

Asymptotically Safe Quantum Gravity: Bimetric Actions, Boundary Terms, & a C-Function

Dissertation

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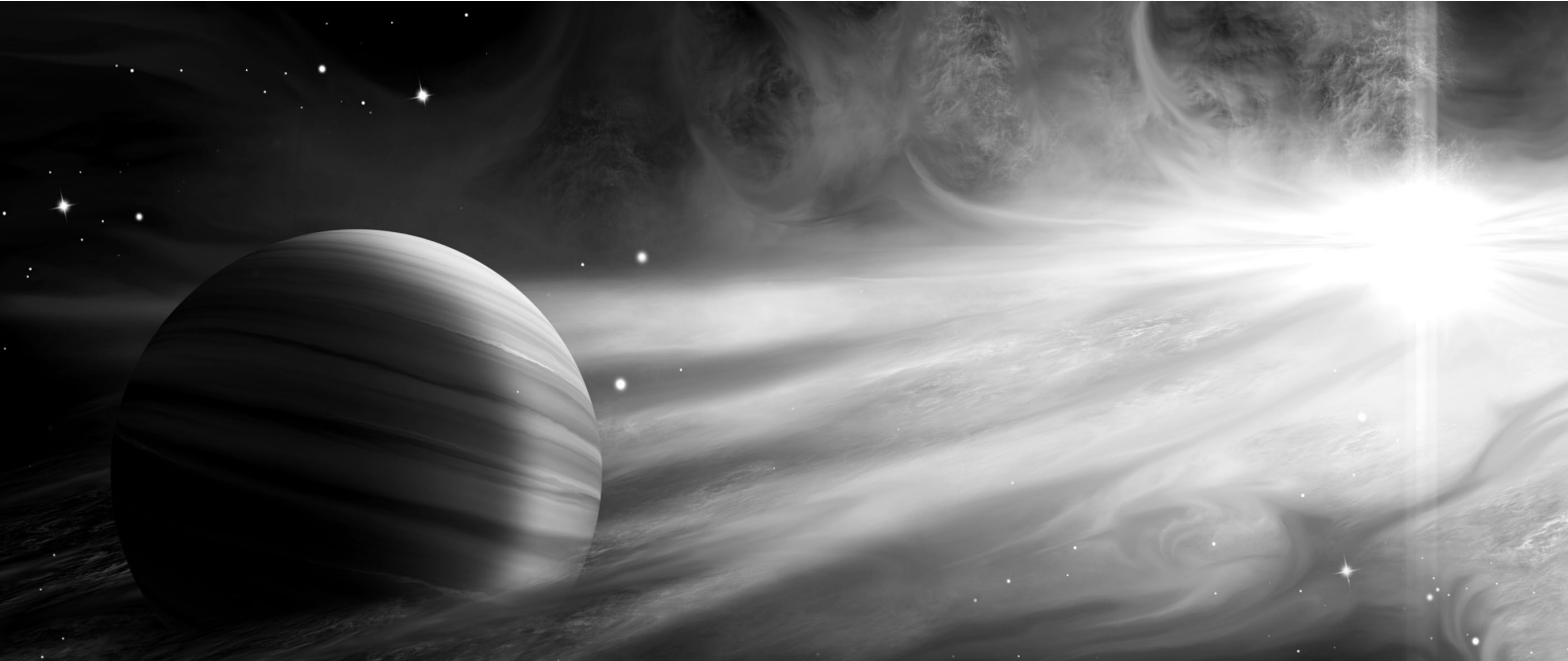
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dedicated to
my family

“ Da steh ich nun, ich armer Tor! Und bin so klug als wie zuvor.
(And here, poor fool, I stand once more, No wiser than I was before.)
J. W. v. Goethe, *Faust I* (transl. W. Arndt)



ABSTRACT

Abstract

Today our understanding of Nature is based on two pillars, the classical theory of General Relativity and the Standard Model of particle physics, each remarkably successful and predictive. However, the coalescence of a classical spacetime with the quantum nature of matter leads to severe inconsistencies which are at the heart of many open problems in modern physics. In constructing a quantum theory of gravity which cures these deficiencies one not only faces the problem that General Relativity is perturbatively non-renormalizable, but one also has to incorporate the fundamental requirement of Background Independence, expressing that no particular spacetime (such as Minkowski space) is singled out a priori.

In this thesis we employ a non-perturbative, universal method which has played an important role in the past decades: the Functional Renormalization Group Equation (FRGE). In the Effective Average Action (EAA) approach the search for a quantum field theory of gravity is guided by the Asymptotic Safety conjecture. Proposed by Steven Weinberg, it employs a generalized notion of renormalizability which goes beyond the standard perturbative ones. The key requirement is the existence of a non-trivial ultraviolet (UV) fixed point of the Renormalization Group (RG) flow which has a finite-dimensional UV-critical hypersurface to ensure predictivity. While classical General Relativity is recovered as an effective description in the infrared (IR), the bare (or ‘classical’) action emerges from the fixed point condition and is thus a prediction rather than an input.

Since the original works on Asymptotic Safety in the 90’s, based on the Einstein-Hilbert truncation, various extensions and generalizations thereof have been studied in the past decades, all of which provide strong evidence for the existence of a suitable non-trivial fixed point with a finite number of relevant directions. Thus, it seems very likely that Quantum Gravity is non-perturbatively renormalizable.

On the other hand, the status of Background Independence in the context of Asymptotic Safety remained largely unclear. It is a requirement on the global properties of the RG flow: there must exist RG trajectories which emanate from the fixed point in the UV and restore the broken split-symmetry (relating background to quantum fields) in the IR.

Employing a ‘bi-metric’ ansatz for the EAA, we demonstrate for the first time in a non-trivial setting that the two key requirements of Background Independence and Asymptotic Safety can be satisfied simultaneously. Taken together they are even found to lead to an increased predictivity. A new powerful calculational scheme (‘deformed $\alpha = 1$ gauge’) is introduced and applied to derive the beta-functions for the bi-metric Einstein-Hilbert truncation including a Gibbons-Hawking-York term.

Exploring further the global properties of the RG flow, a generalized notion of a C -function is proposed and successfully tested for bi-metric RG trajectories consistent with Background Independence. As a byproduct, we also develop a new, completely general testing device to judge the reliability of truncated computations in the FRGE scheme.

Furthermore, we investigate the occurrence of propagating graviton modes in the Asymptotic Safety scenario. The relevant properties of the graviton propagator depend on the sign of the anomalous dimension, η , of Newton’s coupling. While asymptotically safe RG trajectories possess a negative η in the UV, we observe that it turns positive in the semi-classical regime. This feature, not observed in the older single-metric truncations, is found relevant for the scale dependent (non-)occurrence of gravitational waves. The implications for the generation of primordial density perturbations in the early Universe are discussed.

The second main focus of this thesis is the inclusion of boundary terms into the EAA. We present a detailed analysis of the Gibbons-Hawking-York functional and of its RG evolution induced by the bi-metric bulk invariants. This generalization to spacetime topologies with a non-empty boundary is particularly important in black hole thermodynamics. We observe a stabilization of their thermodynamical properties near the Planck scale.

Kurzfassung

Unser heutiges Verständnis der fundamentalen Wechselwirkungen beruht auf zwei sehr erfolgreichen Theorien großer Vorhersagekraft: der Allgemeinen Relativitätstheorie (ART) und dem Standardmodell der Teilchenphysik. Beim Versuch die klassische Raumzeit mit der Quantennatur der Materie zu verbinden, treten grundlegende konzeptionelle Probleme auf, die eng mit zentralen offenen Fragen der modernen Physik zusammenhängen. Eine Lösung dieser Probleme scheint erst in einer Quantentheorie der Gravitation selbst möglich zu sein. In der Konstruktion einer solchen Theorie treten jedoch neben der störungstheoretischen Nichtrenormierbarkeit der ART auch konzeptionelle Schwierigkeiten auf, die von der unabdingbaren Forderung nach Hintergrundunabhängigkeit herrühren.

In dieser Arbeit wenden wir eine universelle, nicht-störungstheoretische Methode an, die in den letzten Jahrzehnten zu weitreichenden Erkenntnissen in den verschiedensten Bereichen der Physik geführt hat: die Funktionale Renormierungsgruppengleichung (FRGE). Basierend auf der “Effective Average Action” (EAA) ist die Suche nach einer geeigneten Quantenfeldtheorie der Gravitation durch die von Steven Weinberg postulierte Forderung nach asymptotischer Sicherheit bestimmt. Hierbei zeichnet sich eine (nicht-)störungstheoretisch renormierbare Theorie dadurch aus, dass ihr UV-Verhalten vollständig durch einen (nicht-)trivialen Fixpunkt (NGFP) des RG-Flusses kontrolliert wird, dessen UV-kritische Hyperfläche endlich-dimensional ist. Dadurch ergibt sich ein klassisches Regime auf makroskopischen Skalen, wohingegen die fundamentale (nackte) Theorie aus der Fixpunktbedingung folgt und damit eine Vorhersage dieses Zugangs ist.

Seit Ende der 90er Jahre die ersten Anzeichen für Asymptotische Sicherheit auf Basis einer Einstein-Hilbert Rechnung gefunden wurden, hat sich in zahlreichen Erweiterungen und Modifikationen der Hinweis auf die Existenz eines nicht-trivialen Fixpunktes der Gravitation stark verdichtet. Demnach scheint die Quantengravitation tatsächlich im nicht-störungstheoretischen Sinn renormierbar zu sein.

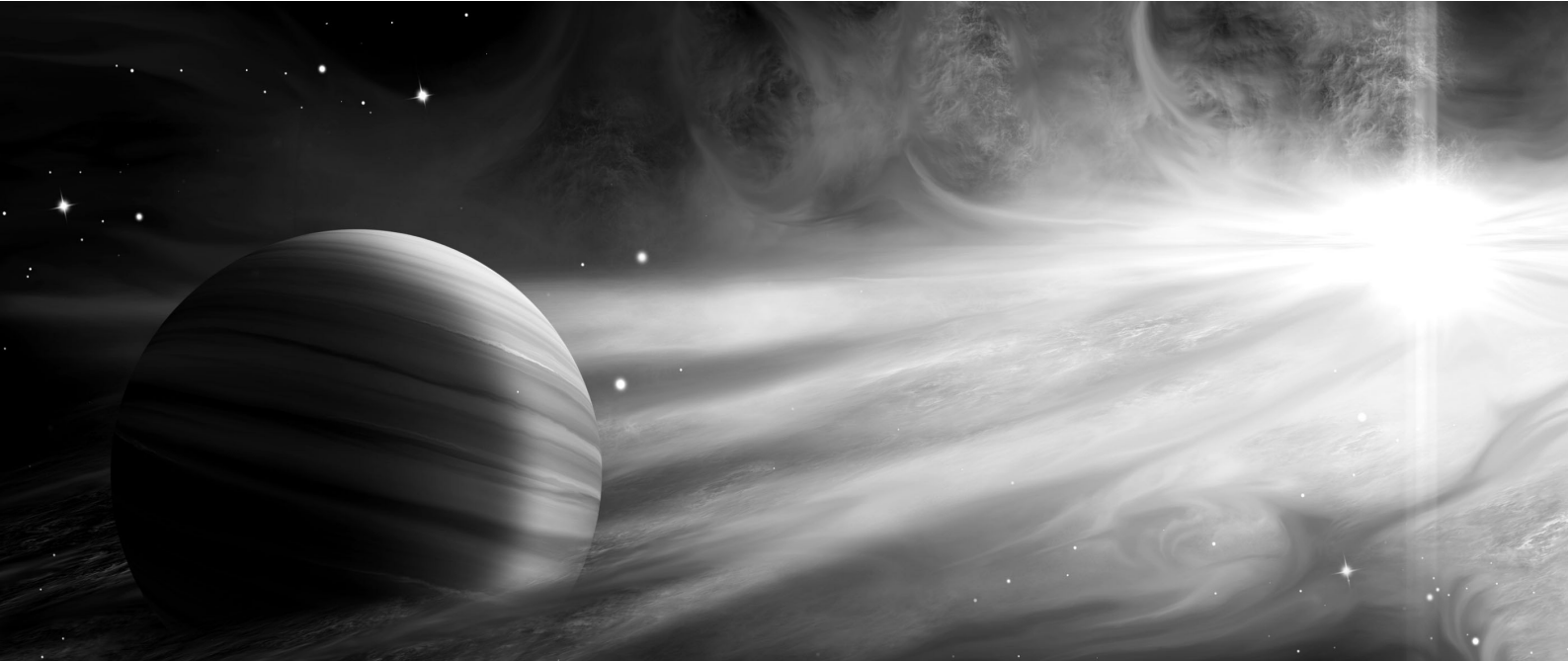
Andererseits ist der Status der Hintergrundunabhängigkeit im Kontext der Asymptotischen Sicherheit noch weitgehend unklar, was maßgeblich damit zusammenhängt, dass diese Frage mit globalen Eigenschaften des RG-Flusses zusammenhängt: Asymptotisch sichere RG-Trajektorien die im UV an einem NGFP beginnen, müssen zusätzlich die sogenannte Split-Symmetrie im IR erfüllen.

Um dieser Fragestellung nachzugehen, betrachten wir in dieser Arbeit einen bi-metrischen Ansatz für die EAA und zeigen erstmalig, dass die Forderungen nach Asymptotischer Sicherheit und Hintergrundunabhängigkeit gleichzeitig erfüllt werden können. Es zeigt sich, dass dadurch die Vorhersagekraft des Zugangs sogar noch größer wird. Wir entwickeln eine neue, leistungsstarke Methode, die auf einer ‘deformierten’ $\alpha = 1$ Eichung basiert und es erlaubt, die RG-Gleichungen der bi-metrischen Einstein-Hilbert Trunkierung, erweitert um einen Gibbons-Hawking-York Term, abzuleiten und zu studieren.

Im zweiten Teil der Arbeit untersuchen wir allgemeine globale Eigenschaften des RG-Flusses über den Rahmen der Quantengravitation hinaus und postulieren eine verallgemeinerte C-Funktion, welche das globale Verhalten der RG-Trajektorien bestimmt. Wir testen diesen Vorschlag erfolgreich anhand der obigen RG-Trajektorien für die Klasse der physikalisch relevanten Theorien mit Split-Symmetrie. Als Nebenprodukt erhalten wir eine allgemeine Methode zur Beurteilung der Güte von beliebigen Näherungslösungen der FRGE.

Anschließend wenden wir die Resultate der obigen bi-metrischen Rechnung auf die Frage nach der Existenz propagierender Graviton-Moden im Rahmen der asymptotischen Sicherheit an. Die Eigenschaften des Gravitonpropagators werden wesentlich durch das Vorzeichen der anomalen Dimension η der Newton-Kopplung bestimmt. Während asymptotisch sichere RG-Trajektorien im UV ein negatives η besitzen, zeigen unsere Untersuchungen, dass η im semiklassischen Bereich positiv wird. Diese Eigenschaft war in vorhergehenden ‘single-metric’ Berechnungen nie gefunden worden. Wir untersuchen, auf welchen Skalen das Auftreten von propagierenden Gravitationswellen zu erwarten ist und diskutieren mögliche Auswirkungen auf die Erzeugung primordialer Dichtefluktuationen im frühen Universum.

Weiterhin betrachten wir Raumzeiten mit Rand, die Trunkierungsansätze für die EAA mit Oberflächentermen erfordert. Wir stellen eine detaillierte Analyse des Gibbons-Hawking-York Funktional vor und untersuchen sein RG-Verhalten, welches durch die bi-metrischen Volumen-Terme induziert wird. Diese Verallgemeinerung des Formalismus ist von zentraler Bedeutung für die Thermodynamik schwarzer Löcher. Wir beobachten eine thermodynamische Stabilisierung schwarzer Löcher, deren Masse sich durch Hawking-Evaporation der Planck-Masse annähert.



PREFACE

“ Brevity is the soul of wit.

This thesis is divided into three main parts, each addressing a conceptually different aspects of the Asymptotic Safety program to Quantum Gravity. We start from scratch and stepwise introduce the ingredients necessary to construct, at least on a formal level, quantum field theories in the Functional Renormalization Group (FRG) approach. The purpose of this first, foundational part goes far beyond a simple introduction. Indeed, it does include standard textbook descriptions of topology, group theory, differential geometry, and measure theory and its relation to physics, well known since decades. While there are excellent books on each of these topics, our emphasis lays in providing a consistent and unified language and thereby fixing the notation and conventions used in this thesis. Our aspiration is a combination of a self-consistent and a transparent formalization of the employed techniques and thus besides paving the grounds for the following chapters and derivations the first part of this thesis can be thought of as a reference resource that can be consulted at any point for clarification. The general structure of this part considers the dependencies of the various relevant subjects, which starts with the most fundamental ingredient, spacetime, and culminates in the universal tool of the Functional Renormalization Group Equation. For the latter we present a coordinate-free version which also allows to study spacetime constructions with a non-vanishing boundary. Readers interested in the foundations and structural aspects of the Functional Renormalization Group approach as well as Asymptotic Safety should focus on this part.

Moving on to the second part, we leave the theoretical description and enter the technical section of this thesis in which the beta-functions of the so-called bi-metric-bulk – pure-background-boundarytruncation are derived within the Effective Average Action approach to Quantum Einstein Gravity. Explicitly, we focus on a bi-metric Einstein-Hilbert ansatz equipped with a set of boundary invariants, as the Gibbons-Hawking-York term. The computation of the associated beta-functions is performed in great detail, showing on several stages possible extensions that can be studied in future work. Therefore, we have kept the notation as general as possible, in particular to modify the class of boundary conditions in subsequent projects. Again,

we stress the assumptions and the range of validity of the results. The reader who is interested in a practical calculation and the conceptual difficulties in evaluating the non-linear functional differential equation, FRGE, is invited to focus on this part.

Finally, the third and concluding part of this thesis sheds light on the implications of the previously derived beta-functions. We study the Asymptotic Safety conjecture within this truncation and confront this non-perturbative generalization of renormalizability with the fundamental requirement of Background Independence within theories of Quantum Gravity. Having discussed the ultraviolet and infrared properties of the results, we propose a generalized mode counting device, in certain features reminiscent of Zamolodchikov's C -function, which would impose global constraints on the RG evolution. As such, it can also be used to test the reliability of truncations and would be very helpful in designing future explorations of the full infinite dimensional theory space. The final chapters of this part are devoted to the concept of propagating gravitons and the study of the boundary sector and its implications on black hole thermodynamics. Readers who are most interested in the results and cosmological applications are invited to browse the corresponding chapters in this part of the thesis.

At this point it should be clear that even though this work is structured such that one can start at page one and continues to the list of abbreviations, it is also possible to sidestep certain aspects at first reading and in case of questions come back or consult the index. Most of the material is presented with an emphasize on transparency and precision, while at the same time respecting the global consistency of the thesis. Therefore, we almost completely abandoned the idea of appendices in order to retain the line of reasoning in the full work and not rip apart conceptually related ideas. For the reader who is interested in a broad overview with a brief but consistent introduction should follow the road map 1 that provides a journey through selected topics of this thesis, relevant for understanding in particular result section.

To further improve the readability of this manuscript, we appended an index of important keywords and keep the list of abbreviations comparatively short. The electronic version is also equipped with a comprehensive hyperlink 'net' that allows to easily transit the thesis. Furthermore, we make use of the following symbols:

- R A short remark
 S A brief summary
 C Chapter based on self-publication

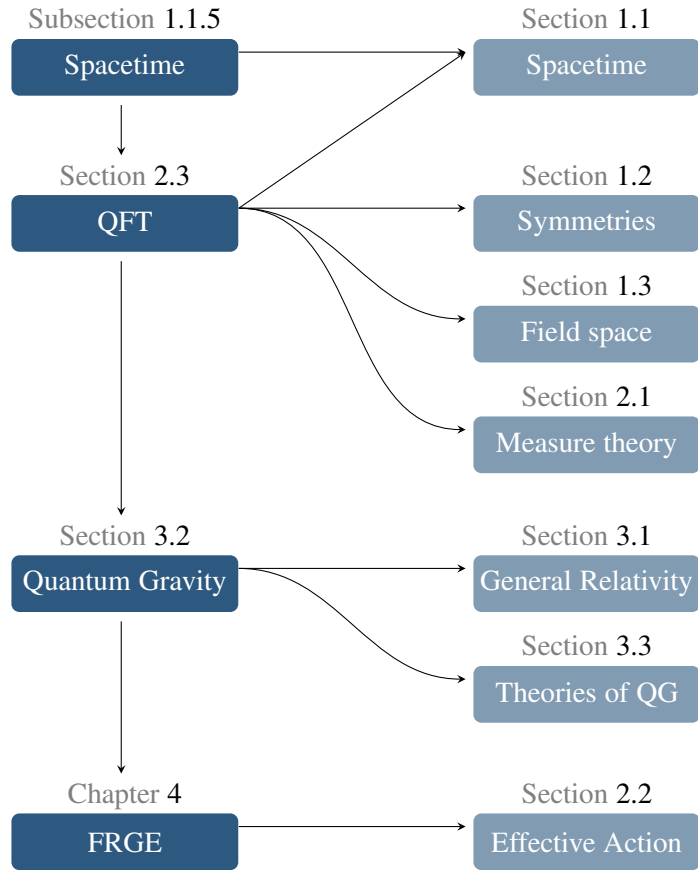
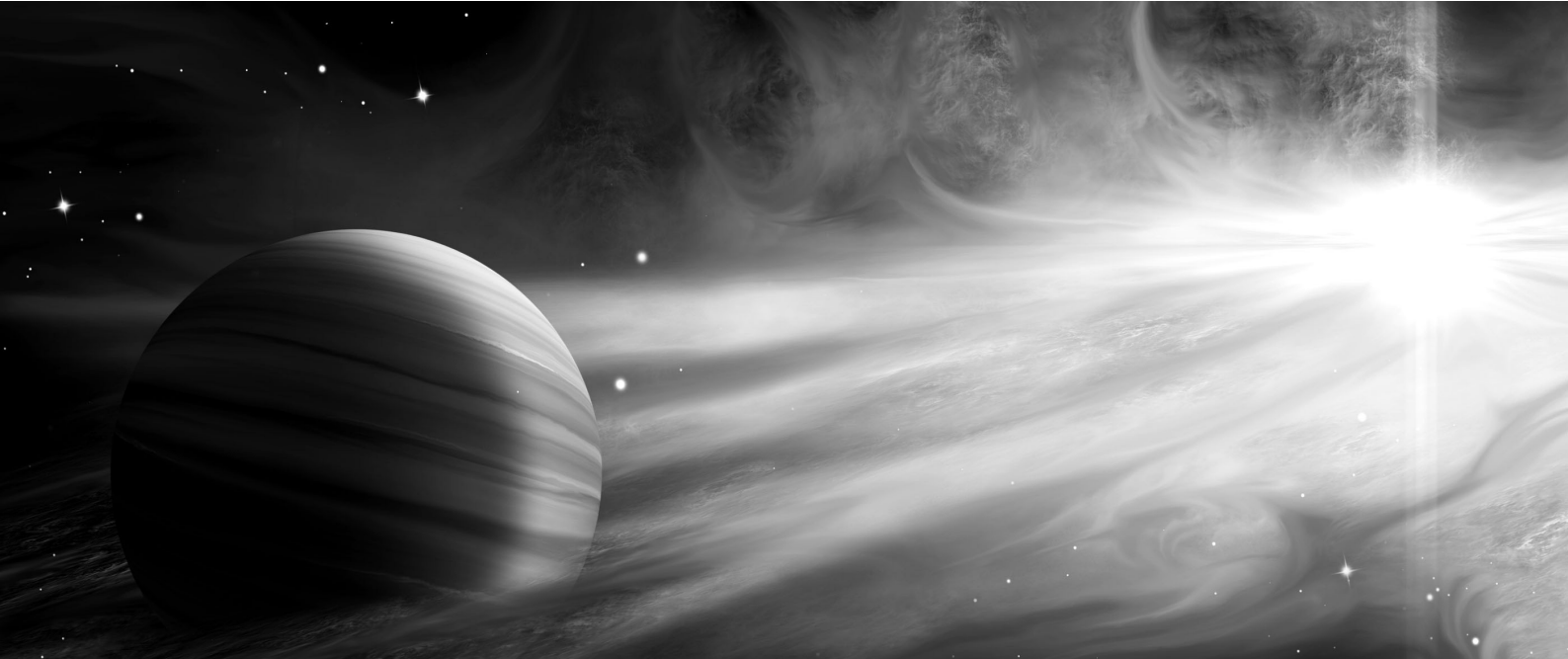


Figure 1: Road map for the first part of this thesis.



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The thesis is divided into three parts, each describing a different aspect of the Functional Renormalization Group approach to Quantum Gravity. This first part investigates its theoretical background in detail, starting from scratch and stepwise develop the necessary machinery to construct this universal tool. While it has an introductory character, presenting some standard textbook results on the various involved branches of physics and mathematics, we intend giving a consistent generalization of those concepts for spacetimes augmented with a boundary. Though more involved, this class of geometries are also well studied in the literature. Rather, we are aiming at a formulation designed to consistently intertwine boundary contributions with the standard description of the Renormalization Group. We further present a (formal) measure theoretic motivation for the Functional Renormalization Group Equation which defines a vector field on theory space describing the Renormalization Group evolution of theories.

The following chapters present a comprehensive description of the ingredients necessary to study the Asymptotic Safety conjecture of Quantum Gravity in the presence of boundaries. The reader who is interested in a condensed introduction may follow the road map in the preface, fig. 1, and may consult the index if details are required.

This part starts in chapter 1 with the three ingredients theory space and in general a standard Quantum Field theory is based on: spacetime, symmetry principles, and field space. We pursue the path of gauge theories and use differential geometry for the definition of the field content leading to a natural combination of symmetries and interactions. With the introduction of operators and functionals field space will become more vivid, allowing for transformations and maps which lead to observables.

In chapter 2 we begin the construction of (probability) theories using functional integration, which has to be understood on a formal level. After some basics on measure theory, we consider the general ‘recipe’ to endow field space with a suitable theoretical framework that is consistent with the symmetries and the underlying spacetime. From this perspective we address the perturbative concept of renormalizability and the general interpretation of perturbation theory along with its range of validity.

Chapter 3 then provides an example of a classical theory which turns out to be incompatible with the standard (perturbative) renormalization procedure, General Relativity. We briefly comment on the general aspects and peculiarities of this interaction and then present an incomplete list of proposals for a quantum theory of gravitation.

The Asymptotic Safety approach to Quantum Gravity is discussed in a separate chapter, 4, where we first introduce the universal applicable Functional Renormalization Group Equation and then describe its application to Quantum Gravity. We conclude with a brief summary on the current status of the AS conjecture.



1.1	Spacetime Point sets Topological spaces Topological manifolds Smooth manifolds Spacetime
1.2	Symmetries Groups Group actions Lie groups & Lie algebras Examples of Groups
1.3	Field space Fiber bundles Field space
1.4	Operators The spectral theorem Trace of operators Determinant of operators
1.5	Functionals Functional derivatives Basis invariants

1. PHYSICAL CONTENT

This first chapter is devoted to a general introduction of the main mathematical concepts that we encounter in the FRG approach to quantum field theories. Besides revealing the conventions and notations used in this thesis, it presents the foundations for the framework on which our theoretical description is based on.

We will start with a very central object in studying field theories, namely the literally ground on which the theory is based, spacetime. In our definition, it corresponds to an equivalence class of smooth manifolds, in general with boundary. The next section presents a summary of group theory with a special emphasis on Lie groups and -algebras. The special status of symmetries in today's understanding of Nature, for instance in the shape of gauge theories, is the reason why group theoretical aspects take a very important place in the introduction of this thesis. Especially universality and the phases of theories are quite essential concepts in studying the FRG and most of the strongest mathematical restrictions are obtained by symmetry observations.

Finally, this chapter concludes by joining smooth manifolds and group theory in the shape of global smooth functions, special subset of them being later consider as the physical field content. Hereby, differential geometry is the mathematical setting of this coalescence and its application to modern QFT has brought fruitful insight into the possible foundations of Nature. This final series of sections contains an introduction to fiber bundles, field space, operators, and functionals.

1.1 Spacetime

Spacetime is the fundamental ingredient our theoretical description is (literally) based on. Starting in the category of sets we continue the journey to the mathematical branch of topology and finally the theory of differential geometry. With each additional structure we attach to the notion of spacetime we are confronted with an important choice about its very nature, and it is the aim of this section to make these choices transparent.

While strengthening the ties between spacetime points, we will see how the rich structure of mathematical description yields redundancies that are related to isomorphisms in the corresponding category. The equivalence relations defined by those isomorphisms will be used to

group indistinguishable structures together and encode the essential information in its associated equivalence classes.

The final notion of spacetime and the summary of this section can be found in subsection 1.1.5. For more details and references to the present discussion see [1–4].

1.1.1 Point sets

The most elementary object in the construction of spacetime is based on the theory of sets. This axiomatic system is situated at the very root of mathematics and is related to its severest limitations in the shape of Gödel's incompleteness theorems [5].¹ Even though sets are intuitively comprehensible, the axiomatics of (standard) modern set theory were established in its full beauty not before 1921s. This systematic approach – pioneered by Zermelo [6] and sharpened by Fraenkel [7] – consists of eight axioms, forming the Zermelo–Fraenkel (ZF) set theory, together with the famous axiom of choice (AC)². The advent of the Zermelo–Fraenkel set theory with the axiom of choice (ZFC) led to an axiomatic description of sets that is so far found to be free of paradoxes, in contrast to naive set theory.³

The precise definition of a set is in fact difficult and we will make use of the following indirect version:

Definition 1.1.1 — Set. Let SET be a class of collections containing distinct objects and which fulfill the axiomatics of ZFC. A *set* with respect to SET is a member of the class SET .

It is now natural to use the class of functions – interconnecting two sets – as morphisms to establish the category of sets.

Definition 1.1.2 — Category of sets. The category of sets $\text{C}_{\text{SET}} \equiv (\text{SET}, \text{m}_{\text{SET}}, \circ)$ is given by the class of sets SET , the class of functions m_{SET} , and the operation $\circ$, being the associative composition of functions in the usual sense.

There is an infinitely large variety of sets one can build within the ZFC. The most often encountered ones are given by sets whose members share a common feature or requirement.

In the construction of spacetime starting with a point set, we assume that the members, i.e. the underlying building blocks of spacetime, do not carry individual information but are essentially the same. In other words, labeling the elements of a set is superfluous in our description of spacetime. In mathematical terms this translates to invariance of spacetime under bijection (the isomorphisms of C_{SET}):

Definition 1.1.3 — Bijection. Let A and B be sets. A function $f : A \rightarrow B$ is a *bijection* $\Leftrightarrow$

$$\forall b \in B \exists a \in A : \quad b = f(a) \quad (\text{surjective}) \quad (1.1a)$$

$$\forall a, c \in A : \quad f(a) = f(c) \Leftrightarrow a = c \quad (\text{injective}) \quad (1.1b)$$

For our purpose two sets give rise to the same spacetime whenever there is a bijection connecting both. Grouping together bijective sets actually defines an equivalence relation $\sim_{\text{SET}}$ within the ZFC. The category of sets can thus be partitioned into equivalence classes of $\sim_{\text{SET}}$ which can be classified by the so called cardinality.

¹In short the theorems state that a set of axiomatics cannot produce all true statements about itself without contradiction.

²In some axiomatic systems of set theory the axiom of choice (AC) is replaced by an equivalent axiom.

³For instance, Russell's paradox exploits the inconsistent formalization of older set theory. Russell constructed a set $S \equiv \{X \mid X \notin X\}$ containing all sets that are not elements of themselves (which was allowed by older set theory). Then, however, $S \in S \Leftrightarrow S \notin S$ and S has to be the empty set for which obviously counterexamples exist.

Definition 1.1.4 — Cardinality. Given the category of sets $\mathcal{C}_{\text{SET}}$, we can partition the sets with the relation $\sim_{\text{SET}}$. Such equivalence classes are classified by the cardinality $\#$ of the contained sets. (In the finite case this corresponds to the number of elements.)

Thus, on the level of point sets distinct spacetime constructions can only differ by their ‘size’ (cardinality). Nevertheless, this already distinguishes in particular the sets of countable and uncountable infinite sets.

R As weak as the selection of a specific point set and thereby a certain cardinality may seem, it is already at this level where the distinction between finite and infinitely extended lattice-based, as well as continuous spacetime construction arises.

In the process of attaching further mathematical structure to spacetime, we have to make sure that the entire prescription retains the invariance under bijections. Lifting this symmetry to other categories corresponds to the invariance under their respective isomorphisms. Hence, topological, or smooth manifolds which are homeomorphic or diffeomorphic, respectively, will give rise to the same spacetime construction.

S In the category of sets, spacetime is classified by its cardinality; it is an element of

$$\text{SET}_{\sim} := \{\#(S) \mid S \text{ is in SET}\} \equiv \{[S]_{\sim_{\text{SET}}} \mid S \text{ is in SET}\}$$

and the morphisms are functions with composition.

1.1.2 Topological spaces

Endowing spacetime with a topology opens up a rich structure of new classification schemes and introduces a first idea of locality. It seems natural to equip spacetime with a formalism that relates its elements in a suitable way. The first step is grouping together distinct points in so called open sets that form a topology on a point set. Abstractly speaking, a topology is a collection of subsets satisfying the following conditions:

Definition 1.1.5 — Topological space. Let X be a set. $\tau_X \subseteq 2^X \equiv \{S \subseteq X\}$ where 2^X is the power set of X . Then $\tau_X \subseteq 2^X$ defines a topology on $X : \Leftrightarrow \emptyset \in \tau_X$ and $X \in \tau_X$ and

$$\begin{aligned} \forall S \subseteq \tau_X : & \quad (\cup_{U \in S} U) \in \tau_X & \quad (\text{closed under union}) \\ \forall S \subseteq \tau_X, \#(S) < \infty : & \quad (\cap_{U \in S} U) \in \tau_X & \quad (\text{closed under finite intersection}) \end{aligned}$$

The ordered pair (X, τ_X) is called a *topological space*, and elements of τ_X are open sets.

This construction gives rise to the notion of neighborhoods and naturally provides concepts of compactness and connectedness, as well as a variety of derived properties. Once we have open sets at our disposal a property exists locally if for each point $x \in X$ there is an open set, $U \in \tau_X$ with $x \in U$, for which this property is satisfied.

Connected spaces

Connectedness is one such topological characteristic with which we can decide whether our concept of spacetime consists of separated (disconnected) parts or is entirely ‘glued together’.

Definition 1.1.6 — (Dis-)Connected space. A topological space (X, τ_X) is disconnected if there exist two disjoint (non-empty) open sets that cover X , i.e.

$$\exists U, V \in \tau_X / \{X, \emptyset\} \text{ with } U \cap V = \emptyset \text{ and } X = U \cup V$$

■ A connected space is a topological space that is not disconnected.

This reflects the intuitive picture of connectedness translated into the language of topology. If we can separate a part of spacetime without affecting its remaining portion we would have at least two distinct non-communicating ‘worlds’. The sets V and U in definition 1.1.6 are open and closed at the same time, a consequence of the very general idea of closed sets in topology: $C \subseteq X$ is a closed set if there exists $U \in \tau_X$ such that $C = X/U$. Only when considering a connected topological space (X, τ_X) we retain the separation of both concepts, by assuring that only X and $\emptyset$ can be both open and closed sets in τ_X . In what follows we will usually assume spacetime to be a connected topological space.

Compact spaces

Another very important concept in the category of topology is compactness, a combination of generalized aspects of finiteness and closeness. It is therefore not astonishing that many theorems rely on this property in one or the other way. Explicitly, we have:

Definition 1.1.7 — Compact space. A topological space (X, τ_X) is a compact space : $\Leftrightarrow$

$$\begin{aligned} &\forall U \subseteq \tau_X, \cup_{A \in U} A = X : \\ &\implies \exists U' \subseteq U, \cup_{A' \in U'} A' = X \text{ with } \#(U') < \infty \quad (\text{finite open subcover of } (X, \tau_X)) \end{aligned}$$

■ That is, every open cover of (X, τ_X) has a finite open subcover.

Usually, whenever infiniteness or a lack of completeness enters the formalism one has to be very careful to transfer well known features of finite sets or $\mathbb{R}$ to the space at hand. In topology the transition from local to global aspects usually invokes patching together an infinite number of open neighborhoods which may spoil local valid properties in the global picture. However, for compact spaces this transition is in a certain sense ‘well-behaved’ for the prescription can always be reduced to a *finite* number of open sets that still cover the entire space. This finiteness may allow to lift local properties to global statements and this is what many theorems are based on.

R Once the underlying point set X has finite cardinality even the discrete topology $\tau_X = 2^X$ will give rise to a compact space. For uncountable sets compactness introduces a notion of finiteness into the infiniteness.

In functional analysis compactness is the crucial ingredient for generalizing theorems on topological spaces, for instance the existence and uniqueness of functions solving differential equations or convergence issues. It is for this reason that in the consideration of, for example, field content the formalism will be sensitive to the status of compactness of the underlying spacetime.

Continuous functions and homeomorphisms

Functions between different sets establish relations and classification schemes in the category of sets. We can use special maps, bijections, to reveal an ordering of sets by their cardinality. In topological spaces, where we have an additional structure consisting of a collection of subsets, the open sets, we have to look for functions that also take care of the topology. To preserve the topological information as well we have to restrict to continuous functions in the first place:

■ **Definition 1.1.8 — Continuous function.** Let $A \equiv (X_A, \tau_A)$ and $B \equiv (X_B, \tau_B)$ be topological

spaces. A function $f : X_A \rightarrow X_B$ is continuous with respect to A and $B : \Leftrightarrow$

$$\forall U \in \tau_B : \quad f^{-1}(U) \equiv \{x \in X_A \mid \exists y \in U : f(x) = y\} \in \tau_A$$

That is, the pre-image of each open set in B is open in A .

This inverted looking definition corresponds to the one in metric spaces where the notion of locality is supplemented with a distance. The true power of continuous functions unfolds when considering the net they spread over the set of topological spaces. It turns out that the composition of continuous functions is again a continuous function, and so is the identity- and constant map, $\text{id}(x) = x$ and $c(x) = c$ for all $x \in X_A$, respectively. This allows to build long chains of continuous functions over the entire class of topological spaces, thereby relating their notions of locality. Some of the main theorems of topologies are concerned with the preservation of topological properties, as e.g. compactness, under continuous maps. It is quite astonishing that a local concept as continuity actually embeds global topological information into its target space and thereby retains compactness and connectedness properties, for instance.

If two topological spaces can be mutually related by continuous functions they locally agree. Extending this concept to the global picture allows us to find topologically identical spaces. As we have taken care of topological information by continuity we have to assure that the set's specific information is as well retained, as is done by bijections. The combination of locality-preserving and bijective maps leads to the isomorphisms of the category of topological spaces: homeomorphisms.

Definition 1.1.9 — Homeomorphism. Let $A \equiv (X_A, \tau_A)$ and $B \equiv (X_B, \tau_B)$ be topological spaces. A function $f : X_A \rightarrow X_B$ is a homeomorphism $:\Leftrightarrow$

$(f \text{ is a bijection}) \quad \wedge \quad (f \text{ and its inverse function } f^{-1} \text{ are continuous})$

Then, A and B are called homeomorphic.

Since our entire formalism is based on the category of sets, all isomorphisms we encounter are subclasses of bijections, preserving additional structures. Each set in a specific cardinality give rise to a variety of topological spaces. For the class of finite sets this is merely a combinatorial task. For the equivalence classes of countable and uncountable sets this will result in an infinite number of topologies we generate from those sets. Homeomorphisms provide the means to clear up the matter and distinguish the different classes of topologies one can obtain for each cardinality.

Two topological spaces of a particular cardinality are the same if they can be transformed homeomorphic into each other and thereby keep their local properties intact everywhere. They share, by definition, the same topological invariants, i.e. quantities that are preserved under homeomorphisms. The set invariants were similarly defined to be the quantities preserved under the isomorphisms of the category of sets, i.e. bijections. The simplicity of sets gives rise to a single invariant: the cardinality. It becomes apparent from the vast number of topological invariants what kind of rich structure is encoded in the category of topology. Besides compactness- and connectedness-properties, several separation-, countability-, and metrizable-conditions were found to be invariant under homeomorphisms. They are usually the starting point to figure out if two topological spaces are *not* homeomorphic. Notice that even though we may find topological spaces that agree in all topological invariants we know they do not have to be homeomorphic. Only in rare cases we can characterize special topological spaces using certain invariants.

S In the category of topological spaces $C_{\text{TOP}} := \{(X, \tau_X) \mid X \text{ is in SET} \wedge \tau_X \text{ is topology on } X\}$ with morphisms $m_{\text{TOP}} = \{f \in m_{\text{SET}} \mid f \text{ is continuous}\}$, spacetime is classified by its car-

dinality and a class of homeomorphic topologies; it is an element of

$$\text{TOP}_\sim := \{[(X, \tau_X)]_{\sim_{\text{TOP}}} \mid (X, \tau_X) \sim_{\text{TOP}} (Y, \tau_Y) :\Leftrightarrow \exists f : X \rightarrow Y \text{ homeomorphism} \}$$

1.1.3 Topological manifolds (with boundary)

A topology attaches a diversity of different structures to the simple equivalence classes of sets – the cardinalities. Thereby, a wide range of new possibilities to classify spacetime arises from the very definition of open sets. In fact, the number of those new constructions is related to the cardinality of the power set and will be infinite for infinite underlying point sets. It is us who have to decide among those topological spaces by gathering an amount of topological properties we want to have satisfied by our model of the Universe. Usually one is keen to stay close to the very familiar Euclidean space and only depart from this guiding principle when it seems unavoidable. Following this path, we focus on topological spaces that reflect our (local) intuitive understanding of the world we are living in, i.e. resemble an Euclidean type structure. Framed in the language of topology, this entails a consideration of so called topological manifolds. Even though the majority of topologies is dropped by those restrictions that keep certain Euclidean features (at least locally) intact, we are still left with a wealth of globally and/or locally distinct spaces.

Before finally giving the definition of topological manifolds (with boundary) we will motivate its additional topological attributes and why they seem to naturally reflect spacetime properties. Thereby, we keep the notation more general in order to include boundaries (in the sense of manifolds) into our description, which will be relevant for the later investigations.

Locally Euclidean space (with boundary)

When perceiving the world with our senses we deduce a continuity of space and time, which seems to be perfect on our scales of observation. Whether or not this description of spacetime holds on a deeper level is one of several questions physicists try to answer by studying theories of quantum gravity. In this work we assume our intuitive (local) picture of the world to hold down to any scales, i.e. we adopt the perspective that spacetime is locally homeomorphic to Euclidean space.

In the construction of those Euclidean spaces two fundamental branches of mathematics are combined, i.e. topology meets linear algebra. Whereas in topology we extend point sets by properties of its subsets, in algebra operations and relations take this role. At this stage fields, vector spaces and in general algebraic structures appear that equip the simple point sets with additional concepts, different to the ones we have encountered so far. It is the purpose of manifolds to enrich topological spaces with those algebraic properties and unfold a variety of possibilities mimicking the Euclidean space.

Let $\mathbb{R} = (R, \cdot, +, \geq, \|\cdot\|_{\mathbb{R}})$ denote the totally ordered, complete field of the real numbers with a norm induced by its multiplication (as scalar product). This is a vector space on its own, endowed with further operations that are familiar to us from every day life. This plenitude of additional structures allows us to introduce a chronology necessary for the description of any evolution process, in particular for the direction of time. It further provides a measure of distance and thereby further orders the unrelated elements of the underlying point set R .

Once we framed this natural appealing space $\mathbb{R}$ into mathematical language, we can derive further algebraic structures from it. The d -dimensional vector spaces $\mathbb{R}^d = (R, +_{\mathbb{R}^d}, \mathbb{R}, \circ_{\mathbb{R} \times \mathbb{R}^d})$ are important examples of this procedure where a first notion of dimensionality appears on the cost of some properties of $\mathbb{R}$. It is most instructive to compare the field $\mathbb{R}$ with the vector spaces $\mathbb{R}^d$.

- A. First of all notice that the underlying point set, more precisely its cardinality, is the same for $\mathbb{R}$ and all $\mathbb{R}^d$ with $d \geq 1$, namely the uncountable infinite sets. Thus, in the category

of sets $\mathbb{R}$ and $\mathbb{R}^d$ are isomorph and the discriminability is only due to their algebraic (operational) properties.

- B. For $d = 1$ the vector space coincides with the field, i.e. $\mathbb{R}^1 \equiv \mathbb{R}$. With $d = 0$ one usually denotes a singleton set A , i.e. a set consisting of a single element: $\#(A) = 1$. Its cardinality obviously does not coincide with the one of $\mathbb{R}$.
- C. In the general case $d > 1$ we have lost the total order property of the field. What still remains is an ordering induced by the metric (distance function) which however is one-dimensional only.⁴ We have the following definition:

Definition 1.1.10 — Metric space. Let X be a set and $d_X : X \times X \rightarrow \mathbb{R}$ a function. The ordered pair (X, d) is a metric space $:\Leftrightarrow d$ is a metric on X , i.e. $\forall x, y, z \in Y$:

$$\begin{aligned} d_X(x, y) &= d_X(y, x) && \text{(symmetric)} \\ d_X(x, y) &= 0 \iff x = y && \text{(equal points)} \\ d_X(x, z) &\leq d_X(x, y) + d_X(y, z) && \text{(triangle inequality)} \end{aligned}$$

The non-negative function d_X thus ‘embeds’ the real numbers along with its rich structure into a subset of X . Any measurement relies on this kind of embedding and is therefore described within a metric space.

- D. A vector space intrinsically carries an algebraic dimension, given by the maximal number of linearly independent elements (vectors). Given a vector space $V \equiv (X_V, \circ_V, \mathbf{e}_V, \mathbb{K}, \circ_{\mathbb{K} \cdot V})$ over the field $\mathbb{K}$, a subset $\mathcal{B} = \{b_j \mid j \in J \subseteq \mathbb{N}\} \subseteq X_V$ is a linearly independent basis $:\Leftrightarrow$

$$a_k \in \mathbb{K}, \forall K \subseteq J \text{ with } \#(K) < \infty : \sum_{k \in K}^{\circ_V} a_k \circ_{\mathbb{K} \cdot V} v_k = \mathbf{e}_V \implies a_k = \mathbf{e}_{\mathbb{K}}, \forall k \in K$$

Each such (basis) vector generates a direction isomorphic to $\mathbb{R}$ and thereby gives rise to the above mentioned one-dimensional ordering. This *algebraic dimension* defines also a *topological* invariant restricted to the category of topological manifolds (with boundary).

Based on these special algebraic objects we continue our journey in the field of topology. In order to define topological spaces that locally reflect the vector- and metric space properties of $\mathbb{R}^d$ we have to equip $\mathbb{R}^d$ with a notion of open sets that respects its algebraic properties. Fortunately, there is a standard way to proceed here, once we have a metric space at our disposal. Though not every topological space can be promoted to a metric space the other way around is straightforward once we have defined a topological basis:

Definition 1.1.11 — Topological basis. Let $N \equiv (X_N, \tau_N)$ be a topological space. A collection of sets $\mathcal{B} \subseteq 2^X$ is a topological basis $:\Leftrightarrow$

$$\begin{aligned} \mathcal{B} &\subseteq \tau_N && \text{(all its elements are open)} \\ \forall U \in \tau_N : \exists \mathcal{B}' \subseteq \mathcal{B} \text{ such that } U &= \cup_{V \in \mathcal{B}'} V && \text{(open sets are unions of } V \in \mathcal{B}) \end{aligned}$$

We say τ_N is generated by $\mathcal{B}$, i.e. $\tau_N = \text{gen } \mathcal{B}$.

The two main benefits a topological basis provides are concerned with the construction of topologies and its reduction of information.

Therefore, let us choose a collection of subsets $\mathcal{B} \in 2^X$ that completely covers X . If it is chosen such that the condition

$$\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2 : \exists B_3 \in \mathcal{B} \text{ with } x \in B_3$$

⁴This is a common practice we will encounter whenever we need an order principle in our formalism. To chronologically order time we have to relate this concept to $\mathbb{R}$. For general orders within a set we have to find an isomorphism to $\mathbb{R}$ in the corresponding category.

holds true, then the collection of all unions of elements of $\mathcal{B}$ uniquely defines a topology on X . The second advantage refers to its generating nature. A topological basis contains the condensed information of all open sets, similar to a linearly independent maximal set of basis vectors allows to reconstruct the full vector space. It therefore usually suffices to test certain properties for the basis only and deduce global statements of the full space. One has to be careful though when the cardinality of the basis is infinite.

R In vector spaces there is a canonical upper limit of basis elements given by the maximal number of linearly independent vectors, which defines the dimension of vector spaces. In contrast, for topological spaces it is the lower bound on the number of basis elements that is of interest. The requirement that a basis should cover the full space X leads to a least cardinality of a basis: the weight of a topological space – invariant under homeomorphisms.

A subset of real space $\mathbb{R}$ is open if it can be written as a union of open intervals, (a, b) with $a \leq b \in \mathbb{R}$. A metric relates the structure of $\mathbb{R}$ to a subset of its elements and thereby allows to inherit its topology. For the real vector spaces $\mathbb{R}^d$ with the standard Euclidean metric $d_{\mathbb{E}^d}$ this leads to the following metric induced topological space:

Definition 1.1.12 — Euclidean space. The ordered pair $\mathbb{E}^d \equiv (\mathbb{R}^d, \tau_{\mathbb{E}^d}, d_{\mathbb{E}^d})$ is the Euclidean space of dimension $d : \Leftrightarrow$

- ▶ $d_{\mathbb{E}^d}$ is the standard metric on $\mathbb{R}^d$, i.e. given a vector space basis $\{v_j \mid j \in [1, d]\}$, for all $q, p \in \mathbb{R}^d$ we have with $q = \sum_{k \in [1, d]} q_k \circ_{\mathbb{R}, \mathbb{R}^d} v_k$:

$$d_{\mathbb{E}^d}(q, p) := \sqrt{\sum_{k \in [1, d]} (q_k - p_k)^2}$$

- ▶ $\tau_{\mathbb{E}^d}$ is the standard topology induced by $d_{\mathbb{E}^d}$, i.e. open balls of varying radii

$$\tau_{\mathbb{E}^d} := \text{gen}\{B_r(p) := \{q \in \mathbb{R}^d \mid d_{\mathbb{E}^d}(p, q) < r\} \mid r \in \mathbb{R}, p \in \mathbb{R}^d\}$$

So far we have modeled the world of our (local) perception with the mathematical concept of a Euclidean space that combines algebraic structures, as the vector- and metric space properties, with a correlative topology. Our objective is the local embedding of the Euclidean richness to more general topological spaces.

To generalize these ideas to include boundaries for future investigation, we have to first introduce a mechanism to construct new topological spaces from existing ones. For our purpose, the subspace topology suffices, for more detailed account see [1].

Definition 1.1.13 — Subspace topology. Let $N \equiv (X_N, \tau_N)$ be a topological space. A collection $\tau_N|_A \subseteq 2^{X_N}$ is the subspace topology for a subset $A \subseteq X_N : \Leftrightarrow$

$$\tau_N|_A := \{U \cap A \mid U \in \tau_N\}$$

The ordered pair $(A, \tau_N|_A)$ is the induced topological subspace of N on A .

If $A \in \tau_N$ is an open set, the very definition of a topology guarantees that also the subspace topology is a subset of the deduced one, i.e. $\tau_N|_A \subseteq \tau_N$. In general however, their intersection may consist of only the empty set and thus $\tau_N|_A \not\subseteq \tau_N$. In order to introduce a boundary into the description we have to confine one direction of the vector space $\mathbb{R}^d$ to the positive (or equivalently negative) real numbers only. Breaking down the topology of the corresponding Euclidean space to the remaining $\mathbb{R}^{d-1} \times \mathbb{R}_0^+$ using the subspace topology yields

Definition 1.1.14 — Euclidean space with boundary. We say the ordered pair $\mathbb{E}_\partial^d \equiv (\mathbb{R}^{d-1} \times \mathbb{R}_0^+, \tau_{\mathbb{E}_\partial^d}, d_{\mathbb{E}_\partial^d})$ is the d -dimensional Euclidean space with boundary : $\Leftrightarrow$

- ▶ $\mathbb{R}_0^+$ is the Euclidean half-space, i.e. its elements are given by $\{p \in \mathbb{R} \mid p \geq 0\}$,
- ▶ $d_{\mathbb{E}}$ is the standard metric on $\mathbb{R}^d$ restricted to the subset $\mathbb{R}^{d-1} \times \mathbb{R}_0^+$,
- ▶ $\tau_{\mathbb{E}_\partial^d}$ is the subspace topology induced by the standard topology $\tau_{\mathbb{E}^d}$ on $\mathbb{E}^d$, i.e.

$$\tau_{\mathbb{E}_\partial^d} := \{U \cap (\mathbb{R}^{d-1} \times \mathbb{R}_0^+) \mid U \in \tau_{\mathbb{E}^d}\}$$

The boundary of this space is given by $\partial \mathbb{E}_\partial^d := \mathbb{E}_\partial^d \cap (\mathbb{R}^{d-1} \times \{0\})$.

For a long time it seemed to be appropriate extending our local intuitive picture of a Euclidean space to a global construction and assuming that spacetime is simply given by $\mathbb{E}^4$. At least since the advent of General Relativity this idea began to unravel and the distinction between locality and the global patch turned out to be of uttermost importance in understanding the fundamentals of space and time. This presumable insight brought physicists to a more tedious but general program, a route we are currently following. Instead of making global statements about spacetime, we are modest and only make assumptions about its local structure:

Definition 1.1.15 — Locally Euclidean space (with boundary). Let $\mathbb{H}^d$ be the Euclidean space $\mathbb{E}^d$ (with boundary $\mathbb{E}_\partial^d$). A topological space $N \equiv (X_N, \tau_N)$ is locally Euclidean (with boundary) : $\Leftrightarrow$

$$\begin{aligned} \exists d \in \mathbb{N}, \forall x \in X_N : \quad & \exists U_x \in \{U \in \tau_N \mid x \in U\} \\ & \exists V \subset \mathbb{H}^d, \exists f_{U_x} \in \{g : U_x \rightarrow V \mid g \text{ is homeomorphism} \} \end{aligned}$$

Then d is called the topological manifold dimension of N and (U_x, f_{U_x}) is a chart of N .

We can thus find open neighborhoods for every point of X_N which are topological equivalent to the same d -dimensional Euclidean space (with boundary). Locally, we regain the nice Euclidean structure along with its vector- and metric space properties, however sewing together the local patches usually results in global networks that topologically differ severely from Euclidean space.

The manifold boundary of topological spaces $N \equiv (X_N, \tau_N)$ that are locally homeomorphic to $\mathbb{E}_\partial^d$ consists of points in the inverse image of Euclidean boundary, i.e.

$$\partial N := \{x \in X \mid \exists (U_x, f_{U_x}) \text{ chart of } x, \exists q \in \partial \mathbb{E}_\partial^d : f_{U_x}(x) = q\} \quad (1.2)$$

Conversely, the union of all pre-images of $\mathbb{E}^d$ defines the manifold interior $\text{Int}(N)$. It seems natural to conclude that the manifold boundary and manifold interior do not overlap, i.e. are disjoint sets whose union compose the entire space. The subtlety in proving this statement lies in the local nature of charts and possible non-empty intersections of them: For a single chart (U, f_U) we have a homeomorphism that excludes manifold boundary points being simultaneously manifold interior points w.r.t. (U, f_U) . However, one has to make sure that for another overlapping chart this separation maintains (the actual proof involves some deep aspects of topology). This confirms the intuitive picture that manifold boundary and -interior are disjoint sets and in fact are topological manifolds without boundary of dimension $(d-1)$ and d , respectively [2].

R Notice that even though every locally Euclidean space is also a locally Euclidean space with boundary, the reverse only holds if the manifold boundary is empty.

At this point we have made our topological space locally Euclidean. Still the global result may reveal certain unexpected properties that does not fit our concept of spacetime. We will therefore add two more restrictions that excludes pathological topological spaces.

Hausdorff spaces

The infiniteness of the underlying point set and the ignorance of topology about measures or distances yield to a peculiar feature of Euclidean spaces for any dimension $d \geq 1$. In Euclidean spaces locality is equal to global picture, meaning that any open set of $\mathbb{E}^d$ is homeomorphic to $\mathbb{E}^d$ itself. For example the unit ball around the origin $B_1(0) := \{q \in \mathbb{R}^d \mid d_{\mathbb{E}^d}(0, q) < 1\}$ is homeomorphic to $\mathbb{E}^d$, and so they are topological equivalent. Open sets in the Euclidean space thus exhibit a self-similarity property:

$$\forall U_V \in V \in \tau_{\mathbb{E}^d} : \quad U_V \cong V \cong \mathbb{E}^d$$

One direct consequence is that there is no smallest open set, in particular single points are not open, $\{p\} \notin \tau_{\mathbb{E}^d}$.

In order to (locally) obtain a duplicate of Euclidean space we have to implement this self-similarity property into the global picture of spacetime. Hence, we require our topological space to fulfill the Hausdorff property:

Definition 1.1.16 — Hausdorff space. Let $N \equiv (X_N, \tau_N)$ be a topological space. N is a Hausdorff space (has the Hausdorff property) $:\Leftrightarrow$

$$\forall x \neq y \in X_N \exists U_x, U_y \in \tau_N \text{ with } x \in U_x, y \in U_y : \quad U_x \cap U_y = \emptyset$$

The Hausdorff property is a topological property that is preserved under homeomorphisms. Besides the closeness of single points this guarantees the uniqueness of limiting points and selects only those topological spaces that have ‘enough’ open sets.

Second-countable spaces

With the Hausdorff property we have a lower bound on the number of open sets that define a topology. However, an upper bound is similarly important to avoid infinity issues. This becomes particularly problematic when considering topological bases. Such a basis generates the topology and contains the essential information of its global features. If the basis is uncountable infinite one has to be very careful in the transition towards the full topology. In Euclidean spaces a countable infinite number of basis elements suffices to represent the collection of all open sets. This property comes under the name of second-countability and is defined as follows:

Definition 1.1.17 — Second-countable space. Let $N \equiv (X_N, \tau_N)$ be a topological space. N is a second-countable space $:\Leftrightarrow$

$$\exists \mathcal{B} \subseteq \tau_N, \#(\mathcal{B}) \leq \#(\mathbb{N}), \forall U \in \tau_N : \exists \mathcal{B}' \subseteq \mathcal{B} \text{ with } U = \cup_{V \in \mathcal{B}'} V$$

With the restriction to second-countable spaces we ensure the generating basis to be ‘well-behaved’ by the existence of an upper bound (though countable infinite in general).

Topological manifolds (with boundary)

The notion of a topological manifold combines the local concept of being Euclidean with global restrictions. All local neighborhoods share the same dimensionality d , thus the local requirement actually contains already some global information. We further mimic real space by the global properties of a Hausdorff- and second-countable space. Together they combine to a topological manifold which forms a generalization of Euclidean spaces to account for the modern

understanding of spacetime.⁵

Definition 1.1.18 — Topological manifold (with boundary). Let $N = (X_N, \tau_N)$ be a topological space. Let $\mathbb{H}^d$ be the Euclidean space $\mathbb{E}^d$ (with boundary $\mathbb{E}_\partial^d$).

N is a *topological manifold (with boundary) of dimension d* $:\Leftrightarrow$

$$\forall x \in X_N, y \in X_N \setminus \{x\} \quad \exists U_x, U_y \subseteq X_N, x \in U_x^\circ, y \in U_y^\circ : \quad U_x \cap U_y = \emptyset \quad (1.3a)$$

$$\begin{aligned} \exists d \in \mathbb{N} \forall x \in X_N : \quad & \exists U_x \in \{U \in \tau_N \mid x \in U\} \\ & \exists V \subset \mathbb{H}^d \exists f_{U_x} \in \{g : U_x \rightarrow V \mid g \text{ is homeomorphism} \} \end{aligned} \quad (1.3b)$$

$$\exists \mathcal{B} \subseteq \tau_N, \#(\mathcal{B}) = \#(\mathbb{N}), \forall U \in \tau_N : \quad \exists \mathcal{B}' \subseteq \mathcal{B} \text{ with } U = \cup_{V \in \mathcal{B}'} V \quad (1.3c)$$

That is, a topological space with Hausdorff property (1.3a), which is locally Euclidean (with boundary) (1.3b) and second-countable (1.3c).

It is apparent that the dimension of the topological manifold (with boundary) is inherited from the local vector space. Notice that the cardinality of the underlying point set is a severe restriction to the possible dimensions that can occur. Whenever the cardinality of the set is finite or countable infinite, the dimension of the vector space and thus the one of the deduced topological manifolds are zero. On the other hand, all real vector spaces of any non-vanishing dimension are obtained by the same equivalence class of uncountable infinite sets. The main concerns in studying Euclidean spaces are therefore sets of uncountably infinite cardinality, its zero dimensional limit includes all the remaining finite and countable infinite cases.

The dimensionality of a manifold, as well as the Hausdorff- and second countable property are topological invariants and are thus preserved under homeomorphisms.⁶ Hence, the class of topological manifolds do not stretch over different topological equivalence classes but rather define a subset in each element of $\text{TOP}_\sim$ separately.

We have further introduced the concept of a manifolds boundary and its interior and thereby laid the grounds for later cosmological spacetime models.

S In the category of topological manifolds (with boundary), spacetime is classified by its cardinality and a class of homeomorphic topologies that are Hausdorff, locally Euclidean (with boundary) of dimension d and second-countable; it is an element of

$$\text{TOP}_\sim^{\text{M}^d} := \{[(X_N, \tau_N)]_{\sim_{\text{TOP}}} \mid \tau_N \text{ topological manifold of dim. } d \}$$

$$\text{TOP}_\sim^{\text{M}_\partial^d} := \{[(X_N, \tau_N)]_{\sim_{\text{TOP}}} \mid \tau_N \text{ topological manifold with boundary of dim. } d \}$$

1.1.4 Smooth manifolds (with boundary)

Our mathematical formulation of our world, $N \equiv (X_N, \tau_N)$, is made of local pieces with Euclidean structure. Every such local patch is described by a chart (U, f_U) , an open set $U \in \tau_N$ with a homeomorphism to $\mathbb{E}_\partial^d$ that locally promotes the multifacetedness of real vector- and metric-spaces to a topological space. The diversity of possible topologies that emerges from this generalization of Euclidean spaces can be attributed to their transition behavior between different patches, i.e. the global variety of topological manifolds arises due to differently gluing together local pieces. While the Hausdorff- and second-countable properties retain certain local features even on the global level, especially non-topological aspects of Euclidean spaces are usually lost. This has in particular consequences for studying the theory of functions in the

⁵Here we denote the topological interior of a set by $A^\circ := \cup_{(U \in \tau_N) \subseteq A} U$.

⁶The theorem stating that different dimensional topological manifolds cannot be homeomorphic is known as invariance of domain. In particular $\mathbb{E}^n$ is not homeomorphic to any $\mathbb{E}^m$ whenever $m \neq n$.

important mathematical branch denoted analysis. Most of the functional calculus that was introduced to describe (smooth) functions on Euclidean space, needs modifications on topological manifolds, at least on a global scale.

Homeomorphisms are insensitive to differential aspects and we have to restrict the possible transitions to spread a consistent calculus over an entire topological space. Therefore, let us choose a collection of charts that cover a topological manifold and form what is known in mathematics as an atlas:⁷

Definition 1.1.19 — Atlas. Let $N \equiv (X_N, \tau_N)$ be a d -dimensional topological manifold.

A collection of ordered pairs $\mathcal{A} \equiv \{(U, f_U)\}$ is a P-atlas on $\mathbb{X} : \Leftrightarrow$

- ▶ Every element of $\mathcal{A}$ is a chart, i.e.
 $\forall c_U = (U, f_U) \in \mathcal{A} : (U \in \tau_X) \wedge (\exists V \in \tau_{\mathbb{E}_0^d} : f : U \rightarrow V \text{ is homeomorphism})$
- ▶ The collection of open sets in $\mathcal{A}$ is an open cover of X , i.e. $\cup_{c_U=(U,f_U) \in \mathcal{A}} U = X$.
- ▶ For any two charts $c_u = (U, f_U), c_w = (W, f_W)$ with intersecting sets, $U \cap W \neq \emptyset$:
 $\forall q \in U \cap V \text{ with } x = f_W(q) : (f_U \circ f_W^{-1} \text{ has P-property at } x)$

The P-property is a placeholder that specifies the kind of changes which are admissible in the transition from one patch to another. For instance, if we consider a smooth (analytic) atlas, this requires the transition functions to be smooth (analytic), etc. This defines what is known as a P-structure on our notion of spacetime. Two P-atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on a topological space are compatible, i.e. they give rise to the same P-structure, if any chart in $\mathcal{A}_2$ fulfills the P-property with any intersecting chart⁸ of $\mathcal{A}_1$. Since this similarity of atlases defines an equivalence relation we can partition the totality of possible P-atlases in equivalence classes, known as maximal P-atlases. The underlying topological manifold determines what kind of different P-structures we can attach to our notion of spacetime, in particular whether there are any p-atlases at all. While for the topological manifolds of dimension less than or equal to 3 there is a unique differential structure [8, 9], higher dimensional classifications are still unresolved. Nevertheless, the non-existence of smooth structures on certain topological manifolds has been shown, in particular for some higher dimensional compact manifolds (for further details consult [2]).

For our purpose it is of uttermost importance to preserve some differential aspects of Euclidean spaces. This requirement translates in a smooth structure we want to endow topological manifolds with, i.e. we require all transition functions $f_U \circ f_W^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ to be smooth in the C^∞ sense. With a selection of a maximal C^∞ -atlas we designate which functions on and to a topological manifold are infinitely differentiable. Distinct C^∞ -structures will alter the notion of differentiability on the manifold.

Spacetimes equipped with a C^∞ -structure are described by smooth⁹ manifolds, i.e. [2]

Definition 1.1.20 — Smooth manifold (with boundary). Let $N = (X_N, \tau_N)$ be a topological manifold (with boundary). The ordered pair $M = [(N, \mathcal{A})]_{\sim_{MC^\infty}}$ is a smooth manifold $: \Leftrightarrow$ $\mathcal{A}$ is a C^∞ -atlas on N .

We have ultimately reached the final mathematical formulation of our spacetime model by the class of smooth manifolds. Thereby we invoke a global differential calculus on the underlying topological space. In order to treat only distinguishable descriptions of spacetime we have absorbed equivalent smooth structures in the concept of a maximal C^∞ -atlas. The confining criterion for this partition is the existence of diffeomorphisms, i.e. smooth homeomorphisms that retain the C^∞ -structure by requiring its inverse to be smooth. This further refines the notion

⁷For each connected component an atlas contains either only one chart or if it has more than one, these charts overlap. Otherwise we would have a contradiction to the connectedness criterion.

⁸With intersecting charts we mean charts whose sets intersect.

⁹Smooth in what follows always refers to C^∞ .

of bijections for smooth manifolds, resulting in the concept of isomorphisms in the category of C^∞ -structures: the diffeomorphisms which intuitively speaking reflects a deformation without cracks.

The standard notion of differentiability we want to promote on the level of topologies refers to Euclidean space *without* boundary. A function is called differentiable if a certain limit exists in an open neighborhood. However, when a topological boundary enters the formalism there appear additional open sets for which this differentiability condition cannot be applied, those containing boundary points. The reason lies in the self-similarity of open sets in $\mathbb{E}^d$ and the prescription of deducing subspace topologies. In order to lift the standard differentiable structure to $\mathbb{E}_\partial^d$ and thereby to topological manifolds with boundary, we simply denote a function f to be differentiable whenever there is an extended C^∞ -function on $\mathbb{E}^d$ that agrees with f on the restricted set. One implication that results from this definition is the invariance of the manifolds boundary under diffeomorphisms.[2]

Attaching an additional differential structure to a topological manifold results in our final definition of spacetime:

S In the category of smooth manifolds (with boundary), spacetime is classified by a topological manifold with a maximal C^∞ -atlas; it is an element of

$$M_\partial^d C^\infty := \{[(X, \tau_X, \mathcal{A})]_{\sim_{MC^\infty}} \mid \mathcal{A} \text{ is } C^\infty\text{-atlas on } (X, \tau_X) \in \text{TOP}_{\sim_\partial}^{M_\partial^d}\}$$

The isomorphisms of this category are the diffeomorphism defining indistinguishable spacetimes. Here $(X, \tau_X, \mathcal{A}_X) \sim_{MC^\infty} (Y, \tau_Y, \mathcal{A}_Y) :\Leftrightarrow (X, \tau_X) \sim_{\text{TOP}} (Y, \tau_Y) \wedge \mathcal{A}_X \text{ compatible with } \mathcal{A}_Y$.

1.1.5 Spacetime

In this section we have undertaken a journey to explore our model of spacetime in more detail. On the way we have encountered three main branches of mathematics that all add their parts to constitute our notion of the arena our entire formalism is based on.

Initially, we started with a point set whose entire information was contained in its cardinality, since we assumed the labeling of spacetime's building blocks to be artificial. The invariance under bijection was one of the main results that propagates through all categories we passed. In topology we enriched the simple sets by exploiting its power set and introduced a first notion of locality. Topological manifolds then intertwine linear algebra and topology by locally translating the 'manifold' properties of Euclidean space (with boundary) to more general topological constructions. In a last step we attached a global calculus by generalizing the standard differentiable structure to those topological manifolds. While homeomorphisms are used to select indistinguishable spacetimes in the category of topological manifolds, the additional requirement of preserving the smooth structure results in the equivalence under diffeomorphisms. The final model of spacetime – translated in the language of mathematics – is then:

Definition 1.1.21 — Spacetime. A spacetime is an equivalence class of smooth manifolds of dimension d defined on a topological space (X, τ_X) , i.e.

$$M \equiv [(X, \tau_X \subseteq 2^X, \mathcal{A} = \{(U \in \tau_X, f_U)\})]_{\sim_{MC^\infty}} \quad (1.4)$$

with the following ingredients:

- ▶ A set X with cardinality $\#(X)$
- ▶ A topology τ_X over X with the properties:
 - A. (X, τ_X) is locally a d -dimensional Euclidean space (with boundary),

- B. (X, τ_X) is a Hausdorff space,
 - C. (X, τ_X) is second countable.
- A maximal C^∞ -atlas $[\mathcal{A}]$ on (X, τ_X) .

The diffeomorphism invariance that appears in our later formalization of quantum gravity precisely reflects this definition where spacetime is actually an equivalence class of indistinguishable smooth manifolds, rather than a single representative. It is important to be aware that the arising symmetry of the formalism is part of the arena itself, rather than being a feature of field space.

Notice that very often spacetime is defined as a Riemannian- or pseudo Riemannian geometry, which is always possible as long as the underlying smooth manifold is paracompact. However, from the perspective presented here, introducing a Riemannian metric defines a connection on the tangent space of M and as such is a constituent of field space.

Furthermore, one can extend the present definition based on real Euclidean models to complex spaces which then results in similar structures that nevertheless reveal some uncommon features not present in the real case. The combination of quantum field theory and Lorentzian metric field content is best studied using complex time coordinates. While the real part still reflects the usual concept of time, its imaginary part can be related to temperature and thus a connection between quantum field theory and statistical physics can be established. Isotropic transformations in this time-temperature subspace are described by Wick-rotations. Whether or not a complexification of spacetime may provide some insights into questions about the attachment of electric charge to massive fields, or the prominent role of the spin group rather than the special orthogonal group which is more directly connected to spacetime, remains to be seen. For more details on complex spacetimes we refer to [10] and references therein.

1.2 Symmetries

One of the guiding principles in the construction of theoretical formulations of Nature are symmetries. Either experimentally observed or theoretically predicted they introduce severe constraints on the class of possible theories. In mathematics symmetries are described within the theory of groups and its affiliation with other branches usually results in strong simplifications and fruitful insights. The strength and wealth of symmetries are founded in its group properties that either act as invariance conditions to reduce complexity or as powerful tools to extract general forms of objects purely by their transformation behavior. Group theory is that important in today's physics, because it allows to assume less and simultaneously deduce more: The diversity of mathematical formalizations of Nature is broken down by invoking additional (physical) constraints, out of which symmetry constraints are usually the most restrictive ones. The glory of the Standard Model of particle physics is strongly correlated with its formulation in terms of symmetry groups that have led to many astonishing predictions verified by experiments.

In this section we touch only those aspects of group theory that are of particular importance in studying QFT. Besides a brief introduction into the basics of group theory, we consider its application to other algebraic structures, as for example vector spaces, and relate some physical notations to the emerging mathematical concepts. Finally, we exemplify this abstract constructions for several isomorphism groups equipped with different algebraic structures.

This section is mainly based on the following references [11–15], to which we refer the reader for more details.

1.2.1 Groups

Nowadays, group theory is incorporated in many fields of mathematics, physics, and chemistry to mention only a few. Compared to the wide-ranging consequences one can deduce from it, its

basic concepts rely on a very simple algebraic structure:

Definition 1.2.1 — Group. Let X be a set. The ordered pair $G \equiv (X, \circ_G)$ is a group $:\Leftrightarrow$ $\circ_G : X \rightarrow X$ is a relation on X with properties:

- ▶ $\forall x, y, z \in X : (x \circ_G y) \circ_G z = x \circ_G (y \circ_G z)$ (associativity)
- ▶ $\exists e_G \in X, \forall x \in X : x \circ_G e_G = e_G \circ_G x = x$ (identity element)
- ▶ $\exists (\cdot)_G^{-1} : X \rightarrow X$ such $\forall x \in X : (x)_G^{-1} \circ_G x = x \circ_G (x)_G^{-1} = e_G$ (existence of inverse)

The beauty that appears once group structures are uncovered within a problem, is related to the invariance and reduction that comes along. Groups are such powerful aspects because they are based on a binary, well-formed operation in the sense of being invertible, stable, and successively applicable. It therefore provides the natural grounds for studying transformations, inevitable in quantum field theory.

The classification of point sets is entirely due to their cardinalities $\#(X)$. When augmented with a binary relation these simple constructions develop a plenitude of $(\#(X))^3$ new structures. Restricting to those relations that fulfill the group properties reduces this number considerably. Two groups $G \equiv (X, \circ_G)$ and $F = (X, \circ_F)$ of the same underlying finite point set X are equivalent if there is an isomorphism in the category of groups that connects both $:\Leftrightarrow$

$$\exists f : X \rightarrow X \text{ bijection, } \forall x, y \in X : f(x \circ_G y) = f(x) \circ_F f(y) \quad (\text{homomorphism})$$

This condition ensures that group structures remain intact under homomorphisms while the bijective property takes care of the point set correspondence. In the case of infinite cardinality homomorphisms are still the valuable maps that categorize equivalent groups, however the number of distinct structures exceeds any limit.

Classification of groups

The classification of groups depends on the properties of their binary relation. There are two options to extract the intrinsic features of a group $G = (X, \circ_G)$ either by directly studying $\circ_G$ or by constructing new group specific (sub-)sets. An example for the first approach is given by the Abelian property with which commutative group structures are labeled, i.e. groups for which $x \circ_G y = y \circ_G x$ holds for all $x, y \in X$.

Concerning the latter classification we may consider the conjugacy classes of $G = (X, \circ_G)$ – subsets of X defined by the equivalence relation of conjugation:

Definition 1.2.2 — Conjugacy class. Let $G \equiv (X, \circ_G)$ be a group. A subset $C_G(x) \subseteq X$ is a conjugacy class of G at $x : \Leftrightarrow$:

$$C_G(x) = \{y \in X \mid \exists g \in G : y = (g \circ_G x) \circ_G (g)_G^{-1}\}$$

The totality of all conjugacy classes $\{C_G(x) \mid x \in X\}$ provides a deep insight into the underlying group structure and will be relevant, in particular, in representation theory. Notice that for Abelian groups each conjugacy class consists of a single element, $C_G(x) = \{x\}$, a feature that holds for $C_G(e_G) = \{e_G\}$ even in the general case.

Groups are something very special in the variety of all possible binary relations and they provide the mechanism to reduce the complexity of problems significantly. Once we hold such a valuable object in our hand it is worth to extract further related groups from it.

Subgroups

The classical way to construct groups from $G \equiv (X, \circ_G)$ is based on searching for closed restrictions of $\circ_G$ for some subset of X , i.e. investigating the symmetries of the binary relation $\circ_G$. The only singleton that forms a group consists of the identity element of G : $\{e_G\} \subseteq X$, the trivial subgroup $(\{e_G\}, \circ_G)$. While $(\{e_G\}, \circ_G)$ satisfies the group properties, the extension to

larger subsets severely depends on $\circ_G$. Adding another element $x \in X$ to the trivial subgroup in general results in introducing a chain of new elements in order to close the algebra and to satisfy Definition 1.2.1. For instance the inverse $(x)_G^{-1}$ usually differs from x itself so does $x \circ_G x$.

Depending on the interaction of a subgroup $H = (S \subseteq X, \circ_G)$ with its parent group $G = (X, \circ_G)$ there are some very special types among them:

Definition 1.2.3 — Normal subgroups. Let $G \equiv (X, \circ_G)$ be a group. The ordered pair $N = (\emptyset \neq S \subseteq X, \circ_G)$ is a normal subgroup of $G : \Leftrightarrow$

$$\begin{aligned} \forall x, y \in S : \quad (x \circ_G y \in S) \quad \wedge \quad ((x)_G^{-1} \in S) & \quad (\text{subgroup of } G) \\ \forall x \in S, g \in G \exists y \in S : \quad (g \circ_G x) \circ_G (g)_G^{-1} = y \in S & \quad (\text{normal}) \end{aligned}$$

One usually emphasizes N being a *normal* subgroup by using the notation $N \trianglelefteq G$.

A normal subgroup is the union of conjugacy classes of its elements and thus closed under conjugation. Notice that in general neither $N \trianglelefteq G$ is Abelian nor is an Abelian subgroup normal. In fact combine the requirement of a conjugate closed and Abelian subgroup yields the center of a group:

Definition 1.2.4 — Center of a group. Let $G \equiv (X, \circ_G)$ be a group. The ordered pair $Z(G) \equiv (S, \circ_G)$ is the center of $G : \Leftrightarrow$

$$S = \{x \in X \mid \forall g \in G : x \circ_G g = g \circ_G x\} \subseteq X$$

Besides its obvious Abelian nature, the center also commutes with the entire group and is therefore trivially normal. Moreover, the center of a group is non-empty since it always contains the identity element and whenever G is Abelian the group coincides with its center.

Finally, let us have a glimpse on a special class of orbits (see below), the cosets of subgroups.

Definition 1.2.5 — Coset. Let $G \equiv (X, \circ_G)$ be a group, $g \in G$, and $H \equiv (S \subseteq X, \circ_G)$ a subgroup.

A subset $gH \subseteq X$ ($Hg \subseteq X$) is a left (right) coset of H in G w.r.t. $g : \Leftrightarrow$

$$gH = \{g \circ_G h \mid h \in H\} \quad (\text{left coset}) \quad Hg = \{h \circ_G g \mid h \in H\} \quad (\text{right coset})$$

In general, a coset endowed with the group operation $\circ_G$ does not lead to a subgroup. Nevertheless, consider a subgroup H for which left cosets and right cosets agree, i.e. $gH = Hg \forall g \in G$. This reflects the property of H to be a normal subgroup and the cosets give rise to a group denoted as factor (or quotient) group.

Product groups

Another way to generate new groups is to extend the underlying point set by direct or semidirect products of existing groups:

Definition 1.2.6 — Semidirect product. Let $N \equiv (A, \circ_N)$ and $H \equiv (B, \circ_H)$ be groups. The ordered pair $G \equiv (N \times H, \circ_G)$ is a semidirect product of N and H w.r.t. $\varphi : \Leftrightarrow$

$$\begin{aligned} \varphi : H \rightarrow \{f : N \rightarrow N \mid f \text{ and } (\forall h \in H : \varphi(h) \equiv \varphi_h) \text{ are group isomorphism}\} \\ \forall g_1 = (n_1, h_1), g_2 = (n_2, h_2) \in G : \quad g_1 \circ_G g_2 = (n_1 \circ_N \varphi_{h_1}(n_2), h_1 \circ_H h_2) \end{aligned}$$

In abbreviated form one writes $G \equiv N \rtimes_{\varphi} H$.

It is straightforward to verify that the group isomorphism property of φ ensures that $G = N \rtimes_{\varphi} H$

factually defines a group.¹⁰ The structure of the original groups N and H is encoded in the subsets $\{(n, e_H) | n \in N\}$ and $\{(e_N, h) | h \in H\}$ that form a normal and a generic subgroup of G , respectively.

There is a shorthand notation to illustrate the interrelation of groups using what is called exact sequences. Within group theory an exact sequence is an ordered set of groups $\{G_j\}$ and homomorphisms $\{f_j\}$ written as

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} G_n \quad \text{with} \quad \text{Im}(f_j) = \ker(f_{j+1}) \quad \forall j$$

In more general cases group homomorphisms are replaced by the morphisms of the corresponding category. Using this abbreviate form the semidirect product $G = N \rtimes_{\varphi} H$ is expressed in terms of

$$1 \rightarrow N \xrightarrow{f_1} G \xrightarrow{f_2} H \rightarrow 1 \quad \text{and} \quad H \xrightarrow{f_2^{-1}} G,$$

where the group homomorphism is given by $\varphi_h(n) = f_1^{-1}(f_2^{-1}(h) \circ_G f_1(n) \circ_G f_2^{-1}((h)_H^{-1}))$. Another example is the central extension given by the following short exact sequence:

$$1 \rightarrow A \subseteq Z(E) \rightarrow E \rightarrow G \rightarrow 1$$

It has some relevance in physics associated with central charges, the generators of the center of the central extension of a (local) symmetry group.

A more common procedure to extend groups is a special case of Definition 1.2.6: the direct (group) product $G = N \times H$ defined by the trivial homomorphism $\varphi_h(n) = n \quad \forall h \in H, n \in N$ or in terms of the exact sequence as follows:

$$1 \rightarrow N \times H \xrightarrow{f_1} G \rightarrow 1$$

It is this construction of direct products that describes the symmetry group of the Standard Model of particle physics, the prosperous candidate for explaining all fundamental interactions excluding gravity. But once symmetries of *spacetime* are considered, especially those of Minkowskian spacetime, we are confronted with (a generalization of) the Poincaré group – a semidirect product of rotation- and translation symmetries.

1.2.2 Group actions

Though powerful on their own, groups unfold their full beauty when revealed within other structures. The general recipe to extract those hidden features is based on group actions.¹¹

Definition 1.2.7 — Group actions. Let $G \equiv (X, \circ_G)$ be a group and S a set. A function $\rho : X \times S \rightarrow S$ is a (left) group action of G on S : $\Leftrightarrow$

$$\begin{aligned} \forall g, h \in X, \forall s \in S : \quad & \rho(g \circ_G h, s) = \rho(g, \rho(h, s)) \\ \forall s \in S : \quad & \rho(e_G, s) = s \end{aligned}$$

The set S is denoted as a G -set. For a right group action the first condition has to be replaced

¹⁰The identity element of the semidirect product is given by $e_G = (e_N, e_H)$ and its inverse operation reads $((n, h))_G^{-1} = (\varphi_{(h)_H^{-1}}((n)_N^{-1}), (h)_H^{-1})$.

¹¹In what follows we focus mainly on left group actions, though the same results apply for the right group actions with some minor adaptations. In particular, every left group action ρ_L gives rise to a right action ρ_R and vice versa: $\rho_R(g, a) := \rho_L((g)_G^{-1}, a)$.

by $\forall g, h \in X, \forall s \in S: \rho(g \circ_G h, s) = \rho(h, \rho(g, s))$ (reversing the order).

There are intrinsic actions of a group onto itself given by the inner-, left-, and right-action fixing a specific member of the group. Now, using group actions we also have a tool to extrinsically attach symmetry properties to other structures with the aid of homomorphisms. This is a very strong mechanism to implement the deep insights we gain from group theory to other objects, in particular those we encounter in physics. Nevertheless, the mere presence of a group action on some set is insufficient to draw conclusions about their intertwined relation.¹² Rather, we have to consider group actions equipped with additional properties that bring those correlations between the underlying group and set to light. In the remainder of this subsection we will emphasize some of the most important features of group actions and finally focus on sets with vector space character that lead to the study of representation theory.

Types of group actions

Group actions as special types of (binary) functions inherit the corresponding characterization scheme of functions, however to stress the group structure in the first argument one uses a different notation.

Definition 1.2.8 — Types of group actions. Let $\rho : X \times S \rightarrow S$ be a left (right) group action of $G = (X, \circ_G)$ on the non-empty set S . The map ρ is ...

... faithful	$:\Leftrightarrow$	$\forall g \in X / \{e_G\} \exists a \in S: \rho(g, a) \neq a$
... free	$:\Leftrightarrow$	$\rho(g, a) = a \implies g = e_G$
... transitive	$:\Leftrightarrow$	$\forall a, b \in S \exists g \in X: \rho(g, a) = b$
... regular	$:\Leftrightarrow$	$\forall a, b \in S, \exists g \in X: \rho(g, a) = b$

Notice that a faithful group action ensures that each group element gives rise to a different map on S , a feature naturally fulfilled by the stronger condition of being free. With this property it is guaranteed that no group element is superfluous, but provides a different transformation rule on S . A free or a transitive group action are the correspondence of injectivity or surjectivity in the language of functions, respectively, while the regular condition combines both concepts and translates bijections into the present context.

In general group actions lack those features and their appearance is very welcome. Once we have establish a free group action on a set we can proceed along the lines of injective function to establish regular (bijective) group actions on subsets of S using orbits. Thereby, the transitive property is gained in the process of generating those subsets by fixing an element of S and let the group act on it, in detail we have:

Definition 1.2.9 — Orbits. Let $G \equiv (X, \circ_G)$ be a group, S a set, and $\rho : X \times S \rightarrow S$ be a left group action. A subset $O_x \subseteq S$ is the orbit of $a \in S$ w.r.t. ρ $:\Leftrightarrow$

$$O_x \equiv \{\rho(g, a) \mid g \in X\} \equiv \text{Im}(\rho(\bullet, a)) \subseteq S$$

This procedure divides the entire set in subsets on which the group acts transitive (surjective) and in fact defines an equivalence relation on S . For a free group action those equivalence classes, i.e. orbits, are the aforementioned partitions on which the group action is regular. In the general case however, the group action will have some fixed points that will spoil its injectivity, being not free. This is similar to functions which have a non-trivial kernel and in the context of group actions we refer to it as having a non-trivial stabilizer subgroup:

¹²For example the trivial group action exists for every group and non-empty set, where each element of the former acts as the identity on the set.

Definition 1.2.10 — Stabilizer subgroups. Let $G \equiv (X, \circ_G)$ be a group, S a set, and $\rho : X \times S \rightarrow S$ be a left group action. The ordered pair $G_a \equiv (A_a, \circ_G)$ is the stabilizer subgroup (isotropy group or little group) of $a \in S$ w.r.t. $\rho : \Leftrightarrow$

$$A_a \equiv \{g \in X \mid \rho(g, a) = a\} \equiv \ker(\rho(\bullet, a)) \subseteq X$$

At this point the group-structure-preserving nature of homomorphisms, as here given by ρ , becomes apparent: As morphisms of group theory they lift the category of sets to the category of groups and in shape of group actions we have seen that orbits and stabilizer subgroups take over the concepts of images and kernels of a map. The additional requirement of a function to be homomorphic actually yielded a partition in equivalence classes of the ‘images’ and ensures that its ‘kernels’ G_a are always *subgroups* of G .

Combining the ideas of orbits and stabilizer subgroups we can form a regular group action at each element of $a \in S$ by dividing out its stabilizer subgroup G_a and finally restrict the domain of $\rho(\bullet, a)$ to the orbit O_a . Especially in gauge theory a suitable knowledge of the underlying orbits and isotropy groups is of particular importance.

Linear representations

Besides specifying group actions by its function properties we can endow the G -set with additional structure and let group homomorphisms preserve it. Since in physics vector spaces play an essential role in understanding Nature, the way groups interact with them is a well studied subject in mathematical physics: the theory of (linear) representations. In fact, from a mathematical perspective representations provide a tool to understand groups in the way they (re-)act with the very familiar branch of linear algebra. Vector spaces come along with fields and a set of algebraic operations including an Abelian one, usually denoted by addition. In order to keep this tied construction intact under group homomorphisms we have to restrict the action of group elements to those functions that are linear, in other words automorphisms in the category of vector spaces:

Definition 1.2.11 — General linear group. Let $V \equiv (S, \circ_V, \mathbb{K}, \circ_{\mathbb{K} \cdot V})$ be a vector space over the field $\mathbb{K}$. A function $f : S \rightarrow S$ is a linear function in $V : \Leftrightarrow$

$$\begin{aligned} \forall a, b \in S : \quad f(a \circ_V b) &= f(a) \circ_V f(b) && \text{(additivity)} \\ \forall a \in S \forall \lambda \in \mathbb{K} : \quad f(\lambda \circ_{\mathbb{K} \cdot V} a) &= \lambda \circ_{\mathbb{K} \cdot V} f(a) && \text{(homogeneity of degree 1)} \end{aligned}$$

This subset of all linear functions of a vector space V forms the general linear group $\text{Gl}(V)$, the automorphism group in the category of linear spaces. While the group operation, given by function composition $\circ$, is closed for all (linear) functions, the existence of an inverse requires the functions to be bijections. $\text{Gl}(V)$ is a significant example of a (non-Abelian) group that is at the very basis of representation theory, as can be seen from the following definition:

Definition 1.2.12 — Linear representation. Let $G \equiv (X, \circ_G)$ be a group, $V \equiv (S, \circ_V, \mathbb{K}, \circ_{\mathbb{K} \cdot V})$ a vector space over the field $\mathbb{K}$. A left group action $\rho : X \times S \rightarrow S$ is a linear representation of G on $V : \Leftrightarrow$

$$\forall g \in X : \rho(g, \bullet) \equiv \rho_g \in \text{Aut}(V) \equiv \text{Gl}(V) := \{f : S \rightarrow S \mid f \text{ is a linear bijection on } V\}$$

Thus, (linear) representation can be understood as group actions that are specially designed to suit the needs of linear algebra.

The structure of a general linear group derives from its associated vector space that is completely characterized by the cardinality of its point set and its vector space dimension, at least in the finite-dimensional case. The variety of seemingly different linear spaces of the same dimen-

sion are only different facets that all correspond to the same equivalence class. When changing the basis one transitions into another representative of this class using an element of the general linear group. For finite-dimensional vector spaces, say $\dim V = d < \infty$, the distinct possibilities for such a basis transformation – hence the size of $\text{Gl}(V) \equiv \text{Gl}(d; \mathbb{K})$ – is also finite and can be represented by invertible $d \times d$ -matrices.¹³

At this point it is instructive to illustrate the coalescence of a free and a faithful group action within the language of representation theory. Since in this context group actions are rather considered as maps from a group into a function space $\text{Gl}(V)$, a faithful representation (group action), which addresses the group argument, actually translates into injectivity, i.e. each group element is mapped to a different function. For a free group action, usually a stronger condition, every group member is associated to a distinct *and injective* function. Within representation theory this does not amount to an additional restriction since all functions are bijections, thus a faithful representation is a free one and vice versa.¹⁴

Representations play an important role in today's physics, as symmetry considerations lie at the very heart of its mathematical foundations. Having established such a connection between a symmetry group G and a vector space V , one is keen to derive new representations from the original one by modifying G and/or V . Hereby, the term ‘new’ refers to distinguishable constructions in the category of representations, i.e. inequivalent linear, bijective group actions:

Definition 1.2.13 — Equivalent representations. Let $G \equiv (X, \circ_G)$ be a group. Further, let $V = (S_V, \circ_V, \mathbb{K}, \circ_{\mathbb{K}.V})$ and $W = (S_W, \circ_W, \mathbb{K}, \circ_{\mathbb{K}.W})$ be vector spaces.

Two (linear) representations $\rho^V : G \rightarrow \text{Gl}(V)$ and $\rho^W : G \rightarrow \text{Gl}(W)$ are equivalent : $\Leftrightarrow$

$$\begin{aligned} \exists f : S_V \rightarrow S_W \text{ linear } \forall g \in X : \quad f \circ \rho_g^V &= \rho_g^W \circ f \quad (\text{equivariant map from } V \text{ to } W) \\ f \text{ is an isomorphism, i.e. } \exists f^{-1} : S_W &\rightarrow S_V \text{ linear} \end{aligned}$$

Notice that equivariance is the natural way to preserve the structure within representation theory and the bijective property lifts this correspondence to isomorphisms. The natural object to distinguish inequivalent representations are their characters:

Definition 1.2.14 — Character of a representation. Let $G \equiv (X, \circ_G)$ be a group, $V = (S_V, \circ_V, \mathbb{K}, \circ_{\mathbb{K}.V})$ a vector space, and $\rho : X \times \text{Gl}(V)$ a (linear) representation of G on V . The function $\chi^\rho : G \rightarrow \mathbb{K}$ is the character of ρ : $\Leftrightarrow$

$$\forall g \in X : \quad \chi^\rho(g) = \text{Tr}(\rho(g)),$$

whereby Tr denotes the trace defined on endomorphisms of V (see Subsection 1.4.2).

Since the trace is cyclic two equivalent representations, $\rho^V = f \circ \rho^W \circ f^{-1}$, clearly have the same character. The treasure provided by group theory is that also the reverse holds true, namely if the character of two representations is the same both are equivalent.¹⁵ Our objective in the following is to extend and simultaneously reduce the representation as much as possible to deduce its inequivalent descendants.

In a first step we focus on a modification of the group and retain the full vector space V for a moment. If we take only a subset of group members to act on V we have to make

¹³At this step $\text{Gl}(V)$ needs vector space properties, which linear functions inherit pointwise from its underlying vector space.

¹⁴Transitivity is also part of the definition of a linear representation and thus the notion of regular group actions also coincides with faithful or free representations in this case.

¹⁵In what follows we always assume that the underlying field has characteristic zero, that is its multiplicative identity does not add up to give the additive identity.

sure that the group operations are closed, i.e. consider subgroups and their restricted group action:

Definition 1.2.15 — Restricted representation. Let $G \equiv (X, \circ_G)$ be a group, V a vector space, and $\rho : X \times \text{Gl}(V)$ a (linear) representation of G on V . Given a subgroup $H \equiv (A \subseteq X, \circ_G)$ of G , the map $\rho_H : A \times \text{Gl}(V)$ is the restricted representation of ρ on $H : \Leftrightarrow$

$$\forall g \in A : \quad \rho_H(g) = \rho(g)$$

This is a very natural restriction of the group action to only a reduced number of symmetry transformations. For the opposite direction there are in principle several ways to extend the linear group actions to a larger group. The induced representation reflects the reverse of a restricted group action, loosely speaking given a subgroup H of G with a representation the representation ρ_H one searches for an extension on G for which ρ_H is a restriction; formally, the induced representation is defined to be the left adjoint functor of ρ_H . Both concepts play an important role in the study of phases and symmetry breaking, as well as in the study of the diffeomorphism group.

Besides generating new representations by altering the group, an even more interesting approach consists in a variation of the underlying vector space. We can gain deep insights of abstract groups by simply investigating their imprint on structures of linear algebra, in particular their actions on linear spaces. The difficulty resides in the search of a suitable vector spaces that is sufficiently condensed while simultaneously being large enough to carry all essential information of the group.

Let us assume we are given two representations ρ^V and ρ^W of a group $G \equiv (X, \circ_G)$ acting on vector spaces V and W , respectively. The direct sum $V \oplus W$ yields a new, extended vector space and we can also endow it with a representation $\rho^{V \oplus W} = (\rho^V, \rho^W)$ of G induced by the existing ones that may result in new insights of the group G . However, the crucial point to notice here is the mutual ignorance of both subspaces V and W under group action – their non-interaction – implying that once we have perfectly understood ρ^V and ρ^W individually, there is no more we can learn from studying $\rho^{V \oplus W}$. This suggests to find the smallest possible building blocks that in its totality still contain the essential information to build up the full group structure. The objective is to spot subspaces that are invariant under group actions and decompose the full vector space using the group action. This procedure will terminate at some irreducible subspaces which – taken together – reveal the abstract group and its properties:

Definition 1.2.16 — (Ir-)reducible representation. Let $V \equiv (S, \circ_V, \mathbb{K}, \circ_{\mathbb{K} \cdot V})$ be a vector space, $G \equiv (X, \circ_G)$ a group. A representation $\rho : X \times \text{Gl}(V)$ of G on V is reducible : $\Leftrightarrow$

$$\begin{aligned} \exists W = (A \subset S, \circ_W, \mathbb{K}, \circ_{\mathbb{K} \cdot W}) \text{ non-trivial subspace of } V, \text{ i.e. } A \neq \{e_V\}, \text{ such that} \\ \forall g \in G \forall a \in A : \quad \rho_g(a) \in A \end{aligned}$$

If ρ has no non-trivial sub-representation (i.e. not reducible) it is called irreducible.

These fundamental building blocks contain all essential information of a group in a very condensed form, i.e. a full knowledge of irreducible representations is equivalent to the full knowledge of all representations which in turn implies understanding the group.

The characterization of inequivalent irreducible representations thus lies at very heart of representation theory and is based on Schur's lemma [16]. It states that for finite dimensional irreducible representations¹⁶ equivariance implies equivalence, i.e. if two irreducible representations are related by an equivariant map f this relation is either an isomorphism or f is the

¹⁶A representation is called (in-)finite dimensional when the associated vector space is (in-)finite dimensional.

zero function. The theory of irreducibility is centered around this lemma and we will come to appreciate it while studying its implications.

We have seen that the content of representation theory is fully described by fundamental building blocks, the irreducible representations of a group, and it is our aim to find and classify all of them. The necessary information needed is the number of those basic ingredients, the associated vector spaces they naturally live in, and their interrelation. We focus on each of these issues in the following list by taking an arbitrary group $G \equiv (X, \circ_G)$:

- **Number of irreducible representations n_{irrep} :** On the search for irreducible representations we first of all have to know how many we have to look for. Fortunately, the number of these group actions is equivalent to a group property: the number of conjugacy classes. As mentioned above, inequivalent representations give rise to different trace functions (characters) and we can thus characterize each of the n_{irrep} inequivalent irreducible representations by exactly those characters $\{\chi^{(j)} \equiv \chi^{\rho^j} \mid \rho^j \text{ irreducible representation of } G\}$.
- **Dimension of irreducible representations:** As soon as we have pinned down the count of different characters and thus the number of irreducible representations, it is important to know their associated vector spaces. In the finite dimensional case, a vector space is fully characterized by its dimensionality. Let us assume a group has n_{irrep} inequivalent irreducible representations $\rho^{(j)}$ that map into vector spaces of dimension $d^{(j)}$, which is denoted the dimension of the representation $\rho^{(j)}$. Now, the dimensionality theorem, which is a direct consequence of the orthogonality theorem we will study in a second, relates the number of elements in G to the sum of all $d^{(j)}$'s:

$$\#(X) = \sum_{j \in [1, n_{\text{irrep}}]} \left(d^{(j)}\right)^2 \quad (1.5)$$

Clearly, if the group is infinite, i.e. it is based on an infinite set, we have little to gain here. Nevertheless, for any finite group we have strong restrictions for the dimension $d^{(j)}$ of each irreducible representation $\rho^{(j)}$.

- **Relation between irreducible representations:** Finally, a great deal of information is obtained by studying the correlation of distinct building blocks. We have already account for the fact that inequivalent representations, in particular irreducible ones, give rise to different characters and are factually encoded in those 1-dimensional representations. From Schur's lemma one can deduce the following important orthogonality theorem:

$$\begin{aligned} \delta^{jk} &= \langle \chi^{(j)} \chi^{(k)*} \rangle_G := \int_{g \in X} d\mu^{(G)}(g) \chi^{(j)}(g) \chi^{(k)*}(g) \\ &\equiv \int_{g \in X} d\mu^{(G)}(g) \chi^{(j)}(g) \chi^{(k)}(g^{-1}) \end{aligned} \quad (1.6)$$

Here, the normalized integral $\int_{g \in X} d\mu^{(G)}(g) = 1$ generalizes the summation over group elements, ' $\frac{1}{\#(X)} \sum_{g \in X}$ ', in order to cover also compact, but infinite groups. (We will have a closer look at measures in section 2.1). The statement of eq. (1.6) is a vital guiding principle to deduce the full structure of irreducible representations using orthogonality of their characters. In addition there is a similar relation between the conjugacy classes of the group. Given a conjugacy class of a group member $g \in X$, i.e. $C_G(g) = \{f \circ_G g \circ_G f^{-1} \mid f \in X\}$, and another group element $h \in X$ we have:

$$\sum_{j \in [1, n_{\text{irrep}}]} \chi^{(j)}(h^{-1}) \int_{f \in C_G(g)} d\mu_G(f) \chi^{(j)}(f) = \begin{cases} 0, & h \notin C_G(g) \\ 1, & h \in C_G(g) \end{cases} \quad (1.7)$$

It is quite astonishing that most of the prescriptions for classifying the fundamental objects of representation theory relies on scalar functions, the characters.

In principle, representation theory reduces to answering the above questions for each group to obtain the full set of inequivalent irreducible representations. Once at hand, any finite dimensional representation $\rho : G \rightarrow \text{Gl}(V)$ of a (finite) group can be completely decomposed into those fundamental blocks (Maschke's theorem [17, 18]):

$$\rho_g = \bigoplus_{j \in [1, n_{\text{irrep}}]} a_{\rho}^{(j)} \rho_g^{(j)}$$

The coefficients $a_{\rho}^{(j)}$ are non-negative integers that stand for the number of times the respective representation $\rho^{(j)}$ is present in the decomposition of ρ . Their actual values can be deduced again with the help of characters and the orthogonality theorem:

$$a_{\rho}^{(j)} = \int_{g \in X} d\mu^G(g) \chi^{\rho}(g) \chi^{(j)}(g^{-1})$$

Taking all together, characters that symbolize irreducible representations play the most important role in studying finite group actions on vector spaces, which in fact reveals a vast amount of the group properties itself. Before coming to an end of this brief introduction of representation theory let us consider some very important application of group actions in physics: symmetry phases.

Phases and phase-transitions

A very important aspect of group actions in physics is associated to symmetry phases and the corresponding phase-transitions [19]. It has many applications in different branches of physics – high- or low energy physic, quantum or classical in nature – and thus underlines the influence of group theory in today's sciences. To fit all those approaches into one general description we have to introduce an abstract notation of constraints defined on a G -set S and some parameter space Λ :

$$(f_j, W_j^{\text{True}} \subset W_j) \quad \text{with} \quad f_j : S \times \Lambda \rightarrow W_j$$

Whenever $f_j(x, u) \in W_j^{\text{True}}$ we say that (x, u) fulfills the constraints. In practical cases f_j will be related to observable quantities and its explicit form will be quite intuitive, see the examples below.

Depending on the value of the parameters u the constraint functions will allow different elements of S , the order parameter space, to be admissible. The collection of all those order parameters $x \in S$ that satisfy the constraints defines the space of solutions:

$$\text{Sol}(u) := \{x \in S \mid f_j(x, u) \in W_j^{\text{True}} \quad \forall j\} \subseteq S \quad (\text{constraints-fulfilling solutions})$$

Assume we have a theory where $U(x, T)$ represents the potential that depends on an external parameter $T \in \mathbb{R}_+$, standing for the temperature of the system, say. Furthermore, let $f : S \times \mathbb{R}_+ \rightarrow W_1 \equiv S$ be defined by $f_1(x, T) = \partial_x U(x, T)$ and $W_1^{\text{True}} = \{0\}$ then all elements of $\text{Sol}(T)$ are extrema of the potential (for minima we need an additional constraint f_2 that restricts to positive second derivatives of $U(x, T)$). Now, the space of solutions sensitively depends on the order parameter space. For instance, let $U(x, T) = (x^3 + x) = U(x)$ be of polynomial type, then $\text{Sol}(T)$ consists of $\{+\iota, -\iota\}$ in case of $S = \mathbb{C}$ while it is empty for the real numbers $\mathbb{R}$.

In the general case, $\text{Sol}(u)$ will also depend on the parameters u and thus it will exhibit different symmetries for different values of u :

$$X^{\text{Sol}}(u) = \{g \in X_G \mid \rho(g, x) \in \text{Sol}(u) \quad \forall x \in \text{Sol}(u)\} \subseteq X_G \quad (\text{symmetry of solutions})$$

Basically, this contains all group elements that map a solution of f_j to another one, and thus it is the intersection of all stabilizing subgroups of elements in $\text{Sol}(u)$. Using this information the symmetry phases partition S by means of an equivalence class of connected spaces on Λ all sharing the same symmetry of solutions. In full generality, the division of the parameter space is encoded in the symmetry phases with respect to the considered constraints, defined as follows:

Definition 1.2.17 — Symmetry phases. Let $G \equiv (X_G, \circ_G)$ be a group, $S = (Y_S, \tau_S)$ and $\Lambda = (Y_\Lambda, \tau_\Lambda)$ be topological spaces. Furthermore, let $\rho : X_G \times S \rightarrow S$ be a (left) group action and $(f_j : Y_S \times \Lambda \rightarrow W_j, W_j^{\text{True}})$ a family of constraints. Define an equivalence relation on the parameter space Λ as follows:

$$v \sim_\Lambda w : \Leftrightarrow \exists U \in \tau_\Lambda, v, u \in \overline{U}, \text{ connected with } \forall w \in U \cup \{v\} : X^{\text{Sol}}(u) \equiv X^{\text{Sol}}(w)$$

The ordered pair (Λ_A, G_A) is a symmetry phase of $(S, \Lambda, G, \{(f_j, W_j^{\text{True}})\}, \rho) : \Leftrightarrow$

$$\exists u \in \Lambda_A : \Lambda_A \equiv \{v \in \Lambda \mid v \sim_\Lambda u\} \quad (\text{equivalence class})$$

$$\forall u \in \Lambda_A : G_A = (X^{\text{Sol}}(u), \circ_G) \subseteq G \quad (\text{symmetry of solutions})$$

The set S is the order parameter-, Λ the parameter-, and $\text{Sol}(\bullet)$ the solution space.

In this general case the constraint functions can assume values in any set and the corresponding constraint is implemented by requiring the image to be in $W_j^{\text{True}} \subset W_j$. Once we are dealing with a normed vector space most often the above form reduces to some constraints over the real space.

The set of solutions $\text{Sol}(u)$ respecting the constraints is in general very sensitive to the parameter choice u and so is the set of its invariant transformations $X^{\text{Sol}}(u)$. In physics the transitions from one symmetry phase to another provides some very interesting insight into the underlying theory and its properties. As we have topological spaces at our disposal these phase transitions occur at the overlap of topological boundaries of two or more phases, say (Λ_A, G_A) and (Λ_B, G_B) :

$$\Lambda_{A \leftrightarrow B} := \overline{\Lambda_A} \cap \overline{\Lambda_B}$$

If this set is empty both phases do not share a direct phase transition. In other cases one distinguishes the change that happens at such a $\Lambda_{A \leftrightarrow B}$ by classifying the continuity of the transition. Basically modern schemes invoke the notation that a discontinuous change of $\text{Sol}(u)$ through the boundary between phases corresponds to a *first order phase transition* while a continuous change is denoted to be a *higher order phase transition*. In the present context, for $\Lambda_{A \leftrightarrow B} \neq \emptyset$, this translates into the following statement for a higher order phase transition:

$$\exists U \in \tau_\Lambda \text{ with } \Lambda_{A \leftrightarrow B} \subseteq U : \{x \in \text{Sol}(u) \mid u \in U\} \in \tau_S$$

Whenever this condition is not fulfilled we encounter a first order phase transition.

We have seen that in the general setting we detect phases of a group action $\rho : S \times G \rightarrow S$ using order parameters and the classification of transitions is due to their continuity along the boundary of phases. In quantum field theory or statistical physics, the field expectation values are usually the constrained solutions that, depending on the explicit form of the measure $\mu(\bullet; \{u\})$, exhibits different symmetry invariances.

It is instructive to give some very simple examples of symmetry phases that however exemplify all the abstract concepts we have introduced previously. Let us define the following system and later specify the missing ingredient, a differentiable function $\eta : S \times \Lambda \rightarrow \mathbb{R}$, to generate all

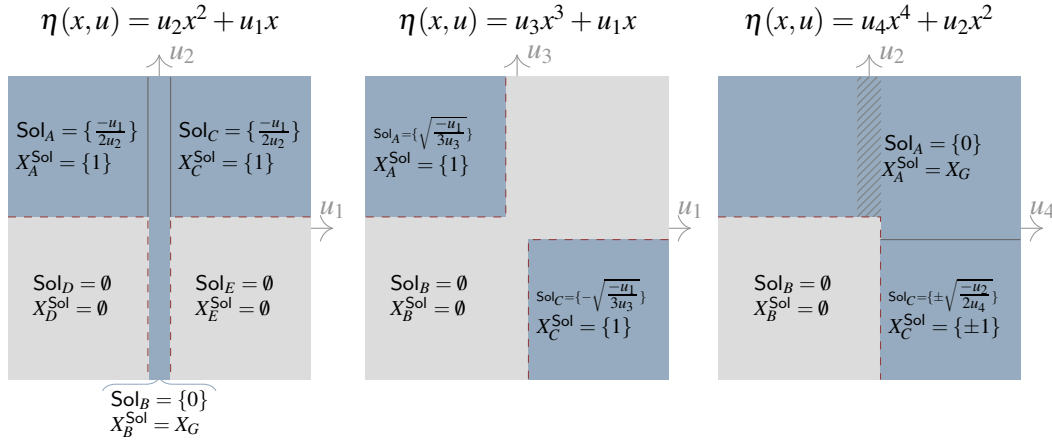


Figure 1.1: The phase diagrams for different choice of $\eta(x, u)$. Symmetry phases are indicated by different shadings. Dashed (solid) lines correspond to first (higher) order phase transitions.

the different examples:

$$\begin{aligned}
 \text{Order parameter space: } S &\equiv \mathbb{R} & \text{Parameter space: } \Lambda &\equiv \mathbb{R} \times \mathbb{R} \\
 \text{Symmetry group: } G &\equiv (\mathbb{R}/\{0\}, \cdot) & \text{Group action: } \rho(g, x) &= g \cdot x \quad \forall g \in G, x \in S \\
 \text{Constraints: } \{(f_1(x, u) &:= \frac{d}{dx}\eta(x, u), W_1^{\text{True}} \equiv \{0\}), (f_2(x, u) = \frac{d^2}{dx^2}\eta(x, u), W_2^{\text{True}} \equiv \mathbb{R}_0^+)\}
 \end{aligned}$$

The constraints are written in the general notation used above, which at this point hides its simple meaning. Actually, the requirements we impose is minimizing the function η , as can be seen by translating them into standard form:

$$\{0\} \ni \frac{d}{dx}\eta(x, u) = 0 \quad \wedge \quad \mathbb{R}_0^+ \ni \frac{d^2}{dx^2}\eta(x, u) \geq 0 \quad \forall x \in \text{Sol}(u)$$

The phase diagrams for three different types of η , thus different types of constraints, are depicted in Figure 1.1.

R Changing the order parameter space from $\mathbb{R}$ to $\mathbb{C}$ will result in very different equivalence classes. In the present case the minima condition is quite restrictive for it involves solutions of polynomials that for some parameters are imaginary and thus are excluded.

1.2.3 Lie groups & Lie algebras

The reason for studying groups in physics is their relation to symmetries which helps revealing the properties of a mathematical structure. Again the cardinality of the point set determines the variety of symmetries that may appear, which is finite for any finite set. In quantum field theory infinities are part of the game and the symmetry groups we encounter are usually based on uncountably infinite sets. Fortunately, the ‘physical’ groups are endowed with additional properties that simplifies matter a lot. The infiltration of topology and smooth structures into group theory gives rise to the concept of Lie groups introducing the notion of continuity and differentiability:

Definition 1.2.18 — Lie groups. Let $(X, \circ_G)$ be a group. The ordered pair $G \equiv (X, \circ_G, \tau_X, \mathcal{A})$

is a Lie group $:\Leftrightarrow$

$\tau_X \subseteq 2^X$ with τ_X a topology

$\mathcal{A} = [\{(U \subseteq X, f_U) \mid (U, f_U) \text{ is } C^\infty\text{-chart}\}]_{\sim_{MC^\infty}}$ is maximal C^∞ -atlas

$\forall x, y \in X : (x, y) \mapsto (x)_G^{-1} \circ_G y$ is C^∞ -function

G is thus a group and a smooth manifold for which the group operations are C^∞ -functions.

Lie groups are such powerful objects because their algebraic construction is compatible with its topological and differential structure. Their inherited features are manifold. We have a closed binary relation that allows to propagate through the set in a continuous and smooth way and there is a local connection to Euclidean space with its properties, in particular a metric- and vector space construct attached to every point in the group. It is the latter, the (local) vector space structure, that ties another very important concept to each local piece of a Lie group, a special vector space named Lie algebra:

Definition 1.2.19 — Lie algebras. Let $V = (X, \circ_V, \mathbb{K}, \circ_{\mathbb{K}, V})$ be a vector space over the field $\mathbb{K}$. The ordered pair $\mathfrak{g} \equiv (V, [\bullet, \bullet]_{\mathfrak{g}} : X \times X \rightarrow X)$ is a Lie algebra $:\Leftrightarrow$

$\forall x \in X : [x, x]_{\mathfrak{g}} = e_V$ (alternating)

$\forall x, y, z \in X, a, b \in \mathbb{K} : [(a \circ_{\mathbb{K}, V} x) \circ_V (b \circ_{\mathbb{K}, V} y), z]_{\mathfrak{g}} = (a \circ_{\mathbb{K}, V} [x, z]_{\mathfrak{g}}) \circ_V (b \circ_{\mathbb{K}, V} [y, z]_{\mathfrak{g}})$
(bilinearity) $[z, (a \circ_{\mathbb{K}, V} x) \circ_V (b \circ_{\mathbb{K}, V} y)]_{\mathfrak{g}} = (a \circ_{\mathbb{K}, V} [z, x]_{\mathfrak{g}}) \circ_V (b \circ_{\mathbb{K}, V} [z, y]_{\mathfrak{g}})$

$\forall x, y, z \in X : [x, [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} \circ_V [z, [x, y]_{\mathfrak{g}}]_{\mathfrak{g}} \circ_V [y, [z, x]_{\mathfrak{g}}]_{\mathfrak{g}} = e_V$

The last property is the Jacobi-identity that is respected by the Lie bracket $[\bullet, \bullet]_{\mathfrak{g}}$.

The connection between a Lie group and its Lie algebra can be best understood within the theory of differential geometry using fiber bundle constructions. At this point we only scratch the surface of this deep relation and postpone a more consistent treatment to section 1.3. The basic idea is to consider the group action w.r.t. a specific element as an isomorphic transformation of the group to itself i.e. maps of the form $\rho : G \rightarrow \{f : G \rightarrow G \mid \text{group isomorphism}\}$, for instance:

... the left action: $g \mapsto L(g) \equiv L_g$ with $L_g(x) = g \circ_G x$ $\forall x \in G$,
 ... the right action: $g \mapsto R(g) \equiv R_g$ with $R_g(x) = x \circ_G g$ $\forall x \in G$,
 ... the inner action: $g \mapsto \text{ad}(g) \equiv \text{ad}_g$ with $\text{ad}_g(x) = (g)_G^{-1} \circ x \circ g$ $\forall x \in G$.

Now, the differential nature of Lie groups gives a meaning to an infinitesimal transformation, that is a group isomorphism whereby the elements are shifted to its closest neighbors. The homomorphism properties of the above actions imply that the local vicinity of the identity element e_G is responsible for these small shifts with the limit of e_G inducing no change at all. Since we have vector space structures locally attached everywhere, we can generate infinitesimal transformation using the vector space at the identity e_G that actually fulfills the Lie algebra properties. Thus, the local vicinity at each point on the group manifold is fully determined by its associated Lie algebra. However the Lie algebra may loose global information of the underlying Lie group, for example its connectedness status. Nevertheless, from a group theoretical point of view there are certain relations that are preserved within the Lie algebra. For instance, in the case of a connected Lie group G the Lie algebra of the (Abelian) subgroup $Z(G)$, the center of G , coincides with the center of the full Lie algebra $\mathfrak{g}$ of G .¹⁷ Hence, whether or not a Lie group is Abelian can be read off by the Abelian property of its Lie algebra, whenever G is connected.

¹⁷It is common to emphasize the allocation of a Lie algebra to a Lie group by choosing small Gothic letters, i.e. the Lie algebra of a Lie group G is denoted by $\mathfrak{g}$.

In conclusion, by fusing group theory with the concept of smooth manifolds we enter the realm of Lie groups, a rich and powerful structure that seems to be at the core of today's understanding of Nature. The theory of Lie algebras is closely attached to these constructions and will be explored later on. For now, we conclude with a classification of those structures.

Classification

While topological features as, for instance, compactness and connectedness determines the global nature of a Lie group, the algebraic information are almost entirely encoded in its associated Lie algebra. Classification of Lie groups thus is closely related to a suitable classification of Lie algebras, at least in the finite dimensional case.¹⁸

For finite groups we have seen that representation theory reveals the group properties by studying its irreducible group actions in a very condensed form using scalar functions, the characters. However, the interesting part of Lie groups concerns infinite groups, where the local vector space dimension is usually finite but non-zero, thus its underlying point set has uncountable infinite cardinality. Therefore, we have to proceed on a different – though closely related – path that is concentrated more on roots and fundamental weights than solely on character functions.

