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Hilbert properties of varieties, rational points, and dynamical systems

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Abstract

An integral variety has the Hilbert property if its rational points are not thin. Corvaja–Zannier showed that a smooth projective integral variety with the Hilbert property over a finitely generated field k of characteristic 0 admits no non-trivial étale covers, motivating the refined notion of the "weak Hilbert property". Conjecturally, every smooth projective integral k-variety with a dense set of k-rational points should have the weak Hilbert property – a question originally posed by Corvaja–Zannier.

This extends to quasi-projective varieties by replacing rational points with near-integral points on arithmetic models. This thesis provides new evidence for this conjecture in the quasi-projective setting. We prove that the Hilbert property and weak Hilbert property for arithmetic schemes are stable under products, generalizing results for varieties by Bary-Soroker–Fehm–Petersen and Corvaja–Demeio–Javanpeykar–Lombardo–Zannier, and other persistence properties. We also prove the conjecture for all algebraic groups, extending known results for linear algebraic groups and abelian varieties.

Combined with a conjecture of Campana, Corvaja–Zannier's question predicts that a variety with a dense set of rational points over a number field satisfies the integral weak Hilbert property even after removing a closed subscheme of codimension at least 2. We verify this "punctured" conjecture for all linear algebraic groups.

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Contents

1 Introduction and overview

The study of rational points on algebraic varieties reveals profound connections between a variety's geometry and the arithmetic of its base field. A central theme in this interplay is the requirement that a variety admit sufficiently many rational points: Results such as Faltings' Theorem, the Hasse principle, and approximation properties exemplify how arithmetic conditions shape their distribution, though density alone is often insufficient for applications.

In some cases, exceptional loci of rational points at which such an interplay fails are given by thin sets, as defined by Serre [74, §3], which are, roughly speaking, "small" sets of rational points. Conversely, if a variety has "many" rational points in the sense that its rational points are not thin, we expect strong relations between its geometry and the arithmetic of its base field. For this reason, the study of varieties whose rational points are not thin – those are said to have the Hilbert property – is an important area of arithmetic geometry and a key aspect of this thesis.

The term "Hilbert property", coined by Serre [74, Definition 3.1.2], comes from Hilbert's famous Irreducibility Theorem: In 1892, Hilbert [45] showed that, given a polynomial $F(x_1, \dots, x_m, y_1, \dots, y_n)$ with integer coefficients and such that F is irreducible in $\mathbb{Q}(x_1, \dots, x_m)[y_1, \dots, y_n]$, there exist infinitely many tuples $(t_1, \dots, t_n)$ of integers such that $F(x_1, \dots, x_m, t_1, \dots, t_n)$ is irreducible in $\mathbb{Q}[x_1, \dots, x_m]$. A similar statement also holds if one considers not one but finitely many irreducible polynomials. Hilbert's theorem has since found many generalizations using the terminology of thin sets and is nowadays usually formulated by saying that, given any integer $n \geq 1$, the $\mathbb{Q}$-rational points in the n-dimensional affine space $\mathbb{A}^n_{\mathbb{Q}}$ are not thin, i.e., $\mathbb{A}^n_{\mathbb{Q}}$ has the Hilbert property. The original motivation for proving this theorem was the Inverse Galois Problem, i.e., the question whether every finite group is realizable as a Galois group over $\mathbb{Q}$. In fact, Hilbert uses this result to prove that every finite symmetric group and every alternating group is realizable as a Galois group over $\mathbb{Q}$, but it also has numerous applications in number theory and arithmetic geometry, such as the construction of elliptic curves of large rank (see [68] and [75, §11]) or toward the construction of injective polynomial maps from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{Q}$ (see [71]). One would obtain a complete positive answer to the Inverse Galois Problem by showing that every unirational variety over $\mathbb{Q}$ has the Hilbert property (see Theorem 2.2.7), which was first asked by Colliot-Thélène–Sansuc [19]. Despite a lot of progress in this area in recent years (which we lay out in detail throughout this first chapter), this is still an unsolved problem.

The aim of this thesis is to provide a detailed overview of the development and current state of research on the Hilbert property and related properties, such as the more refined "weak Hilbert property" introduced by Corvaja–Zannier [26], and to establish novel results. We will explain our main results throughout this chapter after introducing the necessary basic definitions and the conjectures by which they are motivated.

Potential density of rational points can often be approached via dynamical systems, which are dominant rational self-maps of a variety. Such systems arise canonically on algebraic groups through translation, where group operations induce natural self-maps. The resulting orbit structure, described in terms of subgroups and their cosets, forms a key theme in this thesis.

1.1 The Hilbert property, the weak Hilbert property, and the questions of Corvaja–Zannier

A *variety* is a reduced separated scheme of finite type over a field. Let k be a field and let X be an integral variety over k. Roughly speaking, X is said to satisfy the Hilbert property if Hilbert's Irreducibility Theorem generalizes to X. This is made precise in the following definition due to Serre [74, §3].

Definition 1.1.1. Let k be a field, let X be an integral variety over k, and let $\Sigma \subseteq X(k)$ be a subset. Then Σ is *thin (in X or in $X(k)$)* if there exists and integer $n \geq 0$ and a finite family $(\pi_i \colon Y_i \to X)_{i=1}^n$ with Y_i integral k-varieties and π_i dominant generically finite separable morphisms of degree at least 2 such that the set

$$\Sigma \setminus \bigcup_{i=1}^{n} \pi_i(Y_i(k)) = \{x \in \Sigma \mid \forall i = 1, \ldots, n : Y_{i,x}(k) = \emptyset\}$$

is not dense in X. We say that X satisfies the *Hilbert property (over k)* if $X(k)$ is not thin in X.

It is not immediately clear how Definition 1.1.1 relates to the classical formulation of Hilbert's Irreducibility Theorem. When considering finitely many morphisms $Y_i \to X$ as above to a k-variety X, the Hilbert property refers to points $x \in X(k)$ whose fibers $Y_{i,x}$ admit no k-rational points, while Hilbert's Irreducibility Theorem is about points whose fibers are irreducible. In Chapter 2 we will see that replacing the condition that $Y_{i,x}$ have no k-points by the condition that it be irreducible indeed produces an equivalent definition of the Hilbert property. (In fact, this is the definition that Lang gives in [54, §9].) We will also show that we may always assume the varieties Y_i to be normal and geometrically integral and the morphisms π_i to be finite separable surjective. We refer to a finite separable surjective morphism $Y \to X$ of integral varieties as a *cover of* X. (In many cases, our base field will be assumed to be of characteristic 0, so separability will be automatic.)

A field k is called *Hilbertian* if there exists a postive-dimensional integral variety with the Hilbert property over k. It is well-known (cf. [36, Proposition 13.5.3]) that this is equivalent to the affine line $\mathbb{A}^1_k$ having the Hilbert property over k. This allows us to make some easy observations over which fields there are no positive-dimensional varieties with the Hilbert property: If k is algebraically closed, then for any integer $n \geq 2$ coprime to $\mathrm{char}(k)$, the map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ sending x to x^n is a cover of degree n and induces a surjection $\mathbb{A}^1_k(k) \to \mathbb{A}^1_k(k)$, showing that k is not Hilbertian. Moreover, if k is finite, then $\mathbb{A}^1_k(k)$ is finite and hence not dense in $\mathbb{A}^1_k$ (the same holds of course for any positive-dimensional variety over k), so finite fields are not Hilbertian either. Classically well-known Hilbertian fields are global fields (i.e., number fields and functions fields in one variable over a finite field) and finitely generated extensions of transcendence degree > 1 over an infinite field [36, §13]. This includes in particular finitely generated fields of characteristic 0, which play a central role in this thesis. (By a *finitely generated* field we mean a field that is finitely generated over its prime field.) Hilbertianity is also known for the maximal abelian extension and the maximal nilpotent extension of a Hilbertian field [52] and *small* extensions (i.e., algebraic extensions with only finitely many subextensions of any fixed degree) of Hilbertian fields [5].

Corvaja and Zannier showed that the Hilbert property forces topological constraints on a variety. Namely, they proved that a smooth projective variety with the Hilbert property over a number field is algebraically simply connected, i.e., it admits no nontrivial étale covers over the algebraic closure of k [26, Theorem 1.4]. They ask whether this is in fact the only obstruction for a smooth projective variety with a dense of rational points over a number field to satisfy the Hilbert property; we state their question in form of the following conjecture. Corvaja–Zannier's proof extends directly to finitely generated fields of characteristic 0 and even admits an integral analogue (see Theorem B); thus, we state the following conjecture for all finitely generated fields of characteristic 0, whereas the original question due to Corvaja–Zannier is restricted to number fields.

Conjecture 1.1.2 ([26, Question-Conjecture 1])**.** *Let k be a finitely generated field of characteristic 0 and let X be an algebraically simply connected smooth projective integral variety over k. If $X(k)$ is dense in X, then X satisfies the Hilbert property.*

Corvaja–Zannier also ask if a weaker version of Conjecture 1.1.2 holds, in which the assertion that X satisfies the Hilbert property holds only up to replacing k by a finite extension, i.e., X has the *potential Hilbert property.*

Since the Hilbert property is a birational invariant (see Corollary 2.2.4), Conjecture 1.1.2 predicts that any integral variety over a finitely generated field of characteristic 0 with a dense set of rational points that is birational to an algebraically simply connected smooth projective variety has the Hilbert property. A wide range of examples of varieties satisfying the Hilbert property or the potential Hilbert property have been established in recent years, thus providing positive instances of Conjecture 1.1.2 (or its more relaxed version in which X only has the potential Hilbert property). The Hilbert property over any perfect Hilbertian field (and thus also Conjecture 1.1.4) has been proved for reductive groups by Colliot-Thélène–Sansuc [19, Corollary 7.15] and later by Bary-Soroker–Fehm–Petersen for all linear algebraic groups [4, Theorem 4.2]. The first known example of a non-unirational variety with the Hilbert property, due to Corvaja–Zannier [26, Theorem 1.6], is the K3 surface over $\mathbb{Q}$ defined by $x^4+y^4 = z^4+w^4$. The work of Demeio [30, 31] gives examples of varieties satisfying the Hilbert property over a number field, including certain smooth cubic hypersurfaces, K3 surfaces, and Kummer surfaces. All geometrically Kummer surfaces and certain elliptic K3 surfaces over a finitely generated field of characteristic 0 have been shown to have the potential Hilbert property by Gvirtz-Chen–Mezzedimi [41, 42]. Certain del Pezzo surfaces have been shown to admit the Hilbert property by Demeio–Streeter–Winter [28, 29] and Streeter [80], and Loughran–Salgado [58] showed that certain loci of elliptic surfaces are not thin.

In light of their result that the presence of non-trivial étale covers of (a smooth projective model of) a variety over a finitely generated field of characteristic 0 obstructs the Hilbert property, Corvaja–Zannier introduce the more refined notion of the "weak Hilbert property" [26, §2.2], in which only ramified covers are considered. (By a *ramified cover*, we mean a cover $Y \to X$ with X and Y normal integral varieties that is not unramified.) Note that this definition is restricted to the case that the base field is of characteristic 0.

Definition 1.1.3. Let k be a field of characteristic 0, let X be a normal integral variety over k, and let $\Sigma \subseteq X(k)$ be a subset. Then Σ is *strongly thin (in X or in $X(k)$)* if there exists and integer $n \geq 0$ and a finite family $(\pi_i \colon Y_i \to X)_{i=1}^n$ with Y_i normal integral k-varieties and π_i ramified covers such that the set

$$\Sigma \setminus \bigcup_{i=1}^{n} \pi_i(Y_i(k)) = \{x \in \Sigma \mid \forall i = 1, \dots, n : Y_{i,x}(k) = \emptyset\}$$

is not dense in X. We say that X satisfies the *weak Hilbert property (over k)* if $X(k)$ is not strongly thin in X.

To avoid confusion and make a clear distinction between the Hilbert property and the weak Hilbert property, we sometimes refer to the Hilbert property as "the (usual) Hilbert property".

Since ramified covers are of degree at least 2, a strongly thin set is thin. In particular, a normal integral variety over a field of characteristic 0 with the Hilbert property has the weak Hilbert property. As mentioned before, we show that we may assume the dominant generically finite morphisms $\pi\colon Y_i \to X$ in Definition 1.1.1 to be covers with each Y_i normal. Thus, by Corvaja–Zannier's aforementioned theorem, a smooth projective variety has the Hilbert property if and only if it is algebraically simply connected and has the weak Hilbert property. This leads to the following generalization of Conjecture 1.1.2. (Again, in the original question due to Corvaja–Zannier, the base field is assumed to be a number field.)

Conjecture 1.1.4 ([26, Question-Conjecture 2])**.** *Let k be a finitely generated field of characteristic 0 and let X be a smooth integral variety over k. If $X(k)$ is dense in X, then X satisfies the weak Hilbert property.*

Similarly to Conjecture 1.1.2, Corvaja–Zannier also ask if a weaker version of Conjecture 1.1.4 holds, in which the assertion that X satisfies the weak Hilbert property holds only up to replacing k by a finite extension, i.e., X satisfies the *potential weak Hilbert property.*

Recall that a smooth projective surface S is an Enriques surface if its canonical bundle K_S is 2-torsion (i.e. $2K_S = 0$) and $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$. Since every Enriques surface has a non-trivial double étale cover, it follows from Corvaja–Zannier's theorem that they do not have the potential Hilbert property. However, in [7], Bogomolov–Tschinkel showed that every Enriques surface has a potentially dense set of rational points. In particular, we expect that they have the potential *weak* Hilbert property; the latter was verified in work of Gvirtz-Chen–Mezzedimi [41].

If A is an abelian variety with a dense set of rational points over a finitely generated field k of characteristic 0, the prediction of Conjecture 1.1.4 that A satisfies the weak Hilbert property has been verified by Corvaja–Demeio–Javanpeykar–Lombardo–Zannier [25]. However, we can not expect A to have the (usual) Hilbert property if its dimension is positive. Indeed, let $n \geq 2$ be an integer and note that $nA(k)$ is a subgroup of $A(k)$ of finite index, as $A(k)$ is finitely generated by the Mordell–Weil Theorem. Thus, there exist finitely many elements $a_i \in A(k)$ such that $A(k) = \bigcup_i (a_i + nA(k))$.

The morphisms $\pi_i\colon A \to A, x \mapsto a_i + nx$ are covers of degree at least 2, and clearly every k-point of A lifts to a k-point along one of them.

One of the main results of this thesis is that Conjecture 1.1.4 holds for all algebraic groups.

Theorem A (Theorem 4.1.13)**.** *Let k be a finitely generated field of characteristic 0 and let G be a connected algebraic group over k. If $G(k)$ is dense, then G satisfies the weak Hilbert property.*

1.2 Arithmetic schemes and near-integral points

In the same way that we view rational points of a variety X over a field k as solutions to systems of polynomial equations defining X, we are sometimes interested in solutions to those equations that belong to a certain subring R of k. This is a priori not a well-defined notion, since the equations defining X are given only up to isomorphism over k, which might not preserve R. To make this notion well-defined, we use arithmetic schemes.

Definition 1.2.1. Let S be an irreducible scheme with function field $K(S)$ and let $\mathcal{X}$ be a scheme over S. We say that $\mathcal{X}$ is an *arithmetic scheme over S* if $\mathcal{X} \to S$ is dominant, separated, and of finite type and its generic fiber $\mathcal{X}_{K(S)}$ is normal and integral. If $S = \operatorname{Spec} R$, we also refer to an arithmetic scheme over S as an arithmetic scheme over R. By an *arithmetic scheme* we mean an arithmetic scheme over $\mathbb{Z}$.

Essentially, the definition of an arithmetic scheme $\mathcal{X}$ over S is tailored specifically in a way that makes the generic fiber $\mathcal{X}_{K(S)}$ a normal integral $K(S)$-variety. Note that we do not require $\mathcal{X}$ to be irreducible, reduced, or normal globally, but only generically over S. The reason for this is that these properties might not be stable under operations like certain base changes or fiber products. Even if $\mathcal{X}$ is not irreducible, the irreducibility of $\mathcal{X}_{K(S)}$ guarantees that $\mathcal{X}$ has exactly one irreducible component $\mathcal{X}_0$ that dominates S, and we have $\mathcal{X}(K(S)) = \mathcal{X}_0(K(S))$. Thus, when studying (subsets of) $K(S)$-rational points on arithmetic schemes over S, it often suffices to consider irreducible schemes, though we have to allow this more general notion of an arithmetic scheme for technical reasons.

If R is an integral domain with fraction field k and $\mathcal{X}$ is an arithmetic scheme over R with generic fiber $\mathcal{X}_k =: X$, then the set $\mathcal{X}(R)$ of R-integral points of $\mathcal{X}$ provides a well-defined notion of "solutions over R to the polynomial equations defining X". The canonical map $\mathcal{X}(R) \to \mathcal{X}(k) \cong X(k)$ is injective by the valuative criterion of separatedness, so we may in fact view $\mathcal{X}(R)$ as a subset of $X(k)$. Note that $\mathcal{X}(R)$ depends on the choice of $\mathcal{X}$, i.e., a different arithmetic scheme $\mathcal{X}'$ over R with $\mathcal{X}'_k \cong X$ might give us a set of integral points $\mathcal{X}'(R) \neq \mathcal{X}(R)$ (as subsets of $X(k)$). However, since two models of X are generically isomorphic, there exists a dense open subscheme $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathcal{X}_{R'} \cong \mathcal{X}'_{R'}$. This allows us to consider "potential" properties of integral points, i.e., properties that allow for certain extensions of the base ring, without specifying the arithmetic scheme on which the integral points are defined.

The techniques we apply to study thin and strongly thin sets often rely on the properness of a given variety. For example, several of our arguments rely crucially on the Chevalley–Weil Theorem, which ensures that k-rational points lift to L-rational points along finite étale surjective morphisms of proper schemes, for some finite extension L/k, when k is finitely generated of characteristic 0 (see [25, Theorem 3.8]). If k is a number field, the Chevalley–Weil Theorem has an analogue for $\mathcal{O}_{k,S}$-integral points, where S is a finite set of places of k, without the properness assumption. Here, $\mathcal{O}_{k,S}$ denotes the subring of k consisting of all elements that lie in $\mathcal{O}_{k,v}$ for every *finite* place $v \notin S$. (In particular, adding or removing archimedean places from S does not affect $\mathcal{O}_{k,S}$.) For higher-dimensional arithmetic base rings, such a direct analogue need not hold in general. Vojta noticed that the more natural object to study in the case of higher-dimensional base rings is the set of near-integral points, as defined in [81, Definition 4.3] (see also [48, Definition 3.11]).

Definition 1.2.2. Let S be an integral noetherian scheme and let $\mathcal{X} \to S$ be a finite type morphism. We say that a $K(S)$-point P of $\mathcal{X}$ is *near-integral* (or *near-S-integral*) if there is a dense open $U \subseteq S$ whose complement is of codimension at least 2 such that P extends to a morphism $U \to \mathcal{X}$ over S. The set of near-S-integral points of $\mathcal{X}$ is denoted by $\mathcal{X}(S)^{(1)}$. If $T \to S$ is a morphism of integral noetherian schemes, then we define $\mathcal{X}(T)^{(1)} \subseteq \mathcal{X}(K(T)) = \mathcal{X}_T(K(T))$ to be the set of near-T-integral points of $\mathcal{X}_T \to T$. If $S = \operatorname{Spec} R$, we also write $\mathcal{X}(R)^{(1)} := \mathcal{X}(S)^{(1)}$ and refer to near-S-integral points as near-R-integral points.

Note that a morphism $\mathcal{X} \to \mathcal{Y}$ of schemes of finite type over S induces a map $\mathcal{X}(S)^{(1)} \to \mathcal{Y}(S)^{(1)}$. If R is a 1-dimensional integral domain, e.g. $R = \mathcal{O}_{K,S}$ is the ring of S-integers of a number field K for some finite set of places S of K, then the near-R-integral points coincide with the R-integral points. (To be more precise, they coincide with the set of K-points induced by R-integral points. However, if $\mathcal{X} \to \operatorname{Spec} R$ is separated, e.g., if $\mathcal{X}$ is an arithmetic scheme over R, then the map $\mathcal{X}(R) \to \mathcal{X}(K)$ is injective, and we identify the R-integral points with their image in the set of K-rational points.) If $\mathcal{X}$ is a proper arithmetic scheme over a normal $\mathbb{Z}$-finitely generated integral domain R with fraction field k, then the valuative criterion of properness shows that $\mathcal{X}(R)^{(1)} = \mathcal{X}(k)$. Indeed, compatibility with the valuative criteria of separatedness and properness is essentially built into the definition of near-integral points (see also the following remark), allowing for example an analogue to the Chevalley–Weil Theorem (as we will show in Theorem 3.1.10). Thus, they are the natural object for us to study when investigating integral analogues of the (usual and weak) Hilbert property.

Remark 1.2.3. Let S be an integral scheme with $K(S) = k$ and let $\mathcal{X} \to S$ be a morphism of schemes. Then, in [48, Definition 3.11], a k-point $x \in \mathcal{X}(k)$ is defined to be near-S-integral if x extends to a morphism $\operatorname{Spec} \mathcal{O}_{S,s} \to X$ for every point $s \in S$ of codimension 1. If $\mathcal{X} \to S$ is separated and of finite type and S is normal noetherian, then it follows from the valuative criterion of separatedness that this definition is equivalent to Definition 1.2.2 (since the $\mathcal{O}_{S,s}$ are valuation rings in this case).

We extend the definition of the (weak) Hilbert property to arithmetic schemes by requiring their near-integral points to be not (strongly) thin in the generic fiber.

Definition 1.2.4. Let S be a normal noetherian integral scheme with function field $K(S) = k$ and let $\mathcal{X}$ be an arithmetic scheme over S. We say that $\mathcal{X}$ has the *Hilbert property (over S)* if $\mathcal{X}(S)^{(1)}$ is not thin in $\mathcal{X}_k$. If k is of characteristic 0, we say that $\mathcal{X}$ has the *weak Hilbert property (over S)* if $\mathcal{X}(S)^{(1)}$ is not strongly thin in $\mathcal{X}_k$.

If $S = \operatorname{Spec} R$, we also refer to the (weak) Hilbert property over S as the (weak) Hilbert property over R. The Hilbert property over rings has already been implicitly studied in terms of *Hilbertian rings* in [53, §VIII]. Using the above definition, a normal noetherian integral domain R is Hilbertian if and only if $\mathbb{A}^1_R$ has the Hilbert property over R. (To see this, one can apply the algebraic Hartogs' Lemma as given, for example, in [56, §4, Theorem 1.14] to conclude that near-integral and integral points coincide on an affine arithmetic scheme over R.) Hilbertian rings include integral domains that are finitely generated either over $\mathbb{Z}$ or over a field over which their fraction field is transcendental; see [54, §9, Theorem 4.2] or [36, Proposition 13.4.1]. In particular, $\mathbb{A}^1_{\mathbb{Z}}$ has the Hilbert property over $\mathbb{Z}$; this case is already implicit in Lang's proof of Hilbert's Irreducibility Theorem [53, §VIII] and in a different proof due to Serre [74, §9.6]. A formulation of the Hilbert property and the weak Hilbert property for integral points was later given by Corvaja–Zannier [26, §3.2] in the setting of rings of S-integers of number fields, though without using arithmetic schemes to describe the integral points.

Building on the work of Corvaja–Zannier on the algebraic simple connectedness of (smooth projective) varieties with the Hilbert property over a number field, we show that an analogous statement holds for arithmetic schemes:

Theorem B (Theorem 3.1.12). *Let S be a regular integral arithmetic scheme and let $\mathcal{X}$ be a normal integral arithmetic scheme over S. If $\mathcal{X}$ has the Hilbert property* *over S, then $\mathcal{X}$ does not have any non-trivial finite étale covers over S, i.e., any finite étale morphism $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ a normal integral arithmetic scheme over S whose generic fiber $\mathcal{Y}_{K(S)}$ is geometrically connected is an isomorphism.*

Theorem B shows that, like in the case of varieties, the Hilbert property and the weak Hilbert property differ only in the presence of non-trivial étale covers. Corvaja–Zannier's questions could now be naturally extended to arithmetic schemes by asking whether the density of near-integral points implies the weak Hilbert property. In fact, Corvaja and Zannier raise this question for S-integral points of a number field (see [26, Question-Conjecture 1bis] and the remark thereafter). However, as shown by Corvaja–Zannier (in work to appear), the arithmetic surface $\mathcal{X} \subseteq \mathbb{A}^3_{\mathbb{Z}}$ over $\mathbb{Z}$ defined by the equation $x^2 + y^2 = z(xy + 1)$ does not satisfy the Hilbert property over $\mathbb{Z}$, despite $\mathcal{X}(\mathbb{Z})$ being dense and $\mathcal{X}$ being algebraically simply connected (in the sense of Theorem B), thereby providing a negative answer to [26, Question-Conjecture 1bis] in this strong form. Thus, we formulate the following most general conjecture pertaining to near-integral points, where we allow for an extension of the base ring (and its field of fractions).

Conjecture 1.2.5. *Let R be a normal $\mathbb{Z}$-finitely generated integral domain with fraction field k of characteristic 0 and let $\mathcal{X}$ be a regular quasi-projective arithmetic scheme over R. If $\mathcal{X}(R)^{(1)}$ is dense in $\mathcal{X}_k$, then there is a dominant generically finite morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ of normal noetherian integral schemes such that $\mathcal{X}_S$ has the weak Hilbert property over S.*

Conjecture 1.2.5 has been proved over S-integers of a number field by Coccia for smooth cubic surfaces in $\mathbb{A}^3$ [17] and complements of a reduced effective singular anticanonical divisor in a smooth del Pezzo surface [18]. Some results regarding Conjecture 1.2.5 are implicitly proved in [24, 35, 86] and apply to tori over number fields (see [26, §2.3] for details).

Our Theorem A, which shows that algebraic groups satisfy Conjecture 1.1.4, is actually a consequence of our more general result that Conjecture 1.2.5 holds for finite type group schemes, even without extending the fraction field of the base ring.

Theorem C (Theorem 4.1.12)**.** *Let R be a normal $\mathbb{Z}$-finitely generated integral domain with fraction field k of characteristic 0 and let $\mathcal{G}$ be a connected finite type group scheme over R. If $\mathcal{G}(k)$ is dense in $\mathcal{G}$, then there exists a dense open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathcal{G}(R')$ is not strongly thin in $\mathcal{G}_k$. In particular, $\mathcal{G}_{R'}$ has the weak Hilbert property over R'.*

If $\mathcal{X}$ and $\mathcal{Y}$ are arithmetic schemes over an arithmetic scheme S, we are often interested in the product $\mathcal{X} \times_S \mathcal{Y}$. In the case of varieties, it was first asked by Serre

[74, Problem in §3.1] whether the product of two integral varieties with the Hilbert property also has the Hilbert property. (Here it has to be noted that this product is again an integral variety, since an integral variety with the Hilbert property has a dense set of rational points and is thus geometrically integral, which is preserved under products. Similarly, if $\mathcal{X}$ and $\mathcal{Y}$ have dense $K(S)$-points, the product of their generic fibers is integral, so $\mathcal{X} \times_S \mathcal{Y}$ is an arithmetic scheme.)

A positive answer to Serre's question was first given by Bary-Soroker–Fehm–Petersen (see [4]). In fact, Bary-Soroker–Fehm–Petersen prove a more general fibration statement that asserts that, given a variety S with the Hilbert property and a dominant morphism $X \to S$ whose fibers over all rational points have the Hilbert property, then X also has the Hilbert property. We follow this proof closely to show that this fibration argument works for arbitrary subsets of rational points:

Theorem D (Theorem 3.1.3)**.** *Let k be a field, let $X \to S$ be a dominant morphism of integral k-varieties, and let $\Gamma \subseteq X(k)$ and $\Sigma \subseteq S(k)$ be subsets. If Σ is not thin in S and, for every $s \in \Sigma$, the fiber X_s is integral and $\Gamma \cap X_s$ is not a thin subset of X_s, then Γ is not thin in X.*

In particular, we derive from this the persistence of the Hilbert property for arithmetic schemes under products (Theorem 3.1.4).

A first answer to the analogous question for the weak Hilbert property was given by Corvaja–Demeio–Javanpeykar–Lombardo–Zannier [25, Theorem 1.9]. While the proof for the usual Hilbert property does not impose any restrictions on the considered varieties, its analogue for the weak Hilbert property given in *loc. cit.* is restricted to smooth proper varieties over a finitely generated field of characteristic 0. A key ingredient in the proof is the Chevalley–Weil Theorem, which allows for the lifting of rational points in this setting.

Our main theorem concerning products of arithmetic schemes generalizes the result of Corvaja–Demeio–Javanpeykar–Lombardo–Zannier by considering near-integral points in order to apply a Chevalley–Weil type lifting result for near-integral points on varieties over finitely generated fields.

