \to \infty} T.$$

Proof. For $\ell \in \mathbb{N}$ let $Z_\ell := Z_\ell^{(1)} - Z_\ell^{(2)}$, for $j \in \mathbb{N}_0$ let $S_j^{(n)} := S_j^{(n,1)} - S_j^{(n,2)}$ and for $t \geq 0$ let $X_t := X_t^{(1)} - X_t^{(2)}$. Since $\mathbf{P}[Z_1^{(1)} = k] > 0$ for each $k \in \mathbb{Z}$, we have $\mathbf{P}[Z_1 = k] > 0$ for each $k \in \mathbb{Z}$. Due to the independence of $X_1^{(1)}$ and $X_2^{(1)}$ the random variable X_1 has a stable distribution with rate 2. Since the distributions of $Z_1^{(1)}$ and $Z_1^{(2)}$ belong to the normal domain of attraction of an stable distribution with rate 1, it follows from the independence of $Z_1^{(1)}$ and $Z_1^{(2)}$ and Definition 2.17, that Z_1 belongs to the normal domain of attraction of an stable distribution with rate 2. Therefore the result follows from Lemma 2.21. □

Weak convergence of the random walks (Proposition 2.19) and the hitting times (Corollary 2.22) implies convergence of the coalescing system. This is what we now formulate as a corollary. Note that a (discrete-time) vector-valued (respectively set-valued) coalescing system of a finite number of random walks can formally be defined in a completely analogous way as we defined the vector-valued (respectively set-valued) coalescing system of stable processes in the beginning of Section 2.2.

Corollary 2.23. *Let $\alpha \in (1,2)$, $x_1, x_2, \ldots, x_m \in \mathbb{R}$ and let $(\pi_k^{(n)})_{k\in\mathbb{N}_0}$ be the vector-valued coalescing system of $m \in \mathbb{N}$ discrete-time random walks on $\mathbb{Z}$ with*

$$\pi_0^{(n)} = (\lfloor x_1 n^{\frac{1}{\alpha}} \rfloor, \lfloor x_2 n^{\frac{1}{\alpha}} \rfloor, \ldots, \lfloor x_m n^{\frac{1}{\alpha}} \rfloor).$$

Assume that the increments of the random walks are independent copies of a random variable Z satisfying the requirements of Proposition 2.18 with $c = 1$ as well as $\mathbf{P}[Z = k] > 0$ for each $k \in \mathbb{Z}$. Further let $(\zeta_t)_{t\geq 0}$ be the vector-valued system of coalescing α-stable processes starting in $\zeta_0 = (x_1, x_2, \ldots, x_m)$. Then we have

$$\left(\frac{\pi_{\lfloor nt \rfloor}^{(n)}}{n^{\frac{1}{\alpha}}} \right)_{t\geq 0} \xrightarrow{n\to\infty} (\zeta_t)_{t\geq 0} \qquad \text{in } D([0,\infty), \mathbb{R}^m).$$

Proof. The system of coalescing random walks is constructed using m independent random walks and their (in total $\binom{m}{2}$) hitting times (analogous the coalescing system of α-stable particles). Therefore it is enough to show that all these objects converge together in distribution. Since each of the m independent rescaled random walks converges in distribution in $D([0,\infty), \mathbb{R})$ (due to Proposition 2.19), all m processes converge together in $D([0,\infty), \mathbb{R}^m)$ (see for example [Wh02, Theorem 11.6.7]). Due to Corollary 2.22 each of the $\binom{m}{2}$ rescaled hitting times converges in distribution. If we interpret the hitting times (which are random variables with values in $\mathbb{R}$) as constant functions in $D([0,\infty), \mathbb{R})$, we get that the families of rescaled hitting times are C-tight, so [JS03, Corollary VI.3.33] implies that all m rescaled processes and $\binom{m}{2}$ rescaled hitting times converge together in distribution in $D([0,\infty), \mathbb{R}^{m+\binom{m}{2}})$. This implies convergence of the system of coalescing random walks to the system of coalescing α-stable processes in $D([0,\infty), \mathbb{R}^m)$. □

3. Some basic properties

3.1. Formal introduction of the voter model on the real line

In this section we give a formal introduction of the long-range voter model on the real line (with two opinions) using a moment duality with the coalescing stable process $(\Xi_t)_{t\geq 0}$ of index $\alpha \in (1,2]$ which we presented in Section 2.2. The existence result has been proved in [Ev97]. We then show simple consequences of the duality. Let

$$\mathcal{M}_{\leq 1}(\mathbb{R}) := \{u(x)\,dx \,:\, u\colon \mathbb{R} \to [0,1]\,\text{measurable}\} \subset \mathcal{M}(\mathbb{R})$$

where $\mathcal{M}(\mathbb{R})$ denotes the space of all Radon measures on $\mathbb{R}$ which we will topologize with vague convergence. For $u \in \mathcal{M}_{\leq 1}(\mathbb{R})$, we also denote the corresponding density by $u\colon \mathbb{R} \to [0,1]$.
The following existence theorem follows from [Ev97, Theorem 4.1, Proposition 5.1] and [DEF$^+$00, Corollary 7.3] where more general processes are considered for the dual coalescing mechanism. Further [Ev97] deals with infinite opinions, but gives a remark after [Ev97, Theorem 4.1] on how to get a two-opinion process. We present the theorem in the form of [HOV21, Theorem 3.1], where it was formulated for the case $\alpha = 2$.

Theorem 3.1 ([Ev97],[DEF$^+$00])**.** *There exists a unique Feller semigroup on $\mathcal{M}_{\leq 1}(\mathbb{R})$ such that the corresponding Feller process $(u_t)_{t\geq 0}$ is characterized by the following moment duality: For all $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$, $t \geq 0$ and $n \in \mathbb{N}$, we have for Lebesgue-almost all $(x_1, \dots, x_n) \in \mathbb{R}^n$*

$$\mathbf{E}_{u_0}\left[\prod_{i=1}^{n} u_t(x_i)\right] = \mathbf{E}\left[\prod_{y\in\Xi_t^{\{x_1,\dots,x_n\}}} u_0(y)\right].$$

where $\mathbf{P}_{u_0}$ denotes the distribution of $(u_t)_{t\geq 0}$, starting from u_0. For each $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ the process $(u_t)_{t\geq 0}$ has continuous sample paths, and for each fixed $t > 0$, we have $\mathbf{P}_{u_0}$-almost surely

$$u_t(x) \in \{0,1\} \qquad \textit{for Lebesgue-almost all } x \in \mathbb{R}.$$

Remark 3.2. We refer to the measure-valued process $(u_t)_{t\geq 0}$ as the *long-range voter model on the real line.* We think of $u_t(x)$ as the proportion of voters at place $x \in \mathbb{R}$ who have opinion 1 at time $t \geq 0$. Hence the following initial conditions are interesting:

- Let $B \subset \mathbb{R}$ be measurable and bounded, for example $B = [-1,1]$. Let $u_0 = \mathbf{1}_B$. Then the voter model starts with individuals having opinion 1 in B and individuals having opinion 0 in B^c.

- Let $\vartheta \in (0,1)$. In a discrete voter model, the voters' opinions at time $t = 0$ can be distributed independently and identically with a success probability ϑ for opinion 1. This motivates us to consider the initial condition $u_0(x) = \vartheta$ for all $x \in \mathbb{R}$ for the voter model on the real line. However, this initial condition cannot be interpreted formally as in the discrete voter model, since there is an uncountable number of voters.

3.2. Applications of duality

Having introduced the process formally in the previous section, we can now show simple applications of the moment duality. We start to express the probability that there is mass in some set or interval in terms of the dual process. Afterwards we show a scaling property of the total mass that follows from the scaling property of the stable process.

Denote by λ the Lebesgue measure on $\mathbb{R}$. In the next few lemmas and corollaries, we express

$$\mathbf{P}_{u_0}\left[u_t(A) > 0\right] \qquad \text{and} \qquad \mathbf{P}_{u_0}\left[0 < u_t(A) < \lambda(A)\right]$$

for an initial condition $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$, a measurable set $A \subset \mathbb{R}$ and a time $t > 0$ in terms of moments and probabilities concerning the dual process. We start with considering general u_0 and a general set A in Lemma 3.3, in Lemma 3.4 we specify the cases $u_0 = \mathbf{1}_B$ for a measurable set $B \subset \mathbb{R}$ or $u_0(x) = \vartheta$ for $x \in \mathbb{R}$ and some $\vartheta \in (0,1)$ and in Corollary 3.5 we consider the situation when A is an interval.

Lemma 3.3. *Let $A \subset \mathbb{R}$ be a measurable set with $\lambda(A) \in (0,\infty)$. Further let $U_1, U_2, \dots$ be independent random variables that are uniformly distributed on A and independent of $(\Xi_t^{\mathbb{R}})_{t \geq 0}$. Then for each $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ and $t > 0$, we have*

$$\mathbf{P}_{u_0}\left[u_t(A) > 0\right] = 1 - \mathbf{E}\left[\prod_{y \in \Xi_t^{\{U_n\,:\,n \in \mathbb{N}\}}} (1 - u_0(y))\right] \tag{3.1}$$

and

$$\begin{aligned} &\mathbf{P}_{u_0}\left[0 < u_t(A) < \lambda(A)\right] \\ &\quad = 1 - \left(\mathbf{E}\left[\prod_{y \in \Xi_t^{\{U_n\,:\,n \in \mathbb{N}\}}} (1 - u_0(y))\right] + \mathbf{E}\left[\prod_{y \in \Xi_t^{\{U_n\,:\,n \in \mathbb{N}\}}} u_0(y)\right]\right) \\ &\quad \leq \mathbf{P}\left[\left|\Xi_t^{\{U_n\,:\,n \in \mathbb{N}\}}\right| > 1\right]. \end{aligned} \tag{3.2}$$

Proof. For a $[0,1]$-valued random variable X we have

$$\mathbf{P}[X = 0] = \lim_{n \to \infty} \mathbf{E}\left[(1 - X)^n\right] \qquad \text{and} \qquad \mathbf{P}[X = 1] = \lim_{n \to \infty} \mathbf{E}\left[X^n\right].$$

Fix an initial condition $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ and $t > 0$. Since the measure u_t has a density u_t that takes values in $[0,1]$, the random variable $u_t(A)$ takes values in $[0, \lambda(A)]$ and

$\lambda(A)^{-1}u_t(A)$ takes values in $[0,1]$. Thus we get

$$\begin{aligned}\mathbf{P}_{u_0}\left[u_t(A)>0\right] &= 1-\mathbf{P}_{u_0}\left[u_t(A)=0\right]\\ &= 1-\mathbf{P}_{u_0}\left[\lambda(A)^{-1}u_t(A)=0\right]\\ &= 1-\lim_{n\to\infty}\mathbf{E}_{u_0}\left[\left(1-\lambda(A)^{-1}u_t(A)\right)^n\right].\end{aligned}\tag{3.3}$$

Analogously, we have

$$\begin{aligned}&\mathbf{P}_{u_0}\left[0<u_t(A)<\lambda(A)\right]\\ &\quad= 1-\left(\mathbf{P}_{u_0}\left[u_t(A)=0\right]+\mathbf{P}_{u_0}\left[u_t(A)=\lambda(A)\right]\right)\\ &\quad= 1-\left(\mathbf{P}_{u_0}\left[\lambda(A)^{-1}u_t(A)=0\right]+\mathbf{P}_{u_0}\left[\lambda(A)^{-1}u_t(A)=1\right]\right)\\ &\quad= 1-\lim_{n\to\infty}\left(\mathbf{E}_{u_0}\left[\left(1-\lambda(A)^{-1}u_t(A)\right)^n\right]+\mathbf{E}_{u_0}\left[\left(\lambda(A)^{-1}u_t(A)\right)^n\right]\right).\end{aligned}\tag{3.4}$$

For $n\in\mathbb{N}$, we have using the moment duality from Theorem 3.1

$$\begin{aligned}\mathbf{E}_{u_0}\left[\left(\lambda(A)^{-1}u_t(A)\right)^n\right] &= \mathbf{E}_{u_0}\left[\left(\int_A\lambda(A)^{-1}u_t(x)\,dx\right)^n\right]\\ &= \int_{A^n}\lambda(A)^{-n}\,\mathbf{E}_{u_0}\left[\prod_{i=1}^n u_t(x_i)\right]dx\\ &= \int_{A^n}\lambda(A)^{-n}\,\mathbf{E}\left[\prod_{y\in\Xi_t^{\{x_1,\dots,x_n\}}}u_0(y)\right]dx\\ &= \mathbf{E}\left[\prod_{y\in\Xi_t^{\{U_1,\dots,U_n\}}}u_0(y)\right].\end{aligned}$$

and analogously

$$\begin{aligned}\mathbf{E}_{u_0}\left[\left(1-\lambda(A)^{-1}u_t(A)\right)^n\right] &= \int_{A^n}\lambda(A)^{-n}\,\mathbf{E}_{u_0}\left[\prod_{i=1}^n(1-u_t(x_i))\right]dx\\ &= \int_{A^n}\lambda(A)^{-n}\,\mathbf{E}_{1-u_0}\left[\prod_{i=1}^n u_t(x_i)\right]dx\\ &= \int_{A^n}\lambda(A)^{-n}\,\mathbf{E}\left[\prod_{y\in\Xi_t^{\{x_1,\dots,x_n\}}}(1-u_0(y))\right]dx\\ &= \mathbf{E}\left[\prod_{y\in\Xi_t^{\{U_1,\dots,U_n\}}}(1-u_0(y))\right].\end{aligned}$$

Putting the last two identities into (3.3) and (3.4) and passing to the limit gives the equalities in (3.1) and (3.2). The inequality in (3.2) follows from

$$\left(\prod_{y\in\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}}(1-u_0(y))+\prod_{y\in\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}}u_0(y)\right)\mathbf{1}_{\left\{|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}|=1\right\}}=\mathbf{1}_{\left\{|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}|=1\right\}}$$

and

$$\left(\prod_{y\in\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}}(1-u_0(y))+\prod_{y\in\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}}u_0(y)\right)\mathbf{1}_{\left\{|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}|>1\right\}}\geq 0.$$

□

Lemma 3.4. *Let $A\subset\mathbb{R}$ be a measurable set with $\lambda(A)\in(0,\infty)$. Further let $U_1,U_2,\dots$ be independent random variables that are uniformly distributed on A and independent of $(\Xi_t^{\mathbb{R}})_{t\geq 0}$.*

(i) Let $u_0=\mathbf{1}_B$ for some measurable set $B\subset\mathbb{R}$. Then for $t>0$

$$\mathbf{P}_{u_0}\left[u_t(A)>0\right]=\mathbf{P}\left[\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\cap B\neq\emptyset\right]$$

and

$$\begin{aligned}\mathbf{P}_{u_0}\left[0<u_t(A)<\lambda(A)\right]&=\mathbf{P}\left[\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\cap B\neq\emptyset,\ \Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\cap B^c\neq\emptyset\right]\\&\leq\mathbf{P}\left[\left|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\right|>1\right].\end{aligned}$$

(ii) Let $\vartheta\in(0,1)$ and $u_0(x)=\vartheta$ for all $x\in\mathbb{R}$. Then for $t>0$

$$\begin{aligned}\left(1-((1-\vartheta)^2+\vartheta^2)\right)\mathbf{P}\left[\left|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\right|>1\right]&\leq\mathbf{P}_{u_0}\left[0<u_t(A)<\lambda(A)\right]\\&\leq\mathbf{P}\left[\left|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\right|>1\right].\end{aligned}$$

Proof. (i) Since $u_0=\mathbf{1}_B$, we have

$$\mathbf{E}\left[\prod_{y\in\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}}u_0(y)\right]=\mathbf{P}\left[\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\subset B\right]$$

and

$$\mathbf{E}\left[\prod_{y\in\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}}(1-u_0(y))\right]=\mathbf{P}\left[\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}\subset B^c\right].$$

Therefore, the claim follows from Lemma 3.3.

(ii) From Lemma 3.3, we get

$$\mathbf{P}_{u_0}\left[0 < u_t(A) < \lambda(A)\right] = 1 - \left(\mathbf{E}\left[(1-\vartheta)^{|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}|}\right] + \mathbf{E}\left[\vartheta^{|\Xi_t^{\{U_n\,:\,n\in\mathbb{N}\}}|}\right]\right).$$

Write $U := \{U_n \,:\, n \in \mathbb{N}\}$. Then we have

$$\begin{aligned}\mathbf{E}\left[(1-\vartheta)^{|\Xi_t^U|}\right] &= (1-\vartheta)\left(1 - \mathbf{P}\left[\left|\Xi_t^U\right| > 1\right]\right) + \mathbf{E}\left[(1-\vartheta)^{|\Xi_t^U|}\mathbf{1}_{\{|\Xi_t^U|>1\}}\right] \\ &= 1 - \vartheta + \mathbf{E}\left[\left((1-\vartheta)^{|\Xi_t^U|} - (1-\vartheta)\right)\mathbf{1}_{\{|\Xi_t^U|>1\}}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\left[\vartheta^{|\Xi_t^U|}\right] &= \vartheta\left(1 - \mathbf{P}\left[\left|\Xi_t^U\right| > 1\right]\right) + \mathbf{E}\left[\vartheta^{|\Xi_t^U|}\mathbf{1}_{\{|\Xi_t^U|>1\}}\right] \\ &= \vartheta + \mathbf{E}\left[\left(\vartheta^{|\Xi_t^U|} - \vartheta\right)\mathbf{1}_{\{|\Xi_t^U|>1\}}\right].\end{aligned}$$

Hence

$$\mathbf{E}\left[(1-\vartheta)^{|\Xi_t^U|}\right] + \mathbf{E}\left[\vartheta^{|\Xi_t^U|}\right] = 1 - \mathbf{E}\left[\left(1 - \left((1-\vartheta)^{|\Xi_t^U|} + \vartheta^{|\Xi_t^U|}\right)\right)\mathbf{1}_{\{|\Xi_t^U|>1\}}\right]$$

and

$$\mathbf{P}_{u_0}\left[0 < u_t(A) < \lambda(A)\right] = \mathbf{E}\left[\left(1 - \left((1-\vartheta)^{|\Xi_t^U|} + \vartheta^{|\Xi_t^U|}\right)\right)\mathbf{1}_{\{|\Xi_t^U|>1\}}\right].$$

Thus (ii) follows (for the lower bound note $\{|\Xi_t^U| > 1\} = \{|\Xi_t^U| \geq 2\}$). □

Corollary 3.5. *Let $a, b \in \mathbb{R}$ with $a < b$.*

(i) Let $u_0 = \mathbf{1}_B$ for some measurable set $B \subset \mathbb{R}$. Then for $t > 0$,

$$\mathbf{P}_{u_0}\left[u_t([a,b]) > 0\right] = \mathbf{P}\left[\Xi_t^{[a,b]} \cap B \neq \emptyset\right]$$

and

$$\begin{aligned}\mathbf{P}_{u_0}\left[0 < u_t([a,b]) < b - a\right] &= \mathbf{P}\left[\Xi_t^{[a,b]} \cap B \neq \emptyset,\, \Xi_t^{[a,b]} \cap B^c \neq \emptyset\right] \\ &\leq \mathbf{P}\left[\left|\Xi_t^{[a,b]}\right| > 1\right].\end{aligned}$$

(ii) Let $\vartheta \in (0,1)$ and $u_0(x) = \vartheta$ for all $x \in \mathbb{R}$. Then for $t > 0$

$$\begin{aligned}\left(1 - ((1-\vartheta)^2 + \vartheta^2)\right)\mathbf{P}\left[\left|\Xi_t^{[a,b]}\right| > 1\right] &\leq \mathbf{P}_{u_0}\left[0 < u_t([a,b]) < b - a\right] \\ &\leq \mathbf{P}\left[\left|\Xi_t^{[a,b]}\right| > 1\right].\end{aligned}$$

Proof. Let $U_1, U_2, \ldots$ be independent random variables that are uniformly distributed on $[a, b]$ and independent of $(\Xi_t^{\mathbb{R}})_{t\geq 0}$. Due to the lemma of Borel-Cantelli (see [Kl13, Theorem 2.7]), we have for each interval $[c, d] \subset [a, b]$

$$\mathbf{P}[U_n \in [c, d] \text{ for infinitely many } n \in \mathbb{N}] = 1.$$

Hence

$$\begin{aligned}
&\mathbf{P}[\{U_n : n \in \mathbb{N}\} \text{ is a dense subset of } [a, b]] \\
&= \mathbf{P}\left[\bigcap_{\substack{a\leq c<d\leq b \\ c,d\in\mathbb{Q}}} \{U_n \in [c, d] \text{ for infinitely many } n \in \mathbb{N}\} \right] \\
&= 1.
\end{aligned}$$

Thus from Proposition 2.11, we get that $(\Xi_t^{\{U_n : n\in\mathbb{N}\}})_{t>0}$ and $(\Xi_t^{[a,b]})_{t>0}$ have the same distribution and the results follow from Lemma 3.4. □

Now we consider the total mass $u_t(\mathbb{R})$ at time $t \geq 0$.

Corollary 3.6. *Let $B \subset \mathbb{R}$ be a measurable set and let $u_0 = \mathbf{1}_B$. For $t > 0$, we have*

$$\mathbf{P}_{u_0}[u_t(\mathbb{R}) > 0] = \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B \neq \emptyset\right].$$

Proof. We use Corollary 3.5 (i) in the second line and Remark 2.12 in the third line to compute

$$\begin{aligned}
\mathbf{P}_{u_0}[u_t(\mathbb{R}) > 0] &= \lim_{n\to\infty} \mathbf{P}_{u_0}[u_t([-n, n]) > 0] \\
&= \lim_{n\to\infty} \mathbf{P}\left[\Xi_t^{[-n,n]} \cap B \neq \emptyset\right] \\
&= \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B \neq \emptyset\right].
\end{aligned}$$

□

Remark 3.7. Let $t \geq 0$. Shifting a coalescing system of processes is in distribution the same as shifting the initial condition of the system. Therefore, $\Xi_t^{\mathbb{R}}$ is translation invariant. With duality it follows for the voter model that

$$\mathbf{P}_{\mathbf{1}_B}[u_t(\mathbb{R}) > 0] = \mathbf{P}_{\mathbf{1}_{x+B}}[u_t(\mathbb{R}) > 0]$$

for $x \in \mathbb{R}$ and a measurable set $B \subset \mathbb{R}$, where

$$x + B := \{x + y : y \in B\}.$$

Lemma 3.8. *Let $B \subset \mathbb{R}$ be a measurable set and let $t > 0$. Then the scaling property*

$$\mathbf{P}_{\mathbf{1}_B}\left[t^{-\frac{1}{\alpha}} u_t(\mathbb{R}) \in \cdot\right] = \mathbf{P}_{\mathbf{1}_{t^{-1/\alpha}B}}[u_1(\mathbb{R}) \in \cdot]$$

holds.

Proof. In the following we write

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)\,dx$$

for measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$ if the integral exists. Consider some arbitrary initial condition $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$. Let $n \in \mathbb{N}$, $\phi\colon \mathbb{R} \to \mathbb{R}$ be measurable and bounded with compact support and $c > 0$. Using the duality (Theorem 3.1) and the scaling property of the stable process (see (2.3)) we get

$$\begin{aligned}
\mathbf{E}_{u_0}\left[\langle u_t(c^{-1/\alpha}\cdot), \phi\rangle^n\right] &= \int_{\mathbb{R}^n} \mathbf{E}_{u_0}\left[\prod_{i=1}^n u_t\left(c^{-\frac{1}{\alpha}} x_i\right)\right] \prod_{i=1}^n \phi(x_i)\,dx \\
&= \int_{\mathbb{R}^n} \mathbf{E}\left[\prod_{y \in \Xi_t^{c^{-1/\alpha}\{x_1, x_2, \dots, x_n\}}} u_0\left(y\right)\right] \prod_{i=1}^n \phi(x_i)\,dx \\
&= \int_{\mathbb{R}^n} \mathbf{E}\left[\prod_{y \in \Xi_{ct}^{\{x_1, x_2, \dots, x_n\}}} u_0\left(c^{-\frac{1}{\alpha}} y\right)\right] \prod_{i=1}^n \phi(x_i)\,dx \\
&= \int_{\mathbb{R}^n} \mathbf{E}_{u_0\left(c^{-1/\alpha}\cdot\right)}\left[\prod_{i=1}^n u_{ct}\left(x_i\right)\right] \prod_{i=1}^n \phi(x_i)\,dx \\
&= \mathbf{E}_{u_0\left(c^{-1/\alpha}\cdot\right)}\left[\langle u_{ct}, \phi\rangle^n\right].
\end{aligned}$$

Thus [Kl13, Corollary 15.32] gives

$$\mathbf{P}_{u_0}\left[\langle u_t(c^{-1/\alpha}\cdot), \phi\rangle \in \cdot\right] = \mathbf{P}_{u_0\left(c^{-1/\alpha}\cdot\right)}\left[\langle u_{ct}, \phi\rangle \in \cdot\right].$$

This implies for $n \in \mathbb{N}$

$$\mathbf{P}_{u_0}\left[\int_{-n}^{n} u_t(c^{-\frac{1}{\alpha}} x)\,dx \in \cdot\right] = \mathbf{P}_{u_0\left(c^{-1/\alpha}\cdot\right)}\left[\int_{-n}^{n} u_{ct}(x)\,dx \in \cdot\right]$$

and with a substitution we get

$$\mathbf{P}_{u_0}\left[c^{\frac{1}{\alpha}} \int_{-c^{-1/\alpha}n}^{c^{-1/\alpha}n} u_t(x)\,dx \in \cdot\right] = \mathbf{P}_{u_0\left(c^{-1/\alpha}\cdot\right)}\left[\int_{-n}^{n} u_{ct}(x)\,dx \in \cdot\right],$$

hence

$$\mathbf{P}_{u_0}\left[c^{\frac{1}{\alpha}} u_t(\mathbb{R}) \in \cdot\right] = \mathbf{P}_{u_0\left(c^{-1/\alpha}\cdot\right)}\left[u_{ct}(\mathbb{R}) \in \cdot\right].$$

With $c = t^{-1}$ we get

$$\mathbf{P}_{u_0}\left[t^{-\frac{1}{\alpha}} u_t(\mathbb{R}) \in \cdot\right] = \mathbf{P}_{u_0\left(t^{1/\alpha}\cdot\right)}\left[u_1(\mathbb{R}) \in \cdot\right].$$

Plugging in $u_0 = \mathbf{1}_B$ yields

$$\mathbf{P}_{\mathbf{1}_B}\left[t^{-\frac{1}{\alpha}}u_t(\mathbb{R}) \in \cdot\right] = \mathbf{P}_{\mathbf{1}_{t^{-1/\alpha}B}}\left[u_1(\mathbb{R}) \in \cdot\right].$$

□

3.3. Bounded support property

We end the chapter by showing that when the voter model is started with a bounded initial region of opinion 1, then almost surely, the support of opinion 1 is bounded at any fixed deterministic time. The idea is to show that the probability that there is mass far away from the origin is sufficiently small that one can use a Borel-Cantelli argument. Using our results from the previous section to transform the question to the dual picture, this is a corollary of Lemma 2.16. There we gave a bound on the probability that the coalescing system starting in an initial set far away from the origin hits a fixed interval.

For $\mu \in \mathcal{M}(\mathbb{R})$, define the measure-theoretic support of $\mu \in \mathcal{M}(\mathbb{R})$ by

$$\operatorname{supp}(\mu) := \{x \in \mathbb{R} \,:\, \mu(B_\varepsilon(x)) > 0 \text{ for all } \varepsilon > 0\}$$

with

$$B_\varepsilon(x) = \{y \in \mathbb{R} \,:\, |y - x| < \varepsilon\}, \qquad x \in \mathbb{R},\, \varepsilon > 0.$$

Theorem 3.9. *Let* $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ *and assume* $\operatorname{supp}(u_0)$ *is bounded. For each* $t > 0$, $\mathbf{P}_{u_0}$*-almost surely* $\operatorname{supp}(u_t)$ *is a bounded set.*

Proof. Since $\operatorname{supp}(u_0)$ is bounded, there exists $a > 0$ with $\operatorname{supp}(u_0) \subset [-a, a]$, i.e. $u_0 \leq \mathbf{1}_{[-a,a]}$. We recall the formal situation from Lemma 2.16. Let $\frac{\alpha-1}{2} < \eta < \alpha - 1$ and $d := \frac{2}{\eta} > 1$. For $r \in \mathbb{N}$, we defined

$$J_{r,1} = \left[-\sum_{j=1}^{r} j^d, -\sum_{j=1}^{r-1} j^d\right) \qquad \text{and} \qquad J_{r,2} = \left[\sum_{j=1}^{r-1} j^d, \sum_{j=1}^{r} j^d\right).$$

According to Lemma 3.3, Corollary 3.5 (i) and Lemma 2.16, there exist $C_{2.16.1}, C_{2.16.2} > 0$ and $r_0 = r_0(\alpha, a, d) \in \mathbb{N}$ such that, for each $t > 0$, $r \geq r_0$ and $i \in \{1, 2\}$ we have

$$\mathbf{P}_{u_0}\left[u_t(J_{r,i}) > 0\right] \leq \mathbf{P}\left[\Xi_t^{J_{r,i}} \cap [-a, a] \neq \emptyset\right] \leq C_{2.16.1} r^{-2} + C_{2.16.2} t r^{-\alpha}.$$

This implies that for each $t > 0$,

$$\sum_{r=1}^{\infty} \left(\mathbf{P}_{u_0}\left[u_t(J_{r,1}) > 0\right] + \mathbf{P}_{u_0}\left[u_t(J_{r,2}) > 0\right]\right) < \infty.$$

Thus the Borel-Cantelli lemma (see [Kl13, Theorem 2.7]) yields

$$\mathbf{P}_{u_0}\left[u_t(J_{r,i}) > 0 \text{ for at most finitely many } (r,i) \in \mathbb{N} \times \{1,2\}\right] = 1.$$

Since $(J_{r,i})_{r\in\mathbb{N},i\in\{1,2\}}$ is a partition of $\mathbb{R}$, we have shown that

$$\mathbf{P}_{u_0}\left[\text{supp}(u_t) \text{ is a bounded set}\right] = 1.$$

□

4. Hausdorff dimension of the interface – an upper bound

4.1. Main result

One way to study our process is to look at the so called interface which we can interpret as the transition region of the two types. For $u \in \mathcal{M}_{\leq 1}(\mathbb{R})$ (with density $u\colon \mathbb{R} \to [0,1]$) we write $1-u \in \mathcal{M}_{\leq 1}(\mathbb{R})$ for the measure with density $1-u\colon \mathbb{R} \to [0,1]$. Then we define

$$\mathcal{I}(u) := \operatorname{supp}(u) \cap \operatorname{supp}(1-u)$$

as the interface of $u \in \mathcal{M}_{\leq 1}(\mathbb{R})$. For example, if we start the voter model with $u_0 = \mathbf{1}_{[-a,a]}$ for some $a > 0$ we have $\mathcal{I}(u_0) = \{-a, a\}$. In the Brownian case $\alpha = 2$ it is well known that the interface is described by annihilating Brownian motion (see [Tr95] and [HOV18, Theorem 2.12]), a family of independent Brownian motions where processes get killed if they hit each other. Since the behavior of stable processes in the case $\alpha \in (1,2)$ is more chaotic than that of Brownian motion, the interface in that case should be described by the Hausdorff dimension at a fixed time $t > 0$. The appendix (see Appendix A.1) contains a brief overview of the definition of the Hausdorff dimension of a subset of $\mathbb{R}$. The main result of this chapter which we formulate now is an upper bound on the dimension. Getting a lower bound is more difficult. In Section 8.1, we give some heuristics on that. In the following, $\dim_{\mathcal{H}}(A)$ denotes the Hausdorff dimension of a set $A \subset \mathbb{R}$.

Theorem 4.1. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$. For each $t > 0$, we have $\mathbf{P}_{u_0}$-almost surely*

$$\dim_{\mathcal{H}}(\mathcal{I}(u_t)) \leq 1 - (\alpha - 1)^2.$$

Remark 4.2. The dimension bound in Theorem 4.1 agrees with the Brownian case $\alpha = 2$: In this case $\mathcal{I}(u_t)$ is almost surely a locally finite set for fixed $t > 0$ (see [HOV18, Proposition 5.17]). Therefore its dimension is zero. Furthermore

$$(1,2) \ni \alpha \mapsto 1 - (\alpha - 1)^2 = -\alpha^2 + 2\alpha$$

is decreasing. This is something which one would expect for the dimension of the interface since the smaller α is the more heavy are the tails of the α-stable process.

4.2. Proof of Theorem 4.1

The aim of this section is to prove the dimension bound from Theorem 4.1. The main step is to determine the order of the probability that a small interval contains an interface point as a function of the interval diameter. We start with a simple lemma and afterwards we explain the strategy.