Though based on an infinite group, Lie groups come along with a naturally related vector space, their Lie algebra, which provides a direct route to extract the infinite number of irreducible representations using the set of infinitesimal transformations. It is the classification of Lie algebras that will answer the question of irreducible representations for those infinite, but nicely equipped Lie groups.

To characterize Lie algebras, let us start by its vector space properties. The rich algebraic properties of vector spaces allows to reconstruct the full space using a sufficiently small number of elements that form a linearly independent basis, which in the context of Lie algebras is usually referred to as a set of generators. Each basis can be transformed into another one using an element of the associated general linear group and is thus not a unique construct the vector space gives rise to. Nevertheless, since the general linear group consists of the automorphisms, the vector space results are in general preserved under a change of basis.

Though for vector spaces the basis is sufficient for its full reconstruction, a Lie algebra is additionally equipped with a binary operation, the Lie bracket that is not fixing by choosing a basis. Fortunately, a Lie bracket is bilinear and hence for a given set of generators we can encode its complete information into a set of structure functions:

Definition 1.2.20 — Structure functions. Let $V \equiv (X, \circ_V, \mathbb{K}, \circ_{\mathbb{K},V})$ be a vector space of dimension d , $\mathfrak{g} \equiv (V, [\bullet, \bullet]_{\mathfrak{g}} : X \times X \rightarrow X)$ a Lie algebra, and $\mathcal{B} \equiv \{T_j \in X\}_{j=[1,d]}$ a linearly independent basis that spans V .

Functions $f_{jk}^m : [1,d] \times [1,d] \times [1,d] \rightarrow \mathbb{K}$ are the structure function of $\mathfrak{g}$ w.r.t to $\mathcal{B} : \Leftrightarrow$

$$\forall T_j, T_k, T_m \in \mathcal{B} : [T_j, T_k]_{\mathfrak{g}} = \sum_m f_{jk}^m \circ_{\mathbb{K},V} T_m \quad (1.8)$$

The special properties Lie algebras are endowed with allow to reduce its entire information, up to isomorphisms, into a set of d^3 scalar coefficients along with the associated vector space basis, i.e. $\mathfrak{g} \hat{=} (\{T_j\}_{j=[1,d]}, f_{jk}^m)$.

At this step it is instructive to consider the defining equation of the structure function, eq. (1.8) in the light of representation theory. The two necessary ingredients a representation is based on, a group and a vector space, can be conflated into the same object when considering Lie algebras. One intrinsic group action of $\mathfrak{g}$ is given by the adjoint representation

¹⁸There exist infinite dimensional Lie algebras that are not associated to a Lie group, see [20].

where the Abelian group of the underlying vector space induces a natural action using the Lie bracket:

Definition 1.2.21 — Adjoint representation of a Lie algebra. $V \equiv (X, \circ_V, \mathbb{K}, \circ_{\mathbb{K} \cdot V})$ be a vector space of dimension d , $\mathfrak{g} \equiv (V, [\bullet, \bullet]_{\mathfrak{g}} : X \times X \rightarrow X)$ a Lie algebra. The smooth group action $\text{ad} : (X, \circ_V) \rightarrow \text{Gl}(V)$ is the adjoint representation of $\mathfrak{g} : \Leftrightarrow$

$$\forall x \in X : \quad \text{ad} : x \mapsto \text{ad}_x \equiv [x, \bullet]_{\mathfrak{g}}$$

The bilinearity of the Lie bracket manifests an interpretation of structure functions as generators for the adjoint representation by evaluating ‘ad’ on the basis elements:

$$\forall T_j, T_k \in \mathcal{B} : \quad \text{ad}_{T_j}(T_k) = [T_j, T_k]_{\mathfrak{g}} = f_{jk}^m T_m$$

This intrinsically present Lie algebra endomorphisms ad_x is the key to understand the irreducible representations of the associated Lie group. As a particular form of endomorphisms ad_x can be chained and traced over, and when both operations are combined it gives rise to the Killing form:

Definition 1.2.22 — Killing form. Let $\mathfrak{g} \equiv (V, [\bullet, \bullet]_{\mathfrak{g}} : X \times X \rightarrow X)$ be a Lie algebra over the field $\mathbb{K}$. A binary function $K : X \times X \rightarrow \mathbb{K}$ is the Killing form of $\mathfrak{g} : \Leftrightarrow$

$$\forall x, y \in X : \quad K(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y) \equiv \text{tr}([x, [y, \bullet]_{\mathfrak{g}}]_{\mathfrak{g}})$$

The function inherits the bilinearity and symmetry property of the Lie bracket and the trace, respectively. Using the Killing form we obtain a first characterization of Lie algebras:

Definition 1.2.23 — Semisimple Lie algebras. Let $\mathfrak{g} \equiv (V, [\bullet, \bullet]_{\mathfrak{g}} : X \times X \rightarrow X)$ be a Lie algebra over the field $\mathbb{K}$ and K its Killing form. The Lie algebra $\mathfrak{g}$ is semisimple : $\Leftrightarrow$

$$\forall x \in X : \quad K(x, \bullet) \text{ is isomorphism}$$

A more common definition involves the non-existence of non-zero Abelian ideals of $\mathfrak{g}$ which – due to Cartan’s criterion – is equivalent to the more practical form used above. If the restriction to Abelian ideals is dropped then a Lie algebra is simple, meaning it has no non-trivial ideal at all.

Now, Maschke’s theorem concerning the full reducibility of finite groups can be extended to all finite dimensional Lie groups that have a semisimple Lie algebra: A semisimple Lie algebra can be fully decomposed into simple ones that corresponds to a decomposition into a direct sum of irreducible representations. In the language of Lie groups reducible and irreducible representations are replaced by semisimple and simple Lie algebras, respectively. Finally, instead of studying only characters we have to introduce another concept the highest weights which take over the role to classify irreducible subspaces. We now summarize the algorithm to obtain a full characterization of finite-dimensional, irreducible representations of any semisimple Lie algebra $\mathfrak{g} \equiv (V, [\bullet, \bullet]_{\mathfrak{g}} : X \times X \rightarrow X)$ over the complex field $\mathbb{K} \equiv \mathbb{C}$:

A. Semisimple: Check whether or not the Lie algebra is semisimple using the Killing form which has to be non-degenerate:

$$\forall x \in X : \quad K(x, \bullet) \text{ is isomorphism}$$

B. Cartan subalgebra: A maximal Abelian subalgebra $\mathfrak{h} \equiv (V_{\mathfrak{h}}, [\bullet, \bullet]_{\mathfrak{g}} : X_{\mathfrak{h}} \times X_{\mathfrak{h}} \rightarrow X_{\mathfrak{h}}) \subset \mathfrak{g}$:

$$\forall H, H' \in X_{\mathfrak{h}} : \quad [H, H']_{\mathfrak{g}} = e_V$$

$$\forall a \in X : \quad [H, a]_{\mathfrak{g}} = e_V \quad \forall H \in X_{\mathfrak{h}} \implies a \in X_{\mathfrak{h}}$$

is called a Cartan subalgebra of $\mathfrak{g}$ and the dimension of the linear subspace $V_{\mathfrak{h}}$ of V defines its rank $\text{rk}_{\mathfrak{g}}$. The dual space of $V_{\mathfrak{h}}$ denoted $V_{\mathfrak{h}}^*$, in short $\mathfrak{h}^*$, contains all linear functions that map $X_{\mathfrak{h}}$ to the complex numbers:

$$V_{\mathfrak{h}}^* := \{\alpha : X_{\mathfrak{h}} \rightarrow \mathbb{C} \mid \text{linear function}\}$$

C. Root system: Decompose the full Lie algebra $\mathfrak{g}$ into the Cartan subalgebra and associated root spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha} \right)$$

For a d -dimensional (finite) Lie algebra this Cartan decomposition involves only a finite number of $\mathfrak{g}_{\alpha}$'s implying that the cardinality of $R \subset V_{\mathfrak{h}}^*$ is $d - \text{rk}_{\mathfrak{g}}$. Each additional subspace we attach to the Cartan subalgebra has the following property:

$$\mathfrak{g}_{\alpha} = \{E_{\alpha} \in X \mid \exists \alpha \in V_{\mathfrak{h}}^*, \forall H \in X_{\mathfrak{h}} : \text{ad}_H(E_{\alpha}) \equiv [H, E_{\alpha}]_{\mathfrak{g}} = \alpha(H) \circ_{\mathbb{C}, V} E_{\alpha}\}$$

The roots of the Lie algebra $\alpha \in V_{\mathfrak{h}}^*$ can be interpreted as non-zero eigenvalues of the adjoint representation and the corresponding eigenspaces $\mathfrak{g}_{\alpha}$ are referred to as root spaces. One of the fundamental statements for these root spaces is their one-dimensionality and therefore their associated eigenvalues α are all non-degenerate (Cartan's theorem, see ref. [12]). Thus, each $\mathfrak{g}_{\alpha}$ is spanned by a single basis element E_{α} , the root vector, which in physics is usually denoted as step (or ladder) operator.

Furthermore, the roots exhibit a $\mathbb{Z}_2$ -symmetry, meaning that for every $\alpha \in \mathbb{R}$ also its inverse root $(-1) \circ_{\mathbb{C}, V^*} \alpha \equiv -\alpha$ is in $\mathbb{R}$ but at the same time any other multiple $c \circ_{\mathbb{C}, V^*} \alpha \neq \pm \alpha$ is not a root. While the adjoint representation of any E_{β} corresponds to a translation in $\mathfrak{h}^*$ from $\mathfrak{g}_{\alpha}$ to $\mathfrak{g}_{\alpha+\beta}$, its inverse root space $E_{-\beta}$ is mapped to the Cartan subalgebra. It therefore appears that the direct sum

$$\mathfrak{s}_{\beta} \equiv \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{-\beta} \oplus [\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}]_{\mathfrak{g}} \text{ with } [\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}]_{\mathfrak{g}} \equiv \{c \circ_{\mathbb{C}, V} [E_{\beta}, E_{-\beta}]_{\mathfrak{g}} \mid c \in \mathbb{C}\} \subseteq \mathfrak{h}$$

forms a subalgebra of $\mathfrak{g}$ isomorphic to the simple Lie algebra $\mathfrak{su}(2)$ (see below). This has some important and very nice features on which we briefly explore in what follows. First of all, there is a unique element¹⁹

$$H_{\alpha} := \frac{2}{\alpha^2} \sum_{j \in [1, \text{rk}_{\mathfrak{g}}]} \alpha(H_j) \circ_{\mathbb{C}, V} H_j \in [\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}]_{\mathfrak{g}} \subseteq \mathfrak{h}$$

that fulfills the requirement $\alpha(H_{\alpha}) = 2$ and leads to the standard description of $\mathfrak{su}(2)$:

$$[H_{\alpha}, E_{\pm\alpha}]_{\mathfrak{g}} = \pm 2E_{\pm\alpha}, \quad [E_{\alpha}, E_{-\alpha}]_{\mathfrak{g}} = \frac{1}{2}\alpha^2 H_{\alpha}$$

While this shows the closure of the subalgebra $\mathfrak{s}_{\alpha}$, for general root vectors we have $[H_{\alpha}, E_{\beta}]_{\mathfrak{g}} = \frac{2}{\alpha^2} \sum_j \alpha(H_j) \beta(H_j) E_{\beta}$. The revelation of those simple Lie algebras is a great fortune in understanding the irreducible representations of $\mathfrak{g}$. One direct consequence is that for any vector space the representation of H_{α} will have integer eigenvalues.

¹⁹Here and in what follows we use the short hand notation $\alpha^2 = \sum_{j \in [1, \text{rk}_{\mathfrak{g}}]} \alpha(H_j) \cdot \alpha(H_j)$.

D. Simple roots & root lattice: Cartan's decomposition involves a sum over $d - \text{rk}_{\mathfrak{g}}$ roots $\alpha \in V_{\mathfrak{h}}^*$ contained in the selection R , whereas its vector space $V_{\mathfrak{h}}^*$ is only $\text{rk}_{\mathfrak{g}}$ -dimensional. To construct a suitable basis on the root lattice Λ_R , the span of R ,

$$\Lambda_R := \left\{ \sum_{\alpha \in R} c_{\alpha} \circ_{\mathbb{C} \cdot V^*} \alpha \mid c_{\alpha} \in \mathbb{C} \right\} \subseteq V_{\mathfrak{g}}^*,$$

we have to implement an ordering principle on Λ_R to define positive and negative directions within the root lattice. This requires an embedding of the real numbers $\mathbb{R}$ into Λ_R , in fact a linear one to preserve the vector space properties. In detail we have to select a linear function $f_R : \Lambda_R \rightarrow \mathbb{R}$ that gives rise to a splitting:

$$R = R^+ \cup R^-, \quad \text{with } R^+ = \{\alpha \in R \mid f_R(\alpha) > 0\} \text{ and } R^- = \{\alpha \in R \mid f_R(\alpha) < 0\}$$

A direct way to define such a function f_R involves an ordering of the basis elements in $\mathfrak{h}$, that is a labeling $H_1, \dots, H_r$ and then define $f_R(\alpha)$ positive (negative) whenever the first non-vanishing value of $\alpha(H_k)$ has positive (negative) real part. Though f_R introduces an auxiliary ingredient into the prescription, it turns out that the following result does not depend on its particular choice. Having divided the roots in positive $\in R^+$ and negative $\in R^-$ roots what remains is the reduction of R^+ to its basis elements. In a final step we search for simple roots in R^+ which are elements that can not be expressed in terms of two or more other positive roots, i.e.

$$R_S^+ := \{\alpha^S \in R^+ \mid \nexists \beta, \gamma \in R^+ : \alpha^S = \beta \circ_{V^*} \gamma\}$$

It turns out that R_S^+ forms a linearly independent basis for Λ_R and all positive (negative) roots can be written as

$$\alpha = \sum_{j \in [1, \text{rk}_{\mathfrak{g}}]} n_j \circ_{\mathbb{C} \cdot V^*} \alpha_j^S$$

with non-vanishing positive (negative) integers n_j .

There is a nice schematic description to characterize Lie algebras using Dynkin diagrams [14]. We start with the Cartan matrix of simple roots given by

$$M_{jk} := \frac{2}{\alpha_k^S(H_r)} \sum_{r \in [1, \text{rk}_{\mathfrak{g}}]} \alpha_j^S(H_r) \cdot \alpha_k^S(H_r)$$

which is a non-singular square matrix of rank $\text{rk}_{\mathfrak{g}}$. Furthermore, it can be shown – using the Weyl group – that it fulfills the following constraints [12]:

$$\forall j \neq k \in [1, \text{rk}_{\mathfrak{g}}] : \quad M_{jk} \in [0, -1, -2, -3], \quad M_{jk} M_{kj} \in [0, 1, 2, 3], \quad M_{jj} = 2$$

The important information encoded in the non-diagonal elements can be cast into the form of a graph where each circle corresponds to a simple root and the number of connecting lines illustrate their relations, thus represents the off-diagonal values. Hereby the convention is such that whenever $M_{jk} > M_{kj}$ holds true, an arrow is superimposing the lines pointing towards the simple root α_j^S . If and only if the Dynkin diagram is connected we are dealing with a simple Lie algebra which can fall into nine different classes.

As an example consider the Lie algebra $\mathfrak{so}(5)$ with simple roots α_1^S and α_2^S (using the notation that $H_j^*(H_k) = \delta_{jk}$), Cartan matrix $M^{\mathfrak{so}(5)}$, and corresponding Dynkin diagram:

$$\alpha_2^S = H_1^* \circ_{V^*} (-H_2^*) \wedge \alpha_1^S = H_2^* : \quad \Longleftrightarrow \quad M^{\mathfrak{so}(5)} \equiv \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

$$\Longleftrightarrow \quad \text{Dynkin diagram: } \text{---} \bigcirc \text{---} \bigcirc \text{---}$$

Every form can be recovered by any other up to isomorphisms and so they are all equivalent descriptions of the same underlying Lie algebra.

- E. Weights, highest weights & physical charges:** It now appears that once we have succeeded in characterizing the adjoint representation of a Lie algebra (as done in the previous steps) it is straightforward to deduce irreducible representations on any finite-dimensional vector space W . Therefore consider a faithful but otherwise arbitrary representation $\rho : \mathfrak{g} \rightarrow \text{Gl}(W)$. We can adapt the results of the adjoint representation to the more general case while only slightly changing the notation to retain the special status of the root structure.

Let us start by looking for some special vector in W called a highest weight vector:

Definition 1.2.24 — Highest weights. Let $\rho : \mathfrak{g} \rightarrow \text{Gl}(W)$ be a finite-dimensional representation of the Lie algebra $\mathfrak{g}$ on $W \equiv (A, \circ_W, \mathbb{K}, \circ_{\mathbb{K}.W})$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra and $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_{\alpha})$ its decomposition into root spaces $\mathfrak{g}_{\alpha}$ with R^+ its positive roots. Furthermore, let $\mu^{\bullet} : \mathfrak{k} \rightarrow \mathbb{C}$.

An element $w \in W$ is a highest weight vector of $\mathfrak{g}$ on $W : \Leftrightarrow$

$$\begin{aligned} \forall H \in \mathfrak{h} : \quad \rho_H(w) &= \mu^w(H) \circ_{\mathbb{C}.W} w && \text{(eigenvector of } \mathfrak{h}) \\ \forall \alpha \in R^+, : \quad \rho_{E_{\alpha}}(w) &= e_w && (w \text{ is in the kernel of } \mathfrak{g}_{\alpha}) \end{aligned}$$

Any finite-dimensional representation of a semisimple complex Lie algebra gives rise to at least one highest weight vector and if it is in addition irreducible there is a unique such element. Instead of abusing the notation of roots, which are reserved for the adjoint representation, one denotes eigenvalues of ρ_H as weights μ , and in the special case of the highest weight vector it obtains the supplementary adjective ‘highest’. Notice that when choosing the linear combination H_{α} the weights $\mu(H_{\alpha})$ have to be of integer type.

A highest (or dominant) weight vector symbolizes the anchor where our formalism is attached to. While elements of $\mathfrak{h}$ simply scale this vector w , all positive root vectors E_{α} map w to the identity e_w . The only possibility for a non-trivial transformation of a highest weight is an (successive) application of negative root vectors $E_{-\alpha}$ where $\alpha \in R^+$ (so $-\alpha \in R^-$). Remarkably, any irreducible representation of the Lie algebra $\mathfrak{g}$ can be obtained by this latter procedure while restricting to simple negative roots only. Thus given any representation $\rho : \mathfrak{g} \rightarrow \text{Gl}(W)$ and a highest weight vector w w.r.t. $\mu \in \mathfrak{h}^*$. The subspace related to the associated irreducible representation $\rho^{\text{irrep}} : \mathfrak{g} \rightarrow \text{Gl}(W_{\text{irrep}})$ is then given by:

$$W_{\text{irrep}}^{\mu^w} \equiv \{w, \rho_{E_{-\alpha_1^S}}(w), \rho_{E_{-\alpha_2^S}}(w), \dots, \rho_{E_{-n_r\alpha_r^S - \dots - n_1\alpha_1^S}}(w)\}, \quad \alpha_j^S \in R_S^+, n_j \in \mathbb{N}_0$$

The string of simple root compositions $-n_r\alpha_r^S - \dots - n_1\alpha_1^S$ will ultimately terminate for some values of n_j which is bounded by the fact that W_{irrep} is of finite dimension. Due to the uniqueness condition mentioned above each irreducible representation is fully characterized by its highest weight. Any such highest weight can be uniquely rewritten in terms of a linear combination of non-negative integer coefficients factoring the so called fundamental weights given by $\omega_r(H_{\alpha_j^S}) := \delta_{r,j}$. Notice that in physics, highest weights correspond to the respective charges of the underlying gauge theory.

- F. Multiplicity of weights:** Characters provide a condensed form of the information contained in a representation and their knowledge allows us to decompose an arbitrary representation into irreducible ones. In case of a compact Lie algebra, the character is given by Weyl’s character formula, which we will not present here, because far more terminology on e.g. the Weyl group is needed. We focus here only on the multiplicity of a certain

weight within a representation and give the dimensionality associated to them. First of all, the dimension of a finite representation with highest weight μ^W is given by the Weyl dimension formula:

$$\dim(W_{\text{irrep}}^{\mu^W}) \equiv \frac{\prod_{\alpha \in R^+} K(\mu^W + \rho^W, \alpha)}{\prod_{\alpha \in R^+} K(\rho^W, \alpha)}$$

with $\rho^W \equiv \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. This provides the initial point from which to successively derive the multiplicity of the weight spaces associated to a weight μ within $W_{\text{irrep}}^{\mu^W}$, the Freudenthal's multiplicity formula:

$$(\|\mu^W + \rho^W\|^2 - \|\mu + \rho^W\|^2) \cdot \dim(W^\mu) = 2 \sum_{\alpha \in R^+} \sum_{j \geq 1} K(\mu + j \cdot \alpha, \alpha) \dim(W^{\mu + j \cdot \alpha})$$

The full decomposition of any representation into its irreducible counterparts can be carried out by the Steinberg formula [12].

1.2.4 Examples of Groups

In this final remark on symmetries and groups, we will exemplify the above mentioned concepts for some relevant groups essential in the study of physics. We start with finite groups, then move on to infinite groups that however can be associated to a finite dimensional vector space, and finally conclude with the isomorphism group of functions, being infinite with any respect.

Symmetric groups and Young tableaux

The symmetric groups S_n are the prototypes for any finite group due to their relation to the isomorphisms of sets, the bijections:

$$S_n = (\{f : X \rightarrow X \mid f \text{ is bijection}\}, \circ) \quad \text{with} \quad \#(X) = n < \infty$$

Since we have a finite number of elements the set of S_n has also finite cardinality $n!$. The cyclic notation is a very instructive description of bijections $f : X \rightarrow X$ with $\#(X) = n < \infty$ that allows to classify the group elements of S_n . The underlying observation is that any bijection can be represented by a successive application of transpositions or in general permutations in the following way $f \simeq (a_1^{(1)} \dots a_{j_1}^{(1)})(a_1^{(2)} \dots a_{j_2}^{(2)}) \dots$ with $f(a_n^{(r)}) = a_{n+1}^{(r)}$ and $a_{j_1+1}^{(r)} = a_1^{(r)}$. Since interchanging the order of cycles does not affect the function we choose the convention to order them in a natural way by requiring $j_n \geq j_m$ whenever $n < m$.²⁰ The group operation of two bijections g and f on X is the usual composition of functions $\circ_{S_n} \equiv \circ$, thus in cyclic notation we have $(a_1^{(1)} \dots) \dots (a_1^{(r_a)} \dots) \circ (b_1^{(1)} \dots) \dots (b_1^{(r_b)} \dots)$.

For finite groups it is usually the first step to determine the conjugacy classes, since they correspond to the number of inequivalent irreducible representations. It turns out that bijections are categorized under conjugation into several classes each sharing the same partition of n , meaning the possible ways to split n be a sum of real numbers. For a systematic approach we go back to the cyclic notation and order their internal structure such that largest blocks of permutations always stand on the left as mentioned above. In the case of S_6 , for instance, instead of writing $f \simeq (1)(352)(46)$ we reorder this cycle such that the largest permutation comes to the left, the second largest follows, etc. giving $f \simeq (352)(46)(1)$ – an element of the conjugacy class $C_5(3+2+1)$. Hence, bijections that split into different cyclic shapes are elements of different conjugacy classes, e.g.

$$g \simeq (234)(156) \in C_5(3+3) \quad \text{while} \quad h \simeq (156)(24)(3) \in C_5(3+2+1)$$

²⁰Notice that each number from 1 to n can only appear once in this cyclic notation, since f is a bijection.

The labeling of the elements is irrelevant for this classification and usually each conjugacy class is represented in a diagrammatic way by means of Young diagrams. This method is best illustrate listing some examples, say this time for S_4 :

$$\begin{aligned} \lambda_1 \equiv C_{S_4}(4) &\simeq \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}, & \lambda_2 \equiv C_{S_4}(3+1) &\simeq \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, & \lambda_3 \equiv C_{S_4}(2+2) &\simeq \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, \\ \lambda_4 \equiv C_{S_4}(2+1+1) &\simeq \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}, & \lambda_5 \equiv C_{S_4}(1+1+1+1) &\simeq \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \end{aligned} \quad (1.9)$$

A particular representative is given by a filling of the boxes with the elements of X .

The symmetric group is actually an exceptional case for which we can move on and systematically construct its irreducible representations. Therefore, we have to define a standard Young tableaux, some special representative of a conjugacy class whereby the filling is in a particular order, precisely it decreases along the row from left to the right and along a column from top to bottom. For instance $\lambda \equiv C_{S_4}(3+1)$ contains three standard tableaux, denoted $(\lambda)_{ST}$,

$$T_1 \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad T_2 \equiv \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array} \quad T_3 \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}$$

Together they generate the span of the irreducible representation corresponding to $\lambda \equiv C_{S_4}(3+1)$, referred to as Specht module. The number of such standard tableaux can in general be computed directly from the Young diagram itself, using the product of hook length of each box. For a given conjugacy class $\lambda \equiv C_{S_n}(n_1 + \dots + n_r)$ we obtain:

$$d_\lambda^{S(n)} := \#\{T_j \mid T_j \in (\lambda)_{ST}\} \equiv \frac{n!}{\prod_{j,\ell} \text{hook}_{j,\ell}^{(\lambda)}}, \quad \text{hook}_{j,\ell}^{(\lambda)} := (n_j - \ell) + \sum_{k \geq j} \Theta(n_k + 1 - \ell)$$

Thus the Specht module represents a $d_\lambda^{S(n)}$ -dimensional $\mathbb{C}$ -vector space explicitly given by

$$\text{Specht}^{(\lambda)} := \left\{ \sum_{r=1, d_\lambda^{S(n)}} c_r \sum_{\pi \in \text{col}(\lambda)} \text{sgn}(\pi) [\pi \circ T_r \circ \pi^{-1}]_{\sim_{\text{row}}} \mid T_r \in (\lambda)_{ST} \wedge c_r \in \mathbb{C} \right\}$$

whereby $\text{col}(\lambda)$ and $\text{row}(\lambda)$ are the stabilizer group of the Young diagram that interchange elements between columns or rows, respectively, without deforming the shape of λ . Correspondingly $\sim_{\text{row}}$ is the equivalence relation of Young tables under the action of $\text{row}(\lambda)$.

Finally, the full character table can be derived using the Murnaghan–Nakayama rule [21]. It is again based on the Young diagrammatic approach constructing some (non-standard) fillings from which to compute the characters. Basically, for an irreducible representation represented by a Young diagram λ , one obtains the character for a conjugacy class $\mu \equiv C_{S_n}(n_1 + \dots + n_r)$ in the following way. Find the set of possible Young tableaux of λ with n_j boxes filled with j . While equally filled boxes have to be connected one has to avoid 2×2 squares of equal entries, yielding what is known as border-strip tableaux. Then one determines their total heights $H^{(\mu_{bs})} \equiv (-1)^{r - \sum_{j=1, r} H_r^{(\mu_{bs})}}$, whereby $H_j^{(\mu_{bs})}$ agrees with the number of rows the entries j populate. The characters for elements of μ in the representation λ are thus determined by $\chi^\lambda(g \in \mu) \equiv \sum_{\mu_{bs}} H^{(\mu_{bs})}$. This completes the full investigation of the symmetric group for any finite n . We will see that these considerations have also impact on infinite groups, especially those which play an important role in today's physics.

Let us conclude by listing the full character table of S_4 obtained by the described methods. We will use the order of conjugacy classes given in eq. (1.9) so that each χ^{λ_j} defines a tuple with entries $(\chi^{\lambda_j}(g \in \lambda_1), \dots, \chi^{\lambda_j}(g \in \lambda_5))$

$$\begin{aligned}\chi^{\lambda_5} &= (1, 1, 1, 1, 1) && \text{(trivial)} \\ \chi^{\lambda_4} &= (-1, 0, -1, 1, 3) && \text{(standard)} \\ \chi^{\lambda_3} &= (0, -1, 2, 0, 2) \\ \chi^{\lambda_2} &= (1, 0, -1, -1, 3) && \text{(standard + signed)} \\ \chi^{\lambda_1} &= (1, -1, 1, -1, 1) && \text{(signed)}\end{aligned}$$

Similarly, for S_2 we obtain:

$$\begin{aligned}\chi^{\square\square}(g \in \square\square) &= 1 & \chi^{\square\square}(g \in \begin{smallmatrix} \square \\ \square \end{smallmatrix}) &= 1 && \text{(trivial)} \\ \chi^{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}(g \in \square\square) &= -1 & \chi^{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}(g \in \begin{smallmatrix} \square \\ \square \end{smallmatrix}) &= 1 && \text{(signed)}\end{aligned}$$

Notice that in both cases the characters are orthogonal, as required.

Young diagrams are a great tool specially tailored to the need of determine irreducible representation, decomposition, etc. of the symmetric group. We will encounter them at various other occasions, where an intrinsic relation to the symmetric group is present. For a more detail account on Young diagrams, Young tableaux, and Young operators we refer to [11–15]

The general linear group $Gl(d, \mathbb{K})$

Leaving the case of finite cardinality, in the next step we consider isomorphisms on finite-dimensional vector spaces: the group of linear bijections. Finite vector spaces are uniquely characterized by their dimensions and thus the general linear group can be defined on any of its representatives, say V with $\dim(V) = d$:

$$Gl(d; \mathbb{K}) := (\{f : V \rightarrow V \mid f \text{ linear bijection}\}, \circ) \quad \text{with} \quad \dim(V) = d < \infty$$

For $d = 0$ we recover the symmetric group of order $\#(V) = n$.

The (reducible) representations of the linear group $Gl(d; \mathbb{K})$ can be constructed on the basis of tensor products $V \otimes \dots \otimes V \rightarrow \mathbb{K}$ of the associated vector space V of dimension d . Tensors are multilinear maps of vectors defined by some equivalence class over the Cartesian product of V 's. For a rank r tensor $T^{(r)}$, the group action of $g \in Gl(d; \mathbb{K})$ is described by

$$\rho_g(T^{(r)})(v_{j_1}, \dots, v_{j_r}) := T^{(r)}(g(v_{j_1}), \dots, g(v_{j_r})) \quad \forall v_j \in V$$

While the general linear group is unconstrained and there is thus no seed of irreducibility present, one can show that permutations of the arguments actually commute with ρ_g due to the multilinearity of the representations. Explicitly for any tensor $T^{(r)}$ of rank r the (linear) symmetric group action is given by

$$\tilde{\rho}_\sigma(T^{(r)})(v_{j_1}, \dots, v_{j_r}) := T^{(r)}(v_{\sigma(j_1)}, \dots, v_{\sigma(j_r)}) \quad \forall v_j \in V$$

and we denote $T_\sigma^{(r)} \equiv \tilde{\rho}_\sigma(T^{(r)})$ for simplicity. Thus, the irreducible representations of $S(r)$ lead to a reduction of any r -tensor on the basis of some (anti-)symmetrization procedure. In fact, it turns out that this decomposition is actually also irreducible in $Gl(d; \mathbb{K})$ [11]. With this knowledge, we can use all the methods we have at our disposal to characterize the symmetric group

and apply them to $\text{Gl}(d; \mathbb{K})$, in particular the diagrammatic description via Young tableaux. Any tensor of rank r can thus be split into a set of irreducible tensors classified by some associated Specht module. For instance, rank 2 or rank 3 tensors are represented as follows

$$T^{(3)} = T_{\square\square\square} + T_{\square\square} + T_{\square} \quad T^{(2)} = T_{\square\square} + T_{\square}$$

The subscript indicates the invoked symmetrization, meaning the combination of permutations that act on the respective arguments. Thus $T_{\square\square}$ stands for the symmetric rank 2 tensor while $T_{\square}$ represents the anti-symmetric part of $T^{(2)}$. If the underlying field is the space of complex numbers, $\mathbb{K} \equiv \mathbb{C}$, transposed Young diagrams correspond to complex conjugate representations. Notice that the row count of any tensor can not exceed the dimension of the vector space, for then we have at least one anti-symmetrization of equal indices.

What differs from the prescription of $S(n)$ in the present case is the computational rule for the dimensionality of an irreducible representation. The reason is simply that the symmetrization contains only half the truth, while the remaining freedom is hidden in the tensor structure itself, that has in general d^r free parameters. Given an irreducible tensor $T_{\lambda}^{(r)}$ for some Specht module corresponding to the Young diagram $\lambda \equiv C_{S_n}(n_1 + \dots + n_r)$ we obtain the dimensionality as follows:

$$d_{\lambda}^{\text{Gl}(d, \mathbb{K})} \equiv \frac{\prod_j ((d - j + n_j)! / (d - j)!)}{\prod_{j, \ell} \text{hook}_{j, \ell}^{(\lambda)}} \quad \text{with} \quad \text{hook}_{j, \ell}^{(\lambda)} := (n_j - \ell) + \sum_{k \geq j} \Theta(n_k + 1 - \ell)$$

Next, we discuss some other, still uncovered aspect of Young diagrams that is intrinsically encoded in this diagrammatic representation. The so called Clebsch-Gordan series provide a way to decompose inner products of irreducible representations such that the result is again a sum of such fundamental building blocks. By means of Young diagrams this split is easily performed by simply following some very elementary multiplication rules of boxes. One of the diagrams that participate in the inner product is labeled row-wise with distinct numbers or letters. Then one starts to row-wise successively attach boxes of this enumerated diagram to the other while avoiding equal letters or numbers in the same column and constructions that are not Young diagrams at all. In the end one dismisses all diagrams where, counting from right to left and top to bottom, at any point we have more labels of a certain kind than from any preceding ones, i.e. more b 's or c 's then a 's, for instance. Furthermore equally labeled diagrams are condensed to a single tableaux and finally one lifts the enumeration and yields the full Clebsch-Gordan series. If there are more than two irreducible representations involved in the inner product, one proceeds pairwise. As an example consider the following product of $S(2)$:

$$\begin{aligned} \square\square \times \square &\equiv \square\square \times \begin{bmatrix} a \\ b \end{bmatrix} = \left(\square\square a \oplus \begin{bmatrix} \square & \square \\ a & \end{bmatrix} \right) \times \begin{bmatrix} b \end{bmatrix} \\ &= \begin{bmatrix} \square & \square & a \\ b & \end{bmatrix} \oplus \begin{bmatrix} \square & \square \\ a & b \end{bmatrix} \oplus \begin{bmatrix} \square & \square \\ a & b \end{bmatrix} \equiv \begin{bmatrix} \square & \square & \square \\ \square & \end{bmatrix} \oplus \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \oplus \begin{bmatrix} \square & \square \\ \square & \end{bmatrix} \end{aligned}$$

Upon only slightly changing the notation we can thus generate irreducible representations of $\text{Gl}(d; \mathbb{K})$ from existing irreducible tensors of lower rank.

Due to the simplicity of this diagrammatic technique, the geometric side falls a bit behind. In certain situations, however, the topological or differentiable aspect of the Lie structure of $\text{Gl}(d; \mathbb{K})$ is far more appropriate especially in contact with differential geometry. Considered as a Lie group $\text{Gl}(d; \mathbb{K})$ is a d^2 dimensional smooth non-compact manifold. When $\mathbb{K}$ is complex this Lie group is connected, in the real case however there are two disconnected components,

one $\text{Gl}^+(d; \mathbb{K})$ associated to matrices with positive the other $\text{Gl}^-(d; \mathbb{K})$ to matrices with negative determinant. The Lie algebra consists of all $d \times d$ $\mathbb{K}$ -matrices, that do not have to be invertible, and the canonical commutator as Lie bracket:

$$\mathfrak{gl}(d; \mathbb{K}) := (\{g : V \rightarrow V \mid g \text{ is linear} \wedge \dim(V) = d\}, [g, f]_{\mathfrak{gl}} = g \circ f - f \circ g)$$

The intrinsically relation of $\text{Gl}(d; \mathbb{K})$ to vector spaces makes it such an important Lie group and some of its subgroups play a central role in many branches of physics. In particular the maximal compact subgroup $\text{O}(d)$ and $\text{U}(d)$ of $\text{Gl}(d; \mathbb{R})$ and $\text{Gl}(d; \mathbb{C})$, respectively, and their further descendants appear in almost every field of physics. As an example, let us consider the most interesting candidate that appears in the formulation of the Standard Model of particle physics, the special unitary group.

The special unitary group $\text{SU}(d)$ derives from the unitary group $\text{U}(d)$ for which we have to consider a vector space equipped with an inner product $(\bullet, \bullet)_V$. On the basis of this scalar product we can judge whether or not an element of $\text{Gl}(d; \mathbb{C})$ is unitary. Its group structure is given by

$$\text{U}(d) = (\{g \in \text{Gl}(d; \mathbb{C}) \mid (g(v), g(w))_V = (v, w)_V \forall v, w \in V\}, \circ)$$

As such, it is the maximal compact subgroup of $\text{Gl}(d, \mathbb{C})$ as mentioned above and in the finite-dimensional case it can be identified with the matrix algebra of $d \times d$ -unitary matrices. The simplest example, namely $\text{U}(1)$, is an Abelian group famous for its association to Quantum Electrodynamics (QED). Concerning $\text{SU}(d)$, the adjective ‘special’ indicates that it is a subgroup of $\text{U}(d)$ containing only elements with determinant equal to one, i.e.

$$\text{SU}(d) = (\{g \in \text{U}(d) \mid \det(g) = +1\}, \circ)$$

Notice that this condition is the only one that fixes the determinant completely while retaining the group structure of $\text{U}(d)$. Concerning its topological aspects, $\text{SU}(d)$ inherits compactness from $\text{U}(d)$ and is furthermore simply connected as a $d^2 - 1$ real smooth manifold. From the algebraic side its center is isomorphic to the cyclic group $\mathbb{Z}_d$ consisting of the d -square roots of unity times $\mathbb{1}$. Let us now turn our focus on its Lie algebra and classify $\text{SU}(d)$ by means of the properties of $\mathfrak{su}(d)$ given by

$$\text{SU}(d) = (\{g \in \mathfrak{gl}(d; \mathbb{C}) \mid \text{Tr}(g) = 0 \wedge (g^T)^* = A\}, [g, f]_{\mathfrak{gl}})$$

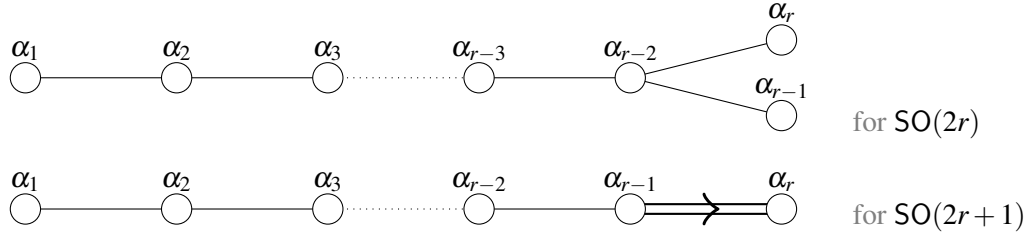
Pursuing the strategy of deducing the root space for $\text{SU}(d)$ requires at first a semisimple Lie algebra, a criterion that is perfectly satisfied for $\mathfrak{su}(d)$ which in fact is also simple. Identifying the subalgebras $\text{SU}(2)$ within $\text{SU}(d)$ is a straightforward calculation that for $d > 1$ yields the following Dynkin diagram



Hereby, α_r denotes the roots, hence the rank of the Lie algebra, i.e. the dimension of its Cartan subalgebra, is equal to $d - 1$. In physics, the Standard Model of particle physics is built on $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$, explaining the prominent status of $\text{SU}(d)$ in physics literature.

While $\text{U}(d)$ and $\text{SU}(d)$ have important applications in modern QFT their real counterparts $\text{O}(d)$ and $\text{SO}(d)$ are closely tied to the geometry of spacetime. They describe certain linear transformation of real vector spaces and as such are very important especially in classical physics or the theory of quantum gravity. Besides these interrelations of (special) orthogonal groups and natural science, the concept of spin in quantum mechanics is constructed by means

of the double universal covering of $SO(d)$, denoted spin group $\text{Spin}(d)$. Their implementation on smooth manifolds M as spin structures, requires the first and second Stiefel-Whitney class of M to vanish. It can be shown that the Lie algebra $\mathfrak{so}(d)$ of $SO(d)$ is semisimple and gives rise to the following Dynkin diagram for :



The Cartan subalgebra has thus rank $d/2$ or $(d-1)/2$ for d even or odd, respectively. Notice that for the orthogonal group $O(d)$ and its descendants there is an additional operator that commutes with the group action, namely the contraction. This is a typical example where irreducible representations of a group (here $Gl(d)$) can be further reduced when restricted to a subgroup (here $O(d)$). For details on the irreducible representations of $O(d)$ we refer to [11]

Diffeomorphism group Diff

We have already mentioned that isomorphisms of a category form a group, that is they can be successively joined to obtain new isomorphisms in the same category and by virtue of their bijectivity there exists an inverse for each of them. In the previous two cases we have encountered two special types of isomorphism groups, the class of finite group over finite sized sets and the class of finite *dimensional* Lie groups over finite vector spaces with infinite cardinality. In the final step of this series of examples we consider the next level of extension for which the group is even not locally compact. This may happen when the dimension of the underlying vector space turns infinite, as for example for function spaces.

We will give a brief overview for the diffeomorphism group, the isomorphisms on a smooth manifold, that has naturally a distinguished status in the present work, however the construction of irreducible representation is beyond the scope of this thesis. So, let us assume a smooth manifold M . The diffeomorphism group acts on the respective equivalence class of M and is defined by

$$\text{Diff}(M) := (\{f : M \rightarrow M \mid f \text{ diffeomorphism} \}, \circ)$$

In order to describe the group structure of $\text{Diff}(M)$ we have to give one more definition for some special topological spaces that is also relevant in the construction of field space:

Definition 1.2.25 — Fréchet-space. Let $V \equiv (X, \circ_V, \mathbb{K}, \circ_{V \cdot \mathbb{K}})$ be a vector space. A topological vector space $\mathbb{V} \equiv (V, \tau_V)$ is a Fréchet-space $:\Leftrightarrow$

$$\exists d_V : V \times V \rightarrow \mathbb{R} \text{ metric, with } d_V(v \circ_V a, w \circ_V a) = d_V(v, w) :$$

$$(V, d_V) \text{ is metric complete} \quad \wedge$$

$$\forall U \in \tau_V \forall u \in U \exists \varepsilon > 0 : \{v \in V \mid d_V(u, v) < \varepsilon\} \subseteq U$$

$$\begin{aligned} \forall U \in \tau_V \text{ with } e_v \in U : & \quad (\forall u, v \in U \forall t \in [0, 1] : (t \circ_{V \cdot \mathbb{K}} u) \circ_V ((1-t) \circ_{V \cdot \mathbb{K}} w) \in U) \wedge \\ & \quad (\forall u \in U \forall t < 0 : (t \circ_{V \cdot \mathbb{K}} u) \in U) \wedge \\ & \quad (\forall v \in V \exists t > 0 : (t \circ_{V \cdot \mathbb{K}} v) \in U) \end{aligned}$$

The last three requirements for neighborhoods of the identity imply that V is a locally convex.

In the same way we introduced a topological manifold along with a smooth differential structure, we can proceed and define a Fréchet manifold that simply replaces the locally Euclidean property with the requirement of being locally a Fréchet space.

It turns out that the diffeomorphism group equipped with its weak topology can be described by a Fréchet manifold if M is compact, thus it carries additional geometrical information that allows to apply the machinery of differential topology. In fact, $\text{Diff}(M)$ for M compact is a regular (Fréchet) Lie group with a Lie algebra equivalent to the space of smooth vector fields on M and the commutator given by the negative Lie bracket. Hence, infinitesimal transformations close to the identity of $\text{Diff}(M)$ can be described by the Lie derivative.

The classification of its irreducible representations is based on the general linear group acting on each tangent space of elements in M and the transitive property of the natural group action. Usually, one considers only compact manifolds M and generates a chain of representations for subgroups of $\text{Diff}(M)$ that ultimately give rise to an induced representation for the whole group. For some d -dimensional connected smooth manifold the orientation preserving diffeomorphisms $\text{Diff}^0(M)$ ²¹ give rise to an equivalence of the underlying space M and the quotient group $\text{Diff}^0(M)/\text{Diff}_p^1(M)$, whereby $p \in M$ and $\text{Diff}_p^1 \equiv \{g \in \text{Diff}^0(M) | g(p) = p\}$ is the stabilizer subgroup of $\text{Diff}^0(M)$ at p . Decomposing smooth functions by means of jet equivalence generates a chain of normal subgroups for which specific representation might be deduced. On the other hand, there are investigations of non-compact manifolds invoking measure theoretical concepts in studying unitary irreducible representations of $\text{Diff}(M)$, usually based on a sequence of compactly supported diffeomorphisms. For more details we refer to [22, 23].

1.3 Field space

We have set the grounds to introduce fields into our formalism. Most of our today's understanding of fundamental physics is closely tied to the revelation of symmetries attached to objects living on spacetime. This combines the algebraic component of group theory with the topological and differential complex of smooth manifolds culminating in the concept of fiber bundles. Lie groups studied previously already build a bridge to this end and it is the aim of this section to extend this connection to smoothly affiliate with the differential structure of spacetime.

In this context a global notion of smoothness for functions can only be answered using fiber bundles and we will start this section by introducing the necessary notation. As a special and important example we consider the tangent space of smooth manifolds and thereby fill the gap between the interrelation of Lie groups and their Lie algebras. Having established the mathematical framework we will see how these concepts fit nicely into the physicist's language of quantum field theory. From the presented perspective it is quite clear what constitutes the differences of interaction mediating- and simple matter fields. In the final part of this section we take a closer look at the resulting field space and the role of diffeomorphism invariance. Once equipped with a thorough understanding of physical fields, the remaining sections of this chapter investigate endomorphisms and functionals of field space, paving the ground to the foundations of quantum field theory.

This section is based on a list of excellent books on field theory or differential geometry, [2–4, 24–28].

²¹Orientation preserving transformations are in the same homotopy class as the identity.

1.3.1 Fiber bundles

Our objective is to define smooth functions on an arbitrary curved spacetime M . The natural candidates to populate spacetime with content are fields, which allocates to each position in space and to each instance of time a particular value in a certain target space. The motivation of fiber bundles is found in the subtleness to consistently combine both differentiable structures to a global notion that will ultimately lead us to the formalism of gauge theory.

In the construction of spacetime the distinction between local and global requirements opened up a variety of new structures described with the concept of smooth manifolds. While the familiar picture of Euclidean space is retained in the vicinity of every point, the global structure in general reveals entirely different properties. All this information is encoded in the set of functions that are called smooth within this manifold, i.e. $C^\infty(M)$ and redefining this class will affect the notion of differentiability on spacetime.

Fields ‘communicate’ between two smooth manifolds, spacetime and a target space which actually defines the physical character of a field (e.g. a Higgs-, scalar-, ... field). It turns out, that in general sewing together spacetime and the target space of fields to a kind of product manifold is not unique at all, but in fact gives rise to a number of distinct topological or smooth manifolds that all differ in their notion of differentiability. The reason for this variety can be found in regions of overlaps on the underlying manifolds, where the global notion of these concepts deviate from the Euclidean versions due to additional twists in the combined total space. In mathematics the notion of product topologies or product manifolds is reserved for the so-called trivial bundle. Given two topological spaces $M \equiv (X_M, \tau_M)$ and $F \equiv (X_F, \tau_F)$, the product space is given by

$$M \times F := (X_M \times X_F, \tau_{M \times F} \equiv \{(U_M, U_F) | U_M \in \tau_M \wedge U_F \in \tau_F\})$$

For products of smooth manifolds an additional Cartesian product of charts has to be implemented.

However, gluing together the *local* structures in general results in a *global* object with a topology and differentiability different to the trivial case. The corresponding information is condensed in the concept of fiber bundles:

Definition 1.3.1 — Fiber bundles. Let $E \equiv (X_E, \tau_E)$, $M \equiv (X_M, \tau_M)$ and $F \equiv (X_F, \tau_F)$ be topological spaces, and $\pi_E : X_E \rightarrow X_M$ be a surjective continuous map. The ordered pair (E, M, F, π_E) is a fiber bundle $:\Leftrightarrow$

$$\begin{aligned} \forall u \in X_E \exists U_M \in \tau_M, \pi_E(u) \in U_M, \exists \vartheta_j : (U_j \equiv \pi_E^{-1}(U_M), \tau_E|_{U_j}) \rightarrow (U_M, \tau_M|_{U_M}) \times F : \\ \vartheta_j \text{ is isomorphism} \quad \wedge \quad \text{proj}_1 \circ \vartheta_j = \pi_E \end{aligned}$$

Here $\text{proj}_j : A_1 \times \dots \times A_n \rightarrow A_j$ with $u = (a_1, \dots, a_n) \mapsto \text{proj}_j(u) = a_j$ is the projection on the j th component of an ordered multiple. A fiber bundle is abbreviated with $E(M, F, \pi_E)$ or $F \rightarrow E \xrightarrow{\pi_E} M$ and E , M , and F are denoted total space, base space and fiber, respectively.

A fiber bundle promotes the concept of a function’s graph from a set theoretic perspective to the category of topological or smooth manifolds. Intuitively speaking, one attaches a fiber (target space) to every spacetime point in such a way that locally it reflects the standard Euclidean picture of continuity and smoothness. Only in the global description fibers can be twisted and deformed with respect to their neighbors.

The collection $\{(\vartheta_j, U_j)\}$ is called a local trivialization of $E(M, F, \pi_E)$ and can in principal be deduced from the comparison of the topologies and smooth structures of E , M , and the fiber F . They imitate the coordinate charts of manifolds and reveal the local embedding of trivial product spaces $(U_M = \pi_E(U_j), \tau_M|_{U_M}) \times F$ within the global description of τ_E .

To extract the set of different fiber bundles compatible with base space M and fiber F , one has to specify when two fiber bundles give rise to the same global differentiability and therefore coincide. This definition reflects the concept of a maximal atlas of smooth manifolds, another indication that fiber bundles are a generalization to products of these objects.

Definition 1.3.2 — Equivalence of fiber bundles. Two fiber bundles $E(M, F, \pi_E)$ and $E'(N, F', \pi'_E)$ are said to be equivalent, i.e. $E \sim_{\text{FB}} E' :\Leftrightarrow$

$$\exists f_E : E \rightarrow E' \text{ isomorphism} \quad \wedge \quad \pi'_N \circ f_E \circ \pi_E^{-1} \text{ isomorphism}$$

Within one equivalence class, all fiber bundles will define the same collection of functions to be smooth and its representatives only differ by a ‘choice of coordinates’. The information of the underlying differential structure, in particular the content of the transition functions, is encoded in a set $\Gamma(E)$ which denotes the collection of (local) sections:

Definition 1.3.3 — Sections. Let $E(M, F, \pi_E)$ be a (smooth) fiber bundle. A morphism (continuous and/or smooth, . . . map) $s : U \subseteq X_M \rightarrow X_E$ is a section of $E(M, F, \pi_E) :\Leftrightarrow$

$$\pi_E \circ s = \text{id}_M|_U$$

The set of smooth (local) sections on E is denoted $\Gamma(E)$. If $U \equiv X_M$, then s defines a global section.

Notice that sections are smooth maps on E that in general fail to be differentiable in the standard Euclidean way, though it holds locally true.

Tensor bundles

Tensor bundles are an illustrative but also very important examples of fiber bundles for they are closely related to the intrinsic properties of a manifold. Given any smooth manifold M it locally coincides with a Euclidean space $\mathbb{R}^d$ by virtue of a set of charts $(U_j, f_{U_j} : U_j \rightarrow \mathbb{R}^d)$ that cover M . Thus, locally M is equipped with vector- and metric properties compatible with the standard notion of differentiability which provides a natural definition of smooth real-valued functions $C^\infty(M, \mathbb{R}) \equiv \{f : M \rightarrow \mathbb{R} \mid f \circ f_{U_j}^{-1} \in C^\infty(\mathbb{R}^d, \mathbb{R}) \forall U_j \in \tau_M\}$. The structural richness of their target space $\mathbb{R}$ gives rise to an algebra on $C^\infty(M, \mathbb{R})$ that is inherited by pointwise evaluation on $\mathbb{R}$:

$$(f \circ_+ g)(p) = f(p) + g(p), \quad (f \circ_\bullet g)(p) = f(p) \cdot g(p), \quad (\alpha \circ_{\mathbb{R}, C^\infty} f)(p) = \alpha \cdot g(p)$$

The generalized notion of a derivative on $C^\infty(M, \mathbb{R})$ is then obtained by a special subset of the dual vectors of $(C^\infty(M, \mathbb{R}), \circ_+, \mathbb{R}, \circ_{\mathbb{R}, C^\infty})$: the derivation.

Definition 1.3.4 — Derivations. Let M be a smooth manifold.

A linear map $D_{[\gamma]}^p : C^\infty(M, \mathbb{R}) \rightarrow \mathbb{R}$ is a derivation at $p \in M :\Leftrightarrow$

$$\forall f, g \in C^\infty(M, \mathbb{R}) : \quad D_{[\gamma]}^p(f \circ_\bullet g) = D_{[\gamma]}^p(f) \cdot g(p) + f(p) \cdot D_{[\gamma]}^p(g) \quad (\text{Leibniz-rule})$$

The additional requirement we impose on the dual vectors $C^\infty(M, \mathbb{R}) \rightarrow \mathbb{R}$ is the well known Leibniz-rule that reflects the standard derivative properties. The space of derivations also inherits further structure from $\mathbb{R}$ which leads to the algebraic definition of tangent spaces:

Definition 1.3.5 — Tangent spaces. Let M be a smooth manifold of dimension d . The

ordered pair $T_p M \equiv (X, \circ_{T_p M}, \mathbb{R}, \circ_{\mathbb{R} \cdot T_p M})$ is the tangent space of M at $p \in M : \Leftrightarrow$

$$\begin{aligned} \forall D_{[\gamma]}^p \in X : \quad & D_{[\gamma]}^p \text{ is derivation at } p \\ \forall D_{[\gamma]}^p, D_{[\sigma]}^p \in X, \forall f \in C^\infty(M, \mathbb{R}), \forall \alpha \in \mathbb{R} : \\ & \left(D_{[\gamma]}^p \circ_{T_p M} D_{[\sigma]}^p \right) (f) = D_{[\gamma]}^p(f) + D_{[\sigma]}^p(f) \quad \wedge \quad \left(\alpha \circ_{\mathbb{R} \cdot T_p M} D_{[\gamma]}^p \right) (f) = \alpha D_{[\gamma]}^p(f) \end{aligned}$$

In other words $T_p M$ is a vector space over $\mathbb{R}$ of dimension d . On the other hand, $T_p M$ can be interpreted as the equivalence class of smooth curves $\gamma : \mathbb{R} \rightarrow M$ passing p :²²

$$\begin{aligned} & \{ [\gamma]_{\sim_{\text{curv}}} \mid \gamma \in C^\infty(\mathbb{R}, M) \} \quad \text{with} \\ & [\gamma]_{\sim_{\text{curv}}} := \left\{ \sigma \in C^\infty(\mathbb{R}, M) \mid \sigma(0) = p \wedge \frac{d}{dt} \gamma(t) \Big|_{t=0} = \frac{d}{dt} \sigma(t) \Big|_{t=0} \right\} \end{aligned}$$

Every equivalence class of curves can be naturally identified with a derivation and vice versa. The connection is established by specifying a map

$$[\gamma]_{\sim_{\text{curv}}} \mapsto D_{[\gamma]}^p : \quad \forall f \in C^\infty(M, \mathbb{R}) \quad D_{[\gamma]}^p(f) := \frac{d}{dt}(f \circ \gamma) \Big|_{t=0}$$

and then check that $D_{[\gamma]}^p$ indeed defines a derivation in the above sense. By virtue of the chain rule this correspondence is well defined, since each element $\sigma \in [\gamma]_{\sim_{\text{curv}}}$ gives rise to the same element $D_{[\sigma]}^p(f) = \frac{d}{dt}(f \circ \sigma) \Big|_{t=0} = f'(\sigma(0)) \frac{d}{dt} \sigma(t) \Big|_{t=0} = f'(\gamma(0)) \frac{d}{dt} \gamma(t) \Big|_{t=0} = D_{[\gamma]}^p(f)$. Therefore, we adopt the short notation $[\dot{\gamma}]_p \equiv D_{[\gamma]}^p$ to describe tangent vectors associated to curves γ at p . Every tangent space describes a vector space over $\mathbb{R}$, whereby the dimension agrees with the one of the underlying manifold; in the finite dimensional case every fiber is thus isomorphic to $\mathbb{R}^d$.

The set of linear functionals, i.e. linear maps from a vector space V to the field $\mathbb{R}$, forms again a vector space, the dual space V^* . We will encounter infinite dimensional dual spaces in section 2.1 when studying measures on field space. In this case the set of continuous linear functionals is only a subspace of V^* . In addition, for infinite vector spaces, subtleties occur in the construction of a suitable basis on V^* . However, for the construction of geometric objects on spacetime a treatment of finite spaces is sufficient. Associated to each tangent space we can thus define a dual description that introduces a differentiable structure on the space of linear functionals on every $T_p M$:

Definition 1.3.6 — Cotangent space. Let M be a smooth manifold of dimension d . Then $T_p^* M \equiv (X, +, \mathbb{R}, \circ_{\mathbb{R} \cdot T_p^* M})$ defines the cotangent space of M at $p : \Leftrightarrow$

$$T_p^* M \equiv (T_p M)^*$$

Any smooth map $f : M \rightarrow N$ can be naturally promoted to a smooth function relating the respective tangent and cotangent spaces. The covariant functor f_* is the so called differential map given by

$$\begin{aligned} f_* : T_p M \rightarrow T_{f(p)} N, \quad [\dot{\gamma}]_p \in T_p M \mapsto (f_*[\dot{\gamma}]_p) \quad \text{with} \\ (f_*[\dot{\gamma}]_p)(g) \equiv [\dot{\gamma}]_p(g \circ f) = \frac{d}{dt}(g \circ f \circ \gamma) \Big|_{t=0} \quad \forall g \in C^\infty(N, \mathbb{R}) \end{aligned}$$

²²Here and in what follows, local charts are implicitly used, as for instance $d\gamma(t)/dt \triangleq d(f_U \circ \gamma(t))/dt$ for some chart (U, f_U) with $\gamma(t) \in U$.

The corresponding contravariant functor f^* , also known as pullback, connects the respective cotangent spaces and by definition reverse the order of composition:

$$f^* : T_{f(p)}^* N \rightarrow T_p^* M, \quad w \in T_{f(p)}^* N \mapsto (f^* w) \quad \text{with} \\ (f^* w)([\dot{\gamma}]_p) = w(f_*[\dot{\gamma}]_p) \quad \forall [\dot{\gamma}]_p \in T_p^* M$$

It is easily verified that for $f_1 : M \rightarrow N$ and $f_2 : N \rightarrow Q$ we obtain $(f_1 \circ f_2)_* = f_{1*} \circ f_{2*}$ and $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$, which indeed classifies f_* and f^* as covariant or contravariant functors, respectively.

Thus, we have a number of natural vector spaces associated to a smooth manifold M at our disposal that can be further combined to a multilinear tensor algebra, as follows. At first, consider the Cartesian or direct product of cotangent and tangent spaces at $p \in M$, i.e. is the ordered pair

$$\underbrace{T_p^* M \times \cdots \times T_p^* M}_{m\text{-times}} \times \underbrace{T_p M \times \cdots \times T_p M}_{n\text{-times}}$$

The tensor product establishes a special equivalence class of elements for this direct product space that gives rise to a multilinear algebra:

$$\begin{aligned} \forall [\dot{\sigma}]_p, [\dot{\gamma}]_p \in T_p M, \forall w', w_k \in T_p^* M, \forall \alpha \in \mathbb{R} : \\ (w_1, \dots, w_\ell, \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\gamma}]_p) +_\otimes (w_1, \dots, w', \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\gamma}]_p) \\ \sim (w_1, \dots, (w_\ell + w'), \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\gamma}]_p) \\ (w_1, \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\gamma}]_p) +_\otimes (w_1, \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\sigma}]_p, \dots, [\dot{\gamma}]_p) \\ \sim (w_1, \dots, w_m, [\dot{\gamma}]_p, \dots, ([\dot{\gamma}]_p \circ_{T_p M} [\dot{\sigma}]_p), \dots, [\dot{\gamma}]_p) \\ \alpha \circ_{\mathbb{R} \cdot \otimes} (w_1, \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\gamma}]_p) \sim (\alpha \circ_{\mathbb{R} \cdot T_p M} w_1, \dots, w_m, [\dot{\gamma}]_p, \dots, [\dot{\gamma}]_p) \\ \sim \cdots \sim (w_1, \dots, w_m, [\dot{\gamma}]_p, \dots, \alpha \circ_{\mathbb{R} \cdot T_p M} [\dot{\gamma}]_p) \end{aligned}$$

The resulting tensor product algebra $T_p^{(n,m)} M \equiv \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{m\text{-times}} \otimes \underbrace{T_p M \otimes \cdots \otimes T_p M}_{n\text{-times}}$ is the

canonical construction of objects that are linear with respect to every constituent while retaining the intrinsic structures of its component spaces. It can be used to define vector bundles over spacetime that are very closely connected to the intrinsic properties of the underlying manifold.

Definition 1.3.7 — Tensor bundles. Let M be a smooth manifold of dimension d .

Then $TM(M, \mathbb{R}^d, \pi_{TM})$ or $T^*M(M, \mathbb{R}^d, \pi_{T^*M})$ defines the tangent or cotangent bundle on M , respectively, $:\Leftrightarrow$

$$TM \equiv \bigsqcup_{p \in M} \{p\} \times T_p M \qquad T^*M \equiv \bigsqcup_{p \in M} \{p\} \times T_p^* M$$

Finally, $T^{(n,m)}M$ is a tensor bundle of type (n,m) on M $:\Leftrightarrow$

$$T^{(n,m)}M = \bigotimes_m T^*M \otimes \bigotimes_n TM \equiv \bigsqcup_{p \in M} \{p\} \times \bigotimes_m T_p^* M \otimes \bigotimes_n T_p M$$

A (global) section on $T^{(n,m)}M$, $TM \equiv T^{(1,0)}M$, and $T^*M \equiv T^{(0,1)}M$ defines a (global) tensor, vector field or differential 1-form, respectively.

Tensor fields take a very prominent role in the formalization of today's physics, for they provide a natural description of movement or change on spacetime that has to be only supplemented with

an interaction to produce any matter fields of Nature. Intrinsically, they carry a rich structure, in particular a concept of linearity, that is inherited by the space of sections, $\Gamma(\mathbb{T}^{(n,m)}M)$. For instance, a vector field $v \in \Gamma(TM)$ defines a smooth collection of derivations

$$v : M \rightarrow TM, \quad \text{with} \quad v_p \equiv [\dot{\gamma}_p]_p \quad \forall p \in M$$

Thus, a vector field associates to every base point an equivalence class of curves. Notice that in general γ_p describe different curves for every p . In the special case that $v_{\gamma(t)} \equiv [\dot{\gamma}]_{\gamma(t)}$ holds true, γ defines the flow of v .

The linearity of $\mathbb{T}_p M$ can be extended to the space of smooth sections $\Gamma(TM)$ in a pointwise manner. In addition, there is a natural commutator for vector fields, the Lie bracket, given by

$$\begin{aligned} [\bullet, \bullet]_{\mathcal{L}} : \Gamma(TM) \times \Gamma(TM) &\rightarrow \Gamma(TM) \\ (v, w) &\mapsto [v, w]_{\mathcal{L}} := v(w(\bullet)) \circ_+ (-1) \circ_{\mathbb{R}, C^\infty} v(w(\bullet)) \end{aligned}$$

The space of k -forms constitutes another class of relevant tensor bundles that frequently appears in physics. It defines the irreducible fully antisymmetric subspace of $\mathbb{T}^{(0,k)}M$ and is denoted $\Lambda^k(M)$, respectively. The associated Young operator that projects $\mathbb{T}^{(0,k)}M$ onto $\Lambda^k(M)$ is given by the epsilon-tensor.

While vector fields and also k -forms are individual algebraic constructions, they all can be embedded into the algebra of tensor fields. Additional operations, as for instance the wedge product, multiplication and addition of tensor fields, allow for transitions from k -forms to $k+r$ -forms, for instance, and also interrelates the important concept of Lie derivatives for these subspaces:

Definition 1.3.8 — Lie derivative. Let M be a smooth manifold and $\mathbb{T}^{(n,m)}M$ the tensor bundle of type (n,m) . A map $\mathcal{L} : \Gamma(TM) \times \Gamma(\mathbb{T}^{(n,m)}M) \rightarrow \Gamma(\mathbb{T}^{(n,m)}M)$ is the Lie derivative on $M : \Leftrightarrow \forall v \in \Gamma(TM)$

$$\begin{aligned} \forall f \in C^\infty(M, \mathbb{R}) : \quad \mathcal{L}_v f &= v(f) \quad \wedge \quad [\mathcal{L}_v, d]_{\mathcal{L}} = 0 \\ \forall t \in \Gamma(\mathbb{T}^{(n,m)}M), w_j \in \Gamma(TM), \alpha_\ell \in \Gamma(\mathbb{T}^*M) : \\ \mathcal{L}_v(t(w_1, \dots, \alpha_n)) &= (\mathcal{L}_v(t))(w_1, \dots, \alpha_n) +_{\otimes} t(\mathcal{L}_v w_1, \dots, \alpha_n) \\ &\quad +_{\otimes} t(w_1, \dots, w_m, \alpha_1, \dots, \mathcal{L}_v \alpha_n) \\ \forall t_1, t_2 \in \Gamma(\mathbb{T}^{(n,m)}M) : \quad \mathcal{L}_v(t_1 \otimes t_2) &= (\mathcal{L}_v(t_1) \otimes t_2) +_{\otimes} (t_1 \otimes \mathcal{L}_v(t_2)) \end{aligned}$$

Notice that for vector fields these properties give rise to an identification of $\mathcal{L}_v(w)$ and the Lie bracket $[v, w]_{\mathcal{L}}$.

A very special type of smooth manifolds are Lie groups that will describe the symmetry principals our fundamental interactions are based on. In the discussion of group theory we already mentioned the close connection between certain Lie algebras and Lie groups that turns out to be a geometric relation. When considering a Lie group G as a smooth manifold the full apparatus of tensor bundles applies. The Lie algebra associated to G defines differentiable changes in the group and thus has to be related to the tangent space of G . In fact, the set of left-invariant vector fields, as well as the tangent space at e_G are all isomorphic to the group theoretical construction of the Lie algebra $\mathfrak{g}$ in section 1.3. We are going to complete the story of the close affiliation of G and $\mathfrak{g}$ when having a more elaborated formalism at hand to study tangent spaces: connections and exponential maps.

Principal bundles

In the context of physics, fields are in general required to be smooth functions – usually equipped with additional symmetry constraints – and thus are sections on the corresponding fiber bundle.

Hence, if we were able to acquire the full knowledge of the set of physical fields, we could deduce the associated equivalence class of global topologies and the differential structures of their target space and spacetime. However, it is usually the other way around in that we first obtain the mathematical description for some very general constraints, derive all admissible theories, and finally reduce this number by additional observable data. Thus, in the maze of all such possibilities, symmetries will guide us to obtain a first (physical) reduction of mathematical formalism, hence, once again, group theory enters the very fundamentals of our description of the Universe. From the very beginning of natural science symmetries have taken an important part in deducing the laws of Nature and lighting the way to the very foundations of physics. With the arising of quantum field theory this tied connection has reached a new level in that interactions and symmetries become severally intertwined. As a result, it is of uttermost importance to keep the notion of smooth functions invariant under the symmetry transformations that are preserved in Nature.

R Let us assume G is a symmetry group realized in Nature and we have found a selection of functions $X \equiv \{f : M \rightarrow F\}$ that represent physical fields and are therefore designated to be smooth. Thus, there should be an equivalence class of fiber bundles $[E(M, F, \pi_E)]_{\sim_{\text{FB}}}$ that define a corresponding notion of differentiability, i.e. $X \subseteq \Gamma(E)$. Under the symmetry action ρ of G on X we obtain a new set of physical fields $gX \equiv \{\rho_g(f) \mid f \in X\}$. While in general $gX \neq X$ may hold, it is important that the transformed set satisfies $gX \subseteq \Gamma(E)$ to assure that gX still describes physically admissible fields. Hence, we have to make the notion of differentiability invariant under the relevant symmetries, since otherwise we may start with a set of physical (i.e. smooth) fields and end up with unphysical (i.e. non-differentiable) ones.

In order to implement symmetry considerations into the formalism, we have to reduce the differential structures to those that are compatible with a selection of group actions related to symmetry constraints observed in Nature. A unified theory of the Universe would consist of a number of fields interacting with each other. Each field is a function from spacetime to a certain target space and thus a section in a specific equivalence class of fiber bundles that differ for the various fields in general. In principal we could decide about the smoothness properties of the distinct field spaces separately but to obtain a consistent picture that gives rise to the same differentiable structure and assures that all fields are invariant under the same symmetry groups we have to merge those smoothness concepts in a natural way. This is accomplished by a principal bundle that, in a nutshell, contains all essential information of a class of associated fiber bundles in a very condensed and universal way.

Definition 1.3.9 — Principal bundles. Let $P \equiv (X_P, \tau_P)$ and $M \equiv (X_M, \tau_M)$ be topological manifolds. Let $G \equiv (X_G, \circ_G, \tau_G)$ be a topological group and $\pi_P : X_P \rightarrow X_M$ a surjective morphism. The ordered set $(P, M, G, \pi_P, \rho_P^R)$, denoted $P = M^{\pi_P} \times_{\rho_P^R} G$, is a principal bundle $:\Leftrightarrow :$

$$\begin{aligned} G &\rightarrow P \xrightarrow{\pi_P} M && \text{(is a fiber bundle)} \\ \rho_P^R : P \times G &\rightarrow P \text{ morphism} \quad \wedge \quad \pi_P \circ \rho_P^R = \pi_P && \text{(preserves fiber)} \\ \forall p, q \in P, \exists g \in X_G : &\quad \rho_P^R(p, g) = q && \text{(regular group action)} \end{aligned}$$

G is called the structure group of the principal bundle. In the category of smooth manifolds we have to replace continuity condition with differentiability.

A section of a principal bundle corresponds to a choice of an identity element over each base space point. It is like fixing the origin of an affine space in a consistent way, meaning that a global differential structure arises. Remarkably, the very existence of a global section on P is

equivalent to P being a trivial principal bundle.

It turns out that a Lie group G , a smooth manifold P , and a free, proper group action $\rho_P^R : P \times G \rightarrow P$ are sufficient to characterize a smooth principal bundle. The missing information can be reconstructed by considering $P/G = \{\{\rho_P^R(p, g) \mid g \in G\} \mid p \in P\}$ as base space and $\pi_P : P \rightarrow P/G$ with $\pi_P(p) = \{\rho_P^R(p, g) \mid g \in G\}$ – the orbit map – as the projection.

From this perspective it is straightforward to reduce the structure group G by or to a subgroup H while retaining the smooth structure of the underlying principal bundle P . The first restriction results in a new principal bundle $P/H = (P/G)^{\pi_{P/H} \times \rho_{P/H}^R} (G/H)$. Hereby, the fiber of this new construction is reduced to the coset space $F \equiv G/H \equiv \{g \circ_G h \mid h \in H\} \mid g \in G\}$ which coincides with the quotient group whenever H is normal w.r.t. G . Using the homomorphism property of ρ_P^R we indeed find that the resulting base space agrees with P/G . On the other hand, it might be possible that all transition functions are compatible with a subgroup H of G , i.e. the structure group can be reduced to H without changing P at all: $P_H = M^{\pi_P} \times_{\rho_{P_H}^R} H \simeq P_G = M^{\pi_P} \times_{\rho_{P_G}^R} G$. This construction which is reasonably denoted the *reduction of the structure group to H* is, if it exists, in general not unique. Prominent examples occur in theories of gravity where one usually reduces the general linear group to its special orthogonal subgroup, for instance, thus the tangent spaces carry a fiberwise inner product.

Principal bundles intertwine the differentiable structure of the basis with a symmetry group in a specific way such that associated fiber bundles share a consistent notion of smoothness. Given a particular principal bundle $P = M^{\pi_P} \times_{\rho_P^R} G$ the set of bundle isomorphisms will generate an entire class of equivalent principal bundles that constitute an equivalence class $[P]_{\sim_{\text{FB}}}$. The natural and important question to ask at this point, is how many non-equivalent principal bundles emanates upon fixing a base space M and a Lie group G , that is we are concerned about the cardinality and structure of the following set:

$$\text{PBs}(M, G) \equiv \{[P = M^{\pi_P} \times_{\rho_P^R} G]_{\sim_{\text{FB}}} \mid P \text{ principal bundle over } M \text{ with structure group } G\}$$

In general there is more than one equivalence class, implying that the full set of distinct differentiable structures has cardinality more than 1. The classification of principal bundles over a given base space is in fact a hot topic in algebraic topology and involves the study of homotopy theory, classifying spaces, and characteristic classes. The latter are elements of cohomology groups giving a measure for the twist of fiber bundles in an invariant way. For instance, the Stiefel-Whitney classes are a collection of topological invariants related to the classifying space of real vector bundles, the respective Grassmannians (generalizations of the projective space). Only if the first and second Stiefel-Whitney class vanish, the base space admits a global spin structure and thus is suitable to construct spinors. A more detailed account on the classification of principal bundles can be found in [4, 29], for instance.

Associated bundles

With the introduction of a structure group we have set the scene to all possible fiber bundles that are compatible with the arising smoothness conditions. Choosing a specific equivalence class out of all principal bundles we end up with a fixed notion of G -invariant differentiability that also specifies the shape of admissible G -invariant transitions. The strategy of producing consistent descendants of such a class of principal bundles is given by the concept of associated bundles.

For a general fiber bundle with typical fiber F transition functions are constraint to be elements of the corresponding isomorphism group in order to maintain the structure of F . Thus, one imposes the symmetry invariant description of a principal bundle to a general fiber bundle by establishing a (faithful) action of the structure group G on the isomorphism group $\text{Iso}(F)$, i.e. $\rho : G \rightarrow \text{Iso}(F)$. We are mainly interested in fields assuming values in a linear space and thus

focus on fibers possessing a vector space structure in what follows. So, let us start by choosing a topological vector space $V \equiv (X_V, \circ_V, \mathbb{F}, \circ_{\mathbb{F}, V})$ that represents the target space F of a certain (physically relevant) field. The domain should be given by the same base space that underlies the principal bundle $P = M^{\pi_P} \times_{\rho_P^R} G$. Hence, the graph of those fields lives in the Cartesian product $X_M \times X_V$. In order to define smooth fields we have to endow $X_M \times X_V$ with a suitable notion of differentiability that in the present case should be compatible with the symmetry constraints encoded in P . Therefore, we have to attach the structure group G to the vector space V by a (faithful) left action $\rho_V : G \times V \rightarrow V$. The remaining ingredients of an associated bundle are ...

► ... the total space, given by

$$E = (P \times V)/G \equiv \{ \{ (\rho_P^R(u, g), \rho_V(g^{-1}, v)) \mid g \in G \} \mid u \in P, v \in V \}$$

► ... the projection, given by

$$\pi_E : E \rightarrow M, \quad \forall q \equiv \{ (\rho_P^R(u_q, g), \rho_V(g^{-1}, v_q)) \mid g \in G \} \in E : \quad \pi_E(q) = \pi_P(u_q)$$

Notice that principal bundles and their associated bundles share the same base space M . Furthermore, π_E is well defined for $\pi_E(q) \equiv \pi_P(u_q) = \pi_P(\rho_P^R(u_q, g)) = \pi_E(q)$ holds true. To emphasize the close connection between an associated and its principal bundle we write $P \tilde{\times}_{\rho_V} V$ for the former.

The most important aspect of associated bundles for gauge theories are the (global) sections, which ultimately defines the matter sector. Sections in $P \tilde{\times}_{\rho_V} V$ have an equivalent description in terms of equivariant maps, i.e. functions that preserve the group actions of domain and target space:

$$f : P \rightarrow V, \quad \text{with} \quad f(\rho_P^R(u, g)) = \rho_V(g^{-1}, f(u)) \quad \forall u \in P, \forall g \in G$$

The correspondence between a section $\sigma : M \rightarrow E$ of an associated bundle $P \tilde{\times}_{\rho_V} V$ and an equivariant map $f : P \rightarrow V$ is established as follows:

$$\sigma_f(\pi_P(u)) := [u, f(u)] \quad \vee \quad [u, f_\sigma(u)] := \sigma(\pi_P(u)) \quad \forall u \in P$$

Especially for computational purposes this provides a more convenient and adaptable way to define and work with physical fields fulfilling a certain symmetry constraint.

Connections and covariant derivatives

So far we have a static construct that will allow us to introduce the physical matter content into our theoretical setting. What is missing constitutes the interactions between matter fields which should result in a change of the initial configuration. In mathematics the notion of ‘change’ is closely tied to ‘differentiation’, thus, we will present this concept for fiber bundles in the following.

For a general smooth manifold M the tangent space fills the role of the linear space needed to define a notion of differentiation. The directional derivative of smooth functions $f : M \rightarrow \mathbb{R}$ at a point $p \in M$ along a certain vector field $v \in \Gamma(TM)$ is given by $v_p(f)$. The specification of a function can be absorbed in a cotangent vector $df \in \Lambda^1(M)$ implicitly defined by $df_p(v) = v_p(f)$ for all $v \in \Gamma(TM)$ and $p \in M$. Tangent spaces contain the local information of M that are essential to extract infinitesimal changes, thus they locally characterize the underlying base space M . Calling to mind that vector fields are equivalence classes of curves, the relation between local patches of a manifold and tangent spaces is even intuitively graspable.

The linear operation d is the key to derive all derivatives of higher rank tensor fields using the Leibniz-rule in a successive manner. Hence, for any principal bundle P seen as a smooth

manifold interactions are related to the corresponding differential d . In this case however a decomposition of the the directional derivatives is more appropriate since a principal bundle is a combination of two individual smooth manifolds that both allow for separate changes. For the overall picture we have a certain freedom in splitting up the distinct contributions to differentiation that is encapsulated in the vertical and horizontal parts of the tangent space TP .

The generation of the vertical subspace $VP \subset TP$ only affects the structure group G . Moving vertical above a fixed base space point $p \in M$ means propagating along the fiber such that $\pi_P(u) = p$. This is naturally provided by the Lie algebra $\mathfrak{g}$ of the structure group that describes infinitesimal displacements vertical to the base space M , i.e.

$$V_u P = \{[\dot{\gamma}]_u \in T_u P \mid \pi_{P*}[\dot{\gamma}]_u = e_{\tau_{\pi_P(u)} M}\} \simeq \mathfrak{g} \simeq T_{e_G} G \quad \Longleftrightarrow \quad VP = \ker(\pi_{P*})$$

This space defines a direction on P perpendicular to the base space. The final equivalence of the vertical tangent space and the Lie algebra of G is generated by fundamental vector fields:

Definition 1.3.10 — Fundamental vector fields. Let $P = M^{\pi_P} \times_{\rho_P^R} G$ be a principal bundle with G a Lie group and $[\dot{\gamma}]_u \in T_{e_G} G \simeq \mathfrak{g}$. An element $v \equiv [\dot{\gamma}]_u^\# \in TP$ is the fundamental vector field w.r.t. $[\dot{\gamma}]_u \Leftrightarrow$

$$\begin{aligned} \forall u \in P: \quad v_u \equiv [\dot{\gamma}]_u^\# &\equiv \rho_P^R(u, \bullet)_* [\dot{\gamma}]_u \in T_u P \quad \text{with} \\ \forall f: P \rightarrow \mathbb{R}: \quad v_u(f) &= \left. \frac{d}{dt} (f \circ \rho_P^R(u, \gamma(t))) \right|_{t=0} \end{aligned}$$

Again, introducing a new structure opens a variety of different possibilities, which in the present case are reflected in the different choices for HP . Fixing a horizontal subspace amounts to defining an orthogonal decomposition of the tangent space w.r.t. VP and thus specifies directions on P which are parallel to the base space. While the vertical subspace is given by the kernel of the projection π_{P*} , the definition of a horizontal counterpart requires an additional piece of information that is not already present in the data of a principal bundle. This missing ingredient is the connection one-form that acts as a projection of $T_u P$ onto its vertical subspace.

Definition 1.3.11 — Principal connection. Let $P = M^{\pi_P} \times_{\rho_P^R} G$ be a principal bundle with G a Lie group. Let $R_g: G \rightarrow G$ be the right group action: $R_g(h) = h \circ_G g$ for all $h, g \in G$. A Lie-algebra valued one-form $\omega: TP \rightarrow \mathfrak{g}$ is a principal connection of $P \Leftrightarrow$

$$\begin{aligned} \forall V \in \mathfrak{g}: \quad \omega(V^\#) &= V && \text{(vertical projection)} \\ \forall g \in G: \quad \omega &= \text{ad}_{g*} (R_g^* \omega) && \text{(equivariant)} \end{aligned}$$

Notice that the second statement of definition 1.3.11 guarantees that horizontal subspaces $H_u P$ and $H_{\rho_P^R(u, g)} P$ on the same fiber are interconnected by a linear right action R_{g*} .

The tangent map π_{P*} and a principal connection define ‘orthogonal’ constructions in the sense that they specify the vertical and horizontal component of $T_u P$ as the kernel of π_{P*} and ω , respectively. A horizontal vector field induces no change when moving along the base space, i.e. explicitly we have

$$H_u P \equiv \{v_u \in T_u P \mid \omega(v_u) = e_g\}$$

For a given principal connection ω a general tangent vector $v_u \in T_u P$ splits uniquely into its vertical and horizontal component:

$$v_u \equiv v_u^H +_{T_u P} v_u^V \quad \text{with} \quad \omega(v_u^H) = e_g \quad \wedge \quad \pi_{P*} v_u^V = e_{\tau_{uP}}$$

A principal connection establishes a consistent classification of change on a principal bundle that leads to a more detailed account on the properties of an arbitrary vector field on P . In particular the projection onto its two constituents, the base space and the structure group, are uniquely defined by means of ω that further introduces the notion of parallelism. For instance, let us consider a curve on the base space M given by $\gamma: \mathbb{R} \rightarrow M$. For each $u_0 \in \pi_P^{-1}(\gamma(0))$ there is a unique curve $\tilde{\gamma} \equiv \hbar(\gamma): \mathbb{R} \rightarrow P$, denoted the horizontal lift of γ , that satisfies

$$\pi_P \circ \hbar(\gamma) = \gamma \quad \wedge \quad \hbar(\gamma)(0) = u_0 \quad \wedge \quad [\dot{\tilde{\gamma}}]_{\tilde{\gamma}(t)} \equiv \left[\frac{d}{dt} \tilde{\gamma}(t) \right] \in H_{\tilde{\gamma}(t)} P$$

Such a curve is parallel to the base space and thus selects a set of elements $\{\hbar(\gamma)(t) | t \in \mathbb{R}\} \subseteq P$ that lie on the same sheet as u_0 . This in fact defines an equivalence relation and the representatives of $[u_0]$ are called the parallel transports of u_0 . Now consider a particular class of smooth curves that closes on M , the so called loops defined by $\gamma: [0, 1] \rightarrow M$ at some point $p = \gamma(0) = \gamma(1)$. Depending on the connection ω on a principal bundle the horizontal lift of a loop in M will not necessarily give rise to a close loop on P but rather introduces a transformation on the fiber over p . By virtue of the Lie group structure in the vertical direction this transformation corresponds to an element of G that in general varies for different loops. Furthermore, the class of all loops emanating at a given point p is denoted

$$C_{\text{Loops}}^\infty(M; p) := \{\gamma: [0, 1] \rightarrow M \mid \gamma \text{ smooth, with } p = \gamma(0) = \gamma(1)\}$$

The set of all possible transformations that arise by the elements of $C_{\text{Loops}}^\infty(M; p)$ results in the holonomy group at $u \in \pi_P^{-1}(p)$, a subgroup of the structure group G :

$$\text{Hol}_u := \{g \in G \mid \exists \gamma \in C_{\text{Loops}}^\infty(M; \pi_P(u)) \quad \text{with} \quad \tilde{\gamma}(0) = u \wedge \tilde{\gamma}(1) = \rho_P^R(u, g)\}$$

As we will see in a moment, the holonomy group is closely tied to the principal connection by means of the Ambrose-Singer theorem.

However, let us first collect all the necessary ingredients to couple fields to interactions via some ω . The first step is the covariant exterior derivative. It provides a consistent notion of change for equivariant functions and general vector-valued forms:

Definition 1.3.12 — Covariant exterior derivative. Let $P = M^{\pi_P} \times_{\rho_P^R} G$ be a principal bundle with G a Lie group and ω a connection one-form on P . Furthermore, let V be a vector space and $\beta \in \Lambda^r(P) \otimes V$ be a V valued one-form on P . Then $(d^\omega \beta) \in \Lambda^{r+1}(P) \otimes V$ is the covariant exterior derivative of β :

$$(d^\omega \beta)(v_1, \dots, v_{r+1}) \equiv (d\beta)(v_1^H, \dots, v_{r+1}^H)$$

The exterior derivative d for $\beta \in \Lambda^r(P) \otimes V$ is given by:

$$\begin{aligned} (d\beta)(v_1, \dots, v_{r+1}) &\equiv \sum_{j \in [1, r+1]} (-1)^{j+1} v_j(\beta(v_1, \dots, \widehat{v_j}, \dots, v_{r+1})) \\ &\quad + \sum_{\ell \in [1, j]} (-1)^{\ell+j} \beta([v_\ell, v_j], v_1, \dots, v_{r+1}) \end{aligned}$$

whereby $\widehat{v_j}$ denotes the omission of a vector field.

Covariant derivatives embed the notion of differentiation on a principal bundle to any associated vector bundle. As a very special but likewise very interesting case we may consider the self-interaction of a principal connection, resulting in the curvature two-form:

Definition 1.3.13 — Curvature two-form. Let $P = M^{\pi_P} \times_{\rho_P^R} G$ be a principal bundle with G a Lie group and ω a principal connection on P . Then, a bilinear, skew-symmetric $\mathfrak{g}$ -valued 2-form $\Omega^\omega \in \Lambda^2(P) \otimes \mathfrak{g}$ is the curvature two-form of $\omega : \Leftrightarrow$

$$\Omega_u^\omega(v, w) = (d^\omega \omega)_u(v, w) \quad \forall u \in P, \forall v, w \in T_u P$$

The Ambrose-Singer theorem states that this curvature two-form contains the entire holonomy information of the connection. More explicitly, the Lie algebra $\mathfrak{h}$ of the holonomy group $H \equiv \text{Hol}_{u_0} \subseteq G$ at $u_0 \in P$ is equivalent to the Lie subalgebra of $\mathfrak{g}$ that is spanned by the values of $\Omega_u^\omega(v, w)$ with curves $\gamma \in C_{\text{Loops}}^\infty(M; \pi_P(u_0))$ such that $\tilde{\gamma}(t) = u$ for some t . In other words, the curvature provides a mean to measure the deviation of the initial and final point of a horizontal lift.

Since the covariant derivative intrinsically carries the information of the underlying principal connection there is a more explicit way to write down the curvature, a form known as Cartan's structure equation:

$$\Omega^\omega \equiv d\omega + \omega \wedge \omega \quad \vee \quad \Omega_u^\omega(v, w) \equiv (d\omega)_u(v, w) + [\omega(v), \omega(w)]_{\mathfrak{g}}$$

The second term originates from the subtraction of vertical vector field contributions to the ordinary differential d . Finally, the Bianchi identities follow from $d^2 = 0$ and the properties of ω , in particular the zero projection of horizontal fields, $\omega_u(v^H) = e_{\mathfrak{g}}$:

$$(d^\omega \Omega_u^\omega)(v, w, y) = (d\Omega_u^\omega)(v^H, w^H, y^H) = (d^2 \omega + 2d\omega \wedge \omega)_u(v^H, w^H, y^H) = e_{\mathfrak{g}}$$

Since this holds true for all tangent vectors $v \in T_u P$ we obtain the identity $d^\omega \Omega^\omega = e_{\Lambda^3(P)} \otimes e_{\mathfrak{g}}$. However, notice that contrary to d the covariant derivative d^ω is in general not nilpotent.

Associated connections

Now, let us assume we are given a smooth manifold M on top of which we have build a principal bundle $P = M^{\pi_P} \times_{\rho_P^R} G$ with a principal connection $\omega : TP \rightarrow \mathfrak{g}$. This suffices to construct an entire class of new associated bundles, related to some vector spaces V , all sharing a consistent notion of differentiability with the induced covariant derivative

$$d^\omega : \Lambda^r(P) \otimes V \rightarrow \Lambda^{r+1}(P) \otimes V$$

The tied bound between a principal bundle and its associated descendants also appears on the level of connections. For any principal connection ω that defines the concept of horizontality on TP there is a unique induced linear connection on E . Therefore, denote $\pi_{PE} : V \times P \rightarrow E$ and $\pi_{PE}^{(v)} : P \rightarrow E$ the canonical projections of P to E , explicitly given by $u \mapsto \pi_{PE}^{(v)}(u) := [(u, v)] \equiv \{(\rho_P^R(u, g), \rho_V(g^{-1}, v))\} \in E$. Then, the induced notion of horizontality on E reads

$$H_a E := \{v_a \in T_a E \mid \exists w_u \in \ker(\omega_u) \equiv H_u P : \pi_{PE*}^{(v)} w_{us} \equiv v_a\} \quad \text{with} \quad \pi_{PE}^{(v)}(u) = a$$

Due to the G -invariance of the horizontal space this definition is independent on the choice of v .

From a geometrical point of view, the most interesting associated bundles are tensor bundles of the base space, in particular TM , for they encode essential information of M itself and are the objects we encounter in Riemannian geometry (smooth manifolds with a metric on its tangent space). A particular well known differential construction is the covariant derivative of tensor fields $\nabla^\omega : \Gamma(TM) \times \Gamma(T^{(p,q)}M) \rightarrow \Gamma(T^{(p,q)}M)$ that we are going to relate to a principal connection ω on P . Essentially, we have to project ω on the tangent space of M using the intrinsic information of P and G .

Let us consider an arbitrary associated vector bundle $E \equiv P \tilde{\times}_\rho V$ for a moment. By the uniqueness of the horizontal lift, a section $s : M \rightarrow P$ provides a local isomorphism between $T_{\gamma(0)}M$ and $H_{s(\gamma(0))}P$, i.e. with $u_0 = s(\gamma(0))$ we have

$$v_{\gamma(0)} \in T_{\gamma(0)}M \quad \leftrightarrow \quad \gamma : \mathbb{R} \rightarrow M \quad \leftrightarrow \quad \hbar^s(\gamma) : \mathbb{R} \rightarrow P \quad \leftrightarrow \quad \hbar_*^s v_{u_0} \in H_{s(\gamma(0))}P$$

Due to the right invariance of horizontal spaces this definition is in fact independent on the section s .

The covariant derivative on E acts on the exterior algebra of V -valued r -forms. Any section $\sigma \in \Gamma(E)$ can be identified by an equivariant function $f_\sigma : P \rightarrow V$, a V -valued 0-form.

$$\begin{aligned} d^\omega f_\sigma &\in \Lambda^1(P) \otimes V, \quad \text{with} \quad [u, f_\sigma(u)] = \sigma(\pi_P(u)) \\ (d^\omega f_\sigma)_u(v) &= (df_\sigma)_u(v^H) \in V \quad \forall v \in \Gamma(TM) \end{aligned}$$

The important observation at this point is the equivalence between the horizontal subspace $H_u P$ and the tangent space $T_{\pi_P(u)}M \simeq \mathbb{R}^d$ of the base space. It suggests the following definition of ∇^ω in case of $E \equiv T^{(p,q)}M$:

$$(\nabla_v^\omega \beta)_{\pi_P(u)} := (d^\omega f_\beta)_u(\hbar_* v) = (df_\beta)_u(\hbar_* v) \quad \forall \beta \in \Gamma(T^{(p,q)}M), v \in \Gamma(TM)$$

Notice for that definition to hold, we need a group action $\rho : G \rightarrow \text{Iso}(\mathbb{R}^d \times \dots \times \mathbb{R}^d)$ with $d = \dim(M)$ that ensures $T^{(p,q)}M$ being an associated bundle of P . Due to its vector space character the natural structure group related to tensor bundles is based on $\text{Gl}(d; \mathbb{R})$.

As an example, let us consider the tangent space TM , its smooth sections being vector fields that locally are described by $v_p \equiv D_{\gamma,p}$. We thus obtain:

$$(\nabla_v^\omega w)_p \equiv (df_w)_{s(p)}(\hbar_*^s(v)) \in T_p M \quad \text{with} \quad [u, f_w] \equiv w(\pi_P(u))$$

This definition provides a means to study the effect of interactions on the tangent space of spacetime or more general of base space. It further allows to establish a relation between similar shaped vector fields in a sense that $\nabla_v^\omega w = e_{\Gamma(TM)}$ holds true, i.e. w is parallel transported along v . Very special vector fields are those based on curves that define the ‘closest’ path between different points on the manifold with respect to the interaction ω , the geodesics which are parallel to themselves everywhere:

Definition 1.3.14 — Geodesics. Let $P \equiv P = M^{\pi_P} \times_{\rho_P^R} G$ be a principal bundle with G a Lie group and ω a principal connection on P . A curve $\gamma : S \subseteq \mathbb{R} \rightarrow M$ is a geodesic w.r.t. $\omega : \Leftrightarrow$

$$\nabla_v^\omega v = e_{\Gamma(TM)} \quad \text{with} \quad v_{\gamma(t)} = [\dot{\gamma}]_{\gamma(t)} \quad \forall t \in S$$

The moment we have a distance measure on M induced by a Riemannian metric that is related to the principal connection such that it fulfills the Levi-Civita property, corresponding geodesics are the shortest paths to interconnect points on base space. Whether or not there is always such a unique shortest path between two points on M and to what extent we can prolong a curve so that it remains geodesic, is up to the manifold and its properties.

Levi-Civita connection and Riemannian metrics

The covariant exterior derivative and the linear connections it induces on associated vector bundles provide a new classification scheme for principal connections. Imposing mathematical constraints on the action of ∇^ω or in general of d^ω restricts the set of all possible connections ω on P to some subset which fulfills these additional requirements. One of the usual implemented assumptions is about the vanishing of the torsion tensor – a concept encountered when considering principal bundles P which are frame bundles of a certain base space M , i.e. its structure

group has to be a subgroup of $\text{Gl}(d; \mathbb{R})$ with $d \equiv \dim(M)$. The special status of frame bundles is due to its close relation to the geometry of M which results in the existence of a Solder- and thus torsion-form on P . Hence, in order to uniquely specify a principal connection on P by its pullback on M we have to impose conditions on the torsion form and its action on a set of smooth global sections

The Levi-Civita constraint is a very prominent example of this identification process where a Riemannian metric, $g \in \text{Riem}(M) \subset \Gamma(\text{T}^{(0,2)}M)$, plays a central role. Thus, let us first define the important concept of a Riemannian manifold:

Definition 1.3.15 — Riemannian manifold. Let $M \equiv [(X_M, \tau_M, \mathcal{A})]_{\sim_{\text{MC}^\infty}}$ be a (real) smooth manifold. Further let $g : X_M \times (\text{TM} \times \text{TM}) \rightarrow \mathbb{R}$ be a smooth function. The ordered pair (M, g) is a Riemannian manifold : $\Leftrightarrow$

$$\begin{aligned} \forall p \in M : \quad & g_p : \text{T}_p M \times \text{T}_p M \rightarrow \mathbb{R} \quad \text{bilinear, with } \forall v_p, w_p \in \text{T}_p M : \\ & g_p(v_p, w_p) = g_p(w_p, v_p) \quad \wedge \quad g_p(v_p, v_p) \geq 0 \quad \wedge \\ & g_p(v_p, v) = 0 \quad \implies \quad v_p \equiv e_{\text{T}_p M} \end{aligned}$$

Then g is called a Riemannian metric on M and defines an inner product on each tangent space $\text{T}_p M$.

In fact, every paracompact smooth manifold admits a Riemannian structure, which is by definition guaranteed in finite dimensions. Riemannian metrics have a special status in physics due to Einstein's theory of General Relativity, which is formulated for (pseudo-)Riemannian metrics, i.e. bilinear forms with Lorentzian signature.

Now, coming back to principal connections and define the Levi-Civita property. First of all, assume that the torsion form vanishes identically. Furthermore, we require the existence of a Riemannian metric g which lies in the kernel of the induced affine connection of ω_g^{LC} , i.e.

$$\nabla_v^{\omega_g^{\text{LC}}} g = 0 \quad \forall v \in \Gamma(\text{TM})$$

This turns out to be sufficient to uniquely specify the underlying principal connection on P .

Conversely, given a torsion free connection ω , it is fully determined by a Riemannian metric whenever its holonomy group is a subgroup of $\text{O}(d)$ [30]:

$$(\text{Hol}_\omega, \circ_{\text{O}(d)}) \subseteq \text{O}(d) \quad \implies \quad \exists g \in \text{Riem}(M) : \quad \nabla_v^\omega g = 0$$

and thus $\omega \equiv \omega_g^{\text{LC}}$. This correspondence is however not unique, for instance any constantly rescaled metric gives rise to the same Levi-Civita connection:

$$\omega_g^{\text{LC}} \equiv \omega_{c \cdot g}^{\text{LC}} \quad \forall c \in \mathbb{R}_+ / \{0\}$$

In theories of gravity, one severally exploits the relation between connections of frame bundles and Riemannian metrics in that ω_g^{LC} is fully replaced by the associated metric g . However, since this relation is not bijective it will produce different results when averaging over the full field space in both cases, since (at least) the entire set of global-conformally related metrics correspond to only one Levi-Civita connection on P .

Exponential map

Before we turn our attention to the very interesting physical aspects of differential geometry, we are going to add yet another piece of the puzzle. It concerns the relation between a smooth manifold and its tangent space and is a very illustrative example of the described fiber bundle techniques. Furthermore, it will play a major role in the study of quantum gravity, where field

space is in fact an infinite dimensional non-trivial manifold that fails to be a global vector space. The local correspondence between tangent space and its manifold allows for a local recovery of the vector space structure that reopens all the possibilities we usually have at our disposal.

Thus, let us consider the part of tangent space that can be used as starting vectors of suitably extended geodesics:

$$\mathcal{E}^M := \{(p, v_p) \mid \exists \gamma: [0, 1] \rightarrow M, p \equiv \gamma(0) : \quad \forall t: v_{\gamma(t)} = [\dot{\gamma}]_{\gamma(t)} \wedge \nabla_v^\omega v = e_{\Gamma(TM)}\}$$

One can show that $\mathcal{E}^M$ forms an open set in the topology of TM and in the case $\mathcal{E}^M \equiv TM$ one denotes M to be geodesically complete. For finite dimensional manifolds without boundary this feature is equivalent to metric completeness of M in terms of Cauchy sequences via the Hopf-Rinow theorem. However, for the infinite dimensional case that we encounter for field space or in the case of spacetimes with boundary this correspondence fails in general and the construction of the set of geodesics cannot be circumvented by means of establishing metric completeness.

With $\mathcal{E}^M$ we have a unique allocation of curves to tangent vectors. Since curves are defined on the manifold M and tangent vectors on the attached linear spaces, this correspondence translates to an isomorphism between a subspace of M and $\mathcal{E}^M$ given by the exponential map:

Definition 1.3.16 — Exponential map. Let $P \equiv P = M^{\pi_P} \times_{\rho_P^R} G$ be a principal bundle with G a Lie group and ω a connection one-form on P . Then, the map $\exp_M^\omega: \mathcal{E}^M \rightarrow M$ is the exponential map on M w.r.t. $\omega: \Leftrightarrow$

$$(p, [\dot{\gamma}]_p) \mapsto \exp_p^\omega([\dot{\gamma}]_p) \equiv \gamma(1) \quad \text{with} \quad \gamma \text{ geodesic}$$

The range where this map provides an isomorphism between the tangent space and a neighborhood of a point p on the manifold is estimated by the injectivity radius:

$$\text{inj}_p(M) := \sup\{r \in \mathbb{R}_+ \mid \exp_p^\omega: B_r(e_{T_p M}) \subset T_p M \rightarrow \text{Im}(\exp_p^\omega) \text{ isomorphism}\}$$

It measures the size of the largest possible ball around the origin in $T_p M$ where we obtain a full geodesic covering. On global scales we can look for the minimal radius that can be spanned by the exponential map, which for a geodesically complete manifold should be half the diameter of the full metric space:

$$\text{inj}(M) := \inf\{\text{inj}_p(M) \mid p \in M\}$$

The larger this quantity is, the more we can cover with a single tangent space $T_p M$ and thus the better M can be approximated as it is diffeomorphic to a linear space.

R In finite dimensions and for empty manifold boundary, the Hopf-Rinow theorem [31] provides a correspondence between metric completeness (a global topological property of metric spaces) and geodesic completeness. Besides this the Cartan-Hadamard theorem can partially substitute this global to local correspondence even in infinite dimensional spaces, as e.g. field space. It states conditions under which the universal cover of a manifold is diffeomorphic to (flat) Euclidean space, explicitly given by the exponential map [4, 32].

Lie groups and Maurer-Cartan form

As a direct application of these concepts consider a Lie group G as a smooth manifold that can be artificially extended to a trivial principal bundle over a single point (a zero-dimensional smooth manifold). The necessary ingredients are given by $P \equiv \{p\} \times G \simeq G$ with $\rho_P^R((p, h), g) \equiv$

$(p, R_g(h)) = (p, h \circ_G g)$ and constant projection $\pi_p(u) = p$. As a result of the simple point structure of the base space the tangent space TP agrees with TG and curves defined on P can only propagate in vertical direction, i.e. for any $\gamma_p : \mathbb{R} \rightarrow P$ there exists a unique $\gamma_G : \mathbb{R} \rightarrow G$ with $\gamma_p(t) = (p, \gamma_G(t))$. In order to construct the set of fundamental vector fields, consider $\mathfrak{g}$ as the tangent space $T_{e_G}G$. For any element $V \equiv [\dot{\gamma}_e]_{e_G} \in T_{e_G}G$ with $\gamma_e(0) = e_G$ the associated fundamental vector field is simply given by its left translation $V_g^\sharp = L_{g*}V$:

$$\begin{aligned} V_g^\sharp(f) &= (\rho_P^R((p, g), \bullet)_* V)(f) = \frac{d}{dt} (f \circ \rho_P^R((p, g), \gamma_e(t))) \big|_{t=0} \\ &= \frac{d}{dt} (f(p, g \circ_G \gamma_e(t))) \big|_{t=0} = (L_{g*}V)(f) \end{aligned}$$

An admissible connection $\theta : TP \rightarrow \mathfrak{g}$ on P has to revert this operation and in addition has to satisfy the equivariance condition, i.e. it should be a solution to the following equations:

$$\omega_g(L_{g*}V) \stackrel{!}{=} V \quad \forall V \in \mathfrak{g} \quad \wedge \quad \omega_g(v) \stackrel{!}{=} \text{ad}_{g^{-1}*} \left(R_{g^{-1}}^* \omega_{e_G}(v) \right) \quad \forall v \in \Gamma(TG)$$

It turns out that the simplest guess gives rise to a unique connection, the so called Maurer-Cartan form or canonical one-form $\theta_g \equiv L_{g^{-1}*}$ [33]. Evaluating the above condition of θ indeed reveals its connection character. For any $f : C^\infty(G; \mathbb{R})$ and any $\gamma_e : \mathbb{R} \rightarrow G$ with $\gamma_e(0) = e_G$ we obtain

$$\begin{aligned} (\theta_g(L_{g*}[\dot{\gamma}_e]_{e_G}))(f) &= (L_{g^{-1}*}L_{g*}[\dot{\gamma}_e]_{e_G})(f) = \frac{d}{dt} (f \circ L_g \circ L_{g^{-1}} \circ \gamma_e(t)) \big|_{t=0} \\ &= \frac{d}{dt} (f \circ \gamma_e(t)) \big|_{t=0} = [\dot{\gamma}_e]_{e_G}(f) \end{aligned}$$

which proves the first statement. In order to check the second condition we consider any tangent vector $[\dot{\gamma}]_g \in T_gG$ and any smooth function $f \in C^\infty(G; \mathbb{R})$ and evaluate both sides:

$$\begin{aligned} (\theta_g([\dot{\gamma}]_g))(f) &= (L_{g^{-1}*}[\dot{\gamma}]_g)(f) = \frac{d}{dt} (f \circ L_{g^{-1}} \circ \gamma(t)) \big|_{t=0} = \frac{d}{dt} (f(g^{-1} \circ_G \gamma(t))) \big|_{t=0} \\ &= \frac{d}{dt} (f(g^{-1} \circ_G \gamma(t) \circ_G g \circ_G g^{-1})) \big|_{t=0} = \frac{d}{dt} (f \circ R_{g^{-1}} \circ \text{ad}_{g^{-1}} \circ \gamma(t)) \big|_{t=0} \\ &= \text{ad}_{g^{-1}*} \left(R_{g^{-1}}^* L_{e_G*}([\dot{\gamma}]_g) \right) = \text{ad}_{g^{-1}*} \left(R_{g^{-1}}^* \theta_{e_G}([\dot{\gamma}]_g) \right) \end{aligned}$$

In the second last step we artificially inserted L_{e_G} which then finally leads to the required equality.

Hence, every Lie group has its unique connection, the Maurer-Cartan form that we now analyze further. With a principal connection at hand, we should now be able to decompose the tangent space in its vertical and horizontal part. The vertical subspace of TP is defined by the kernel of the pushforward projection π_{P*} while the horizontal part is given by the kernel of θ .

$$V_P P = \{[\dot{\gamma}]_P \in T_P P \mid \forall f \in C^\infty(P, \mathbb{R}) : \frac{d}{dt} (f \circ \pi_P \circ \gamma(t)) \stackrel{!}{=} 0\} \equiv T_P P$$

The identification of the vertical and full tangent space is due to the fact that any $\gamma_p : \mathbb{R} \rightarrow P$ can be uniquely written as $\gamma_p(t) = (p, \gamma_G(t))$. Under projection, $\pi_P(\gamma_p(t)) = \pi_P((p, \gamma_G(t))) = p$, we obtain a constant function, hence the restricting condition is redundant yielding ‘0 = 0’. On the other hand, the triviality of the horizontal space is demonstrated using an auxiliary function $f \in C^\infty(G; \mathbb{R})$:

$$H_P P = \ker \theta = \{[\dot{\gamma}]_P \in T_P P \mid \theta([\dot{\gamma}]_P)(f) = \frac{d}{dt} (f \circ (p, L_{g^{-1}} \gamma(t))) \stackrel{!}{=} 0\} \equiv \{e_{T_P P}\}$$

The necessary condition for tangent vectors to be horizontal can only be satisfied for constant curves $\gamma_p(t) = \gamma_p$, since the left-action on G is a group isomorphism. A direct consequence is the vanishing of the curvature two-form since $d\theta$ is bilinear in its arguments which by definition are horizontally projected. The result are the well known Maurer-Cartan structure equations:

$$\Omega^\theta = 0 = d\theta + \theta \wedge \theta$$

The missing ingredient, the exponential map of a Lie group, is however not associated to the Maurer-Cartan form, for in this case the base space is the trivial point $\{p\}$. Therefore, we have to reverse the description and define a connection with the Lie group as base space. The important fact here is that a Lie group is always trivializable and thus its tangent space TG can be written as the direct product $TG \simeq G \times T_e G \simeq G \times \mathfrak{g}$ by means of $L_{g*} : \mathfrak{g} \rightarrow T_g G$. This correspondence extends further in that the Lie algebra $\mathfrak{g}$ is isomorphic to the space of left-invariant vector fields:

$$\mathfrak{g} \simeq \{v^L \in \Gamma(TG) \mid L_{g*} v_h^L = v_{g \circ_G h}^L \quad \forall g, h \in G\}$$

The canonical construction on Lie groups with a bi-invariant form is based on a left-invariant Levi-Civita connection resulting in geodesics starting at $\gamma(0) = e_G$ that are group homomorphisms from $(\mathbb{R}, +)$ to G , i.e.

$$\gamma(t_1) \circ_G \gamma(t_2) = \gamma(t_1 + t_2) = \gamma(t_2 + t_1) = \gamma(t_2) \circ_G \gamma(t_1) \quad \wedge \quad \gamma(0) = e_G$$

Hence, the exponential map is simply given by the extension of curves on G associated to some left-invariant vector field:

$$\exp_G : T_{e_G} G \simeq \mathfrak{g} \rightarrow G, \quad \exp_G(t \cdot [\gamma]_{e_G}) = \gamma(t) \quad \text{with} \quad \gamma(t_1) \circ_G \gamma(t_2) = \gamma(t_1 + t_2)$$

Furthermore, fundamental vector fields on a principal bundle P over some Lie group G are generated by elements of $\mathfrak{g}$ and the exponential map of G :

$$V_u^\# \equiv \rho_P^R(u, \bullet)_* V \equiv \frac{d}{dt} (u \circ_G \exp_G(t \cdot V)) \Big|_{t=0} \quad \forall V \in T_{e_G} G \simeq \mathfrak{g}$$

Every vertical subspace $V_u P$ is thus isomorphic to the Lie algebra $\mathfrak{g}$. Notice that the exponential map commutes with the adjoint action and that in addition the following identity holds [24, 25]:

$$\rho_P^R(\bullet, g)_* (V^\#) = (\text{ad}_{g^{-1}}(V))^\#$$

As an example, consider the group $\text{Gl}(d; \mathbb{R})$ consisting of all real $d \times d$ matrices, $\text{Mat}(d; \mathbb{R})$, that are invertible with $\circ_G$ the usual multiplication. The left-invariant vector fields are isomorphic to the Lie algebra consisting of

$$\begin{aligned} \mathfrak{gl}(d; \mathbb{R}) &\equiv (\{A_\gamma \in \text{Mat}(d; \mathbb{R}) \mid \gamma(0) = \mathbb{1} \wedge \gamma(\varepsilon) \in \text{Gl}(d; \mathbb{R})\}, +, [\bullet, \bullet]_{\mathfrak{gl}} \equiv [\bullet, \bullet]) \\ &= (\text{Mat}(d; \mathbb{R}), +, [\bullet, \bullet]) \end{aligned}$$

In this case, the exponential map is simply given by the exponential series of a matrix:

$$\exp_{\text{Gl}}(t \cdot A) \equiv \sum_{j \in \mathbb{N}_0} \frac{t^j}{j!} A^j \quad \forall A \in \mathfrak{gl}(d; \mathbb{R})$$

and the fundamental vector fields are determined as

$$A_u^\# \equiv \frac{d}{dt} (u \circ_G \exp_{\text{Gl}}(t \cdot A)) \Big|_{t=0} = u \circ_G \frac{d}{dt} \left(\sum_{j \in \mathbb{N}_0} \frac{t^j}{j!} A^j \right) \Big|_{t=0} = u \circ_G A$$

Summary

Fiber bundles introduce the possibility to enlarge our understanding of continuity and differentiability and free those concepts globally from the standard Euclidean picture while retaining most of its valuable features due to its local correspondence. Removing the global restrictions provides a multitude of new structures that locally agree with the standard notion but globally produce new and interesting properties that turn out to be at the heart of today's physics. We have to be aware that in curved spacetimes smoothness is a more flexible notion than a simple non-interrupted curvy function. Indeed, even fields that have jumps or obvious zigzag patterns may show up to be perfectly admissible when considered in their natural environment, i.e. a suitable fiber bundle. The confusion that may arise is only due to the projection into the familiar Euclidean structure that we should abandon whenever we leave the realm of locality. The several patches of spacetime (the base space) that have to be glued together already reflects this formalism, and here we extended this prescription to the space of functions.

While there is a variety of possible differential structures a target F and a domain space M can be combined to, we have to focus on inequivalent ones that respect a certain symmetry constraint, leading to the concept of principal bundles. The trivial case coincides with the product topology $\tau_{M \times F}$ where all transition functions are identities and the only source of discontinuities in the standard sense, i.e. compared with $\mathbb{R}^d$, is due to a possibly non-trivial structure of M . Taken together, we have to decide about the differentiable structure of M , F , and $E(M, F, \pi_E)$ independently in order to fix the notion of smooth functions $f : X_M \rightarrow X_F$ in the bundle context. For our purpose, the most important examples of fiber bundles are tensor bundles. Combining different fiber bundles that share the same symmetry group G requires a consistent notion of differentiability compatible with G . Principal bundles, i.e. $P = M^{\pi_P} \times_{\rho_P^R} G$, provide this universal notion and give rise to a class of descendants, the associated bundles, which are naturally connected in their concept of smoothness.

An additional ingredient that has to be defined on top of a principal bundle are connections. They define a zero line on the tangent space of a principal bundle from which we measure deviations, by decomposing vector fields into horizontal and vertical components. Such connections are the natural candidates to describe interactions in physics, for they introduce a consistent notion of change into the framework of fiber bundles. A derived concept that attaches these interactions to ordinary fields is the covariant derivative that we will encounter at several steps of our investigations. Following this short summary, we consider next the implementation of differential geometry into the context of field theory.

1.3.2 Field space

We have seen so far three main ingredients of the mathematical theory that turn out to be at the root of the functional renormalization group approach to quantum gravity. The first addresses topology and differentiability in the shape of the base space our physical field content lives in. It sets the arena in which all the physics takes places. The second ingredient is a remarkably simple yet very powerful branch of mathematics that reflects symmetries. They appear as structure groups in the construction of principal bundles and are directly related to the fundamental laws of Nature under which the physical content interacts. In many ways group theory is an excellent guiding principle to extract very strong constraints on mathematical objects and nowadays it seems that the interplay on an elementary level – that gives rise to the diversity of phenomena we observe around us – actually emanates from some very simple symmetry principle. Fiber bundles are the natural construction to consistently combine symmetry constraints of interactions and differential geometry, thus it is not astonishing that the very successful Standard Model of particle physics is presented in this language. The necessary connection to quantum field theory is established by giving the mathematical objects in the theory of fiber bundles a

physical meaning, which leads to what is known as gauge theories.

At this point, we are going to present the missing link to physics by introducing the basics of gauge theory. It turns out that the set of physical fields can be usually embedded into a smooth structure, a Fréchet manifold. This equips field space with a large variety of additional properties, in particular it is locally described by a vector space. In addition lifting the collection of fields from a mere set to a Fréchet space allows to consider a notion of change within field space itself. Thus, functional variation can be defined at least on a local patch generated by the so called background field method, which provides the transition from dynamical to fluctuation fields. The necessity of this patchwork is demonstrated in the presence of boundary conditions.

Fields in gauge theories

The two basic ingredients for the construction of a principal bundle are the base space and the structure group. In physics, *spacetime* M constitutes the ground on which physics takes place and the structure group is given by some symmetry $G \equiv (X_G, \circ_G)$ most often denoted the *gauge* or *symmetry group* of the theory. The latter can be a combination of direct and semi-direct products of groups but it is usually assumed to be a Lie group. This information is enough to generate the set of inequivalent principal bundles:

$$\text{PBs}(M, G) \equiv \{[P = M^{\pi_P} \times_{\rho_P^R} G]_{\sim_{\text{FB}}}\}$$

Each equivalence class of principal bundles, an element of $\text{PBs}(M, G)$, gives rise to a different set of smooth functions and hence to different theories. The classification of all principal bundles for some base space M and structure group G is a hot topic in differential geometry and is based on the homotopy group of some classifying space that exists whenever the base space is paracompact. In fact, the homotopy properties of the classifying space are a direct consequence of the homotopy groups of base space M and structure group G . For instance contractible base manifolds give rise to only trivial principal bundles and depending on compactness and connectedness properties of M and G further classifications can be deduced (see ref. [4, 29] for further details).

At this point we have to make a suitable choice for $[P]$ in order to proceed. Having selected $[P]_{\sim_{\text{FB}}} \in \text{PBs}(M, G)$ the set of (local) smooth sections over some region $U \subseteq X_M$ defines the admissible (local) *gauges*

$$s \in \Gamma(P) \quad \text{defines a gauge}$$

Each (local) section defines a (local) reference scale and corresponds to a *gauge choice* in physics. A global section however is only possible if P is trivial. Nevertheless, it is usually sufficient to transfer the global theory into local coordinates using a covering of charts on M . The theory should be *gauge invariant* under a change of section (gauge) which is generated by a *gauge transformation*, an element of

$$\mathcal{G} \equiv \{f \in \text{Iso}(\Gamma(P))\} \equiv \{f \in \text{Iso}(P) \mid \pi_P \circ f = \pi_P \wedge \rho_P^R(f(u), g) = f(\rho_P^R(u, g))\}$$

This second equivalence renders the group of gauge transformation as the vertical automorphisms of P in that they shuffle only the structure group while acting as the identity on space-time. A key requirement in gauge theory is the invariance of observables, especially expectation fields, under $\mathcal{G}$.²³

R Notice that the horizontal automorphisms of P are given by the diffeomorphisms of M . Whenever a special coordination of the spacetime is invoked, i.e. a representative of

²³A *pure gauge transformation* describes the action of $\mathcal{G}$ on a zero-section, i.e. $s(p) = (p, e_G)$.