Theorem E (Theorem 3.2.6)**.** *Let S be a regular integral arithmetic scheme with function field k, let $\mathcal{X}, \mathcal{Y}$ be quasi-projective arithmetic schemes over S, and define $X = \mathcal{X}_k, Y = \mathcal{Y}_k$. Furthermore, let $p\colon X \times Y \to X$ denote the projection morphism, let $\Sigma \subseteq (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)}$ be a subset, and define $\Sigma_X = p(\Sigma) \subseteq \mathcal{X}(S)^{(1)}$. For $x \in X(k)$, define $\Sigma_x = \Sigma \cap (\{x\} \times Y)$. If Σ_X is not strongly thin in X and, for every $x \in \Sigma_X$, the set Σ_x is not strongly thin in $\{x\} \times Y$, then Σ is not strongly thin in $X \times Y$.*

An immediate consequence of Theorem E is that the product of two arithmetic schemes with the weak Hilbert property has the weak Hilbert property. Note, however, that the set Σ is not necessarily given as a product of two sets of near-integral points; this will be crucial for our results on punctured linear algebraic groups (see Theorem H).

We also prove some other persistence properties of the usual and weak Hilbert property for arithmetic schemes; namely:

- They are invariant under proper birational morphisms (Theorems 3.1.5 and 3.2.9);
- they are preserved by finite type flat base change (Theorems 3.1.8 and 3.2.16);
- the weak Hilbert property descends along smooth proper morphisms (Proposition 3.2.12).

While the weak Hilbert property descends along smooth proper morphisms, it does not ascend along them (i.e., the converse to Proposition 3.2.12 is false). In fact, this fails even for finite étale morphisms of varieties, see [25, Remark 3.5]. However, in the case of smooth proper varieties, the weak Hilbert property does ascend along finite étale morphisms *up to a finite extension of the base field* [25, Theorem 3.16]. We give an analogue of this result for arithmetic schemes:

Theorem F (Theorem 3.2.15)**.** *Let S be a regular integral arithmetic scheme and $\varphi\colon \mathcal{X} \to \mathcal{Y}$ a finite étale morphism of arithmetic schemes over S. If $\mathcal{Y}$ satisfies the weak Hilbert property over S and $\mathcal{X}_{K(S)}$ is geometrically integral, then there exists a finite étale morphism $S' \to S$ of regular integral arithmetic schemes such that $\mathcal{X}_{S'}$ is an arithmetic scheme over S' with the weak Hilbert property over S'.*

We stress that Theorem F says that the *potential* weak Hilbert property is invariant under finite étale morphisms of arithmetic schemes.

The last persistence property we prove for the weak Hilbert property concerns fibrations. Recall that the product theorem for the Hilbert property due to Bary-Soroker–Fehm–Petersen is a consequence of their more general fibration theorem for the Hilbert property [4, Theorem 1.1] (see also Theorem D). While a full analogue to this fibration theorem for the weak Hilbert property seems out of reach at this time, a fibration theorem in which the base is assumed to have the *weak* Hilbert property and the fibers have the *usual* Hilbert property has been given by Javanpeykar [49, Theorem 1.3]. This "mixed fibration theorem" assumes that the base field is finitely generated and both varieties are projective. We prove an extension eliminating both these assumptions:

Theorem G (Theorem 3.3.2)**.** *Let k be a field of characteristic* 0 *and let $X \to S$ be a dominant morphism of normal integral varieties over k. Let $\Gamma \subseteq X(k)$ and $\Sigma \subseteq S(k)$ be subsets such that Σ is not strongly thin in S and, for every $s \in \Sigma$, the fiber X_s is a normal integral k-variety and $\Gamma \cap X_s(k)$ is not thin in X_s. Then Γ is not strongly thin in X.*

We may expect that in Theorem G, the assumption that $\Gamma \cap X_s(k)$ is not thin in X_s may be replaced with "$\Gamma \cap X_s(k)$ is not *strongly* thin in X_s". This version remains currently unknown, but has recently been proved by Petersen [69, Theorem 1.12] for near-integral points in fibrations which admit a section.

1.3 Corvaja–Zannier's questions and Campana's puncturing conjecture

Conjecture 1.2.5 predicts that the density of near-integral points on a smooth integral variety over a finitely generated field of characteristic 0 implies their non-strongly-thinness, up to a suitable extension of the base. The density of near-integral points is also subject of another conjecture due to Campana, which we explain here. We start with some terminology given in [3].

Let k be an algebraically closed of characteristic 0. An *snc pair* over k is a pair (X, D) consisting of a smooth proper integral variety X over k and a simple normal crossings divisor $D \subseteq X$. For such a pair, let $\Omega^1_X(\log D) \subseteq \Omega^1_X$ be the subsheaf of differential forms with log poles along X (cf. [47, §11]), and for $p \geq 2$ set $\Omega^p_X(\log D) = \Lambda^p \Omega^1_X(\log D)$. If $\mathcal{L}$ is a line bundle on X admitting a non-zero morphism $\mathcal{L} \to \Omega^p_X(\log D)$, the Iitaka dimension $\kappa(\mathcal{L})$ satisfies $\kappa(\mathcal{L}) \leq p$ by a theorem of Bogomolov (see [6, §12, Theorem 4] and [33, Corollary 6.9]). The following definition says that an snc pair is *special* if this upper bound is never attained, and the notion of specialness for a variety is defined through an snc model:

Definition 1.3.1. Let k be an algebraically closed field of characteristic 0. An snc pair (X, D) over k is *special* if, for every integer $p = 1, \dots, \dim X$, there is no line bundle $\mathcal{L}$ on X that admits a non-zero morphism $\mathcal{L} \to \Omega^p_X(\log D)$ and satisfies $\kappa(\mathcal{L}) = p$.

Definition 1.3.2. Let k be an algebraically closed field of characteristic 0 and let X be an integral variety over k. Then X is *special* if there exists a resolution of singularities $Y \to X$ and a smooth projective compactification $\overline{Y}$ of Y such that $(\overline{Y}, \overline{Y} \setminus Y)$ is a special snc pair over k.

The notion of specialness satisfies several formal properties similar to the weak Hilbert property, such as closedness under products [12, Example 2.1.2], invariance under proper birational morphisms [3, Lemma 2.1], and invariance under étale covers [13, Théorème 10.11]. (See [3, Lemma 2.8] for the case of varieties.)

Campana [12, Conjecture 9.20] conjectured that an integral variety over an algebraically closed field of characteristic 0 is special if and only if it is "arithmetically special", i.e., admits a dense set of near-integral points in the following sense.

Conjecture 1.3.3 (Campana). *Let k be an algebraically closed field of characteristic 0 and let X be an integral variety over k. Then X is special over k if and only if there exists a $\mathbb{Z}$-finitely generated subring $R \subsetneq k$ and a finite type separated integral scheme $\mathcal{X}$ over R such that $\mathcal{X}_K = X$ and $\mathcal{X}(R)^{(1)}$ is dense in X.*

To avoid having to refer to specific arithmetic models of a given variety, we make the following definition for the sake of readability.

Definition 1.3.4. Let R be a normal $\mathbb{Z}$-finitely generated integral domain with fraction field k of characteristic 0 and let X be a normal integral k-variety. We say that X *has a dense set of R-integral (resp. near-R-integral) points* if there exists an arithmetic scheme $\mathcal{X}$ over R with $\mathcal{X}_k \cong X$ such that $\mathcal{X}(R)$ (resp. $\mathcal{X}(R)^{(1)}$) is dense in X. We say that X has the *Hilbert property over R* (resp. the *weak Hilbert property over R*) if there exists an arithmetic scheme $\mathcal{X}'$ over R with $\mathcal{X}'_k \cong X$ such that $\mathcal{X}'(R)^{(1)}$ is not thin (resp. not strongly thin) in X.

Combining Campana's conjecture with Conjecture 1.2.5, we expect an integral variety X over a number field K to be special over $\overline{K}$ if and only if there exists a number field L/K and a finite set T of places of L such that X_L satisfies the weak Hilbert property over $\mathcal{O}_{L,T}$. Since connected algebraic groups over a number field are special over the algebraic closure (see [1, Lemma 2.11]), our Theorem C shows that this prediction holds true for all algebraic groups. (Note that algebraic groups have a potentially dense set of rational points, which is required to apply Theorem C here.)

A smooth special variety remains special after removing a closed subset of codimension at least 2 (see [3, Theorem G]). Thus, the combination of Campana's conjecture and Corvaja–Zannier's question predicts the following "puncturing conjecture":

Conjecture 1.3.5. *Let X be a smooth integral variety over a number field K. Then the following are equivalent.*

(1) $X_{\overline{K}}$ is special.

(2) There is a number field L/K and a finite set of places T of L such that X_L has a dense set of $\mathcal{O}_{L,T}$-integral points.

(3) *For every closed subscheme $Z \subseteq X$ of codimension at least 2, there is a number field L/K and a finite set of places T of L such that $(X \setminus Z)_L$ has a dense set of $\mathcal{O}_{L,T}$-integral points.*

(4) *There is a number field L/K and a finite set of places T of L such that X_L satisfies the weak Hilbert property over $\mathcal{O}_{L,T}$.*

(5) *For every closed subscheme $Z \subseteq X$ of codimension at least 2, there is a number field L and a finite set of places T of L such that $(X \setminus Z)_L$ satisfies the weak Hilbert property over $\mathcal{O}_{L,T}$.*

Note that (5) implies (2)-(4). The equivalence of (1) and (2) is Campana's conjecture [12, Conjecture 9.20], and the equivalence of (2) and (4) (resp. (3) and (5)) is Corvaja–Zannier's question for the variety X (resp. $X \setminus Z$). The equivalence of (1), (2) and (3) is again a special case of Campana's conjectures (by the aforementioned result that $(X \setminus Z)_{\overline{K}}$ is special). Part of this equivalence was also conjectured by Hassett–Tschinkel [44, Problem 2.13] with the weaker assumption that X has only canonical singularities (in which case there are counterexamples, see [3, Theorem 1.19]).

The potential density of integral points on punctured varieties has been studied by Hassett and Tschinkel in [44], where they prove part (3) of Conjecture 1.3.5 for toric varieties and del Pezzo surfaces with potentially dense rational points. Analogous results for a large number of examples of varieties of dimension up to 3 have been found by McKinnon–Roth [63].

As mentioned before, if G is a connected algebraic group over a number field, then G is special over the algebraic closure (i.e., satisfies (1)), satisfies (4) by Theorem C, and thus also satisfies (2). Our main result concerning Conjecture 1.3.5 is that connected linear algebraic groups satisfy (5) and therefore shows that Conjecture 1.3.5 holds for all linear algebraic groups:

Theorem H (Theorem 4.2.23)**.** *Let G be a connected linear algebraic group over a number field K and let $Z \subseteq G$ be a closed subscheme of codimension at least 2. Then there exists a finite field extension L/K and a finite set of places T of L such that $(G \setminus Z)_L$ satisfies the weak Hilbert property over $\mathcal{O}_{L,T}$.*

Let us sketch the strategy behind the proof of Theorem H. Using standard structure results for algebraic groups together with the persistence properties of the usual and weak Hilbert property proved in Chapter 3, we reduce to punctured products of three basic types of groups: unipotent groups, semi-simple simply connected groups, and tori. Our product theorem for the weak Hilbert property (Theorem E) requires that every fiber of the relevant projection morphism contain a non-strongly-thin set of integral points. The crucial step is to show that, in our situation, the presence of just a single integral point in a fiber already guarantees the existence of such a non-strongly-thin set. For tori, this reduction is achieved via an approximation argument of Hassett–Tschinkel [44] (attributed to McKinnon) combined with the strong form of Hilbert irreducibility for algebraic groups that we prove in Section 4.1. For unipotent and semi-simple groups, it follows from the fact that these satisfy the so-called strong approximation property, even after puncturing, due to the results of Cao–Huang [14], Cao–Liang–Xu [15], and Cao–Xu [16].

Roughly speaking, strong approximation, as well as the related property weak approximation, concerns whether global rational points can be closely approximated by local solutions at various places of a number field. Their connection to Hilbert-type phenomena goes back to work of Ekedahl [32], who showed that strong approximation implies the Hilbert property for rational points on varieties. This has since been generalized to broader notions of arithmetic points, such as "Campana points" by Nakahara–Streeter [67] and "$\mathcal{M}$-points" by Moerman [65]. Although not directly relevant to this thesis beyond Theorem H, approximation properties form a central theme in arithmetic geometry, with contributions by many authors; see, for example, [10, 19, 20, 21, 27, 28, 37, 57, 64, 74, 85].

1.4 Outline of thesis

This thesis is divided into chapters, sections, and subsections. In Chapter 2, we lay out the relation between the Hilbert property and irreducible fibers and prove in particular the equivalence of Serre's and Lang's definitions of the Hilbert property. In the course of this proof, we show some basic properties of the Hilbert property that will be often used throughout the thesis, such as its birationality and an alternative formulation using covers instead of dominant generically finite morphisms.

In Chapter 3, we prove several persistence properties of thin and strongly thin sets, including the persistence of the usual and weak Hilbert property for arithmetic schemes under products, the invariance of the potential weak Hilbert property under finite étale covers, and our theorem on mixed fibrations.

Chapter 4 is dedicated to the special case of algebraic groups. Here we answer positively Corvaja–Zannier's question for all algebraic groups and prove the combined "puncturing conjecture" of Campana and Corvaja–Zannier for linear algebraic groups.

Definitions, theorems, lemmas, and similar statements in the following chapters are numbered within sections. The main theorems presented in this introduction are labeled Theorems A–H and are restated later in the thesis, immediately before their proofs. In each restated version, the theorem's original label (e.g., Theorem A) is shown alongside its internal numbering (e.g., Theorem 4.1.12), indicating where in the thesis the proof appears.

1.5 Disclaimer

Material included in this dissertation is adapted from the author's published works [59, 60, 61, 62], with editorial changes.

2 Irreducible fibers and Lang's definition of the Hilbert property

Given an integral variety X over a field k and a family $(Y_i \to X)_{i=1}^n$ of dominant generically finite morphisms of degree at least 2, Serre's definition of the Hilbert property is expressed by the fibers $Y_{i,x}$ over many points $x \in X(k)$ admitting no rational points. Hilbert's original theorem [45] instead refers to the irreducibility of many fibers $Y_{i,x}$. These two properties are closely related; in fact, Lang [54, §9] uses the latter to give an alternative definition of the Hilbert property, which we refer to as "Lang's Hilbert property". To prove the equivalence of Serre's and Lang's definition, we use the natural extension of Lang's notion to define a "thin set in the sense of Lang", so that a variety will have Lang's Hilbert property if its rational points are not thin in the sense of Lang.

Definition 2.0.1. Let k be a field, let X be an integral variety over k, and let $\Sigma \subseteq X(k)$ be a subset of the k-rational points of X. We say that Σ is *thin (in X or in $X(k)$) in the sense of Lang* if there exists a finite family $(Y_i \to X)_{i=1}^n$ of covers such that the set

$$\{x \in \Sigma \mid \forall i = 1, \ldots, n : Y_{i,x} \text{ is irreducible}\} \subseteq X(k)$$

is not dense in X. We say that X satisfies *Lang's Hilbert property (over k)* if $X(k)$ is not thin in X in the sense of Lang.

In this chapter, we show that a subset of the rational points of variety is thin if and only if it is thin in the sense of Lang (in particular, Serre's and Lang's definition of the Hilbert property are equivalent). To do this, we consider Galois covers and use the action of their Galois groups on fibers.

2.1 Galois actions on fibers

We start the proof of the equivalence of Serre's and Lang's definition of the Hilbert property by investigating the action that the Galois group of a Galois cover has on fibers over the étale locus. The following proposition shows that this action is transitive.

Proposition 2.1.1. *Let A be a normal integral domain with fraction field k. Let L/k be a Galois extension with Galois group $G = \mathrm{Gal}(L/k)$ and let B be the integral closure of A in L. Then the following holds.*

(1) For any two primes $\mathfrak{q}, \mathfrak{q}' \subseteq B$ lying over the same prime in A, there exists an element $\sigma \in G$ with $\sigma(\mathfrak{q}) = \mathfrak{q}'$.

(2) Let $\mathfrak{q} \subseteq B$ be a prime lying over a prime $\mathfrak{p} \subseteq A$. Then $\kappa(\mathfrak{q})$ is an algebraic normal extension of $\kappa(\mathfrak{p})$ and the map

$$D = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\} \to \mathrm{Aut}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))$$

is surjective.

Proof. This is [79, Tag 0BRK]. □

Proposition 2.1.2. *Let k be a field, let $Y \to X$ be a Galois cover of normal integral k-varieties with group G, and let $x \in X(k)$ be a point over which $Y \to X$ is étale. Then Y_x is a disjoint union of r copies of $\operatorname{Spec} L$, where L/k is a Galois extension. The Galois group of L/k can be identified with a subgroup $H \subseteq G$ of index r and the fiber $(Y/H)_x$ of the quotient $Y/H \to X$ admits a k-point.*

Proof. Let $U = \operatorname{Spec} A$ be an affine open neighborhood of x with A a normal integral domain having fraction field $k(X)$. Then the pullback of U along $Y \to X$ is given by an affine scheme $\operatorname{Spec} B$, with B a normal integral domain with fraction field $k(Y)$. Since $A \subseteq B$ is a finite extension and B is normal, B is the integral closure of A in $k(Y)$. Let $\mathfrak{m} \subseteq A$ denote the maximal ideal corresponding to x. By étaleness of $Y \to X$, the fiber Y_x is given by $Y_x = \operatorname{Spec} \prod_{i=1}^{r} B/\mathfrak{m}_i$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals of B lying over $\mathfrak{m}$. By Proposition 2.1.1, the group G acts transitively on the set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. In particular, given any two maximal ideals $\mathfrak{m}_i, \mathfrak{m}_j$, an element $\sigma \in G$ satisfying $\sigma(\mathfrak{m}_i) = \mathfrak{m}_j$ induces an isomorphism $B/\mathfrak{m}_i \cong B/\mathfrak{m}_j$. This shows that $Y_x = \operatorname{Spec} \prod_{i=1}^{r} L$ for some finite field extension $L \cong B/\mathfrak{m}_1 \cong \ldots \cong B/\mathfrak{m}_r$ of $\kappa(x) = k$ satisfying $r[L : k] = \deg(Y \to X) = [k(Y) : k(X)] = \#G$.

By Proposition 2.1.1, L/k is Galois (as separability follows from étaleness) and the map

$$H := \{\sigma \in G \mid \sigma(\mathfrak{m}_1) = \mathfrak{m}_1\} \to \mathrm{Gal}(L/k)$$

is surjective. By the transitivity of the G-action on Y_x, we have $Y_x \cong G/H$ as a set, from which it follows that $r\#H = \#G$. Since $r[L : k] = \#G$, this implies that $\#H = [L : k]$, i.e., the map $H \to \mathrm{Gal}(L/k)$ is an isomorphism, thus identifying $\mathrm{Gal}(L/k)$ with a subgroup of G.

By construction of Y/H, the pullback of $U = \operatorname{Spec} A$ along $Y/H \to X$ is given by $\operatorname{Spec} B^H$. Note that B^H is a normal integral domain with fraction field $k(Y)^H$ and the extension $k(Y)/k(Y)^H$ is Galois with group H.

Define $\mathfrak{n} = \mathfrak{m}_1 \cap B^H$ and note that $\mathfrak{n}$ is a maximal ideal of B^H lying under $\mathfrak{m}_1$. Since H acts transitively on the maximal ideals of B lying over $\mathfrak{n}$ but fixes $\mathfrak{m}_1$ by definition, $\mathfrak{m}_1$ is the only maximal ideal lying over $\mathfrak{n}$. Thus, by the arguments from above, we have a tower of field extensions $k \subseteq B^H/\mathfrak{n} \subseteq L$ with $B^H/\mathfrak{n} \subseteq L$ Galois with Galois group H. But since $H = \operatorname{Gal}(L/k)$, this shows that $B/\mathfrak{n} = k$. This concludes the proof. □

Corollary 2.1.3. *Let k be a field, let $Y \to X$ be a Galois cover of normal integral k-varieties with Galois group G, and let $x \in X(k)$ be a point over which $Y \to X$ is étale. If, for every proper subgroup $H \subsetneq G$, the fiber $(Y/H)_x$ of the quotient $Y/H \to X$ has no k-rational point, then Y_x is irreducible.*

Proof. If Y_x is not irreducible, then, by Proposition 2.1.2, it is a disjoint union of $r \geq 2$ copies of $\operatorname{Spec} L$, where L/k is Galois, its Galois group is isomorphic to a subgroup H of G of index $r \geq 2$ (i.e., $H \subsetneq G$ is a proper subgroup), and the fiber $(Y/H)_x$ admits a k-point. □

2.2 Equivalence of Serre's and Lang's definitions

In order to use the results of the previous section to prove the equivalence of Serre's and Lang's definition of the Hilbert property, we first show some basic properties of thin sets that allow us to reduce to the case of Galois covers.

The Hilbert property is often stated in terms of covers by normal (geometrically integral) varieties instead of dominant generically finite morphisms. The following theorem shows the equivalence of these definitions, which we will use frequently throughout this thesis.

Theorem 2.2.1. *Let k be a field, let X be an integral variety over k, and let $\Sigma \subseteq X(k)$ be a subset. Then Σ is thin in X if and only if there exists a finite family of covers $(\pi_i \colon Y_i \to X)_{i=1}^n$ of degree at least 2 with each Y_i a normal geometrically integral variety over k such that $\Sigma \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ is not dense in X.*

Proof. We only have to show "only if", so we assume that Σ is thin, i.e., there exists a finite family $(\pi_i \colon Y_i \to X)_{i=1}^n$ of dominant generically finite separable morphisms of degree at least 2 such that $\Sigma \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ is not dense in X. For every $i = 1, \ldots, n$, let $\overline{\pi_i} \colon \overline{Y_i} \to X$ be the normalization of X in $k(Y)$. Then $\overline{Y_i}$ is normal and integral (see [79, Tag 035L]) and $\overline{\pi_i}$ is finite separable surjective. Moreover, note that Y_i and $\overline{Y_i}$ are birational over X, as their function fields are isomorphic over $k(X)$, so $\overline{\pi_i}$ is of the same degree as π_i and the images $\pi_i(Y_i(k))$ and $\overline{\pi_i}(\overline{Y_i}(k))$ agree outside of a proper closed subscheme of X. Thus, it follows from the non-density of $\Sigma \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ that $\Sigma \setminus \bigcup_{i=1}^n \overline{\pi_i}(\overline{Y_i}(k))$ is not dense in X.

To finish the proof, assume that $\overline{Y_i}$ is not geometrically integral for some index i, i.e., $\overline{Y_i}$ is irreducible and not geometrically irreducible. Then, by [79, Tag 0G69], the k-points on $\overline{Y_i}$ are not dense. In particular, the image $\overline{\pi_i}(\overline{Y_i}(k))$ is not dense in X. This shows that $\Sigma \setminus \bigcup_j \overline{\pi_j}(\overline{Y_j}(k))$ is not dense in X, where the union is taken only over the indices j with $\overline{Y_j}$ geometrically integral. This concludes the proof. □

Remark 2.2.2. A similar argument shows that thinness in the sense of Lang can be checked using only normal geometrically integral covers, i.e., that we may assume all the varieties Y_i in Definition 2.0.1 to be normal and geometrically integral. Indeed, given a cover $Y_i \to X$, let $\overline{Y_i} \to Y_i$ be the normalization of Y_i in its function field; then $\overline{Y_i} \to Y_i$ is finite surjective birational, so $\overline{Y_i} \to X$ is a cover and we have an isomorphism of fibers $Y_{i,x} \cong (\overline{Y_i})_x$ for all x in some dense open subscheme of X.

Theorem 2.2.1 allows us to reduce the investigation of thin sets to covers $Y_i \to X$ with Y_i normal (and geometrically integral). To furthermore reduce to the case that X is also normal, we show that thinness (and thinness in the sense of Lang) is birational in the sense that we may replace X with a dense open.

Proposition 2.2.3. *Let X be an integral variety over a field k, let $U \subseteq X$ be a dense open, and let $\Sigma \subseteq X(k)$ be a subset. Then Σ is thin (resp. thin in the sense of Lang) in X if and only if $\Sigma \cap U$ is thin (resp. thin in the sense of Lang) in X. Moreover, Σ is thin (resp. thin in the sense of Lang) in X if and only if $\Sigma \cap U$ is thin (resp. thin in the sense of Lang) in U.*

Proof. The first part is clear since the intersection of a dense subset and a dense open is dense. In particular, for the second part, we may and do assume that $\Sigma \subseteq U(k)$.

Assume first that Σ is thin in X in the sense of Lang, so there exists a finite family $(Y_i \to X)_{i=1}^n$ of covers such that the set

$$\Phi := \{x \in \Sigma \mid \forall i : Y_{i,x} \text{ is irreducible}\}$$

is not dense in X (and thus not dense in U). For every $i = 1, \ldots, n$, let $Y_{i,U} \to U$ be the pullback of $Y_i \to X$ along $U \to X$. Then $Y_{i,U} \to U$ is a cover and, for every $x \in \Sigma$, we have an isomorphism of fibers $Y_{i,U,x} \cong Y_{i,x}$. Thus, the set of $x \in \Sigma$ for which all fibers $Y_{i,U,x}$ are irreducible is equal to Φ and therefore not dense in U, which shows that Σ is thin in U in the sense of Lang.

For the converse implication, assume that Σ is thin in U in the sense of Lang, so there exists a finite family $(Y_i' \to U)_{i=1}^n$ of covers such that the set

$$\Phi' := \left\{x \in \Sigma \;\middle|\; \forall i : Y_{i,x}' \text{ is irreducible}\right\}$$

is not dense in U (and thus not dense in X). As we have shown in the proof of Theorem 2.2.1, taking the normalization of X in $k(Y_i')$ gives us a finite family of covers $(\overline{Y_i'} \to X)_{i=1}^n$ where each $\overline{Y_i'}$ is birational to Y_i'. Thus, the set of $x \in \Sigma$ with $(\overline{Y_i'})_x$ irreducible for all i agrees with Φ' outside of a proper closed subset of X; in particular, it is not dense in X, which shows that Σ is not thin in X in the sense of Lang, as required.

The proof that Σ is thin in X if and only if it is thin in U is exactly the same as for thinness in the sense of Lang. (In fact, the "only if part" can also be shown more easily, since we can simply consider the composition $Y_i' \to U \to X$ instead of $\overline{Y_i'} \to X$.) □

Corollary 2.2.4. *Let k be a field and let X and X' be birational integral varieties over k. Then X satisfies the Hilbert property over k if and only if X' satisfies the Hilbert property over k.*

After these reduction steps, we can now show the equivalence of thinness and thinness in the sense of Lang using Galois closures and the action of their Galois groups on the fibers.

Theorem 2.2.5. *Let k be a field, let X be an integral variety over k, and let $\Sigma \subseteq X(k)$ be a subset. Then Σ is thin if and only if it is thin in the sense of Lang.*

Proof. Assume first that Σ is thin. By Theorem 2.2.1, we find a finite family of covers $(Y_i \to X)_{i=1}^n$ with each Y_i a normal (geometrically) integral k-variety and $Y_i \to X$ a cover of degree at least 2 such that

$$\Phi := \{x \in \Sigma \mid \forall i = 1, \dots, n : Y_{i,x}(k) = \emptyset\}$$

is not dense in X. Define

$$\Phi' := \{x \in \Sigma \mid \forall i = 1, \dots, n : Y_{i,x} \text{ is irreducible}\}.$$

Since the covers $Y_i \to X$ are separable and thus generically étale, there exists a dense open $U \subseteq X$ such that, for every $i = 1, \dots, n$ and every $x \in U$, the fiber $Y_{i,x}$ is finite étale over $\kappa(x)$. Note that $Y_{i,x} \to \operatorname{Spec}\kappa(x)$ is of the same degree as $Y_i \to Y$. Let $x \in U \cap \Phi'$. By étaleness, each fiber $Y_{i,x}$ is a disjoint union of spectra of field extensions of k, so by irreducibility, $Y_{i,x}$ is the spectrum of a field extension L_i/k of degree equal to the degree of $Y_i \to Y$, i.e., at least 2. Thus, $Y_{i,x}$ has no k-point, showing that $U \cap \Phi' \subseteq \Phi$. Since U is a dense open and Φ is not dense, we conclude that Φ' is not dense in X, i.e., Σ is thin in the sense of Lang.