Lemma 4.3. *Let* $u \in \mathcal{M}_{\leq 1}(\mathbb{R})$ *and* $a, b \in \mathbb{R}$ *with* $a < b$. *The following statements are equivalent:*

(i) $\mathcal{I}(u) \cap (a,b) \neq \emptyset$

(ii) $0 < \int_a^b u(y)\,dy < b - a$

Proof. Assume (i) holds. This implies that there exists $x \in (a,b)$ such that $\int_{x-\delta}^{x+\delta} u(y)\,dy > 0$ and $\int_{x-\delta}^{x+\delta}(1-u(y))\,dy > 0$ for each $\delta > 0$. Thus (ii) follows from (i) via choosing $\delta > 0$ small enough such that $[x-\delta, x+\delta] \subset (a,b)$.
Now assume (i) does not hold. Then for each $x \in (a,b)$, there is a $\delta = \delta(x) > 0$ such that either

$$\int_{x-\delta}^{x+\delta} u(y)\,dy = 0 \qquad \text{or} \qquad \int_{x-\delta}^{x+\delta} (1-u(y))\,dy = 0. \tag{4.1}$$

Fix some $x_0 \in (a,b)$ and assume $\int_{x_0-\delta}^{x_0+\delta} u(y)\,dy = 0$ for some $\delta = \delta(x_0) > 0$. Let

$$x_1 := \inf\left\{ r \leq x_0 \,:\, \int_r^{x_0} u(y)\,dy = 0 \right\}.$$

For $\varepsilon > 0$, we have

$$\int_{x_1-\varepsilon}^{x_1+\varepsilon} u(y)\,dy > 0$$

and

$$\int_{x_1-\varepsilon}^{x_1+\varepsilon} u(y)\,dy \leq \varepsilon < 2\varepsilon$$

what is equivalent to

$$\int_{x_1-\varepsilon}^{x_1+\varepsilon} (1-u(y))\,dy > 0.$$

Thus, due to (4.1), we get $x_1 \notin (a,b)$, which implies $x_1 \leq a$. Now let

$$x_2 := \sup\left\{ s \geq x_0 \,:\, \int_{x_0}^{s} u(y)\,dy = 0 \right\}.$$

With an analogous argument as above, we get $x_2 \geq b$. This implies

$$\int_a^b u(y)\,dy = 0.$$

In the case $\int_{x_0-\delta}^{x_0+\delta}(1-u(y))\,dy = 0$ for some $\delta = \delta(x_0) > 0$ one can show with the same argument $\int_a^b u(y)\,dy = b-a$. We conclude that (ii) does not hold. $\square$

With Lemma 4.3 we get from duality (Lemma 3.3) for $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$, $t > 0$ and $a, b \in \mathbb{R}$ with $a < b$

$$\mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap (a,b) \neq \emptyset\right] = \mathbf{P}_{u_0}\left[0 < u_t(([a,b]) < b-a\right] \leq \mathbf{P}\left[\left|\Xi_t^{[a,b]}\right| > 1\right].$$

Thus in the dual picture we have to bound the probability that the coalescing system starting in an interval $[a,b]$ has not coalesced down to one particle up to time t. In the Brownian case $\alpha = 2$, this is the probability that the two extremal processes started in a and b have not met up to time t since the paths of Brownian motion are continuous. For the α-stable process with $\alpha < 2$, this is not the case and thus the question is more difficult to answer. We start with bounds for a finite initial number of particles: In Lemma 4.4 we prepare this via giving bounds on the non-hitting-probability of two arbitrary particles. Then we consider the case of 2^n particles for $n \in \mathbb{N}$ in Lemma 4.5 and Lemma 4.6 and then deduce a bound for any finite number $N \in \mathbb{N}$ of particles in Corollary 4.7. With the help of Corollary 2.15, which is a consequence of [EMS13, Lemma 6.10], we can then get results for the situation where the coalescing system starts with infinitely many particles which we formulate in Lemma 4.8. Then we can prove Theorem 4.1.

Recall from Section 2.2 the construction of the vector-valued coalescing process ζ^σ for bijections $\sigma\colon [N] \to [N]$ starting with $N \in \mathbb{N} \cup \{\infty\}$ particles. For $i \in [N]$ we denoted by τ_i the first coalescence time of particle number $\sigma(i)$. For $i, j \in [N]$ with $i < j$, define the stopping time

$$T^{(i,j)} := \inf\left\{t \geq 0 \,:\, \zeta_t^{\mathrm{id},i} = \zeta_t^{\mathrm{id},j}\right\}.$$

This is the first time when the path that particle i follows hits the path that particle j follows.

Lemma 4.4. *For $\xi_0 \in \mathbb{R}^N$, $i, j \in [N]$ with $i < j$ and $t > 0$, we have*

$$\begin{aligned} C_{4.4.1} \min&\left\{C_{4.4.2} t^{-\left(1-\frac{1}{\alpha}\right)} \left|\xi_0^{(j)} - \xi_0^{(i)}\right|^{\alpha-1}, 1\right\} \\ &\leq \mathbf{P}\left[T^{(i,j)} \geq t\right] \\ &\leq \min\left\{C_{4.4.3} t^{-\left(1-\frac{1}{\alpha}\right)} \left|\xi_0^{(j)} - \xi_0^{(i)}\right|^{\alpha-1}, 1\right\} \end{aligned}$$

for some constants $C_{4.4.1} = C_{4.4.1}(\alpha) > 0$, $C_{4.4.2} = C_{4.4.2}(\alpha) > 0$ and $C_{4.4.3} = C_{4.4.3}(\alpha) > 0$.

Proof. Fix $i, j \in [N]$ with $i < j$. Let $\sigma\colon [N] \to [N]$ be a bijection with $\sigma(1) = i$ and

$\sigma(2) = j$ and let $t > 0$.

$$\begin{aligned}
\mathbf{P}\left[T^{(i,j)} \geq t\right] &= \mathbf{P}\left[\zeta_s^{\mathrm{id},i} \neq \zeta_s^{\mathrm{id},j} \text{ for all } s < t\right] \\
&= \mathbf{P}\left[\zeta_s^{\sigma,i} \neq \zeta_s^{\sigma,j} \text{ for all } s < t\right] \\
&= \mathbf{P}\left[\zeta_s^{\sigma,\sigma(1)} \neq \zeta_s^{\sigma,\sigma(2)} \text{ for all } s < t\right] \\
&= \mathbf{P}\left[\xi_s^{(\sigma(1))} \neq \xi_s^{(\sigma(2))} \text{ for all } s < t\right] \\
&= \mathbf{P}\left[\xi_s^{(j)} - \xi_s^{(i)} \neq 0 \text{ for all } s < t\right].
\end{aligned}$$

In the second equality we used that the distribution of ζ^σ does not depend on the choice of the bijection $\sigma\colon [N] \to [N]$ (see Lemma 2.9). Since the difference of two independent stable processes is again a stable process, we can now use the bounds of the non-hitting-probability of the stable process from Proposition 2.5 (ii). □

Lemma 4.5. *Let $n \in \mathbb{N}$ and $A \subset \mathbb{R}$ with $|A| = 2^n$. For $t > 0$, we have*

$$\mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] \leq C_{4.4.3} t^{-\left(1-\frac{1}{\alpha}\right)} \sum_{j=1}^{n} \sum_{i=1}^{2^{n-j}} \left|\xi_0^{(1+(i-1)2^j+2^{j-1})} - \xi_0^{(1+(i-1)2^j)}\right|^{\alpha-1}.$$

Proof. The idea is to represent the event that all processes have coalesced down to one particle as an intersection of hitting events of pairs of paths. Recall due to the definition of $T^{(i,j)}$ for $i, j \in \{1, 2, \ldots, 2^n\}$ with $i < j$ that after the coalescence that particle j with the larger number follows particle i with the smaller number. We explain the strategy informally for $n = 3$, so think of $2^3 = 8$ initial particles. Assume that the paths of particles 1 and 2, 3 and 4, 5 and 6 and 7 and 8 have met up to time t. Since we consider the dependent paths the particles follow and not the independent processes think from now on of each of these four pairs as one particle (or one block) and therefore imagine particles 2, 4, 6 and 8 are from now on not there. In the next step assume the paths of particles 1 and 3 and particles 5 and 7 have met up to time t and again think of each of these pairs as one block. If we further assume that additionally the paths of particles 1 and 5 have met up to time t we know that there is only one particle left. Formally, consider the following events in the general situation with 2^n particles:

$$\begin{aligned}
A_1(t) &= \left\{T^{(1,2)} < t, T^{(3,4)} < t, \ldots, T^{(2^n-1,2^n)} < t\right\}, \\
A_2(t) &= \left\{T^{(1,3)} < t, T^{(5,7)} < t, \ldots, T^{(2^n-3,2^n-1)} < t\right\}, \\
A_3(t) &= \left\{T^{(1,5)} < t, T^{(9,13)} < t, \ldots, T^{(2^n-7,2^n-3)} < t\right\}, \\
&\vdots \\
A_{n-1}(t) &= \left\{T^{(1,1+2^{n-2})} < t, T^{(1+2^{n-1},1+2^{n-1}+2^{n-2})} < t\right\} \\
A_n(t) &= \left\{T^{(1,1+2^{n-1})} < t\right\}.
\end{aligned}$$

So we can write for $j \in \{1,2,\dots,n\}$

$$A_j(t) = \bigcap_{i=1}^{2^{n-j}} \left\{ T^{(1+(i-1)2^j, 1+(i-1)2^j+2^{j-1})} < t \right\}.$$

By construction, we have

$$\bigcap_{j=1}^{n} A_j(t) = \left\{ \left| \Xi_t^A \right| = 1 \right\}.$$

Lemma 4.4 yields

$$\begin{aligned}
\mathbf{P}\left[\left| \Xi_t^A \right| > 1 \right] = \mathbf{P}\left[\bigcup_{j=1}^{n} A_j(t)^c \right] &\le \sum_{j=1}^{n} \sum_{i=1}^{2^{n-j}} \mathbf{P}\left[T^{(1+(i-1)2^j, 1+(i-1)2^j+2^{j-1})} \ge t \right] \\
&\le C_{4.4.3} t^{-\left(1-\frac{1}{\alpha}\right)} \sum_{j=1}^{n} \sum_{i=1}^{2^{n-j}} \left| \xi_0^{(1+(i-1)2^j+2^{j-1})} - \xi_0^{(1+(i-1)2^j)} \right|^{\alpha-1}.
\end{aligned}$$

□

Lemma 4.6. *Let $\ell > 0$, $n \in \mathbb{N}$ and $A \subset \mathbb{R}$ with $\operatorname{diam}(A) \le \ell$ and $|A| = 2^n$. One has for $t > 0$*

$$\mathbf{P}\left[\left| \Xi_t^A \right| > 1 \right] \le C_{4.6} (2^n)^{2-\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1}$$

for some constant $C_{4.6} = C_{4.6}(\alpha) > 0$.

Proof. We will use Lemma 4.5. To control the sum on the right hand side of the bound in Lemma 4.5 we will add particles in a way that we have a uniform bound on the distance of two neighbouring particles. Due to our assumptions on the set A there exist $a, x_1, x_2, \dots, x_{2^n} \in \mathbb{R}$ such that $A = \{x_1, x_2, \dots, x_{2^n}\}$ and $A \subset [a, a+\ell]$. Define

$$\widetilde{\xi_0} := \left(x_1, x_2, \dots, x_{2^n}, a, a + \frac{\ell}{2^n - 1}, a + 2 \cdot \frac{\ell}{2^n - 1}, \dots, a + (2^n - 1) \cdot \frac{\ell}{2^n - 1} \right).$$

Sort the 2^{n+1} entries of $\widetilde{\xi_0} \in [a, a+\ell]^{2^{n+1}}$ in increasing order and call the resulting vector

$$\widehat{\xi_0} = \left(\widehat{\xi_0}^{(1)}, \widehat{\xi_0}^{(2)}, \dots, \widehat{\xi_0}^{(2^{n+1})} \right) \in [a, a+\ell]^{2^{n+1}}.$$

By construction, for $i \in \{1, 2, \dots, 2^{n+1} - 1\}$ we have

$$\left| \widehat{\xi_0}^{(i+1)} - \widehat{\xi_0}^{(i)} \right| = \widehat{\xi_0}^{(i+1)} - \widehat{\xi_0}^{(i)} \le \frac{\ell}{2^n - 1} \le \frac{\ell}{2^{n-1}}.$$

This implies for $i, j \in \{1, 2, \dots, 2^{n+1}\}$ with $i < j$

$$\left| \widehat{\xi_0}^{(j)} - \widehat{\xi_0}^{(i)} \right| \le (j-i) \frac{\ell}{2^{n-1}}$$

and for $j \in \{1, 2, \ldots, n+1\}$ and $i \in \{1, 2, \ldots, 2^{n+1-j}\}$

$$\left|\widehat{\xi_0}^{(1+(i-1)2^j+2^{j-1})} - \widehat{\xi_0}^{(1+(i-1)2^j)}\right| \leq 2^{j-1}\frac{\ell}{2^{n-1}}. \tag{4.2}$$

Recall that we introduced the notation $\Xi_t^A = \Xi_t^{(x_1,x_2,\ldots,x_{2^n})}$. By construction of the coalescing system and using Remark 2.12 we can assume

$$\Xi_t^{(x_1,x_2,\ldots,x_{2^n})} \subset \Xi_t^{\widehat{\xi_0}}.$$

Using this, Lemma 4.5 with 2^{n+1} initial particles, and (4.2), we get

$$\begin{aligned}
\mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] &\leq \mathbf{P}\left[\left|\Xi_t^{\widehat{\xi_0}}\right| > 1\right] \\
&\leq C_{4.4.3} t^{-\left(1-\frac{1}{\alpha}\right)} \sum_{j=1}^{n+1} \sum_{i=1}^{2^{n+1-j}} \left|\widehat{\xi_0}^{(1+(i-1)2^j+2^{j-1})} - \widehat{\xi_0}^{(1+(i-1)2^j)}\right|^{\alpha-1} \\
&\leq C_{4.4.3} t^{-\left(1-\frac{1}{\alpha}\right)} \left(\frac{\ell}{2^{n-1}}\right)^{\alpha-1} \sum_{j=1}^{n+1} \sum_{i=1}^{2^{n+1-j}} 2^{(j-1)(\alpha-1)} \\
&= C_{4.4.3} t^{-\left(1-\frac{1}{\alpha}\right)} \left(\frac{\ell}{2^{n-1}}\right)^{\alpha-1} 2^n \sum_{j=1}^{n+1} 2^{1-j} 2^{(j-1)(\alpha-1)} \\
&= C_{4.4.3} 2^{\alpha-1} (2^n)^{2-\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1} \sum_{j=1}^{n+1} \left(\left(\frac{1}{2}\right)^{2-\alpha}\right)^{j-1} \\
&\leq C_{4.4.3} \frac{2^{\alpha-1}}{1-\left(\frac{1}{2}\right)^{2-\alpha}} (2^n)^{2-\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1}.
\end{aligned}$$

□

Corollary 4.7. *Let $\ell > 0$, $N \in \mathbb{N}$ and $A \subset \mathbb{R}$ with* $\operatorname{diam}(A) \leq \ell$ *and $|A| = N$. For $t > 0$, we have*

$$\mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] \leq C_{4.7} N^{2-\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1}$$

for some constant $C_{4.7} = C_{4.7}(\alpha) > 0$.

Proof. We add so many particles that the number of particles is a power of two: Since we start with N particles we consider instead $2^{\lceil \log_2(N) \rceil} \geq N$ particles and get with Remark 2.12

$$\begin{aligned}
\mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] &\leq \sup_{\substack{\widetilde{A} \subset \mathbb{R} \\ \operatorname{diam}(\widetilde{A}) \leq \ell, \left|\widetilde{A}\right| = 2^{\lceil \log_2(N) \rceil}}} \mathbf{P}\left[\left|\Xi_t^{\widetilde{A}}\right| > 1\right] \\
&\leq C_{4.6} \left(2^{\lceil \log_2(N) \rceil}\right)^{2-\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1} \\
&\leq C_{4.6} 2^{2-\alpha} N^{2-\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1}.
\end{aligned}$$

In the second inequality we used Lemma 4.6 and in the third inequality $2^{\lceil \log_2(N) \rceil} \leq 2 \cdot 2^{\log_2(N)} = 2N$. $\square$

Now we consider the case of an infinite number of initial particles.

Lemma 4.8. *There is a $t_0 > 0$ and for each $t > 0$ and $\eta \in \left(\frac{\alpha-1}{2}, \alpha - 1\right)$ there is a constant $C_{4.8} = C_{4.8}(\alpha, \eta, t) > 0$ such that we have for all $\ell \in (0, \min\{(\frac{t}{2t_0})^{\frac{1}{\alpha}}, 1\})$*

$$\sup_{\substack{n\in\mathbb{N}, A\subset\mathbb{R} \\ \operatorname{diam}(A)\leq \ell, |A|=n}} \mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] \leq C_{4.8}\ell^{(\alpha-1)\frac{\eta}{2-\alpha+\eta}}.$$

Proof. The idea is the following: We have to make sure that the infinitely many particles quickly coalesce down to a finite number $\lceil \gamma^k \rceil$ of particles (for some $\gamma > 1$ and $k \in \mathbb{N}$) and during that time no particle leaves an ℓ-fattening of the starting set A of diameter ℓ. This good event will be denoted by $B_k^{A,m,\gamma,\ell}$. Corollary 2.15 says that this event has a high probability. In that situation we can use the strong Markov property of the process. Thus, instead of bounding the probability that there is more than one particle left for infinitely many particles, we have to bound the probability in the case of $\lceil \gamma^k \rceil$ particles starting from a set $\widetilde{A}$ of diameter 3ℓ. Then we use Corollary 4.7. Here are the details: According to Corollary 2.15, there are $t_0 > 0$, $\gamma > 1$ and a constant $C_{2.15} = C_{2.15}(\eta, \alpha) > 0$ such that for each $m \in \mathbb{N}_0$, $\ell > 0$, $A \subset \mathbb{R}$ with $\operatorname{diam}(A) \leq \ell$ and $|A| = \lceil \gamma^m \rceil$ and each $k \in \mathbb{N}_0$ with $k \leq m$

$$\mathbf{P}\left[\left(B_k^{A,m,\gamma,\ell}\right)^c\right] \leq C_{2.15}\gamma^{-\eta k} \tag{4.3}$$

with

$$B_k^{A,m,\gamma,\ell} = \left\{\tau^A_{\lceil \gamma^k \rceil} \leq \ell^\alpha t_0\right\} \cap \left\{\mathcal{R}(A; [0, \tau^A_{\lceil \gamma^k \rceil}]) \subseteq (A)^\ell\right\}.$$

Let $\ell < \min\{(\frac{t}{2t_0})^{\frac{1}{\alpha}}, 1\}$. On the event on $B_k^{A,m,\gamma,\ell}$ we have

$$t - \tau^A_{\lceil \gamma^k \rceil} \geq t - \ell^\alpha t_0 > \frac{t}{2}.$$

Thus, using the decomposition

$$\begin{aligned}
&\sup_{\substack{n\in\mathbb{N}, A\subset\mathbb{R} \\ \operatorname{diam}(A)\leq \ell, |A|=n}} \mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] \\
&= \sup_{\substack{m\in\mathbb{N}_0, A\subset\mathbb{R} \\ \operatorname{diam}(A)\leq \ell, |A|=\lceil \gamma^m \rceil}} \mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] \\
&= \sup_{\substack{m\in\mathbb{N}_0, A\subset\mathbb{R} \\ \operatorname{diam}(A)\leq \ell, |A|=\lceil \gamma^m \rceil}} \left(\mathbf{P}\left[\left\{\left|\Xi_t^A\right| > 1\right\} \cap B_k^{A,m,\gamma,\ell}\right] + \mathbf{P}\left[\left\{\left|\Xi_t^A\right| > 1\right\} \cap \left(B_k^{A,m,\gamma,\ell}\right)^c\right]\right)
\end{aligned}$$

the strong Markov property of $\left(\Xi_t^A\right)_{t\geq 0}$ (see Corollary 2.13), Corollary 4.7 and (4.3) yield

$$\begin{aligned}
&\sup_{\substack{n\in\mathbb{N},A\subset\mathbb{R}\\ \operatorname{diam}(A)\leq\ell,|A|=n}} \mathbf{P}\left[\left|\Xi_t^A\right|>1\right] \\
&\leq \left(\sup_{\substack{\widetilde{A}\subset\mathbb{R}\\ \operatorname{diam}(\widetilde{A})\leq 3\ell,\left|\widetilde{A}\right|=\lceil\gamma^k\rceil}} \mathbf{P}\left[\left|\Xi_{t/2}^{\widetilde{A}}\right|>1\right]\right) + C_{2.15}\gamma^{-\eta k} \qquad (4.4)\\
&\leq C_{4.7}t^{-\left(1-\frac{1}{\alpha}\right)}\left(\lceil\gamma^k\rceil\right)^{2-\alpha}\ell^{\alpha-1} + C_{2.15}\gamma^{-\eta k}\\
&\leq 2^{2-\alpha}C_{4.7}t^{-\left(1-\frac{1}{\alpha}\right)}\gamma^{k(2-\alpha)}\ell^{\alpha-1} + C_{2.15}\gamma^{-\eta k}
\end{aligned}$$

Now choose $k\in\mathbb{N}$ such that $\gamma^k\approx\ell^{-\delta}$ for some $\delta>0$, i.e. let $k=\lceil\delta\log_\gamma(\ell^{-1})\rceil$. Using $\lceil x\rceil\leq x+1$ for $x>0$, we get

$$\begin{aligned}
\gamma^{k(2-\alpha)}\ell^{\alpha-1} &\leq \gamma^{\delta\log_\gamma(\ell^{-1})(2-\alpha)+(2-\alpha)}\ell^{\alpha-1} = \gamma^{2-\alpha}\ell^{-(2-\alpha)\delta}\ell^{\alpha-1}\\
&= \gamma^{2-\alpha}\ell^{\alpha-1-(2-\alpha)\delta}
\end{aligned}$$

and using $\lceil x\rceil\geq x$ for $x>0$

$$\gamma^{-\eta k}\leq\gamma^{-\eta\delta\log_\gamma(\ell^{-1})}=\ell^{\eta\delta}.$$

Putting the last two inequalities into (4.4) we get for each $\delta\in\left(0,\frac{\alpha-1}{2-\alpha}\right)$ (recall $\ell\leq 1$)

$$\begin{aligned}
\sup_{\substack{n\in\mathbb{N},A\subset\mathbb{R}\\ \operatorname{diam}(A)\leq\ell,|A|=n}} \mathbf{P}\left[\left|\Xi_t^A\right|>1\right] &\leq 2^{2-\alpha}C_{4.7}t^{-\left(1-\frac{1}{\alpha}\right)}\gamma^{2-\alpha}\ell^{\alpha-1-(2-\alpha)\delta}+C_{2.15}\ell^{\eta\delta}\\
&\leq C_{4.8}\ell^{\min\{\alpha-1-(2-\alpha)\delta,\eta\delta\}}
\end{aligned}$$

for some constant $C_{4.8}=C_{4.8}(\alpha,\eta,\gamma,t)>0$ (recall that the constant $C_{2.15}$ depends on η). Note that $\alpha-1-(2-\alpha)\delta=\eta\delta$ if and only if $\delta=\frac{\alpha-1}{2-\alpha+\eta}$, so

$$\min\{\alpha-1-(2-\alpha)\delta,\eta\delta\}=\begin{cases}\eta\delta, & \text{if } \delta\in\left(0,\frac{\alpha-1}{2-\alpha+\eta}\right],\\ \alpha-1-(2-\alpha)\delta, & \text{if } \delta\in\left(\frac{\alpha-1}{2-\alpha+\eta},\frac{\alpha-1}{2-\alpha}\right).\end{cases}$$

To obtain an optimal bound we choose $\delta=\frac{\alpha-1}{2-\alpha+\eta}$ and conclude

$$\sup_{\substack{n\in\mathbb{N},A\subset\mathbb{R}\\ \operatorname{diam}(A)\leq\ell,|A|=n}} \mathbf{P}\left[\left|\Xi_t^A\right|>1\right]\leq C_{4.8}\ell^{(\alpha-1)\frac{\eta}{2-\alpha+\eta}}.$$

□

Corollary 4.9. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$, $x \in \mathbb{R}$ and $t > 0$. Then we have*

$$\mathbf{P}_{u_0}\left[x \in \mathcal{I}(u_t)\right] = 0.$$

Proof. Let $\eta \in \left(\frac{\alpha-1}{2}, \alpha - 1\right)$. Using Lemma 4.3 in the second equality, Lemma 3.3 (duality) in the first inequality and Lemma 4.8 in the last inequality we get

$$\begin{aligned}
\mathbf{P}_{u_0}\left[x \in \mathcal{I}(u_t)\right] &= \lim_{m\to\infty} \mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \neq \emptyset\right] \\
&= \lim_{m\to\infty} \mathbf{P}_{u_0}\left[0 < u_t\left(\left[x - \frac{1}{m}, x + \frac{1}{m}\right]\right) < \frac{2}{m}\right] \\
&\leq \limsup_{m\to\infty} \mathbf{P}\left[\left|\Xi_t^{[x-\frac{1}{m}, x+\frac{1}{m}]}\right| > 1\right] \\
&\leq \lim_{m\to\infty} C_{4.8}\left(\frac{2}{m}\right)^{(\alpha-1)\frac{\eta}{2-\alpha+\eta}} = 0.
\end{aligned} \tag{4.5}$$

□

Corollary 4.10. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$. There is a $t_0 > 0$ and for each $t > 0$ and $\eta \in \left(\frac{\alpha-1}{2}, \alpha - 1\right)$ there is a constant $C_{4.8} = C_{4.8}(\alpha, \eta, t) > 0$ such that we have for all $\ell \in (0, \min\{(\frac{t}{2t_0})^{\frac{1}{\alpha}}, 1\})$*

$$\sup_{x\in\mathbb{R}} \mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap [x, x+\ell] \neq \emptyset\right] \leq C_{4.8}\ell^{(\alpha-1)\frac{\eta}{2-\alpha+\eta}}.$$

Proof. Let $x \in \mathbb{R}$, $t > 0$ and $\ell > 0$. Analogously to the computation in (4.5), we get

$$\begin{aligned}
\mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap [x, x+\ell] \neq \emptyset\right] &= \mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap (x, x+\ell) \neq \emptyset\right] \\
&= \mathbf{P}_{u_0}\left[0 < u_t\left([x, x+\ell]\right) < \ell\right] \\
&\leq \mathbf{P}\left[\left|\Xi_t^{[x,x+\ell]}\right| > 1\right]
\end{aligned}$$

where we used Corollary 4.9 in the first equality. Now Lemma 4.8 gives the desired bound. □

Now we can complete the proof of the main theorem.

Proof of Theorem 4.1. Let $t > 0$. Using the countable stability-property of the Hausdorff dimension it suffices to show that for each $x \in \mathbb{R}$, $\mathbf{P}_{u_0}$-almost surely

$$\dim_{\mathcal{H}}\left(\mathcal{I}(u_t) \cap [x, x+1]\right) \leq 1 - (\alpha - 1)^2.$$

Let $x \in \mathbb{R}$ and $\eta \in (\frac{\alpha-1}{2}, \alpha - 1)$. Define $s(\alpha, \eta) := 1 - (\alpha - 1)\frac{\eta}{2-\alpha+\eta}$. For $n \in \mathbb{N}$, we can bound

$$\mathcal{H}_{\frac{1}{n}}^{s(\alpha,\eta)}\left(\mathcal{I}(u_t) \cap [x, x+1]\right) \leq \sum_{i=1}^{n} n^{-s(\alpha,\eta)} \mathbf{1}_{\left\{\mathcal{I}(u_t)\cap\left[x+\frac{i-1}{n}, x+\frac{i}{n}\right]\neq\emptyset\right\}}.$$

According to Corollary 4.10 there are $t_0 > 0$ and a constant $C_{4.8} = C_{4.8}(\alpha, \eta, t) > 0$ such that we have for all $n \in \mathbb{N}$ with $n > (\min\{(\frac{t}{2t_0})^{\frac{1}{\alpha}}, 1\})^{-1}$

$$\begin{aligned}\mathbf{E}_{u_0}\left[\mathcal{H}^{s(\alpha,\eta)}_{\frac{1}{n}}\left(\mathcal{I}(u_t)\cap[x,x+1]\right)\right] &\leq \sum_{i=1}^{n} n^{-s(\alpha,\eta)}\mathbf{P}_{u_0}\left[\mathcal{I}(u_t)\cap\left[x+\frac{i-1}{n}, x+\frac{i}{n}\right]\neq\emptyset\right]\\ &\leq C_{4.8}\sum_{i=1}^{n} n^{-s(\alpha,\eta)}\left(\frac{1}{n}\right)^{1-s(\alpha,\eta)}\\ &= C_{4.8}.\end{aligned}$$

Thus Fatou's lemma (see [Kl13, Theorem 4.21]) yields

$$\mathbf{E}_{u_0}\left[\mathcal{H}^{s(\alpha,\eta)}\left(\mathcal{I}(u_t)\cap[x,x+1]\right)\right] \leq \liminf_{n\to\infty}\mathbf{E}_{u_0}\left[\mathcal{H}^{s(\alpha,\eta)}_{\frac{1}{n}}\left(\mathcal{I}(u_t)\cap[x,x+1]\right)\right] \leq C_{4.8} < \infty.$$

This implies we have $\mathbf{P}_{u_0}$-almost surely $\mathcal{H}^{s(\alpha,\eta)}\left(\mathcal{I}(u_t)\cap[x,x+1]\right) < \infty$, i.e. $\mathbf{P}_{u_0}$-almost surely

$$\dim_{\mathcal{H}}\left(\mathcal{I}(u_t)\cap[x,x+1]\right) \leq s(\alpha,\eta) = 1-(\alpha-1)\frac{\eta}{2-\alpha+\eta}$$

for each $\eta \in (\frac{\alpha-1}{2}, \alpha-1)$. Since $(0,\infty) \ni \eta \mapsto \frac{\eta}{2-\alpha+\eta}$ is increasing, we conclude that $\mathbf{P}_{u_0}$-almost surely

$$\dim_{\mathcal{H}}\left(\mathcal{I}(u_t)\cap[x,x+1]\right) \leq \lim_{\eta\nearrow\alpha-1}\left(1-(\alpha-1)\frac{\eta}{2-\alpha+\eta}\right) = 1-(\alpha-1)^2.$$

□

5. Lifetime of the colonies

5.1. Main results

In this chapter we investigate the total mass $u_t(\mathbb{R})$ of the voter model at time $t \geq 0$. Particularly we are interested in the lifetime

$$\tau := \inf \{t \geq 0 \, : \, u_t(\mathbb{R}) = 0\} \tag{5.1}$$

of the voter model if we start with a bounded set of opinion 1. Formally, let $u_0 = \mathbf{1}_B$ where $B \subset \mathbb{R}$ is a measurable and bounded set with $\lambda(B) > 0$. We are especially concerned the case where B is a small interval. The aim is to show asymptotic bounds for the survival probability at time $t > 0$

$$\mathbf{P}_{u_0}\left[u_t(\mathbb{R}) > 0\right] = \mathbf{P}_{u_0}\left[\tau > t\right].$$

We will formulate these bounds in the next two theorems and state afterwards in a corollary that the voter model dies out with probability 1. This suggests to imagine the support of the voter model (viewed in time) as a *colony* of opinion 1. In Section 5.2, we prove Theorem 5.1 (and obtain that the total mass is a martingale) and in Section 5.3 we prove Theorem 5.2. Later in Corollary 7.21 we show that if u_0 has a continuous density and $u_0(\mathbb{R}) < \infty$, then $(u_t(\mathbb{R}))_{t \geq 0}$ has almost surely continuous sample paths.

Theorem 5.1. *Let $B \subset \mathbb{R}$ be a measurable bounded set with $\lambda(B) > 0$ and $u_0 = \mathbf{1}_B$. For $t > 0$, we have*

$$\mathbf{P}_{u_0}\left[u_t(\mathbb{R}) > 0\right] > 0.$$

Furthermore, there exists a constant $C_{5.1} = C_{5.1}(\alpha) > 0$ such that for each $t > 0$ and $a \in \mathbb{R}$, we have

$$\liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[u_t(\mathbb{R}) > 0\right] \geq C_{5.1}\, t^{-\frac{1}{\alpha}}.$$

Theorem 5.2. *There is a $C_{5.2} = C_{5.2}(\alpha) \in (0, \infty)$ such that for each $a \in \mathbb{R}$, $\varepsilon > 0$ and $t > 0$ it holds*

$$\mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[u_t(\mathbb{R}) > 0\right] \leq C_{5.2}\, \varepsilon\, t^{-\frac{1}{\alpha}}.$$

In particular

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[u_t(\mathbb{R}) > 0\right] \leq C_{5.2}\, t^{-\frac{1}{\alpha}}.$$

Remark 5.3. The results of Theorem 5.1 and Theorem 5.2 agree with the Brownian case $\alpha = 2$: As we mentioned in Section 4.1, the interface is described by a system of annihilating Brownian motions. If we start the voter model with $u_0 = \mathbf{1}_{[a,a+\varepsilon]}$ for some $a \in \mathbb{R}$ and $\varepsilon > 0$, the probability that opinion 1 is still alive at time $t > 0$ is the probability that two independent Brownian motions, started in a and $a+\varepsilon$, have not met up to time t. This probability is of order $\varepsilon t^{-\frac{1}{2}}$ (see Proposition 2.5).