$M \equiv [M]_{\sim \text{Diff}(M)}$ is chosen, we have to assure that the formalism is invariant under the full diffeomorphism group of P . While for the general description it is mandatory to define equivalence classes, in practical applications one usually resolves the equivalence character, but ensures the results be independent on a particular choice.

In most of practical calculations gauge invariance is described or tested using infinitesimal gauge transformations which form a Lie algebra. The transition to $\mathcal{G}$ is provided by the exponential map, though in general the Lie algebra spanned by its generators covers only part of the full gauge group, in particular only the identity component. Hence, we have set the scene to introduce interactions in a smooth and symmetry preserving way, before we finally let matter fields enter the game.

Interactions In quite general terms, interactions change the status quo depending on the underlying rules. The diversity of opportunities one has in constructing these fundamental laws is huge, however those that preserve the differential structure and map physical fields into new, still physical, ones is severally constrained. Given a certain principal bundle P the collection of distinct principal connections are exactly the different possibilities in defining smooth symmetry invariant notions of change:

$$\begin{aligned} X_{\mathcal{C}(P)} &\equiv \{ \omega : TP \rightarrow \mathfrak{g} \mid \omega \text{ is principal connection} \} \\ X_{[\mathcal{C}](P)} &\equiv X_{\mathcal{C}}/\mathcal{G} \equiv \{ \{ f^* \omega \mid f \in \mathcal{G} \} \mid \omega \in \mathcal{C}(P) \} \end{aligned}$$

Again, the mathematical redundancy of physically equivalent constructions is removed by building equivalence classes of principal connections. At this step additional (mathematical) requirements can be imposed, for instance a vanishing torsion or the Levi-Civita constraint, which may single out a unique representative of $X_{[\mathcal{C}](P)}$. For (anti-)self-dual principal connections one refers to ω as *instanton*.

Each connection defines a notion of interaction and goes under the name of *gauge connection* in the physics literature. In a local frame $U \subseteq M$ with a trivializing cover $\{(U_j, f_j)\}$ and $s_j \in \Gamma(P)$ local sections, the (local) *gauge potential* related to $\omega \in X_{\mathcal{C}(P)}$ is given by

$$\mathcal{A}_j : TU \rightarrow \mathfrak{g} \quad \text{with} \quad \mathcal{A}_j := s_j^* \omega$$

Notice that $\mathcal{A}_j$ is in general a local object that can be combined to a global smooth 1-form only on the principal bundle. In general, we have to combine different smooth functions $\mathcal{A}_\ell$ using the local sections s_ℓ , which however have to fulfill a consistency condition in order to give rise to the same ω :

$$\mathcal{A}_j = \text{ad}_{\kappa_{\ell j}^{-1}} \mathcal{A}_\ell +_{\mathfrak{g}} \kappa_{\ell j}^* L_{\bullet^{-1}}^* \quad \forall j, \ell : U_j \cap U_\ell \neq \emptyset$$

Here $\kappa_{\ell j}(p) : G \rightarrow G$ are the transition functions, defined by $f_\ell \circ f_j(p, g) \equiv (p, \kappa_{\ell j}(p) \circ_G g)$ whenever $p \in U_j \cap U_\ell$. The above transformation is obviously not linear, since it contains the Maurer-Cartan form of G as an additional translation term. For a matrix Lie group G the embedding of the Lie algebra into G is trivial and the gauge transformation of the gauge potential assumes the form:

$$\mathcal{A}_j = \kappa_{\ell j}^{-1} \circ_G \mathcal{A}_\ell \circ \kappa_{\ell j} +_{\mathfrak{g}} \kappa_{\ell j}^{-1} \circ d \kappa_{\ell j} \quad \forall j, \ell : U_j \cap U_\ell \neq \emptyset$$

Furthermore, the curvature Ω^ω in a local gauge is the well known (local) *field strength*

$$\mathcal{F}_j^\omega = s_j^* \Omega^\omega$$

Imposing some restrictions on Ω^ω is another way to reduce the set of allowed connections. However, usually these constraints only appear in combination with some even stronger condition, as for instance the requirement of ω being Levi-Civita, i.e. related to a Riemannian metric on M , which is encountered in theories of quantum gravity for instance.

Matter fields With a principal connection ω we have introduced change and thus interactions into the description. The missing ingredient are the matter fields, the physical content that interacts with each other. Notice however, that matter fields on their own do not require a principal connection but are independent concepts that can be defined whenever a principal bundle is selected.

Matter fields usually live in a certain vector space over spacetime, and if they take part in the interaction defined by ω it should be a (local) section of an associated bundle, i.e. a *matter field* of a gauge theory is given by

$$\hat{\Phi} \in X_{F_M} := \{\hat{\Phi} \in C^\infty(P, V) \mid \rho_V(g^{-1}, \hat{\Phi}(u)) = \hat{\Phi}(\rho_P^R(u, g))\} \simeq \{\sigma \in \Gamma(E) \mid E \equiv P \tilde{\times}_\rho V\}$$

Hereby, the *internal space* V stands for a vector space, that when specified further gives some particular meaning to the related matter fields. For instance, if $V \equiv \mathbb{R}$ we speak of a *scalar field*, more general terms are related to *n-component vector fields* in case of $V \equiv \mathbb{C}^n$, and for $V \equiv \mathfrak{g}$ and the adjoint representation $\rho \equiv \text{ad}$ physicists denote $\hat{\Phi}$ as *Higgs field*.

Gauge transformation equivalent matter fields represent the same physical field, so in principal we work with an equivalence class of sections, that constitute the actual field content of the system. Let $\{V_j\}_{j \in J}$ be the collection of all target spaces in which the relevant matter fields are defined by some associated group action ρ_j . The full set of independent physical fields constitute the matter part of field space, and with $V \equiv V_1 \times \cdots \times V_J$ and $\rho \equiv \rho_1 \otimes \cdots \otimes \rho_J$, we define

$$X_{[F_M]} := X_{F_M} / \mathcal{G} \equiv \{[\hat{\Phi}] \mid \hat{\Phi} \in X_{F_M}\} \equiv \{\{\hat{\Phi} \circ f \mid f \in \mathcal{G}\} \mid \hat{\Phi} \in X_{F_M}\}$$

They interact with each other via the covariant derivative induced by elements of $X_{[\mathcal{G}](P)}$.

Summary Gauge theory, based on symmetry principles, is one of the great achievements of the last century, translating QFT into the language of differential geometry. By this, remarkable insight into the geometrical structure of fundamental theories might be obtained. The information we need to specify the field content of a gauge theory consists of

- **Spacetime:** M an equivalence class of smooth manifolds
- **Symmetry group:** G , acts as the structure group
- **Principal bundle:** $[P = M^{\pi_P} \times_{\rho_P^R} G]_{\sim_{FB}} \in \text{PBs}(M, G)$, defines a G -invariant notion of differentiability and thus smooth functions
- **Gauge fields:** $[\omega] \in X_{[\mathcal{G}](P)}$, a principal connection on P that defines interaction
- **Matter fields:** V and $\rho_V : G \times V \rightarrow V$, a vector space (target space) and group action that defines an associated bundle along with admissible matter fields $X_{[F_M]}$

The remaining ingredients can be reconstructed by the above structure, in particular the gauges and the associated gauge group of transformations.

R For Levi-Civita connections, which are often considered in quantum gravity, the gauge fields can be rewritten in terms of ‘matter fields’, namely as a Riemannian metric. In those cases, listing the gauge field is redundant and is usually omitted, though it is intrinsically present.

Field space

We have already seen that the set of smooth functions or vector fields can be endowed with additional structure and at least locally they define vector spaces. It turns out that the collection of matter and gauge fields inherits a plenitude of rich structure from the fiber bundle construction and the base space properties. In physical applications, one usually has supplementary constraints on the true set of relevant fields, for example in the shape of boundary conditions, that restricts $X_{[\mathcal{F}_M]}$ and $X_{[\mathcal{G}]}$ further, which may either spoil or enrich certain properties of field spaces.

In order to implement a notion of differentiation on field space, which is nothing else than variation w.r.t. fields, a suitable topological structure is necessary. The actual choice severally depends on the underlying spacetime and most of the discussion presented here is only valid for closed manifolds, i.e. for compact manifolds with vanishing boundary. Notice however that there exists extensions to non-compact manifolds and cases of non-vanishing boundary along similar lines, usually based on a different topology. In most applications, the sets $X_{\mathcal{F}_M}$ and $X_{\mathcal{G}}$ are endowed with an extension of the Whitney C^∞ -topology for smooth functions $C^\infty(P, V)$ where the tensor structure is trivially added. Basically, the infinite dimensional space $C^\infty(P, V)$ is reduced to a finite dimensional jet space using an equivalence relation similar to the Taylor expansion. Assume we are given a function $\sigma \in C^\infty(P, V)$ that in local coordinates (U_P, f_{U_P}) and (U_V, f_{U_V}) can be expanded around a point $f_{U_P}(u) \equiv x_0 \in \mathbb{R}^d$ with $u \in U \subseteq P$:

$$\begin{aligned} \tilde{\sigma}(x) &\equiv (f_{U_V} \circ \sigma \circ f_{U_P}^{-1})(x) = (j_{x_0}^k \tilde{\sigma})(x) + \mathcal{O}(x^{k+1}) \quad \text{with} \\ (j_{x_0}^k \tilde{\sigma})(x) &:= \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha \tilde{\sigma}(x_0) (x - x_0)^\alpha \end{aligned}$$

Hereby, we have used the notation of multivariate Taylor expansion with $\alpha \in \mathbb{R}^d$ and $\partial^\alpha f(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f(x)$. The map $j_{x_0}^k$ simply reduces a function to its k th-order Taylor expansion, which gives rise to an equivalence relation on $C^\infty(P, V)$, the k th-jet space:

$$J_u^k(P, V) := \{ \{ \sigma_2 \in C^\infty(P, V) \mid (j_{f_{U_P}(u)}^k \tilde{\sigma}_1)(\bullet) = (j_{f_{U_P}(u)}^k \tilde{\sigma}_2)(\bullet) \} \mid \sigma_1 \in C^\infty(P, V) \}$$

It turns out that $J_u^k(P, V)$ is a real, smooth manifold with dimension $\dim(P) + \dim(V) \cdot (\dim(P) + k)! / (\dim(P)! \cdot k!)$. The Whitney C^k -topology is then obtained using the topology of V :²⁴

$$\tau_{C^k} := \text{gen} \{ \sigma \in C^\infty(P, V) \mid (j_{x_0}^k \tilde{\sigma})(x) \} \quad \text{and} \quad \tau_{C^\infty} = \text{gen} \left\{ \bigcup_{k \in \mathbb{N}_0} \tau_{C^k} \right\}$$

Lifting this topology to $X_{\mathcal{F}_M}$ and $X_{\mathcal{G}}$ gives rise to infinite-dimensional topological manifolds $\mathcal{F}_M \equiv (X_{\mathcal{F}_M}, \tau_{C^\infty})$ and $\mathcal{G} \equiv (X_{\mathcal{G}}, \tau_{C^\infty})$, respectively. The sensitivity of this construction on the properties of spacetime is encoded in the jet space. For non-compact or otherwise generalized spacetimes the properties of $J_u^k(P, V)$ may not be sufficient to deduce a well behaved topology on $C^\infty(P, V)$.

Once we have introduced a suitable topology on field space with a local vector space structure and Hausdorff character, there are several derived concepts at our disposal. With an additional differentiable structure a notion of field variation, which is of particular importance in QFT, is introduced, that further supplements the construction of field space. For this, let us first consider the space of connections²⁵ and follow the lines of ref. [34, 35]. In ref. [36] the author demonstrates that the group of gauge transformation can be used to define a fibration on $\mathcal{G}$

²⁴Here $\text{gen}\{U\}$ denotes the topology generated by $\{U\}$, see definition 1.1.11.

²⁵As in [34] we confine $[\mathcal{G}]$ to the space of generic connections. These are connections for which $d^{\omega*} d^\omega$ has trivial kernel.

with the canonical map $\pi_{\mathcal{G}} : \omega \in \mathcal{G} \mapsto [\omega] \in [\mathcal{G}]$.²⁶ The space of equivalence classes $[\mathcal{G}]$ turns out to be a smooth manifold with an additional Riemannian structure [36–38]. The associated Riemannian metric – a bilinear, symmetric, non-degenerated form on the tangent space of M – is a consequence of the underlying constituents of $\mathcal{G}$, namely a metric $\bar{g}$ on M and the Killing form K of G . To obtain its explicit form, notice that $[\mathcal{G}]$ carries an affine structure modeled on $\Gamma(\Lambda^1(M) \otimes E)$, the space of 1-forms on M with values in the adjoint bundle $E \equiv P \tilde{\times}_{\text{adg}}$. A translation of any connection ω along $\alpha \in \Lambda^1(M) \otimes \Gamma(E)$ gives rise to a new principal connection ω' and thus the tangent space of $\mathcal{G}$ is identical to $\Lambda^1(M) \otimes \Gamma(E)$, i.e. $T_{\omega}\mathcal{G} \simeq \Lambda^1(M) \otimes \Gamma(E)$. The Riemannian metric $\mathfrak{G}$ on $\mathcal{G}$ is induced by the inner product on $A^k \equiv \Lambda^k(M) \otimes \Gamma(E)$ restricted to $k = 1$:

$$(\alpha_1, \alpha_2)_{A^k} := \int_{x \in M} d^d x K(\alpha_1 \wedge \star \alpha_2) \quad \forall \alpha_1, \alpha_2 \in A^k$$

This inner product and thus the Riemannian metric on $\mathcal{G}$ are gauge invariant by construction. As we will see, an inner product introduces a classification scheme on the space of operators, in particular it gives rise to the adjoint operator of d^{ω} denoted $d^{\omega*}$. A natural definition of horizontality on $T_{\omega}\mathcal{G}$ is then provided by a principal connection on field space:

$$\chi_{\mathcal{G}}^{\omega} := (d^{\omega*} d^{\omega})^{-1} d^{\omega*} \quad \text{and} \quad H_{\omega}[\mathcal{G}] := \ker(\chi_{\mathcal{G}}^{\omega})$$

Horizontal vector fields on $\mathcal{G}$ generate physically non-equivalent gauge fields, while the vertical translations do not change the equivalence class $[\omega]$ at all. The projection onto the horizontal component explicitly depends on the field space connection $\chi_{\mathcal{G}}^{\omega}$ and is given by

$$\Pi_{\omega}^H : T_{\omega}\mathcal{G} \rightarrow H_{\omega}\mathcal{G}, \quad \Pi_{\omega}^H \equiv \text{id} - d^{\omega} \chi_{\mathcal{G}}^{\omega}$$

which is easily shown to be idempotent and in fact also self-adjoint.

From now on, we assume that the fibration of $\mathcal{G}(P)$ actually defines a principal bundle $\mathcal{G} = [\mathcal{G}]^{\pi_{\mathcal{G}}} \times_{\rho_{\mathcal{G}}^h} \mathcal{G}$ with the base space consisting of physically equivalent gauge fields. All theories we are going to construct actually live on this base space, however for practical calculations and general considerations it is usually convenient to rather work in the total space $\mathcal{G}(P)$ and ensure the formalism being invariant under $\mathcal{G}$. While it simplifies the evaluation of the path integral, at the same time this transition may introduce regularization issues and ambiguities, due to the non-existence of a global section in $[\mathcal{G}]$. These artifacts of gauge invariance cause several problems in QFT and are very prominent for theories of quantum gravity. The actual transition of theories defined on $[\mathcal{G}]$ to the total space $\mathcal{G}$ is postponed until we have the full device of measure theory at our disposal, see subsection 2.1.5.

Whilst there are in general no global sections on $\mathcal{G}$ a horizontal local section is a set of principal connections which have no vertical displacement, thus satisfy

$$s^{\omega_0} : U_0 \subseteq [\mathcal{G}] \rightarrow \mathcal{G} \quad \text{with} \quad [\omega_A] \mapsto \pi_{\mathcal{G}}^{-1}([\omega_A]) \cap S^{\omega_0} \quad \wedge \quad S^{\omega_0} := \{\omega \mid d^{\omega_0*}(\omega - \omega_0) = 0\}$$

This defines the covariant background gauge condition and fixes the gauge *in field space*, at least locally. Based on this local section, an induced Riemannian metric on $[\mathcal{G}]$ is thus obtained by a horizontal lift of the scalar product on A^1 at some $\omega \in \pi_{\mathcal{G}}^{-1}([\omega])$. Given two vector fields on base space $v, w \in \Gamma(T[\mathcal{G}])$ there are horizontal vector fields $\tilde{v}$ and $\tilde{w}$ on $T\mathcal{G}$ which locally describe the respective lifts. Then, pointwise we have:

$$\begin{aligned} (v_{[\omega]}, w_{[\omega]})_{T_{[\omega]}[\mathcal{G}]} &:= (\tilde{v}_{\omega}, \tilde{w}_{\omega})_{A^1} \equiv (\tilde{v}_{\omega}, \Pi_{\omega}^H \tilde{w}_{\omega})_{A^1} \\ &\equiv (\tilde{v}_{\omega}, \Pi_{\omega_0}^H \Pi_{\omega}^H \Pi_{\omega_0}^H \tilde{w}_{\omega})_{A^1} \end{aligned}$$

²⁶Actually, $\mathcal{G}$ has to be reduced by its center to provide a consistent fibration. We will assume this in what follows.

Due to the gauge invariance and the properties of the horizontal lift, this definition is independent on the choice of ω and thus well suited for our purpose. However, this coordinate choice fails whenever the Faddeev-Popov operator $\mathcal{M}(\omega, \omega_0) := d\omega_0^* d\omega$ has a non-trivial kernel, i.e. $\ker(\mathcal{M}) \neq \{e_{T_{\omega_0}\mathcal{C}}\}$ which corresponds to the appearance of the Gribov ambiguity. Notice that $\Pi_{\omega_0}^H \Pi_{\omega}^H \Pi_{\omega_0}^H$ is the local coordinate form of the metric induced on $[\mathcal{C}]$. With $(\omega - \omega_0) \equiv v_{\omega_0} \in T_{\omega_0}\mathcal{C}$ this local expression can be recast in the form

$$\Pi_{\omega_0}^H \Pi_{\omega}^H \Pi_{\omega_0}^H \equiv \Pi_{\omega_0}^H \left(\text{id} - K_{v_{\omega_0}} (d\omega^* d\omega)^{-1} K_{v_{\omega_0}}^* \right) \Pi_{\omega_0}^H \quad \text{with} \quad K_{v_{\omega_0}}(\xi) := [v_{\omega_0}, \xi]$$

Here, $K_{v_{\omega_0}} : A^0 \rightarrow A^1$ denotes the infinitesimal generators of the gauge transformation, and in the case of metric gravity it corresponds to $\mathcal{L}_{v_{\omega_0}} \xi$. Hence, the second term in the bracket subtracts the vertical component as required.

If the horizontal space S^{ω_0} intersects each orbit on $\mathcal{C}$ it actually defines a global normal coordinate system [34]. In fact, it turns out that geodesics are ‘straight lines’, namely curves of the form $\omega_0 + t \cdot v_{\omega_0}$ and once it is perpendicular to a specific orbit it will remain so for any other orbit intersecting with S^{ω_0} . Hence, the exponential map at ω_0 is simply given by

$$\exp_{\omega_0}^{\chi_{\mathcal{C}}^{\omega_0}}(t \cdot v_{\omega_0}) \equiv \pi_{\mathcal{C}}(\omega_0 + t \cdot v_{\omega_0})$$

The exponential map becomes singular at $\omega_S = \omega_0 + t_S \cdot v_{\omega_0}$ exactly if $\mathcal{M}(\omega, \omega_0) + t_S \cdot d\omega_0^* K_{v_{\omega_0}}$ has a non-trivial kernel, which in any case will be finite dimensional. The associated conjugate points ω_S mark the Gribov horizon of ω_0 along v_{ω_0} and their occurrence spoils the diffeomorphic identification of $\mathcal{C}$ with its tangent space at ω_0 . Explicitly, ω_S corresponds to a $\xi \in A^0$ with $\mathcal{M}(\omega, \omega_0)\xi = 0$ and $\pi_{\mathcal{C}*}(d\omega\xi) = 0$, hence to a vanishing Faddeev-Popov determinant. As such the existence of Gribov ambiguities and horizons is an intrinsic artifact of the theory.

For matter fields similar arguments holds, at least if the full set of sections is considered. Then, F_M has a fibration $\pi_{F_M} : \hat{\Phi} \mapsto [\hat{\Phi}] \in [F_M]$ which in fact provides a fiber bundle construction under some very weak restrictions:

$$F_M = [F_M]^{\pi_{F_M} \times \rho_{F_M}^R} \mathcal{G}$$

As long as there are no further constraints imposed on the matter fields, the full set of sections inherits many, very nice properties of the function space, in particular vector space and Riemannian structures. Hence, the usual L^2 -scalar product can be extended to the space of matter fields by means of the Killing form K . Global sections provide a way to evaluate the generating functional in QFT for the set of physically inequivalent fields, i.e. $[F_M]$. However, usually additional restrictions are invoked, as for instance boundary conditions in the case of $\partial M \neq \emptyset$, that may spoil the trivial nature of field space. Some of the severe challenges on the way to a quantum theory of gravity can be traced back to the non-trivial fiber bundle structure of F_M compared to the Standard Model of particle physics for instance. To maintain the covariant and global descriptions (on field space) the formalism turns out to be far more complicated and actual evaluation of the generating functional seem to be impossible. As history has given many examples, it might be exactly this difficulty that will provide further insight into the underlying principles of Nature for all interactions.

Background field method

We have seen that the space of principal connections and sections of associated bundles has a wealth of mathematical structure that allows to introduce almost the entire apparatus of differential geometry. In particular its fiber bundle and Riemannian manifold description are of uttermost importance to define a suitable notion of field variation. While $\mathcal{C}$ carries an affine

structure, the matter sector is usually given by a global smooth vector space, hence a trivial principal bundle with structure group $\mathcal{G}$. Though in general, we have to keep the notion of connections and matter fields separately, for simplicity we combine both field spaces into a single one, denoted Y , and only remove this covering cloak when a distinction seems to be unavoidable. One can also think about Y as the direct product of $\mathcal{C}$ with F_M giving rise to a product principal bundle

$$Y = [Y]^{\pi_Y \times \rho_Y^R} \mathcal{G} \quad \text{with} \quad [Y] \equiv \{ \{ \rho_Y^R(\hat{\Phi}, f) \mid f \in \mathcal{G} \} \mid \hat{\Phi} \in Y \}$$

It inherits all topological, differentiable and bundle properties from its constituents, in particular the projection π_Y and the group action ρ_F^R are naturally induced by the respective objects on $\mathcal{C}$ and F_M .

For the gauge fields, we have already seen that there is no global vector space structure present, however due to its affine nature it gives rise to a trivial principal bundle and hence – without additional constraints – Y is also trivial. This provides a global section and $[Y]$ can be nicely embedded in Y , which is particularly relevant for the construction of a generating functional in QFT. Unfortunately, it turns out that Nature seems to favor only a subset of all mathematically possible principal connections and sections of associated bundles. Usually, this restriction is best implemented into the formalism using an operator on Y with non-trivial kernel:

$$\mathcal{B} : Y \rightarrow Y \quad \text{with} \quad \ker(\mathcal{B}) \neq \{e_Y\}$$

One then defines the true set of physical fields by the kernel of these constraining operators, i.e.

$$X_F := \{ \hat{\Phi} \in Y \mid \hat{\Phi} \in \bigcap_j \ker(\mathcal{B}_j) \}$$

In general, the reduction of the fiber bundle Y to the subset X_F will spoil topological and some derived properties, in particular the concept of smoothness and fibration. In order to maintain the calculus of variation on X_F one has to start from scratch and rebuild, if possible, a new differentiable structure that hopefully promotes X_F to a new equally equipped principal bundle. While for a generic $\mathcal{B}$ this procedure might fail, it seems that most of the constraints we encounter in Nature lead to a field space that is open in Y , i.e. $X_F \in \tau_Y$. The induced subspace topology, resulting in F , keeps certain differentiable and fiber bundle properties intact, however algebraic structures might differ for Y and F , as for instance vector space properties. Depending on the explicit form of $\mathcal{B}_j$ this can have further (global) consequences for F , in particular the triviality of principal bundle may be lost, as for example in the case of Levi-Civita connections.

Some very well known examples of constraining operators are boundary conditions. In case of spacetimes with non-vanishing boundary $\partial M \neq \emptyset$ differential operators require that the fields of interest fulfill some conditions on ∂M that render certain relevant operators self-adjoint or elliptic. The most famous choices are Dirichlet or Neumann boundary conditions that correspond to the following class of constraint operators:²⁷

$$\begin{aligned} \mathcal{B}_{(D);\Psi}^{\partial M}(\hat{\Phi}) &:= (\hat{\Phi} - \Psi)(\bullet) \delta_{\partial M}(\bullet) \quad \text{with} \quad \ker(\mathcal{B}_{(D);\Psi}^{\partial M}) \equiv \{ \hat{\Phi} \in Y \mid \hat{\Phi}|_{\partial M} = \Psi|_{\partial M} \} \\ \mathcal{B}_{(N);\Psi}^{\partial M}(\hat{\Phi}) &:= (d^{\omega_0} \hat{\Phi} - \Psi)(\bullet) \delta_{\partial M}(\bullet) \quad \text{with} \quad \ker(\mathcal{B}_{(N);\Psi}^{\partial M}) \equiv \{ \hat{\Phi} \in Y \mid d^{\omega_0} \hat{\Phi}|_{\partial M} = \Psi|_{\partial M} \} \end{aligned}$$

There are also e.g. mixed or non-local boundary conditions which seem relevant in physical applications. However, it is already for these two simple and nicely behaved operators that

²⁷Here $\delta_{\partial M}(\bullet)$ denotes the restriction to the boundary of M , hence a zero (identity) map for all interior (boundary) points of M .

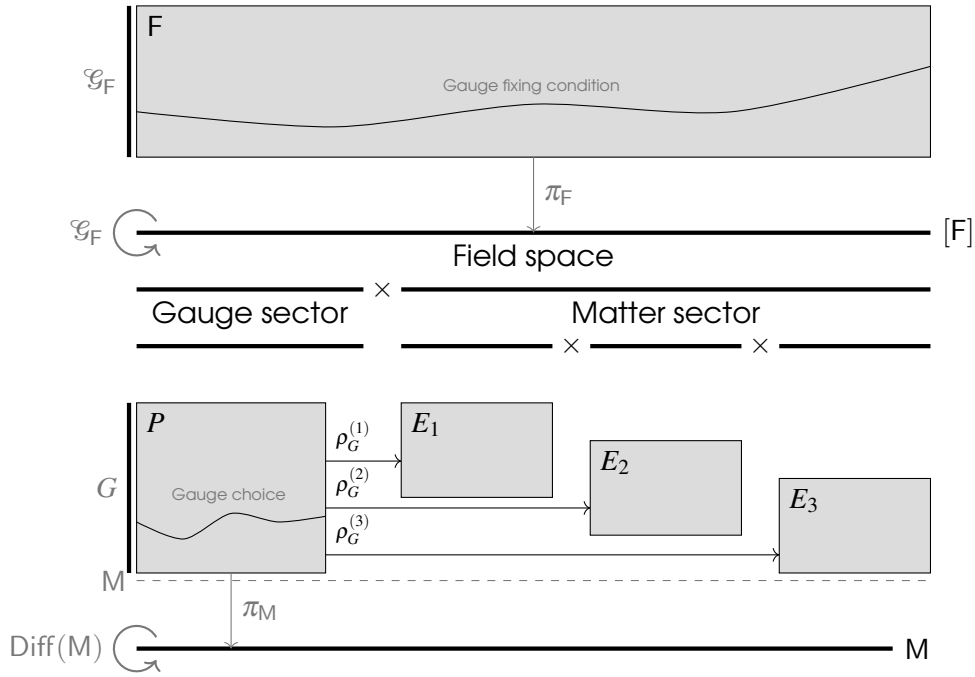


Figure 1.2: The fiber bundle construction of field space based on the gauge theory over space-time M and symmetry group G . The principal field space bundle F consists of base space $[F]$, inequivalent fields, and structure group $\mathcal{G}_F$.

certain structures of field space are lost. Therefore, notice that for any choice of $\Psi \neq e_Y$ the Dirichlet and Neumann boundary conditions are not linear but only affine operators. Reducing field space by these supplementary constraints results at most in an affine space, even for the matter sector. While this is no major drawback one can imagine that non-local or non-affine operators will severely reduce the structure of F . Most important, due to the diffeomorphism invariance of a manifold's boundary the smoothness structure is kept intact, though one has to adapt the group of gauge transformation appropriately, i.e. such that $\rho_Y^R(X_F, \mathcal{G}_F) \subseteq X_F$. An illustration of field space and its constituents is given in fig. 1.2.

Besides the apparent negative effects that come along with $\mathcal{B}_{(D);\Psi}^{\partial M}$ and $\mathcal{B}_{(N);\Psi}^{\partial M}$, there are some very welcome implications on the level of operators, in particular concerning their spectra, see section 1.4. The Riemannian manifold property of Y is nevertheless maintained in the transition from Y to F and in fact imposing only Dirichlet or Neumann boundary conditions to obtain field space does not affect the trivial nature of the principal bundle, even though it lacks a global notion of linearity. Hence, in this case we obtain

$$F = [F]^{\pi_F} \times_{\rho_F^R} \mathcal{G}_F \equiv [F] \times \mathcal{G}_F$$

with a flat field space connection and a trivial exponential map that provides a diffeomorphism for any $T_{\Phi}F$ to F .

We have seen that the explicit form of the boundary operator, which in the above case defines an affine transformation, decides about the inherited properties of F . For nicely formed constraint operators F is a trivial principal bundle with an affine structure that in particular allows to cover F by one of its tangent spaces, say at Φ . For more complex restrictions that still fulfill $\ker(\mathcal{B}) \in \tau_Y$ non-trivial fiber bundles emerge and also the affine property of F is lost. Hence, in general we only retain a Riemannian manifold F with some fibration and the injectivity region of the (non-trivial) exponential map might not cover full field space. In these

cases we need several base points $\bar{\Phi}_j$ and a partition of unity to span F by some $U_j \subseteq T_{\bar{\Phi}}F$. Whenever field space turns out to be non-trivial, i.e. it carries a non-flat principal connection χ_F , we will encounter a lot of difficulties, especially in the construction of a covariant definition of field variations. Fortunately, for the Standard Model of particle physics on flat spacetime on which most of our today's understanding of Nature is based on, field space seems to be perfectly trivial and no such problems occur.

However, attempts to construct a quantum version of General Relativity revealed the non-trivial field space character and thus a non-vanishing curvature tensor. In what follows we consider metric gravity, where we have a non-linear constraint on the vector space of symmetric 2-tensor fields. Besides boundary conditions in the case of non-vanishing ∂M , the class of (pseudo-)Riemannian metrics is not only symmetric but has a fixed signature along with certain non-degeneracy properties. The latter constraints turn the subset of (pseudo-)Riemannian metrics into a non-trivial fiber bundle, which is still a Fréchet manifold and can be equipped with a 'Riemannian' metric [39–41]. Conveniently, the field space connection is chosen to be of Levi-Civita type w.r.t. a field space metric $\mathfrak{G}$. In addition the physical assumptions concerning the nature of free theories impose further requirements on the connection – and thus on $\mathfrak{G}$ – in particular to be ultra-local (not-containing any field derivatives), trivial for free theories, and fully determined by the classical action functional. The field space metric exhibits all properties of a Riemannian metric. However, later we will enlarge field space to contain also anti-commuting Faddeev-Popov ghost fields and thus its symmetry properties have to be slightly modified. This also takes place when fermionic field content is added, for then additional factors of (-1) appear when two anti-commuting fields change position. Otherwise, we have a symmetric, bilinear function which is in addition non-degenerate:

$$\mathfrak{G}_{\bar{\Phi}}(v, w) = \mathfrak{G}_{\bar{\Phi}}(w, v) \quad \forall v, w \in T_{\bar{\Phi}}F, \quad \mathfrak{G}_{\bar{\Phi}}(v, v) = 0 \quad \implies \quad v = e_{T_{\bar{\Phi}}F}$$

To establish a covariant description of field space based on a Riemannian metric, F has to be (locally) represented by a vector space. At least in the vicinity of some $\bar{\Phi}$ we can approximate field space by its associated linear space $T_{\bar{\Phi}}F$. In quite generality, the background field method is a way to cover field space in terms of these local vector spaces in such a way that derived concepts will be independent of the specific covering. Then the full apparatus of differentiation and superposition applies and a smooth notion of integration can be derived. The local diffeomorphism equivalence is provided by the exponential map.

Definition 1.3.17 — Background field covering. Let M be a smooth (infinite Fréchet) manifold. The collection of ordered pairs $\mathcal{U}_{\text{BFC}} \equiv \{(p_j, U_j)\}$ with $\#(\mathcal{U}_{\text{BFC}}) < \infty$ is a background field covering of $M : \Leftrightarrow$

$$M = \bigcup_{(p_j, U_j) \in \mathcal{U}_{\text{BFC}}} U_j \quad \text{with} \quad U_j \in \tau_M \quad (\{U_j\} \text{ is an open cover of } M)$$

$$\forall p_j : \quad \exp_{p_j}^{\chi_M} : B_{\text{inj}_{p_j}(M)}(e_{T_{p_j}M}) \subseteq T_{p_j}M \rightarrow U_j \quad \text{isomorphism}$$

When $M \equiv F$ one refers to $\mathcal{U}_{\text{BFC}}$ as the background field covering of field space and the tangent vectors $\hat{\phi} \in T_{\bar{\Phi}}M$ at some background field $\bar{\Phi}$ are called fluctuation fields, while $\hat{\Phi}$ is referred to as dynamical field.

Hence, the existence of a partition of unity and geodesic completeness of field space is certainly important in defining a suitable background field method. A detail account on the necessary properties is presented in [37]. If field space is paracompact, then any finite open cover $\mathcal{U} \equiv \{U_i\}$ gives rise to a partition of unity that ultimately provides a background field covering of F . This results in a partition of unity in such a way that we sum locally over subsets of vector

spaces which ultimately allows to define the generating functional of a Quantum Field theory. Very often geodesic completeness can be ensured by metric completeness via the Hopf-Rinow theorem. For infinite-dimensional manifolds, which is the case for F , this correspondence however fails in general. Thus, an explicit derivation of the geodesics is usually unavoidable, though in certain cases other theorems, as e.g. the Cartan-Hadamard theorem, can be exploited. Otherwise, a suitable field space metric $\mathfrak{G}$ is the starting point to develop a (local) Levi-Civita connection on F and determine the normal coordinate form of geodesics along with the exponential map and its properties. In what follows, we assume for simplicity that field space can be covered by a single background field cover denoted $\mathcal{U}_{\text{BFC}} \equiv (\tilde{\Phi}, F)$ or $[\mathcal{U}_{\text{BFC}}] \equiv ([\tilde{\Phi}], [F])$, respectively.

Intrinsic and extrinsic symmetries

To avoid confusion, especially in the case of quantum gravity, it is instructive to distinguish the symmetries on field space from the symmetries of the coordinate version of a field. We have seen that $[F]$ allows for a principal bundle construction F with the group of gauge transformation $\mathcal{G}_F$ as its symmetry group. Physically distinguished fields live in different equivalence classes $[\hat{\Phi}]$ or $[\hat{\phi}]$ respectively. While for practical purposes it is more convenient to work with a special representative $\hat{\Phi} \in [\hat{\Phi}]$, one has to assure that later results do not depend on this particular choice but hold for the entire equivalence class $[\hat{\Phi}]$. In the case of field space one speaks about invariance under gauge transformation $\mathcal{G}_F$, an intrinsic symmetry of the formalism, and a specific choice of gauge condition fixes the ‘coordinates’. It is a vertical automorphism such that it does not affect the domain part of the fields, i.e. the spacetime coordinates.

$$X_{[F]} := X_F / \mathcal{G}_F \equiv \{[\hat{\Phi}] \mid \hat{\Phi} \in X_F\} \equiv \{\{\rho_F^R(\hat{\Phi}, f) \mid f \in \mathcal{G}_F\} \mid \hat{\Phi} \in X_F\}$$

with $\mathcal{G}_F \equiv \{f \in \mathcal{G} \mid \mathcal{B}_F(\rho_F^R(\bullet, f)) \equiv \rho_F^R(\mathcal{B}_F(\bullet), f)\}$

The symmetry on field space is referred to as *intrinsic symmetry*.

On the other hand we have a spacetime coordinate invariance of the entire formalism implemented in the construction of M . This symmetry reflects the missing individuality of spacetime points and hence spacetime is also an equivalence class of smooth manifolds, $M \equiv [M]_{\sim \text{Diff}(M)}$. Dissolving the abstract notation by choosing a particular ‘coordinatization’ of spacetime is unavoidable in practical calculations. Hence, giving an explicit coordinate form of fields, either $[\hat{\Phi}](x)$ or $\hat{\Phi}(x)$ is associated to a specific representative $M \in M \equiv [M]_{\sim \text{Diff}(M)}$. Again, we have to ensure that the results are independent of the particular choice of M rendering the formalism diffeomorphism invariant, $\text{Diff}(M)$. Thus, different coordinate versions of fields $\hat{\Phi}(x)$ and $(\hat{\Phi} \circ f_M)(x) \equiv \hat{\Phi}(f_M(x))$ with $f_M \in \text{Diff}(M)$ should be indistinguishable, however now due to the symmetry of spacetime. This ‘horizontal’ invariance is referred to as *extrinsic symmetry*. Both concepts have to be carefully told apart for they reflect completely different symmetry principles.

In summary, the space of physical fields is described by an equivalence class with respect to the group of gauge transformation. The remaining invariance, the extrinsic symmetry, generated by $f_M \in \text{Diff}(M)$ is entirely due to spacetime and does not further specify physical fields, but it labels spacetime points:

$$X_{[F]}(M) \equiv \{[\hat{\Phi}](x) \mid \hat{\Phi} \in X_F\} \equiv \{\{\rho_F^R(\hat{\Phi} \circ f_M, f_F) \mid f_F \in \mathcal{G}_F \wedge f_M \in \text{Diff}(M)\} \mid \hat{\Phi} \in X_F\}$$

Under a simultaneous choice of ‘coordinates’ the spacetime invariance reduces to the stabilizer subgroup of the field with respect to the induced group action of $\text{Diff}(M)$. Both intrinsic and extrinsic symmetries have to be clearly distinguished, since especially in quantum gravity it may otherwise cause some confusions about invariance requirements.

R Consider for example the space of Lorentzian metrics on TM and select the equivalence class containing the Minkowski metric: $[\eta]_{\sim[F]}$. On field space all metrics $\rho_F^K(\eta, f_F) \equiv f_F^{-1*} \circ \eta$ are equivalent. As a representative of this equivalence relation we may select η and make sure that any field transformation with respect to f_F does not alter the result. The moment we dissolve the equivalence class of smooth manifolds that constitute spacetime we have to additionally ensure the invariance of spacetime coordinates while retaining the field space representative η . This exactly reduces to an invariance under the stabilizer subgroup of η w.r.t. pullbacks f_M^* of $f_M \in \text{Diff}(M)$. For the Minkowski metric this is the famous Poincaré group.

1.4 Operators

A profound understanding of field space is provided by studying the effect of operators on it. This reflects the way we investigated groups by their action on vector spaces, however in a kind of inverted sense. Representation theory is actually closely tied to operator theory and in fact a very prominent set of operators are symmetry transformations of fields that link group theory and functional analysis. It is therefore not astonishing that we will meet several concepts of representation theory in this broader concept again. From a physical perspective, transformations are the natural way to implement physical constraints and interpretations into the mathematical description. The reason is that rather than obtaining the full content of a field at once, our observations project out only part of the information, usually those visible in their effects and interactions with the environment. Most of the material presented here is a brief summary of ref. [3, 26, 28, 42, 43].

For our purpose, an operator maps a field to a new element of field space and is thus an endomorphism in the respective category. The more general definition allows for several extension and can be described as follows:

Definition 1.4.1 — Operators. Let X and Y be sets possibly equipped with further structure.

A function $T : X \rightarrow Y$ is an operator on X to Y : $\Leftrightarrow$

T is a morphism (structure preserving map) in the respective category of X and Y and $\text{Dom}(T) \equiv X$ is the domain of T .

Usually we will focus on special operators that are endomorphisms of field space, hence morphisms where target and domain space coincide. Depending on which structural part of field space the operator acts on, it has to preserve the vector space, smoothness, or/and topological properties and will be thus denoted as a linear, differentiable, or continuous operator.

1.4.1 The spectral theorem

Fortunately, we usually have linear- and smooth-properties at least locally at our disposal and are thus mainly focused on linear differentiable operators in what follows. At this point the virtue of the background field method becomes apparent, for it diffeomorphically relates a non-vector space with a local patch of a linear space. Operators then act on the tangent space rather than on field space itself, which in the case of flat field spaces turns out to be a trivial identification. However, in the general case one has to pay special attention to the range of validity and to regions of overlap. In the remainder of this section we will consider a (local) Hilbert space as part of field space, that adds an inner product to the list of features we have at our disposal:

Definition 1.4.2 — Hilbert space. Let $V \equiv (X, \circ_V, \mathbb{F} \subseteq \mathbb{C}, \circ_{\mathbb{F}V})$ be a vector space and

$(\cdot, \cdot)_V : X \times X \rightarrow \mathbb{F}$ a map. The ordered pair $\mathcal{H}_V \equiv (V, (\cdot, \cdot)_V)$ is a Hilbert space $:\Leftrightarrow$

$$\begin{aligned} \forall x, y \in X : \quad (x, y)_V &= (y, x)_V^* && \text{(conjugate symmetry)} \\ \forall x \in X : \quad (x, x)_V &\geq 0 \quad \wedge \quad (x, x) = 0 \implies x = e_V && \text{(positive-definite)} \\ \forall x, y, z \in X, \forall \alpha \in \mathbb{F} : \\ (\alpha \circ_{\mathbb{F}V} x) \circ_V z, y)_V &= \alpha \cdot (x, y)_V + (z, y)_V && \text{(linear in first argument)} \\ \forall x, y \in X : \quad d_V : (x, y) &\mapsto \sqrt{(x \circ_V (-y), x \circ_V (-y))_V} && (V, d_V) \text{ metric complete} \end{aligned}$$

Thus, a Hilbert space is a complete inner product space.

Notice that the inner product $(\cdot, \cdot)_V$ is a sesquilinear semi-positive definite form and in the real case bilinear.

Once the target and domain space of an operator exhibits a Hilbert space structure, or more general are densely embedded in such a rich structured space, we have many possibilities to characterize operators using the respective scalar products. In fact the inner product on a Hilbert space already classifies the full set of dual vectors, i.e. linear operators from the Hilbert space to the underlying field $\mathfrak{K}$. This deep connection is provided by representation theorems as the Riesz or the Lax-Milgram representation theorem, stating that

$$\begin{aligned} \forall v \in V : \quad T_v : \text{Dom}(T_v) \subseteq V &\rightarrow \mathbb{C} \quad \text{with} \quad T_v := (\bullet, v)_V \text{ is linear operator} \quad \wedge \\ \forall T : \text{Dom}(T) \subseteq V &\rightarrow \mathbb{C} \text{ linear with } \overline{\text{Dom}(T)} \equiv V, \exists v_T \in V : \\ (w, v_T)_V &\equiv T(w) \quad \forall w \in \text{Dom}(T) \end{aligned}$$

In order that v_T is unique we have to require that the domain of T is dense in the Hilbert space.

In what follows we will consider two Hilbert spaces $\mathcal{H}_V$ and $\mathcal{H}_W$ over the associated vector spaces V and W , respectively. The corresponding scalar products provide a way to measure the spreading effect of an operator in the target space. Therefore, we define the operator norm of $T : \text{Dom}(T) \subseteq V \rightarrow W$ via the scalar products of domain and target space:

$$\|T\|_{\text{Op}} := \inf\{c \geq 0 \mid (T(v), T(v))_W \leq c \cdot (v, v)_V \quad \forall v \in V\}$$

If $\|T\|_{\text{Op}} < \infty$ we say that T is bounded which in fact is equivalent to T being continuous with respect to the topologies defined by both scalar products. Bounded operators form a very important class of transformations, for they avoid infinities by restricting its image to a ‘finite size’ region. For instance, in the case of an isometric operator $(T(v), T(v))_W = (v, v)_V$ we obtain $\|T\|_{\text{Op}} = 1$.

Unfortunately, there are many important examples of unbounded operators that play an important role in mathematics and physics. Therefore, consider the subspace of $\text{Dom}(T)$ where the operator $T : \text{Dom}(T) \subseteq V \rightarrow W$ is bounded and denote it

$$\text{Dom}_{\text{bd}}(T) \equiv \{v \in \text{Dom}(T) \mid (T(v), T(v))_W < \infty\}$$

On this subspace we can uniquely identify an adjoint operator $T^* : \text{Dom}(T^*) \subseteq V \rightarrow V$ that is naturally related to T for it represents T in the target space. The adjoint operator is implicitly defined for the bounded domain, i.e. for all $v \in \text{Dom}_{\text{bd}}(T)$, by

$$(T^*(w), v)_V := (w, T(v))_W \quad \forall w \in \{w \in W \mid \exists v \in \text{Dom}_{\text{bd}}(T) : T(v) = w\}$$

Surely, our main concern lies on operators that are endomorphisms, meaning for which target and domain spaces are subsets of the same Hilbert space $\mathcal{H}_V$. This allows a comparison of $T : \text{Dom}(T) \subseteq V \rightarrow V$ and its adjoint T^* on a direct level, in particular if both coincide. The following presents a list of definitions depending on the relation between an endomorphism and its adjoint:

Definition 1.4.3 — Operator classification. Let $\mathcal{H}_V$ be a Hilbert space. A linear operator $T : \text{Dom}(T) \subseteq V \rightarrow V$ is ...

... symmetric (hermitian)	$:\Leftrightarrow \forall v \in \text{Dom}_{\text{bd}}(T) : T^*(v) = T(v)$
... essentially self-adjoint	$:\Leftrightarrow \forall v \in \text{Dom}_{\text{bd}}(T^*) : T^{**}(v) = T^*(v)$
... self-adjoint	$:\Leftrightarrow \forall v \in \text{Dom}_{\text{bd}}(T) \equiv V : T^*(v) = T(v)$
... normal	$:\Leftrightarrow \forall v \in \text{Dom}_{\text{bd}}(T) : T^* \circ T(v) = T \circ T^*(v)$
... unitary	$:\Leftrightarrow \forall v \in \text{Dom}_{\text{bd}}(T) : T^* \circ T(v) = T \circ T^*(v) = v$
... orthogonal projection	$:\Leftrightarrow \forall v \in \text{Dom}_{\text{bd}}(T) : T^*(v) = T(v) = T \circ T(v)$

Symmetric or the even stronger restricted class of self-adjoint operators, the latter being always bounded by the Hellinger-Toeplitz theorem, play an important role as observables in physics. The reason is the existence of a powerful theorem that provides criteria for operators to be mutually diagonalizable with real eigenvalues. In its precise form one version of the spectral theorem states:²⁸

Theorem 1.4.1 — Spectral theorem. Let $\mathcal{H}_V$ be a Hilbert space and let

$$\mathcal{N}_{\text{bd}} := \{T : V \rightarrow V \mid \forall v \in V : T^* \circ T(v) = T \circ T^*(v) \text{ with } \text{Dom}_{\text{bd}}(T) \equiv V\}$$

be the set of bounded normal operators on $\mathcal{H}_V$.

Then, for any family of commuting normal operators $S \equiv \{T_j\} \subseteq \mathcal{N}_{\text{bd}}$, i.e. for all $T_j \in S$ and $T_k \in S$ with $T_j \circ T_k \equiv T_k \circ T_j$, the following holds true:

$$\begin{aligned} \exists (Y, \Sigma_Y, \mu : \Sigma_Y \rightarrow \mathbb{R}) \quad \wedge \quad U : L^2(Y, \Sigma_Y, \mu; \mathbb{C}) \rightarrow V : \\ \forall T_j \in S \exists f_j \in L^\infty(Y, \Sigma_Y, \mu; \mathbb{C}) \quad U^* \circ T_j \circ U \equiv f_j \end{aligned}$$

If T_j is self-adjoint then f_j is real valued.

In other words, the family S of operators is simultaneously diagonalizable. This theorem has several extensions, generalizations and alterations, for example in case of self-adjoint compact operators there exist eigenvalues and eigenfunctions that span the Hilbert space.

Since being self-adjoint is central in many aspects of operator theory and in the study of physical observables, it is important to understand when a symmetric operator can be extended to a self-adjoint one. The relevant information is encoded in the defect space along with its defect indices, consisting of the eigenspaces of some complex eigenvalue:

$$\text{defect}_T(z) := \{v \in V \mid T^*(v) = z \cdot v\} \quad (\dim(\text{defect}_T(z)), \dim(\text{defect}_{T^*}(z)))$$

For a self-adjoint operator purely imaginary eigenvalues are prohibited and thus $\dim(\text{defect}_T(i)) = \dim(\text{defect}_T(-i)) = 0$.²⁹ This is also reflected in the resolvent set $\rho_{\text{Op}}(T)$ and the spectrum $\sigma_{\text{Op}}(T)$ of T defined by

$$\begin{aligned} \rho_{\text{Op}}(T) &:= \{z \in \mathbb{C} \mid (T - z \cdot \text{id}_V) : V \rightarrow V \text{ is bijective}\}, \\ \sigma_{\text{Op}}(T) &:= \mathbb{C} / \rho_{\text{Op}}(T) \equiv \{z \in \mathbb{C} \mid \ker(T - z \cdot \text{id}_V) \neq \{e_v\} \vee \text{Im}(T - z \cdot \text{id}_V) \neq V\} \end{aligned}$$

The statement that T is (essentially) self-adjoint is equivalent to $\ker(T - (\pm i) \cdot \text{id}_V) = \{e_v\}$ and $\text{Im}(T - z \cdot \text{id}_V)$ is (densely) equivalent to V .

²⁸For the details of measure theory we refer to section 2.1.

²⁹It suffices to consider either $+i$ or $-i$, for $\dim(\text{defect}_T(z))$ is constant on the upper and lower complex half plane [28].

Now, let us go back to a general operator T_0 and systematically approach the question when T_0 gives rise to a self-adjoint extension, meaning that it is essentially a restriction of a self-adjoint operator T , in the sense that

$$\text{Graph}(T_0) := \{(v, T_0(v)) \mid v \in \text{Dom}(T_0)\} \subseteq \{(v, T(v)) \mid v \in \text{Dom}(T)\} \equiv \text{Graph}(T)$$

The necessary and sufficient conditions can be expressed on the basis of the defect indices of T_0 . If $\dim \text{defect}_T(\iota)$ and $\dim \text{defect}_T(-\iota)$ are strictly positive or agree, i.e. $\dim \text{defect}_T(\iota) = \dim \text{defect}_T(-\iota)$, then T_0 can be extended to a symmetric or self-adjoint operator T , respectively.

The enlargement of the domain space in order to promote a symmetric operator to a self-adjoint one, is tightly connected with a suitable choice of boundary conditions. In particular for the differential operators we encounter later on, linear maps $L : \Gamma(E_1) \rightarrow \Gamma(E_2)$, we have to impose suitable boundary conditions to apply the machinery of the spectral theorem.

For the remainder of this discussion on operator theory, let us consider a specific class of differential operators, the elliptic ones, for they are of uttermost importance when studying and understanding quantum gravity. In a sense, they describe some well behaved differential operator that among all the infinities they are surrounded by, they may have a finite dimensional kernel. A differential operator can be judged by its principal symbol, defined for some $f \in C^\infty(M, R)$ with $df(p) \in T_p^*M$ and $p \in M$ by

$$\sigma(L, df)s(p) := \frac{1}{N!} L(f^N \cdot s)|_p$$

whereby N represents the order of L , the highest power of differentiation in local coordinates. If $\sigma(L, df)$ is a linear isomorphism for any $p \in M$ and any $df(p) \in T_p^*M / \{e_{T_p^*M}\}$ we say that L is weakly elliptic. In case of $E_2 = E_1$ an operator is additionally strongly elliptic whenever it is positive, thus fulfills the condition

$$\exists c > 0 : \quad (\sigma(L, df)(v), v)_{\Gamma(E_1)} \geq c \cdot (v, v)_{\Gamma(E_1)} \quad \forall v \in \Gamma(E_1), \quad \|df\|_{T^*M} = 1$$

Such a differential operator L is a kind of generalized Laplacian operator that shares many features with this famous map. Strong ellipticity is in fact restricted to operators of even order, though a weakly elliptic odd operator, as the Dirac one, may be composed with itself to fulfill the stronger condition. While there are many applications of weakly elliptic operators – in particular the Atiyah-Singer index theorem – the full range of implications, as for example the discreteness of all eigenvalues, is only guaranteed for the strong elliptic ones. What is even more important for practical applications, is the existence of the elliptic regularity theorem. It assures that solutions v of any differential equation of an elliptic operator L , say $L(v) = f$ are at least as smooth as the contained coefficients and the resulting function f . For symmetric operators, strong ellipticity further guarantees that there is only a finite number of negative and zero eigenfunctions, which is of particular importance for the existence of the heat kernel trace evaluation.

The spectral theorem introduces some severe constraints on the set of possible eigenvalues of a linear operator. In case of (weakly) elliptic operators we can finally give a classification of its zero-eigenspace that turns out to be entirely described by topological properties of the base space. Therefore, consider a Fredholm operator, meaning an elliptic linear map that has finite kernel and co-kernel, for which the so called analytical index is well-defined:

$$\begin{aligned} \text{index}(L) &:= \dim \ker L - \dim \text{coker} L && \text{with} \quad \ker L \equiv \{s \in \Gamma(E_1) \mid L(s) = e_{\Gamma(E_2)}\}, \\ &&& \wedge \quad \text{coker} L \equiv \Gamma(E_2) / \text{Im}(L) \end{aligned}$$

Whenever M is compact, this analytical property is tied to a topological index that classifies the base space. This famous relation, that has many descendants as for example the Gauß-Bonnet

theorem, goes under the name of the Atiyah-Singer index theorem [44]. As it stands it provides an astonishing bridge between topology and analysis with many fruitful results in mathematics and physics, in particular the study of instanton solutions.

1.4.2 Trace of operators

For the definition of a trace, we restrict to operators that act as endomorphisms on a single, linear space V . In this case there is an equivalence between the set of finite rank endomorphisms $\text{End}_{<\infty}(V)$ and the tensor product space of V and its continuous dual space V^* , i.e. $V \otimes V^* \simeq \text{End}_{<\infty}(V)$ [45]. Explicitly, this correspondence is induced by a map $r : \text{End}_{<\infty}(V) \rightarrow V \otimes V^*$ with

$$E \mapsto r(E) = v_E \otimes \beta_E \quad \text{with} \quad r(E)(u) = (\beta_E(u)) \cdot v_E := E(u) \quad \forall u \in V$$

Thereby, v_E and β_E are implicitly defined by their action on functions and vector fields, respectively. Furthermore, the linear map on the direct product space $t : V \times V^* \rightarrow \mathbb{R}$ given by $(v, \beta) \mapsto t(v, \beta) = \beta(v)$ can be naturally extended to the associated tensor product space $V \otimes V^*$ by means of the universal property of the tensor algebra:³⁰

$$T : V \otimes V^* \rightarrow \mathbb{R}, \quad \text{with} \quad v \otimes \beta \mapsto T(v \otimes \beta) = t(v, \beta) = \beta(v)$$

Finally, the composition of T and r yields a linear functional on the space of endomorphisms, the trace:

$$\text{Tr} : \text{End}(V) \rightarrow \mathbb{R}, \quad \text{with} \quad E \mapsto \text{Tr}[E] = T \circ r(E) = T(r(E)) \equiv \beta_E(v_E) \in \mathbb{R} \quad (1.10)$$

The last equality holds for $r(E) = v_E \otimes \beta_E$ and provides a straightforward transition to a coordinate formalization. However, as it is apparent from the above coordinate-free construction the concept of the trace of a finite-rank operator is basis independent. In general the basis independence can be guaranteed for all trace class operators for which additionally the trace converges absolutely. The trace class of bounded operators $\{T\}$ defined over a separable Hilbert space consists of those linear maps for which $\text{Tr}[(T^* \circ T)^{1/2}]$ is finite.

The cyclic property of Tr follows immediately from the previous considerations, for let E_1, E_2 , and E_4 be suitable endomorphisms with $r(E_n) = v_{E_n} \otimes \beta_{E_n}$. For any permutation of composition we obtain

$$\begin{aligned} r(E_n \circ E_m \circ E_\ell) &= (v_{E_n} \otimes \beta_{E_n}) \circ (v_{E_m} \otimes \beta_{E_m}) \circ (v_{E_\ell} \otimes \beta_{E_\ell}) \\ &= \beta_{E_n}(v_{E_m}) \cdot \beta_{E_m}(v_{E_\ell}) \cdot (v_{E_n} \otimes \beta_{E_\ell}) \quad n, m, \ell \in \{1, 2, 3\} \end{aligned}$$

The linearity of the tensor product transfers to the linearity of T which thus produces the cyclic property of the operator trace:

$$\begin{aligned} \text{Tr}[E_n \circ E_m \circ E_\ell] &= \beta_{E_n}(v_{E_m}) \cdot \beta_{E_m}(v_{E_\ell}) \cdot \text{Tr}[v_{E_n} \otimes \beta_{E_\ell}] = \beta_{E_n}(v_{E_m}) \cdot \beta_{E_m}(v_{E_\ell}) \cdot \beta_{E_\ell}(v_{E_n}) \\ &= \beta_{E_\ell}(v_{E_n}) \cdot \beta_{E_n}(v_{E_m}) \cdot \text{Tr}[v_{E_\ell} \otimes \beta_{E_m}] = \text{Tr}[E_\ell \circ E_n \circ E_m] \end{aligned}$$

In case of operators acting on a supermodule over some superalgebra the usual trace Tr has to be replaced by the super-trace STr for which the commutation rules produce some additional factors of (-1) . This becomes relevant when field space contains fermionic or ghost fields.

³⁰A universal property is a simplified but still unique criterion for the construction of certain mathematical objects, while the actual explicit construction may be cumbersome. For instance, the tensor algebra of a vector space V is characterized by the statement that any linear function from the vector space V to some algebra A over the same field, can be uniquely extended to a homomorphism between the algebras of the tensor space and A .

1.4.3 Determinant of operators

In the Hilbert spaces we are most interested in, infinities plague the standard definition of matrices and certain regularization techniques have to be (implicitly) used to generalize these concepts for arbitrary operators. Determinants of operators are no exception to this requirement, for they consist of the product of eigenvalues which in the infinite case will in general diverge. There are several procedures to extend this defining property, among which the mathematical most rigorous one is based on the zeta function:

$$\det_{\zeta}(T) := \exp\left(-\frac{d}{ds}\zeta_T(s)\Big|_{s=0}\right), \quad \text{with} \quad \zeta_T(s) := \text{Tr}[T^{-s}] = \sum_{\lambda_j \in \sigma_{\text{Op}}(T)} \lambda_j^{-s}$$

Another version of functional determinant is given by the Fredholm definition for a special class of bounded invertible operators of the form $\text{id} + T$ where T is of trace class. Extending the Hilbert space to its exterior power the associated operator description gives rise to the so called Fredholm determinant:

$$\det_{\text{Fd}}(\text{id} + T) := \sum_{r=0}^{\infty} \text{Tr}_{(r)}[\underbrace{T(\bullet) \wedge \cdots \wedge T(\bullet)}_{r\text{-times}}]$$

In both cases we see the implicit dependence on the trace definition as expected from its finite dimensional analog. In fact for T being trace class and $z \in \mathbb{C}$, we have the following identities:

$$\det_{\text{Fd}}(\exp(T)) = \exp_{\mathbb{C}}(\text{Tr}[T]), \quad \log \det_{\text{Fd}}(\text{id} + z \cdot T) = \text{Tr}[\log(\text{id} + z \cdot T)]$$

Finally, let us state a representation of the functional determinant that is more convenient for practical purposes in quantum field theory. It relates to the properties of the Gaußian measure that we discuss in detail in section 2.1. So assume we have a free theory given by a Gaußian measure $\mu_G^{(S)}$ over some vector space V based on some positive self-adjoint operator S (as can be read of from translation transformation). A related (non-normalized) Gaußian measure results from the Radon-Nikodym theorem with respect to the function $g(\Phi) = \exp(-\frac{1}{2}(\Phi, T(\Phi))_{\text{T}_{\Phi}[F]})$ for some positive self-adjoint operator T . The functional determinant is then given by

$$\frac{\det(S+T)}{\det(S)} \mu_G^{(S)}(U) \equiv \int_{\varphi \in U \subseteq \text{T}_{\Phi}[F]} \mu_G^{(S)} \exp(-\frac{1}{2}(\varphi, T(\varphi))_{\text{T}_{\Phi}[F]})$$

Notice that only the relative size of determinants is determined in this way and for $U \equiv \text{T}_{\Phi}[F]$ $\mu_G^{(S)} = 1$. Furthermore, if we choose a $S = \text{id}$ and T to be trace class, then this path integral representation relates to the Fredholm determinant.

1.5 Functionals

Fields unfold their full beauty in combination with a physical theory that introduces interactions. Usually the concrete information about the theory is encoded in an action functional that transfers the field content into a measurable quantity, for instance a real (or complex) number.

Definition 1.5.1 — Functionals. Let $V \equiv (X_V, \circ_V, \mathbb{K}, \circ_{KV})$ be a vector space over the field $\mathbb{K}$. A map A is a functional on $V : \Leftrightarrow A : X_V \rightarrow \mathbb{K}$.

This general definition gives rise to a huge class of different functionals, among which we have to select the physical ones by some suitable constraints.³¹ For example, action functionals

³¹Notice that dual vectors, i.e. elements of V^* , are special functionals in that they additionally have to be linear, which is not assumed to hold for a generic functional A .

are usually assumed to have an integral representation in terms of a Lagrangian density over spacetime. However, since we will work in an effective picture later on, we do not have to impose any restrictions in this direction yet. Rather, we will use the virtue of group theory again to reduce the number of candidates by symmetry constraints.

Nevertheless, let us consider some basic applications of generic functionals for a moment, before let group theory infiltrate. The entire mathematical framework is supposed to be establish on field space $[F]$ or the total space F . In many regards it turns out to be convenient to convert $[F]$ into a (local) vector space by virtue of the background field method based on some background field $\mathcal{U}_{\text{BFC}} \equiv (\bar{\Phi}, F)$ or $[\mathcal{U}_{\text{BFC}}] \equiv ([\bar{\Phi}], [F])$ and the respective exponential map. Thus, in terms of fluctuation and background fields a general functional is given by:³²

$$\begin{aligned} A : F \rightarrow \mathbb{K}, \quad \hat{\Phi} \mapsto A[\hat{\Phi}] &\equiv A[\exp_{\bar{\Phi}}^{\chi}(\hat{\Phi}(\hat{\Phi}, \bar{\Phi}))] \equiv A[\hat{\Phi}(\hat{\Phi}, \bar{\Phi}); \bar{\Phi}] \quad \text{with} \\ A : T_{\bar{\Phi}}F \times F &\rightarrow \mathbb{K}, \quad A[\bullet; \bar{\Phi}] : T_{\bar{\Phi}}F \rightarrow \mathbb{K}, \quad \hat{\Phi} \mapsto A[\hat{\Phi}; \bar{\Phi}] \end{aligned}$$

Notice that $\hat{\Phi}$ is the tangent vector that generates a geodesic connecting $\hat{\Phi}$ and $\bar{\Phi}$ with respect to the field space connection χ . In the context of field theory, the set of functionals respecting a certain symmetry invariance is denoted as theory space:

$$\mathcal{T}_F \equiv \{A : F \rightarrow \mathbb{K} \mid A[\rho_F^R(\hat{\Phi}, f_{\mathcal{G}})] = A[\hat{\Phi}] \quad \forall f_{\mathcal{G}} \in \mathcal{G}_F \quad \forall \hat{\Phi} \in F\}$$

As we will see, besides a differential structure, there is a subspace of $\mathcal{T}$ that can be described by a vector space containing all local monomials based on the field content.

The benefit of working with functionals is due to the properties they may inherit from their well equipped domain and target spaces. In particular, the space of linear functionals turns out to be a vector space under pointwise evaluation. The smooth inner product space in which the fluctuation fields live provides several important operations concerning linearity, topology, convergence, and differentiability, while the complex (or real) numbers have an even richer structure. One aspect that needs many of those properties concerns the dependence of a functional result on its field argument, the functional derivative.

1.5.1 Functional derivatives

The functional derivative addresses one of the most interesting question about functionals except perhaps their symmetry properties. It gives a way to measure the sensitivity of a functional A on changes of the fluctuation fields. While a functional that is constant on the entire field space is of subordinate relevance, symmetry constraints may require a vanishing functional derivative of A along a certain group orbit. This establishes the concept of functional derivatives in the following sense:

Definition 1.5.2 — Functional derivative. Let F be field space principal bundle with χ a principal connection. Then $\mathfrak{d}_v A[\hat{\Phi}]$ is the (covariant) functional derivative of $A : F \rightarrow \mathbb{C} : \Leftrightarrow$

$$\mathfrak{d}_v A[\hat{\Phi}] \equiv \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{A[\exp_{\bar{\Phi}}^{\chi}(\hat{\Phi} + v)] - A[\exp_{\bar{\Phi}}^{\chi}(\hat{\Phi})]\} \quad \text{with} \quad \hat{\Phi} \equiv \hat{\Phi}(\hat{\Phi}, \bar{\Phi}) \in T_{\bar{\Phi}}F$$

The functional derivative is a covariant derivative on the space of functionals. In the background field language it linearly translates the fluctuation field such that

$$\mathfrak{d}_v A[\hat{\Phi}; \bar{\Phi}] \equiv \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{A[\hat{\Phi} + v; \bar{\Phi}] - A[\hat{\Phi}; \bar{\Phi}]\}$$

³²We use the convention that arguments of action functionals are written in square brackets. For convenience we use the same symbol for maps on field space and the maps for the background field method, distinguishing both by a semicolon $A[\hat{\Phi}] \equiv A[\hat{\Phi}(\hat{\Phi}, \bar{\Phi}); \bar{\Phi}]$.

Since field space constitutes a Riemannian manifold, functional derivations reflect the usual differentiation of functions on differentiable structures. In particular, the chain rule applies and for some functional $F : T_{\bar{\Phi}}^* F \rightarrow \mathbb{C}$ reads:

$$\mathfrak{d}_v F[J(\hat{\Phi})] = \mathfrak{d}_{\mathfrak{d}_v J(\hat{\Phi})} F[J] \quad \text{with} \quad v \in T_{\bar{\Phi}} F$$

Notice that the interrelation of fields is provided by the transformation properties of operators. A functional derivative on the other hand is an operation that maps a functional to another functional. A very important concept is the second functional derivative that gives rise to the Hessian and possibly to the Hessian operator of a functional:

Definition 1.5.3 — Hessian, and Hessian operator. Let F be the principal bundle associated to field space with χ a principal connection, $\mathfrak{d}$ the variation operator, and $\mathfrak{G}$ the field space metric. Then, $\text{Hess}_{\hat{\Phi}}[A[\hat{\Phi}; \bar{\Phi}]]$ is the Hessian of a functional $A : F \rightarrow \mathbb{C} : \Leftrightarrow$

$$\text{Hess}_{\hat{\Phi}}[A[\hat{\Phi}; \bar{\Phi}]](v, w) \equiv \frac{1}{2} (\mathfrak{d}_w (\mathfrak{d}_v A[\hat{\Phi}; \bar{\Phi}]) + \mathfrak{d}_v (\mathfrak{d}_w A[\hat{\Phi}; \bar{\Phi}])) \quad \forall v, w \in T_{\bar{\Phi}} F$$

If it exists, then $\widehat{\text{Hess}}_{\hat{\Phi}}[A]$ is the Hessian operator associated to $A : \Leftrightarrow$

$$\text{Hess}_{\hat{\Phi}}[A[\hat{\Phi}; \bar{\Phi}]](v, w) \equiv \mathfrak{G}_{\bar{\Phi}}(v, \widehat{\text{Hess}}_{\hat{\Phi}}[A[\hat{\Phi}; \bar{\Phi}]] w) \quad \forall v, w \in T_{\bar{\Phi}} F$$

The self-adjoint property of the Hessian operator is some very important requirement in QFT for instance. Especially in the presence of spacetime boundaries additional assumptions on field space are needed in order to extract a suitable Hessian operator for the action functional.

From the definition of the Hessian operator convexity properties can be read off, stating that when the operator is (positive) non-negative an associated real functional is (strictly) convex, i.e. it fulfills:

$$A[t \cdot \hat{\Phi}_1 +_{\text{TF}} (1-t) \cdot \hat{\Phi}_2; \bar{\Phi}] \leq t \cdot A[\hat{\Phi}_1; \bar{\Phi}] + (1-t) \cdot A[\hat{\Phi}_2; \bar{\Phi}]$$

with $<$ replacing $\leq$ in the case of strong convexity. The main advantage of convex functions is the uniqueness of a (possible) extremum and the vast material that is available for its differential calculus. In particular the information preserving Legendre transform is a central tool in QFT that interrelates the generating functional with an effective action.

1.5.2 Basis invariants

In the usual calculus of functions it is very convenient to introduce a basis in which each and every function can be expanded. The corresponding algebra is conventionally established in an infinite-dimensional Hilbert space with some self-adjoint operator, a typical example being the Fourier basis. Similarly, we can ask whether or not there is an analog basis on the space of functionals, in particular on theory space $\mathcal{T}_F$. For functionals based on some non-local operator, this is quite difficult to answer, however for the remaining subspace, which we consider in the sequel, there is a stepwise construction procedure based on Young diagrams at least for Levi-Civita connections $\bar{D}$ for some metric $\bar{g}$ [46]. We have already seen that the irreducible components of tensors are obtained using permutations and partitions. For the basis invariants that constitute the integral kernels of $A \in \mathcal{T}_F$ the crucial observation is that

$$\underbrace{\bar{D}_{\mu_5} \cdots \bar{D}_{\mu_{p+4}}}_{p\text{-times}} \bar{R}_{\mu_1 \mu_2 \mu_3 \mu_4} \quad \Longleftrightarrow \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & \cdots & \cdots & (p+4) & \\ \hline & & & & & & & \\ \hline \end{array}$$

Here $\bar{R}$ is the Riemann tensor with all indices lowered by $\bar{g}$. Each tensor that is a combination of such derivatives of $\bar{R}$ can be uniquely decomposed into irreducible representations using Young

diagrams. The association of partitions and tensor structures is not as straightforward, but at least the implications of the group theoretical considerations of the symmetric group about dimensionality and the number of derivatives is a great guiding principle.

Since we are only interested in the reduction of tensors to invariant scalars of the Riemannian metric $\bar{g}$, for each partition a unique tensor structure survives the decomposition procedure and in fact it is classified by all rows being of even length. In addition, as mentioned for the general linear group, diagrams with more than d rows vanish identically. Thus, the number of different basis invariance is sensitive to the spacetime dimension, which has strong implications for theories of gravity especially in $d = 2$.

Keeping these simple rules in mind one can deduce the possible invariants of particular order in the derivatives which actually give rise to a basis for the local part of $\mathcal{T}_F$. For more details we refer to [46].

R As an example, let us consider the decomposition of $\bar{R}_{\mu_1\mu_2\mu_3\mu_4} \equiv \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$ combined with $\bar{D}_{v_5}\bar{D}_{v_6}\bar{R}_{v_1v_2v_3v_4} \equiv \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$:

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$$

whereby all contributions with at least one odd row length have been neglected. Depending on the dimensionality of M we obtain no, one, three, or four independent basis monomials contained in $R(\bar{g})\bar{D}\bar{D}R(\bar{g})$, namely:

$\begin{array}{ c c c c } \hline \square & \square & \square & \square \\ \hline \end{array}$	$\bar{R}\bar{D}^2\bar{R}$	for $d \leq 2$
$\begin{array}{ c c c c } \hline \square & \square & \square & \square \\ \hline \end{array}$	$\bar{R}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu\bar{R}$	for $d \leq 3$
$\begin{array}{ c c c } \hline \square & \square & \square \\ \hline \end{array}$	$\bar{R}^{\mu\nu}\bar{D}^2\bar{R}_{\mu\nu}$	for $d \leq 3$
$\begin{array}{ c c c c } \hline \square & \square & \square & \square \\ \hline \end{array}$	$\bar{R}^{\mu_1\mu_2\mu_3\mu_4}\bar{D}_{\mu_3}\bar{D}_{\mu_4}\bar{R}_{\mu_1\mu_2}$	for $d \leq 4$

Especially for the heat kernel expansion this technique provides a very instructive way to construct a suitable basis for $\mathcal{T}_F$ in case of quantum gravity.



2.1	Measure theory
	Measurable spaces
	Normalization
	Observables
	Relation between measures
	Measures for gauge theories
	Interpretation
2.2	Effective action
	Schwinger functional
	Effective action
	Free theory
2.3	Quantum field theories
	Quantum field theories
	Theories, phases, and anomalies
	Perturbation theory
	Euclidean quantum field theory

2. QUANTUM FIELD THEORY

In chapter 1 we considered the ingredients our theoretical description of Nature is based on. While mathematics provide an infinity of different possible structures we have focused on those which seem to be favored from a physical perspective. Assuming spacetime to be continuous and equipped with a differential construction reflects our intuitive (local) picture of what reality seems to be. By this we have fixed the notion of locality in a very particular way. Next, we introduced fields as the mediators of interactions and we were led in the realm of differential geometry and fiber bundles. Again from a mathematical point of view there were plenty of opportunities however the most conservative one stuck to differential objects based on Lie groups. The reason for this choice may reflect the beauty of the Universe to possess a set of symmetries that may be tied up with our notion of interactions. We always followed the lines of what seemed most natural and stay close to the path of our intuitive understanding. Generations of physicists have enlarged those basic concepts to take care of predictions and experiments that could not be described within this simple ideas.

Still, we have to decide about another fundamental assumption that builds up our description of Nature: the theoretical framework. The question that we have yet to answer concerns the way we combine the ingredients to form a consistent theoretical setting that covers the observations and predicts the unobserved phenomena.

So, let us begin this chapter on theories and rules with some philosophical considerations. This brings us back to the basic question physics is concerned with, namely ‘what is Nature?’. Centuries of great thinkers inquired into the origin and cause of the Universe, whereby logic has played a very important role. Over the time mathematics has emerged as the language of science, for its ‘objective’ and changeless character. Whether or not Nature speaks in this language is a mystery, for us it will be only one more assumption we have to live with. But even after the restriction to logic a vast regime remains, thus further assumptions are needed. With a theoretical construction we somehow fix the grammar and the kind of words Nature is allowed to use and then verify if the sentences give rise to different facets that can be observed. Hence, once we have mathematical consistency, everything reduces to observations and the value of a theoretical setup is measured in its reproduction of known and its prediction of unknown effects.

The guiding principle to obtain suitable frameworks is surely our perception, however the history of science has revealed that this might be misleading. Nowadays, consistency, simplicity, and aesthetics of the mathematical description feature prominently in staying on the track, and here again symmetries stand out.

Observations symbolizes the lighthouses in the construction of theories and lead to a basic concept of quantum field theory: (re-)normalization. Here, the vicious cycle is that in order to draw conclusions from observations we have to frame them into a particular theory. Thus, both concepts are strongly interwoven and only inconsistencies may help to unravel this knot. The question what observations can tell us about reality and its rules is related to the question of determinism and probability. While a deterministic philosophy assumes a single observation contains all information of its underlying principle, the probabilistic image of the world takes the position that only a (possible infinite) number of identical observations may reveal Nature's secrets. For long it had been the conception that determinism is at the heart of all rules, but quantum physics severally shook this confidence. It is still an open issue whether the quantum world is the door to some new deterministic theory or if probabilities take part in Nature's game. The lessons we learn from this is rather of practical than of philosophical character and involves the mathematical theory of probability. This assumption still covers both, deterministic and stochastic concepts, and will naturally give rise to a statistical description closely related to the path integral formalism of quantum mechanics. Only at the very end when choosing a certain configuration and interpret the results one has to take sides of determinism or indeterminism.

To establish such a broadly applicable concept, we will start with a short summary of measure theory. Subsequently, we will put the notion of theory and observables on a more solid ground and have a look at systematic constructions of the former. The remaining sections of this chapter are then devoted to allocate physics to this mathematical framework and introduce some simplified methods to evaluate observable quantities. With this at hand, we are then prepared to study a powerful and universal tool in the next chapter, the renormalization group.¹

2.1 Measure theory

Probability theory has found its way into physics at the latest in shape of the Wiener process that describes Brownian motions relying on concepts of measure theory. Its greatest success and the foundation of this formalization at the root of quantum field theory were laid by Dirac and some years later by Feynman, see ref. [28]. Till now the generalization of the so called path integrals play a major role in describing the fundamental forces of Nature. The mathematical aspects of this section can be found for instance in [47] while the relation to physics is based on [28].

2.1.1 Measurable spaces

The basic common idea regardless of whether applied to condensed matter physics, quantum field theory, or societal statistical problems is to sum over all possibilities and get out the most likeliest candidate.

In the present case, we have imposed certain symmetry and further physically motivated restrictions to construct field space, the set of all admissible global sections on a certain fiber bundle. It is this space our operators act on and in which we have to search for suitable solutions to deduce real observables. Hence, we have to somehow weight portions of field space differently depending on their physical importance or likelihood. Measure theory establishes

¹The conceptual program of this chapter is mainly based on a lecture which Maximilian Demmel and myself delivered in the winter term 2014/2015 at the University of Mainz. It was a real inspiration and full of fruitful discussion for which I like to thank Maximilian and the audience, giving me this opportunity.

a framework that takes care of the mathematical side of this procedure while on the physical side we still have to choose among the different possible weighting methods by means of a consistent theory.

In order to assign a weight or size to each collection of fields we have to endow its subsets with a suitable algebra that can be related with the real numbers, in particular with its order principle. Therefore, we have to endow it with a commutative associative operation that is closed and invertible and that can be scaled in a certain way. The combination of both requirements give rise to the definition of σ -algebras:

Definition 2.1.1 — σ -algebras. Let X be a set, 2^X its power set. A collection of subsets $\Sigma \subseteq 2^X$ is a σ -algebra on X : $\Leftrightarrow$

$$\begin{aligned} X, \emptyset &\in \Sigma && (\implies \text{existence of inverse \& closure}) \\ \forall S \subseteq \Sigma, \#(S) \leq \#(\mathbb{N}) : & (\cup_{U \in S} U) \in \Sigma && (\text{closed under countably infinite unions}) \\ \forall A \in \Sigma : & X/A \in \Sigma && (\text{closed under complement}) \end{aligned}$$

Elements $A \in \Sigma$ are denoted as measurable sets and (X, Σ) is called a measurable space.

Cast in this form the required additive and scaling operations are disguised in this more convenient definition. The algebraic construct that reflects addition of real numbers is the symmetric difference, given by

$$\forall A, B \in \Sigma : \quad A \Delta B = (A/B) \cup (B/A)$$

with identity element $\emptyset$ and A being its own inverse. Thus (Σ, Δ) forms an Abelian group and is the counterpart of $(\mathbb{R}, +)$. On the other hand, $\cap$ can be regarded as the multiplication of subsets and $(\Sigma, \cap)$ forms a commutative monoid with identity X , but lacks an inverse. Furthermore, $X/\bullet$ interpreted as a scalar multiplication of $\{X\}$ on Σ completes the picture of an algebraic structure on 2^X that models $\mathbb{R}$ in a suitable way. We have seen that for a consistent closure of Σ we have to include X and $\emptyset$. Thus σ -algebras really satisfy the requirements of an algebra.²

In the construction of spacetime we have encountered a similar prescription to introduce locality into the physics: topology. Topologies and σ -algebras are constructed by endowing subsets with a specific structure and in fact they share some of their defining properties. First they distinguish X and $\emptyset$ as open and measurable sets, respectively, and secondly they are both closed under finite intersection (for σ -algebras that is a direct consequence of the remaining properties). Apart from that, both concepts lay focus on different aspects that fulfill their respective purposes. While topology allows for any, even uncountably infinite, unions of its elements, σ -algebras asks only for a countably infinite closure but additionally includes all complements into its algebraic structure by definition. So in general a topology might not be a σ -algebra, in fact every Hausdorff space fails to include its complements, and vice versa. Nevertheless, the fusion of openness and measurability for a certain collection of subsets is a very interesting and worthwhile problem that is of particular importance in the theory of Lebesgue integration. The general question is treated in the study of Borel σ -algebras that define measurable sets as generated from a topology, which is not in general possible [47].

The preparation of those subsets in the way of σ -algebras is only the first step in establishing a thorough treatment of probability theory. Next, we have to define the root of a theory that allocates those measurable sets a particular value in the real numbers, the measure

²The prefix σ indicates that we have more than a usual algebra of finitary operations, but in fact allow for a closure of countably infinite unions. This additional restriction takes care of part of the infinities that naturally arise in the theory of integration.

itself.³

Definition 2.1.2 — Measures. Let X be a set, $\Sigma \subseteq 2^X$ a σ -algebra. A function $\mu : \Sigma \rightarrow [0, +\infty]$ defines a measure on Σ $:\Leftrightarrow \mu(\emptyset) = 0$

$$\forall S \subseteq \Sigma, \#(S) \leq \#(N) \text{ with } A \cap B = \emptyset \forall A, B \in S :$$

$$\mu(\cup_{A \in S} A) = \sum_{A \in S} \mu(A) \quad (\text{Countable additivity})$$

The ordered pair (X, Σ, μ) is called a measure space. Given a measurable space (X, Σ) the set of well defined measures (with $\mu(X) = 1$) is denoted $\text{Meas}(X, \Sigma)$ ($\text{Meas}_P(X, \Sigma)$).

One can extend these non-negative measures to signed or complex ones by Hahn's decomposition theorem [47], whereby special care has to be taken when treating positive and negative infinities to preserve the ordering and multiplication principles. Most of the following results can be extended to signed and complex measures, but we will not mention them any further. Though important in today's understanding of physics, due to the seemingly Lorentzian signature of the gravitational field, in this work we focus on Euclidean theories that allow for a probabilistic interpretation. The clash between the observational data and this assumption can be dissolved whenever the Osterwalder-Schrader axioms apply, allowing for a transition from Euclidean to Lorentzian theories. We will come back to this point in chapter 3 where the full motivation for the Euclidean treatment is discussed.

The way we constructed measurable sets and the measure itself, allows to endow the class of all measures on (X, Σ) with the structure of a linear space, (scalar multiplication, additivity). For signed and complex measure we can even extend this linear space to a vector space since then the identity and inverse elements exist.

2.1.2 Normalization

Having established a collection of measurable subsets we encode their relative importance in a suitable measure. This function takes care of weighting the physically relevant subsets the most and the irrelevant ones least. Thereby one usually assigns to infinite sized subsets (infinite cardinality) a finite value in accordance with countably additivity of the measure. This inherited property of $\mathbb{R}$ yields a well-behaved regularization procedure and provides a consistent way to weigh uncountable infinite sets.

An even deeper regularization is obtained by a probability measure that normalizes the weight of the entire space to one, i.e. $\mu(X) = 1$. This choice of normalization is a particular suitable one, for it takes care of another source of infinities by invoking an additional external requirement, here the observation that all events (measurable sets) taken together cover all possibilities that might appear. With this constraint the number of possible measures reduces significantly, at least if Σ is of infinite cardinality.

In general, we have a series of mathematical conditions our theory, the measure, has to satisfy. Among them, three additional constraints on μ are of particular importance for the reflect the integration properties on Euclidean space:

Definition 2.1.3 — Measure properties. Let (X, Σ, μ) be a measure space. Let $G = (Y, \circ_G)$ be a group and $\rho : G \rightarrow \text{Aut}(X)$ be a free group action on X .

³For this general definition we have to extend the real numbers by positive infinity. The extension of the algebra is based on the following rules: $0 + \infty = \infty + 0 = \infty$ and $0 \cdot \infty = \infty \cdot 0 = 0$.

The measure μ is denoted ...

$$\begin{aligned}
 \dots \sigma\text{-finite} & \quad :\Leftrightarrow \quad \exists S \subseteq \Sigma, \#(S) \leq \#(\mathbb{N}) \wedge X = \bigcup_{A \in S} A : \quad \mu(A \in S) < \infty \\
 \dots \text{complete} & \quad :\Leftrightarrow \quad \forall S \subseteq N \in \Sigma \wedge \mu(A) = 0 \implies S \in \Sigma \\
 \dots \rho\text{-invariant} & \quad :\Leftrightarrow \quad \forall g \in Y, \forall A \in \Sigma : \\
 & \quad [\rho_g(A) := \{\rho_g(a) \mid a \in A\} \in \Sigma] \wedge [\mu(A) = \mu(\rho_g(A))]
 \end{aligned}$$

Probability measures are severely constraint by the simultaneous normalization and regularization condition $\mu(X) = 1$.⁴ However, in most cases we do not have a way to measure the ‘size’ of the full space X , hence to construct a scale relative to the overall size is not feasible. For several purposes it seems more natural to invoke the weaker form of σ -finiteness on a measure, that introduces a less strong regularization on μ , but leaves the normalization open. Though the entire space X may have a countable infinite ‘size’, it is composed of finite subsets that allow for a successive treatment. It is this condition that we naturally use when defining a (local) length scale.

Completeness of a measure refers to the concept of null sets, subsets that have no influence on the integration process, for the do not contribute at all. In physical terms, the contained fields are excluded in the evaluation and may be denoted as unphysical. Statements that hold true except for some null sets of μ , termed μ -almost everywhere, acquire full validity within measure theory. A thorough treatment of null sets is vital when playing around with measure spaces, for even though they are invisible on their own, they can have an impact when considering products of measures. The consistent way to take care of these cases is by completing the measure to include all subsets of its null sets, or using a complete measure from the outset. Another related concept is the support of a measure, that is the collection of all sets which have non-vanishing measure.

The third condition can be thought of as the most extensive one, for it implements symmetry constraints, a very powerful, multi-faceted, and aesthetic construction. The invariance under a certain group action severely restricts the class of possible measures on a given σ -algebra and it will play may be the most important role in forming a sufficiently reduced mathematical description of Nature. Still, even with very strong constraints from the mathematical side, in the end one has to build a bridge from theory to reality with the aid of a physical motivated normalization. This, then hopefully selects a unique theory from all remaining ones, and only for probability like measures the mathematical regularization may come along with a normalization.

R An important example is the Lebesgue measure on the Borel sets of real numbers, (the σ -algebra generated from the standard topology on $\mathbb{R}$). It is the natural candidate for a length (or more generally volume) measuring device, based on three principles: σ -finiteness, completeness, and invariance under translation ($\rho_{a \in \mathbb{R}}(b \in \mathbb{R}) = b + a$). These mathematical constraints leave only a one-parameter family of measures

$$\mu_L(A = (a, b); \lambda) = \lambda \cdot (a +_{\mathbb{R}^d} (-b))$$

The transition to reality is given by fixing the non-negative parameter λ when assigning a particular length scale to a standard object.

By introducing a measure we have defined a theory that has to be matched with all observable phenomena it is supposed to describe. One such observation may be sufficient to pin down the mathematical freedom by the corresponding normalization condition. It is at this step were

⁴Every finite measure, i.e. $0 \neq \mu(X) < \infty$, can be transformed into a probability measure by an additional normalization: $\mu_P(A) = \frac{\mu(A)}{\mu(X)}$.

theory and experiment meet in order to dissolve the remaining uncertainties of the mathematical formalism and distinguish whether or not we have followed the right path. Every observation that contains some notion of size, intrinsically depend on a normalization, and the appearance of a unit is an indicator for a non pure-mathematical result. We will later see how these normalization, i.e. the mapping of observations to theoretical parameters, can be used as a guiding principle in deducing fundamental theories.

2.1.3 Observables

Though fields are thought of as the fundamental building blocks that mediate interactions and form the matter content of the Universe, we usually only observe their effects rather than having direct access to their shapes. In addition, for observations in the realm of quantum field theory we choose a measure, and thereby a particular realization of a theory, that judges the influence of an entire subclass of fields instead of evaluating a single field configuration. Thus, having a measure spaces that yields a full description of the weighting of field space is crucial, but it is only half the story. In order to deduce predictions and establish a normalization condition we have to provide the missing link to observables. These are functions that map a specific field configuration to an observable value, which is then weighted by the importance of this field using the underlying measure. It is a kind of very general averaging of functionals on field space, covered in the concept of measurable functions:

Definition 2.1.4 — Measurable functions. Let (X, Σ_X, μ_X) and (Y, Σ_Y, μ_Y) be measure spaces. A function $f : X \rightarrow Y$ is a measurable function : $\Leftrightarrow$

$$\forall A_Y \in \Sigma_Y : f^{-1}(A_Y) \equiv \{\varphi \in X \mid f(\varphi) \in A_Y\} \in \Sigma_X$$

If (X, Σ_X, μ_X) stands for a theoretical description f is called an observable.

The definition of measurable functions, or observables, mirrors the way we defined continuous functions in the category of topologies. In fact, if both σ -algebras are generated by a topology, the concept of measurable and continuous functions coincide and we have an intuitive understanding of what observables may look like.

Most of the time the target space will be a real finite-dimensional Euclidean space equipped with a Lebesgue measure. This is due to our understanding of the world that is basically constructed on top of the real numbers and leads to a certain limitations in extracting observables by experimental devices. We therefore usually stick to the case of $f : X \rightarrow \mathbb{R}^d$.

Now, the evaluation of an observable on the basis of a certain theory $T_X \equiv (X, \Sigma, \mu)$ proceeds as follows. First, consider a very simple class of observables, the indicator functions, given by

$$\chi : \Sigma \times X \rightarrow \{0, 1\} \subset \mathbb{R}, \quad \chi_A : X \rightarrow \{0, 1\}, \quad \varphi \mapsto \chi_A(\varphi) = \begin{cases} 1 & \varphi \in A \in \Sigma \\ 0 & \varphi \notin A \in \Sigma \end{cases}$$

As its name suggests, χ_A indicates whether or not a field is contained in the respective measurable set $A \in \Sigma$. They can be used as a basis to build more general measurable functions by exploiting the rich structure of $\mathbb{R}$ again. Besides multiplication by a (complex) constant we may add, subtract, or take the product of indicator functions to obtain an extended class of new measurable functions, denoted simple:

Definition 2.1.5 — Simple functions. Let (X, Σ_X, μ_X) and $(\mathbb{C}, \Sigma_{\mathbb{C}} \equiv \text{gen}\{\tau_{\mathbb{C}}\}, \mu_{L;\mathbb{C}})$ be

(complex) measure spaces. A function $f : X \rightarrow \mathbb{C}$ is a simple function $:\Leftrightarrow$

$$\exists S \equiv \{A_j\} \subseteq \Sigma, \#(S) < \infty \wedge A_j \cap A_{k \neq j} = \emptyset, a_j \in \mathbb{C} : \quad f(\varphi) = \sum_{A_j \in S}^{+\mathbb{C}} \chi_{A_j}(\varphi) \circ_{\mathbb{C}} a_j$$

The set of all simple functions on Σ is denoted S_Σ , its restriction to non-negative ($0 \leq a_j \in \mathbb{R}$) ones S_Σ^+ . It gives rise to a commutative algebra over the complex numbers and can be extended to describe all measurable functions. Explicitly, given a measure space (X, Σ, μ) , any measurable function $f : X \rightarrow [0, \infty]$ can be expressed in terms of a sequence $(f_\ell)_{\ell \in \mathbb{N}}$ of simple functions on Σ :

$$\forall \varphi \in X : \quad f(\varphi) = \lim_{\ell \rightarrow \infty} f_\ell(\varphi)$$

This is a remarkable result that distinguish simple functions as fundamental objects from which we can deduce an even larger class of functions. In a last generalization matched for our need to derive intermediate results in field space, we consider any measurable function $f : X \rightarrow Y$ on associated σ -algebras Σ_X and Σ_Y , respectively. As long as the target space of f , i.e. Y , is a normed vector space over the complex or real numbers with a suitable notion of convergence we can express f in terms of simple real-valued functions, as follows:

$$f(\varphi) = \lim_{\ell \rightarrow \infty} \left(\sum_j^{+\mathbb{C}} \chi_{A_j^{(\ell)}}(\varphi) \circ_{\mathbb{R}Y} a_j^{(\ell)} \right), \quad \text{where} \quad a_j^{(\ell)} \in Y, \forall \varphi \in X$$

Finally, to keep sight of our objective to evaluate ‘averaged’ observables we apply the decomposition of measurable functions into sums of indicator functions. This leads to the well known concept of integration within the abstract field of measure theory:

Definition 2.1.6 — Integral. Let (X, Σ_X, μ) and (Y, Σ_Y, μ_Y) be measure spaces, whereby Y is endowed with a $\mathbb{R}$ -vector space structure. Further, let $f : X \rightarrow \mathbb{R}$ be a non-negative measurable function.

The map $\langle f \rangle_{\mu|_B} \equiv \int_{\varphi \in B} d\mu(\varphi) f(\varphi)$ is the integral of f over $B \in \Sigma : \Leftrightarrow$

$$\langle f \rangle_{\mu|_B} \equiv \int_{\varphi \in B} d\mu(\varphi) f(\varphi) := \sup_{g \in S_\Sigma^+} \left\{ \sum_{A_j \in S}^{+\mathbb{C}} \mu(A_j \cap B) \circ_{\mathbb{C}} a_j \mid \sum_{A_j \in S}^{+\mathbb{C}} \chi_{A_j}(\varphi) \circ_{\mathbb{C}} a_j \equiv g \leq f \right\}$$

If $\langle f \rangle_{\mu|_B} < \infty$ we say f is integrable and an element of $L^1(X, \Sigma, \mu; Y)$.

In the context of physics the integral of an observable is usually denoted as its expectation value. Since many of the relevant measurable functions also contain negative parts there are extensions to general real or complex valued functions using a decomposition like $f(\varphi) = f^+(\varphi) + (-f^-(\varphi))$, whereby f^+ and f^- have to be non-negative integrable functions. Similarly, functions with values in an arbitrary vector space over $\mathbb{C}$ can be decomposed in this form and are therefore equipped with a notion of integrability.

It turns out that the integral is a linear functional on the space of measurable functions $L^n(X, \Sigma, \mu; \mathbb{C})$, for it linearly maps the vectors f to its scalars, the real (or complex) numbers. One can show [47] that the integral does not depend on a specific decomposition into simple functions. Thus, we have found the solid mathematical grounds to treat physical observables consistently with the underlying theory μ . For a particular operator on field space X the integral provides an instrument to extract its averaged expectation value that may correspond to observable data and is thus helpful in verifying the theory.

2.1.4 Relation between measures

Fields as the natural ingredients for quantum field theory live in what we have called field space $[F]$. Equipped with a reasonable σ -algebra $\Sigma_{[F]}$ we can construct the class of all theories by defining measures on $\Sigma_{[F]}$ and giving a physical meaning to certain observable operators on $[F]$. This leads to a plenitude of different theories among which we have to select the one realized in Nature. To reduce the number of possibilities one usually invokes physically motivated, mathematical constraints that hopefully leaves only a finite number of free parameters that have to be fixed by normalization conditions. In one or the other way we need to translate the mathematical side of measure theory into the language of physics. To this end, we will introduce a very natural link between mathematics and physics given by the Gaußian measure μ_G that we define to be a free, non-interacting theory. Having established the first contact we use this distinguish element as a reference point to spread a net over the full class of theories. The threats we spin are based on the Radon-Nikodym theorem that filtrates all σ -finite measures and allows for a first classification of the space of measures on $\Sigma_{[F]}$.

Gaußian measure

In order to implement the interface between quantum field theory and measure spaces, we have to clarify the notion of non-interaction in both cases. From a mathematical perspective, given a measure space $([F], \Sigma_{[F]}, \mu)$ this corresponds to the question of full independence of the elements in $[F]$ within the theory μ . The basic idea is to fully decompose the measure in a product of one-dimensional non-interacting theories without loosing information and retaining the measure specific structure. While there is a more general description of independence, we only consider a special class of non-interacting measures, the Gaußian ones, for its status in physics as a free field theory.

First, let us build new measures from existing ones in the most natural way:

Definition 2.1.7 — Product measure. Let (X, Σ_X, μ_X) and (Y, Σ_Y, μ_Y) be measure spaces. The measure space $(X \times Y, \Sigma_{X \times Y}, \mu_{X \times Y})$ is a product measure space $:\Leftrightarrow$

$$\begin{aligned} \Sigma_{X \times Y} &\equiv \text{gen}\{(A_X, A_Y) \mid A_X \in \Sigma_X \wedge A_Y \in \Sigma_Y\} \\ \forall B \in \Sigma_{X \times Y} : \quad \mu_{X \times Y} &= \int_{x \in X} d\mu_X(x) \mu_Y(\{y \in Y \mid (x, y) \in B\}) \\ &= \int_{y \in Y} d\mu_Y(y) \mu_X(\{x \in X \mid (x, y) \in B\}) \end{aligned}$$

We emphasize the relation between μ_X , μ_Y , and $\mu_{X \times Y}$ by writing $\mu_{X \times Y} \equiv \mu_X \times \mu_Y$.

Notice that the very existence of the product measure results from the Hahn–Kolmogorov theorem, while its uniqueness is in general only valid for σ -finite measures μ_X and μ_Y . Indeed, as we attempt to establish a measure theoretical approach to field space, an infinite dimensional manifold that only locally agrees with a suitable vector space, the main focus lays on controlling the arising infinities. Again a decomposition into an infinite number of constitutions may spoil their joined properties and in fact the most natural candidate for a measure on field space, an infinite Lebesgue measure fails to exist at all. For any separable Banach space (a complete normed topological vector space) of infinite dimension there does not exist a translation-invariant Borel measure, in particular, the infinite product of Lebesgue measures fails to be of Lebesgue type. Moreover, any infinite product of the same finite measure μ on (X, Σ_X) is either trivial, infinite, or a probability measure if and only if $\mu(X)$ is less, larger, or equal 1, respectively. Thus, the natural candidates to extrapolate existing measures to field space are the probabilistic measures, of which the Gaußian one are particular suitable for our purpose.

For the construction of this class of theories, we have to introduce some notations that relate

measures with each other on a formal level. The first concerns their equivalence:

Definition 2.1.8 — Equivalent measures. Let (X, Σ) be a measurable space and $\mu : \Sigma \rightarrow \mathbb{R}$, $\nu : \Sigma \rightarrow \mathbb{R}$ be two measures on Σ . We say $\mu \sim \nu$, i.e. μ and ν are equivalent $:\Leftrightarrow$

$$\forall N \in \Sigma : (\mu(N) = 0 \implies \nu(N) = 0) \wedge (\nu(N) = 0 \implies \mu(N) = 0)$$

That is, μ is absolutely continuous w.r.t. ν and vice versa.

The crucial information is thus encoded in the null sets of μ and ν . Only if the null sets of two different measures agree both are considered equivalent in this respect which defines an equivalence class on the associated linear space. This definition is also illustrative from the physicist's perspective, since non-relevant fields of a theory should not become relevant in a different representation.

The equivalence relation $\sim$ interconnects measures on the same underlying σ -algebra. However, we want to go beyond a single measurable space and study how seemingly different theories turn out to be the image of one another and thus the same. The associated technical term pushforward measure actually refers to the commonly known concept of a change of variables in which a suitable transformation of the underlying set is promoted to a transition of entire measure spaces (in form of a covariant functor):

Definition 2.1.9 — Pushforward measures. Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces and $\mu_X : \Sigma_X \rightarrow \mathbb{R}$ a measure. A measure $\nu_Y : \Sigma_Y \rightarrow \mathbb{R}$ is the pushforward measure of a measurable function $f : X \rightarrow Y$ $:\Leftrightarrow$

$$\forall U_Y \in \Sigma_Y : \nu_Y(U_Y) = \mu_X(f^{-1}(U_Y) \equiv \{x \in X \mid f(x) \in U_Y\})$$

One usually emphasizes the relation between μ_X and ν_Y by writing $\nu_Y \equiv f_*(\mu_X)$.

This is clearly what we have in mind when performing a change of variables in the usual finite dimensional Lebesgue integration. The tied connection between μ and its pushforward measures $f_*(\mu)$ is also evident by the identification of their respective integrable functions:

$$\begin{aligned} g \in L^1(Y, \Sigma_Y, f_*(\mu_X); \mathbb{C}) &\iff g \circ f \in L^1(X, \Sigma_X, \mu_X; \mathbb{C}) \\ \forall g \in L^1(Y, \Sigma_Y, f_*(\mu_X); \mathbb{C}) : &\int_{Y \in U_Y} d[f_*(\mu)](y) g(y) = \int_{x \in f^{-1}(U_Y)} d\mu_X(x) g(f(x)) \end{aligned}$$

Now, with these tools at hand we are able to define Gaussian measures in a very general sense using its properties on the real numbers:

Definition 2.1.10 — Gaussian measure. Let $V \equiv (X, \tau_X, \circ_X)$ be a topological vector space, V^* its dual, and $V^* := \{w \in V^* \mid w \text{ continuous}\} \subseteq V^*$ the restriction to *continuous* dual vectors. Further let (X, Σ) be a measurable space. A measure $\mu : \Sigma \rightarrow \mathbb{C}$ is a Gaussian measure on (V, Σ) $:\Leftrightarrow$

$$\text{gen}\{U_X \in \tau_X\} \subseteq \Sigma \quad \wedge \quad \mu(X) = 1 \quad (\text{Borel probability measure})$$

$$\forall w \in V^* \exists \sigma, \lambda \in \mathbb{R} : w_*(\mu) : \mathbb{R} \rightarrow \mathbb{R},$$

$$w_*(\mu)(U_{\mathbb{R}}) = \int_{x \in U_{\mathbb{R}}} d\mu_L(x) \frac{\exp(-\|x - \lambda\|_{\mathbb{R}}^2 / 2\sigma^2)}{\sqrt{2\pi\sigma^2}} \quad (\text{Gaussian distributions})$$

Here μ_L denotes the Lebesgue measure on $(\mathbb{R}, \Sigma_{\mathbb{R}})$.

While the Lebesgue measure in infinite dimension does not exist, the Gaussian does for it fails to be translational invariant. For Banach spaces, it follows as a special case of the Cameron-

Martin theorem that under group translations, $\rho_a : V \rightarrow V$ with $\rho_a(v) = v \circ_V a$, for any Gaussian measure μ_G there exists a self-adjoint positive operator $S_{\mu_G} : V \rightarrow V$ such that for all $U \in \Sigma$

$$\mu_G^{\rho_a}(U) \equiv \mu_G(\rho_a(U)) = \int_{x \in U} d\mu_G(x) \exp\left(-\frac{1}{2}(a, S_{\mu_G}(a))_V + (x, S_{\mu_G}(a))_V\right) \quad (2.1)$$

whenever a is a square integrable function of V , otherwise both measures are not equivalent and no such relation can be established. Thus, for any Gaussian measure there is a certain admissible operator S_{μ_G} implicitly defined by translations and vice versa. Notice however, that the zero operator is excluded for then the corresponding measure would be translational invariant, thus ill-defined in infinite dimensions. Furthermore, it is important to keep in mind that the Gaussian measure is only defined on the subspace V^* of dual vectors.

One of the great advantages of Gaussian measures is the possibility to characterize even infinite dimensional measures with the aid of a positive semi-definite, symmetric, bilinear form q on V^* (see for example [48]):

$$\int_{x \in X} d\mu_G(x) e^{iw(x)} = \exp_{\mathbb{C}}\left(-\frac{1}{2}q(w, w)\right) \quad (2.2)$$

The full power of this relation will become clear when studying the effective action approach to quantum field theory, but it should be noted here that eq. (2.2) plays a very fundamental role in understanding physics. Notice, that every positive bilinear q form can be uniquely identified with a positive self-adjoint operator $T_q : V^* \rightarrow V^*$, see ref. [28], and thus eq. (2.2) can be rewritten in terms of the usual inner product on V^* , i.e.

$$\begin{aligned} \int_{x \in X} d\mu_G(x) e^{iw(x)} &= \exp_{\mathbb{C}}\left(-\frac{1}{2}(w, T_q(w))_{V^*}\right) && \text{with } T_q = T_q^* \\ &&& \text{and } (w, T_q(w))_{V^*} \geq 0 \quad \forall w \in V^* \end{aligned}$$

The Gaussian measure is the ideal prototype for a free theory. If there are no dominant contributions that result in non-equally distributed effects, everything smears out and what remains is a free Gaussian measure (the statement of the central limit theorem). The natural construction of free theories is thus built on Gaussian distributions which provides the missing link between physics and the mathematical theory of measures:

Definition 2.1.11 — Free theory. Let $V \equiv (X, \tau_X, \circ_X)$ be a topological vector space, V^* its dual, and as above $V^* := \{w \in V^* \mid w \text{ continuous}\} \subseteq V^*$. Further let $\text{Sym}_+^2(V)$ denote the space of symmetric, positive semidefinite, bilinear forms on V^* .

The measure space $T \equiv (X, \Sigma, \mu)$ is called a free theory $:\Leftrightarrow$

μ is a Gaussian measure

$$\forall w \in V^*, \exists q \in \text{Sym}_+^2(V) \text{ diagonalizable} : \int_{x \in X} d\mu(x) e^{iw(x)} = \exp_{\mathbb{C}}\left(-\frac{1}{2}q(w, w)\right)$$

Thus, a free theory is described by a Gaussian measure that can be split into a complete product of Gaussian measures on the eigenspaces of an operator on V^* .

Under this choice the basis of the (linear) observables is fully uncorrelated and thus the associated rays are independent of each other. Whenever a theory is found to be described by a Gaussian measure we will denote it as a free theory and interpret the underlying fields as non-interacting.

Radon-Nikodym theorem

We have given the class of Gaußian measures a physical interpretation as free, non-interacting theories. In a sense, we have normalized the space of measures with a very convenient assumption. From this distinguished elements onwards we can establish further relations between the mathematical objects and physical explanations. Basically, there are two possibilities to propagate through the space of theories.

The first method concerns a transformation of the underlying field space as in the case of integration by substitution. When this change of ‘coordinates’ is based on isomorphisms in the category of measure spaces, i.e. measurable, bijective functions, it allows us to identify indistinguishable theories. It was for this reason that we had to set up a more elaborated definition for free theories in order to assure that simple transformations of coordinates do not cloak their free characters.

The above discussed deformation of field space allows to connect different measure spaces via pushforwards, and in fact identify them. A second method that is more relevant to actual get an understanding of the physical content of a theory relates measures in the same σ -algebra, the famous Radon-Nikodym theorem:

Theorem 2.1.1 — Radon-Nikodym theorem. Let (X, Σ) be a measurable space, $\mu : \Sigma \rightarrow [0, \infty)$ a σ -finite positive measure, and $\nu : \Sigma \rightarrow \mathbb{C}$ a σ -finite (complex or signed) measure with $\forall A \in \Sigma : \mu(A) = 0 \implies \nu(A) = 0$. Then the Radon-Nikodym theorem states:

$$\exists g \in L^1(X, \Sigma, \mu; \mathbb{C}) : \quad \nu(A) = \int_{\varphi \in A} d\mu(\varphi) g(\varphi) \quad \forall A \in \Sigma \quad (2.3)$$

The function g is unique μ -almost everywhere and is called the Radon-Nikodym derivative, sometimes denoted $g = \frac{d\nu}{d\mu}$.

This generalization of the Jacobi-determinant is the central statement on which we will build our entire construction and deduction of theories in what follows. Especially we can now establish a connection of Gaußian measures to more general theories ν , whereby the Radon-Nikodym derivative g reflects the deviation from a free theory and thus contains the contributions of non-trivial interactions. A sophisticated theory usually involves a sufficient amount of these correlations that renders the evaluation process of observables arbitrarily difficult. Exploiting the Radon-Nikodym theorem we have a way to study those interactions as fluctuations of a free theory which is thus the origin of perturbation theory. In addition, we obtain the possibility for transitions from positive to complex measures that may help understanding the Euclidean path integral approach in the light of standard quantum field theory. As if that were not enough, also for non-perturbative approaches it provides a mechanism to encode the structure of field space in an effective description on which the functional renormalization group technique heavily depend on and thus lays the ground for this thesis. To obtain an effective description we have to introduce the Fourier transform of a measure, also known as the characteristic functional.

Fourier transform of a measure

We have introduced the framework of measure theory in a general way without referring to specific properties of the underlying set $[F]$ on which we build the σ -algebra and define a theory. The introduction of the concept of Fourier transforms however requires $[F]$ to be endowed at least with an additional vector space structure over the complex numbers and thus we have to make an exception to this rule. In order to model field space in what follows, this vector space is usually infinite dimensional that brings in a certain amount of complications and the notion of basis and topologies have to be reconsidered thoroughly. However, we will present a rather incomplete picture and only give a mere motivation and analogy for the given definition. For a

more detail account on the appearing subtleties we refer to [28, 47–49].

The general purpose of a Fourier transform is to analyze a vector space in the light of its dual space. In most cases the full information of a function is also encoded in its Fourier transform and both can be reconstructed from the other in a unique way. By definition maps a measure subsets of field space to specific weights that are related to the relevance of the contained fields. This relevance has to be physically motivated even though there are additional mathematical constraints that reduces the possibilities. In real life, we are equipped with a certain amount of observable data and the objective is to deduce the underlying theory, the measure, that matches those findings. But rather than guessing a measure by brute force we can revert the description and use the analysis of expectation values to successively shape a class of suitable theories. The Fourier transform of a measure is the initial step for this procedure and formally defined by

Definition 2.1.12 — Fourier transform of measures. Let $([F], \Sigma, \mu)$ be a measure space and $[F]$ a (infinite-dimensional) vector space over $\mathbb{C}$ with dual $[F]^* = \{f : [F] \rightarrow \mathbb{C} \mid f \text{ linear}\}$. Let $z \in \mathbb{C}$ be a complex number. A functional $\hat{\mu} : [F]^* \rightarrow \mathbb{C}$ is the Fourier transform of $\mu : \Leftrightarrow$

$$\hat{\mu}_A(f) \equiv \int_{\varphi \in A} d\mu(\varphi) \exp_{\mathbb{C}}(zf(\varphi)), \quad \forall A \in \Sigma, \forall f \in [F]^*$$

The subscript A indicates the range of integration and is omitted in case of $A \equiv X$. For $A \equiv X$ it is further custom to denote $Z(f) := \hat{\mu}_X(f)$ as the generating or characteristic functional which is more common, especially in physics literature.

We have encountered an example of a Fourier transform already in eq. (2.2) when stating a particular transformation property of the Gaußian measure. Notice, that each dual vector $f \in [F]^*$ yields a map $\hat{\mu}_{\bullet}(f) : \Sigma \rightarrow \mathbb{C}$ that defines a measure on $([F], \Sigma)$. Whenever μ is a σ -finite positive measure the Radon-Nikodym theorem applies since by virtue of the properties of the exponential map, the Radon-Nikodym derivative is a nowhere vanishing measurable function, and thus $\hat{\mu}_{\bullet}(f)$ is a σ -finite complex measure. The standard Laplace and Fourier transform are recovered by the choice of $z = -1$ and $z = i$, respectively. In the more general context, it is difficult to extract the full set of properties for those Fourier transforms. Nevertheless, we will mention some of them along with their additional assumptions and the range of admissible z -values.

There is an intensive study of separable Hilbert spaces containing only functions with compact support. The dual space – on which $\hat{\mu}$ depends on – is a special set of linear, continuous functionals on field space: the distributions. In this case the uniqueness and existence of the Fourier transform with $z \equiv i$ is guaranteed and in fact the above relation can be inverted using Minlos' theorem [28]. This states that given a functional $w : Y \rightarrow \mathbb{R}$ on the set of compactly supported functions $Y = \{J : \mathbb{R}^d \rightarrow \mathbb{C}\}$ satisfying

$$\begin{aligned} w(0_Y) &= 1 && \text{(normalized)} \\ \forall c_r \in \mathbb{C}, \forall J_r \in Y : \sum_{r,s} \bar{c}_s \cdot c_r \cdot w(J_r - \bar{J}_s) &\geq 0 && \text{(positivity)} \\ w : Y \rightarrow \mathbb{R} &\text{ is continuous} && \text{(continuity)} \end{aligned} \tag{2.4}$$

there exists a unique Borel probability measure μ on the space of associated distributions with

$$w(J) = \int_{\varphi \in X} d\mu(\varphi) e^{i\varphi(J)} \quad \forall J \in Y$$

Notice that positive definiteness of $w(J)$ only implies that $w(0_Y) \geq \|w(J)\|_{\mathbb{R}} \geq 0$ holds true, with 0_Y the zero map. Therefore, the additional condition of normalization, $w(0_Y) = 1$, ensures that

μ is a probability measure. The requirements stated in eq. (2.4) are properties that are fulfilled by any Fourier transform of Borel probability measures as long as the conditions of the Minlos'- or more elementary the Bochner theorem are satisfied. In the context of Euclidean quantum field theory these constraints build the basis of the Osterwalder-Schrader axioms [50, 51]. The lesson we can learn at this point is that the Fourier transform is a useful method to characterize measures over a certain σ -algebra and fully encode the theories information into a functional of its dual space.

This close connection that is established between a vector space and its continuous dual space can also be generalized to locally compact groups using the Pontryagin duality theorem [52]. For any such topological group $G \equiv (X, \circ_G, \tau_X)$ that is locally compact and Hausdorff, there exists a one-parameter family of measures $\mu_H^{(G)}(\bullet; \lambda)$ on the Borel sets Σ_G of (X, τ_X) that satisfies:

$$\begin{aligned} \forall g \in X, \forall A \in \Sigma_G : \quad \mu_H^{(G)}(g \circ_G A; \lambda) &= \langle A \rangle_{\lambda} && \text{(left-} G \text{ invariant)} \\ \forall C \in \Sigma_G \text{ compact} : \quad \mu_H^{(G)}(C; \lambda) &< \infty && \text{(compactly finite)} \\ \forall U \in \tau_X : \quad \mu_H^{(G)}(U; \lambda) &= \inf\{\mu_H^{(G)}(C; \lambda) \mid C \subseteq U \wedge C \text{ compact}\} && \text{(inner regular)} \\ \forall A \in \Sigma_G : \quad \mu_H^{(G)}(A; \lambda) &= \inf\{\mu_H^{(G)}(U; \lambda) \mid A \subseteq U \in \tau_X\} && \text{(outer regular)} \end{aligned}$$

This measure that we have encountered previously in studying group representations is called the left Haar measure of G . In the context of group theory the Fourier transform provides a natural correspondence between the group and its dual group G^* , i.e. the set of characters $\{\chi : G \rightarrow \{e^{iz} \mid z \in \mathbb{R}\}\}$ that forms a locally compact Abelian group. The Pontryagin duality ensures that this dual group contains all information of G on basis of $G^{**} \simeq G$. In other words, applying the Fourier transform on the Haar measure of a group results in the character group equipped with its (up to scaling unique) Haar measure $\mu_H^{(G^*)}$ without losing information. Thus the Fourier transform of $\mu_H^{(G^*)}$ yields the original measure we started with [52].

The generalized Fourier transformations occupy an important place in the practical treatment of problems in physics and related subjects, because it has the feature to convert differential- in algebraic equations and vice versa. In a broader context it allows to interchange a vector space with its dual one that in combination with measure theory provides a way to study field spaces and the associated theory by investigating the space of observables. In particular for quantum field theories this translates into the concept of Schwinger functionals and effective actions that carry the entire information of the underlying measure in terms of its Fourier decomposition. And finally, we will see how the framework of renormalization group exploits the properties of Fourier transforms to determine algebraic (integral) objects as the measures by a (functional) differential equation.

2.1.5 Measures for gauge theories

All the described techniques in this section are at the very basis of QFT where the measure is defined over an infinite dimensional manifold, field space. This general formalism of probability theory plays a crucial role in understanding the fundamental interactions of Nature. Together with the symmetry construction of fiber bundles it gives rise to a universal language in which those theories seem to be describable. Basically, the transition from the mathematical concepts to physics is provided by some change of vocabulary, namely interpreting the partition function as the theory under investigation, the integration variables as dynamical fields and its associated expectation values as observables of certain fundamental processes. This slight modification of notation has to be supplemented with some measure theoretic construction that usually only become relevant when non-trivial field spaces are considered, thus in particular in the case of quantum gravity.

We will start by introducing a well known technique for integration on manifolds: the partition of unity. While for flat field spaces this amounts to a trivial identification of F and $T_{\bar{\Phi}}F$ for a single arbitrary background field $\bar{\Phi}$, in the general case we have to look for some geodesic cover, a background field covering, that yields a diffeomorphic equivalence between local regions and their tangent spaces via a (non-trivial) exponential map. Next, we establish a link between theories defined for the physical fields, i.e. elements of $[F]$, and their counterparts defined on F , the space of all mathematically possible fields. The route we are going to follow describes the general idea of Becchi, Rouet, Stora and Tyutin in a geometrical frame. One explicit construction, suitable for a large class of theories, is the Faddeev-Popov method for which we state some general conditions under which it becomes admissible even in the case of spacetimes with non-vanishing boundary.