Now assume that Σ is not thin; we claim that Σ is not thin in the sense of Lang. By Proposition 2.2.3, we may and do replace X by a dense open if necessary to assume that X is normal. Let $(Y_i' \to X)_{i=1}^n$ be a finite family of covers. Since a cover of X of degree 1 is an isomorphism by Zariski's Main Theorem and thus all of its fibers are trivially irreducible, we may and do assume that all covers $Y_i' \to X$ are of degree at least 2. For every $i = 1, \ldots, n$, let $\widehat{Y_i} \to Y_i' \to X$ denote the Galois closure of $Y_i' \to X$. Consider the finite family $(\widehat{Y_i}/H \to X)$, where $i = 1, \ldots, n$ and H goes through all proper subgroups of the Galois group of $\widehat{Y_i} \to X$. Note that $\widehat{Y_i}/H \to X$ is a cover of degree at least 2 for every i and H. Thus, by the non-thinness of Σ, the set

$$\Phi'' := \left\{ x \in \Sigma \;\middle|\; \forall i = 1, \ldots, n : \forall H : (\widehat{Y_i}/H)_x(k) = \emptyset \right\}$$

is dense in X. Since the $Y_i \to X$ and thus their Galois closures $\widehat{Y_i}/H \to X$ are generically étale, we find a dense open $V \subseteq X$ such that all the covers $\widehat{Y_i}/H \to X$ are étale over V. Note that $\Phi'' \cap V$ is dense in X. By Corollary 2.1.3, for every $x \in \Phi'' \cap V$ and every $i = 1, \ldots, n$, the fiber $(\widehat{Y_i})_x$ is irreducible, so $Y_{i,x}'$ is irreducible by the surjectivity of $\widehat{Y_i} \to Y_i'$. This shows that Σ is not thin in X in the sense of Lang, as required. ☐

Corollary 2.2.6. *Let X be an integral variety over a field k. Then X satisfies the Hilbert property if and only if X satisfies Lang's Hilbert property.*

One reobtains Hilbert's result that every symmetric group S_n is representable as a Galois group over $\mathbb{Q}$ by considering the cover $\mathbb{A}^n_{\mathbb{Q}} \to \mathbb{A}^n_{\mathbb{Q}}/S_n$, where S_n acts on $\mathbb{A}^n_{\mathbb{Q}}$ by permuting coordinates. Here we use that $\mathbb{A}^n_{\mathbb{Q}}/S_n$ is rational and thus satisfies (Lang's version of) the Hilbert property. For a general finite group G acting on $\mathbb{A}^n_{\mathbb{Q}}$, the quotient $\mathbb{A}^n_{\mathbb{Q}}/G$ need not be rational, though it is unirational. The question if every unirational integral variety over $\mathbb{Q}$ has the Hilbert property, as predicted by Conjecture 1.1.2, was asked by Colliot-Thélène–Sansuc [19, p. 190]. It is still open and a positive answer would imply a positive answer to the Inverse Galois Problem (see also [74, Corollary 3.3.2]):

Theorem 2.2.7. *Assume that every unirational integral variety over $\mathbb{Q}$ has the Hilbert property. Then every finite group is representable as a Galois group over $\mathbb{Q}$.*

Proof. Let G be a non-trivial finite group of cardinality n. Since every element of G acts on G by translation, we can identify G with a subgroup of the symmetric group $\operatorname{Sym}(G) \cong S_n$. Under this identification, let G act on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ and its field of fractions $\mathbb{Q}(x_1, \ldots, x_n)$ by permuting the variables. Denote by $\mathbb{Q}[x_1, \ldots, x_n]^G$ (resp. $\mathbb{Q}(x_1, \ldots, x_n)^G$) the ring (resp. field) of G-invariants. Then $\mathbb{Q}(x_1, \ldots, x_n)^G$ is the fraction field of $\mathbb{Q}[x_1, \ldots, x_n]^G$ and the field extension $\mathbb{Q}(x_1, \ldots, x_n)^G \subseteq \mathbb{Q}(x_1, \ldots, x_n)$ is a Galois extension with Galois group G. Let $Y = \operatorname{Spec} \mathbb{Q}[x_1, \ldots, x_n]$ and $X = \operatorname{Spec} \mathbb{Q}[x_1, \ldots, x_n]^G$. Then X is unirational and $Y \to X$ is a Galois cover with Galois group G. By the assumption, X satisfies the Hilbert property (and thus Lang's Hilbert property), so the set of $x \in X(\mathbb{Q})$ with irreducible fiber Y_x is dense in X. In particular, there exists a point $x \in X(\mathbb{Q})$ such that Y_x is irreducible and $Y \to X$ is étale over x. By Proposition 2.1.2, the fiber Y_x is the spectrum of a Galois field extension $L/\mathbb{Q}$ whose Galois group is isomorphic to G. This proves the claim. □

2.3 Fibers of ramified covers

Before proceeding, we recall that any finite surjective morphism $Y \to X$ of integral schemes with X normal and noetherian is either ramified or étale (see e.g. [25, Lemma 2.3]). In particular, if X is a normal integral variety, every cover of X is either ramified or étale. We will use this fact freely throughout this thesis.

While the Hilbert property can be formulated in terms of the irreducibility of many fibers, this is not the case for the weak Hilbert property. Indeed, two important steps in the proof of Theorem 2.2.5 fail if one only considers ramified covers. First, if $Y_i \to X$ are Galois ramified covers and H is a subgroup of the Galois group of $Y_i \to X$, then the cover $Y_i/H \to X$ might not be ramified. If, however, all of the covers $Y_i \to X$ have no non-trivial étale subcovers, we obtain the following related observation (which we give in [62, Proposition 4.1]).

Proposition 2.3.1. *Let X be a normal integral variety over a field k of characteristic 0, let $\Sigma \subseteq X(k)$ be a subset, and let $(Y_i \to X)_{i=1}^n$ be a finite family of Galois ramified covers admitting no non-trivial étale subcovers. If Σ is not strongly thin, then there exists a dense subset of points $x \in \Sigma$ such that, for every $i = 1, \ldots, n$, the fiber $Y_{i,x}$ is irreducible.*

Proof. Consider the finite family of covers $(Y_i/H \to X)$, where $i = 1, \ldots, n$ and H goes through all proper subgroups of the Galois group of $Y_i \to X$. By the assumption that $Y_i \to X$ has no non-trivial étale subcovers, this is a family of ramified covers.

Thus, since Σ is not strongly thin, the set

$$\Phi := \{x \in \Sigma \mid \forall i = 1, \ldots, n : \forall H : (Y_i/H)_x(k) = \emptyset\}$$

is dense in X. Let $U \subseteq X$ be a dense open over which all covers $Y_i \to X$ are étale. Then $\Phi \cap U$ is dense in X and, for every $x \in \Phi \cap U$ and every $i = 1, \ldots, n$, the fiber $Y_{i,x}$ is irreducible by Corollary 2.1.3. □

Proposition 2.3.1 shows that non-strongly-thinness can be expressed in terms of irreducible fibers if one considers Galois ramified covers that admit no non-trivial étale subcovers. The second important step in proving that the Hilbert property can be expressed in terms of irreducible fibers that fails for the weak Hilbert property is the reduction to the Galois case. The following example (which we give in [62, Remark 4.2]) shows that, if a ramified cover admits no non-trivial étale subcovers, the same is not necessarily true for its Galois closure.

Example 2.3.2. Let $A = B = \mathbb{P}^1_{\mathbb{Q}}$ and let $\pi\colon B \to A$ be a cover of degree 3 that is ramified over 3 distinct $\mathbb{Q}$-points of A and not Galois. Let $C \to B \to A$ be the Galois closure of π, so that $C \to A$ is of degree 6, and let $C \to D \to A$ be a subcover with $D \to A$ of degree 2. Note that $D \to A$ can only be ramified over branch points of π, i.e., $D \to A$ has at most 3 branch points. By the Riemann–Hurwitz formula, this implies that D is of genus 0, so $D_{\overline{\mathbb{Q}}} \to A_{\overline{\mathbb{Q}}}$ has exactly 2 branch points. From this it follows that the étale locus $X \subseteq A$ of $D \to A$ contains a branch point of π. Let $Y := B \times_A X$ and $\widehat{Y} := C \times_A X$. Then $Y \to X$ is ramified (as X contains a branch point of π) and admits no non-trivial subcovers since its degree is prime. The Galois closure of $Y \to X$ is $\widehat{Y} \to Y \to X$, and $\widehat{Y} \to X$ factorizes over $D \times_A X \to X$, which is a non-trivial étale subcover.

3 Products, fibrations, and persistence properties

In this chapter, we establish the fundamental persistence properties of the Hilbert property and the weak Hilbert property discussed in the introduction. These include:

- the algebraic simple connectedness of arithmetic schemes with the Hilbert property (Theorem B);
- the persistence of both the usual and weak Hilbert properties under products (via the more general Theorems D and E);
- the invariance of the potential weak Hilbert property under finite étale covers (Theorem F);
- Theorem G on mixed fibrations.

We first address results concerning thin sets and the usual Hilbert property, then turn to strongly thin sets and the weak Hilbert property, and conclude with the mixed fibration theorem.

3.1 Thin sets

Up to editorial changes, the results of this section, except for Theorem 3.1.10 and Corollary 3.1.11 (which have been published in [62]), have been published in [59].

3.1.1 Products and fibrations

To positively answer Serre's question on the persistence of the Hilbert property under products, Bary-Soroker–Fehm–Petersen proved the following fibration theorem:

Theorem 3.1.1 ([4, Theorem 1.1]). *Let $X \to S$ be a dominant morphism of integral varieties over a field k. Assume that the set of $s \in S(k)$ for which the fiber X_s is integral and satisfies the Hilbert property over k is not thin. Then X satisfies the Hilbert property over k.*

If X and Y are integral varieties with the Hilbert property over a field k, applying Theorem 3.1.1 to the morphism $X \times Y \to X$ immediately shows that $X \times Y$ has the Hilbert property over k. (Note that density of rational points implies that X and Y, and thus also $X \times Y$, are geometrically integral.) In order to generalize this to arithmetic schemes, our Theorem D works with arbitrary subsets of rational points. The proof of Theorem D follows the proof of Theorem 3.1.1 closely. In fact, Bary-Soroker–Fehm–Petersen's proof already contains all the necessary arguments for our generalization, we repeat them in the necessary generality for the sake of completeness. We start with the following generalization of [4, Lemma 3.2].

Lemma 3.1.2. *Let k be a field, let $f\colon X \to S$ be a dominant morphism of normal integral varieties over k, and let $\Gamma \subseteq X(k)$ and $\Sigma \subseteq S(k)$ be subsets with Σ not thin in S. Assume that, for every $s \in \Sigma$, the fiber X_s is integral and $\Gamma \cap X_s$ is not a thin subset of X_s. Let $Y_1, \ldots, Y_n$ be integral k-varieties and let $\pi_i\colon Y_i \to X$ be finite étale morphisms of degree at least 2. Then*

$$\Gamma \not\subseteq \bigcup_{i=1}^{n} \pi_i(Y_i(k)).$$

Proof. For $i = 1, \ldots, n$, consider the composite morphism $\varphi_i := f \circ \pi_i\colon Y_i \to S$. By [51, Lemma 9], there exists a dense open subscheme $U_i \subseteq S$ and a factorization

$$\varphi_i^{-1}(U_i) \xrightarrow{g_i} T_i \xrightarrow{r_i} U_i$$

of φ_i such that g_i has geometrically irreducible fibers, r_i is finite étale, and T_i is an integral k-variety. Let $U = \bigcap_{i=1}^{n} U_i$, so the dominant morphism $r_i\colon T_i \to S$ induced by the inclusion of U_i in S is finite étale over U for every i. (In particular, it is a dominant generically finite morphism.)

Set $J = \{i \mid \deg r_i \geq 2\} \subseteq \{1, \ldots, n\}$ and let I be its complement.

For every $s \in \Sigma$ and $i \in \{1, \ldots, n\}$, the morphism $\pi_{i,s}\colon Y_{i,s} \to X_s$ induced by π_i is finite étale of the same degree as π_i, the fiber $Y_{i,s}$ is a reduced scheme over k, and we have the following commutative diagram.

$$\begin{tikzcd}
Y_{i,s} \arrow[r] \arrow[d, "\pi_{i,s}"'] & Y_i \arrow[r, "g_i"] \arrow[d, "\pi_i"'] \arrow[dr, "\varphi_i"] & T_i \arrow[d, "r_i"] \\
X_s \arrow[r] & X \arrow[r, "f"] & S
\end{tikzcd}$$

Since $\Sigma \cap U$ is not thin in S (by Proposition 2.2.3) and r_i is of degree at least 2 for every $i \in J$, there exists a point

$$s \in (\Sigma \cap U) \setminus \bigcup_{i \in J} r_i(T_i(k)).$$

For this point s and every $i \in J$, we have $T_{i,s}(k) = \emptyset$ and thus $\bigcup_{i \in J} Y_{i,s}(k) = \emptyset$. Furthermore, for every $i \in I$, the map r_i is finite étale of degree 1 (i.e. an isomorphism) over U, so we have $T_{i,s} = \operatorname{Spec} k$ and thus, the reduced scheme $Y_{i,s}$ is (geometrically) irreducible by the definition of g_i, i.e., $Y_{i,s}$ is an integral k-variety. Thus, since $\Gamma \cap X_s$ not thin in X_s, there exists an element $x \in \Gamma \cap X_s$ such that

$$x \notin \bigcup_{i \in I} \pi_{i,s}(Y_{i,s}(k)).$$

Since $\bigcup_{i \in J} Y_{i,s}(k) = \emptyset$, this shows that

$$x \notin \bigcup_{i \in I} \pi_{i,s}(Y_{i,s}(k)) = \bigcup_{i=1}^{n} \pi_{i,s}(Y_{i,s}(k)).$$

We have thus found an element $x \in \Gamma$ with $x \notin \bigcup_{i=1}^{n} \pi_i(Y_i(k))$, as required. □

In the situation of Theorem D, Lemma 3.1.2 shows that the set Γ can not be completely covered by finite étale covers (of degree at least 2), provided that X and S are normal. The proof of Theorem D is now a reduction to this case.

Theorem 3.1.3 (Theorem D). *Let k be a field, let $X \to S$ be a dominant morphism of integral k-varieties, and let $\Gamma \subseteq X(k)$ and $\Sigma \subseteq S(k)$ be subsets. If Σ is not thin in S and, for every $s \in \Sigma$, the fiber X_s is integral and $\Gamma \cap X_s$ is not a thin subset of X_s, then Γ is not thin in X.*

Proof. Let $C \subsetneq X$ be a proper closed subset and let $(\pi_i \colon Y_i \to X)_{i=1}^n$ be a finite collection of covers of degree at least 2 (cf. Theorem 2.2.1). We have to prove that $\Gamma \nsubseteq C \cup \bigcup_{i=1}^{n} \pi_i(Y_i(k))$.

Since the covers π_i are separable and thus generically étale, we find a dense open $X' \subseteq X \setminus C$ such that X' is normal and the restriction $\pi_i^{-1}(X') \to X'$ is finite étale for every $i = 1, \dots, n$. The image of X' in S is dense and constructible, so it contains a dense open $S' \subseteq S$. Replacing S' with a dense open normal subscheme and X' with $X' \cap f^{-1}(S')$ if necessary, the restriction of f to $X' \to S'$ is a surjective morphism of normal integral k-varieties. By Proposition 2.2.3, $\Sigma \cap S'(k)$ is not thin in S' and, for every $s \in \Sigma \cap S'$, the fiber X'_s is a dense open X_s (hence it is integral) and $\Gamma \cap X'_s$ is not thin in X'_s.

For every $i = 1, \dots, n$, set $Y_i' = \pi_i^{-1}(X') \subseteq Y_i$ and denote the restriction of π_i to Y_i' by $\pi_i' \colon Y_i' \to X'$. Then π_i' is finite étale of the same degree as π_i (i.e. of degree at least 2) by the definition of X'. Thus, by Lemma 3.1.2 applied to $X' \to S'$, the subsets $\Gamma \cap X'$ and $\Sigma \cap S'$, and the finite étale covers π_i', we have

$$\Gamma \cap X' \not\subseteq \bigcup_{i=1}^{n} \pi_i'(Y_i'(k)).$$

Therefore, $(\Gamma \cap X') \setminus \bigcup_{i=1}^{n} \pi_i'(Y_i'(k)) \neq \emptyset$. Moreover, we obviously have

$$\emptyset \neq (\Gamma \cap X') \setminus \bigcup_{i=1}^{n} \pi_i'(Y_i'(k)) = (\Gamma \cap X') \setminus \bigcup_{i=1}^{n} \pi_i(Y_i(k)).$$

Finally, since $X' \subseteq X \setminus C$, we obtain that

$$\emptyset \neq (\Gamma \cap X') \setminus \bigcup_{i=1}^{n} \pi_i(Y_i(k)) \subseteq \Gamma \setminus \left(C \cup \bigcup_{i=1}^{n} \pi_i(Y_i(k)) \right),$$

and thus $\Gamma \not\subseteq C \cup \bigcup_{i=1}^{n} \pi_i(Y_i(k))$, as required. ☐

The persistence of the Hilbert property for arithmetic schemes under products is an immediate consequence of Theorem D.

Theorem 3.1.4. *Let S be a normal noetherian integral scheme and let $\mathcal{X}$ and $\mathcal{Y}$ be arithmetic schemes over S. If $\mathcal{X}$ and $\mathcal{Y}$ have the Hilbert property over S, then $\mathcal{X} \times_S \mathcal{Y}$ has the Hilbert property over S.*

Proof. Let $\Gamma = (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)}$ and $\Sigma = \mathcal{Y}(S)^{(1)}$. Since the intersection of two dense open subschemes of S with complement of codimension at least 2 also has complement of codimension at least 2, we have $\Gamma = \mathcal{X}(S)^{(1)} \times \mathcal{Y}(S)^{(1)}$. The claim thus follows from Theorem D applied to the morphism $\mathcal{X}_k \times \mathcal{Y}_k \to \mathcal{Y}_k$ and the subsets Γ and Σ. ☐

3.1.2 Birationality and base change

The Hilbert property over a field is a birational invariant of integral varieties (Corollary 2.2.4). We proved this by showing that a non-thin set remains non-thin after intersection with (the set of rational points of) a dense open subscheme. However, if we consider a near-integral point of an arithmetic scheme which is a rational point of some dense open subscheme, it is not necessarily a near-integral point of this subscheme. Birational invariance of the Hilbert property for arithmetic schemes holds only in the following "strict-birational" sense.

Theorem 3.1.5. *Let S be a normal noetherian integral scheme and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper birational morphism of arithmetic schemes over S. Then $\mathcal{X}$ satisfies the Hilbert property over S if and only if $\mathcal{Y}$ satisfies the Hilbert property over S.*

Proof. Define $k = K(S)$ and let $X = \mathcal{X}_k, Y = \mathcal{Y}_k$. Let $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$ be dense opens such that f induces an isomorphism $\mathcal{U} \to \mathcal{V}$. Define $U = \mathcal{U}_k$ and $V = \mathcal{V}_k$. Clearly, by this isomorphism, $\mathcal{X}(S)^{(1)} \cap U(k)$ is thin in U if and only if $f(\mathcal{X}(S)^{(1)} \cap U(k))$ is thin in V.

We claim that $f(\mathcal{X}(S)^{(1)} \cap U(k)) = \mathcal{Y}(S)^{(1)} \cap V(k)$. Since the inclusion from left to right is trivial, let $y \in \mathcal{Y}(S)^{(1)} \cap V(k)$ and let $x \in U(k) = \mathcal{U}(k)$ such that $f(x) = y$. Then, for every $s \in S$ of codimension 1, the map $\operatorname{Spec} \mathcal{O}_{S,s} \to \mathcal{Y}$ defined by y lifts to an $\mathcal{O}_{S,s}$-point of $\mathcal{X}$ extending x by the valuative criterion of properness, so we indeed have $x \in \mathcal{X}(S)^{(1)}$ (cf. Remark 1.2.3).

Thus, $\mathcal{X}(S)^{(1)} \cap U(k)$ is thin in U if and only if $\mathcal{Y}(S)^{(1)} \cap V(k)$ is thin in V. By Proposition 2.2.3, it follows that $\mathcal{X}(S)^{(1)}$ is thin in X if and only if $\mathcal{Y}(S)^{(1)}$ is thin in Y, which concludes the proof. ☐

We shall see later (Theorem 3.2.9) that a similar statement holds for the weak Hilbert property. The proof is slightly more involved, since birationality of strongly-thinness does not hold in the same strong sense as for thinness (see Proposition 2.2.3).

We will prove stability under extensions of the base field for both the Hilbert property and the weak Hilbert property simultaneously. The first result in this direction is due to Serre, who showed that a non-thin subset of rational points of a quasi-projective variety remains non-thin after finite base change.

Proposition 3.1.6 ([74, Proposition 3.2.1])**.** *Let V be a quasi-projective geometrically integral variety over a field k, let L/k be a finite extension, and let $A \subseteq V(L) = V_L(L)$ be a subset. If A is thin in V_L, then $A \cap V(k)$ is thin in V.*

If V is normal and k of characteristic 0, then we have the following strengthening of Proposition 3.1.6 to finitely generated extensions due to Bary-Soroker–Fehm–Petersen. Their result even holds for *strongly* thin sets.

Proposition 3.1.7 ([5, Proposition 3.2])**.** *Let k be a field of characteristic 0, let V be a normal geometrically integral variety over k, let L/k be a finitely generated extension, and let $A \subseteq V(L) = V_L(L)$ be a subset. If A is thin (resp. strongly thin) in V_L, then $A \cap V(k)$ is thin (resp. strongly thin) in V.*

In order to apply Proposition 3.1.7 to the (usual or weak) Hilbert property of arithmetic schemes under a base change $S' \to S$, one needs a condition to ensure that the above intersection contains all near-S-integral points. This is the case if $S' \to S$ is finite type flat.

Theorem 3.1.8. *Let $S' \to S$ be a finite type flat morphism of normal integral noetherian schemes with function field of characteristic 0. Let $\mathcal{X}$ be an arithmetic scheme over S. If $\mathcal{X}$ has the Hilbert property (resp. the weak Hilbert property) over S, then $\mathcal{X}_{S'}$ has the Hilbert property (resp. the weak Hilbert property) over S'.*

Proof. Note first that $\mathcal{X}_{S'}$ is indeed an arithmetic scheme over S' since $\mathcal{X}_{K(S)}$ is geometrically integral by [79, Tag 0G69]. Since finite type flat morphisms are open, the image S_0 of S' in S is a dense open. Clearly, if $\mathcal{X}$ has the Hilbert property (resp. the weak Hilbert property) over S, then $\mathcal{X}_{S_0}$ has the Hilbert property (resp. the weak Hilbert property) over S_0, since the set of near-integral points can only become larger after this base change. By [39, Corollaire 6.1.4], pullback along $S' \to S_0$ preserves codimension of closed subsets, so the inclusion $\mathcal{X}(S_0)^{(1)} \subseteq \mathcal{X}(S')^{(1)}$ holds. Let k and L denote the function fields of S_0 and S', respectively, and define $T = \mathcal{X}(S')^{(1)}$. If $\mathcal{X}$ satisfies the Hilbert property (resp. weak Hilbert property) over S, and thus over S_0 (as noted before), then $\mathcal{X}(S_0)^{(1)}$ is not thin (resp. not strongly thin) in $\mathcal{X}_k$. Since $\mathcal{X}(S_0)^{(1)} \subseteq T \cap \mathcal{X}_k(k)$, it follows that $T \cap \mathcal{X}_k(k)$ is not thin (resp. not strongly thin) in $\mathcal{X}_k$, so T is not thin (resp. not strongly thin) in $\mathcal{X}_L$ by Proposition 3.1.7, i.e., $\mathcal{X}_{S'}$ has the Hilbert property (resp. weak Hilbert property) over S'. □

3.1.3 Algebraic simple connectedness

If $\pi\colon Y \to X$ is a finite étale morphism of smooth projective integral varieties over a number field k, then by the Chevalley–Weil Theorem (see e.g. [25, Theorem 3.8]), there exists a number field k'/k such that $X(k) \subseteq \pi(Y(k'))$. If, moreover, the degree of π is at least 2, Corvaja–Zannier [26, Proposition 1.7] used the Chevalley–Weil Theorem to show that there exists a finite family $(\pi_i\colon Y_i \to X)_{i=1}^n$ of covers of degree at least 2 such that $X(k) \subseteq \bigcup_{i=1}^n \pi_i(Y_i(k))$. Since the existence of such covers obstructs the Hilbert property of X over k, this implies that, if X has the Hilbert property over k, then it admits no non-trivial finite étale cover over k. Moreover, as the Hilbert property is preserved under finite base change, it follows that X has no non-trivial finite étale cover over the algebraic closure $\overline{k}$, i.e., X is algebraically simply connected (see [26, Theorem 1.4]). In particular, a smooth projective variety over a number field satisfies the Hilbert property if and only if it is algebraically simply connected and satisfies the weak Hilbert property. Using Corvaja–Zannier's proposition and an analogue of the Chevalley–Weil Theorem for near-integral points, we extend this result to arithmetic schemes (see Theorem B).

We start by stating the aforementioned proposition due to Corvaja–Zannier. It is stated in [26] for number fields, but the proof works for all fields, provided that the extension k'/k is separable.

Proposition 3.1.9 ([26, Proposition 1.7])**.** *Let k'/k be a finite separable field extension, let $\pi\colon Y \to X$ be a cover of quasi-projective integral k-varieties of degree at least 2, and let $T \subseteq Y(k')$ be a subset such that $\pi(T) \subseteq X(k)$. Then there exists a finite family $(\pi_i\colon Y_i \to X)_{i=1}^n$ of covers of quasi-projective integral k-varieties of degree at least 2 such that $\pi(T) \subseteq \bigcup_{i=1}^n \pi_i(Y_i(k))$.*

In the case of an étale cover $Y \to X$, the existence of a field extension as in the assumptions of the proposition is deduced in Corvaja–Zannier's proof from the Chevalley–Weil Theorem. In order to generalize their result to arithmetic schemes, we use a version of this for near-integral points. The following theorem and its corollary are published in [62]; the proof follows the proof of the Chevalley–Weil Theorem for finitely generated fields of characteristic 0 given in [25, Theorem 3.8] closely.

Theorem 3.1.10. *Let S be a regular integral arithmetic scheme and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite étale morphism of integral finite type schemes over S. Then there exists a finite étale morphism $S' \to S$ of regular integral arithmetic schemes such that the inclusion $\mathcal{Y}(S)^{(1)} \subseteq f(\mathcal{X}(S')^{(1)})$ holds.*

Proof. Let $y \in \mathcal{Y}(S)^{(1)}$ be a near-integral point, so there exists a dense open subscheme $U_y \subseteq S$ with complement of codimension at least 2 such that y is induced by a morphism $U_y \to \mathcal{Y}$. Let V_y be a connected component of $U_y \times_{\mathcal{Y}} \mathcal{X} \to U_y$ and note that $V_y \to U_y$ is finite étale surjective and of degree at most $\deg(f)$. Let $\overline{V_y} \to S$ be the normalization of S in the function field of V_y.

We claim that $\overline{V_y} \setminus V_y$ is of codimension at least 2 in $\overline{V_y}$. Indeed, let $D \subseteq \overline{V_y} \setminus V_y$ be an irreducible component. Note that D maps finitely to $S \setminus U_y$, so we have

$$\dim D = \dim(S \setminus U_y) \leq \dim S - 2.$$

Since $\dim S = \dim \overline{V_y}$ by the finiteness of $\overline{V_y} \to S$, this proves the claim.

Since $\overline{V_y} \setminus V_y$ is of codimension at least 2 in $\overline{V_y}$, by purity of the branch locus [38, Théorème X.3.1], it follows that $\overline{V_y} \to S$ is étale.

Since there are only finitely many isomorphism classes of finite étale covers of S with bounded degree [43, Theorem 2.9], the set of isomorphism classes of the morphisms $\overline{V_y} \to S$ with $y \in \mathcal{Y}(S)^{(1)}$ is finite. Thus, there exists a finite étale morphism $S' \to S$ of integral noetherian schemes such that, for every $y \in \mathcal{Y}(S)^{(1)}$, the morphism $S' \to S$ factors over $\overline{V_y}$, i.e., we have $\mathcal{Y}(S)^{(1)} \subseteq f(\mathcal{X}(S')^{(1)})$. Note that the composed morphism

$S' \to S \to \operatorname{Spec}\mathbb{Z}$ is dominant of finite type and that S' is regular since $S' \to S$ is finite étale and S is regular. Thus, S' is a regular integral arithmetic scheme, as required. ☐

Occasionally, when applying Theorem 3.1.10, we are only interested in the function field of S', so we give the following corollary.

Corollary 3.1.11. *Let S be a regular integral arithmetic scheme, let $\mathcal{Y}$ be an integral finite type scheme over S, and let $f\colon X \to \mathcal{Y}_{K(S)}$ be a finite étale morphism of integral $K(S)$-schemes. Then there exists a finite field extension L/k such that $\mathcal{Y}(S)^{(1)} \subseteq f(X(L))$.*

Proof. By spreading out schemes and finite étale morphisms (see [72, Theorem 3.2.1]), there exists a dense open subscheme $S' \subseteq S$, an integral scheme $\mathcal{X}$ of finite type over S' satisfying $\mathcal{X}_{K(S')} = X$, and a finite étale morphism $\mathcal{X} \to \mathcal{Y}_{S'}$ extending f. By Theorem 3.1.10, there exists a finite étale (in particular, finite surjective) morphism $S'' \to S'$ of integral arithmetic schemes such that $\mathcal{Y}_{S'}(S')^{(1)} \subseteq f(\mathcal{X}(S'')^{(1)})$. Since $\mathcal{Y}(S)^{(1)} \subseteq \mathcal{Y}_{S'}(S')^{(1)}$ (as $S' \subseteq S$ is a dense open subscheme), the claim follows by taking $L = K(S'')$. ☐

Just like in the proof of Corvaja–Zannier, the algebraic simple connectedness of an arithmetic scheme with the Hilbert property over a suitable base scheme is now an easy consequence of Proposition 3.1.9 and (the above version of) the Chevalley–Weil Theorem.