Corollary 5.4. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ and assume* $\operatorname{supp}(u_0)$ *is bounded.* $\mathbf{P}_{u_0}$*-almost surely $\tau < \infty$ and $u_t(\mathbb{R}) = 0$ for all $t \geq \tau$.*

Proof. Since $\operatorname{supp}(u_0)$ is bounded, there exist $a, b \in \mathbb{R}$ with $a < b$ such that $u_0 \leq \mathbf{1}_{[a,b]}$, i.e. using duality (Lemma 3.3 and Corollary 3.6) for each $t > 0$, we have

$$\mathbf{P}_{u_0}\left[u_t(\mathbb{R}) > 0\right] \leq \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a,b] \neq \emptyset\right] = \mathbf{P}_{\mathbf{1}_{[a,b]}}\left[u_t(\mathbb{R}) > 0\right].$$

This implies using Theorem 5.2 (recall the lifetime τ from (5.1))

$$\begin{aligned}
\mathbf{P}_{u_0}\left[\tau < \infty\right] &= 1 - \lim_{n\to\infty} \mathbf{P}_{u_0}\left[\tau > n\right] \\
&= 1 - \lim_{n\to\infty} \mathbf{P}_{u_0}\left[u_n(\mathbb{R}) > 0\right] \\
&\geq 1 - \lim_{n\to\infty} \mathbf{P}_{\mathbf{1}_{[a,b]}}\left[u_n(\mathbb{R}) > 0\right] \\
&\geq 1 - C_{5.2}(b-a) \lim_{n\to\infty} n^{-\frac{1}{\alpha}} \\
&= 1.
\end{aligned}$$

This means $\tau < \infty$ $\mathbf{P}_{u_0}$-almost surely. Since the process is Feller (see Theorem 3.1) it has the strong Markov property (see [Ka97, Theorem 17.17]). This implies for each $t > 0$, that

$$\mathbf{E}_{u_0}\left[u_{\tau+t}(\mathbb{R})\right] = \mathbf{E}_{\mathbf{1}_{\emptyset}}\left[u_t(\mathbb{R})\right] = 0$$

due to the moment duality from Theorem 3.1. Therefore we get $u_t(\mathbb{R}) = 0$ for each $t \geq \tau$ $\mathbf{P}_{u_0}$-almost surely. □

Conjecture 5.5. It is possible that the statement of Corollary 5.4 is also true under the weaker assumption $u_0(\mathbb{R}) < \infty$ instead of $\operatorname{supp}(u_0)$ being bounded. However, it is not clear how to prove it. $u_0(\mathbb{R}) < \infty$ allows that the initial mass of opinion 1 may not lie in a compact interval, but is scattered over the real line. It is technically difficult to handle this situation, since there could be arbitrarily many colonies that are arbitrarily far away from the origin: On the one hand smaller colonies extinct faster. But on the other hand, each of the small colonies has a chance to get large. The same question arises in Theorem 3.9, where the boundedness of the support was discussed.

5.2. Lower bound for the survival probability

The aim of this section is to prove Theorem 5.1, i.e., we want to get a lower bound on the survival probability at a fixed time $t > 0$. To this end we use the Paley-Zygmund inequality (see [MP10, Lemma 3.23]) and first and second moment bounds of the total mass $u_t(\mathbb{R})$.

Lemma 5.6. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$. For each $t > 0$, we have*

$$\mathbf{E}_{u_0}\left[u_t(\mathbb{R})\right] = u_0(\mathbb{R}).$$

If $u_0(\mathbb{R}) < \infty$, then $(u_t(\mathbb{R}))_{t\geq 0}$ is a martingale.

Proof. Let $t > 0$, let $A \subset \mathbb{R}$ be a measurable bounded set and let $(X_t)_{t\geq 0}$ be a standard stable process. We use the moment duality (Theorem 3.1) and Fubini's theorem to compute

$$\begin{aligned}
\mathbf{E}_{u_0}\left[u_t(A)\right] &= \int_A \mathbf{E}_{u_0}\left[u_t(x)\right]\,dx \\
&= \int_A \mathbf{E}_x\left[u_0(X_t)\right]\,dx \\
&= \int_A \int_{\mathbb{R}} u_0(y) p_t^{(\alpha)}(y-x)\,dy\,dx \\
&= \int_{\mathbb{R}} u_0(y) \int_A p_t^{(\alpha)}(x-y)\,dx\,dy
\end{aligned} \tag{5.2}$$

Now exhaust $\mathbb{R}$ with bounded intervals. We use the fact, that $p_t^{(\alpha)}(\cdot - y)$ is a probability density, to infer from (5.2) that

$$\mathbf{E}_{u_0}\left[u_t(\mathbb{R})\right] = \int_{\mathbb{R}} u_0(y)\,dy = u_0(\mathbb{R}).$$

Now assume $u_0(\mathbb{R}) < \infty$. What we have just shown implies that $\mathbf{E}_{u_0}\left[u_t(\mathbb{R})\right] < \infty$. Due to Theorem 3.1, $(u_t(\mathbb{R}))_{t\geq 0}$ is a Markov process with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$. Using this and (5.2) we get for measurable bounded $A \subset \mathbb{R}$ and an independent copy $(\widetilde{u}_t)_{t\geq 0}$ of $(u_t)_{t\geq 0}$ $\mathbf{P}_{u_0}$-almost surely for $0 \leq s \leq t$

$$\mathbf{E}_{u_0}\left[u_t(A)\,\middle|\,\mathcal{F}_s\right] = \mathbf{E}_{u_s}\left[\widetilde{u}_{t-s}(A)\right] = \int_{\mathbb{R}} u_s(y) \int_A p_{t-s}^{(\alpha)}(x-y)\,dx\,dy$$

Again exhausting $\mathbb{R}$ with bounded intervals, we conclude that $\mathbf{P}_{u_0}$-almost surely

$$\mathbf{E}_{u_0}\left[u_t(\mathbb{R})\,\middle|\,\mathcal{F}_s\right] = \int_{\mathbb{R}} u_s(y)\,dy = u_s(\mathbb{R}).$$

□

Lemma 5.7. *Let $B \subset \mathbb{R}$ be a measurable bounded set with $\lambda(B) > 0$ and $u_0 = \mathbf{1}_B$. For $t > 0$, we have*

$$\mathbf{E}_{u_0}\left[u_t(\mathbb{R})^2\right] \leq C_{5.7.1}\,\lambda(B)^2\,\mathrm{diam}(B)^{\alpha-1}\,t^{-(1-\frac{1}{\alpha})} + C_{5.7.2}\,\lambda(B)\,t^{\frac{1}{\alpha}}$$

for some constants $C_{5.7.1} = C_{5.7.1}(\alpha) \in (0,\infty)$ and $C_{5.7.2} = C_{5.7.2}(\alpha) \in (0,\infty)$.

Proof. We will use time reversal arguments. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two independent standard stable processes. We use the duality (Theorem 3.1) and the time reversal formula (2.5) from Proposition 2.4 to compute

$$
\begin{aligned}
\mathbf{E}_{u_0}\left[u_t(\mathbb{R})^2\right] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{E}_{u_0}\left[u_t(x)u_t(y)\right]\,dy\,dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{P}\left[\Xi_t^{\{x,y\}} \subset B\right]\,dy\,dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{P}_{(x,y)}\left[X_s \neq Y_s \text{ for all } s \leq t, X_t, Y_t \in B\right]\,dy\,dx \\
&\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{P}_{(x,y)}\left[X_s = Y_s \text{ for some } s \leq t, X_t \in B\right]\,dy\,dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{P}_{(x,y)}\left[X_{t-s} \neq Y_{t-s} \text{ for all } s \leq t, X_0, Y_0 \in B\right]\,dy\,dx \\
&\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{P}_{(x,y)}\left[X_{t-s} = Y_{t-s} \text{ for some } s \leq t, X_0 \in B\right]\,dy\,dx \\
&= \int_{B}\int_{B} \mathbf{P}_{(x,y)}\left[X_s \neq Y_s \text{ for all } s \leq t\right]\,dy\,dx \\
&\quad + \int_{B}\int_{\mathbb{R}} \mathbf{P}_{(x,y)}\left[X_s = Y_s \text{ for some } s \leq t\right]\,dy\,dx.
\end{aligned}
\tag{5.3}
$$

Since $(Y_t - X_t)_{t\geq 0}$ is a stable process with rate 2, the probability of non-hitting (or hitting) of Y and X up to time t is the same as the probability of non-hitting (or hitting) of 0 of an standard stable process started in $y-x$ up to time $2t$. Therefore, we can take use of Proposition 2.5 where we cited a bound for the non-hitting-probability and a formula for the hitting probability of a standard stable process: There exists a constant $C_{5.7.1} = C_{5.7.1}(\alpha) > 0$ such that

$$
\mathbf{P}_{(x,y)}\left[X_s \neq Y_s \text{ for all } s \leq t\right] \leq C_{5.7.1} t^{-\left(1-\frac{1}{\alpha}\right)} |y-x|^{\alpha-1}
$$

and

$$
\begin{aligned}
\mathbf{P}_{(x,y)}\left[X_s = Y_s \text{ for some } s \leq t\right] &= \int_0^1 \frac{(1-s)^{-\left(1-\frac{1}{\alpha}\right)} s^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)} \cdot \frac{p_1^{(\alpha)}\left((y-x)(2ts)^{-\frac{1}{\alpha}}\right)}{p_1^{(\alpha)}(0)}\,ds \\
&= t^{\frac{1}{\alpha}} \int_0^1 \frac{(1-s)^{-\left(1-\frac{1}{\alpha}\right)} s^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)} \cdot \frac{p_t^{(\alpha)}\left((y-x)(2s)^{-\frac{1}{\alpha}}\right)}{p_1^{(\alpha)}(0)}\,ds
\end{aligned}
$$

where we used the scaling property of the stable density in the last equality. Now we bound both summands in (5.3) separately. For the first summand, we get

$$
\begin{aligned}
&\int_B\int_B \mathbf{P}_{(x,y)}\left[X_s \neq Y_s \text{ for all } s \leq t\right]\,dy\,dx \\
&\leq C_{5.7.1} t^{-\left(1-\frac{1}{\alpha}\right)} \int_B\int_B |y-x|^{\alpha-1}\,dy\,dx \\
&\leq C_{5.7.1} \lambda(B)^2 \operatorname{diam}(B)^{\alpha-1} t^{-\left(1-\frac{1}{\alpha}\right)}.
\end{aligned}
$$

For the second summand we get for a constant $C_{5.7.3} = C_{5.7.3}(\alpha) > 0$ and $C_{5.7.2} := \alpha \cdot C_{5.7.3}$

$$\begin{aligned}
&\int_B \int_{\mathbb{R}} \mathbf{P}_{(x,y)} \left[X_s = Y_s \text{ for some } s \leq t \right] dy\, dx \\
&\quad = t^{\frac{1}{\alpha}} \int_B \int_0^1 \frac{(1-s)^{-\left(1-\frac{1}{\alpha}\right)} s^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(1-\frac{1}{\alpha}\right) p_1^{(\alpha)}(0)} \int_{\mathbb{R}} p_t^{(\alpha)}\left((y-x)(2s)^{-\frac{1}{\alpha}}\right) dy\, ds\, dx \\
&\quad = C_{5.7.3} t^{\frac{1}{\alpha}} \int_B \int_0^1 (1-s)^{-\left(1-\frac{1}{\alpha}\right)} \int_{\mathbb{R}} p_t^{(\alpha)}(z)\, dz\, ds\, dx \\
&\quad = C_{5.7.2} \lambda(B) t^{\frac{1}{\alpha}}.
\end{aligned}$$

We used

$$\int_0^1 (1-s)^{-\left(1-\frac{1}{\alpha}\right)} ds = \int_0^1 s^{-\left(1-\frac{1}{\alpha}\right)} ds = \alpha$$

in the last equality due to $\alpha \in (1, 2]$, i.e. $(1 - \frac{1}{\alpha}) \in (0, \frac{1}{2}]$. Finally, (5.3) and the last two computations yield

$$\mathbf{E}_{u_0}\left[u_t(\mathbb{R})^2\right] \leq C_{5.7.1} \lambda(B)^2 \text{diam}(B)^{\alpha-1} t^{-\left(1-\frac{1}{\alpha}\right)} + C_{5.7.2} \lambda(B) t^{\frac{1}{\alpha}}.$$

□

With the previous lemma we can now prove Theorem 5.1.

Proof of Theorem 5.1. We use the Paley–Zygmund inequality ([MP10, Lemma 3.23]), Lemma 5.6 and Lemma 5.7. We get with $u_0 = \mathbf{1}_B$ for some measurable $B \subset \mathbb{R}$ with $\lambda(B) > 0$

$$\mathbf{P}_{u_0}\left[u_t(\mathbb{R}) > 0\right] \geq \frac{\mathbf{E}_{u_0}\left[u_t(\mathbb{R})\right]^2}{\mathbf{E}_{u_0}\left[u_t(\mathbb{R})^2\right]} \geq \frac{\lambda(B)^2}{C_{5.7.1} \lambda(B)^2 \text{diam}(B)^{\alpha-1} t^{-\left(1-\frac{1}{\alpha}\right)} + C_{5.7.2} \lambda(B) t^{\frac{1}{\alpha}}} > 0.$$

Now let $t > 0$, $a \in \mathbb{R}$, $\varepsilon > 0$ and $u_0 = \mathbf{1}_{[a,a+\varepsilon]}$. We have

$$\begin{aligned}
\mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[u_t(\mathbb{R}) > 0\right] &\geq \frac{\varepsilon^2}{C_{5.7.1} \varepsilon^{\alpha+1} t^{-\left(1-\frac{1}{\alpha}\right)} + C_{5.7.2} \varepsilon t^{\frac{1}{\alpha}}} \\
&= \frac{\varepsilon}{C_{5.7.1} \varepsilon^{\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} + C_{5.7.2} t^{\frac{1}{\alpha}}},
\end{aligned}$$

thus

$$\begin{aligned}
\liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[u_t(\mathbb{R}) > 0\right] &\geq \lim_{\varepsilon \searrow 0} \frac{1}{C_{5.7.1} \varepsilon^{\alpha} t^{-\left(1-\frac{1}{\alpha}\right)} + C_{5.7.2} t^{\frac{1}{\alpha}}} \\
&= C_{5.7.2}^{-1} t^{-\frac{1}{\alpha}}.
\end{aligned}$$

□

5.3. Upper bound for the survival probability

In this section we prove Theorem 5.2, i.e. an upper bound for the survival probability, if we start the voter model with an initial condition of the form $u_0 = \mathbf{1}_{[a,a+\varepsilon]}$ for some $a \in \mathbb{R}$ and $\varepsilon > 0$. To explain informally the strategy of the proof assume for the moment $a = 0$ and $\varepsilon \in (0,1)$. Due to the scaling property of the total mass (see Lemma 3.8) it is enough to show that the survival probability at time $t = 1$ is bounded by a constant times ε. Using the moment duality (Corollary 3.6), we get

$$\mathbf{P}_{\mathbf{1}_{[0,\varepsilon]}}\left[u_1(\mathbb{R}) > 0\right] = \mathbf{P}\left[\Xi_1^{\mathbb{R}} \cap [0,\varepsilon] \neq \emptyset\right].$$

The idea is to split the unit interval into intervals of size ε and show that the events

$$\left\{\Xi_1^{\mathbb{R}} \cap [0,\varepsilon] \neq \emptyset\right\}, \left\{\Xi_1^{\mathbb{R}} \cap [\varepsilon,2\varepsilon] \neq \emptyset\right\}, \left\{\Xi_1^{\mathbb{R}} \cap [2\varepsilon,3\varepsilon] \neq \emptyset\right\}, \ldots$$

are negatively correlated. Since all this events have the same probability we obtain an upper bound for $\mathbf{P}\left[\Xi_1^{\mathbb{R}} \cap [0,\varepsilon] \neq \emptyset\right]$ provided that the probability

$$\mathbf{P}_{\mathbf{1}_{[0,1]}}\left[u_1(\mathbb{R}) > 0\right] = \mathbf{P}\left[\Xi_1^{\mathbb{R}} \cap [0,1] \neq \emptyset\right].$$

is non-trivial. We still know that the latter probability is positive (see Theorem 5.1), therefore we only need to show that it is smaller than one. That is, we have to show that the voter model dies out with positive probability. We start in Lemma 5.8 showing this. Then we state in Corollary 5.10 and Corollary 5.11 for the coalescing stable process, that hitting disjoint intervals is negatively correlated. Here we use that this is already known for coalescing random walks (see [Ar81], [NRS05]) and an approximation of the coalescing stable process by a coalescing random walk (see Section 2.3). In Lemma 5.12 we finally put the things together.

Lemma 5.8. *Let $a, b \in \mathbb{R}$ with $a < b$ and $u_0 = \mathbf{1}_{[a,b]}$. For $t > 0$, we have*

$$\mathbf{P}_{u_0}\left[u_t(\mathbb{R}) > 0\right] = \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a,b] \neq \emptyset\right] < 1.$$

Proof. Note that the first equality follows from duality (Corollary 3.6). The strategy of the proof is to decompose the real line into disjoint sets and to compare the coalescing system starting on the complete real line with independent coalescing systems starting in the sets of the partition. This allows us to get a lower bound on the non-hitting-probability of $[a,b]$ which can be written as an infinite product. Finally, we have to show that this infinite product is positive. Since we know from Lemma 2.16 how to bound the hitting probability of the coalescing system started in certain intervals far away from the origin, we will use these intervals for the decomposition: Let $\frac{\alpha-1}{2} < \eta < \alpha - 1$, $d := \frac{2}{\eta} > 1$.

For $r \in \mathbb{N}$, define

$$J_{r,1} := \left[-\sum_{j=1}^{r} j^d, -\sum_{j=1}^{r-1} j^d \right), \qquad J_{r,2} := \left[\sum_{j=1}^{r-1} j^d, \sum_{j=1}^{r} j^d \right)$$

and let $J_r := J_{r,1} \cup J_{r,2}$. According to Lemma 2.16, there exists $R_0 = R_0(\alpha, d, b-a) \in \mathbb{N}$ such that for each $t > 0$, there is a constant $C_{5.8} = C_{5.8}(\alpha, d, t) > 0$ such that for all $r \geq R_0$, we have

$$\mathbf{P}\Big[\Xi_t^{\overline{J_r}} \cap [a,b] \neq \emptyset\Big] \leq C_{5.8} r^{-\alpha} < 1. \tag{5.4}$$

Let $J_0 := \bigcup_{r=1}^{R_0-1} J_r$ and $t > 0$. Now we show $\mathbf{P}[\Xi_t^{\overline{J_0}} \cap [a,b] \neq \emptyset] < 1$: Due to [EMS13, Theorem 6.1] the set $\Xi_{t/2}^{\overline{J_0}}$ is finite $\mathbf{P}$-almost surely, hence

$$\mathbf{P}\left[\left|\Xi_{t/2}^{\overline{J_0}}\right| = 1, \Xi_{t/2}^{\overline{J_0}} \subset [b+1, \infty)\right] > 0. \tag{5.5}$$

Let $(X_t)_{t \geq 0}$ be a standard stable process. Using the Markov property of the coalescing system (see Corollary 2.13) we have

$$\mathbf{P}\Big[\Xi_t^{\overline{J_0}} \cap (-\infty, b] \neq \emptyset \,\Big|\, \left|\Xi_{t/2}^{\overline{J_0}}\right| = 1, \Xi_{t/2}^{\overline{J_0}} \subset [b+1, \infty)\Big] \leq \mathbf{P}_{b+1}\Big[X_{t/2} \leq b\Big] < 1. \tag{5.6}$$

Note that for two arbitrary events A and B with $\mathbf{P}[A] > 0$ and $\mathbf{P}[B \,|\, A] < 1$, we have

$$\mathbf{P}[B] \leq \mathbf{P}[B \,|\, A] \cdot \mathbf{P}[A] + \mathbf{P}[A^c] < 1.$$

Hence from (5.5) and (5.6), we may infer

$$\mathbf{P}\Big[\Xi_t^{\overline{J_0}} \cap [a,b] \neq \emptyset\Big] \leq \mathbf{P}\Big[\Xi_t^{\overline{J_0}} \cap (-\infty, b] \neq \emptyset\Big] < 1. \tag{5.7}$$

Fix $N \in \mathbb{N}$ with $N \geq R_0$ and let

$$A^{(N)} := J_0 \cup \left(\bigcup_{r=R_0}^{N} J_r \right).$$

We modify the coalescing system $\Xi^{\overline{A^{(N)}}}$ in the following way: Let $\widetilde{\Xi}^{\overline{A^{(N)}}}$ be the coalescing system where two particles are only allowed to coalesce if both particles started in the same set J_r with $r \in \{0, R_0, R_0+1, \ldots, N\}$. Since the coalescing stable process is constructed using independent stable processes we get that the paths of different particles in this new coalescing system are independent if they start in different intervals J_r with $r \in \{0, R_0, R_0+1, \ldots, N\}$. Due to [EMS13, Section 2.3] it is possible to construct these systems such that $\Xi_t^{\overline{A(N)}} \subset \widetilde{\Xi}_t^{\overline{A(N)}}$ **P**-almost surely for $t \geq 0$. This implies

$$\begin{aligned}
\mathbf{P}\Big[\Xi_t^{\overline{A(N)}} \cap [a,b] = \emptyset\Big] &\geq \mathbf{P}\Big[\widetilde{\Xi}_t^{\overline{A(N)}} \cap [a,b] = \emptyset\Big] \\
&= \mathbf{P}\Big[\Xi_t^{\overline{J_0}} \cap [a,b] = \emptyset\Big] \cdot \prod_{r=R_0}^{N} \mathbf{P}\Big[\Xi_t^{\overline{J_r}} \cap [a,b] = \emptyset\Big] \\
&\geq \Big(1 - \mathbf{P}\Big[\Xi_t^{\overline{J_0}} \cap [a,b] \neq \emptyset\Big]\Big) \cdot \prod_{r=R_0}^{N} \left(1 - Cr^{-\alpha}\right)
\end{aligned}$$

where we used (5.4) in the last inequality. Since all factors in the latter product are positive (according to (5.7) and (5.4)) and since $\sum_{r=R_0}^{\infty} r^{-\alpha} < \infty$ due to $\alpha \in (1,2]$ the product converges to a positive limit as $N \longrightarrow \infty$ and we get

$$\begin{aligned}
\mathbf{P}\Big[\Xi_t^{\mathbb{R}} \cap [a,b] = \emptyset\Big] &= \lim_{N\to\infty} \mathbf{P}\Big[\Xi_t^{\overline{A(N)}} \cap [a,b] = \emptyset\Big] \\
&\geq \Big(1 - \mathbf{P}\Big[\Xi_t^{\overline{J_0}} \cap [a,b] \neq \emptyset\Big]\Big) \cdot \prod_{r=R_0}^{\infty} \left(1 - Cr^{-\alpha}\right) > 0.
\end{aligned}$$

This implies as desired

$$\mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a,b] \neq \emptyset\right] < 1.$$

□

One important tool for proving the upper bound for the survival probability is to show that in the dual picture hitting two disjoint intervals is negatively correlated. This is known for coalescing (recurrent) random walks due to [Ar81, Lemma 1] in continuous time and due to [NRS05, Lemma 2.8] in discrete time. Recall that a random walk is recurrent if it almost surely returns to its starting point. We state now the discrete time version.

Proposition 5.9 ([NRS05])**.** *Let $(\Pi_k^A)_{k\in\mathbb{N}_0}$ be the set-valued coalescing system of some recurrent random walk on $\mathbb{Z}$ starting in some finite set $A \subset \mathbb{Z}$. Let $B, C \subset \mathbb{Z}$ be two disjoint sets. For $k \in \mathbb{N}$ we have*

$$\begin{aligned}
&\mathbf{P}\left[\Pi_k^A \cap B \neq \emptyset,\, \Pi_k^A \cap C \neq \emptyset\right] \\
&\qquad \leq \mathbf{P}\left[\Pi_k^A \cap B \neq \emptyset\right] \cdot \mathbf{P}\left[\Pi_k^A \cap C \neq \emptyset\right].
\end{aligned}$$

Proof. This is [NRS05, Lemma 2.8]. □

Since we discussed in Section 2.3 the approximation of the coalescing stable process by certain coalescing random walks, the negative-correlation property from the last proposition thus transfers to the coalescing stable process.

Corollary 5.10. *Let $A \subset \mathbb{R}$ be a nonempty and closed set. Further let $t > 0$ and $a, b, c, d \in \mathbb{R}$ with $a < b \leq c < d$. We have*

$$\begin{aligned}\mathbf{P}\left[\Xi_t^A \cap [a,b] \neq \emptyset,\, \Xi_t^A \cap [c,d] \neq \emptyset\right] \\ \leq \mathbf{P}\left[\Xi_t^A \cap [a,b] \neq \emptyset\right] \cdot \mathbf{P}\left[\Xi_t^A \cap [c,d] \neq \emptyset\right].\end{aligned}$$

Proof. We will assume $b < c$, since the coalescing system hits fixed points with probability zero. For finite sets A the result follows from Proposition 5.9 via approximation with coalescing random walks (see Corollary 2.23). If A is not finite this holds using approximations with finite sets (see Remark 2.12). □

Corollary 5.11. *Let $A \subset \mathbb{R}$ be a nonempty and closed set. Further let $t > 0$ and $a, b, c, d \in \mathbb{R}$ with $a < b \leq c < d$. We have*

$$\begin{aligned}\mathbf{P}\left[\Xi_t^A \cap [a,b] = \emptyset,\, \Xi_t^A \cap [c,d] = \emptyset\right] \\ \leq \mathbf{P}\left[\Xi_t^A \cap [a,b] = \emptyset\right] \cdot \mathbf{P}\left[\Xi_t^A \cap [c,d] = \emptyset\right].\end{aligned}$$

Proof. Note that for events A_1 and A_2 with $\mathbf{P}[A_1 \cap A_2] \leq \mathbf{P}[A_1] \cdot \mathbf{P}[A_2]$ we have

$$\begin{aligned}\mathbf{P}[A_1^c \cap A_2^c] &= 1 - \mathbf{P}[A_1 \cup A_2] \\ &= 1 - (\mathbf{P}[A_1] + \mathbf{P}[A_2] - \mathbf{P}[A_1 \cap A_2]) \\ &= 1 - \mathbf{P}[A_1] - \mathbf{P}[A_2] + \mathbf{P}[A_1 \cap A_2] \\ &\leq 1 - \mathbf{P}[A_1] - \mathbf{P}[A_2] + \mathbf{P}[A_1] \cdot \mathbf{P}[A_2] \\ &= (1 - \mathbf{P}[A_1]) \cdot (1 - \mathbf{P}[A_2]) \\ &= \mathbf{P}[A_1^c] \cdot \mathbf{P}[A_2^c]\end{aligned}$$

and so the assertion follows from Corollary 5.10. □

Now we are ready to show that the survival probability of the voter model of a colony of size ε at some fixed time is bounded by a constant times ε. This is an application of the negative-correlation-property from Corollary 5.11.

Lemma 5.12. *Let $t > 0$. There is $C_{5.12} = C_{5.12}(\alpha, t) > 0$ such that for each $a \in \mathbb{R}$ and $\varepsilon > 0$ it holds with $u_0 = \mathbf{1}_{[a,a+\varepsilon]}$*

$$\mathbf{P}_{u_0}\left[u_t(\mathbb{R}) > 0\right] = \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a, a+\varepsilon] \neq \emptyset\right] \leq C_{5.12}\varepsilon.$$

Proof. The first equality follows from duality (Corollary 3.6). Now assume $\varepsilon \in (0,1)$ (at the end we can replace the constant $C_{5.12}$ by $\max\{C_{5.12}, 1\}$ to cover the case $\varepsilon \geq 1$). For $i \in \{1, 2, \ldots, \lceil \varepsilon^{-1} \rceil\}$ define $B_i^{(\varepsilon)} := [a + (i-1) \cdot \varepsilon, a + i \cdot \varepsilon]$. Then

$$\bigcup_{i=1}^{\lceil \varepsilon^{-1} \rceil} B_i^{(\varepsilon)} \subset [a, a+2].$$

Note that due to Remark 3.7

$$\mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B_j^{(\varepsilon)} \neq \emptyset\right] = \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B_i^{(\varepsilon)} \neq \emptyset\right]$$

for all $i, j \in \{1, 2, \ldots, \lceil \varepsilon^{-1} \rceil\}$. Using these two facts and applying Corollary 5.11 iteratively gives

$$\begin{aligned}
\mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a, a+2] \neq \emptyset\right] &\geq \mathbf{P}\left[\bigcup_{i=1}^{\lceil \varepsilon^{-1} \rceil} \left\{\Xi_t^{\mathbb{R}} \cap B_i^{(\varepsilon)} \neq \emptyset\right\}\right] \\
&= 1 - \mathbf{P}\left[\bigcap_{i=1}^{\lceil \varepsilon^{-1} \rceil} \left\{\Xi_t^{\mathbb{R}} \cap B_i^{(\varepsilon)} = \emptyset\right\}\right] \\
&\geq 1 - \prod_{i=1}^{\lceil \varepsilon^{-1} \rceil} \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B_i^{(\varepsilon)} = \emptyset\right] \\
&= 1 - \prod_{i=1}^{\lceil \varepsilon^{-1} \rceil} \left(1 - \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B_i^{(\varepsilon)} \neq \emptyset\right]\right) \\
&= 1 - \left(1 - \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B_1^{(\varepsilon)} \neq \emptyset\right]\right)^{\lceil \varepsilon^{-1} \rceil}.
\end{aligned}$$

This implies

$$\left(1 - \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a, a+\varepsilon] \neq \emptyset\right]\right)^{\lceil \varepsilon^{-1} \rceil} \geq 1 - \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a, a+2] \neq \emptyset\right].$$

Define

$$p := p(\alpha, t) := \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a, a+2] \neq \emptyset\right]$$

and

$$p_\varepsilon := p_\varepsilon(\alpha, t) := \mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap [a, a+\varepsilon] \neq \emptyset\right].$$

We have $p, p_\varepsilon \in (0, 1)$, since we know from duality (Corollary 3.6) and Theorem 5.1 that hitting probabilities are positive and due to Lemma 5.8 that they are smaller than one. With this notation the last inequality can be shortly written as

$$(1 - p_\varepsilon)^{\lceil \varepsilon^{-1} \rceil} \geq 1 - p.$$

Let $\widetilde{\varepsilon} := \lceil \varepsilon^{-1} \rceil^{-1} \leq \varepsilon$. Then

$$1 - p_\varepsilon \geq (1 - p)^{\widetilde{\varepsilon}}.$$

Hence (using $1 - e^{-x} \leq x$ for $x \in \mathbb{R}$)

$$\begin{aligned}
\mathbf{P}\left[\Xi_t^{\mathbb{R}} \cap B \neq \emptyset\right] &= p_\varepsilon \\
&\leq 1 - (1-p)^{\widetilde{\varepsilon}} = 1 - e^{-\ln\left(\frac{1}{1-p}\right)\widetilde{\varepsilon}} \leq \ln\left(\frac{1}{1-p}\right)\widetilde{\varepsilon} \leq \ln\left(\frac{1}{1-p}\right)\varepsilon.
\end{aligned}$$

So we can choose $C_{5.12} := C_{5.12}(\alpha, t) := \max\left\{\ln((1 - p(\alpha, t))^{-1}), 1\right\}$. □

Now we can conclude the proof of Theorem 5.2.

Proof of Theorem 5.2. Consider the constant $C_{5.12}(\alpha, 1)$ from Lemma 5.12. With the scaling property of the total mass (Lemma 3.8) in the second equality we get for each $a \in \mathbb{R}$, $\varepsilon > 0$ and $t > 0$

$$\begin{aligned}\mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[u_t(\mathbb{R}) > 0\right] &= \mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}}\left[t^{-\frac{1}{\alpha}} u_t(\mathbb{R}) > 0\right] \\ &= \mathbf{P}_{\mathbf{1}_{t^{-1/\alpha}[a,a+\varepsilon]}}\left[u_1(\mathbb{R}) > 0\right] \\ &\le C_{5.12}(\alpha, 1)\varepsilon t^{-\frac{1}{\alpha}}.\end{aligned}$$

□

6. A toy model for exceptional time points

6.1. Description of the model and main result

In Theorem 3.9 we have seen that the support of the voter model stays almost surely bounded at fixed deterministic times if the initial condition has bounded support. A question to ask is whether there are random times when the support is unbounded. Those time points we call *exceptional time points*. The aim of this chapter is to compute the Hausdorff dimension of that set of time points in a simpler toy model. In Section 8.2 we discuss and formulate conjectures for the real voter model.

Having the discrete voter model in mind one would like to think of the long-range voter model on the real line as distributing infinitesimally small colonies of opinion 1 (according to a stable transition kernel) in space. Therefore we want to consider a toy model where colonies are represented by random time intervals (having as length the lifetime of a voter colony) and the intervals are placed in the space-time-plane according to a Poisson point process. Since each time interval has a starting time point, a spatial position and a length or lifetime, we consider a Poisson point process N on $[0,1] \times \mathbb{R} \times (0,1)$ with intensity measure

$$\mu^{(\alpha,\gamma)}(ds, dx, du) = \lambda(ds) \otimes \nu^{(\alpha)}(dx) \otimes u^{-\gamma}\, du.$$

with $\nu^{(\alpha)}(dx) = p_1^{(\alpha)}(x)\, dx$, $\alpha \in (1,2]$ and $\gamma \in (1,2)$. Due to Theorem 5.1 and Theorem 5.2 for each $a \in \mathbb{R}$, the voter model satisfies

$$\liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}} [u_t(\mathbb{R}) > 0] \geq C_{5.1} t^{-\frac{1}{\alpha}}$$

and

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbf{P}_{\mathbf{1}_{[a,a+\varepsilon]}} [u_t(\mathbb{R}) > 0] \leq C_{5.2} t^{-\frac{1}{\alpha}}.$$

This motivates to consider $\gamma = 1 + \frac{1}{\alpha}$ (since the derivative of $t^{-\frac{1}{\alpha}}$ is a multiple of $t^{-(1+\frac{1}{\alpha})}$). However, we allow a general $\gamma \in (1,2)$. Further we only consider interval lengths in $(0,1)$ as the small colonies typically live for only a short time. We refer to the introduced point process N as the toy model for the long-range voter model on the real line.

Now we define the set of exceptional time points for the toy model. Let

$$\begin{aligned} E_T &= \{t \in [0,1] \,:\, t \text{ is contained in an interval infinitely far away from the origin}\} \\ &= \{t \in [0,1] \,:\, N\left(\{(s,x,u) \,:\, s < t,\, |x| \geq n, u > t-s\}\right) \geq 1 \text{ for each } n \in \mathbb{N}\}\,. \end{aligned}$$

The main result of this chapter is the following.