Partition of unity

In general, field space will be a smooth (infinite-dimensional) manifold that is only locally diffeomorphic to a vector space. However, the definition of a Fourier transform requires a vector space on a global scale. In fact, there is quite a number of reasons to rather use the tangential vectors than the fields itself, for example to invoke operator constraints. But surely the most important motivation to have a vector space at our disposal is the local existence of Fourier decomposition, a very powerful tool that is especially relevant in the construction of a generating functional. Fortunately, there is at least a local notion of vector spaces attached to field space and we have an intuitive understanding how to formally identify those regions using the background field covering 1.3.2. In this program we select one element of field space, construct its tangential space and determine the patch of base space that we can diffeomorphically identify with this local vector space. By increasing the number of basis elements we finally cover the entire field space that is then represented by $\{(\bar{\Phi}, U_{\bar{\Phi}}, T_{\bar{\Phi}}[F])\}$ with $\cup_{\bar{\Phi}} U_{\bar{\Phi}} = [F]$. Then, on each local patch the full formalism of measure theory including the definition of Fourier transforms is at our disposal.

Nevertheless, one difficult task remains, namely the recombination of those pieces into a global object, while assuring no over-counting of fields occurs. One method to obtain a global object from a collection of local ones is described by the gluing lemma. It provides a method that relies on a set of smooth functions $\{f_j\}$ defined over some chart of $[F]$. If all functions agree on regions of overlap, then there exists a global smooth function that piecewise coincides with the f_j 's. While this is a strong statement, it is far too restrictive for the present case: we have to find a background field covering that on all regions of overlap fully agrees. Fortunately, there exists another strategy which makes the full agreement of functions f_j on large portions of field space expendable while ensuring a consistent recombination, the partition of unity.

Definition 2.1.13 — Partition of unity. Let $M \equiv (X_M, \tau_M, \mathcal{A})$ be a smooth paracompact manifold and $\{U_j \in \tau_M\}_{j \in A}$ be an open cover of X_M . Furthermore, let $\{\varsigma_j : X_M \rightarrow \mathbb{R}\}_{j \in A}$ be a collection of smooth functions. The set of ordered pairs $\{(U_j, \varsigma_j)\}$ is called a smooth partition of unity on M $\Leftrightarrow$:

$$\begin{aligned} \forall j \in A : \quad \text{supp}(\varsigma_j) &:= \overline{\{p \in X_M \mid \varsigma_j(p) \neq 0\}} \subseteq U_j \\ \forall j \in A, \forall p \in X_M : \quad 0 &\leq \varsigma_j \leq 1 \quad \wedge \quad \sum_{j \in A} \varsigma_j(p) = 1 \\ \forall p \in X_M : \quad \#(\{j \in A \mid p &\in \text{supp}(\varsigma_j)\}) < \infty \end{aligned}$$

In the finite dimensional case, smooth manifolds are always paracompact which guarantees the existence of a partition of unity for any open cover of X_M . However, in the case of infinite dimensions, the paracompactness property of the cover is very important in order to let the sum

$\sum_{j \in A} \varsigma_j(p)$ converge [53]. Assuming the existence of a smooth partition of unity, we can split a measure μ on field space into small portions covered by the support of a suitable measurable partitioning function:

$$\mu([F]) = \sum_{j \in A} \int_{p \in U_j} d\mu(p) \varsigma_j(p)$$

Apart from its obvious application to measures on $[F]$, it is also relevant for some approximation techniques in the perturbative treatment of QFT.

Maybe the most prominent application in QFT is the background field method. Hereby, the cover is such that it coincides with a background field covering for a set of background fields $\bar{\Phi}$ and the open sets are generated by the exponential map, i.e.

$$\mu([F]) = \sum_{j \in A} \int_{\hat{\phi} \in T_{\bar{\Phi}_j}[F]} d\mu(\exp_{\bar{\Phi}_j}(\hat{\phi})) \varsigma_j(\hat{\phi})$$

This defines a very general prescription to obtain a local vector space formalization of field space on the level of measure theory. For the later discussion we usually omit the partition of unity but assume that there is a single background field $\bar{\Phi}$ from which the entire field space can be constructed using the exponential map. Whenever this is known to be only approximated true, one has to come back to this general construction and define a suitable cover with the prescribed techniques.

Measure theory on field space

In the mathematical treatment considered so far, we have seen that equivalence relations take a very prominent role in reducing the number of redundancies due to some symmetry principle. For instance, spacetime represents an entire equivalence class of diffeomorphically identical smooth structures consistent with a certain point set and topology. This superfluous data reduced by the equivalence relation of diffeomorphisms can be traced back to indistinguishable constituents (points) of spacetime and thus to the symmetry under relabeling of ‘coordinates’. Similarly, at the very heart of the definition of physical fields via the fiber bundle construction used in gauge theory, symmetries are to the fore. Redundancies are generated by the group of gauge transformations and only unrelated global sections of some associated bundle give rise to different physical objects. While on an abstract level, for instance in the general measure theoretical implementation of QFT, working with the space of physical indistinguishable fields $[F]$ is unproblematic, practical calculation are usually based on coordinates, both in spacetime and in field space. Thus the formalism has to be presented on the basis of some particular representative to make the mathematical objects usable for computations. Nevertheless, though we might introduce some coordinates at intermediate steps of the evaluation, the final results should be independent on a specific choice and thus it has to be invariant under the symmetry principle the equivalence relations are based on. It is actually one of the severest problems in modern field theories to define a systematic and still practical suitable transition from $[F]$ to F .

In a moderate form this issue is also present in quantum mechanics, where physical states are associated with an infinite number of mathematical elements combined in what is called a ray in a Hilbert space. There are different ways to implement the symmetry of physical ‘reality’ into the mathematical formalism that contains redundancies in this sense. In case of quantum mechanics the generic way is the insertion of an arbitrary phase factor into the description while ensuring that all observable results are independent on a specific phase choice (still phase differences are in principle observable). In a sense this can be compared with the Faddeev-Popov method, where so called ghost fields generate the gauge equivalent objects and the final observable results are required to be independent on these fields. In principle, it is possible to

work on an abstract level and keep the formalism such that it can be consistently defined using equivalence classes only. From the perspective of quantum mechanics this requires a well-behaved operator calculus on some projective space, which however is far more involved from a practical viewpoint. Thus, it is not astonishing that most of the explicit calculations ignore the abstract description and employ a specific choice of ‘coordinates’. Nevertheless in the end, the validity of the results has to be checked by verifying the invariance under the associated symmetry transformations.

Once a fiber bundle description of the underlying space of equivalence classes, $[F]$, exists, there is a general treatment to relate measures of the (full) mathematical- with the (restricted) physical-field space. For field space, such a fiber bundle description was established in subsection 1.3.2, whereby the physical sector $[F]$ represents the base space of a principal bundle F with structure group $\mathcal{G}$, the group of gauge transformations. Hereby, $P = M^{\pi_P} \times_{\rho_P^R} G$ denotes the underlying principal bundle of the interaction symmetry group, for instance $SU(3) \times SU(2) \times U(1)$ for the Standard Model of particle physics. Thus, there is a natural, in general non-linear, embedding of $[F]$ in F given by global sections, $\chi : [F] \rightarrow F \in \Gamma(F)$, which in the physical literature goes under the name of gauge fixing conditions. For generic field spaces the existence of global sections is however not guaranteed at all, but in fact an indication for a trivial bundle. In the non-trivial case local sections have to be combined with suitable transition functions, as usually, which is at the heart of Gribov copies [54].

In order to understand the necessity of BRST invariance for gauge field theories it is crucial to distinguish between the group of gauge transformations acting on a single physical field, and the group of transformations acting on the set of gauge fixing conditions:

$$\begin{aligned}\mathcal{G} &\equiv \{f \in \text{Iso}(\Gamma(P))\} \equiv \{f \in \text{Iso}(P) \mid \pi_P \circ f = \pi_P \wedge \rho_P^R(f(u), g) = f(\rho_P^R(u, g))\} \\ \mathcal{G}_{\text{gf}} &\equiv \{f \in \text{Iso}(\Gamma(F))\} \equiv \{f \in \text{Iso}(F) \mid \pi_F \circ f = \pi_F \wedge \rho_F^R(f(\hat{\Phi}), g) = f(\rho_F^R(\hat{\Phi}, g))\}\end{aligned}$$

On the field space F the structure group $\mathcal{G}$ acts on each element individually such that $\hat{\Phi}$ and $f_{\mathcal{G}}(\hat{\Phi})$ share the same equivalence class and hence are physically indistinguishable. The invariance under $\mathcal{G}$ is important in the construction of an action functional that should satisfy

$$S : F \rightarrow \mathbb{R}, \quad \hat{\Phi} \mapsto S[\hat{\Phi}] \quad \text{with} \quad S[f_{\mathcal{G}}(\hat{\Phi})] = S[\hat{\Phi}] \quad \forall f_{\mathcal{G}} \in \mathcal{G}$$

This group is associated to the properties of the interaction bundle P rather than field space F and transforms sections on P to other sections on the same bundle.

Concerning gauge fixing conditions, which are by definition sections on F , i.e. $\chi \in \Gamma(F)$, in the general case there exist gauge transformations which do not respect the smooth structure on F , for instance

$$f_{\mathcal{G}} \circ \chi \notin \Gamma(F) \quad \text{in general}$$

Thus $f_{\mathcal{G}}$ keeps the notion of a physical field intact but in principle it destroys the concept of gauge fixing conditions.

On the other hand, $\mathcal{G}_{\text{gf}}$ is explicitly constructed such that it preserves the notion of gauge fixing conditions, as well as the equivalence class of physical fields. To understand the scope of application for $\mathcal{G}_{\text{gf}}$ let us have a look at the usual way gauge fixing conditions appear in physical theories:

$$m_{\text{BRST}} : \Gamma(F) \times \text{Meas}_P(F, \Sigma_F) \rightarrow \text{Meas}_P([F], \Sigma_{[F]}) \quad \text{with} \quad \mu_F \mapsto m_{\text{BRST}}(\chi, \mu_F) = \mu_{[F]}^{(\chi)}$$

The map m_{BRST} is a certain – yet to defined – method that reduces the measure, or generating functional, from field space to the physical sector. A possible realization may assume the form

of the Faddeev-Popov method, i.e.

$$\mu_F \equiv \int_{\hat{\phi} \in T_{\hat{\Phi}} F} d\mu_G(\hat{\phi}; \bar{\Phi}) \rho(\hat{\phi}; \bar{\Phi}) \mapsto \int_{\hat{\phi} \in T_{\hat{\Phi}} F} d\mu_G(\hat{\phi}; \bar{\Phi}) \rho(\hat{\phi}; \bar{\Phi}) \mathcal{N}_\delta(\chi) \delta(\chi([\hat{\phi}])) \equiv \mu_{[F]}^{(\chi)}$$

Notice that $\mathcal{N}_\delta(\chi) \delta(\chi([\hat{\phi}]))$ represents the invariant delta distribution on field space, with $\mathcal{N}_\delta(\chi)$ being the Jacobian determinant associated to this hyperplane embedding. Depending on the explicit implementation method m_{BRST} , the resulting theory on the physical sector will depend on the gauge fixing condition and thus $\mu_{[F]}$ generates different theories for different embeddings of $[F]$ in F . However, a suitable map m_{BRST} has to give rise to a unique theory in the physical sector, which corresponds to $\mu_{[F]}^{(\chi)} \equiv \mu_{[F]}$ being independent on χ . On the level of m_{BRST} the invariance under transformations of the gauge fixing condition reads:

$$m_{\text{BRST}}(\chi, \mu_F) \stackrel{!}{=} m_{\text{BRST}}(f_{\mathcal{G}_{\text{gf}}}(\chi), \mu_F)$$

Hence, a systematic procedure to define m_{BRST} should be based on the underlying fiber bundle construction and emerges from purely differential geometry considerations. This program, known as BRST-invariant gauge fixing of a theory, goes back to independent works of Becchi, Rouet, Stora [55] and Tyutin. Since then, anti-BRST as well as BRST transformations have been understood in the light of Chevalley-Eilenberg cohomology [56] of the Lie algebra associated to the adjoint bundle of F . In this geometrical framework BRST transformations are simply related to the coboundary operator of this cohomology, and are denoted s_{BRST} . This is a very important fact, for it tells us that a successive application of BRST operations vanishes, i.e. $s_{\text{BRST}} \circ s_{\text{BRST}} = 0$, which is also known as Wess-Zumino consistency condition. For details we refer to [57, 58] which provides a mathematical overview.

For our purpose it suffices to know that the structure of the principle bundle F gives rise to a nilpotent operation s_{BRST} . In order to give a precise meaning to m_{BRST} that can be easily extended to the generating functional and its derived concepts, let us decompose field space F into its vertical and horizontal components.

$$T_{\hat{\Phi}} F \equiv V_{\hat{\Phi}} F \oplus H_{\hat{\Phi}} F \simeq \text{LieAlg}(\mathcal{G}) \times T_{\hat{\Phi}}[F]$$

This split corresponds to the phase representation of Hilbert rays, now in a more general setting. In particular it is based on a (non-trivial) field space connection $\chi_F : TF \rightarrow \text{LieAlg}(\mathcal{G})$ that defines the horizontal space by its kernel. Basically, we have encountered many of the currently discussed aspects already in subsection 1.3.2 but with a different emphasis. Central for the following discussion is the decomposition of the fluctuation fields into physical ones and those that generate the Lie algebra of gauge transformations. Explicitly it states:

$$\hat{\phi} \equiv \rho_{F*}^R(\hat{\phi}, C)$$

Furthermore the connection defined on $\mathcal{G}$ is called the *Faddeev-Popov ghost* η and represents the Maurer-Cartan form, thus $\eta(C) = C$ holds true for all $C \in \text{LieAlg}(\mathcal{G})$. Ghost fields violate the spin-statistic theorem and are as such unphysical quantities. They are represented by anticommuting vector fields assuming values in $\text{LieAlg}(\mathcal{G})$. Thus the direction of physically indistinguishable and distinguishable fields is nicely separated, which also extends to the measure spaces, i.e.

$$\mu_{T_{\hat{\Phi}} F}(U_F \equiv (U_{[F]}, U_C)) \equiv \mu_{T_{\hat{\Phi}}[F]}(U_{[F]}) \times \mu_{\text{LieAlg}(\mathcal{G})}(U_C)$$

For the reduction to the physical sector, we have to apply a suitable projection m_{BRST} which should be BRST invariant for this implies invariance under $\mathcal{G}_{\text{gf}}$. The nilpotent character of

the BRST transformation allows for any insertion that can be expressed as the image of s_{BRST} . Furthermore, we have to ensure that $\rho(\hat{\phi}; \bar{\Phi})$ is in the kernel of s_{BRST} , which in fact is naturally satisfied whenever ρ is gauge invariant. Hence, two theories that differ by a total derivative w.r.t. s_{BRST} are BRST equivalent and therefore agree on the physical sector:⁵

$$\mu_F \equiv \int_{\hat{\phi} \in T_{\bar{\Phi}} F} d\mu(\hat{\phi}; \bar{\Phi}) \simeq \int_{\hat{\phi} \in T_{\bar{\Phi}} F} d\mu(\hat{\phi}; \bar{\Phi}) d[C] s_{\text{BRST}}(\Psi(\hat{\phi}, C; \bar{\Phi})) \quad \text{with} \quad \hat{\phi} \equiv ([\hat{\phi}], C)$$

Now, in order to cancel the gauge orbit integral by means of a suitable implementation of gauge fixing conditions, we consider a specific, s_{BRST} -exact insertion:

$$\rho_{\text{BRST}}(\hat{\phi}, C; \bar{\Phi}; \chi) \quad \text{with} \quad s_{\text{BRST}}(\rho_{\text{BRST}}(\hat{\phi}, C; \bar{\Phi}; \chi)) = 0$$

Besides these geometric construction yielding the physical- and ghost-fields by the fiber bundle formalism, there is an analytic part of the BRST method introducing the anti-ghost field C and the so-called Nakanishi-Lautrup field $B \in C^\infty(M)$. Therefore, let us assume we have a gauge fixing condition $\chi \in \Gamma(F)$ that – consider as a hyperplane embedded in field space – corresponds to the a function

$$\mathcal{F} : F \rightarrow C^\infty(M) \quad \text{with} \quad \ker(\mathcal{F}[\bar{\Phi}](\hat{\phi}) - B) \equiv \text{Im}(\chi)$$

Here the artificial Nakanishi-Lautrup field B appears in the formalism, representing the right-hand-side (RHS) of the hyperplane equation $\mathcal{F}[\bar{\Phi}](\hat{\phi}) = B$. It allows for a smooth implementation and a more flexible gauge fixing construction. Now, we have to adapt B and later $\bar{C}$ such that neither BRST invariance nor other important properties are lost, in particular we require

$$s_{\text{BRST}}(B) \stackrel{!}{=} 0 \quad \implies \quad s_{\text{BRST}}\mathcal{F}[\bar{\Phi}](\hat{\phi}) = 0 \quad \text{for } \hat{\phi} \in \chi([\hat{\phi}])$$

Notice that the constraint for the gauge fixing condition is only satisfied for fields on the hyperplane, while in general $s_{\text{BRST}}\mathcal{F}[\bar{\Phi}](\hat{\phi}) \neq 0$.

Supplementary to B we introduce the anti-ghost field $\bar{C}$, an anti-commuting dual vector field that in a sense represents the (unphysical) partner of C . It acts as a Lagrange multiplier that installs the hyperplane condition into the generating functional. The BRST transformation is chosen such that it linearly relates to B , i.e. $s_{\text{BRST}}(\bar{C}) = B$. In total, assuming $[F]$ to describe only the gauge (interaction) sector, i.e. neglecting matter fields for a moment, we have the following set of rules for the four types of fields:

$$\begin{aligned} s_{\text{BRST}}(\hat{\phi}) &= d\eta + \mathcal{L}_\eta \hat{\phi} & \wedge & \quad s_{\text{BRST}}(\eta) = -\frac{1}{2}[\eta, \eta]_{\text{LieAlg}(\text{LieAlg}(\mathcal{G}))} \\ s_{\text{BRST}}(\bar{C}) &= B & \wedge & \quad s_{\text{BRST}}(B) = 0 \end{aligned} \quad (2.5)$$

As long as the Lie algebra $\text{LieAlg}(\mathcal{G})$ closes and thus fulfills the Jacobi-identity, the action of s_{BRST} on the field content is indeed nilpotent. Hence, m_{BRST} is defined as

$$m_{\text{BRST}}(\mu) := \int_{([\hat{\phi}], C, \bar{C}, B) \in T_{\bar{\Phi}} F \times U_{\bar{C}, B}} d\mu(\hat{\phi}; \bar{\Phi}) d[C] d[\bar{C}] d[B] s_{\text{BRST}}(\rho_{\text{BRST}}([\hat{\phi}], C, \bar{C}, B; \bar{\Phi})) \quad (2.6)$$

Here $v(U) \equiv \int_U d[C] d[\bar{C}] d[B] \rho_{\text{BRST}}([\hat{\phi}], C, \bar{C}, B; \bar{\Phi})$ defines a suitable measure for the supplementary (unphysical) fields.

It should be mentioned that there is also a geometric construction involving the anti-BRST transformations, where the roles of C and $\bar{C}$ are interchanged. Invariance of m_{BRST} under both

⁵In the language of differential geometry, the set of physically equivalent theories is a s_{BRST} -closed but not exact form.

BRST- and anti-BRST transformations is very closely related to the background field method and its associated background Ward identities [59].

Before we are going to present the Faddeev-Popov method as a specific example, let us make a final remark on how to control the BRST invariance of the theory on the level of the expectation values. Since on the effective side all fields are integrated out, we need source terms for every BRST transformation thus the additional term

$$\begin{aligned} & \exp \left(K_{\hat{\phi}}(s_{\text{BRST}} \hat{\phi}) + K_C(s_{\text{BRST}} C) + K_{\bar{C}}(s_{\text{BRST}} \bar{C}) + K_B(s_{\text{BRST}} B) \right) \\ & \equiv \exp \left(K_{\hat{\phi}}(s_{\text{BRST}} \hat{\phi}) + K_C(s_{\text{BRST}} C) + K_{\bar{C}}(B) \right) \end{aligned}$$

In the last step we inserted the trivial BRST transformations of $\bar{C}$ and B . One then incorporates this kind of Fourier transformation of the BRST field transformations into the generating functional and hence the variation w.r.t. the BRST sources $K_{\bullet}$ gives rise to expectation values of the corresponding operation $s_{\text{BRST}}(\bullet)$. On the level of the effective action, which will be introduced in the next section, the BRST symmetry is intrinsically present in form of the Slavnov-Taylor identity:

$$\sum_{\phi \in \{\hat{\phi}, C, \bar{C}\}} \text{Tr} \left[(\partial_{\delta K_{\phi}} \Gamma[\hat{\phi}; \bar{\Phi}|K]) (\partial_{\delta \phi} \Gamma[\hat{\phi}; \bar{\Phi}|K]) \right] = 0$$

There is no summation over B due to the trivial BRST transformation and thus its vanishing source term.

Before the advent of the concept of BRST invariance and its geometrical interpretation, Faddeev and Popov introduced a very successful method to treat functional integrals for gauge theories [60]. It is in fact a special case for m_{BRST} that is suitable for a large class of QFT and given by

$$\rho_{\text{BRST}}^{\text{FP}}([\hat{\phi}], C, \bar{C}, B; \bar{\Phi}) \equiv \exp \left(\int_{\text{M}} d^d x \sqrt{\bar{g}} \left(\bar{C} \mathcal{F}[\bar{\Phi}](\hat{\phi}) - \frac{1}{2} \bar{\alpha} \bar{C} B \right) \right)$$

Under the application of s_{BRST} the exponential reduces to a sum over the familiar gauge fixing- and ghost-functionals given by

$$\begin{aligned} S_{\text{gf}}[\hat{\phi}, C, \bar{C}; \bar{\Phi}] & \equiv + \int_{\text{M}} d^d x \sqrt{\bar{g}} \left(-\frac{1}{2} \bar{\alpha} (B)^2 + B \mathcal{F}[\bar{\Phi}](\hat{\phi}) \right) \\ S_{\text{gh}}[\hat{\phi}; \bar{\Phi}] & \equiv - \int_{\text{M}} d^d x \sqrt{\bar{g}} \left(\bar{C} s_{\text{BRST}} \mathcal{F}[\bar{\Phi}](\hat{\phi}) \right) \end{aligned}$$

The ghost term depends on $\bar{C}$ as well as on C , due to s_{BRST} . It represents the functional determinant over anti-commuting fields which is associated to the Jacobian of the hyperplane embedding, i.e. $\mathcal{N}_{\delta}(\chi)$. On the other hand the gauge fixing term can be associated to a smeared delta distribution which becomes more explicit once we perform the Gaussian integration $d[B] \exp(S_{\text{gf}})$:

$$S_{\text{gf}}[\hat{\phi}; \bar{\Phi}] \equiv + \int_{\text{M}} d^d x \sqrt{\bar{g}} \left(-\frac{1}{2} \bar{\alpha}^{-1} (\mathcal{F}[\bar{\Phi}](\hat{\phi}))^2 \right)$$

Thus, for a set of field space configurations an explicit implementation of m_{BRST} is given by

$$m_{\text{BRST}}(\mu) := \int_{([\hat{\phi}], C, \bar{C}) \in \text{T}_{\bar{\Phi}} \text{F} \times \bar{U}_{\bar{C}}} d\mu([\hat{\phi}], C; \bar{\Phi}) d[\bar{C}] \exp \left(-S_{\text{gf}}[\hat{\phi}; \bar{\Phi}] - S_{\text{gh}}[\hat{\phi}, C, \bar{C}; \bar{\Phi}] \right) \quad (2.7)$$

It is BRST invariant and corresponds to a unique measure on $[\text{F}]$, as required. In the following sections of part I we will implicitly assume, without emphasizing, that we can freely transit between measures on F and $[\text{F}]$ based on some appropriate m_{BRST} .

In the presence of a non-vanishing spacetime boundary, reference [61] lists additional requirements that have to be satisfied in order to apply the Faddeev-Popov method. The first necessary ingredients are gauge invariant boundary conditions for field space and the ghost sector. Furthermore, the given boundary conditions should be such that $\mathcal{F}[\bar{\Phi}](\hat{\phi} + \mathcal{L}_C \hat{\phi}) = 0$ gives rise to a unique solution for C w.r.t. any $\hat{\phi}$. Once these requirements are fulfilled, the Faddeev-Popov method can be self-consistently implemented even in the case of $\partial M \neq \emptyset$.

2.1.6 Interpretation

We have started this chapter by leaving the interpretation of the chosen probabilistic way open and we will continue to do so. Nevertheless, it is very instructive to emphasize the universal character of this measure theoretical approach and see how the different philosophical concepts emerge.

Obviously, this method is best suited for a stochastic treatment of Nature, where events take place with a certain probability. However, already at this step we meet determinism in the shape of effective theories and upon implementing experimental data. No matter what the sources of uncertainties are, whether they are intrinsically present in the laws of Nature or if they are due to a lack of experimental precision, the formalism does not care. Mathematics on its own is free from any interpretation and the true genius of natural science is precisely to link numbers, letters and formula with real life. Uncertainties nowadays play a very important role, at least since the advent of quantum mechanics that gave them a prominent place even in the theoretical description. The future generations of physicists have to figure out if this feature is merely an artifact of an effective description or truly imprinted in the laws of physics.

Surely, there are methods more appropriate to study deterministic theories. It is yet very convenient to recover it within the same tool used to study generically different approaches, especially for comparison reasons. On the basis of measure theory the transition of a probabilistic to a full deterministic setting is carried out by simply restricting the class of admissible measures. While in the general case we will invoke only symmetry constraints and some plausible regularization conditions, determinism requires a very special class of measure spaces where only a single field contributes non-trivially. This corresponds to a measure comparable to a delta distribution over the realized field. Hence, one may suggest that the more a single field dominates a measure on field space, the closer we come to a deterministic description.

In this section we have met a very powerful tool that may help us in understanding the foundations of Nature. Covering both, deterministic and general probabilistic theories, measure theory provides a universal approach that takes care of experimental and theoretical uncertainties on the same footing. In the remainder of this chapter we will see how this prescription naturally fits in the beautiful concept of quantum field theories.

2.2 Effective action

The Fourier transform we considered in the end of section 2.1 provides a very practical tool to investigate the space of theories that culminates in the effective action formalism of quantum field theories. It relates the (tangent of) field space to its dual space and this connection can be extended to express a theory, defined by a suitable measure, in terms of its field expectation values. This is a more natural construction that reflects the actual process of deducing theories from observable data. On the level of expectation values we are able to address questions in a more instructive way which are obscured when considered from the perspective of probability measures.

2.2.1 Schwinger functional

Once we decided about a particular theory by means of a probability measure the main focus lays on the evaluation of observables to relate theory with experiments. These measurable functions are naturally defined over field space or in case of the background field method over the respective tangent space. When evaluated for the corresponding theory, we obtain its expectation values given by

$$\langle f(\hat{\Phi}) \rangle_\mu := \int_{\hat{\Phi} \in [F]} d\mu(\hat{\Phi}) f(\hat{\Phi}) \quad \vee \quad \langle f(\hat{\phi}) \rangle_\mu := \sum_{\hat{\Phi}} \int_{\hat{\phi} \in T_{\hat{\Phi}}[F]} d\mu(\hat{\phi}; \hat{\Phi}) f(\hat{\phi}; \hat{\Phi})$$

Instead of evaluating every observable separately, we can draw on a set of basis functions and build all measurable functions on top of them. On such collection contains the monomials of fields that constitutes a large class of observables, in particular the polynomials. Each basis element is thus a power of the dynamical or fluctuation field with respect to the pointwise product of fields, i.e.

$$\hat{\Phi}, \hat{\Phi}^2 \equiv \hat{\Phi} \otimes \hat{\Phi}, \hat{\Phi}^3, \dots \quad \vee \quad \hat{\phi}, \hat{\phi}^2 \equiv \hat{\phi} \otimes \hat{\phi}, \hat{\phi}^3, \dots$$

Measurable functions form an associative, commutative algebra under pointwise and tensorial operations, and as such their expectation values can be recovered by a suitable linear combination of field monomials, which are classified by their degree of homogeneity n and also denoted n -point functions. The most significant role is played by those of degree one, which naturally relate the dynamical or fluctuation fields with their expectation valued counterparts:

$$\Phi := \langle \hat{\Phi} \rangle_\mu \equiv \int_{\hat{\Phi} \in [F]} d\mu(\hat{\Phi}) \hat{\Phi} \quad \vee \quad \phi := \langle \hat{\phi} \rangle_\mu \equiv \sum_{\hat{\Phi}} \int_{\hat{\phi} \in T_{\hat{\Phi}}[F]} d\mu(\hat{\phi}; \hat{\Phi}) \hat{\phi}$$

They describe the most likeliest field configuration for the respective theory defined by the measure μ on which they depend. For each μ there is a unique field ϕ and a variation of this expectation value is equal to a change of the underlying theory. This is the aforementioned duality between expectation values and measures which allows us to attack the construction of suitable theories from a different perspective.

To this end, let us start with a particular field space $[F]$ and define a suitable σ -algebra, most preferable of Borel type to have a coincidence between measurable and continuous functions. For this specified measurable space $([F], \Sigma_{[F]})$ the set of probability measures is given by

$$\text{Meas}_P([F], \Sigma_{[F]}) \equiv \{\mu : \Sigma_{[F]} \rightarrow \mathbb{R} \mid \mu([F]) = 1 \wedge \mu \text{ measure}\}$$

This space is more then just a simple set, but is equipped with algebraic operations that are related to averaging processes. Every measure $\mu \in \text{Meas}_P([F], \Sigma_{[F]})$ defines its associated expectation field $\Phi \equiv \langle \hat{\Phi} \rangle_\mu$ in a unique way yielding the total set of all possible Φ in $([F], \Sigma_{[F]})$:

$$\langle [F] \rangle := \{\Phi \equiv \langle \hat{\Phi} \rangle_\mu \mid \mu \in \text{Meas}_P([F], \Sigma_{[F]})\}$$

The linearity of integration and of field space assure that every Φ is a scalar multiple of a specific element on $[F]$, i.e. for every μ there exists an element in field space $\hat{\Phi}_\mu$ and a (positive real) number $z \in \mathbb{R}$ such that $\Phi \equiv \langle \hat{\Phi} \rangle_\mu = z \circ_{\mathbb{R}, [F]} \hat{\Phi}_\mu$. Whenever $[F]$ is closed under this scalar multiplication, necessarily it has to be defined over the field $\mathbb{R}$, the expectation fields $\langle [F] \rangle$ occupy a subspace of $[F]$, i.e. $\langle [F] \rangle \subseteq [F]$.

R Notice that in the case of $[F] \subseteq \mathbb{N}$ a probability measure usually produces non-integer expectation values and thus $\langle [F] \rangle \not\subseteq [F]$.

Even though a measure uniquely determines its associated expectation value, we cannot expect that the converse holds true. There may be an infinity of different measures that result in the same Φ , hence injectivity is lost. However, under certain circumstances the full variety of expectation values for field monomials allows to uniquely reconstruct the measure from its observables. The missing link is provided by the Fourier transform of a measure μ that fully classifies the underlying theory. As it stands, it relates two measures by a Radon-Nikodym derivative containing an exponential linear in the fluctuation field, by virtue of a dual continuous vector J . When the integration is performed of the entire field space the measure specific information are then encoded in the resulting functional, its Fourier transform. Expanding the Radon-Nikodym derivative in a power series reveals the knowledge contained in the set of monomial expectation values:

$$\begin{aligned} Z[J] &\equiv \hat{\mu}_{T_{\Phi}[F]}(J; \bar{\Phi}) = \int_{\varphi \in T_{\Phi}[F]} d\mu(\hat{\varphi}) \exp_{\mathbb{C}}(z \cdot J(\hat{\varphi})) = \int_{\varphi \in T_{\Phi}[F]} d\mu(\hat{\varphi}) \sum_{n=0}^{\infty} \frac{z^n}{n!} (J(\hat{\varphi}))^n \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \underbrace{J \otimes \cdots \otimes J}_{n\text{-times}} \int_{\varphi \in T_{\Phi}[F]} d\mu(\hat{\varphi}) \underbrace{\hat{\varphi} \otimes \cdots \otimes \hat{\varphi}}_{n\text{-times}} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \underbrace{J \otimes \cdots \otimes J}_{n\text{-times}} \langle \hat{\varphi} \otimes \cdots \otimes \hat{\varphi} \rangle_{\mu} \end{aligned}$$

The critical point in this demonstration is the interchange of integration and summation which may yield indefinite intermediate results. Nevertheless, it is instructive to note that the Taylor-expansion of the exponential can be used to deduce expectation values of field monomials. In this light, we can extract any n -point function from the generating functional $Z[J]$ by means of taking n -times the derivative with respect to J and then setting $J = e_{T_{\Phi}[F]}$ to get rid of the higher order terms. The basis of this expansion is the Euler-MacLaurin formula yielding

$$\underbrace{\partial_{\delta J} \cdots \partial_{\delta J}}_{\ell\text{-times}} Z[J; \bar{\Phi}] \Big|_{J=e_{T_{\Phi}[F]}} = z^{\ell} \underbrace{\delta J \otimes \cdots \otimes \delta J}_{\ell\text{-times}} \underbrace{\langle \hat{\varphi} \otimes \cdots \otimes \hat{\varphi} \rangle_{\mu}}_{\ell\text{-times}}$$

For the Gaußian measure the interchange of Taylor-expansion and integration turns out to be well behaved due to the particular nice structure of its Fourier transform:

$$\hat{\mu}_G[J; \bar{\Phi}] = \exp_{\mathbb{C}} \left(-\frac{1}{2} \mathfrak{G}^{-1}(J, A_{\bar{\Phi}} J) \right)$$

A self-adjoint, positive $A_{\bar{\Phi}}$ corresponds to a free theory, for which we can directly compute its n -point functions. Therefore, let us start taking the respective number of functional derivatives of the generating functional:

$$\underbrace{\partial_{\delta J} \cdots \partial_{\delta J}}_{\ell\text{-times}} Z[J; \bar{\Phi}] \Big|_{J=e_{T_{\Phi}[F]}} = \begin{cases} \frac{\ell!}{(\ell/2)!} \left(-\frac{1}{2}\right)^{\ell} \mathfrak{G}^{-1}(\delta J, A_{\bar{\Phi}} \delta J)^{\otimes(\ell/2)} & \text{for } \ell \in 2\mathbb{N} \\ 0 & \text{for } \ell \in 2\mathbb{N} + 1 \end{cases}$$

Since $A_{\bar{\Phi}}$ is self-adjoint and for ℓ even, n -point functions are given by

$$\langle \hat{\varphi}^{\ell} \rangle_{\mu_G} = \left(\frac{1}{2z} \right)^{\ell} \frac{\ell!}{(\ell/2)!} \mathfrak{G}^{-1}(\bullet, A_{\bar{\Phi}} \bullet)^{\otimes(\ell/2)} \quad \text{for } \ell \in 2\mathbb{N}$$

As a consequence the propagator, the 2-point function, which is a measure for the relation of two fields, assumes the form: $\langle \hat{\varphi} \otimes \hat{\varphi} \rangle_{\mu_G} = z^{-1} \mathfrak{G}^{-1}(\bullet, A_{\bar{\Phi}} \bullet)$, while the mean field $\langle \hat{\varphi} \rangle_{\mu_G} = e_{T_{\Phi}[F]}$ vanishes.

A more instructive way to present the n -point functions is the associated Schwinger functional. We have seen that even in the free theory there appear relations between the non-interacting fields due to the exponential properties. To extract the connected propagator along with the higher connected n -point functions we choose a representation of $Z[J; \bar{\Phi}]$ that transforms the multiplicative exponential in an additive construction:

Definition 2.2.1 — Schwinger functional. Let $([F], \Sigma_{[F]}, \mu)$ describe a probability theory. Assume $\bar{\Phi}$ defines a suitable background field covering. Then the functional $W[\bullet; \bar{\Phi}] : T_{\bar{\Phi}}^*[F] \rightarrow \mathbb{C}$ is the Schwinger functional of $\mu : \Leftrightarrow$

$$W[J; \bar{\Phi}] \equiv \frac{1}{z} \ln \hat{\mu}(J; \bar{\Phi}) = \frac{1}{z} \ln Z[J; \bar{\Phi}]$$

The connected n -point functions are implicitly defined by

$$\underbrace{\partial_{\delta J} \cdots \partial_{\delta J}}_{\ell\text{-times}} W[J; \bar{\Phi}] \Big|_{J=e_{T_{\bar{\Phi}}^*[F]}} := z^{\ell-1} \underbrace{\delta J \otimes \cdots \otimes \delta J}_{\ell\text{-times}} \underbrace{\langle \hat{\phi} \otimes \cdots \otimes \hat{\phi} \rangle_{\mu}^{\text{con}}}_{\ell\text{-times}}$$

The logarithm acts on a real positive or complex number, depending on the type of Fourier transform and the measure μ . Now, in the transition from the generating to the Schwinger functional we get rid of an enormous number of redundant information and retain only the more elementary building blocks that in their mutual combination will reproduce the original n -point functions. Therefore, this condensed construction is very helpful for systematically understanding the full theory μ . The mechanism of reduction becomes apparent when considering the connected n -point functions and their relation with the original ones. For instance, the connected propagator, likewise defined by the second variation of W and a subsequent evaluation at $J = e_{T_{\bar{\Phi}}^*[F]}$, assumes the form:

$$\begin{aligned} \langle \hat{\phi} \otimes \hat{\phi} \rangle_W^{\text{con}} &= z^{-1} \partial_{\bullet} \partial_{\bullet} W[J; \bar{\Phi}] \Big|_{J=e_{T_{\bar{\Phi}}^*[F]}} = z^{-2} \left(\partial_{\bullet} \partial_{\bullet} Z[J; \bar{\Phi}] - (\partial_{\bullet} Z[J; \bar{\Phi}])^{\otimes 2} \right) \Big|_{J=e_{T_{\bar{\Phi}}^*[F]}} \\ &= \langle \hat{\phi} \otimes \hat{\phi} \rangle_{\mu} - \langle \hat{\phi} \rangle_{\mu} \otimes \langle \hat{\phi} \rangle_{\mu} \end{aligned}$$

Notice, since the logarithm is an isomorphism on $[0, 1] \subset \mathbb{R}$ we are not loosing any information when moving to the Schwinger functional.

The amount of superfluous data in the generating functional is best illustrated on the basis of a free theory for which the Schwinger functional is given by

$$W[J; \bar{\Phi}] = -\frac{1}{2} z^{-1} \mathfrak{G}^{-1}(J, A_{\bar{\Phi}} J)$$

Hence, the infinite number of non-vanishing n -point functions is only due to a single connected expectation value, the propagator while all higher correlation functions are identically zero:

$$\langle \hat{\phi} \otimes \hat{\phi} \rangle_W^{\text{con}} \equiv \langle \hat{\phi} \otimes \hat{\phi} \rangle_{\mu_G} = z^{-2} \mathfrak{G}^{-1}(\bullet, A_{\bar{\Phi}} \bullet)$$

We are now going to proceed to yet another representation of the Fourier transform in which the theory specific information are cast into a more natural form. While there is no further reduction of information, we merely replace the sources J by their associated expectation values.

2.2.2 Effective action

In a first step we have transferred the measure specific information into its characteristic functional, which is the Fourier transform of the measure. Under certain condition, this transition is an isomorphism and thus preserves all information of the underlying theory. It turns out to be convenient to rephrase the Fourier transform into the Schwinger functional that reduces the

amount of data by retaining only the connected building blocks. We are now going to absorb the content of $W[J; \bar{\Phi}]$ into a better accessible object that fits more our needs: the effective action. In this case the space of measures comes to the fore, which turns out to be the basic ingredient for the functional renormalization group. For this purpose we have to introduce a suitable method to transfer the full information of a functional into another one defined over the dual space.

Legendre-Fenchel transform

The transition from the dual vectors J to a functional of expectation fields is given by the Legendre-Fenchel transform.

Definition 2.2.2 — Legendre-Fenchel transform. Let V be a smooth (infinite dimensional) vector space and V^* the associated continuous dual space. A functional $B : V \rightarrow \mathbb{R}$ is the Legendre-Fenchel transform of a functional $A : V^* \rightarrow \mathbb{R} : \Leftrightarrow :$

$$B[b] \equiv \text{LFT}(A)[b] = \sup_{a \in V^*} \{a(b) - A[a]\} \quad \forall b \in V \quad (2.8)$$

Two functionals that are related by a Legendre-Fenchel transform possess a close mutual link which, under a set of conditions, is so strong that one can uniquely reconstruct one from the other. In this case both contain the same information and for them the Legendre-Fenchel transform defines an involution. The class of functionals that fulfill these requirements is suitable to reformulate the Schwinger functional in terms of the effective action and thereby replace the sources, elements $J \in T_{\bar{\Phi}}^*[F]$, by the more physically accessible expectation fields, i.e. fields like $\varphi \in T_{\bar{\Phi}}\langle[F]\rangle$.

Definition 2.2.3 — Effective action. Let $([F], \Sigma_{[F]})$ describe a measurable space and denote $\text{Meas}_{\mathcal{P}}([F], \Sigma_{[F]})$ the space of probability theories. Let $\mu \in \text{Meas}_{\mathcal{P}}([F], \Sigma_{[F]})$ be a probability measure and $W[\bullet; \bar{\Phi}]$ the corresponding Schwinger functional with $\bar{\Phi}$ defining a suitable background field covering. Then, the functional $\Gamma : T_{\bar{\Phi}}\langle[F]\rangle \rightarrow \mathbb{C}$ is the effective action of $W : \Leftrightarrow$

$$\Gamma[\varphi; \bar{\Phi}] \equiv \text{LFT}(W[\bullet; \bar{\Phi}])[\varphi; \bar{\Phi}] \equiv \sup_{J \in T_{\bar{\Phi}}^*[F]} \{J(\varphi) - W[J; \bar{\Phi}]\}$$

The associated n -point functions are called one-particle irreducible correlation functions and are given by

$$\underbrace{\partial_{\delta\varphi} \cdots \partial_{\delta\varphi}}_{\ell\text{-times}} \Gamma[\varphi; \bar{\Phi}] \Big|_{\varphi = e_{T_{\bar{\Phi}}\langle[F]\rangle}} := z^{\ell-1} \underbrace{\delta\varphi \otimes \cdots \otimes \delta\varphi}_{\ell\text{-times}} \underbrace{\langle \hat{\varphi} \otimes \cdots \otimes \hat{\varphi} \rangle_{\mu}^{1\text{PI}}}_{\ell\text{-times}}$$

In what follows, we work out the hidden connection between a functional and its Legendre-Fenchel transform along with the requirements we need for an information preserving transition. This entire formalism is not specific to the effective action approach, but is a general property of the Legendre-Fenchel transform that is here only exemplified for the physical quantities of interest. For convenience we are going to abbreviate $W \equiv W[\bullet; \bar{\Phi}]$ and $\Gamma \equiv \Gamma[\bullet; \bar{\Phi}]$ in the following derivation.

Thus, let us start with a Schwinger functional $W : T_{\bar{\Phi}}^*[F] \rightarrow \mathbb{C}$ and consider its Legendre-Fenchel transform $\text{LFT}(W) \equiv \Gamma : T_{\bar{\Phi}}\langle[F]\rangle \rightarrow \mathbb{C}$. In general the supremum may be not contained in the co-domain of W . However, in case that W is second order differentiable and strictly convex, meaning that its Hessian is positive definite, the supremum condition reduces to the Legendre transform evaluate at a unique global maximum of W . We denote the position of the

supremum $J_m \equiv \vartheta(\varphi) \in T_{\Phi}^*[F]$ to indicate that it depends on the value of φ and write:

$$\begin{aligned} \text{LFT}(W)[\varphi] &= \sup_{J \in T_{\Phi}^*[F]} \{J(\varphi) - W[J]\} = J_m(\varphi) - W[J_m] = (\vartheta(\varphi))(\varphi) - W[\vartheta(\varphi)] \equiv \Gamma[\varphi] \\ \text{with } \partial_w W[\vartheta(\varphi)] &= w(\varphi) \quad \forall w \in T_{\Phi}^*[F] \end{aligned} \quad (2.9)$$

Whenever W is convex, the second variation of $(J(\varphi) - W[J])$ is negative and thus $\vartheta(\varphi)$ describes a maximum. While in general W may not fulfill the strong convexity condition, its Legendre-Fenchel transform is at least convex, as can be seen as follows: First note that $\sup_{x \in X \subseteq Y} \{f(x)\} \leq \sup_{x \in Y} \{f(x)\}$ holds true that further implies

$$\begin{aligned} \sup_{x \in X} \{f(x) + g(x)\} &= \sup_{(x_1, x_2) \in \{(x, x) | x \in X\}} \{f(x_1) + g(x_2)\} \\ &\leq \sup_{(x_1, x_2) \in X \times X} \{f(x_1) + g(x_2)\} = \sup_{x \in X} \{f(x)\} + \sup_{x \in X} \{g(x)\} \end{aligned}$$

Hence, with $\varphi \equiv (1 - \lambda) \circ_{\mathbb{R} \cdot T_{\Phi} \langle [F] \rangle} \varphi_1 +_{T_{\Phi} \langle [F] \rangle} \lambda \circ_{\mathbb{R} \cdot T_{\Phi} \langle [F] \rangle} \varphi_2$ for all $\varphi_1, \varphi_2 \in T_{\Phi} \langle [F] \rangle$ and $\lambda \in [0, 1]$ we can exploit the linearity of J and artificially insert a zero $0 = \lambda W[J] - \lambda W[J]$ to obtain:

$$\begin{aligned} \Gamma[\varphi] &= \sup_{J \in T_{\Phi}^*[F]} \{J(\varphi) - W[J]\} = \sup_{J \in T_{\Phi}^*[F]} \{(1 - \lambda)J(\varphi_1) + \lambda J(\varphi_2) - (1 - \lambda)W[J] - \lambda W[J]\} \\ &= \sup_{J \in T_{\Phi}^*[F]} \{(1 - \lambda)[J(\varphi_1) - W[J]] + \lambda[J(\varphi_1) - W[J]]\} \\ &\leq (1 - \lambda) \sup_{J \in T_{\Phi}^*[F]} \{J(\varphi_1) - W[J]\} + \lambda \sup_{J \in T_{\Phi}^*[F]} \{J(\varphi_2) - W[J]\} = (1 - \lambda)\Gamma[\varphi_1] + \lambda\Gamma[\varphi_2] \end{aligned}$$

Thus $\Gamma[(1 - \lambda) \circ_{\mathbb{R} \cdot T_{\Phi} \langle [F] \rangle} \varphi_1 +_{T_{\Phi} \langle [F] \rangle} \lambda \circ_{\mathbb{R} \cdot T_{\Phi} \langle [F] \rangle} \varphi_2] \leq (1 - \lambda)\Gamma[\varphi_1] + \lambda\Gamma[\varphi_2]$ proves the convexity of the effective action, or the Legendre-Fenchel transform in general.

Under the above assumptions of strict convexity and differentiability of the Schwinger functional, $\text{LFT}(W)$ has a very important property that is crucial in successively relating W with the effective action Γ . If we perform a second Legendre transformation, now on Γ , the solution to the associated supremum condition again depends on the evaluation value, i.e. $\varphi_m = \gamma(J) \in T_{\Phi}[F]$:

$$\begin{aligned} \text{LFT}(\Gamma)[J] &= \sup_{\varphi \in T_{\Phi}[F]} \{J(\varphi) - \Gamma[\varphi]\} = J(\varphi_m) - \Gamma[\varphi_m] = J(\gamma(J)) - \Gamma[\gamma(J)] \\ \text{with } \partial_v \Gamma[\gamma(J)] &= J(v) \quad \forall v \in T_{\Phi} \langle [F] \rangle \end{aligned} \quad (2.10)$$

The constraint that results from the supremum condition for the Legendre transform is exactly the effective field equations stating that the maximum of Γ is positioned at the source, i.e. $\partial_v \Gamma[\gamma(J)] = J(v)$ for all $v \in T_{\Phi} \langle [F] \rangle$ and $J \in T_{\Phi}^*[F]$. Next, we replace Γ by its explicit form given in eq. (2.9) that results in

$$\begin{aligned} \text{LFT}(\Gamma)[J] &= J(\gamma(J)) - ((\vartheta(\varphi))(\varphi) - W[\vartheta(\varphi)])_{\varphi=\gamma(J)} \\ &= J(\gamma(J)) - (\vartheta(\gamma(J))) (\gamma(J)) + W[\vartheta(\gamma(J))] \\ \text{with } \partial_v \Gamma[\gamma(J)] &= J(v) \wedge \partial_w W[\vartheta(\varphi)] = w(\varphi) \quad \forall v \in T_{\Phi} \langle [F] \rangle, w \in T_{\Phi}^*[F] \end{aligned}$$

The important observation is that ϑ and γ are not independent but in fact inverse operations of each other. This can be seen by evaluating the defining condition of $\gamma(J)$ using both parts of eq. (2.9), linearity of dual vectors, and the chain rule.

$$\begin{aligned} J(v) &= \partial_v \Gamma[\gamma(J)] = \partial_v ((\vartheta(\varphi))(\varphi) - W[\vartheta(\varphi)])_{\varphi=\gamma(J)} \\ &= \left((\partial_v \vartheta(\varphi))(\varphi) + (\vartheta(\varphi))(v) - \partial_w W[\vartheta(\varphi)]|_{w=\partial_v \vartheta(\varphi)} \right)_{b=\gamma(J)} \\ &= ((\partial_v \vartheta(\varphi))(\varphi) + (\vartheta(\varphi))(v) - (\partial_v \vartheta(\varphi))(\varphi))_{\varphi=\gamma(J)} = (\vartheta(\gamma(J)))(v) \end{aligned}$$

In the second last step we simplified the equality using $\partial_w W[\vartheta(\varphi)] = w(\varphi)$ for $w = \partial_v \vartheta(\varphi)$. Since $J(v) = (\vartheta(\gamma(J))) (v)$ holds for all $v \in T_{\bar{\Phi}} \langle [F] \rangle$ we conclude $J = (\vartheta \circ \gamma)(J)$ which confirms the inverse behavior of ϑ and γ considered as functions $T_{\bar{\Phi}} \langle [F] \rangle \rightarrow T_{\bar{\Phi}}^*[F]$ and $T_{\bar{\Phi}}^*[F] \rightarrow T_{\bar{\Phi}} \langle [F] \rangle$, respectively. Along the same lines one establishes the reversed relation, so that we obtain:

$$\vartheta \circ \gamma = \text{id}_{T_{\bar{\Phi}}^*[F]} \quad \wedge \quad \gamma \circ \vartheta = \text{id}_{T_{\bar{\Phi}} \langle [F] \rangle} \quad (2.11)$$

On a deeper level this connection implies a similar relation between the associated Hessian operators of a functional W and its Legendre-Fenchel transform Γ , in case they exist. To reveal this missing link we go back to eq. (2.11) and determine its Hessian with the aid of $\partial_v \Gamma[\varphi] = (\vartheta(\varphi)) (v)$:

$$\begin{aligned} \text{Hess}_{\gamma(J)} [\Gamma] (v_1, v_2) &= \frac{1}{2} [\partial_{v_2} \partial_{v_1} \Gamma[\varphi] + \partial_{v_1} \partial_{v_2} \Gamma[\varphi]]_{\varphi=\gamma(J)} \\ &= \frac{1}{2} [(\partial_{v_2} \vartheta(\varphi)) (v_1) + \partial_{v_1} (\vartheta(\varphi)) (v_2)]_{\varphi=\gamma(J)} \\ &= \frac{1}{2} [\mathfrak{G}(\mathfrak{G}^{-1}(\bullet, \partial_{v_2} \vartheta(\varphi)), v_1) + \mathfrak{G}(\mathfrak{G}^{-1}(\bullet, \partial_{v_1} \vartheta(\varphi)), v_2)]_{\varphi=\gamma(J)} \end{aligned} \quad (2.12)$$

At this stage we have to assume that both W and Γ give rise to a well behaved Hessian operator in order to proceed. It suffices to assure that the operator $v \mapsto \mathfrak{G}^{-1}(\bullet, \partial_v \vartheta(\varphi))$ is self-adjoint to obtain:

$$\begin{aligned} \text{Hess}_{\gamma(J)} [\Gamma] (v_1, v_2) &= \mathfrak{G}(\mathfrak{G}^{-1}(\bullet, \partial_{v_2} \vartheta(\varphi)), v_1) \big|_{\varphi=\gamma(J)} \\ &= \mathfrak{G}(\mathfrak{G}^{-1}(\bullet, \partial_{v_1} \vartheta(\varphi)), v_2) \big|_{\varphi=\gamma(J)} \end{aligned}$$

If this is the case, the Hessian operator associated to Γ is simply given by

$$\widehat{\text{Hess}}_{\gamma(J)} [\Gamma] v = \mathfrak{G}^{-1}(\bullet, \partial_v \vartheta(\varphi)) \big|_{\varphi=\gamma(J)} \quad (\widehat{\text{Hess}}_{\gamma(J)} [\Gamma] : T_{\bar{\Phi}} \langle [F] \rangle \rightarrow T_{\bar{\Phi}} \langle [F] \rangle) \quad (2.13)$$

Along similar lines and with an equivalent assumption on the Hessian of W we obtain:

$$\widehat{\text{Hess}}_{\vartheta(\varphi)} [W] w = \mathfrak{G}(\bullet, \partial_w \gamma(J)) \big|_{J=\vartheta(\varphi)} \quad (\widehat{\text{Hess}}_{\vartheta(\varphi)} [W] : T_{\bar{\Phi}}^*[F] \rightarrow T_{\bar{\Phi}}^*[F]) \quad (2.14)$$

Both equations (2.13) and (2.14) can be reverted due to the non-degeneracy of the field space metric, yielding

$$\partial_v \vartheta(\varphi) \big|_{J=\gamma(J)} = \mathfrak{G}(\bullet, \widehat{\text{Hess}}_{\gamma(J)} [\Gamma] v) \quad (2.15a)$$

$$\partial_w \gamma(J) \big|_{\varphi=\vartheta(\varphi)} = \mathfrak{G}^{-1}(\bullet, \widehat{\text{Hess}}_{\vartheta(\varphi)} [W] w) \quad (2.15b)$$

In the final step we go back to eq. (2.11), take its functional derivative, and insert the appropriate equation of (2.15):

$$\begin{aligned} w = \partial_w J \big|_{J=\vartheta(\varphi)} &= \partial_w \vartheta(\gamma(J)) \big|_{J=\vartheta(\varphi)} = \partial_{\partial_w \gamma(J)} \vartheta(\varphi) \big|_{\varphi=\gamma(J); J=\vartheta(\varphi)} \\ &= \mathfrak{G}(\bullet, \widehat{\text{Hess}}_{\gamma(J)} [\Gamma] \partial_w \gamma(J) \big|_{J=\vartheta(\varphi)}) \\ &= \mathfrak{G}(\bullet, \widehat{\text{Hess}}_{\gamma(J)} [\Gamma] \mathfrak{G}^{-1}(\bullet, \widehat{\text{Hess}}_{\vartheta(\varphi)} [W] w)) \end{aligned} \quad (2.16)$$

Similarly, the second equality of eq. (2.15) produces the inverted result:

$$v = \partial_v \varphi \big|_{\varphi=\gamma(J)} = \partial_v \vartheta(\varphi) \big|_{\varphi=\gamma(J)} = \partial_{\partial_v \vartheta(\varphi)} \gamma(J) \big|_{J=\vartheta(\varphi); \varphi=\gamma(J)} \quad (2.17)$$

$$\begin{aligned} &= \mathfrak{G}^{-1}(\bullet, \widehat{\text{Hess}}_{\vartheta(\varphi)} [W] \partial_v \vartheta(\varphi) \big|_{\varphi=\gamma(J)}) \\ &= \mathfrak{G}^{-1}(\bullet, \widehat{\text{Hess}}_{\vartheta(\varphi)} [W] \mathfrak{G}(\bullet, \widehat{\text{Hess}}_{\gamma(J)} [\Gamma] v)) \end{aligned} \quad (2.18)$$

These are exactly the identities we were looking for, namely relations connecting the Hessian operators of a functional and its associated Legendre-Fenchel transform. Except for some factors of the field space metric both are their mutual inverse operators, i.e.

$$\widehat{\text{Hess}}_b[\Gamma] = \mathfrak{G}^{-1}(\bullet, (\widehat{\text{Hess}}_{\vartheta(\varphi)}[W])^{-1} \mathfrak{G}) \quad (2.19a)$$

$$\widehat{\text{Hess}}_a[W] = \mathfrak{G}(\bullet, (\widehat{\text{Hess}}_{\gamma(J)}[\Gamma])^{-1} \mathfrak{G}^{-1}) \quad (2.19b)$$

The Legendre-Fenchel transform thus not only establish the bridge between the Schwinger functional and the effective action, but also provides the link between the connected and the irreducible propagators. In case of a strictly convex and differentiable $W[J; \bar{\Phi}]$ the correspondence is in fact an isomorphism and thus all information of the Schwinger functional is also contained in $\Gamma[\varphi; \bar{\Phi}]$. If both allow for a self-adjoint Hessian operator than the property of Legendre transforms ensures that their propagators are indeed the inverse of each other which has further implications on the relation of higher n -point functions.

The advantage of working with the effective formalism instead of using connected correlation functions becomes apparent when systematically approaching the underlying *theory* or when using non-perturbative approximations. While the Schwinger functional is concerned with field space itself, the effective action depends on the expectation values and may be interpreted as a functional of theories for a specific field space. Varying the expectation value for fixed sources corresponds to altering the measure and thereby the theory itself. It turns out to be of great benefit to work in the effective picture whenever the theory rather than the field space is the objects of interest. This reflects the distinction between $[F]$ and $\langle[F]\rangle$ meaning that perturbation of fluctuations are something fundamentally different than perturbations of expectation values. This has to be kept in mind when introducing approximations in one or the other description.

2.2.3 Free theory

At this point it is very instructive to exemplify the described techniques for a free theory. By definition each non-interacting theory is classified by a linear, positive, self-adjoint operator $A_{\bar{\Phi}} = A_{\bar{\Phi}}^*$ over field space, for which the generating functional can be cast into the following form:

$$Z[J; \bar{\Phi}] = \exp\left(-\frac{1}{2}\mathfrak{G}^{-1}(J, A_{\bar{\Phi}}J)\right) \implies W[J; \bar{\Phi}] = \frac{1}{z} \ln Z[J; \bar{\Phi}] = -\frac{1}{2z}\mathfrak{G}^{-1}(J, A_{\bar{\Phi}}J)$$

Notice that the Schwinger functional is bilinear in the sources and symmetric due to the self-adjoint property of $A_{\bar{\Phi}}$. As stated above, the effective theory is defined by the Legendre-Fenchel transformation of the Schwinger functional and explicitly given by

$$\Gamma[\varphi; \bar{\Phi}] = \sup_{J \in T_{\bar{\Phi}}^*[F]} \{J(\varphi) - W[J; \bar{\Phi}]\} = \sup_{J \in T_{\bar{\Phi}}^*[F]} \{J(\varphi) + \frac{1}{2}\mathfrak{G}^{-1}(J, A_{\bar{\Phi}}J)\}$$

Due to the positive definiteness of $A_{\bar{\Phi}}$ the Schwinger functional $W[J; \bar{\Phi}]$ is strictly convex and surely differentiable, so we can evaluate the supremum by determine its global maximum:

$$0 = \partial_v(\vartheta(\varphi) - W[\vartheta(\varphi); \bar{\Phi}]) = \partial_v \vartheta(\varphi) - \partial_v W[\vartheta(\varphi); \bar{\Phi}]$$

The maximum function ϑ maps to a dual vector which is linear in the expectation fields and thus its functional derivative is simply given by

$$\partial_v J(\varphi) = v(\varphi) = \mathfrak{G}^{-1}(v, \mathfrak{G}(\bullet, \varphi))$$

In the last step used the isomorphism provided by the field space metric to artificially rewrite the real number $v(\varphi)$. The second contribution originates from the functional variation of W , i.e.

$$\begin{aligned}\partial_v W[J; \bar{\Phi}] &= -\frac{1}{2z} \mathfrak{G}^{-1}(v, A_{\bar{\Phi}} J) - \frac{1}{2z} \mathfrak{G}^{-1}(J, A_{\bar{\Phi}} v) = -\frac{1}{2z} \mathfrak{G}^{-1}(v, (A_{\bar{\Phi}} + {}_{op} A_{\bar{\Phi}}^*) J) \\ &= -z^{-1} \mathfrak{G}^{-1}(v, A_{\bar{\Phi}} J)\end{aligned}$$

Here we used the fact that $A_{\bar{\Phi}}$ is self-adjoint and the field space metric symmetric in its arguments. Combining both results to deduce the maximum yields

$$\begin{aligned}0 &= v(\varphi) + z^{-1} \mathfrak{G}^{-1}(v, A_{\bar{\Phi}} \vartheta(\varphi)) = \mathfrak{G}^{-1}(v, \mathfrak{G}(\bullet, \varphi)) + z^{-1} \mathfrak{G}^{-1}(v, A_{\bar{\Phi}} \vartheta(\varphi)) \\ &= \mathfrak{G}^{-1}(v, \mathfrak{G}(\bullet, \varphi) + z^{-1} A_{\bar{\Phi}} \vartheta(\varphi))\end{aligned}$$

The field space metric is a non-degenerated symmetric bilinear form and the above condition holds for all $v \in T_{\bar{\Phi}} \langle [F] \rangle$. Hence the second argument has to be the identity $0 \equiv e_{T_{\bar{\Phi}} \langle [F] \rangle}$ which further implies

$$\vartheta(\varphi) = -z A_{\bar{\Phi}}^{-1} \mathfrak{G}(\bullet, \varphi) \iff \varphi = -z^{-1} \mathfrak{G}^{-1}(\bullet, A_{\bar{\Phi}} \vartheta(\varphi)) \quad (2.20)$$

This reveals a special feature for a free theory, namely the linear relation between the maximum field ϑ and the mean fluctuations φ . While $\vartheta(\varphi)(v)$ is a dual vector and thus linear in v , the explicit dependence on φ is in general not linear. Next, we evaluate the supremum condition for $\Gamma[\bullet; \bar{\Phi}]$ using the relation between ϑ and φ :

$$\begin{aligned}\Gamma[\varphi; \bar{\Phi}] &= J(\varphi)|_{J=\vartheta(\varphi)} + \frac{1}{2z} \mathfrak{G}^{-1}(\vartheta(\varphi), A_{\bar{\Phi}} \vartheta(\varphi)) \\ &= \mathfrak{G}^{-1}(\vartheta(\varphi), \mathfrak{G}(\bullet, \varphi)) + \frac{1}{2z} \mathfrak{G}^{-1}(\vartheta(\varphi), A_{\bar{\Phi}} \vartheta(\varphi)) \\ &= -\frac{1}{z} \mathfrak{G}^{-1}(\vartheta(\varphi), A_{\bar{\Phi}} \vartheta(\varphi)) + \frac{1}{2z} \mathfrak{G}^{-1}(\vartheta(\varphi), A_{\bar{\Phi}} \vartheta(\varphi)) \\ &= -\frac{1}{2z} \mathfrak{G}^{-1}(\vartheta(\varphi), A_{\bar{\Phi}} \vartheta(\varphi))\end{aligned}$$

This results in a bilinear expression in ϑ which is by eq. (2.20) also linear in φ . We conversion to the bilinear form of $\Gamma[\bullet; \bar{\Phi}]$ in φ is found to be

$$\begin{aligned}\Gamma[\varphi; \bar{\Phi}] &= -\frac{1}{2z} \mathfrak{G}^{-1}(\vartheta(\varphi), A_{\bar{\Phi}} \vartheta(\varphi)) \\ &= -\frac{z}{2} \mathfrak{G}^{-1}(A_{\bar{\Phi}}^{-1} \mathfrak{G}(\bullet, \varphi), A_{\bar{\Phi}}^{-1} \mathfrak{G}(\bullet, \varphi)) \\ &= -\frac{z}{2} \mathfrak{G}^{-1}(\mathfrak{G}(\bullet, \varphi), A_{\bar{\Phi}}^{-1} \mathfrak{G}(\bullet, \varphi)) \\ &= -\frac{z}{2} \mathfrak{G}(\varphi, \mathfrak{G}^{-1}(\bullet, A_{\bar{\Phi}}^{-1} \mathfrak{G}) \varphi)\end{aligned} \quad (2.21)$$

This confirms the general inversion relation between the Hessian operator of $W[\bullet; \bar{\Phi}]$ and the respective one of $\Gamma[\bullet; \bar{\Phi}]$ found in eq. (2.19), here explicitly given by

$$\widehat{\text{Hess}}_J[W[\bullet; \bar{\Phi}]] = -z^{-1} A_{\bar{\Phi}} = \mathfrak{G}(\bullet, \left(\widehat{\text{Hess}}_{\bar{\Phi}}[\Gamma[\bullet; \bar{\Phi}]] \right)^{-1} \mathfrak{G}^{-1})$$

For a free theory, the bilinearity of the Schwinger functional implies:

$$\Gamma[\varphi; \bar{\Phi}] = \frac{1}{2} \mathfrak{G}(\varphi, \widehat{\text{Hess}}_{\bar{\Phi}}[\Gamma[\bullet; \bar{\Phi}]] \varphi) \quad \text{with} \quad \widehat{\text{Hess}}_{\bar{\Phi}}[\Gamma[\bullet; \bar{\Phi}]] \equiv -z \mathfrak{G}^{-1}(\bullet, A_{\bar{\Phi}}^{-1} \mathfrak{G})$$

Depending on the form of the field space metric this Hessian operator is diagonal whenever $A_{\bar{\Phi}}$ is diagonal. When $\mathfrak{G}$ contains non-diagonal elements it is still non-degenerated and a change of basis in $T_{\bar{\Phi}} \langle [F] \rangle$ will yield a diagonal $\widehat{\text{Hess}}_{\bar{\Phi}}[\Gamma[\bullet; \bar{\Phi}]]$. Thus, for any free theory, the one-particle and connected propagators are diagonalizable and in fact the only non-vanishing connected n -point functions.

We are now going to leave the more technical side of field theory and solidify the physical part of quantum field theory. We will see how the mathematical considerations yield to a straightforward description of probabilistic theories and how it provides a deep understanding of the physical concepts.

2.3 Quantum field theories

The beginning of the 20th century witnessed the birth of a new era in physics that shook science to its very foundations. Hundred years later, our description and understanding of Nature has undergone a severe change in which generations of physicists and mathematicians have re-built a seemingly solid basement for future investigations. This transformation that took place at the turn of the century culminated in works of Planck and Einstein and from then on resulted in deep insights of the nature of particles and fields that constitute our Universe, culminating in the formulation of the Standard Model of particle physics in the late 60' s. One can say that this time span of natural science was the golden age of particle physics where we gained remarkable precisions between experiments and theory on these energy scales. Furthermore, the inconsistencies and problems that arise in the Standard Model of particle physics can be mainly consolidated to a single interaction, gravitation, and thus some of them disappear whenever gravitational effects are negligible. Nevertheless, the great success of recent decades provides only a kind of affirmation that there is some hidden truth in what we have derived. The incorporation of gravitational effects is surely one of the most exciting questions future generations of scientists have to answer. Fortunately, the scales at which gravity becomes relevant or dominates the structure of the Universe are two-sided: On one hand quantum effects of the gravitational force are assumed to strongly participate in the fusion of interactions at the Planck scale, thus at very high energies. On the other hand we observe its importance on cosmological scales where all other interactions have smeared out and what is left is a (effective) gravitational theory well described by General Relativity. This dual importance in describing infrared and ultraviolet phenomena may help us to deduce a fundamental theoretical description of gravity by means of cosmological data and it seems that with the advent of the new millennium the era of cosmology revives in order to understand the origin of the small gap that remains in the (possibly) dense blanket of our today's knowledge. As usually, smallness is only relative and time will tell what is hidden on the next, deeper level.

We have prepared the mathematical grounds to present the framework of quantum field theory in the light of symmetry principles and measure theory. The missing mathematical rigor in some parts of this presentation is partially due to focus on the essential concepts relevant for physics and partially because some of the interesting constructions that appear in physics are yet to be put on solid mathematical grounds. Therefore, the following discussion has to be understood on a formal level. In this section we will continue to translate measure theoretical concepts into the language of quantum field theory, state its Euclidean axioms, perform symmetry analysis, and present some perturbative techniques to evaluate observables within this framework. Finally, we conclude with a short summary of the Standard Model of particle physics before introducing the main, non-perturbative tool for later calculations: the Functional Renormalization Group Equation. For references we refer to [24, 25, 28].

2.3.1 Quantum field theories

Quantum field theory is an attempt to reconcile the theory of special relativity with quantum mechanics, which is indicated by its incorporation of the physical constants c and $\hbar$, closely related to the two very successful theoretical descriptions. Within the formulation of gauge theory, QFT had its greatest success so far in the shape of the SM that has not only recovered the classical observations in its low energy effective limit, but has also made remarkably precise predictions that led to new discoveries and a deep understanding of Nature. Before its formulation in terms of gauge theory, there had been several attempts to extend the classical world to be emerging from the laws of quantum physics. In this process it is very difficult for scientists to not go astray due to the large amount of possibilities of how to extend a certain mathematical formalism. There are several distinct routes one can pursue, which all lead to the same classical

limit but are fundamentally different beyond this classical regime.⁶ In order to deduce the ‘right’ mechanism or strategy one has to distinguish between the various approaches by verifying or falsifying their predictions using experiments, or by revealing flaws in the mathematical description. Nowadays, we meet the same problem at the heart of constructing a theory of quantum gravity, however with the difference that so far there is no real experimental guiding line that helps in singling out certain theories. Overcoming the mathematical issues that appear when applying the standard methods of QFT to gravity turns out to be a severe challenge, and all approaches that either suggest cures on the level of the theory itself or extend the general framework still struggle with their own flaws emerging due to their particular generalization ideas. Thus, in comparison to the strong- and electroweak forces there is so far no fully satisfactory program that describes a quantum version of the classical gravitational interaction. The usual procedure that provides a systematic way to construct a quantum field theory from a classical Hamiltonian or Lagrangian formalism is denoted *quantization*. Though it is surely not the way Nature works, i.e. generating the more fundamental laws from effective ones, it at least ensures that all experiments of the ‘classical world’ are also described by its quantum theory. This is one of the pillars a solid theory is based on, the verification and prediction of observations. The other one is its mathematical consistency, which is a fundamental necessity as long as one assumes Nature to be written in this language.

From the various, under certain assumptions equivalent, description of quantum theories, in the sequel we are following the route of the generating functional approach to QFT. We basically differentiate between two approaches: the ‘deductive’ and ‘inductive’ strategy. Both invoke the same mathematical machinery, namely measure theory, and in addition both rely on some spacetime manifold over which a suitable field space in accordance with a set of symmetries is constructed. What differs is the perspective – or direction – from where the methods to deduce results are applied. In the ‘deductive’ approach one sets up a theory, a measure space $([F], \Sigma_{[F]}, \mu)$, over some fundamental fields right at the beginning and walks through all the prescribed steps to obtain the associated observables as expectation values, that can be measured in certain, especially adapted experiments. The difficulty resides in the fact that usually there is a large class of theories which fulfill the mathematical and physically motivated restrictions. Thus, finding the physically realized measure out of all possible ones is a laborious task involving a large number of calculations whereby observables from the theory have to be compared to real experiments. The more mathematical constraints are available the less trial and error is needed.

On the contrary, in the ‘inductive’ approach one assumes a familiarity with the observable data and starts out with the expectation fields $\Phi \in \langle [F] \rangle$ from which one tries to infer the underlying theory by reverting each step towards a finally admissible measure space $([F], \Sigma_{[F]}, \mu)$. This second approach stays closer to the experimental side and has to assume much less at the very beginning, at least if it is based on real physical data. Its difficulty however is the process of guessing the right measure by its produced results. This is a quite complex task since different measure spaces may produce the same expectation values for a set of observables. Hence, only the full knowledge of all experimental outcomes may reveal the right, realized theory consisting of field space $[F]$ and a measure μ .

While the two strategies start from opposite directions and have in principle different partial objections, in practice there is not a clear separation. Instead one employs deduction and induction techniques alternately to ultimately find the fundamental theory underlying Nature. This synthesis is found in the renormalization procedure in perturbative methods, for instance. Before we continue to discuss the mixed forms of both strategies suitable for practical applications,

⁶This already happens on a classical level, where different theories describe the same so far observed phenomena successfully, for instance in the theory of gravity.

we present their uncombined versions first.

The ‘deductive’ approach

Starting from a theory and deducing the experimental observables as predictions is surely the ultimate dream of physicists. Therefore, we need at least some mathematical constraints that guide us on this difficult journey. Nevertheless, in the end we have to utilize experiments to uniquely distinguish between the different mathematical constructions. In our setting based on measure theoretical concepts, the ‘deductive’ path proceeds along the following lines:

- A. Spacetime** Define a set of admissible spacetimes as smooth manifolds over some topology, possibly with additional constraints (e.g. compactness,...):

$$\text{spacetimes} \equiv \{[M]_{\sim_{\text{Diff}(M)}} \mid M \text{ smooth manifold} + \text{constraints}\}$$

In this step one has to choose a *cardinality* of spacetime, a *topological class*, and a *differential structure* to uniquely specify the associated equivalence class of smooth manifolds.

- B. Symmetries** Select a symmetry, a (Lie) group $G \equiv (X_G, \circ_G)$, that should be preserved by the theory. It may consist of several basic groups that are combined by means of a (semi-)direct product. For a choice of spacetime, construct the class of principal fiber bundles:

$$\text{PBs}(M, G) \equiv \{[P = M^{\pi_P} \times_{\rho_P^R} G]_{\sim_{\text{FB}}}\}$$

Thus, in this second step we choose the symmetry group G and the underlying differential structure by means of an equivalence class of $\text{PBs}(M, G)$.

- C. Field space** For each class of principal bundles we can construct the set of compatible principal connections that constitute the gauge fields:

$$[\mathcal{G}] \equiv \{\{f_{\mathcal{G}}^* \omega \mid f_{\mathcal{G}} \in \mathcal{G}\} \mid \omega : TP \rightarrow \mathfrak{g} \text{ principal connection}\}$$

Here and in what follows $\mathcal{G}$ represents the group of gauge transformations.

Additionally, define a set of representations ρ_j of G on some vector spaces V_j with $V \equiv V_1 \times V_2 \times \dots$ to obtain the matter sector:

$$[F_M] \equiv \{\{f_V \circ \phi \circ f_{\mathcal{G}} \mid f_{\mathcal{G}} \in \mathcal{G} \wedge f_V \in \text{Iso}(V)\} \mid \phi : P \rightarrow V \\ \text{with } \phi(\rho_P^R(\bullet, g)) = \rho(g^{-1}, \phi(\bullet))\}$$

Next, we have to transform the space $[F]$, being defined as a suitable combination of $[\mathcal{G}]$ and $[F_M]$, into a Fréchet manifold F sliced by the respective diffeomorphism group, which is usually equipped with a field space metric $\mathfrak{G}$. Therefore, it is convenient to employ the background field covering to express F in terms of its tangent spaces. In this setting, we can apply geometric methods to field space, in particular we can establish a principal bundle construction based on all mathematical fields with the group of gauge transformation as its structure group. In the non-trivial case, which appears in theories of QG the corresponding principal connections and smooth structures are also non-trivial. Hence, in this step we choose the field content of our theory by fixing a connection and determine a number of representations.

- D. Measure space** We use the topology of $[F]$ to define the associated Borel σ -algebra $\Sigma_{[F]}$. The class of theories is then given by the set of all measures on $([F], \Sigma_{[F]})$:

$$\text{Meas}([F], \Sigma_{[F]}) \equiv \{([F], \Sigma_{[F]}, \mu) \mid \mu \text{ measure on } ([F], \Sigma_{[F]})\}$$

Usually it is appropriate to restrict to measures that are Radon-Nikodym transforms of some Gaussian measure μ_G , i.e.

$$\{([F], \Sigma_{[F]}, \mu) \in \text{Meas}([F], \Sigma_{[F]}) \mid \mu(U \in \Sigma_{[F]}) \equiv \int_{x \in U} \mu_G(u) \rho(u)\}$$

Then, normalizing the Gaussian measures such that $\mu_G([F]) \equiv 1$ holds true determines the notion of a free theory, see [62]. Fixing the measure corresponds to a choice of theory.

In case of gauge theories, we usually replace the physical field space by the full mathematical space F using a gauge fixing independent description, for instance by requiring BRST invariance of the measure, see subsection 2.1.5 for details.

E. Observables To extract observables from a theory it is convenient to introduce the Fourier transform of the measure and therefore express field space as a patchwork of its tangent spaces by means of a background field covering $\mathcal{U}_{\text{BFC}} \equiv \{(\bar{\Phi}_j, U_j)\}$ and a partition of unity $\{(U_j, \varsigma_j)\}$.

$$Z[J; \mathcal{U}_{\text{BFC}}] = \mu([F])^{-1} \cdot \sum_{(\bar{\Phi}_j, U_j) \in \mathcal{U}_{\text{BFC}}} \int_{\hat{\phi} \in U_j} d\mu(\exp_{\bar{\Phi}_j}(\hat{\phi})) \varsigma_j(\hat{\phi}) \exp_{\mathbb{C}}(zJ(\hat{\phi}))$$

To be consistent, observables should not depend on the choice of $\mathcal{U}_{\text{BFC}}$. In what follows, we assume that $[F]$ can be covered by a single tangent space at $\bar{\Phi}$, say, and thus we obtain a simplified expression for the generating functional:

$$Z[J; \bar{\Phi}] = \mu([F])^{-1} \cdot \int_{\hat{\phi} \in T_{\bar{\Phi}}[F]} d\mu(\exp_{\bar{\Phi}}(\hat{\phi})) \exp_{\mathbb{C}}(zJ(\hat{\phi})) \quad (2.22)$$

Notice that we have formally normalized the theory with the factor $\mu([F])^{-1}$ in order to describe a probability setting, i.e. $Z[0; \bar{\Phi}] = 1$ and thus a vanishing Schwinger functional for source-free constellations, $W[0; \bar{\Phi}] = \ln(Z[0; \bar{\Phi}]) = 0$. If we proceed to the effective action, $\Gamma[\varphi; \bar{\Phi}] = \sup_J \{J(\varphi) - W[J; \bar{\Phi}]\}$, we have to ensure that the Schwinger functional is strictly convex to retain all information during the transition. Notice, that the effective action is a functional of the expectation fields φ rather than the quantum analogs $\hat{\phi}$, which in principle can live in different spaces.⁷

One then selects a set of observables, represented by some functional on F , and evaluates them using functional variation with respect to the sources. This establishes the link to the experimental side of physics where predictions derived from the underlying theory have to be tested. Hence, the final step consists in testing a number of experimental observables against theoretical predictions:

$$\text{Obs} \equiv \{O : [F] \rightarrow \mathbb{R} \mid O \text{ measurable} \}$$

Likewise, one can work on the level of $\Gamma[\varphi; \bar{\Phi}]$ directly and relate its vertices to observables.

We have seen that at several points we have to make a decision where physical input is needed to distinguish between the mathematical concepts. Thus, this ‘deductive’ method is already far from being purely deductive.

⁷In particular, this appears for probability theories for which F is not closed under scalar multiplication of $\mathbb{R}$. For instance, the expected number one rolls with a fair dice is $\Phi \equiv \langle \hat{\Phi} \rangle_{\text{dice}} = 7/2$ which is a fraction rather than an integer.

The ‘inductive’ approach

The somewhat opposite direction is the ‘inductive’ method, whereby initially one observes Nature and then deduces the mathematical structure hidden behind the curtain of phenomena. Even though some steps seem to be equivalent to the ‘deductive’ approach there is a subtle difference as we will see.

- A. Spacetime** In fact, the first step actually coincides with the ‘deductive’ one. We define the set of admissible spacetimes, being smooth manifolds over some topology that may fulfill additional constraints:

$$\text{spacetimes} \equiv \{[M]_{\sim_{\text{Diff}(M)}} \mid M \text{ smooth manifold} + \text{constraints}\}$$

Again one decides about the *cardinality* of spacetime, the *topological class*, and the *differential structure* when selecting an equivalence class of smooth manifolds.

- B. Symmetries** The order in which we present the ingredients of the theory in both cases, in the ‘deductive’ or ‘inductive’ approach, already indicates the central role of symmetries in determining the laws of Nature. There is however a difference in the interpretation of the postulated symmetries. While the former describes invariance of the fundamental theory, the symmetries we introduce in the present context concern observables and appear on an effective level. The two symmetry concepts may not agree and symmetries we deduce from observing Nature may differ from those actually realized. On the one hand the averaging procedure may smear out certain impurities thus giving rise to new symmetries on macroscopic scales. On the other hand, the fundamental theory can exhibit symmetries only for some relevant subclasses of fields, for instance the classical sector, while on other measurable sets they disappear. The overall sum, the integral, can result in observables that cloak the fundamental invariance and the so-called anomalies arise. Usually, we expect symmetries to survive the construction of expectation values and assume that the fundamental symmetries are also indirectly visible in the effective action. Given a choice of spacetime and the symmetry $G_{\langle[F]\rangle}$, we construct the class of principal fiber bundles:

$$\text{PBs}(M, G_{\langle[F]\rangle}) \equiv \{[P = M^{\pi_P} \times_{\rho_P^R} G_{\langle[F]\rangle}]_{\sim_{\text{FB}}}\}$$

An element of $\text{PBs}(M, G_{\langle[F]\rangle})$ defines the differential structure. In this second step we have to select the symmetry group $G_{\langle[F]\rangle}$ of expectation fields.

- C. Effective field space** Once a specific class of (observable) symmetries $G_{\langle[F]\rangle} \equiv (X_G, \circ_G)$ is fixed the field space is recovered in the usual way by fiber bundle techniques. We start by defining the set of all principal connections compatible with a choice of $P \in \text{PBs}(M, G_{\langle[F]\rangle})$.

$$[\langle \mathcal{G} \rangle] \equiv \{ \{ f_{\mathcal{G}}^* \omega \mid f_{\mathcal{G}} \in \mathcal{G} \} \mid \omega : TP \rightarrow \mathfrak{g} \text{ principal connection} \}$$

Like their quantum analogs, they can be understood as gauge fields that mediate the interactions, however here they have a different interpretation as expectation fields. Again, the physical field content is obtained by an equivalence class of principal connections generated by the group of gauge transformations, now with respect to $G_{\langle[F]\rangle}$ rather than G .

Similarly, the space of matter fields on the level of expectation values, is defined in an analogous way to its quantum counterpart. We first of all choose a set of representations ρ_j of $G_{\langle[F]\rangle}$ on some vector spaces V_j with $V \equiv V_1 \times V_2 \times \dots$ and then consider the collection of smooth sections as matter fields:

$$[\langle F_M \rangle] \equiv \{ \{ f_V \circ \phi \circ f_{\mathcal{G}} \mid f_{\mathcal{G}} \in \mathcal{G} \wedge f_V \in \text{Iso}(V) \} \mid \phi : P \rightarrow V \\ \text{with } \phi(\rho_P^R(\bullet, g)) = \rho(g^{-1}, \phi(\bullet)) \}$$

Dropping the discrimination between matter and gauge fields and introduce the combined space $\langle [F] \rangle$, we proceed along the same lines as previously, however with a different interpretation for the underlying fields. First, the extended space $\langle F \rangle$ is transferred into a Fréchet manifold sliced by the respective diffeomorphism group. It is then usually possible to equip $\langle F \rangle$ with a field space metric $\mathcal{G}$, whereby the background field method is applied in order to diffeomorphically relate $\langle F \rangle$ with its tangent spaces. The resulting principal bundle structure can then be studied concerning its (non-)triviality and a connection on field space has to be selected in order to define field variations.

Hence, in this step we choose the field content of our theory by fixing a connection and determine a number of representations. We further establish a differential geometry description of $\langle F \rangle$.

D. Observables In this setting, observables are maps from the space of expectation fields to the real or complex numbers:

$$\langle \text{Obs} \rangle \equiv \{ \langle O \rangle : \langle [F] \rangle \rightarrow \mathbb{C} \mid \langle O \rangle \text{ smooth} \}$$

At least the subspace of convex functions is closely related to the concept of (convex) observables in the deductive approach. Given a probability theory μ , then Jensen's inequality provides a relation between a convex (quantum) observable $O \in \text{Obs}$ and an associated convex element $\langle A \rangle \in \langle \text{Obs} \rangle$:

$$\langle O \rangle(\varphi) \equiv \langle O \rangle(\langle \hat{\varphi} \rangle_\mu) \leq \langle O(\hat{\varphi}) \rangle_\mu$$

The equality is satisfied only in case of linear, convex functions A , thus in particular we have $\varphi \equiv \langle [\hat{\varphi}] \rangle_\mu$ for a probability theory.

E. Effective Theories Next, we have to establish a theory describing the properties and phenomena of observables. Instead of integrating or averaging over the fundamental degrees of freedom, we consider a prescription build on the space of expectation fields $\langle [F] \rangle$ which matches the observables. Therefore, we start with the effective action $\Gamma[\varphi; \bar{\Phi}]$ and 'deduce' its structure from the observed phenomena. Usually, however, one starts with a very general ansatz for $\Gamma[\varphi; \bar{\Phi}]$ based on the symmetry $G_{\langle [F] \rangle}$ and then matches the coefficients or couplings with experimental results:

$$\underbrace{\partial_{\delta\varphi} \cdots \partial_{\delta\varphi}}_{\ell\text{-times}} \Gamma[\varphi; \bar{\Phi}] \Big|_{\varphi=\varphi_{\text{Exp}}} \equiv O(\varphi_{\text{Exp}})$$

An effective action for which all n -point functions can be simultaneously diagonalized describes a free theory.

F. Reconstruction of the measure If the effective action is based on a fundamental theory described by the integral representation à la Feynman, then we should be able to revert the steps from $Z[J; \bar{\Phi}]$ to $\Gamma[\varphi; \bar{\Phi}]$. Therefore, we have to ensure that $\Gamma[\varphi; \bar{\Phi}]$ is convex in order to be the Legendre-Fenchel transform of a Schwinger functional $W[J; \bar{\Phi}]$:

$$W_\Gamma[J; \bar{\Phi}] = \text{LFT}(\Gamma[\bullet; \bar{\Phi}])[J; \bar{\Phi}] \equiv \sup_{\varphi \in T_{\bar{\Phi}} \langle [F] \rangle} \{ J(\varphi) - \Gamma[\varphi; \bar{\Phi}] \}$$

Whereas the possible convexity of $W[J; \bar{\Phi}]$ is an important, even though not necessary feature that ensures all information is available on the level of the effective action, the convexity of $\Gamma[\varphi; \bar{\Phi}]$ is mandatory. For any $\Gamma[\varphi; \bar{\Phi}]$ which derives from a Schwinger functional $W[J; \bar{\Phi}]$ convexity is an inevitable requirement one has to impose. All deviations from $\Gamma[\varphi; \bar{\Phi}]$ being convex result in some ignorance of the fundamental theory towards certain observable effects and hence the fundamental nature of the theory is questionable.

In the next step, $W_\Gamma[J; \bar{\Phi}]$ has to be converted into the generating functional, which in the usual case is trivial:

$$Z_\Gamma[J; \bar{\Phi}] \equiv \exp(W_\Gamma[J; \bar{\Phi}])$$

The consistency of the formalism now requires that if $\Gamma[\varphi; \bar{\Phi}]$ is a true effective action derived from a fundamental measure μ in the usual way, then $Z_\Gamma[J; \bar{\Phi}]$ has to be the Fourier transform of μ . As such, it should exhibit all the properties a generic Fourier transform satisfies, in particular those described by the Minlos- or Bochner theorem mentioned in subsection 2.1.4, i.e.

$$\begin{aligned} Z_\Gamma[0; \bar{\Phi}] &= 1 && \text{normalized,} && Z_\Gamma[\bullet; \bar{\Phi}] && \text{continuous} \\ \forall c_r \in \mathbb{C}, \forall J_r \in T_{\bar{\Phi}}^* F : & \sum_{r,s} \bar{c}_s \cdot c_r \cdot Z_\Gamma[J_r - J_s^*; \bar{\Phi}] \geq 0 && \text{positive definite} \end{aligned}$$

Notice that a consequence of positive-definiteness are the following two necessary, but not sufficient, conditions: $Z_\Gamma[0; \bar{\Phi}] > 0$ and $|Z_\Gamma[J; \bar{\Phi}]| \leq Z_\Gamma[0; \bar{\Phi}]$. Any admissible effective action Γ should be related to a Z_Γ which satisfies the above conditions, i.e. which is the Fourier transform of a suitable probability measure. Parts of the Osterwalder-Schrader axioms are related to exactly those requirements.

If Z_Γ is well behaved in the above context, then we can apply the inverse Fourier transform and obtain the underlying measure μ . However, the actual field content is still unknown, since there might be theories that can be described by the same effective action albeit based on different measurable spaces. This concludes the second strategy, a route from observations to a fundamental theory.

A comparison

Even though it is instructive to keep the ‘deductive’ and ‘inductive’ ideas kind of separated, for practical applications a combination of both approaches is necessary and one encounters ‘smooth transitions’ between them in any construction of a theory. A ‘deductive’ method always requires experimental input to define the mathematical boundaries and in a sense makes a choice of what is real and what is pure imagination. Mathematics is usually not the ground to decide about this, but rather provides insights why the Universe may resolve the way it does. On the other hand, the ‘inductive’ approach is of no need if it is purely for the sake of describing observed experiments without producing independent predictions. At a certain point one has to leave either of these approaches and start working in the opposite direction to gain a true insight into the very foundations of Nature. The functional renormalization group method is actually a remarkable example where one starts by some very basic experimental input on the level of the effective action and derives a mathematically consistent theory which then is used to produce new predictions from fundamental principles.

We have seen that the relation in the path integral approach is given by the transition from fundamental fields $\hat{\phi}$ to its expectation values $\varphi \equiv \langle \hat{\phi} \rangle_\mu$ and from a measure μ to an effective action Γ . As should be emphasized again, the space of expectation fields will in general differ from the one describing the actual degrees of freedom we averaged over. It may even affect the boundary conditions the fields satisfy. Hence, one has to be particularly careful which side of the theory one tries to interpret, after or before integration. Approximation schemes applied to the effective action rather concern variations of the measure itself, while perturbation approaches of $Z[J; \bar{\Phi}]$ rather address small deviations of the underlying field space. We will put some special emphasis on this point when considering perturbation theory in the measure-based context.

2.3.2 Theories, phases, and anomalies

No matter if one starts with the ‘deductive’ or ‘inductive’ framework, there is a general procedure to construct theories, quite similar to fitting curves to observable data. The initial ingredients are given by a symmetry group G and a concept of spacetime M , which might be discrete or continuous (here we prefer the latter case). Notice that this choice is usually guided by our observations and thus inductive in nature.

Once these basic conditions are settled, we determine the most general smooth functional $A : F \rightarrow \mathbb{R}$ compatible with the invariance conditions and the spacetime structure:⁸

$$A[\sigma(g, \hat{\Phi})] = A[\hat{\Phi}] \quad \text{and} \quad A[f_{M*} \hat{\Phi}] = A[\hat{\Phi}] \quad (2.23)$$

Here, $\sigma : G \times F \rightarrow F$ stands for the G -representation on F and f_M denotes an element of the diffeomorphism group acting on $\hat{\Phi}$. The solution to this equation is by no means unique and additional initial conditions have to be chosen. In the theory of differential equations, parameters usually substitute the remaining freedom and likewise we can define a family of functionals that solve eq. (2.23), each of its elements denote an action functional:

$$S : F \times \Lambda \rightarrow \mathbb{R}, \quad \text{with} \quad S[\sigma(g, \hat{\Phi}); \{u\}] = S[\hat{\Phi}; \{u\}] \quad \text{and} \quad S[f_{M*} \hat{\Phi}; \{u\}] = S[\hat{\Phi}; \{u\}]$$

Here, Λ represents the space of free parameters. The more supplementary mathematical constraints we can physically justify, the less parameters $u \in \Lambda$ appear. Each left-over parameter u has to be fixed by experimental data in the way a generic family of functions has to be matched to an experimentally observed curve. This amounts to imposing initial conditions that are compatible with the phenomena we experience in Nature.

Similar to the family of action functionals, we can define a quantum field theoretical version based on a family of probability measures satisfying certain symmetry and diffeomorphism properties:

$$\mu : \Sigma_F \times \Lambda \rightarrow \mathbb{R}, \quad \text{with} \quad \mu(\bullet; \{u\}) \in \text{Meas}_P(F, \Sigma_F) \quad \text{and further constraints}$$

A particular example is the Lebesgue measure for which a single parameter ‘survives’ the mathematical constraints and ultimately has to be fixed by a normalization condition. In the general case, where we have a possibly large number of free parameters that define the most general theory compatible with the invariance requirements, we have to evaluate observables $O : \Sigma_F \rightarrow \mathbb{R}$ each representing a particular experimental setting. For any free parameter, we need at least one observation O_{Exp} to fix the initial conditions, i.e.

$$O_{\text{Exp}} \stackrel{!}{=} \langle O \rangle_{\mu; \{u\}} \equiv \sum_{\ell=0}^{\infty} \alpha_O^{(\ell)} \cdot \underbrace{\partial_{J_{\text{Exp}}} \cdots \partial_{J_{\text{Exp}}}}_{\ell\text{-times}} Z[J; \bar{\Phi}; \{u\}] \Big|_{J=0}$$

Notice the dependence of the RHS on the free parameters $\{u\}$, which are in general not observables but can be inferred from the set of observable equations. In order to have full predictivity of the theory there should be only a finite number of independent parameters $\{u\}$. Hence, either we have strong enough mathematical constraints at the very beginning, or we have to impose relations between the different parameters in order to reduce their number to be finite. If this is the case, a probability theory is denoted renormalizable. The origin of this word is found in perturbation theory, which is discussed in the next subsection. A concept of renormalizability that extends the perturbative notion is later given in the context of Asymptotic Safety.

⁸In the following we consider the measure-based description and refer for its effective action counterpart to the discussion of the Functional Renormalization Group in chapter 4.

R Notice that besides consistency conflicts that appear on a mathematical level, observations can falsify the assumptions on G and M , retrospectively.

Now, let us assume we are given a set of experimental observations $\{O_{\text{Exp}}^{(j)}\}$ that should match with a certain class of theories, $\mu(\bullet; \{u\})$, compatible with a symmetry group G and spacetime M , and its number should be sufficient to fix all free parameters $\{u\}$. Then, if possible, we can use the standard methods of fitting curves to data, which usually defines a kind of iterative procedure, as for instance the Gauß-Newton algorithm. In general, this amounts to a sequence in the space of probability measures $\text{Meas}_P(F, \Sigma_F)$ that ultimately converges to the possible real theory. Initially, one has to choose any allowed set of parameters $\{u^{(0)}\}$, evaluate the predictions for the experimental results $\langle O \rangle_{\mu; \{u^{(0)}\}}$ and apply a suitable algorithm to reduce the error by adjustment of $\{u^{(0)}\}$:

$$\mu(\bullet; \{u^{(0)}\}) \xrightarrow{\text{algorithm}} \mu(\bullet; \{u^{(1)}\}) \xrightarrow{\text{algorithm}} \dots \xrightarrow{\text{algorithm}} \mu(\bullet; \{u^{\text{Exp}}\})$$

The convergence and number of iterative steps of this series depends on the starting point $\{u^{(0)}\}$ and hence physically reasonable choices of these initial values, as for instance results from the classical theory, may severely reduce the work.

R Notice that in each step, the choice of $\{u^{(j)}\}$ fixes a measure, i.e. $\mu_{(j)}(\bullet) \equiv \mu(\bullet; \{u^{(j)}\})$ and thus gives rise to new expectation values and observable results. The procedure of adjusting (or normalizing) the theory w.r.t. $\{O_{\text{Exp}}^{(j)}\}$ is referred to as (re-)normalization in certain contexts.

Besides this experimental matching, there is another very useful approach that addresses the symmetry properties of $\text{Meas}_P(F, \Sigma_F)$ depending on the set of parameters. The characterization of the family of admissible theories nicely demonstrates the significance of studying symmetry phases and phase transitions. We first focus our attention on the density $\rho(\hat{\Phi}; \{u\})$ and afterwards consider the full measure $\int d\mu_G \rho(\hat{\Phi}; \{u\})$. While the general constraint imposed on the class of admissible theories is given by the group action of G , for certain measures, for instance the trivial one $\rho(\hat{\Phi}; \{u\}) = 1$, the system will exhibit further invariances. Explicitly, let us categorize the space of probability densities $\rho(\hat{\Phi}; \{u\})$ under a particular symmetry group H . The parameter space Λ is given by different choices for $\vec{u} \equiv \{u\}$ and for the density function the order space will be the (quantum) field space, i.e. $S \equiv F$. Depending on the constraint-functions $f^{(j)}$, we investigate different properties of the probability densities concerning their symmetry behavior. Typical examples address the transformation of the density itself and/or solutions to the field equations:

$$\begin{aligned} f^{(0)}(\hat{\Phi}) &= \rho(\sigma(h, \hat{\Phi}); \{u\}) - \rho(\hat{\Phi}; \{u\}) \stackrel{!}{=} 0 & \forall h \in H \\ f^{(1)}(\hat{\Phi}) &= \mathfrak{d}_v \rho(\hat{\Phi}; \{u\}) \stackrel{!}{=} 0 & \forall v \in T_{\hat{\Phi}} F \end{aligned}$$

The space of solutions to these constraints is sensitive to the particular choice of $\vec{u} \equiv \{u\}$ and will in general be only a subspace of $S \equiv F$, denoted $\text{Sol}(\vec{u})$. The properties of $\text{Sol}(\vec{u})$ characterize the phase in which the corresponding theory lives in. Important choices of H are the conformal group, or $O(N)$, where N denotes the number scalar fields, for instance. The first condition $f^{(0)}$ specifies the subspace $\text{Sol}(\vec{u})$ for which the associated density $\rho(\hat{\Phi}; \{u\})$ is conformally invariant. If for a particular choice of $\vec{u}$ the full space satisfies this criterion, i.e. $\text{Sol}(\vec{u}) \equiv S \equiv F$ holds true, then the density is conformally invariant, irrespectively of its argument. In another example, based on the cyclic symmetry group $H \equiv \mathbb{Z}_2$, let us consider the following

two theories, specified by, say, $\vec{u}^{(a)} = (2, 1, -3, 0, 0 \dots)$ and $\vec{u}^{(b)} = (2, 0, -3, 0, 0 \dots)$ that are special cases of the general family of polynomial functionals (here $u_{\geq 3}^{(\bullet)} = 0$):

$$\begin{aligned} \rho(\hat{\Phi}; \vec{u}^{(a)}) &\equiv u_0^{(a)} + u_1^{(a)} \hat{\Phi} + u_2^{(a)} \hat{\Phi}^2 &\implies & \text{Sol}(\vec{u}^{(a)}) = \{e_F\} \\ \rho(\hat{\Phi}; \vec{u}^{(b)}) &\equiv u_0^{(b)} + 0 \cdot \hat{\Phi} + u_2^{(b)} \hat{\Phi}^2 &\implies & \text{Sol}(\vec{u}^{(b)}) = F \end{aligned}$$

Here, we assumed that F is vector space and e_F denotes its identity element. Thus, we observe that both theories live in different phases. In a similar way one can study the properties of the solutions of the field equations and classify theories according to their solution spaces.

In most approaches, the density is given by a classical analog of the action functional and thus is invariant under classical symmetries, i.e.

$$\rho(\hat{\Phi}; \vec{u}) \equiv \exp(-S[\hat{\Phi}; \vec{u}])$$

Nevertheless, the quantum theory includes a measure μ_G and may violate the invariance principle on the quantum level. That is, while the density exhibits a certain symmetry, the full measure does not:

$$\int_{\sigma(h, U)} \mu_G(\hat{\Phi}) \rho(\hat{\Phi}; \vec{u}) \neq \int_U \mu_G(\hat{\Phi}) \rho(\hat{\Phi}; \vec{u}) \quad \text{while} \quad \rho(\sigma(h, \hat{\Phi}); \{u\}) \equiv \rho(\hat{\Phi}; \{u\})$$

In the transition from a classical theory to its quantum version the breaking of a classical symmetry is referred to as the appearance of an anomaly. This phenomenon may occur for any probability density, not necessarily restricted to exponential types. A very famous example is the conformal anomaly, where in addition to the failure of the quantum theory to exhibit conformal invariance, the trace of the energy-momentum tensor has a non-vanishing expectation value.

2.3.3 Perturbation theory

The reason why we put so much emphasis on symmetries and group theoretical aspects in the introduction is essentially the important status of these concepts in identifying suitable theories. It is one of the most instructive guiding principals that helps in deducing the laws of Nature from experiments. In the study of the renormalization group, symmetries again play a crucial role. They will identify the behavior of entire universality classes under the renormalization group flow and thus provide the foundations of what follows.

No matter from which direction we hunt the fundamental theory of Nature, whether we start from some theoretical assumptions or from certain deductions of observations, it is always the aim to reduce the amount of input as far as possible. Symmetries surely play a central role, due to the powerful constraints they introduce, nevertheless at a certain point we have to make a connection to reality in order to uniquely determine the right theory. The famous example we have already encountered is the Lebesgue measure which is defined on the basis of some mathematical constraints that seem plausible for measuring lengths. The result is a family of measures parametrized by a single variable λ , i.e.

$$\mu_L(A = (a, b); \lambda) = \lambda \cdot (a +_{\mathbb{R}^d} (-b))$$

This remaining freedom still needs one more physical information we have to add to our description to uniquely determine the correct measure. In fact, in this case it corresponds to fixing a physical unit, meaning that we have to perform a measurement and allocate a specific value to this length. This normalization allows to express all other sizes in terms of multiples of this prototype unit (also compare the definition of the international standard units).

The same ideas apply to the concepts of quantum field theory. At the very outset, we collect a set of mathematical requirements that we assume to be physically relevant. The most general form of theory that respects the demands involves a set of free parameters. While on paper we have a clear separation of ‘inductive’ and ‘deductive’ methods, in praxis the applied techniques are similar to each other. In both cases invoking (for instance symmetry) constraints, we are left with a family of theories; each member being a suitable candidate from a mathematical point of view:

$$Z[J; \bar{\Phi}; \{u\}] \equiv \int_{\hat{\Phi}} d\mu_G(\hat{\Phi}; \bar{\Phi}) \rho(\hat{\Phi}; \bar{\Phi}; \{u\}) \quad \text{or} \quad \Gamma[\varphi; \bar{\Phi}; \{u\}]$$

In this way we can make the construction systematic in either of the approaches and start with mathematical assumptions before incorporating experimental results. To recognize the hidden symmetries that underlie the laws of Nature requires deep insight and is in fact either some fortunate coincidence or true stroke of genius. Once restricted to a class of suitable theories, the next step involves evaluating observables. In this general setting the actual value is determined up to the free parameters $\{u\}$ we introduced previously, thus every allocation to experimental data will fix at least one of them.

$$\langle O(\hat{\Phi}) \rangle_{\mu; \{u\}} \equiv \sum_{\bar{\Phi}} \int_{\hat{\Phi} \in T_{\bar{\Phi}}[F]} d\mu(\hat{\Phi}; \bar{\Phi}; \{u\}) O(\hat{\Phi}; \bar{\Phi}) \quad \text{or} \quad \langle O(\hat{\Phi}) \rangle_{\mu; \{u\}}^{1Pl}$$

This process, known as normalization, may also come along with mathematical restrictions due to regularity aspects. This bridge between theory and experiment can be summarized by $u \equiv u(O_{\text{Exp}}(\hat{\Phi}))$, i.e. indicating the dependence of the free parameters on certain observable data.

While the general techniques are similar on the level of Z and Γ , the interpretation, in particular of approximation schemes, will differ severely. As long as the theories are treated on an exact level, we have the chain of equivalences on which the entire effective action approach is based on:

$$\mu(\bullet; \bar{\Phi} | \{u\}) \iff Z[J; \bar{\Phi}; \{u\}] \iff W[J; \bar{\Phi}; \{u\}] \iff \Gamma[\varphi; \bar{\Phi}; \{u\}]$$

Likewise the expectation values given by their respective functional derivatives contain the same information, though in different formats. However, the exact level is far from being reached in almost any interesting QFT and hence a generation of physicists derived a number of different approximation techniques that resolve certain obstruction, however by paying the price of losing certain information. Depending for which purpose they are made, these methods either work on the level of the generating functional or the effective action. It is crucial to understand the differences between those schemes whenever interpretation or validity issues are concerned. For instance, employing perturbation methods on $Z[J; \bar{\Phi}; \{u\}]$ in which one modifies the measure while field space is untouched, may correspond to small perturbations around a fixed expectation field $\Phi = \exp_{\bar{\Phi}}(\varphi)$ in $\Gamma[\varphi; \bar{\Phi}; \{u\}]$. Similarly, small fluctuations around Φ on the side of the effective action may still belong to a full integration over $[F]$, however evaluated only over a very small class of different measures instead of all possible ones. We will see that theories converging to a free field theory can be usually well described within perturbation approaches of $Z[J; \bar{\Phi}; \{u\}]$, while otherwise additional tools are in need, sometimes more appropriately stated for the ‘dual’ object $\Gamma[\varphi; \bar{\Phi}; \{u\}]$.

The necessity of approximation schemes is seen as soon as one leaves the formal description of QFT and works on explicit calculations to deduce observable quantities of Nature. Apart from the Gaußian measure there are only very rare and exotic cases for which the functional integral is understood and analytically solved. In particular the generating functional, i.e. the

Fourier transform of the measure, is in most discussions only a formal object. Thus, over the years a full landscape of different approximation techniques have emerged which partially reveal the hidden aspects of possibly fundamental theories. Almost exclusively, exponential types of probability densities are studied, $\rho(\hat{\Phi}; \vec{u}) \equiv \exp(-S[\hat{\Phi}; \vec{u}])$, whereby the exponent usually mirrors the classical action functional. Thus, one sometimes refers to it as the path integral quantization of classical theories. However, in general the relation will differ and only in a certain limit the classical theory is recovered by its quantum version, whereby S denotes the bare action. The reconstruction of S by observable data is a very difficult task and involves the steps described in when discussing the ‘inductive’ approach.

In what follows, we address perturbation theory in a very broad sense, where the probability density is approximated in a particular manner to allow for a partial integration. This almost exclusively corresponds to reducing the theory to a Gaussian integral, for its distinguished status in functional integration. Hence, perturbation theory deals with a sequence of measures that ultimately should converge towards the true theory:

$$\mu_{(0)} \xrightarrow{\text{algorithm}} \mu_{(1)} \xrightarrow{\text{algorithm}} \dots \xrightarrow{\text{algorithm}} \mu \quad (2.24)$$

The idea is to start with a measure $\mu_{(0)}$ that is as close as possible to the true theory μ while being as simple as possible, meaning close to Gaussian. In certain settings, both requirements are mutually exclusive since the theory turns out to be highly non-trivial. So, let us assume we are given a theory $\mu = \int d\mu_G \rho(\hat{\Phi})$ then its crudest approximation is setting $\rho(\hat{\Phi}) = 1$, which corresponds to the associated free theory. In this case we can compute in principal any correlation function and observable, though the results may have little to do with the actual theory μ . Nevertheless, there might be regions on field space $U_0 \in \Sigma_F$ where $\rho(\hat{\Phi}) = 1$ is approximately fulfilled, thus

$$\mu(U_0) = \int_{U_0} d\mu_G \rho(\hat{\Phi}) \approx \int_{U_0} d\mu_G \equiv \mu_G(U_0) \quad \text{if} \quad \rho(\hat{\Phi}) \approx 1 \quad \forall \hat{\Phi} \in U_0$$

In the above sequence (2.24) the measure $\mu_{(0)}(E \in \Sigma_F) \equiv \mu(E \cap U_0)$ may correspond to a starting point of the perturbative algorithm.⁹ One then extends U_0 to a larger region $U_0 \subset U_1 \in \Sigma_F$. However, in order to be reliably describing μ one also has to adapt the probability density, by expanding $\rho(\hat{\Phi}) \approx 1 + f(\hat{\Phi}) + \dots$, for instance in a Taylor series w.r.t. the fields. One arrives at

$$\mu_{(1)}(E) \equiv \int_{E \cap U_1} d\mu_G \cdot (1 + f(\hat{\Phi})) \equiv \mu_G(E) + \langle f(\hat{\Phi}) \rangle_{\mu_G(E)}$$

This algorithm can be pursued until the computational abilities are reached or the theory seems to converge. The latter is a quite difficult and rarely appearing case, where the fundamental theory turns out to be almost Gaussian, i.e. free. However, depending on the explicit expansion scheme and the initial region U_0 one may have a well defined and proper description of the theory under certain conditions where the above approximation is valid. For instance, expanding in momentum modes may show that the theory is close to trivial in the ultraviolet (UV) while on large scale structures it is highly non-Gaussian. In the following we present some examples of approximation schemes: the method of steepest descent, and perturbation in small parameters or couplings.

R There is a very important difference concerning the perturbation technique and the Renormalization Group approach, though both define a sequence of measures by means of a filtration on the σ -algebra. The former is restricted to a region in field space that is well-described by a free theory, while the latter ultimately extends to the full space and rather replaces the functional integral by a systematic sequence on the space of measures.

⁹In fact $E \cap U_0$ is an element of Σ_F by the properties of σ -algebras.

Method of steepest descent

While equally in their pattern, the three approximation techniques differ in their interpretation and, depending on the theory μ , in their range of validity. The method of steepest descent (saddle point approximation) searches for a particular convenient region U_0 in field space in which one expands the probability density around its maximum $\partial_\nu \rho(\hat{\Phi}_{\max}) = 0$ for all ν :

$$\rho(\hat{\Phi}) = \rho(\hat{\Phi}_{\max}) + \mathfrak{d}_{(\hat{\Phi}-\hat{\Phi}_{\max})}^{(2)} \rho(\hat{\Phi}_{\max}) + \mathcal{O}((\hat{\Phi} - \hat{\Phi}_{\max})^3)$$

Notice that while the first term is a constant under the functional integral, the linear term vanishes due to the maximum condition and what remains are corrections quadratic in the deviation of the maximum position. Hence, for this approximation to work, we choose $\hat{\Phi}_{\max} \in U_0$ and restrict the integration measure such that $\mathcal{O}((\hat{\Phi} - \hat{\Phi}_{\max})^3)$ are subordinate:

$$\mu(E) = \int_{\hat{\Phi} \in E} d\mu_G(\hat{\Phi}) \rho(\hat{\Phi}) \approx \rho(\hat{\Phi}_{\max}) \int_{\hat{\Phi} \in E} d\mu_G(\hat{\Phi}) \mathfrak{d}_{(\hat{\Phi}-\hat{\Phi}_{\max})}^{(2)} \rho(\hat{\Phi}_{\max}) + \mathcal{O}((\hat{\Phi} - \hat{\Phi}_{\max})^3)$$

The more pronounced the maximum of ρ , or likewise the minimum of S , is, the better the range of possible validity.

Parameter expansion

A similar technique involves a small parameter that factors the additional term in an exponential type probability density, say $\rho(\hat{\phi}; \bar{\Phi}) = \exp(-\lambda V[\hat{\phi}; \bar{\Phi}])$. As long as $V[\hat{\phi}; \bar{\Phi}]$ is comparably small w.r.t. λ the generating functional can be expanded in powers of λ :

$$Z[J; \bar{\Phi}] = \int_{\mathbb{T}_{\bar{\Phi}} F} d\mu_G(\hat{\phi}; \bar{\Phi}) \exp(-\lambda V[\hat{\phi}; \bar{\Phi}] + J(\hat{\phi})) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \lambda^\ell \langle V[\hat{\phi}; \bar{\Phi}]^\ell \rangle_{\mu_G}$$

Notice that the expectation values are computed for the Gaussian measure. This method is appropriate if $\lambda V[\hat{\phi}; \bar{\Phi}] \ll 1$ and thus the series converges rapidly enough. If this condition is only fulfilled for some subset of F the range of validity has to be reduced to this region.

Expansion in $\hbar$

We should mention another technique that not directly relates to the above approximation schemes but involves a constant of Nature. Here, one reestablishes Planck's constant in the description and expands the theory in powers of this constant $\hbar$. Assume we are given a probability density of exponential type, then the Schwinger functional in units of $\hbar$ is given by

$$\exp(-\frac{1}{\hbar} W[J; \bar{\Phi}]) \equiv \int_{\mathbb{T}_{\bar{\Phi}} F} d[\hat{\phi}] \exp(-\frac{1}{\hbar} S[\hat{\phi}; \bar{\Phi}] + J(\hat{\phi}))$$

Converting this measure using the Legendre transform and the expression for the effective action one obtains

$$\exp(-\frac{1}{\hbar} \Gamma[\varphi; \bar{\Phi}]) \equiv \int_{\mathbb{T}_{\bar{\Phi}} F} d[\hat{\phi}] \exp(-\frac{1}{\hbar} S[\hat{\phi}; \bar{\Phi}] + \mathfrak{d}_{(\hat{\phi}-\varphi)} \Gamma[\varphi])$$

Finally, one expands the effective action in a series in $\hbar$, i.e. $\Gamma[\varphi; \bar{\Phi}] = S[\varphi; \bar{\Phi}] + \sum_{\ell=1}^{\infty} \hbar^\ell \Gamma_\ell[\varphi; \bar{\Phi}]$. On the left-hand-side (LHS) this is straightforward, however on the RHS we have to compute the expectation value of the reduced density, which may need further simplifications. Now, comparison of the coefficients for each order in $\hbar$ results in

$$\Gamma[\varphi; \bar{\Phi}] = S[\varphi; \bar{\Phi}] + \frac{\hbar}{2} \text{Tr} \ln S^{(2)}[\varphi; \bar{\Phi}] + \mathcal{O}(\hbar^2)$$

Notice that compared to the other methods, this technique is more concerned with the effective action and thus the expectation fields.

R Under certain circumstances this method can be referred to as loop expansion, though in general they do not agree [63, 64].

Perturbative renormalizability and Regularization

The problem with the above approximation techniques descends from the subtlety to split a full measure into parts and evaluate possibly undefined quantities. From a mathematical point these systematic expansions are difficult to control and in general on each intermediate step infinities may appear.¹⁰ Hence, all the prescribed perturbative methods have to be understood as an asymptotic expansion which might not converge to the full theory, but nevertheless provides insight into its structure and properties.

The failure of perturbation methods or of a naive choice for the fundamental theory which results in divergences is usually cured by introducing a regulator scheme, that spots the deficiency of the description and cuts off the problematic terms in a suitable manner. In a sense it remedies the artificially introduced divergences by modifying the perturbation method such that potential infinities are replaced by a finite cutoff Λ . In the limit of removing the regulator Λ the divergences reappear. The advantage of this procedure is control over the problematic terms and a finite description on each step of the evaluation. While the regulator is constructed to eliminate infinities by means of a finite upper or lower bound on intermediate steps, one assumes that for the full theory this regulator is superfluous and in fact only an indication of either a flawed evaluation technique or an insufficient initial theory. Though general results may depend on Λ , it is desirable that predictions of the regulated theory do not depend on the particular regularization scheme employed. Hence, regularization should be understood as a mathematical trick to compensate for impurities of the applied (perturbation) technique, for instance by means of a suitable adaption of the range of integration. As such it should be understood to be mandatory to regularize the theory in each step of the perturbative expansion.

Once, the theory is regularized and cured from infinities, one has to bridge the mathematical theory with physical observables and thereby fix the free parameters. This procedure is known as normalization and provides the actual testing ground for theories, which are so far only mathematical objects based on some (physically motivated) symmetry principle.

R Exactly this point, where theory meets experiments and mathematical objects, like expectation values, are related to observable quantities, describes the appearance of physical units, or scales, in the mathematical formalism. Experiments are comparative processes, where some quantity is measured with respect to another. Remarkably, there are only seven basis units defined in the International System of Units, hence there are (theoretically) at most seven comparative measurements necessary to fix a quantity uniquely. However, most of the basis units are in fact related by some constant of Nature and thus are not independent.

In the perturbative approach normalization actually corresponds to a (re-)normalization¹¹ algorithm, as can be understood as follows. Let us take a family of measures that fulfill some symmetry based constraint and thus contains a number of free parameters that have to be specified by experimental input, hence $Z[J; \{u\}]$. Since this quantity and its expectation values are in general not computable, one relies on a perturbative treatment and has the sequences of generating functionals, or likewise measures, now suitably regularized by a regulator Λ :

$$Z_{(0)}^\Lambda[J; \{u\}] \xrightarrow{\text{algorithm}} Z_{(1)}^\Lambda[J; \{u\}] \xrightarrow{\text{algorithm}} \dots \xrightarrow{\text{algorithm}} Z[J; \{u\}] \quad (2.25)$$

At each level we consider a different class of theories and thus a different set of expectation values and observable predictions. If we want to match these theories to experiments in order

¹⁰Even a well defined probability theory μ may suffer from those infinities when some approximation scheme is applied to evaluate its correlation functions.