Theorem 3.1.12 (Theorem B)**.** *Let S be a regular integral arithmetic scheme and let $\mathcal{X}$ be a normal integral arithmetic scheme over S. If $\mathcal{X}$ has the Hilbert property over S, then $\mathcal{X}$ does not have any non-trivial finite étale covers over S, i.e., any finite étale morphism $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ a normal integral arithmetic scheme over S whose generic fiber $\mathcal{Y}_{K(S)}$ is geometrically connected is an isomorphism.*

Proof. Let $\pi\colon \mathcal{Y} \to \mathcal{X}$ be a finite étale morphism with $\mathcal{Y}$ a normal integral arithmetic scheme over S and $\mathcal{Y}_{K(S)}$ geometrically connected. Assume that π is not an isomorphism, so we have $\deg \pi \geq 2$. By Theorem 3.1.10, there exists a finite étale morphism $S' \to S$ of regular integral arithmetic schemes such that $\mathcal{X}(S)^{(1)} \subseteq \pi(\mathcal{Y}(S')^{(1)})$. Let $k = K(S)$ and $k' = K(S')$, define $X = \mathcal{X}_k$ and $Y = \mathcal{Y}_k$, and let $\pi_k\colon Y \to X$ denote the base change of π. Note that π_k is of degree at least 2 and $\mathcal{X}(S)^{(1)} \subseteq \pi_k(Y(k'))$. Let $X' \subseteq X$ be a dense open affine subscheme and let $\pi'\colon Y' = \pi_k^{-1}(X') \to X'$ denote the restriction of π. Then π' is a cover of quasi-projective integral k-varieties of degree at least 2 and satisfies $\mathcal{X}(S)^{(1)} \cap X'(k) \subseteq \pi'(Y'(k'))$. Setting

$$T := (\pi')^{-1}\left(\mathcal{X}(S)^{(1)} \cap X'(k)\right) \cap Y'(k') \subseteq Y'(k'),$$

we thus have

$$\pi'(T) = \mathcal{X}(S)^{(1)} \cap X'(k) \subseteq X'(k).$$

By Proposition 3.1.9, there exists a finite family $(\pi_i' \colon Y_i \to X')_{i=1}^n$ of covers of integral k-varieties of degree at least 2 such that $\mathcal{X}(S)^{(1)} \cap X'(k) \subseteq \bigcup_{i=1}^n \pi_i'(Y_i(k))$. In particular, letting $\pi_i \colon Y_i \to X$ denote the composition of π_i' and the inclusion of X' in X, each π_i is a dominant generically finite separable morphism of degree at least 2 and

$$\mathcal{X}(S)^{(1)} \setminus \bigcup_{i=1}^n \pi_i(Y_i(k)) \subseteq X(k) \setminus X'(k)$$

is not dense in X, i.e., $\mathcal{X}$ does not have the Hilbert property over S. This concludes the proof. □

3.2 Strongly thin sets

When trying to adapt the strategy of Bary-Soroker–Fehm–Petersen's fibration theorem for the Hilbert property toward the weak Hilbert property, one runs into the issue that a ramified cover $Z \to X \times Y$ of a product of varieties with the weak Hilbert property might induce *étale* covers $Z_x \to \{x\} \times Y \cong Y$ for most $x \in X(k)$, in which case the weak Hilbert property of Y can not be applied to those covers. To handle such "vertically ramified covers" of a product $X \times Y$, Corvaja–Demeio–Javanpeykar–Lombardo–Zannier [25] use "arithmetic refinements", where a vertically ramified cover $Z_i \to X \times Y$ is replaced by finitely many covers $W_{ij} \to Z_i$ that can be handled more easily. The construction of such covers relies on the base field being finitely generated, the explicit structure of $X \times Y$ as a product (instead of a more general fibration), and the properness of X and Y. Thus, the resulting product theorem [25, Theorem 1.9] is not a consequence of a more general fibration theorem, and it requires the base field to be finitely generated and X and Y to be proper.

In this section, we generalize the product theorem of Corvaja–Demeio–Javanpeykar–Lombardo–Zannier to arithmetic schemes which are not necessarily proper. The key technical improvement here is the handling of vertically ramified covers without the properness assumption on the varieties (see Lemma 3.2.1). We also show the other persistence properties mentioned in the introduction. Up to editorial changes, the results of this section have been published in [62].

3.2.1 Vertically ramified covers and arithmetic refinements

If X and Y are normal integral varieties over a field k of characteristic 0 and their product is an integral variety, we say that a cover $Z \to X \times Y$ is *vertically ramified* over X if there exists a dense open $U \subseteq X$ such that $Z \to X \times Y$ is étale over $U \times Y$. If X and Y are smooth proper, it is shown in [25, Lemma 2.17] that a cover $Z \to X \times Y$ vertically ramified over X splits as a product of a ramified cover of X and an étale cover of Y, up to a finite étale cover.

The following lemma provides a generalization of this result that eliminates the properness assumptions and is the key technical result of this section.

Lemma 3.2.1. *Let k be a field of characteristic 0, let X, Y, Z be normal integral varieties over k with $Z(k)$ dense, and let $\pi\colon Z \to X \times Y$ a ramified cover. Assume that there exists a dense open subscheme $U \subseteq X$ such that π is unramified over $U \times Y$ (i.e., π is vertically ramified over X). Let $Y \to \overline{Y}$ be an open immersion into a normal integral proper variety $\overline{Y}$ over k and let $\overline{Z} \to X \times \overline{Y}$ be the normalization of $X \times \overline{Y}$ in Z. Assume that the fibers of $\overline{Z} \to X$ are geometrically connected. Then there exists a commutative diagram*

$$\begin{tikzcd}
 & \overline{Z'} \arrow[dl] \arrow[d] & Z' \cong X' \times Y' \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\
X' \arrow[dr] & X \times \overline{Y'} \arrow[d] & X \times Y' \arrow[l] \arrow[r] \arrow[d] & X \times Y \arrow[d] \\
 & X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

such that

(1) $X', Y', Z', \overline{Y'}, \overline{Z'}$ are normal integral varieties over k;

(2) $X' \to X$ is a ramified cover;

(3) $Y' \to Y$ and $Z' \to Z$ are finite étale;

(4) Z' is a connected component of the fiber product $Z \times_Y Y'$;

(5) $\overline{Y'}$ is the normalization of $\overline{Y}$ in Y';

(6) $\overline{Z'}$ is the normalization of $X \times \overline{Y'}$ in Z';

(7) $\overline{Z'} \to X' \to X$ is the Stein factorization of $\overline{Z'} \to X$.

Proof. Let $\overline{k}$ be the algebraic closure of k. Since $Z(k)$ is dense in Z and thus $X(k)$ and $Y(k)$ are dense in X and Y, respectively, it follows from [79, Tag 0G69] that $X_{\overline{k}}$ and $Y_{\overline{k}}$ are irreducible. Since $\overline{k}$ is algebraically closed of characteristic 0, the étale fundamental group of $Y_{\overline{k}}$ is topologically finitely generated (see [73, Théorème 2.3.1]), so there are only finitely many $Y_{\overline{k}}$-isomorphism classes of finite étale morphisms $V \to Y_{\overline{k}}$ of bounded degree. Since $\overline{Z}$ is normal and k is of characteristic 0, we may shrink U if necessary to assume that $(\overline{Z})_x$ is normal for every $x \in U(k)$. In particular, since $(\overline{Z})_x$ is geometrically connected, this implies that Z_x is normal geometrically connected (and thus geometrically integral) for every $x \in U(k)$. Let $Z^1 \to Y_{\overline{k}}, \dots, Z^n \to Y_{\overline{k}}$ be representatives of the classes of finite étale morphisms of degree $\deg(Z \to X \times Y)$. For $x \in X(k)$, let $\overline{x}\colon \operatorname{Spec} \overline{k} \to X$ be the induced geometric point. Define $\Sigma_i \subseteq U(k)$ to be the subset of $x \in U(k)$ such that the finite étale morphism

$$Z_{\overline{x}} = (Z_x)_{\overline{k}} \to (\{x\} \times Y)_{\overline{k}} \cong Y_{\overline{k}}$$

is $Y_{\overline{k}}$-isomorphic to Z^i. Note that $U(k) = \Sigma_1 \cup \ldots \cup \Sigma_n$.

Since $U(k)$ is dense in X, there exists an index i such that Σ_i is dense in X; we define $\Sigma_i =: \Sigma$. Fix some point $x' \in \Sigma$ and define $Y' = Z_{x'}$. Then Y' is a normal geometrically connected k-variety, $Y' \to Y$ is finite étale, and Σ is a dense subset of $U(k)$ such that, for every $x \in \Sigma$, there exists a $Y_{\overline{k}}$-isomorphism $Y'_{\overline{k}} \to Z_{\overline{x}}$.

Let $Z'' := Z \times_Y Y' = Z \times_{X \times Y} (X \times Y')$. Note that Z'' is normal, as the projection morphism to Z is finite étale and Z is normal. Moreover, since X is normal geometrically connected and Y' is normal connected, $X \times Y'$ is normal connected (and thus integral). Therefore, every connected component of Z'' is finite surjective over $X \times Y'$ and finite étale over Z. Let $Z''_1, \dots, Z''_t$ be the connected components of Z'' and $I = \{1, \dots, t\}$. For every $x \in \Sigma$, we have $Z''_{\overline{x}} = \coprod_{j \in I} Z''_{j,\overline{x}}$, and by choice of Σ, there exists an isomorphism $\{\overline{x}\} \times Y' \to Z_{\overline{x}}$ over $\{\overline{x}\} \times Y$, which induces a section of $Z''_{\overline{x}} \to \{\overline{x}\} \times Y'$ by the universal property of the fiber product. Since Y' is geometrically irreducible, there exists $j(x) \in I$ such that this section restricts to a section of $Z''_{j(x),\overline{x}} \to \{\overline{x}\} \times Y'$. By the density of Σ in $U(k)$, there exists $j_0 \in I$ such that the fiber of $j\colon \Sigma \to I$ over j_0 is dense in $U(k)$. Define $\Sigma' = j^{-1}(j_0)$ and $Z' := Z''_{j_0}$. Then Σ' is a subset of Σ such that Σ' is dense in $U(k)$ and, for every $x \in \Sigma'$, the morphism $Z'_{\overline{x}} \to \{\overline{x}\} \times Y'$ is finite étale (since $Z \to X \times Y$ is étale over $U \times Y$) and has a section.

Let $\overline{Y'} \to \overline{Y}$ be the normalization of $\overline{Y}$ in Y'. Let $\overline{Z'} \to X \times \overline{Y'}$ be the normalization of $X \times \overline{Y'}$ in Z' and note that this is étale over $U \times \overline{Y'}$. Finally, let $\overline{Z'} \to X' \to X$ be the Stein factorization of the composition $\overline{Z'} \to X \times \overline{Y'} \to X$ and note that X' is normal connected (thus integral). We have the following commutative diagram.

$$\begin{tikzcd}
 & \overline{Z'} \arrow[dl] \arrow[d] & Z' \arrow[l] \arrow[r, "\text{finite étale}"] \arrow[d] & Z \arrow[d] \\
X' \arrow[dr] & X \times \overline{Y'} \arrow[d] & X \times Y' \arrow[l] \arrow[r, "\text{finite étale}"] \arrow[d] & X \times Y \arrow[d] \\
 & X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd} \tag{3.2.2}$$

Let $Z' \to X' \times Y' = X' \times_X (X \times Y')$ be the natural morphism. We claim that $Z' \to X' \times Y'$ is an isomorphism and start by showing finiteness and surjectivity. By the definition of Stein factorization, the morphism $X' \to X$ is finite, and it is surjective since $Z' \to X$ is surjective and factorizes through it. Since both $X' \times Y'$ and Z' are finite over $X \times Y'$, the morphism $Z' \to X' \times Y'$ is also finite.

Moreover, since both $Z' \to X \times Y'$ and $X' \times Y' \to X \times Y'$ are finite surjective, we have

$$\dim(Z') = \dim(X \times Y') = \dim(X' \times Y'),$$

and the image of Z' in $X' \times Y'$ is of dimension $\dim(X' \times Y')$. Since X' is normal connected and Y' is normal geometrically connected, $X' \times Y'$ is normal and connected, so $Z' \to X' \times Y'$ is dominant and thus (by its finiteness) surjective, as claimed.

Next we prove that $Z' \to X' \times Y'$ is birational. Let $\overline{f}$ denote the morphism $\overline{Z'} \to X'$ and $f = \overline{f}|_{Z'} \colon Z' \to X'$. Let $V \subseteq X'$ be a dense open subscheme such that the fibers of $\overline{f}^{-1}(V) \to V$ are normal. Since they are geometrically connected by the defining properties of Stein factorization, they are geometrically integral. By generic flatness and since a flat morphism of finite presentation is open, we may shrink V further if necessary to assume that $f^{-1}(V) \to V$ is surjective. Thus, the fibers of $f^{-1}(V) \to V$ are nonempty and open in the fibers in $\overline{f}^{-1}(V) \to V$. Since the latter are normal and geometrically integral, the fibers of $f^{-1}(V) \to V$ are also normal and geometrically integral. The complement of V in X' maps to a proper closed subscheme of X. Let U' denote the intersection of the complement of this closed subscheme with U. Then U' is a dense open of X, contained in U, and since $X'_{U'} \subseteq V$, the fibers of $Z'_{U'} \to X'_{U'}$ are normal and geometrically connected. By generic étaleness and shrinking U' further if necessary, we may and do assume that $X'_{U'} \to U'$ is étale.

Let $x \in U'(k) \cap \Sigma'$ be a point. Then $X'_{\overline{x}}$ is finite étale over $\{\overline{x}\} = \operatorname{Spec} \overline{k}$, and thus it is the disjoint union $\coprod_{\overline{x}' \in X'_{\overline{x}}} \{\overline{x}'\}$, where each $\{\overline{x}'\}$ is a copy of $\operatorname{Spec} \overline{k}$. Let g and h denote the morphisms $Z' \to X' \times Y'$ and $Z' \to X \times Y'$, respectively, and let $g_{\overline{x}} \colon Z'_{\overline{x}} \to X'_{\overline{x}} \times Y'$ and $h_{\overline{x}} \colon Z'_{\overline{x}} \to \{\overline{x}\} \times Y'$ denote the respective base changes. We then have the following commutative diagram.

$$\begin{tikzcd}
\coprod_{\overline{x}'\in X'_{\overline{x}}} g_{\overline{x}}^{-1}(\{\overline{x}'\}\times Y') \arrow[r, equal] & Z'_{\overline{x}} \arrow[d, "g_{\overline{x}}"'] \arrow[rd, "h_{\overline{x}}"] & \\
\coprod_{\overline{x}'\in X'_{\overline{x}}}(\{\overline{x}'\}\times Y') \arrow[r, equal] & X'_{\overline{x}}\times Y' \arrow[r] \arrow[d] & \{\overline{x}\}\times Y' \arrow[d] \\
\coprod_{\overline{x}'\in X'_{\overline{x}}}\{\overline{x}'\} \arrow[r, equal] & X'_{\overline{x}} \arrow[r] & \{\overline{x}\}
\end{tikzcd}$$

Since $Z'\to X'$ has geometrically connected fibers over U', the $g_{\overline{x}}^{-1}(\{\overline{x}'\}\times Y')$ are connected, so they are the connected components of $Z'_{\overline{x}}$. Moreover, by our choice of Σ', the morphism $h_{\overline{x}}$ is finite étale and has a section. Thus, there exists a connected component W of $Z'_{\overline{x}}$ such that $h_{\overline{x}}$ restricts to an isomorphism $h_{\overline{x}}|_W\colon W\to\{\overline{x}\}\times Y'$. Let $\overline{x}'\in X'_{\overline{x}}$ such that $g_{\overline{x}}^{-1}(\{\overline{x}'\}\times Y')=W$, i.e., $g_{\overline{x}}^{-1}(\{\overline{x}'\}\times Y')\to\{\overline{x}\}\times Y'$ is an isomorphism. Choose a point $\overline{p}\in\{\overline{x}'\}\times Y'(\overline{k})$. Since both $\{\overline{x}'\}\times Y'\to\{\overline{x}\}\times Y'$ and $h_{\overline{x}}|_W$ are isomorphisms, there exists exactly one point in $W(\overline{k})$ over $\overline{p}$, i.e., we have $|g_{\overline{x}}^{-1}(\overline{p})(\overline{k})|=1$. Thus, $\overline{p}$ is a point of $(X'\times Y')(\overline{k})$ over U' such that $g^{-1}(\overline{p})(\overline{k})$ has cardinality 1. Now, since $g_{U'}\colon Z'_{U'}\to X'_{U'}\times Y'$ is finite étale, it follows that $g_{U'}$ is an isomorphism and we have shown birationality of g. As a finite surjective birational morphism of normal connected varieties, g is an isomorphism, as claimed.

We have now constructed the desired diagram. All of the claimed properties except for (2) are true by construction or proved in the discussion above, so it only remains to verify that $X'\to X$ is a ramified cover. It is finite (as the finite part of the Stein factorization of $\overline{Z'}\to X$) and surjective (by the surjectivity of $Z'\to X$). Thus, it remains to show that $X'\to X$ is ramified. For a contradiction, assume that it is unramified (and therefore étale). Then $Z'\cong X'\times Y'\to X\times Y'\to X\times Y$ and thus $Z'\to Z\to X\times Y$ would be étale. Since $Z'\to Z$ is étale surjective, π would be étale by the cancellation property of étale morphisms, contradiction. $\square$

If S is a normal noetherian integral scheme with function field k of characteristic 0 and $\mathcal{X}$ is an arithmetic scheme over S, we "test" the weak Hilbert property of $\mathcal{X}$ over S by considering finite families of ramified covers $(\pi_i\colon Z_i\to\mathcal{X}_k)_{i=1}^n$. When doing this, it might be beneficial to replace a given cover π_i by taking finitely many covers $(f_{ij}\colon W_{ij}\to Z_i)_{j\in J_i}$ such that

$$\mathcal{X}(S)^{(1)}\setminus\bigcup_{i=1}^n\pi_i(Z_i(k))=\mathcal{X}(S)^{(1)}\setminus\bigcup_{i=1}^n\bigcup_{j\in J_i}(\pi_i\circ f_{ij})(W_{ij}(k)).$$

This is achieved, for example, if $Z_i(k) = \bigcup_{j\in J_i} f_{ij}(W_{ij}(k))$, in which case the collection $(f_{ij}\colon W_{ij} \to Z_i)_{j\in J_i}$ is called an *arithmetic refinement* of the variety Z_i in [25, §3.2]. Arithmetic refinements of smooth proper varieties are constructed in *loc. cit.* as an application of the Chevalley–Weil Theorem. Without the properness assumption, the Chevalley–Weil Theorem does not allow us to lift all k-rational points, but its near-integral analogue (see Theorem 3.1.10 and Corollary 3.1.11) lets us lift near-integral points. This leads us to the following definition.

Definition 3.2.3. Let S be a normal noetherian integral scheme with function field k of characteristic 0 and let $\mathcal{Z}$ be an arithmetic scheme over S. An *arithmetic refinement* of $\mathcal{Z}$ is a finite family $(f_j\colon W_j \to \mathcal{Z}_k)_{j\in J}$ of covers with W_j normal and integral over k such that $\mathcal{Z}(S)^{(1)} \subseteq \bigcup_{j\in J} \psi_j(W_j(k))$.

Using Lemma 3.2.1 and the near-integral analogue of Chevalley–Weil Theorem, we now prove a generalization of [25, Theorem 3.9] that allows us to replace a vertically ramified cover by a suitable arithmetic refinement (of an arithmetic model) when testing for the weak Hilbert property of a product of arithmetic schemes.

Theorem 3.2.4. *Let S be a regular integral arithmetic scheme with function field k, let $\mathcal{X}$ and $\mathcal{Y}$ be quasi-projective arithmetic schemes over S, and define $X := \mathcal{X}_k$ and $Y := \mathcal{Y}_k$. Let $\pi\colon Z \to X \times Y$ be a cover which is vertically ramified over X and assume that $Z(k)$ is dense in Z, so that in particular $X(k)$ and $Y(k)$ are dense in X and Y, respectively. Let $Y \to \overline{Y}$ be an open immersion into a normal projective integral variety over k and let $\overline{Z} \to X \times \overline{Y}$ be the normalization of $X \times \overline{Y}$ in Z. Then there exists a finite family of covers $(f_j\colon W_j \to Z)_{j\in J}$ such that*

$$(\mathcal{X} \times_S \mathcal{Y})(S)^{(1)} \setminus \pi(Z(k)) = (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)} \setminus \bigcup_{j\in J} (\pi \circ f_j)(W_j(k))$$

and, for every $j \in J$, the Stein factorization of $\overline{W_j} \to X \times \overline{Y} \to X$ is ramified over X, where $\overline{W_j} \to \overline{Z}$ is the normalization of $\overline{Z}$ in W_j.

Proof. Let $\overline{Z} \to T \to X$ be the Stein factorization of the composition $\overline{Z} \to X \times \overline{Y} \to X$ and note that T is a normal geometrically integral k-variety. If $T \to X$ is ramified, the family consisting of the identity map $f = \mathrm{id}\colon W = Z \to Z$ proves the claim, so we may assume that $T \to X$ is étale.

The morphism $\overline{Z} \to X \times \overline{Y}$ factors over a morphism $\overline{Z} \to T \times \overline{Y}$. Since $\overline{Z}$ is the normalization of $X \times \overline{Y}$ in Z, we have that $\overline{Z}$ is also the normalization of $T \times \overline{Y}$ in Z. Moreover, $\overline{Z} \to T \times \overline{Y}$ is proper and quasi-finite by the finiteness of both $\overline{Z} \to X \times \overline{Y}$ and $T \times \overline{Y} \to X \times \overline{Y}$, so it is finite surjective. Let $\tilde{Z} = \overline{Z} \times_{T \times \overline{Y}} (T \times Y)$. Note that

$\tilde{Z} = \overline{Z} \times_{X \times \overline{Y}} (X \times Y)$, so there is a natural morphism $Z \to \tilde{Z}$, which is proper (and in particular closed) by the properness of $Z \to X \times Y$ [79, Tag 01W6]. Moreover, since $Y \to \overline{Y}$ (and thus $\tilde{Z} \to \overline{Z}$) and $Z \to \overline{Z}$ are open immersions, it follows that $Z \to \tilde{Z}$ is an open immersion. By the connectedness of $\tilde{Z}$, we conclude that $Z \to \tilde{Z}$ is a surjective open immersion and therefore an isomorphism. In particular, $Z \to T \times Y$ is finite surjective. As $T \times Y \to X \times Y$ is étale and $Z \to X \times Y$ is ramified, $Z \to T \times Y$ is ramified. Moreover, by the étaleness of Z over $U \times Y$ and the cancellation law for étale morphisms, Z is étale over $T_U \times Y$, i.e., $Z \to T \times Y$ is a cover vertically ramified over T.

Since the composition $\overline{Z} \to T \times \overline{Y} \to T$ has geometrically connected fibers by the defining properties of Stein factorization and $Z(k)$ is dense, Lemma 3.2.1 gives us a diagram

$$\begin{tikzcd}
& \overline{Z'} \arrow[dl] \arrow[d] & Z' \cong T' \times Y' \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\
T' \arrow[dr] & T \times \overline{Y'} \arrow[d] & T \times Y' \arrow[l] \arrow[r] \arrow[d] & T \times Y \arrow[d] \\
& T \arrow[r, equal] & T \arrow[r, equal] & T
\end{tikzcd} \tag{3.2.5}$$

of varieties over k such that $T' \to T$ is a ramified cover, $Y' \to Y$ and $Z' \to Z$ are finite étale, $\overline{Y'}$ is the normalization of $\overline{Y}$ in Y', and $\overline{Z'}$ is the normalization of $T \times \overline{Y'}$ in Z'. Let ψ denote the morphism $Z' \to Z$.

By standard spreading out schemes and finite étale (resp. finite surjective) morphisms (see [72, Theorem 3.2.1]), there exist a regular dense open subscheme $S' \subseteq S$ and an integral finite type separated S'-scheme $\mathcal{Z}$ whose generic fiber over S' is Z, and such that ψ and π extend to a finite étale (hence surjective, since $\mathcal{Z}$ is connected) morphism $\mathcal{Z}' \to \mathcal{Z}$ and a finite surjective morphism $\mathcal{Z} \to (\mathcal{X} \times \mathcal{Y})_{S'}$, respectively. If $x \in (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)}$ is a near-S-integral point that lifts along π to a k-rational point of Z, then, by the valuative criterion of properness, we have $x \in \pi(\mathcal{Z}(S')^{(1)})$.

By Corollary 3.1.11, there exists a finite field extension L/k such that we have $\mathcal{Z}(S')^{(1)} \subseteq \psi(Z'(L))$. In the next steps, we follow the arguments of [25, Theorem 3.9] to construct an arithmetic refinement of $\mathcal{Z}$. Consider the induced morphism of Weil restriction of scalars (cf. [9])

$$R_{L/k}(\psi_L) \colon R_{L/k}(Z'_L) \to R_{L/k}(Z_L),$$

which is finite étale (as ψ is). Let $\Delta \colon Z \to R_{L/k}(Z_L)$ be the diagonal morphism and define $W = Z \times_{\Delta, R_{L/k}(Z_L)} R_{L/k}(Z'_L)$, so we have the following cartesian diagram.

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & Z \arrow[d, "\Delta"] \\
R_{L/k}(Z'_L) \arrow[r, "R_{L/k}(\psi_L)"] & R_{L/k}(Z_L)
\end{tikzcd}$$

Since ψ (and thus $R_{L/k}(\psi_L)$) is finite étale, W is finite étale over Z, so it is normal. Let $W_j, j \in J$, be the connected components of W that satisfy $W_j(k) \neq \emptyset$ and note that these are normal, geometrically connected (thus geometrically integral), and finite étale over Z.

The map of L-points $Z'(L) \to Z(L)$ induced by ψ is identified by $R_{L/k}$ with $R_{L/k}(Z'_L)(k) \to R_{L/k}(Z_L)(k)$ and the map $Z(k) \to R_{L/k}(Z_L)(k) \cong Z(L)$ induced by Δ is the inclusion map. Thus, for every point $x \in Z(k)$ that lifts along ψ to $Z'(L)$, there exists a point in $R_{L/k}(Z'_L)(k)$ lying over $\Delta(x)$, so x lifts to $W(k)$. In particular, $\mathcal{Z}(S)^{(1)}$ is contained in the image of $W(k)$ in $Z(k)$, i.e., the family $(f_j \colon W_j \to Z)_{j \in J}$ is an arithmetic refinement of $\mathcal{Z}$ and we have

$$\begin{aligned}(\mathcal{X} \times_S \mathcal{Y})(S)^{(1)} \setminus \pi(Z(k)) &= (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)} \setminus \pi(\mathcal{Z}(S')^{(1)}) \\ &= (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)} \setminus \bigcup_{j \in J} (\pi \circ f_j)(W_j(k)).\end{aligned}$$

To finish the proof, it remains to show that the Stein factorization of the composed morphism $\overline{W_j} \to X \times \overline{Y} \to X$ is ramified. Consider the composed morphism $W \to Z \to T$ and, for every $j \in J$, let T_j denote the normalization of T in W_j. Note that T_j is normal and integral by [79, Tag 035L] (in particular, $T_j \to T$ is a cover) and, since $T \to X$ is finite, T_j is also the normalization of X in W_j.

The base change of $W = Z \times_{\Delta, R_{L/k}(Z_L)} R_{L/k}(Z'_L)$ to the algebraic closure $\overline{k}$ is given by $W_{\overline{k}} \cong Z'_{\overline{k}} \times_{Z_{\overline{k}}} \ldots \times_{Z_{\overline{k}}} Z'_{\overline{k}}$, so we have a finite étale morphism $W_{\overline{k}} \to Z'_{\overline{k}}$ over $Z_{\overline{k}}$, and similarly a finite étale morphism $W_{j,\overline{k}} \to Z'_{\overline{k}}$ over $Z_{\overline{k}}$ for every $j \in J$.

Since normalization commutes with flat base change (in particular, base change from k to $\overline{k}$), $T_{j,\overline{k}}$ is the normalization of $T_{\overline{k}}$ in $W_{j,\overline{k}}$. By the universal mapping property of normalization [79, Tag 035I], there is a finite surjective morphism $T_{j,\overline{k}} \to T'_{\overline{k}}$ over $T_{\overline{k}}$. Étaleness can be checked after base change to $\overline{k}$, so $T'_{\overline{k}} \to T_{\overline{k}}$ is ramified (as $T' \to T$ is). If $T_{j,\overline{k}} \to T'_{\overline{k}}$ is ramified, then $T_{j,\overline{k}} \to T_{\overline{k}}$ is ramified by [79, Tag 02GG], and if $T_{j,\overline{k}} \to T'_{\overline{k}}$ is étale, then $T_{j,\overline{k}} \to T_{\overline{k}}$ is ramified by [79, Tag 02K6]. We conclude that $T_{j,\overline{k}} \to T_{\overline{k}}$ is ramified. This implies that $T_j \to T$ and thus the composition $T_j \to T \to X$ is ramified.