Theorem 6.1. *We have*

$$\dim_{\mathcal{H}}(E_T) = \gamma - 1 \qquad a.s.$$

Remark 6.2. If we choose $\gamma = 1 + \frac{1}{\alpha}$, which seems to be a plausible choice for the voter model as described above, Theorem 6.1 gives

$$\dim_{\mathcal{H}}(E_T) = \frac{1}{\alpha}.$$

6.2. Proof of Theorem 6.1

In this section we prove the dimension result from Theorem 6.1. The main idea is to compare the exceptional time points with the time points that are contained in infinitely many intervals because the Hausdorff dimension of this set is known due to results of [FW04] and essentially depends on the decay of the lengths of the intervals. In Lemma 6.3 and Corollary 6.4 we discuss the connection to the time points that are covered by infinitely many intervals via introducing some modified point process. In Lemma 6.5 we determine the order of the decay of the interval length and in Corollary 6.6 we finally compute the dimension of the time points that are infinitely often covered for the modified process. This finally allows us to get the dimension formula of the exceptional time points in Theorem 6.1.

We see in the next lemma that N produces with positive probability infinitely many intervals in each spatial set even in bounded sets. To get a lower bound on the Hausdorff dimension we want to distinguish between exceptional time points and the time points that are covered by infinitely many intervals. To do this, we throw away some intervals. Only those intervals remain included where the interval length does not become too small. The lengths are coupled to the spatial position in such a way that the further away the intervals are from the origin, the shorter they are allowed to be. The resulting point process still produces infinitely many intervals on the total real line, but the number of intervals in each bounded set is almost surely finite. Thus for that point process the exceptional time points and the time points that are covered by infinitely many intervals coincide almost surely. Formally, we introduce a point process $N^{(\eta)}$ for some $\eta \in [0, \gamma - 1)$ defined by

$$N^{(\eta)} := N\left(\cdot \cap B^{(\alpha,\gamma,\eta)}\right)$$

with

$$B^{(\alpha,\gamma,\eta)} = \left\{(s,x,u) \in [0,1] \times \mathbb{R} \times (0,1) \,:\, \nu^{(\alpha)}((-\infty,-x] \cup [x,\infty)) < 2u^{\eta}\right\}.$$

This means $N^{(\eta)}$ counts only the points (s, x, u) of N that fulfill $\nu^{(\alpha)}((-\infty, -x] \cup [x, \infty)) < 2u^\eta$. Note that $N^{(0)} = N$. In the following lemma we collect some properties of the intensities of the introduced point processes that imply the latter statements. Recall that $\mu^{(\alpha,\gamma)}$ is the intensity measure of N.

Lemma 6.3. *Let* $\eta \in [0, \gamma - 1)$.

(i) $N^{(\eta)}$ *is a Poisson point process on* $[0,1] \times \mathbb{R} \times (0,1)$ *with intensity measure*

$$\mu^{(\alpha,\gamma,\eta)}(ds, dx, du) = \lambda(ds) \otimes \nu^{(\alpha)}(dx) \otimes u^{-\gamma}\, du\, \mathbf{1}_{B(\alpha,\gamma,\eta)}(s, x, u).$$

(ii) $\widetilde{N}^{(\eta)}$ *is a Poisson point process on* $[0,1] \times (0,1)$ *with intensity measure*

$$\widetilde{\mu}^{(\alpha,\gamma,\eta)}(ds, du) = \lambda(ds) \otimes u^{-(\gamma-\eta)}\, du.$$

(iii) $\mu^{(\alpha,\gamma)}([0,1] \times A \times (0,1)) = \infty$ *for each measurable set* $A \subset \mathbb{R}$ *with* $\nu^{(\alpha)}(A) > 0$.

(iv) $\mu^{(\alpha,\gamma,\eta)}([0,1] \times \mathbb{R} \times (0,1)) = \infty$.

(v) If $\eta \in (0, \gamma - 1)$, *then* $\mu^{(\alpha,\gamma,\eta)}([0,1] \times A \times (0,1)) < \infty$ *for each bounded measurable set* $A \subset \mathbb{R}$.

Proof.

(i) This follows from the definition of $N^{(\eta)}$ since

$$N^{(\eta)}(ds, dx, du) = \mathbf{1}_{B(\alpha,\gamma,\eta)}(s, x, u)\, N(ds, dx, du).$$

(ii) Due to the mapping theorem for Poisson point processes (see [Kl13, Theorem 24.16]) $\widetilde{N}^{(\eta)}$ is a Poisson point process on $[0,1] \times (0,1)$ with intensity measure $\widetilde{\mu}^{(\alpha,\gamma,\eta)} = \mu^{(\alpha,\gamma,\eta)} \circ f^{-1}$. Thus we compute for $s_1, s_2 \in [0,1]$ with $s_1 < s_2$ and $u_1, u_2 \in (0,1)$ with $u_1 < u_2$

$$\begin{aligned}
&\widetilde{\mu}^{(\alpha,\gamma,\eta)}([s_1, s_2] \times [u_1, u_2]) \\
&\quad = \mu^{(\alpha,\gamma,\eta)}([s_1, s_2] \times \mathbb{R} \times [u_1, u_2]) \\
&\quad = \int_{s_1}^{s_2} \lambda(ds) \int_{\mathbb{R}} \nu^{(\alpha)}(dx) \int_{u_1}^{u_2} du\, u^{-\gamma} \mathbf{1}_{B(\alpha,\gamma,\eta)}(s, x, u) \\
&\quad = \int_{s_1}^{s_2} \lambda(ds) \int_{u_1}^{u_2} \nu^{(\alpha)} \left(\left\{ x \in \mathbb{R} : \nu^{(\alpha)}((-\infty, -x] \cup [x, \infty)) < 2u^\eta \right\} \right) u^{-\gamma}\, du \\
&\quad = \int_{s_1}^{s_2} \lambda(ds) \int_{u_1}^{u_2} u^{-(\gamma-\eta)}\, du
\end{aligned}$$

where we used

$$\begin{aligned}&\nu^{(\alpha)}\left(\left\{x\in\mathbb{R}\,:\,\nu^{(\alpha)}((-\infty,-x]\cup[x,\infty))<2u^{\eta}\right\}\right)\\&=\nu^{(\alpha)}\left(\left\{x\in\mathbb{R}\,:\,\nu^{(\alpha)}([x,\infty))<u^{\eta}\right\}\right)\\&=u^{\eta}\end{aligned}$$

in the last equality. Thus the intensity measure has the stated form.

(iii) Let $A\subset\mathbb{R}$ be a measurable set. We compute

$$\mu^{(\alpha,\gamma)}([0,1]\times A\times(0,1))=\nu^{(\alpha)}(A)\int_0^1 u^{-\gamma}\,du=\infty$$

since $\gamma>1$.

(iv) We compute analogously to the calculation in (ii)

$$\mu^{(\alpha,\gamma,\eta)}([0,1]\times\mathbb{R}\times(0,1))=\int_0^1 u^{-(\gamma-\eta)}\,du=\infty$$

since $\gamma-\eta>1$.

(v) Now let $\eta>0$ and assume $A\subset\mathbb{R}$ is a bounded measurable set. Choose $K>0$ such that $A\subset[-K,K]$. Then we get due to the definition of $B^{(\alpha,\gamma,\eta)}$

$$\begin{aligned}&\mu^{(\alpha,\gamma,\eta)}([0,1]\times A\times(0,1))\\&\le\mu^{(\alpha,\gamma,\eta)}([0,1]\times[-K,K]\times(0,1))\\&=\int_0^1\lambda(ds)\int_{-K}^{K}\nu^{(\alpha)}(dx)\int_{(\frac{1}{2}\nu^{(\alpha)}((-\infty,-x]\cup[x,\infty)))^{\frac{1}{\eta}}}^{1}du\,u^{-\gamma}\\&=\frac{1}{\gamma-1}\int_{-K}^{K}\nu^{(\alpha)}(dx)\left(\left(\frac{1}{2}\nu^{(\alpha)}((-\infty,-x]\cup[x,\infty))\right)^{-\frac{\gamma-1}{\eta}}-1\right)\\&\le\frac{\nu^{(\alpha)}([-K,K])}{\gamma-1}\left(\left(\frac{1}{2}\nu^{(\alpha)}((-\infty,-K]\cup[K,\infty))\right)^{-\frac{\gamma-1}{\eta}}-1\right)<\infty.\end{aligned}$$

□

Now let

$$E_T^{\eta,\infty}:=\left\{t\in[0,1]\,:\,\widetilde{N}^{(\eta)}\left(\{(s,u)\,:\,s<t,u>t-s\}\right)=\infty\right\}$$

be the set of time points (of $N^{(\eta)}$) that are contained in infinitely many intervals (note that this set does not depend on the spatial position of the intervals, this is why we consider $\widetilde{N}^{(\eta)}$ instead of $N^{(\eta)}$ in the definition of $E_T^{\eta,\infty}$). As a corollary to the previous lemma we can compare $E_T^{\eta,\infty}$ with the set E_T of exceptional time points of the original point process N.

Corollary 6.4. *For each $\eta \in (0, \gamma - 1)$, we have almost surely*

$$E_T^{\eta,\infty} \subset E_T \subset E_T^{0,\infty}.$$

Proof. First note that $E_T \subset E_T^{0,\infty}$ is clear: If a time point is an exceptional time point, then it is contained in infinitely many time intervals. For the other inclusion we use Lemma 6.3: Let

$$B := \left\{ N^{(\eta)}([0,1] \times [-n,n] \times (0,1)) < \infty \text{ for all } n \in \mathbb{N} \right\}.$$

Due to Lemma 6.3 (v) we have $\mathbf{P}[B] = 1$. Further let

$$E_T^{\eta} := \left\{ t \in [0,1] \,:\, N^{(\eta)}\left(\{(s,x,u) \,:\, s < t,\, |x| \geq n, u > t - s\}\right) \geq 1 \text{ for each } n \in \mathbb{N} \right\}$$

be the set of exceptional time points of $N^{(\eta)}$. On the one hand we have $E_T^{\eta} \subset E_T$ since the point process $N^{(\eta)}$ is constructed via throwing away points of N. On the other hand on B the time points (of $N^{(\eta)}$) that are contained in infinitely many intervals coincide with the exceptional time points (of $N^{(\eta)}$). Thus

$$B \subset \{E_T^{\eta,\infty} = E_T^{\eta} \subset E_T\}$$

and we get

$$\mathbf{P}\left[E_T^{\eta,\infty} \subset E_T\right] = 1.$$

□

The inclusions in Corollary 6.4 motivate to compute $\dim_{\mathcal{H}}(E_T^{\eta,\infty})$ for $\eta \in [0, \gamma - 1)$. Since $E_T^{\eta,\infty}$ does not depend on the spatial positions of the intervals we can think of placing random intervals in $[0,1]$. Covering problems like that have been studied first in [FW04]. They proved a formula for the Hausdorff dimension of the points on the torus that are covered by infinitely many random arcs. [FJJS18] considered this problem for more general sets than the torus. The results of [FW04] and [FJJS18] depend on the decay of the lengths of the intervals. We therefore order the points of $\widetilde{N}^{(\eta)}$ in descending order with respect to the second coordinate (the interval lengths). Denote the lengths by $\ell_1^{\eta} \geq \ell_2^{\eta} \geq \ell_3^{\eta} \geq \dots$.

Lemma 6.5. *For each $\eta \in [0, \gamma - 1)$, we have*

$$\frac{\ell_n^{\eta}}{n^{-\frac{1}{\gamma-\eta-1}}} \xrightarrow{n\to\infty} (\gamma - \eta - 1)^{-\frac{1}{\gamma-\eta-1}} \qquad a.s.$$

Proof. In this proof we consider $\widetilde{N}^{(\eta)}$ to be a point process on $[0,1] \times (0,\infty)$ instead of $[0,1] \times (0,1)$. Note that this does not change the order of decay of lengths of the intervals since there are only finitely many intervals with a length that is larger than 1. Let

$$g \colon [0,1] \times (0,\infty) \to (0,\infty), \qquad g((s,u)) = (\gamma - \eta - 1)^{-1} u^{-(\gamma-\eta-1)}$$

and define $\widehat{N}^{(\eta)} := \widetilde{N}^{(\eta)} \circ g^{-1}$. Due to the mapping theorem for Poisson point processes (see [Kl13, Theorem 24.16]) $\widehat{N}^{(\eta)}$ is a Poisson point process on $(0,\infty)$ with intensity measure $\widehat{\mu}^{(\alpha,\gamma,\eta)} = \widetilde{\mu}^{(\alpha,\gamma,\eta)} \circ g^{-1}$. This is the Lebesgue measure on $(0,\infty)$ since for $u_1, u_2 \in (0,\infty)$ with $u_1 < u_2$, we compute

$$\begin{aligned}
&\widehat{\mu}^{(\alpha,\gamma,\eta)}([u_1,u_2]) \\
&= \int_0^\infty du\, u^{-(\gamma-\eta)} \mathbf{1}_{[u_1,u_2]}((\gamma-\eta-1)^{-1} u^{-(\gamma-\eta-1)}) \\
&= \int_{((\gamma-\eta-1)u_2)^{\frac{1}{(\gamma-\eta-1)}}}^{((\gamma-\eta-1)u_1)^{\frac{1}{(\gamma-\eta-1)}}} u^{-(\gamma-\eta)}\, du \\
&= u_2 - u_1.
\end{aligned}$$

Let $k_n := g(0,\ell_n^\eta)$ for $n \in \mathbb{N}$. Since ℓ_n^η is the n-th largest point of $\widetilde{N}^{(\eta)}$, k_n is the n-th smallest point of $\widehat{N}^{(\eta)}$, which we have identified as a Poisson process on $(0,\infty)$ with rate 1. Therefore k_n is a sum of n independent exponentially distributed random variables with rate 1. Thus the strong law of large numbers (see [Kl13, Theorem 5.17]) yields

$$\frac{k_n}{n} \xrightarrow{n\to\infty} 1 \qquad \text{a.s.}$$

Since $k_n = (\gamma-\eta-1)^{-1} (\ell_n^\eta)^{-(\gamma-\eta-1)}$, the result follows. □

Corollary 6.6. *For each $\eta \in [0,\gamma-1)$ we have almost surely*

$$\dim_{\mathcal{H}}(E_T^{\eta,\infty}) = \gamma - \eta - 1.$$

Proof. Let $\eta \in [0,\gamma-1)$. Due to Lemma 6.5 there is a constant $C_{6.6} = C_{6.6}(\eta,\gamma) > 0$ such that

$$\ell_n^\eta \sim C_{6.6} n^{-\frac{1}{\gamma-\eta-1}} \quad \text{as } n \longrightarrow \infty.$$

Note that $\frac{1}{\gamma-\eta-1} > 1$ since $\gamma-\eta-1 \le \gamma-1 < 1$ (due to the assumption $\gamma < 2$). Therefore almost surely

$$\sum_{n=1}^\infty \ell_n^\eta < \infty$$

Using [FJJS18, Theorem 1.1] (which is the general version of the result of [FW04]), we get

$$\dim_{\mathcal{H}}(E_T^{\eta,\infty}) = \gamma - \eta - 1 \qquad \text{a.s.}$$

□

Proof of Theorem 6.1. Let $\eta \in (0,\gamma-1)$. Apply Corollary 6.4 and Corollary 6.6 to infer

$$\gamma - \eta - 1 \le \dim_{\mathcal{H}}(E_T) \le \gamma - 1 \qquad \text{a.s.}$$

This implies the desired result as $\eta \searrow 0$. □

7. The finite rate model - Existence, duality, convergence

7.1. Overview

Consider the following stochastic partial differential equation (SPDE)

$$\partial_t u_t^{[\gamma]}(x) = \left(\mathcal{L}_\alpha u_t^{[\gamma]}\right)(x) + \sqrt{\gamma u_t^{[\gamma]}(x)(1-u_t^{[\gamma]}(x))}\dot{W}_{t,x}, \qquad t \geq 0,\, x \in \mathbb{R} \tag{7.1}$$

where $\mathcal{L}_\alpha := -(-\Delta)^{\alpha/2}$ is the fractional Laplacian operator with $\alpha \in (1,2]$, $\gamma > 0$ and $\dot{W}$ is a space-time white noise. The aim is to show that (7.1) has a unique weak solution and that the long-range voter model on the real line arises as a limit of that SPDE as $\gamma \to \infty$. For $\gamma < \infty$, we call the solution of (7.1) the *finite rate model* with rate γ or the *continuous-space stepping stone model* with α-stable motion.

In the case $\alpha = 2$, the existence of solutions of (7.1), taking values in the space of continuous functions from $\mathbb{R}$ to $[0,1]$, was investigated in [Re89] and [Sh94]. Uniqueness in distribution follows from a moment duality with delayed coalescing Brownian motions (see [Sh88]). [EVY20] shows for the general situation $\alpha \in (1,2]$ that the finite rate model appears as a scaling limit of a spatial Lambda-Fleming-Viot process.

[Tr95] examined the convergence in distribution of the interface of a space-time-rescaling of the finite rate model in the Brownian case $\alpha = 2$. For suitable initial conditions, sending the scaling parameter to ∞ is equivalent to sending $\gamma \to \infty$ (see [BHO16, Section 1.1] and [EF04, Lemma 8]). In [HOV18, Theorem 2.8, Remark 2.9] and [BO21, Proposition 5.4] it was shown that solutions of (7.1) converge to the nearest-neighbour voter model on the real line as $\gamma \to \infty$. The approximating process and the limiting process were considered as continuous processes taking values in the Radon measures. Convergence was shown with respect to the uniform convergence on compact sets, endowing the space of Radon measures with the topology of vague convergence.

In this chapter several function and measure spaces appear, therefore we introduce some notations. Let E be a metric space and I an interval. Define

- $\mathcal{B}(E) = \sigma(\{A \subset E \,:\, A \text{ is open}\})$
- $\mathcal{C}(E) = \{f \colon E \to \mathbb{R} \,:\, f \text{ is continuous}\}$
- $\mathcal{C}^\infty(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) \,:\, f \text{ is infinitely often continuously differentiable}\}$

- $\mathcal{C}_b(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) \,:\, f \text{ is bounded}\}$
- $\mathcal{C}_0(\mathbb{R}) = \left\{f \in \mathcal{C}(\mathbb{R}) \,:\, \lim_{|x|\to\infty} f(x) = 0\right\}$
- $\mathcal{C}_c(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) \,:\, f \text{ has compact support}\}$
- $\mathcal{C}_c^\infty(\mathbb{R}) = \mathcal{C}^\infty(\mathbb{R}) \cap \mathcal{C}_c(\mathbb{R})$
- $\mathcal{C}(I, E) = \{f\colon I \to E \,:\, f \text{ is continuous}\}$
- $D([0,\infty), E) = \{f\colon [0,\infty) \to E \,:\, f \text{ is càdlàg}\}$
- $\mathcal{M}(E) = \{\mu\colon \mathcal{B}(E) \to [0,\infty] \,:\, \mu \text{ is a Radon measure}\}$
- $\mathcal{M}_1(E) = \{\mu\colon \mathcal{B}(E) \to [0,1] \,:\, \mu \text{ is a probability measure}\}$
- $\mathcal{M}_{\leq 1}(\mathbb{R}) = \{u(x)\,dx \,:\, u\colon \mathbb{R} \to [0,1] \,\text{measurable}\} \subset \mathcal{M}(\mathbb{R})$

If $X, X_1, X_2, \ldots$ are random variables with values in E, we write $X_n \xRightarrow{n\to\infty} X$ if $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X. Let E be a Polish space. Recall from the beginning of Section 3.1 that we topologize $\mathcal{M}(E)$ and $\mathcal{M}_{\leq 1}(\mathbb{R})$ with the topology of vague convergence. $\mathcal{M}(E)$ is again a Polish space (see [Ka86, Section 15.7]). We equip $\mathcal{C}(I, E)$ with the topology of uniform convergence on compact subsets of I and $D([0,\infty), E)$ with the Skorokhod topology (see [EK86, Section 3.5] or [JS03, Chapter VI]). If $E \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ and $p \geq 1$ let

$$L^p(E) = \left\{f\colon E \to \mathbb{R} \,:\, f \text{ is measurable with } \int_E |f(x)|^p \, dx < \infty\right\}$$

and for $f \in L^p(E)$ write

$$\|f\|_{L^p(E)} := \left(\int_E |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

For $f, g \in L^2(\mathbb{R})$ define

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)g(x)\,dx.$$

If μ is a measure and $f\colon \mathbb{R} \to \mathbb{R}$ measurable define

$$\mu(f) := \int_{\mathbb{R}} f(x)\,\mu(dx)$$

if the integral exists. Further write

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$$

if the latter is finite. Finally, we define

$$\left(S_0 f\right)(x) := f(x), \qquad \left(S_t f\right)(x) := \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x) f(y)\,dy,$$

if the integral exists. On $\mathcal{C}_0(\mathbb{R})$ the family of operators $(S_t)_{t\geq 0}$ is the corresponding semigroup of a standard stable process.

Before stating the main results we recap some facts that we need to give the above SPDE a precise formal meaning (see [IW89], [Wa86] and [DKM$^+$09] for further details): We say a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfied the usual conditions, if $\mathcal{F}_0$ contains each $\mathbf{P}$-zero set and if the filtration is right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$$

for each $t \geq 0$.

Let $\lambda^{(2)}$ be the Lebesgue measure on $\mathbb{R}^2$ and define

$$\mathcal{B}_f([0,\infty)\times\mathbb{R}) := \left\{ A \in \mathcal{B}([0,\infty)\times\mathbb{R}) \,:\, \lambda^{(2)}(A) < \infty \right\}.$$

A space-time white noise is an $(\mathcal{F}_t)_{t\geq 0}$-adapted centered Gaussian process $(W(A))_{A\in\mathcal{B}_f([0,\infty)\times\mathbb{R})}$ with

$$\mathbf{E}\left[W(A_1)W(A_2)\right] = \lambda^{(2)}(A_1 \cap A_2)$$

for all sets $A_1, A_2 \in \mathcal{B}_f([0,\infty)\times\mathbb{R})$. $(\mathcal{F}_t)_{t\geq 0}$-adapted means that $(W([0,t]\times A))_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ for each $A \in \mathcal{B}_f([0,\infty)\times\mathbb{R})$. We call a function $\psi\colon [0,\infty)\times\mathbb{R}\times\Omega \to \mathbb{R}$ elementary with respect to $(\mathcal{F}_t)_{t\geq 0}$ if

$$\psi(s,y,\omega) = X(\omega)\mathbf{1}_{(s_1,s_2]}(s)\mathbf{1}_A(y)$$

for some bounded and $\mathcal{F}_{s_1}$-measurable X, $0 \leq s_1 \leq s_2$ and $A \in \mathcal{B}(\mathbb{R})$. We call a function $\psi\colon [0,\infty)\times\mathbb{R}\times\Omega \to \mathbb{R}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable if it is measurable with respect to the σ-algebra generated by finite sums of elementary functions. In the sequel we ignore the dependence of ω in our notation. For every $(\mathcal{F}_t)_{t\geq 0}$-predictable function ψ with

$$\mathbf{E}\left[\int_0^t \int_{-\infty}^{\infty} \psi(s,y)^2\, dy\, ds\right] < \infty$$

one can define a stochastic integral $(\psi \cdot W)_t$ of ψ with respect to W at time $t \geq 0$ in the sense of John Walsh (see [Wa86, Theorem 2.5]) which is a continuous $((\mathcal{F}_t)_{t\geq 0})$-martingale with quadratic variation process

$$\langle \psi \cdot W \rangle_t = \int_0^t \int_{-\infty}^{\infty} \psi(s,y)^2\, dy\, ds.$$

We will use the notation

$$(\psi \cdot W)_t = \int_0^t \int_{-\infty}^{\infty} \psi(s,y)\, W(ds,dy).$$

Let $a \in \mathcal{C}_b(\mathbb{R})$. In the sequel we will show existence of so called *mild* solutions of SPDEs of type

$$\partial_t v_t(x) = (\mathcal{L}_\alpha v_t)(x) + a(v_t(x))\dot{W}_{t,x}, \qquad t \geq 0,\ x \in \mathbb{R}$$

for suitable functions a. By this we mean an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(v_t)_{t\geq 0}$ that fulfills $\mathbf{P}$-almost surely for each $t > 0$ and $x \in \mathbb{R}$

$$v_t(x) = \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x)v_0(y)\,dy + \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)a(v_s(y))\,W(ds,dy).$$

If there exists such a process $(v_t)_{t\geq 0}$ for each filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions and each $(\mathcal{F}_t)_{t\geq 0}$-adapted space-time white noise W, we say the solution is *strong* (in the sense of probability theory). If there exists a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions, an $(\mathcal{F}_t)_{t\geq 0}$-adapted space-time white noise W and an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(v_t)_{t\geq 0}$ satisfying the latter equation, we say it is a *weak* solution (in the sense of probability theory).

Now we present the two main results. Theorem 7.1 is partially included in [EVY20, Theorem 1.14], but we give a completely different proof.

Theorem 7.1. *Let $u_0 \colon \mathbb{R} \to [0,1]$ be a continuous function and fix $\gamma > 0$. Then there exists a mild solution to (7.1), which is weak in the sense of probability theory, i.e.: There exists a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions, a space-time white noise W and an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(u_t^{[\gamma]})_{t\geq 0}$ such that we have $\mathbf{P}$-almost surely for each $t > 0$ and $x \in \mathbb{R}$*

$$u_t^{[\gamma]}(x) = \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x)u_0^{[\gamma]}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\,W(ds,dy) \tag{7.2}$$

and $u_0^{[\gamma]} = u_0$. $(u_t^{[\gamma]})_{t\geq 0}$ is jointly continuous in both variables $t \in [0,\infty)$ and $x \in \mathbb{R}$. Furthermore the distribution of $(u_t^{[\gamma]})_{t\geq 0}$ is uniquely determined and it is a strong Markov process that takes values in $\mathcal{C}([0,\infty) \times \mathbb{R})$.

Theorem 7.2. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ with continuous density. Let $(u_t^{[\gamma]})_{t\geq 0}$ be the weak solution of (7.1) from Theorem 7.1 with initial condition u_0. Write $u_t^{[\gamma]}(dx) = u_t^{[\gamma]}(x)\,dx$. Further let $(u_t)_{t\geq 0}$ be the long-range voter model on the real line from Theorem 3.1 with initial condition u_0. Then the convergence*

$$u^{[\gamma]} \xRightarrow{\gamma\to\infty} u \qquad \text{in } \mathcal{C}((0,\infty), \mathcal{M}_{\leq 1}(\mathbb{R})).$$

holds.

The proof strategy of the theorems is as follows: We show in Section 7.2 existence of a jointly continuous strong solution to (7.1) in the case of Lipschitz-coefficients instead of the square root term. In Section 7.3 we obtain a weak solution to (7.1) via approximation by solutions with Lipschitz-coefficients. This proves the existence-part in Theorem 7.1.

The uniqueness we obtain in Section 7.4 via showing a moment duality of the finite rate model with a coalescing system of stable processes where the processes coalesce with some delay. In order to prove Theorem 7.2 we show tightness in Section 7.5 and finally we identify sub-sequential limits as $\gamma \to \infty$ in Section 7.6 with the long-range voter model on the real line.

7.2. SPDE with Lipschitz-coefficients

Let us consider the SPDE (7.1) with bounded Lipschitz-coefficients instead of the square-root-term, that is

$$\partial_t u_t(x) = (\mathcal{L}_\alpha u_t)(x) + a(u_t(x))\dot{W}_{t,x}, \qquad t \geq 0,\, x \in \mathbb{R} \tag{7.3}$$

with $a\colon \mathbb{R} \to \mathbb{R}$ being bounded and Lipschitz-continuous. With abuse of notation we also denote solutions to that SPDE in this section with $(u_t)_{t\geq 0}$ (outside of this section $(u_t)_{t\geq 0}$ denotes the long-range voter model on the real line). We want to show that there is a strong solution to (7.3) and that there is a continuous modification of the solution. We start with a technical lemma and show afterwards in Proposition 7.4 that a certain stochastic integral has a continuous modification. The main step in the proof of Proposition 7.4 is to show the requirements of Kolmogorov's continuity theorem (see for example [Ka97, Theorem 2.23]). In Lemma 7.5 we quote a Gronwall-type lemma from [Wa86, Lemma 3.3]. Then, in Proposition 7.6, we obtain a strong solution via Picard's iteration. Finally in Corollary 7.7, we get the existence of a continuous modification as a corollary of Proposition 7.4. In this section we follow [DKM$^+$09, Part I, Theorems 6.7] and [Wa86, Theorem 3.2] where similar results for the case $\alpha = 2$ have been proved.

Lemma 7.3. *There is a constant $C_{7.3} = C_{7.3}(\alpha) > 0$ such that for $0 \leq t \leq t' < \infty$ and $x \in \mathbb{R}$ one has*

$$\int_t^{t'} \int_{-\infty}^{\infty} p^{(\alpha)}_{t'-s}(y-x)^2 \, dy \, ds = C_{7.3}(t'-t)^{1-1/\alpha}.$$

Proof. Using the semigroup property and the scaling property of the stable density we can calculate

$$\begin{aligned}
\int_t^{t'} \int_{-\infty}^{\infty} p^{(\alpha)}_{t'-s}(y-x)^2 \, dy \, ds &= \int_t^{t'} \int_{-\infty}^{\infty} p^{(\alpha)}_{t'-s}(y-x) p^{(\alpha)}_{t'-s}(x-y) \, dy \, ds \\
&= \int_t^{t'} p^{(\alpha)}_{2(t'-s)}(0) \, ds \\
&= \int_0^{t'-t} p^{(\alpha)}_{2s}(0) \, ds \\
&= p^{(\alpha)}_1(0) \int_0^{t'-t} (2s)^{-1/\alpha} \, ds \\
&= C_{7.3}(t'-t)^{1-1/\alpha}.
\end{aligned}$$

for some constant $C_{7.3} = C_{7.3}(\alpha) > 0$. Note: Since $\alpha \in (1, 2]$ we have $\frac{1}{\alpha} \in \left[\frac{1}{2}, 1\right)$ and $1 - \frac{1}{\alpha} \in \left(0, \frac{1}{2}\right]$. □

The proof of the next result follows [DKM+09, Part I, Theorem 6.7].

Proposition 7.4. *Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions and let W be a space-time white noise that is adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$. Further let $a \in \mathcal{C}_b(\mathbb{R})$ and let $(v_t)_{t\geq 0}$ be $(\mathcal{F}_t)_{t\geq 0}$-predictable. Then,*

$$V_t(x) := \begin{cases} \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x) a(v_s(y))\, W(ds, dy), & \textit{if } (t,x) \in (0,\infty) \times \mathbb{R}, \\ 0, & \textit{if } (t,x) \in \{0\} \times \mathbb{R}, \end{cases}$$

is well defined, and there is a constant $C_{7.4.1} = C_{7.4.1}(a, \alpha) \in (0, \infty)$, such that for each $T > 0$

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbf{E}\left[(V_t(x))^2\right] \leq C_{7.4.1} T^{1-1/\alpha} < \infty.$$

Furthermore, for each $p > \frac{4}{1-1/\alpha} > 8$ and $T > 0$, there exist constants $\beta = \beta(\alpha, p) > 0$ and $C_{7.4.2} = C_{7.4.2}(\alpha, p, T) > 0$ such that for every $t, t' \in [0, T]$ and every $x, \widetilde{x} \in \mathbb{R}$ with $|\widetilde{x} - x| \leq 1$ we have

$$\mathbf{E}\left[|V_{t'}(\widetilde{x}) - V_t(x)|^p\right] \leq C_{7.4.2} \|a\|_\infty^p \left(|t'-t|^{2+\beta} + |\widetilde{x} - x|^{2+\beta}\right). \tag{7.4}$$

Thus, there exists a modification of $(V_t)_{t\geq 0}$ which is jointly continuous in both variables $t \in [0, \infty)$ and $x \in \mathbb{R}$, and $(V_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable.