¹¹We write (re-)normalization in this context to distinguish the concept from the one appearing in the Renormalization Group approach.

to reduce the number of free parameters, we have to compute specific correlation functions and evaluate them for certain physical conditions that match the experimental setting:

$$\mathcal{O}_{\text{Exp}} \stackrel{!}{=} \langle \mathcal{O} \rangle_{\mu_{(j)}^\Lambda; \{u\}} \equiv \sum_{\ell=0}^{\infty} \alpha_{\mathcal{O}}^{(\ell)} \cdot \mathfrak{d}_{J_{\text{Exp}}}^{(\ell)} Z_{(j)}^\Lambda[J; \Phi; \{u\}] \Big|_{J=0} \quad (2.26)$$

For, say, n free parameters in the theory we need at least n such equations and therefore n experiments. Then, in principle we can solve for the unknowns and fix the theory completely, at least if n is finite. This normalization of the theory is a physical process that actually addresses $Z[J; \{u\}]$. Applied to some approximate form of $Z[J; \{u\}]$, say of level j , we denote the set of parameters that solves n equations of the type (2.26) by $\{u_{(j)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}$. At level j , the unique (approximated) measure is thus given by

$$Z_{(j)}^\Lambda[J; \{u_{(j)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}] \quad \text{with} \quad \mathcal{O}_{\text{Exp}}^{(r)} \stackrel{!}{=} \langle \mathcal{O}^{(r)} \rangle_{\mu_{(j)}^\Lambda; \{u\}} \quad r \in 1, \dots, n$$

whereby now, there is no remaining free parameter. Remarkably, the matching of finite physical values usually results in absorption of the cutoff dependence into the normalization procedure.

If we perform the same steps on the next level of the perturbative expansion, the theory, thus the expectation values, and therefore the solutions $\{u_{(j+1)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}$ will differ in general from $\{u_{(j)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}$. This explains the notion of (re-)normalization since the previous normalized, or physically matched theory $Z_{(j)}^\Lambda[J; \{u_{(j)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}]$ gets replaced by a new one $Z_{(j+1)}^\Lambda[J; \{u_{(j+1)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}]$ with a (re-)normalized set of parameters. The crucial requirement one imposes on a perturbatively renormalizable theory is that the number of free parameters converges to a finite value, thus at a certain order in the expansion one expects that no new terms become relevant but rather that all effects can be absorbed in the already present coefficients. This corresponds to studying a general family of measures and (re-)normalizing the parameters only:

$$Z^\Lambda[J; \{u_{(0)}\}] \xrightarrow{\text{algorithm}} Z^\Lambda[J; \{u_{(1)}\}] \xrightarrow{\text{algorithm}} \dots \xrightarrow{\text{algorithm}} Z[J; \{u_{\text{Exp}}\}]$$

For instance, the initial theory may start with all $\{u\}$ vanishing and then during the process one switches on the contributions from certain invariants and observes their influence on the results.

Definition 2.3.1 — Perturbative (re-)normalizability. Let (F, Σ_F) be a measurable space and $\mu(\bullet; \{u\})$ be a family of probability measures. Then we say that $\mu(\bullet; \{u\})$ is perturbatively renormalizable w.r.t. the algorithm $\mathcal{A} : \Leftrightarrow$

- The sequence $(\{u_{(\ell)}(\mathcal{O}_{\text{Exp}}; \Lambda)\})_{\ell \in \mathbb{N}_0}$ converges to $\{u_{\text{Exp}}\}$ within the validity of the perturbative method $\mathcal{A}$. Hereby, $\{u_{(\ell)}(\mathcal{O}_{\text{Exp}}; \Lambda)\}$ is defined by

$$Z^\Lambda[J; \{u_{(0)}\}] \xrightarrow{\mathcal{A}} Z^\Lambda[J; \{u_{(1)}\}] \xrightarrow{\mathcal{A}} \dots \xrightarrow{\mathcal{A}} Z[J; \{u_{\text{Exp}}\}]$$

- The number of free parameters in the theory $\mu(\bullet; \{u\})$ converges to a finite value, i.e.

$$\lim_{\ell \rightarrow \infty} \# \{u_{(\ell)}(\mathcal{O}_{\text{Exp}}; \Lambda)\} = n < \infty$$

While the first criterion ensures that the chosen perturbation technique defines a valid approximation for μ , the second refers to the predictivity of the theory, stating that perturbatively (re-)normalizable theories only need a finite number of experiments to be uniquely determined.

R Notice that even though a theory may be found to be perturbatively non-(re-)normalizable it is not in general doomed to failure. Perturbative non-(re-)normalizability can be an artifact of the applied perturbation technique which might be incompatible with the underlying theory. We will encounter a generalization of this perturbative concept in the context of the Renormalization Group.

2.3.4 Euclidean quantum field theory

Einstein's theory of Special Relativity is based on a Minkowski spacetime and since all fundamental interactions are nowadays constructed over these Lorentzian spacetimes the general requirement of Lorentz invariance is at the roots of QFT. Furthermore, Einstein's later generalization to General Relativity locally embeds the concepts of special relativity and thus the classical theory of gravitation also relies on a pseudo-Riemannian geometry. However, in the functional integral approach to study field theories the non-compactness of the Lorentz group is found to be a problematic aspect of Lorentzian theories that leads to some very undesirable difficulties in a mathematical treatment of the Feynman integral.

A remedy to this obstruction, at least on a technical level, was introduced by Schwinger using analytical continuation to complex time. This procedure is known as Wick rotation $t_L \rightarrow it_E$ whereby the vacuum correlation functions get replaced by their Euclidean version that usually fulfill better controlled differential equations.¹² On the level of functional integrals, this amounts to converting the Feynman-Kac integral by a Wiener (probability) measure, for instance, and the ordinary Fourier transform $z = i$ is substituted by its Laplace counterpart $z = -1$. If the density of the measure was previously given by $\rho_L = e^{iS}$ it assumes a non-oscillating form $\rho_E = e^{-S}$ in the Euclidean framework and thus one has to ensure that the action functional is bounded below. The great advantage of this method is founded in the invariance requirements of spacetimes, where analytical continuation replaces the non-compact Lorentz group by its compact Euclidean counterpart. This basically explains the interest in Euclidean Quantum Field theory (EQFT), in particular for a better controlled mathematical treatment of functional integrations [28].

In the fifties Wightman proposed a set of axioms which translate the general idea into a rigorous prescription [65]. Some twenty years later the Osterwalder-Schrader axioms presented a catalog of criteria that specify the condition under which an EQFT describes a well-behaved Lorentzian theory [28, 50, 51]. For a given probability theory μ its generating functional $Z[J]$ is required to fulfill the following axioms:

- A. Analyticity** The generating functional $Z[J]$ has to be analytic.
This ensures analytic properties of the Minkowski theory.
- B. Regularity** The generating functional $Z[J]$ and its correlation functions have to be regular.
This severely restricts emergence of local singularities in the Minkowski case
- C. Euclidean invariance** Let G_E be the group of Euclidean transformations. Then one requires the theory to be invariant under G_E , i.e. $Z[\sigma_E(g_E, J)] = Z[J]$ for all $g_E \in G_E$.
This corresponds to invariance under the Lorentz group.
- D. Reflection positivity** Let Θ_- denote the time reflection operation sending t to $\Theta_-(t) = -t$. Then, the Euclidean theory should be positive w.r.t. time reflection, i.e.

$$M_{j\ell} := Z[J_j - \Theta_-(J_\ell)] \quad \text{for which} \quad M\phi = \lambda\phi \quad \implies \quad \lambda \geq 0$$

Thus M is a positive matrix having only non-negative eigenvalues.

This ensures positivity of the inner product in the Hilbert space the Minkowski theory acts on.

¹² If one replace the time coordinate (at least locally) with a complex version $\tau = t + iT$ a compactification of the complex direction leads to a thermodynamical interpretation of the description, where T represents the temperature of the system.

E. Ergodicity Denote Θ_s the operation of time translations. Then the ergodicity condition has to be satisfied:

$$\lim_{t_E \rightarrow \infty} \frac{1}{t_E} \int_0^{t_E} ds \Theta_s O(\hat{\phi}(x)) \Theta_{-s} \equiv \langle O(\hat{\phi}) \rangle_\mu \quad \forall O \in \text{Obs}$$

Thus, time averaging of expectation values corresponds to averaging over field space.

This corresponds to the uniqueness of the vacuum in the Minkowski case.

Hence, the above requirements imposed on the Euclidean theory ensure the validity of the Wightman axioms in its Minkowski counterpart [28].

While this prescription is well suited for flat spacetimes, it already appears that for general curved geometries modifications are in need. In particular the reflection positivity and ergodicity make explicit use of a (global) time-like Killing vector field which does not exist in the general case. Still the mathematical formalism of functional integration can be promoted to general spacetimes; the Wightman- and Osterwalder-Schrader axioms need a generalization in terms of a covariant formalism that describes the continuation from Euclidean to Lorentzian theories in this more general setting. We briefly comment on this issue in chapter 3 and refer to ref.[26, 66, 67] for more details.



3.1	General Relativity
3.2	General aspects of Quantum Gravity A motivation for theories of Quantum Gravity Field space of Riemannian metrics Background Independence Boundaries in Quantum Gravity Euclidean Quantum Gravity
3.3	Theories of Quantum Gravity String theory Loop Quantum Gravity Causal Dynamical Triangulation Experimental verification

3. QUANTUM GRAVITY

On the path towards a deep understanding of Nature physicists are confronted with two theoretical concepts, the Standard Model of particle physics and General Relativity, each outstanding in its predictivity and precision. In quest of a unified theory that intertwine the quantum nature of matter with the geometrical construction of spacetime, the search still continues. Decades of research have brought deep insights into both concepts, have revealed their deficiencies and their mathematical foundations. Many physicists nowadays believe that a solid theory of quantum gravity may cure most, or even all, of the so far encountered issues. Promoting GR to a quantum level is a very sophisticated task, since most of the methods available in the construction of previous quantum field theories turned out to be inapplicable.

One of the reasons may be that, considered as a gauge theory, the set of mediating gauge fields defines a non-trivial field space with non-vanishing curvature, hence the simple vector space identification has to be replaced by the geometric background field method. Thereby, Background Independence, the requirement that the formulation should be independent on any – possibly prescribed – background geometry, becomes central to all constructions of quantum theories of gravity [39, 68, 69].

In addition, it turned out that from the perspective of conventional perturbative approaches quantum gravity seems to be non-renormalizable [70]. While GR provides an excellent description of classical gravitational physics it seems to be only an effective description on our energy scales. The task is thus to extend and generalize the classical theory to fit into the framework of quantum field theory while recovering the GR limit at sufficient low energy scales. Due to the lack of experimental results, a large variety of approaches towards a quantum theory of gravity are pursued by different communities all sharing the same ultimate goal of a unified language (not necessarily theory) in which Nature can be described.

In this chapter, we are going to present the general setting of GR in a nutshell along with the necessary implementations for spacetimes with non-vanishing boundary. Next, we take a look at the reasons why most physicists believe that finding a solid theory of quantum gravity is the most important task in theoretical physics. Besides of aesthetic arguments, there are some severe problems in having non-unified descriptions for the SM and gravity, most of them known

since many decades and steadily at the top of the list of unsolved problems in physics. After briefly comment on some general aspects of Quantum Gravity and the importance of boundaries in cosmology, we briefly present an incomplete summary of some very prominent approaches to a possible fundamental theory of gravity. For more details on this subject and a thorough introduction to the various approaches we refer to [26, 39, 68, 69, 71–76].

In the next chapter, we will put the scene for studying the effective average action approach to QG based on the FRGE. This in particular requires a characterization of field space and the underlying symmetry principle, whereby we restrict to the metric setting, the field content we considering later on in a particular bi-metric truncation.

3.1 General Relativity

Gravitation is one of the fundamental interactions of Nature that we experience very pronounced in our every-day life. Considered a bit closer its actual influence seems to be restricted to vertical accelerations and cosmic phenomena, while most of the effects we perceive with our senses are mainly due to electromagnetism. The conception of the world before the discovery of electromagnetism was based on Newton's laws of mechanics, where there was a general setting describing changes of momenta with respect to four axioms, introducing the concept of forces. This construction was highly successful in explaining our macroscopic world. However it remained an effective description which was already apparent due to the number of different forces which did not derive from first principle, but rather were derived purely from observations. This theoretical framework has survived generations of physicists and is still capable of explaining most of the phenomena around us.

But, the moment scientists left the region of perception and dug deeper into the structure of matter, new unseen features of Nature emerged which did not fit into the general description of Newton. And in fact, the explanation of light, a particular expression of this newly discovered interaction, had already undergone many conceptual changes over the centuries before it was apparently consistently described within Maxwell's equations. What this story and many others tells us, is that the ignorance of either experimentally or theoretically perceiving Nature sets natural limitations on the range of validity for our current model of the Universe. Even though we have a very powerful and extremely predictive theory with the Standard Model of particle physics, we can only be sure about its validity on the energy scales we have tested it. The same holds true for GR, which has so far explained all deviations of observations from classical mechanics, however we can only be sure about its precision within the scales accessible by today's experiments. Hence, observations are an extremely valuable treasure that helps in understanding the world. Whenever experimental data is unavailable, it is important to check the inner consistency of theories while at the same time one has to invent new techniques to make significant observations accessible.

The history of gravitation has experienced phases of stagnation and also sudden progress, as for instance with the invention of better telescopes that opened the cosmos to a deeper search for the fundamental nature of the Universe. This interplay between experiments and theories will once answer the question, concerning the gravitational interaction. From the experimental perspective the current (classical) theory of gravity, i.e. General Relativity, is remarkably precise in its predictions and explanations and we rely on them whenever we make use of a GPS device, say. On the other hand, there are theoretical inconsistencies that are difficult to cure within the theory itself and there are also aesthetic mathematical indications that there might be a theory beyond GR. However, we postpone this subject to the next section and first consider the famous theory of GR.

In 1905, Albert Einstein started to revolutionize the concept of space and time by combining them into a unified geometrical object, denoted spacetime. His theory of SR challenged

Newton's perspective of the nature of the Universe and with this the classical conception of the world. By changing the assumption on the symmetry properties of space and time had remarkable consequences which however became only relevant when velocities close to the upper limit, given by the speed of light, are reached. In the spirit of this conceptually new picture of space-time, Einstein worked for ten years on translating Newton's law of gravity into the language of SR. In 1915, after ten years of effort, Einstein finally proposed a new theory of gravitation, General Relativity [77], that in fact provided also a generalization of SR on global scales, while retaining its local validity. In Einstein's understanding, the geometry of spacetime is best described by a (pseudo-)Riemannian manifold (M, g) and the gravitational interaction is related to the corresponding Levi-Civita connection. A property of (pseudo-)Riemannian manifolds is the existence of normal coordinates in which the metric becomes flat and thus for the respective signature of g Special Relativity is recovered in the local vicinity of every point on M . The deviation from g being completely flat, is encapsulated in the Riemann curvature tensor, and thus on global scales one may encounter non-trivial effects which are not described by SR. This is a mathematical consequence of Einstein's equivalence principle that loosely speaking equals the trajectory of an observer in a homogeneous gravitational field with one under uniform acceleration. The geometric formulation of the gravitational interaction is described by Einstein's field equation, which related the curvature of spacetime's geometry to interaction with matter:

$$G^{\mu\nu} + g^{\mu\nu}\Lambda = 8\pi G_N T^{\mu\nu} \quad (3.1)$$

The LHS contains the gravitational contribution, where $G^{\mu\nu} \equiv (R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)$ denotes the Einstein tensor. On the RHS the energy momentum tensor $T^{\mu\nu}$ is coupled via Newton's constant G_N to the geometry of spacetime and thus gives rise to an interaction of matter fields and the metric g .

Equation (3.1) defines a non-linear differential equation with $T^{\mu\nu}$ representing a source term and Λ a free parameter, the cosmological constant, which in cosmology is related to the vacuum energy of the Universe. Thus, solutions to Einstein's field equation for some matter content $T^{\mu\nu}$ are not closed under superposition since linearity is missing.

A very important consequence of the geometrical description of field equations is a requirement imposed on the energy momentum tensor $T^{\mu\nu}$. By definition $G^{\mu\nu}$ is symmetric and by virtue of the Bianchi identities it is also divergenceless, i.e. $D_\mu G^{\mu\nu} = 0$. Hence, only for symmetric source terms with vanishing divergence, i.e. $T^{\mu\nu} = T^{\nu\mu}$ and $D_\mu T^{\mu\nu} = 0$, there exist solutions to equation (3.1). (Notice that $g^{\mu\nu}\Lambda$ is symmetric and by metric compatibility of D also divergenceless.) A particular well studied class of solutions are those for vacuum field content, where $T^{\mu\nu} = 0$ vanishes. For instance in four dimensions vacuum solutions with self-dual curvature tensor define the class of instantons. Another prominent example is given by the Schwarzschild metric, which – according to Birkhoff's theorem – describes the most general vacuum solution with spherically symmetry and $\Lambda = 0$. It has a single free parameter, which sometimes is referred to as the mass of a black hole situated at the origin.

As on cosmological scales all other fundamental interactions have compensated, what remains is the comparatively weak gravitational 'force'. Thus, the greatest impact of GR is found in the depth of our cosmos, in explaining the movement of the planets in our solar system, gravitational lensing, large scale structures or in short a deep insight into the structure of the Universe. Constructing a cosmological model that reflects the homogeneity and isotropy properties of space observed in studying the night sky, the statement of the cosmological principle, it turns out that these requirements result in the famous Friedmann–Lemaître–Robertson–Walker (FLRW) metric that includes a time dependent scale factor $a(t)$ for the space foliations:

$$g_{\text{FLRW}} = -dt \otimes dt + a(t)^2 ((1 - k \cdot r^2)^{-1} dr \otimes dr + r^2 d\Omega^2) \quad (3.2)$$

Here $d\Omega^2 = d\theta \otimes d\theta + \sin^2(\theta)d\phi \otimes d\phi$ denotes the spherical part of the metric. Furthermore, the constant $k \in \{-1, 0, +1\}$ represents the topological class of negative, zero, or positive curvature, respectively. In order for g_{FLRW} to be a solution to Einstein's field equation a suitable energy momentum tensor must exhibit the same symmetry properties in the spatial part. For the Friedmann equations one assume matter to be described by a perfect fluid for which $T^{\mu\nu}$ assumes the form:

$$T = (\rho(t) + p(t))dt \otimes dt + p(t)g \quad \text{with} \quad p(\rho) = w \cdot \rho$$

The additional relation between the density ρ and the pressure p , given by $p(\rho) = w \cdot \rho$, is a special case of an equation of state. Depending on the 'fluids' properties and main constituents w will assume different values during the phases of evolution of our Universe. Finally, one can show that g_{FLRW} defines an analytic solution for Einstein's field equation, the so-called Friedmann equations, which describe the time evolution of the scale factor $a(t)$ with respect to the sources $\rho(t)$ and $p(t)$:

$$\dot{\rho}(t) = -3 \left(\frac{\dot{a}(t)}{a(t)} \right) \cdot (\rho(t) + p(t)) \quad (3.3a)$$

$$\left(\frac{\ddot{a}(t)}{a(t)} \right) = -\frac{4}{3}\pi G_N (\rho(t) + 3p(t)) + \frac{1}{3}\Lambda \quad (3.3b)$$

Most of the cosmological considerations are derived from these equations. Matching the free functions and parameters to observations, the Friedmann metric gives rise to an space-expansion and probably (re-)collapse of the Universe and it predicts a time-like singularity, the Big Bang. With this application of Einstein's theory of GR to cosmology, it plays a central role in theoretical physics also in the beginning of the twenty-first century.

3.2 General aspects of Quantum Gravity

3.2.1 A motivation for theories of Quantum Gravity

The form in which Albert Einstein presented his theory of GR describes the field equations for a (pseudo-)Riemannian metric that corresponds to the geometry of spacetime and encodes the gravitational interaction in a non-trivial curvature. Though, extremely successful in its tested ranges, there are several attempts to reformulate GR in terms of different field content, more general topological settings, and find a more fundamental principle from which Einstein's field equation derive naturally. All of these approaches have to cope with the tremendous success of GR and should at least reproduce all those results in a particular limit.

Examples of the non-metric field content is the Einstein-Cartan theory, where the gravitational connection is more generally described by a principal connection on a frame bundle where torsion does not have to vanish and compatibility with a Riemannian metric is only a special case. Changes of the topology involve a discretization of spacetime and thus will only yield GR in an effective limit. The third motivation is rather concerned with a general prescription that explains the origin of Einstein's field equations from a more fundamental principle. In fact there are various theories, which may combine two or even all of these three reformulation attempts, which are close enough to compete with the experimental predictivity and validity of GR. Besides this, some approaches even cure some consistency problems intrinsically present in General Relativity.

Before motivating a theory of Quantum Gravity let us consider a very prominent example to reformulate GR in terms of a variational principle of an action functional, the Einstein-Hilbert action here stated for d -dimensions:

$$S_{\text{EH}}[g] = \frac{1}{16\pi G_N} \int_M d^d x \sqrt{|g|} (R - 2\Lambda) \quad (3.4)$$

If one adds a general action functional for the matter content $S_{\text{Mat}}[\phi; g] = \int d^d x \sqrt{|g|} \mathcal{L}_{\text{Mat}}$ the variational principle w.r.t. the metric assumes the form:

$$\delta_{\delta g} (S_{\text{EH}}[g] + S_{\text{Mat}}[\phi; g]) = 0 \quad \implies \quad G^{\mu\nu} + g^{\mu\nu} \Lambda = 8\pi G_N T^{\mu\nu} \quad (3.5)$$

Hereby, $T^{\mu\nu} h_{\mu\nu} = -2\sqrt{|g|}^{-1} \delta_h(\sqrt{|g|} \mathcal{L}_{\text{Mat}})$ directly derives from the matter Lagrangian. This relation between the Einstein-Hilbert action and GR was proposed by David Hilbert already in 1915, and provides a more elegant way to relate gravity with other matter interaction. However, generalizing GR usually comes along with new issues. In the present case the variational principle requires M to be a compact manifold without boundary. Any deviation from these additional assumptions will spoil the derivation of Einstein's field equation. In particular for the important class of compact spacetimes with a non-empty boundary, the variation of the Einstein-Hilbert action generates boundary terms for both Dirichlet or Neumann field constraints. The cure to this problem in case of Dirichlet boundary conditions was introduced by York, Gibbons and Hawking [78, 79], who suggest to add another compensating boundary term to the action functional, the Gibbons-Hawking-York term:

$$S_{\text{GHY}}[g] = \frac{1}{8\pi G_N} \int_{\partial M} d^{d-1}x \sqrt{|H|} K \quad (3.6)$$

The appearing geometrical objects describe the properties of the embedded boundary ∂M . Usually, one starts with an embedding hyperplane equation, $\Sigma(x) = 0$ whenever $x \in \partial M$. The derivative of this hyperplane describes deviations from the boundary and points in the normal direction w.r.t. Σ . It defines the outward pointing normal vector field $n^\mu = \partial^\mu \Sigma(x)$ that can be used to project tensor fields at $x \in M$ tangent to the boundary. One especially important 2-tensor field is the tangent metric $H^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$ which can be translated into the induced metric defined on the manifold ∂M . Finally, the trace of the extrinsic curvature $K^{\mu\nu} = D^\mu n^\nu$ is the source of the compensating derivatives $S_{\text{GHY}}[g]$ that is needed in order to compensate the obstructing boundary terms in the variational procedure of the Einstein-Hilbert action. It turns out that there is a fixed ratio for the coefficients of both terms that leads to a full cancellation of all boundary contributions, namely:

$$\int_M d^d x \sqrt{|g|} R + 2 \cdot \int_{\partial M} d^{d-1}x \sqrt{|H|} K$$

The additional factor of 2 in front of the Gibbons-Hawking-York term is mandatory in the classical theory.

As we have seen, General Relativity combines the geometry of spacetime with theories of matter and thus is a central piece in the formulation of Nature in terms of fundamental interactions. If GR is really part of the fundamental theories there are very severe constraints and requirements it has to pass. This involves consistency, as well as aesthetic conditions in order to match the quantum field character of the other 'forces'. While from an experimental perspective, there is no evidence which suggests a breakdown or an effective nature of GR, from a theoretical point of view there are quite a number of indications that we are only dealing with an extremely successful effective description. Therefore, let us consider Einstein's field equation again:

$$\underbrace{G^{\mu\nu} + g^{\mu\nu} \Lambda}_{\text{geometry}} \quad \underbrace{=}_{\text{relation}} \quad \underbrace{8\pi G_N T^{\mu\nu}}_{\text{QFT}} \quad (3.7)$$

There are three main statements in this equality. First of all, the geometric structure of spacetime is fully described by a suitable metric and a constant parameter Λ , which constitutes the

LHS. Second, the matter content, which in our today's understanding is formulated in terms of a QFT represent the RHS and is factor by a constant G_N . Finally, we have an equality which establishes a link between spacetime's geometry and the fundamental matter interaction. Intertwining different theories has important consequences, in particular if singularities or inconsistencies appear, since they ultimately affect both sides of the equation.

In our current understanding of Nature, the quantum character of the electroweak and strong interaction has undergone many experimental tests, so far outstandingly successful, in particular quantum electrodynamics. Nowadays, the Standard Model of particle physics seems to be fully confirmed within its range of validity, having found probably all predicted particles and revealing an extremely accurate coincidence of theory and experiment. This however describes a first, more conceptual, problem within equation (3.7). While the RHS seems to be best written in the language of QFT, the LHS is purely classical with no occurrence of $\hbar$, indicating that those contributions have to be taken care of in matter-matter interactions only. This is partly acceptable since gravity is special among the fundamental interactions in that it is interwoven with the geometry of spacetime, but still from an aesthetic point of view and the strong restrictions on the quantum nature of matter suggest to question this classical to quantum correspondence.

If we further take a look at a list of unsolved problems in physics we will see that many of the stated issues address the equality of eq. (3.7) where in a general or cosmological setting a theory of matter is connected to the curvature on spacetime. For instance, the hierarchy problem addresses the coupling constant G_N on the RHS of eq. (3.7). It defines the strength with which the gravitational interaction effects the matter theory and vice versa. Compared to the other fundamental interactions, gravity is by magnitudes weaker and only becomes relevant when all other effects have smeared out, as for instance on cosmological scales. Another interesting consequence of the equality sign concerns the cosmological constant Λ . There seems to be a clash between observational data setting an upper limit of $|\Lambda| < 10^{-52} \text{m}^2$ with the theoretical prediction one obtains when interpreting Λ as the quantum vacuum energy of some QFT in the matter sector. The discrepancy is such huge that there seems to be no way to justify the severe fine-tuning of Λ on the basis of the matter content.

Probably the strongest indication that at least some part of eq. (3.7) needs to be improved is the problem of dark matter and energy. Based on some cosmological model that derives from Einstein's field equations and involved techniques to chart the Universe one predicts the matter distribution in galaxies and beyond. These results however seem to be in sharp contrast with observations, for instance the flattening of the velocity for galaxy rotation curves which requires quite an amount of matter at 'dark' regions, meaning that the gravitational objects necessary to account for these effects have to interact very weakly, if at all, with respect to the electromagnetism, thus photons. From the field content of the Standard Model of particle physics there is no appropriate candidate that could be in charge of almost 27% of the energy content of the Universe, while the visible ordinary matter constitutes only about 5%. The missing energy 68% is also unexplained by the standard concepts and is denoted dark energy, which only interacts gravitationally. Introducing dark energy seemed to be unavoidable to match the cosmological observations of an almost flat Universe with an accelerated expansion, with theoretical predictions based on the observed (electromagnetically interacting) energy distribution. Again, one part of Einstein's field equations has to be modified in order to account for around 95% of the gravitational effects. One explanation is given by fields which are not yet described within the SM. This would suggest a refinement of the RHS of eq. (3.7), i.e. the matter content of the theory, for instance by introducing new particles. On the other hand, the geometrical side of Einstein's field equations could be only an effective description to which further terms should be added or even for which a completely different formulation is needed. This shows some very profound issues in the relation of QFT and GR on cosmological scales.

Finally, let us focus our attention on the geometrical part of Einstein's field equation and consider the properties of solutions for some very general matter sector. It turns out that under very natural conditions GR will be plagued by singularities, in that geodesic completeness is lost and thus GR predicts its own breakdown. This was first studied by Penrose and Hawking who gave certain conditions under which space- or time-like singularities form in GR [80–82]. Basically, one imposes a set of (causal) conditions on the global properties of spacetime, assumes the existence of a trapped region where gravity is such strong that light rays can not escape, and requires some energy condition to be satisfied by the matter content. For instance the weak energy condition which states that *light rays converge caused by gravitational effects*, (in other words the energy of matter is non-negative) is some very natural condition that seems to be respected during the entire evolution of our Universe, even in a possible inflationary phase. Combined with some natural causal requirements of spacetime the interior of a black hole for example will contain a singularity. Thus, rather than some exotic case in which GR experiences its limitations, singularities appear under very mild and natural conditions indicating that for strong gravitational effects modifications might become relevant.

Hence, there is some inconsistency within the current form of Einstein's field equations, either hidden in our understanding of the QFT character of matter, the geometrical description of spacetime or in relating both. In any way cosmology and even structural aspects suggest to improve Einstein's field equations in one or the other direction. Furthermore, mathematical aesthetics argue in favor of searching for a quantum theory of gravity to translate this interaction into the language of the others. Though it might well be possible that we have to further work on the matter content, one should keep in mind that gravity is only studied in a small window of scales, where all other interactions have smeared out. It would be a remarkable coincidence that Einstein found a theory that could be extended to higher and lower energy scales without any modifications just by observing a small fraction of scales and extrapolate the full landscape of gravitational physics. It is similar to considering a segment of a function's graph and guess its outer shape with almost no guiding principle. So far, we have not even seen which fundamental phenomena arise from gravitational interactions but only tested and deduced rules from a small accessible spectrum. The great virtue of gravity is that although high energy physics is still out of reach, it is possible to similarly test its fundamental structure encoded in the large scale structures of the Universe and find some further advice if GR is really fundamental or only effective in nature.

3.2.2 Field space of Riemannian metrics

One of the difficulties in studying quantum theories of gravity in the conventional way, is that the set of all gravitational fields, denoted field space $\mathcal{F}$, has a non-trivial structure when considered as a manifold. In this subsection, we briefly comment on this important aspect of Quantum Gravity which is responsible for some issues encountered in 'quantizing' General Relativity. For further details on the space of Riemannian metrics, we refer to [83] and references therein.

The most conservative assumption on the field content of gravity is the set of Riemannian metrics for a compact, oriented smooth manifold M . Therefore, consider the spaces of smooth tensor fields over M endowed with the Whitney C^∞ -topology, whereby only the second order covariant tensor fields $T^{(0,2)}M \equiv \Gamma(T^*M \otimes T^*M)$ will be relevant in what follows. Since M has a well-defined orientation, we consider the group action of $GL^+(d; \mathbb{R})$ on $T^{(0,2)}M$ and decompose this space in the respective irreducible components, i.e. into an anti-symmetric and symmetric part. The latter is part of the definition of a metric tensor field and in fact defines a smooth vector space which is globally flat from a geometrical point of view. In order to obtain a (pseudo-)Riemannian metric that defines an inner product on each tangent space of M , we have to impose an additional constraint that selects only the non-degenerate symmetric tensor fields. Thus, for

all $p \in M$, we define the following operator:

$$(\mathcal{B}_p(g))(V, W) \equiv (1 - |\text{sgn}(\det(g_p))|) + |g_p(V, W) - g_p(W, V)| \quad \forall V, W \in T_p M$$

The kernel of $\mathcal{B}_p(g)$ gives rise to the space of Riemannian metrics, whereby the second term is redundant if we already restrict to symmetric tensor fields only. The crucial ingredient in this constraint is the first term that requires a non-vanishing determinant and thus a non-degenerate bilinear form:

$$\text{Riem}(M) := \{g \in T^{(0,2)}M \mid g \in \ker(\mathcal{B})\} \quad (3.8)$$

This space of Riemannian metrics on M is constructed by a highly non-linear operator constraint, and thus we expect that $\text{Riem}(M)$ does not retain the global vector space properties of $T^{(0,2)}M$ or of its symmetric subspace. In fact, it turns out that $\text{Riem}(M)$ defines an open (convex) cone within the space of symmetric 2-tensor fields $\Gamma(S^2 T^*M)$. Though it is in general non-compact, this subregion is path connected and in case of Euclidean signature it is also simply-connected [83, 84].

For a given Riemannian metric, one can construct an inner product on $T^{(0,2)}M$ which for elements of $\text{Riem}(M)$ becomes positive definite:

$$(t_1, t_2)_g := \int_M d^d x \sqrt{g} \text{Tr}_g [t_1 t_2^T] \quad \forall t_1, t_2 \in T^{(0,2)}M$$

In combination with the Fréchet space property of $\text{Riem}(M)$ we can endow the space of Riemannian metrics with a Riemannian structure on its own, in particular the above inner product defines a natural Riemannian metric $\mathcal{G}$ on $\text{Riem}(M)$. The main difference to ordinary field spaces is the non-trivial structure of $(\text{Riem}(M), \mathcal{G})$ that gives rise to non-trivial geodesics and the necessity of the background field method. Furthermore, field space exhibits very interesting aspects related to the Levi-Civita connection associated to $\mathcal{G}$ which in the present case results in a non-vanishing field space curvature. The interesting geometry of metric field space is one reason why theories of gravitation are far more involved than standard QFTs one encounters usually.

R Notice that metric fluctuations live in the tangent space to some background field $\bar{g}$ which is isomorphic to $\Gamma(S^2 T^*M)$, since $\text{Riem}(M)$ is an open subset.

Remarkably, in the Euclidean case the exponential map is a diffeomorphism and thus a single background field $\bar{g}$ suffices to cover the full field space [84].

The physical field content, also denoted the space of geometries, corresponds to the base space of $\text{Riem}(M)$ considered as a principal bundle with structure group $\text{Diff}^+(M)$, the orientation preserving diffeomorphisms on M . Basically, one defines the group action of $\text{Diff}^+(M)$ on $\text{Riem}(M)$ by pullback and establishes an isomorphism between the space of vector fields and the Lie algebra of $\text{Diff}^+(M)$ using the Lie derivative. Then, the space of geometries $\text{Riem}(M)' / \text{Diff}^+(M)$ satisfies a number of differential topological properties, in particular there exists an induced Riemannian structure with an induced metric.¹ These smooth embedding of the space of geometries into $\text{Riem}(M)$ is guaranteed by Ebin's slice theorem [40].

There are many subtleties in studying $\text{Riem}(M)$, most of them are due to the infinite dimensionality of this space.² Thus, usually, compactness of M is required to ensure a kind of finiteness and all so far stated results are presented under this particular assumption [83].

¹Here, $\text{Riem}(M)'$ denotes the space of Riemannian metrics that admit no non-trivial isometries.

²Notice however that the space of Einstein metrics in $\text{Riem}(M) / \text{Diff}(M)$ is only finite dimensional.

Fortunately, there are extensions to pseudo-Riemannian metrics with compact support and most important in the current context to compact manifolds with boundary [85]. In this latter case one has to impose boundary conditions on the fields and the diffeomorphism group, which in ref. [85] were chosen to be of Dirichlet type. A similar slicing theorem was recovered and the above concepts could be extended to spacetimes with non-vanishing boundary.

3.2.3 Background Independence

Since in Quantum Gravity it is the geometry of spacetime itself which becomes dynamically, there is no a priori fixed background in which the field content evolves. Missing the arena a standard QFT description is placed in, already very basic constructions, as for instance integration over spacetime points, becomes conceptually difficult. There are several techniques how to implement this requirement of Background Independence into a consistent and still practical formalism. There are basically two directions one can pursue at this point:

The first possibility is literally abandon the idea of introducing any background at all and let spacetime emerge by itself. The advantage of this approach is the intrinsic implementation of Background Independence. However, at the same time one has to leave the realm of standard QFT which makes heavily use of a predefined geometry.

In the second case, one actually introduces a background field, $\bar{g}$, and applies the standard techniques, treating the metric fluctuations in the same way as matter fields on a rigid geometry. If field space allows for a smooth structure, this corresponds to the background field method [68]. However, in the end, one has to ensure that the explicit form of $\bar{g}$ does not affect the obtained results, but in fact can be understood as a mere technical trick to recover the standard QFT picture on intermediate steps. This describes both, the advantage and disadvantage of the second approach, where we have all the standard tools of QFT at our disposal, but have to put effort in re-establishing Background Independence, in this sense requiring that physics is literally independent on the background $\bar{g}$.

3.2.4 Boundaries in Quantum Gravity

Boundaries play a very important role in physical problems, for they represent regions, usually in spacetime, for which we have a certain amount of information at our disposal. A boundary, B is essentially a collection of points that defines an interior and possibly an exterior component, which in general describe conceptually different phases of the problem.

If these points can be combined in a smooth fashion that gives rise to a manifold these kind of problems are particularly interesting in studying differential equations. So, let us assume we can find an embedding of the boundary, B , in the Riemannian manifold (M, g) where g denotes a metric tensor field. Usually, this is based on a smooth hypersurface condition $\Sigma(x)$, with $x \in M$, a straightforward generalization of plane equations.

In order to geometrically classify the embedding defined by $\ker(\Sigma) \simeq B$, it is convenient to introduce the normal vector field (in our convention assumed to point outwards):

$$n^\mu(y) \equiv \partial^\mu \Sigma(y) \quad \text{with} \quad y \in \ker(\Sigma) \simeq B \quad (3.9)$$

Moving along the normal vector implies leaving the hypersurface, while in the orthogonal direction to n one explores Σ in more detail. This yields to a decomposition into tangent and normal vector fields, the former described by the tangential metric $H^{\mu\nu} \equiv \bar{g}^{\mu\nu} - n^\mu n^\nu$ which is simply the projection of the Riemannian metric g onto Σ .

A measure of the deformation Σ experiences within M is given by the *extrinsic curvature* tensor, that determines the rate of change in n :

$$K^{\mu\nu} = D^\mu n^\nu \quad \text{and} \quad K = \bar{D}_\mu n^\mu \quad (3.10)$$

Here D_μ denotes the covariant derivative w.r.t. g . As its name suggests, $K^{\mu\nu}$ defines the curvature or bending of the hypersurface as embedded into M . It has to be carefully distinguished from the intrinsic geodesic structure of B , the *intrinsic curvature* associated to the Riemann tensor, $R_{\mu\nu\rho\sigma}$, that is independent of any embedding. From the Gauß-Codazzi equation both concepts of curvature can be related, for instance as follows

$${}^dR|_\Sigma = {}^{d-1}R + (K^2 - K^{\mu\nu}K_{\mu\nu}) + 2D_\nu(n^\mu D_\mu n^\nu - n^\nu \bar{D}_\mu n^\mu) \quad (3.11)$$

Thus, the four-dimensional geometry of M , given by R , when evaluated on Σ , decomposes into the intrinsic and extrinsic curvature of Σ and a further total derivative. Thus, without knowing the topological structure of B , we cannot say which part of the curvature is due to the embedding and which part is intrinsically present in B .

This geometrical picture of hypersurfaces and thus general notions of boundaries, allows to study initial value problems in any (smooth) branch of physics. In the usual cases where the underlying space(time) is flat, $R = 0$, and B is a subset of the Euclidean space with a trivial embedding, the curvature of Σ can be completely associated to a non-vanishing $K^{\mu\nu}$. We then recover the generic sets of differential equations that have to be evaluated for a specific field configuration, or boundary condition. Thereby the boundary B or its embedding Σ has usually a physical significance. For instance B may be associated to the contact layer of two materials with distinct properties, to be explicit a conductor and an isolator. Or it may reflect the symmetry property of the configuration, e.g. spheres in an isotropic problem. In any case, the boundary describes a ‘region of knowledge’, meaning that we have a suitable amount of information on B which restricts the set of solutions of some (differential) equation, usually in a unique way.

For simplicity let us start with some differential equation $\partial_x^2 f(x) = J(x)$, which allows for, let say, an infinite number of solutions $f_{\text{sol}} \in \text{Sol}$. If this equation is related to a physical problem, we have to implement further constraints, either by experimental observation or additional theoretical requirements, in order to obtain a fully predictive description, i.e. $\#(\text{Sol}_{\text{phys.}}) = 1$. Depending on the explicit setting, we thus define a hypersurface Σ on M where we have access to further information, for instance we know that it represents the surface of a conductor, and then evaluate the class of mathematical solutions $f_{\text{sol}} \in \text{Sol}$ on Σ to see if they satisfy the additional constraint. If $\text{Sol} \equiv \emptyset$, the system is overdetermined and either the underlying physical law does not described Nature, or there is some inconsistency in the experimental results.

Thus, boundaries are not at all restricted to describe spatial edges of spacetime, but in fact occur in almost any problem based on differential equations, as an example. It represents a way to implement physical configurations or observations into the mathematical framework by feeding the system with an additional amount of data. The imposed boundary conditions can assume any form, however the most familiar ones are of Dirichlet and Neumann type, where either the solution itself or its derivative is fixed on Σ . Apart from local generalizations, so called mixed conditions, there are also non-local forms, where the integrated value of the solution is fixed, for instance the total electric charge on a surface. In the realm of QFT the most popular example of boundary conditions is the Casimir effect, where two conduction planes are ‘exposed’ to the vacuum fluctuations of the electromagnetic field and thereby attract each other.

In the context of this thesis, the most interesting applications are found in cosmology. There, boundaries usually occur either as real spatial regions in our four dimensional world, or as a Cauchy hypersurface, which describes an ‘instant of time’. A vividly discussed proposal of Hartle and Hawking [86] suggests to study a kind of quantum ground state of the Universe, $\Psi_0(H)$, which is a sum over all spatially closed geometries, $\text{Riem}(M)^{\text{sp.cl.}}$ described by metrics g that give rise to the induced (spatial) metric H on some Cauchy surface Σ_{t_i} :

$$\Psi_0(H) \propto \int_{g \in \text{Riem}(M)^{\text{sp.cl.}} \wedge g|_{\Sigma_{t_i}} \equiv H} d[g] \exp(-S_E[g]) \quad (3.12)$$

This describes a boundary value problem related to the Wheeler-DeWitt equation [86] which then should yield the full evolution of the Universe.

Another important example is found in the correspondence of the statistical mechanics of black holes and its thermodynamic description, relating the entropy to the area of the horizon. The latter represents a natural separation between two different ‘phases’, the inner and outer region and their corresponding sets of solutions.

Studying boundaries is thus a very central aspects in understanding mathematical equations for physical applications. A theory of Quantum Gravity that incorporates the possibility to treat those generalized spacetime geometries is thus an essential tool to describe quantum effects in cosmology and study the evolution and history of our Universe. For further reading on the value of boundaries in physics we refer to [26].

3.2.5 Euclidean Quantum Gravity

Finally, let us briefly comment on the Euclidean formalization of Quantum Gravity. For flat spacetime, there is a general framework how to transit from the Lorentzian picture to its Euclidean analog and vice versa. The Wick rotation $t_L = it_E$, of an Euclidean theory will result in a proper QFT if the Osterwalder-Schrader axioms are satisfied. The advantage of the Euclidean formulation in favor of the Lorentzian description is found in the properties of the respective symmetry group, which is only compact in the former case. In particular from the mathematical perspective, the Euclidean construction is thus much better under control, in particular the study of boundary value problems for elliptic operators becomes accessible. The problem is that Nature seems to be based on a Lorentzian geometry and thus we have to convert the Euclidean results back to spacetimes of pseudo-Riemannian structure. While as mentioned above this prescription is well understood on flat space, in Quantum Gravity we are dealing with a generic metric g for which there will be no global time-like Killing vector field. This however, is crucial for the standard Wick rotation and thus the relation between both description in the general case is still subject of discussion [26].

3.3 Theories of Quantum Gravity

One remedy to cure the problems intrinsically present in Einstein’s field equation is to reconcile quantum physics with General Relativity by means of a quantum theory of gravity. There are standard techniques to quantize a classical theory, which turn out to be inappropriate for gravity. The obstruction can be traced back to the (re-)normalization requirements one imposes on suitable QFT, which on a perturbative level is incompatible with a quantum version of GR. A simple power-counting argument demonstrates that the theory is perturbative non-(re-)normalizable due to the negative mass dimension $(2 - d)$ of Newton’s coupling G_N . In the traditional sense a QFT should require only a finite number of counterterms that remove divergent contributions in loop integrals, which corresponds to only a finite number of free parameters that have to be fixed by experiments. In case of gravity, this is perturbative procedure fails, for in each higher loop correction supplementary counterterms are needed to compensate divergences. Hence, while towards higher energy scales the number of free parameters approaches infinity, the predictive power of the quantum theory tends to zero [70, 87, 88].

After accepting that the standard procedure to link a classical theory with its quantum analog is not applicable for GR, there are several ways to pursue to address the problems describe in the previous section. First, one may focus on the matter sector instead and retain General Relativity in its classical shape. Despite of this, one can also think about modifications of GR by introducing higher order curvature terms, for instance, and then apply the standard quantization schemes. It turns out that in certain cases indeed perturbative (re-)normalizability is reestab-

lished however with the price of loosing unitarity. The great virtue and also drawback of the classical theory is the diversity of different mathematical formulations and physical approaches that are consistent with the experimental results and somehow are closely related to GR. Therefore, one has various opportunities from which to construct a theory of quantum gravity and if one direction fails, one may try a different route. In the following we give a incomplete list of approaches that all share the same objective, namely reconciling the laws of quantum physics with the gravitational interaction, while the applied methods differ sometimes quite significantly. Thus, we encounter concepts which stay very close to the standard procedure while modifying the classical field content severally, or in contrast constructions which question the quantization scheme while being conservative with the general setting of the physical content. Besides the ultimate goal to interconnect the light and small scales of Nature described by quantum physics with its long and heavy scales covered by GR, all these approaches have in common that they give rise to a new perspective on the whole of physics, new mathematical techniques, and interpretation of the geometric structure of spacetime. In particular objects that combine the heavy and small features, as for instance black holes, are probably the best candidates to test and ultimately distinguish between the various directions in order to bring scientists back on the right track.

3.3.1 String theory

The basic idea of string theory is that the elementary particles that appear in the SM can be divided one step further, revealing a one-dimensional string structure rather than a simple zero-dimensional point [89–92]. The necessary resolution to its discovery would involve detections on the level of Planck length scales above which the standard picture of SM is recovered. The vibration of the strings then correspond to the different particles which also includes the mediator of gravity, the graviton. String theory turns out to be a very involved formalism that needs highly advanced mathematical concepts, which has led to fruitful contributions in both mathematics and physics in general. In particular the discovery of duality within string theory, most famous the AdS/CFT correspondence, has brought new aspects and insights by relating seemingly different theories by means of a duality transformation that ensures certain aspects being equivalently described in one or the other form. However, very often the mathematical level is also a drawback since most of the necessary techniques are still under development. For instance, the scattering of strings is so far based on perturbative methods and the reliability of those results has to be confirmed by a non-perturbative treatment. Besides the replacement of point particles, by closed or open strings the theoretical setting is based on supersymmetry, implementing fermions and bosons in a symmetric way. In addition, in order to have a massless photon, for instance, spacetime has to have at least 11 dimensions, where the additional, unobserved dimensions are either compactified on very small scales (loosely speaking rolled up) or thought of as being quite big whereby our familiar four dimensional spacetime is only a (boundary) slice, the so-called braneworld theory. These somehow exotic supplementary ingredients and the small predictivity power due to the emergence of a huge number of equally likely universes are the most severe difficulties string theory has to deal with.

3.3.2 Loop Quantum Gravity

We have already mentioned that there is an alternative formulation of GR in terms of non-metric field content, that even allows for torsion and a more general connection: the Einstein-Cartan theory. Thereby, the Einstein-Hilbert action gets replaced by the Palatini-Holst functional, i.e.

$$S_{\text{P-H}}[e, \omega] \equiv \frac{1}{32\pi G_N} \int \varepsilon_{jkl n} e^j \wedge e^k \wedge (\Omega^\omega)^{ln} - \frac{1}{16\pi\gamma} \int e^j \wedge e^k \wedge (\Omega^\omega)_{jk}$$

Here, e represents the tetrad, ω the connection, $\Omega^\omega = d\omega + \omega \wedge \omega$ the curvature 2-form and γ the Immirzi parameter factoring the Holst action part of $S_{\text{P-H}}[e, \omega]$. Based on this description of the classical theory of gravity, Loop Quantum Gravity (LQG) applies a canonical quantization using Ashtekar variables which basically puts quantizing matter and spacetime on an equal footing [72, 93]. Therefore, one uses a Arnowitt-Deser-Misner (ADM) foliation splitting the four spacetime into three dimensional space slices with induced metric H_{ab} and a time-like direction including a shift and lapse function $\vec{N}$ and N , respectively. The evolution of the space foliations is then described by commutation relations between the (densitized) triads of H_{ab} and a $SU(2)$ related spin connection A . Before the discovery of the $SU(2)$ gauge group within the ADM formalism by Ashtekar there was little to no progress in this field due to the very difficult nature of the Hamiltonian constraints. The introduction of the Ashtekar variables lead to a significant simplification and thus brought LQG back to the game of describing a theory of gravity. The great advantage of this approach is its Background Independence and its non-perturbative nature. Following LQG space can be understood as some spin network that evolves with time or in a covariant formalism, as a spin foam that exhibits a porous structure of spacetime. The challenge of LQG is to recover the low energy (effective) theory from its fundamental principles. While in general this is still a very difficult task, for cosmological models involving homogeneous and isotropic space constraints there are predictions based on LQG calculations which for instance change the picture of the big bang to a big bounce and thereby removing the corresponding singularity.

3.3.3 Causal Dynamical Triangulation

A background independent approach denoted Causal Dynamical Triangulations (CDT) attempts to derive the structure of spacetime from first principle [94–98]. Basically, spacetime is modeled by a set of discrete causal simplices which describe a local patch of flatness while (causally) gluing them together yields a non-trivial global geometry. One initially replaces the formal path integral for GR with a lattice regularized version involving the Regge-action and a sum over all causal triangulations.

$$\int d[g] \exp(\iota S_{\text{EH}}[g]) \xrightarrow{\text{CDT}} Z[\{\kappa_0, \kappa_4, \Delta\}] \equiv \sum_{\tau_{\text{causal}}} \exp(-S_{\text{R}}[\tau; \{\kappa_0, \kappa_4, \Delta\}])$$

Hereby, the causality requirement significantly restricts the number of possible triangulations. Furthermore the global proper time foliation is used to perform a Wick rotation thus that the exponent is a real valued action functional, consisting of:

$$S_{\text{R}}[\tau; \{\kappa_0, \kappa_4, \Delta\}] = -(\kappa_0 + 6\Delta)N_0(\tau) + \kappa_4(N_4^{(4,1)}(\tau) + N_4^{(3,2)}(\tau)) + \Delta(2N_4^{(4,1)}(\tau) + N_4^{(3,2)}(\tau))$$

The coupling constants κ_0 , κ_4 , and Δ correspond to $1/G_{\text{N}}$, Λ/G_{N} , and an indicator for an asymmetry of time- and space-like link lengths, respectively. Furthermore, $N_0(\tau)$ denotes the total number of vertices in the triangulation, while $N_4^{(\bullet)}$ represents the number of $(\bullet)$ simplices within τ . The function $Z[\{\kappa_0, \kappa_4, \Delta\}]$ for arbitrary dimensions d is then computed using Monte Carlo simulations, which reveals a phase diagram with three different phases. Remarkably a classical four-dimensional de Sitter universe is discovered, which was not implemented by the algorithm itself but emerged from the causal structure of triangulations. Future Monte-Carlo simulations can hopefully extend this region to a second order phase transition at which a suitable continuum limit can be performed. Conceptually, CDT has a certain overlap with LQG and even qualitatively with the Effective Average Action (EAA) approach we study in this thesis, for the latter see [99, 100].

In the next chapter we present framework to construct a theory of quantum gravity that is based on the Effective Average Action approach employing the Functional Renormalization

Group Equation to study the Asymptotic Safety conjecture for Quantum Gravity. The following two parts of this thesis are devoted to an in depth study of a metric based calculation within the Asymptotic Safety program. At the same time the derivation of the Renormalization Group evolution demonstrates quite nicely where the technical difficulties reside in studying theories of quantum gravity and which part is conceptually universal and thus applicable in various branches of physics.

3.3.4 Experimental verification

Before we conclude our incomplete list by stating the current results of the Asymptotic Safety program to QG in the next chapter, let us shortly comment on experimental search for physics beyond classical General Relativity, see ref. [101–104] for further details. The diversity and difference of the various approaches towards a quantum theory of gravity already indicates that there is a significant lack of experimental data, which would exclude at least a certain number of theoretical directions. Opposite to many other inventions in theoretical physics, the concept of Quantum Gravity is motivated more on a technical and mathematical ground than on a failure on the phenomenological side. Although there is a sign of incompleteness in Einstein's field equations, for instance the prediction of dark matter and -energy, it is so far completely unclear whether or not the discrepancy is due to the geometrical part of the equation. Nevertheless, one attempt of every quantum version of GR is the understanding and possible resolution of the prescribed classical issues. From the theoretical perspective it seems plausible to go beyond GR and reconcile quantum physics with the geometry of spacetime. However, in absence of any experimental justification all descriptions are mere hypothesis or speculations. The only guiding principle that remains is intrinsic consistency and the emergence of GR as an approximation to the low energy physics.

While the quantum nature of all other interactions contained in the Standard Model of particle physics have passed several tests on energy scales now up to the $\sim 10\text{TeV}$ scale, the difficulty with gravity is related to the hierarchy problem, namely the comparatively weak coupling of gravity. In fact, one assumes that quantum corrections for the gravitational interaction become relevant only at the Planck scale, i.e. $\ell_{\text{Planck}} = \sqrt{G_N}$ or $m_{\text{Planck}} = 1/\sqrt{G_N}$. If the classical value of Newton's constant is inserted one obtains a length scale of around 10^{-35}m and an energy scale of about 10^{16}TeV . With a current maximum collision energy of 13TeV there is little hope that on-Earth experiments may answer the puzzle of a quantum nature of gravity within the next decades, if at all. Nevertheless, a number of quantum theories of gravity change the classical value of the Planck scale, which seems to be natural since an observational result is extrapolated over 15 orders of magnitude, an assumption rather unlikely. Hence, there are even theories that predict measurable effects on Large Hadron Collider (LHC) energy scales which should be confirmed or excluded in the near future.

Fortunately, the cosmos provides another playground to test gravity, for it contains many objects that combine heavy and small properties and thus involve large curvature deformation of spacetime. Furthermore, on cosmological scales, the gravitational interaction is in a sense isolated from the other interactions, which have smeared out due to their compensating charges. However, compared to on-Earth experiments, there is no control over the experimental setting which has to be modeled on some cosmological assumptions and the reproducibility is usually not given. For instance, supernovae observations or lensing effects due to massive black holes are good candidates to study the effect of strong spacetime deformations.

The Cosmic Microwave Background (CMB) is an example where large scale structures may reveal the nature of the smallest quantum fluctuation of gravity. It is a 'footprint' of the very early form of the Universe before a possible inflationary phase sets in and leads to a rapid expansion. The remarkable observation in the temperature density pattern of the CMB is its

uniformity, indicating that even the most departed galaxies were once in equilibrium with each other and thus causally connected. (Inflation was actually introduced to account for this connection by a phase of rapid expansion.) Still this sea of uniformity contained very tiny seeds of impurities causing the emergence of galaxies and its clusters, which are assumed to represent the ‘frozen’ quantum fluctuations at the moment when inflation sets in. Thus, deep inside the CMB we can search for imprints of quantum fluctuations. In addition, we may find answers about the field content of the classical (effective) theory, by detection of gravitational waves that requires tensor modes.

These are the most promising candidates to search for experimental evidence of theories of quantum gravity: either we look for space-like strong curvatures, or we study the physics close to a possible Big Bang by considering primordial remnants.

- R** On the other hand, observations that constraint the classical theory will also affect quantum theories for they have to reproduce this effective theory in a certain limit. Thus, experiments that underpin or disprove the very fundamental requirements of GR have also some distinguishing properties for the variety of quantum theories of gravity. For instance a violation of Lorentz invariance of the ground state of gravity would imply quite different quantum generalizations than those discussed usually.



4.1	Renormalization group Theory space General aspects of the Renormalization Group
4.2	Infrared-cutoff implementation Eigenmode expansion Rescaling relation between $\bar{\Phi}$ and τ Smooth cutoff implementation
4.3	The Functional Renormalization Group Equation The derivation Properties of the FRGE The normalization factor
4.4	Renormalizability & Asymptotic Safety
4.5	Truncations

4. FUNCTIONAL RENORMALIZATION GROUP

The success of the generating functional approach in studying field theories can be mainly attributed to its universal applicability. It is used to describe the Standard Model of particle physics and at the same time it is suitable to discuss statistical field theories on very different scales. The abstract and general structure of this approach is at the same time the source of its severest problems which bother physicists since its arising. Measure theory in combination with the infinite dimensional field space demands a thorough regularization procedure and the practical manipulation of observables is usually beyond the scope of today's mathematics. Thus, instead of direct computations addressing the full integration, one systematically reduced the difficulties of its evaluation by approximating the underlying measures in a suitable way. The hope is that extrapolating the obtained results may lead to predictions far beyond the involved scales. With those techniques, known under the name of perturbation theory, we have gained enormous and very fruitful insights into all interactions up to the TeV scale, except for gravity. While for the Standard model interactions the described approximation methods seem to be appropriate, at least on certain scales, there is a severe conceptual problem in studying a quantum theory analog of General Relativity. The prominent role gravity plays in current theoretical physics is also partly due to the insight one gains in revealing the limitations of the standard descriptions and in the encouraging work to put the current techniques on a solid ground. From our point of view it is not that the present results suggest gravity to be not renormalizable, but rather that we have to go beyond the standard, perturbative, techniques and enter the realm of non-perturbative effects to judge the ultraviolet behavior of theories.

In this chapter we present a brief introduction into a very promising, intrinsically non-perturbative method related to the block spin transformations of Kadanoff [105] and its generalization by Wilson [106–108] and Polchinski [109]. Together they lay the ground for a very powerful, universal, though highly non-trivial functional differential equation, the FRGE, which – since almost 20 years – is also available for non-perturbative studies of quantum gravity, with remarkable success.

We focus here on its implementation based on the Effective Average Action (EAA) [110–114] and refer the reader to the extensive literature on the full variety of this subject. An excel-

lent review on the foundations and different shapes of Exact Renormalization Group Equation can be found in [115], for instance. At the very beginning we introduce the main idea of the Renormalization Group (RG) and give a definition of RG steps along the lines of the previous chapters. In the original form, these RG transformations correspond to a sharp cutoff. In order to obtain a function differential flow equation it has to be replaced by a smooth implementation on which we focus next. Subsequently, we come to the main part of this chapter and study the Functional Renormalization Group Equation, aka flow equation. We first establish a relation between this highly non-trivial differential equation for the EAA and the path integral procedure based on the generating functional. We briefly comment on its properties and present a short discussion on the flow on probability theories, which involve an additional normalization factor. To cope with its universal character, we keep the notation in this section very general and postpone the particular important case of Quantum Gravity to a later section. In addition we allow for spacetime manifolds with a non-empty boundary. We then address the notion of renormalizability in the context of the FRGE and a brief summary on the results for Quantum Gravity. This chapter and the entire first part of this thesis concludes with some remarks on truncations which we study in detail for the bi-metric-bulk – pure-background-boundary ansatz in part II.

4.1 Renormalization group

In studying Nature we see that it reveals itself in very different shapes depending on the scales of observation. While in our everyday life we are mainly confronted with effective electromagnetic and a bit of gravitational forces, exploring the world towards larger or smaller scales shows fascinating new phenomena, either dominated by gravity or by the joint product of electroweak and strong interactions, respectively. This scale dependence of the Universe is expressed in terms of effective descriptions which model part of a more fundamental underlying theory. Like the narrow window we cover with our sense of sight, the visible part of the electromagnetic spectrum, most of our today's understanding of Nature relies on relatively small fractions of its full scale structure. Fortunately, we can observe signs of the underlying fundamental interactions by their influence on macroscopic scales and by testing the limits of our current models. Furthermore, there are also promising indications from the opposite direction, suggesting that on cosmological scales, the Universe behaves as a very symmetric isotropic and homogeneous system. Nevertheless, scientists are always searching for a way to extend the boundaries of knowledge by digging deeper into the structures and foundations. Depending on how stable the physical laws behave under a change of scale, we will be relative successful in extrapolating our macroscopic conception to larger and smaller structures. Usually, the range of validity is quite restricted and we rapidly touch the edge where conceptual new effects become important.

A very successful guiding principle was discovered by the path integral approach that in certain cases promotes a classical theory to its quantum counterpart and thus translates physics on macroscopic scales to a microscopic level. However, the technical difficulties have shown that in order to take account of the scale dependence of Nature there are different methods necessary that are particularly designed to give rise to a mathematical realizations of the 'scale-evolution' of physical laws. As an example consider Quantum chromodynamics (QCD), the theory of strong interactions. On a fundamental level, in the ultraviolet, the associated path integral $Z_{UV}[J; \bar{\Phi}]$ is situated in the vicinity of the free theory which makes perturbative studies perfectly suitable. Decreasing the resolution we recover effective theories which exhibit very different structures, $Z_{p^2}[J; \bar{\Phi}]$, until in the infrared its elementary constituents given by quarks and gluons are tightly bound by what is known as confinement, $Z_{p^2=0}[J; \bar{\Phi}]$. In this way, we have a set of theories most suitable in certain branches of energy or momentum scales. Though seemingly very different, they all descent from the same fundamental interaction $Z_{UV}[J; \bar{\Phi}]$ and at

least in principle can be recovered from this theory. This patchwork of effective descriptions can be made systematically in that we construct a map $p^2 \mapsto Z_{p^2}[J; \bar{\Phi}]$ that interpolates between the UV- and IR-formulation of the underlying interaction. If all functionals happen to be defined over the same field space and symmetry class, this amounts to a sequence of flow on theory space to which we turn next.

4.1.1 Theory space

As we described in the chapter on QFT, 2, the necessary ingredients to specify a gauge theory are given by a concept of spacetime M , a symmetry principles G , and a set of vector spaces which can be geometrically promoted to a smooth structure F defining field space. Under these assumptions any theory which is compatible with this physical content can be associated to a measure μ , or a generating functional $Z[J; \bar{\Phi}]$, on the σ -algebra (e.g. Borel sets) of F . For later convenience suppose we can transfer all information contained in $Z[J; \bar{\Phi}]$ to its effective action $\Gamma[\varphi; \bar{\Phi}]$ by means of the Legendre-transform. The respective set of all $\Gamma[\varphi; \bar{\Phi}]$ that are related to some suitable $Z[J; \bar{\Phi}]$ defines what is known as theory space:

$$\mathcal{T}(M, G, F) := \{A[\varphi; \bar{\Phi}] \mid \text{invariant under } G, \text{Diff}(M), \text{ and } \mathcal{G}\} \quad (4.1)$$

A similar concept, requires in addition that the underlying measure μ is probabilistic, implying in particular $\mu(F) = 1$. If we impose this further constraint theory space reduces to

$$\mathcal{T}_p(M, G, F) := \{A[\varphi; \bar{\Phi}] \in \mathcal{T} \mid \partial_\nu A[\varphi; \bar{\Phi}]|_{\varphi_0} = 0 \Rightarrow A[\varphi_0; \bar{\Phi}] = 0\} \quad (4.2)$$

Both spaces are in general infinite dimensional and inherit, at least on finite subsets, a smooth vector space structure from its target space $\mathbb{R}$. By specifying a basis on $\mathcal{T}$ it is thus possible to distinguish different theories by their coefficients. For a scalar functional, a basis $\{\mathcal{P}[\varphi; \bar{\Phi}]\}$ is constructed by means of field monomials which are compatible with the symmetry requirements. If we have a scalar field content, ϕ , with a trivial tensor structure, then the non-local part of theory space can be expanded in the basis invariants $\{\int \phi, \int \phi^2, \int \phi^3, \dots, \int \phi \bar{D}^2 \phi, \int \phi^2 \bar{D}^2 \phi, \dots\}$. Notice that depending on the symmetry group G and the presence of a non-empty boundary $\partial M \neq \emptyset$, this set reduces or extends. However, for general field content, this procedure is far less intuitive. Due to the absence of a suitable inner product on $\mathcal{T}$ iteratively deriving a linear independent basis $\{\mathcal{P}[\varphi; \bar{\Phi}]\}$ is more elaborated. There is a different method that makes use of other well studied vector spaces, for instance $\text{Gl}(d)$, to explore $\mathcal{T}$. We have briefly mentioned a technique based on Young diagrams [46] to systematically classification a set of linear independent $\mathcal{P}[\varphi; \bar{\Phi}]$'s that can be used to study metric based theory spaces, see section 1.5.

Each allowed action functional $A[\varphi; \bar{\Phi}] \in \mathcal{T}$ (or $\in \mathcal{T}_p$) defines a point in theory space, which for a specific set of basis invariants $\{\mathcal{P}_{(\alpha)}[\varphi; \bar{\Phi}]\}$ can be, at least formally, expressed in terms of their associated coefficients $\{u^{(\alpha)}\}$, i.e.

$$A[\varphi; \bar{\Phi}] \simeq \sum_{\alpha} u^{(\alpha)} \mathcal{P}_{(\alpha)}[\varphi; \bar{\Phi}] \quad (4.3)$$

Different action functionals $A[\varphi; \bar{\Phi}]$ and $A'[\varphi; \bar{\Phi}] \in \mathcal{T}$ give rise to different coordinates $\{u^{(\alpha)}\}$ and $\{u'^{(\alpha)}\}$, respectively. If the chosen basis $\{\mathcal{P}_{(\alpha)}[\varphi; \bar{\Phi}]\}$ is implicitly understood, we can thus use the parameter notation $A[\varphi; \bar{\Phi} | \{u^{(\alpha)}\}]$ and $A'[\varphi; \bar{\Phi} | \{u'^{(\alpha)}\}]$ to denote $A[\varphi; \bar{\Phi}]$ and $A'[\varphi; \bar{\Phi}]$, respectively. In chapter 2 we made heavily use of this fact without explicitly stating the underlying class of basis invariants.

R Notice that $\{u^{(\alpha)}\}$ are combinations of the coupling constants encountered in QFT and as such they measure the importance of the related field monomial in the overall theory.

This formal description of the class of theories allows to study sequences of measures in a topological and sometimes smooth way. Perturbative methods provide examples of this prescription where a family of theories $Z[J; \bar{\Phi} | \{u\}]$ is studied by defining a map for its coordinates $\{u\}$. If this amounts to a continuous, parametrized construction it gives rise to a connected path on $\mathcal{T}(M, G, F)$. In fact, at the end of this chapter, we will focus on smooth trajectories on theory space which solve a functional differential equation. It provides a way to systematically explore theory space and its (non-perturbative) renormalizable regions.

R Notice, that in the perturbative case, one concentrates on a small (local) region of the Gaussian measure and then one tries to extrapolate the fate of the underlying sequence to global scales. Thus one only studies a fraction of all theories.

Finally, let us mention that with the requirements of symmetry and spacetime invariants, we have severely restricted the class of possible theories, still there remain in general an infinite number of linearly independent basis invariants. Each of its elements defines a perfectly suitable theory from what we have discussed till now. By means of experiments, we have further constraints which may help identifying the ‘preferred coordinates’ of the theory realized in Nature. With each measurement we can fix a coefficient for some of the field monomials until (in the finite case) there is no freedom left. If we can impose additional constraints for which only a finite number of free coefficients remain the theory can be considered as complete or predictive.

In the sequel we study a method that is particular suitable in exploring the structure of theory space, the Renormalization Group approach and then classify trajectories (evolution of theories) according to their limiting behavior. In the end, we describe the notion of renormalizability that generalizes the perturbative concept in a very natural way which in fact gives rise to strong evidence for a non-perturbatively renormalizable theory of QG.

4.1.2 General aspects of the Renormalization Group

There are different implementation of the Renormalization Group procedure which have in common that they describe a path in $\mathcal{T}$ connecting the fundamental UV theory with its full effective description. For the block spin transformation and the Wilsonian RG approach, the idea is to iteratively evaluate a preexisting UV theory by absorbing the microscopic effects up to a scale τ in an effective, averaged theory and continue this procedure in a subsequent manner down to the infrared. We will see that introducing the Effective Average Action and computing its RG evolution using the FRGE actually leads to a prediction of the fundamental UV theory, related to the bare action.

However, let us first consider the general receipt to construct suitable sequences $Z_\tau[J; \bar{\Phi}]$ or $\Gamma_\tau[\hat{\phi}; \bar{\Phi}]$ in a systematic and consistent way. The corresponding path is parametrized by the dimensionless renormalization group scale, $\tau \in S \subseteq \mathbb{R}$, which is related to its dimensionful counterpart k in a canonical way, i.e. $\tau = k/k_0$. If we agree on a particular basis in theory space $\mathcal{T}$ or in its quantum counterpart, $\hat{\mathcal{T}} \ni Z_\tau[J; \bar{\Phi}]$, we may express each of its elements by a full set of coefficients or couplings, i.e.

$$Z[J; \bar{\Phi} | \{u_j\}] \equiv \mu(F | J; \bar{\Phi} | \{u_j\}) = \int_{\hat{\phi} \in T_{\hat{\phi}} F} d\mu_G(\hat{\phi}; \bar{\Phi}) \rho(\hat{\phi}; \bar{\Phi} | \{u_j\}) e^{J(\hat{\phi})}$$

Instead of approximating $\rho(\hat{\phi}; \bar{\Phi} | \{u_j\})$ by $1 + \dots$ as in the perturbative case, which usually leads to undefined intermediate results, we proceed in a different direction and follow a path that seems more under our control. The basic idea is to establish a flow on the space of theories by endowing the σ -algebra of field space with an order structure. That is, we define a filtration

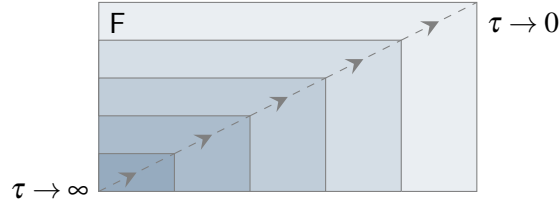


Figure 4.1: The RG procedure visualized as a filtration over field space. With decreasing scale τ we include more fields in the functional integral.

on Σ_F that controls the amount of field modes integrated out:

$$\text{rg} : S \subseteq \mathbb{R} \rightarrow \Sigma_F, \quad \tau \mapsto \text{rg}(\tau) := U_\tau \in \Sigma_F \quad \text{with} \quad U_{\tau'} \subseteq U_\tau \quad \forall \tau' > \tau \wedge U_{\inf\{S\}} = F$$

Loosely speaking, one chooses a sequence of measurable sets $\{U_\tau\}$ that decrease in their size when τ is increased and for which every set is contained in its precursor, see fig. 4.1. This procedure defines a ‘coarse-graining’ of the functional integration, we will consider in a moment.

A Renormalization Group step consists of a coarse-graining and a subsequent field rescaling part. In technical terms, we define the following τ -dependent generating functional:¹

$$Z_\tau[J; \bar{\Phi} | \{u_j\}] := N_\tau[\bar{\Phi} | \{u_j\}] \int_{\hat{\Phi} \in U_\tau} d\mu_G(\hat{\Phi}; \bar{\Phi}) \rho(\hat{\Phi}; \bar{\Phi} | \{u_j\}) e^{I(\hat{\Phi})} \quad (4.4)$$

The differences to the ordinary Fourier transform of μ , i.e. $Z_\tau[J; \bar{\Phi} | \{u_j\}]$, are the coarse-graining contribution, resulting in the changed integration range, $U_\tau \subseteq T_{\bar{\Phi}}F$, and the subsequent rescaling of the fields which is covered by the pre-factor $N_\tau[\bar{\Phi} | \{u_j\}] =: N_\tau[\bar{\Phi}]$ which satisfies $N_0[\bar{\Phi}] = 1$. Notice that in the limit $\tau \rightarrow 0$ we recover $Z[J; \bar{\Phi} | \{u_j\}]$ since then both RG effects vanish.² Likewise, on the level of the theory, the measure μ , we obtain the following sequence

$$\tau \mapsto \mu_\tau \quad \text{with} \quad \mu_\tau(A) = N_\tau[\bar{\Phi}] \int_{x \in A} d\mu(x) \chi_{U_\tau}(x) = \int_{x \in A \cap U_\tau} d\mu(x) \quad \forall A \in \Sigma$$

The measures, μ_τ and μ are in general not equivalent. In fact, all μ_τ are absolutely continuous with respect to the original (positive) theory μ meaning that for all A with $\mu(A) = 0$ it follows that $\mu_\tau(A) = 0$. If we omit $N_\tau[\bar{\Phi}]$, then, due to countable additivity of μ , the RG filtration generates a monotone sequence on $\Sigma_{[F]}$:

$$\mu(A) \equiv \mu_0(A) \geq \mu_{\tau_1} \geq \mu_{\tau_2} \geq \dots \geq \mu_{\sup\{S\}} \quad \text{with} \quad \tau_1 \leq \tau_2 \leq \dots \in S$$

These results also translates to the τ -dependent generating functionals. A very important observation for theories that are restricted to be probabilistic is that $N_\tau[\bar{\Phi}]$ is a mandatory τ -dependent normalization factor which ensures that for each τ we have $Z_\tau[0; \bar{\Phi} | \{u_j\}] = 1$. In this way theories at different RG scales can be directly compared. Furthermore, expectation values of constant observables are unaffected, for instance $\langle \bar{\Phi} \rangle_{\mu_\tau} \equiv \bar{\Phi}$, while in case of $N_\tau[\bar{\Phi}] \equiv 1$ we would obtain $\langle \bar{\Phi} \rangle_{\mu_\tau} \equiv \bar{\Phi} Z_\tau[0; \bar{\Phi} | \{u_j\}]$ which now becomes τ -dependent. However, in theories for which Background Independence is an issue, $N_\tau[\bar{\Phi}]$ is an additional source of violation, though vanishing for $\tau = 0$.

At this point it is instructive to notice the difference between perturbative approximations of μ , the Wilsonian program and the here presented RG technique. While in the former two cases

¹In this discussion we omit a possible necessary UV-cutoff Λ .

²In the following we assume $\inf(S) = 0$ without loss of generality.

the density $\rho[\hat{\phi}; \bar{\Phi}]$ is deformed, in the present case, except for a global pre-factor, the actual measure is kept intact but the range of integration is modified. By changing the scale τ we can thus study the relative importance of particular subsets on field space to the overall theory. In other words, we can ‘switch on’ certain fields to contribute to the theory and likewise suppress others from taking part in the interaction.

In summary, we can say that one introduces an order on field space that is described by the RG scale $\tau \in S$. Decreasing τ amounts to interpolating between the fundamental theory and its IR effective description.

4.2 Infrared-cutoff implementation

The technique introduced by the RG framework is based on a filtration of Σ_F that acts as a partial evaluation of a theory, more precise of its generating functional. In principle, any such iterative definition of measurable sets $\{U_\tau\}_{\tau \in S}$ is admissible, as long as the integration over the whole field space is recovered in the limit $\lim_{\tau \rightarrow 0} U_\tau = F$. Thus, it is up to us to decide how to explicitly implement the RG procedure, though the preferred choice are filtrations with a rich mathematical structure, as for instance smooth and well understood sequences. In addition, it is convenient to relate these RG steps to some physical relevant object, even though we will see that in general only the interpretation of the sequence’s limits might be meaningful.

4.2.1 Eigenmode expansion

The most common and instructive approach is based on a decomposition of field space using a suitable operator $T_{\bar{\Phi}}$ which besides the background fields might also depend on the gauge fields of the theory. In this case the discussion follows the same lines, with only changing the operators target and domain space. Non-negative, self-adjoint operators naturally categorize field space in terms of the real numbers exploiting their eigenspace decomposition:

$$T_{\bar{\Phi}}(\psi_\lambda) = \lambda \psi_\lambda$$

The spectral theorem provides a criterion when $\{\psi_\lambda\}_\lambda$ defines a basis for the space of fluctuation fields at $\bar{\Phi}$ and the index theorem classifies the zero-mode eigenspace related to $\lambda \geq 0$, which we assume to be trivial for simplification. On the basis of this eigenfunction decomposition, the natural candidate to introduce an order on the space of measurable sets, and thus a notion of RG steps, is given by

$$\text{rg}_{T_{\bar{\Phi}}} : S \subseteq \mathbb{R} \rightarrow \Sigma_F, \quad \tau \mapsto \text{rg}_{T_{\bar{\Phi}}}(\tau) = \text{span}\{\psi_\lambda \mid b \cdot \lambda^\ell \geq \tau\}$$

This seemingly innocent allocation of the RG scale to some field operator can result in strong correlations between (background) field rescaling and a corresponding rescaling of τ . Since U_τ is actually defined via the operator $T_{\bar{\Phi}}$ and hence closely tied to the background fields any transformation of $\bar{\Phi}$ affects U_τ and all derived concepts.

Therefore, let us assume that we perform a field rescaling $\bar{\Phi} \equiv c \cdot \bar{\Phi}'$ with $c > 0$ that results in some rescaled operator $T_{\bar{\Phi}} = T_{c \cdot \bar{\Phi}'} = c^{n_T} T'_{\bar{\Phi}'}$. This simple (homogeneous) transformation law for $T_{\bar{\Phi}}$ (which is for instance satisfied for the Laplacian $\bar{D}^2$) implies that $T_{\bar{\Phi}}$ and $T'_{\bar{\Phi}'}$ share the same eigenvectors, however for different eigenvalues:

$$T'_{\bar{\Phi}'}(\psi'_{\lambda'}) = \lambda' \psi'_{\lambda'}, \quad T'_{\bar{\Phi}'} \equiv c^{-n_T} T_{\bar{\Phi}}, \quad \psi'_{\lambda'} \equiv \psi_\lambda, \quad \lambda' \equiv c^{-n_T} \lambda$$

Depending on which operator the RG filtration is based on, we obtain different measurable sets, which however are not unrelated. In fact, whenever the operator behaves homogeneous under

(constant) rescaling of the fields, there is actually a simple identity fulfilled for the RG sets:

$$\begin{aligned} \text{rg}_{T_{\Phi'}}(\tau) &= \text{span}\{\psi'_{\lambda'} \mid b \cdot (\lambda')^\ell \geq \tau\} \\ &= \text{span}\{\psi_\lambda \mid b \cdot (c^{-n_T} \cdot \lambda)^\ell \geq \tau\} \\ &= \text{span}\{\psi_\lambda \mid b \cdot \lambda^\ell \geq c^{\ell \cdot n_T} \cdot \tau\} \equiv \text{rg}_{T_{\Phi}}(c^{\ell \cdot n_T} \cdot \tau) \end{aligned} \quad (4.5)$$

Hence, a rescaling of the background fields can be compensated by a simultaneous rescaling of the RG scale, keeping the eigenspace decomposition unchanged, i.e.

$$\text{rg}_{T_{\Phi}}(\tau) \equiv \text{rg}_{T_{\Phi}}(c^{\ell \cdot n_T} \cdot \tau) \quad (4.6)$$

If the spectrum is continuous we can in fact use the same field space filtration $\{U_\tau\}$ for all rescaling-related operators.

This method of using the eigenspace of an operator to obtain a suitable RG implementation is a very efficient and well studied technique, among which the Fourier expansion w.r.t. $-\partial^2$ is a prominent example. Nevertheless, as usual, simplifications have a price, which in this case is given by a strong relation between τ and the background fields $\bar{\Phi}$. This renders the RG scale to be field dependent which will explicitly break the background field independence, at least on intermediate RG scales.

4.2.2 Rescaling relation between $\bar{\Phi}$ and τ

If we formally solve eq. (4.6) for the background field we find

$$\bar{\Phi}(\tau) = c^{-1} \cdot \bar{\Phi}(c^{\ell \cdot n_T} \cdot \tau) = \bar{\Phi}(k_0) \cdot \left(\frac{k}{k_0}\right)^{-d_{\bar{\Phi}}}, \quad \text{with} \quad d_{\bar{\Phi}} := -(\ell \cdot n_T)^{-1}, \quad \forall \tau \neq 0$$

In the final step we have used $\tau = k/k_0$. While mathematics on its own is fully dimensionless, in physics it is convenient to define relative quantities in terms of which all fields and other mathematical objects are measured. The canonical factorization of the background field structure, contained in $\bar{\Phi}(\tau_0)$, and its τ -dependence is usually exploited to define τ -independent – but now dimensionful – fields $\tau_0^{d_{\bar{\Phi}}} \cdot \bar{\Phi}(\tau_0)$. This turns out to be convenient since we can separate the RG-scale dependence of a function from the geometrical information encoded in the fields, which is the origin of dimensionful coupling constants.

Using the above operator-based sequence allows to express the field space of fluctuations in terms of eigenfunctions with respect to the operator T . A very natural, non-negative, self-adjoint operator associated to a physical theory is the one that defines the underlying free theory and classifies a Gaussian measure. A further advantage in using this special operator is the feature that at each RG step we have a natural (well-posed) measure theoretical interpretation, since the restriction of a Gaussian measure to some subspace is again Gaussian and therefore defines a free theory.

R For the just described reasons, a very extensively used operator is the Laplacian, $\Delta = -\bar{D}^2$ of the background metric, based on a uniquely related Levi-Civita connection:

$$\bar{D}_{\bar{g}}^2 = \bar{g}^{-1}(\bar{D}, \bar{D}) \quad \text{with} \quad \bar{D}_{c \cdot \bar{g}}^2 = c^{-1} \cdot \bar{g}^{-1}(\bar{D}, \bar{D}) = c^{-1} \bar{D}_{\bar{g}}^2 \implies n_{\bar{D}^2} = -1$$

The implementation of the ordering principle is specified by the parameters b and ℓ , where the latter is conventionally chosen to be $\ell = 1/2$ and b is some positive number usually set to $b = +1$. For the Laplacian we find $n_{\bar{D}^2} = -1$ and hence the field dimension of the (inverse) background metric can be read off to be $d_{\bar{\Phi}} = 2$ (and $d_{\bar{\Phi}-1} = -2$).

4.2.3 Smooth cutoff implementation

So far, we have a sharp RG implementation in that all eigenfunctions associated to eigenvalues below a certain scale τ are completely neglected in the functional integral. It is for several reasons more convenient to have a smooth RG procedure that allows for differentiation w.r.t. τ , in particular. The basic idea is to replace the indicator function $\chi_{U_\tau}(\hat{\phi})$ in the definition of eq. (4.4) by a smeared version $\exp(-\Delta S_\tau[\hat{\phi}; \bar{\Phi}])$, i.e.

$$\mu_\tau(A) = \int_{\hat{\phi} \in A \cap U_\tau} d\mu(\hat{\phi}; \bar{\Phi}) = \int_{\hat{\phi} \in A} d\mu(\hat{\phi}; \bar{\Phi}) \chi_{U_\tau}(\hat{\phi}) \rightarrow \int_{\hat{\phi} \in A} d\mu(\hat{\phi}) \exp(-\Delta S_\tau[\hat{\phi}; \bar{\Phi}])$$

This additional contribution should be designed such that eigenfunctions of the self-adjoint operator associated to the Gaußian measure, say $T_{\bar{\Phi}}$, are suppressed if they fall below a certain threshold. Technically, it should describe the following replacement:³

$$T \rightarrow (T_{\bar{\Phi}} + \mathfrak{R}_\tau(T_{\bar{\Phi}})) \quad \text{with} \quad \mathfrak{R}_\tau(\lambda) = \begin{cases} 0 & \text{for } b \cdot \lambda^\ell \gg \tau \\ \tau^{1/\ell} & \text{for } b \cdot \lambda^\ell \ll \tau \end{cases} \quad (4.7)$$

whereby $\mathfrak{R}_\tau(\lambda)$ is (anti-)self-adjoint for (anti-)commuting fluctuation fields, w.r.t. the field space metric $\mathfrak{G}_{\bar{\Phi}}$. This is naturally provided, if $T_{\bar{\Phi}}$ fulfills these requirements and $\mathfrak{R}_\tau[\bar{\Phi}]$ is a function of this operator only. Furthermore, let us assume $\mathfrak{R}_\tau(\lambda)$ to be a monotone function in both arguments, ensuring that the original idea of a filtration is still valid.

The full structure of the cutoff action derives from the defining requirement (4.7) yielding a bi-linear functional of the fluctuation fields:

$$\Delta S_\tau[\hat{\phi}; \bar{\Phi}] = \frac{1}{2} \mathfrak{G}_{\bar{\Phi}}(\hat{\phi}, \mathfrak{R}_\tau[\bar{\Phi}]\hat{\phi}) \quad (4.8)$$

In principle, the smearing function related to $\mathfrak{R}_\tau[\bar{\Phi}]$ is only subject to some asymptotic conditions, to monotonicity and to be (anti-)self-adjoint for (anti-)commuting fields. Nevertheless, it is convenient to keep its explicit form close to the sharp cutoff such that its derivative $\partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]$ is peaked around τ .

From the scaling properties discussed in the previous subsections we impose the following additional requirement on the cutoff action:

$$\Delta S_{c\tau}[c^{d_\phi} \phi; c^{d_{\bar{\Phi}}} \bar{\Phi}] = \Delta S_\tau[\phi; \bar{\Phi}] \quad (4.9)$$

The transformation of the fluctuation fields does not change the range of integration since we assume a vector space, for which holds:

$$\text{rg}_{T_{\bar{\Phi}}}(\tau) = \text{span}\{\psi_\lambda \mid b \cdot (\lambda)^\ell \geq \tau\} = \text{span}\{c \cdot \psi_\lambda \mid b \cdot (\lambda)^\ell \geq \tau\} \equiv c \cdot \text{rg}_{T_{\bar{\Phi}}}(\tau)$$

As an example, consider the Laplace-Beltrami operator $\Delta \equiv -\bar{D}$ associated to some background metric $\bar{g}$. Its tensor structure is trivial and relates to the identity map on the respective space of fluctuations. Under suitable boundary conditions, e.g. of Dirichlet type on some compact space M , this operator is self-adjoint and strongly elliptic giving rise to a non-negative spectrum with only a finite number of zero-eigenmodes. In this case, $\tau \in \mathbb{R}_+$ is sufficient to recover the entire field space integration. Let us choose $\ell = 1/2$ and $b = +1$. Then, the Gaußian part of the functional integration is replaced by

$$\Delta \rightarrow (\Delta + \mathfrak{R}_\tau(\Delta))$$

³In this discussion we omit the normalization constant for the sake of brevity.

Expanding $\hat{\phi}$ in a complete basis of eigenfunctions ψ_λ of Δ and using a relatively sharp cutoff for simplicity, leads to

$$\begin{aligned} (\Delta + \mathfrak{R}_\tau(\Delta)) \sum_{\lambda \in \mathbb{R}_+} \alpha_\lambda \psi_\lambda &= \sum_{\lambda \in \mathbb{R}_+} (\lambda + \mathfrak{R}_\tau(\lambda)) \alpha_\lambda \psi_\lambda \\ &\approx \sum_{\lambda > \tau} \lambda \alpha_\lambda \psi_\lambda + \sum_{\lambda < \tau} (\lambda + \tau^2) \alpha_\lambda \psi_\lambda \\ &= \Delta \hat{\phi}_{\lambda > \tau} + \tau^2 \hat{\phi}_{\lambda < \tau} \end{aligned}$$

This illustrates the low-eigenmode suppression of the smooth cutoff procedure.

Since the tensor structure, here $\mathbb{1}$, and pre-factors, u , are usually inherited by the associated operator $T_{\bar{\Phi}}$, it is convenient to introduce the cutoff shape function which contains the essence of the cutoff-implementation:

$$\mathfrak{R}_\tau[\bar{\Phi}] := R_\tau^{(0)}(T_{\bar{\Phi}}) \cdot u \mathbb{1} = \tau^{1/\ell} R^{(0)}(T_{\bar{\Phi}}/\tau^{1/\ell}) \cdot u \mathbb{1} \quad (4.10)$$

Notice that $R^{(0)}(T_{\bar{\Phi}}/\tau^{1/\ell})$ is invariant under a simultaneous rescaling of τ and $\bar{\Phi}$ in the preferred ratio.

4.3 The Functional Renormalization Group Equation

In this section we perform a coordinate free derivation of the Functional Renormalization Group Equation [112, 116–118] from the generating functional under very general conditions. We extend the derivation by an additional normalization factor, $N_\tau^\Lambda[\bar{\Phi}]$, which is τ - and $\bar{g}$ -dependent. Due to its dependence of $\bar{g}$ which, in addition to $\Delta S_\tau[\hat{\phi}; \bar{\Phi}]$ and the gauge fixing sector breaks the split-symmetry, it is actually a source of possible issues. Nevertheless, since this factor is quite interesting from a mathematical point of view, we keep it in the derivation. Anyhow, the traditional result is recovered by $N_\tau^\Lambda[\bar{\Phi}] = 1$. We will discuss its influence at the end of this section. Furthermore, we will also briefly comment on the general properties of the FRGE, irrespectively on the choice of $N_\tau^\Lambda[\bar{\Phi}]$.

4.3.1 The derivation

In this subsection we establish the link between the generating functional $Z[J; \bar{\Phi}]$ and the Functional Renormalization Group Equation, in a coordinate-free version. We initially start by introducing the τ -dependent construction $Z[J; \bar{\Phi}]$ by means of a smooth cutoff action $\Delta S_\tau[\hat{\phi}; \bar{\Phi}]$ which is assume to be a bilinear functional in its first argument. Explicitly it is given by $\Delta S_\tau[\hat{\phi}; \bar{\Phi}] = \frac{1}{2} \mathfrak{G}_{\bar{\Phi}}(\hat{\phi}, \mathfrak{R}_\tau[\bar{\Phi}]\hat{\phi})$. Along the lines of the previous discussions transfer $Z[J; \bar{\Phi}]$ to a sequence of functionals $Z_\tau^\Lambda[J; \bar{\Phi}]$ with a ultraviolet cutoff Λ , yielding

$$Z_\tau^\Lambda[J; \bar{\Phi}] = N_\tau^\Lambda[\bar{\Phi}] \int_{\hat{\phi} \in U_\Lambda[\bar{\Phi}]} d\mu(\hat{\phi}; \bar{\Phi}) \exp(-\Delta S_\tau[\hat{\phi}; \bar{\Phi}] + J(\hat{\phi})) \quad (4.11)$$

Here, we retained the τ -dependent normalization factor $N_\tau^\Lambda[\bar{\Phi}]$ for later convenience. However, since $N_\tau^\Lambda[\bar{\Phi}]$ is independent on J it does not influence the main part of the derivation and the usual result is recovered by setting $N_\tau[\bar{\Phi}] = \text{const}$. So, let us write $Z_\tau^\Lambda[J; \bar{\Phi}] \equiv N_\tau^\Lambda[\bar{\Phi}] z_\tau^\Lambda[J; \bar{\Phi}]$ with $N_\tau^\Lambda[\bar{\Phi}] \equiv (z_\tau^\Lambda[0; \bar{\Phi}])^{-1}$ such that in each RG step the generating functional corresponds to a probability measure, $Z_\tau^\Lambda[0; \bar{\Phi}] = 1$.

The sequence of Schwinger functionals $W_\tau^\Lambda[J; \bar{\Phi}]$ is given by the logarithm of $Z_\tau^\Lambda[J; \bar{\Phi}] \in \mathbb{R}$. In the above notation, $W_\tau^\Lambda[J; \bar{\Phi}]$ assumes the following form:

$$W_\tau^\Lambda[J; \bar{\Phi}] \equiv \ln Z_\tau^\Lambda[J; \bar{\Phi}] = \ln z_\tau^\Lambda[J; \bar{\Phi}] + \ln N_\tau^\Lambda[\bar{\Phi}] \quad (4.12)$$

Notice that $N_\tau^\Lambda[\bar{\Phi}]$ does not affect the correlation functions, but ensures that for $J = 0$ the Schwinger functional vanishes, i.e. $W_\tau^\Lambda[0; \bar{\Phi}] = 0$. Assuming that the underlying theory gives rise to a continuous differential and convex $W_\tau^\Lambda[J; \bar{\Phi}]$, we can transfer all its information into the Legendre-transform, which in the limit $\tau \rightarrow 0$ corresponds to the effective action. For its τ -dependence we obtain

$$\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] \equiv \text{LFT}(W_\tau^\Lambda[\bullet; \bar{\Phi}])[\varphi] \equiv \sup_{J \in U_\Lambda^*[\bar{\Phi}]} \{J(\varphi) - W_\tau^\Lambda[J; \bar{\Phi}]\} \quad (4.13)$$

In terms of the normalization factor, this expression can be recast into the following form, making again use of $N_\tau^\Lambda[\bar{\Phi}]$ being independent on J :

$$\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] = \sup_{J \in U_\Lambda^*[\bar{\Phi}]} \{J(\varphi) - \ln z_\tau^\Lambda[J; \bar{\Phi}]\} - \ln N_\tau^\Lambda[\bar{\Phi}] \quad (4.14)$$

Under the above assumptions, the supremum can be evaluated at the global maximum of the bracket, $J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]$, which solves

$$v(\varphi) = \partial_v W_\tau^\Lambda[J; \bar{\Phi}] = \partial_v \ln z_\tau^\Lambda[J; \bar{\Phi}] \quad v \in \text{TU}_\Lambda^*[\bar{\Phi}] \quad (4.15)$$

At this point we suppose that we can invert this relation to obtain $J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]$ and insert this solution into eq. (4.14) that results in:

$$\begin{aligned} \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] &= J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}](\varphi) - W_\tau^\Lambda[J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]; \bar{\Phi}] \\ &= J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}](\varphi) - \ln z_\tau^\Lambda[J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]; \bar{\Phi}] - \ln N_\tau^\Lambda[\bar{\Phi}] \end{aligned} \quad (4.16)$$

In the later derivation, it will prove very useful that $J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]$ is a solution of eq. (4.15). For simplicity, let us denote $J_w \equiv J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]$, whereby w absorbs the full set of dependencies. Then the application of the chain rule w.r.t. the derivative of w results in⁴

$$\begin{aligned} \partial_w (J_w(\varphi) - W_\tau^\Lambda[J_w; \bar{\Phi}]) &= \partial_w J_w(\varphi) - \partial_w W_\tau^\Lambda[J_w; \bar{\Phi}] \\ &= [(\partial_w J_w)(\varphi) - \partial_{\partial_w J_w} W_\tau^\Lambda[J_w; \bar{\Phi}]] + J_w(\partial_w \varphi) - \partial_w W_\tau^\Lambda[J; \bar{\Phi}]|_{J=J_w} \\ &= J_w(\partial_w \varphi) - \partial_w W_\tau^\Lambda[J; \bar{\Phi}]|_{J=J_w} \end{aligned}$$

In the last step we made use of eq. (4.15) for $v \equiv \partial_w J_w$ to cancel the bracket. Notice that the LHS of eq. (4.17a) corresponds to the derivative of the Legendre transform $\partial_w \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]$. For our purpose the differentiation of $\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]$ w.r.t. φ and τ are most important. For these special cases, eq. (4.17a) reads

$$\partial_{\delta\varphi} \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] = J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}](\delta\varphi) \quad (4.17a)$$

$$\tau \partial_\tau \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] = -\tau \partial_\tau W_\tau^\Lambda[J; \bar{\Phi}]|_{J=J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]} \quad (4.17b)$$

The first expression can be thought of as a τ -dependent effective field equation.

In the sequel, we focus on the second equality, eq. (4.17b), which represents the starting point for the derivation of the FRGE. If we insert the relation between $W_\tau^\Lambda[J; \bar{\Phi}]$ and the generating functional, we can extend this chain of dependencies up to the definition of the underlying theory, i.e.

$$\tau \partial_\tau \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] = -\tau \partial_\tau W_\tau^\Lambda[J; \bar{\Phi}]|_{J=J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]} = -\tau \partial_\tau \ln z_\tau^\Lambda[J; \bar{\Phi}]|_{J=J_{\tau;\Lambda}^{\max}[\varphi; \bar{\Phi}]} - \tau \partial_\tau \ln N_\tau^\Lambda[\bar{\Phi}] \quad (4.18)$$

⁴For simplicity, we use the shorthand notation ∂_w to denote both, functional variation (in case of $\bar{\Phi}$ or φ) and ordinary differentiation (if τ or Λ are involved).

Thus, once we know the RG behavior of $z_\tau^\Lambda[J; \bar{\Phi}]$ we can directly compute the τ -dependence of $\bar{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]$.⁵ In the remainder of this section, we cast the RHS of eq. (4.18) into a more convenient, and above all, closed form that gives rise to the definition of the Effective Average Action (EAA), $\Gamma_\tau^\Lambda[\varphi; \bar{\Phi}]$.

Therefore, let us consider $z_\tau^\Lambda[J; \bar{\Phi}]$ and notice that its entire τ -dependence is due to the cutoff action $\Delta S_\tau[\hat{\phi}; \bar{\Phi}]$, a bilinear functional of $\hat{\phi}$:

$$z_\tau^\Lambda[J; \bar{\Phi}] = \int_{\hat{\phi} \in U_\Lambda[\bar{\Phi}]} d\mu(\hat{\phi}; \bar{\Phi}) \exp(-\Delta S_\tau[\hat{\phi}; \bar{\Phi}] + J(\hat{\phi})) \quad (4.19)$$

Hence, the application of $\tau \partial_\tau$ on $z_\tau^\Lambda[J; \bar{\Phi}]$ affects only the exponential and corresponds to the evaluation of a ‘propagator-like’ expectation value $-\langle \tau \partial_\tau \Delta S_\tau[\hat{\phi}; \bar{\Phi}] \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]}$.

$$\begin{aligned} \tau \partial_\tau z_\tau^\Lambda[J; \bar{\Phi}] &= - \int_{\hat{\phi} \in U_\Lambda[\bar{\Phi}]} d\mu(\hat{\phi}; \bar{\Phi}) \exp(-\Delta S_\tau[\hat{\phi}; \bar{\Phi}] + J(\hat{\phi})) \tau \partial_\tau \Delta S_\tau[\hat{\phi}; \bar{\Phi}] \\ &= - \frac{1}{2} \int_{\hat{\phi} \in U_\Lambda[\bar{\Phi}]} d\mu(\hat{\phi}; \bar{\Phi}) \exp(-\Delta S_\tau[\hat{\phi}; \bar{\Phi}] + J(\hat{\phi})) \mathfrak{G}_\Phi(\hat{\phi}, \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]\hat{\phi}) \end{aligned} \quad (4.20)$$

In the second step, we introduced the defining expression for the cutoff operator. Notice that $\mathfrak{R}_\tau[\bar{\Phi}]$ satisfies a generalized symmetry property $\mathfrak{G}_\Phi(v, \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]w) = \mathfrak{G}_\Phi(w, \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]v)$, which is equivalent to $\mathfrak{R}_\tau[\bar{\Phi}]$ being (anti-)self-adjoint in case of (anti-)commuting fields.

In order to obtain a more convenient form of eq. (4.20) it will be useful to rewrite the expectation value $\langle \mathfrak{G}_\Phi(\hat{\phi}, \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]\hat{\phi}) \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]}$ in terms of derivatives of J . Therefore, consider the second variation, associated to the Hessian, of $z_\tau^\Lambda[J; \bar{\Phi}]$ w.r.t. the sources v and w .

$$\begin{aligned} \mathfrak{D}_v \mathfrak{D}_w z_\tau^\Lambda[J; \bar{\Phi}] &= \langle v(\hat{\phi}) w(\hat{\phi}) \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]} = \langle \mathfrak{G}_\Phi(v^*, \hat{\phi}) \cdot w(\hat{\phi}) \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]} \\ &= \langle \mathfrak{G}_\Phi(v^* w(\hat{\phi}), \hat{\phi}) \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]} \end{aligned}$$

Here, we have introduced the dual vector v^* in the canonical way, see Lax-Milgram theorem in section 1.4, $v = \mathfrak{G}_\Phi(v^*, \bullet)$, and made use of the bilinearity of $\mathfrak{G}_\Phi$ noticing that $w(\hat{\phi}) \in \mathbb{C}$. Recapitulating the definition of the trace we notice that $v^* w(\hat{\phi})$ can be interpreted as an endomorphism $(v^* \otimes w)\hat{\phi}$, which due to the properties of w and v is also linear. Hence, if we choose $w = u \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]$, we obtain an expression which is quite similar to eq. (4.20):

$$\begin{aligned} \mathfrak{D}_v \mathfrak{D}_{u \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]} z_\tau^\Lambda[J; \bar{\Phi}] &= \langle \mathfrak{G}_\Phi((v^* \otimes u \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}])\hat{\phi}, \hat{\phi}) \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]} \\ &= \langle \mathfrak{G}_\Phi((v^* \otimes u) \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}](\hat{\phi}), \hat{\phi}) \rangle_{z_\tau^\Lambda[J; \bar{\Phi}]} \end{aligned} \quad (4.21)$$

If we symmetrize this relation, we obtain the Hessian of $z_\tau^\Lambda[J; \bar{\Phi}]$ that is associated to the Hessian operator in the following way

$$\text{Hess}_J[z_\tau^\Lambda[J; \bar{\Phi}]](v, w) \equiv \frac{1}{2} (\mathfrak{D}_v \mathfrak{D}_w + \mathfrak{D}_w \mathfrak{D}_v) z_\tau^\Lambda[J; \bar{\Phi}] \equiv \mathfrak{G}_\Phi(w, \widehat{\text{Hess}_J[z_\tau^\Lambda[J; \bar{\Phi}]]} v) \quad (4.22)$$

In a final step utilize the definition of the trace given in subsection 1.4.2 to get rid of the extra tensor structure $v^* \otimes u$ and make use of the (anti-)self-adjointness of $\mathfrak{R}_\tau[\bar{\Phi}]$:

$$\tau \partial_\tau z_\tau^\Lambda[J; \bar{\Phi}] = - \frac{1}{2} \text{STr} \left[\widehat{\text{Hess}_J[z_\tau^\Lambda[J; \bar{\Phi}]]} \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] \quad (4.23)$$

⁵If the τ -dependence of $N_\tau^\Lambda[\bar{\Phi}]$ is retained, hence $Z_\tau^\Lambda[J; \bar{\Phi}]$ is normalized, notice that $N_\tau^\Lambda[\bar{\Phi}] = z_\tau^\Lambda[0; \bar{\Phi}]$ provides the missing information to solve eq. (4.18).

Here, for brevity we omitted the necessary field space metric insertions, writing $\tau\partial_\tau\mathfrak{R}_\tau[\bar{\Phi}]$ instead of $\mathfrak{G}(\bullet, \mathfrak{R}_\tau[\bar{\Phi}]\mathfrak{G}^{-1})$. Equation (4.23) is a central result in our derivation and we should pause for a moment and subsume the necessary assumptions we had to impose. First of all, we required $\mathfrak{R}_\tau[\bar{\Phi}]$ to be (anti-)self-adjoint, which compensates the extra minus sign of the supertrace in case of anti-commuting fields. Considered as a L^2 -operator it yields to restrictions on the set of allowed boundary conditions if $M \neq \emptyset$. Similarly, in case of non-empty boundary the construction of the Hessian operator is usually more involved. Either field space is such that all surface contribution cancel or one has to rely on more elaborated techniques in which bulk-boundary terms appear, see ref. e.g. [119, 120].

In eq. (4.23) we have found a closed RG flow equation for $z_\tau^\Lambda[J; \bar{\Phi}]$ that in principle can be inserted on the RHS of (4.18). However, we are more interested in deriving a closed differential equation for the effective (average) action and therefore going to rewrite the trace in (4.23) in terms of the Hessian of $\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]$. To this end, we first have to relate the Hessian of $z_\tau^\Lambda[J; \bar{\Phi}]$ with the Schwinger functional counterpart. From the chain rule and definition (4.12) we obtain

$$\begin{aligned} \partial_\nu \partial_w W_\tau^\Lambda[J; \bar{\Phi}] &= \partial_\nu \partial_w \ln z_\tau^\Lambda[J; \bar{\Phi}] + \partial_\nu \partial_w \ln N_\tau^\Lambda[\bar{\Phi}] \\ &= (z_\tau^\Lambda[J; \bar{\Phi}])^{-1} \partial_\nu \partial_w z_\tau^\Lambda[J; \bar{\Phi}] - (z_\tau^\Lambda[J; \bar{\Phi}])^{-2} \partial_w z_\tau^\Lambda[J; \bar{\Phi}] \partial_\nu z_\tau^\Lambda[J; \bar{\Phi}] \end{aligned} \quad (4.24)$$

Notice that $N_\tau^\Lambda[\bar{\Phi}]$ does not affect this relation for it is independent of J . For the Hessian, we have to symmetrize this relation. The first term in eq. (4.24) gives rise to the Hessian of $z_\tau^\Lambda[J; \bar{\Phi}]$ while the second contribution has first to be converted into a more convenient form:

$$\begin{aligned} \partial_w W_\tau^\Lambda[J; \bar{\Phi}] \partial_\nu W_\tau^\Lambda[J; \bar{\Phi}] &= (z_\tau^\Lambda[J; \bar{\Phi}])^{-2} \partial_w z_\tau^\Lambda[J; \bar{\Phi}] \partial_\nu z_\tau^\Lambda[J; \bar{\Phi}] \\ &= \langle w(\hat{\phi}) \rangle_{W_\tau^\Lambda[J; \bar{\Phi}]} \langle v(\hat{\phi}) \rangle_{W_\tau^\Lambda[J; \bar{\Phi}]} \\ &\equiv (w \otimes v)(\langle \hat{\phi} \rangle_{W_\tau^\Lambda[J; \bar{\Phi}]}, \langle \hat{\phi} \rangle_{W_\tau^\Lambda[J; \bar{\Phi}]}) \end{aligned}$$

Evaluated under the trace of (4.23), we find

$$\begin{aligned} (z_\tau^\Lambda[J; \bar{\Phi}])^{-1} \text{STr} \left[\widehat{\text{Hess}}_J [z_\tau^\Lambda[J; \bar{\Phi}]] \circ \tau\partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] &= \text{STr} \left[\widehat{\text{Hess}}_J [W_\tau^\Lambda[J; \bar{\Phi}]] \circ \tau\partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] \\ &\quad + \mathfrak{G}_\Phi(\langle \hat{\phi} \rangle_{W_\tau^\Lambda}, \mathfrak{R}_\tau[\bar{\Phi}] \langle \hat{\phi} \rangle_{W_\tau^\Lambda}) \end{aligned} \quad (4.25)$$

The second contribution results from converting v to v^* as above and the self-adjointness of $\mathfrak{R}_\tau[\bar{\Phi}]$. Notice that this term represents the $\tau\partial_\tau$ derivative of the cutoff action evaluated for the connected expectation values of $\hat{\phi}$. Hence, the flow equation for $W_\tau^\Lambda[J; \bar{\Phi}]$ reads

$$\tau\partial_\tau W_\tau^\Lambda[J; \bar{\Phi}] = -\frac{1}{2} \text{STr} \left[\widehat{\text{Hess}}_J [W_\tau^\Lambda[J; \bar{\Phi}]] \circ \tau\partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] - \tau\partial_\tau \left(\Delta S_\tau(\langle \hat{\phi} \rangle_{W_\tau^\Lambda}; \bar{\Phi}) - N_\tau^\Lambda[\bar{\Phi}] \right) \quad (4.26)$$

Here, we used $\tau\partial_\tau W_\tau^\Lambda[J; \bar{\Phi}] = (z_\tau^\Lambda[J; \bar{\Phi}])^{-1} \tau\partial_\tau z_\tau^\Lambda[J; \bar{\Phi}]$ to cancel a global factor of $(z_\tau^\Lambda[J; \bar{\Phi}])^{-1}$ on both sides and consider the normalization factor.

For the final step, we need to translate the Hessian of the Schwinger functional to its Legendre transform. The derivation of this relation is found in section 2.2, while we here only state its result

$$\widehat{\text{Hess}}_J [W_\tau^\Lambda[J; \bar{\Phi}]] \Big|_{J=J_{\tau;\Lambda}^{\text{max}}[\varphi; \bar{\Phi}]} = \widehat{\text{Hess}}_\varphi [\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]]^{-1} \quad (4.27)$$

Here we omit insertions of the field space metric. Keeping in mind that under all traces $\tau\partial_\tau \mathfrak{R}_\tau[\bar{\Phi}]$ actually stands for $\mathfrak{G}(\bullet, \mathfrak{R}_\tau[\bar{\Phi}]\mathfrak{G}^{-1})$ we see that the final result is free of additional contributions of $\mathfrak{G}_\Phi$. Furthermore notice that $\langle \hat{\phi} \rangle_{W_\tau^\Lambda[J; \bar{\Phi}]}|_{J=J_{\tau;\Lambda}^{\text{max}}[\varphi; \bar{\Phi}]} \equiv \varphi$ reduces to the expectation value of the fluctuation field, thus the argument of $\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]$.

From eq. (4.18) and relation (4.27) we obtain a closed flow equation for the Legendre transform of $W_\tau^\Lambda[J; \bar{\Phi}]$, which reads

$$\tau \partial_\tau \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi [\tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}]] \right)^{-1} \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] + \tau \partial_\tau (\Delta S_\tau[\varphi; \bar{\Phi}] - \ln N_\tau^\Lambda[\bar{\Phi}]) \quad (4.28)$$

A more convenient form of this expression can be obtained by absorbing the additional $\tau \partial_\tau$ derivative in eq. (4.28) into a new functional which is denoted the Effective Average Action (EAA) [110–114]:

$$\Gamma_\tau^\Lambda[\varphi; \bar{\Phi}] := \tilde{\Gamma}_\tau^\Lambda[\varphi; \bar{\Phi}] - \Delta S_\tau[\varphi; \bar{\Phi}] \quad (4.29)$$

Since the Hessian operator is linear in its functional argument and furthermore that for the cutoff action we have $\widehat{\text{Hess}}_\varphi [\Delta S_\tau[\varphi; \bar{\Phi}]] \equiv \mathfrak{R}_\tau[\bar{\Phi}]$ we succeeded in a coordinate-free derivation of the Functional Renormalization Group Equation (FRGE) for the EAA:

$$\tau \partial_\tau \Gamma_\tau[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi [\Gamma_\tau[\varphi; \bar{\Phi}]] + \mathfrak{R}_\tau[\bar{\Phi}] \right)^{-1} \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] - \ln N_\tau^\Lambda[\bar{\Phi}] \quad (4.30)$$

Compared to the standard form found in the literature, we have an additional term $\ln N_\tau^\Lambda[\bar{\Phi}]$ that originates from the normalization of $Z_\tau^\Lambda[J; \bar{\Phi}]$. If this factor is set to $N_\tau^\Lambda[\bar{\Phi}] = 1$, as in the standard derivation, we recover the familiar definition of the FRGE:

$$\tau \partial_\tau \Gamma_\tau[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi [\Gamma_\tau[\varphi; \bar{\Phi}]] + \mathfrak{R}_\tau[\bar{\Phi}] \right)^{-1} \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right] \quad (4.31)$$

Equation (4.31) was first established by Wetterich in [112] and has found many applications in various fields. It is quite remarkable that the Renormalization Group procedure gives rise to a closed and differential equation that covers in a very universal way the RG behavior of field theories. Notice, in particular, that as it stands, it is an exact reformulation of the underlying theory described by the generating functional and as such is non-perturbative. Furthermore, the structure of this equation and the required properties of $\mathfrak{R}_\tau[\bar{\Phi}]$ justify the removal of the ultraviolet cutoff Λ by taking the limit $\Lambda \rightarrow \infty$ on both sides. We will briefly comment on this aspect in the next subsection and conclude with some remarks on the normalization factor.

4.3.2 Properties of the FRGE

The FRGE is an exact equation and a particular realization of the generating functional for QFT or statistical systems. As such, it is non-perturbative in nature and the question of renormalizability can be studied on the basis of different approximation schemes than usually employed within the perturbative methods. From the mathematical point of view, we are confronted with a highly complicated expression, a functional non-linear differential equation. In a suitable basis of field monomials, this corresponds to an infinite number of coupled ODE. In its final form, eq. (4.31), it describes the RG evolution of the EAA, a functional related to the familiar effective action and as such it is defined on the field space of expectation values.

Now, let us comment on the limit $\Lambda \rightarrow \infty$. Since we are dealing with a differential equation, both sides of eq. (4.31) describe the change w.r.t. the RG scale τ . So far we have not yet stated a suitable criterion that distinguishes the predictivity and regularity of solutions to eq. (4.31) we will discuss this important aspect in the next section. Here, our focus is on whether or not we can safely remove the ultraviolet cutoff from the differential equation. It turns out that the way we implemented the cutoff operator $\mathfrak{R}_\tau$ amounts to a sharp localization of eq. (4.31) at τ . Therefore notice that the inverse operator that appears under the trace of the FRGE

contains a the cutoff operator $\mathfrak{R}_\tau[\bar{\Phi}]$ which acts as an infrared regulator for small eigenvalues. On the other hand, $\mathfrak{R}_\tau[\bar{\Phi}]$ is chosen such that it only changes significantly within in a small region of τ . For instance, if $\mathfrak{R}_\tau[\bar{\Phi}]$ is associated to Laplacian-type operator, we may write $\tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \approx \delta(\tau^{1/\ell} - \Delta)$ and thus eq. (4.31) is also UV-finite.

Next, let us consider the limit $\tau \rightarrow \infty$. Therefore, we make use of an equivalent description of the FRGE (4.30) in terms of a functional integro-differential equation:

$$e^{-\Gamma_\tau[\varphi; \bar{\Phi}]} = \int [d\hat{\varphi}] \exp \left(-S[\hat{\varphi}; \bar{\Phi}] - \Delta S_\tau[\hat{\varphi}; \bar{\Phi}] + \int d^d x \hat{\varphi}(x) \frac{\delta \Gamma_\tau}{\delta \varphi(x)}[\varphi; \bar{\Phi}] \right) \quad (4.32)$$

The saddle point approximation for fluctuations around φ yields, under the assumption that $\frac{\delta \Gamma_\tau}{\delta \varphi(x)}$ does not diverge,

$$\lim_{\tau \rightarrow \infty} \Gamma_\tau[\varphi; \bar{\Phi}] = S[\varphi; \bar{\Phi}] + \dots \quad (4.33)$$

Here, the dots represent higher order loop corrections [121].

4.3.3 The normalization factor

Notice that $N_\tau^\Lambda[\bar{\Phi}]$ does not contribute to the RHS of the FRGE for it is independent on the fluctuation fields. It only appears as an extra contribution in the definition of the EAA. In later considerations we always set $N_\tau^\Lambda[\bar{\Phi}] = 1$, resulting in the standard formulation of the flow equation. The τ -dependence of $N_\tau^\Lambda[\bar{\Phi}] \equiv (z_\tau^\Lambda[0; \bar{\Phi}])^{-1}$ is obtained from eq. (4.23)

$$\tau \partial_\tau N_\tau^\Lambda[\bar{\Phi}] = -\frac{1}{2} \text{STr} \left[\widehat{\text{Hess}}_J [z_\tau^\Lambda[J; \bar{\Phi}]] \circ \tau \partial_\tau \mathfrak{R}_\tau[\bar{\Phi}] \right]_{J=0} \quad (4.34)$$

If included in the definition of $Z_\tau[J; \bar{\Phi}]$ it ensures that

It is instructive to present a specific example that demonstrates the effect of $N_\tau^\Lambda[\bar{\Phi}]$. Therefore, consider the free Gaußian theory of a scalar field based on the Laplacian operator $\Delta = -\bar{D}^2$. The corresponding generating functional is given by

$$Z[J; \bar{\Phi}] \equiv \int_{\hat{\varphi} \in \mathcal{T}_{\bar{\Phi}} \hat{\varphi}} d\mu_G(\hat{\varphi}; \bar{\Phi}) e^{J(\hat{\varphi})} = \exp \left(\frac{1}{2} \int d^d x \sqrt{\bar{g}} J(x) \Delta^{-1} J(x) \right)$$

The implementation of the RG scale amounts to replace the Laplacian with $\Delta + \mathfrak{R}_\tau[\bar{g}]$ and further add a global factor $N_\tau[\bar{g}]$ to the measure. The Fourier transform of μ_G^Λ with the additional $\Delta + \mathfrak{R}_\tau[\bar{g}]$ Hence, we obtain

$$\begin{aligned} Z_\tau[J; \bar{\Phi}] &\equiv \int_{\hat{\varphi} \in \mathcal{T}_{\bar{\Phi}} \hat{\varphi}} d\mu_G(\hat{\varphi}; \bar{\Phi}) e^{J(\hat{\varphi})} e^{-\Delta S_\tau[\hat{\varphi}; \bar{\Phi}]} \\ &= N_\tau[\bar{g}] \sqrt{\frac{\det(\Delta)}{\det(\Delta + \mathfrak{R}_\tau[\bar{g}])}} \exp \left(\frac{1}{2} \int d^d x \sqrt{\bar{g}} J(x) (\Delta + \mathfrak{R}_\tau[\bar{g}])^{-1} J(x) \right) \end{aligned}$$

In the following we consider the two cases of $N_\tau[\bar{g}] = 1$ and $N_\tau[\bar{g}] = \sqrt{\frac{\det(\Delta + \mathfrak{R}_\tau[\bar{g}])}{\det(\Delta)}}$, the latter yielding a suitable probability theory. It is straightforward, see section 2.2, to derive the associated effective action which then gives rise to $\Gamma_\tau[\varphi; \bar{\Phi}]$:

$$\Gamma_\tau[\varphi; \bar{\Phi}] = \frac{1}{2} \int d^d x \sqrt{\bar{g}} \varphi(x) \Delta \varphi(x) + \ln \left(N_\tau[\bar{g}] \sqrt{\frac{\det(\Delta + \mathfrak{R}_\tau[\bar{g}])}{\det(\Delta)}} \right)$$

Notice that if we retain the probabilistic interpretation the second term vanishes and what remains is the bilinear τ -independent term. In this case the RHS of eq. (4.30) vanishes, that is $\tau \partial_\tau \Gamma_\tau[\varphi; \bar{\Phi}] = 0$ and the RG effects of the trace can be entirely attributed to the τ -dependence of the normalization factor $N_\tau[\bar{g}]$. On the other hand, if we set $N_\tau[\bar{g}] = 1$ than the free, scale invariant theory shows an τ evolution.