Let $\overline{W_j} \to \overline{T_j} \to X$ be the Stein factorization of $\overline{W_j} \to X$, i.e., $\overline{T_j}$ is the normalization of X in $\overline{W_j}$. Since $W_j \subseteq \overline{W_j}$ is an open subscheme, we have an open immersion

$T_j \hookrightarrow \overline{T_j}$ over X. Thus, as $T_j \to X$ is ramified, we conclude that $\overline{T_j} \to X$ is ramified, as required. □

3.2.2 Product theorem

We are now ready to prove that the product of two quasi-projective arithmetic schemes with the weak Hilbert property over a regular integral arithmetic scheme satisfies the weak Hilbert property (Theorem E). This generalizes the special case of smooth proper varieties due to Corvaja–Demeio–Javanpeykar–Lombardo–Zannier [25, Theorem 1.9]. More generally, we consider subsets of near-integral points of such products that are not necessarily given as a product of two sets of near-integral points, which is crucial to proving the results of the next chapter; the persistence of the weak Hilbert property under products is then an immediate consequence of this. To prove Theorem E, we can now follow the proof of [25, Theorem 1.9] closely, as the key improvements necessary to generalize *loc. cit.* are already contained in Lemma 3.2.1 and Theorem 3.2.4.

Theorem 3.2.6 (Theorem E)**.** *Let S be a regular integral arithmetic scheme with function field k, let $\mathcal{X}, \mathcal{Y}$ be quasi-projective arithmetic schemes over S, and define $X = \mathcal{X}_k, Y = \mathcal{Y}_k$. Furthermore, let $p\colon X \times Y \to X$ denote the projection morphism, let $\Sigma \subseteq (\mathcal{X} \times_S \mathcal{Y})(S)^{(1)}$ be a subset, and define $\Sigma_X = p(\Sigma) \subseteq \mathcal{X}(S)^{(1)}$. For $x \in X(k)$, define $\Sigma_x = \Sigma \cap (\{x\} \times Y)$. If Σ_X is not strongly thin in X and, for every $x \in \Sigma_X$, the set Σ_x is not strongly thin in $\{x\} \times Y$, then Σ is not strongly thin in $X \times Y$.*

Proof. Note first that X and Y have a dense set of k-rational points by the assumptions, so they are geometrically irreducible by [79, Tag 0G69], and hence $X \times Y$ is a normal (geometrically) integral k-variety. Let $(\pi_i\colon Z_i \to X \times Y)_{i=1}^n$ be a finite family of ramified covers with Z_i normal geometrically integral varieties over k. Without loss of generality, we may assume that $Z_i(k)$ is dense in Z_i for every i (as otherwise the image $\pi_i(Z_i(k))$ is not dense in $X \times Y$). Let $Y \to \overline{Y}$ be an open immersion with $\overline{Y}$ a normal projective geometrically integral variety over k, and let $\overline{Z_i} \to X \times \overline{Y}$ be the normalization of $X \times \overline{Y}$ in Z_i. Let $\psi_i\colon T_i \to X$ denote the Stein factorization of $\overline{Z_i} \to X$.

Let $i \in \{1, \dots, n\}$ be an index such that $Z_i \to X \times Y$ is vertically ramified over X. By Theorem 3.2.4, there exists a finite family of covers $(f_{ij}\colon W_{ij} \to Z_i)_{j \in J_i}$ such that

$$\Sigma \setminus \pi_i(Z_i(k)) = \Sigma \setminus \bigcup_{j \in J_i} (\pi_i \circ f_{ij})(W_{ij}(k))$$

and, for every $j \in J_i$, the Stein factorization of $\overline{W_{ij}} \to X \times \overline{Y} \to X$ is ramified over X, where $\overline{W_{ij}} \to \overline{Z_i}$ is the normalization of $\overline{Z_i}$ in W_{ij}. Thus, we may replace $Z_i \to X \times Y$

by the finitely many covers $W_{ij} \to X \times Y$ to assume that ψ_i is ramified. Up to reindexing, we may now assume that there exists an integer $m \in \{1, \dots, n\}$ such that

(1) ψ_i is ramified for all $i = 1, \dots, m$,

(2) ψ_i is étale and the branch locus of π_i dominates X for all $i = m+1, \dots, n$.

Let

$$\Omega := \Sigma_X \setminus \bigcup_{i=1}^{m} \psi_i(T_i(k)),$$

which is dense in X since Σ_X is not strongly thin. To conclude the proof, we show that

$$\Psi := \bigcup_{x \in \Omega} \left(\Sigma_x \setminus \bigcup_{i=m+1}^{n} \pi_{i,x}(Z_{i,x}(k)) \right) \subseteq \Sigma \setminus \bigcup_{i=1}^{n} \pi_i(Z_i(k))$$

is dense in $X \times Y$.

Let $V \subseteq X$ be a dense open subscheme such that, for every x in $V(k)$ and every $i \in \{m+1, \dots, n\}$, the fiber $(\overline{Z_i})_x$ is normal. Recall that, for $i \in \{m+1, \dots, n\}$, the ramification locus $D_i \subseteq X \times Y$ of π_i dominates X. Thus, the image of D_i in X is a dense constructible set and therefore contains a dense open subset $U_i \subseteq X$. Let $U = V \cap \bigcap_{i=m+1}^{n} U_i$.

Let $\pi_{i,T_i} \colon Z_{i,T_i} \to T_i \times Y$ denote the pullback of π_i along $\psi_{i,Y} \colon T_i \times Y \to X \times Y$. Since $\psi_{i,Y}$ is étale, the ramification locus $D_{i,T_i} \subseteq T_i \times Y$ of π_{i,T_i} is given by the pullback of D_i along $\psi_{i,Y}$.

By the density of $\Omega \cap U$ in X, we obtain the desired density of Ψ by showing that, for every $x \in \Omega \cap U$, the set $\Sigma_x \setminus \bigcup_{i=m+1}^{n} \pi_{i,x}(Z_{i,x}(k))$ is dense in $\{x\} \times Y \cong Y$. Since Σ_x is not strongly thin, this follows after showing that, for $m+1 \leq i \leq n$ and $x \in U(k)$, and for every irreducible component Z' of $Z_{i,x}$, the restriction of $\pi_{i,x}$ to Z' is a ramified cover.

Let $i \in \{m+1, \dots, n\}$ and fix $x \in U(k)$. Let $\psi_i^{-1}(x) = t_1 \sqcup \dots \sqcup t_r$ be the decomposition into isolated points (recall that ψ_i is étale). By the normality of $(\overline{Z_i})_x$, it decomposes as a disjoint union of normal varieties $(\overline{Z_i})_{t_1} \sqcup \dots \sqcup (\overline{Z_i})_{t_r}$. Since the fibers of $\overline{Z_i} \to T_i$ are geometrically connected, the varieties $(\overline{Z_i})_{t_j}$ are connected (and thus integral). Let Z' be an irreducible component of $Z_{i,x}$. Then $Z' = Z_{i,t_j} \subseteq (\overline{Z_i})_{t_j}$ is an open subscheme for some point $t_j \in T_{i,x}$. Since D_{i,T_i} is given by the pullback of D_i along $\psi_{i,Y}$ and $x \in U(k)$ lies in the image of D_i inside X, the universal property of fiber products gives us a point of D_{i,T_i} lying over t_j. This implies that $Z' = Z_{i,t_j} \to \{t_j\} \times Y$ and thus $Z' \to \{x\} \times Y$ ramifies, as claimed. ☐

As mentioned above, the persistence of the weak Hilbert property under products is a direct consequence of Theorem E.

Theorem 3.2.7. *Let S be a regular integral arithmetic scheme and let $\mathcal{X}$ and $\mathcal{Y}$ be quasi-projective arithmetic schemes over S. If both $\mathcal{X}$ and $\mathcal{Y}$ satisfy the weak Hilbert property over S, then $\mathcal{X} \times_S \mathcal{Y}$ also satisfies the weak Hilbert property over S.*

Proof. Define $\Sigma = (\mathcal{X} \times \mathcal{Y})(S)^{(1)} = \big(\mathcal{X}(S)^{(1)}\big) \times \big(\mathcal{Y}(S)^{(1)}\big)$ and let $X = \mathcal{X}_{K(S)}$ and $Y = \mathcal{Y}_{K(S)}$. Then the image Σ_X of Σ under the projection $X \times Y \to X$ equals $\mathcal{X}(S)^{(1)}$, which is not strongly thin in X by the weak Hilbert property of $\mathcal{X}$. For every $x \in \Sigma_X$, we have $\Sigma \cap (\{x\} \times Y) = \{x\} \times \mathcal{Y}(S)^{(1)}$, which is not strongly thin in $\{x\} \times Y$ by the weak Hilbert property of Y. Thus, by Theorem E, Σ is not strongly thin in $X \times Y$, i.e., $\mathcal{X} \times_S \mathcal{Y}$ has the weak Hilbert property over S. □

3.2.3 Birational invariance

The weak Hilbert property is shown to be a birational invariant among smooth proper varieties over finitely generated fields of characteristic 0 in [25, Proposition 3.1]. Slightly adapting the arguments of the proof gives us the following related result.

Lemma 3.2.8. *Let $\pi\colon X' \to X$ be a proper birational morphism of smooth integral varieties over a field k of characteristic 0, let $\Sigma_X \subseteq X(k)$ be a subset, and define $\Sigma_{X'}$ to be the preimage of Σ_X under the induced morphism $X'(k) \to X(k)$. Then Σ_X is strongly thin in X if and only if $\Sigma_{X'}$ is strongly thin in X'.*

Proof. Let $\mathrm{Cov}(X)$ (resp. $\mathrm{Cov}(X')$) denote the class of covers $Y \to X$ (resp. $Y \to X'$) with Y a normal integral k-variety. Define the maps $N'\colon \mathrm{Cov}(X) \to \mathrm{Cov}(X')$ and $N\colon \mathrm{Cov}(X') \to \mathrm{Cov}(X)$ as follows:

If $Y' \to X'$ is a cover, let $N(Y') \to X$ be the normalization of X in Y'. If $Y \to X$ is a cover, note that the generic fiber of $Y \times_X X' \to X'$ is isomorphic to the generic fiber of $Y \to X$ (since $X' \to X$ is generically an isomorphism) and thus irreducible. Therefore, $Y \times_X X'$ has exactly one irreducible component Y'_0 dominating X'. The morphism $Y'_0 \to Y$ is birational and $Y'_0 \to X$ is finite surjective since $Y'_0 \to Y \times_X X'$ is a closed immersion and thus finite. We now let $N'(Y) \to X'$ be the normalization of X' in Y'_0. Note that both N and N' preserve the degree of a cover by the birationality of X' and X.

We claim that N and N' are inverse to each other. To see this, let $Y \to X$ be a cover with Y normal and define Y'_0 as above. By the universal property of normalization, the inclusion $Y'_0 \to Y \times_X X'$ factors as $Y'_0 \to N'(Y) \to Y \times_X X'$ over X'. In particular, this induces a factorization $N'(Y) \to Y \to X$ of the composition $N'(Y) \to X' \to X$. Therefore, again by the universal property of normalization, we have a morphism $N(N'(Y)) \to Y$ over X, which is finite (by the finiteness of $N(N'(Y)) \to X$) and birational (by the degrees over X being equal), and thus an isomorphism by Zariski's main theorem. Conversely, if $Y' \to X'$ is a cover with Y' normal, we obtain a finite birational morphism $N'(N(Y')) \to Y'$ over X', which again is an isomorphism by Zariski's main theorem, as required.

Next we show that N and N' send étale covers to étale covers. If $Y \to X$ is an étale cover, then $Y \times_X X'$ is étale over X' and thus normal. Hence, the irreducible component Y_0' dominating X' is normal and étale over X' and we have $N'(Y) = Y_0'$.

For the converse, let $Y' \to X'$ be an étale cover and consider the cover $N(Y') \to X$. Since $X' \to X$ is proper birational, by [56, Corollary 4.4.3], there exists a dense open $U \subseteq X$ such that $U' := \pi^{-1}(U) \to U$ is an isomorphism and the codimension of $X \setminus U$ in X is at least 2. The pullback $N(Y') \times_X U \to U$ is étale, as its pullback along the isomorphism $U' \to U$ is $Y' \times_{X'} U' \to U'$ (which is étale). Since $X \setminus U$ is of codimension at least 2, it follows from purity of the branch locus [38, Théorème X.3.1] that $N(Y') \to X$ is étale, as claimed. We conclude that N and N' are inverse to each other and send étale covers to étale covers and thus ramified covers to ramified covers.

To finish the proof, since a subset is strongly thin if and only if its intersection with some dense open subscheme is strongly thin, we may assume that $\Sigma_X \subseteq U(k)$ (and thus $\Sigma_{X'} \subseteq U'(k)$). Given a finite family of ramified covers $(\pi_i \colon Y_i \to X)_{i=1}^n$ (resp. $(\pi_i' \colon Y_i' \to X)_{i=1}^n$), we define $Y_i' = N'(Y_i)$ (resp. $Y_i = N(Y_i')$) to obtain a finite family of ramified covers $(\pi_i' \colon Y_i' \to X)_{i=1}^n$ (resp. $(\pi_i \colon Y_i \to X)_{i=1}^n$) such that π induces a bijection

$$\Sigma_{X'} \setminus \bigcup_{i=1}^n \pi_i'(Y_i'(k)) \cong \Sigma_X \setminus \bigcup_{i=1}^n \pi_i(Y_i(k)).$$

In particular, the left side is dense in X' if and only if the right side is dense in X, which finishes the proof. □

The invariance of the weak Hilbert property for arithmetic schemes under proper birational morphisms now follows similarly to the analogue of the usual Hilbert property (Theorem 3.1.5).

Theorem 3.2.9. *Let $\mathcal{X}, \mathcal{X}'$ be arithmetic schemes over a normal integral arithmetic scheme S. Assume that $\mathcal{X}_{K(S)}$ and $\mathcal{X}'_{K(S)}$ are smooth and that there exists a proper birational morphism $\pi \colon \mathcal{X}' \to \mathcal{X}$ over S. Then $\mathcal{X}$ has the weak Hilbert property over S if and only if $\mathcal{X}'$ has the weak Hilbert property over S.*

Proof. Let $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{U}' \subseteq \mathcal{X}'$ be dense open subschemes such that the morphism $\mathcal{X}' \to \mathcal{X}$ induces an isomorphism $\mathcal{U}' \to \mathcal{U}$. If k is the function field of S, then the induced map $\mathcal{X}'(S)^{(1)} \cap \mathcal{U}'(k) \to \mathcal{X}(S)^{(1)} \cap \mathcal{U}(k)$ is surjective. Indeed, for a near-integral point $s \in \mathcal{X}(S)^{(1)} \cap \mathcal{U}(k)$, there exists a point $s' \in \mathcal{U}'(k)$ lying over s, and the valuative criterion of properness shows that $s' \in \mathcal{X}'(S)^{(1)}$. Thus, we have

$$\pi^{-1}\left(\mathcal{X}(S)^{(1)} \cap \mathcal{U}(k)\right) = \mathcal{X}'(S)^{(1)} \cap \mathcal{U}'(k).$$

Since the intersection of a dense subset and a dense open subset is dense, it is clear that $\mathcal{X}(S)^{(1)}$ is strongly thin in $\mathcal{X}_k$ if and only if $\mathcal{X}(S)^{(1)} \cap \mathcal{U}(k)$ is strongly thin in $\mathcal{X}_k$, and analogously for $\mathcal{X}'$. Therefore, since $\mathcal{X}'_k \to \mathcal{X}_k$ is a proper birational morphism of smooth k-varieties, the claim follows from Lemma 3.2.8. ☐

Example 3.2.10. To see that the properness assumption on the birational morphism in Theorem 3.2.9 cannot be dropped, consider the open inclusion of arithmetic schemes over $\mathbb{Z}$

$$\mathbb{G}_{\mathrm{m},\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x, x^{-1}] \hookrightarrow \operatorname{Spec} \mathbb{Z}[x] = \mathbb{A}^1_{\mathbb{Z}}.$$

Here the right side satisfies the weak Hilbert property by the Hilbertianity of $\mathbb{Z}$, but the left side has only finitely many integral points. We give the following example to show that dropping the properness assumption will also not work up to an extension of the base scheme. Let K be a number field and S a finite set of places of K such that the unit group $\mathcal{O}^*_{K,S}$ of the ring of S-integers $R = \mathcal{O}_{K,S}$ is infinite (i.e., S contains at least one finite place, or K is neither $\mathbb{Q}$ nor imaginary quadratic). Consider the arithmetic R-schemes

$$\mathcal{X} = \mathbb{G}_{\mathrm{m},R} = \operatorname{Spec} R[x, x^{-1}] \quad \text{and} \quad \mathcal{X}' = \mathcal{X} \setminus \{1\} = \operatorname{Spec} R[x, x^{-1}, (x-1)^{-1}]$$

with the obvious open immersion $\mathcal{X}' \hookrightarrow \mathcal{X}$. Since $\mathcal{X}(R) = \mathcal{O}^*_{K,S}$ is infinite by assumption and finitely generated by Dirichlet's theorem on S-units, it has an infinite (thus dense) cyclic subgroup Ω, and it follows from Zannier's work [86] on tori that Ω is not strongly thin, i.e., $\mathcal{X}$ has the weak Hilbert property over R. (This also follows directly from Corollary 4.2.7; see Definition 4.2.1 for the definition of the relevant property (OPS), which we apply here to the empty subscheme of $\mathcal{X}$.) However, the set $\mathcal{X}'(R)$ is finite. Indeed, it is given by the set of units $x \in \mathcal{O}^*_{K,S}$ for which $x - 1$ is also a unit, which we can identify with the set of pairs $(x, y) \in (\mathcal{O}^*_{K,S})^2$ satisfying $x - y = 1$. This set of pairs is finite by a theorem of Siegel–Mahler–Lang [53, §VII.4].

3.2.4 Smooth proper images

The weak Hilbert property for smooth proper varieties descends along smooth proper morphisms, as shown in [25, Theorem 3.7]. The proof does not require properness of the varieties (only of the morphism) and thus readily implies that the weak Hilbert property descends along smooth proper morphisms of regular arithmetic schemes (note that regularity implies generic smoothness in characteristic 0). Using the following more general result given in [5] allows for an even stronger conclusion on arithmetic schemes.

Lemma 3.2.11 ([5, Lemma 5.3]). *Let* $\varphi\colon X \to Y$ *be a smooth surjective morphism of normal integral varieties over a field* k *of characteristic* 0 *and* $T \subseteq Y(k)$ *a strongly thin subset. If the generic fiber of* φ *is geometrically irreducible or if* φ *is proper, then the preimage of* T *in* $X(k)$ *is strongly thin.*

As a direct consequence, we obtain that the weak Hilbert property descends along proper morphisms of arithmetic schemes which are smooth on the generic fiber.

Proposition 3.2.12. *Let* S *be a normal noetherian integral scheme with function field* k *of characteristic* 0 *and let* $\varphi\colon \mathcal{X} \to \mathcal{Y}$ *be a surjective morphism of arithmetic schemes over* S *such that* $\varphi_k\colon \mathcal{X}_k \to \mathcal{Y}_k$ *is smooth. Assume that* φ_k *is proper or its generic fiber is geometrically irreducible. If* $\mathcal{X}$ *has the weak Hilbert property over* S*, then* $\mathcal{Y}$ *has the weak Hilbert property over* S*.*

Proof. Define $X = \mathcal{X}_k, Y = \mathcal{Y}_k$ and $T = \varphi(\mathcal{X}(S)^{(1)}) \subseteq \mathcal{Y}(S)^{(1)}$. Since $\mathcal{X}(S)^{(1)}$ is contained in the preimage of T under the base change $\varphi_k\colon X \to Y$ and not strongly thin in X by the weak Hilbert property of $\mathcal{X}$, Lemma 3.2.11 shows that T is not strongly thin in Y. In particular, $\mathcal{Y}$ satisfies the weak Hilbert property over S. ☐

3.2.5 Finite étale covers and extensions of the base

While the weak Hilbert property descends along smooth proper morphisms, it does not ascend along them (i.e., the converse to Proposition 3.2.12 is false). In fact, this fails even for finite étale morphisms of varieties, see [25, Remark 3.5]. However, in the case of smooth proper varieties, the weak Hilbert property does ascend along smooth proper morphisms *up to a finite extension of the base field* [25, Theorem 3.16]. To prove an analogous statement for arithmetic schemes, we first show the following analogue of [25, Lemma 3.14].

Lemma 3.2.13. *Let* L/k *be a finite extension of fields of characteristic* 0*, let* X *be a normal integral variety over* L*, let* Y *be a normal integral variety over* k*, and let* $\pi\colon X \to Y$ *be a finite étale morphism of* k*-varieties. Furthermore, let* $\Sigma_X \subseteq X(L)$ *and* $\Sigma_Y \subseteq Y(k)$ *be subsets such that* $\Sigma_Y \subseteq \pi(\Sigma_X)$ *and* Σ_Y *is not strongly thin in* Y*. Then* Σ_X *is not strongly thin in* X*.*

Proof. Since Σ_Y is not strongly thin in Y, it follows from [5, Lemma 2.1] that Σ_Y and thus $\pi(\Sigma_X)$ are not strongly thin in Y_L. Thus, we may and do assume that $k = L$. Let $(\pi_i \colon Z_i \to X)_{i=1}^n$ be a finite collection of ramified covers and note that each $\pi \circ \pi_i \colon Z_i \to Y$ is a ramified cover. Since Σ_Y is not strongly thin in Y, the set $T := \Sigma_Y \setminus \bigcup_{i=1}^n \pi(\pi_i(Z_i(k)))$ is dense in Y. Let $y \in T$. Since $\Sigma_Y \subseteq \pi(\Sigma_X)$, there exists an element $x \in \Sigma_X$ such that $\pi(x) = y$, and clearly $x \notin \pi_i(Z_i(k))$ for any i, as otherwise $y \in \pi(\pi_i(Z_i(k)))$. Thus, we have

$$T \subseteq \pi\left(\Sigma_X \setminus \bigcup_{i=1}^n \pi_i(Z_i(k))\right).$$

Since T is dense in Y and π is finite surjective, we conclude that $\Sigma_X \setminus \bigcup_{i=1}^n \pi_i(Z_i(k))$ is dense in X, so Σ_X is not strongly thin in X, as claimed. ☐

Applied to near-integral points of arithmetic schemes, Lemma 3.2.13 immediately implies the following.

Proposition 3.2.14. *Let $S' \to S$ be a finite surjective morphism of normal integral noetherian schemes with function field of characteristic 0, let $\mathcal{X}$ be an arithmetic scheme over S', let $\mathcal{Y}$ be an arithmetic scheme over S, and let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a finite étale morphism of S-schemes. If $\mathcal{Y}$ has the weak Hilbert property over S and $\mathcal{Y}(S)^{(1)} \subseteq \pi(\mathcal{X}(S')^{(1)})$, then $\mathcal{X}$ has the weak Hilbert property over S'.*

Proof. This follows directly from Lemma 3.2.13 applied to the finite étale morphism $\mathcal{X}_{K(S')} \to \mathcal{Y}_{K(S)}$ and the sets $\Sigma_X := \mathcal{X}(S')^{(1)}$ and $\Sigma_Y := \mathcal{Y}(S)^{(1)}$. ☐

With the same arguments as in the proof of [25, Theorem 3.16], combining the Chevalley–Weil type lifting argument for near-integral points with the above proposition, we can now show that the weak Hilbert property ascends along finite étale morphisms of arithmetic schemes, up to a finite étale cover of the base.

Theorem 3.2.15 (Theorem F)**.** *Let S be a regular integral arithmetic scheme and $\varphi \colon \mathcal{X} \to \mathcal{Y}$ a finite étale morphism of arithmetic schemes over S. If $\mathcal{Y}$ satisfies the weak Hilbert property over S and $\mathcal{X}_{K(S)}$ is geometrically integral, then there exists a finite étale morphism $S' \to S$ of regular integral arithmetic schemes such that $\mathcal{X}_{S'}$ is an arithmetic scheme over S' with the weak Hilbert property over S'.*

Proof. By Theorem 3.1.10, there exists a finite étale morphism $S' \to S$ of regular integral arithmetic schemes such that $\mathcal{Y}(S)^{(1)} \subseteq \varphi(\mathcal{X}(S')^{(1)})$. Since $\mathcal{X}_{K(S')}$ is normal integral, $\mathcal{X}_{S'}$ is an arithmetic scheme over S'. Let $\pi \colon \mathcal{X}_{S'} \to \mathcal{Y}$ be the composition $\mathcal{X}_{S'} \to \mathcal{X} \to \mathcal{Y}$. Then π is finite étale (hence surjective), so $\mathcal{X}_{S'}$ has the weak Hilbert property over S' by Proposition 3.2.14. ☐

As mentioned above, the weak Hilbert property does not in general ascend along finite étale morphisms without allowing for an extension of the base (even in the case of varieties, cf. [25, Remark 3.5]). In particular, in Theorem F, we can not expect $S' \to S$ to be an isomorphism. An explicit example of this phenomenon is given in [34], based on an example of Silverman [77, Example X.4.10]; here Y is an elliptic curve over $\mathbb{Q}$ with positive Mordell–Weil rank (and hence satisfying the weak Hilbert property), while X is a homogeneous space for Y with $X(\mathbb{Q}) = \emptyset$, and $X \to Y$ is finite étale.

As a further corollary to Proposition 3.2.14 and an analogue to [25, Proposition 3.15], it is easy to see that the weak Hilbert property of an arithmetic scheme $\mathcal{X}$ persists under base change along a finite étale morphism $S' \to S$ by applying Proposition 3.2.14 to the morphism $\mathcal{X}_{S'} \to \mathcal{X}$. However, [5, Proposition 3.2] already implies the following stronger result.

Theorem 3.2.16. *Let $S' \to S$ be a finite type flat morphism of normal integral noetherian schemes with function field of characteristic 0. Let $\mathcal{X}$ be an arithmetic scheme over S. If $\mathcal{X}$ has the weak Hilbert property over S, then $\mathcal{X}_{S'}$ has the weak Hilbert property over S'.*

Proof. Note first that $\mathcal{X}_{S'}$ is indeed an arithmetic scheme over S' since $\mathcal{X}_{K(S)}$ is geometrically integral by [79, Tag 0G69]. Since finite type flat morphisms are open, the image S_0 of S' in S is a dense open. Clearly, we have $\mathcal{X}(S)^{(1)} \subseteq \mathcal{X}(S_0)^{(1)}$, so $\mathcal{X}_{S_0}$ has the weak Hilbert property over S_0. By [39, Corollaire 6.1.4], pullback along $S' \to S_0$ preserves codimension of closed subsets, so the inclusion $\mathcal{X}(S_0)^{(1)} \subseteq \mathcal{X}(S')^{(1)}$ holds. Let k and L denote the function fields of S_0 and S', respectively, and define $T = \mathcal{X}(S')^{(1)}$. Since $T \cap \mathcal{X}_k(k) \supseteq \mathcal{X}(S_0)^{(1)}$ is not strongly thin in $\mathcal{X}_k$, it follows from [5, Proposition 3.2] that T is not strongly thin in $\mathcal{X}_L$, i.e., $\mathcal{X}_{S'}$ has the weak Hilbert property over S'. □

Remark 3.2.17. Theorem 3.2.16 also holds if $S' \to S$ is flat surjective (instead of finite type flat) and the extension $K(S')/K(S)$ is small (in the sense of [5]) by repeating the proof and replacing [5, Proposition 3.2] with [5, Proposition 4.2].

3.3 Mixed fibrations

The result of Bary-Soroker–Fehm–Petersen that the product of two varieties with the Hilbert property has the Hilbert property [4, Corollary 3.4] is a consequence of their more general fibration theorem [4, Theorem 1.1]: If $X \to S$ is a dominant morphism of integral varieties over a field k with S satisfying the Hilbert property and, for all

$s \in S(k)$, the fiber X_s is an integral variety with the Hilbert property, then X satisfies the Hilbert property. The persistence under products can then be seen by applying this fibration theorem to the projection morphism $X \times Y \to Y$.

The analogous product theorem for the weak Hilbert property of proper varieties (over a finitely generated field of characteristic 0) due to Corvaja–Demeio–Javanpeykar–Lombardo–Zannier [25, Theorem 1.9] on the other hand is *not* a consequence of a more general fibration theorem. Even in our generalization of this product theorem to arithmetic schemes (Theorem E), where the subset $\Sigma \subseteq (\mathcal{X} \times \mathcal{Y})(S)^{(1)}$ that is shown to be not strongly thin is not necessarily a product of subsets of $\mathcal{X}(S)^{(1)}$ and $\mathcal{Y}(S)^{(1)}$, we still have to work with a product of arithmetic schemes instead of a more general fibration.

A "mixed fibration theorem" similar to the result of Bary-Soroker–Fehm–Petersen for the weak Hilbert property was established in [49], where the base has the *weak* Hilbert property and the fibers have the *usual* Hilbert property:

Theorem 3.3.1 ([49, Theorem 1.3]). *Let k be a number field and let $X \to S$ be a morphism of normal integral projective k-varieties. Let $\Omega \subseteq S(k)$ be a subset that is not strongly thin in S and such that, for every $s \in \Omega$, the fiber X_s is a normal integral variety satisfying the Hilbert property over k. Then X satisfies the weak Hilbert property over k.*

The restriction in Theorem 3.3.1 that k is a number field and X and S are projective is due to use of Corvaja–Zannier's result [26, Theorem 1.4] that a normal integral projective variety with the Hilbert property over a number field is algebraically simply connected, which allows to conclude that the covers of the fibers X_s occurring in the proof are ramified, and thus of degree at least 2. In the proof of our more general Theorem G, we avoid this argument by comparing the degrees of these covers of X_s to the degrees of other non-trivial covers, and by using Nagata compactification to use Stein factorization without the projectivity assumption.