Proof. Due to Lemma 7.3, for $t > 0$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} &\mathbf{E}\left[\int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)^2 a(v_s(y))^2\, dy\, ds\right] \\ &\quad \leq \|a\|_\infty^2 \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)^2\, dy\, ds \\ &\quad = \|a\|_\infty^2\, C_{7.3} t^{1-1/\alpha}. \end{aligned}$$

Then V is well defined and for each $T > 0$, we have with $C_{7.4.1} := \|a\|_\infty^2\, C_{7.3}$

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbf{E}\left[(V_t(x))^2\right] \leq C_{7.4.1} T^{1-1/\alpha} < \infty.$$

In the following the aim is to use Kolmogorov's continuity criterion to show the existence of a continuous modification. Let us first prove that for $p \geq 2$ and $T > 0$ there exists a constant $C_{7.4.3} = C_{7.4.3}(\alpha, p, T) > 0$ such that for every $t, t' \in [0, T]$ and $x \in \mathbb{R}$, we have

$$\mathbf{E}\left[|V_{t'}(x) - V_t(x)|^p\right] \leq C_{7.4.3} \|a\|_\infty^p |t'-t|^{(1-\frac{1}{\alpha})\frac{p}{2}}. \tag{7.5}$$

To this end let $p \geq 2$, $T > 0$, $t, t' \in [0, T]$ and $x \in \mathbb{R}$ be arbitrarily chosen und let us assume $0 < t < t'$. Using the inequality $|a+b|^p \le 2^p\left(|a|^p + |b|^p\right)$ for $a, b \in \mathbb{R}$ and Burkholder's inequality (see [IW89, Chapter 3, Theorem 3.1]) we obtain for some constant $C_{7.4.4} = C_{7.4.4}(p) > 0$

$$\begin{aligned}
&\mathbf{E}\left[|V_{t'}(x) - V_t(x)|^p\right] \\
&\quad \le 2^p \Bigg(\mathbf{E}\left[\left| \int_0^t \int_{-\infty}^{\infty} \left(p^{(\alpha)}_{t'-s}(y-x) - p^{(\alpha)}_{t-s}(y-x)\right) a(v_s(y))\, W(ds,dy) \right|^p \right] \\
&\qquad\qquad + \mathbf{E}\left[\left| \int_t^{t'} \int_{-\infty}^{\infty} p^{(\alpha)}_{t'-s}(y-x) a(v_s(y))\, W(ds,dy) \right|^p \right] \Bigg) \\
&\quad \le C_{7.4.4} 2^p \Bigg(\mathbf{E}\left[\left| \int_0^t \int_{-\infty}^{\infty} \left| p^{(\alpha)}_{t'-s}(y-x) - p^{(\alpha)}_{t-s}(y-x)\right|^2 a(v_s(y))^2\, dy\, ds \right|^{p/2} \right] \\
&\qquad\qquad + \mathbf{E}\left[\left| \int_t^{t'} \int_{-\infty}^{\infty} p^{(\alpha)}_{t'-s}(y-x)^2 a(v_s(y))^2\, dy\, ds \right|^{p/2} \right] \Bigg)
\end{aligned}$$

Since a is bounded, we have

$$\begin{aligned}
&\mathbf{E}\left[|V_{t'}(x) - V_t(x)|^p\right] \\
&\quad \le C_{7.4.4} \left(2 \|a\|_\infty\right)^p \Bigg(\left(\int_0^t \int_{-\infty}^{\infty} \left(p^{(\alpha)}_{t'-s}(y-x) - p^{(\alpha)}_{t-s}(y-x)\right)^2 dy\, ds \right)^{p/2} \\
&\qquad\qquad + \left(\int_t^{t'} \int_{-\infty}^{\infty} p^{(\alpha)}_{t'-s}(y-x)^2\, dy\, ds \right)^{p/2} \Bigg) \\
&\quad =: C_{7.4.4} \left(2 \|a\|_\infty\right)^p \left(I_1^{p/2} + I_2^{p/2} \right).
\end{aligned} \tag{7.6}$$

According to Lemma 7.3 there is a constant $C_{7.3} = C_{7.3}(\alpha) > 0$ such that $I_2 = C_{7.3} |t'-t|^{1-1/\alpha}$, thus

$$I_2^{p/2} = C_{7.3}^{p/2} \left|t'-t\right|^{(1-\frac{1}{\alpha})\frac{p}{2}}. \tag{7.7}$$

Now we derive an appropriate estimate for I_1. For $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we denote the Fourier transform of g by $\mathcal{F}g$, i.e.

$$(\mathcal{F}g)(\xi) = \int_{-\infty}^{\infty} g(x) e^{i\xi x}\, dx, \qquad \xi \in \mathbb{R}.$$

By Plancherel's theorem, we have

$$\int_{-\infty}^{\infty} |g(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |(\mathcal{F}g)(\xi)|^2\, d\xi$$

and we know $(\mathcal{F}p_t)(\xi) = e^{-t|\xi|^\alpha}$ as well as $(\mathcal{F}(x \mapsto g(x+a)))(\xi) = e^{-i\xi a}(\mathcal{F}g)(\xi)$ for $a \in \mathbb{R}$, i.e.

$$\int_{-\infty}^{\infty}\left(p^{(\alpha)}_{t'-s}(y-x)-p^{(\alpha)}_{t-s}(y-x)\right)^2 dy=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{-(t'-s)|\xi|^\alpha}-e^{-(t-s)|\xi|^\alpha}\right)^2 d\xi. \quad (7.8)$$

for $s\in[0,t]$, since $\mathcal{F}$ is linear. One can compute

$$\left(e^{-(t'-s)|\xi|^\alpha}-e^{-(t-s)|\xi|^\alpha}\right)^2=e^{-2(t-s)|\xi|^\alpha}\left(1-e^{-(t'-t)|\xi|^\alpha}\right)^2 \quad (7.9)$$

and

$$\int_0^t e^{-2(t-s)|\xi|^\alpha}\,ds=\int_0^t e^{-2|\xi|^\alpha s}\,ds=\frac{(1-e^{-2t|\xi|^\alpha})}{2\,|\xi|^\alpha}. \quad (7.10)$$

Using (7.8), (7.9), Fubini's theorem and (7.10) we get

$$\begin{aligned}
I_1&=\int_0^t\int_{-\infty}^{\infty}\left(p^{(\alpha)}_{t'-s}(y-x)-p^{(\alpha)}_{t-s}(y-x)\right)^2 dy\,ds\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^t e^{-2(t-s)|\xi|^\alpha}\,ds\right)\left(1-e^{-(t'-t)|\xi|^\alpha}\right)^2 d\xi\\
&=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1-e^{-2t|\xi|^\alpha}}{|\xi|^\alpha}\left(1-e^{-(t'-t)|\xi|^\alpha}\right)^2 d\xi\\
&=\frac{1}{2\pi}\int_0^{\infty}\frac{1-e^{-2t|\xi|^\alpha}}{|\xi|^\alpha}\left(1-e^{-(t'-t)|\xi|^\alpha}\right)^2 d\xi,
\end{aligned}$$

where the last equality holds since the integrand on the right-hand side is a even function in the integration variable ξ. Note that

$$\left(1-e^{-(t'-t)|\xi|^\alpha}\right)^2\le 2\min\left\{(t'-t)\,|\xi|^\alpha,1\right\}.$$

Moreover, there exists a constant $C_{7.4.5}=C_{7.4.5}(T)>0$ such that we have uniformly in $t\in[0,T]$

$$\frac{(1-e^{-2t|\xi|^\alpha})}{|\xi|^\alpha}\le\frac{C_{7.4.5}}{1+|\xi|^\alpha}.$$

This implies

$$\begin{aligned}
I_1&\le\frac{C_{7.4.5}}{\pi}\int_0^{\infty}\frac{\min\left\{(t'-t)\,|\xi|^\alpha,1\right\}}{1+|\xi|^\alpha}\,d\xi\\
&=\frac{C_{7.4.5}}{\pi}\left(\int_0^{|t'-t|^{-1/\alpha}}\frac{(t'-t)\,|\xi|^\alpha}{1+|\xi|^\alpha}\,d\xi+\int_{|t'-t|^{-1/\alpha}}^{\infty}\frac{1}{1+|\xi|^\alpha}\,d\xi\right).
\end{aligned} \quad (7.11)$$

Since $\frac{|\xi|^\alpha}{1+|\xi|^\alpha}\le 1$ we have

$$\int_0^{|t'-t|^{-1/\alpha}}\frac{(t'-t)\,|\xi|^\alpha}{1+|\xi|^\alpha}\,d\xi\le\left|t'-t\right|^{1-1/\alpha}. \quad (7.12)$$

Moreover we compute (note that $1-\alpha \in [-1,0)$ because of $\alpha \in (1,2]$)

$$\int_{|t'-t|^{-1/\alpha}}^{\infty} \frac{1}{1+|\xi|^{\alpha}}\, d\xi \leq \int_{|t'-t|^{-1/\alpha}}^{\infty} \xi^{-\alpha}\, d\xi = \frac{1}{\alpha-1}\, |t'-t|^{1-1/\alpha}\,. \tag{7.13}$$

Using equations (7.11), (7.12), (7.13) and $\alpha - 1 \leq 1$, we get

$$I_1 \leq \frac{C_{7.4.5}}{\pi}\left(|t'-t|^{1-1/\alpha} + \frac{1}{\alpha-1}\,|t'-t|^{1-1/\alpha}\right) \leq C_{7.4.6}\,|t'-t|^{1-1/\alpha}\,,$$

for some constant $C_{7.4.6} = C_{7.4.6}(\alpha, T) > 0$, hence

$$I_1^{p/2} \leq (C_{7.4.6})^{p/2}\,|t'-t|^{(1-\frac{1}{\alpha})\frac{p}{2}}\,. \tag{7.14}$$

Finally, (7.6), (7.7) and (7.14) show that there is a constant $C_{7.4.3} = C_{7.4.3}(\alpha, p, T) > 0$ such that

$$\mathbf{E}\left[|V_{t'}(x) - V_t(x)|^p\right] \leq C_{7.4.3}\,\|a\|_\infty^p\,|t'-t|^{(1-\frac{1}{\alpha})\frac{p}{2}}\,.$$

This is (7.5). Note that (7.5) is also true for $t = 0$, since the calculation can be done analogously to the derivation of (7.7). I_1 vanishes in this case. Next, we show that for $p \geq 2$ there exists a constant $C_{7.4.7} = C_{7.4.7}(\alpha, p) > 0$ such that for every $t \geq 0$ and $x, \widetilde{x} \in \mathbb{R}$ we have

$$\mathbf{E}\left[|V_t(\widetilde{x}) - V_t(x)|^p\right] \leq C_{7.4.7}\,\|a\|_\infty^p\,|\widetilde{x}-x|^{(\alpha-1)\frac{p}{2}}\,. \tag{7.15}$$

For this purpose let $p \geq 2$, $t > 0$ and $x, \widetilde{x} \in \mathbb{R}$ be arbitrarily chosen and let us assume $x < \widetilde{x}$. Using Burkholder's inequality and the boundedness of a we obtain

$$\begin{aligned}
\mathbf{E}\left[|V_t(\widetilde{x}) - V_t(x)|^p\right] &= \mathbf{E}\left[\left|\int_0^t \int_{-\infty}^{\infty} \left(p_{t-s}^{(\alpha)}(y-\widetilde{x}) - p_{t-s}^{(\alpha)}(y-x)\right) a(v_s(y))\, W(ds, dy)\right|^p\right] \\
&\leq C_{7.4.4}\mathbf{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} \left(p_{t-s}^{(\alpha)}(y-\widetilde{x}) - p_{t-s}^{(\alpha)}(y-x)\right)^2 a(v_s(y))^2\, dy\, ds\right)^{p/2}\right] \\
&\leq C_{7.4.4}\,\|a\|_\infty^p \left(\int_0^t \int_{-\infty}^{\infty} \left(p_{t-s}^{(\alpha)}(y-\widetilde{x}) - p_{t-s}^{(\alpha)}(y-x)\right)^2 dy\, ds\right)^{p/2} \\
&=: C_{7.4.4}\,\|a\|_\infty^p\, I_3^{p/2}.
\end{aligned} \tag{7.16}$$

Using a substitution we obtain

$$\begin{aligned}
I_3 &= \int_0^t \int_{-\infty}^{\infty} \left(p_{t-s}^{(\alpha)}(y-\widetilde{x}) - p_{t-s}^{(\alpha)}(y-x)\right)^2 dy\, ds \\
&= \int_0^t \int_{-\infty}^{\infty} \left(p_{t-s}^{(\alpha)}(y-\widetilde{x}) - p_{t-s}^{(\alpha)}(y-\widetilde{x}+\widetilde{x}-x)\right)^2 dy\, ds \\
&= \int_0^t \int_{-\infty}^{\infty} \left(p_{t-s}^{(\alpha)}(y) - p_{t-s}^{(\alpha)}(y+\widetilde{x}-x)\right)^2 dy\, ds.
\end{aligned}$$

With the help of the properties of the Fourier transform, with which (7.8) was derived, we can calculate

$$\begin{aligned}\int_{-\infty}^{\infty}\left(p_{t-s}^{(\alpha)}(y)-p_{t-s}^{(\alpha)}(y+\widetilde{x}-x)\right)^2 dy &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{-(t-s)|\xi|^\alpha}-e^{-i\xi(\widetilde{x}-x)}e^{-(t-s)|\xi|^\alpha}\right|^2 d\xi\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{-(t-s)|\xi|^\alpha}\left(1-e^{-i\xi(\widetilde{x}-x)}\right)\right|^2 d\xi\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-2(t-s)|\xi|^\alpha}\left|1-e^{-i\xi(\widetilde{x}-x)}\right|^2 d\xi.\end{aligned}$$

It follows

$$\begin{aligned}I_3 &= \int_0^t\int_{-\infty}^{\infty}\left(p_{t-s}^{(\alpha)}(y)-p_{t-s}^{(\alpha)}(y+\widetilde{x}-x)\right)^2 dy\,ds\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^t e^{-2(t-s)|\xi|^\alpha}\,ds\right)\left|1-e^{-i\xi(\widetilde{x}-x)}\right|^2 d\xi\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1-e^{-2t|\xi|^\alpha}}{|\xi|^\alpha}\left|1-e^{-i\xi(\widetilde{x}-x)}\right|^2 d\xi.\end{aligned}$$

where we used Fubini's theorem in the second step and equation (7.10) in the last step. Noticing that $\left|1-e^{-2t|\xi|^\alpha}\right|\le 1$ and that for $\theta\in\mathbb{R}$

$$\left|1-e^{-i\theta}\right|^2=(1-e^{-i\theta})(1-e^{i\theta})=2-e^{i\theta}-e^{-i\theta}=2(1-\cos(\theta))\le 2\min\left\{1,\theta^2\right\},$$

we get

$$\begin{aligned}|I_3| &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{(1-e^{-2t|\xi|^\alpha})}{|\xi|^\alpha}\left|1-e^{-i\xi(\widetilde{x}-x)}\right|^2 d\xi\\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\min\left\{1,\xi^2(\widetilde{x}-x)^2\right\}}{|\xi|^\alpha}d\xi\\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\min\left\{1,\xi^2(\widetilde{x}-x)^2\right\}}{|\xi|^\alpha}d\xi\\ &= \frac{1}{\pi}\left(\int_0^{|\widetilde{x}-x|^{-1}}\frac{\xi^2\,|\widetilde{x}-x|^2}{|\xi|^\alpha}d\xi+\int_{|\widetilde{x}-x|^{-1}}^{\infty}\frac{1}{|\xi|^\alpha}d\xi\right).\end{aligned}\tag{7.17}$$

We have

$$\begin{aligned}\int_0^{|\widetilde{x}-x|^{-1}}\frac{\xi^2\,|\widetilde{x}-x|^2}{|\xi|^\alpha}d\xi &= |\widetilde{x}-x|^2\int_0^{|\widetilde{x}-x|^{-1}}\xi^{2-\alpha}\,d\xi\\ &= |\widetilde{x}-x|^2\,\frac{1}{3-\alpha}\xi^{3-\alpha}\Big|_{\xi=0}^{\xi=|\widetilde{x}-x|^{-1}}\\ &= \frac{1}{3-\alpha}\,|\widetilde{x}-x|^{\alpha-1}\end{aligned}\tag{7.18}$$

and

$$\begin{aligned}\int_{|\widetilde{x}-x|^{-1}}^{\infty} \frac{1}{|\xi|^{\alpha}}\, d\xi &= \int_{|\widetilde{x}-x|^{-1}}^{\infty} \xi^{-\alpha}\, d\xi \\ &= \frac{1}{1-\alpha} \xi^{1-\alpha} \Big|_{\xi=|\widetilde{x}-x|^{-1}}^{\xi=\infty} \\ &= \frac{1}{\alpha-1} |\widetilde{x}-x|^{\alpha-1}.\end{aligned} \tag{7.19}$$

Using equations (7.17), (7.18), (7.19) and $\alpha - 1 \leq 3 - \alpha$ (since $\alpha \leq 2$), we conclude

$$\begin{aligned}|I_3| &\leq \frac{1}{\pi}\left(\frac{1}{3-\alpha}|\widetilde{x}-x|^{\alpha-1} + \frac{1}{\alpha-1}|\widetilde{x}-x|^{\alpha-1}\right) \\ &\leq C_{7.4.8}\, |\widetilde{x}-x|^{\alpha-1},\end{aligned}$$

for some constant $C_{7.4.8} = C_{7.4.8}(\alpha) > 0$, hence

$$|I_3|^{p/2} \leq C_{7.4.8}^{p/2}\, |\widetilde{x}-x|^{(\alpha-1)\frac{p}{2}}. \tag{7.20}$$

Finally (7.16) and (7.20) show that there is a constant $C_{7.4.7} = C_{7.4.7}(\alpha, p) > 0$ such that

$$\mathbf{E}\left[|V_t(\widetilde{x}) - V_t(x)|^p\right] \leq C_{7.4.7}\, \|a\|_\infty^p\, |\widetilde{x}-x|^{(\alpha-1)\frac{p}{2}}.$$

This is (7.15). Note that (7.15) is also true for $t = 0$. Summing up, (7.5), (7.15) and the inequality $|a+b|^p \leq 2^p\left(|a|^p + |b|^p\right)$ for $a, b \in \mathbb{R}$ show, for each $p \geq 2$ and $T > 0$, there exists a constant $C_{7.4.2} = C_{7.4.2}(\alpha, p, T) > 0$ such that for every $t, t' \in [0, T]$ and every $x, \widetilde{x} \in \mathbb{R}$ we have

$$\mathbf{E}\left[|V_{t'}(\widetilde{x}) - V_t(x)|^p\right] \leq C_{7.4.2}\, \|a\|_\infty^p \left(|t'-t|^{(1-\frac{1}{\alpha})\frac{p}{2}} + |\widetilde{x}-x|^{(\alpha-1)\frac{p}{2}}\right). \tag{7.21}$$

Now let $x, \widetilde{x} \in \mathbb{R}$ with $|\widetilde{x} - x| \leq 1$. Since $1 - \frac{1}{\alpha} \leq \alpha - 1$ from (7.21) it follows that

$$\mathbf{E}\left[|V_{t'}(\widetilde{x}) - V_t(x)|^p\right] \leq C_{7.4.2}\, \|a\|_\infty^p \left(|t'-t|^{(1-\frac{1}{\alpha})\frac{p}{2}} + |\widetilde{x}-x|^{(1-\frac{1}{\alpha})\frac{p}{2}}\right).$$

If $p > \frac{4}{1-1/\alpha}$, then $(1-\frac{1}{\alpha})\frac{p}{2} > 2$ and we can define $\beta := (1-\frac{1}{\alpha})\frac{p}{2} - 2 > 0$. This implies that: For $p > \frac{4}{1-1/\alpha} > 8$, there exists a constant $\beta = \beta(\alpha, p) > 0$, such that

$$\mathbf{E}\left[|V_{t'}(\widetilde{x}) - V_t(x)|^p\right] \leq C_{7.4.2}\, \|a\|_\infty^p \left(|t'-t|^{2+\beta} + |\widetilde{x}-x|^{2+\beta}\right)$$

for each $t, t' \in [0, T]$ and every $x, \widetilde{x} \in \mathbb{R}$ with $|\widetilde{x} - x| \leq 1$. This is (7.4). Thus, the process $(V_t)_{t\geq 0}$ fulfills the prerequisites of Kolmogorov's continuity theorem (see for example [Ka97, Theorem 2.23]) which shows that there exists a jointly continuous modification of $(V_t)_{t\geq 0}$. The latter implies $(\mathcal{F}_t)_{t\geq 0}$-predictability of $(V_t)_{t\geq 0}$. □

Lemma 7.5 ([Wa86]). *Let $T > 0$ and let $h_0, h_1, h_2, \ldots : [0,T] \to [0,\infty)$ be a sequence of functions. Suppose that h_0 is bounded and, for some $a > -1$ there exists a constant $C_{7.5.1} \in (0,\infty)$ such that for all $m, n \in \mathbb{N}$ and $t \in [0,T]$,*

$$h_n(t) \leq C_{7.5.1} \int_0^t h_{n-1}(s)(t-s)^a \, ds.$$

Then there is a constant $C_{7.5.2} \in (0,\infty)$ and an integer $k \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ and $t \in [0,T]$,

$$h_{n+mk}(t) \leq \frac{C_{7.5.2}^m}{(m-1)!} \int_0^t h_n(s)(t-s) \, ds.$$

Proof. This is [Wa86, Lemma 3.3]. Note that [Wa86, Lemma 3.3] states the result for each $a > 1$, but proves it for each $a > -1$. □

Now we formulate the existence result of a strong solution to (7.3). The proof follows [Wa86, Theorem 3.2] and [Zä04, Theorem 6.8].

Proposition 7.6. *Let $a\colon \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz-continuous and let $u_0\colon \mathbb{R} \to [0,1]$ be a continuous function. Then there exists a mild solution to (7.3), which is strong in the sense of probability theory, i.e.: Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions and let W be a space-time white noise that is adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$. There exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(u_t)_{t\geq 0}$ such that we have $\mathbf{P}$-almost surely for each $t > 0$ and $x \in \mathbb{R}$*

$$u_t(x) = \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x) u_0(y) \, dy + \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x) a(u_s(y)) \, W(ds, dy). \tag{7.22}$$

Further we have for $T > 0$

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbf{E}\left[(u_t(x))^2\right] < \infty. \tag{7.23}$$

Proof. We use Picard's iteration scheme. Define for $(t,x) \in [0,\infty) \times \mathbb{R}$

$$u_t^{(0)}(x) := \begin{cases} \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x) u_0(y) \, dy, & \text{if } (t,x) \in (0,\infty) \times \mathbb{R}, \\ u_0(x), & \text{if } (t,x) \in \{0\} \times \mathbb{R}, \end{cases}$$

and for $n \in \mathbb{N}_0$

$$u_t^{(n+1)}(x) := \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x) u_0(y) \, dy + \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x) a(u_s^{(n)}(y)) \, W(ds, dy)$$

for $(t,x) \in (0,\infty) \times \mathbb{R}$, and $u_0^{(n+1)}(x) := u_0(x)$ for $x \in \mathbb{R}$. Denote by $\mathcal{P}^2$ the space of all

$(\mathcal{F}_t)_{t\geq 0}$-predictable $(\psi_t)_{t\geq 0}$ that satisfy

$$\|\psi\|_T := \left(\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}} \mathbf{E}\left[\psi(t,x)^2\right]\right)^{1/2} < \infty$$

for each $T > 0$. Identify $\psi, \widetilde{\psi} \in \mathcal{P}^2$ if $\psi(t,x) = \widetilde{\psi}(t,x)$ holds $\mathbf{P}$-almost surely for each fixed $(t,x) \in [0,\infty) \times \mathbb{R}$. Define a metric on $\mathcal{P}^2$ via

$$d_{\mathcal{P}^2}(\psi,\widetilde{\psi}) := \sum_{T=1}^{\infty} 2^{-T} \min\left\{\left\|\widetilde{\psi} - \psi\right\|_T, 1\right\}, \qquad \psi, \widetilde{\psi} \in \mathcal{P}^2.$$

Then, $\mathcal{P}^2$ is a complete metric space with respect to $d_{\mathcal{P}^2}$. Proposition 7.4 ensures for each $n \in \mathbb{N}$, that $u^{(n)}$ is well defined with $u^{(n)} \in \mathcal{P}^2$. Because of the continuity of u_0 and Proposition 7.4, we can assume that $u^{(n)}$ takes values in $C([0,\infty) \times \mathbb{R})$. We would now like to show that $(u^{(n)})_{n\in\mathbb{N}}$ converges in $\mathcal{P}^2$ to some limit. For $n \in \mathbb{N}_0$ and $t \geq 0$, let

$$h_n(t) := \sup_{x\in\mathbb{R}} \mathbf{E}\left[\left(\left|u_t^{(n+1)}(x) - u_t^{(n)}(x)\right|\right)^2\right].$$

Note that Lemma 7.3 ensures that for each $T > 0$

$$\sup_{n\in\mathbb{N}_0}\sup_{t\in[0,T]} h_n(t) < \infty.$$

The Lipschitz-continuity of a and an analogous calculation as in the proof of Lemma 7.3 show for $n \in \mathbb{N}$, $t > 0$ and $x \in \mathbb{R}$

$$\begin{aligned}
&\mathbf{E}\left[\left(\int_0^t\int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)\left(a(u_s^{(n)}(y)) - a(u_s^{(n-1)}(y))\right) W(ds,dy)\right)^2\right] \\
&\quad = \int_0^t\int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)^2 \mathbf{E}\left[\left(a(u_s^{(n)}(y)) - a(u_s^{(n-1)}(y))\right)^2\right] dy\,ds \\
&\quad \leq C_{7.6.1}\int_0^t h_{n-1}(s)\int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)^2\,dy\,ds \\
&\quad = C_{7.6.1}\int_0^t h_{n-1}(s) p_{2(t-s)}^{(\alpha)}(0)\,ds \\
&\quad = C_{7.6.2}\int_0^t \frac{h_{n-1}(s)}{(t-s)^{1/\alpha}}\,ds
\end{aligned} \tag{7.24}$$

for some constants $C_{7.6.1} = C_{7.6.1}(a) > 0, C_{7.6.2} = C_{7.6.2}(a,\alpha) > 0$. Thus, for $n \in \mathbb{N}$ and $t \geq 0$, we get

$$h_n(t) \leq C_{7.6.2}\int_0^t \frac{h_{n-1}(s)}{(t-s)^{1/\alpha}}\,ds.$$

Let $T > 0$. Due to Lemma 7.5 there exists a constant $C_{7.6.3} = C_{7.6.3}(a,\alpha) > 0$ and an integer $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ and $t \in [0,T]$

$$h_{n+mk}(t) \leq \frac{C_{7.5.3}^m}{(m-1)!} \int_0^t h_n(s)(t-s)\,ds.$$

Thus for each $n \in \mathbb{N}$,

$$\sum_{m=0}^{\infty} \left(\sup_{t \in [0,T]} h_{n+mk}(t) \right)^{1/2} < \infty,$$

which implies for each $T > 0$

$$\sum_{n=0}^{\infty} \left\| u^{(n+1)} - u^{(n)} \right\|_T = \sum_{n=0}^{\infty} \left(\sup_{t \in [0,T]} h_n(t) \right)^{1/2} < \infty.$$

Therefore $(u^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{P}^2$ and converges to some $u \in \mathcal{P}^2$ as $n \longrightarrow \infty$. Following the calculation in (7.24), for $n \in \mathbb{N}$, $t \in [0,T]$ and $x \in \mathbb{R}$, we get

$$\begin{aligned} &\mathbf{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x) \left(a(u_s^{(n)}(y)) - a(u_s(y)) \right) W(ds, dy) \right)^2 \right] \\ &\quad \leq C_{7.6.4} \left\| u^{(n)} - u \right\|_T \end{aligned}$$

for some constant $C_{7.6.4} = C_{7.6.4}(a, \alpha, T) \in (0, \infty)$. Finally, (7.22) and Equation (7.23) hold as claimed. □

Corollary 7.7. *Under the assumptions of Proposition 7.6 let $(u_t)_{t \geq 0}$ be a mild solution of (7.3). There exists a modification of $(u_t)_{t \geq 0}$ in $C([0,\infty) \times \mathbb{R})$.*

Proof. Using the representation (7.22), we have **P**-almost surely for each $t \geq 0$ and $x \in \mathbb{R}$

$$u_t(x) = U_t^{(1)}(x) + U_t^{(2)}(x)$$

with

$$U_t^{(1)}(x) := \begin{cases} \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x) u_0(y)\,dy, & \text{if } (t,x) \in (0,\infty) \times \mathbb{R}, \\ u_0(x), & \text{if } (t,x) \in \{0\} \times \mathbb{R}, \end{cases}$$

and

$$U_t^{(2)}(x) := \begin{cases} \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x) a(u_s(y))\, W(ds,dy), & \text{if } (t,x) \in (0,\infty) \times \mathbb{R}, \\ 0, & \text{if } (t,x) \in \{0\} \times \mathbb{R}. \end{cases}$$

$(U_t^{(1)})_{t \geq 0}$ is jointly continuous in $t \in [0,\infty)$ and $x \in \mathbb{R}$, since u_0 is continuous. According to Proposition 7.4 there exists a jointly continuous modification of $(U_t^{(2)})_{t \geq 0}$. □

7.3. Solution for finite γ

In this section we prove the existence of the finite rate model with rate $\gamma > 0$, i.e., we obtain a weak solution to (7.1) via approximation by solutions with Lipschitz-coefficients (this is the proof of the existence-result of Theorem 7.1). Then we show in Proposition 7.8 that the constructed solution solves a martingale problem.

Proof of Theorem 7.1 (Existence). Let $u_0 \colon \mathbb{R} \to [0,1]$ be a continuous function. Define $a \colon \mathbb{R} \to \mathbb{R}$ via

$$a(x) := \begin{cases} \sqrt{\gamma x(1-x)}, & \text{if } x \in [0,1], \\ 0, & \text{otherwise,} \end{cases}$$

Let us approximate $a \in \mathcal{C}_c(\mathbb{R})$ by a sequence $(a_n)_{n\in\mathbb{N}}$ of Lipschitz-continuous functions using smoothing and convolution (see [Ev10, Appendix C.4]): For $n \in \mathbb{N}$ define functions $\eta, \eta_n \in \mathcal{C}_c^\infty(\mathbb{R})$ by

$$\eta(x) := \begin{cases} b \exp\left(\frac{1}{|x|^2-1}\right), & \text{if } x \in (-1,1), \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \eta_n(x) := n\eta(nx),$$

where the constant $b > 0$ is chosen so that $\int_{-\infty}^{\infty} \eta(x)\,dx = 1$. Let

$$a_n(x) := \int_{-\infty}^{\infty} \eta_n(y-x)a(y)\,dy, \qquad x \in \mathbb{R}.$$

Then we have $a_n \in \mathcal{C}_c^\infty(\mathbb{R})$, in particular a_n is Lipschitz-continuous, and

$$\|a - a_n\|_\infty \xrightarrow{n\to\infty} 0.$$

Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions and let W be a space-time white noise. According to Proposition 7.6, for each $n \in \mathbb{N}$, there exists an adapted process $(u_t^{(n)})_{t\geq 0}$ which is a mild solution of

$$\partial_t u_t^{(n)}(x) = \left(-(-\Delta)^{\alpha/2} u_t^{(n)}\right)(x) + a_n(u_t^{(n)}(x))\dot{W}_{t,x}, \qquad t \geq 0,\, x \in \mathbb{R}$$

with continuous and bounded initial condition $u_0^{(n)} := u_0$. That is, $\mathbf{P}$-almost surely for each $t > 0$ and $x \in \mathbb{R}$ we have

$$u_t^{(n)}(x) = \int_{-\infty}^{\infty} p_t^{(\alpha)}(y-x)u_0(y)\,dy + \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)a_n(u_s^{(n)}(y))\,W(ds,dy). \quad (7.25)$$

Using Corollary 7.7 we can further assume that $(u_t^{(n)})_{t\geq 0}$ takes values in $\mathcal{C}([0,\infty)\times\mathbb{R})$. Define

$$U_t^{(n)}(x) := \begin{cases} \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)a_n(u_s^{(n)}(y))\,W(ds,dy), & \text{if } (t,x) \in (0,\infty)\times\mathbb{R}, \\ 0, & \text{if } (t,x) \in \{0\}\times\mathbb{R}. \end{cases}$$

For $p > \frac{4}{1-1/\alpha} > 8$ and $T > 0$, according to (7.4) in Proposition 7.4, there exist constants $\beta = \beta(\alpha, p) > 0$ and $C_{7.4.2} = C_{7.4.2}(\alpha, p, T) > 0$, such that

$$\mathbf{E}\left[\left|U_{t'}^{(n)}(\widetilde{x}) - U_t^{(n)}(x)\right|^p\right] \leq C_{7.4.2}\, \|a_n\|_\infty^p \left(|t'-t|^{2+\beta} + |\widetilde{x}-x|^{2+\beta}\right)$$

for each $n \in \mathbb{N}$, $t, t' \in [0,T]$ and every $x, \widetilde{x} \in \mathbb{R}$ with $|\widetilde{x} - x| \leq 1$. Since a_n converges uniformly to a, we have

$$\sup_{n\in\mathbb{N}} \|a_n\|_\infty < \infty.$$

Thus [Ka97, Corollary 14.9] yields tightness of $\{\mathcal{L}(u^{(n)}) : n \in \mathbb{N}\}$ in $\mathcal{M}_1(\mathcal{C}([0,\infty)\times\mathbb{R}))$. The white noise W can be reconstructed from

$$\widetilde{W} := (W([t_1,t_2]\times[a,b]))_{t_1,t_2,a,b\in\mathbb{Q}, t_1\leq t_2, a\leq b},$$

which we can interpret as a random variable taking values in the Polish space $\mathbb{R}^{\mathbb{N}}$. Using [Kl13, Remark 13.27] we obtain tightness of the only one element containing set $\{\mathcal{L}(\widetilde{W})\}$ in $\mathcal{M}_1(\mathbb{R}^{\mathbb{N}})$. The two tightness results imply that given $\varepsilon > 0$ there exist a compact set $K_{1,\varepsilon}$ in $\mathcal{C}([0,\infty)\times\mathbb{R})$ and a compact set $K_{2,\varepsilon}$ in $\mathbb{R}^{\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}} \mathbf{P}[u^{(n)} \notin K_{1,\varepsilon}] \leq \frac{\varepsilon}{2} \qquad \text{and} \qquad \mathbf{P}[\widetilde{W} \notin K_{2,\varepsilon}] \leq \frac{\varepsilon}{2}.$$

This implies

$$\sup_{n\in\mathbb{N}} \mathbf{P}\left[\left(u^{(n)}, \widetilde{W}\right) \notin K_{1,\varepsilon}\times K_{2,\varepsilon}\right] \leq \varepsilon,$$

i.e. the set

$$\left\{\mathcal{L}\left(\left(u^{(n)}, \widetilde{W}\right)\right) : n\in\mathbb{N}\right\}$$

is tight in $\mathcal{M}_1((\mathcal{C}([0,\infty)\times\mathbb{R}))\times\mathbb{R}^{\mathbb{N}})$. Thus by Prohorov's theorem relative compact (see [Kl13, Theorem 13.29]). Therefore there is a subsequence of $(u^{(n)})_{n\in\mathbb{N}}$ which we will denote also by $(u^{(n)})_{n\in\mathbb{N}}$ and a random variable $u^{[\gamma]}$ with values in $\mathcal{C}([0,\infty)\times\mathbb{R})$ such that

$$\left(u^{(n)}, \widetilde{W}\right) \xRightarrow{n\to\infty} \left(u^{[\gamma]}, \widetilde{W}\right).$$

Using Skorokhod's theorem (see [Ka97, Theorem 3.30]) there is a probability space which we will denote again by $(\Omega, \mathcal{A}, \mathbf{P})$ such that the convergence holds $\mathbf{P}$-almost surely. As already mentioned, we can reconstruct the space-time white noise W from $\widetilde{W}$ on that probability space. For $t \geq 0$ define

$$\mathcal{H}_t := \sigma\left(\int_0^t \int_{-\infty}^{\infty} \psi(s,y)\, W(ds,dy) : \psi \in L^2([0,\infty)\times\mathbb{R})\right)$$

and

$$\mathcal{G}_t := \bigcap_{n\in\mathbb{N}} \sigma\left(\mathcal{H}_{t+\frac{1}{n}} \cup \widetilde{\mathcal{A}}\right)$$

where $\widetilde{\mathcal{A}}$ is the σ-algebra generated by the $\mathbf{P}$-zero sets of $\mathcal{A}$. Then $(\Omega, \mathcal{A}, (\mathcal{G}_t)_{t\geq 0}, \mathbf{P})$ is a filtered probability space satisfying the usual conditions.
For $s \geq 0$ and $y \in \mathbb{R}$, we have $\mathbf{P}$-almost surely

$$\left|a(u_s^{[\gamma]}(y)) - a_n(u_s^{(n)}(y))\right| \leq \left|a(u_s^{[\gamma]}(y)) - a(u_s^{(n)}(y))\right| + \left|a(u_s^{(n)}(y)) - a_n(u_s^{(n)}(y))\right| \xrightarrow{n\to\infty} 0,$$

where we used the $\mathbf{P}$-a.s.-convergence of $(u^{(n)})_{n\in\mathbb{N}}$ and continuity of the function a for the first term on the right-hand side and the uniform convergence of $(a_n)_{n\in\mathbb{N}}$ for the second term. Burkholder's inequality and dominated convergence (for the bound use Lemma 7.3) give for $t > 0$ and $x \in \mathbb{R}$

$$\begin{aligned}\mathbf{E}&\left[\left(\int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)(a(u_s^{[\gamma]}(y)) - a(u_s^{(n)}(y)))\, W(ds, dy)\right)^2\right] \\ &\leq C_{7.1}\mathbf{E}\left[\int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)^2 \left|a(u_s^{[\gamma]}(y)) - a_n(u_s^{(n)}(y))\right|^2 dy\, ds\right] \xrightarrow{n\to\infty} 0.\end{aligned}$$

for some constant $C_{7.1} > 0$. This implies that, for a suitable sub-sequence, we have $\mathbf{P}$-almost surely

$$\int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)a(u_s^{(n)}(y))\, W(ds, dy) \xrightarrow{n\to\infty} \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x)a(u_s^{[\gamma]}(y))\, W(ds, dy).$$

Since $u_t^{(n)}(x)$ converges $\mathbf{P}$-almost surely to $u_t^{[\gamma]}(x)$ and $u_t^{(n)}(x)$ satisfies (7.25), we have that $u^{[\gamma]}$ satisfies (7.2) and thus we are done. □

Next we show that the solution of the SPDE (7.1) solves a martingale problem. Recall that $\mathcal{L}_\alpha := -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian. The following proposition and its proof are an adaption of [Zä04, Propositions 6.4, 6.7] where in the case $\alpha = 2$ more general SPDEs were considered.