4.4 Renormalizability & Asymptotic Safety

Renormalizability is a condition on a mathematical theory which requires that by removing the ultraviolet cutoff the description remains regular with only a finite number of free parameters which have to be fixed by experiments. In this section we are going to classify solutions of the FRGE according to their UV behavior. We will make use of the renormalization condition in the form stated above and show its implementation into the RG language that significantly reduce the number of ‘proper’ theories in $\mathcal{T}$. We will see that the notion of perturbative renormalizability is only a special case of a more general class of renormalizable theories.

Special, renormalizable points in theory space are those which are fixed points of the RG procedure and therefore scale independent. For the EAA we have to perform a joint transformation of the RG scale τ and a field rescaling whereby the associated dimensionalities of the fields are obtained from the scaling property of the cutoff action:

$$\Delta S_{c\tau}[c^{d_\varphi}\varphi; c^{d_\Phi}\bar{\Phi}] = \Delta S_\tau[\varphi; \bar{\Phi}] \quad (4.35)$$

Formally expanding the EAA in terms of basis invariants we can read off the canonical scaling behavior of the coefficients $\{u^\alpha(\tau)\}$. At scale invariant theories the following requirement has to hold:

$$u^{(\alpha)}(\tau[\bar{\Phi}])\mathcal{P}_{(\alpha)}[\varphi; \bar{\Phi}] \stackrel{!}{=} u^{(\alpha)}(\tau[c^{d_\Phi}\bar{\Phi}])\mathcal{P}_{(\alpha)}[c^{d_\varphi}\varphi; c^{d_\Phi}\bar{\Phi}] \quad \forall \alpha \quad (4.36)$$

A fixed point under the RG evolution is thus given by the following condition:

$$\Gamma_\tau[\varphi; \bar{\Phi}] \equiv \Gamma[\tau^{d_\varphi}\varphi; \tau^{d_\Phi}\bar{\Phi}] \quad (4.37)$$

Notice that in particular $\partial_\tau \Gamma_\tau[\varphi; \bar{\Phi}] \neq 0$ at fixed points! If we rewrite the flow equation in terms of dimensionless couplings we recover the familiar picture that critical points of the dimensionless beta-functions signal the occurrence of fixed points of the RG flow.

The important observation is that fixed points are natural candidates to describe a renormalizable field theory. Due to their scale independence, we can safely remove the UV cutoff Λ . Furthermore, notice that the coordinates of the fixed point are determined by the zero conditions and thus there are no further experiments needed to specify the underlying theory.

In the space of dimensionless couplings, fixed points correspond to zeros of the RG vector field and as such determine the topological landscape of the space of solutions. The fate of RG trajectories is very sensitive to the way they approach or depart from these critical points. The linearized regime of a fixed point already furnishes very valuable particulars about its structure and the behavior of bypassing trajectories. In fact the critical exponents, see chapter 9, already contain all essential information that characterize the RG flow in the vicinity of the fixed point. The observation that conceptual very different models of Nature describing very distinct physical systems, may approach a fixed point of the same structure is encoded in the concept of universality classes. That is, while on general scales the RG flow for these two systems exhibits very contrary properties and in fact may be defined on completely different theory spaces, it may happen that in the vicinity of a fixed point both are described by the same linearized differential equations and thus are closely related on these scales.

Besides revealing connections of in general unrelated systems at criticality, the linearized regime of a fixed point contains essential information what happens to RG trajectory that enter its realm. In any case the RG solution feels the presence of a fixed point and is either attracted or repulsed from its position. Associated to each fixed point there are (ir-)relevant directions in which it acts as an (repeller) attractor when increasing the scale τ . While for relevant directions the trajectory is always pulled towards the fixed point, for irrelevant directions only those which admit the fixed point value are not repelled, but stay in the close vicinity of this critical point.

If this happens to be the case for all irrelevant directions the UV behavior of the corresponding RG trajectory is fully controlled by the fixed point which it approaches. Thus, the underlying theory shows more and more signs of scale-invariance and converges to a point in theory space, which is nothing else than the condition of renormalizability given above. Hence, from this perspective a theory has a well behaved UV limit if it is pulled towards a fixed point for increasing τ . For a specific fixed point the collection of all these RG-trajectories defines its UV-critical hypersurface, $\mathcal{S}_{UV}$. Its linearization closed to the fixed point coincides with the hypersurface spanned by the relevant field monomials. All trajectories not contained in $\mathcal{S}_{UV}$ will depart from the corresponding fixed point and their UV fate is either untamed or under the control of a different critical point. From this perspective it is clear that the dimensionality of $\mathcal{S}_{UV}$ is related to the predictivity of the RG flow in that for each relevant direction (equivalent for each dimension of $\mathcal{S}_{UV}$) there is one free parameter that has to be fixed by additional constraints. Besides experimental results this includes other fundamental requirements of the underlying setting, for instance Background Independence in case of Quantum Gravity. Hence, the fewer the number of fixed points that are compatible with the classical limit and the lower the dimensionality of the associated UV-critical hypersurfaces, the higher the predictivity of the RG flow. The other way around, if $\dim \mathcal{S}_{UV}$ is infinite and no further mathematical constraint can be imposed that reduce the physical subspace to finite dimension, we cannot obtain full predictivity from the results since an infinite number of experiments would be needed in order to uniquely identify the RG trajectory realized in Nature.

In conclusion, we are prepared to present the concept of renormalizability in the context of the Functional Renormalization Group:

Definition 4.4.1 — Renormalizability of RG trajectories. Let $\Gamma_\tau[\varphi; \bar{\Phi}]$ describe the RG solution of the FRGE subject to a certain theory space $\mathcal{T}(M, G, F)$. The RG trajectory corresponds to a (non-perturbative) renormalizable QFT, if and only if it is contained within a finite-dimensional $\mathcal{S}_{UV}$ -critical hypersurface of a (non-)Gaussian, aka (non-)trivial, fixed point of the RG flow.

We distinguish the type of renormalizability by the fixed point properties which controls the UV of a RG-trajectory. The first classification refers to the underlying theory of the critical point itself, i.e. we denote a fixed point (non-)trivial or being (non-)Gaussian if it corresponds to a (interacting) free theory. In this case, the critical exponents reflect the canonical scaling of the dimensionful couplings. Based on this distinction we have the following notation for renormalizable theories, see fig. 4.2:

Definition 4.4.2 — Classification of Renormalizability. Let $\Gamma_\tau[\varphi; \bar{\Phi}]$ be renormalizable in the above sense, thus the RG-trajectory lies within a UV-critical hypersurface of a fixed point FP. We say that $\Gamma_\tau[\varphi; \bar{\Phi}]$ is ...

- ▶ ... **trivial** : $\Leftrightarrow$
FP is Gaussian and or all $A[\hat{\varphi}; \bar{\Phi}] \in \mathcal{S}_{UV}$ $A[\hat{\varphi}; \bar{\Phi}]$ are Gaussian (triviality problem)
- ▶ ... **asymptotically free** : $\Leftrightarrow$
FP is Gaussian and there exists an interacting relevant direction
- ▶ ... **asymptotically safe** : $\Leftrightarrow$ FP is non-Gaussian

Notice the subtle difference between triviality and asymptotic freedom. In the former case, all trajectories that are attracted towards the Gaussian fixed point correspond to free theories. On the contrary, asymptotic free theories become only Gaussian in the UV limit but are interacting on lower scales. A prominent example of asymptotic freedom is given by QCD. Notice that perturbation theory describes the local vicinity of Gaussian fixed points and perturbative renormalizable theories lay within their UV-critical hypersurface. Thus, if we perturbative results

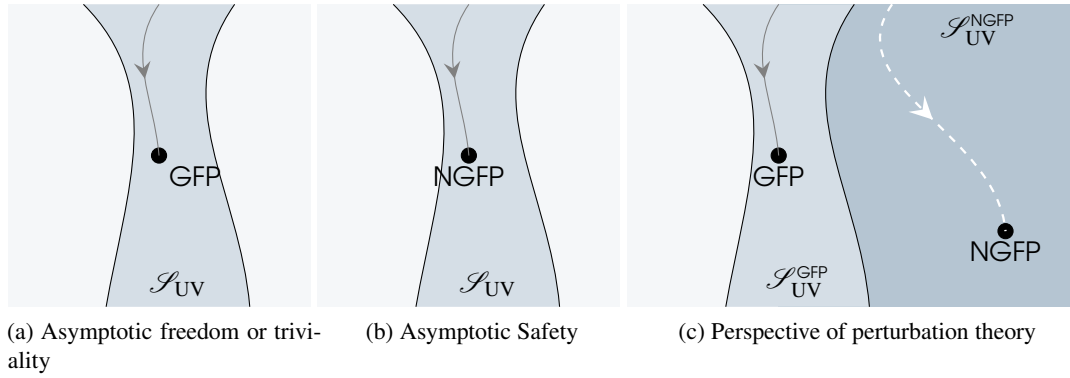


Figure 4.2: These plots describe the types of renormalizable RG-trajectories. If the UV-critical hypersurface, $\mathcal{S}_{UV}$, is spanned by only non-interacting relevant directions $\mathcal{S}_{UV}$ describes a regime of triviality. Otherwise, a theory is asymptotically (safe) free if it is contained in the critical manifold of a (non-)Gaussian fixed point. On the right, we have sketch a RG trajectory that is perturbatively non-renormalizable but turns out to be non-perturbatively renormalizable for it approaches a non-Gaussian fixed point.

suggest that a theory is perturbative non-renormalizable it simply states the corresponding RG-trajectory does not emanate from the trivial fixed points. Whether or not its UV behavior is controlled by a different (non-trivial) critical point depends on the structure of the RG flow.

The last case of Asymptotic Safety is particular important for studying Quantum Gravity. There, exactly the above described scenario arises: from perturbative analysis we observe that associated RG-trajectories pass by the Gaussian fixed point and are repelled towards the UV. The crucial question that people try to answer is whether or not there is another, non-trivial fixed point that takes over this role and renders at least a single RG-trajectory asymptotically safe.

Up to now there has been strong evidence for the existence of such a non-trivial fixed point, first obtained in the Einstein-Hilbert truncation almost twenty years ago [122], for the concept of truncations see the next subsection. This picture of an asymptotically safe theory of Quantum Gravity has prevailed in a number of different extensions of the groundbreaking work. Till now, there are confirmation of the existence of a suitable fixed points from various directions, including $f(R)$ -truncations [123–129], the square of the Weyl tensor [128] and also by coupling matter to gravity [130, 131]. In addition the influence of the ghost and gauge sector has been analyzed [132–134] and [135]. While these confirmations are based on single-metric computations, first studies in taken the full bi-metric character of the EAA into account became available recently, [136, 137], again confirming the general results. Studies on the difference of the usual Euclidean truncations and the Lorentzian case revealed re-ensuring confidence in the previous results [138, 139] While one focuses mainly on metric field content there is also evidence based on different field spaces, as for instance the Einstein-Cartan formalism [140–143] tetrad formulation [144]. In conclusion one can say that taken together the Asymptotic Safety conjecture for Quantum Gravity has a very solid foundation.

4.5 Truncations

The Functional Renormalization Group Equation is an exact representation of the path integral and thus intrinsically non-perturbative. From the technical point of view it combines the

difficulties of functional differential equations with the problem of non-linearity:

$$\partial_t \Gamma_\tau[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi[\Gamma_\tau[\varphi; \bar{\Phi}]] + \mathfrak{R}_\tau[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_\tau[\bar{\Phi}] \right] \quad (4.38)$$

Notice that we have introduced the RG-time $t \equiv \ln \tau = \ln k/k_0$. In order to evaluate this differential equations one usual employs its coordinatized version, expanding $\Gamma_\tau[\varphi; \bar{\Phi}]$ in terms of a full linearly independent basis of field monomials $\{\mathcal{P}_a[\varphi; \bar{\Phi}]\}$:

$$\mathcal{T} \ni A[\varphi; \bar{\Phi}] \equiv \sum_a \bar{u}^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

We can employ this expansion formally into an infinite number of ordinary differential equations for the coefficients using the coordinate expansion. The LHS of the flow equation (4.38) assumes the form

$$\text{LHS : } \quad \partial_t \Gamma_\tau[\varphi; \bar{\Phi}] = \partial_t \left(\sum_a \bar{u}_\tau^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right) = \sum_a (\partial_t \bar{u}_\tau^a) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

Since we assume that $\{\mathcal{P}_a[\varphi; \bar{\Phi}]\}$ forms a complete, linearly independent basis, the RHS of eq. (4.38) can also be expanded onto the basis invariants $\{\mathcal{P}_a[\varphi; \bar{\Phi}]\}$ in a unique way. This defines the dimensionful beta-functions $\bar{\beta}$:

$$\text{RHS : } \quad \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi[\Gamma_\tau[\varphi; \bar{\Phi}]] + \mathfrak{R}_\tau[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_\tau[\bar{\Phi}] \right] =: \sum_a \bar{\beta}^a(\{\bar{u}_\tau^b\}; \tau) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

Since the $\mathcal{P}$'s are linearly independent we can compare coefficients and find

$$\partial_t \bar{u}_\tau^a = \bar{\beta}^a(\{\bar{u}_\tau^b\}; \tau) \quad \forall a$$

This defines an infinite set of coupled differential equations which is no improvement compared to a single functional differential equation. As long as there are no systematic techniques to, for instance iteratively, attack this differential system, we have to rely on approximations to study the flow equations. The idea is to project this equation onto a (possibly still infinite) subset of field monomials, which is denoted a truncation. However, such a truncated theory space is in general not invariant under the RG flow, since the non-linearity together with the second variation of the basis invariants will lead to a generation of terms on the RHS which are not present in the original ansatz for Γ_τ .

In general, we introduce a projection operator that reduces an action functional to a specific direction of theory space:

$$\Pi_{\mathcal{P}_b}(A) \equiv \Pi_{\mathcal{P}_b} \left(\sum_a \bar{u}^a \cdot \mathcal{P}_a \right) := \bar{u}^b \cdot \mathcal{P}_b$$

Assume that we can find a suitable linearly independent basis of field monomials on $\mathcal{T}$, in the general case this space will not be endowed with an inner product and projection operators are difficult to obtain. Still, let us proceed on a formal level and apply $\Pi_{\mathcal{P}_b}$ to the FRGE:

$$\Pi_{\mathcal{P}_b}(\partial_t \Gamma_\tau[\varphi; \bar{\Phi}]) = \frac{1}{2} \Pi_{\mathcal{P}_b} \left(\text{STr} \left[\left(\widehat{\text{Hess}}_\varphi[\Gamma_\tau[\varphi; \bar{\Phi}]] + \mathfrak{R}_\tau[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_\tau[\bar{\Phi}] \right] \right)$$

Since for dimensionful coefficients the fields and thus the basis invariants are τ -independent. Hence, ∂_t and $\Pi_{\mathcal{P}_b}$ commute and we can change the order on the LHS. In the coordinatized version we find

$$(\partial_t \bar{u}_\tau^b) \cdot \mathcal{P}_b[\varphi; \bar{\Phi}] = \frac{1}{2} \Pi_{\mathcal{P}_b} \left(\text{STr} \left[\left(\widehat{\text{Hess}}_\varphi \left[\sum_a \bar{u}_\tau^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right] + \mathfrak{R}_\tau[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_\tau[\bar{\Phi}] \right] \right)$$

The supertrace gives rise to the dimensionful beta-functions and thus the application of $\Pi_{\mathcal{P}_b}$ reproduces the differential equation for the coefficient $\bar{u}_\tau^b$:

$$(\partial_t \bar{u}_\tau^b) \cdot \mathcal{P}_b[\varphi; \bar{\Phi}] = \Pi_{\mathcal{P}_b} \left(\sum_a \bar{\beta}^a(\{\bar{u}_\tau^b\}; \tau) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right) = \beta^b(\{\bar{u}_\tau^b\}; \tau) \cdot \mathcal{P}_b[\varphi; \bar{\Phi}]$$

Thus, even if we are only interested in the RG evolution of a single basis invariant, we still have to compute the Hessian of a generic action functional in *full* theory space, for the projection is applied only after evaluating the trace. The true approximation, or truncation, is employed if we project the EAA on the RHS before computing the Hessian operator, i.e.

$$\Pi_{\text{Trunc}}^S(\Gamma_k) = \Pi_{\text{Trunc}}^S \left(\sum_a \bar{u}_\tau^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right) := \sum_{a \in S} \bar{u}_\tau^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

Here S is usually a finite subset of basis invariants which span the truncated theory space. The evaluation of the flow equation is then given by

$$\begin{aligned} \partial_t \Pi_{\text{Trunc}}^S(\Gamma_\tau[\varphi; \bar{\Phi}]) &= \frac{1}{2} \Pi_{\text{Trunc}}^S \left(\text{STr} \left[\widehat{\text{Hess}}_\varphi \left[\Pi_{\text{Trunc}}^S(\Gamma_\tau[\varphi; \bar{\Phi}]) \right] + \mathfrak{R}_\tau[\bar{\Phi}] \right]^{-1} \circ \partial_t \mathfrak{R}_\tau[\bar{\Phi}] \right) \\ &\quad + \text{corrections} \end{aligned}$$

Notice, that the corrections describe the contribution from the infinite number of neglected directions and indicate that the RG flow is not closed. Improving truncations and test their stability for different cutoff operators $\mathfrak{R}_\tau[\bar{\Phi}]$, for instance, provides a way to test the reliability of the results.



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The first part of this thesis focused on the conceptual aspects of the Functional Renormalization Group Equation (FRGE) along with the general framework it is embedded in. We have seen that it provides a very universal and – for the present case more important – non-perturbative method based on the Effective Average Action (EAA) approach to Quantum Field theory (QFT).

In the Functional Renormalization Group formulation, a QFT is renormalizable in a more general, non-perturbative sense, namely if the underlying probability theory assumes a fixed point under the Renormalization Group (RG) evolution. The usual notion of perturbative renormalizability is then recovered in the vicinity of a Gaussian fixed point, however in the general case a non-trivial fixed point with a finite number of relevant operators suffices to render a theory asymptotically safe and thus (non-perturbative) renormalizable.

In the FRGE setting, the difficulty in proving that a QFT is asymptotically safe, resides in the nonlinear, functional nature of the underlying differential equation. For a specific field content and symmetry constraint, this amounts to an infinite dimensional theory space $\mathcal{T}$ with the compatible field monomials as basis and the couplings related to the coefficients. In absence of any systematic treatment of the resulting infinite dimensional ordinary but coupled differential system, one has to rely on truncations that investigate only a (probably finite dimensional) subset of $\mathcal{T}$. Notice that truncations are performed on the level of the EAA and thus are intrinsically non-perturbative and in fact correspond to a functional integration over the full field space, in particular over all fluctuations. Rather one should visualize this approximation to the complete RG flow as restricting the analysis of allowed theories (measures) to only a subset, by projecting the FRGE to the corresponding basis invariants. By construction projections only reveal part of the information and in fact can lead to ‘optical illusions’ that can destroy or simulate certain features which are or are not present in the RG evolution on full theory space. Thus, as for any other approximation technique, the significance of truncations has to be analyzed and confirmed in order to define the range of validity of the employed truncation. Basically, one either tests the truncations to possess certain fundamental properties of the full solution that can be read off by the structure of the FRGE, or by stability analysis whereby the truncation is deformed in one or the other direction. Concerning the first technique that investigates the consistency of truncations, in part III of this thesis we present and study a monotonicity property in the sense of Zamolodchikov’s C -function to test and design truncations, which is non-perturbative and universal in nature. Furthermore, the Ward Identities for split-symmetry (WISS) and other exact equations can in principle, though usually very difficult, be utilized to set the range of reliability. The second way to judge the validity of truncations involves the stability under changes of field space, symmetries, and general topological modifications, as well as increasing the number of basis invariants.

While theories of Quantum Gravity (QG), in particular those based on General Relativity (GR), are known to either spoil unitarity or renormalizability on the perturbative level, the hope is to cure those fundamental issues when leaving the realm of perturbation theory and employ non-perturbative techniques, as for instance the FRGE. In this setting, the Asymptotic Safety conjecture of Quantum Gravity states that even though QG fails to be perturbative renormalizable, there exists a non-trivial fixed point of the RG evolution with a finite dimensional ultraviolet (UV)-critical hypersurface [122, 145]. As an additional initial condition one requires a classical regime in infrared (IR) where GR or a consistent classical modification is recovered. In the pioneering work [122] a first (single-metric) truncation of Quantum Einstein Gravity (QEG) based on a metric field content and diffeomorphism symmetry was studied for the Einstein-Hilbert functional. The promising result of the existence of a non-Gaussian fixed point (NGFP) – hence the first evidence of the Asymptotic Safety conjecture – has been consolidated since then in various different truncations and modifications [99, 100, 121, 123, 126,

130–135, 138, 139, 141, 142, 146–171]. Besides Einstein-Cartan formulations, cutoff-scheme dependency checks, and even infinite dimensional $f(R)$ -truncations based on single-metric techniques, there is also evidence from bi-metric calculations [136, 137, 172, 173]. As Background Independence (BI) is very central to Quantum Gravity bi-metric studies are in fact mandatory, however the difficulties that arise once the EAA depends on two metrics, a dynamical g and a background field $\bar{g}$, has led almost exclusively to developments in the single-metric approximation scheme.

This part of the thesis is therefore devoted to an in depth bi-metric calculation of the FRGE of Quantum Einstein Gravity (QEG) that can be used to address the question of coexistence of Background Independence and Asymptotic Safety. We further allow for a topological modification that is quite important, for instance in cosmology, namely we allow spacetime to have a non-vanishing boundary. Both, performing a bi-metric calculation and a changing the topology to spacetime with non-vanishing boundary, are non-trivial extensions to the well-studied cases, and thus we expect subtle issues to arise due to the non-linearity and functional nature of the FRGE and in constructing the Hessian operator.

We start this part of the thesis by introducing theory space and then present the steps leading from the truncation ansatz for the Effective Average Action to the beta-functions for the running couplings it contains. In order to carefully disentangle fluctuation and background fields, study a truncated theory space spanned by two separate Einstein-Hilbert actions for the dynamical and the background metric, respectively, with a background Gibbons-Hawking-York functional. A new powerful method is used to derive the corresponding RG equations for the Newton- and cosmological constant, both in the dynamical and the background boundary and bulk sector.

In part III we classify and analyze the solutions of the derived beta-functions of this part, determine their fixed point structure, and identify an attractor mechanism which turns out instrumental in the split-symmetry restoration. This then allows to study global properties of the RG flow in particular the coexistence of Background Independence and Asymptotic Safety. We then propose a C -function like property of the FRGE and analyze the beta-functions of the present (bulk part of the) truncation, as well as for a different bi-metric calculation based on a full transverse-traceless (TT)-decomposition, and the single-metric approximation. The same beta-functions are used for study applications to propagating gravitons and to Black Hole thermodynamics. For the latter case the running of Gibbons-Hawking-York boundary term is essential.



5.1	Theory space
	Spacetime
	Field space
	Space of functionals
	Notation and convention
5.2	Preliminary truncation
	Gravitational functional
	Gauge fixing functional
	Ghost functional
5.3	The final truncation
	Split-symmetry
	Coefficients and couplings
5.4	Cutoff-operator

5. THE TRUNCATION

In this chapter we present the details of the considered bi-metric Einstein-Hilbert–Gibbons-Hawking-York truncation. We start by specifying theory space and then described the preliminary truncation ansatz containing a total of eight coupling constants in the gravitational sector plus gauge fixing and ghost parameters. During the process of evaluation, we slightly modify this initial ansatz in order to simplify the nonlinear functional trace on the right-hand-side (RHS) of the flow equation. The actual analyzed truncated theory space then consists of six running couplings all appearing as coefficients of invariants in the gravitational functional. We describe its explicit form in the background-dynamical as well as the level-language.

5.1 Theory space

In the FRG approach there are three essential ingredients that specify the entire RG evolution. The first concerns spacetime, the second field space, and the third the operators that are about to appear in the theory. In this section we present these general constituents of theory space $\mathcal{T}$ step by step. Our objection is a theory space build on a manifold with non-vanishing boundary, metric field content, a Levi-Civita field space connection, and with diffeomorphisms as symmetry group.

5.1.1 Spacetime

In our setting, spacetime does not contain an a priori geometry, in a sense that it is naturally equipped with a Riemannian metric. Rather, it is a special class of topological manifolds with boundary endowed with a smooth structure, for details we refer to section 1.1. Only when the question about the nature of spacetime is settled, we can move on an define fields and operators on top. It is thus a very central ingredient on which all the following concepts sensitively depend on.

In what follows, we actually consider an entire collection of spacetimes that share certain important features, while others are kept arbitrary for a moment and are represented by a parameter that is fixed only for practical applications, for instance the dimension of spacetime.

Since we assume spacetime to be continuous rather than lattice-like having discrete building blocks, the underlying point set X_M has uncountably infinite cardinality. Furthermore, we only consider inequivalent topologies τ_M which are locally Euclidean with boundary, Hausdorff and second countable, hence which define a topological manifold with boundary. In addition, we attach a maximal smooth atlas to the topological space (X_M, τ_X) that results in a smooth manifold with boundary. These requirements still allow for a large class of different spacetimes, in particular the constraints do not fix the spacetime dimension d . Though for a full specification of theory space this is a necessary ingredient, for truncated subspaces a general dimension d suffices in deducing the RG evolution for several choices of d at once. However, for later convenience we restrict d being finite and larger equal 2.

Since a manifold with boundary is more general than one without, the presence of a non-vanishing boundary in the following calculation generalizes the usual considered truncations where ∂M is empty. These special cases are recovered in setting ∂M along with its boundary invariants and coefficients to zero. The price we have to pay in order to extend spacetime to account for non-vanishing boundaries appear on the level of field space and operators. Most of the very technical details in constructing a suitable smooth structure on the set of fields is only proven for $\partial M = \emptyset$. However, in particular assuming spacetime to be compact is usually sufficient to recover most of those features even in the presence of boundaries. Replacing compactness with a weaker condition might also be suitable in certain cases. In particular, in frameworks for which the rigorous mathematical treatment is yet to be established one considers the results to be valid even without compactness. Then future work especially theorems in mathematics have to either verify or reduce the validity of these results. In this work, at least formally we assume a compact manifold whenever boundary effects are studied.

S Thus, the class of spacetimes, $\{M\}$, we consider in this thesis are d -dimensional smooth manifolds with non-vanishing boundary, whereby $2 \leq d < \infty$ holds true and M is assumed to be compact whenever $\partial M \neq \emptyset$.

5.1.2 Field space

Now that the arena is set by a class of smooth manifolds, we can consider field space, F , defined over some M . Since we have only put a lower bound on the dimensionality of spacetime, F will also depends on d .

Even though the original theory of GR is based on a metric field content, this could in fact be an artifact of an effective description and various generalizations are possible. For instance, the Einstein-Cartan formalism is a simple extension of GR whereby the field content is not necessarily related to a Levi-Civita connection but based on a tetrad formulation of gravity. Studies of the Asymptotic Safety conjecture for QG pursuing the Einstein-Cartan formalism were performed in [141–144, 162, 163, 174].

In this thesis, we consider a field content that is given by the set of Riemannian metrics, $\text{Riem}(\bullet)$, defined over M . Since spacetime is assumed finite, it is always paracompact and thus admits a Riemannian structure. Every Riemannian metric represents a section of an associated bundle of a principal bundle with structure group related to $\text{Gl}(d; \mathbb{R})$. The group of gauge transformations $\mathcal{G}$ is thus generated by infinitesimal diffeomorphisms defining the set of invariant action functionals. Hence, the physical field space $[F]$ consists of equivalence classes on $\text{Riem}(\bullet)$ identified by the action of the diffeomorphism group. As already mentioned $\text{Riem}(\bullet)$ can be endowed with a principal bundle structure where the base space constitutes the physical sector $[\text{Riem}(\bullet)]$ and the structure group is given by $\mathcal{G}$. In order to avoid the use of equivalence

classes, we introduce (anti-)ghost fields η and $\bar{\eta}$ and define F to be

$$F = \text{Riem}(M) \times \theta \cdot \Gamma(TM) \times \theta' \cdot \Gamma(T^*M) \quad (5.1)$$

The (anti-)ghost field represents an anti-commuting (dual) vector field, whereby θ (or θ') indicates that the algebraic field of these vector spaces is replaced by anti-commuting numbers.

R Notice that we define the space of expectation fields rather than the fundamental field content. Depending on the actual measure (theory) both coincide or differ, for instance when bosonization of the field content take places in the transition from μ to Γ . All assumptions we impose on $\langle [F] \rangle$ are confined to the arguments of the effective action and the very difficult task is then to extract the bare action functional along with the fundamental field content from its precise structure.

Whereas in the ghost sector, the Maurer-Cartan form acts as a (trivial) connection, the space of Riemannian metrics is non-linearly constrained due to the non-degenerate property of its elements and turns out to have a non-vanishing curvature in general. This complicates matters a lot and we have to rely on a geometrical definition of functional variation, involving the non-trivial connection. Though we are not going to make explicit use of this connection, one can in principal define the Levi-Civita connection of associated to some field space metric $\mathfrak{G}_{\bar{\Phi}} : T_{\bar{\Phi}}F \times T_{\bar{\Phi}}F \rightarrow \mathbb{R}$. Here, we choose the following class of field space metrics [39]:

$$\mathfrak{G}_{\bar{\Phi}}(v, w) = \int_M d^d x \sqrt{\bar{g}} \begin{pmatrix} (v_h)_{\mu\nu} \\ (v_{\xi})^{\mu} \\ (v_{\bar{\xi}})_{\mu} \end{pmatrix} \begin{pmatrix} \mathfrak{g}^{\mu\nu\rho\sigma} & \mathfrak{g}^{\mu\nu}_{\rho} & \mathfrak{g}^{\mu\nu\rho} \\ \mathfrak{g}_{\mu}^{\rho\sigma} & \mathfrak{g}_{\mu\rho} & \mathfrak{g}_{\mu}^{\rho} \\ \mathfrak{g}^{\mu\rho\sigma} & \mathfrak{g}^{\mu}_{\rho} & \mathfrak{g}^{\mu\rho} \end{pmatrix} \begin{pmatrix} (w_h)_{\rho\sigma} \\ (w_{\xi})^{\rho} \\ (w_{\bar{\xi}})_{\rho} \end{pmatrix}$$

Hereby, $\bar{\Phi} = (\bar{g}, \Xi, \bar{\Xi})^T$ represents the set of background fields.

The field space metric assumes the following form in the metric-metric component evaluated at $\bar{g}$, see DeWitt [39]:

$$\mathfrak{g}^{\mu\nu\rho\sigma} = \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - \varpi \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \quad (5.2)$$

The second term affects only the trace part of the metric fluctuations and contains a free parameter ϖ , which in GR assumes the value $\varpi = 1/2$. Notice that the inverse of $\mathfrak{g}^{\mu\nu\rho\sigma}$ is given by

$$\mathfrak{g}_{\mu\nu\rho\sigma}^{-1} = \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} + \frac{\varpi}{(1-d \cdot \varpi)} \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} \quad \forall \varpi \neq 1/d$$

Except for the metric-ghost entries, which are irrelevant in the present context, the remaining structure of $\mathfrak{G}$ is described by the ghost components. Since the ghost fields are Grassmann valued and thus anti-commute, in particular $\bar{\xi} \xi = -\xi \bar{\xi}$ holds true, $\mathfrak{G}$ will be degenerate and anti-symmetric in the ghost sector, while its associated matrix representation is symmetric:

$$\begin{pmatrix} \mathfrak{g}_{\mu\rho} & \mathfrak{g}_{\mu}^{\rho} \\ \mathfrak{g}^{\mu}_{\rho} & \mathfrak{g}^{\mu\rho} \end{pmatrix} = \begin{pmatrix} \bar{g}_{\mu\rho} & \delta_{\mu}^{\rho} \\ \delta^{\mu}_{\rho} & \bar{g}^{\mu\rho} \end{pmatrix} \quad (5.3)$$

This subtlety will be relevant when evaluating the traces of the Hessian operators and in the definition of the cutoff-operator.

Due to the non-vanishing boundary of M we have to impose supplementary constraints on field space in order to obtain self-adjoint differential operators. For a moment we assume that a particular choice is made, however only after computing the Hessian, we explicitly fix those to be of Dirichlet type. Up to this point the considerations involve a completely general boundary operator $\mathcal{B}^{\partial M}$ whose kernel defines field space.

R Notice that the fluctuation fields $\varphi \in T_{\bar{\Phi}}F$ are constraint by the tangent map of this operator, in particular for Dirichlet conditions this results in

$$\mathcal{B}_{(D);\bar{\Phi}}^{\partial M}(\Phi) := (\Phi - \bar{\Phi})(\bullet)\delta_{\partial M}(\bullet) \text{ with } \ker(\mathcal{B}_{(D);\bar{\Phi}}^{\partial M}) \equiv \{\Phi | \Phi|_{\partial M} = \bar{\Phi}|_{\partial M}\}$$

Obviously, while F does not define a vector space since the sum of two fields fulfills different boundary conditions, its tangent space $T_{\bar{\Phi}}F$ is indeed restricted by the linear operator $\mathcal{B}_{(D);\bar{\Phi}}^{\partial M}$ with $\ker(\mathcal{B}_{(D);\bar{g}}^{\partial M}) = \{\varphi \in T_{\bar{\Phi}}F | \varphi|_{\partial M} = 0\}$.

5.1.3 Space of functionals

Finally, theory space can be constructed on the basis of certain (differential) operators $\bar{D}$ and D , the field content F , and spacetime M . In order to give rise to functionals that are independent on the gauge fixing condition, we require theory space to consist of diffeomorphism invariant functionals only.

$$\mathcal{T}(M, F, \{\bar{D}, D\}) = \{A[g, \bar{g}, \eta, \bar{\eta}] | \text{diffeomorphism invariant}\}$$

On the level of the Effective Average Action, which is an element of $\mathcal{T}$, the cutoff implementation and the gauge fixing contribution introduce a true ‘bi-metric’ dependence of theory space that in general spoils split-symmetry, the manifestation of Background Independence in the present context. Thus, in generic elements of theory space are bi-metric functionals with two independent metric arguments: the dynamical- g and the background metric $\bar{g}$. Likewise, utilizing the background field method, we can express $\mathcal{T}$ in terms of fluctuation fields, which reads

$$\mathcal{T}(M, F, \{\bar{D}, D\}) = \{A[h, \xi, \bar{\xi}; \bar{g}] | (h, \xi, \bar{\xi}) \in T_{\bar{\Phi}}F \wedge \text{diffeomorphism invariant}\}$$

In addition to the diffeomorphism constraint we will later impose boundary conditions on the fields, that leads to a vanishing of certain field monomials.

The local part of theory space can be systematically expanded in terms of basis invariants with increasing number of covariant derivatives. For a single metric ansatz, the field monomials up to fourth order in D are given by

$$\begin{array}{lll} \int_M d^d x \sqrt{g} & \int_M d^d x \sqrt{g} R & \int_M d^d x \sqrt{g} \bar{D}^2 \bar{R} \\ \int_M d^d x \sqrt{g} \bar{R}^2 & \int_M d^d x \sqrt{g} \bar{R}^{\mu\nu} \bar{R}_{\mu\nu} & \int_M d^d x \sqrt{g} \bar{R}^{\mu\nu\rho\sigma} \bar{R}_{\mu\nu\rho\sigma} \end{array}$$

In case of non-vanishing boundaries, we have supplementary monomials on ∂M . The boundary conditions for the fields will decide whether or not they appear as basis invariants in the corresponding theory space.

$$\begin{array}{lll} \int_{\partial M} d^{d-1} x \sqrt{H} & \int_{\partial M} d^{d-1} x \sqrt{H} K & \int_{\partial M} d^{d-1} x \sqrt{H} K^2 \\ \int_{\partial M} d^{d-1} x \sqrt{H} K_{\mu\nu} K^{\mu\nu} & \int_{\partial M} d^{d-1} x \sqrt{H} R & \int_{\partial M} d^{d-1} x \sqrt{H} n^\mu n^\nu R_{\mu\nu} \end{array}$$

Here, $n^\mu \partial_\mu$ denotes the *outward-pointing* normal vector, $K^{\mu\nu} = D^\mu n^\nu$ the extrinsic curvature tensor, and $H^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$ the induced metric on ∂M .

Once a second metric enters the game, as the background field $\bar{g}$ in the present case, an infinite number of mixed invariants arise in each order of D and $\bar{D}$, for instance

$$\int_M d^d x \sqrt{\bar{g}} \left(\frac{\sqrt{g}}{\sqrt{\bar{g}}} \right)^j \quad \int_M d^d x \sqrt{\bar{g}} R(g) \quad \int_M d^d x \sqrt{g} \bar{R}(\bar{g})$$

Thus, Background Independence is a severe restriction on the set of functionals that constitute theory space.

5.1.4 Notation and convention

In this part of the thesis, we use the following list of conventions and notations.

- **Einstein sum convention & indices:** If not stated otherwise, an implicit summation over (double) repeated indices is assumed. Small Greek letters denote spacetime indices with values $\mu \in [1, d]$. In case of symmetric 2-tensor fields we sometimes use capital Latin symbols $A \equiv \mu\nu$ to represent its pair of indices.
- **Curvature tensor fields:** In our convention, the Riemann-, Ricci-, and scalar curvature tensor are locally defined as follows:

$$\begin{aligned} R^\sigma{}_{\rho\mu\nu} &= \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\rho\nu} \Gamma^\sigma_{\lambda\mu} - \Gamma^\lambda_{\rho\mu} \Gamma^\sigma_{\lambda\nu} \\ R_{\mu\nu} &= R^\sigma{}_{\rho\sigma\nu} \\ R &= g^{\mu\nu} R_{\mu\nu} \end{aligned}$$

- **Boundary geometry:** In our setting, the normal vector field n of ∂M is pointing outwards. If the normal vector is normalized using the dynamical g or the background metric $\bar{g}$, we write n or $\bar{n}$, respectively. However, in case of Dirichlet boundary conditions $n = \bar{n}$ holds true. The projection of the metric tensor field to the boundary is defined by $H_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu$. Furthermore, the extrinsic curvature tensor assumes the form $K^{\mu\nu} \equiv D^\mu n^\nu$.
- **Field space:** We assume a single background field $\bar{\Phi} = (\bar{g}, \bar{\Xi}, \bar{\Xi})$ is sufficient to cover the entire field space. Thus, the field space metric is evaluated at $\bar{\Phi}$ and we usually write $\mathfrak{G} \equiv \mathfrak{G}_{\bar{\Phi}}$. Furthermore, the dynamical fields are denoted $\Phi = (g, \eta, \bar{\eta})$ and its fluctuation fields $\varphi = (h, \xi, \bar{\xi})$ are given by $\Phi = \exp_{\bar{\Phi}}(\varphi)$.

5.2 Preliminary truncation

To understand the actual process of evaluating truncations it is very illuminating to start with a preliminary ansatz for the class of action functionals that constitute the truncated theory space and then present the final ansatz that is most suitable for practical reasons, we encounter later on. The purpose of this section is to motivate the initial truncation ansatz and define the general theme that should be preserved in the final form of truncated theory space. It is therefore instructive to start once more with the properties of field space in order to decompose theory space into conceptual different portions.

In the previous section we have introduced the frame in which our investigation takes place by specifying theory space, $\mathcal{T}$. Therefore, we defined spacetime, the space of physical fields, and the invariance group of compatible action functionals. Usually, instead of working with the physical relevant fields and operators, one considers an enlarged field space while reducing the set of operators by certain symmetry constraints, as e.g. Becchi, Rouet, Stora and Tyutin (BRST) invariance. To account for the additional (infinite¹) contribution to the path integral measure, one typically implements the Faddeev-Popov insertion by restricting the path integral to a hypersurface that intersects equivalence classes of physical fields exactly once.² This is realized by adding a gauge fixing and a corresponding ghost term to the action, see subsection 2.1.5.

However, there is a subtlety that occurs when variations on the field space appear in the formalism. One source, a covariant way to define the above mentioned gauge fixing condition, is the background field method, see subsection 2.1.5. Thereby, the dynamical field $\hat{\Phi}$ (dynamical metric $\hat{g}$) is expressed in terms of a fixed, but arbitrary background field $\bar{\Phi}$ (background metric

¹The factor is related to the size of an orbit and thus it depends on the symmetry group whether it is finite or infinite.

²The existence of such global hypersurfaces is in general not given [36, 54].

$\bar{g}$) and a fluctuation field $\hat{\phi}$ (or $\hat{h}$) being an element of the tangential space at $\bar{\Phi}$ (or $\bar{g}$), where the identification is given by the exponential map, i.e. for metric field content $\hat{g} = \exp_{\bar{g}}(\hat{h})$ or $\hat{g} = \bar{g} + \hat{h}[\hat{g}, \bar{g}]$. In general, this relation is valid only in a local region of $\bar{\Phi}$ (or $\bar{g}$) and thus we need several patches to cover the entire field space. (On the level of the path integral this is reflected in a summation over different background fields which carries over to the definition of the effective action, depending now on the expectation value of the dynamical metric $g = \langle \hat{g} \rangle$ and $\bar{g} = \langle \bar{g} \rangle$ or likewise $h = \langle \hat{h} \rangle$.) However, in certain cases the field space topology allows to identify each dynamical field with a tangential vector *globally*; then the path integral can be either written in terms of $\hat{g}$ or h and the derived effective action depends on a single field only.

In Quantum Einstein Gravity the physical field content is given by the equivalence classes of Riemannian metrics under the action of the diffeomorphism group. Unfortunately, this space turns out to be curved and we have to related the dynamical and the background fields in a non-trivial way. This results in a theory space spanned by all diffeomorphism invariant field monomials $\mathcal{P}_{(\alpha)}$ depending on a dynamical metric g , a background field $\bar{g}$, and ghost fields ξ , $\bar{\xi}$ which are in total ghost number neutral.³ Each point in theory space can thus be described by an effective action of the form $\Gamma[g, \xi, \bar{\xi}, \bar{g}] = \sum_{\mathcal{I}} \bar{u}^I \mathcal{P}^I[g, \xi, \bar{\xi}, \bar{g}]$. Since we consider a pure gravity truncation, there are no additional matter fields present, though there is no obstruction for including them, see ref. [175]. Hence, the present theory space can be decomposed into pure gravitational contributions, gauge fixing terms and ghost functionals:

$$\Gamma_k[\varphi; \bar{\Phi}] = \Gamma_k^{\text{grav}}[\varphi; \bar{\Phi}] + \Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}] + \Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}] \quad (5.4)$$

Here $\Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}] \equiv \Gamma_k^{\text{gh}}[h, \xi, \bar{\xi}; \bar{g}]$ is the only contribution that contains basis invariants of ghost fields.

R Notice that from the perspective of functionals $\Gamma_k^{\text{grav}}[\varphi; \bar{\Phi}]$ and $\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]$ are indistinguishable, for they describe actions with the same field arguments. However, the relation between ghost- and gauge fixing functional allow to separate the latter from the gravitational contribution, at least formally.

The general way to design a truncation is to put a special emphasize on one of the three constituents of eq. (5.4) and modify the remaining parts as needed. Surely, the most interesting part is the RG evolution of the gravitational functional whereby the gauge fixing- and ghost terms are kept as simple as possible.⁴ The word ‘simple’, however, is very sensitive to the actual truncation ansatz considered and thus it will only become clear during the evaluation process which modifications turn out to be extremely helpful.

In the remainder of this section we specify the initial truncation ansatz in this threefold manner, starting with the gravitational contribution and then define a gauge fixing condition, from which we deduce the ghost functional.

5.2.1 Gravitational functional

In this thesis the emphasize lies on the gravitational part of theory space, $\Gamma_k^{\text{grav}}[\varphi; \bar{\Phi}] \equiv \Gamma_k^{\text{grav}}[h; \bar{g}]$. We want to consolidate a bi-metric calculation on the bulk with boundary terms that seem to be interesting from the cosmological point of view. In its full beauty the (untruncated) Effective Average Action contains an infinite number of different basis invariants of any order in the ghost fields, curvatures and fluctuation- and background fields.

³Here, we make use of the identification of the ghost background fields and their tangent vectors.

⁴Different directions in which the running of the ghost or gauge fixing sector is studied can be found in ref. [132–134] and ref. [135], respectively.

The Einstein-Hilbert truncation, a central part of the following derivation, is for at least two reasons, a suitable choice or testing ground. First of all, in an successive derivative expansion it constitutes all invariants up to second order on manifolds without boundary. Secondly, General Relativity is found to be a very good description of IR physics of gravitational interaction. Thus, its invariants should be included in any truncation that wants to reproduce GR as an effective field theory at small scales k .

In the bi-metric setting and for manifolds with non-vanishing boundaries, theory spaces gets significantly extended, for there are now two metrics and two covariant derivatives which allow for an infinite number of field monomials at each level of the derivative expansion. For instance, the Einstein-Hilbert term is now represented by $\sqrt{\bar{g}}\bar{R}$, $\sqrt{g}R$, $\sqrt{\bar{g}}R$, $\sqrt{g}\bar{R}$ or in general $\sqrt{\bar{g}}^{n+1}/\sqrt{g}^n R$. Furthermore, there are now additional basis invariants defined on the boundary of spacetime, as for instance the Gibbons-Hawking-York term, particular important to obtain Einstein's field equations by a variational principle in the presence of $\partial M \neq \emptyset$:

$$S_{\text{GHY}}[g] = -\frac{1}{8\pi G_N} \int_{\partial M} d^{d-1}x \sqrt{H} K(H)$$

Notice that compared to the Einstein-Hilbert action $S_{\text{EH}}[g] = -\frac{1}{16\pi G_N} \int_M d^d x \sqrt{g} R(g)$ there is a relative prefactor of 2 multiplying the extrinsic curvature K .

In the sequel, we will focus on fully disentangled dynamical and background terms and neglect the basis directions of the 'mixed' interactions. The thus considered subspace is described by EAA's of the form:

$$\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}] \equiv \Gamma_k^{\text{grav};D}[g] + \Gamma_k^{\text{grav};B}[\bar{g}] \quad (5.5)$$

The single-field functionals $\Gamma_k^{\text{grav};D}[g]$ and $\Gamma_k^{\text{grav};B}[\bar{g}]$ only differ by their coefficients. Our truncation ansatz includes all non-mixed basis invariants up to second order in the bulk and correspondingly first order on the boundary, thus a bulk and boundary volume term as well as the Einstein-Hilbert- and the Gibbons-Hawking-York functional, yielding four field monomials for each metric, hence a total of eight basis invariants:

$$\begin{aligned} \Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}] = & (\bar{u}_k^D \bar{\lambda}_k^D) \cdot \Gamma_{(A)}^{\text{grav}}[g] + (\bar{u}_k^D) \cdot \Gamma_{(B)}^{\text{grav}}[g] + (\bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D}) \cdot \Gamma_{(\partial A)}^{\text{grav}}[g] + (\bar{u}_k^{\partial D}) \cdot \Gamma_{(\partial B)}^{\text{grav}}[g] \\ & + (\bar{u}_k^B \bar{\lambda}_k^B) \cdot \Gamma_{(A)}^{\text{grav}}[\bar{g}] + (\bar{u}_k^B) \cdot \Gamma_{(B)}^{\text{grav}}[\bar{g}] + (\bar{u}_k^{\partial B} \bar{\lambda}_k^{\partial B}) \cdot \Gamma_{(\partial A)}^{\text{grav}}[\bar{g}] + (\bar{u}_k^{\partial B}) \cdot \Gamma_{(\partial B)}^{\text{grav}}[\bar{g}] \end{aligned} \quad (5.6)$$

Hereby, $\bar{u}_k^\bullet$ and $\bar{\lambda}_k^\bullet$ are dimensionful coefficients which are subject to the RG flow and related to Newton's coupling and the cosmological constant, respectively. In increasing order in the covariant derivatives D or $\bar{D}$ we have the following list of basis invariants:

$$\Gamma_{(A)}^{\text{grav}}[g] \equiv +2 \cdot \int_M d^d x \sqrt{g} \quad (5.7a)$$

$$\Gamma_{(\partial A)}^{\text{grav}}[g] \equiv +2 \cdot \int_{\partial M} d^{d-1}x \sqrt{H} \quad (5.7b)$$

$$\Gamma_{(\partial B)}^{\text{grav}}[g] \equiv -2 \cdot \int_{\partial M} d^{d-1}x \sqrt{H} K(H) \quad (5.7c)$$

$$\Gamma_{(B)}^{\text{grav}}[g] \equiv - \int_M d^d x \sqrt{g} R(g) \quad (5.7d)$$

Notice that we have defined the coefficients of the Einstein-Hilbert- ($\bar{u}_k^D$) and the Gibbons-Hawking-York term ($\bar{u}_k^{\partial D}$) such that if both agree we recover the classical matching condition that results in the standard field equations.

5.2.2 Gauge fixing functional

To specify the second term, the gauge fixing contribution, we have to define the operator $\mathcal{F}[g, \bar{g}]$ acting on the fluctuation field h . While the entire derivation of the Hessian operator is valid for an arbitrary linear $\mathcal{F}[g, \bar{g}]$ that ‘commutes’ with the background metric $\bar{g}$, there are general arguments in favor of a certain class of gauge fixing conditions where $\mathcal{F}[\bar{g}]$ is only background dependent. Therefore, consider the following.

Whenever the background field method gives rise to a bi-field formulation of the path integral, the gauge fixing hyperplane can be a functional of $\hat{g}$ as well as $\bar{g}$. In the bi-field construction, infinitesimal gauge transformation of the dynamical field, $\delta_v \hat{g} \equiv \mathcal{L}_v \hat{g}$, assumes the form

$$\delta_v \hat{g} = \delta_v \bar{g} + \delta_v \hat{h}[\hat{g}, \bar{g}] \quad \text{for all } v \in \Gamma(TM) \quad (5.8)$$

In decomposing the full dynamical field into a background and a fluctuation field one has to decide in which way the gauge transformation affect $\bar{g}$. A natural choice are the quantum gauge transformations where the background field is unaffected by δ_v :

$$\delta_v \bar{g} = 0 \quad \delta_v \hat{h}[\hat{g}, \bar{g}] \equiv \mathcal{L}_v \hat{g} = \mathcal{L}_v (\bar{g} + \hat{h}[\hat{g}, \bar{g}]) \quad \text{for all } v \in \Gamma(TM) \quad (5.9)$$

However, for generic linear $\mathcal{F}[g, \bar{g}]$ based on the quantum gauge transformation the EAA is not manifestly diffeomorphism invariant off-shell, though on-shell the Ward identities ensure coordinate invariance. Thus, instead of absorbing the full gauge transformation into the fluctuation fields, one may equally distribute its action on both, the fluctuation and background field:

$$\delta_v \bar{g} \equiv \mathcal{L}_v \bar{g} \quad \delta_v \hat{h}[\hat{g}, \bar{g}] \equiv \mathcal{L}_v \hat{h}[\hat{g}, \bar{g}] \quad \text{for all } v \in \Gamma(TM) \quad (5.10)$$

This split is usually denoted background gauge transformation that keeps eq. (5.8) intact. In sharp contrast to its ‘quantum’ counterpart gauge fixing condition which are invariant under background gauge transformations result in a $\bar{g}$ -covariant Effective Average Action suitable to study QEG. The corresponding gauge fixing functional is given by

$$\begin{aligned} \Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}] &\equiv \frac{1}{2} \bar{u}_k^{\text{gf}} \mathfrak{G}(\mathcal{F}[\bar{g}](h) \cdot \hat{e}_{\bar{g}}, \mathcal{F}[\bar{g}](h) \cdot \hat{e}_{\bar{g}}) \\ &= \frac{1}{2} \bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}[\bar{g}]_{\mu}(h) \bar{g}^{\mu\nu} \mathcal{F}[\bar{g}]_{\nu}(h) \end{aligned} \quad (5.11)$$

It is bilinear in the fluctuation fields and thus contributes to the propagator and hence the Hessian operator. Here $\bar{u}_k^{\text{gf}} = Z_k^{\text{D}} / \alpha_k$ contains a running α -parameter, which will be constraint in the end.

A family of appropriate gauge fixing conditions, invariant under background gauge transformations and linear, is the family of generalized harmonic gauge conditions that we will use later on

$$\mathcal{F}[\bar{g}]_{\mu}^{\rho\sigma} = \left(\bar{g}^{\alpha\rho} \bar{g}^{\beta\sigma} - \varpi \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} \right) \bar{g}_{\alpha\mu} \bar{D}_{\beta} \quad (5.12)$$

The kernel of $\mathcal{F}[\bar{g}]$ yields the gauge fixing hyperplane which includes the zero-fluctuation corresponding to $g_{\mu\nu} = \bar{g}_{\mu\nu}$. Notice that this general ansatz, with an up to now free parameter ϖ , covers the harmonic gauge ($\varpi = 1/2$) as well as the ‘anharmonic gauge’, used in [172], for instance, which has $\varpi = 1/d$, respectively.

5.2.3 Ghost functional

The third term in eq. (5.4) is correspondingly the ghost functional containing in principle arbitrary powers of the ghost fields. However, we retain only a bilinear term which is associated to

the Faddeev-Popov trick w.r.t. the gauge condition $\mathcal{F}_\mu^{\alpha\beta}[\bar{g}]h^{\mu\nu}$.

$$\begin{aligned}\Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}] &= -\sqrt{2}\bar{u}_k^{\text{gh}} \mathcal{G}(\varphi_{\bar{\xi}} \cdot \hat{e}_{\bar{\xi}}, \mathcal{M}[g, \bar{g}](\varphi_{\xi})) \\ &= -\sqrt{2}\bar{u}_k^{\text{gh}} \int_M d^d x \sqrt{\bar{g}} \bar{\xi}_\mu \mathcal{M}[g, \bar{g}](\varphi_{\xi})\end{aligned}\quad (5.13)$$

Here, $\mathcal{M}[g, \bar{g}]$ denotes the Faddeev-Popov operator that acts on contravariant vectors according to

$$\mathcal{M}[g, \bar{g}](\xi) = \mathcal{F}[\bar{g}](\mathcal{L}_\xi g) \cdot \hat{e}_{\bar{\xi}} \quad (5.14)$$

Thus, in our setting the ghost functional is fully determined once we specify the gauge fixing condition.

5.3 The final truncation

The final form of the truncation ansatz is mainly oriented by the structure of the Hessian operator that has to be simplified in order to apply the standard heat kernel techniques for evaluating Laplacian like operators. Since we are most interested in the purely gravitational part of theory space most of the modifications affect the gauge and ghost sector, while $\Gamma_k^{\text{grav}}[g, \bar{g}]$ is kept as close as possible to the initial ansatz (5.6).

The RHS of the flow equation involves the second variation of Γ_k w.r.t. the field content and thus it is convenient to expand the action functionals in terms of the fluctuation fields. For the gravitational part, we obtain

$$\Gamma_k^{\text{grav}}[g, \bar{g}] \equiv \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] + (\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \frac{1}{2}(\partial_h \partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \mathcal{O}(h^3)$$

This expansion defines a sum of functionals with increasing degree of homogeneity in h , denoted the level. Hence, $\Gamma_k^{\text{grav}}[\bar{g}, \bar{g}]$ or $(\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}}$ correspond to level-(0) or level-(1), respectively, and the associated k -dependent coefficients play different roles in physics; the level-(1) couplings appear in the field equations for self-consistent backgrounds, while level-(2) coefficients affect the propagator for instance. Employing the linear field parametrization $\partial_v g = v$ and imposing Dirichlet boundary conditions $h|_{\partial M} = 0$, eq. (5.6) is almost identical to the following level-expansion

$$\begin{aligned}\Gamma_k^{\text{grav}}[h; \bar{g}] &= -\bar{u}_k^{(0)} \cdot \int_M d^d x \sqrt{\bar{g}} (\bar{R} - 2\bar{\lambda}_k^{(0)}) - \bar{u}_k^{(0)} \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} (2\bar{K} - 2\bar{\lambda}_k^{(0)}) \\ &\quad + \bar{u}_k^{(1)} \cdot \int_M d^d x \sqrt{\bar{g}} \{ \bar{G}^{\mu\nu} + \bar{\lambda}_k^{(1)} \bar{g}^{\mu\nu} \} h_{\mu\nu} \\ &\quad + \bar{u}_k^{(1)} \cdot \int_M d^d x \sqrt{\bar{g}} h_{\mu\nu} \left\{ \frac{1}{4} \{ -\bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \bar{\mathcal{U}}^{\mu\nu\rho\sigma} \} \right. \\ &\quad \left. - \frac{1}{2} \bar{\lambda}_k^{(1)} \cdot (\bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu}) \right\} h_{\rho\sigma}\end{aligned}\quad (5.15)$$

Here, $\bar{G}^{\mu\nu} \equiv \bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R}$ denotes the Einstein tensor. The two operators that appear in the bilinear part of $\Gamma_k^{\text{grav}}[h; \bar{g}]$ are a kinetic-like and a potential-like contribution, $\bar{\mathcal{K}}^{\mu\nu\rho\sigma} \equiv \mathcal{K}^{\mu\nu\rho\sigma}(\bar{g})$ and $\bar{\mathcal{U}}^{\mu\nu\rho\sigma} \equiv \mathcal{U}^{\mu\nu\rho\sigma}(\bar{g})$, respectively. They are defined as

$$\mathcal{K}^{\mu\nu\rho\sigma}(g) \equiv (g^{\mu\nu} D^\rho D^\sigma + g^{\rho\sigma} D^\mu D^\nu - 2g^{\nu\sigma} D^\rho D^\mu) - (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) D^2 \quad (5.16a)$$

$$\mathcal{U}^{\mu\nu\rho\sigma}(g) \equiv g^{\rho\sigma} G^{\mu\nu} + g^{\mu\nu} R^{\rho\sigma} + R g^{\nu\rho} g^{\mu\sigma} - 2g^{\mu\rho} R^{\nu\sigma} - 2g^{\nu\rho} R^{\mu\sigma} \quad (5.16b)$$

The relation between the level-(p) and the ‘D’/‘B’ coefficients used in eq. (5.15) and (5.6), respectively, will be spelled out in a moment. First of all, notice that the difference between the

final (5.6) and preliminary gravitational part of the truncation (5.15) consists of a surface term, which is omitted in the final result omitted in order to apply the conformal projection technique and evaluate the functional trace. We have the following correspondence between the preliminary and the final gravitational action functional for the linear field parametrization:

$$\Gamma_k^{\text{grav};\text{init}}[h; \bar{g}] = \Gamma_k^{\text{grav};\text{final}}[h; \bar{g}] + (\bar{u}_k^{\partial(1)} - \bar{u}_k^{(1)}) \cdot \int_{\partial M} d^{d-1}x \sqrt{\bar{H}} (\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) h_{\mu\nu}$$

The additional boundary term is linear in h and thus does not affect the Hessian operator. It is proportional to the discrepancy of the classical matching condition for the Einstein-Hilbert and Gibbons-Hawking-York action: $\bar{u}_k^{\partial(1)} = \bar{u}_k^{(1)}$. Thus, if the classical fine tuning survives the RG evolution this term will be absent for all trajectories that preserve this fixed ratio of boundary and bulk term. While the heat kernel expansion comes with the right relative factors for $\int_M \sqrt{\bar{g}} \bar{R}$ and $\int_{\partial M} \sqrt{\bar{H}} \bar{K}$, one would expect that the imbalance of bulk and boundary contributions in the Hessian operator will spoil the matching in general.

Notice that for the choice of Dirichlet conditions, i.e. $h|_{\partial M} \equiv 0$, the set of boundary invariants reduces significantly, in particular the boundary volume element is identical to the background counterpart:

$$\int_{\partial M} d^{d-1}x \sqrt{H}|_{(D)} \equiv \int_{\partial M} d^{d-1}x \sqrt{\bar{H}} \quad \text{since} \quad g_{\mu\nu}|_{\partial M} = \bar{g}_{\mu\nu}|_{\partial M}$$

Hence, once we impose Dirichlet boundary conditions for the metric fluctuations there is literally no corresponding volume term in level-(1) or higher levels. This has to be clearly distinguished from the omitted level-(1) Gibbons-Hawking-York contribution, for which the invariant itself does not vanish.

In this final setting we have translated the original coefficients, related to the background or dynamical functionals in eq. (5.6), into the level-language. The new coefficients are combinations of the original couplings and in each step one can easily switch between both descriptions. For the level-(0) monomials the correspondence is given by:

$$\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)} := \bar{u}_k^D \bar{\lambda}_k^D + \bar{u}_k^B \bar{\lambda}_k^B \quad \bar{u}_k^{(0)} := \bar{u}_k^D + \bar{u}_k^B \quad (5.17a)$$

$$\bar{u}_k^{\partial(0)} \bar{\lambda}_k^{\partial(0)} := \bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D} + \bar{u}_k^{\partial B} \bar{\lambda}_k^{\partial B} \quad \bar{u}_k^{\partial(0)} := \bar{u}_k^{\partial D} + \bar{u}_k^{\partial B} \quad (5.17b)$$

For the invariants of higher orders in h , there is no contribution from the background sector and only the D-coefficients appear. This will be different if we include mixed field monomials in the truncation ansatz, as e.g. $\sqrt{\bar{g}} \bar{R}$. Nevertheless, in the present case we have the following correspondence between the D- and the higher level-coefficients:

$$\bar{u}_k^{(p)} \bar{\lambda}_k^{(p)} := \bar{u}_k^D \bar{\lambda}_k^D \quad \bar{u}_k^{(p)} := \bar{u}_k^D \quad \forall p \geq 1 \quad (5.18a)$$

$$\bar{u}_k^{\partial(p)} \bar{\lambda}_k^{\partial(p)} := \bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D} \quad \bar{u}_k^{\partial(p)} := \bar{u}_k^{\partial D} \quad \forall p \geq 1 \quad (5.18b)$$

Notice, that again due to Dirichlet conditions for the fluctuation fields, certain boundary monomials of non-zeroth order in h vanish in $\Gamma_k[\varphi; \bar{\Phi}]$ and thus do not define independent basis directions in the field space of Dirichlet constraints.

Next, let us consider the gauge fixing functional, which is now evaluated for the harmonic gauge fixing condition with the k -dependence of $\bar{u}_k^{\text{gf}}$ fixed to the gravitational propagator coefficient $\bar{u}_k^{(2)} \equiv \bar{u}_k^{(1)}$, i.e. $\bar{u}_k^{\text{gf}} = \alpha^{-1} \bar{u}_k^{(1)}$. For the harmonic gauge condition, $\mathcal{F}_\mu^{\rho\sigma} = (\bar{g}^{\alpha\rho} \bar{g}^{\beta\sigma} - \varpi \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma}) \bar{g}_{\alpha\mu} \bar{D}_\beta$, we obtain the following type of functionals

$$\Gamma^{\text{gf}}[h; \bar{g}] = \alpha^{-1} \bar{u}_k^{(1)} \int_M d^d x \sqrt{\bar{g}} h_{\mu\nu} \left\{ \frac{1}{2} \bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} - \bar{R}^{\rho\mu\nu\sigma} - \frac{1}{2} (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}) \bar{D}^2 \right\} h_{\rho\sigma} \quad (5.19)$$

Notice that the RG flow is ‘artificially’ constrained along the corresponding basis invariants by allocating a specific k -dependence for the coefficient. The resolution of this field monomial would require a k -dependent α_k , while in the present case it could only be used as a consistency check for the identification $\bar{u}_k^{\text{gf}} = \alpha^{-1} \bar{u}_k^{(1)}$.

The associated ghost functional is projected on a k -independent direction of the corresponding basis invariant, i.e. we choose $\bar{u}_k^{\text{gh}} = \bar{u}^{\text{gh}}$ constant. This results in the following truncation of the ghost sector

$$\Gamma^{\text{gh}}[\varphi; \bar{g}] = -\frac{1}{2} \sqrt{2} \bar{u}^{\text{gh}} \int_{\text{M}} d^d x \sqrt{\bar{g}} \bar{\xi}_\mu \left\{ \bar{g}^{\mu\nu} \mathcal{F}_\nu^{\rho\sigma} (\bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \right\} \xi^\lambda \quad (5.20)$$

As already mentioned, our emphasis is laid on the gravitational sector and we use the gauge fixing and ghost functional to simplify the trace evaluation of the RHS of the flow equation:

$$\partial_t \Gamma_k[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi[\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right]$$

As we see, we make heavy use of modifying the gauge sector, which will become clear in the process of evaluation.

This concludes the final truncation ansatz for which we later compute the beta-functions. Next, we discuss what split-symmetry means for this truncated theory space and then present the conversion of the coefficients into more familiar quantities, as the Newton and cosmological couplings.

5.3.1 Split-symmetry

One of our objectives to study bi-metric truncations is the question of whether or not Asymptotic Safety is compatible with the fundamental requirement of Background Independence in theories of Quantum Gravity. In the EAA approach based on the FRGE this translates to the investigating the status of split-symmetry of the BRST invariant part of theory space, here $\Gamma_k^{\text{grav}}[g, \bar{g}]$. Technically speaking, split-symmetry is the condition that $\Gamma_k^{\text{grav}}[g, \bar{g}]$ reduces to a functional of the dynamical metric g , i.e. $\Gamma_k^{\text{grav}}[g]$ or in terms of the fluctuation fields $\Gamma_k^{\text{grav}}[h; \bar{g}] \equiv \Gamma_k^{\text{grav}}[\exp_{\bar{g}}(h)] \equiv \Gamma_k^{\text{grav}}[\bar{g} + h]$, whereby the last equality holds true for the linear field parametrization.

For the truncation ansatz (5.15) notice that $\Gamma_k^{\text{grav}}[h; \bar{g}]$ corresponds to the Volterra expansion of a single-field functional $\Gamma_k^{\text{grav}}[g]$ if all coefficients of different levels agree. For the bulk invariants these conditions read

$$\bar{u}_k^{(0)} = \bar{u}_k^{(1)} = \bar{u}_k^{(2)} = \dots = \bar{u}_k^{(p)} \quad \text{for all } p \geq 0 \quad (5.21a)$$

$$\bar{\lambda}_k^{(0)} = \bar{\lambda}_k^{(1)} = \bar{\lambda}_k^{(2)} = \dots = \bar{\lambda}_k^{(p)} \quad \text{for all } p \geq 0 \quad (5.21b)$$

Part of these requirements are already implemented into the truncation ansatz by identifying all level- (p) coefficients for $p \geq 1$. While in general one expects that this connection breaks the considered truncation is anyway insensitive to a separation of the corresponding basis invariants. Compared to the usually employed single-metric calculations the disentanglement of level-(0) and level-(1) coefficients is an important progress in understanding the infinite number of Newton-type couplings isolate their respective RG evolution.

For the boundary coefficients, there is a subtlety involved, since for Dirichlet boundary conditions all field monomials associated to $\int_{\partial\text{M}} \sqrt{H}$ vanish above level-(0). Hence, $\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)}$ can assume any value without spoiling split-symmetry. For the curvature boundary term resulting from the Gibbons-Hawking-York invariant, $\int_{\partial\text{M}} \sqrt{H} K$, Dirichlet conditions give rise to two independent field monomials, one of level-(0) and one of level-(1). However, the latter is

not resolved within the present truncation and thus we have to focus the study of Background Independence on the bulk part of this truncation only.

A different way to present split-symmetry is given by the D/B description, where $\bar{u}_k^D \equiv \bar{u}_k^{(p)}$ equals the level-coefficients for all $p \geq 1$, for instance. The case of $\Gamma_k^{\text{grov}}[g, \bar{g}]$ loosing its ‘extra’ $\bar{g}$ -dependence, [122], is equal to the vanishing of all background coefficients, i.e. the following conditions have to hold true:

$$\bar{u}_k^B = 0 \quad \wedge \quad \bar{u}_k^B \bar{\lambda}_k^B = 0 \quad \wedge \quad \bar{u}_k^{\partial B} = 0 \quad \wedge \quad \bar{u}_k^{\partial B} \bar{\lambda}_k^{\partial B} = 0 \quad (\text{split sym.}) \quad (5.22)$$

In general, due to the cutoff-action and the gauge fixing contribution that explicitly introduce extra factors of $\bar{g}$ that do not combine to purely dynamical invariants, split-symmetry will be broken along RG trajectories. In fact, even for $k = 0$ where the cutoff-dependence vanishes, the gauge fixing functional is still a potential source of split-symmetry violation.

R For single-metric calculations, split-symmetry is assumed to be preserved by the full RG evolution. This is a very severe constraint, a priori not justified by any algebraic or analytic argument. Indeed, we will later demonstrate that there bi-metric calculation seem to support the statement that there exists not a single trajectory for which split-symmetry is intact for all k . Consequences of these violation of the single-metric assumption are regions in theory space where the single-metric results are unreliable and in fact exhibit properties incompatible with exact solutions.

5.3.2 Coefficients and couplings

In order to make contact to the couplings encountered in classical General Relativity let us reformulate the coefficients $\bar{u}_k$ in terms of the dimensionful Newton-type couplings G_k :

$$\bar{u}_k^I \equiv \frac{1}{16\pi G_k^I} \quad \text{for all} \quad I \in \{(p), \partial(p), B, \partial B, D, \partial D\} \quad (5.23)$$

Within these more familiar couplings, our truncation ansatz (5.15) reads

$$\begin{aligned} \Gamma_k^{\text{grov}}[h, \bar{g}] = & -\frac{1}{16\pi G_k^{(0)}} \cdot \int_M d^d x \sqrt{\bar{g}} \left(\bar{R} - 2\bar{\lambda}_k^{(0)} \right) - \frac{1}{16\pi G_k^{\partial(0)}} \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left(2\bar{K} - 2\bar{\lambda}_k^{\partial(0)} \right) \\ & + \frac{1}{16\pi G_k^{(1)}} \cdot \int_M d^d x \sqrt{\bar{g}} \left\{ \bar{G}^{\mu\nu} + \bar{\lambda}_k^{(0)} \bar{g}^{\mu\nu} \right\} h_{\mu\nu} \\ & + \frac{1}{16\pi G_k^{(1)}} \cdot \int_M d^d x \sqrt{\bar{g}} \left\{ h_{\mu\nu} \left[\frac{1}{4} \left\{ -\bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \bar{\mathcal{U}}^{\mu\nu\rho\sigma} \right\} \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \bar{\lambda}_k^{(2)} \cdot \left(\bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right) \right] h_{\rho\sigma} + \mathcal{O}(h^3) \right\} \end{aligned} \quad (5.24)$$

The relation between the level- and D/B-formulation is given by the following conversion rules:

$$\frac{1}{G_k^{(0)}} = \frac{1}{G_k^B} + \frac{1}{G_k^D} \quad \wedge \quad \frac{\bar{\lambda}_k^{(0)}}{G_k^{(0)}} = \frac{\bar{\lambda}_k^B}{G_k^B} + \frac{\bar{\lambda}_k^D}{G_k^D} \quad \wedge \quad G_k^{(1)} = G_k^{(2)} = \dots = G_k^D \quad (5.25)$$

And similarly for the boundary couplings. Hence, a truncation ansatz with the six independent couplings $G_k^B, \bar{\lambda}_k^B, G_k^D, \bar{\lambda}_k^D, G_k^{\partial D}, \bar{\lambda}_k^{\partial D}$ is equivalent to an ansatz of the type (5.24) whose a priori infinitely many couplings $G_k^{(p)}, \bar{\lambda}_k^{(p)}, \dots$ are not independent but satisfy

$$G_k^{(0)} = \frac{G_k^D G_k^B}{G_k^D + G_k^B}, \quad G_k^{(1)} = G_k^{(2)} = \dots = G_k^D \quad (5.26)$$

and likewise for the cosmological constant type couplings:

$$\bar{\lambda}_k^{(0)} = \frac{\bar{\lambda}^D G_k^B + \bar{\lambda}^B G_k^D}{(G_k^D + G_k^B)}, \quad \bar{\lambda}^{(1)} = \bar{\lambda}^{(2)} = \dots = \bar{\lambda}^D \quad (5.27)$$

It will be instructive to switch back and forth between the $(g, \bar{g})$ -language employing the ‘B’ and ‘D’ parameters, and the $(h; \bar{g})$ -language with $G_k^{(p)}, \bar{\lambda}_k^{(p)}, p = 0, 1, 2, \dots$.

In case of intact split-symmetry where the different levels stem from a Volterra expansion we have the following set of relations among the various Newton- and cosmological constant type couplings:

$$G_k^{(0)} = G_k^{(1)} = G_k^{(2)} = \dots, \quad \text{and} \quad \bar{\lambda}_k^{(0)} = \bar{\lambda}_k^{(1)} = \bar{\lambda}_k^{(2)} = \dots \quad (\text{split sym.}) \quad (5.28)$$

In the $(g, \bar{g})$ -language these relations are satisfied if the ‘B’ part of the EAA completely vanishes

$$\frac{1}{G_k^B} = 0, \quad \text{and} \quad \frac{\bar{\lambda}^B}{G_k^B} = 0 \quad (\text{split sym.}) \quad (5.29)$$

or, stated differently when $\Gamma_k^{\text{grav}}[g, \bar{g}]$ is actually independent of its second argument.

Again, we should mention that there is no guarantee for the existence of trajectories satisfying condition (5.28) or likewise (5.29). However, there are points in theory space where these conditions are fulfilled or are closed to be. In these regions a single metric description is well applicable. From eq. (5.29) we see that if $G_k^B \gg G_k^D$ and $\bar{\lambda}^B \approx \bar{\lambda}^D$ split-symmetry is approximately restored.

5.4 Cutoff-operator

The FRGE defines a vector field on theory space which in turn describes the RG evolution of certain action functionals. The RG flow that is described by a sequence of functionals $\Gamma_k[\Phi, \bar{\Phi}]$ is triggered by the cutoff operator $\mathfrak{R}_k[\Phi]$, which loosely speaking gives a k -dependent mass to all fluctuation fields. Its implementation is based on a bilinear functional, the so-called cutoff action:

$$\Delta S_k[\varphi; \bar{\Phi}] = \frac{1}{2} \mathfrak{G}(\varphi, \mathfrak{R}_k[\bar{\Phi}] \varphi) \quad (5.30)$$

This action, essentially the cutoff-operator $\mathfrak{R}_k$, divides field space into suppressed and unsuppressed field modes, depending on k . Lowering k corresponds to gradually increasing the amount of fields one integrates over and in the limit $k \rightarrow 0$ one recovers the usual effective action $\lim_{k \rightarrow 0} \Gamma_k = \Gamma$.

In the sequel, we use the Laplacian operator of the background metric $\Delta \equiv -\bar{D}^2$ to express field space in terms of its eigenfunctions. For Dirichlet boundary conditions, Δ is strongly elliptic, which implies that all its eigenvalues are non-negative, and there is only a finite number of zero eigenmodes. Hence, when relating the cutoff procedure to the eigenfunction expansion of Δ , the scale k assumes values from 0 to ∞ only.

The Gaussian part associated to Δ in the present truncation reads

$$\begin{aligned} \Gamma_k^{\text{free}}[\varphi; \bar{\Phi}] = & + \frac{1}{2} \bar{u}_k^D \int_{\mathcal{M}} d^d x \sqrt{\bar{g}} h_{\mu\nu} (\mathfrak{g}_{hh})^{\mu\nu\rho\sigma} (-\bar{D}^2) h_{\rho\sigma} \\ & - \frac{1}{2} \sqrt{2} \bar{u}^{\text{gh}} \int_{\mathcal{M}} d^d x \sqrt{\bar{g}} \left(\bar{\xi}_\nu (\mathfrak{g}_{\bar{\xi}\bar{\xi}})^\nu{}_\mu (-\bar{D}^2) \xi^\mu - \xi^\mu (\mathfrak{g}_{\xi\xi})_\mu{}^\nu (-\bar{D}^2) \bar{\xi}_\nu \right) \end{aligned}$$

whereby $\mathfrak{g} \equiv \mathfrak{g}[\bar{g}]$ denotes the local part of the field space metric, i.e. $(\mathfrak{g}_{hh})^{\mu\nu\rho\sigma} \equiv \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}$ for the gravitational sector (for the harmonic choice $\bar{\omega} = 1/2$) and $(\mathfrak{g}_{\xi\xi})^\nu_\mu \equiv \delta^\nu_\mu$ or $(\mathfrak{g}_{\xi\xi})_\mu^\nu \equiv \delta_\mu^\nu$ for the ghost parts.

The cutoff procedure is now understood as replacing Δ with $\Delta + R_k^{(0)}(\Delta)$, whereby $R_k^{(0)}(\Delta) \equiv k^2 R^{(0)}(\Delta/k^2)$ denotes the shape-function. The matrix form related to $\Gamma_k^{\text{free}}[\varphi; \bar{\Phi}]$ is thus given by

$$\mathfrak{R}_k[\bar{g}] = \begin{pmatrix} \mathfrak{R}_k^{\text{grav}}[\bar{g}] & 0 & 0 \\ 0 & 0 & \mathfrak{R}_k[\bar{g}]^{\text{gh}} \\ 0 & \mathfrak{R}_k[\bar{g}]^{\text{gh}} & 0 \end{pmatrix} \quad (5.31)$$

A comparison with eq. (5.30) reveals the explicit structure of the L^2 -operators in terms of the shape-function:

$$\mathfrak{R}_k^{\text{grav}}[\bar{g}] = +\bar{u}_k^{\text{D}} R_k^{(0)}(-\bar{D}^2) \cdot \mathbb{1}_{[h]} \quad (5.32a)$$

$$\mathfrak{R}_k^{\text{gh}}[\bar{g}] = -\sqrt{2}\bar{u}^{\text{gh}} R_k^{(0)}(-\bar{D}^2) \cdot \mathbb{1}_{[\xi]} \quad (5.32b)$$

$$\mathfrak{R}_k^{\text{gh}}[\bar{g}] = +\sqrt{2}\bar{u}^{\text{gh}} R_k^{(0)}(-\bar{D}^2) \cdot \mathbb{1}_{[\xi]} \quad (5.32c)$$

For this choices, the required ‘symmetry’ property of the cutoff-operator is indeed satisfied:

$$\mathfrak{G}(v, \mathfrak{R}_k[\bar{\Phi}] w) = \mathfrak{G}(w, \mathfrak{R}_k[\bar{\Phi}] v) \quad \forall v, w \in T_{\bar{\Phi}} F,$$

R Notice that the cutoff operator does not depend on the background fields of the ghost sector. Thus, there is no violation of ‘ghost split-symmetry’ between the background and dynamical ghost fields. The situation is quite different for the metric field content, explaining one source of split-symmetry violation which however only appears for $k \neq 0$.

6.1	Hessian of the cutoff action
6.2	Hessian of the gauge fixing functional
6.3	Hessian of the ghost functional
6.4	Hessian of the gravitational functional
6.5	Hessian of the full truncation ansatz
6.6	First variation formula
6.7	Implementation of the gauge fixing condition
6.8	The level expansion

6. THE HESSIAN OPERATOR

The vital piece in the evaluation of the RHS of the Functional Renormalization Group Equation is the Hessian operator of the Effective Average Action. Besides the non-linearity of the FRGE that emanates from the inversion of this operator, determining the trace of the Hessian operator represents the major difficulty in the derivation of beta-functions especially in the present of boundaries for spacetime. In this chapter we compute the second variations of the preliminary truncation ansatz under very general conditions, in particular for generic boundary constraints, any linear gauge fixing condition, and arbitrary field parametrization. This allows future work to modify the settings in the range of validity and deduce new results with less effort. Towards the end of the chapter, we then evaluate the Hessian operators for Dirichlet boundary conditions and for the family of generalized harmonic gauge conditions that will be used in the sequel of the derivation.

6.1 Hessian of the cutoff action

The cutoff action, as defined in eq. (5.30), is a bilinear functional of the fluctuation field $\varphi \equiv \varphi(\hat{\Phi}, \bar{\Phi})$ that defines a geodesic $\nabla_{\varphi}^{[F]} \varphi = 0$ on field space F interconnecting a base point $\bar{\Phi}$ with the dynamical field $\hat{\Phi}$:

$$\Delta S_k[\varphi; \bar{\Phi}] = \frac{1}{2} \mathfrak{G}(\varphi, \mathfrak{R}_k[\bar{\Phi}] \varphi) \text{ with } \mathfrak{G}(v, \mathfrak{R}_k[\bar{\Phi}] w) = \mathfrak{G}(w, \mathfrak{R}_k[\bar{\Phi}] v) \quad \forall v, w \in T_{\bar{\Phi}} F$$

The latter condition is equivalent to $\mathfrak{R}_k[\bar{\Phi}]$ being (anti-)self-adjoint for (Grassmannian) fields.

Since $\mathfrak{R}_k[\bar{\Phi}]$ and the field space metric $\mathfrak{G}$ are linear the functional variation of $\Delta S_k[\varphi; \bar{\Phi}]$ with respect to the fluctuation field φ is a straightforward computation. Since the tangent space is a usual vector space we have the trivial identification $T_{\bar{\Phi}} F \simeq T_{\varphi} T_{\bar{\Phi}} F$ with exponential map $\exp_{\varphi}(tv) \equiv \varphi +_{TF} \iota(v)$, whereby $\iota : T_{\varphi} T_{\bar{\Phi}} F \rightarrow T_{\bar{\Phi}} F$ denotes the inclusion map. Hence, any functional variation on $T_{\bar{\Phi}} F$ reduces to a simple Gateaux derivative. We are going to demonstrate the general procedure of functional variation for this particular elementary example making each step explicit. In the later derivation of the Hessian operators we present a reduced form in which some of the rather technical details are omitted.

For now, let $v \in T_\varphi T_{\bar{\Phi}}F$ be any tangent vector of φ with $\tilde{v} \equiv \iota(v)$ its natural identification in $T_{\bar{\Phi}}F$. The first variation of the cutoff action yields

$$\begin{aligned} \mathfrak{d}_v \Delta S_k[\varphi; \bar{\Phi}] &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\Delta S_k[\varphi +_{TF} (\varepsilon \cdot \tilde{v}); \bar{\Phi}] - \Delta S_k[\varphi; \bar{\Phi}]) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} (\mathfrak{G}(\varphi +_{TF} (\varepsilon \cdot \tilde{v}), \mathfrak{R}_k[\bar{\Phi}] \varphi +_{TF} (\varepsilon \cdot \tilde{v})) - \mathfrak{G}(\varphi, \mathfrak{R}_k[\bar{\Phi}] \varphi)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} (\varepsilon \cdot \mathfrak{G}(\varphi, \mathfrak{R}_k[\bar{\Phi}] \tilde{v}) + \varepsilon \cdot \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \varphi) + \varepsilon^2 \cdot \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \tilde{v})) \\ &= \frac{1}{2} \mathfrak{G}(\varphi, \mathfrak{R}_k[\bar{\Phi}] \tilde{v}) + \frac{1}{2} \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \varphi) \end{aligned}$$

As expected, the resulting functional $\mathfrak{d}_v \Delta S_k[\varphi; \bar{\Phi}]$ is linear in φ and $\tilde{v}$. Thus the second field derivative w.r.t. $w \in T_\varphi T_{\bar{\Phi}}F$, whereby $\tilde{w} \equiv \iota(w)$, assumes the form:

$$\begin{aligned} \mathfrak{d}_w (\mathfrak{d}_v \Delta S_k[\varphi; \bar{\Phi}]) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left((\mathfrak{d}_v \Delta S_k[\varphi; \bar{\Phi}])|_{\varphi=\varphi+_{TF}(\varepsilon \cdot \tilde{w})} - \mathfrak{d}_v \Delta S_k[\varphi; \bar{\Phi}] \right) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} (\varepsilon \cdot \mathfrak{G}(\tilde{w}, \mathfrak{R}_k[\bar{\Phi}] \tilde{v}) + \varepsilon \cdot \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \tilde{w})) \\ &= \frac{1}{2} \mathfrak{G}(\tilde{w}, \mathfrak{R}_k[\bar{\Phi}] \tilde{v}) + \frac{1}{2} \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \tilde{w}) \end{aligned}$$

The second functional variation of the cutoff action is already symmetric and hence agrees with the Hessian of $\Delta S_k[\varphi; \bar{\Phi}]$. Using the (anti-)self-adjoint property of the cutoff operator $\mathfrak{R}_k[\bar{\Phi}]$ the previous result further simplifies to

$$\begin{aligned} \text{Hess}_\varphi [\Delta S_k[\varphi; \bar{\Phi}]](v, w) &\equiv \frac{1}{2} (\mathfrak{d}_w (\mathfrak{d}_v \Delta S_k[\varphi; \bar{\Phi}]) + \mathfrak{d}_v (\mathfrak{d}_w \Delta S_k[\varphi; \bar{\Phi}])) \\ &= \frac{1}{2} \mathfrak{G}(\tilde{w}, \mathfrak{R}_k[\bar{\Phi}] \tilde{v}) + \frac{1}{2} \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \tilde{w}) \\ &= \mathfrak{G}(\tilde{w}, \mathfrak{R}_k[\bar{\Phi}] \tilde{v}) \equiv \mathfrak{G}(\tilde{v}, \mathfrak{R}_k[\bar{\Phi}] \tilde{w}) \end{aligned} \quad (6.1)$$

The second equality of eq. (6.1) is the very definition of the associated Hessian operator which in this case coincides with the cutoff operator:

$$\widehat{\text{Hess}}_\varphi [\Delta S_k[\varphi; \bar{\Phi}]] = \mathfrak{R}_k[\bar{\Phi}] \quad (6.2)$$

Crucial in the derivation of this Hessian operator is the (anti-)self-adjoint property of $\mathfrak{R}_k[\bar{\Phi}]$ that has to be satisfied for any explicit choice.

6.2 Hessian of the gauge fixing functional

Working with an enlarged field space F instead of the set of physical indistinguishable fields $[F]$ we have to introduce a gauge fixing condition that selects a single representative Φ for each $\bar{\Phi}$. On the level of the EAA a very general form of the gauge fixing contribution is given by

$$\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}] \equiv \frac{1}{2} \bar{u}_k^{\text{gf}} \mathfrak{G}(\mathcal{F}[\bar{g}](h) \cdot \hat{e}_{\bar{\xi}}, \mathcal{F}[\bar{g}](h) \cdot \hat{e}_{\bar{\xi}})$$

While the full transformation $\mathcal{F}[\bar{g}](\bullet) \cdot \hat{e}_{\bar{\xi}} \otimes \hat{e}_h^*$ is indeed an operator on $T_{\bar{\Phi}}[F]$ in the usual sense, its field coordinate form $\mathcal{F} \equiv \mathcal{F}[\bar{g}]$ is not, for it lacks the endomorphism property mapping metric fluctuation into dual vectors: $T_{\bar{g}}F \rightarrow \Gamma(T^*M)$. Hence, the adjoint map $\mathcal{F}^*[\bar{g}] : \Gamma(TM) \rightarrow T_{\bar{g}}F$ has to be lifted to a transformation on field space – rather than one on its components – to give rise to the adjoint operator $\mathcal{F}^*[\bar{g}](\bullet) \cdot \hat{e}_h \otimes \hat{e}_{\bar{\xi}}^*$.

For the sake of generality, we will keep the explicit form of $\mathcal{F}[\bar{g}]$ arbitrary for a moment, however besides locality we assume it satisfies the following requirements:

$$\begin{aligned}\mathcal{F}[\bar{g}](c \cdot h_1 + h_2) &\equiv c \cdot \mathcal{F}[\bar{g}](h_1) + \mathcal{F}[\bar{g}](h_2) && \text{(linear)} \\ [\mathcal{F}[\bar{g}](\bullet), \bar{g}] &= 0 && \text{(commutes with } \bar{g})\end{aligned}$$

This still give rise to a very large class of gauge fixing functionals built from the covariant derivative $\bar{D}$ and the background field $\bar{g}$. In this general setting, consider the L^2 operator associated to the gauge fixing condition and determine its adjoint form, which is given by

$$(\mathcal{F}^*[\bar{g}])_A^\mu = -\frac{1}{\sqrt{\bar{g}}} \mathfrak{g}_{AB}^{-1} \mathcal{F}_V^B (\sqrt{\bar{g}} \bar{g}^{\mu\nu} \bullet) \equiv -\mathfrak{g}_{AB}^{-1} \mathcal{F}_V^B (\mathfrak{g}^{\mu\nu} \bullet)$$

Now, the linearity of $\mathcal{F}[\bar{g}]$ results in a bilinearity of $\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]$ which in turn makes the evaluation of the Hessian very simple:

$$\begin{aligned}\text{Hess}_\varphi [\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]](v, w) &\equiv \frac{1}{2} (\mathfrak{d}_w (\mathfrak{d}_v \Gamma_k^{\text{gf}}[h; \bar{g}]) + \mathfrak{d}_v (\mathfrak{d}_w \Gamma_k^{\text{gf}}[h; \bar{g}])) \\ &= \bar{u}_k^{\text{gf}} \mathfrak{G}(w, \hat{e}_h \cdot (\mathcal{F}^*[\bar{g}] \circ \mathcal{F}[\bar{g}]) (\hat{e}_h \cdot v)) \\ &\quad + \bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\mu^A (\bar{g}^{\mu\nu} (w_h)_A \mathcal{F}_V^B (v_h)_B) \quad || \triangleq \mathcal{J}^{\text{gf}}[v, w]\end{aligned} \tag{6.3}$$

The second term in this final result stems from the chain rule and can be usually transformed into a boundary contribution, depending on the explicit form of $\mathcal{F}_V^B$. This additional boundary piece in the Hessian is an obstruction to deduce a Hessian operator. Hence, assuming that in following we find a way to absorb or eliminate this second contribution, we formally define the Hessian operator of $\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]$ in the usual way, i.e.

$$\widehat{\text{Hess}}_\varphi [\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]](v, w) = \mathfrak{G}(w, \widehat{\text{Hess}}_\varphi [\Gamma_k^{\text{gf}}[h; \bar{g}]](v)) + \mathcal{J}^{\text{gf}}[v, w] \tag{6.4}$$

By a comparison of eq. (6.3) and eq. (6.4) we obtain the matrix form in field space coordinates:

$$\widehat{\text{Hess}}_\varphi [\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]] \equiv \bar{u}_k^{\text{gf}} \cdot \begin{pmatrix} \widehat{\text{Hess}}_{hh}^{\text{gh}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Obviously, this operator on its own is not invertible for its vanishing in the ghost sector. The only non-vanishing contribution is a pure background field dependent L^2 -operator given by

$$\left(\widehat{\text{Hess}}_{hh}^{\text{gf}} \right)_B^A \equiv (\mathcal{F}^*[\bar{g}] \circ \mathcal{F}[\bar{g}])_B^A \equiv -\mathfrak{g}_{BC}^{-1} \mathcal{F}_\mu^C (\mathfrak{g}^{\mu\nu} \mathcal{F}_V^A \bullet)$$

Once we have collected all the contributions to the Hessian, we are going to consider a very important class of gauge fixing conditions. Then, the explicit form of $\widehat{\text{Hess}}_{hh}^{\text{gh}}$ and $\mathcal{J}^{\text{gf}}[v, w]$ for these choices will be inserted.

6.3 Hessian of the ghost functional

In order to extract the superfluous information of the gravitational functional that is due to an over-counting of gauge equivalent fields we had to add the ghost action which couples the gravitational fluctuation field with the corresponding fluctuations of the ghost sector. Using

the metric on field space this final contribution of Faddeev-Popov method can be cast into the following form:

$$\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}] = -\sqrt{2\bar{u}_k^{\text{gh}}} \mathfrak{G}(\varphi_{\bar{\xi}} \cdot \hat{e}_{\bar{\xi}}, \mathcal{M}[g, \bar{g}](\varphi_{\xi})) \quad (6.5)$$

The appearing Faddeev-Popov operator, $\mathcal{M}[g, \bar{g}]$, considered as a function on full field space, is based on the not yet specified gauge fixing condition and the infinitesimal gauge transformations:

$$\mathcal{M}[g, \bar{g}](\xi) = \mathcal{F}[\bar{g}](\mathcal{L}_{\xi} g) \cdot \hat{e}_{\bar{\xi}} \quad (6.6)$$

Notice however that we assume a linear, local gauge fixing condition. This term is the only source of graviton-ghost interactions and as such will generate the ghost sector on the RHS of the flow equation.

For the derivation of the Hessian of $\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]$ we can use its linearity w.r.t. every field, yielding three contributions which we consider separately in what follows:

$$\begin{aligned} \text{Hess}_{\varphi}[\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]](v, w) &= -\sqrt{2\bar{u}_k^{\text{gh}}} \mathfrak{G}(\partial_v \varphi_{\bar{\xi}} \cdot \hat{e}_{\bar{\xi}}, \partial_w \mathcal{M}[g, \bar{g}](\xi)) && \parallel \triangleq \mathcal{N}_I(v, w) \\ &\quad - \sqrt{2\bar{u}_k^{\text{gh}}} \mathfrak{G}(\partial_w \varphi_{\bar{\xi}} \cdot \hat{e}_{\bar{\xi}}, \partial_v \mathcal{M}[g, \bar{g}](\xi)) && \parallel \triangleq \mathcal{N}_I(w, v) \\ &\quad - \sqrt{2\bar{u}_k^{\text{gh}}} \mathfrak{G}(\varphi_{\bar{\xi}} \cdot \hat{e}_{\bar{\xi}}, \tfrac{1}{2}(\partial_v \partial_w + \partial_w \partial_v) \mathcal{M}[g, \bar{g}](\xi)) && \parallel \triangleq \mathcal{N}_{II}(v, w) \end{aligned}$$

Due to the linear structure, the Hessian will contain only off-diagonal terms and furthermore both $\mathcal{N}_I$ and $\mathcal{N}_{II}$ involve derivatives of the Faddeev-Popov operator. Since $\mathcal{F}[\bar{g}]$ is independent on the fluctuation fields and linear, the functional variation of $\mathcal{M}[g, \bar{g}](\xi)$ affects only the Lie derivative:

$$\partial_v \mathcal{M}[g, \bar{g}](\xi) = \mathcal{F}[\bar{g}](\partial_v \mathcal{L}_{\xi} g) \cdot \hat{e}_{\bar{\xi}}$$

At this point the ghost field ξ and the dynamical metric g appear as different constituents in the Lie derivative. Again linearity in both arguments allows the following expression of the functional variation of $\mathcal{L}_{\xi} g$:

$$\begin{aligned} \partial_v (\mathcal{L}_{\xi} g) &= \mathcal{L}_{\xi} \partial_v g + \mathcal{L}_{v_{\xi}} h \\ \partial_w \partial_v (\mathcal{L}_{\xi} g) &= \mathcal{L}_{\xi} \partial_w \partial_v g + \mathcal{L}_{w_{\xi}} \partial_v g + \mathcal{L}_{v_{\xi}} \partial_w g \end{aligned}$$

For a symmetric two-tensor, as encountered for g , the Lie derivative can be written using the covariant derivative of the background metric $\bar{g}$:

$$\begin{aligned} (\mathcal{L}_{\xi} g)_{\rho\sigma} &= \xi^{\mu} \partial_{\mu} g_{\rho\sigma} + g_{\mu\rho} \partial_{\sigma} \xi^{\mu} + g_{\mu\sigma} \partial_{\rho} \xi^{\mu} \\ &= \xi^{\mu} \bar{D}_{\mu} g_{\rho\sigma} + g_{\mu\rho} \bar{D}_{\sigma} \xi^{\mu} + g_{\mu\sigma} \bar{D}_{\rho} \xi^{\mu} \end{aligned}$$

Hereby, we substituted $\bar{D}_{\mu} g_{\rho\sigma} = \partial_{\mu} g_{\rho\sigma} - \bar{\Gamma}_{\mu\rho}^{\lambda} g_{\lambda\sigma} - \bar{\Gamma}_{\mu\sigma}^{\lambda} g_{\lambda\rho}$ in the last step and notice that all connection coefficients $\bar{\Gamma}$ cancel each other. It is now convenient to express the functional variation of the Lie derivative using its explicit action on the fluctuation field leading to¹

$$\begin{aligned} \partial_v (\mathcal{L}_{\xi} g)_{\rho\sigma} &= \xi^{\mu} \bar{D}_{\mu} (\partial_v g)_{\rho\sigma} + (\partial_v g)_{\mu\rho} \bar{D}_{\sigma} \xi^{\mu} + (\partial_v g)_{\mu\sigma} \bar{D}_{\rho} \xi^{\mu} \\ &\quad + v_{\xi}^{\mu} \bar{D}_{\mu} g_{\rho\sigma} + g_{\mu\rho} \bar{D}_{\sigma} v_{\xi}^{\mu} + g_{\mu\sigma} \bar{D}_{\rho} v_{\xi}^{\mu} \end{aligned} \quad (6.7a)$$

$$\begin{aligned} \partial_w \partial_v (\mathcal{L}_{\xi} g)_{\rho\sigma} &= w_{\xi}^{\mu} \bar{D}_{\mu} (\partial_v g)_{\rho\sigma} + (\partial_v g)_{\mu\rho} \bar{D}_{\sigma} w_{\xi}^{\mu} + (\partial_v g)_{\mu\sigma} \bar{D}_{\rho} w_{\xi}^{\mu} \\ &\quad + v_{\xi}^{\mu} \bar{D}_{\mu} (\partial_w g)_{\rho\sigma} + (\partial_w g)_{\mu\rho} \bar{D}_{\sigma} v_{\xi}^{\mu} + (\partial_w g)_{\mu\sigma} \bar{D}_{\rho} v_{\xi}^{\mu} \end{aligned} \quad (6.7b)$$

¹In the following we omit contributions of $\partial_w \partial_v g$. While in the linear parametrization employed later on this terms identically vanish, in a geometric parametrization those terms contribute, however only to the ghost sector.

This information is sufficient to deduce the Hessian of the ghost functional and as mentioned above we will proceed in three steps. The main effort originates from rewriting the second variation of $\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]$ into a symmetric expression suitable to read off the Hessian operator, i.e. $\mathfrak{G}(w, \widehat{\text{Hess}}_\phi[\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]](v))$. Due to the non-empty boundary of the manifold there will be obstructions in form of boundary contributions that we have to get rid off by imposing appropriate constraints on field space.

6.3.1 Hessian contribution of $\mathcal{N}_I(v, w)$

The first two terms $\mathcal{N}_I(v, w)$ and $\mathcal{N}_I(w, v)$ involve field variations for the anti-ghost field $\bar{\xi}$. In the former case, the one we consider first, we have to revert the order of v and w using the chain rule and the assumption of $\mathcal{F}_V^A$ commuting with $\bar{g}$:

$$\begin{aligned} (-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_I(v, w) &\equiv \int_M d^d x \sqrt{\bar{g}} (v_{\bar{\xi}})_\mu \bar{g}^{\mu\nu} \mathcal{F}_V (\partial_w \mathcal{L}_\xi g) \\ &= \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^A \left((v_{\bar{\xi}})_\mu \bar{g}^{\mu\nu} (\partial_w \mathcal{L}_\xi g)_A \right) \quad \parallel \triangleq \mathcal{N}_{\text{III}}(v, w) \\ &\quad + \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^A \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \partial_w (\mathcal{L}_\xi g)_A \end{aligned}$$

In most cases $\mathcal{N}_{\text{III}}(v, w)$ can be transferred into a boundary term and thus introduces some additional constraints on field space. Once we specify the gauge fixing condition we are going to evaluate this piece of contribution. So far, let us focus on the second part and insert the functional variation of the Lie derivative from eq. (6.7a) that yields

$$\begin{aligned} &(-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_I(v, w) - \mathcal{N}_{\text{III}}(v, w) \\ &= \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \left[\xi^\lambda \bar{D}_\lambda (\partial_w g)_{\rho\sigma} + (\partial_w g)_{\lambda\rho} \bar{D}_\sigma \xi^\lambda + (\partial_w g)_{\lambda\sigma} \bar{D}_\rho \xi^\lambda \right] \\ &\quad + \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \left[w_\xi^\lambda \bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma w_\xi^\lambda + g_{\lambda\sigma} \bar{D}_\rho w_\xi^\lambda \right] \end{aligned}$$

To establish the required order of the variation fields, we have to integrate by parts contributions containing covariant derivatives of w . Basically, these are of two types: the first affects the gravitational sector and assumes the form:

$$\begin{aligned} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \xi^\lambda \bar{D}_\lambda (\partial_w g)_{\rho\sigma} &= \int_{\partial M} d^{d-1} x \sqrt{H} n_\lambda \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \xi^\lambda (\partial_w g)_{\rho\sigma} \\ &\quad - \int_M d^d x \sqrt{\bar{g}} (\partial_w g)_{\rho\sigma} \left(\bar{D}_\lambda \xi^\lambda \right) \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \\ &\quad - \int_M d^d x \sqrt{\bar{g}} (\partial_w g)_{\rho\sigma} \xi^\lambda \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} \bar{D}_\lambda (v_{\bar{\xi}})_\mu \right) \end{aligned}$$

The second type consists of covariant derivatives acting on the ghost component ξ which has to be converted using integration by parts:

$$\begin{aligned} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) g_{\lambda\rho} \bar{D}_\sigma w_\xi^\lambda &= \int_{\partial M} d^{d-1} x \sqrt{H} n_\sigma \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) g_{\lambda\rho} w_\xi^\lambda \\ &\quad - \int_M d^d x \sqrt{\bar{g}} w_\xi^\lambda \left(\bar{D}_\sigma g_{\lambda\rho} \right) \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \\ &\quad - \int_M d^d x \sqrt{\bar{g}} w_\xi^\lambda g_{\lambda\rho} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} \bar{D}_\sigma (v_{\bar{\xi}})_\mu \right) \end{aligned}$$

Notice that in both cases we interchanged anti-commuting fields, $v_{\bar{\xi}}$ with ξ or w_ξ , producing the minus sign in the bulk contributions. In addition, due to integration by parts we encounter further boundary terms, which in general obstruct the construction of a Hessian operator.

Combining the previous results we carefully reorder all integrals such that we obtain the desired form by introducing additional signs whenever anti-commuting fields are swapped. The final result for $\mathcal{N}_I(v, w)$ including boundary contributions affects two different entries in the field space matrix of a possible Hessian operator, namely the off-diagonal ξ - $\bar{\xi}$ and h - $\bar{\xi}$ entries:

$$\begin{aligned} & (-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_I(v, w) - \mathcal{N}_{III}(v, w) \\ &= - \int_M d^d x \sqrt{\bar{g}} w_\xi^\lambda \left\{ (\bar{D}_\lambda g_{\rho\sigma} + \bar{D}_\sigma g_{\lambda\rho} + \bar{D}_\rho g_{\lambda\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \mathcal{F}_v^{\rho\sigma} \bar{g}^{\mu\nu} \right\} (v_{\bar{\xi}})_\mu \\ & \quad - \int_M d^d x \sqrt{\bar{g}} (\partial_w g)_{\alpha\beta} \left\{ (\bar{D}_\sigma \xi^\alpha \delta_\rho^\beta + \bar{D}_\rho \xi^\alpha \delta_\sigma^\beta + \delta_\rho^\alpha \delta_\sigma^\beta (\bar{D}_\lambda \xi^\lambda + \xi^\lambda \bar{D}_\lambda)) \mathcal{F}_v^{\rho\sigma} \bar{g}^{\mu\nu} \right\} (v_{\bar{\xi}})_\mu \\ & \quad + \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu) \left(n_\sigma g_{\lambda\rho} w_\xi^\lambda + n_\rho g_{\lambda\sigma} w_\xi^\lambda + n_\lambda \xi^\lambda (\partial_w g)_{\rho\sigma} \right) \end{aligned}$$

While the first contribution, the one appearing in the ξ - $\bar{\xi}$ component, does not contain any Ghost- or Anti-ghost fields, the second contribution is linear in ξ . Hence, it is a possible source of ghost-functionals on the RHS of the FRGE.

6.3.2 Hessian contribution of $\mathcal{N}_I(w, v)$

The next piece in the derivation of the Hessian operator is actually already cast in the right form and does not involve any further manipulation. It suffices to insert the explicit form of eq. (6.7a) into $\mathcal{N}_I(w, v)$ which leads to:

$$\begin{aligned} & (-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_I(w, v) \equiv \int_M d^d x \sqrt{\bar{g}} (w_{\bar{\xi}})_\mu \bar{g}^{\mu\nu} \mathcal{F}_v (\partial_v \mathcal{L}_\xi g) \\ &= \int_M d^d x \sqrt{\bar{g}} (w_{\bar{\xi}})_\mu \left\{ \bar{g}^{\mu\nu} \mathcal{F}_v^{\rho\sigma} (\xi^\lambda \bar{D}_\lambda \delta_\rho^\alpha \delta_\sigma^\beta + \bar{D}_\sigma \xi^\alpha \delta_\rho^\beta + \bar{D}_\rho \xi^\alpha \delta_\sigma^\beta) \right\} (\partial_v g)_{\alpha\beta} \\ & \quad + \int_M d^d x \sqrt{\bar{g}} (w_{\bar{\xi}})_\mu \left\{ \bar{g}^{\mu\nu} \mathcal{F}_v^{\rho\sigma} (\bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \right\} v_\xi^\lambda \end{aligned}$$

Notice that there is no boundary term appearing. Furthermore, the above contributions populate the reverted entries in the field space matrix of the Hessian operator, i.e. the components $\bar{\xi}$ - h and $\bar{\xi}$ - ξ , respectively. While the latter is independent on the ghost fields, the former explicitly depends linearly on ξ and affects the ghost sector in the RG flow.

6.3.3 Hessian contribution of $\mathcal{N}_{II}(v, w)$

Finally, we turn our attention to the last ingredient $\mathcal{N}_{II}(v, w)$ which is obviously a linear functional in $\bar{\xi}$. Since in our final truncation we ignore the running of the ghost sector $\mathcal{N}_{II}(v, w)$ is actually projected to zero in the final result. Nevertheless it is worth going through the details of its derivation to pave the ground for later enlarged truncations.

From the second variation of the Lie derivative of eq. (6.7b), which is central in the present contribution, we observe that the result is already symmetric in v and w . Thus, it suffices to consider only $\partial_w \partial_v$ -terms instead of their symmetrized versions. Applying the chain rule and again using the commutativity of $\mathcal{F}_v$ with $\bar{g}$ we obtain:

$$\begin{aligned} & (-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_{II}(v, w) \equiv \int_M d^d x \sqrt{\bar{g}} \bar{\xi}_\mu \bar{g}^{\mu\nu} \mathcal{F}_v (\partial_w \partial_v \mathcal{L}_\xi g) \\ &= \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^A (\bar{\xi}_\mu \bar{g}^{\mu\nu} \partial_w \partial_v (\mathcal{L}_\xi g)_A) \quad \parallel \triangleq \mathcal{N}_{IV}(v, w) \\ & \quad + \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^A (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \partial_w \partial_v (\mathcal{L}_\xi g)_A \end{aligned}$$

Usually, a choice for the gauge fixing condition transforms the first summand $\mathcal{N}_{\text{IV}}(v, w)$ into a boundary contribution. Inserting the explicit form for $\mathfrak{d}_w \mathfrak{d}_v (\mathcal{L}_\xi g)$ of eq. (6.7b) we find

$$\begin{aligned} & (-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_{\text{II}}(v, w) - \mathcal{N}_{\text{IV}}(v, w) \\ &= \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left[w_\xi^\lambda \bar{D}_\lambda (\mathfrak{d}_v g)_{\rho\sigma} + (\mathfrak{d}_w g)_{\lambda\rho} \bar{D}_\sigma v_\xi^\lambda + (\mathfrak{d}_w g)_{\lambda\sigma} \bar{D}_\rho v_\xi^\lambda \right] \\ &+ \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left[v_\xi^\lambda \bar{D}_\lambda (\mathfrak{d}_w g)_{\rho\sigma} + (\mathfrak{d}_v g)_{\lambda\rho} \bar{D}_\sigma w_\xi^\lambda + (\mathfrak{d}_v g)_{\lambda\sigma} \bar{D}_\rho w_\xi^\lambda \right] \end{aligned}$$

In the second integral all terms appear in the non-desired order, thus we have to revert it using integration by parts. Concerning the components in the metric sector we can derive the following identity

$$\begin{aligned} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) v_\xi^\lambda \bar{D}_\lambda (\mathfrak{d}_w g)_{\rho\sigma} &= \int_{\partial M} d^{d-1} x \sqrt{H} n_\lambda \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) v_\xi^\lambda (\mathfrak{d}_w g)_{\rho\sigma} \\ &+ \int_M d^d x \sqrt{\bar{g}} (\mathfrak{d}_w g)_{\rho\sigma} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{D}_\lambda \bar{\xi}_\mu) v_\xi^\lambda \\ &+ \int_M d^d x \sqrt{\bar{g}} (\mathfrak{d}_w g)_{\rho\sigma} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \bar{D}_\lambda v_\xi^\lambda \end{aligned}$$

Besides a boundary contribution we have two bulk terms whereby no need of interchanging anti-commuting fields was necessary. On the other hand, for the remaining terms a reordering of the ghost variations will generate additional signs yielding

$$\begin{aligned} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) (\mathfrak{d}_v g)_{\lambda\rho} \bar{D}_\sigma w_\xi^\lambda &= \int_{\partial M} d^{d-1} x \sqrt{H} n_\sigma \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) (\mathfrak{d}_v g)_{\lambda\rho} w_\xi^\lambda \\ &- \int_M d^d x \sqrt{\bar{g}} w_\xi^\lambda \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{D}_\sigma \bar{\xi}_\mu) (\mathfrak{d}_v g)_{\lambda\rho} \\ &- \int_M d^d x \sqrt{\bar{g}} w_\xi^\lambda \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \bar{D}_\sigma (\mathfrak{d}_v g)_{\lambda\rho} \end{aligned}$$

This identity has to be substituted twice into the previous result giving rise to two additional boundary contributions. Thus, finally we arrive at the overall result for the last ingredient of the Hessian:

$$\begin{aligned} & (-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{N}_{\text{II}}(v, w) - \mathcal{N}_{\text{IV}}(v, w) \\ &= - \int_M d^d x \sqrt{\bar{g}} w_\xi^\lambda \left\{ \left(\delta_\rho^\beta \delta_\gamma^\sigma + \delta_\rho^\beta \delta_\gamma^\sigma \right) \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{D}_\gamma \bar{\xi}_\mu) \delta_\lambda^\alpha \right. \\ &\quad \left. + \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left(\bar{D}_\lambda \delta_\rho^\alpha \delta_\sigma^\beta + \bar{D}_\sigma \delta_\rho^\alpha \delta_\lambda^\beta + \bar{D}_\rho \delta_\lambda^\alpha \delta_\sigma^\beta \right) \right\} (\mathfrak{d}_v g)_{\alpha\beta} \\ &+ \int_M d^d x \sqrt{\bar{g}} (\mathfrak{d}_w g)_{\alpha\beta} \left\{ \mathcal{F}_v^{\alpha\beta} (\bar{g}^{\mu\nu} \bar{D}_\lambda \bar{\xi}_\mu) \right. \\ &\quad \left. + \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left(\delta_\lambda^\alpha \delta_\rho^\beta \bar{D}_\sigma + \delta_\lambda^\alpha \delta_\sigma^\beta \bar{D}_\rho + \delta_\sigma^\alpha \delta_\rho^\beta \bar{D}_\lambda \right) \right\} v_\xi^\lambda \\ &+ \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left(n_\sigma (\mathfrak{d}_v g)_{\lambda\rho} w_\xi^\lambda + n_\rho (\mathfrak{d}_v g)_{\lambda\sigma} w_\xi^\lambda + n_\lambda v_\xi^\lambda (\mathfrak{d}_w g)_{\rho\sigma} \right) \end{aligned}$$

It fills the up to now empty off-diagonal entries in the field space matrix of the Hessian operator and introduces some more boundary terms that have to vanish in its combined form.

6.3.4 Hessian for the ghost functional

The ghost functional is a vital piece of the effective action. Together with the gauge fixing condition, it ensures that the redundancies in the description are canceled and condense the

theory to the physical content. Hence, it is also an important ingredient in the effective action though its general form may vary significantly from its ‘bare’ counterpart. Quite general, the Hessian of the ghost functional $\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]$ is given by

$$\text{Hess}_\varphi[\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]](v, w) = \mathfrak{G}(w, \widehat{\text{Hess}}_\varphi[\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]](v)) + \mathcal{J}^{\text{gh}}[v, w]$$

Besides the formal Hessian operator there are usually boundary contributions, here subsumed into $\mathcal{J}^{\text{gh}}[v, w]$. Only for suitable boundary conditions for the fluctuation fields we can actually arrange for a well-behaved Hessian operator. For linear gauge fixing conditions, the associated field space matrix contains only off-diagonal entries denoted by

$$\widehat{\text{Hess}}_\varphi[\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]] \equiv -\sqrt{2}\bar{u}_k^{\text{gh}} \cdot \begin{pmatrix} 0 & \widehat{\text{Hess}}_{h\xi}^{\text{gh}} & \widehat{\text{Hess}}_{h\bar{\xi}}^{\text{gh}} \\ \widehat{\text{Hess}}_{\xi h}^{\text{gh}} & 0 & \widehat{\text{Hess}}_{\xi\bar{\xi}}^{\text{gh}} \\ \widehat{\text{Hess}}_{\bar{\xi} h}^{\text{gh}} & \widehat{\text{Hess}}_{\bar{\xi}\xi}^{\text{gh}} & 0 \end{pmatrix}$$

For our later purpose only the ghost block turns out to be relevant for the respective operators are ghost field independent and thus generate terms on the RHS of the FRGE that affect the gravitational or gauge fixing terms only. These two relevant contributions can be read off to be

$$(\widehat{\text{Hess}}_{\xi\bar{\xi}}^{\text{gh}})^\mu = -(\mathfrak{g}_{\xi\bar{\xi}}^{-1})^\lambda (\bar{D}_\lambda g_{\rho\sigma} + \bar{D}_\sigma g_{\lambda\rho} + \bar{D}_\rho g_{\lambda\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \mathcal{F}_v^{\rho\sigma} \bar{g}^{\mu\nu} \quad (6.8a)$$

$$(\widehat{\text{Hess}}_{\xi\xi}^{\text{gh}})_\lambda = +(\mathfrak{g}_{\xi\xi}^{-1})^\bullet_\mu \left\{ \bar{g}^{\mu\nu} \mathcal{F}_v^{\rho\sigma} (\bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \right\} \quad (6.8b)$$

The remaining terms which interrelate metric and ghost variations are linear in either ξ or $\bar{\xi}$ and hence are ultimately projected to zero in the final truncation. Nevertheless, we list them here for the linear parametrization $\mathfrak{d}_v g = v$:

$$(\widehat{\text{Hess}}_{\xi h}^{\text{gh}})^{\alpha\beta} = -(\mathfrak{g}_{h\xi}^{-1})^\lambda_\bullet \left\{ (\delta_\rho^\beta \delta_\gamma^\sigma + \delta_\rho^\sigma \delta_\gamma^\beta) \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{D}_\gamma \bar{\xi}_\mu) \delta_\lambda^\alpha \right. \\ \left. + \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) (\bar{D}_\lambda \delta_\rho^\alpha \delta_\sigma^\beta + \bar{D}_\sigma \delta_\rho^\alpha \delta_\lambda^\beta + \bar{D}_\rho \delta_\lambda^\alpha \delta_\sigma^\beta) \right\} \quad (6.9a)$$

$$(\widehat{\text{Hess}}_{h\xi}^{\text{gh}})_\lambda = +(\mathfrak{g}_{h\xi}^{-1})^\bullet_{\alpha\beta} \left\{ \mathcal{F}_v^{\alpha\beta} (\bar{g}^{\mu\nu} \bar{D}_\lambda \bar{\xi}_\mu) \right. \\ \left. + \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) (\delta_\lambda^\alpha \delta_\rho^\beta \bar{D}_\sigma + \delta_\lambda^\alpha \delta_\sigma^\beta \bar{D}_\rho + \delta_\sigma^\alpha \delta_\rho^\beta \bar{D}_\lambda) \right\} \quad (6.9b)$$

$$(\widehat{\text{Hess}}_{h\bar{\xi}}^{\text{gh}})^\mu = -(\mathfrak{g}_{h\bar{\xi}}^{-1})^\bullet_{\alpha\beta} \left\{ (\bar{D}_\sigma \xi^\alpha \delta_\rho^\beta + \bar{D}_\rho \xi^\alpha \delta_\sigma^\beta \right. \\ \left. + \delta_\rho^\alpha \delta_\sigma^\beta (\bar{D}_\lambda \xi^\lambda + \xi^\lambda \bar{D}_\lambda)) \mathcal{F}_v^{\rho\sigma} \bar{g}^{\mu\nu} \right\} \quad (6.9c)$$

$$(\widehat{\text{Hess}}_{\bar{\xi} h}^{\text{gh}})^{\alpha\beta} = +(\mathfrak{g}_{h\bar{\xi}}^{-1})^\bullet_{\alpha\mu} \left\{ \bar{g}^{\mu\nu} \mathcal{F}_v^{\rho\sigma} (\xi^\lambda \bar{D}_\lambda \delta_\rho^\alpha \delta_\sigma^\beta + \bar{D}_\sigma \xi^\alpha \delta_\rho^\beta + \bar{D}_\rho \xi^\alpha \delta_\sigma^\beta) \right\} \quad (6.9d)$$

The next step would be to get rid of the boundary contributions by somehow constraining the fluctuation fields. The total obstruction to a Hessian operator of $\Gamma_k^{\text{gh}}[\Phi, \bar{\Phi}]$ is absorbed into the following functional:

$$(-\sqrt{2}\bar{u}_k^{\text{gh}})^{-1} \mathcal{J}^{\text{gh}}[v, w] \\ \equiv \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_v^A \left(\bar{\xi}_\mu \bar{g}^{\mu\nu} \mathfrak{d}_w \mathfrak{d}_v (\mathcal{L}_\xi g)_A + (v_{\bar{\xi}})_\mu \bar{g}^{\mu\nu} (\mathfrak{d}_w \mathcal{L}_\xi g)_A \right) \\ + \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu) \left(n_\sigma g_{\lambda\rho} w_\xi^\lambda + n_\rho g_{\lambda\sigma} w_\xi^\lambda + n_\lambda \xi^\lambda (\mathfrak{d}_w g)_{\rho\sigma} \right) \\ + \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_v^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left(n_\sigma (\mathfrak{d}_v g)_{\lambda\rho} w_\xi^\lambda + n_\rho (\mathfrak{d}_v g)_{\lambda\sigma} w_\xi^\lambda + n_\lambda v_\xi^\lambda (\mathfrak{d}_w g)_{\rho\sigma} \right)$$

For the last two integrals Dirichlet boundary conditions are sufficient for their vanishing, however for the first integral the explicit form of the gauge fixing condition is crucial. We will come back to this issue when having the full Hessian at our disposal.