Theorem 3.3.2 (Theorem G). *Let k be a field of characteristic 0 and let $X \to S$ be a dominant morphism of normal integral varieties over k. Let $\Gamma \subseteq X(k)$ and $\Sigma \subseteq S(k)$ be subsets such that Σ is not strongly thin in S and, for every $s \in \Sigma$, the fiber X_s is a normal integral k-variety and $\Gamma \cap X_s(k)$ is not thin in X_s. Then Γ is not strongly thin in X.*

Proof. Let $(\pi_i \colon Y_i \to X)_{i=1}^n$ be a finite collection of ramified covers. To prove the claim, we have to show that $\Gamma \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ is dense in X. Let $X \to \overline{X} \to S$ be a factorization of $X \to S$ such that $\overline{X}$ is a normal integral k-variety, $X \to \overline{X}$ is an open immersion, and $\overline{X} \to S$ is proper. (This can be achieved by taking a Nagata compactification of $X \to S$ [79, Tag 0F3T] and normalization.) For every $i = 1, \ldots, n$, let $\overline{\pi_i}\colon \overline{Y_i} \to \overline{X}$ be the normalization of $\overline{X}$ in Y_i and let $\overline{Y_i} \to T_i \xrightarrow{\psi_i} S$ be the Stein factorization of $\overline{Y_i} \to S$ (see [79, Tag 03H0]), so ψ_i is finite, the fibers of $\overline{Y_i} \to T_i$ are geometrically connected, and T_i is the normalization of S in $\overline{Y_i}$. We have the following commutative diagram.

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d, "\pi_i"'] & \overline{Y_i} \arrow[d, "\overline{\pi_i}"'] \arrow[dr] & \\
X \arrow[r] \arrow[dr] & \overline{X} \arrow[d] & T_i \arrow[dl, "\psi_i"] \\
& S &
\end{tikzcd}$$

Since $\overline{Y_i}$ is normal and integral, it follows from [79, Tag 035L] that T_i is normal and integral.

Let $U \subseteq X$ be a dense open subscheme such that, for every i, the restriction morphism $Y_{i,U} := Y_i \times_X U \to U$ is étale. Moreover, let $V \subseteq S$ be a dense open such that, for every $s \in V$, we have $U_s \subseteq X_s$ dense open. (Such V exists by [79, Tag 0573] since X and the fibers X_s are reduced, which implies that scheme-theoretic density and Zariski-density of open subschemes are equivalent by [79, Tag 056D].)

Up to reindexing, we may and do assume that there exists an integer $r \in \{1, \ldots, n\}$ such that ψ_i is unramified (hence étale) for $i = 1, \ldots, r$ and ramified for $i = r+1, \ldots, n$. Let $V' \subseteq S$ be a dense open subscheme such that, for every $i \leq r$ and every $s \in V'$, the fiber $(\overline{Y_i})_s$ is normal. Define

$$\Omega = (\Sigma \cap V \cap V') \setminus \bigcup_{i=r+1}^{n} \psi_i(T_i(k)).$$

Since Σ is not strongly thin in S and V, V' are dense opens, Ω is dense in S. Moreover, for every $s \in \Omega$ and every $i \geq r+1$, we have $Y_{i,s}(k) = \emptyset$ (as $T_{i,s}(k)$ is empty). Let

$$\Psi = \bigcup_{s \in \Omega} ((\Gamma \cap X_s) \setminus \cup_{i=1}^{r} \pi_{i,s}(Y_{i,s}(k))) \subseteq \Gamma \setminus \bigcup_{i=1}^{n} \pi_i(Y_i(k)).$$

Since Ω is dense in S, to conclude the proof, it suffices to show that, for every $s \in \Omega$, the set $(\Gamma \cap X_s) \setminus \cup_{i=1}^{r} \pi_{i,s}(Y_{i,s}(k))$ is dense in X_s.

Let $i \leq r$ and $s \in \Omega$. By the étaleness of ψ_i, the fiber $T_{i,s}$ is a disjoint union of closed points, say $T_{i,s} = t_1 \sqcup \ldots \sqcup t_N$. We then have $(\overline{Y_i})_s = \sqcup_{j=1}^{N} (\overline{Y_i})_{t_j}$ with $(\overline{Y_i})_{t_j}$ normal and geometrically connected, and thus geometrically integral, over the residue field $\kappa(t_j)$. Moreover, we have $Y_{i,s} = \sqcup_{j=1}^{N} Y_{i,t_j}$ and, for every j, the fiber Y_{i,t_j} is either empty or a dense open of $(\overline{Y_i})_{t_j}$ (in which case it is normal and geometrically integral over $\kappa(t_j)$). Clearly, if $\kappa(t_j) \neq k$, then $Y_{i,t_j}(k) = \emptyset$. Therefore, since $\Gamma \cap X_s$ is not thin in X_s, it suffices to show that, for every j with $\kappa(t_j) = k$ and $Y_{i,t_j} \subseteq (\overline{Y_i})_{t_j}$ dense open, the degree of $Y_{i,t_j} \to X_s$ is at least 2.

Let $X_{T_i} = X \times_S T_i$ and $Y_{i,T_i} = Y_i \times_S T_i$. We have $T_i \to S$ finite étale, so $X_{T_i} \to X$ is finite étale. It follows that the induced morphism $Y_i \to X_{T_i}$ is not étale (since $Y_i \to X$ would be étale otherwise). Since $X_{T_i} \to X$ and $Y_i \to X$ are finite, $Y_i \to X_{T_i}$ is also finite [79, Tag 035D]. Moreover, $Y_i \to X_{T_i}$ is a left factor of the surjective morphism $Y_{i,T_i} \to X_{T_i}$ and therefore surjective, so X_{T_i} is connected. By Zariski's main theorem [79, Tag 0AB1] and the fact that $Y_i \to X_{T_i}$ is not étale (in particular not an isomorphism), it follows that $Y_i \to X_{T_i}$ is of degree at least 2.

Let $t_j \in T_{i,s}(k)$ such that $Y_{i,t_j} \neq \emptyset$. To conclude the proof, we show that the degree of $Y_{i,t_j} \to X_s$ is at least 2. Note that the composition of $t_j\colon \operatorname{Spec} k \to T_i$ and ψ_i is s, so we have $X_{T_{i,t_j}} = X_s$. Let $U_{T_i} = U \times_S T_i$. Then $Y_{i,U} \subseteq Y_i$ and $U_{T_i} \subseteq X_{T_i}$ are dense opens and the induced morphism $Y_{i,U} \to U_{T_i}$ has the same degree as $Y_i \to X_{T_i}$, which is at least 2. By the definition of U, the morphism $Y_{i,U} \to U$ is étale, and since $U_{T_i} \to U$ is étale, it follows from [79, Tag 02GW] that $Y_{i,U} \to U_{T_i}$ is étale, hence flat. By flatness, the degree of the induced morphism on fibers $(Y_{i,U})_{t_j} \to U_s$ equals the degree of $Y_{i,U} \to U_{T_i}$, i.e., is at least 2. Since $(Y_{i,U})_{t_j} \to U_s$ is the pullback of $Y_{i,t_j} \to X_s$ along $U_s \to X_S$ and $U_s \subseteq X_s$ is a dense open, this shows that $Y_{i,t_j} \to X_s$ is of degree at least 2, as required. ☐

Theorem 3.3.1 was proved to establish the potential weak Hilbert property for a smooth projective variety X over a number field k with a nef tangent bundle (see [49, Theorem 1.7 and 1.8]): After possibly replacing k by a finite extension and X by a finite étale cover, one obtains a smooth surjective map $X \to \mathrm{Alb}(X)$, where $\mathrm{Alb}(X)$ is the Albanese variety of X and the fibers are Fano varieties with nef tangent bundle. The Campana–Peternell conjecture (see [49, Conjecture 1.6]) predicts that Fano varieties with nef tangent bundle are homogeneous spaces under a connected linear algebraic group, implying that they satisfy the Hilbert property over a finite extension of k (independent of the fiber); see [49, Theorem 1.4]. The result then follows from Theorem 3.3.1 and the potential weak Hilbert property for the abelian variety $\mathrm{Alb}(X)$, provided that the Campana–Peternell conjecture holds for the fibers of $X \to \mathrm{Alb}(X)$. Since the Campana–Peternell conjecture is known for dimension at most 5 by [50, 82], X satisfies the potential weak Hilbert property if $\dim X \leq 5$; see [49, Theorem 1.7].

Our extension of Theorem 3.3.1 to non-proper varieties (Theorem G) was motivated by potential applications to the conjectures of Campana and Corvaja–Zannier. This generalization has since been applied to prove the $\mathbb{P}^1$-invariance of the weak Hilbert property for symmetric products (see [3, Lemma 7.7] and [3, Corollary 7.8]) and the potential weak Hilbert property for sufficiently high symmetric powers of ruled surfaces (see [3, Theorem 7.9]).

4 Integral points on algebraic groups

In this chapter, we investigate integral and rational points on algebraic groups. By an *algebraic group* we mean a group scheme of finite type over a field. Since any group scheme over a field is separated [79, Tag 047L] and an algebraic group over a field of characteristic 0 is smooth [79, Tag 047N], it is in particular a normal variety.

Our first main result (Theorem A) is that algebraic groups satisfy Conjecture 1.1.4, i.e., a connected algebraic group G with a dense set of rational points over a finitely generated field k of characteristic 0 satisfies the weak Hilbert property over k. This is inspired by the work of Bary-Soroker–Fehm–Petersen [4] and Corvaja–Demeio–Javanpeykar–Lombardo–Zannier [25], who proved Theorem A in the special case that G is a linear algebraic group or an abelian variety, respectively. More generally, we prove that there exists a $\mathbb{Z}$-finitely generated domain $R \subseteq k$ with fraction field k such that G admits a non-strongly-thin set of R-integral points with respect to an appropriate model of G over R (see Theorem C). In particular, the special case that k is a number field shows that algebraic groups satisfy part (4) of Conjecture 1.3.5. (Note also that (1) is satisfied by [1, Lemma 2.11] and that (2) is implied by (4).) Finally, our last main theorem is that part (5) of Conjecture 1.3.5 (and thus the full conjecture) holds for all linear algebraic groups, see Theorem H.

The contents of Sections 4.1 and 4.2, up to editorial changes, have been published in [62, §5] and [60], respectively.

4.1 The weak Hilbert property for algebraic groups

In order to prove the non-strongly-thinness of integral points on algebraic groups (Theorem C), we combine results from [25] on abelian varieties and [55] on linear algebraic groups using a fibration theorem due to Liu [55, Proposition 2.15]. We briefly recall some terminology used in *loc. cit.*

While the usual Hilbert property can be formulated using the integrality of many fibers, this may not be done for the weak Hilbert property for arbitrary ramified covers (recall Proposition 2.3.1). This was first noted by Zannier in [86], leading to the definition of so-called *(PB)-covers* (where the abbreviation (PB) refers to pullback). These are defined in [86] for connected commutative algebraic groups over number fields, in [25] for abelian varieties over fields of characteristic 0, and in [55] for connected algebraic groups over finitely generated fields of characteristic 0. We give the definition provided in [55]:

Definition 4.1.1. Let k be a finitely generated field of characteristic 0, let G be a connected algebraic group over k, and let Y be a normal geometrically irreducible variety over k. A cover $\pi\colon Y \to G$ satisfies the property *(PB)* (or *is (PB)* or a *(PB)-cover*) if, for every isogeny $G' \to G_{\overline{k}}$ of connected algebraic groups over $\overline{k}$, the fiber product $G' \times_{G_{\overline{k}}} Y_{\overline{k}}$ is irreducible.

As is shown in [55, Lemma 2.8] (and similarly in [86, Proposition 2.1] and [25, Lemma 4.4]), a cover of an algebraic group over a finitely generated field of characteristic 0 is (PB) if and only if it does not admit any non-trivial étale subcover. Similarly to how we can define a non-thin set in terms of many fibers being irreducible, we can now ask if, in a given subset of rational points, many points admit irreducible fibers under a (PB)-cover. Similarly to [55, Definition 2.9], we make the following definition:

Definition 4.1.2. Let k be a finitely generated field of characteristic 0, let G be a connected algebraic group over k, and let $\Omega \subseteq G(k)$ be a (Zariski) dense subgroup. The pair (G, Ω) satisfies the property *(HIT) (over k)* if, for every (PB)-cover $\pi\colon Y \to G$ and every finite field extension k'/k, the set of $x \in \Omega$ such that $Y_{x,k'}$ is irreducible is dense in G.

The property (HIT), indicating that an algebraic group satisfies a generalization of Hilbert's Irreducibility Theorem, follows easily from the results of [25] and [55] for all pairs (G, Ω), where G is an abelian variety or a connected linear algebraic group and $\Omega \subseteq G(k)$ is dense:

Proposition 4.1.3. *Let k be a finitely generated field of characteristic 0, let G be a connected linear algebraic group or an abelian variety over k, and let $\Omega \subseteq G(k)$ be a dense subgroup. Then (G, Ω) satisfies (HIT).*

Proof. Let $Y \to G$ be a (PB)-cover and k'/k a finite field extension. By definition, the base change $Y_{k'} \to G_{k'}$ is a (PB)-cover. If G is an abelian variety (resp. a connected linear algebraic group), then it follows from [25, Theorem 1.4] (resp. [55, Theorem 1.1]) that the set of $x \in \Omega$ for which $Y_{k',x} \cong Y_{x,k'}$ is irreducible is dense. □

If G is an abelian variety over a finitely generated field k of characteristic 0 and Ω is a dense subgroup of $G(k)$, then [25, Theorem 1.4] provides an even stronger conclusion. Namely, a subset above which the fibers of a given (PB)-cover are irreducible can be given as a finite index coset of Ω. Similarly to this, Liu shows that the property (HIT) may be stated in terms of finite index cosets if Ω is finitely generated (which is automatically the case if G is an abelian variety).

Lemma 4.1.4. *Let k be a finitely generated field of characteristic 0, let G be a connected algebraic group over k, let $\Omega \subseteq G(k)$ be a finitely generated dense subgroup, and let $Y \to G$ be a (PB)-cover. If (G, Ω) satisfies (HIT), then there exists a finite index coset C of Ω such that, for all $c \in C$, the fiber Y_c is irreducible.*

Proof. A full proof of this result is given in [55, Lemma 5.2] under the assumption that G is linear. We briefly explain how the arguments in this proof extend to the general case. Liu only uses the linearity of G at two points: First, he reduces to the case that $Y \to G$ is Galois by applying [55, Lemma 4.2], which holds for arbitrary algebraic groups as noted directly above *loc. cit.* The second and last use of linearity of G is to appeal to Chebotarev's density theorem [76, Theorem 9.11], but this holds in the generality we need as well, cf. [70, Theorem B.9]. □

To go from irreducible fibers of (PB)-covers to fibers of ramified covers without rational points, we will work over the étale locus of the considered covers. Under the assumptions that the considered subgroup Ω is finitely generated, the following lemma allows us to do this by showing there exists a finite index coset of Ω avoiding the branch locus of the (PB)-cover at hand. This is an extension of [25, Lemma 4.6], where G is assumed to be an abelian variety. The proof of *loc. cit.* generalizes to our situation; we include it for the sake of completeness.

Lemma 4.1.5. *Let k be a finitely generated field of characteristic 0, let G be a connected algebraic group over k, let $\Omega \subseteq G(k)$ be a finitely generated dense subgroup, and let $Z \subsetneq G$ be a closed subscheme. Then there exists a finite index coset C of Ω such that $C \cap Z = \emptyset$.*

Proof. By the density of Ω, there exists a point $P \in \Omega$ with $P \notin Z$. By spreading out the group scheme G over k, there exists a $\mathbb{Z}$-finitely generated integral domain R with fraction field k and a connected finite type group scheme $\mathcal{G}$ over R satisfying $\mathcal{G}_k = G$. Spreading out the closed subscheme Z, extending R if necessary, we find a closed subscheme $\mathcal{Z} \subseteq \mathcal{G}$ with $\mathcal{Z}_k = Z$. Furthermore, by spreading out the finitely many generators of Ω and the point P, we may extend R if necessary such that there is a section $\sigma\colon \operatorname{Spec} R \to \mathcal{G}$ which induces P in $G(k)$ and the generators of Ω are contained in $\mathcal{G}(R)$. Since $\mathcal{G}(R)$ is a subgroup of $G(k)$, this implies that $\Omega \subseteq \mathcal{G}(R)$.

Since $P \notin Z$, the preimage $\sigma^*\mathcal{Z}$ is a proper closed subset of $\operatorname{Spec} R$ and there exists a point $p \in \operatorname{Spec} R$ that satisfies $P \bmod p = \sigma(\operatorname{Spec} \kappa(p)) \notin \mathcal{Z}$. Since $\kappa(p)$ is finite, $\mathcal{G}(\kappa(p))$ is a finite group, and the kernel of the specialization map $\mathcal{G}(R) \to \mathcal{G}(\kappa(p))$ is a finite index subgroup of $\mathcal{G}(R)$. Let Ω' be the intersection of this kernel with Ω, and note that Ω' is a finite index subgroup of Ω. Consider the finite index coset $C := P\Omega' \subseteq \Omega$.

Then, for every $c \in C$, we have $c \equiv P \bmod p$ and therefore $c \bmod p \notin \mathcal{Z}$. Since $Z = \mathcal{Z} \times_{\mathcal{G}} G$, we obtain $c \notin Z$ and thus $C \cap Z = \emptyset$, as claimed. $\square$

4.1.1 The property (HIT) for many covers

In order to deduce the weak Hilbert property for algebraic groups from the property (HIT), the goal of this subsection is to show that the property (HIT) can be stated in terms of finitely many covers instead of only one. We fix the following datum throughout this subsection:

- a finitely generated field k of characteristic 0,
- a connected algebraic group G over k,
- a connected linear algebraic group H over k,
- an abelian variety A over k,
- an exact sequence
$$0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0$$
of algebraic groups over k,
- a finitely generated dense subgroup $\Omega \subseteq G(k)$ such that $\Omega \cap H(k)$ is dense in H.

The following proposition shows that these assumptions imply the property (HIT) for (G, Ω) and that they are still satisfied if we replace Ω by a finite index subgroup.

Proposition 4.1.6. *The pair (G, Ω) satisfies (HIT). Moreover, if $\Omega' \subseteq \Omega$ is a finite index subgroup, then Ω' is dense in G and $\Omega' \cap H(k)$ is dense in H, i.e., the assumptions of this subsection are still satisfied after replacing Ω by Ω'. In particular, (G, Ω') satisfies (HIT).*

Proof. Let L/k be a finite field extension, let $\tilde{G}$ be a connected algebraic group over L, and let $\tilde{\Omega} \subseteq \tilde{G}(L)$ be a subgroup with $\tilde{\Omega}$ dense in $\tilde{G}$. Assume that there exists an isogeny $\tilde{G} \to H_L$ or an isogeny $\tilde{G} \to A_L$. In particular, $\tilde{G}$ is a connected linear algebraic group or an abelian variety over L. Then, by Proposition 4.1.3, $(\tilde{G}, \tilde{\Omega})$ satisfies (HIT). By [55, Proposition 2.15], this implies that (G, Ω) satisfies (HIT).

Now let $\Omega' \subseteq \Omega$ be a finite index subgroup, so there are finitely many elements $\omega_i \in \Omega$ such that $\Omega = \bigcup_i \omega_i \Omega'$. By the irreducibility of G and the density of Ω in G, this implies that there is an index i with $\omega_i \Omega'$ dense in G. Multiplication with ω_i^{-1} is an automorphism of G, so we conclude that $\Omega' = \omega_i^{-1} \omega_i \Omega'$ is dense in G.

By the same argument applied to H and $\Omega \cap H(k)$, it now suffices to show that $\Omega' \cap H(k)$ is of finite index in $\Omega \cap H(k)$. To see this, consider the composed morphism $\Omega \cap H(k) \to \Omega \to \Omega/\Omega'$. Its kernel is $\Omega' \cap H(k)$, so it induces an injective homomorphism

$$(\Omega \cap H(k))/(\Omega' \cap H(k)) \hookrightarrow \Omega/\Omega'.$$

Since the right-hand side is finite, so is the left-hand side, which proves the claim. □

With the observation from above, the fact that (HIT) may be formulated for many covers follows by an easy induction. The arguments for this are the same as in the proof of [25, Lemma 4.14] on a similar statement for abelian varieties.

Lemma 4.1.7. *Let* $(\pi_i\colon Y_i \to G)_{i=1}^n$ *be a finite family of (PB)-covers. Then there exists a finite index coset* C *of* Ω *such that, for all* $c \in C$ *and all* $i = 1, \dots, n$*, the fiber* $Y_{i,c}$ *is irreducible.*

Proof. We argue by induction on n. If $n = 1$, the claim follows from Lemma 4.1.4. Now assume that the statement holds for $(\pi_i)_{i=1}^{n-1}$, so there exists a finite index coset $C_{n-1} \subseteq \Omega$ such that, for all $c \in C_{n-1}$ and all $i = 1, \dots, n-1$, the fiber $Y_{i,c}$ is irreducible. Choose an element $c_{n-1} \in C_{n-1}$ and a finite index subgroup $\Omega' \subseteq \Omega$ such that $C_{n-1} = c_{n-1}\Omega'$. For every $i = 1, \dots, n$, let $\pi_i'\colon Y_i \to G$ be the composition of π_i and the translation on G by c_{n-1}^{-1}. In particular, for $i = 1, \dots, n-1$ and $c \in \Omega'$, the fiber of π_i' over c is irreducible.

Since (G, Ω') satisfies (HIT) by Proposition 4.1.6, there exists a finite index coset C of Ω' such that, for every $c \in C$, the fiber of π_n' over c is irreducible. It now follows that the finite index coset $c_{n-1}C$ of Ω satisfies the claim. □

4.1.2 From (HIT) to the weak Hilbert property

We keep the notation and assumptions of the previous subsection. To show that Ω is not strongly thin in G, we first consider a single ramified cover of G, combining the arguments of [25, Proposition 4.17] with the Chevalley–Weil type lifting result Theorem 3.1.10. We stress that to do this, it is crucial to work with near-integral points.

The following lemma is an extension of a well-known result for abelian varieties [66, §IV.18] to the case of all algebraic groups.

Lemma 4.1.8. *Let* K *be a field of characteristic* 0*, let* $\tilde{G}$ *be a connected algebraic group over* K*, and let* $\lambda\colon X \to \tilde{G}$ *be a finite étale morphism of integral* K*-varieties with* X *geometrically connected over* K*. Then there is a finite field extension* L/K *such that*

X_L has the structure of a connected algebraic group over L that makes $\lambda_L\colon X_L \to \tilde{G}_L$ an isogeny.

Proof. Since $\tilde{G}$ is connected with a K-rational point, it is geometrically connected over K by [39, Corollaire 4.5.13.1(i)], so X, being finite étale over $\tilde{G}$, is normal (even smooth) and geometrically connected over K. Since $\overline{K}$ is of characteristic 0, the étale fundamental group $\pi_1^{\text{ét}}(\tilde{G}_{\overline{K}})$ is commutative by [11, Proposition 1.1]. In particular, every étale cover of $\tilde{G}_{\overline{K}}$ corresponds to a normal subgroup of $\pi_1^{\text{ét}}(\tilde{G}_{\overline{K}})$ and is thus Galois. Thus, it follows from [11, Proposition 1.1] that $X_{\overline{K}}$ carries the structure of an algebraic group over $\overline{K}$ that makes $\lambda_{\overline{K}}$ an isogeny. The claim now follows by descending the group structure of $\tilde{G}_{\overline{K}}$ and the isogeny $\lambda_{\overline{K}}$ to a finite extension L of K. ☐

The following proposition shows that, given a single ramified cover of G, we find a finite index coset of Ω over which the fibers admit no k-point.

Proposition 4.1.9. *With the notation and assumptions of the previous subsection, let $\pi\colon Y \to G$ be a ramified cover. Then there exists a finite index coset C of Ω such that $Y_c(k) = \emptyset$ for all $c \in C$.*

Proof. Since the claim is trivially true if $Y(k) = \emptyset$, we may and do assume that Y admits a k-point and is thus geometrically connected over k. Moreover, it suffices to prove the claim after replacing k by a finite extension. Thus, by Lemma 4.1.8, up to a finite extension of k, we may and do assume that there exists a factorization $Y \xrightarrow{\mu} G' \xrightarrow{\lambda} G$ of π such that λ is an étale subcover of π of maximal degree and $G' \to G$ is an isogeny of algebraic groups. Note that μ is a (PB)-cover of G' by [55, Lemma 2.8]. By Lemma 4.1.5, up to replacing Ω by a finite index coset and composing π with a translation on G that maps this coset to a subgroup, we may and do assume that Ω is contained in the étale locus of π.

By spreading out, there exists a normal $\mathbb{Z}$-finitely generated integral domain R with fraction field k, a connected linear finite type group scheme $\mathcal{H}$ over R, and a connected finite type group scheme $\mathcal{G}$ over R such that $\mathcal{H}_k = H$ and $\mathcal{G}_k = G$. By [24, Proposition 4.4], there exists a finitely generated subgroup $\tilde{\Omega} \subseteq \Omega \cap H(k)$ that is dense in H. Enlarging R if necessary, we may and do assume that the finitely many generators of Ω and $\tilde{\Omega}$ are contained in the groups $\mathcal{G}(R)$ and $\mathcal{H}(R)$, respectively. From this we obtain $\Omega \subseteq \mathcal{G}(R)$ and $\tilde{\Omega} \subseteq \mathcal{H}(R)$. By [55, Lemma 2.13], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H' \arrow[r] \arrow[d, "h"] & G' \arrow[r] \arrow[d, "\lambda"] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & G \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

of connected algebraic groups over k such that every row is exact and every column is an isogeny. Let L/k be a finite extension such that the kernel $\Gamma := \ker(\lambda_L)$ of $\lambda_L \colon G'_L \to G_L$ consists only of L-rational points. Enlarging L further if necessary, by Corollary 3.1.11, we may and do assume that $\mathcal{G}(R) \subseteq \lambda(G'(L))$ and $\mathcal{H}(R) \subseteq h(H'(L))$.

Since a finite base change of a (PB)-cover is (PB), the cover $\mu_L \colon Y_L \to G'_L$ is (PB). For every $\sigma \in \Gamma$, let $\mu_\sigma = \sigma \circ \mu_L \colon Y_L \to G'_L$ denote the (PB)-cover given as the composition of μ_L and translation on G'_L by σ.

Define $\Omega' := \lambda^{-1}(\Omega) \cap G'(L)$. By our choice of L, the isogeny λ_L induces a surjective homomorphism $\Omega' \to \Omega$ with finite kernel. Thus, Ω' is an extension of finitely generated groups and therefore finitely generated. Moreover, $\Omega' \cap H'(L)$ is dense in H', as its image in H_L contains $\tilde{\Omega}$ by the choice of L. Thus, by Lemma 4.1.7, there exists a finite index coset C' of Ω' such that, for every $c' \in C'$ and every $\sigma \in \Gamma$, the fiber $\mu_\sigma^{-1}(c')$ is irreducible. Let $C = \lambda(C')$, which by the surjectivity of $\Omega' \to \Omega$ is a finite index coset of Ω.

We claim that C satisfies the claim of the proposition. Let $c \in C$; we have to show that $Y_c(k) = \emptyset$. If there is no $x \in G'(k)$ with $\lambda(x) = c$, then there is nothing to show, so we assume that there exists an $x \in G'(k)$ such that $\lambda(x) = c$. To prove the claim, it suffices to show that the fiber $Y_x = \mu^{-1}(x)$ has no k-rational point. By the definition of C, there exists a point $c' \in G'_c(L)$ such that $c' \in C'$. Since Γ acts transitively on the fibers of $G'_L \to G_L$, there exists an element $\sigma \in \Gamma$ such that $\sigma x = c'$ (as L-points of G'_L). In particular, we have $\mu_\sigma^{-1}(c') = Y_{x,L}$. Since $c' \in C'$, we have that $Y_{x,L}$ is irreducible, and thus Y_x is irreducible. By our assumption that π is étale over every element of Ω and Y_x is irreducible, Y_x is the spectrum of a field extension of $\kappa(x) = k$ of degree $\deg \mu$. Since π is ramified and λ is étale, we have $\deg \mu \geq 2$. This shows that $Y_x(k) = \emptyset$ for every $x \in G'_c(k)$ and thus $Y_c(k) = \emptyset$, as claimed. □

To conclude that Ω is not strongly thin, we have to go from one ramified cover to many; this step is the same induction as in Lemma 4.1.7.

Proposition 4.1.10. *Let $(\pi_i \colon Y_i \to G)_{i=1}^n$ be a finite family of ramified covers. Then there exists a finite index coset C of Ω such that, for all $c \in C$ and all $i = 1, \ldots, n$, we have $Y_{i,c}(k) = \emptyset$. In particular, Ω is not strongly thin in G.*

Proof. We repeat the arguments from Lemma 4.1.7 to prove the claim by induction, with the case $n = 1$ being true by Proposition 4.1.9.