Proposition 7.8. *Consider the mild solution $(u_t^{[\gamma]})_{t\geq 0}$ of (7.1) from Theorem 7.1. Then $(u_t^{[\gamma]})_{t\geq 0}$ solves $\mathbf{P}$-almost surely for each $t \geq 0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$*

$$\left\langle u_t^{[\gamma]}, \phi\right\rangle = \left\langle u_0^{[\gamma]}, \phi\right\rangle + \int_0^t \left\langle u_s^{[\gamma]}, \mathcal{L}_\alpha\phi\right\rangle ds + \int_0^t \int_{-\infty}^{\infty} \sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\phi(y)\, W(ds, dy). \tag{7.26}$$

Let further

$$M_t(\phi) := \left\langle u_t^{[\gamma]}, \phi\right\rangle - \left\langle u_0^{[\gamma]}, \phi\right\rangle - \int_0^t \left\langle u_s^{[\gamma]}, \mathcal{L}_\alpha\phi\right\rangle ds.$$

Then $(M_t(\phi))_{t\geq 0}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \int_{-\infty}^{\infty} \gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))(\phi(y))^2\, dy\, ds.$$

Proof. First note that the martingale property follows immediately from (7.26). Now define again

$$a(x) := \sqrt{\gamma x(1-x)}$$

and let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be arbitrary. Due to Lemma A.3 we have $\mathcal{L}_\alpha \phi \in L^1(\mathbb{R})$. We can write the mild formulation (7.2) as

$$u_t^{[\gamma]}(x) = (S_t u_0^{[\gamma]})(x) + \int_0^t \int_{-\infty}^{\infty} p_{t-s}^{(\alpha)}(y-x) a(u_s^{[\gamma]}(y))\, W(ds, dy).$$

This yields to

$$\begin{aligned} \int_0^t \left\langle u_s^{[\gamma]}, \mathcal{L}_\alpha \phi \right\rangle \, ds &= \int_0^t \left\langle S_s u_0^{[\gamma]}, \mathcal{L}_\alpha \phi \right\rangle \, ds \\ &\quad + \int_0^t \int_0^s \int_{-\infty}^{\infty} (S_{s-r}(\mathcal{L}_\alpha \phi))(y)\, a(u_r^{[\gamma]}(y))\, W(dr, dy)\, ds. \end{aligned} \tag{7.27}$$

Using

$$(S_t \phi)(x) - \phi(x) = \int_0^t (S_s(\mathcal{L}_\alpha \phi))(x)\, ds, \qquad x \in \mathbb{R} \tag{7.28}$$

from [EK86, Proposition 1.1.5] we can calculate for the first term on the right-hand side of (7.27)

$$\begin{aligned} \int_0^t \left\langle S_s u_0^{[\gamma]}, \mathcal{L}_\alpha \phi \right\rangle \, ds &= \int_0^t \left\langle u_0^{[\gamma]}, S_s(\mathcal{L}_\alpha \phi) \right\rangle \, ds \\ &= \left\langle u_0^{[\gamma]}, \int_0^t S_s(\mathcal{L}_\alpha \phi)\, ds \right\rangle \\ &= \left\langle u_0^{[\gamma]}, S_t \phi \right\rangle - \left\langle u_0^{[\gamma]}, \phi \right\rangle. \end{aligned} \tag{7.29}$$

With Fubini's theorem for stochastic integrals (see [Wa86, Theorem 2.6]) we have for the second term

$$\begin{aligned} &\int_0^t \int_0^s \int_{-\infty}^{\infty} (S_{s-r}(\mathcal{L}_\alpha \phi))(y)\, a(u_r^{[\gamma]}(y))\, W(dr, dy)\, ds \\ &\quad = \int_0^t \int_{-\infty}^{\infty} \int_r^t (S_{s-r}(\mathcal{L}_\alpha \phi))(y)\, ds\, a(u_r^{[\gamma]}(y))\, W(dr, dy) \\ &\quad = \int_0^t \int_{-\infty}^{\infty} \int_0^{t-r} (S_s(\mathcal{L}_\alpha \phi))(y)\, ds\, a(u_r^{[\gamma]}(y))\, W(dr, dy) \\ &\quad = \int_0^t \int_{-\infty}^{\infty} [(S_{t-r}\phi)(y) - \phi(y)]\, a(u_r^{[\gamma]}(y))\, W(dr, dy), \end{aligned} \tag{7.30}$$

where we used (7.28) in the last equality. Putting (7.29) and (7.30) into (7.27) we get

$$\begin{aligned} \int_0^t \left\langle u_s^{[\gamma]}, \mathcal{L}_\alpha \phi \right\rangle \, ds &= \left\langle u_0^{[\gamma]}, S_t \phi \right\rangle - \left\langle u_0^{[\gamma]}, \phi \right\rangle \\ &\quad + \int_0^t \int_{-\infty}^{\infty} [(S_{t-r}\phi)(y) - \phi(y)]\, a(u_r^{[\gamma]}(y))\, W(dr, dy). \end{aligned} \tag{7.31}$$

In addition the mild formulation yields

$$\begin{aligned}\left\langle u_t^{[\gamma]},\phi\right\rangle &= \left\langle S_t u_0^{[\gamma]},\phi\right\rangle + \left\langle \int_0^t\int_{-\infty}^{\infty} p_{t-r}^{(\alpha)}(y-\cdot)a(u_r^{[\gamma]}(y))\,W(dr,dy),\phi\right\rangle \\ &= \langle u_0, S_t\phi\rangle + \int_0^t\int_{-\infty}^{\infty}(S_{t-r}\phi)(y)a(u_r^{[\gamma]}(y))\,W(dr,dy).\end{aligned} \tag{7.32}$$

Subtracting (7.31) from (7.32) shows finally

$$\left\langle u_t^{[\gamma]},\phi\right\rangle - \int_0^t \left\langle u_s^{[\gamma]},\mathcal{L}_\alpha\phi\right\rangle\,ds = \left\langle u_0^{[\gamma]},\phi\right\rangle + \int_0^t\int_{-\infty}^{\infty} a(u_r^{[\gamma]}(y))\phi(y)\,W(dr,dy)$$

as claimed. □

7.4. Duality with a coalescing system with delay

The aim of this section is to show the uniqueness-part of Theorem 7.1. Therefore we show a moment duality relation with a system of coalescing stable processes where two processes do not coalesce immediately when they hit each other, but they coalesce when the local time of their difference in 0 exceeds the value of an exponentially distributed random variable with parameter γ. Thus we have a coalescing system with delay. Such systems have been studied for example by [EF96, Section 3.1] where coalescing Lévy processes on a hierarchical group were considered. Further the Brownian case $\alpha = 2$ has been analyzed in [Sh88], [Tr95], [AT00] and [DF16, Section 8.1]. After introducing the coalescing system we prove in Proposition 7.9 the moment duality and afterwards get the uniqueness-part of Theorem 7.1.

Let $N \in \mathbb{N}$ and let $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(N)}$ be independent standard α-stable processes of index $\alpha \in (1,2)$. Let $(e^{(i,j)})_{1\le i<j\le N}$ be independent exponentially distributed random variables with parameter 1. Denote by $L^{(i,j)}$ the local time of the rate-2-stable process $\xi^{(j)} - \xi^{(i)}$ in 0. We give a construction of a vector-valued system $\zeta^{[\gamma]} = (\zeta^{[\gamma,1]}, \zeta^{[\gamma,2]}, \ldots, \zeta^{[\gamma,N]})$ together with the lifetimes $\tau^{[\gamma,1]}, \tau^{[\gamma,2]}, \ldots, \tau^{[\gamma,N]}$ for $\gamma \in (0,\infty]$. Let $\tau^{[\gamma,1]} = \infty$ and $\zeta^{[\gamma,1]} = \xi^{(1)}$. Assume that $\zeta^{[\gamma,1]}, \zeta^{[\gamma,2]}, \ldots, \zeta^{[\gamma,i-1]}$ and $\tau^{[\gamma,1]}, \tau^{[\gamma,2]}, \ldots, \tau^{[\gamma,i-1]}$ are already constructed for some $i \in \{1,2,\ldots,N-1\}$. For $j \in \{1,\ldots,i-1\}$ define (in the case $\gamma = \infty$ we read $\frac{1}{\infty} = 0$)

$$\tau^{[\gamma,j,i]} = \inf\left\{t \ge 0 \,:\, L^{(j,i)}_{t\wedge\tau^{[\gamma,j]}} > \frac{1}{\gamma}e^{(j,i)}\right\}$$

as well as

$$\tau^{[\gamma,i]} = \min\left\{\tau^{[\gamma,j,i]} \,:\, j \in \{1,2,\ldots,i-1\}\right\}$$

and

$$J^{[\gamma,i]} := \min\left\{j \in \{1,2,\ldots,i-1\} \,:\, \tau^{[\gamma,i]} = \tau^{[\gamma,j,i]}\right\}$$

Thus particle number $J^{[\gamma,i]}$ is the coalescence partner of particle i and they coalesce at time $\tau^{[\gamma,i]} = \tau^{[\gamma,J^{[\gamma,i]},i]}$. For $t \geq 0$ let

$$\zeta_t^{[\gamma,i]} := \begin{cases} \xi_t^{(i)}, & \text{if } t < \tau^{[\gamma,i]}, \\ \zeta_t^{[\gamma,J^{[\gamma,i]}]}, & \text{if } t \geq \tau^{[\gamma,i]}. \end{cases}$$

Note that in the case $\gamma = \infty$ the introduced coalescing system coincides with the instantaneous coalescing system from Section 2.2. In the following notation we suppress for the sake of clarity the dependence of γ (since we need it only for fixed $\gamma \in (0,\infty)$ in this section). Define for $t \geq 0$

$$N_t := \left|\left\{ i \in \{1,2,\ldots,N\} \, : \, \tau^{[\gamma,i]} > t \right\}\right|$$

the number of particles that are still alive at time t and let

$$\eta_t := \left(\zeta_t^{[\gamma,i]}\right)_{1\leq i\leq N, \tau^{[\gamma,i]}>t} = \left(\xi_t^{(i)}\right)_{1\leq i\leq N, \tau^{[\gamma,i]}>t},$$

such that $\eta_t = (\eta_t^{(1)}, \eta_t^{(2)}, \ldots, \eta_t^{(N_t)})$ is a N_t-dimensional vector (recall that $\zeta^{[\gamma]} = (\zeta_t^{[\gamma,1]}, \zeta_t^{[\gamma,2]}, \ldots, \zeta_t^{[\gamma,N]})$ is a N-dimensional vector).

Now we state the moment duality result in terms of $(\eta_t)_{t\geq 0}$. The proof follows [DF16, Section 8.1].

Proposition 7.9. *Let $\gamma \in (0,\infty)$. Consider the solution $(u_t^{[\gamma]})_{t\geq 0}$ of (7.1) from Theorem 7.1 and a system of coalescing stable processes with delay $(\eta_t)_{t\geq 0}$ with initial condition $\eta_0 = (x_1, \ldots, x_n) \in \mathbb{R}^n$ for some $n \in \mathbb{N}$. For each $t \geq 0$, we have*

$$\mathbf{E}\left[\prod_{i=1}^{n} u_t^{[\gamma]}(x_i)\right] = \mathbf{E}\left[\prod_{i=1}^{N_t} u_0^{[\gamma]}\left(\eta_t^{(i)}\right)\right]. \tag{7.33}$$

Proof. In this proof we drop the index γ and write u instead of $u^{[\gamma]}$. For $\varepsilon > 0$ and $x \in \mathbb{R}$, define

$$\phi^{(\varepsilon,x)}(y) := \frac{1}{\sqrt{2\pi\varepsilon}} e^{-(y-x)^2/(2\varepsilon)}$$

and for every bounded measurable function $f\colon \mathbb{R} \to \mathbb{R}$ let

$$f^{(\varepsilon)}(x) := \int_{-\infty}^{\infty} f(y)\phi^{(\varepsilon,x)}(y)\, dy.$$

Using the weak formulation (7.26) from Proposition 7.8 with $\phi = \phi^{(\varepsilon,x)}$ we get

$$u_t^{(\varepsilon)}(x) = u_0^{(\varepsilon)}(x) + \int_0^t (\mathcal{L}_\alpha u_s^{(\varepsilon)})(x)\, ds + \int_0^t \int_{-\infty}^{\infty} \sqrt{\gamma u_s(y)(1-u_s(y))}\phi^{(\varepsilon,x)}(y)\, W(ds,dy).$$

Ito's formula (see [IW89, Chapter 2, Theorem 5.1]) yields for $m \in \mathbb{N}$ and $z_1, \dots, z_m \in \mathbb{R}$

$$\begin{aligned}
\prod_{i=1}^m u_t^{(\varepsilon)}(z_i) &= \prod_{i=1}^m u_0^{(\varepsilon)}(z_i) + \sum_{i=1}^m \int_0^t \prod_{\substack{j=1\\ j\neq i}}^m u_s^{(\varepsilon)}(z_j)\, du_s^{(\varepsilon)}(z_i) \\
&\quad + \frac{1}{2}\sum_{i=1}^m \sum_{\substack{j=1\\ j\neq i}}^m \int_0^t \prod_{\substack{k=1\\ k\neq i,j}}^m u_s^{(\varepsilon)}(z_k)\, d\langle u^{(\varepsilon)}(z_i), u^{(\varepsilon)}(z_j)\rangle_s \\
&= \prod_{i=1}^m u_0^{(\varepsilon)}(z_i) + \sum_{i=1}^m \int_0^t \prod_{\substack{j=1\\ j\neq i}}^m u_s^{(\varepsilon)}(z_j)(\mathcal{L}_\alpha u_s^{(\varepsilon)})(z_i)\, ds \\
&\quad + \sum_{i=1}^m \int_0^t \int_{-\infty}^\infty \prod_{\substack{j=1\\ j\neq i}}^m u_s^{(\varepsilon)}(z_j)\sqrt{\gamma u_s(y)(1-u_s(y))}\phi^{(\varepsilon,z_i)}(y)\, W(ds,dy) \\
&\quad + \sum_{i=1}^{m-1} \sum_{j=i+1}^m \int_0^t \prod_{\substack{k=1\\ k\neq i,j}}^m u_s^{(\varepsilon)}(z_k) \int_{-\infty}^\infty \gamma u_s(y)(1-u_s(y))\phi^{(\varepsilon,z_i)}(y)\phi^{(\varepsilon,z_j)}(y)\, dy\, ds.
\end{aligned} \tag{7.34}$$

Denote by $\mathcal{A}_\alpha$ the generator of $(\eta_t)_{t\geq 0}$ and by $L^{(z,i,j)}$ the local time process of the difference of $\eta^{(j)} - \eta^{(i)}$ in $z \in \mathbb{R}$. Further let $L^{(i,j)} := L^{(0,i,j)}$. Let $f\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$ be a measurable function such that $f_t := f(t,\cdot)$ is an element of $\mathcal{C}_b(\mathbb{R})$ and contained in the domain of the fractional Laplacian for each $i \in \mathbb{N}$ and $t \geq 0$. For $x \in \bigcup_{m\in\mathbb{N}} \mathbb{R}^m$ with $x = (x_1, \dots, x_m)$ for some $m \in \mathbb{N}$ write $f_t(x) := \prod_{i=1}^m f_t(x_i)$. Then the coalescing system satisfies

$$\begin{aligned}
&\int_0^t (\mathcal{A}_\alpha f_s)(\eta_s^{(1)}, \dots, \eta_s^{(N_s)})\, ds \\
&= \int_0^t \sum_{i=1}^{N_s} (\mathcal{L}_\alpha f_s)(\eta_s^{(i)}) \prod_{\substack{j=1\\ j\neq i}}^{N_s} f_s(\eta_s^{(j)})\, ds \\
&\quad + \int_0^t \sum_{i=1}^{N_s-1} \sum_{j=i+1}^{N_s} \gamma \left(f_s(\eta_s^{(i)}) \prod_{\substack{k=1\\ k\neq i,j}}^{N_s} f_s(\eta_s^{(k)}) - f_s(\eta_s^{(i)}) f_s(\eta_s^{(j)}) \prod_{\substack{k=1\\ k\neq i,j}}^{N_s} f_s(\eta_s^{(k)}) \right) dL_s^{(i,j)} \qquad (7.35) \\
&= \int_0^t \sum_{i=1}^{N_s} (\mathcal{L}_\alpha f_s)(\eta_s^{(i)}) \prod_{\substack{j=1\\ j\neq i}}^{N_s} f_s(\eta_s^{(j)})\, ds \\
&\quad + \int_0^t \sum_{i=1}^{N_s-1} \sum_{j=i+1}^{N_s} \gamma \left(f_s(\eta_s^{(i)})(1 - f_s(\eta_s^{(j)})) \prod_{\substack{k=1\\ k\neq i,j}}^{N_s} f_s(\eta_s^{(k)}) \right) dL_s^{(i,j)}.
\end{aligned}$$

If we define for $\varepsilon > 0$, $v, \widetilde{v} \in \mathcal{C}_b(\mathbb{R})$ and $z \in \bigcup_{m\in\mathbb{N}} \mathbb{R}^m$ with $z = (z_1, \dots, z_m)$ for some $m \in \mathbb{N}$

$$F(z, \widetilde{v}) := \prod_{i=1}^{m} \widetilde{v}(z_i)$$

and

$$G(z, \widetilde{v}) := \left(\mathcal{A}_\alpha F(\cdot, \widetilde{v})\right)(z)$$

and if we let

$$\begin{aligned} H^{(\varepsilon)}(z, v, \widetilde{v}) &:= \sum_{i=1}^{m} \prod_{\substack{j=1 \\ j\neq i}}^{m} \widetilde{v}(z_j)(\mathcal{L}_\alpha \widetilde{v})(z_i) \\ &+ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \prod_{\substack{k=1 \\ k\neq i,j}}^{m} \widetilde{v}(z_k) \int_{-\infty}^{\infty} \gamma v(y)(1 - v(y))\phi^{(\varepsilon,z_i)}(y)\phi^{(\varepsilon,z_j)}(y)\, dy \end{aligned}$$

it follows on the one hand from general theory on Markov processes (see [EK86, Proposition 4.1.7]) and on the other hand from equation (7.34) that

$$F(\eta_t, \widetilde{v}) - F(\eta_0, \widetilde{v}) - \int_0^t G(\eta, \widetilde{v})\, ds$$

and

$$F(z, u_t^{(\varepsilon)}) - F(z, u_0^{(\varepsilon)}) - \int_0^t H^{(\varepsilon)}(z, u_s, u_s^{(\varepsilon)})\, ds$$

are martingales seen as process in $t \geq 0$. [EK86, Theorem 4.4.11] thus yields

$$\begin{aligned} &\mathbf{E}\left[F(\eta_t, u_0^{(\varepsilon)})\right] - \mathbf{E}\left[F(\eta_0, u_t^{(\varepsilon)})\right] \\ &\quad = \mathbf{E}\left[\int_0^t G(\eta_s, u_{t-s}^{(\varepsilon)}) - H^{(\varepsilon)}(\eta_s, u_{t-s}, u_{t-s}^{(\varepsilon)})\, ds\right]. \end{aligned} \tag{7.36}$$

Note that $H^{(\varepsilon)}$ depands on u and $u^{(\varepsilon)}$ in contrast to the formulation in [EK86, Theorem 4.4.11], but that does not change the result. Since $x \mapsto u_t(x)$ is continuous we have using dominated convergence (since $u_t(x) \in [0, 1]$ for each $x \in \mathbb{R}$)

$$\begin{aligned} &\mathbf{E}\left[F(\eta_t, u_0^{(\varepsilon)})\right] - \mathbf{E}\left[F(\eta_0, u_t^{(\varepsilon)})\right] \\ &\quad \xrightarrow{\varepsilon\searrow 0} \quad \mathbf{E}\left[F(\eta_t, u_0)\right] - \mathbf{E}\left[F(\eta_0, u_t)\right] \\ &\quad = \mathbf{E}\left[\prod_{i=1}^{N_t} u_0\left(\eta_t^{(i)}\right)\right] - \mathbf{E}\left[\prod_{i=1}^{n} u_t\left(x_i\right)\right]. \end{aligned}$$

Hence to finish the proof we have to show that the right-hand side of (7.36) goes to zero as ε goes to zero. Using (7.35) and the definition of $H^{(\varepsilon)}$ we have

$$
\begin{aligned}
&\mathbf{E}\left[\int_0^t G(\eta_s, u^{(\varepsilon)}_{t-s}) - H^{(\varepsilon)}(\eta_s, u_{t-s}, u^{(\varepsilon)}_{t-s})\, ds\right] \\
&= \mathbf{E}\left[\int_0^t \sum_{i=1}^{N_s-1} \sum_{j=i+1}^{N_s} \gamma \left(u^{(\varepsilon)}_{t-s}(\eta^{(i)}_s)(1 - u^{(\varepsilon)}_{t-s}(\eta^{(j)}_s)) \prod_{\substack{k=1\\ k\neq i,j}}^{N_s} u^{(\varepsilon)}_{t-s}(\eta^{(k)}_s)\right) dL^{(0,i,j)}_s\right] \\
&\quad - \mathbf{E}\Bigg[\int_0^t \sum_{i=1}^{N_s-1} \sum_{j=i+1}^{N_s} \prod_{\substack{k=1\\ k\neq i,j}}^{N_s} u^{(\varepsilon)}_{t-s}(\eta^{(k)}_s) \\
&\qquad\qquad \int_{-\infty}^{\infty} \gamma u_{t-s}(y)(1-u_{t-s}(y))\phi^{(\varepsilon,\eta^{(i)}_s)}(y)\phi^{(\varepsilon,\eta^{(j)}_s)}(y)\, dy\, ds\Bigg].
\end{aligned}
\tag{7.37}
$$

Note that for each $t \geq 0$

$$
N_t \leq n, \qquad \sup_{x\in\mathbb{R}} u_t(x) \leq 1, \qquad \sup_{x\in\mathbb{R}} u^{(\varepsilon)}_t(x) \leq 1. \tag{7.38}
$$

Further from the occupation time formula (2.6) it follows that

$$
L^{(z,i,j)}_t = \lim_{\delta\searrow 0} \frac{1}{2\delta}\int_0^t \mathbf{1}_{[z-\delta,z+\delta]}(\eta^{(i)}_s - \eta^{(j)}_s)\, ds.
$$

Using Fatou's lemma, $p^{(\alpha)}_t(x) \leq c_2(\alpha)t^{-\frac{1}{\alpha}}$ for $x\in\mathbb{R}$ from (2.2) and $\int_0^t s^{-\frac{1}{\alpha}}\, ds < \infty$ (due to $\alpha \in (1,2]$) this implies

$$
\sup_{z\in\mathbb{R}} \mathbf{E}[L^{(z,i,j)}_t] < \infty. \tag{7.39}
$$

Let us define for $y \in \mathbb{R}$ and $\varepsilon \geq 0$

$$
Y^{(\varepsilon,i,j)}_{s,t}(y) := \gamma \left(\prod_{\substack{k=1\\ k\neq i,j}}^{N_s} u^{(\varepsilon)}_{t-s}(\eta^{(k)}_s)\right) u_{t-s}(y+\eta^{(i)}_s)\left(1 - u_{t-s}(y+\eta^{(i)}_s)\right)\mathbf{1}_{A^{(i,j)}_s}
$$

where $A^{(i,j)}_s$ is the event that particle i and j are still alive at time s and where we read $u^{(0)}_t := u_t$ in the case $\varepsilon = 0$. The first term on the right-hand side of (7.37) converges to

$$
\mathbf{E}\left[\int_0^t \sum_{i=1}^{N_s-1}\sum_{j=i+1}^{N_s} Y^{(0,i,j)}_{s,t}(0)\, dL^{(0,i,j)}_s\right] \tag{7.40}
$$

as ε goes to zero where we used dominated convergence (using (7.38) and (7.39) for the domination). Thus we have to show that the second term on the right-hand side of (7.37) converges also to (7.40). Using the substitution $y \mapsto y + \eta_s^{(i)}$ this second term is equal to

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t \sum_{i=1}^{N_s-1} \sum_{j=i+1}^{N_s} \int_{-\infty}^{\infty} \phi^{(\varepsilon,0)}(y)\phi^{(\varepsilon,0)}(y+\eta_s^{(i)}-\eta_s^{(j)})Y_{s,t}^{(\varepsilon,i,j)}(y)\,dy\,ds\right] \\
&\quad = \mathbf{E}\left[\int_{-\infty}^{\infty}\int_0^t \phi^{(\varepsilon,0)}(y)\phi^{(\varepsilon,0)}(y+\eta_s^{(i)}-\eta_s^{(j)})\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Y_{s,t}^{(\varepsilon,i,j)}(y)\,ds\,dy\right] \\
&\quad = \mathbf{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi^{(\varepsilon,0)}(y)\phi^{(\varepsilon,0)}(y+z)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\int_0^t Y_{s,t}^{(\varepsilon,i,j)}(y)\,dL_s^{(z,i,j)}\,dz\,dy\right]
\end{aligned}$$

where we used Fubini's theorem and the indicator on $A_s^{(i,j)}$ in the definition of $Y_{s,t}^{(\varepsilon,i,j)}(y)$ in the first equality and the occupation time formula (Lemma 2.7) in the last equality. Using the fact that $\phi^{(\varepsilon,0)}$ is a probability density and the continuity of $y \mapsto Y_{s,t}^{(\varepsilon,i,j)}(y)$ and $z \mapsto L_t^{(z,i,j)}$ we get that the latter converges to (7.40) as ε goes to zero using again dominated convergence (with (7.38) and (7.39) for the domination). □

Now we can complete the proof of Theorem 7.1.

Proof of Theorem 7.1 (Uniqueness & Strong Markov property). Weak solutions of (7.1) from Theorem 7.1 solve a martingale problem due to Proposition 7.8. The moment duality from Proposition 7.9 determines the one-dimensional distributions of such solutions. Thus the uniqueness follows from [EK86, Theorem 4.4.2 (a)] and the strong Markov property from [EK86, Theorem 4.4.2 (c)]. Note that [EK86, Theorem 4.4.2 (c)] requires measurability in the initial value of the distribution of the solution. This is true because the Picard iteration in the proof of Proposition 7.6 is measurable in the initial value by construction. Thus, the limits in the proof of Proposition 7.6 and in the proof of existence of Theorem 7.1 are also measurable. □

7.5. Tightness

Consider the mild solution $(u_t^{[\gamma]})_{t\geq 0}$ of (7.1) from Theorem 7.1. For $t \geq 0$ write

$$u_t^{[\gamma]}(dx) = u_t^{[\gamma]}(x)\,dx.$$

That is, we interpret $u_t^{[\gamma]}$ as a random variable with values in $\mathcal{M}_{\leq 1}(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$. The aim of this and the next section is to prove Theorem 7.2. To this end, in this section we will show the tightness of $(u^{[\gamma]})_{\gamma>0}$ in $\mathcal{C}((0,\infty),\mathcal{M}_{\leq 1}(\mathbb{R}))$, i.e., we will show tightness in $\mathcal{C}([\varepsilon,\infty),\mathcal{M}(\mathbb{R}))$ for each $\varepsilon > 0$ (note that $\mathcal{M}_{\leq 1}(\mathbb{R})$ is a compact subset of $\mathcal{M}(\mathbb{R})$ due to [HOV21, Lemma A.1]). Since $(u_t^{[\gamma]})_{t\geq 0}$ has continuous sample paths it is enough to show tightness in the space of càdlàg-paths with the Skorokhod topology (see Remark 7.16 for details).

The main step is to show tightness of $(u^{[\gamma]}(\phi))_{\gamma>0}$ where $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Let $t > 0$. Due to (7.2) one has

$$u_t^{[\gamma]}(\phi) = \left\langle u_0^{[\gamma]}, S_t\phi \right\rangle + U_t^{[\gamma]}(\phi) = \langle u_0, S_t\phi\rangle + U_t^{[\gamma]}(\phi) \tag{7.41}$$

with

$$U_t^{[\gamma]}(\phi) := \int_0^t \int_{-\infty}^\infty (S_{t-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy).$$

Further, for $x \in \mathbb{R}$, let

$$U_t^{[\gamma]}(x) := \int_0^t \int_{-\infty}^\infty p_{t-s}^{(\alpha)}(x-y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy).$$

The crucial point is a second moment bound of

$$\begin{aligned} U_{t'}^{[\gamma]}(\phi) - U_t^{[\gamma]}(\phi) = &\int_0^t \int_{-\infty}^\infty (S_{t'-s}\phi - S_{t-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy) \\ &+ \int_t^{t'} \int_{-\infty}^\infty (S_{t'-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy) \end{aligned} \tag{7.42}$$

where $\varepsilon \le t < t'$.

Here is the outline for this section: We start in Lemma 7.10 and Corollary 7.11 with bounds on the first summand of the decomposition of the right-hand side of (7.42). Then we will prove some intermediate results to obtain with the duality from the last section a second moment bound of the second summand in Corollary 7.14 which leads to a bound on $U_{t'}^{[\gamma]}(\phi) - U_t^{[\gamma]}(\phi)$ in Corollary 7.15. In Proposition 7.17 we will get tightness of $(u^{[\gamma]}(\phi))_{\gamma>0}$ by Aldous' criterion (see [Da93, Theorem 3.6.5]) and in Proposition 7.18 and Corollary 7.19 tightness of the measure-valued process $(u^{[\gamma]})_{\gamma>0}$ with Jakubowski's criterion (see [Da93, Theorem 3.6.4]).