6.4 Hessian of the gravitational functional

Except for the ghost functional, in all previous derivations of Hessian operators, the underlying functionals were defined in terms of the fluctuation fields, i.e. elements of TF. For the gravitational contribution the full dynamical fields Φ are generally used to define the corresponding functional $\Gamma_k^{\text{grav}}[\Phi, \varphi]$. This approach is very suitable to interpret the chosen truncation for the appearing invariants a well studied geometrical quantities in classical gravitational theory. However, in the light of deducing the Hessian it has the main disadvantage that the functional variation is based on geodesics on field space. Due to the chain rule, the occurring complications of this geodesic approach can be absorbed into the variation of the dynamical fields only. In the present context, the ghost part of field space is trivial and only the metric field content requires this involved technique, hence

$$\partial_\nu \Phi = \partial_\nu (g, \eta, \bar{\eta})^T = (\partial_{\nu_h} g, \nu_\xi, \nu_{\bar{\xi}})^T \in T_\Phi F$$

Notice that $\nu \in T_\Phi F$ is a tangent vector at Φ contrary to the usual case of being in the tangent space of $\bar{\Phi}$. While the geometric construction of $\partial_{\nu_h} g$ based on geodesics through g retains the meaning or interpretation of $\Gamma_k^{\text{grav}}[\Phi, \varphi]$ different prescription for $\partial_{\nu_h} g$ give rise to a different truncation ansatz. Hence, the interpretation of $\Gamma_k^{\text{grav}}[\Phi, \varphi]$ in terms of the Einstein-Hilbert functional is only approximately true, while the considered truncation is a valid independent on this choice. Nevertheless, once we leave the realm of truncations or aim for a more systematic treatment of the latter, it is crucial to rely on the geometrical formalism.

In this section we start very generally and derive the Hessian for arbitrary boundary conditions and choices of $\partial_{\nu_h} g$.² This will allow future work to draw on at this point by slightly changing one or the other assumption and test the sensitivity of the derived results on these choices. However, the present context, we are ultimately going to simplify the structure of the Hessian employing a linear geodesic flow, i.e. we use $\partial_{\nu_h} g \equiv \nu_h$.

In the present truncation we consider only invariants that are either constructed by the background or the dynamical metric. In the general case, on the level of the effective action, merged monomials may appear of which there is an infinite number. Due to the complexity of the FRGE we focus here on only two Einstein-Hilbert like monomials with respective Gibbons-Hawking-York like terms and cosmological volume elements:

$$\begin{aligned} \Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}] &\equiv \Gamma_k^{\text{grav}}[g] + \Gamma_k^{\text{grav}}[\bar{g}] \\ &= (\bar{u}_k^D \bar{\lambda}_k^D) \cdot \Gamma_{(A)}^{\text{grav}}[g] + (\bar{u}_k^D) \cdot \Gamma_{(B)}^{\text{grav}}[g] + (\bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D}) \cdot \Gamma_{(\partial A)}^{\text{grav}}[g] + (\bar{u}_k^{\partial D}) \cdot \Gamma_{(\partial B)}^{\text{grav}}[g] \\ &\quad + (\bar{u}_k^B \bar{\lambda}_k^B) \cdot \Gamma_{(A)}^{\text{grav}}[\bar{g}] + (\bar{u}_k^B) \cdot \Gamma_{(B)}^{\text{grav}}[\bar{g}] + (\bar{u}_k^{\partial B} \bar{\lambda}_k^{\partial B}) \cdot \Gamma_{(\partial A)}^{\text{grav}}[\bar{g}] + (\bar{u}_k^{\partial B}) \cdot \Gamma_{(\partial B)}^{\text{grav}}[\bar{g}] \end{aligned} \quad (6.10)$$

Notice, that this truncation consists of two equivalent contributions of background and dynamical metric, however each factored with a different coefficient. Hereby, the following abbrevia-

²Except for omitting terms $\partial_{w_h} \partial_{\nu_h} g$ in the ghost functional, which anyway are quadratic in the ghost fields.

tions are used

$$\Gamma_{(A)}^{\text{grav}}[g] \equiv +2 \cdot \int_M d^d x \sqrt{g} \quad (6.11a)$$

$$\Gamma_{(\partial A)}^{\text{grav}}[g] \equiv +2 \cdot \int_{\partial M} d^{d-1} x \sqrt{H} \quad (6.11b)$$

$$\Gamma_{(B)}^{\text{grav}}[g] \equiv - \int_M d^d x \sqrt{g} R(g) \quad (6.11c)$$

$$\Gamma_{(\partial B)}^{\text{grav}}[g] \equiv -2 \cdot \int_{\partial M} d^{d-1} x \sqrt{H} K(H) \quad (6.11d)$$

Another very important remark concerns the bulk-boundary matching which is usually introduced on the classical level to assure a well-behaved variational principle. In the present ansatz we explicitly distinguish the coefficients of Einstein-Hilbert and Gibbons-Hawking-York term in order to obtain their RG evolution and study whether or not the classical correspondence can be recovered on the effective quantum level. Therefore, we have to deduce the Hessian operator of $\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]$ which by construction contributes only in the metric sector of field space. The background terms do not affect the RHS of the FRGE for it vanishes under functional variation w.r.t. the dynamical fields. We thus proceed by evaluating the Hessians for each term of eq. (6.11) separately. After combining these intermediate results to form the complete Hessian of the gravitational functional, we also state the principal of variation for the present truncation ansatz.

6.4.1 Hessian for the volume term

The cosmological constant in the classical theory factors the volume element of the dynamical metric. In our notation it corresponds to $\Gamma_{(A)}^{\text{grav}}[g]$ where the entire field dependence is contained in the square root of the determinant of g . Its functional variation can be translated into a variation of $(\partial_{v_h} g)$ that we are going to replace only in the very end:

$$\partial_{v_h} \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} (\partial_{v_h} g)_{\mu\nu} \quad \implies \quad \partial_{v_h} (\Gamma_{(A)}^{\text{grav}}[g]) = + \int_M d^d x \sqrt{g} g^{\mu\nu} (\partial_{v_h} g)_{\mu\nu}$$

The second variation involves the functional derivative of the inverse metric which follows from the identity $\delta_V^\mu = g^{\mu\rho} g_{\rho\nu}$:

$$(\partial_{v_h} g^{-1})^{\mu\nu} \equiv \partial_{v_h} g^{\mu\nu} = -g^{\rho\mu} g^{\sigma\nu} (\partial_{v_h} g)_{\rho\sigma}$$

The Leibniz-rule applied to $\partial_{v_h} \sqrt{g}$ results in

$$\partial_{w_h} \partial_{v_h} \sqrt{g} = (\partial_{w_h} g)_{\rho\sigma} \left\{ -\frac{1}{2} \sqrt{g} (g^{\rho\mu} g^{\sigma\nu} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}) \right\} (\partial_{v_h} g)_{\mu\nu} - \frac{1}{2} \sqrt{g} g^{\mu\nu} (\partial_{w_h} \partial_{v_h} g)_{\mu\nu}$$

The term in brackets $(\dots)$ is the standard form of the DeWitt field space metric with $\varpi = -1/2$. In the generic – geometrical – way the field variation $(\partial_{v_h} g)$ is still dependent on g and hence $(\partial_{w_h} \partial_{v_h} g) \neq 0$. However, later we will project the preliminary ansatz using $(\partial_{v_h} g) = v_h$ and $(\partial_{w_h} \partial_{v_h} g) = 0$ onto the final truncation which simplifies the preceding calculations.

Combining the above results gives rise to the Hessian of the volume contribution in the gravitational sector:

$$\begin{aligned} \text{Hess}_\varphi [\Gamma_{(A)}^{\text{grav}}[g]](v, w) = & - \int_M d^d x \sqrt{g} (\partial_{w_h} g)_{\rho\sigma} \left\{ (g^{\rho\mu} g^{\sigma\nu} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}) \right\} (\partial_{v_h} g)_{\mu\nu} \\ & - \int_M d^d x \sqrt{g} g^{\mu\nu} \left(\frac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu} \end{aligned}$$

As long as $(\partial_{v_h} g)$ does not involve any derivative operators there are in fact no obstructions in reading off the associated Hessian operator:

$$\text{Hess}_\varphi [\Gamma_{(A)}^{\text{grav}}[g]](v, w) = \mathfrak{G}(w, \widehat{\text{Hess}_\varphi [\Gamma_{(A)}^{\text{grav}}[g]]}(v))$$

Notice that for $\varpi = -1/2$ the first term in the Hessian operator is proportional to the identity in the metric sector and vanishes elsewhere. In the current state of the analysis we still have to specify $(\partial_{v_h} g)$ by means of a field space connection which we however postpone to the very end of the gravitational derivation part.

6.4.2 Hessian for the boundary volume term

Due to the smoothness of the Riemannian metric and the manifold character of the boundary ∂M , one can equip ∂M with a Riemannian metric that is induced by g (or $\bar{g}$) and defines a symmetric tensor field in $d - 1$ dimensions. The boundary of a manifold brings in another piece of information into the spacetime formalism usually presented in form of a hyperplane equation in some embedding space. Here, we skip all intermediate steps and only work with an *outward* pointing normal vector field n ($\bar{n}$) which should be normalized w.r.t. the dynamical (background) metric g (or $\bar{g}$), i.e. $1 = g(n, n) \stackrel{\text{loc.}}{=} n^\mu g_{\mu\nu} n^\nu$. Rather than working with the induced metric directly, we introduce a projection operator for all $p \in \partial M$ that projects onto the boundary's tangent hyperplane:

$$\Pi_{\partial M} : T_p M \rightarrow T_p \partial M, \quad v \mapsto \Pi_{\partial M}(v) \stackrel{\text{loc.}}{=} (\Pi_{\partial M})^\mu_\nu v^\nu \partial_\mu \equiv (\delta^\mu_\nu - n^\mu n_\nu) v^\nu \partial_\mu$$

Hereby, we lower indices with the respective metric field, in the present case the dynamical one: $n_\nu = g_{\nu\mu} n^\mu$. The induced metric can thus be related to a projected version of g by means of the pullback, which locally agrees with subtracting the normal directions of vector fields:

$$H := \Pi_{\partial M}^*(g) \quad \text{locally:} \quad H_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu \quad \wedge \quad H^{\mu\nu} \equiv g^{\mu\nu} - n^\mu n^\nu$$

Notice that this indeed defines a suitable metric only on the boundary, for it is degenerate in the perpendicular direction in the sense that $H(n, \bullet) = 0 \stackrel{\text{loc.}}{=} H_{\mu\nu} n^\mu$ vanishes.

The boundary volume term, abbreviated $\Gamma_{(\partial A)}^{\text{grav}}[g]$, is the analog of the usual bulk volume contribution now projected onto ∂M . It involves the determinant of the projected metric H and hence its field derivative requires the variation of the normal vector field. To this end, we exploit the properties of n that should be kept intact after the variation procedure, in particular its normalization:

$$0 = (\partial_{v_h} 1) = (\partial_{v_h} n^\mu n_\mu) = n_\mu \partial_{v_h} n^\mu + n^\mu \partial_{v_h} n_\mu$$

This provides a mean to interchange the variation with respect to n and its dual object $g(n, \bullet)$, but still does not associate one or the other with the variation of the metric field. Therefore, we start with the first summand $n_\mu \partial_{v_h} n^\mu$ and artificially convert it to depend only on the dual $g(n, \bullet)$:

$$\begin{aligned} n_\mu \partial_{v_h} n^\mu &= n_\mu (\partial_{v_h} (g^{\mu\nu} n_\nu)) = n_\mu (-n_\nu g^{\rho\mu} g^{\sigma\nu} (\partial_{v_h} g)_{\rho\sigma} + g^{\mu\nu} \partial_{v_h} n_\nu) \\ &= -n^\sigma n^\rho (\partial_{v_h} g)_{\rho\sigma} + n^\mu \partial_{v_h} n_\mu \\ &= -n^\sigma n^\rho (\partial_{v_h} g)_{\rho\sigma} - n_\mu \partial_{v_h} n^\mu \end{aligned}$$

In the final step we inserted the above derived relation $n_\mu \partial_{v_h} n^\mu = -n^\mu \partial_{v_h} n_\mu$. Hence, rearranging the last equality yields the field variation of the normal vector field n and thus its dual

correspondence $g(n, \bullet)$:

$$\begin{aligned} (\partial_{v_h} n)^\mu &\equiv \partial_{v_h} n^\mu = -\frac{1}{2} n^\mu n^\sigma n^\rho (\partial_{v_h} g)_{\sigma\rho} \\ (\partial_{v_h} g(n, \bullet))_\mu &\equiv \partial_{v_h} n_\mu = \frac{1}{2} n_\mu n^\sigma n^\rho (\partial_{v_h} g)_{\sigma\rho} \end{aligned}$$

For the projected boundary metric H we only have to substitute the results for the variation of g and $g(n, \bullet)$ successively and likewise for its dual version $H^* \stackrel{\text{loc.}}{=} H^{\mu\nu}$:³

$$\begin{aligned} (\partial_{v_h} H)_{\mu\nu} &\equiv \partial_{v_h} H_{\mu\nu} = (\partial_v g)_{\mu\nu} - (\partial_v n_\mu) n_\nu - n_\mu (\partial_v n_\nu) \\ &= (\partial_{v_h} g)_{\mu\nu} - n_\nu n_\mu n^\sigma n^\rho (\partial_{v_h} g)_{\sigma\rho} \\ (\partial_{v_h} H^*)^{\mu\nu} &\equiv \partial_{v_h} H^{\mu\nu} = \partial_v g^{\mu\nu} - (\partial_v n^\mu) n^\nu - n^\mu (\partial_v n^\nu) \\ &= -g^{\mu\rho} g^{\nu\sigma} (\partial_{v_h} g)_{\sigma\rho} + n^\nu n^\mu n^\sigma n^\rho (\partial_{v_h} g)_{\sigma\rho} \end{aligned}$$

The central geometrical object of interest in the currently considered invariant is determinant of H . Taking the functional derivative amounts to a variation of the inverse boundary metric H and thus yields

$$\begin{aligned} \partial_v \sqrt{H} &= -\frac{1}{2} \sqrt{H} H_{\mu\nu} \partial_v H^{\mu\nu} = -\frac{1}{2} \sqrt{H} H_{\mu\nu} (-g^{\mu\rho} g^{\nu\sigma} (\partial_{v_h} g)_{\sigma\rho} + n^\nu n^\mu n^\sigma n^\rho (\partial_{v_h} g)_{\sigma\rho}) \\ &= +\frac{1}{2} \sqrt{H} H_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} (\partial_{v_h} g)_{\sigma\rho} \equiv +\frac{1}{2} \sqrt{H} H^{\mu\nu} (\partial_{v_h} g)_{\mu\nu} \end{aligned}$$

In the last step we used the transverse property of H , i.e. $H_{\mu\nu} n^\nu = 0$, hence only the first summand remains. The volume boundary invariant adds the following contribution to the first variation of the preliminary truncation ansatz:

$$(\partial_{v_h} \Gamma_{(\partial A)}^{\text{grav}}[g]) = + \int_{\partial M} d^{d-1} x \sqrt{H} H^{\mu\nu} (\partial_{v_h} g)_{\mu\nu}$$

In combination with all other gravitational terms it has to cancel whenever a well-behaved variational principle is required to hold true.

$$\begin{aligned} \partial_w \partial_v \sqrt{H} &= (\partial_{w_h} g)_{\sigma\rho} \left\{ \frac{1}{2} \sqrt{H} \left(\frac{1}{2} H^{\rho\sigma} H^{\mu\nu} - g^{\mu\rho} g^{\nu\sigma} + n^\nu n^\mu n^\sigma n^\rho \right) \right\} (\partial_{v_h} g)_{\mu\nu} \\ &\quad + \frac{1}{2} \sqrt{H} H^{\mu\nu} (\partial_{w_h} \partial_{v_h} g)_{\mu\nu} \end{aligned}$$

Again, in the case of a true geometrical variation the last term in general produces a non-vanishing contribution, except for special choices of boundary conditions. It is very instructive to rewrite the first expression of this result using $H^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$:

$$\begin{aligned} \partial_w \partial_v \sqrt{H} &= (\partial_{w_h} g)_{\sigma\rho} \left\{ \frac{1}{2} \sqrt{H} \left(- (H^{\mu\rho} H^{\nu\sigma} - \frac{1}{2} H^{\rho\sigma} H^{\mu\nu}) - H^{\mu\rho} n^\nu n^\sigma - H^{\nu\sigma} n^\rho n^\mu \right) \right\} (\partial_{v_h} g)_{\mu\nu} \\ &\quad + \frac{1}{2} \sqrt{H} H^{\mu\nu} (\partial_{w_h} \partial_{v_h} g)_{\mu\nu} \end{aligned}$$

What appears is the gravitational part of the field space metric for $\varpi = -1/2$ projected onto the boundary with two additional contributions in the orthogonal direction. In conclusion, the Hessian of the boundary volume invariant assumes the form:

$$\begin{aligned} \text{Hess}_\varphi [\Gamma_{(\partial A)}^{\text{grav}}[g]](v, w) &= - \int_{\partial M} d^{d-1} x \sqrt{H} (\partial_{w_h} g)_{\sigma\rho} \left\{ (H^{\mu\rho} H^{\nu\sigma} - \frac{1}{2} H^{\rho\sigma} H^{\mu\nu}) \right. \\ &\quad \left. + H^{\mu\rho} n^\nu n^\sigma + H^{\nu\sigma} n^\rho n^\mu \right\} (\partial_{v_h} g)_{\mu\nu} \\ &\quad - \int_{\partial M} d^{d-1} x \sqrt{H} H^{\mu\nu} \left(\frac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu} \end{aligned}$$

³ Notice that H^* is not the inverse of H in the general sense. In fact H as such has no inverse due to its degeneracy, however restricted to the boundary (thus considered as the induced metric) the induced form of H^* coincides with the inverse of the induced form of H .

In contrast to the bulk volume term the second variation is a purely boundary contribution as expected. While there are techniques to study generalized Hessian operators that contain bulk-boundary terms [119, 120] it is not clear how these methods translate into the FRG setting. Rather, we require that all boundary terms have to vanish, either by suitable boundary conditions or by matching certain coefficients, as in the classical Gibbons-Hawking-York case. However, the latter method may be incompatible with the RG evolution and thus should be omitted whenever possible.

6.4.3 Hessian for the curvature term

When curvature enters the game, derivatives appear which are a potential source of boundary terms in the Hessian of bulk invariants. Besides its appearance as a central ingredient of General Relativity in the shape of the Einstein-Hilbert action, $\Gamma_{(B)}^{\text{grav}}[g] = -\int_M d^d x \sqrt{g} R(g)$ is also special for it is the only bulk invariant containing two derivatives D , assuming $d \geq 2$. In our conventions the scalar curvature is the full contraction of the Riemannian tensor in local coordinates given by

$$R_{\mu\nu\lambda}{}^\sigma = -\partial_\mu \Gamma_{\nu\lambda}^\sigma + \partial_\nu \Gamma_{\mu\lambda}^\sigma + \Gamma_{\mu\lambda}^\kappa \Gamma_{\nu\kappa}^\sigma - \Gamma_{\nu\lambda}^\kappa \Gamma_{\mu\kappa}^\sigma,$$

$$R_{\mu\nu} = \delta_\sigma^\nu R_{\mu\nu\lambda}{}^\sigma \quad \text{and} \quad R = g^{\mu\nu} R_{\mu\nu}$$

Comparing different literature on General Relativity or on its quantum versions exhibits various sign conventions that may alter certain results, so one has to be careful in this process.

Based on the above form of the scalar curvature, the Ricci tensor, and the Riemann tensor we now deduce the Hessian of the Einstein-Hilbert functional in the presence of boundaries. The crucial constituent is the covariant derivative D which builds up these second derivative tensors. Under field variation the partial derivative is unaffected, while the connection coefficients⁴ transform as

$$\partial_{v_h} \Gamma_{\mu\nu}^\lambda = \frac{1}{2} \left(g^{\lambda\rho} \delta_\nu^\sigma D_\mu + g^{\lambda\rho} \delta_\mu^\sigma D_\nu - g^{\lambda\kappa} \delta_\mu^\sigma \delta_\nu^\rho D_\kappa \right) (\partial_{v_h} g)_{\sigma\rho}$$

This result suffices to arrive at the functional variation of the Riemann tensor which can be directly contracted to the respective form for the Ricci tensor. Finally, the behavior of the scalar curvature under infinitesimal changes of the metric is deduced from the Leibniz-rule acting on $g^{\mu\nu} R_{\mu\nu} \equiv R$, i.e.

$$\partial_{v_h} R_{\rho\sigma} = \frac{1}{2} \left(D^\mu \left(\delta_\sigma^\nu D_\rho + \delta_\rho^\nu D_\sigma \right) - g^{\mu\nu} D_\rho D_\sigma - \delta_\rho^\mu \delta_\sigma^\nu D^2 \right) (\partial_{v_h} g)_{\mu\nu}$$

$$\partial_{v_h} R = \left(-R^{\mu\nu} + D^\mu D^\nu - g^{\mu\nu} D^2 \right) (\partial_{v_h} g)_{\mu\nu}$$

In combination with the outcome of the metric variation of $\sqrt{g}$ we derived previously the first order field derivative of the Einstein-Hilbert functional is given by

$$\begin{aligned} (\partial_{v_h} \Gamma_{(B)}^{\text{grav}}[g]) &= - \int_M d^d x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} + D^\mu D^\nu - g^{\mu\nu} D^2 \right) (\partial_{v_h} g)_{\mu\nu} \\ &= + \int_M d^d x \sqrt{g} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) (\partial_{v_h} g)_{\mu\nu} \\ &\quad - \int_{\partial M} d^{d-1} x \sqrt{H} \left(n^\mu D^\nu - g^{\mu\nu} n^\rho D_\rho \right) (\partial_{v_h} g)_{\mu\nu} \end{aligned}$$

In the last step we converted the total derivatives into some boundary contributions. The remaining bulk term constitutes the Einstein tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ and is one of the central

⁴In the present case we have a torsion free connection, for which the connection coefficients coincides with the Christoffel symbols. The entire derivation is based on this assumption.

objects in the field equations for General Relativity. The obstruction is exactly the boundary term that in a classical theory is usually canceled by introducing the Gibbons-Hawking-York functional with an appropriate coefficient. We come back to this point in the end of this section after deriving the full Hessian of $\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]$.

The second variation of $\Gamma_{(B)}^{\text{grav}}[g]$ that contributes to the RHS of the FRGE is treated in two steps, the first focusing on the bulk, the second on the boundary term:

$$\begin{aligned} (\partial_{w_h} \partial_{v_h} \Gamma_{(B)}^{\text{grav}}[g]) &= + \partial_{w_h} \int_M d^d x \sqrt{g} (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) (\partial_{v_h} g)_{\mu\nu} & \parallel \triangleq \mathcal{N}_V(v, w) \\ &- \partial_{w_h} \int_{\partial M} d^{d-1} x \sqrt{H} (n^\mu D^\nu - g^{\mu\nu} n^\rho D_\rho) (\partial_{v_h} g)_{\mu\nu} & \parallel \triangleq \mathcal{N}_{VI}(v, w) \end{aligned}$$

In fact, we have all the variations of the individual constituents at our disposal which – together with the Leibniz-rule – yields the final result for the Hessian of the Einstein-Hilbert functional. Though tedious it is straightforward to determine its two contributions from the previous functional variations. An important intermediate result is the derivative of the Einstein tensor:

$$\begin{aligned} (\partial_{w_h} G^{\mu\nu}) &= \tfrac{1}{2} \{ -\mathcal{K}^{\mu\nu\rho\sigma} + (g^{\mu\nu} R^{\rho\sigma} + R g^{\nu\rho} g^{\mu\sigma} - 2g^{\mu\rho} R^{\nu\sigma} - 2g^{\nu\rho} R^{\mu\sigma}) \} (\partial_{w_h} g)_{\rho\sigma} \\ \text{with } \mathcal{K}^{\mu\nu\rho\sigma} &\equiv (g^{\mu\nu} D^\rho D^\sigma + g^{\rho\sigma} D^\mu D^\nu - 2g^{\nu\sigma} D^\rho D^\mu) - (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) D^2 \end{aligned}$$

Here we introduced the operator $\mathcal{K}_A^B \equiv g_{AC}^{-1} \mathcal{K}^{CB}$, the propagator of the field modes. It is the only source of not ultra-local contributions to the gravitational Hessian except for possible boundary terms that we assume to vanish in the very end. Now, the full bulk variation splits into three components, one which disappears whenever we rely on a flat field space connection:

$$\begin{aligned} \mathcal{N}_V(v, w) &= + \int_M d^d x \sqrt{g} \left(\sqrt{g}^{-1} (\partial_{w_h} \sqrt{g}) G^{\mu\nu} + (\partial_{w_h} G^{\mu\nu}) \right) (\partial_{v_h} g)_{\mu\nu} \\ &+ \int_M d^d x \sqrt{g} G^{\mu\nu} (\partial_{w_h} \partial_{v_h} g)_{\mu\nu} \end{aligned}$$

Substituting the variation of the determinant of g the first integral reduces to

$$\begin{aligned} \mathcal{N}_V(v, w) &= + \int_M d^d x \sqrt{g} (\partial_{v_h} g)_{\mu\nu} \left\{ \tfrac{1}{2} (-\mathcal{K}^{\mu\nu\rho\sigma} + \mathcal{U}^{\mu\nu\rho\sigma}) \right\} (\partial_{w_h} g)_{\rho\sigma} \\ &+ \int_M d^d x \sqrt{g} G^{\mu\nu} (\partial_{w_h} \partial_{v_h} g)_{\mu\nu} \end{aligned}$$

whereby we condensed the notation using the following abbreviation for the completely ultra-local (potential like) tensor

$$\mathcal{U}^{\mu\nu\rho\sigma}(g) \equiv g^{\rho\sigma} G^{\mu\nu} + g^{\mu\nu} R^{\rho\sigma} + R g^{\nu\rho} g^{\mu\sigma} - 2g^{\mu\rho} R^{\nu\sigma} - 2g^{\nu\rho} R^{\mu\sigma} \quad (6.12)$$

In the Hessian the symmetric version of $\partial_{w_h} \partial_{v_h}$ is relevant which requires both $\mathcal{K}$ and $\mathcal{U}$ being self-adjoint. While for the ultra-local part this simply translates in considering the symmetric combination $\tfrac{1}{2} (\mathcal{U}^{AB} + \mathcal{U}^{BA})$ instead, for the ‘kinetic’ term additional boundary conditions are in need. In detail we are looking for a way to express the symmetric sum $\mathcal{N}_V(v, w) + \mathcal{N}_V(w, v)$ as an operator relation giving rise to a suitable Hessian operator for the gravitational curvature term:

$$\begin{aligned} (\mathcal{N}_V(v, w) + \mathcal{N}_V(w, v)) &= + \int_M d^d x \sqrt{g} (\partial_{v_h} g)_{\mu\nu} \left\{ \tfrac{1}{2} (-\mathcal{K}^{\mu\nu\rho\sigma} + \mathcal{U}^{\mu\nu\rho\sigma}) \right\} (\partial_{w_h} g)_{\rho\sigma} \\ &+ \int_M d^d x \sqrt{g} (\partial_{w_h} g)_{\rho\sigma} \left\{ \tfrac{1}{2} (-\mathcal{K}^{\rho\sigma\mu\nu} + \mathcal{U}^{\rho\sigma\mu\nu}) \right\} (\partial_{v_h} g)_{\mu\nu} \\ &+ \int_M d^d x \sqrt{g} G^{\mu\nu} \left(\tfrac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu} \end{aligned}$$

For a natural symmetric operator, we reorder the first term such that all the operators act on $(\mathfrak{d}_{v_h}g)$. As mentioned above, for the ultra-local part this is trivial in that we only have to relabel the indices. Concerning $\mathcal{K}^{\mu\nu\rho\sigma}$ a stepwise derivation reveals the appearing of boundary terms. First of all, consider the adjoint of the following integral using integration by parts twice

$$\begin{aligned} \int_M d^d x \sqrt{g} (\mathfrak{d}_{v_h}g)_{\mu\nu} \left(g^{\alpha\beta} D^\kappa D^\lambda \right) (\mathfrak{d}_{w_h}g)_{\rho\sigma} = & + \int_M d^d x \sqrt{g} (\mathfrak{d}_{w_h}g)_{\rho\sigma} \left(g^{\alpha\beta} D^\lambda D^\kappa \right) (\mathfrak{d}_{v_h}g)_{\mu\nu} \\ & + \int_{\partial M} d^{d-1} x \sqrt{H} g^{\alpha\beta} n^\kappa (\mathfrak{d}_{v_h}g)_{\mu\nu} D^\lambda (\mathfrak{d}_{w_h}g)_{\rho\sigma} \\ & - \int_{\partial M} d^{d-1} x \sqrt{H} g^{\alpha\beta} n^\lambda (\mathfrak{d}_{w_h}g)_{\rho\sigma} D^\kappa (\mathfrak{d}_{v_h}g)_{\mu\nu} \end{aligned}$$

Notice that in the absence of spacetime boundary $\mathcal{K}^{\mu\nu\rho\sigma}$ is symmetric and under suitable boundary conditions, as for instance Dirichlet constraints, it even holds if $\partial M \neq \emptyset$.

$$\int_M d^d x \sqrt{g} (\mathfrak{d}_{v_h}g)_A \mathcal{K}^{AB} (\mathfrak{d}_{w_h}g)_B = + \int_M d^d x \sqrt{g} (\mathfrak{d}_{w_h}g)_B \mathcal{K}^{BA} (\mathfrak{d}_{v_h}g)_A + \mathcal{N}_{\text{VII}}(v, w)$$

This last contribution, $\mathcal{N}_{\text{VII}}(v, w)$, represents the boundary terms arising from integration by parts, explicitly given by

$$\begin{aligned} \mathcal{N}_{\text{VII}}(v, w) = & + \int_{\partial M} d^{d-1} x \sqrt{H} (\mathfrak{d}_{v_h}g)_{\mu\nu} (g^{\mu\nu} n^\rho D^\sigma - 2g^{\nu\sigma} n^\nu D^\rho + g^{\rho\sigma} n^\nu D^\mu) (\mathfrak{d}_{w_h}g)_{\rho\sigma} \\ & - \int_{\partial M} d^{d-1} x \sqrt{H} (\mathfrak{d}_{w_h}g)_{\rho\sigma} (g^{\mu\nu} n^\sigma D^\rho - 2g^{\nu\sigma} n^\rho D^\mu + g^{\rho\sigma} n^\mu D^\nu) (\mathfrak{d}_{v_h}g)_{\mu\nu} \\ & + \int_{\partial M} d^{d-1} x \sqrt{H} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \cdot \left((\mathfrak{d}_{v_h}g)_{\mu\nu} n^\lambda D_\lambda (\mathfrak{d}_{w_h}g)_{\rho\sigma} \right. \\ & \quad \left. - (\mathfrak{d}_{w_h}g)_{\rho\sigma} n^\lambda D_\lambda (\mathfrak{d}_{v_h}g)_{\mu\nu} \right) \end{aligned}$$

We thus arrive at the final symmetric version of the gravitational bulk contribution to the Hessian, which reads

$$\begin{aligned} & \left(\frac{1}{2} \mathcal{N}_V(v, w) + \frac{1}{2} \mathcal{N}_V(w, v) \right) \\ & = -\frac{1}{2} \int_M d^d x \sqrt{g} (\mathfrak{d}_{v_h}g)_{\mu\nu} \{ \mathcal{K}^{\mu\nu\rho\sigma} \} (\mathfrak{d}_{w_h}g)_{\rho\sigma} \\ & \quad + \frac{1}{2} \int_M d^d x \sqrt{g} (\mathfrak{d}_{w_h}g)_{\rho\sigma} \{ \mathcal{U}^{\mu\nu\rho\sigma} \} (\mathfrak{d}_{v_h}g)_{\mu\nu} \\ & \quad + \frac{1}{2} \int_M d^d x \sqrt{g} G^{\mu\nu} \left(\frac{1}{2} (\mathfrak{d}_{w_h} \mathfrak{d}_{v_h} + \mathfrak{d}_{v_h} \mathfrak{d}_{w_h}) g \right)_{\mu\nu} + \frac{1}{2} \mathcal{N}_{\text{VII}}(v, w) \end{aligned}$$

The missing part constitutes a pure boundary term containing the normal vector field n and covariant derivatives acting on the metric fluctuation. For general Dirichlet boundary conditions this term spoils the variational principle; it has to be canceled by either Neumann conditions or by introducing the matched version of the Gibbons-Hawking-York term. Nevertheless, as we will see in a moment, its second variation vanishes under either Dirichlet or Neumann boundary constraints. Besides the field derivative of $\sqrt{H}$, we have to determine the variation of $n^\sigma g^{\alpha\beta} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu}$, which is given by

$$\begin{aligned} \mathfrak{d}_{w_h} (n^\sigma g^{\alpha\beta} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu}) & = (\mathfrak{d}_{w_h} n^\sigma) g^{\alpha\beta} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu} + n^\sigma (\mathfrak{d}_{w_h} g^{\alpha\beta}) D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu} \\ & \quad + n^\sigma g^{\alpha\beta} (\mathfrak{d}_{w_h} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu}) \\ & = -n^\sigma (\mathfrak{d}_{w_h}g)_{\kappa\lambda} \left(\frac{1}{2} n^\lambda n^\kappa g^{\alpha\beta} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu} + g^{\kappa\alpha} g^{\lambda\beta} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu} \right) \\ & \quad + n^\sigma g^{\alpha\beta} (\mathfrak{d}_{w_h} D_\rho (\mathfrak{d}_{v_h}g)_{\mu\nu}) \end{aligned}$$

Hence it necessary to compute the variation of the covariant derivative acting on $(\partial_{v_h} g)$:

$$\begin{aligned}\partial_{w_h}(D_\rho(\partial_{v_h} g)_{\mu\nu}) &= \partial_{w_h} \left(\partial_\rho(\partial_{v_h} g)_{\mu\nu} - \Gamma_{\rho\mu}^\lambda(\partial_{v_h} g)_{\lambda\nu} - \Gamma_{\rho\nu}^\lambda(\partial_{v_h} g)_{\mu\lambda} \right) \\ &= D_\rho(\partial_{w_h} \partial_{v_h} g)_{\mu\nu} - (\partial_{v_h} g)_{\lambda\nu}(\partial_{w_h} \Gamma_{\rho\mu}^\lambda) - (\partial_{v_h} g)_{\mu\lambda}(\partial_{w_h} \Gamma_{\rho\nu}^\lambda)\end{aligned}$$

For brevity we are not going to further expand $(\partial_{w_h} \Gamma_{\rho\mu}^\lambda)$ but notice that each of its terms is proportional to some $D_\kappa(\partial_{w_h} g)$, hence the corresponding boundary terms vanish whenever Dirichlet or Neumann conditions are imposed. The complete variation of the boundary contribution is thus given by

$$\begin{aligned}\mathcal{N}_{\text{VI}}(v, w) &= + \int_{\partial M} d^{d-1}x \sqrt{H} (\partial_{w_h} g)_{\rho\sigma} \left\{ \frac{1}{2} H^{\rho\sigma} \left(-n^\mu D^\nu + g^{\mu\nu} n^\lambda D_\lambda \right) \right\} (\partial_{v_h} g)_{\mu\nu} \\ &\quad + \int_{\partial M} d^{d-1}x \sqrt{H} (\partial_{w_h} g)_{\rho\sigma} \left\{ \frac{1}{2} n^\sigma n^\rho \left(n^\mu D^\nu - g^{\mu\nu} n^\lambda D_\lambda \right) \right. \\ &\quad \quad \left. + g^{\rho\nu} \left(n^\mu D^\sigma - g^{\sigma\mu} n^\lambda D_\lambda \right) \right\} (\partial_{v_h} g)_{\mu\nu} \\ &\quad + \int_{\partial M} d^{d-1}x \sqrt{H} \left(n^\lambda g^{\mu\nu} - n^\mu g^{\nu\lambda} \right) \left(D_\lambda (\partial_{w_h} \partial_{v_h} g)_{\mu\nu} - 2(\partial_{v_h} g)_{\alpha\nu} (\partial_{w_h} \Gamma_{\lambda\mu}^\alpha) \right)\end{aligned}$$

This concludes the derivation of the Hessian associated to the bulk curvature invariant. The symmetric second variation of $\Gamma_{(B)}^{\text{grav}}[g]$ reads

$$\begin{aligned}\text{Hess}_\varphi[\Gamma_{(B)}^{\text{grav}}[g]](v, w) &= -\frac{1}{2} \int_M d^d x \sqrt{g} (\partial_{v_h} g)_{\mu\nu} \{ \mathcal{K}^{\mu\nu\rho\sigma} \} (\partial_{w_h} g)_{\rho\sigma} \\ &\quad + \frac{1}{2} \int_M d^d x \sqrt{g} (\partial_{w_h} g)_{\rho\sigma} \{ \mathcal{U}^{\mu\nu\rho\sigma} \} (\partial_{v_h} g)_{\mu\nu} \\ &\quad + \frac{1}{2} \int_M d^d x \sqrt{g} G^{\mu\nu} \left(\frac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu} + \mathcal{N}_{\text{VI}}(v, w) + \frac{1}{2} \mathcal{N}_{\text{VII}}(v, w)\end{aligned}$$

Hereby, a symmetrization of the boundary terms $\mathcal{N}_{\text{VI}}(v, w)$ and $\mathcal{N}_{\text{VII}}(v, w)$ is implicitly understood but omitted for the sake of brevity. We are now going to derive the final piece in the gravitational functional that on a classical level was introduced to get rid of the boundary terms appearing in the construction of field equations for GR: the Gibbons-Hawking-York term.

6.4.4 Hessian for the boundary curvature term

Finally, we arrive at the boundary curvature term $\Gamma_{(\partial B)}^{\text{grav}}[g] = -2 \int_{\partial M} d^{d-1}x \sqrt{H} K(H)$ of the present truncation, a Gibbons-Hawking-York like invariant including the trace of the extrinsic curvature $K = D_\mu n^\mu$. There are various ways to derive the variation of this field monomial all related by some algebraic transformations involving identities of the form $n_\mu n^\mu = 1$ and $H_{\mu\nu} n^\nu = 0$, for instance. For our purpose we will follow a path that directly yields the standard shape of the functional derivative for K such that the resulting boundary contributions are comparable with those occurring in the derivation of the Einstein-Hilbert functional. To this end, we present two of the previously obtained results in a new fashion that is more suitable for the current analysis. First of all, we replace $n_\sigma n^\mu = \delta_\sigma^\mu - H^\sigma_\mu$ in the variation of $g(n, \bullet)$ which leads to

$$(\partial_{v_h} n_\sigma) = \frac{1}{2} \delta_\sigma^\mu n^\nu (\partial_{v_h} g)_{\mu\nu} + b_\sigma \quad \text{with} \quad b_\sigma \equiv -\frac{1}{2} g_{\sigma\lambda} H^{\lambda\mu} n^\nu (\partial_{v_h} g)_{\mu\nu}$$

Here, we introduced a co-vector field b for reasons that will be obvious in the final result. Moreover, we consider the contracted form of the variation of the Christoffel symbol w.r.t. g

yielding

$$g^{\rho\sigma}(\partial_{v_h}\Gamma_{\rho\sigma}^\lambda) = \left(g^{\lambda\mu}D^\nu - \frac{1}{2}g^{\mu\nu}D^\lambda\right)(\partial_{v_h}g_{\mu\nu})$$

Both become relevant, when we insert an artificial inverse metric in K that lowers the indices of the normal vector field such that the above equation applies:

$$(\partial_{v_h}K) = (\partial_{v_h}g^{\rho\sigma}D_\rho n_\sigma) = -g^{\rho\mu}g^{\sigma\nu}(\partial_{v_h}g_{\mu\nu})D_\rho n_\sigma + g^{\rho\sigma}D_\rho(\partial_{v_h}n_\sigma) - g^{\rho\sigma}n_\lambda(\partial_{v_h}\Gamma_{\rho\sigma}^\lambda)$$

While the first term reduces to the extrinsic curvature $K^{\mu\nu} = D^\mu n^\nu$ the second and third contribution are expanded in terms of the above identities, i.e.

$$\begin{aligned}(\partial_{v_h}K) &= -K^{\mu\nu}(\partial_{v_h}g_{\mu\nu}) + \frac{1}{2}D^\mu(n^\nu(\partial_{v_h}g)_{\mu\nu}) + D_\rho b^\rho - \left(n^\mu D^\nu - \frac{1}{2}n^\lambda g^{\mu\nu}D_\lambda\right)(\partial_{v_h}g_{\mu\nu}) \\ &= -\frac{1}{2}K^{\mu\nu}(\partial_{v_h}g_{\mu\nu}) - \frac{1}{2}\left(n^\mu D^\nu - n^\lambda g^{\mu\nu}D_\lambda\right)(\partial_{v_h}g_{\mu\nu}) + D_\rho b^\rho\end{aligned}$$

In the last equality we used the Leibniz rule and replaced $D^\mu n^\nu$ by the extrinsic curvature tensor $K^{\mu\nu}$. The second term is exactly the boundary contribution that appears in the variation of the Einstein-Hilbert functional however with opposite sign. Hence, adapting the coefficients of Gibbons-Hawking-York and the Einstein-Hilbert functional cancels all the boundary integrals containing derivatives of the metric fluctuations, implying that Dirichlet conditions are sufficient to generate a well behaved variational principle. However, there is an additional term $D_\rho b^\rho$ which seems to spoil this nice relation but it turns out to be purely of Dirichlet type with an additional total derivative on the boundary. To see this explicitly, let us integrate this part over the boundary manifold ∂M :

$$\begin{aligned}\int_{\partial M} d^{d-1}x\sqrt{H}D_\rho b^\rho &= \int_{\partial M} d^{d-1}x\sqrt{H}(\partial_\rho b^\rho + \Gamma_{\mu\rho}^\mu b^\rho) \\ &= \int_{\partial M} d^{d-1}x\partial_\rho(\sqrt{H}b^\rho) - \int_{\partial M} d^{d-1}x\sqrt{H}\left(\frac{1}{\sqrt{H}}b^\rho\partial_\rho\sqrt{H} - \Gamma_{\mu\rho}^\mu b^\rho\right)\end{aligned}$$

The first term represents a total derivative and thus can be converted to a boundary integral over $\partial\partial M \equiv \emptyset$, thus vanishes. For the remaining integral notice that $\Gamma_{\mu\rho}^\mu = \frac{1}{2}g^{\mu\nu}\partial_\rho g_{\mu\nu}$ holds true and furthermore that $\partial_\rho\sqrt{H} = \frac{1}{2}\sqrt{H}H^{\mu\nu}\partial_\rho H_{\mu\nu} \equiv \frac{1}{2}\sqrt{H}H^{\mu\nu}\partial_\rho g_{\mu\nu}$ due to $n^\mu H_{\mu\nu} = 0$. Hence we obtain:

$$\begin{aligned}\int_{\partial M} d^{d-1}x\sqrt{H}D_\rho b^\rho &= -\int_{\partial M} d^{d-1}x\sqrt{H}\left(\frac{1}{2}H^{\mu\nu}\partial_\rho H_{\mu\nu} - \frac{1}{2}g^{\mu\nu}\partial_\rho g_{\mu\nu}\right)b^\rho \\ &= -\int_{\partial M} d^{d-1}x\sqrt{H}\frac{1}{2}(H^{\mu\nu} - g^{\mu\nu})(\partial_\rho g_{\mu\nu})b^\rho \\ &= +\int_{\partial M} d^{d-1}x\sqrt{H}\frac{1}{2}n^\mu n^\nu(\partial_\rho g_{\mu\nu})b^\rho\end{aligned}$$

Apparently, $D_\rho b^\rho$ really stands for some contribution that vanishes whenever Dirichlet boundary conditions are imposed, and in fact using the orthogonality of H and n again, we can write $D_\rho b^\rho \equiv D_\rho^{(H)}b^\rho + (n^\lambda\partial_\rho n_\lambda)b^\rho$.

Going back to the outcome of the variation for $\sqrt{H}$, the first field derivative of the Gibbons-Hawking-York functional is thus given by

$$\begin{aligned}(\partial_{v_h}\Gamma_{(\partial B)}^{\text{grav}}[g]) &= -2\cdot\int_{\partial M} d^{d-1}x\sqrt{H}\left(\frac{1}{2}H^{\mu\nu}K(\partial_{v_h}g)_{\mu\nu} + (\partial_{v_h}K)\right) \\ &= +\int_{\partial M} d^{d-1}x\sqrt{H}\left\{(K^{\mu\nu} - H^{\mu\nu}K) + \left(n^\mu D^\nu - n^\lambda g^{\mu\nu}D_\lambda\right)\right\}(\partial_{v_h}g)_{\mu\nu} \\ &\quad +\int_{\partial M} d^{d-1}x\sqrt{H}(n^\lambda\partial_\rho n_\lambda)H^{\rho\mu}n^\nu(\partial_{v_h}g)_{\mu\nu}\end{aligned}$$

In order to deduce the Hessian we have to compute the symmetric version of the functional derivative of this equation, explicitly:

$$\begin{aligned}
(\mathfrak{d}_{w_h} \mathfrak{d}_{v_h} \Gamma_{(\partial B)}^{\text{grav}}[g]) &= + \mathfrak{d}_{w_h} \int_{\partial M} d^{d-1} x \sqrt{H} (K^{\mu\nu} - H^{\mu\nu} K) (\mathfrak{d}_{v_h} g)_{\mu\nu} & \| \triangleq \mathcal{N}_{\text{VIII}}(v, w) \\
&+ \mathfrak{d}_{w_h} \int_{\partial M} d^{d-1} x \sqrt{H} (n^\mu D^\nu - n^\lambda g^{\mu\nu} D_\lambda) (\mathfrak{d}_{v_h} g)_{\mu\nu} & \| \triangleq -\mathcal{N}_{\text{VI}}(v, w) \\
&+ \mathfrak{d}_{w_h} \int_{\partial M} d^{d-1} x \sqrt{H} (n^\lambda \partial_\rho n_\lambda) H^{\rho\mu} n^\nu (\mathfrak{d}_{v_h} g)_{\mu\nu} & \| \triangleq \mathcal{N}_{\text{IX}}(v, w)
\end{aligned}$$

Notice that the second contribution, $\mathcal{N}_{\text{VI}}(v, w)$, was already determined for the Einstein-Hilbert functional and represents exactly the boundary contribution consisting of only derivatives of the metric variation which under a suitable matching of coefficients is the reason to add the Gibbons-Hawking-York term in order to obtain a well defined variational principle. Hence, we only have to consider $\mathcal{N}_{\text{VIII}}(v, w)$ and $\mathcal{N}_{\text{IX}}(v, w)$ in what follows and reuse the result of the previous subsection for $\mathcal{N}_{\text{VI}}(v, w)$.

Aiming at $\mathcal{N}_{\text{VIII}}(v, w)$, we first present another variation of K which is equivalent to the above construction but more appropriate for the purpose of deriving the Hessian:

$$\begin{aligned}
(\mathfrak{d}_{w_h} K) &= -\frac{1}{2} D_\lambda (n^\lambda n^\rho n^\sigma (\mathfrak{d}_{w_h} g)_{\rho\sigma}) + \frac{1}{2} n^\kappa g^{\rho\sigma} D_\kappa (\mathfrak{d}_{w_h} g)_{\rho\sigma} \\
&= -\frac{1}{2} (n^\rho n^\sigma K + 2n^\rho n_\lambda K^{\lambda\sigma} - H^{\rho\sigma} n^\lambda D_\lambda) (\mathfrak{d}_{w_h} g)_{\rho\sigma}
\end{aligned}$$

In addition, we have to compute the variation of the non-contracted form of the extrinsic curvature which involves a further inverse metric:

$$(\mathfrak{d}_{w_h} K^{\mu\nu}) = (\mathfrak{d}_{w_h} g^{\mu\lambda} D_\lambda n^\nu) = -g^{\mu\rho} (D^\sigma n^\nu) (\mathfrak{d}_{w_h} g)_{\rho\sigma} + D^\mu (\mathfrak{d}_{w_h} n^\nu) + g^{\mu\lambda} (\mathfrak{d}_{w_h} \Gamma_{\lambda\alpha}^\nu) n^\alpha$$

Upon substituting all partial results and expressing $D^\alpha n^\beta$ as the extrinsic curvature, we arrive at

$$\begin{aligned}
(\mathfrak{d}_{w_h} K^{\mu\nu}) &= - (g^{\mu\rho} K^{\sigma\nu} + n^\mu n^\rho K^{\nu\sigma} + \frac{1}{2} K^{\mu\nu} n^\sigma n^\rho) (\mathfrak{d}_{w_h} g)_{\rho\sigma} \\
&- \frac{1}{2} n^\nu n^\sigma n^\rho D^\mu (\mathfrak{d}_{w_h} g)_{\rho\sigma} + \frac{1}{2} g^{\nu\rho} g^{\mu\sigma} n^\lambda D_\lambda (\mathfrak{d}_{w_h} g)_{\rho\sigma} \\
&= - \left((g^{\mu\rho} + n^\mu n^\rho) K^{\sigma\nu} + \frac{1}{2} K^{\mu\nu} n^\sigma n^\rho + \frac{1}{2} (n^\nu n^\sigma n^\rho D^\mu - g^{\nu\rho} g^{\mu\sigma} n^\lambda D_\lambda) \right) (\mathfrak{d}_{w_h} g)_{\rho\sigma}
\end{aligned}$$

The integral kernel of $\mathcal{N}_{\text{VIII}}(v, w)$ is based on these ingredients and has a lengthy but very simple structure:

$$\begin{aligned}
\mathfrak{d}_{w_h} (K^{\mu\nu} - H^{\mu\nu} K) &= (\mathfrak{d}_{w_h} K^{\mu\nu}) - H^{\mu\nu} (\mathfrak{d}_{w_h} K) - (\mathfrak{d}_{w_h} H^{\mu\nu}) K \\
&= - ((g^{\mu\rho} + n^\mu n^\rho) K^{\sigma\nu} + \frac{1}{2} (K^{\mu\nu} - H^{\mu\nu} K) n^\sigma n^\rho + (n^\nu n^\mu n^\sigma n^\rho - g^{\mu\rho} g^{\nu\sigma}) K) (\mathfrak{d}_{w_h} g)_{\rho\sigma} \\
&+ \left(H^{\mu\nu} (n^\rho n_\lambda K^{\lambda\sigma} - \frac{1}{2} H^{\rho\sigma} n^\lambda D_\lambda) - \frac{1}{2} (n^\nu n^\sigma n^\rho D^\mu - g^{\nu\rho} g^{\mu\sigma} n^\lambda D_\lambda) \right) (\mathfrak{d}_{w_h} g)_{\rho\sigma}
\end{aligned}$$

Notice that only the last three terms of the second contribution contains derivatives of the metric fluctuation. Including the field derivative acting on the volume element of the integration we arrive at

$$\begin{aligned}
\mathcal{N}_{\text{VIII}}(v, w) &= + \int_{\partial M} d^{d-1} x \sqrt{H} (\mathfrak{d}_{w_h} g)_{\rho\sigma} \frac{1}{2} H^{\rho\sigma} (K^{\mu\nu} - H^{\mu\nu} K) (\mathfrak{d}_{v_h} g)_{\mu\nu} \\
&+ \int_{\partial M} d^{d-1} x \sqrt{H} (\mathfrak{d}_{v_h} g)_{\mu\nu} \mathfrak{d}_{w_h} (K^{\mu\nu} - H^{\mu\nu} K) \\
&+ \int_{\partial M} d^{d-1} x \sqrt{H} (K^{\mu\nu} - H^{\mu\nu} K) (\mathfrak{d}_{w_h} \mathfrak{d}_{v_h} g)_{\mu\nu}
\end{aligned}$$

Hence, we are ready to compute the final piece of the full Hessian, in fact a contribution that we are keen to remove in the very end by imposing suitable boundary conditions. While we have already encountered several boundary terms in the Hessian that are non-vanishing under Neumann conditions for the metric fluctuation, the very nature of our truncation ansatz ensures that Dirichlet conditions remove all these obstructions on the level of the Hessian. Similarly, for $\mathcal{N}_{\text{IX}}(v, w)$ we find some contributions that contain derivatives of metric fluctuations, and some which do not:

$$\begin{aligned}\mathcal{N}_{\text{IX}}(v, w) = & + \int_{\partial M} d^{d-1}x \sqrt{H} (\partial_{w_h} g)_{\rho\sigma} \frac{1}{2} H^{\rho\sigma} (n^\alpha \partial_\lambda n_\alpha) H^{\lambda\mu} n^\nu (\partial_{v_h} g)_{\mu\nu} \\ & + \int_{\partial M} d^{d-1}x \sqrt{H} \partial_{w_h} (H^{\lambda\mu} n^\nu n^\alpha \partial_\lambda n_\alpha) (\partial_{v_h} g)_{\mu\nu} \\ & + \int_{\partial M} d^{d-1}x \sqrt{H} (n^\alpha \partial_\lambda n_\alpha) H^{\lambda\mu} n^\nu (\partial_{w_h} \partial_{v_h} g)_{\mu\nu}\end{aligned}$$

We omitted to present the explicit form of the variation of the second term, but emphasize at this point that it is proportional to $D_\lambda \partial_{w_h} g$. Notice however that being multiplied by another fluctuation $(\partial_{v_h} g)_{\mu\nu}$ here again Dirichlet conditions appear to be a proper choice for a full cancellation of all boundary contributions.

6.4.5 Hessian for the gravitational functional

The gravitational constituent of the effective average action is independent of the ghosts functional that hence contributes only in the metric block to the Hessian operator. Without specifying boundary conditions and/or the explicit form of the exponential map contained in $(\partial_{v_h} g)_{\mu\nu}$ no Hessian operator can be deduced.

Thus, let us first present the full Hessian of the gravitational functional and then proceed by defining a concrete truncation ansatz with a simple choice for $\partial_{v_h} g_{\mu\nu}$. Collecting the previous results from the volume and curvature functionals the Hessian is given by

$$\begin{aligned}\text{Hess}_\varphi [\Gamma^{\text{grav}}[\Phi, \bar{\Phi}]](v, w) &= \int_M d^d x \sqrt{g} (\partial_{v_h} g)_{\mu\nu} \left\{ -(\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot (g^{\rho\mu} g^{\sigma\nu} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}) \right. \\ &\quad \left. + \frac{1}{2} (\bar{u}_k^{\text{D}}) \cdot \{ -\mathcal{K}^{\mu\nu\rho\sigma} + \mathcal{U}^{\mu\nu\rho\sigma} \} \right\} (\partial_{w_h} g)_{\rho\sigma} \\ &+ \int_M d^d x \sqrt{g} \left(\frac{1}{2} (\bar{u}_k^{\text{D}}) \cdot G^{\mu\nu} - (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot g^{\mu\nu} \right) \left(\frac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu}\end{aligned}\quad (6.13)$$

A treatment of the full Einstein-Hilbert functional requires a geometrical connection for which $(\partial_{v_h} g)$ would be still a function of g due to a non-trivial field space connection in the metric component. Choosing a different, for example flat, connection corresponds to some truncation which is only equivalent to the Einstein-Hilbert functional for small (effective) perturbations around g . This is a very important remark, for it allows to simplify the Hessian at the expense of loosing some of its interpretation. While in this thesis, we are going to consider a truncation resulting from the Einstein-Hilbert functional for $(\partial_{v_h} g) = v_h$ future work can draw on equation (6.13) and insert different geodesic solutions which may fully reproduce the Einstein-Hilbert case. Under the choice $(\partial_{v_h} g) = v_h$ we see that the last term actually does not contribute to the Hessian for then $(\partial_{w_h} \partial_{v_h} g) = (\partial_{w_h} v_h) = 0$. If the field space connection – implicitly present in $(\partial_{v_h} g)$ – is ultra-local, thus does not contain any derivatives, eq. (6.13) always gives rise to a Hessian operator as long as the boundary terms vanish under suitable conditions on the

fluctuation fields:

$$\begin{aligned}
\mathcal{J}^{\text{grav}} = & -(\bar{u}_k^{\partial\text{D}} \bar{\lambda}_k^{\partial\text{D}}) \cdot \int_{\partial M} d^{d-1}x \sqrt{H} (\partial_{w_h} g)_{\sigma\rho} \left\{ (H^{\mu\rho} H^{\nu\sigma} - \frac{1}{2} H^{\rho\sigma} H^{\mu\nu}) \right. \\
& \left. + H^{\mu\rho} n^\nu n^\sigma + H^{\nu\sigma} n^\rho n^\mu \right\} (\partial_{v_h} g)_{\mu\nu} \\
& - (\bar{u}_k^{\partial\text{D}} \bar{\lambda}_k^{\partial\text{D}}) \cdot \int_{\partial M} d^{d-1}x \sqrt{H} H^{\mu\nu} \left(\frac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu} \\
& + (\bar{u}_k^{\text{D}} - \bar{u}_k^{\partial\text{D}}) \cdot \mathcal{N}_{\text{VI}}(v, w) + \frac{1}{2} (\bar{u}_k^{\text{D}}) \cdot \mathcal{N}_{\text{VII}}(v, w) + (\bar{u}_k^{\partial\text{D}}) \cdot (\mathcal{N}_{\text{VIII}}(v, w) + \mathcal{N}_{\text{IX}}(v, w))
\end{aligned}$$

The very nature of the present truncation assures that all these contributions disappear for Dirichlet boundary conditions. The reason is encoded in the maximal number of covariant derivatives that constitutes the various invariants. On the bulk, we restricted ourselves to two derivatives and on the boundary we consider only field monomials with at most one covariant derivative. Each boundary contribution that originates from some bulk invariant, due to integration by parts, converts one of its derivatives to a normal vector field. Hence, any boundary term irrespectively of its origin contains at most one derivative. For the Hessian this implies that out of the two fluctuations there is always at least one fluctuation field without a derivative acting on it: therefore any boundary term vanishes under Dirichlet conditions in the present truncation.

This simple argument naturally does not generalize to truncations including higher order derivative invariants, still a thorough treatment of manifolds with boundary in the FRGE setting requires a Hessian operator. In these cases matching of coefficients along with some choice of – possibly non-standard – boundary conditions would be appropriate to continue the evaluation of the RHS of the FRGE. Surely, in the very end the RG solutions reveal whether or not the chosen matching condition is consistent with the RG flow on theory space.

In what follows we always assume Dirichlet boundary conditions to hold, i.e. for the entire (effective) field space we consider a fixed field all over ∂M :

$$\Phi|_{\partial M} = \Psi \equiv \bar{\Phi}|_{\partial M} \quad \text{and} \quad \varphi|_{\partial M} = 0$$

The related effective fluctuation fields thus have to vanish on the boundary.

6.4.6 Linear field parametrization

In most of the truncations for theories of quantum gravity field space was approximated to be flat resulting in what is called the linear field parametrization of metric fluctuations:

$$(\partial_{v_h} g) = v_h \tag{6.14}$$

As already mentioned at several steps in the derivation, this transition from dynamical fields Φ to fluctuation fields φ is unavoidable in the presence of field variations. All the supplementary functionals – the gauge-, ghost-, and cutoff-action – that are necessary to obtain a well behaved quantum field theory, are by definition formulated in terms of φ using some background field $\bar{\Phi}$. For the gravitational contribution we could have started at first place with this set of dependencies using the exponential map for a transition between both functionals:

$$\Gamma^{\text{grav}}[\Phi, \bar{\Phi}] \equiv \Gamma^{\text{grav}}[\exp_{\bar{\Phi}}(\varphi(\Phi, \bar{\Phi})), \bar{\Phi}] \equiv \Gamma^{\text{grav}}[\varphi; \bar{\Phi}]$$

Notice that on the effective level we in general encounter a bi-field functional, being individually dependent on the dynamical and the background field, Φ and $\bar{\Phi}$, respectively. While on the fundamental level we usually assume $\rho([\hat{\Phi}])$ being background independent, the gauge fixing and cutoff functional contribute with background non-invariant insertions to the effective action resulting in the bi-field nature of Γ_k . When approximating $\exp_{\bar{\Phi}}(\varphi(\Phi, \bar{\Phi}))$ in a linear fashion,

i.e. $\varphi = \Phi - \bar{\Phi}$, the nice correspondence between $\Gamma^{\text{grav}}[\varphi; \bar{\Phi}]$ and $\Gamma^{\text{grav}}[\Phi, \bar{\Phi}]$ will in general be lost. This is not problematic at all in case of truncations on the effective level for then we are just considering a modified ansatz for $\Gamma_k^{\text{grav}}[\varphi; \bar{\Phi}]$ which is still compatible with the symmetry and field content, though one may spoil gauge invariance. The mere disadvantage in using non-geometrical relations for the geodesics on $\langle [F] \rangle$ is the status of interpretation and the systematic generalization of the ‘new’ truncation ansatz. So one may start at first place with an Einstein-Hilbert like truncation ansatz for $\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]$ but substituting $\Phi = \bar{\Phi} + \varphi$ only approximately reproduces its analog $\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]$. Nevertheless, this linear parametrization is at least suitable for almost flat regions in the vicinity of normal geodesic coordinates of Φ . In other words restricting the *effective* fluctuations φ to be small, i.e. considering small derivations of the associated measure, retains the Einstein-Hilbert interpretation while larger fluctuations will in general result in some modified class of measures.

Combining all the previous results for the gravitational functional and insert the linear parametrization $(\mathfrak{d}_{v_h} g) = v_h$ we obtain a Hessian operator of the following form:

$$\widehat{\text{Hess}}_\varphi [\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]] \equiv \begin{pmatrix} \widehat{\text{Hess}}_{hh}^{\text{grav}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6.15)$$

Since the gravitational contribution is a functional of only metric field content, the Hessian is degenerated in the ghost sector and contains only one non-vanishing component, explicitly given by a combination of second-order differential and ultra-local terms:

$$\begin{aligned} g^{\mu\nu A} \left(\widehat{\text{Hess}}_{hh}^{\text{grav}} \right)_A^{\rho\sigma} &= -(\bar{u}_k^D \bar{\lambda}_k^D) \cdot \frac{\sqrt{\bar{g}}}{\sqrt{g}} (g^{\rho\mu} g^{\sigma\nu} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}) \\ &\quad + \frac{1}{2} (\bar{u}_k^D) \cdot \frac{\sqrt{\bar{g}}}{\sqrt{g}} \{ -\mathcal{K}^{\mu\nu\rho\sigma} + \mathcal{U}^{\mu\nu\rho\sigma} \} \end{aligned} \quad (6.16)$$

It is part of the propagator for the effective gravitational fluctuation field, for which ghost and gauge fixing corrections have to be added. The later analysis will depend on this Hessian structure employing the linear parametrization.

6.4.7 Geometric field parametrization

We have already mentioned that the linear parametrization may spoil the original interpretation of the truncation ansatz, for one actually considers a (non-geometrically) reduced functional that only corresponds to the initial object for small effective fluctuation fields h . Nevertheless, it is not a perturbative approximation at all, for it only gives rise to a different truncation ansatz with a possible different interpretation for large h . A more sophisticated and natural procedure relies on a geometric parametrization of the dynamical field g with h and the background metric $\bar{g}$, usually based on some specific connection on field space. This general treatment that actually corresponds to the covariant formalism of field space we introduced earlier, goes back to work of DeWitt [39, 68, 69] and has found attention in the mathematical literature on metric field spaces. It is the geometric method of background field covering that relates the dynamical field g with the background one $\bar{g}$ by geodesics, corresponding to the fluctuation fields. The problem is merely on a practical level, for the associated exponential map is usually non-local and highly non-linear, which adds another difficulty into the list of challenges one encounters in evaluating the FRGE.

Recently, there has been progress towards a more geometric parametrization which is local but still non-linear [84]. The underlying field connection is ultra-local on F and the exponential map reduces to matrix exponentiation given by

$$g_{\mu\nu} = \bar{g}_{\mu\rho} (e^h)^\rho_\nu$$

The non-linear character of this parametrization yields in particular additional contributions from terms $\mathfrak{D}_{w_h} \mathfrak{D}_{v_h} g$ which vanish for the linear choice. Future works should in addition employ this geometric ansatz and a comparison between the linear and this geometric parametrization might reveal the interpretation status of the former one. The here derived Hessian is thus a suitable starting point to proceed in both directions.

6.5 Hessian of the full truncation ansatz

Finally, we combine the results of the preceding sections to obtain the full Hessian of our truncation ansatz within the linear parametrization. While the cutoff-, the ghost-, and gauge-functional are purely defined in terms of fluctuation h and background fields $\bar{g}$, the gravitational part is a functional of the dynamical field g . Our truncation thus relies on a specific connection on field space and in the present non-geometric identification, we actually compute the FRGE for the following ansatz:

$$\Gamma_k[\varphi; \bar{\Phi}] = \Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}] + \Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}] + \sum_{n=0}^{\infty} \partial_s^n \Gamma_k^{\text{grav}}[\bar{\Phi} + s\varphi, \bar{\Phi}]|_{s=0} \quad (6.17)$$

In a covariant formalism the (flat) Taylor expansion has to be replaced by the exponential map on field space. We are going to simplify this ansatz even further by choosing a suitable linear gauge fixing condition and neglect the running of the ghost sector later on.

6.5.1 Boundary conditions

In order to extract a Hessian operator for the above truncation, the first step is to insert $(\mathfrak{D}_{v_h} g) = v_h$ into the Hessian and impose a set of boundary conditions on the fluctuation fields in order to get rid of the impeding boundary terms. However, different to the usual constraints in order to obtain a suitable variational principle, we rather focus our attention on the second variation in order to find the FRGE applicable. In total we have three sources of boundary contributions associated to each of the functionals in the truncation ansatz:

$$\mathcal{F} = \mathcal{F}^{\text{gh}}[v, w] + \mathcal{F}^{\text{gf}}[v, w] + \mathcal{F}^{\text{grav}}[v, w]$$

The very existence of a Hessian operator for $\Gamma_k[\varphi; \bar{\Phi}]$ requires the vanishing of $\mathcal{F}$. Due to the restricted number of at most two derivatives in the invariants of our ansatz, Dirichlet boundary conditions are sufficient to cancel all terms individually, as long as $\mathcal{F}$ is linear and contains at most two $\bar{D}$. In the presence of a non-vanishing boundary of spacetime constraining field space on ∂M is mandatory to obtain ellipticity for the set of operators defining the RG steps. Though in the general case one is interested in preserving ellipticity and symmetries of the formalism, in particular BRST invariance, there are indications that it might be difficult to retain both in the case of QG. However, on the level of truncations ellipticity is far more important property, for it assures a proper heat kernel expansion, or in more general terms a well behaved cutoff operator based on the Laplacian. In this respect, Dirichlet conditions are a very natural choice and are also admissible in the sense of Faddeev-Popov as stated by Vassilevich [61]. Nevertheless, quite different boundary constraints may be imposed and the current status of derivation is such general that future investigations based on Neumann-, or boundary conditions can be applied as long as $\mathcal{F}$ is removed from the Hessian. While Dirichlet boundary conditions actually cancel each of the arising boundary terms separately, other constraints might additionally assume a matching of certain bulk and boundary couplings which then has to be justified by the RG evolution. In fact, for local, linear gauge fixing conditions, we have a total set of seven equations constituting the boundary operator, whose kernel yields the valid field space.

$$\mathcal{B}_{(D);\Psi}^{\partial M}(\Phi) := (\Phi - \Psi)(\bullet) \delta_{\partial M}(\bullet) \quad \text{with} \quad \ker(\mathcal{B}_{(D);\Psi}^{\partial M}) \equiv \{\Phi \in Y \mid \Phi|_{\partial M} = \Psi|_{\partial M}\} \quad (6.18)$$

This boundary operator is equivalent to the previously described condition:

$$\Phi|_{\partial M} = \Psi \equiv \bar{\Phi}|_{\partial M} \quad \text{and} \quad \varphi|_{\partial M} = 0 \quad (6.19)$$

Notice that once we implement the background field covering on field space Ψ is encoded in $\bar{\Phi}$ and thus redundant.

The only source for the ghost sector and its coupling to the metric field content is the ghost functional coming along with boundary terms:⁵

$$\begin{aligned} \mathcal{F}^{\text{gh}}[v, w] &\equiv -\sqrt{2}\bar{u}_k^{\text{gh}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_V^A \left(\bar{\xi}_\mu \bar{g}^{\mu\nu} \partial_w \partial_v (\mathcal{L}_\xi g)_A + (v_{\bar{\xi}})_\mu \bar{g}^{\mu\nu} (\partial_w \mathcal{L}_\xi g)_A \right) \\ &\quad - \sqrt{2}\bar{u}_k^{\text{gh}} \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} (v_{\bar{\xi}})_\mu \right) \left(n_\sigma g_{\lambda\rho} w_\xi^\lambda + n_\rho g_{\lambda\sigma} w_\xi^\lambda + n_\lambda \xi^\lambda (\partial_w g)_{\rho\sigma} \right) \\ &\quad - \sqrt{2}\bar{u}_k^{\text{gh}} \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_V^{\rho\sigma} \left(\bar{g}^{\mu\nu} \bar{\xi}_\mu \right) \left(n_\sigma (v_g)_{\lambda\rho} w_\xi^\lambda + n_\rho (v_g)_{\lambda\sigma} w_\xi^\lambda + n_\lambda v_\xi^\lambda (\partial_w g)_{\rho\sigma} \right) \end{aligned}$$

For suitable choices of gauge fixing condition, it is in fact sufficient to impose Dirichlet conditions on ghost and metric fields to get rid of these terms, i.e. for all $\Phi \in \ker(\mathcal{B}_{(D);\Psi}^{\partial M})$ and $v, w \in T_\Phi F$ we have $\mathcal{F}^{\text{gh}}[v, w] = 0$.

The gauge fixing functional gives rise to a single contribution restricting only the metric part of field space, i.e.

$$\mathcal{F}^{\text{gf}}[v, w] \equiv \bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{g} \mathcal{F}_\mu^A \left(\bar{g}^{\mu\nu} (w_h)_A \mathcal{F}_\nu^B (v_h)_B \right)$$

We are going to implement a class of gauge fixing conditions in a moment, however under the above mentioned assumptions of locality, linearity, and at most second order in the derivatives, any $\mathcal{F}$ will result in a contribution that vanishes on the boundary for $\Phi \in \ker(\mathcal{B}_{(D);\Psi}^{\partial M})$.

The remaining part originates from the gravitational functional and is either related to some integration by parts of certain bulk terms or by the implemented variations of the Gibbons-Hawking-York like terms in the truncation ansatz. Its total contribution to the Hessian reads

$$\begin{aligned} \mathcal{F}^{\text{grav}}[v, w] &= -(\bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D}) \cdot \int_{\partial M} d^{d-1} x \sqrt{H} (\partial_{w_h} g)_{\sigma\rho} \left\{ (H^{\mu\rho} H^{\nu\sigma} - \frac{1}{2} H^{\rho\sigma} H^{\mu\nu}) \right. \\ &\quad \left. + H^{\mu\rho} n^\nu n^\sigma + H^{\nu\sigma} n^\rho n^\mu \right\} (\partial_{v_h} g)_{\mu\nu} \\ &\quad - (\bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D}) \cdot \int_{\partial M} d^{d-1} x \sqrt{H} H^{\mu\nu} \left(\frac{1}{2} (\partial_{w_h} \partial_{v_h} + \partial_{v_h} \partial_{w_h}) g \right)_{\mu\nu} \\ &\quad + (\bar{u}_k^D - \bar{u}_k^{\partial D}) \mathcal{N}_{\text{VI}}(v, w) + \frac{1}{2} (\bar{u}_k^D) \mathcal{N}_{\text{VII}}(v, w) + (\bar{u}_k^{\partial D}) (\mathcal{N}_{\text{III}}(v, w) + \mathcal{N}_{\text{IX}}(v, w)) \end{aligned}$$

Notice that for the truncation ansatz (6.17) the second integral vanishes identically for any choice of boundary conditions. Furthermore, while Dirichlet constraints yield a cancellation of every term irrespective of the coefficients, a matching of $\bar{u}_k^D$ and $\bar{u}_k^{\partial D}$ as required for the construction of field equations by means of the variational principle will at least cancel $\mathcal{N}_{\text{VI}}(v, w)$.

In conclusion we have to either restrict field space by some adapted choice of boundary conditions for the fluctuation fields that cancel $\mathcal{F}$ or imposing certain boundary conditions and adapt the explicit form of $\Gamma_k[\varphi; \bar{\Phi}]$ such that the non-vanishing terms in $\mathcal{F}$ cancel each other. In the present work, we follow the first approach for the structure of the truncation ansatz favor Dirichlet boundary conditions and allows for an application of the heat kernel technique. Thus,

⁵Notice that for brevity we neglect are not presenting the symmetric version of the boundary terms that appears in the Hessian. The actual result is given by $\frac{1}{2} \mathcal{F}^{\text{gh}}[v, w] + \frac{1}{2} \mathcal{F}^{\text{gh}}[w, v]$.

as already mentioned in the gravitational part, in the sequel we impose the boundary constraints of Dirichlet type that gives rise to a cancellation of the entire functional $\mathcal{J}$, as required. This step marks the end of the general status of our results, so that from now on the statements are only valid within the assumption of Dirichlet conditions.

6.5.2 The Hessian operator

Now that we have imposed suitable boundary conditions for which $\mathcal{J}$ vanishes identically, the full Hessian gives rise to a Hessian operator consisting of the gauge fixing, ghost, and gravitational contributions:

$$\widehat{\text{Hess}}_\varphi[\Gamma_k[\varphi; \bar{\Phi}]] = \widehat{\text{Hess}}_\varphi[\Gamma_k^{\text{gf}}[\varphi; \bar{\Phi}]] + \widehat{\text{Hess}}_\varphi[\Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}]] + \widehat{\text{Hess}}_\varphi[\Gamma_k^{\text{grav}}[\varphi; \bar{\Phi}]]$$

Its matrix structure in field space has only two vanishing entries, the diagonal elements of the ghost block. The remaining components give rise to the following matrix form:

$$\widehat{\text{Hess}}_\varphi[\Gamma_k[\varphi; \bar{\Phi}]] \equiv \begin{pmatrix} \widehat{\text{Hess}}_{hh} & \widehat{\text{Hess}}_{h\xi} & \widehat{\text{Hess}}_{h\bar{\xi}} \\ \widehat{\text{Hess}}_{\xi h} & 0 & \widehat{\text{Hess}}_{\xi\bar{\xi}} \\ \widehat{\text{Hess}}_{\bar{\xi} h} & \widehat{\text{Hess}}_{\bar{\xi}\xi} & 0 \end{pmatrix} \quad (6.20)$$

All its entries are L^2 -operators that in general will not commute with each other. For the later truncation the relevant contributions are those which are independent on the ghost and anti-ghost fields, thus the metric component consisting of the gravitational and the gauge fixing Hessian:

$$\begin{aligned} g^{\mu\nu B} (\widehat{\text{Hess}}_{hh})_B^{\rho\sigma} &= -(\bar{u}_k^D \bar{\lambda}_k^D) \cdot \frac{\sqrt{g}}{\sqrt{g}} (g^{\rho\mu} g^{\sigma\nu} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}) \\ &\quad + \frac{1}{2} (\bar{u}_k^D) \cdot \frac{\sqrt{g}}{\sqrt{g}} \{ -\mathcal{H}^{\mu\nu\rho\sigma} + \mathcal{U}^{\mu\nu\rho\sigma} \} \\ &\quad - \bar{u}_k^{\text{gf}} \cdot \mathcal{F}_\alpha^{\mu\nu}[\bar{g}] \left(\mathfrak{g}^{\alpha\beta} \mathcal{F}_\beta^{\rho\sigma}[\bar{g}] \right) \end{aligned} \quad (6.21)$$

In addition, due to the linearity of the ghost functional in ξ and $\bar{\xi}$ the entries in the ghost block are also independent on the ghost fluctuations. They exclusively originate from $\Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}]$ and assume the form:

$$\begin{aligned} (\widehat{\text{Hess}}_{\xi\bar{\xi}})_\alpha^\mu &= +\sqrt{2} \bar{u}_k^{\text{gh}} (\mathfrak{g}_{\xi\bar{\xi}}^{-1})_\alpha^\lambda (\bar{D}_\lambda g_{\rho\sigma} + \bar{D}_\sigma g_{\lambda\rho} + \bar{D}_\rho g_{\lambda\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \mathcal{F}_\nu^{\rho\sigma} \bar{g}^{\mu\nu} \\ (\widehat{\text{Hess}}_{\bar{\xi}\xi})_\lambda^\alpha &= -\sqrt{2} \bar{u}_k^{\text{gh}} (\mathfrak{g}_{\xi\bar{\xi}}^{-1})_\mu^\alpha \left\{ \bar{g}^{\mu\nu} \mathcal{F}_\nu^{\rho\sigma} (\bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \right\} \end{aligned} \quad (6.22)$$

The components that interrelate metric fluctuation with the ghost sector all depend on either $\bar{\xi}$ and ξ . They emanate from the ghost functional and can be summarized as follows:

$$(\widehat{\text{Hess}}_{\xi h})^{\alpha\beta} = -\sqrt{2} \bar{u}_k^{\text{gh}} \cdot (\widehat{\text{Hess}}_{\xi h}^{\text{gh}})^{\alpha\beta} \quad (\widehat{\text{Hess}}_{h\xi})_\lambda = -\sqrt{2} \bar{u}_k^{\text{gh}} \cdot (\widehat{\text{Hess}}_{h\xi}^{\text{gh}})_\lambda \quad (6.23a)$$

$$(\widehat{\text{Hess}}_{h\bar{\xi}})^\mu = -\sqrt{2} \bar{u}_k^{\text{gh}} \cdot (\widehat{\text{Hess}}_{h\bar{\xi}}^{\text{gh}})^\mu \quad (\widehat{\text{Hess}}_{\bar{\xi} h})^{\alpha\beta} = -\sqrt{2} \bar{u}_k^{\text{gh}} \cdot (\widehat{\text{Hess}}_{\bar{\xi} h}^{\text{gh}})^{\alpha\beta} \quad (6.23b)$$

Their explicit form is found in equation (6.9). Later on, neglecting those contributions simplifies the RHS of the FRGE significantly, which corresponds to projecting out the running of the ghost sector.

6.6 First variation formula

The original motivation to include the Gibbons-Hawking-York term to the Einstein-Hilbert functional was motivated by the variational principle in the presence of boundaries [79]. In the

present context we expect that this correspondence is only satisfied under additional assumption, since quantum corrections in principle spoil this classical matching.

Even though our main focus is the derivation of the RG flow, it is worth considering the conditions under which a well stated variational principle can be established. In the metric sector the gauge fixing functional produces the following boundary term under first variation:

$$0 \stackrel{!}{=} \bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\mu^A (\bar{g}^{\mu\nu} (v_h)_A \mathcal{F}_\nu^B (h)_B)$$

Whenever $\mathcal{F}$ is linear in the covariant derivative $\bar{D}$ Dirichlet and Neumann boundary constraints make this term disappear. In particular the class of gauge fixing conditions we finally implement satisfies this property. Furthermore, we have to assure that the gravitational contribution also vanishes, for which however mixed boundary conditions are necessary:

$$\begin{aligned} 0 \stackrel{!}{=} & + \left\{ (\bar{u}_k^{\partial\text{D}} \bar{\lambda}_k^{\partial\text{D}}) \cdot H^{\mu\nu} - (\bar{u}_k^{\partial\text{D}}) \cdot \left((n^\lambda \partial_\rho n_\lambda) H^{\rho\mu} n^\nu + K^{\mu\nu} - H^{\mu\nu} K \right) \right\} (\partial_{v_h} g)_{\mu\nu} \\ & + \left\{ (\bar{u}_k^{\text{D}} - \bar{u}_k^{\partial\text{D}}) \cdot (n^\mu D^\nu - g^{\mu\nu} n^\rho D_\rho) \right\} (\partial_{v_h} g)_{\mu\nu} \end{aligned}$$

In the linear field parametrization we employ here, this requirement translates into

$$\begin{aligned} 0 \stackrel{!}{=} & + \left\{ (\bar{u}_k^{\partial\text{D}} \bar{\lambda}_k^{\partial\text{D}}) \cdot H^{\mu\nu} - (\bar{u}_k^{\partial\text{D}}) \cdot \left((n^\lambda \partial_\rho n_\lambda) H^{\rho\mu} n^\nu + K^{\mu\nu} - H^{\mu\nu} K \right) \right\} (v_h)_{\mu\nu} \\ & + \left\{ (\bar{u}_k^{\text{D}} - \bar{u}_k^{\partial\text{D}}) \cdot (n^\mu D^\nu - g^{\mu\nu} n^\rho D_\rho) \right\} (v_h)_{\mu\nu} \end{aligned}$$

Notice that the first (second) term is zero for Dirichlet (Neumann) constraints for the metric fluctuations thus an additional matching of the couplings would be necessary. Apparently, the ‘classical’ choice of $\bar{u}_k^{\text{D}} \equiv \bar{u}_k^{\partial\text{D}}$ suffices to get rid of the entire boundary contribution in case of Dirichlet type constraints.

Finally, the variation of the ghost functional generates two more boundary integrals that however vanish for the above restrictions of the gauge fixing condition and imposing Dirichlet constraints on the fluctuation fields:

$$\begin{aligned} 0 \stackrel{!}{=} & + \bar{u}_k^{\text{gh}} \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_\nu^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left(n_\sigma g_{\lambda\rho} v_\xi^\lambda + n_\rho g_{\lambda\sigma} v_\xi^\lambda + n_\lambda \xi^\lambda (\partial_\nu g)_{\rho\sigma} \right) \\ & + \bar{u}_k^{\text{gh}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\nu^A (\bar{\xi}_\mu \bar{g}^{\mu\nu} (\partial_\nu \mathcal{L}_\xi g)_A) \end{aligned}$$

However, for gauge fixing conditions that are of degree two or higher in the derivatives $\bar{D}$, the vanishing of these boundary terms is not in general given and a case by case study is demanded.

6.6.1 Stationary point solutions

Assuming that all boundary contributions dissolve under a suitable choice of constraints on field space, stationary point solutions for the effective fluctuations can be deduced from a variational principle. They read as follows:

► Anti-ghost field variation

$$0 = \left\{ \bar{g}^{\mu\nu} \mathcal{F}_\nu^{\rho\sigma} (\bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \right\} \xi^\lambda \equiv \mathcal{M}[g, \bar{g}](\xi)^\mu$$

Thus, solutions ξ which satisfy the stationary point equation should be orthogonal to the image of the Faddeev-Popov operator $\mathcal{M}[g, \bar{g}]$. Due to the linearity of the field equation and the lack of further constraints on ξ , the trivial solution $\xi = 0$ exists. Since the kernel of $\mathcal{M}[g, \bar{g}]$ is assumed to be trivial this is actually the only possible solution.

► **Ghost field variation**

$$0 = (\bar{D}_\lambda g_{\rho\sigma} + \bar{D}_\sigma g_{\lambda\rho} + \bar{D}_\rho g_{\lambda\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \mathcal{F}_\nu^{\rho\sigma} \bar{g}^{\mu\nu} \bar{\xi}_\mu$$

This equation represents the adjoint of the Faddeev-Popov operator acting on the anti-ghost field. Similar arguments as in the previous case apply, hence favoring the trivial solution $\bar{\xi} = 0$.

► **Metric field variation**

$$0 = +2\sqrt{2}\bar{u}_k^{\text{gh}} \left\{ \left(\bar{D}_\sigma \xi^\alpha \delta_\rho^\beta + \bar{D}_\rho \xi^\alpha \delta_\sigma^\beta + \delta_\rho^\alpha \delta_\sigma^\beta \left(\bar{D}_\lambda \xi^\lambda + \xi^\lambda \bar{D}_\lambda \right) \right) \mathcal{F}_\nu^{\rho\sigma} \bar{g}^{\mu\nu} \right\} \bar{\xi}_\mu \\ + \left\{ -\bar{u}_k^{\text{gf}} \left(\mathcal{F}_\mu^{\alpha\beta} \bar{g}^{\mu\nu} \mathcal{F}_\nu^A \right) h_A \right\} \quad (6.24)$$

$$+ \frac{\sqrt{\bar{g}}}{\sqrt{g}} \left\{ (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot g^{\alpha\beta} + (\bar{u}_k^{\text{D}}) \cdot \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \right\} \quad (6.25)$$

The variation w.r.t. metric fluctuations obtains contributions from each functional in our truncation ansatz, except the purely background terms inserted to study Background Independence. Here the linear parametrization was employed giving rise to the modified Einstein field equations in the last line. The remaining contribution to the metric field equations are due to the gauge fixing and ghost functional. Inserting the solutions from the previous field equations we get

$$0 = \left\{ -\bar{u}_k^{\text{gf}} \left(\mathcal{F}_\mu^{\alpha\beta} \bar{g}^{\mu\nu} \mathcal{F}_\nu^A \right) h_A \right\} + \frac{\sqrt{\bar{g}}}{\sqrt{g}} \left\{ (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot g^{\alpha\beta} + (\bar{u}_k^{\text{D}}) \cdot \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \right\}$$

Notice that self-consistent background fields satisfying $(\partial_\nu \Gamma_k[\varphi; \bar{\Phi}])_{\varphi=0; \bar{\Phi}=\bar{\Phi}^{\text{s.c.}}} = 0$ are solutions to the modified Einstein field equations only, for in this case the gauge fixing term drops out. Contrary to the ghost sector, we are considering a mixed field equations based on the fluctuation and the dynamical field h and g , respectively. While $h = 0$ is allowed and exactly gives rise to the self-consistent background solutions, setting $g = 0$ would solve the equation as it stands, however is not a metric field anymore and thus prohibited. Rather, we may consider instanton solutions for either $\bar{g}$ or g , satisfying the above modified Einstein equations.

Let us emphasize again that we are studying the effective action and thus the effective dynamical fields and fluctuations, for which the above derived stationary point equations hold true. Whenever we are interested in interpreting those results, we have to keep this subtlety in mind. In particular the imposed boundary conditions apply to φ rather than $\bar{\varphi}$ and will constraint the space of measures yielding expectation fields with, for instance, vanishing boundary values. From this perspective, even requiring Dirichlet and Neumann conditions to be satisfied simultaneously, is not over-determining but in fact amounts to a restriction of the class of theories keeping the underlying field space perfectly well-defined.

6.7 Implementation of the gauge fixing condition

In this section we give a more restricted representation for the missing ingredient in our truncation ansatz: the gauge fixing condition. So far, all results only depend on a few conditions on $\mathcal{F}[\bar{\Phi}]$, the commutativity with the background metric, linearity, and the vanishing of the following term under Dirichlet conditions:

$$\mathcal{F}^{\text{gf}}[v, w] \equiv \bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\mu^A \left(\bar{g}^{\mu\nu} (w_h)_A \right) \mathcal{F}_\nu^B (v_h)_B$$

We already mentioned that a sufficient condition is a gauge fixing condition linear in $\bar{D}$.

In the sequel of this discussion we work with a generalized harmonic gauge condition which is compatible with the Faddeev-Popov method:

$$\mathcal{F}[\bar{g}]_{\mu}^{\rho\sigma} = \left(\bar{g}^{\alpha\rho} \bar{g}^{\beta\sigma} - \varpi \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} \right) \bar{g}_{\alpha\mu} \bar{D}_{\beta} \quad (6.26)$$

Notice that the first factor has the same tensor structure than the field space metric in the gravitational sector.

A particular important, and well studied case of this family of gauge fixing conditions is the harmonic one with $\varpi = 1/2$, sometimes referred to as de Donder gauge. In a local chart this corresponds to $\partial_{\mu}(\sqrt{\bar{g}} \bar{g}^{\mu\nu}) = 0$ and defines the harmonic coordinates, a kind of analog to the Lorentz gauge condition for Yang-Mills connections. Harmonic coordinates can be ascribed to any Riemannian manifold at least in the local vicinity of any spacetime point $p \in M$. Their existence is by no means a specific feature of a certain class of geometries but in fact a general result obtained in the theory of elliptic differential operators associated to the Laplacian. In addition, the elliptic regularity theorem and the Hölder condition ensure that these coordinates are in fact regular and thus have a special status in a mathematical description of gauge fixing conditions.

Nevertheless, for a moment we are going to use the family of gauge fixing conditions defined in eq. (6.26) to reduce the Hessian operator and present an explicit form for the effective field equations.

6.7.1 Hessian operator

Before we are going to simplify the Hessian operator, let us first check the validity of the generalized harmonic gauge condition in the present context. Apparently, eq. (6.26) defines a linear operator, which – due to the metric compatibility of $\bar{D}$ – commutes with the background metric, i.e.

$$\mathcal{F}[\bar{g}]_{\mu}^{\rho\sigma} (c \cdot h)_{\rho\sigma} = c \cdot \mathcal{F}[\bar{g}]_{\mu}^{\rho\sigma} (h)_{\rho\sigma} \quad \text{and} \quad [\mathcal{F}[\bar{g}]_{\mu}^{\rho\sigma}, \bar{g}] = 0$$

Next, we verify the vanishing of $\mathcal{F}^{\text{gf}}[v, w]$ under Dirichlet conditions:

$$\begin{aligned} \mathcal{F}^{\text{gf}}[v, w] &= \bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{\bar{g}} \left(\bar{g}^{\nu\rho} \bar{g}^{\beta\sigma} - \varpi \bar{g}^{\nu\beta} \bar{g}^{\rho\sigma} \right) \bar{D}_{\beta} \left((w_h)_A \mathcal{F}_v^B (v_h)_B \right) \\ &= \bar{u}_k^{\text{gf}} \int_{\partial M} d^{d-1} x \sqrt{H} \left(\bar{g}^{\nu\rho} \bar{g}^{\beta\sigma} - \varpi \bar{g}^{\nu\beta} \bar{g}^{\rho\sigma} \right) n_{\beta} \left((w_h)_A \mathcal{F}_v^B (v_h)_B \right) = 0 \end{aligned}$$

In the first step the total derivative is transformed into a boundary integral, containing an undifferentiated w_h which vanishes therefore.

Hence, the family of gauge fixing conditions allows for a Hessian operator of the form given in eq. (6.20). For later reasons, we focus only on the entries of $\widehat{\text{Hess}}_{\varphi}[\Gamma_k[\varphi; \bar{\Phi}]]$ independent on the ghost fields. Therefore, consider the contraction of the gauge fixing condition occurring in the Hessian part of the gauge fixing functional:

$$\begin{aligned} \mathcal{F}_{\alpha}^{\nu\mu}[\bar{g}] \left(\mathfrak{g}^{\alpha\beta} \mathcal{F}_{\beta}^{\sigma\rho}[\bar{g}] \right) &= (\bar{g}^{\nu\alpha} \bar{g}^{\mu\kappa} - \varpi \bar{g}^{\mu\nu} \bar{g}^{\kappa\alpha}) \bar{g}_{\alpha\beta} \bar{D}_{\kappa} \left(\bar{g}^{\sigma\beta} \bar{g}^{\rho\lambda} - \varpi \bar{g}^{\rho\sigma} \bar{g}^{\lambda\beta} \right) \bar{D}_{\lambda} \\ &= \bar{g}^{\sigma\nu} \bar{D}^{\mu} \bar{D}^{\rho} - \varpi (\bar{g}^{\rho\sigma} \bar{D}^{\nu} \bar{D}^{\mu} + \bar{g}^{\mu\nu} \bar{D}^{\sigma} \bar{D}^{\rho}) + \varpi^2 \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \bar{D}^2 \end{aligned}$$

It turns out convenient to express the symmetrized version of this operator – w.r.t. $(\mu\nu)$ and $(\rho\sigma)$ – in terms of the kinetic operator appearing in the Hessian of the gravitational functional:

$$\mathcal{K}(g)^{\mu\nu\rho\sigma} \equiv (g^{\mu\nu} D^{\rho} D^{\sigma} - 2g^{\nu\sigma} D^{\rho} D^{\mu} + g^{\rho\sigma} D^{\mu} D^{\nu}) - (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) D^2$$

Let us denote $\bar{\mathcal{K}} := \mathcal{K}(\bar{g})$, then we have the following identity

$$\begin{aligned} \mathcal{F}_\alpha^{\nu\mu}[\bar{g}] \left(\mathfrak{g}^{\alpha\beta} \mathcal{F}_\beta^{\rho\sigma}[\bar{g}] \right) &= -\frac{1}{2} \bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \left(\frac{1}{2} - \varpi \right) (\bar{g}^{\mu\nu} \bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^\mu \bar{D}^\nu) \\ &\quad + \bar{g}^{\nu\sigma} (\bar{D}^\mu \bar{D}^\rho - \bar{D}^\rho \bar{D}^\mu) + \left(\frac{1}{2} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} + (\varpi^2 - \frac{1}{2}) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \right) \bar{D}^2 \end{aligned}$$

Here again, an implicit symmetrization in both $(\mu\nu)$ and $(\rho\sigma)$ is understood. Notice that $\bar{g}^{\nu\sigma} [\bar{D}^\mu, \bar{D}^\rho] v_{\rho\sigma} \equiv -(\bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} + \bar{R}^{\rho\mu\nu\sigma}) v_{\rho\sigma}$ holds true for any symmetric 2-tensor field.⁶ Thus, the final form of the squared gauge fixing operator assumes the form

$$\begin{aligned} \mathcal{F}_\alpha^{\nu\mu}[\bar{g}] \left(\mathfrak{g}^{\alpha\beta} \mathcal{F}_\beta^{\rho\sigma}[\bar{g}] \right) &= -\frac{1}{2} \bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \left(\frac{1}{2} - \varpi \right) (\bar{g}^{\mu\nu} \bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^\mu \bar{D}^\nu) \\ &\quad - \bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} - \bar{R}^{\rho\mu\nu\sigma} + \left(\frac{1}{2} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} + (\varpi^2 - \frac{1}{2}) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \right) \bar{D}^2 \end{aligned}$$

The first line contains uncontracted covariant derivatives while the remaining contributions are either ultra-local operators or combine to the Laplacian. Remarkably, for $\varpi = 1/2$, the harmonic gauge condition, the additional uncontracted derivatives vanish. Next, we insert this result into the metric-metric entry of the Hessian operator which is now explicitly given by

$$\begin{aligned} \mathfrak{g}^{\mu\nu B} (\widehat{\text{Hess}}_{hh})_B^{\rho\sigma} &= -(\bar{u}_k^D \bar{\lambda}_k^D) \cdot \frac{\sqrt{\bar{g}}}{\sqrt{g}} (g^{\rho\mu} g^{\sigma\nu} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu}) \\ &\quad + \bar{u}_k^{\text{gf}} \cdot (\bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} + \bar{R}^{\rho\mu\nu\sigma}) \\ &\quad + \frac{1}{2} (\bar{u}_k^D) \cdot \frac{\sqrt{\bar{g}}}{\sqrt{g}} \{ -\bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \mathcal{U}^{\mu\nu\rho\sigma} \} + \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \bar{\mathcal{K}}^{\mu\nu\rho\sigma} \\ &\quad - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (1 - 2\varpi) (\bar{g}^{\mu\nu} \bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^\mu \bar{D}^\nu) \\ &\quad - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - (1 - 2\varpi^2) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}) \bar{D}^2 \end{aligned} \quad (6.27)$$

We omit to explicitly compute the ghost sector contributions to the Hessian and retain their general form for a moment, i.e.

$$\begin{aligned} (\widehat{\text{Hess}}_{\xi\xi})_\alpha^\mu &= +\sqrt{2} \bar{u}_k^{\text{gh}} (\mathfrak{g}_{\xi\xi}^{-1})_\alpha^\lambda (\bar{D}_\lambda g_{\rho\sigma} + \bar{D}_\sigma g_{\lambda\rho} + \bar{D}_\rho g_{\lambda\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \mathcal{F}_\nu^{\rho\sigma} \bar{g}^{\mu\nu} \\ (\widehat{\text{Hess}}_{\xi\xi})_\lambda^\alpha &= -\sqrt{2} \bar{u}_k^{\text{gh}} (\mathfrak{g}_{\xi\xi}^{-1})_\mu^\alpha \left\{ \bar{g}^{\mu\nu} \mathcal{F}_\nu^{\rho\sigma} (\bar{D}_\lambda g_{\rho\sigma} + g_{\lambda\rho} \bar{D}_\sigma + g_{\lambda\sigma} \bar{D}_\rho) \right\} \end{aligned}$$

Retrospectively, this turns out to be much more convenient, since most of the terms drop out in the projection of the RHS of the FRGE.

6.7.2 First variation formula

We are now going to revisit the metric part of the stationary point solutions using the explicit form of the gauge fixing condition. We have already stated that for the effective field equations to emerge from a variational principle the following terms have to vanish:

$$\begin{aligned} 0 &\stackrel{!}{=} + \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\mu^A (\bar{g}^{\mu\nu} (v_h)_A \mathcal{F}_\nu^B (h)_B) \\ 0 &\stackrel{!}{=} + \int_{\partial M} d^{d-1} x \sqrt{H} \mathcal{F}_\nu^{\rho\sigma} (\bar{g}^{\mu\nu} \bar{\xi}_\mu) \left(n_\sigma h_{\lambda\rho} v_\xi^\lambda + n_\rho h_{\lambda\sigma} v_\xi^\lambda + n_\lambda \xi^\lambda (v_h)_{\rho\sigma} \right) \\ &\quad + \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\nu^A (\bar{\xi}_\mu \bar{g}^{\mu\nu} (\partial_\nu \mathcal{L}_\xi h)_A) \end{aligned}$$

Fortunately, in the present context $\mathcal{F}$ is chosen linear in $\bar{D}$ and thus all boundary terms contain at least one non-differentiated fluctuation field which vanishes due to Dirichlet conditions. Thus,

⁶ In general the commutator of covariant derivatives acting on a 2-tensor field $v_{\rho\sigma}$ is given by $[D_\mu, D_\nu] v_{\rho\sigma} = -R^\lambda{}_{\rho\mu\nu} v_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu} v_{\rho\lambda}$.

in particular the metric field equations are well defined. Inserting the result from the previous subsection into eq. (6.25) yields

$$\begin{aligned}
0 = & +2\sqrt{2}\bar{u}_k^{\text{gh}} \left\{ \left(\bar{D}_\sigma \xi^\alpha \delta_\rho^\beta + \bar{D}_\rho \xi^\alpha \delta_\sigma^\beta + \delta_\rho^\alpha \delta_\sigma^\beta \left(\bar{D}_\lambda \xi^\lambda + \xi^\lambda \bar{D}_\lambda \right) \right) \mathcal{F}_\nu^{\rho\sigma} \bar{g}^{\mu\nu} \right\} \bar{\xi}_\mu \\
& + \left\{ -\bar{u}_k^{\text{gf}} \left(\mathcal{F}_\mu^{\alpha\beta} \bar{g}^{\mu\nu} \mathcal{F}_\nu^A \right) h_A \right\} \\
& + \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \bar{\mathcal{K}}^{\alpha\beta\rho\sigma} h_{\rho\sigma} - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (1 - 2\varpi) \left(\bar{g}^{\alpha\beta} \bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^\alpha \bar{D}^\beta \right) h_{\rho\sigma} \\
& - \bar{u}_k^{\text{gf}} \cdot \left(\bar{g}^{\beta\sigma} \bar{R}^{\alpha\rho} + \bar{R}^{\rho\alpha\beta\sigma} \right) h_{\rho\sigma} - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \left(\bar{g}^{\alpha\rho} \bar{g}^{\beta\sigma} - (1 - 2\varpi^2) \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} \right) \bar{D}^2 h_{\rho\sigma} \\
& + \frac{\sqrt{\bar{g}}}{\sqrt{\bar{g}}} \left\{ (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot \bar{g}^{\alpha\beta} + (\bar{u}_k^{\text{D}}) \cdot \left(R^{\alpha\beta} - \frac{1}{2} \bar{g}^{\alpha\beta} R \right) \right\}
\end{aligned}$$

Again, the harmonic gauge condition is quite special for it leads to the cancellation of uncontracted derivatives in the field equations. Furthermore notice that all terms proportional to $\bar{u}_k^{\text{gf}}$ are linear in h and thus indeed vanish for self-consistent background solutions.

6.8 The level expansion

In this section we give a representation of the current state of our truncation in the level expansion. Geometrically, this corresponds to expanding the exponential map, whereby the dynamical field is expressed in terms of background- and fluctuation fields. The only functional that is defined by means of the dynamical fields is the gravitational term, thus the present discussion involves only $\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]$. Besides a more consistent description in which the arguments of all functionals agree, it also helps in clarifying the several modifications we already applied to the initial truncation ansatz. Thereby, the separation between simplifications in determining the Hessian operator and evaluating the functional trace becomes more transparent.

The Hessian operator is the central ingredient we need in evaluating the RHS of the FRGE. What remains is essentially a successive application of mathematical techniques to simplify the operator structure of the Hessian. Nevertheless, in these final steps we have to severely adapt the truncation ansatz, since the mathematical apparatus to evaluate the non-linear operator product is very limited. However, already at the present stage we have implicitly modified the initial truncation ansatz by imposing Dirichlet conditions on the fluctuation fields, use a specific class of gauge fixing conditions, and – more severally – choose a linear parametrization scheme.

The first modification actually affects theory space itself, namely the set of boundary conditions we imposed on field space. It turns out that Dirichlet boundary conditions are particular suitable in the present context, for they give rise to an strongly elliptic Laplacian and a well-defined Hessian operator. In general, we can impose different boundary conditions as long as they are consistent with the gauge condition and allow for the construction of a Hessian operator. But in any case field space has to be restricted by fixing the behavior of the fields on ∂M . For the choice of Dirichlet conditions, i.e. $h|_{\partial M} \equiv 0$, the set of boundary invariants reduces significantly, in particular the boundary volume element is identical to the background counterpart:

$$\int_{\partial M} d^{d-1}x \sqrt{H}|_{(\text{D})} \equiv \int_{\partial M} d^{d-1}x \sqrt{\bar{H}} \quad \text{since} \quad g_{\mu\nu}|_{\partial M} = \bar{g}_{\mu\nu}|_{\partial M}$$

On the other hand, the Gibbons-Hawking-York invariant contains the extrinsic curvature, a function of the derivatives of g , which in general differ from those of $\bar{g}$, i.e. $\partial_\rho g_{\mu\nu}|_{\partial M} \neq \partial_\rho \bar{g}_{\mu\nu}|_{\partial M}$. While g agrees with $\bar{g}$ on the boundary it will differ on the manifold interior and thus

$$K(g) = D_\mu n^\mu = \bar{D}_\mu n^\mu + \bar{g}^{\mu\kappa} (\partial_\rho g_{\mu\kappa} - \partial_\rho \bar{g}_{\mu\kappa}) n^\rho = \bar{K}(\bar{g}) + \bar{g}^{\mu\kappa} (\partial_\rho g_{\mu\kappa} - \partial_\rho \bar{g}_{\mu\kappa}) n^\rho$$

Hence, the Gibbons-Hawking-York field monomial based on $\bar{g}$ and g give rise to different basis invariants that can be distinguished under Dirichlet boundary conditions.

The second adaption concerns the gauge fixing functional which is now based on the class of generalized harmonic gauge conditions. However, this is mainly in view of the upcoming analysis, thus due to the trace evaluation. In order to construct a suitable Hessian operator it suffices that $\mathcal{F}$ is consistent with the boundary conditions of the fluctuation fields, that commutes with $\bar{g}$ and is linear in h .

The so far presented adaption of field space and the initial truncation ansatz, seemed natural and in fact do not alter the interpretation of the gravitational sector in terms of an Einstein-Hilbert and Gibbons-Hawking-York functional. However, for the sake of simplicity we have applied the linear parametrization $\partial_{v_h} g = v_h$, instead of a geometric one, consistent with the fiber bundle structure of field space. Nevertheless, it simply describes a different truncation ansatz compared to the geometrical expansion of $\Gamma_k^{\text{grav}}[\Phi, \bar{\Phi}]$.

Now, let us consider the level expansion based on the linear parametrization scheme. The general description reflects a Taylor expansion in the fluctuation fields and formally reads

$$\Gamma_k[\Phi, \bar{\Phi}] = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_k^{(n)}[\bar{\Phi}] (\varphi)^n \equiv \Gamma_k[\varphi; \bar{\Phi}]$$

Focusing on the gravitational functional we obtain the following structure of the level expansion up to second order in h :

$$\Gamma_k^{\text{grav}}[g, \bar{g}] \equiv \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] + (\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \frac{1}{2} (\partial_h \partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \mathcal{O}(h^3)$$

Based on the linear parametrization that we applied in the previous sections, i.e. $\partial_h g = h$, we can explicitly derive the various terms in the expansion. In the first contribution both the background invariants and the dynamical ones enter, which yields

$$\begin{aligned} \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] = & +2 \cdot \int_M d^d x \sqrt{\bar{g}} \left((\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) + (\bar{u}_k^{\text{B}} \bar{\lambda}_k^{\text{B}}) \right) \\ & - \int_M d^d x \sqrt{\bar{g}} \bar{R} ((\bar{u}_k^{\text{D}}) + (\bar{u}_k^{\text{B}})) \\ & + 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left((\bar{u}_k^{\partial \text{D}} \bar{\lambda}_k^{\partial \text{D}}) + (\bar{u}_k^{\partial \text{B}} \bar{\lambda}_k^{\partial \text{B}}) \right) \\ & - 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \bar{K}(\bar{H}) \left((\bar{u}_k^{\partial \text{D}}) + (\bar{u}_k^{\partial \text{B}}) \right) \end{aligned} \quad (6.28)$$

Notice that either $\bar{\lambda}_k^{\partial \text{D}}$ or $\bar{\lambda}_k^{\partial \text{B}}$ is superfluous, since the dynamical boundary volume element agrees with its background counterpart due to Dirichlet conditions.

Now, all the remaining terms in the expansion are entirely due to the dynamical invariants. The term linear in h comprises the gravitational contribution to the metric field equations with additional boundary terms:

$$\begin{aligned} (\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} = & + \int_M d^d x \sqrt{\bar{g}} \left\{ (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot \bar{g}^{\mu\nu} + (\bar{u}_k^{\text{D}}) \cdot (\bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R}) \right\} h_{\mu\nu} \\ & - \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left\{ (\bar{u}_k^{\text{D}} - \bar{u}_k^{\partial \text{D}}) \cdot (\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) \right\} h_{\mu\nu} \end{aligned} \quad (6.29)$$

Apparently, the coefficient of the boundary volume invariant disappears, which holds true for all higher orders in h . Hence, there is no chance to distinguish between $\bar{\lambda}_k^{\partial \text{D}}$ and $\bar{\lambda}_k^{\partial \text{B}}$ for Dirichlet conditions.

Finally, the second order in h is given by the gravitational contribution to the Hessian operator, listed in eq. (6.13). Evaluated at $v_h = w_h = h$ and for $g = \bar{g}$ it assumes the following form:

$$\begin{aligned} \frac{1}{2}(\partial_h \partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} &= \frac{1}{2} \text{Hess}_{\bar{\phi}}[\Gamma^{\text{grav}}[\Phi, \bar{\Phi}]](h, h)|_{g=\bar{g}} \\ &= \frac{1}{4} \cdot \int_M d^d x \sqrt{\bar{g}} h_{\mu\nu} \left\{ -2(\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot (\bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu}) \right. \\ &\quad \left. + (\bar{u}_k^{\text{D}}) \cdot \{ -\bar{\mathcal{K}}^{\mu\nu\rho\sigma} + \bar{\mathcal{U}}^{\mu\nu\rho\sigma} \} \right\} h_{\rho\sigma} \end{aligned} \quad (6.30)$$

In combination with the gauge fixing- and ghost functional, which are already quadratic and linear in h , respectively, this concludes the level-expansion up to second order.

In this setting the coefficients of the invariants are combinations of the original couplings, related to the background or dynamical functionals. For the level-(0) monomials we introduce the following new coefficients:

$$\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)} := (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) + (\bar{u}_k^{\text{B}} \bar{\lambda}_k^{\text{B}}) \quad \bar{u}_k^{(0)} := (\bar{u}_k^{\text{D}}) + (\bar{u}_k^{\text{B}}) \quad (6.31a)$$

$$\bar{u}_k^{\partial(0)} \bar{\lambda}_k^{\partial(0)} := (\bar{u}_k^{\partial\text{D}} \bar{\lambda}_k^{\partial\text{D}}) + (\bar{u}_k^{\partial\text{B}} \bar{\lambda}_k^{\partial\text{B}}) \quad \bar{u}_k^{\partial(0)} := (\bar{u}_k^{\partial\text{D}}) + (\bar{u}_k^{\partial\text{B}}) \quad (6.31b)$$

For the invariants of higher orders in h , there is no contribution from the background sector. This will be different if we would include mixed field monomials in the truncation ansatz, as e.g. $\sqrt{g}\bar{R}$. Thus, we have the following correspondence between the previous- and the level-coefficients:

$$\bar{u}_k^{(p)} \bar{\lambda}_k^{(p)} := (\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \quad \bar{u}_k^{(p)} := (\bar{u}_k^{\text{D}}) \quad \forall p \geq 1 \quad (6.32a)$$

$$\bar{u}_k^{\partial(p)} \bar{\lambda}_k^{\partial(p)} := (\bar{u}_k^{\partial\text{D}} \bar{\lambda}_k^{\partial\text{D}}) \quad \bar{u}_k^{\partial(p)} := (\bar{u}_k^{\partial\text{D}}) \quad \forall p \geq 1 \quad (6.32b)$$

Notice, that again due to Dirichlet conditions for the fluctuation fields, boundary monomials vanish in the higher order terms of $\Gamma_k[\varphi; \bar{\Phi}]$ and thus do not define basis directions in the field space of Dirichlet constraints.



7.1	Inversion of the Hessian operator Inversion of a matrix with non-commuting entries
7.2	Decomposition of symmetric 2-tensor fields Hodge theorem for k -forms with $\partial M \neq \emptyset$ Decomposition theorem for symmetric 2-tensors for $\partial M \neq \emptyset$ The transverse-traceless decomposition
7.3	Projection techniques for truncations Fixing the ghost sector Conformal projection technique The Hessian in the trace-traceless decomposition Projection to maximally symmetric spaces
7.4	The 'Ω deformed' $\alpha = 1$ harmonic gauge
7.5	Heat kernel Mellin transform Heat kernel expansion Tensor space traces

7. TRACE EVALUATION

This chapter describes in great detail the technique applied in order to evaluate the functional traces. While the derivation of the Hessian operator was performed for very general settings, it is in this chapter that we have to specify the truncation ansatz and employ certain modifications that ultimately lead to the difference between the preliminary version and the final form of truncated theory space.

We start in section 7.1 with the derivation of a general prescription to invert matrices of operators. In section 7.2 we present the generalization necessary to employ a decomposition of symmetric 2-tensor fields in the presence of boundaries. While we make use of only a fraction of the full decomposition, future work can make use of these results to study extended truncations.

In the next section 7.3 we actually modify the truncation ansatz by exploiting the freedom in the gauge and ghost sector. Furthermore, we describe in detail the various projection techniques that become useful when attacking the functional trace on the RHS of the flow equation.

In section 7.4 we develop a new computational strategy for deriving the explicit beta-functions which is based on the conformal projection technique and on a specific choice of the gauge sector. Thereby, we can circumvent the problematic aspects encountered when studying bi-metric truncations. The problem thereby resides in the dependence of the FRGE on *two* independent metrics. For this case there exist basically no standard tools (such as general heat kernel expansions, etc.) to evaluate the functional trace on the RHS. This is one of the reasons why, despite their obvious significance, there are still almost no results on bi-metric truncations available today. In a previous work [172] a transverse-traceless (TT) decomposition of the fluctuation field $h_{\mu\nu}$ was used, which compared to the present technique amounts to an considerable increase of computational effort.

We conclude this chapter using the heat kernel expansion in section 7.5 to project the remaining L^2 -traces onto the basis invariants of the truncation ansatz.

7.1 Inversion of the Hessian operator

In the derivation of the Hessian operator, assuming Dirichlet boundary conditions, we finally arrived at a field space operator of the following form:

$$\widehat{\text{Hess}}_{\hat{\phi}}[\Gamma_k[\hat{\phi}; \bar{\Phi}]] \equiv \begin{pmatrix} \widehat{\text{Hess}}_{hh} & \widehat{\text{Hess}}_{h\xi} & \widehat{\text{Hess}}_{h\bar{\xi}} \\ \widehat{\text{Hess}}_{\xi h} & 0 & \widehat{\text{Hess}}_{\xi\bar{\xi}} \\ \widehat{\text{Hess}}_{\bar{\xi} h} & \widehat{\text{Hess}}_{\bar{\xi}\xi} & 0 \end{pmatrix}$$

Every entry represents a L^2 transformation in the respective component on field space, thus $\widehat{\text{Hess}}_{\hat{\phi}}[\Gamma_k[\hat{\phi}; \bar{\Phi}]]$ constitutes a matrix with non-commuting entries in general. This requires a more elaborated mechanism to invert the Hessian operator necessary to determine the RHS of the FRGE, for besides an inversion on the level of its L^2 -components we have to invert its associated matrix on field space. Different to the standard case of commuting entries additional terms will appear that turns this simple procedure into a very lengthy one. However, as we will see, already some slight modification of the truncation ansatz will cure most of the occurring difficulties with only a small price to pay.

7.1.1 Inversion of a matrix with non-commuting entries

In this section we derive the matrix inverse for a symmetric $n \times n$ matrix M_n with non-commuting entries. If it exists, the defining property for an inverse is twofold, namely under matrix multiplication it should map M_n to the identity from the left and from the right:

$$(M_n)^{-1} M_n = \mathbb{1}_n, \quad M_n (M_n)^{-1} = \mathbb{1}_n$$

In particular these conditions ensure that $(M_n)^{-1}$ and M_n trivially commute.

Now, in order to determine the general form of $(M_n)^{-1}$ let us start with the simple two-dimensional case and then iteratively deduce the structure in the higher dimensional case. A generic 2×2 matrix M_2 and its inverse $(M_2)^{-1}$ are given by:

$$M_2 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \quad \text{and} \quad (M_2)^{-1} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$$

The defining conditions invoke constraints on the elements of $(M_2)^{-1}$ with respect to those of M_2 and give rise to eight relations:

$$\begin{pmatrix} \beta_1 \alpha_1 + \beta_2 \alpha_3 & \beta_1 \alpha_2 + \beta_2 \alpha_4 \\ \beta_3 \alpha_1 + \beta_4 \alpha_3 & \beta_3 \alpha_2 + \beta_4 \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 + \alpha_2 \beta_3 & \alpha_1 \beta_2 + \alpha_2 \beta_4 \\ \alpha_3 \beta_1 + \alpha_4 \beta_3 & \alpha_3 \beta_2 + \alpha_4 \beta_4 \end{pmatrix}$$

The ordering of the L^2 -operators is important and has not to be changed without care.

The following procedure now depends on what entities we assume to be invertible. For our purpose it is most practicable to assume α_1 and α_4 to be invertible also in the sense of L^2 -operators and we denote their inverse by α_1^{-1} and α_4^{-1} , respectively. Later we will see that in addition $(\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)$ has to be invertible. In the next steps, we successively express the components of $(M_2)^{-1}$ in terms of the known entries of M_2 , i.e. the non-commuting operators $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1^{-1}$, and α_4^{-1} .