Assume that the statement holds for $(\pi_i)_{i=1}^{n-1}$, i.e., there exists a finite index coset $C_{n-1} \subseteq \Omega$ such that, for all $c \in C_{n-1}$ and all $i = 1, \ldots, n-1$, we have $Y_{i,c}(k) = \emptyset$. Let $c_{n-1} \in C_{n-1}$ be an element and $\Omega' \subseteq \Omega$ a finite index subgroup such that $C_{n-1} = c_{n-1}\Omega'$. For every $i = 1, \ldots, n$, let $\pi'_i \colon Y_i \to G$ be the composition of π_i and the translation on

G by c_{n-1}^{-1}. Then, for every $i = 1, \dots, n-1$ and every $c \in \Omega'$, the fiber of π_i' over c admits no k-point.

By Proposition 4.1.6, the pair (G, Ω') satisfies (HIT), so by Proposition 4.1.9, there exists a finite index coset C of Ω' such that, for every $c \in C$, the fiber of π_n' over c has no k-point. Hence, the finite index coset $c_{n-1}C$ of Ω satisfies the claim. □

4.1.3 Proof of the weak Hilbert property

In this subsection, we conclude the proof of the weak Hilbert property for algebraic groups. We now drop the notation fixed in the previous subsections. We first show that we may always reduce to the case of finitely generated subgroups.

Proposition 4.1.11. *Let k be a finitely generated field of characteristic 0 and let*

$$0 \longrightarrow H \longrightarrow G \xrightarrow{p} A \longrightarrow 0$$

be an exact sequence of connected algebraic groups over k, where H is linear and A is an abelian variety. Let $\Omega \subseteq G(k)$ be a subgroup that is dense in G and such that $\Omega \cap H(k)$ is dense in H. Then there exists a finitely generated subgroup Ω' of Ω such that Ω' is dense in G and $\Omega' \cap H(k)$ is dense in H.

Proof. By [24, Proposition 4.4], there is a finitely generated subgroup Ω_H of $\Omega \cap H(k)$ that is dense in H, say $\Omega_H = \langle h_1, \dots, h_m \rangle$. Moreover, since $A(k)$ is finitely generated by the Mordell–Weil Theorem and commutative, its subgroup $p(\Omega)$ is finitely generated. Let $a_1, \dots, a_n \in \Omega$ be elements such that $p(\Omega)$ is generated by $p(a_1), \dots, p(a_n)$, and define $\Omega' = \langle h_1, \dots, h_m, a_1, \dots, a_n \rangle \subseteq \Omega$.

We first claim that $\Omega = \langle a_1, \dots, a_n \rangle (\Omega \cap H(k))$. Since the the inclusion from right to left is obvious, we have to prove the inclusion from left to right. For any $x \in \Omega$, we find $r_1, \dots, r_n \in \mathbb{Z}$ such that $p(x) = p(a_1)^{r_1} \cdots p(a_n)^{r_n}$. Define $y = a_1^{r_1} \cdots a_n^{r_n} \in \Omega$ and note that $y^{-1}x \in \Omega \cap \ker(p) = \Omega \cap H(k)$, which implies that $x \in y(\Omega \cap H(k))$. Since $y \in \langle a_1, \dots, a_n \rangle$, this proves the claim.

We now show that the Zariski closure of Ω' in G contains Ω, which by the density of Ω in G concludes the proof. Given a subset $V \subseteq G$, let $\mathrm{cl}\,(V)$ denote the (Zariski) closure of V in G. Then the inclusion

$$\mathrm{cl}\,(\Omega') \supseteq \mathrm{cl}\left(\bigcup_{a \in \langle a_1, \dots, a_n \rangle} a\Omega_H \right) \supseteq \bigcup_{a \in \langle a_1, \dots, a_n \rangle} \mathrm{cl}\,(a\Omega_H)$$

of subsets of G holds. Since multiplication by an element $a \in G$ on G is a homeomorphism, we have

$$\mathrm{cl}\,(a\Omega_H) = a\mathrm{cl}\,(\Omega_H) \supseteq aH \supseteq a(\Omega \cap H(k)).$$

This implies that

$$\mathrm{cl}\,(\Omega') \supseteq \bigcup_{a \in \langle a_1, \ldots, a_n \rangle} a(\Omega \cap H(k)) = \langle a_1, \ldots, a_n \rangle (\Omega \cap H(k)) = \Omega,$$

as required. □

We are now ready to prove Theorem C.

Theorem 4.1.12 (Theorem C)**.** *Let R be a normal $\mathbb{Z}$-finitely generated integral domain with fraction field k of characteristic* 0 *and let $\mathcal{G}$ be a connected finite type group scheme over R. If $\mathcal{G}(k)$ is dense in $\mathcal{G}$, then there exists a dense open* $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ *such that $\mathcal{G}(R')$ is not strongly thin in $\mathcal{G}_k$. In particular, $\mathcal{G}_{R'}$ has the weak Hilbert property over R'.*

Proof. Let k be the fraction field of R and $G = \mathcal{G}_k$. Note that, by [79, Tag 047N], G is smooth over k. Therefore, by [22, Theorem 1.1], there exists an exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0$$

of connected algebraic groups over k with H linear and A an abelian variety. By [8, Corollary 18.3], $H(k)$ is dense in H. Thus, by Proposition 4.1.11, there exists a finitely generated dense subgroup Ω of $G(k)$ such that $\Omega \cap H(k)$ is dense in H. Let $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ be a dense open such that the generators of Ω are defined over R', i.e., $\Omega \subseteq \mathcal{G}_{R'}(R')$. Since the assumptions of the previous subsections are satisfied for G, H, A, Ω, it follows from Proposition 4.1.10 that Ω is not strongly thin in G. In particular, $\mathcal{G}_{R'}(R')$ is not strongly thin in G, so $\mathcal{G}_{R'}$ satisfies the weak Hilbert property over R'. □

Theorem C and standard spreading out arguments directly imply Theorem A:

Theorem 4.1.13 (Theorem A)**.** *Let k be a finitely generated field of characteristic* 0 *and let G be a connected algebraic group over k. If $G(k)$ is dense, then G satisfies the weak Hilbert property.*

Proof. By spreading out the scheme G and its group structure (see [72, Theorem 3.2.1]), there exists a normal $\mathbb{Z}$-finitely generated integral domain R with fraction field k and a connected finite type group scheme $\mathcal{G}$ over R such that $G = \mathcal{G}_k$. By Theorem C, there exists a dense open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathcal{G}(R')$ is not strongly thin in G.

Thus, $G(k) \supseteq \mathcal{G}(R')$ is not strongly thin, i.e., G satisfies the weak Hilbert property over k. $\square$

4.2 Punctured linear algebraic groups

Throughout this section, we let K denote a number field, S a finite set of places of K, and $R = \mathcal{O}_{K,S}$ the ring of S-integers of K. The goal of this section is to verify Conjecture 1.3.5 (i.e., prove part (5) of it) for all linear algebraic groups. Note that R is 1-dimensional, so the set $\mathcal{X}(R)^{(1)}$ of near-R-integral points on an arithmetic scheme $\mathcal{X}$ over R equals its set $\mathcal{X}(R)$ of R-integral points (as subsets of $\mathcal{X}(K)$).

4.2.1 One point suffices

In order to to verify part (5) of Conjecture 1.3.5 for certain varieties, we investigate arithmetic schemes for which the existence of one integral point in the complement of a closed subscheme implies the non-strongly-thinness of such points. We will say that such arithmetic schemes satisfy the property "(OPS)", for "one point suffices". If this property is obtained after an enlargement of S (and preserved under further enlargements), we refer to it as "(OPSE)", for "one point suffices after extension (of the base ring)".

Definition 4.2.1. Let $\mathcal{X}$ be an arithmetic scheme over R and let $c \geq 1$ be an integer. We say that $\mathcal{X}$ satisfies *(OPS) for codimension at least c (over R)* if $\mathcal{X}(R) \neq \emptyset$ and, for every closed subscheme $\mathcal{Z} \subseteq \mathcal{X}$ such that $\mathcal{Z}_K \subseteq \mathcal{X}_K$ is of codimension at least c and $(\mathcal{X} \setminus \mathcal{Z})(R) \neq \emptyset$, the set $(\mathcal{X} \setminus \mathcal{Z})(R)$ is not strongly thin in $\mathcal{X}_K$. We say that $\mathcal{X}$ satisfies *(OPSE) for codimension at least c* if there exists a finite set of places S' of K containing S such that, for every finite set of places S'' of K containing S', the scheme $\mathcal{X}_{\mathcal{O}_{K,S''}}$ satisfies (OPS) for codimension at least c.

In the definition of (OPS) we require $\mathcal{X}(R) \neq \emptyset$ to avoid tautologies for schemes without integral points. Note that, if $(\mathcal{X} \setminus \mathcal{Z})(R)$ is not strongly thin in $\mathcal{X}_K$, then it is also not strongly thin in $\mathcal{X}_K \setminus \mathcal{Z}_K$. (This can be seen by normalizing a given ramified cover of $\mathcal{X}_K \setminus \mathcal{Z}_K$ in $\mathcal{X}_K$, which produces a ramified cover of $\mathcal{X}_K$.) The property (OPS) as defined above is not necessarily stable under enlargements of S (though it will be for all schemes that we consider in this thesis), which is why in the definition of (OPSE) we require (OPS) to hold for all enlargements S'' of S'. An arithmetic scheme satisfying (OPS) or (OPSE) (for any codimension) has a dense set of rational points, since the empty subscheme is of codimension ∞.

Given a closed subscheme $\mathcal{Z} \subseteq \mathcal{X}$ and a finite set of places S' of K containing S, we have

$$(\mathcal{X}_{\mathcal{O}_{K,S'}} \setminus \mathcal{Z}_{\mathcal{O}_{K,S'}})(\mathcal{O}_{K,S'}) = (\mathcal{X} \setminus \mathcal{Z})_{\mathcal{O}_{K,S'}}(\mathcal{O}_{K,S'}) = (\mathcal{X} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$$

as subsets of $\mathcal{X}_K(K)$; we make this identification tacitly when using the property (OPS) for $\mathcal{X}_{\mathcal{O}_{K,S'}}$.

In this subsection, we investigate integral points on products whose factors satisfy (OPSE). The main tool to do this is our strong form of the product theorem for the weak Hilbert property, Theorem E.

The following result shows that a punctured product of schemes satisfying (OPSE) admits a non-strongly-thin set of integral points, up to enlarging the base ring.

Theorem 4.2.2. *Let $N \geq 1$ be an integer, let $\mathcal{X} = \mathcal{X}_1 \times_R \ldots \times_R \mathcal{X}_N$ with each $\mathcal{X}_i$ a normal quasi-projective arithmetic scheme over R satisfying (OPSE) for codimension at least c_i, and let $c = \max_{i=1,\ldots,N} c_i$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subscheme such that $\mathcal{Z}_K \subseteq \mathcal{X}_K$ is of codimension at least c. Then there exists a finite set of places S' of K containing S such that $(\mathcal{X} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$ is not strongly thin in $\mathcal{X}_K$.*

Proof. Enlarging S if necessary, we may assume that each $\mathcal{X}_{i,\mathcal{O}_{K,S'}}$ satisfies (OPS) for codimension at least c_i for every finite set of places S' containing S. We argue by induction on N. Assume that $N = 1$. By the property (OPS), $\mathcal{X}(K)$ is dense, so $\mathcal{X} \setminus \mathcal{Z}$ admits a K-point p. Let $S' \supseteq S$ be a finite set of places such that $p \in (\mathcal{X} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$. Since $\mathcal{X}_{\mathcal{O}_{K,S'}}$ satisfies (OPS) for codimension at least c and $(\mathcal{X} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$ is non-empty, it follows that $(\mathcal{X} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$ is not strongly thin in $\mathcal{X}_K$, as required.

Now assume that $N > 1$ and define $\mathcal{X}' = \mathcal{X}_1 \times_R \ldots \times_R \mathcal{X}_{N-1}$. Let $\pi' \colon \mathcal{X} \to \mathcal{X}'$ be the projection onto $\mathcal{X}'$ and let $\pi_N \colon \mathcal{X} \to \mathcal{X}_N$ be the projection onto the last factor. Let π'_K and $\pi_{N,K}$ denote the base change of π' and π_N along $\operatorname{Spec} K \to \operatorname{Spec} R$, respectively, and note that π'_K and $\pi_{N,K}$ are faithfully flat.

Since the $\mathcal{X}_{i,K}$ are integral and have a dense set of K-rational points by the property (OPS), they are geometrically integral. By [15, Proposition 3.5], the codimension condition spreads out; more precisely, there exist dense opens $U' \subseteq \mathcal{X}'_K$ and $U_N \subseteq \mathcal{X}_{N,K}$ such that, for every $x' \in U'(K)$ and for every $x_N \in U_N(K)$, the codimension of $(\pi'_K)^{-1}(x') \cap \mathcal{Z}_K$ in $(\pi'_K)^{-1}(x')$ and the codimension of $(\pi_{N,K})^{-1}(x_N) \cap \mathcal{Z}_K$ in $(\pi_{N,K})^{-1}(x_N)$ are at least c.

Since $\mathcal{X}(K)$ is dense in $\mathcal{X}$, there exists a K-point $p = (p', p_N) \in (\mathcal{X} \setminus \mathcal{Z})(K)$ with $p' \in U'(K)$ and $p_N \in U_N(K)$. Let $S' \supseteq S$ be a finite set of places of K such that $p \in (\mathcal{X} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$ and define $R' = \mathcal{O}_{K,S'}$. Let $\pi_{N,R'}$ and $\pi'_{R'}$ be the pullback of π_N and π' along $\operatorname{Spec} R' \to \operatorname{Spec} R$, respectively, identify $\pi_{N,R'}^{-1}(p_N)$ with $\mathcal{X}'_{R'}$, and write $\mathcal{Z}' := \mathcal{Z}_{R'} \cap \pi_{N,R'}^{-1}(p_N)$. By the choice of p_N, the codimension of $\mathcal{Z}'_K$ in $\mathcal{X}'_K$ is at least c, so by the induction hypothesis, there exists a finite set of places S'' of K containing S' such that $(\mathcal{X}' \setminus \mathcal{Z}')(\mathcal{O}_{K,S''})$ is not strongly thin in $\mathcal{X}'_K$. Define $R'' = \mathcal{O}_{K,S''}$ and

$$\Gamma := (\mathcal{X}'_{R'} \setminus \mathcal{Z}')(R'') \cap U'(K),$$

which is (also) not strongly thin in $\mathcal{X}'_K$ since $U' \subseteq \mathcal{X}'_K$ is a dense open. Let $s \in \Gamma$ and let $F_s := (\pi'_{R''})^{-1}(s) \cong \mathcal{X}_{N,R''}$. Let $T_s = \mathcal{Z}_{R''} \cap F_s$ be the intersection of F_s and $\mathcal{Z}_{R''}$ in $\mathcal{X}_{R''}$.

Note that $F_{s,K} \cap \mathcal{Z}_K$ has codimension at least c in $F_{s,K}$ by the choice of U', and that F_s satisfies (OPS) for codimension at least c_N (and thus for codimension at least c). Thus, since $(s, p_N) \in (F_s \setminus T_s)(R'')$, we obtain that $(F_s \setminus T_s)(R'')$ is not strongly thin in F_s. Define

$$\Sigma = \bigcup_{s\in\Gamma} (F_s \setminus T_s)(R'') \subseteq (\mathcal{X} \setminus \mathcal{Z})(R'').$$

Since $\Gamma = \pi'(\Sigma)$ is not strongly thin in $\mathcal{X}'_K$ and, for every element $s \in \Gamma$, the set $\Sigma \cap (\{s\} \times \mathcal{X}_{N,K}) = (F_s \setminus T_s)(R'')$ is not strongly thin in $F_{s,K} = \{s\} \times \mathcal{X}_{N,K}$, it follows from Theorem E that Σ (and thus $(\mathcal{X} \setminus \mathcal{Z})(R'')$) is not strongly thin in $\mathcal{X}_K$, as required. □

One might ask whether Theorem 4.2.2 holds with $R'' = R$ under the additional assumption that $(\mathcal{X} \setminus \mathcal{Z})(R)$ is dense in $\mathcal{X}$. Indeed, repeating the proof in this case, we still find a point $p = (p', p_N) \in (\mathcal{X} \setminus \mathcal{Z})(R)$ such that $p' \in U'(K)$ and $p_N \in U_N(K)$. While the fiber $\mathcal{X}' \setminus \mathcal{Z}'$ does admit an R-point, it is not clear why its R-points should be dense without enlarging R, so one would not be able to conclude that they are not strongly thin. However, this implication of course does hold if $\mathcal{X}' = \mathcal{X}_1 \times_R \ldots \times_R X_{N-1}$ satisfies (OPS) for codimension at least c. Since we are not able to show that a product of schemes with (OPS) satisfies (OPS), the same arguments in the proof of Theorem 4.2.2 only show the following statement (in which we only consider products of *two* schemes):

Theorem 4.2.3. *Let $c \geq 1$ be an integer, let $\mathcal{X}_1$ and $\mathcal{X}_2$ be normal quasi-projective arithmetic schemes over R satisfying (OPS) for codimension at least c, denote their product by $\mathcal{X} = \mathcal{X}_1 \times_R \mathcal{X}_2$, and let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subscheme such that $\mathcal{Z}_K \subseteq \mathcal{X}_K$ is of codimension at least c. If $(\mathcal{X} \setminus \mathcal{Z})(R)$ is dense, then it is not strongly thin in $\mathcal{X}_K$.*

Proof. We adapt the arguments of the proof of Theorem 4.2.2. Let π_1 and π_2 be the projection from $\mathcal{X}$ to $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, and let $\pi_{1,K}$ and $\pi_{2,K}$ denote their base change along $\operatorname{Spec} K \to \operatorname{Spec} R$, respectively.

Let $i \in \{1, 2\}$ and note that $\pi_{i,K}$ is faithfully flat. Moreover, $\mathcal{X}_{i,K}$ is integral with a dense set of K-rational points by the property (OPS), so it is geometrically integral.

This shows that $\mathcal{X}_K$ is also geometrically integral. Therefore, by [15, Proposition 3.5], there exists a dense open $U_i \subseteq \mathcal{X}_{i,K}$ such that, for every $x \in U_i(K)$, the codimension of $(\pi_{i,K})^{-1}(x) \cap \mathcal{Z}_K$ in $(\pi_{i,K})^{-1}(x)$ is at least c.

By the density of $(\mathcal{X} \setminus \mathcal{Z})(R)$, there exists a point $p = (p_1, p_2) \in (\mathcal{X} \setminus \mathcal{Z})(R)$ with $p_1 \in U_1(K)$ and $p_2 \in U_2(K)$. Identify $\pi_2^{-1}(p_2)$ with $\mathcal{X}_1$ and write $\mathcal{Z}_1 := \mathcal{Z} \cap \pi_2^{-1}(p_2)$. By the choice of p_2, the codimension of $\mathcal{Z}_{1,K}$ in $\mathcal{X}_{1,K}$ is at least c. Thus, since $\mathcal{X}_1$ satisfies (OPS) for codimension at least c and $p_1 \in (\mathcal{X}_1 \setminus \mathcal{Z}_1)(R)$, we obtain that $(\mathcal{X}_1 \setminus \mathcal{Z}_1)(R)$ is not strongly thin in $\mathcal{X}_{1,K}$.

Define

$$\Gamma := (\mathcal{X}_1 \setminus \mathcal{Z}_1)(R) \cap U_1(K),$$

which is (also) not strongly thin in $\mathcal{X}_{1,K}$ since $U_1 \subseteq \mathcal{X}_{1,K}$ is a dense open. Let $s \in \Gamma$ and let $F_s := (\pi_1)^{-1}(s) \cong \mathcal{X}_2$. Let $T_s = \mathcal{Z} \cap F_s$ be the intersection of F_s and $\mathcal{Z}$ in $\mathcal{X}$. Note that $F_{s,K} \cap \mathcal{Z}_K$ has codimension at least c in $F_{s,K}$ by the choice of U_1, and that F_s satisfies (OPS) for codimension at least c. Thus, since $(s, p_2) \in (F_s \setminus T_s)(R)$, we obtain that $(F_s \setminus T_s)(R)$ is not strongly thin in F_s. Define

$$\Sigma = \bigcup_{s \in \Gamma} (F_s \setminus T_s)(R) \subseteq (\mathcal{X} \setminus \mathcal{Z})(R).$$

Since $\Gamma = \pi_1(\Sigma)$ is not strongly thin in $\mathcal{X}'_K$ and, for every element $s \in \Gamma$, the set $\Sigma \cap (\{s\} \times \mathcal{X}_{2,K}) = (F_s \setminus T_s)(R)$ is not strongly thin in $F_{s,K} = \{s\} \times \mathcal{X}_{2,K}$, it follows from Theorem E that Σ (and thus $(\mathcal{X} \setminus \mathcal{Z})(R)$) is not strongly thin in $\mathcal{X}_K$, as required. □

4.2.2 (OPS) for commutative group schemes

Recall that $R = \mathcal{O}_{K,S}$. If $\mathcal{X}$ is a finite type R-scheme and $\mathcal{Z} \subseteq \mathcal{X}$ is a closed subscheme, we say that $\mathcal{Z}$ is *vertical* if its image in $\operatorname{Spec} R$ is a finite set of closed points (i.e., $\mathcal{Z}_K = \emptyset$, as follows, for example, from [79, Tag 055H]).

In [44, Proposition 4.1], Hassett and Tschinkel use an approximation argument to prove potential density of integral points on punctured tori (which they attribute to McKinnon). This argument shows the following.

Lemma 4.2.4. *Let $\mathcal{G}$ be a finite type group scheme over R. Let $\mathcal{Z}$ be a vertical closed subscheme of $\mathcal{G}$. For every $p \in (\mathcal{G} \setminus \mathcal{Z})(R)$, there is an integer $m \geq 1$ such that $p \cdot \mathcal{G}(R)^m \subseteq (\mathcal{G} \setminus \mathcal{Z})(R)$.*

Proof. Since $\mathcal{Z}$ is vertical, its image in $\operatorname{Spec} R$ is finite. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the primes lying under $\mathcal{Z}$. Let m_i be the order of the (finite) group $\mathcal{G}(k(\mathfrak{p}_i))$ and define m as the least common multiple of the m_i. Note that, for every g in $\mathcal{G}(R)^m \subseteq \mathcal{G}(R)$, the element $p \cdot g$ lies in $(\mathcal{G} \setminus \mathcal{Z})(R)$. Indeed, for every $i = 1, \dots, n$, the image of $p \cdot g$ in $\mathcal{G}(k(\mathfrak{p}_i))$ equals the image of p (as g reduces to the identity element). This concludes the proof. $\square$

Remark 4.2.5. Let $\mathcal{G}$ be a connected finite type group scheme over R and let $\mathcal{Z}$ be a vertical closed subscheme of $\mathcal{G}$. If $\mathcal{G}(R)$ is dense and $(\mathcal{G} \setminus \mathcal{Z})(R)$ is non-empty, then $(\mathcal{G} \setminus \mathcal{Z})(R)$ is dense. Indeed, let $p \in (\mathcal{G} \setminus \mathcal{Z})(R)$ and choose $m \geq 1$ such that $p \cdot \mathcal{G}(R)^m \subseteq (\mathcal{G} \setminus \mathcal{Z})(R)$. Let $[m]\colon \mathcal{G}_K \to \mathcal{G}_K$ denote the m-power map. By [79, Tag 0BF5], the tangent map induced by $[m]$ is the multiplication by m and thus surjective. Since $\mathcal{G}_K$ is smooth over K [79, Tag 047N], it follows from [2, Lemma 3.3] that $[m]$ is dominant. Thus, by the density of $\mathcal{G}(R)$, its image $\mathcal{G}(R)^m$ under $[m]$ is dense, so $(\mathcal{G} \setminus \mathcal{Z})(R)$ contains the dense subset $p \cdot \mathcal{G}(R)^m$ and is thus dense.

Combining the approximation argument in Lemma 4.2.4 with the strong form of Hilbert irreducibility given in Proposition 4.1.10, we can strengthen the density conclusion of the above remark to a non-strongly-thinness conclusion.

Proposition 4.2.6. *Let $\mathcal{G}$ be a connected commutative finite type group scheme over R and let*

$$0 \longrightarrow H \longrightarrow \mathcal{G}_K \longrightarrow A \longrightarrow 0$$

be an exact sequence of connected algebraic groups over K with H linear and A an abelian variety. Assume that $\mathcal{G}(R)$ is dense in $\mathcal{G}_K$ and that $\mathcal{G}(R) \cap H(K)$ is dense in H. Then $\mathcal{G}$ satisfies (OPS) for codimension at least $\dim \mathcal{G}_K + 1$.

Proof. Let $\mathcal{Z} \subseteq \mathcal{G}$ be a closed subscheme with $(\mathcal{G} \setminus \mathcal{Z})(R) \neq \emptyset$ and such that $\mathcal{Z}_K$ is of codimension at least $\dim \mathcal{G}_K + 1$ in $\mathcal{G}_K$, i.e. $\mathcal{Z}_K = \emptyset$, so $\mathcal{Z}$ is a vertical closed subscheme of $\mathcal{G}$. Let $p \in (\mathcal{G} \setminus \mathcal{Z})(R)$. By Lemma 4.2.4, we may choose $m \geq 1$ such that $p \cdot \mathcal{G}(R)^m \subseteq (\mathcal{G} \setminus \mathcal{Z})(R)$. To prove the claim, it suffices to show that $\Omega := \mathcal{G}(R)^m$ is not strongly thin is $\mathcal{G}_K$ (as is evident by composing ramified covers of $\mathcal{G}_K$ with translation by p). Note that Ω is dense in $\mathcal{G}_K$ (cf. Remark 4.2.5) and, similarly, $\Omega \cap H(K) \supseteq (\mathcal{G}(R) \cap H(K))^m$ is dense in H. Moreover, Ω is a subgroup of $\mathcal{G}(K)$ by the commutativity of $\mathcal{G}$. Thus, by Proposition 4.1.11, there exists a finitely generated subgroup Ω' of Ω such that Ω' is dense in $\mathcal{G}$ and $\Omega' \cap H(K)$ is dense in H. By Proposition 4.1.10, this implies that Ω' (and thus Ω) is not strongly thin in $\mathcal{G}_K$. $\square$

We stress that $\mathcal{G}$ satisfying (OPS) for codimension at least $\dim \mathcal{G}_K + 1$ is not an empty statement, as the codimension of the closed subscheme $\mathcal{Z} \subseteq \mathcal{G}$ that is removed is taken in the generic fiber, so $\mathcal{Z}$ is not necessarily empty, as it could be a vertical subscheme of $\mathcal{G}$.

Corollary 4.2.7. *Let $\mathcal{G}$ be either an affine connected commutative finite type group scheme over R or an abelian scheme over R. If $\mathcal{G}(R)$ is dense, then $\mathcal{G}$ satisfies (OPS) for codimension at least* $\dim \mathcal{G}_K + 1$.

Proof. Apply Proposition 4.2.6 to the obvious exact sequence in which $H = 0$ (if $\mathcal{G}$ is an abelian scheme) or $A = 0$ (if $\mathcal{G}$ is affine). (Note that an affine algebraic group over K is linear [8, Proposition 1.10].) ☐

Corollary 4.2.8. *Let $\mathcal{G}$ be a connected commutative finite type group scheme over R with $\mathcal{G}(K)$ dense. Then $\mathcal{G}$ satisfies (OPSE) for codimension at least* $\dim \mathcal{G}_K + 1$.

Proof. Note that $\mathcal{G}_K$ is smooth over K by [79, Tag 047N]. Thus, by [22, Theorem 1.1], there exists an exact sequence

$$0 \longrightarrow H \longrightarrow \mathcal{G}_K \longrightarrow A \longrightarrow 0$$

of connected algebraic groups over K with H linear and A an abelian variety. By assumption, $\Omega := \mathcal{G}(K)$ is dense in $\mathcal{G}_K$, and by [8, Corollary 18.3], $H(K) = \Omega \cap H(K)$ is dense in H. Thus, by Proposition 4.1.11, there exists a finitely generated dense subgroup $\Omega' \subseteq \Omega$ such that $\Omega' \cap H(K)$ is dense in H. Let $R' = \mathcal{O}_{K,S'}$ for S' a finite set of places of K containing S such that the generators of Ω' lie in $\mathcal{G}(R')$, i.e., $\Omega' \subseteq \mathcal{G}(R')$. Let S'' be any finite set of places of K containing S' and define $R'' = \mathcal{O}_{K,S''}$. Since $\mathcal{G}(R'') \supseteq \mathcal{G}(R')$, we see that $\mathcal{G}(R'') \supseteq \mathcal{G}(R')$ is dense in $\mathcal{G}_K$ and $\mathcal{G}(R'') \cap H(K)$ is dense in H, so $\mathcal{G}_{R''}$ satisfies (OPS) for codimension at least $\dim \mathcal{G}_K + 1$ by Proposition 4.2.6. ☐

The following example uses the above corollaries to highlight the difference between Theorem 4.2.2 and Theorem 4.2.3.

Example 4.2.9. Let $\mathcal{G} = \mathbb{G}_{\mathrm{m},R}^4$ and let $\mathcal{Z} \subseteq \mathcal{G}$ be a closed subscheme.