Lemma 7.10. *For each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ and for measurable and bounded $f\colon \mathbb{R} \to \mathbb{R}$ we have*

$$|\langle f, (S_{t'} - S_t)\phi\rangle| \le \|f\|_\infty \|\mathcal{L}_\alpha \phi\|_{L^1(\mathbb{R})} |t'-t|, \qquad t,t' \ge 0.$$

Proof. Since $\mathcal{L}_\alpha\phi \in L^1(\mathbb{R})$ (see Lemma A.3), we compute using [EK86, Proposition 1.1.5] in the first equality

$$\begin{aligned} |\langle f, (S_{t'} - S_t)\phi\rangle| &= \left| \left\langle f, \int_t^{t'} S_s(\mathcal{L}_\alpha \phi)\, ds \right\rangle \right| \\ &\le \|f\|_\infty \int_t^{t'} \int_{-\infty}^\infty |(S_s(\mathcal{L}_\alpha\phi))(x)|\, dx\, ds. \end{aligned}$$

S_s is a $L^1(\mathbb{R})$-contraction, i.e. $\|S_s(\mathcal{L}_\alpha\phi)\|_{L^1(\mathbb{R})} \le \|\mathcal{L}_\alpha\phi\|_{L^1(\mathbb{R})}$. This implies the result. $\square$

Corollary 7.11. *For each $\gamma > 0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ one has* $\mathbf{P}$*-almost surely for* $0 \le t < t'$

$$\left|\int_0^t \int_{-\infty}^\infty (S_{t'-s}\phi - S_{t-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\right| \le 2\left\|\mathcal{L}_\alpha \phi\right\|_{L^1(\mathbb{R})} \left|t'-t\right|.$$

Proof. Use the following computation and Lemma 7.10 (note that $\|U_t^{[\gamma]}\|_\infty \le 2$ $\mathbf{P}$-almost surely (due to the definition of $U_t^{[\gamma]}$ and (7.2)).

$$\begin{aligned}
&\int_0^t \int_{-\infty}^\infty (S_{t'-s}\phi - S_{t-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\\
&\quad = \int_0^t \int_{-\infty}^\infty (S_{t-s}(S_{t'-t} - S_0)\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\\
&\quad = U_t^{[\gamma]}((S_{t'-t} - S_0)\phi)\\
&\quad = \left\langle U_t^{[\gamma]}, (S_{t'-t} - S_0)\phi\right\rangle.
\end{aligned}$$

□

Lemma 7.12. *For $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, there is a constant $C_{7.12} = C_{7.12}(\alpha,\phi) > 0$ such that one has for $0 < t < t'$*

$$\int_t^{t'} \int_{-\infty}^\infty (S_{t'-s}\phi)(y)^2\, dy\, ds \le C_{7.12}\left|t'-t\right|^{1-1/\alpha}.$$

Proof. Using Plancherel's theorem and the convolution-theorem for the Fourier transform, we get

$$\int_t^{t'} \int_{-\infty}^\infty (S_{t'-s}\phi)(y)^2\, dy\, ds \le C_{7.12.1}\int_t^{t'} \int_{-\infty}^\infty p_{t'-s}^{(\alpha)}(y)^2\, dy\, ds$$

for some constant $C_{7.12.1} = C_{7.12.1}(\phi) > 0$. Then use Lemma 7.3. □

Lemma 7.13. *There is a constant $C_{7.13} = C_{7.13}(\alpha) > 0$ such that for $s > 0$, $y \in \mathbb{R}$ and $\gamma > 0$ one has*

$$\gamma \mathbf{E}\left[u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))\right] \le C_{7.13} s^{-(1-\frac{1}{\alpha})}.$$

Proof. The idea is to apply the moment duality from Proposition 7.9 in the case of $n = 2$ particles. Let us recall the notation of the dual process from the beginning of Section 7.4: Let $\xi^{(1)}$ and $\xi^{(2)}$ be two independent standard stable processes with $\xi_0^{(1)} = \xi_0^{(2)} = y$, denote by $L^{(1,2)}$ the local time process of $\xi^{(2)} - \xi^{(1)}$ in 0. Note that $\xi_t^{(2)} - \xi_t^{(1)} \overset{d}{=} X_{2t}$ where X is a standard stable process with $X_0 = 0$. If we denote its local time process in 0 by L we have the identity $L_t^{(1,2)} \overset{d}{=} \frac{1}{2}L_{2t}$. This follows from the occupation time formula (2.6). So according to Lemma 2.8 there exists a constant $C_{7.13} = C_{7.13}(\alpha) > 0$ (namely $C_{7.13} = 2 \cdot 2^{-(1-\frac{1}{\alpha})} \cdot C_{2.8}$) such that

$$\mathbf{E}\left[e^{-\gamma L_s^{(1,2)}}\right] \le \frac{C_{7.13}}{\gamma} s^{-(1-\frac{1}{\alpha})}.$$

Now let $e^{(1,2)}$ be an exponential distributed random variable with parameter 1 and define

$$\tau^{[\gamma,1,2]} = \inf\left\{t \geq 0 \,:\, L_t^{(1,2)} \geq \frac{1}{\gamma} e^{(1,2)}\right\}.$$

Let $\tau^{[\gamma,1]} = \infty$ and $\tau^{[\gamma,2]} = \tau^{[\gamma,1,2]}$. We compute

$$\begin{aligned}\mathbf{P}[\tau^{[\gamma,2]} > s] &= \mathbf{P}[e^{(1,2)} > \gamma L_s^{(1,2)}] = \mathbf{E}\left[\mathbf{P}[\, e^{(1,2)} > \gamma L_s^{(1,2)} \mid L_s^{(1,2)}\,]\right] = \mathbf{E}\left[e^{-\gamma L_s^{(1,2)}}\right] \\ &\leq \frac{C_{7.13}}{\gamma} s^{-\left(1-\frac{1}{\alpha}\right)}.\end{aligned}$$

The moment duality (see Proposition 7.9) yields

$$\begin{aligned}\gamma \mathbf{E}\left[u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))\right] &= \gamma\left(\mathbf{E}\left[u_s^{[\gamma]}(y)\right] - \mathbf{E}\left[u_s^{[\gamma]}(y)^2\right]\right) \\ &= \gamma\left(\mathbf{E}\left[u_0(\xi_s^{(1)})\right] - \mathbf{E}\left[\prod_{\substack{i=1 \\ \tau^{[\gamma,i]}>s}}^{2} u_0(\xi_s^{(i)})\right]\right) \\ &= \gamma \mathbf{E}\left[u_0(\xi_s^{(1)})(1-u_0(\xi_s^{(2)}))\mathbf{1}_{\{\tau^{[\gamma,2]}>s\}}\right] \\ &\leq \gamma \mathbf{P}[\tau^{[\gamma,2]} > s] \\ &\leq C_{7.13} s^{-\left(1-\frac{1}{\alpha}\right)}.\end{aligned}$$

□

Corollary 7.14. *Let $\varepsilon > 0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. There is a constant $C_{7.14} = C_{7.14}(\alpha, \varepsilon, \phi) > 0$ such that for each $\gamma > 0$ and $\varepsilon \leq t < t'$ one has*

$$\begin{aligned}\mathbf{E}&\left[\left(\int_t^{t'}\int_{-\infty}^{\infty} (S_{t'-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\right)^2\right] \\ &\leq C_{7.14}\left|t'-t\right|^{1-1/\alpha}.\end{aligned}$$

Proof. We compute

$$\begin{aligned}\mathbf{E}&\left[\left(\int_t^{t'}\int_{-\infty}^{\infty} (S_{t'-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\right)^2\right] \\ &= \int_t^{t'}\int_{-\infty}^{\infty} (S_{t'-s}\phi)(y)^2 \gamma \mathbf{E}\left[u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))\right] dy\, ds \\ &\leq C_{7.13}\varepsilon^{-(1-\frac{1}{\alpha})}\int_t^{t'}\int_{-\infty}^{\infty} (S_{t'-s}\phi)(y)^2\, dy\, ds \\ &\leq C_{7.13}\varepsilon^{-(1-\frac{1}{\alpha})} C_{7.12}\left|t'-t\right|^{1-1/\alpha}.\end{aligned}$$

In the second inequality we used Lemma 7.13 (with $C_{7.13} = C_{7.13}(\alpha) > 0$) and in the third inequality Lemma 7.12 (with $C_{7.12} = C_{7.12}(\alpha, \phi) > 0$). Now we can define $C_{7.14} := C_{7.14}(\alpha, \varepsilon, \phi) := C_{7.13}\varepsilon^{-(1-\frac{1}{\alpha})}C_{7.12}$ and we are done. □

Corollary 7.15. *Let $\varepsilon > 0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. There is a constant $C_{7.15} = C_{7.15}(\alpha, \varepsilon, \phi) > 0$ such that for each $\gamma > 0$ and $\varepsilon \le t < t'$ with $|t' - t| \le 1$ one has*

$$\mathbf{E}\left[\left(U_{t'}^{[\gamma]}(\phi) - U_t^{[\gamma]}(\phi)\right)^2\right] \le C_{7.15}\left|t' - t\right|^{1-1/\alpha}.$$

Proof. From Corollary 7.11 we know that

$$\begin{aligned}\mathbf{E}&\left[\left(\int_0^t \int_{-\infty}^{\infty} (S_{t'-s}\phi - S_{t-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\right)^2\right] \\ &\le 4\left\|\mathcal{L}_\alpha\phi\right\|_{L^1(\mathbb{R})}^2\left|t'-t\right|^2.\end{aligned}$$

Using the decomposition (7.42) and the previous result (Corollary 7.14) we conclude

$$\begin{aligned}\mathbf{E}&\left[\left(U_{t'}^{[\gamma]}(\phi) - U_t^{[\gamma]}(\phi)\right)^2\right] \\ &\le 2\mathbf{E}\left[\left(\int_0^t \int_{-\infty}^{\infty} (S_{t'-s}\phi - S_{t-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\right)^2\right] \\ &\quad + 2\mathbf{E}\left[\left(\int_t^{t'} \int_{-\infty}^{\infty} (S_{t'-s}\phi)(y)\sqrt{\gamma u_s^{[\gamma]}(y)(1-u_s^{[\gamma]}(y))}\, W(ds,dy)\right)^2\right] \\ &\le 8\left(\left\|\mathcal{L}_\alpha\phi\right\|_{L^1(\mathbb{R})}^2\left|t'-t\right|^2\right) + 2C_{7.14}\left|t'-t\right|^{1-1/\alpha} \\ &\le C_{7.15}\left|t'-t\right|^{1-1/\alpha}\end{aligned}$$

for some constant $C_{7.15} = C_{7.15}(\alpha, \varepsilon, \phi) > 0$ (recall $C_{7.14} = C_{7.14}(\alpha, \varepsilon, \phi) > 0$). In the last inequality we used $|t' - t| \le 1$. □

Remark 7.16. Let E be a metric space. Recall from Section 7.1 that we equip $\mathcal{C}([0,\infty), \mathcal{M}(\mathbb{R}))$ with the topology of uniform convergence on compact subsets of $[0,\infty)$ and $D([0,\infty), E)$ with the Skorokhod topology. Let $X_1, X_2, \dots$ be random variables with values in $\mathcal{C}([0,\infty), E) \subset D([0,\infty), E)$. Further let X be a random variable with values in $D([0,\infty), E)$.

- If X takes values in $\mathcal{C}([0,\infty), E)$ and $X_n \xRightarrow{n\to\infty} X$ in $\mathcal{C}([0,\infty), E)$, [EK86, Proposition 3.5.3] implies that $X_n \xRightarrow{n\to\infty} X$ in $D([0,\infty), E)$.
- If $X_n \xRightarrow{n\to\infty} X$ in $D([0,\infty), E)$, [EK86, Theorem 3.10.2] implies that $X \in \mathcal{C}([0,\infty), E)$ almost surely due to the assumption that $X_1, X_2, \dots$ take values in $\mathcal{C}([0,\infty), E)$. The latter and [EK86, Lemma 3.10.1] show that $X_n \xRightarrow{n\to\infty} X$ in $\mathcal{C}([0,\infty), E)$.

Recall again from Section 7.1 that we topologize $\mathcal{M}(\mathbb{R})$ with the topology of vague convergence and that this is a Polish space. Therefore $\mathcal{C}([0,\infty),\mathcal{M}(\mathbb{R}))$ and $D([0,\infty),\mathcal{M}(\mathbb{R}))$ are Polish spaces (see [EK86, Theorem 3.5.6]). We deduce from the latter facts and Prohorov's theorem (see [Kl13, Theorem 13.29]) that for processes with values in $\mathcal{C}([0,\infty),\mathcal{M}(\mathbb{R}))$ it is equivalent to show tightness in $\mathcal{C}([0,\infty),\mathcal{M}(\mathbb{R}))$ or $D([0,\infty),\mathcal{M}(\mathbb{R}))$.

Proposition 7.17. *Let $\varepsilon > 0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. The family $(u^{[\gamma]}(\phi))_{\gamma>0}$ is tight in $\mathcal{C}([\varepsilon,\infty),\mathbb{R})$.*

Proof. Pick a sequence $(\gamma_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $\lim_{k\to\infty}\gamma_k = \infty$ and define

$$v_t^{[\varepsilon,k]}(\phi) := u_{t+\varepsilon}^{[\gamma_k]}(\phi), \qquad t \geq 0.$$

According to Remark 7.16 it is enough to show tightness of $(v^{[\varepsilon,k]}(\phi))_{k\in\mathbb{N}}$ in $D([0,\infty),\mathbb{R})$ and therefore we will use Aldous' criterion [Da93, Theorem 3.6.5]. We have to check two conditions. The first is that for fixed $t \in [0,\infty)\cap\mathbb{Q}$, the family of real valued random variables $(v_t^{[\varepsilon,k]}(\phi))_{k\in\mathbb{N}}$ is tight, but this is obvious since $\left|u_t^{[\gamma]}(x)\right| \leq 1$ almost surely for each $\gamma > 0$, $t \geq 0$ and $x \in \mathbb{R}$ and therefore

$$v_t^{[\varepsilon,k]}(\phi) \in [-\|\phi\|_\infty, \|\phi\|_\infty]$$

almost surely for each $\gamma > 0$ and $t \geq 0$. The second condition we have to check is that for each sequence $(\tau^{(k)})_{k\in\mathbb{N}}$ of finite stopping times and each $\eta > 0$, we have

$$\lim_{\delta\searrow 0}\limsup_{k\to\infty}\mathbf{P}\left[\left|v_{\tau^{(k)}+\delta}^{[\varepsilon,k]}(\phi) - v_{\tau^{(k)}}^{[\varepsilon,k]}(\phi)\right| \geq \eta\right] = 0. \tag{7.43}$$

For each $\gamma > 0$, let $\widetilde{u}^{[\gamma]}$ be an independent copy of $u^{[\gamma]}$. Using the Markov inequality, the strong Markov property (see Theorem 7.1) and the decomposition (7.41) we can compute

$$\begin{aligned}
\mathbf{P}\left[\left|v_{\tau^{(k)}+\delta}^{[\varepsilon,k]}(\phi) - v_{\tau^{(k)}}^{[\varepsilon,k]}(\phi)\right| \geq \eta\right] &\leq \frac{1}{\eta^2}\mathbf{E}\left[\left(u_{\tau^{(k)}+\varepsilon+\delta}^{[\gamma_k]}(\phi) - u_{\tau^{(k)}+\varepsilon}^{[\gamma_k]}(\phi)\right)^2\right] \\
&= \frac{1}{\eta^2}\mathbf{E}\left[\mathbf{E}\left[\left(\widetilde{u}_{\varepsilon+\delta}^{[\gamma_k]}(\phi) - \widetilde{u}_{\varepsilon}^{[\gamma_k]}(\phi)\right)^2 \,\middle|\, \widetilde{u}_0^{[\gamma_k]} = u_{\tau^{(k)}}^{[\gamma_k]}\right]\right] \\
&\leq \frac{2}{\eta^2}\Bigg(\mathbf{E}\left[\mathbf{E}\left[\left\langle \widetilde{u}_0^{[\gamma_k]}, (S_{\varepsilon+\delta} - S_\varepsilon)\phi\right\rangle^2 \,\middle|\, \widetilde{u}_0^{[\gamma_k]} = u_{\tau^{(k)}}^{[\gamma_k]}\right]\right] \\
&\qquad + \mathbf{E}\left[\mathbf{E}\left[\left(\widetilde{U}_{\varepsilon+\delta}^{[\gamma_k]}(\phi) - \widetilde{U}_{\varepsilon}^{[\gamma_k]}(\phi)\right)^2 \,\middle|\, \widetilde{u}_0^{[\gamma_k]} = u_{\tau^{(k)}}^{[\gamma_k]}\right]\right]\Bigg)
\end{aligned}$$

According to Lemma 7.10 and Corollary 7.15 (with the constant $C_{7.15} = C_{7.15}(\alpha,\varepsilon,\phi) > 0$)

we have for each $k \in \mathbb{N}$ and $\delta \in (0,1]$ the bound

$$\mathbf{P}\Big[\Big|v^{[\varepsilon,k]}_{\tau^{(k)}+\delta}(\phi) - v^{[\varepsilon,k]}_{\tau^{(k)}}(\phi)\Big| \geq \eta\Big] \leq \frac{2}{\eta^2}\left(\|\mathcal{L}_\alpha \phi\|^2_{L^1(\mathbb{R})}\,\delta^2 + C_{7.15}\delta^{1-1/\alpha}\right)$$

which shows (7.43). □

Proposition 7.18. *Let $\varepsilon > 0$. $(u^{[\gamma]})_{\gamma>0}$ is tight in $\mathcal{C}([\varepsilon,\infty),\mathcal{M}(\mathbb{R}))$.*

Proof. Again pick a sequence $(\gamma_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $\lim_{k\to\infty}\gamma_k = \infty$ and define

$$v^{[\varepsilon,k]}_t := u^{[\gamma_k]}_{t+\varepsilon}, \qquad t \geq 0.$$

According to Remark 7.16 it is enough to show tightness of $(v^{[\varepsilon,k]})_{k\in\mathbb{N}}$ in $D([0,\infty),\mathcal{M}(\mathbb{R}))$ and therefore we will use the criterion of Jakubowski [Da93, Theorem 3.6.4]. We have to check two conditions. The first is the following compact containment condition: For each $T > 0$ and $\eta > 0$ we have to show the existence of a compact set $K_{T,\eta} \subset \mathcal{M}(\mathbb{R})$ such that

$$\inf_{k\in\mathbb{N}} \mathbf{P}\Big[v^{[\varepsilon,k]}_t \in K_{T,\eta} \text{ for all } t \in [0,T]\Big] \geq 1-\eta. \tag{7.44}$$

As mentioned in the beginning of this section, $u^{[\gamma]}_t$ is a random variable with values in the compact set $\mathcal{M}_{\leq 1}(\mathbb{R})$ for each $\gamma > 0$ and $t \geq 0$ (see [HOV21, Lemma A.1] for the compactness of that space). Therefore we can choose $K_{T,\eta} = \mathcal{M}_{\leq 1}(\mathbb{R})$ for each $T > 0$ and $\eta > 0$ and with this choice (7.44) is fulfilled. The second condition of Jakubowski's criterion is tightness of $(v^{[\varepsilon,k]}(\phi))_{k\in\mathbb{N}}$ in $D([0,\infty),\mathbb{R})$ for each $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ which we proved in Proposition 7.17 (see also Remark 7.16). □

For the next corollary, recall from Section 7.1 that we equip $\mathcal{C}((0,\infty),\mathcal{M}(\mathbb{R}))$ with the topology of uniform convergence on compact subsets of $(0,\infty)$. It should be emphasized that the convergence only holds on compact sets that do not contain 0.

Corollary 7.19. *$(u^{[\gamma]})_{\gamma>0}$ is tight in $\mathcal{C}((0,\infty),\mathcal{M}_{\leq 1}(\mathbb{R}))$.*

Proof. From Proposition 7.18 and a diagonal sequence argument we get tightness of $(u^{[\gamma]})_{\gamma>0}$ in $\mathcal{C}((0,\infty),\mathcal{M}(\mathbb{R}))$ and since $\mathcal{M}_{\leq 1}(\mathbb{R})$ is a compact subset of $\mathcal{M}(\mathbb{R})$ the claim follows. □

7.6. Characterization of limit points

In the last section we showed tightness of $(u^{[\gamma]})_{\gamma>0}$ in $\mathcal{C}((0,\infty),\mathcal{M}_{\leq 1}(\mathbb{R}))$, therefore Prohorov's theorem (see [Kl13, Theorem 13.29]) implies existence of weak limit points. Now we want to conclude the proof of Theorem 7.2 via showing that limit points satisfy the moment duality with the coalescing stable process from Theorem 3.1 to identify them with the long-range voter model on a real line. At the end of the section in Corollary 7.21 we deduce from the results in this chapter that $(u_t(\mathbb{R}))_{t\geq 0}$ has almost surely continuous sample paths. We start with a lemma concerning a convergence result on the hitting times of the dual process (recall the notations from the beginning of Section 7.4). In the following we write $A\Delta B = (A\setminus B)\cup(B\setminus A)$ for sets A and B.

Lemma 7.20. *For each $N\in\mathbb{N}$, $i\in\{1,\dots,N\}$ and $t\geq 0$ we have*

$$\lim_{\gamma\to\infty}\mathbf{P}\left[\left\{\tau^{[\gamma,i]}>t\right\}\Delta\left\{\tau^{[\infty,i]}>t\right\}\right]=0.$$

Proof. Let

$$\Gamma_t := \inf\left\{\gamma>0\ :\ \mathbf{1}_{\left\{\tau^{[\gamma,i]}>t\right\}}=\mathbf{1}_{\left\{\tau^{[\infty,i]}>t\right\}}\text{ for all } i\in\{1,2,\dots,N\}\right\}.$$

Due to the proof of [EF96, Proposition 20] (where coalescing systems of where coalescing Lévy processes on a hierarchical group were considered) one has

$$\mathbf{P}\left[\Gamma_t<\infty,\,\mathbf{1}_{\left\{\tau^{[\gamma,i]}>t\right\}}=\mathbf{1}_{\left\{\tau^{[\infty,i]}>t\right\}}\text{ for all } i\in\{1,2,\dots,N\}\text{ and }\gamma\geq\Gamma_t\right]=1.$$

Thus we get

$$\begin{aligned}&\lim_{\gamma\to\infty}\mathbf{P}\left[\left\{\tau^{[\gamma,i]}>t\right\}\Delta\left\{\tau^{[\infty,i]}>t\right\}\right]\\&=\lim_{\gamma\to\infty}\mathbf{P}\left[\left\{\tau^{[\gamma,i]}>t\right\}\Delta\left\{\tau^{[\infty,i]}>t\right\},\gamma\geq\Gamma_t\right]=0.\end{aligned}$$

□

Now we can conclude the proof of Theorem 7.2.

Proof of Theorem 7.2. Let $(\gamma_k)_{k\in\mathbb{N}}\subset(0,\infty)$ with $\lim_{k\to\infty}\gamma_k=\infty$. According to Corollary 7.19 and Prohorov's theorem (see [Kl13, Theorem 13.29]) there is a subsequence which we will denote again by $(\gamma_k)_{k\in\mathbb{N}}$ and a random variable $\widetilde{u}$ taking values in $\mathcal{C}((0,\infty),\mathcal{M}_{\leq 1}(\mathbb{R}))$ such that we have the convergence

$$u^{[\gamma_k]}\xRightarrow{k\to\infty}\widetilde{u}\qquad\text{in }\mathcal{C}((0,\infty),\mathcal{M}_{\leq 1}(\mathbb{R})).\tag{7.45}$$

Define $\widetilde{u}_0:=u_0$ and write $\widetilde{u}_t$ for the density of $\widetilde{u}_t$. The main aspect we have to do is showing that $\widetilde{u}$ satisfies the moment duality of the voter model, that is we want to show that we have for each $n\in\mathbb{N}$ and Lebesgue-almost all $x=(x_1,\dots,x_n)\in\mathbb{R}^n$

$$\mathbf{E}\left[\prod_{i=1}^{n}\widetilde{u}_t(x_i)\right]=\mathbf{E}_{(x_1,\dots,x_n)}\left[\prod_{\substack{i=1\\ \tau^{[\infty,i]}>t}}^{n}\widetilde{u}_0(\xi_t^{(i)})\right],\qquad t\geq 0.\tag{7.46}$$

where $\xi^{(1)},\xi^{(2)},\dots,\xi^{(n)}$ are independent standard stable processes with $\xi_0^{(i)}=x_i$ for $i\in\{1,2,\dots,n\}$ and $\mathbf{P}_{(x_1,\dots,x_n)}$ denotes the law of $(\xi^{(1)},\xi^{(2)},\dots,\xi^{(n)})$ (again recall the notations regarding the dual process and the hitting times $\tau^{[\gamma,i]}$ for $i \in \{1, 2, \ldots, n\}$ and $\gamma \in (0, \infty]$ from the beginning of Section 7.4).

For $t = 0$ it is clear that (7.46) holds. Let $t > 0$. First note that there is a constant $C_{7.2} = C_{7.2}(n) > 0$ such that we have for each $k \in \mathbb{N}$

$$\left| \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\infty,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right] - \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\gamma_k,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right] \right| \\ \leq C_{7.2} \sum_{i=1}^{n} \mathbf{P} \left[\left\{ \tau^{[\gamma_k,i]} > t \right\} \Delta \left\{ \tau^{[\infty,i]} > t \right\} \right]$$

which implies using Lemma 7.20

$$\lim_{k\to\infty} \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\gamma_k,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right] = \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\infty,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right]. \tag{7.47}$$

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Using the duality relation from Proposition 7.9 we have for each $k \in \mathbb{N}$

$$\int_{\mathbb{R}^n} \phi(x)\, \mathbf{E} \left[\prod_{i=1}^{n} u_t^{[\gamma_k]}(x_i) \right] dx = \int_{\mathbb{R}^n} \phi(x)\, \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\gamma_k,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right] dx.$$

The convergence (7.45) for the left-hand side and (7.47) for the right-hand side of the last equality imply

$$\int_{\mathbb{R}^n} \phi(x)\, \mathbf{E} \left[\prod_{i=1}^{n} \widetilde{u}_t(x_i) \right] dx = \int_{\mathbb{R}^n} \phi(x)\, \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\infty,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right] dx.$$

Since this holds for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and since

$$\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto \mathbf{E}_{(x_1,\ldots,x_n)} \left[\prod_{\substack{i=1 \\ \tau^{[\infty,i]}>t}}^{n} \widetilde{u}_0(\xi_t^{(i)}) \right] - \mathbf{E} \left[\prod_{i=1}^{n} \widetilde{u}_t(x_i) \right] \in [-1, 1]$$

is a locally integrable function the fundamental lemma of calculus of variations yields that (7.46) holds for Lebesgue-almost all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. As proven in [Ev97, Theorem 4.1] the duality relation (7.46) determines a unique Feller semigroup and we can identify the law of $\widetilde{u}$ with that of the long-range voter model on the real line from Theorem 3.1, that is $\widetilde{u} \overset{d}{=} u$. □

Corollary 7.21. *Let $u_0 \in \mathcal{M}_{\leq 1}(\mathbb{R})$ with continuous density and $u_0(\mathbb{R}) < \infty$. Then $(u_t(\mathbb{R}))_{t\geq 0}$ has almost surely continuous sample paths.*

Proof. First note that using Lemma 5.6 we have $\mathbf{E}_{u_0}[u_t(\mathbb{R})] = u_0(\mathbb{R}) < \infty$ for $t \geq 0$. For each $t \geq 0$ and $n \in \mathbb{N}$ we can decompose

$$u_t(\mathbb{R}) = u_t([-n,n]) + u_t([-n,n]^c).$$

Due to Theorem 3.1, $(u_t)_{t\geq 0}$ has almost surely continuous sample paths with respect to the topology of vague convergence. Thus, $(u_t([-n,n]))_{t\geq 0}$ has almost surely continuous sample paths for each $n \in \mathbb{N}$ and it remains to show that, for each $T > 0$ and $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}_{u_0}\left[\sup_{t\in[0,T]} u_t([-n,n]^c) > \varepsilon\right] = 0.$$

We will show in the following

$$\lim_{n\to\infty} \mathbf{P}_{u_0}\left[\sup_{t\in[0,T]} u_t((n,\infty)) > \varepsilon\right] = 0. \tag{7.48}$$

The proof with $u_t((-\infty,-n))$ instead of $u_t((n,\infty))$ works analogously. Define $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$ by

$$\eta(x) := \begin{cases} b\exp\left(\frac{1}{|x|^2-1}\right), & \text{if } x \in (-1,1), \\ 0, & \text{otherwise,} \end{cases}$$

where the constant $b > 0$ is chosen so that $\int_{-\infty}^{\infty} \eta(x)\,dx = 1$. For $n, N \in \mathbb{N}_0$ with $N > n$ and $x \in \mathbb{R}$ let

$$\phi_{n,N}(x) := \int_n^N 2\,\eta(y-x)\,dy, \qquad \phi_n(x) := \int_n^\infty 2\,\eta(y-x)\,dy.$$

We have $\phi_{n,N} \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\mathbf{1}_{(n,N)} \leq \phi_{n,N}$ and $\phi_n \in \mathcal{C}^\infty(\mathbb{R})$ with $\mathbf{1}_{(n,\infty)} \leq \phi_n$. Therefore, to show (7.48) it is enough to show that, for each $T > 0$ and $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}_{u_0}\left[\sup_{t\in[0,T]} u_t(\phi_n) > \varepsilon\right] = 0. \tag{7.49}$$

We use a martingale argument: We can decompose for $n \in \mathbb{N}_0$

$$u_t(\phi_n) = u_0(\phi_n) + M_t(\phi_n) + C_t(\phi_n) \tag{7.50}$$

with

$$M_t(\phi_n) := u_t(\phi_n) - u_0(\phi_n) - \int_0^t \langle u_s, \mathcal{L}_\alpha \phi_n\rangle\ ds \tag{7.51}$$

and

$$C_t(\phi_n) = \int_0^t \langle u_s, \mathcal{L}_\alpha \phi_n\rangle\ ds. \tag{7.52}$$

We briefly discuss the well-definedness of the last integral and show that all summands in (7.50), (7.51) and (7.52) have first moments. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be twice differentiable with $\int_{\mathbb{R}} \frac{|\phi(x)|}{(1+|x|)^{\alpha+1}}\,dx < \infty$. Then, $\mathcal{L}_\alpha \phi$ is well defined with

$$|(\mathcal{L}_\alpha\phi)(x)| \leq C_{7.21.1} \int_{x-r}^{x+r} \frac{|\phi''(y)|}{|y-x|^{\alpha-1}}\,dy + C_{7.21.1} \int_{\mathbb{R}\setminus(x-r,x+r)} \frac{|\phi(y)-\phi(x)|}{|y-x|^{\alpha+1}}\,dy \tag{7.53}$$

for each $r > 0$ and $x \in \mathbb{R}$ with some constant $C_{7.21.1} = C_{7.21.1}(\alpha) \in (0,\infty)$ (see [Kw19, Section 1, equation (4)]). Since $\|\phi_n\|_\infty \leq 2 < \infty$, $\mathcal{L}_\alpha \phi_n$ is well defined and for each $r > 0$ we have

$$\begin{aligned}
&\sup_{n\in\mathbb{N}_0} \|\mathcal{L}_\alpha\phi_n\|_\infty \\
&\quad\leq \sup_{n\in\mathbb{N}_0}\sup_{x\in\mathbb{R}} \left(C_{7.21.1} \int_{-r}^{r} \frac{|\phi_n''(z+x)|}{|z|^{\alpha-1}}\,dz + C_{7.21.1} \int_{\mathbb{R}\setminus(-r,r)} \frac{|\phi_n(z+x)-\phi_n(x)|}{|z|^{\alpha+1}}\,dz \right) \\
&\quad\leq 2C_{7.21.1} \|\eta''\|_{L^1(\mathbb{R})} \int_{-r}^{r} \frac{1}{|z|^{\alpha-1}}\,dz + 4C_{7.21.1} \|\eta\|_{L^1(\mathbb{R})} \int_{\mathbb{R}\setminus(-r,r)} \frac{1}{|z|^{\alpha+1}}\,dz \\
&\quad=: C_{7.21.2} < \infty
\end{aligned}$$

with $C_{7.21.2} = C_{7.21.2}(\alpha, r, \eta) \in (0,\infty)$. Thus,

$$\sup_{n\in\mathbb{N}_0} \mathbf{E}\left[\left|\int_0^t \langle u_s, \mathcal{L}_\alpha\phi_n\rangle\,ds\right|\right] \leq C_{7.21.2}\, u_0(\mathbb{R})\, t < \infty.$$

for each $t \geq 0$. Furthermore,

$$\sup_{n\in\mathbb{N}_0} \mathbf{E}\left[|u_t(\phi_n)|\right] \leq 2u_0(\mathbb{R}) < \infty$$

for each $t \geq 0$. If we replace ϕ_n by $\phi_{n,N}$ with $N > n$ in (7.50), (7.51) and (7.52), the last estimates work in exactly the same way. Let $(u_t^{[\gamma]})_{t\geq 0}$ be as in Theorem 7.2 for $\gamma > 0$. We can also replace $(u_t)_{t\geq 0}$ with $(u_t^{[\gamma]})_{t\geq 0}$ in the last estimates, since $\mathbf{E}_{u_0}[u_t^{[\gamma]}(\mathbb{R})] = u_0(\mathbb{R}) < \infty$ for $t \geq 0$ which follows from the duality (see Proposition 7.9) with the same proof as the proof of Lemma 5.6.

Due to Proposition 7.8, for each $\gamma > 0$, $(M_t^{[\gamma]}(\phi_{n,N}))_{t\geq 0}$ with

$$M_t^{[\gamma]}(\phi_{n,N}) := u_t^{[\gamma]}(\phi_{n,N}) - u_0(\phi_{n,N}) - \int_0^t \left\langle u_s^{[\gamma]}, \mathcal{L}_\alpha\phi_{n,N}\right\rangle ds$$

is a martingale. Theorem 7.2 implies that on some probability space we have almost surely for each $t > 0$ and $\phi \in \mathcal{C}_c(\mathbb{R})$

$$u_t^{[\gamma]}(\phi) \xrightarrow{\gamma\to\infty} u_t(\phi). \tag{7.54}$$

The latter convergence is also true for $t = 0$ since $u_0^{[\gamma]} = u_0$ for each $\gamma > 0$. Due to Lemma A.4 (7.54) holds for all $\phi \in L^1(\mathbb{R})$. Since $\phi_{n,N} \in \mathcal{C}_c^\infty(\mathbb{R})$, we have $\mathcal{L}_\alpha \phi_{n,N} \in L^1(\mathbb{R})$ (see Lemma A.3). Thus, with dominated convergence, $(M_t(\phi_{n,N}))_{t\geq 0}$ with

$$M_t(\phi_{n,N}) := u_t(\phi_{n,N}) - u_0(\phi_{n,N}) - \int_0^t \langle u_s, \mathcal{L}_\alpha \phi_{n,N} \rangle \, ds$$

is a martingale. With (7.53), for $x \in \mathbb{R}$, we get

$$\begin{aligned}
&|\mathcal{L}_\alpha \phi_n(x) - \mathcal{L}_\alpha \phi_{n,N}(x)| \\
&\quad = |\mathcal{L}_\alpha \phi_N(x)| \\
&\quad \leq C_{7.21.1} \int_{-r}^{r} \frac{|\phi_N''(z+x)|}{|z|^{\alpha-1}} \, dz + C_{7.21.1} \int_{\mathbb{R}\setminus(-r,r)} \frac{|\phi_N(z+x) - \phi_N(x)|}{|z|^{\alpha+1}} \, dz \\
&\quad \leq C_{7.21.1} \int_{-r}^{r} \frac{|\phi_N''(z+x)|}{|z|^{\alpha-1}} \, dz + C_{7.21.1} \int_{\mathbb{R}\setminus(-r,r)} \frac{|\phi_N(z+x)| + |\phi_N(x)|}{|z|^{\alpha+1}} \, dz.
\end{aligned}$$

Since $\phi_N(x)$ and $\phi_N''(x)$ converge to zero as $N \longrightarrow \infty$ for each $x \in \mathbb{R}$, dominated convergence implies

$$\lim_{N\to\infty} \mathbf{E}_{u_0} \left[\left| \int_0^t \langle u_s, \mathcal{L}_\alpha \phi_n \rangle \, ds - \int_0^t \langle u_s, \mathcal{L}_\alpha \phi_{n,N} \rangle \, ds \right| \right] = 0$$

and we can follow, that $(M_t(\phi_n))_{t\geq 0}$ is a martingale. Let $\varepsilon > 0$ and $T > 0$. The decomposition (7.50), Doob's inequality (see [KS91, Theorem 1.3.8]) and the Markov inequality (see [Kl13, Theorem 5.11]) yield

$$\begin{aligned}
\mathbf{P}_{u_0} \left[\sup_{t\in[0,T]} u_t(\phi_n) > \varepsilon \right] &\leq \frac{3}{\varepsilon} \left(u_0(\phi_n) + \mathbf{E}_{u_0} \left[|M_T(\phi_n)| \right] + \mathbf{E}_{u_0} \left[\sup_{t\in[0,T]} |C_t(\phi_n)| \right] \right) \\
&\leq \frac{3}{\varepsilon} \left(u_0(\phi_n) + \mathbf{E}_{u_0} \left[|M_T(\phi_n)| \right] + \mathbf{E}_{u_0} \left[\int_0^T |\langle u_s, \mathcal{L}_\alpha \phi_n \rangle| \, ds \right] \right).
\end{aligned}$$

Using dominated convergence, the right hand side of the last inequality converges to zero as $n \longrightarrow \infty$. This is (7.49). □

8. Conjectures and outlook

8.1. Lower bound on the Hausdorff dimension of the interface

In Theorem 4.1 we proved that the Hausdorff dimension of the interface $\mathcal{I}(u_t)$ at a fixed time $t > 0$ is almost surely bounded above by $1 - (\alpha - 1)^2$. In this section we give some heuristics on a lower dimension bound.

Lemma 8.1. *There is a constant $C_{8.1} = C_{8.1}(\alpha) > 0$, and for each $t > 0$, there is an $\ell_0 = \ell_0(\alpha, t) > 0$ such that for all $\ell \in (0, \ell_0)$, we have*

$$\inf_{\substack{n\in\mathbb{N}\setminus\{1\}, A\subset\mathbb{R} \\ \operatorname{diam}(A)=\ell, |A|=n}} \mathbf{P}\left[\left|\Xi_t^A\right| > 1\right] \geq C_{8.1} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1}.$$

Proof. For each $A \subset \mathbb{R}$ with $|A| = n \in \mathbb{N} \setminus \{1\}$ we have

$$\mathbf{P}\left[T^{(1,n)} > t\right] \leq \mathbf{P}\left[\left|\Xi_t^A\right| > 1\right]$$

where $T^{(1,n)}$ is the hitting time of the two processes started in the smallest and largest point of A. If $\operatorname{diam}(A) = \ell$, we have $\left|\xi_0^{(n)} - \xi_0^{(1)}\right| = \ell$ and the assertion follows directly from the lower bound of the hitting time (see Lemma 4.4). □

Corollary 8.2. *Let $\vartheta \in (0,1)$ and $u_0(x) = \vartheta$ for all $x \in \mathbb{R}$. There is a constant $C_{8.2} = C_{8.2}(\alpha, \vartheta) > 0$, and for each $t > 0$, there is an $\ell_0 = \ell_0(\alpha, t) > 0$ such that for all $\ell \in (0, \ell_0)$, we have*

$$\inf_{x\in\mathbb{R}} \mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap \left(x - \frac{\ell}{2}, x + \frac{\ell}{2}\right) \neq \emptyset\right] \geq C_{8.2} t^{-\left(1-\frac{1}{\alpha}\right)} \ell^{\alpha-1}.$$

Proof. Let $x \in \mathbb{R}$, $t > 0$ and $\ell > 0$. We compute

$$\begin{aligned}
&\mathbf{P}_{u_0}\left[\mathcal{I}(u_t) \cap \left(x - \frac{\ell}{2}, x + \frac{\ell}{2}\right) \neq \emptyset\right] \\
&= \mathbf{P}_{u_0}\left[0 < \int_{x-\frac{\ell}{2}}^{x+\frac{\ell}{2}} u_t(y)\, dy < \ell\right] \\
&= \mathbf{P}_{u_0}\left[0 < u_t\left(\left[x - \frac{\ell}{2}, x + \frac{\ell}{2}\right]\right) < \ell\right] \\
&\geq \left(1 - ((1-\vartheta)^2 + \vartheta^2)\right) \mathbf{P}\left[\left|\Xi_t^{[x-\frac{\ell}{2}, x+\frac{\ell}{2}]}\right| > 1\right]
\end{aligned}$$

where we used Lemma 4.3 in the first equality and Corollary 3.5 (ii) in the last inequality. Now use Lemma 8.1 which concludes the proof. □

Remark 8.3. Let $A \subset \mathbb{R}$ be measurable and $s \geq 0$. If $\mu \in \mathcal{M}(\mathbb{R})$ is a probability measure supported on A we call

$$I_s(\mu) = \int_A \int_A |x-y|^{-s} \, \mu(dy)\, \mu(dx)$$

the s-energy of μ and we define the Riesz-s-capacity, or simply the s-capacity, of A via

$$\mathrm{Cap}_s(A) := \sup \Big\{ I_s(\mu)^{-1} \,:\, \mu \in \mathcal{M}(\mathbb{R}) \text{ is a probability measure supported on } A \Big\}.$$

In the appendix (Appendix A.1) we provide a brief overview of the connection between the Riesz-capacity and Hausdorff dimension. Note that if A is bounded we have $I_s(\mu) \geq \mathrm{diam}(A)^{-s}$ for each probability measure $\mu \in \mathcal{M}(\mathbb{R})$ supported on A, hence $\mathrm{Cap}_s(A) \leq \mathrm{diam}(A)^s$. Therefore we get from Corollary 8.2 for $u_0 = \vartheta \in (0,1)$, each $t > 0$ and each bounded open interval $A \subset \mathbb{R}$ with sufficiently small diameter

$$\mathbf{P}[\mathcal{I}(u_t) \cap A \neq \emptyset] \geq C_{8.3} \, \mathrm{Cap}_{\alpha-1}(A) \tag{8.1}$$

for some constant $C_{8.3} = C_{8.3}(\alpha, \vartheta, t) > 0$. The latter inequality also holds if A is closed. (8.1) an indication that the Hausdorff dimension of the interface could be at least $1-(\alpha-1) = 2-\alpha$, but it is not clear how to prove it. It would be sufficient to have (8.1) for arbitrary analytic sets $A \subset [-K, K]$ with $C_{8.2}$ may depend on K and arbitrary $K > 0$. This would be sufficient to get with positive probability a lower bound on the Hausdorff dimension (see [MMP17, Section 5.1] using methods of [Ha75]). The difficulty is that if A is not an interval it is not clear how to get a lower bound on the probability that there is an interface point in A since Lemma 4.3 only works for intervals. For example think of A as a union of two closed disjoint intervals. Using inclusion-exclusion and the moment duality one has to deal with the non-hitting probability of paths of two pairs of particles of the coalescing system, one pair starting in the first interval, the other pair starting in the second. This problem occurs also if one asks the question whether Lemma 8.1 is sharp or not: It seams to be a very rough bound because only the two outer particles are considered. Since α-stable processes make jumps it is doubtful that the distribution of the last coalescence is determined by the two outer particles. If one uses inclusion-exclusion for a bound one has to deal again with the non-hitting probability of paths of two pairs of particles.

We conclude this section with the following Lemma 8.4 in which we give a bound on the non-hitting probability of two pairs for the special case that there is a gap between the pairs. This is not enough for our purposes, but it is a non-trivial upper bound on the probability, at least in this special case. Let $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}$ be independent standard stable processes starting in some points $\xi_0^{(1)}, \xi_0^{(2)}, \xi_0^{(3)}, \xi_0^{(4)} \in \mathbb{R}$ with $\xi_0^{(1)} < \xi_0^{(2)} < \xi_0^{(3)} < \xi_0^{(4)}$ and consider the corresponding vector-valued coalescing system $(\zeta^{\mathrm{id},1}, \zeta^{\mathrm{id},2}, \zeta^{\mathrm{id},3}, \zeta^{\mathrm{id},4})$ (recall the beginning of Section 2.2 for the construction). For $i, j \in \{1,2,3,4\}$ with $i \neq j$ let

$$T^{(i,j)} := \inf\left\{t \geq 0 \,:\, \zeta_t^{\mathrm{id},i} = \zeta_t^{\mathrm{id},j}\right\}, \qquad \tau^{(i,j)} := \inf\left\{t \geq 0 \,:\, \xi_t^{(i)} = \xi_t^{(j)}\right\}.$$

Thus $T^{(i,j)}$ is the first time when the coalescing particles number i and j meet and $\tau^{(i,j)}$ is the first time when the independent particles number i and j meet. The idea of proof of the following result is based partly on of the proof of [Tr95, Lemma 2.1 (c)].

Lemma 8.4. *Let $\delta > \frac{1}{2}(\alpha - 1)$. For each $t > 0$, there is an $\ell_0 = \ell_0(\alpha, \delta) \in (0,1)$ and a constant $C_{8.4} = C_{8.4}(\alpha, t) > 0$ such that for each $\ell \in (0, \ell_0)$ and $\xi_0^{(1)}, \xi_0^{(2)}, \xi_0^{(3)}, \xi_0^{(4)} \in \mathbb{R}$ with $\xi_0^{(2)} - \xi_0^{(1)} = \xi_0^{(4)} - \xi_0^{(3)} = \ell^{1+\delta}$ and $\xi_0^{(3)} - \xi_0^{(2)} = \ell$,*

$$\mathbf{P}\left[T^{(1,2)} > t, T^{(3,4)} > t\right] \leq C_{8.4}\ell^{(1+\delta)(\alpha-1)(1+\varepsilon)}$$

with $\varepsilon = \frac{1}{\alpha} - \frac{\alpha+1}{2\alpha(1+\delta)} > 0$.

Proof. From Proposition 2.5 and (2.2), we know for $i \neq j$ and $t > 0$,

$$\mathbf{P}\left[\tau^{(i,j)} \leq t\right] \leq C_{8.4.1}\left(\xi_0^{(j)} - \xi_0^{(i)}\right)^{-\alpha-1} t^{1+\frac{1}{\alpha}}$$

and

$$\mathbf{P}\left[\tau^{(i,j)} > t\right] \leq C_{8.4.2}\left(\xi_0^{(j)} - \xi_0^{(i)}\right)^{\alpha-1} t^{-\left(1-\frac{1}{\alpha}\right)}$$

for constants $C_{8.4.1} = C_{8.4.1}(\alpha) > 0$ and $C_{8.4.2} = C_{8.4.2}(\alpha) > 0$. Now we decompose the event of non-coalescence of the two pairs in the case where particles from the first pair hit particles of the second pair in a short time and the situation where this does not happen. In the latter case we can regard the two pairs as independent up to this short time. Let $\delta, \beta > 0$ and let $\ell \in (0,1)$ be such that $\ell^{(1+\delta)\beta} \leq t$. Assume that $\xi_0^{(2)} - \xi_0^{(1)} = \xi_0^{(4)} - \xi_0^{(3)} = \ell^{1+\delta}$ and $\xi_0^{(3)} - \xi_0^{(2)} = \ell$. We compute

$$\begin{aligned}
&\mathbf{P}\left[T^{(1,2)} > t, T^{(3,4)} > t\right] \\
&\quad \leq \mathbf{P}\left[\min\left\{\tau^{(1,3)}, \tau^{(1,4)}, \tau^{(2,3)}, \tau^{(2,4)}\right\} \leq \ell^{(1+\delta)\beta}\right] + \mathbf{P}\left[\tau^{(1,2)} > t\right] \cdot \mathbf{P}\left[\tau^{(3,4)} > \ell^{(1+\delta)\beta}\right] \\
&\quad \leq 4\mathbf{P}\left[\tau^{(2,3)} \leq \ell^{(1+\delta)\beta}\right] + \mathbf{P}\left[\tau^{(1,2)} > t\right] \cdot \mathbf{P}[\tau^{(3,4)} > \ell^{(1+\delta)\beta}] \\
&\quad \leq C_{8.4.3}\left(\ell^{-\alpha-1}\ell^{(1+\delta)\beta(1+\frac{1}{\alpha})} + \ell^{(1+\delta)(\alpha-1)}\ell^{(1+\delta)(\alpha-1)}\ell^{-(1+\delta)\beta(1-\frac{1}{\alpha})}\right)
\end{aligned}$$

for some constant $C_{8.4.3} = C_{8.4.3}(\alpha, t) > 0$. Solving the system of linear equations

$$\begin{cases} -\alpha - 1 + (1+\delta)\beta(1 + \frac{1}{\alpha}) = (1+\delta)(\alpha-1)(1+\varepsilon) \\ 2(1+\delta)(\alpha-1) - (1+\delta)\beta(1-\frac{1}{\alpha}) = (1+\delta)(\alpha-1)(1+\varepsilon) \end{cases}$$

in the variables $\beta > 0$ and $\varepsilon \in (0,1)$ leads to

$$\beta = \alpha - 1 + \frac{\alpha+1}{2(1+\delta)}, \quad \varepsilon = \frac{1}{\alpha} - \frac{\alpha+1}{2\alpha(1+\delta)}.$$

Note that $\varepsilon > 0$ if and only if $\delta > \frac{1}{2}(\alpha - 1)$. □

8.2. Hausdorff dimension of the exceptional time points

In Chapter 6 we computed the Hausdorff dimension of the set of time points of unbounded support (which we call *exceptional time points*) of a Poisson point process-toy model for the voter model. A natural goal is to show existence of such points for the actual voter model and compute its Hausdorff dimension. According to Theorem 6.1 and Remark 6.2, one could guess it should be almost surely $\frac{1}{\alpha}$.

The approach from Chapter 6 was to assume, as a simplification, that colonies of opinion 1 would rain down onto the space-time plane and that is how the toy model came about. In order to be able to make corresponding dimension calculations for the actual voter model, on the one hand, one needs a colony description of the voter model and on the other hand one would have to compare this colony process with another process with more independence structure in space and time (since we have a lot of independence in the toy model). In addition, we would need a coupling between the colony model and the voter model, which we introduced (only implicitly) via a moment duality. However, even for the discrete voter model it is not clear whether such a colony description exists.

If such a description of colonies (and a coupling) existed, one could compare the voter model with a model in which colonies independently rain onto the space-time plane for an upper dimensional bound. Then one could get an upper bound with the help of an analogous argument as in Chapter 6 since we know from Chapter 5 the lifetime of the colonies. For a lower bound one could try to throw away sufficiently few colonies, so that one gets more independence structure, but without changing the dimension of the exceptional time points.

A. Appendix

A.1. Hausdorff dimension

We remark some facts about the Hausdorff dimension (see for example [Ma95, Chapter 4], [Fa90, Chapter 2] or [MP10, Section 4.1.2]). For $E \subset \mathbb{R}$ we denote by

$$\mathrm{diam}(E) := \sup_{x,y \in E} |y - x|$$

the diameter of E. For $s \geq 0$, $\delta > 0$ and $E \subset \mathbb{R}$, let

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(E_i))^s \,:\, \bigcup_{i=1}^{\infty} E_i \supset E,\, \mathrm{diam}(E_i) \leq \delta \text{ for all } i \in \mathbb{N} \right\}.$$

Then

$$\mathcal{H}^s(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E) = \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(E)$$

is the s-dimensional Hausdorff measure of the set E. The Hausdorff dimension of E is

$$\begin{aligned} \dim_{\mathcal{H}}(E) &= \inf \{ s \geq 0 \,:\, \mathcal{H}^s(E) = 0 \} = \inf \{ s \geq 0 \,:\, \mathcal{H}^s(E) < \infty \} \\ &= \sup \{ s \geq 0 \,:\, \mathcal{H}^s(E) > 0 \} = \sup \{ s \geq 0 \,:\, \mathcal{H}^s(E) = \infty \}. \end{aligned}$$

Hence $\mathcal{H}^s(E) < \infty$ for some $s \geq 0$ implies $\dim_{\mathcal{H}}(E) \leq s$. Further, the Hausdorff dimension has the countable stability-property: For $E_1, E_2, \ldots \subset \mathbb{R}$ we have

$$\dim_{\mathcal{H}} \left(\bigcup_{i \in \mathbb{N}} E_i \right) = \sup_{i \in \mathbb{N}} \dim_{\mathcal{H}}(E_i).$$

We end this section with some classic results on the relationship between the Hausdorff dimension and the Riesz capacity (see [Ma95, Chapter 8] or [MP10, Section 4.3, 4.4]): Let $E \subset \mathbb{R}$ be measurable and $s \geq 0$. If $\mu \in \mathcal{M}(\mathbb{R})$ is a probability measure supported on E we call

$$I_s(\mu) = \int_E \int_E |x - y|^{-s} \, \mu(dy) \, \mu(dx)$$

the s-energy of μ and we define the Riesz-s-capacity, or simply the s-capacity, of E via

$$\mathrm{Cap}_s(E) := \sup \left\{ I_s(\mu)^{-1} \,:\, \mu \in \mathcal{M}(\mathbb{R}) \text{ is a probability measure supported on } E \right\}.$$

We first cite the so-called energy method, with which one can obtain lower bounds on the Hausdorff dimension.

Proposition A.1 ([MP10]). *Let $E \subset \mathbb{R}$ be measurable, let $s \geq 0$ and let $\mu \in \mathcal{M}(\mathbb{R})$ be a probability measure supported on E. If $I_s(\mu) < \infty$ then $\mathcal{H}^s(E) = \infty$ and, in particular, $\dim_{\mathcal{H}}(E) \geq s$.*

Proof. This is proved in [MP10, Theorem 4.27]. □

The latter result shows that a set of positive s-capacity has at least Hausdorff dimension s. Frostman's lemma (see [MP10, Theorem 4.30]) can be used to prove an upper bound for closed sets. Both statements can be summarized as follows.

Proposition A.2 ([MP10]). *Let $E \subset \mathbb{R}$ be a closed set. We have*

$$\dim_{\mathcal{H}}(E) = \sup\left\{s > 0 \,:\, \mathrm{Cap}_s(E) > 0\right\}.$$

Proof. This is proved in [MP10, Theorem 4.32]. □

A.2. Some technical lemmas

Lemma A.3. *Let $\alpha \in (1,2)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then, $\mathcal{L}_\alpha \phi \in L^1(\mathbb{R})$.*

Proof. Due to [Kw19, Section 1, equation (4)], we have

$$|(\mathcal{L}_\alpha \phi)(x)| \leq C_{A.3} \int_{x-r}^{x+r} \frac{|\phi''(y)|}{|y-x|^{\alpha-1}}\, dy + C_{A.3} \int_{\mathbb{R}\setminus(x-r,x+r)} \frac{|\phi(y)-\phi(x)|}{|y-x|^{\alpha+1}}\, dy$$

for each $r > 0$ and $x \in \mathbb{R}$ with some constant $C_{A.3} = C_{A.3}(\alpha) \in (0,\infty)$ (see [Kw19, Section 1, equation (4)]). Let $K > 0$ such that $\operatorname{supp} \phi \subset [-K,K]$. Further let $x \in \mathbb{R}$ and $r > 0$. We can bound

$$\begin{aligned}
|(\mathcal{L}_\alpha \phi)(x)| &\leq C_{A.3} \int_{-r}^{r} \frac{|\phi''(z+x)|}{|z|^{\alpha-1}}\, dz + C_{A.3} \int_{\mathbb{R}\setminus(-r,r)} \frac{|\phi(z+x)-\phi(x)|}{|z|^{\alpha+1}}\, dz \\
&\leq C_{A.3} \left\|\phi''\right\|_\infty \int_{-r}^{r} \frac{\mathbf{1}_{[-K-z,K-z]}(x)}{|z|^{\alpha-1}}\, dz \\
&\qquad + C_{A.3} \left\|\phi\right\|_\infty \int_{\mathbb{R}\setminus(-r,r)} \frac{\mathbf{1}_{[-K-z,K-z]}(x) + \mathbf{1}_{[-K,K]}(x)}{|z|^{\alpha+1}}\, dz.
\end{aligned}$$

Thus, with Fubini's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}} |(\mathcal{L}_\alpha \phi)(x)|\, dx &\leq C_{A.3} \left\|\phi''\right\|_\infty 2K \int_{-r}^{r} \frac{1}{|z|^{\alpha-1}}\, dz + 2C_{A.3} \left\|\phi\right\|_\infty 2K \int_{\mathbb{R}\setminus(-r,r)} \frac{1}{|z|^{\alpha+1}}\, dz \\
&= 4C_{A.3}K \left\|\phi''\right\|_\infty \int_0^r z^{-\alpha+1}\, dz + 8C_{A.3}K \left\|\phi\right\|_\infty \int_r^\infty z^{-\alpha-1}\, dz \\
&= \frac{4C_{A.3}K \left\|\phi''\right\|_\infty}{2-\alpha} r^{2-\alpha} + \frac{8C_{A.3}K \left\|\phi\right\|_\infty}{\alpha} r^{-\alpha} < \infty.
\end{aligned}$$

□

Lemma A.4. *Let* $u, u_1, u_2, \ldots \in \mathcal{M}_{\leq 1}(\mathbb{R})$ *with*

$$\lim_{n\to\infty} u_n(\phi) = u(\phi) \tag{A.1}$$

for each $\phi \in \mathcal{C}_c(\mathbb{R})$. *Then* (A.1) *holds for each* $\phi \in L^1(\mathbb{R})$.

Proof. Let $\phi \in L^1(\mathbb{R})$ and $\varepsilon > 0$. Since $\mathcal{C}_c(\mathbb{R})$ is a dense subset of $L^1(\mathbb{R})$ with respect to $\|\cdot\|_{L^1(\mathbb{R})}$, there is $\psi \in \mathcal{C}_c(\mathbb{R})$ such that

$$\|\phi - \psi\|_{L^1(\mathbb{R})} \leq \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$. Using $u, u_n \in \mathcal{M}_{\leq 1}(\mathbb{R})$, we get

$$\begin{aligned} |u(\phi) - u_n(\phi)| &\leq |u(\phi) - u(\psi)| + |u(\psi) - u_n(\psi)| + |u_n(\psi) - u_n(\phi)| \\ &\leq 2\,\|\phi - \psi\|_{L^1(\mathbb{R})} + |u(\psi) - u_n(\psi)| \\ &\leq \varepsilon + |u(\psi) - u_n(\psi)|\,. \end{aligned}$$

This implies

$$\limsup_{n\to\infty} |u(\phi) - u_n(\phi)| \leq \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, the result follows. □

Bibliography

[Ar79] R. Arratia. Coalescing Brownian motions on the line. *Ph.D. Thesis*, 1979.

[Ar81] R. Arratia. Limiting Point Processes for Rescalings of Coalescing and Annihilating Random Walks on $\mathbb{Z}^d$. *Ann. Probab.*, 9(6):909-936, 1981.

[AS11] S. Athreya, R. Sun. One-dimensional voter model interface revisited. *Elect. Comm. in Probab.*, 16(70):792-800, 2011.

[AT00] S. Athreya, R. Tribe: Uniqueness for a class of one dimensional stochastic PDEs using moment duality *Ann. Probab.*, 28(4):1711-1734, 2000.

[Ba88] M. T. Barlow. Necessary and Sufficient Conditions for the Continuity of Local Time of Lévy Processes. *Ann. Probab.*, 16(4):1389-1427, 1988.

[Be96] J. Bertoin. Lévy processes. *Cambridge University Press*, 1996.

[Be99] J. Bertoin. Subordinators: Examples and Applications. *Bernard, P. (eds) Lectures on Probability Theory and Statistics. Lecture Notes in Mathematics, vol 1717. Springer*, 1999.

[Bi99] P. Billingsley. Convergence of Probability Measures. *Second Edition, John Wiley & Sons Inc.*, 1995.

[BHO16] J. Blath, M. Hammer, M. Ortgiese. The scaling limit of the interface of the continuous-space symbiotic branching model. *Ann. Probab.*, 44(2):807-866, 2016.

[BO21] J. Blath, M. Ortgiese. The symbiotic branching model: Duality and Interfaces. *Probabilistic Structures in Evolution (E. Baake and A. Wakolbinger, eds.), EMS Press, Berlin*, 311-336, 2021.

[BG60] R. Blumenthal, R. Getoor. Some theorems on stable processes. *Transactions of the American Mathematical Society 95.*, 263-273, 1960.

[BJ07] K. Bogdan, T. Jakubowski. Estimates of Heat Kernel of Fractional Laplacian Perturbed by Gradient Operators. *Commun. Math. Phys.*, 271:179-198, 2007.

[Bo64] E. S. Boylan. Local times for a class of Markoff processes. *Illinois J. Math.*, 8(1):19-39, 1964.

[Br71] J. Bretagnolle. Résultats de Kesten sur les processus à accroissements indépendants. *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969-1970)*, pages 21–36. Lecture Notes in Math., vol. 191. Springer, Berlin, 1971.

[CS73] P. Clifford, A. Sudbury. A Model for Spatial Conflict. *Biometrika*, 60(3):581-588, 1973.

[DKM+09] R. C. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, Y. Xiao. A Minicourse on Stochastic Partial Differential Equations. *Springer*, 2009.

[Da93] D. A. Dawson. Measure-valued Markov processes. In P.L. Hennequin, Editor, *École d'Été de Probabilités de Saint-Flour XXI—1991*, Volume 1541 of *Lecture Notes in Mathematics*, pages 1–260, Berlin, Springer, 1993.

[DEF+00] P. Donnelly, S. N. Evans, K. Fleischmann, T. G. Kurtz, and X. Zhou. Continuum-sites stepping stone models, coalescing exchangeable partitions and random trees. *Ann. Probab.*, 28(3):1063-1100, 2000.

[Du88] R. Durrett. Lecture Notes on Particle Systems and Percolation. *Wadsworth & Brooks/Cole Advanced Books & Software*, 1988.

[DF16] R. Durett, W.-T. Fan. Genealogies in Expanding Populations. *The Annals of Applied Probability*, 26(6):3456–3490, 2016.

[EK86] S. N. Ethier, T. G. Kurtz. Markov Processes. Characterization and Convergence. *John Wiley & Sons Inc.*, New York, 1986.

[EF04] A. M. Etheridge, K. Fleischmann. Compact interface property for symbiotic branching. *Stochastic Process. Appl.*, 114:127-160, 2004.

[EVY20] A. M. Etheridge, A. Véber, F. Yu. Rescaling limits of the spatial Lambda-Fleming-Viot process with selection. *Electron. J. Probab.*, 25:1-89, 2020.

[Ev10] L. C. Evans. Partial Differential Equations, *American Mathematical Society*, 2nd edition, 2010.

[Ev97] S. N. Evans. Coalescing Markov labelled partitions and a continuous sites genetics model with infinitely many types. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(3):339-358, 1997.

[EF96] S. N. Evans, K. Fleischmann. Cluster formation in a stepping-stone model with continuous, hierarchically structured sites. *Ann. Probab.*, 24(4), 1926-1952, 1996.

[EMS13] S. N. Evans, B. Morris, A. Sen. Coalescing systems of non-Brownian particles. *Probab. Theory Relat. Fields*, 156:307–342, 2013.

[Fa90] K. Falconer. Fractal Geometry. *John Wiley & Sons Ltd.*, 1990.

[FW04] A. Fan, J. Wu. On the covering by small random intervals. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(1):125-131, 2004.

[FJJS18] D.J. Feng, E. Järvenpää, M. Järvenpää, V. Suomala. Dimensions of random covering sets in Riemann manifolds. *Ann. Probab.*, 46(3):1542-1596, 2018.

[GK54] B. V. Gnedenko, A. N. Kolmogorov. Limit distributions for sums of independent random variables. *Addison-Wesley Publishing Company, Inc., Cambridge, Mass.*, 1954.

[GSW16] A. Greven, R. Sun, A. Winter. Continuum space limit of the genealogies of interacting Fleming-viot processes on $\mathbb{Z}$. *Electron. J. Probab.*, 21(58):1-64, 2016.

[GR17] T. Grzywny, M. Ryznar. Hitting Times of Points and Intervals for Symmetric Lévy Processes. *Potential Analysis*, 46:739-777, 2017.

[HOV18] M. Hammer, M. Ortgiese, F. Völlering. A new look at duality for the symbiotic branching model. *Ann. Probab.*, 46(5):2800-2862, 2018.

[HOV21] M. Hammer, M. Ortgiese, F. Völlering. Entrance laws for annihilating Brownian motions and the continuous-space voter model. *Stochastic Processes and their Applications*, 134:240-264, 2021.

[Ha75] J. Hawkes. On the Potential Theory of Subordinators. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 33:113-132, 1975.

[HL75] R. A. Holley, T. M. Liggett. Ergodic Theorems for Weakly Interacting Infinite Systems and the Voter Model. *Ann. Probab.*, 3(4):643-663, 1975.

[IL71] I. A. Ibragimov, Yu. V. Linnik. Independent and stationary sequences of variables random. *Groningen, Wolters-Noordhoff*, 1971.

[IW89] N. Ikeda, S. Watanabe. Stochastic Differential Equations and Diffusion Processes. *North-Holland Publishing Company, Kodansha Scientific Books*, 2nd Edition, New York, 1989.

[JS03] J. Jacod, A. N. Shiryaev. Limit theorems for stochastic processes. *Volume 288 von Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Berlin, Springer, 2. Edition*, 2003.

[Ka86] O. Kallenberg. Random measures. *4th edn., Akademie Berlin*, 1986.

[Ka97] O. Kallenberg. Foundations of Modern Probability. *Springer*, 1997.

[KS91] I. Karatzas, S. E. Shreve. Brownian motion and stochastic calculus. *Graduate Texts in Mathematics. Vol. 113 (Second edition of 1988 original ed.), New York*, 1991.

[Ke69] H. Kesten. Hitting probabilities of single points for processes with stationary independent increments. *Memoirs of the American Mathematical Society, No. 93. American Mathematical Society, Providence, R.I.*, 1969.

[Kh11] D. Khoshnevisan. Topics in Probability: Lévy Processes. *Lecture Notes*, 2011.

[Kl13] A. Klenke. Probability Theory *Springer*, 2nd Edition, 2013.

[Kw19] M. Kwaśnicki. Fractional Laplace operator and its properties. *Volume 1 Basic Theory, edited by Anatoly Kochubei and Yuri Luchko, Berlin, Boston: De Gruyter*, 159-194, 2019.

[Kw17] M. Kwaśnicki. Ten equivalent definitions of the fractional laplace operator. *Fractional Calculus and Applied Analysis*, 20(1):7-51, 2017.

[Kyp14] A. E. Kyprianou. Fluctuations of Lévy Processes with Applications. *Universitext. Springer-Verlag Berlin Heidelberg, 2 edition*, 2014.

[Li85] T. M. Liggett. Interacting Particle Systems. *Springer*, 1985.

[Ma95] P. Mattila. Geometry of Sets and Measures in Euclidean Spaces - Fractals and Rectifiability, *Cambridge University Press*, 1995.

[MP10] P. Mörters, Y. Peres. Brownian Motion, *Cambridge University Press*, 2010.

[MRV19] T. Mountford, K. Ravishankar, G. Valle. A construction of the Stable Web. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 16:787-907, 2019.

[MMP17] C. Mueller, L. Mytnik, E. Perkins. On the boundary of the support of Super-Brownian Motion. *Ann. Probab.*, 45(6A):3481-3534, 2017.

[NRS05] C. M. Newman, K. Ravishankar, R. Sun. Convergence of Coalescing Nonsimple Random Walks to the Brownian Web. *Electron. J. Probab.*, 10:21-60, 2005.

[Po67] S. C. Port. Hitting times and potentials for recurrent stable processes. *J. Anal. Math.*, 20:371-395, 1967.

[Re89] M. Reimers. One dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields*, 81:319-340, 1989.

[Sc78] D. Schwartz. On Hitting Probabilities for an Annihilating Particle Model. *Ann. Probab.*, 6(3):398-403, 1978.

[Sh88] T. Shiga. Stepping stone models in population genetics and population dynamics. *Stochastic Processes in Physics and Engineering. Albevario, S., et al. (eds.)*, 1988.

[Sh94] T. Shiga. Two contrasting properties of solutions for one dimensional stochastic partial differential equations. *Canad. J. Math.*, 46(2):415-437, 1994.

[Sk57] A. V. Skorokhod. Limit Theorems for Stochastic Processes with Independent Increments. *Theory of Probability & Its Applications*, 2(2):138-171, 1957.

[Tr95] R. Tribe. Large time behavior of interface solutions to the heat equation with Fisher-Wright white noise. *Probab. Theory Related Fields*, 102:289–311, 1995.

[Wa86] J. Walsh. An introduction to stochastic partial differential equations. *Lecture Notes in Mathematics*, 1180:265-439, 1986.

[Wh02] W. Whitt. Stochastic-Process Limits. *Springer*, 2002.

[YYY09] K. Yano, Y. Yano, M. Yor. On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes. *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 187-227, *Springer*, 2009.

[Zä04] H. Zähle. Stochastic heat equation and catalytic super-Brownian motion. *PhD Thesis, Technical University Berlin*, 2004

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