1. By Corollary 4.2.8, each factor $\mathbb{G}_{\mathrm{m},R}$ satisfies (OPSE) for codimension at least 2. Thus, if $\mathcal{Z}_K \subseteq \mathcal{G}_K$ is of codimension at least 2, then by Theorem 4.2.2, there exists a finite set of places S' containing S such that $(\mathcal{G} \setminus \mathcal{Z})(\mathcal{O}_{K,S'})$ is not strongly thin in $\mathcal{G}_K$.

2. Assume that $(\mathcal{G} \setminus \mathcal{Z})(R)$ is dense and that $\mathcal{Z}_K \subseteq \mathcal{G}_K$ is of codimension at least 3. Write $\mathcal{G} \cong \mathbb{G}_{\mathrm{m},R}^2 \times_R \mathbb{G}_{\mathrm{m},R}^2$. Then $\mathbb{G}_{\mathrm{m},R}^2(R)$ is dense in $\mathbb{G}_{\mathrm{m},K}^2$, so both factors satisfy (OPS) for codimension at least 3 by Corollary 4.2.7. Thus, $(\mathcal{G} \setminus \mathcal{Z})(R)$ is already not strongly thin in $\mathcal{G}_K$ by Theorem 4.2.3.

4.2.3 Background on weak and strong approximation

Among the many arithmetic properties studied in relation to rational points are weak and strong approximation, which, roughly speaking, concern whether global rational points can be closely approximated by local solutions at various places of a number field. These approximation properties have been shown to imply the Hilbert property in certain settings, first by Ekedahl [32] for rational points on varieties, and they play a central role in the results developed in the remainder of this section. In this subsection, we give a brief overview about weak and strong approximation to provide the necessary background. While only strong approximation is directly relevant to the results of this thesis, we also include a short discussion of weak approximation for completeness. Our main references for this are [20, 74].

Recall that K is a number field and S is a finite set of places of K (though the following definitions also extend to all global fields). We let Ω denote the set of places of K and $\Omega' \subseteq \Omega$ the subset of finite (i.e., non-archimedian) places.

If v is a place of K, then K_v carries its natural v-adic topology. Moreover, if X is a variety over K, then $X(K_v)$ is naturally equipped with a v-adic topology and we have a map $X(K) \to X(K_v)$ induced by the inclusion $K \hookrightarrow K_v$.

Definition 4.2.10 (see [74, §3.5])**.** Let X be a variety over K and let $S' \subseteq \Omega$ be a finite set of places. We say that X satisfies *weak approximation for* S' if the image of the diagonal map $X(K) \to \prod_{v \in S'} X(K_v)$ is dense.

Weak approximation for S' means that, for any tuple $(\alpha_v)_{v \in S'} \in \prod_{v \in S'} X(K_v)$, all points α_v can be approximated simultaneously by a series of K-rational points in the respective v-adic topology. We say that X satisfies weak approximation if this holds for every finite set of places S':

Definition 4.2.11 (see [20, Definition 12.2.2], [74, Definition 3.5.6])**.** A variety X over K satisfies *weak approximation* if X satisfies weak approximation for S' for every finite set of places $S' \subseteq \Omega$.

A weaker version of weak approximation is weak weak approximation, where points may only be approximated in the v-adic topology for v outside of a given finite set of places:

Definition 4.2.12 (see [20, Definition 12.2.4], [74, Definition 3.5.6])**.** A variety X over K satisfies *weak weak approximation* if there exists a finite set of places $T \subseteq \Omega$ such that X satisfies weak approximation for S' for every finite set of places $S' \subseteq \Omega \backslash T$.

The weak approximation property is known for rational varieties and weak weak approximation is known, for example, for tori [74, §3.5].

By a theorem of Ekedahl [32], an integral variety satisfying weak weak approximation satisfies the Hilbert property. A conjecture due to Colliot-Thélène (see [74, Conjecture 3.5.8]) predicts that weak weak approximation holds for every smooth unirational variety:

Conjecture 4.2.13 (Colliot-Thélène). *Let X be a smooth unirational variety over K. Then X satisfies weak weak approximation.*

In particular, by Ekedahl's theorem, a proof of Conjecture 4.2.13 would imply that every smooth unirational variety over $\mathbb{Q}$ (even every unirational variety since unirationality is birational) has the Hilbert property and thus provide a positive answer to the Inverse Galois Problem (see Theorem 2.2.7).

Weak approximation tells us that $X(K)$ can approximate local points at any finite set of places. Working with arithmetic schemes over $\mathcal{O}_{K,S}$, we are not only interested in prescribing behavior in finitely many places, but also requiring good reduction at all the other (finite) places simultaneously. The natural way to encode this global information is through the space of *adelic points*.

In all of the following definitions of this subsection, we assume that S contains all archimedian places of K.

Definition 4.2.14. The ring of *adeles* of K is the subring $\mathbb{A}_K \subseteq \prod_{v\in\Omega} K_v$ consisting of the elements of the form $(\lambda_v)_{v\in\Omega}$ such that $\lambda_v \in \mathcal{O}_{K,v}$ for all except finitely many $v \in \Omega'$. The ring of *S-adeles* of K is the subring

$$\mathbb{A}_{K,S} = \prod_{v\in S} K_v \times \prod_{v\in\Omega\setminus S} \mathcal{O}_{K,v} \subseteq \mathbb{A}_K.$$

If $T \subseteq \Omega$ is a finite set of places of K, then the ring $\mathbb{A}_K^T$ of *adeles away from T* (resp. the ring $\mathbb{A}_{K,S}^T$ of *S-adeles away from T*) is the image of $\mathbb{A}_K$ (resp. of $\mathbb{A}_{K,S}$) under the projection $\prod_{v\in\Omega} K_v \to \prod_{v\in\Omega\setminus T} K_v$.

Alternatively, one can define the adeles (and S-adeles) away from T similarly to the adeles (and S-adeles) by omitting all places $v \in T$ in the product. The ring of (S-)adeles (away from T) is endowed with the subtopology induced by the product topology, with each component carrying its natural v-adic topology.

Note that K embeds diagonally into $\mathbb{A}_K$ since every element of K is integral with respect to almost every finite place of K. In particular, if X is a variety over K, we obtain a map from $X(K)$ to the space of *adelic points* $X(\mathbb{A}_K)$ of X and a natural topology on $X(\mathbb{A}_K)$, called the *adelic topology*. Similarly, we have a natural topology induced on the space $X(\mathbb{A}_K^T)$ of *adelic points away from* T on X.

Definition 4.2.15 (see [20, Definition 12.2.7])**.** Let X be a variety over K and let $T \subseteq \Omega$ be a finite set of places of K. Then X satisfies *strong approximation off* T if the image of $X(K)$ in $X(\mathbb{A}_K^T)$ is dense.

If X is proper over K, we have $X(\mathbb{A}_K^T) = \prod_{v \in \Omega \setminus T} X(K_v)$, so if X satisfies weak approximation, then X satisfies strong approximation off any finite set $T \subseteq \Omega$.

There is however no diagonal embedding of K into $\mathbb{A}_{K,S}$. Considering integral points instead of rational points, we obtain an analogue of the strong approximation property by using the diagonal embedding of $\mathcal{O}_{K,S}$ into $\mathbb{A}_{K,S}$ (or into the S-adeles away from some finite set of places). We give the following definition in the context of arithmetic schemes.

Definition 4.2.16. Let $\mathcal{X}$ be an arithmetic scheme over $\mathcal{O}_{K,S}$ and let $T \subseteq \Omega$ be a finite set of places. We say that $\mathcal{X}$ satisfies *integral strong approximation off* T if (the diagonal image of) $\mathcal{X}(\mathcal{O}_{K,S\cup T})$ is dense in $\mathcal{X}(\mathbb{A}_{K,S}^T)$.

If $\mathcal{X}_K$ is smooth, this property of $\mathcal{X}$ is also referred to as integral strong approximation off T in [64, Definition 2.21]. We will use the term integral strong approximation without the smoothness assumption. If S contains T and $\mathcal{X}$ is normal, integral strong approximation off T is a special case of *Campana weak weak approximation*, as defined in [67, Definition 2.7] (note that $\mathcal{X}$ is dominant and therefore flat over the Dedekind domain $\mathcal{O}_{K,S}$, which is part of the definition given by Nakahara–Streeter). If $\mathcal{X}$ is normal, it is shown in [67, Theorem 1.1] that Campana weak weak approximation implies the *Campana Hilbert property* (which is the usual Hilbert property in our case). This theorem of Nakahara–Streeter has been further generalized by Moerman [65, Theorem 1.1], where normality of $\mathcal{X}$ is not required. By Moerman's theorem, an arithmetic scheme satisfying integral strong approximation has the Hilbert property, provided that it admits at least one integral point. This gives us the following proposition.

Proposition 4.2.17. *Let T be a finite set of places of K, let $\mathcal{X}$ be an arithmetic scheme over $\mathcal{O}_{K,S}$, and assume that $\mathcal{X}$ satisfies integral strong approximation off T, i.e., $\mathcal{X}(\mathcal{O}_{K,S\cup T})$ is dense in $\mathcal{X}(\mathbb{A}_{K,S\cup T}^T)$. If $\mathcal{X}(\mathcal{O}_{K,S\cup T})$ is not empty, then it is not thin in $\mathcal{X}_K$.*

Proof. To see that this is a special case of [65, Theorem 1.1], we write the integral points of $\mathcal{X}$ using the notion of $\mathcal{M}$-points introduced in [65, Definition 3.12]. By the universal property of fiber products, we may replace $\mathcal{X}$ by $\mathcal{X}_{\mathcal{O}_{K,S\cup T}}$ to assume that S contains T.

Let $\mathcal{X} \to \overline{\mathcal{X}}$ be an open immersion of arithmetic $\mathcal{O}_{K,S}$-schemes with $\overline{\mathcal{X}}$ proper over $\mathcal{O}_{K,S}$ and define $\mathcal{D} = \overline{\mathcal{X}} \setminus \mathcal{X}$. Let $X, \overline{X}, D$ denote the generic fiber of $\mathcal{X}, \overline{\mathcal{X}}, \mathcal{D}$, respectively. We consider the set of multiplicities $\mathfrak{M} = \{(0)\}$ and define $\mathcal{M} = ((\mathcal{D}), \mathfrak{M})$ and $M = ((D), \mathfrak{M})$. Then $\overline{X}$ is a proper variety over the PF field $(K, \operatorname{Spec} \mathcal{O}_K)$ and the pair $(\overline{\mathcal{X}}, \mathcal{M})$ is an integral model of the pair $(\overline{X}, M)$ over $\operatorname{Spec} \mathcal{O}_{K,S}$ (see [65, Definition 3.1 and Definition 3.4]). Moreover, for the pair $(\overline{\mathcal{X}}, \mathcal{M})$, the set of $\mathcal{M}$-integral points is $\mathcal{X}(\mathcal{O}_{K,S})$ [65, §3.3, Example (1)], and the set of integral adelic $\mathcal{M}$-points is $\mathcal{X}(\mathbb{A}^T_{K,S\cup T})$ [65, Example 4.4].

Our assumption now says that $(\overline{\mathcal{X}}, \mathcal{M})$ admits an $\mathcal{M}$-integral point and that its $\mathcal{M}$-integral points are dense in its integral adelic $\mathcal{M}$-points, i.e., $(\overline{\mathcal{X}}, \mathcal{M})$ satisfies integral $\mathcal{M}$-approximation off T [65, Definition 4.8]. Therefore, by [65, Theorem 1.1], $\mathcal{X}(\mathcal{O}_{K,S})$ is not thin in $\overline{X}$, and thus not in X, as required. □

As a direct corollary, we see that an arithmetic scheme over $\mathcal{O}_{K,S}$ has the Hilbert property if it admits an integral point and its generic fiber satisfies strong approximation.

Corollary 4.2.18. *Let T be a finite set of places of K contained in S, let X be an integral K-variety satisfying strong approximation off T, and let $\mathcal{X}$ be a arithmetic scheme over $\mathcal{O}_{K,S}$ such that $\mathcal{X}_K = X$. If $\mathcal{X}(\mathcal{O}_{K,S})$ is non-empty, then it is not thin in X.*

Proof. As $\mathcal{X}(\mathcal{O}_{K,S})$ is non-empty, $\mathcal{X}(\mathbb{A}^T_{K,S})$ is a non-empty open subset of $X(\mathbb{A}^T_K)$. By strong approximation off T, the diagonal image of $X(K)$ inside $X(\mathbb{A}^T_K)$ is dense. Therefore, the intersection $\mathcal{X}(\mathbb{A}^T_{K,S}) \cap X(K)$ is dense in $\mathcal{X}(\mathbb{A}^T_{K,S})$. Since every K-point of X inside $\mathcal{X}(\mathbb{A}^T_{K,S})$ is S-integral, this shows that the diagonal image of $\mathcal{X}(\mathcal{O}_{K,S})$ in $\mathcal{X}(\mathbb{A}^T_{K,S})$ is dense. Thus, it follows from Proposition 4.2.17 that $\mathcal{X}(\mathcal{O}_{K,S})$ is not thin in X, as claimed. □

4.2.4 Strong approximation and the punctured Hilbert property

If X is a variety over K and X satisfies strong approximation off a finite set of places T of K, it was asked by Wittenberg [84, Question 2.11] if $X \setminus Z$ satisfies strong approximation off T for every closed subscheme $Z \subseteq X$ of codimension at least 2. If this condition is satisfied for a given normal integral variety X and $\mathcal{X}$ is a model of X with an integral point over $\mathcal{O}_{K,S}$, then Corollary 4.2.18 implies that $\mathcal{X}$ satisfies (OPS) for codimension at least 2.

For $T \neq \emptyset$, a positive answer to Wittenberg's question was given for the affine space $X = \mathbb{A}^n_K$ by Cao–Xu [16, Proposition 3.6] and Wei [83, Lemma 2.1], and for X a connected semi-simple simply connected quasi-split linear algebraic group by Cao–Liang–Xu [15, Theorem 1.1]. The latter result has been further extended by Cao–Huang [14, Theorem 1.4] to semi-simple simply connected almost simple algebraic groups which are isotropic. As a direct corollary of these results on strong approximation and the work of Moerman and Nakahara–Streeter, we obtain the following.

Corollary 4.2.19. *Let $\mathcal{G}$ be an arithmetic scheme over $\mathcal{O}_{K,S}$ such that $\mathcal{G}_K$ is a quasi-split semi-simple simply connected linear algebraic group over K or a unipotent group over K. If $\mathcal{G}(\mathcal{O}_{K,S}) \neq \emptyset$, then $\mathcal{G}$ satisfies (OPS) for codimension at least* 2.

Proof. We may and do assume that S contains all archimedean places of K (recall that this does not affect $\mathcal{O}_{K,S}$). In particular, this implies that $S \neq \emptyset$. Let $\mathcal{Z} \subseteq \mathcal{G}$ be a closed subscheme such that $\mathcal{Z}_K \subseteq \mathcal{G}_K$ is of codimension at least 2. Define $\mathcal{X} = \mathcal{G} \setminus \mathcal{Z}$ and $X = \mathcal{X}_K$ and assume that $\mathcal{X}(\mathcal{O}_{K,S}) \neq \emptyset$. If $\mathcal{G}_K$ is a semi-simple simply connected quasi-split linear algebraic group over K, then X satisfies strong approximation off S by [15, Theorem 1.1]. If $\mathcal{G}_K$ is a unipotent group over K, then it is isomorphic (as a variety) to $\mathbb{A}^n_K$, and thus X satisfies strong approximation off S by [16, Proposition 3.6] (see also [83, Lemma 2.1]). Therefore, it follows from Corollary 4.2.18 (taking $T = S$) that $\mathcal{X}(\mathcal{O}_{K,S})$ is not thin in X and thus not thin in $\mathcal{G}_K$. ☐

Using universal coverings and the Chevalley–Weil type lifting arguments for integral points, we can drop the simply-connectedness assumption in Corollary 4.2.19 by replacing (OPS) with (OPSE).

Proposition 4.2.20. *Let $\mathcal{G}$ be an arithmetic scheme over $\mathcal{O}_{K,S}$ such that $\mathcal{G}_K$ is a semi-simple quasi-split linear algebraic group over K or a unipotent group over K. Then $\mathcal{G}$ satisfies (OPSE) for codimension at least* 2.

Proof. Let $G = \mathcal{G}_K$. Assume first that G is unipotent. Let S' be a finite set of places of K containing S such that $\mathcal{G}(\mathcal{O}_{K,S'}) \neq \emptyset$. Then, by Corollary 4.2.19, $\mathcal{G}_{\mathcal{O}_{K,S''}}$ satisfies (OPS) for codimension at least 2 for every finite set of places S'' of K containing S', i.e., $\mathcal{G}$ satisfies (OPSE) for codimension at least 2.

Now assume that G is semi-simple quasi-split. By [23, Corollary A.4.11], there exists an isogeny $\lambda\colon G' \to G$ with G' a connected semi-simple simply connected linear algebraic group over K. Enlarging S if necessary, there exists a model $\mathcal{G}'$ of G' over $R = \mathcal{O}_{K,S}$ such that λ extends to a finite étale morphism $\bar{\lambda}\colon \mathcal{G}' \to \mathcal{G}$. Let S' be any finite set of places of K containing S. To prove the claim, we may replace S by S' (and $\mathcal{G}$ and $\mathcal{G}'$ by $\mathcal{G}_{\mathcal{O}_{K,S'}}$ and $\mathcal{G}_{\mathcal{O}_{K,S'}}$ accordingly) and show that $\mathcal{G}$ satisfies (OPS) for codimension at least 2.

By Theorem 3.1.10, there exists a finite étale morphism $\operatorname{Spec} T \to \operatorname{Spec} R$ such that $\mathcal{G}(R)$ is contained in $\bar{\lambda}(\mathcal{G}'(T))$. Let L denote the fraction field of T. We modify the Weil restriction argument from Theorem 3.2.4 to integral points to replace the T-integral points of $\mathcal{G}'$ by R-integral points of finitely many twists of $\mathcal{G}'$. Consider the induced morphism of Weil restrictions

$$\operatorname{Res}_{T/R}(\bar{\lambda}_T)\colon \operatorname{Res}_{T/R}(\mathcal{G}'_T) \to \operatorname{Res}_{T/R}(\mathcal{G}_T).$$

Since Weil restrictions are compatible with fiber products (see the discussion following Lemma 1 of [9, §7.6]), we have that

$$(\operatorname{Res}_{T/R}(\bar{\lambda}_T))_K = \operatorname{Res}_{L/K}(\lambda_L)\colon \operatorname{Res}_{L/K}(G'_L) \to \operatorname{Res}_{L/K}(G_L).$$

Let $\Delta\colon \mathcal{G} \to \operatorname{Res}_{T/R}(\mathcal{G}_T)$ denote the diagonal embedding. Let $\mathcal{G}_1, \ldots, \mathcal{G}_d$ be the connected components of $\mathcal{G} \times_{\Delta, \operatorname{Res}_{T/R}(\mathcal{G}_T)} \operatorname{Res}_{T/R}(\mathcal{G}'_T)$ that satisfy $\mathcal{G}_i(R) \neq \emptyset$ and let $G_i = \mathcal{G}_{i,K}$. Note that the natural morphisms $\lambda_i\colon G_i \to G$ are finite étale and extend to the natural morphisms $\bar{\lambda}_i\colon \mathcal{G}_i \to \mathcal{G}$, so in particular, we have $\lambda_i(\mathcal{G}_i(R)) \subseteq \mathcal{G}(R)$. Since G_i is normal connected with a K-rational point, it is geometrically integral. Moreover, over the algebraic closure $\overline{K}$ of K, each G_i is isomorphic to $G'_{\overline{K}}$, with the base change of λ_i over $\overline{K}$ being identified with $\lambda_{\overline{K}}$. In particular, the G_i are connected semi-simple simply connected linear algebraic groups over K. Since G is quasi-split, so are the G_i, since quasi-splitness is invariant under isogeny. (In fact, the image of a Borel subgroup under a surjective homomorphism of linear algebraic groups is a Borel subgroup, see [78, Corollary 6.2.8].) Thus, by Corollary 4.2.19, all $\mathcal{G}_i$ satisfy (OPS) for codimension at least 2.

Let $\mathcal{Z} \subseteq \mathcal{G}$ be a closed subscheme with $\mathcal{Z}_K \subseteq G$ of codimension at least 2 and $(\mathcal{G} \setminus \mathcal{Z})(R) \neq \emptyset$; we have to prove that $(\mathcal{G} \setminus \mathcal{Z})(R)$ is not strongly thin in G. Let $p \in (\mathcal{G} \setminus \mathcal{Z})(R)$. Since $\mathcal{G}(R)$ is contained in $\lambda(\mathcal{G}'(T))$, by the defining properties of Weil restriction, p lifts to an R-integral point of $\operatorname{Res}_{T/R}(\mathcal{G}'_T)$, and thus to a point $p' \in \mathcal{G}_i(R)$ for some i. Let $\mathcal{Z}'$ denote the inverse image of $\mathcal{Z}$ in $\mathcal{G}_i$. By [39, Corollaire 6.1.4], $\mathcal{Z}'_K \subseteq G_i$ is of codimension at least 2. Since $p' \in (\mathcal{G}_i \setminus \mathcal{Z}')(R)$ and $\mathcal{G}_i$ satisfies (OPS) for codimension at least 2, the set $(\mathcal{G}_i \setminus \mathcal{Z}')(R)$ is not strongly thin in G_i. Moreover, its image under the finite étale morphism λ_i is contained in $(\mathcal{G} \setminus \mathcal{Z})(R)$. By [5, Lemma 5.3(b)], this implies that $(\mathcal{G} \setminus \mathcal{Z})(R)$ is not strongly thin in G, as required. $\square$

Proposition 4.2.20 says that connected semi-simple quasi-split linear algebraic groups and unipotent groups over K satisfy the "punctured integral Hilbert property", i.e., part (5) of Conjecture 1.3.5 (even without enlarging K). Combining this with Theorem 4.2.2 and Corollary 4.2.8, we obtain the same result for products of those groups and 1-dimensional connected commutative algebraic groups with a dense set of rational points.

Corollary 4.2.21. *Let* $G = \prod_{i=1}^{N} G_i$ *be an algebraic group over* K *with each* G_i *either isomorphic to* $\mathbb{G}_{\mathrm{a}}$*,* $\mathbb{G}_{\mathrm{m}}$*, a connected semi-simple quasi-split linear algebraic group, a unipotent group, or an elliptic curve over* K*. Let* $Z \subseteq G$ *be a closed subscheme of codimension at least* 2*. If* $G(K)$ *is dense, then there exists a finite set* T *of places of* K *such that* $G \setminus Z$ *satisfies the weak Hilbert property over* $\mathcal{O}_{K,T}$*. In particular,* G *satisfies part (5) (and thus also (2)-(4)) of Conjecture 1.3.5.*

Proof. Let T be a finite set of places of K such that, for each i, there exists a normal quasi-projective model $\mathcal{G}_i$ of G_i over $\mathcal{O}_{K,T}$, and $\mathcal{G}_i$ is a commutative group scheme over $\mathcal{O}_{K,T}$ for all i for which $\dim G_i = 1$ (i.e., for all except the semi-simple quasi-split and the unipotent case). Let $R' = \mathcal{O}_{K,T}$, consider the model $\mathcal{G} = \mathcal{G}_1 \times_{R'} \ldots \times_{R'} \mathcal{G}_N$ of G over R', and let $\mathcal{Z} \subseteq \mathcal{G}$ be the closure of Z in $\mathcal{G}$. By Corollary 4.2.8 and Proposition 4.2.20, each $\mathcal{G}_i$ satisfies (OPSE) for codimension at least 2, so the claim follows from Theorem 4.2.2. □

4.2.5 Application to linear algebraic groups

We are now ready to prove Theorem H, which says that linear algebraic groups satisfy part (5) Conjecture 1.3.5 and thus the full conjecture. The following descent argument reduces the proof of Theorem H to products of algebraic groups that satisfy (OPSE).

Lemma 4.2.22. *Let* $\varphi\colon X' \to X$ *be a smooth proper morphism of normal integral varieties over* K *and let* $c \geq 1$ *be an integer. Assume that, for every closed subscheme* $Z' \subseteq X'$ *of codimension at least* c*, there is a number field* L *and a finite set of places* T *of* L *such that* $(X' \setminus Z')_L$ *satisfies the weak Hilbert property over* $\mathcal{O}_{L,T}$*. Then, for every closed subscheme* $Z \subseteq X$ *of codimension at least* c*, there is a number field* L *and a finite set of places* T *of* L *such that* $(X \setminus Z)_L$ *satisfies the weak Hilbert property over* $\mathcal{O}_{L,T}$*.*

Proof. Let $Z \subseteq X$ be a closed subscheme of codimension at least c and let $Z' = \varphi^{-1}(Z)$. By [39, Corollaire 6.1.4], $Z' \subseteq X'$ has codimension at least c. By the assumption on X', there is a number field L, a finite set of places T of L, and a model $\mathcal{U}'$ of $(X' \setminus Z')_L$ over $\mathcal{O}_{L,T}$ such that $\mathcal{U}'(\mathcal{O}_{L,T})$ is not strongly thin in $\mathcal{U}'_L$. Enlarging T if necessary, there exists a model $\mathcal{U}$ of $(X \setminus Z)_L$ over $\mathcal{O}_{L,T}$ such that the smooth proper morphism $\varphi'\colon \mathcal{U}'_L \to \mathcal{U}_L$ induced by φ extends to a morphism $\mathcal{U}' \to \mathcal{U}$ over $\mathcal{O}_{L,T}$. In particular, the image $\varphi'(\mathcal{U}'(\mathcal{O}_{L,T}))$ is contained in $\mathcal{U}(\mathcal{O}_{L,T})$. Since $\mathcal{U}'(\mathcal{O}_{L,T})$ is not strongly thin in $\mathcal{U}'_L$, it follows from [5, Lemma 5.3(b)] that $\varphi'(\mathcal{U}'(\mathcal{O}_{L,T}))$ (and thus $\mathcal{U}(\mathcal{O}_{L,T})$) is not strongly thin in $\mathcal{U}_L = (X \setminus Z)_L$, i.e., $(X \setminus Z)_L$ satisfies the weak Hilbert property over $\mathcal{O}_{L,T}$. ☐

Note that Lemma 4.2.22 says in particular that condition (5) of Conjecture 1.3.5 descends along smooth proper morphisms. (The same also holds for smooth surjective morphisms with geometrically integral generic fiber by [5, Lemma 5.3(a)].)

Theorem 4.2.23 (Theorem H)**.** *Let G be a connected linear algebraic group over a number field K and let $Z \subseteq G$ be a closed subscheme of codimension at least 2. Then there exists a finite field extension L/K and a finite set of places T of L such that $(G \setminus Z)_L$ satisfies the weak Hilbert property over $\mathcal{O}_{L,T}$.*

Proof. By Mostow's Theorem [46, §VIII, Theorem 4.3], there exists a unipotent group U and a reductive group P over K such that G is the semi-direct product of U and P. In particular, G is isomorphic to $U \times P$ as a K-variety.

Let $\mathrm{Rad}(P)$ denote the radical of P and let P^{der} denote the derived subgroup of P. By [40, Exposé XXII, Proposition 6.2.4], there exists an isogeny (in particular, a finite étale cover) $\mathrm{Rad}(P) \times P^{\mathrm{der}} \to P$. Since $\mathrm{Rad}(P)$ is a torus, there exists a finite field extension L/K and an integer $n \geq 0$ such that $\mathrm{Rad}(P)_L \cong \mathbb{G}^n_{\mathrm{m},L}$. The derived subgroup P^{der} of P is semi-simple, so by [23, Corollary A.4.11] there exists a connected semi-simple simply connected linear algebraic group G' over K and a finite étale cover $G' \to P^{\mathrm{der}}$. Enlarging L if necessary, we may further assume that G'_L is split. Define $G'' := U_L \times \mathbb{G}^n_{\mathrm{m},L} \times G'_L$ and note that we have a finite étale morphism $G'' \to G_L$.

We claim that the theorem holds for G''. Let $Z'' \subseteq G''$ be a closed subscheme of codimension at least 2. Let $T \neq \emptyset$ be a finite set of places of L such that there exist normal quasi-projective models $\mathcal{U}$ of U_L and $\mathcal{G}'$ of G' over $R' := \mathcal{O}_{L,T}$. Let $\mathcal{Z}''$ denote the closure of Z'' in the model $\mathcal{G}'' := \mathcal{U} \times_{R'} \mathbb{G}^n_{\mathrm{m},R'} \times_{R'} \mathcal{G}'$ of G'' over R'. Since $\mathbb{G}_{\mathrm{m},R'}$ satisfies (OPSE) for codimension at least 2 by Corollary 4.2.8 and $\mathcal{U}$ and $\mathcal{G}''$ satisfy (OPSE) for codimension at least 2 by Corollary 4.2.19, it follows from Theorem 4.2.2 that there exists a finite set of places T' of L containing T such that $(\mathcal{G}'' \setminus \mathcal{Z}'')(\mathcal{O}_{L,T'})$ is not strongly thin in G'', i.e., the theorem holds for G''. By Lemma 4.2.22, this implies that it also holds for G. ☐

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