- At first, we are going to solve $\beta_3 \alpha_1 + \beta_4 \alpha_3 = 0$ for β_3 , resulting in $\beta_3 = -\beta_4 \alpha_3 \alpha_1^{-1}$ and then substitute the outcome into $\beta_3 \alpha_2 + \beta_4 \alpha_4 = 1$ yielding:

$$1 = \beta_3 \alpha_2 + \beta_4 \alpha_4 (-\beta_4 \alpha_3 \alpha_1^{-1}) \alpha_2 + \beta_4 \alpha_4 = \beta_4 (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)$$

At this point we have to assume that $(\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)$ to be also invertible, as mentioned above. Thus, we arrive at the following intermediate result:

$$\beta_4 = (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1}$$

- This immediately leads to an expression for β_3 invoking the previously derived relation $\beta_3 = -\beta_4 \alpha_3 \alpha_1^{-1}$:

$$\beta_3 = -(\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1} \alpha_3 \alpha_1^{-1} = -\alpha_4 (\mathbb{1}_{\text{rk}(\alpha_4)} - \alpha_3 \alpha_1^{-1} \alpha_2 \alpha_4^{-1}) \alpha_3 \alpha_1^{-1}$$

Hereby, in the second step, we used $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices A and B . Furthermore, $\mathbb{1}_{\text{rk}(\alpha_4)}$ denotes the identity matrix of rank equal to the rank of α_4 , here identical to id_{L^2} .

- Concerning the remaining two entries, we start with the identity $\alpha_1 \beta_1 + \alpha_2 \beta_3 = 1$ and solve for β_1 . Then, inserting the previously derived result for β_3 leads to

$$\beta_1 = \alpha_1^{-1} (1 - \alpha_2 \beta_3) = \alpha_1^{-1} + \alpha_1^{-1} \alpha_2 (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1} \alpha_3 \alpha_1^{-1}$$

- Finally, we can use any of the remaining equalities involving β_2 , for instance $\alpha_1 \beta_2 + \alpha_2 \beta_4 = 0$, to complete the set of expressions for the entries in the inverse matrix $(M_2)^{-1}$. We thus find

$$\beta_2 = -\alpha_1^{-1} \alpha_2 \beta_4 = -\alpha_1^{-1} \alpha_2 (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1}$$

Notice that $(\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)$ appears in every component and its invertible character is crucial in the derivation of the final result.

Combining all these results ultimately gives rise to the full expression for $(M_2)^{-1}$ in terms of M_2 for non-commuting entries:

$$(M_2)^{-1} = \begin{pmatrix} \alpha_1^{-1} + \alpha_1^{-1} \alpha_2 (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1} \alpha_3 \alpha_1^{-1} & -\alpha_1^{-1} \alpha_2 (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1} \\ -(\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1} \alpha_3 \alpha_1^{-1} & (\alpha_4 - \alpha_3 \alpha_1^{-1} \alpha_2)^{-1} \end{pmatrix}$$

Though derivation was based on a 2×2 matrix, we actually never made any assumptions on the substructure of its entries, and in fact kept the formalism very general. Generalization from $(M_2)^{-1}$ to higher rank matrices $(M_n)^{-1}$ are thus straightforward. Identifying α_1 and α_4 to be invertible $(n-r) \times (n-r)$ and $r \times r$ matrices A and D , respectively, requires α_2 and α_3 to be of $(n-r) \times r$ or $r \times (n-r)$ rank. For this setting, we obtain the following statement: Given

$$M_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

a rank $n \geq 2$ matrix, if A , D and $(D - CA^{-1}B)$ are invertible then its inverse is given by

$$(M_n)^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \quad (7.1)$$

Notice that we do not have to concern about inverses of non-squared matrices, since our only assumptions affect the diagonal elements. For $n > 2$ one then repeat this procedure to derive the inverse for A and D , as well as of $(D - CA^{-1}B)$, which in general yield an iterative description.

Field space matrix of rank 3

While applicable for any rank matrix satisfying the above criteria, in the present context field space operators are given by 3×3 matrices. Thus, we exemplify the described procedure for a rank 3 matrix and use the result for the Hessian operator derived previously.

$$M_3 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

For later convenience we choose the following decomposition reflecting the separation of metric and ghost fluctuations and corresponding to $r = \text{rk}(D) = 2$:

$$A = \begin{pmatrix} \alpha_1 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_2 & \alpha_3 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_4 \\ \alpha_5 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha_6 & \alpha_7 \\ \alpha_8 & \alpha_9 \end{pmatrix}$$

Hereby, α_j represents L^2 -operators. The necessary conditions that have to be fulfilled in order to apply the inversion rule for non-commuting matrices affects only the entries A , D , and the combination $N \equiv (D - CA^{-1}B)$, which all have to be invertible. In fact, on the level of operators we demand α_1 , α_7 , and α_8 to have well defined inverse. Under this assumption, the inverted matrix reads

$$(M_3)^{-1} = \begin{pmatrix} \alpha_1^{-1} + A^{-1}BN^{-1}C\alpha_1^{-1} & -\alpha_1^{-1}BN^{-1} \\ -N^{-1}C\alpha_1^{-1} & N^{-1} \end{pmatrix}$$

The remaining evaluation is simple matrix multiplication except for the inversion of N which again is a matrix of non-commuting components, this time of rank 2. Since we for reasons that become apparent later on, we have chosen α_7 and α_8 to be invertible, the following identity turns out to be very useful:

$$N = \begin{pmatrix} \alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2 & \alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3 \\ \alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2 & \alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2 & \alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3 \\ \alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2 & \alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3 \end{pmatrix}$$

Since inversion is a kind of right action that acts on each matrix factor separately while reverting their order, N^{-1} can be computed stepwise. Therefore, notice that the artificial traceless matrix factor is its own inverse, i.e.

$$\mathbb{1}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{thus} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This construction has the advantage that it becomes evident that for α_1 , α_7 , and α_8 being invertible in general also N will have an inverse, namely

$$\begin{aligned} N^{-1} &= \begin{pmatrix} \alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2 & \alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3 \\ \alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2 & \alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2 & \alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3 \\ \alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2 & \alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Its explicit form can now in principle be deduce with the above prescription, now for 2×2 matrices. However, for our purpose this form is already sufficient, for we are going to trace this kind of matrix in the very end. Notice in particular that the off-diagonal entries play introduce most of the non-standard contributions for the case of non-commuting components.

The matrix trace of field space

The whole purpose for these derivations is found on RHS of the Functional Renormalization Group Equation where the trace over an inverted operator on field space has to be computed, i.e.

$$\text{STr}_F \left[\left(\widehat{\text{Hess}}_\varphi [\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right]$$

Thus, we are now going to apply the just derived results to this special case, whereby in the previous notation the field space operators are 3×3 matrices with non-commuting entries:

$$\text{STr}_F \left[(M_3)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \quad \text{with} \quad M_3 \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \widehat{\text{Hess}}_\varphi [\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}]$$

The aim of this section is the reduction of the functional trace STr_F to a mere trace over L^2 -operators STr_{L^2} . Again, we choose a convenient separation of field space operators in blocks of at most 2×2 matrices and label the operator entries according to the convention used above, i.e.

$$A = \begin{pmatrix} \alpha_1 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_2 & \alpha_3 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_4 \\ \alpha_5 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha_6 & \alpha_7 \\ \alpha_8 & \alpha_9 \end{pmatrix}$$

The cutoff operator is essentially orthogonal in its components, with the following block structure

$$\mathfrak{R}_k[\bar{\Phi}] = \begin{pmatrix} \mathfrak{R}_k^{(\alpha_1)} & 0_{1 \times 2} \\ 0_{2 \times 1} & \mathfrak{R}_k^{(D)} \end{pmatrix} \quad \text{with} \quad \mathfrak{R}_k^{(D)} = \begin{pmatrix} 0 & \mathfrak{R}_k^{(\alpha_7)} \\ \mathfrak{R}_k^{(\alpha_8)} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} \mathfrak{R}_k^{(\alpha_8)} & 0 \\ 0 & \mathfrak{R}_k^{(\alpha_7)} \end{pmatrix}$$

The last equality will turn out useful in later considerations. Now, the derivative of this operator w.r.t. the RG time t is composed with the inverse of M_3 by means of matrix multiplication. Employing the explicit form of $(M_3)^{-1}$ we obtain

$$\begin{aligned} (M_3)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] &= \begin{pmatrix} A^{-1} + A^{-1} B N^{-1} C A^{-1} & -A^{-1} B N^{-1} \\ -N^{-1} C A^{-1} & N^{-1} \end{pmatrix} \circ \begin{pmatrix} \partial_t \mathfrak{R}_k^{(\alpha_1)} & 0_{1 \times 2} \\ 0_{2 \times 1} & \partial_t \mathfrak{R}_k^{(D)} \end{pmatrix} \\ &= \begin{pmatrix} [A^{-1} + A^{-1} B N^{-1} C A^{-1}] \partial_t \mathfrak{R}_k^{(\alpha_1)} & -A^{-1} B N^{-1} \partial_t \mathfrak{R}_k^{(D)} \\ -N^{-1} C A^{-1} \partial_t \mathfrak{R}_k^{(\alpha_1)} & N^{-1} \partial_t \mathfrak{R}_k^{(D)} \end{pmatrix} \end{aligned}$$

Analog to the previous notation we have introduced an invertible matrix $N \equiv (D - C A^{-1} B)$ which is related to the determinant factor appearing in the commuting case. In the resulting matrix operator the inverse of N has to be inserted in each component which will blow up the equation even further. However, a slight reduction is obtained when taking the trace of this matrix, whereby the off-diagonal elements disappear. This is exactly the procedure we have to apply in order to evaluate the RHS of the FRGE, which in the present case gives rise to

$$\begin{aligned} \text{STr}_F \left[(M_3)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] &= \text{STr}_{1 \times 1 | L^2} \left[(A^{-1} + A^{-1} B N^{-1} C A^{-1}) \partial_t \mathfrak{R}_k^{(\alpha_1)} \right] \\ &\quad + \text{STr}_{2 \times 2 | L^2} \left[N^{-1} \partial_t \mathfrak{R}_k^{(D)} \right] \end{aligned}$$

Here, $\text{STr}_{n \times n | L^2}$ denotes the supertrace over $n \times n$ matrices with L^2 -operators as entries, thus it corresponds to a successive evaluation of matrix trace and STr_{L^2} . Notice that for this truncation

$\text{STr}_F \equiv \text{STr}_{3 \times 3|L^2}$ holds true. Hence, the first trace $\text{STr}_{1 \times 1|L^2}[\dots]$ is identical to $\text{STr}_{L^2}[\dots]$. For the second contribution, before focusing on the operator trace we have to compute the trace of the 2×2 matrix

$$\begin{aligned} N^{-1} \partial_t \mathfrak{R}_k^{(D)} &= \begin{pmatrix} \alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2 & \alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3 \\ \alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2 & \alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3 \end{pmatrix}^{-1} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \circ \begin{pmatrix} 0 & \partial_t \mathfrak{R}_k^{(\alpha_7)} \\ \partial_t \mathfrak{R}_k^{(\alpha_8)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2 & \alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3 \\ \alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2 & \alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3 \end{pmatrix}^{-1} \circ \begin{pmatrix} \partial_t \mathfrak{R}_k^{(\alpha_8)} & 0 \\ 0 & \partial_t \mathfrak{R}_k^{(\alpha_7)} \end{pmatrix} \end{aligned} \quad (7.2)$$

Due to the diagonal character of the remaining matrix multiplication and in view of the trace evaluation, we only have to derive the diagonal elements of the inverse matrix. Proceeding along the prescription for non-commuting matrices, we thus arrive at

$$\begin{aligned} \text{STr}_{2 \times 2|L^2} \left[N^{-1} \partial_t \mathfrak{R}_k^{(D)} \right] &= \text{STr}_{L^2} \left[\left(\gamma_\xi^{-1} + \gamma_\xi^{-1} (\alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3) \gamma_\xi^{-1} (\alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2) \gamma_\xi^{-1} \right) \partial_t \mathfrak{R}_k^{(\alpha_8)} \right] \\ &\quad + \text{STr}_{L^2} \left[\gamma_\xi^{-1} \partial_t \mathfrak{R}_k^{(\alpha_7)} \right] \end{aligned} \quad (7.3)$$

For this inversion to hold we need two more assumptions, namely that both of the following operators are invertible:

$$\gamma_\xi = (\alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2) \quad (7.4a)$$

$$\gamma_\xi = (\alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3) - (\alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2) \gamma_\xi^{-1} (\alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3) \quad (7.4b)$$

Combining this result with the previous one, finally yields the reduction of the functional trace on field space to operator traces:

$$\begin{aligned} \text{STr}_F \left[(M_3)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] &= \text{STr}_{L^2}^{[h]} \left[(\alpha_1^{-1} + \alpha_1^{-1} B N^{-1} C A^{-1}) \partial_t \mathfrak{R}_k^{(\alpha_1)} \right] + \text{STr}_{L^2}^{[\xi]} \left[\gamma_\xi^{-1} \partial_t \mathfrak{R}_k^{(\alpha_7)} \right] \\ &\quad + \text{STr}_{L^2}^{[\xi]} \left[\left(\gamma_\xi^{-1} + \gamma_\xi^{-1} (\alpha_9 - \alpha_5 \alpha_1^{-1} \alpha_3) \gamma_\xi^{-1} (\alpha_6 - \alpha_4 \alpha_1^{-1} \alpha_2) \gamma_\xi^{-1} \right) \partial_t \mathfrak{R}_k^{(\alpha_8)} \right] \end{aligned} \quad (7.5)$$

Here, we introduced an additional superscript for the L^2 -traces indicating over which part of field space it has to be evaluated. However, due to the domain and target spaces associated to each operator this is a somehow redundant information, nevertheless it might be helpful.

Actually, for the considered truncation, equation (7.5) can be further simplified due to the vanishing of the diagonal terms in the ghost sector of the Hessian and the cutoff operator. The sum of both operators defines M_3 which for this special case assumes the form

$$M_3 \equiv \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & 0 & \alpha_7 \\ \alpha_5 & \alpha_8 & 0 \end{pmatrix} \equiv \begin{pmatrix} \widehat{\text{Hess}}_{hh} + \mathfrak{R}_k^{\alpha_1} & \widehat{\text{Hess}}_{h\xi} & \widehat{\text{Hess}}_{h\bar{\xi}} \\ \widehat{\text{Hess}}_{\xi h} & 0 & \widehat{\text{Hess}}_{\xi\bar{\xi}} + \mathfrak{R}_k^{\alpha_7} \\ \widehat{\text{Hess}}_{\bar{\xi} h} & \widehat{\text{Hess}}_{\bar{\xi}\xi} + \mathfrak{R}_k^{\alpha_8} & 0 \end{pmatrix}$$

Thus, in all derived solutions we can set α_6 and α_9 to zero. This implies the following reduced form of eq. (7.4) for the functional trace of M_3 :

$$\begin{aligned} \text{STr}_F \left[(M_3)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] &= \text{STr}_{L^2}^{[h]} \left[(\alpha_1^{-1} + \alpha_1^{-1} B N^{-1} C A^{-1}) \partial_t \mathfrak{R}_k^{(\alpha_1)} \right] + \text{STr}_{L^2}^{[\xi]} \left[\gamma_\xi^{-1} \partial_t \mathfrak{R}_k^{(\alpha_7)} \right] \\ &\quad + \text{STr}_{L^2}^{[\xi]} \left[\left(\gamma_\xi^{-1} + \gamma_\xi^{-1} (\alpha_5 \alpha_1^{-1} \alpha_3) \gamma_\xi^{-1} (\alpha_4 \alpha_1^{-1} \alpha_2) \gamma_\xi^{-1} \right) \partial_t \mathfrak{R}_k^{(\alpha_8)} \right] \end{aligned} \quad (7.6)$$

Notice that also the abbreviated operators γ_ξ and γ_ξ^{-1} simplify, for they explicitly depend on the diagonal elements of the ghost block. Their reduced expression is given by

$$\gamma_\xi = (\alpha_8 - \alpha_5 \alpha_1^{-1} \alpha_2) \quad (7.7a)$$

$$\gamma_\xi^{-1} = (\alpha_7 - \alpha_4 \alpha_1^{-1} \alpha_3) - (\alpha_4 \alpha_1^{-1} \alpha_2) \gamma_\xi^{-1} (\alpha_5 \alpha_1^{-1} \alpha_3) \quad (7.7b)$$

Nevertheless, this seems more to be a drop in the ocean than a true simplification. The entire expression still consists of several products of operators, some of them being inverted, and there is thus quite a lot of effort we have to put into the evaluation of the L^2 -traces.

7.2 Decomposition of symmetric 2-tensor fields

Group theoretical methods are found in various branches of physics and usually come along with severe simplifications even in very practical applications. Most often the applied techniques have some group theoretical background which however is hidden in the details of the method. The current status of analyzing the RHS of the FRGE for our specific truncation ansatz consists of a set of L^2 -operators acting on the space of either metric or ghost type tensor fields. While the ghost sector is unconstrained, the metric component of field space is a subset of symmetric 2-tensor fields fulfilling the non-linear condition of being non-degenerate. For the fluctuation fields this additional requirement drops out and what remains is an element of the symmetric subset of $T^{(0,2)}M$. The aim of this section is a decomposition of these tensor fields such that the action of the Hessian operator becomes of Laplace type. This technique goes back to works of Berger and Ebin [176], as well as York [177] and is intrinsically linked with differential geometry and group theory. Both combine in the construction of fiber bundles, here explicitly given by tensor bundles. Furthermore, we consider only operators that arise from a scalar functional by means of second variation. This implies that the symmetries of the fields and the invariance of the Γ_k decide about the variety of allowed operators we may encounter. We start with the decomposition of a symmetric 2-tensor field based on the irreducible subspaces of $Gl(d; \mathbb{R})$. Since we have a metric tensor at our disposal, we can further split the resulting components using the restricted group action of $O(d)$ and $SO(d)$, for which in particular contraction is defined. This principle is first applied to a generic k -form in the presence of boundaries and then presented in detail for the symmetric part of a 2-tensor field. The result is ultimately implemented into our current form of the operator trace. For more details, references, and derivations, as well as a mathematical discussion we refer to [178, 179] for the Hodge theorem and [176, 177, 180–182] for the symmetric tensor field decomposition, respectively.

As a preliminary remark, let us consider a generic tensor field $t \in T^{(p,q)}M$ which by definition has at most d^{p+q} independent components. Imposing symmetry restrictions or other constraints will in general reduce this number. In the course of this section we will see how certain symmetry conditions affect the overall structure of tensor fields. For instance, choosing $p = 0$ and $q = 2$, we showed in subsection 1.2.4 that an arbitrary $t \in T^{(0,2)}M$ consist of two irreducible components w.r.t. the general linear group $Gl(d; \mathbb{R})$: a symmetric and an anti-symmetric part, thus

$$t_{\mu\nu} = t_{\mu\nu}^S + t_{\mu\nu}^A \quad \text{with} \quad t_{\mu\nu}^S = t_{\nu\mu}^S \quad \wedge \quad t_{\mu\nu}^A = -t_{\nu\mu}^A \quad (7.8)$$

We thus replaced an unconstrained generic tensor field by two constrained tensor fields of the same degree, whereby the total number of independent components is unchanged. This becomes in particular useful if we consider transformations of t , for instance a symmetric, linear transformation $\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$:

$$\hat{T}^{\mu\nu} t_{\mu\nu} = \hat{T}^{\mu\nu} t_{\mu\nu}^S + \hat{T}^{\mu\nu} t_{\mu\nu}^A \equiv \hat{T}^{\mu\nu} t_{\mu\nu}^S$$

In the last step the main advantage of the decomposition method becomes apparent: the anti-symmetric component vanishes under the action of a symmetric transformation. Any further analysis based on this contraction involves only the independent components of $t_{\mu\nu}^S$, thus we have a reduction from d^2 to $d(d+1)/2$ independent quantities. This suggests that a decision about the most efficient decomposition is very sensitive on the symmetry properties of the associated operator. Hence most of the decomposition theorems rely on a specific setting as for instance Riemannian geometry with its natural differential operators.

In the sequel we describe a systematic decomposition method built on the covariant derivative $\bar{D}$ for a smooth, connected and compact Riemannian manifold with in general non-vanishing boundary. Its original formulation was given for manifolds without boundary [176], though generalizations for ∂M are known since decades, for instance [181]. Given a scalar product on the tensor algebra we can uniquely define the transverse and longitudinal part of a tensor field:

Definition 7.2.1 — Transverse & longitudinal tensors. Let M be a smooth manifold and $T^{(p,q)}M$ the space of tensor fields of type (p,q) . Furthermore, let $(\cdot, \cdot)_{T^{(p,q)}M} : T^{(p,q)}M \times T^{(p,q)}M \rightarrow \mathbb{R}$ be the inner product on $T^{(p,q)}M$.

A tensor field $t \in T^{(p,q)}M$, in a local chart given by $t_{\alpha_1 \dots \alpha_q}^{\beta_1 \dots \beta_p}(x)$, is called ...

$$\begin{aligned} \dots \text{transverse in } \alpha_j (\beta_j) &: \Leftrightarrow \bar{D}^\mu t_{\alpha_1 \dots \mu \dots \alpha_q}^{\beta_1 \dots \beta_j \dots \beta_p}(x) = 0 \text{ resp. } \bar{D}_\mu t_{\alpha_1 \dots \alpha_j \dots \alpha_q}^{\beta_1 \dots \mu \dots \beta_p}(x) = 0 \\ \dots \text{longitudinal in } \alpha_j (\beta_j) &: \Leftrightarrow \forall v \in T^{(p,q)}M \text{ transverse in } \alpha_j (\beta_j) : \\ &\quad (v, t)_{T^{(p,q)}M} = 0 \end{aligned}$$

The longitudinal tensor fields form the orthogonal complement of the transverse sector.

A simple example is given by tensors of type $(1,0)$, i.e. the set of vector fields, which can be decomposed as $t = (t^T) + (t^\perp)$ with $\bar{D}_\mu (t^T)^\mu = 0$ and $\bar{g}(t^T, t^\perp) = 0$. Orthogonality will turn out to be very useful due to the decoupling of the different contributions in the decomposition.

7.2.1 Hodge theorem for k -forms with $\partial M \neq \emptyset$

The general procedure to decompose tensor fields – and thus also suitable for its fully anti-symmetric subsets the k -forms – is based on the kernel of a certain differential operator. As mentioned above we are going to employ a Hodge-decomposition associated to the covariant derivative. In order to keep the notation as general as possible, we introduce a new symbol d , thus one can replace $\bar{D}$ with different operators as long as they share similar properties. Furthermore, in the present context it is worth emphasizing the degree of differential k -form by writing $\omega \equiv \overset{k}{\omega} \in \Lambda^k(M)$. Using the Riemannian metric $\bar{g}$ as the generator of the inner products on $T^{(p,q)}M$ we define

$$d : \Lambda^k(M) \rightarrow \Lambda^{k+1}(M) \quad \text{with} \quad d \overset{k}{\omega} = \bar{D}_\bullet \overset{k}{\omega} \in \Lambda^{k+1}(M) \quad \forall \overset{k}{\omega} \in \Lambda^k(M)$$

Equipped with an inner product, there is an associated – in a sense reverted – adjoint operator which is also known as co-differential:

$$\delta : \Lambda^k(M) \rightarrow \Lambda^{k-1}(M) \quad \text{with} \quad (\overset{k}{\omega}, d \overset{k-1}{v})_{T^{(0,k)}M} \equiv (\delta \overset{k}{\omega}, \overset{k-1}{v})_{T^{(0,k-1)}M}$$

The properties of the inner product can be absorbed in what is called the Hodge-star operator:

$$(\star \overset{k}{\omega})_{i_1, \dots, i_{d-k}} \equiv \frac{1}{k!} \overset{k}{\omega}^{j_1, \dots, j_k} \sqrt{\bar{g}} \varepsilon_{j_1, \dots, j_k, i_1, \dots, i_{d-k}}$$

Here, we have introduced the fully anti-symmetric epsilon tensor ε representing the Young operator of the anti-symmetric component. This allows for a more convenient expression of the co-differential δ in terms of $\star$ and d :

$$\delta \stackrel{k}{\omega} \equiv (-1)^{d(k+1)} (\star d \star) \stackrel{k}{\omega}$$

Hodge theorem

In case of manifolds without boundary the 'Hodge theorem' provides a suitable decomposition of k -forms into independent constituents based on the image and kernels of d and the co-differential δ :

$$\Lambda^k(M) = \ker(d\delta + \delta d) \oplus d\Lambda^{k-1}(M) \oplus \delta\Lambda^{k+1}(M) \quad (\text{Hodge decomposition}) \quad (7.9)$$

Any k -form can thus be expressed as a sum of some transverse and longitudinal terms, whereby the first summand represents a harmonic k -form:

$$\text{Harm}^k \equiv \ker(\Delta \equiv d\delta + \delta d) \equiv \left\{ \stackrel{k}{\omega} \mid (d\delta + \delta d) \stackrel{k}{\omega} = 0 \right\}$$

Here, $\Delta \equiv d\delta + \delta d$ defines the usual Laplacian operator, which is elliptic in case of Dirichlet boundary conditions.

Hodge-Morrey theorem

The Hodge-decomposition is a very useful result which finds application in many branches of mathematics and physics, hence generalizations to manifolds with non-vanishing boundary were of great interest. However, in this latter case neither $\delta = d^{**}$ holds true nor are the split components orthogonal, due to the appearance of boundary contributions when integrating by parts. Fortunately, it is sufficient to slightly change the former decomposition and exchange the set of harmonic k -forms with the set of harmonic k -fields, introduced by Kodaira [183]:¹ For $\partial M \neq \emptyset$ boundary terms will in general spoil this identity.

$$\mathcal{H}^k \equiv \left\{ \stackrel{k}{\omega} \mid d \stackrel{k}{\omega} = \delta \stackrel{k}{\omega} = 0 \right\}$$

In addition to this replacement one has to implement boundary conditions on the k -forms which are either of Dirichlet or Neumann type, respectively denoted as

$$\Lambda_{(D)}^k(M) \equiv \left\{ \stackrel{k}{\omega} \mid \stackrel{k}{\omega} \big|_{\partial M} = 0 \right\} \quad \text{and} \quad \Lambda_{(N)}^k(M) \equiv \left\{ \stackrel{k}{\omega} \mid \left(\star \stackrel{k}{\omega} \right) \big|_{\partial M} = 0 \right\}$$

These modification give rise to the 'Hodge-Morrey decomposition', the generalization of eq. (7.9) for the case of manifolds with non-vanishing boundary:

$$\Lambda^k(M) = \mathcal{H}^k(M) \oplus d\Lambda_{(D)}^{k-1}(M) \oplus \delta\Lambda_{(N)}^{k+1}(M) \quad (\text{Hodge-Morrey decomposition}) \quad (7.10)$$

The boundary conditions ensure the orthogonality of this splitting, necessary for a suitable decomposition of the underlying vector space. Notice, if the boundary of M happens to be empty, $\partial M = \emptyset$, the Hodge-Morrey theorem reduces to eq. (7.9), as required.

¹Notice that the harmonic fields are a subset of harmonic forms, i.e. $\mathcal{H}^k \subseteq \text{Harm}^k$. In case of vanishing boundary, $\partial M = \emptyset$, in fact equality holds which can be seen by

$$0 = \left(\stackrel{k}{\omega}, (d\delta + \delta d) \stackrel{k}{\omega} \right) = \left(d \stackrel{k}{\omega}, d \stackrel{k}{\omega} \right) + \left(\delta \stackrel{k}{\omega}, \delta \stackrel{k}{\omega} \right) = \|d \stackrel{k}{\omega}\|^2 + \|\delta \stackrel{k}{\omega}\|^2 \Rightarrow d \stackrel{k}{\omega} = \delta \stackrel{k}{\omega} = 0.$$

Hodge-Morrey decomposition for 1-forms

For the final splitting of symmetric 2-tensor fields we will only need the explicit decomposition of dual vectors, i.e. 1-forms. From the Hodge-Morrey theorem we can deduce the following mutually orthogonal decomposition with respect to the above defined inner product:

$$\Lambda^1(M) = \mathcal{H}^1(M) \oplus d\Lambda_{(D)}^0(M) \oplus \delta\Lambda_{(N)}^2(M)$$

The first constituent of a dual vector is the harmonic field component, i.e. an element of $\mathcal{H}^1(M)$ satisfying the closeness ($d \overset{1}{v} = 0$) and co-closeness condition ($\delta \overset{1}{v} = 0$). The next constituent $\Lambda_{(D)}^0(M)$ is equivalent to the space of continuous differentiable functions which in addition satisfy Dirichlet-boundary conditions. Its contribution to the orthogonal decomposition is locally given by $\bar{D}_\mu \overset{0}{\sigma} dx^\mu$. Finally, the last ingredient is constructed by the application of the co-differential on a 2-form respecting Neumann conditions on ∂M , i.e.

$$\delta \overset{2}{\eta} \equiv (-1)^d (\star d \star) \overset{2}{\eta} = (-1)^d \bar{D}^\nu \overset{2}{\eta}_{\mu\nu} dx^\mu$$

To obtain the second equality one actually has to determine the explicit form of the co-differential acting on $\Lambda_{(N)}^2(M)$. Therefore, we computation rules for the epsilon tensor are used to arrive at $(\delta \overset{2}{\eta})_\mu = (-1)^d \bar{D}^\nu \overset{2}{\eta}_{\mu\nu}$.

Due to the Hodge-Morrey theorem we can thus represent any generic 1-form as a sum of three constrained constituents

$$\overset{1}{\omega} = \overset{1}{\omega}_\mu dx^\mu = \left(\overset{1}{v}_\mu + \bar{D}_\mu \overset{0}{\sigma} + (-1)^d \bar{D}^\nu \overset{2}{\eta}_{\mu\nu} \right) dx^\mu \quad (7.11)$$

While the first term underlies the harmonic field constraint, the remaining parts only fulfill boundary conditions, i.e.

$$d \overset{1}{v} = \bar{D}_\nu \overset{1}{v}_\mu = 0 = \delta \overset{1}{v} = \bar{D}^\mu \overset{1}{v}_\mu \quad \wedge \quad \overset{0}{\sigma}|_{\partial M} = 0 \quad \wedge \quad \bar{D}^\nu \overset{2}{\eta}_{\mu\nu}|_{\partial M} = 0 \quad (7.12)$$

In the sequel, a simplified notation involving only two constituents will be sufficient. Therefore, let us combine the harmonic field and the image of the co-differential into a new – kind of longitudinal – term $\overset{1}{\xi} = \overset{1}{v} + \delta \overset{2}{\eta}$:

$$\overset{1}{\omega} = \left(\overset{1}{\xi}_\mu + \bar{D}_\mu \overset{0}{\sigma} \right) dx^\mu \quad \text{with} \quad \delta \overset{1}{\xi} = \delta \overset{1}{v} + \delta \circ \delta \overset{2}{\eta} = 0 = \bar{D}^\mu \overset{1}{\xi}_\mu \quad \wedge \quad \sigma|_{\partial M} = 0 \quad (7.13)$$

Since $\overset{1}{\xi}_\mu|_{\partial M} = (\overset{1}{v}_\mu + \bar{D}^\nu \overset{2}{\eta}_{\mu\nu})|_{\partial M} = \overset{1}{v}_\mu|_{\partial M}$ holds true, the new field $\overset{1}{\xi}$ is unconstrained on the boundary of M .

7.2.2 Decomposition theorem for symmetric 2-tensors for $\partial M \neq \emptyset$

At the very beginning of this section, we have stated a result from group theory that a generic 2-tensor field $t \in T^{(0,2)}M$ can be partitioned into its irreducible components w.r.t. $Gl(d; \mathbb{R})$, namely $t_{\mu\nu} = t_{\mu\nu}^S + t_{\mu\nu}^A$, or in more general terms:

$$T^{(0,2)}M \equiv \text{Sym}^2(M) + \Lambda^2(M) \quad \text{with} \quad \text{Sym}^2(M) \equiv \left\{ s \in T^{(0,2)}M \mid s_{\mu\nu} = s_{\nu\mu} \right\} \quad (7.14)$$

This is a first, however in general not orthogonal, decomposition based on the general linear group. Once an inner product is at our disposal we can further decompose the symmetric and anti-symmetric tensor field looking for the irreducible parts of the restricted representations

for the orthogonal group $O(d)$ which resulted in the famous Hodge-Morrey theorem for the anti-symmetric tensor $t^A \in \Lambda^2(M)$.

In the sequel of this section we present a similar orthogonal decomposition for $\text{Sym}^2(M)$, whereby the previous results for k -forms turn out to be very useful. Instead of directly introducing the operator $d_{(s)}$ that provides the decomposition scheme for $\text{Sym}^2(M)$, we take a look at the more intuitive co-differential, or adjoint operator. It represents the counterpart of the fully anti-symmetric $\delta \overset{1}{\omega}$ in that we have to replace the Young operator of $Gl(d; \mathbb{R})$ with its symmetric analog yielding

$$d_{(s)}^* \overset{1}{\omega} \equiv 2 \left(\bar{D}_\mu \overset{1}{\omega}_\nu + \bar{D}_\nu \overset{1}{\omega}_\mu \right) dx^\mu \otimes dx^\nu = \mathcal{L}_\bullet \overset{1}{\omega} \in \text{Sym}^2(M)$$

Remarkably, it is the Lie derivative of a 1-form that generates a symmetric tensor field. Using the inner product related to the field space metric, the associated differential operator reads

$$d_{(s)} s \equiv dx^\beta \left(-2g^{\rho\sigma\mu\nu} \bar{g}_{\beta\nu} \bar{D}_\mu \right) s_{\rho\sigma} \equiv -2 \left(\bar{D}^\rho s_{\rho\beta} + \bar{\omega} \bar{D}_\beta \bar{g}^{\rho\sigma} s_{\rho\sigma} \right) dx^\beta$$

Since the manifold is endowed with a boundary, it will in general be necessary to impose boundary conditions in order to establish the close bond of adjointness between $d_{(s)}$ and $d_{(s)}^*$. In particular Dirichlet constraints suffices to get rid of the boundary terms that occur when integrating by parts.

For this natural choice of operators there exists a Hodge-Morrey like theorem for $\text{Sym}^2(M)$: Again, let us assume M to be a compact Riemannian manifold with regular, orientable boundary $\partial M \neq \emptyset$. Then, the decomposition with respect to $d_{(s)}$ for symmetric 2-tensor fields satisfying Dirichlet boundary conditions is given by:

$$\text{Sym}_{(D)}^2(M) = d_{(s)}^* \Lambda_{(D)}^1(M) \oplus \ker(d_{(s)}) \quad (7.15)$$

Thus, based on this theorem a generic symmetric 2-tensor field $s \in \text{Sym}_{(D)}^2(M)$ that vanishes on ∂M can be decomposed as follows

$$s = s_{\mu\nu} dx^\mu \otimes dx^\nu = \left(\bar{D}_\mu \omega_\nu + \bar{D}_\nu \omega_\mu + z_{\mu\nu} \right) dx^\mu \otimes dx^\nu$$

Hereby the left-hand-side (LHS) is solely constrained to be symmetric and of Dirichlet type, while the RHS contains tensor fields obeying additional conditions, namely

$$z_{\mu\nu} = z_{\nu\mu} \quad \wedge \quad (g^{\rho\sigma\mu\nu} \bar{g}_{\beta\nu} \bar{D}_\mu) z_{\rho\sigma} = 0 \quad \wedge \quad \overset{1}{\omega}|_{\partial M} = 0$$

Finally, we can apply the Hodge-Morrey decomposition (7.13) to $\overset{1}{\omega}$ which yields a representation based on two more restricted tensor fields ξ and σ . Due to the identity $\bar{D}_\mu \bar{D}_\nu \sigma = \bar{D}_\nu \bar{D}_\mu \sigma$ for scalar fields this gives rise to

$$s = s_{\mu\nu} dx^\mu \otimes dx^\nu = \left(\bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu + 2\bar{D}_\mu \bar{D}_\nu \sigma + z_{\mu\nu} \right) dx^\mu \otimes dx^\nu \quad (7.16)$$

Instead of a single symmetric 2-tensor field s we have thus a total number of three constrained tensors of different degree, each carrying a certain aspect of s . In detail, we have the following list of conditions that have to be satisfied by z , $\overset{1}{\xi}$ and $\overset{0}{\sigma}$:

$$\begin{aligned} \delta \overset{1}{\xi} = \bar{D}^\mu \xi_\mu = 0 & \quad \wedge \quad \xi_\nu|_{\partial M} = -\bar{D}_\nu \sigma|_{\partial M} & \quad \wedge \quad \sigma|_{\partial M} = 0 \\ z_{\mu\nu} = z_{\nu\mu} & \quad \wedge \quad (g^{\rho\sigma\mu\nu} \bar{g}_{\beta\nu} \bar{D}_\mu) z_{\rho\sigma} = 0 \end{aligned} \quad (7.17)$$

Notice that, if necessary, $\overset{1}{\xi}$ can be further orthogonally decomposed as $\overset{1}{\xi} = \overset{1}{\psi} + \delta \overset{2}{\eta}$.

7.2.3 The transverse-traceless decomposition

There are several ways to reduce the tensor structure of the Hessian to some simple scalar operator, one using a full decomposition of the metric fluctuation as presented in eq. (7.16). However, this projection technique implies a whole sequence of traces over constrained fields which then have to be individually evaluated using different heat kernel expansions. Later on we are going to introduce another method that at least in the present truncation simplifies the trace evaluation a lot. Basically, it employs not the full metric field decomposition, but only a split into trace and traceless part, i.e.

$$s_{\mu\nu} \equiv s_{\mu\nu}^{\text{noTr}} + \frac{1}{d} \bar{g}_{\mu\nu} s^{\text{Tr}} \quad (7.18)$$

whereby $s^{\text{Tr}} = b g^{\mu\nu} s_{\mu\nu}$ denotes the trace part of s and s^{noTr} the remaining traceless component, satisfying $\bar{g}^{\mu\nu} s_{\mu\nu}^{\text{noTr}} = 0$. It is also very instructive to combine both decompositions which results in the transverse-traceless decomposition. Therefore notice that from eq. (7.16) using the constraints of (7.17), we express the trace part of s in terms of the tensor fields z , ξ and σ , i.e.

$$s^{\text{Tr}} = \bar{g}^{\mu\nu} s_{\mu\nu}^{\text{Tr}} = 2\bar{D}^\mu \xi_\mu + 2\bar{D}^2 \sigma + \bar{g}^{\mu\nu} z_{\mu\nu} = 2\bar{D}^2 \sigma + z^{\text{Tr}}$$

In the second step we made use of the transverse property of ξ , i.e. $\bar{D}^\mu \xi_\mu = 0$. Hence, the traceless part assumes the form

$$\begin{aligned} s_{\mu\nu}^{\text{noTr}} &= s_{\mu\nu} - \frac{1}{d} \bar{g}^{\mu\nu} s^{\text{Tr}} = \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu + 2\bar{D}_\mu \bar{D}_\nu \sigma + z_{\mu\nu} - \frac{1}{d} \bar{g}_{\mu\nu} (2\bar{D}^2 \sigma + z^{\text{Tr}}) \\ &= \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu + 2(g_{\rho\mu} g_{\nu\sigma} - \frac{1}{d} g_{\mu\nu} g_{\rho\sigma}) \bar{D}^\rho \bar{D}^\lambda \sigma + (g_{\rho\mu} g_{\nu\sigma} - \frac{1}{d} g_{\mu\nu} g_{\rho\sigma}) z^{\rho\lambda} \end{aligned}$$

Remarkably, the brackets in the last two terms are reminiscent of the local field space metric, $g^{\mu\nu\rho\lambda}$, for $\varpi = -1/d$. Whenever this matching holds true, z^{Tr} is not only traceless but in fact also transverse as is $(\bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu)$ however for any choice of ϖ .

In the present case, we follow a path that circumvents the full decomposition of the tensor fields by using the freedom of the gauge fixing condition and only split the metric fluctuations into its trace- and traceless part. However, this new, time-saving technique might be suitable only for a class of truncations, in particular it might be limited to an ansatz with a sufficient small number of derivatives in the invariants. Thus, further investigations for manifolds with non-vanishing boundary may rely on the full decomposition procedure presented here.

7.3 Projection techniques for truncations

The Functional Renormalization Group Equation combines the difficulties of functional differential equations with the problem of non-linearity:

$$\partial_t \Gamma_k[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi[\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \quad (7.19)$$

Exact as it stands, the usual employed simplifications are based on a projection of theory space $\mathcal{T}$ to its coordinatized version, making use of a full set of independent basis invariants $\{\mathcal{P}_a[\varphi; \bar{\Phi}]\}$. Formally, an action functional living on theory space $\mathcal{T}$ can be expanded in this basis, whereby the coefficients are associated to combinations of couplings:

$$\mathcal{T} \ni A[\varphi; \bar{\Phi}] \equiv \sum_a \bar{u}^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

At least on a formal level, we can use this expansion to rewrite the single functional differential equation in terms of an infinite number of ordinary differential equations for the coefficients. Therefore, we insert the expanded form of the effective average action, $\Gamma_k[\varphi; \bar{\Phi}] \equiv \sum_a \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$, into the FRGE.

$$\text{LHS : } \quad \partial_t \Gamma_k[\varphi; \bar{\Phi}] = \partial_t \left(\sum_a \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right) = \sum_a (\partial_t \bar{u}_k^a) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

Furthermore, since we assume $\{\mathcal{P}_a[\hat{\varphi}; \bar{\Phi}]\}$ to form a complete, linearly independent basis, the RHS of eq. (7.19) can be projected onto the basis invariants $\{\mathcal{P}_a[\hat{\varphi}; \bar{\Phi}]\}$ in a unique way, which defines the dimensionful beta-functions $\bar{\beta}$:

$$\text{RHS : } \quad \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi [\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{K}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{K}_k[\bar{\Phi}] \right] =: \sum_a \bar{\beta}^a(\{\bar{u}_k^b\}; k) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

In order to obtain the differential equations for the (dimensionful) coefficients, we have to combine both sides yielding

$$\begin{aligned} \sum_a (\partial_t \bar{u}_k^a) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] &= \sum_a \bar{\beta}^a(\{\bar{u}_k^b\}; k) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \quad \Longleftrightarrow \\ \sum_a \left(\partial_t \bar{u}_k^a - \bar{\beta}^a(\{\bar{u}_k^b\}; k) \right) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] &= 0 \end{aligned}$$

Since the $\mathcal{P}$'s are linearly independent, this implies that each bracket has to vanish identically. This is the aforementioned infinite number of differential equations for the $\bar{u}$'s, each describing the RG evolution along a certain direction in theory space, associated to a specific basis invariant.

$$\partial_t \bar{u}_k^a = \bar{\beta}^a(\{\bar{u}_k^b\}; k) \quad \forall a$$

Due to the second functional variation that constitutes the operator on the RHS, the resulting differential equations are in general coupled in an hierarchical order, which is indicated by the explicit dependence of $\bar{\beta}$ on the set of coefficients. Furthermore, due to the connection between the cutoff operator and the background metric, there is an additional contribution in the dimensionful version of the EAA which gives rise to a non-autonomous system of differential equations, thus the explicit dependence of $\bar{\beta}$ on k .

In this way, we get rid of the functional nature of the non-linear differential equation, however we are now plagued by the problem to simultaneously solve an infinite number of coupled ordinary differential equations. As long as we do not have a systematic technique to, for instance iteratively, attack this mathematical problem, we have to rely on approximations where only a (possibly still infinite) subset of field monomials is considered. However, such a truncated theory space is in general not invariant under the RG flow, since the non-linearity together with the second variation of the basis invariants will lead to a generation of terms on the RHS which are not present in the original ansatz for Γ_k . Nevertheless, once we consider a subset which exhibits an additional symmetry, it might happen that the FRGE preserves this symmetry and the RG evolution closes for the most general truncation ansatz of this kind.

To get a better understanding of the approximation involved when truncating theory space, let us introduce a projection operator

$$\Pi_{\mathcal{P}_b}(A) \equiv \Pi_{\mathcal{P}_b} \left(\sum_a \bar{u}^a \cdot \mathcal{P}_a \right) := \bar{u}^b \cdot \mathcal{P}_b$$

Surely, if it exists, $\Pi_{\mathcal{P}_b}$ is idempotent and thus in fact a projection. The crucial point here is the infinite dimensional character of theory space. Assume that we can find a suitable linearly independent basis of field monomials on $\mathcal{T}$, in the general case this space will not be endowed with an inner product and projection operators are difficult to obtain. Still, let us proceed on a formal level and apply $\Pi_{\mathcal{P}_b}$ to the FRGE:

$$\Pi_{\mathcal{P}_b}(\partial_t \Gamma_k[\varphi; \bar{\Phi}]) = \frac{1}{2} \Pi_{\mathcal{P}_b} \left(\text{STr} \left[\left(\widehat{\text{Hess}}_{\varphi} [\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \right)$$

Notice that the basis invariants are k -independent in the dimensionful description. Thus, ∂_t and $\Pi_{\mathcal{P}_b}$ commute and we can change the order on the LHS. Again, we insert the coordinatized version of the effective average action $\Gamma_k[\varphi; \bar{\Phi}] = \sum_a \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$, now on both sides, yielding

$$(\partial_t \bar{u}_k^b) \cdot \mathcal{P}_b[\varphi; \bar{\Phi}] = \frac{1}{2} \Pi_{\mathcal{P}_b} \left(\text{STr} \left[\left(\widehat{\text{Hess}}_{\varphi} \left[\sum_a \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \right)$$

The supertrace gives rise to the dimensionful beta-functions and thus the application of $\Pi_{\mathcal{P}_b}$ reproduces the differential equation for the coefficient $\bar{u}_k^b$:

$$(\partial_t \bar{u}_k^b) \cdot \mathcal{P}_b[\varphi; \bar{\Phi}] = \Pi_{\mathcal{P}_b} \left(\sum_a \bar{\beta}^a(\{\bar{u}_k^b\}; k) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right) = \beta^b(\{\bar{u}_k^b\}; k) \cdot \mathcal{P}_b[\varphi; \bar{\Phi}]$$

Hence, even if we are only interested in the RG evolution of a single basis invariant, we still have to compute the Hessian of a generic action functional in *full* theory space, for the projection is applied only after evaluating the trace.

In order to actually reduce theory space in a practical manner, we have to simplify the RHS before expanding the trace or even before computing the Hessian operator. This kind of truncation is in general only an approximated solution to the exact FRGE but it is so far unavoidable. Thus, one of the crucial, yet very complicated, parts in the evaluation of truncations is an estimate of the validity of the obtained results. We come back to this point when studying the generalized C -function like proposal. For now, let us choose a specific truncation in the shape of a projection operator

$$\Pi_{\text{Trunc}}^S(\Gamma_k) = \Pi_{\text{Trunc}}^S \left(\sum_a \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] \right) := \sum_{a \in S} \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]$$

Usually S is a finite set, though even in theory spaces of metric quantum gravity there systematic treatments of some infinite subsets of $\mathcal{T}$ were successfully carried out. The approximation arises once we substitute $\Pi_{\text{Trunc}}^S(\Gamma_k)$ for Γ_k in the Hessian operator on the RHS. Thereby, we neglect the additional contributions to the functional trace which originate from the omitted invariants. Technically, this approximation can be understood as a ‘sandwiched’ type of projection, i.e.

$$\begin{aligned} \partial_t \Pi_{\text{Trunc}}^S(\Gamma_k[\varphi; \bar{\Phi}]) &= \frac{1}{2} \Pi_{\text{Trunc}}^S \left(\text{STr} \left[\left(\widehat{\text{Hess}}_{\varphi} [\Pi_{\text{Trunc}}^S(\Gamma_k[\varphi; \bar{\Phi}])] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \right) \\ &\quad + \text{corrections} \end{aligned}$$

The corrections appear not due to the outer projection but the inner one, which in general neglects an infinite number of contributions. In the present case the RHS of the FRGE contains invariants with higher powers of R , $\bar{R}$, $(\xi \bar{\xi})$, $\dots$, for instance, while its LHS consists of only a few of them, namely those invariants retained in the Einstein-Hilbert and Gibbons-Hawking-York functional. Thus, in order to complete the specification of the truncation it is necessary to define a projection of the RHS onto exactly those field monomials.

Besides a physical motivated ansatz for Γ_k , the main focus in the development of suitable truncation schemes is the simplification of the Hessian operator appearing on the RHS. Since it is difficult to predict issues that may arise during the evaluation process, the general procedure is to initially start with a certain class of action functionals that have some physical or mathematical relevance. Then one has to compute and invert the Hessian operator before finally taking the trace over the composed endomorphism. Whenever severe problems in one of these steps occur one adapts slight modifications of the truncation ansatz such that standard techniques can be applied, as for example heat kernel techniques. Usually, these modifications change the truncated theory space by fixing the k -dependence of certain coefficients and thus modify the projection operator in the following sense:

$$\begin{aligned}\Pi_{\text{Trunc}}^{(S, S_{\text{fix}})}(\Gamma_k[\varphi; \bar{\Phi}]) &\equiv \Pi_{\text{Trunc}}^{(S, S_{\text{fix}})}\left(\sum_a \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]\right) \\ &\equiv \sum_{a \in S} \bar{u}_k^a \cdot \mathcal{P}_a[\varphi; \bar{\Phi}] + \sum_{a \in S_{\text{fix}}} c^a(\{\bar{u}_k^b\}; k) \cdot \mathcal{P}_a[\varphi; \bar{\Phi}]\end{aligned}$$

Notice that the fixed coefficients $c^a(\{\bar{u}_k^b\}; k)$ are in general artificially matched to a combination of the running couplings and still k -dependent. The price one has to pay for simplifying the RHS of the FRGE is reflected by the reduced dimensionality of truncated theory space, which is determined by $\dim(S)$ only. With new techniques one then can study the validity of the previously considered truncation by including the fixed basis invariants and look for solutions that stay close or even agree with the choice $c^a(\{\bar{u}_k^b\}; k)$.

Apart from an actual modification of the truncation ansatz, we can make use of all simplifying techniques that commute with the truncation projection $\Pi_{\text{Trunc}}^{(S, S_{\text{fix}})}(\Gamma_k[\varphi; \bar{\Phi}])$ and the functional trace. We will encounter a particular example of this method applied to the ghost sector. We have seen that the inversion of a matrix with non-commuting entries results in a very complicated expression, which however reduces significantly once the matrix assumes a block diagonal form. All the problematic contributions are in fact proportional either to $\bar{\xi}$ or ξ . Thus fixing and thereby modifying the truncation such that the ghost invariants have a fixed (not independently running) coefficient results in a projection operator $\Pi_{\text{Trunc}}^{(S, S_{\text{fix}})}(\Gamma_k[\varphi; \bar{\Phi}])$ that commutes with $\bar{\xi} = 0 = \xi$. Even changing invariants on intermediate steps is acceptable and does not reduce the considered truncated space as long as the included basis invariants can be uniquely resolved (see for instance the maximally symmetric projection technique).

In the following, we are going to simplify the very general form of the present truncation ansatz such that we can ultimately apply the heat kernel expansion to determine the dimensional beta-functions. First of all, based on the results of the matrix inversion formula, we will fix the ghost sector by omitting the k -dependence of its coefficient.

7.3.1 Fixing the ghost sector

Let us now consider truncations for which the ghost sector is kept fixed, that is it contributes to the RG evolution of the metric fluctuations but does not feel any RG effect for its own. For the exact flow, one expects there is no apparent reason why we should assume a trivial RG behavior of the ghost sector, however when approximating the exact Effective Average Action it is always a question which contributions to omit. In fact, there have been investigations of the ghost sector and its RG dependence [132–134], however in most cases the community focuses on the gravitational part alone. The fixation of the ghost sector can be easily arranged by keeping its coefficient artificially k independent, meaning $\bar{u}_k^{\text{gh}} \equiv \bar{u}^{\text{gh}} = \text{const.}$ Up to now, all results are such general that neglecting the ghost sector in the RG evolution might seem most unnecessarily. The reason is closely related to the complicated structure of the trace which we are going to evaluate in the final step using heat kernel methods. While the derivation of the

Hessian operator, the inversion of the matrix operator, and the evaluation of the matrix trace were shown for a large class of gauge fixing functionals and including the full ghost sector, this final step in computing the remaining L^2 traces requires a list of further simplifications. Fortunately, we thus have a lot of freedom by either adapting the gauge fixing condition to fit our needs or modifying the truncation ansatz.

R Notice that keeping the ghost sector fixed under RG evolution does not correspond to a vanishing of $\Gamma_k^{\text{gh}}[\varphi; \bar{\Phi}]$. In the latter case, being equivalent to $\bar{u}_k^{\text{gh}} = \bar{u}^{\text{gh}} \equiv 0$, there will be no contributions from the ghost sector to the gravitational RG solutions. For general $\bar{u}_k^{\text{gh}}$ however, we simply do not resolve the running which is perfectly fine in the vicinity of a possible fixed point and in regions where $\bar{u}_k^{\text{gh}}$ is almost k independent.

Hence, let us have a look at what kind of reduction might appear once we keep $\bar{u}_k^{\text{gh}} \equiv \bar{u}^{\text{gh}}$ fixed. While the RHS of the FRGE is almost unaffected by this choice, all ghost invariants disappear on the LHS, since ∂_t now acts on a constant ghost sector. Thus, the LHS, $\partial_t \Gamma_k[\varphi; \bar{\Phi}]$, is independent on ξ and $\bar{\xi}$. This in particular implies that all monomials that are generated in the RG procedure and containing ghost fields are projected out. In other words, the projection of the RHS onto our truncation ansatz does not coincide with the identity map, indicating that the truncation is not closed and further invariants have to be added.² The great advantage once we purely focus on the RG evolution of the gravitational sector is thus the irrelevance of field monomials on the RHS containing ξ or $\bar{\xi}$. As a natural consequence all contributions to the Hessian operator which are ghost field dependent can be neglected, for they are finally projected out. In the present case this amounts to the following reduction:

$$M_3|_{\bar{\xi}=0=\xi} \equiv \left(\begin{array}{ccc} \widehat{\text{Hess}}_{hh} + \mathfrak{R}_k^{\alpha_1} & 0 & 0 \\ 0 & 0 & \widehat{\text{Hess}}_{\xi\bar{\xi}} + \mathfrak{R}_k^{\alpha_7} \\ 0 & \widehat{\text{Hess}}_{\bar{\xi}\xi} + \mathfrak{R}_k^{\alpha_8} & 0 \end{array} \right) \Big|_{\bar{\xi}=0=\xi} \equiv \left(\begin{array}{ccc} \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_7 \\ 0 & \alpha_8 & 0 \end{array} \right)$$

At this point it becomes apparent why we have chosen α_1 , α_7 , and α_8 to be invertible while all other operators were kept unconstrained. This constellation is indeed necessary, for all other operators vanishes under projecting out ghost invariants, in particular the operator matrices B and C , which become non-invertible. The great virtue of being ignorant against the RG evolution for the ghost invariants appears on the level of the L^2 -trace. Adding eq. (7.6), the functional trace reduces quite efficiently to

$$\begin{aligned} \text{STr}_F \left[(M_3)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \Big|_{\bar{\xi}=0=\xi} & \quad (7.20) \\ = \text{STr}_{L^2}^{[h]} \left[(\alpha_1^{-1}) \partial_t \mathfrak{R}_k^{(\alpha_1)} \right] & + \text{STr}_{L^2}^{[\bar{\xi}]} \left[(\alpha_7^{-1}) \partial_t \mathfrak{R}_k^{(\alpha_7)} \right] + \text{STr}_{L^2}^{[\xi]} \left[(\alpha_8^{-1}) \partial_t \mathfrak{R}_k^{(\alpha_8)} \right] \end{aligned}$$

Hence, the orthogonality of the Hessian and cutoff operator within this class of truncations yields a sum over three non-entangled operators. It is for this severe simplification that we reduce our truncation ansatz such that it becomes insensitive of the RG flow of the ghost sector. Nevertheless, it should be stated that all truncations including ghost invariants have not altered the qualitative results, in particular they agree with the Asymptotic Safety conjecture for Quantum Gravity [132–134].

R While $\bar{u}_k^{\text{gh}} \equiv \bar{u}^{\text{gh}}$ corresponds to an actual reduction of the truncated theory space, thus modifies the truncation ansatz, the evaluation of $\bar{\xi} = 0 = \xi$ describes a projection technique $\Pi_{\bar{\xi}=0=\xi}$ that commutes with the modified projection operator $\Pi_{\text{Trunc}}^{(S, S_{\text{fix}})}(\Gamma_k[\varphi; \bar{\Phi}])$.

²Nevertheless, this also happens in the metric sector due to the appearance of higher derivative terms which we are not going to resolve.

7.3.2 Conformal projection technique

The remaining trace evaluation is only concerned with L^2 -operator based on the covariant derivatives w.r.t. $\bar{g}$ and g . However, the preferred structure of these operators would be a function of a single Laplacian only, since otherwise we have to deal with the non-commuting operators $\bar{D}$ and D . The general procedure would be to expand the Hessian operator, which is a function of $\bar{g}$ and g , in powers of h and then evaluate the trace level by level in the fluctuation field h :

$$\begin{aligned} \text{LHS : } \partial_t \Gamma_k^{\text{grav}}[g, \bar{g}] &\equiv \partial_t \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] + \partial_t \mathfrak{d}_h \Gamma_k^{\text{grav}}[g, \bar{g}]|_{g=\bar{g}} + \mathcal{O}(h^2) \\ \text{RHS : } \text{STr}[F[g, \bar{g}]] &\equiv \text{STr}[F[\bar{g}, \bar{g}] + \mathfrak{d}_h F[g, \bar{g}]|_{g=\bar{g}} + \mathcal{O}(h)] \end{aligned} \quad (7.21)$$

For the present truncation ansatz, in which all level- (p) with $p \geq 1$ coefficients are identified, it suffices to compare only the first two orders in h . Thus at each order of the expansion D is identified with $\bar{D}$ at the expense of an additional fluctuation field h and a more difficult tensor structure. While this cures the problem of non-commutation, we will depart further from a purely Laplacian operator form.

Thus, we are need of a different technique that solves both problems at once but still retains the essential information of the truncation ansatz. This is actually the main problem in simplifying the Hessian operator, namely that the same method has to be applied also on the LHS which may in general alter the truncation ansatz. As described in the introduction of this section, a suitable projection technique has to commute with $\Pi_{\text{trunc}}(\bullet)$ and in case it does not modifications of the truncated subset arise. In the current analysis, the gravitational sector should remain unaltered, while adaptations to the ghost and gauge fixing functionals are acceptable. This was already utilized in setting $\bar{u}_k^{\text{gh}}$ to a fixed value in order to apply the zero-ghost projection technique.

The single-metric projection

First, let us consider a very simple projection namely $\Pi_{g=\bar{g}}(F[g, \bar{g}]) \equiv F[g, \bar{g}]|_{g=\bar{g}}$ which corresponds to the zeroth order in the expansion of eq. (7.21). Then, naturally all covariant derivatives D are replaced by $\bar{D}$ while the original tensor structure remains unchanged, thus yielding a RHS that is perfectly suitable for the further analysis. However, this projection does not commute even with the reduction onto the gravitational part of the truncation ansatz, which consists of seven basis invariants, either purely constructed w.r.t. $\bar{g}$ or g :

$$\begin{array}{cccc} \int_M d^d x \sqrt{\bar{g}} \bar{R} & \int_M d^d x \sqrt{\bar{g}} & \int_{\partial M} d^{d-1} \sqrt{\bar{H}} \bar{K} & \int_{\partial M} d^{d-1} \sqrt{\bar{H}} \\ \int_M d^d x \sqrt{g} R & \int_M d^d x \sqrt{g} & \int_{\partial M} d^{d-1} x \sqrt{H} K & \end{array}$$

Under the action of $\Pi_{g=\bar{g}}$ onto the FRGE the Hessian operator gets sufficiently simplified, however all dynamical invariants reduce to their background counterparts and thus cannot be resolved in the RG evolution:

$$\begin{aligned} \Pi_{g=\bar{g}}(\text{LHS}) : \quad \Pi_{g=\bar{g}}(\partial_t \Gamma_k^{\text{grav}}[g, \bar{g}]) &\equiv \partial_t \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] \\ \Pi_{g=\bar{g}}(\text{RHS}) : \quad \text{STr}[\Pi_{g=\bar{g}}(F[g, \bar{g}])] &\equiv \text{STr}[F[\bar{g}, \bar{g}]] \end{aligned}$$

While at the beginning we distinguished between seven basis invariants constructed with the background and dynamical field $\bar{g}$ and g , respectively, the projection $\text{proj}_{g=\bar{g}}$ retains only four distinguishable monomials. In particular for the Einstein-Hilbert functionals we have

$$\begin{aligned} \Pi_{g=\bar{g}} \left(\partial_t \bar{u}_k^1 \cdot \int \sqrt{g} R + \partial_t \bar{u}_k^2 \cdot \int \sqrt{g} \bar{R} \right) &= \Pi_{g=\bar{g}} \left(\bar{\beta}^1 \cdot \int \sqrt{g} R + \bar{\beta}^2 \cdot \int \sqrt{g} \bar{R} \right) \\ \implies \partial_t (\bar{u}_k^1 + \bar{u}_k^2) &= (\bar{\beta}^1 + \bar{\beta}^2) \end{aligned}$$

This demonstrates that $\Pi_{g=\bar{g}}$ is incapable of resolving the background and dynamical RG evolutions and thus inappropriate for bi-metric truncations.

The bi-metric conformal projection

We now demonstrate that for $\partial M = \emptyset$ a slight modification of $\Pi_{g=\bar{g}}$ is sufficient to extract the full set of beta-functions described by eq. (7.21) while keeping the tensor structure as simple as for $g = \bar{g}$. The cure that circumvents both problems at once is the conformal projection methods, which in the case of vanishing boundary was successfully applied in ref. [172, 173] to study bi-metric truncations. The essential step is instead of identifying g and $\bar{g}$ one conformally relates the two metrics as follows: $g_{\mu\nu}(x) = e^{2\Omega} \bar{g}_{\mu\nu}(x)$. Then, after having eliminated $g_{\mu\nu}$ in favor of Ω and $\bar{g}_{\mu\nu}$, the associated projection operator Π_Ω is used to expand the supertrace in the flow equation in powers of the x -independent conformal factor Ω . For $\Omega = 0$ we obtain the invariants built from $\bar{g}$, whereas the level-(1) invariants $\propto h$ are identified as those linear in Ω . All terms which are at least quadratic in Ω are irrelevant in the present context, since we identified all couplings of level-(1) and higher.

Indeed $\Pi_\Omega(F[g, \bar{g}]) := F[e^{2\Omega} \bar{g}, \bar{g}]$ separates the contributions to background and dynamical invariants. When applied to the FRGE it results in the following LHS and RHS:

$$\begin{aligned} \Pi_\Omega(\text{LHS}) : \quad & \Pi_\Omega(\partial_t \Gamma_k^{\text{grav}}[g, \bar{g}]) \equiv \partial_t \Gamma_k^{\text{grav}}[e^{2\Omega} \bar{g}, \bar{g}] \\ \Pi_\Omega(\text{RHS}) : \quad & \text{STr}[\Pi_\Omega(F[g, \bar{g}])] \equiv \text{STr}[F[e^{2\Omega} \bar{g}, \bar{g}]] \end{aligned} \quad (7.22)$$

In fact, expanding both sides in powers of Ω yields an expression quite similar to eq. (7.21) with $h_{\mu\nu} \equiv (e^{2\Omega} - 1)\bar{g}_{\mu\nu}$:

$$\begin{aligned} \Pi_\Omega(\text{LHS}) : \quad & \partial_t \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] + \partial_t(\partial_\Omega \Gamma_k^{\text{grav}}[e^{2\Omega} \bar{g}, \bar{g}]|_{\Omega=0}) \cdot \Omega + \mathcal{O}(\Omega^2) \\ \Pi_\Omega(\text{RHS}) : \quad & \text{STr}[F[\bar{g}, \bar{g}] + \partial_\Omega F[e^{2\Omega} \bar{g}, \bar{g}]|_{\Omega=0} \cdot \Omega + \mathcal{O}(\Omega^2)] \end{aligned} \quad (7.23)$$

On intermediate steps Ω thus acts as a separating device that keeps the terms associated to level-(0) or higher orders in h , since the former are independent of h and hence of Ω . For the Einstein-Hilbert functionals this separation reads as follows:

$$\begin{aligned} \Pi_\Omega \left(\partial_t \bar{u}_k^1 \cdot \int \sqrt{\bar{g}} R + \partial_t \bar{u}_k^2 \cdot \int \sqrt{\bar{g}} \bar{R} \right) &= \Pi_\Omega \left(\bar{\beta}^1 \cdot \int \sqrt{\bar{g}} R + \bar{\beta}^2 \cdot \int \sqrt{\bar{g}} \bar{R} \right) \\ \Rightarrow \quad \partial_t (\bar{u}_k^1 + \bar{u}_k^2) + \Omega \cdot ((d-4)\bar{u}_k^1) + \mathcal{O}(\Omega^2) &= (\bar{\beta}^1 + \bar{\beta}^2) + \Omega \cdot ((d-4)\bar{\beta}^1) + \mathcal{O}(\Omega^2) \end{aligned}$$

Hence, while $\Omega = 0$ gives rise to the single-metric projection including the background and dynamical beta-functions, the first order in Ω contains only the dynamical terms and thus we have two independent equations to solve for $\bar{u}^1$ and $\bar{u}^2$, separately.

While the gravitational functional on the bulk is preserved under conformal projection, the gauge fixing functional is projected out on the LHS of the flow equation:

$$\Pi_\Omega(\mathcal{F}_\mu[\bar{g}](h)) = (e^{2\Omega} - 1)\mathcal{F}_\mu[\bar{g}](\bar{g}) = 2\Omega \left(\bar{g}^{\alpha\rho} \bar{g}^{\beta\sigma} - \varpi \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} \right) \bar{g}_{\alpha\mu} \bar{D}_\beta \bar{g}_{\rho\sigma} = 0$$

Hence, the gauge fixing functional which is quadratic in $\mathcal{F}[\bar{g}](h)$ vanishes $\Pi_\Omega(\Gamma_k^{\text{gf}}[h; \bar{g}]) = 0$, a consequence of $\mathcal{F}[\bar{g}]$ being a function of $\bar{D}$ only, which is a metric compatible derivative. Once we apply the conformal projection technique there the RG evolution of $\bar{u}_k^{\text{gf}}$ is not resolvable anymore. We come back to this point after considering a much severe issue concerning boundary invariants.

R One should keep in mind that we introduced this bookkeeping constant Ω only as a mere projection tool which is ultimately set to zero and has no physical meaning. All other techniques that are compatible with the truncation ansatz can be used and should result in the same set of beta-functions.

Conformal projection in the presence of boundary invariants

While this conformal projection method works very well for $\partial M = \emptyset$, there is a subtlety involved when considering manifolds with non-vanishing boundary. Only if $\Omega = 0$ the dynamical g and the background $\bar{g}$ field fulfill the same boundary conditions and thus share the same field space. In particular, the conformal projection requires to set $h|_{\partial M} \equiv (e^{2\Omega} - 1)\bar{g}|_{\partial M}$ on intermediate steps which corresponds to a combination of Dirichlet and Neumann condition rather than pure Dirichlet boundary conditions:

$$h_{\mu\nu}|_{\partial M} \equiv (e^{2\Omega} - 1)\bar{g}_{\mu\nu}|_{\partial M} \quad \implies \quad \bar{D}_\rho h_{\mu\nu}|_{\partial M} \equiv 0$$

These modified boundary condition for h will in general spoil the arguments that led to the Hessian operator we derived for Dirichlet conditions and more severely does not preserve the gravitational truncation ansatz when applied to the dynamical boundary invariant. This basically corresponds to imposing Dirichlet and Neumann conditions simultaneously, while at the same time distinguish between contributions from $\sqrt{H}K$ and $\sqrt{H}\bar{K}$. However, these are mutually exclusive constraints and thus Π_Ω is not compatible with a dynamical Gibbons-Hawking-York term in the presence of Dirichlet conditions on field space:

$$\Pi_\Omega(K|_{g=\bar{g}}) = \Pi_\Omega\left(\bar{K} - \frac{1}{2}(n^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} n^\lambda \bar{D}_\lambda)h_{\mu\nu}\right) = \bar{K}$$

In the final step we used the fact that $\bar{D}\bar{g} = 0$ and Ω is an x -independent constant. Thus, solving for both $\bar{u}^{\partial B}$ and $\bar{u}^{\partial D}$ becomes impossible once the conformal projection technique is used.

At this point there are basically two possibilities how to continue to resolve the RG evolution of the dynamical Gibbons-Hawking-York term: either choose different boundary conditions for field space, for instance Neumann conditions which are perfectly compatible with the conformal projection technique, or consider a different projection method. Concerning the first option, notice that Dirichlet conditions were found to be essential in deriving a well-defined Hessian operator. For instance, in the case of Neumann conditions, $\bar{D}h|_{\partial M} = 0$, several boundary terms remained in the second variation of Γ_k and thus no Hessian operator can be deduced. Only when evaluated for the conformal projection, i.e. additionally imposing $h = (e^{2\Omega} - 1)\bar{g}$, these terms result in negligible contributions of second order in Ω :

$$\begin{aligned} \mathcal{J}^\Omega \equiv & +2(d-1)(d-3) \cdot (\bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D}) \cdot \int_{\partial M} d^{d-1}x \sqrt{H} (\Omega^2 + \mathcal{O}(\Omega^3)) \\ & + \frac{1}{2}(\bar{u}_k^{\partial D}) \cdot \int_{\partial M} d^{d-1}x \sqrt{H} ((d^2 + 2d)\bar{K} + 2(d-2)\bar{K}^{\mu\nu} \bar{n}_\mu \bar{n}_\nu) (\Omega^2 + \mathcal{O}(\Omega^3)) \end{aligned}$$

Here, we have introduced $\bar{n}^\mu \equiv e^\Omega n^\mu$ as the normal vector of the boundary that is normalized with respect to the background metric $\bar{g}$. The first term originates from the volume element on the boundary constructed by the dynamical field and in fact is absent in case of Dirichlet conditions, $\Omega = 0$. Similarly, for the second term which is associated to the Gibbons-Hawking-York functional. But at the point we utilize the conformal projection technique we already should have a suitable Hessian operator. Thus, even though different boundary conditions are compatible with Π_Ω they introduce much severe and fundamental problems on the level of the FRGE. This suggests that we should rather look for a different projection technique.

An alternative, that at first sounds very promising, is an x -dependent conformal projection. Thus we replace the original constant parameter Ω with a function $\Omega(x)$ that respects Dirichlet boundary conditions:

$$\Pi_{\Omega(x)}(\bullet) := (\bullet)|_{g=e^{2\Omega(x)}\bar{g}} \quad \text{with} \quad \Omega(x)|_{\partial M} = 0$$

Obviously, now g and $\bar{g}$ fulfill the same boundary conditions and thus are elements of the same field space. In particular, the fluctuation field h vanishes on ∂M but has a non-trivial derivative in general:

$$\begin{aligned}\Pi_{\Omega(x)}(h_{\mu\nu})|_{\partial M} &= (e^{2\Omega(x)} - 1)\bar{g}_{\mu\nu}|_{\partial M} = (2\Omega(x) + \mathcal{O}(\Omega^2))\bar{g}_{\mu\nu}|_{\partial M} = 0 \quad \text{and} \\ \Pi_{\Omega(x)}(\bar{D}_\rho h_{\mu\nu})|_{\partial M} &= \bar{D}_\rho(e^{2\Omega(x)} - 1)\bar{g}_{\mu\nu}|_{\partial M} = 2\bar{g}_{\mu\nu}(\bar{D}\Omega(x)) + \mathcal{O}(\Omega^2)|_{\partial M} \neq 0\end{aligned}$$

This seems encouraging since an x -dependent conformal projection can be consistently combined with Dirichlet boundary conditions. By construction, it will preserve the dynamical Gibbons-Hawking-York term on the LHS of the flow equation, which in the level expansion reads

$$\partial_t \Gamma_k^{\text{grav}}[g, \bar{g}] \equiv \partial_t \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] + \partial_t (\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \frac{1}{2} \partial_t (\partial_h \partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \mathcal{O}(h^3)$$

For the present truncation, for which only the zeroth and first order couplings differ, it is actually sufficient to reduce the FRGE up to first order in h . Once we employ the x -dependent conformal projection method, this corresponds to focusing on the zeroth and first order in $\Omega(x)$ only, i.e.

$$\begin{aligned}\Pi_{\Omega(x)}(\partial_t \Gamma_k^{\text{grav}}[g, \bar{g}]) &= \Pi_{\Omega(x)}(\partial_t \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}]) + \Pi_{\Omega(x)}(\partial_t (\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}}) + \mathcal{O}(\Omega^2) \\ &= \partial_t \Gamma_k^{\text{grav}}[\bar{g}, \bar{g}] + \partial_t (\partial_{2\Omega(x)\bar{g}} \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} + \mathcal{O}(\Omega^2)\end{aligned}$$

To obtain the second equality, we notice that the first term is independent on h and thus invariant under $\Pi_{\Omega(x)}$. For the second term, representing the h -linear contribution, we expanded the exponential $\exp(2\Omega(x)) = 1 + 2\Omega(x) + \mathcal{O}(\Omega^2)$ to first order. Next, we insert its explicit form given in eq. (6.29) that leads to

$$\begin{aligned}\Pi_{\Omega(x)} \left((\partial_h \Gamma_k^{\text{grav}}[g, \bar{g}])|_{g=\bar{g}} \right) &= + \int_M d^d x \sqrt{\bar{g}} \left\{ (\bar{u}_k^\mathcal{D} \bar{\lambda}_k^\mathcal{D}) \cdot \bar{g}^{\mu\nu} + (\bar{u}_k^\mathcal{D}) \cdot (\bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R}) \right\} \Omega(x) \bar{g}_{\mu\nu} \\ &\quad - \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left\{ (\bar{u}_k^\mathcal{D} - \bar{u}_k^{\partial\mathcal{D}}) \cdot (\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) \right\} \Omega(x) \bar{g}_{\mu\nu} \\ &= + \int_M d^d x \sqrt{\bar{g}} \Omega(x) \left\{ (\bar{u}_k^\mathcal{D} \bar{\lambda}_k^\mathcal{D}) \cdot 2d - (d-2)(\bar{u}_k^\mathcal{D}) \cdot \bar{R} \right\} \\ &\quad + (d-1) \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left\{ (\bar{u}_k^\mathcal{D} - \bar{u}_k^{\partial\mathcal{D}}) \cdot \bar{n}^\rho \bar{D}_\rho \Omega(x) \right\}\end{aligned} \tag{7.24}$$

This equation illustrates the problem of choosing an x -independent conformal projection. While the bulk invariant does not feel any difference for Ω and $\Omega(x)$, the boundary contribution that originates from the dynamical Gibbons-Hawking-York functional survives only for $\Omega(x)$. In combination with the zeroth order projection term, eq. (6.28), that contains also background couplings, the x -dependent conformal projection of the LHS of the flow equation yields, to the required order

$$\begin{aligned}\Pi_{\Omega(x)}(\Gamma_k^{\text{grav}}[g, \bar{g}]) &= +2 \cdot \int_M d^d x \sqrt{\bar{g}} \left((1 + d \cdot \Omega(x))(\bar{u}_k^\mathcal{D} \bar{\lambda}_k^\mathcal{D}) + (\bar{u}_k^\mathcal{B} \bar{\lambda}_k^\mathcal{B}) \right) \\ &\quad - \int_M d^d x \sqrt{\bar{g}} \bar{R} \left((1 + (d-2)\Omega(x))(\bar{u}_k^\mathcal{D}) + (\bar{u}_k^\mathcal{B}) \right) \\ &\quad + 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left((\bar{u}_k^{\partial\mathcal{D}} \bar{\lambda}_k^{\partial\mathcal{D}}) + (\bar{u}_k^{\partial\mathcal{B}} \bar{\lambda}_k^{\partial\mathcal{B}}) \right) \\ &\quad - 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \bar{K}(\bar{H}) \left((\bar{u}_k^{\partial\mathcal{D}}) + (\bar{u}_k^{\partial\mathcal{B}}) \right) \\ &\quad + (d-1) \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left\{ (\bar{u}_k^\mathcal{D} - \bar{u}_k^{\partial\mathcal{D}}) \cdot \bar{n}^\rho \bar{D}_\rho \Omega(x) \right\} + \mathcal{O}(\Omega^2)\end{aligned} \tag{7.25}$$

This indeed is a sufficient generalization of the conformal technique that comprises all seven basis invariants.

- R** The the dynamical boundary volume element, $\int_{\partial M} \sqrt{H}$, agrees with its background counterpart due to Dirichlet conditions. Thus either $\bar{\lambda}_k^{\partial D}$ or $\bar{\lambda}_k^{\partial B}$ can be resolved but not both, independent on the conformal projection technique.

Notice that even in the case of a generalized conformal projection the gauge fixing action does not contribute. While for $\Omega(x) = \Omega$ constant, $\Gamma_k^{\text{gf}}[e^{2\Omega}\bar{g}, \bar{g}] = 0$ is identical zero, in the present case the linearity of $\mathcal{F}[\bar{g}](h)$ in the fluctuation field h implies that the gauge fixing functional is of second order in $\Omega(x)$:

$$\Gamma_k^{\text{gf}}[2\Omega(x)\bar{g}; \bar{g}] = \frac{1}{2}\bar{u}_k^{\text{gf}} \int_M d^d x \sqrt{\bar{g}} \mathcal{F}_\mu(2\Omega(x)\bar{g}) \bar{g}^{\mu\nu} \mathcal{F}_\nu(2\Omega(x)\bar{g}) = \mathcal{O}(\Omega(x)^2)$$

Thus, the running of the gauge parameter $\bar{u}_k^{\text{gf}}$ cannot be resolved with this method, even in the x -dependent case.

Finally, in the ghost sector the LHS of the flow equation reduces under the x -dependent conformal projection to

$$\begin{aligned} \partial_t \Gamma_k^{\text{gh}}[2\Omega\bar{g}, \xi, \bar{\xi}; \bar{g}] & \\ &= -\sqrt{2}(\partial_t \bar{u}_k^{\text{gh}}) \int d^d x \sqrt{\bar{g}} (1 + 2\Omega(x)) \bar{\xi}_\mu \left\{ \delta_\nu^\mu \bar{D}^2 + (1 - 2\varpi) \bar{D}^\mu \bar{D}_\nu + \bar{R}_\nu^\mu \right\} \xi^\nu \\ &\quad - 2\sqrt{2}(\partial_t \bar{u}_k^{\text{gh}}) \int d^d x \sqrt{\bar{g}} \bar{\xi}_\mu \left\{ (2 - d \cdot \varpi) \left(\xi^\lambda \bar{D}^\mu \bar{D}_\lambda \Omega(x) + (\bar{D}^\mu \xi^\lambda) \bar{D}_\lambda \Omega(x) \right) \right. \\ &\quad \left. + (\bar{D}_\lambda \xi^\mu) \bar{D}^\lambda \Omega(x) - 2\varpi (\bar{D}_\lambda \xi^\lambda) \bar{D}^\mu \Omega(x) \right\} \end{aligned} \quad (7.26)$$

Incidentally, note that for constant $\Omega(x) = \Omega$ and the harmonic gauge condition, corresponding to $\varpi = 1/2$, the ghost action, and therefore the entire truncation ansatz on the bulk, does not contain any covariant derivative $\bar{D}_\mu$ that would not be contracted to a Laplacian $\bar{D}^2 \equiv \bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$. However, since $\bar{u}_k^{\text{gh}}$ is chosen k -independent the ghost sector vanishes on the LHS.

The drawback of an x -dependent conformal method appears on the RHS of the flow equation. Initially, the conformal projection technique was introduced to circumvent a h -expansion, since in the latter case additional not contracted derivatives are generated preventing a straightforward heat kernel expansion. Though in principal there are methods available that treat this general case, we would need a full decomposition of the fluctuation fields and ultimately rely on off-diagonal heat kernel techniques to evaluate the trace. Already in the case of manifolds with vanishing boundary this turns out to be quite sophisticated and time-consuming, whereas for non-vanishing ∂M additional complication arises. For instance the supplementary boundary constraints have to be implemented and furthermore additional heat kernel coefficients appear that generate the boundary invariants.

Thus, simplified projection techniques are desirable as for instance the just described x -dependent conformal projection method. In fact this technique can be understood as keeping track of the $h^{\text{Tr}} \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$ part only, corresponding to the trace contribution in the h -expansion, which for the present truncation is sufficient. While this resolves a number of unwanted terms in the Hessian, there is still a sufficient amount of not fully contracted derivatives acting on $\Omega(x)$ that requires the full apparatus of off-diagonal heat kernel techniques described above. One can further specify the function $\Omega(x)$ to best fit the needs of the present truncation. Nevertheless, this turns out cumbersome since the requirement to retain the boundary term in question already

excludes the most simple choices. Basically, the following conditions should be satisfied by $\Omega(x)$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \Omega(x)\bar{g}_{\mu\nu} \quad \bar{n}^\rho \bar{D}_\rho \Omega(x) \neq 0 \quad \Omega(x)|_{\partial M} = 0$$

Besides these requirements, it is desirable to obtain a simple as possible Hessian operator w.r.t. $\Omega(x)$.

R It should be noted that the appearance of $\bar{D}\Omega(x)$ in the Hessian is in fact an indication for a non-trivial RG evolution of $\bar{u}^{\partial D}$, though for computational purposes it makes things unnecessary complicated.

The bi-metric-bulk – pure-background-boundary truncation

In this thesis, we have prepared the ground for an in depth study of a bi-metric Einstein-Hilbert–Gibbons-Hawking-York truncation. We have derived the Hessian for general boundary conditions and arbitrary parametrization schemes and only afterwards we have specified to a particular class of gauge fixing conditions, the linear parametrization, and Dirichlet boundary conditions. Similarly, we have provided the general inversion rule for the matrix in field space, while only later we ignored the running of the ghost sector. Thus, future work may enter at any step and change or generalize the setting at will to explore the RG evolution of boundary terms in quantum gravity to its full beauty. In the sequel, we consider the a single-metric ansatz for the boundary gravitational part in order to utilize the standard Ω -expansion. In a next step, one might proceed in using the prescribed x -dependent generalization to resolve the dynamical and background running independently.

For now, we employ the projection Π_Ω instead of $\Pi_{\Omega(x)}$ for which the Effective Average Action on the LHS of the flow equation reduces to

$$\begin{aligned} \Pi_\Omega(\Gamma_k[g, \bar{g}]) = & +2 \cdot \int_M d^d x \sqrt{\bar{g}} [(1 + d \cdot \Omega) \bar{u}_k^D \bar{\lambda}_k^D + \bar{u}_k^B \bar{\lambda}_k^B] \\ & - \int_M d^d x \sqrt{\bar{g}} \bar{R} [(1 + (d-2)\Omega) \bar{u}_k^D + \bar{u}_k^B] \\ & + 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} [\bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D} + \bar{u}_k^{\partial B} \bar{\lambda}_k^{\partial B}] \\ & - 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \bar{K}(\bar{H}) [\bar{u}_k^{\partial D} + \bar{u}_k^{\partial B}] + \mathcal{O}(\Omega^2) \end{aligned} \quad (7.27)$$

Thus, a total of six independent couplings is left, that we are going to resolve in what follows. Notice that the ghost as well as the gauge fixing coefficients drop out, however for different reasons. While the former is set to a constant value, the latter basis invariant turns zero under conformal projection with $\bar{u}_k^{\text{gf}}$ unspecified so far. From now on, there will be no more reduction of the truncated theory space, but nevertheless we will exploit the freedom in choosing $\bar{u}_k^{\text{gf}}$ to get rid of certain undesirable terms.

The Hessian operator on the RHS of the flow equation has to be equally projected by means of Π_Ω . Therefore, let us first list the conformal projection of several tensor fields appearing in the Hessian operator:

$$\begin{aligned} \Pi_\Omega(g_{\mu\nu}) &= e^{2\Omega} \bar{g}_{\mu\nu} & \Pi_\Omega(g^{\mu\nu}) &= e^{-2\Omega} \bar{g}^{\mu\nu} & \Pi_\Omega(\sqrt{g}) &= e^{d\Omega} \sqrt{\bar{g}} \\ \Pi_\Omega(R^{\mu\nu\rho\sigma}) &= e^{-6\Omega} \bar{R}^{\mu\nu\rho\sigma} & \Pi_\Omega(R^{\mu\nu}) &= e^{-4\Omega} \bar{R}^{\mu\nu} & \Pi_\Omega(R) &= e^{-2\Omega} \bar{R} \end{aligned}$$

In the gravitational sector of the Hessian operator, the conformal projection technique only affects the contributions due to Γ_k^{grav} , since the additional terms of the gauge fixing functional

are already independent on h . For a constant parameter Ω we obtain

$$\begin{aligned}\Pi_\Omega(\mathfrak{g}^{\mu\nu B}(\widehat{\text{Hess}}_{hh})_B^{\rho\sigma}) = & -(\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot (\bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu}) \cdot e^{(d-4)\Omega} \\ & + \bar{u}_k^{\text{gf}} \cdot (\bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} + \bar{R}^{\rho\mu\nu\sigma}) \\ & + \frac{1}{2} (\bar{u}_k^{\text{D}}) \cdot e^{(d-6)\Omega} \{ -\bar{\mathcal{H}}^{\mu\nu\rho\sigma} + \bar{\mathcal{U}}^{\mu\nu\rho\sigma} \} + \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \bar{\mathcal{H}}^{\mu\nu\rho\sigma} \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (1 - 2\varpi) (\bar{g}^{\mu\nu} \bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^\mu \bar{D}^\nu) \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - (1 - 2\varpi^2) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}) \bar{D}^2\end{aligned}$$

Since on the LHS of the flow equation it is sufficient to expand up to first order in Ω and retain all gravitational invariants (except the dynamical Gibbons-Hawking-York term), we can neglect all higher order contributions in the Hessian, too. This gives rise to the following simplification

$$\begin{aligned}\Pi_\Omega(\mathfrak{g}^{\mu\nu B}(\widehat{\text{Hess}}_{hh})_B^{\rho\sigma}) = & -[(\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}}) \cdot (1 + (d-4)\Omega)] (\bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu}) \\ & + \bar{u}_k^{\text{gf}} \cdot (\bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} + \bar{R}^{\rho\mu\nu\sigma}) \\ & + \frac{1}{2} [(1 + (d-6)\Omega)(\bar{u}_k^{\text{D}})] \cdot \bar{\mathcal{U}}^{\mu\nu\rho\sigma} \\ & - \frac{1}{2} [(1 + (d-6)\Omega)(\bar{u}_k^{\text{D}}) - \bar{u}_k^{\text{gf}}] \bar{\mathcal{H}}^{\mu\nu\rho\sigma} \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (1 - 2\varpi) (\bar{g}^{\mu\nu} \bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^\mu \bar{D}^\nu) \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - (1 - 2\varpi^2) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}) \bar{D}^2\end{aligned}\tag{7.28}$$

The first three terms are ultra-local tensor structures, and are almost in shape for a heat kernel expansion, while the remaining terms contain free derivatives. Out of them, only the last contribution is of Laplacian type and needs no more further modification.

Next, consider the ghost sector in the Hessian operator. Since $\Pi_\Omega(h_{\mu\nu}) = 2\Omega \bar{g}_{\mu\nu}$ holds true all contributions that involve a derivative of h w.r.t. $\bar{D}$ vanish. This leads to a strong reduction under which the ghost part of the Hessian operator assumes the form:

$$\begin{aligned}(\widehat{\text{Hess}}_{\xi\xi})_\alpha^\mu = & +\sqrt{2}(1+2\Omega) \cdot \bar{u}^{\text{gh}}(\mathfrak{g}_{\xi\xi}^{-1})_\alpha^\lambda (\bar{g}_{\lambda\rho} \bar{D}_\sigma + \bar{g}_{\lambda\sigma} \bar{D}_\rho) \mathcal{F}_\nu^{\rho\sigma} \bar{g}^{\mu\nu} \\ = & +\sqrt{2}(1+2\Omega) \cdot \bar{u}^{\text{gh}}(\mathfrak{g}_{\xi\xi}^{-1})_\alpha^\lambda (\delta_\lambda^\mu \bar{D}^2 + (1-2\varpi) \bar{D}^\mu \bar{D}_\lambda + \bar{R}_\lambda^\mu)\end{aligned}\tag{7.29a}$$

$$\begin{aligned}(\widehat{\text{Hess}}_{\xi\xi})_\lambda^\alpha = & -\sqrt{2}(1+2\Omega) \cdot \bar{u}^{\text{gh}}(\mathfrak{g}_{\xi\xi}^{-1})_\mu^\alpha \left\{ \bar{g}^{\mu\nu} \mathcal{F}_\nu^{\rho\sigma} (\bar{g}_{\lambda\rho} \bar{D}_\sigma + \bar{g}_{\lambda\sigma} \bar{D}_\rho) \right\} \\ = & -\sqrt{2}(1+2\Omega) \cdot \bar{u}^{\text{gh}}(\mathfrak{g}_{\xi\xi}^{-1})_\mu^\alpha (\delta_\lambda^\mu \bar{D}^2 + (1-2\varpi) \bar{D}^\mu \bar{D}_\lambda + \bar{R}_\lambda^\mu)\end{aligned}\tag{7.29b}$$

Hereby, on an intermediate step, we used twice $[\bar{D}^\mu, \bar{D}_\lambda]_{\nu\mu} = \bar{R}^{\mu\lambda}{}_{\nu\mu}$ for dual vector fields, to simplify the result. A close look at the structure of these equations already gives a hint what choices for the gauge fixing condition and $\bar{\alpha}_k$ are most favorable. For the moment, we turn our attention to the ultra-local contributions and come back to the issue of uncontracted derivatives at the very end.

R On has to keep in mind that the constant Ω is an artificial parameter which is only introduced in order to distinguish between purely background contributions and terms of dynamical origin. It has no physical relevance and does not alter field space or the truncation ansatz at all.

7.3.3 The Hessian in the trace-traceless decomposition

Previously, we described a technique to decompose symmetric 2-tensor fields with respect to a derivative operator $\bar{D}$ and a scalar product, here given by $\bar{g}$. We are now going to apply only

its simplest form, namely the trace-traceless splitting of the metric part of the Hessian operator. The corresponding projection operators are given by

$$(\Pi_{\text{Tr}})^{\mu\nu}_{\rho\sigma} := \frac{1}{d} \bar{g}_{\rho\sigma} \bar{g}^{\mu\nu} \quad (7.30a)$$

$$(\Pi_{\text{noTr}})^{\mu\nu}_{\rho\sigma} := (\mathbb{1}_{\rho\sigma}^{\mu\nu} - (\Pi_{\text{Tr}})^{\mu\nu}_{\rho\sigma}) \equiv \frac{1}{2} (\delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} + \delta_{\rho}^{\nu} \delta_{\sigma}^{\mu} - \frac{2}{d} \bar{g}_{\rho\sigma} \bar{g}^{\mu\nu}) \quad (7.30b)$$

With the above normalization, both operators are idempotent and thus define projections. Very important is the decomposition behavior of the field space metric under the trace-traceless split induced by eq. (7.30). It turns out that $\mathfrak{g}^{AB}$ is orthogonal w.r.t. (Π_{Tr}) and (Π_{noTr}) and hence it commutes with the projections:

$$(\Pi_{\text{noTr}})_A \mathfrak{g}^{AB} (\Pi_{\text{Tr}})_B = 0 = (\Pi_{\text{Tr}})_A \mathfrak{g}^{AB} (\Pi_{\text{noTr}})_B \quad \wedge \quad \mathfrak{g}^{AB} (\Pi_{\text{Tr/noTr}})_B^C = (\Pi_{\text{Tr/noTr}})_B^A \mathfrak{g}^{BC}$$

The traceless and trace component of the gravitational part of the field space metric assumes the form:

$$\mathfrak{g}^{\mu\nu B} (\Pi_{\text{noTr}})_B^{\rho\sigma} = \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - (\frac{1}{d} - 2\varpi) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \quad \wedge \quad \mathfrak{g}^{\mu\nu B} (\Pi_{\text{Tr}})_B^{\rho\sigma} = (\frac{1}{d} - \varpi) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}$$

Notice that for $\varpi \equiv 1/d$ the metric vanishes in the trace sector and thus is not invertible for general symmetric 2-tensor fields. Avoiding this special case, let us consider the projection of the inverted field space metric $\mathfrak{g}_{\mu\nu\rho\sigma}^{-1} \equiv \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} + \frac{\varpi}{(1-d\varpi)} \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma}$:

$$\mathfrak{g}_{\mu\nu B}^{-1} (\Pi_{\text{noTr}})_B^{\rho\sigma} = \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \frac{1}{d} \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} \equiv (\Pi_{\text{noTr}})_{\mu\nu}^{\alpha\beta} \cdot (\bar{g}_{\alpha\rho} \bar{g}_{\beta\sigma}) \quad (7.31a)$$

$$\mathfrak{g}_{\mu\nu B}^{-1} (\Pi_{\text{Tr}})_B^{\rho\sigma} = (\frac{1}{d} - \varpi) \cdot (\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}) \equiv (\Pi_{\text{Tr}})_{\mu\nu}^{\alpha\beta} \cdot \left(\frac{1}{d(1-d\varpi)} \bar{g}_{\alpha\beta} \bar{g}_{\rho\sigma} \right) \quad (7.31b)$$

This is a quite important result for it shows the decomposition of the field space metric under the trace-traceless split. Its relevance in the present case is due to the fact that in the previous expressions for the Hessian operator we used the matrix $\mathfrak{g}^{AB} (\widehat{\text{Hess}}_{hh})_B^C$ rather than the operator $\widehat{\text{Hess}}_{hh}$ itself.

However, before we are going to derive the projections of the Hessian operator, let us first decompose the gravitational cutoff operator $\mathfrak{R}_k^{\text{grav}}[\bar{g}]$. By definition its tensor structure is given by the identity of the respective field subspace, here $\mathbb{1}_{\rho\sigma}^{\mu\nu} = \frac{1}{2} (\delta_{\sigma}^{\mu} \delta_{\rho}^{\nu} + \delta_{\rho}^{\mu} \delta_{\sigma}^{\nu})$. Thus, the cutoff operator commutes with the trace-traceless decomposition and its orthogonal components are given by:

$$\begin{aligned} \mathfrak{R}_k^{\text{grav}}[\bar{g}] &= (\Pi_{\text{Tr}} + \Pi_{\text{noTr}}) \mathfrak{R}_k^{\text{grav}}[\bar{g}] (\Pi_{\text{Tr}} + \Pi_{\text{noTr}}) \\ &= \Pi_{\text{Tr}} \circ \mathfrak{R}_k^{\text{grav}}[\bar{g}] \circ \Pi_{\text{Tr}} + \Pi_{\text{noTr}} \circ \mathfrak{R}_k^{\text{grav}}[\bar{g}] \circ \Pi_{\text{noTr}} \end{aligned} \quad (7.32)$$

Next, consider eq. (7.28) that describes the Hessian operator conformally projected and with a family of generalized harmonic gauge fixing conditions inserted. Let us contract this Hessian matrix on the left with the inverse metric $\mathfrak{g}_{AB}^{-1}$. While the LHS of the FRGE is completely unaffected by this split due to the disappearance of metric fluctuations under conformal projection, the RHS decomposes in an non-orthogonal way.

$$\begin{aligned} \Pi_{\Omega}(\widehat{\text{Hess}}_{hh})_B^A &= (\Pi_{\text{noTr}})_B^C (\widehat{\text{Hess}}_{hh}^{\text{noTr}})_C^D (\Pi_{\text{noTr}})_D^A + (\Pi_{\text{Tr}})_B^C (\widehat{\text{Hess}}_{hh}^{\text{Tr}})_C^D (\Pi_{\text{Tr}})_D^A \\ &\quad + (\Pi_{\text{noTr}})_B^C (\widehat{\text{Hess}}_{hh}^{\text{noTr-Tr}})_C^D (\Pi_{\text{Tr}})_D^A + (\Pi_{\text{Tr}})_B^C (\widehat{\text{Hess}}_{hh}^{\text{Tr-noTr}})_C^D (\Pi_{\text{noTr}})_D^A \end{aligned} \quad (7.33)$$

Since the field space metric is diagonal in the trace-traceless split, we make use of eq. (7.31) and stepwise evaluate the operator parts in this decomposition. Therefore consider first the

contribution to the Hessian operator originating from the volume functional:

$$\begin{aligned} \mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \left\{ \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right\} &= \mathfrak{g}_{\alpha\beta}^{-1} \left((\Pi_{\text{noTr}})^B_{\mu\nu} + (\Pi_{\text{Tr}})^B_{\mu\nu} \right) \left\{ \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right\} \\ &= + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \bar{g}_{\kappa\mu} \bar{g}_{\lambda\nu} \left\{ \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right\} \\ &\quad + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \frac{1}{d(1-d\varpi)} \bar{g}_{\kappa\lambda} \bar{g}_{\mu\nu} \left\{ \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right\} \end{aligned}$$

Utilizing the idempotency of the projection operator and further simplify the expressions yields:

$$\begin{aligned} \mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \left\{ \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right\} &= + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \left(\frac{2-d}{2-2d\varpi} \right) \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \end{aligned}$$

Notice that this part of the operator is diagonal, in fact proportional to the identity, in the trace-traceless decomposition and the pre-factor of the trace part turns to one if $\varpi = 1/2$.

The next term in $(\widehat{\text{Hess}}_{hh})$ contains a Ricci- and a Riemann tensor. The first of these contributions is based on the curvature tensors which originally were associated to the dynamical metric. We abbreviate the calculation and only state the result:

$$\mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \left\{ \bar{g}^{\nu\sigma} \bar{R}^{\mu\rho} + \bar{R}^{\rho\mu\nu\sigma} \right\} = (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \bar{R}_{\kappa}^{\mu} \delta_{\lambda}^{\nu} + \bar{R}_{\kappa\lambda}^{\mu\nu} \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu}$$

Both, mixed and purely trace invariants vanish due to the symmetry properties of the Riemann tensor. Thus the result describes an operator restricted to the traceless part of metric field space only and therefore is also diagonal in the trace-traceless decomposition.

Now, the missing ultra-local term $\bar{\mathcal{U}}^{\mu\nu\rho\sigma}$ is evaluated for trace-traceless split. It contains combinations of Ricci- and scalar curvature tensors and decomposes as follows

$$\begin{aligned} \mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \left\{ \bar{\mathcal{U}}^{\mu\nu\rho\sigma} \right\} &= + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \left(\frac{2-d}{2-2d\varpi} \right) \left(\frac{d-4}{d} \right) \bar{R} \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \frac{(d-4)}{d(1-d\varpi)} \bar{g}_{\kappa\lambda} \bar{R}^{\mu\nu} \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \frac{(d-4)}{d} \bar{R}_{\kappa\lambda} \bar{g}^{\mu\nu} \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \bar{R} \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} - 4 \bar{R}_{\kappa}^{\mu} \delta_{\lambda}^{\nu} \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu} \end{aligned}$$

Here, for the first time we encounter off-diagonal contributions given by the second and third term. Similarly, the decomposition of $\bar{\mathcal{K}}^{\mu\nu\rho\sigma}$ splits non-diagonally whereby these terms in addition consist of uncontracted derivatives:

$$\begin{aligned} \mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \left\{ \bar{\mathcal{K}}^{\mu\nu\rho\sigma} \right\} &= + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ 2 \left(\frac{2-d}{2-2d\varpi} \right) \left(\frac{d-1}{d} \right) \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \bar{D}^2 \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \frac{(d-2)}{d(1-d\varpi)} \bar{g}_{\kappa\lambda} \bar{D}^{\mu} \bar{D}^{\nu} \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \frac{(d-2)}{d} \bar{g}^{\mu\nu} \bar{D}_{\kappa} \bar{D}_{\lambda} \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \\ &\quad + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ -2 \delta_{\lambda}^{\nu} \bar{D}^{\mu} \bar{D}_{\kappa} + \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \bar{D}^2 \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu} \end{aligned}$$

However, we wont be concerned with these terms for now, since they are treated later quite effectively using the freedom in the gauge fixing condition.

We are left with two contributions associated to the gauge fixing functional, the first which reads

$$\begin{aligned} \mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \left\{ (1-2\varpi) (\bar{g}^{\mu\nu} \bar{D}^{\rho} \bar{D}^{\sigma} + \bar{g}^{\rho\sigma} \bar{D}^{\mu} \bar{D}^{\nu}) \right\} \\ = + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ 2 \left(\frac{1-2\varpi}{1-d\varpi} \right) \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \bar{D}^2 \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \\ + (\Pi_{\text{Tr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ \left(\frac{1-2\varpi}{1-d\varpi} \right) \bar{g}_{\kappa\lambda} \bar{D}^{\mu} \bar{D}^{\nu} \right\} \cdot (\Pi_{\text{noTr}})^{\rho\sigma}_{\mu\nu} \\ + (\Pi_{\text{noTr}})^{\kappa\lambda}_{\alpha\beta} \cdot \left\{ (1-2\varpi) \bar{g}^{\mu\nu} \bar{D}_{\kappa} \bar{D}_{\lambda} \right\} \cdot (\Pi_{\text{Tr}})^{\rho\sigma}_{\mu\nu} \end{aligned}$$

While this expression does not contain a traceless-traceless term in the decomposition further off-diagonal contributions appear. Notice that for the harmonic gauge condition this entire expression vanishes.

Finally, the remaining term is again related to Γ_k^{gf} . While it is not proportional to $(1 - 2\varpi)$, it is of Laplacian type and needs no further simplification:

$$\mathfrak{g}_{\alpha\beta\mu\nu}^{-1} \{ (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - (1 - 2\varpi^2) \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma}) \bar{D}^2 \} = (\Pi_{\text{Tr}})_{\alpha\beta}^{\kappa\lambda} \cdot \left\{ \left(\frac{2d\varpi^2 - d + 1}{1 - d\varpi} \right) \delta_\kappa^\nu \delta_\lambda^\mu \bar{D}^2 \right\} \cdot (\Pi_{\text{Tr}})_{\mu\nu}^{\rho\sigma} \\ + (\Pi_{\text{noTr}})_{\alpha\beta}^{\kappa\lambda} \cdot \{ \delta_\kappa^\nu \delta_\lambda^\mu \bar{D}^2 \} \cdot (\Pi_{\text{noTr}})_{\mu\nu}^{\rho\sigma}$$

Here again, we observe a diagonal decomposition under the trace-traceless splitting.

We can now read off the components in the trace-traceless decomposition of the Hessian operator defined in eq. (7.33).

► **Traceless-Traceless term:**

$$(\widehat{\text{Hess}}_{hh}^{\text{noTr}})^{\mu\nu}_{\kappa\lambda} = - [\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}} \cdot (1 + (d - 4)\Omega)] \cdot (\delta_\kappa^\nu \delta_\lambda^\mu) \\ + \bar{u}_k^{\text{gf}} \cdot (\bar{R}_\kappa^\mu \delta_\lambda^\nu + \bar{R}^\mu{}_{\kappa\lambda}{}^\nu - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (\delta_\kappa^\nu \delta_\lambda^\mu \bar{D}^2)) \\ + \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}}] \cdot (\bar{R} \delta_\kappa^\nu \delta_\lambda^\mu - 4 \bar{R}_\kappa^\mu \delta_\lambda^\nu) \\ - \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot (-2 \delta_\lambda^\nu \bar{D}^\mu \bar{D}_\kappa + \delta_\kappa^\nu \delta_\lambda^\mu \bar{D}^2)$$

► **Trace-Trace term:**

$$(\widehat{\text{Hess}}_{hh}^{\text{Tr}})^{\mu\nu}_{\kappa\lambda} = - [\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}} \cdot (1 + (d - 4)\Omega)] \cdot \left(\frac{2 - d}{2 - 2d\varpi} \right) \delta_\kappa^\nu \delta_\lambda^\mu \\ + \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}}] \cdot \left(\frac{2 - d}{2 - 2d\varpi} \right) \left(\frac{d - 4}{d} \right) \cdot \bar{R} \delta_\kappa^\nu \delta_\lambda^\mu \\ - [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot \left(\left(\frac{2 - d}{2 - 2d\varpi} \right) \left(\frac{d - 1}{d} \right) \bar{D}^2 \right) \delta_\kappa^\nu \delta_\lambda^\mu \\ - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \left(\frac{2d\varpi^2 - d + 1}{1 - d\varpi} \right) \delta_\kappa^\nu \delta_\lambda^\mu \bar{D}^2$$

► **Traceless-Trace term:**

$$(\widehat{\text{Hess}}_{hh}^{\text{noTr-Tr}})^{\mu\nu}_{\kappa\lambda} = + \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}}] \cdot \left(\frac{d - 4}{d} \right) \bar{R}_{\kappa\lambda} \bar{g}^{\mu\nu} \\ - \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot \left(\frac{d - 2}{d} \right) \bar{g}^{\mu\nu} \bar{D}_\kappa \bar{D}_\lambda \\ - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (1 - 2\varpi) \bar{g}^{\mu\nu} \bar{D}_\kappa \bar{D}_\lambda$$

► **Trace-Traceless term:**

$$(\widehat{\text{Hess}}_{hh}^{\text{Tr-noTr}})^{\mu\nu}_{\kappa\lambda} = + \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}}] \cdot \left(\frac{d - 4}{d(1 - d\varpi)} \right) \bar{g}_{\kappa\lambda} \bar{R}^{\mu\nu} \\ - \frac{1}{2} [(1 + (d - 6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot \left(\frac{d - 2}{d(1 - d\varpi)} \right) \bar{g}_{\kappa\lambda} \bar{D}^\mu \bar{D}^\nu \\ - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \left(\frac{1 - 2\varpi}{1 - d\varpi} \right) \bar{g}_{\kappa\lambda} \bar{D}^\mu \bar{D}^\nu$$

Except for the terms related to $\bar{\mathcal{K}}$ and those proportional to $(1 - 2\varpi)$, there are only two additional contributions in the off-diagonal sector. Whereas the former two expression can be taken care of using the freedom in $\bar{u}_k^{\text{gf}}$ and ϖ , the latter which is ultra-local requires a different projection technique that we are going to exploit in the next subsection.

7.3.4 Projection to maximally symmetric spaces

The Hessian operator contains terms dependent on the Ricci- and Riemann-tensors, which – similar to the scalar curvature – are of second order in the covariant derivatives $\bar{D}$. Since this

corresponds to the maximal degree of $\bar{D}$ contained in the present truncation ansatz, these contributions either lead to higher order basis invariants or condense to the scalar curvature invariant which is the only basis monomial of degree $\mathcal{O}(\bar{D}^2)$.

A very convenient way to extract these contributions in the projection of the functional trace is based on maximally symmetric spaces. It uses the fact that the basis invariants on the LHS transform trivially when background geometries are inserted for which the Riemann- and Ricci-tensors are proportional to the constant scalar curvature, i.e.

$$\Pi_{\text{m.s.}}(\bar{R}_{\mu\nu\lambda\sigma}) = \frac{1}{d(d-1)} (\bar{g}_{\mu\lambda}\bar{g}_{\nu\sigma} - \bar{g}_{\nu\lambda}\bar{g}_{\mu\sigma}) C^2 \quad \text{and} \quad \Pi_{\text{m.s.}}(\bar{R}_{\mu\nu}) = \frac{1}{d}\bar{g}_{\mu\nu}C^2 \quad (7.38)$$

This is the definition of a maximally symmetric spaces, whereby $C^2 \equiv \Pi_{\text{m.s.}}(\bar{R})$ is a parameter representing the constant curvature. Notice that for higher derivative invariants the projection operator does not in general commute with the restriction to maximally symmetric spaces and thus this technique will be insufficient for a full resolution of the truncation, i.e.

$$\Pi_{\text{m.s.}}(\Pi_{\text{Trunc}}^S(\Gamma_k[\varphi; \bar{\Phi}])) \neq \Pi_{\text{Trunc}}^S(\Pi_{\text{m.s.}}(\Gamma_k[\varphi; \bar{\Phi}])) \quad \text{in general}$$

This is already the case for the basis invariants of $\mathcal{O}(\bar{D}^4)$ which are at most four in number. Under projection onto maximally symmetric spaces a generic functional of these four field monomials condenses to a single term given by

$$\Pi_{\text{m.s.}}(\bar{u}_k^1 \cdot \bar{D}^2 \bar{R} + \bar{u}_k^2 \cdot \bar{R}^2 + \bar{u}_k^3 \cdot \bar{R}^{\mu\nu} \bar{R}_{\mu\nu} + \bar{u}_k^4 \cdot \bar{R}^{\mu\nu\rho\sigma} \bar{R}_{\mu\nu\rho\sigma}) = \left(\bar{u}_k^2 + \frac{1}{d}\bar{u}_k^3 + \frac{2}{d(d-1)}\bar{u}_k^4 \right) \cdot C^4$$

This map is no isomorphism and thus inappropriate to distinguish between the RG evolution of the four couplings. Notice that the invariant associated to $\bar{u}_k^1$ even vanishes identically under the projection $\Pi_{\text{m.s.}}$, due to the fact that C is constant.

However, for the family of $f(\bar{R})$ truncations, of which the current Einstein-Hilbert functional is a special case, an isomorphism can be established and thus the trace evaluation for maximally symmetric spaces is just a simple procedure to extract the scalar part contained in $\bar{R}_{\mu\nu\lambda\sigma}$ and $\bar{R}_{\mu\nu}$. As an example, consider a polynomial-like function in $\bar{R}$ and apply the maximally symmetric space projection:

$$\Pi_{\text{m.s.}}\left(\sum_{n=0}^{\infty} \bar{u}_k^{(n)} \cdot \bar{R}^n\right) = \sum_{n=0}^{\infty} \bar{u}_k^{(n)} \cdot C^{2 \cdot n}$$

Instead of a polynomial in the scalar curvature $\bar{R}$ we have now a series of same degree in a constant value C^0 , in which one can straightforwardly expand. This demonstrates the isomorphism property of $\Pi_{\text{m.s.}}$ in case of $f(\bar{R})$ -theories.

Concerning the boundary invariants, we need a further specification of the embedded boundary to connect the extrinsic curvature tensor to C . A suitable choice would be the d -dimensional half-sphere that has S^{d-1} as its boundary. From the Gauß-Codazzi equations, for instance in the form of eq. (3.11), that we obtain [184]

$$\Pi_{\text{m.s.}}(\bar{K}_{\mu\nu}) = \frac{1}{\sqrt{d(d-1)}} \bar{H}_{\mu\nu} C \quad \Pi_{\text{m.s.}}(\bar{K}) = \sqrt{\frac{d-1}{d}} C$$

For the present truncation, this additional projection can be omitted, since the Hessian operator is of even power in C and will only contribute to boundary contributions of $\sqrt{\bar{H}}$ - or $\sqrt{\bar{H}}\bar{K}^2$ -type, for instance. Nevertheless, we adopt this general projection technique in what follows.

The $\Pi_{\text{m.s.}}$ -isomorphism applied to the LHS of the flow equation amounts to a simple replacement of the scalar- and extrinsic curvature tensor by powers of the constant C . Explicitly,

the projection of eq. (7.27) using the prescribed isomorphism results in

$$\begin{aligned} \Pi_{\text{m.s.}} \circ \Pi_{\Omega} (\Gamma_k^{\text{grav}} [g, \bar{g}]) = & +2 \cdot \int_M d^d x \sqrt{\bar{g}} \left((1 + d \cdot \Omega) \bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}} + (\bar{u}_k^{\text{B}} \bar{\lambda}_k^{\text{B}}) \right) \\ & - \int_M d^d x \sqrt{\bar{g}} C^2 \left((1 + (d-2)\Omega) \bar{u}_k^{\text{D}} + (\bar{u}_k^{\text{B}}) \right) \\ & + 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \left((\bar{u}_k^{\partial \text{D}} \bar{\lambda}_k^{\partial \text{D}}) + (\bar{u}_k^{\partial \text{B}} \bar{\lambda}_k^{\partial \text{B}}) \right) \\ & - 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \sqrt{\frac{(d-1)}{d}} C \left((\bar{u}_k^{\partial \text{D}}) + (\bar{u}_k^{\partial \text{B}}) \right) + \mathcal{O}(\Omega^2) \end{aligned} \quad (7.39)$$

On the RHS we the Riemann- and Ricci-tensor reduces to a combination of metric tensor structures. Thus, in particular the contraction with the traceless component yields further simplifications. In the gravitational sector, the application of $\Pi_{\text{m.s.}}$ gives rise to the following structures:³

► **Traceless-Traceless term:**

$$\begin{aligned} \Pi_{\text{m.s.}} (\widehat{\text{Hess}}_{hh}^{\text{noTr}})^{\mu\nu}_{\kappa\lambda} = & - [\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}} \cdot (1 + (d-4)\Omega)] \cdot (\delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu}) \\ & + \bar{u}_k^{\text{gf}} \cdot \left(\frac{1}{(d-1)} \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} C^2 \right) - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (\delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \bar{D}^2) \\ & + \frac{1}{2} [(1 + (d-6)\Omega) \bar{u}_k^{\text{D}}] \cdot \left(\frac{(d-4)}{d} \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} C^2 \right) \\ & - \frac{1}{2} [(1 + (d-6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot (-2 \delta_{\lambda}^{\nu} \bar{D}^{\mu} \bar{D}_{\kappa} + \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \bar{D}^2) \end{aligned} \quad (7.40)$$

► **Trace-Trace term:**

$$\begin{aligned} \Pi_{\text{m.s.}} (\widehat{\text{Hess}}_{hh}^{\text{Tr}})^{\mu\nu}_{\kappa\lambda} = & - [\bar{u}_k^{\text{D}} \bar{\lambda}_k^{\text{D}} \cdot (1 + (d-4)\Omega)] \cdot \left(\frac{2-d}{2-2d-\varpi} \right) \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \\ & + \frac{1}{2} [(1 + (d-6)\Omega) \bar{u}_k^{\text{D}}] \cdot \left(\frac{2-d}{2-2d-\varpi} \right) \left(\frac{d-4}{d} \right) \cdot C^2 \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \\ & - [(1 + (d-6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot \left(\left(\frac{2-d}{2-2d-\varpi} \right) \left(\frac{d-1}{d} \right) \bar{D}^2 \right) \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \left(\frac{2d\varpi^2-d+1}{(1-d\varpi)} \right) \delta_{\kappa}^{\nu} \delta_{\lambda}^{\mu} \bar{D}^2 \end{aligned} \quad (7.41)$$

► **Traceless-Trace term:**

$$\begin{aligned} \Pi_{\text{m.s.}} (\widehat{\text{Hess}}_{hh}^{\text{noTr-Tr}})^{\mu\nu}_{\kappa\lambda} = & - \frac{1}{2} [(1 + (d-6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot \frac{(d-2)}{d} \bar{g}^{\mu\nu} \bar{D}_{\kappa} \bar{D}_{\lambda} \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot (1 - 2\varpi) \bar{g}^{\mu\nu} \bar{D}_{\kappa} \bar{D}_{\lambda} \end{aligned} \quad (7.42)$$

► **Trace-Traceless term:**

$$\begin{aligned} \Pi_{\text{m.s.}} (\widehat{\text{Hess}}_{hh}^{\text{Tr-noTr}})^{\mu\nu}_{\kappa\lambda} = & - \frac{1}{2} [(1 + (d-6)\Omega) \bar{u}_k^{\text{D}} - \bar{u}_k^{\text{gf}}] \cdot \left(\frac{(d-2)}{d(1-d\varpi)} \bar{g}_{\kappa\lambda} \bar{D}^{\mu} \bar{D}^{\nu} \right) \bar{g}_{\mu\nu} \\ & - \frac{1}{2} \bar{u}_k^{\text{gf}} \cdot \left(\frac{1-2\varpi}{(1-d\varpi)} \right) \bar{g}_{\kappa\lambda} \bar{D}^{\mu} \bar{D}^{\nu} \end{aligned} \quad (7.43)$$

Notice that the existence of off-diagonal terms in the trace-traceless decomposition is now closely linked to the appearance of uncontracted derivatives.

Finally, the ghost sector appears a slight modification in that the contained Ricci-tensor is simply replaced by its maximally symmetric version, yielding

$$\Pi_{\text{m.s.}} ((\widehat{\text{Hess}}_{\xi\xi})_{\alpha}^{\mu}) = +\sqrt{2}(1+2\Omega) \cdot \bar{u}^{\text{gh}} (\mathfrak{g}_{\xi\xi}^{-1})_{\alpha}^{\lambda} (\delta_{\lambda}^{\mu} (\bar{D}^2 + \frac{1}{d}C^2) + (1-2\varpi)\bar{D}^{\mu}\bar{D}_{\lambda}) \quad (7.44a)$$

$$\Pi_{\text{m.s.}} ((\widehat{\text{Hess}}_{\xi\xi})_{\lambda}^{\alpha}) = -\sqrt{2}(1+2\Omega) \cdot \bar{u}^{\text{gh}} (\mathfrak{g}_{\xi\xi}^{-1})_{\mu}^{\alpha} (\delta_{\lambda}^{\mu} (\bar{D}^2 + \frac{1}{d}C^2) + (1-2\varpi)\bar{D}^{\mu}\bar{D}_{\lambda}) \quad (7.44b)$$

³From now on we usually omit $\mathcal{O}(\Omega^2)$ which has to be understood implicitly.

In order to apply the standard heat kernel expansion to evaluate the L^2 -operator trace, we finally have to solve the problem of consistently removing the uncontracted derivatives in the Hessian operator. This will remove both remaining issues, namely the off-diagonal terms in the trace-traceless decomposition will vanish and the Hessian operator becomes a function of the Laplacian $\bar{D}^2$ only.

R Notice that we are not fixing the background metric at all, but only use the prescribed technique to evaluate the functional trace. Different methods, which make no use of any special form of $\bar{g}$ at all, will give rise to the same results.

7.4 The ‘ Ω deformed’ $\alpha = 1$ harmonic gauge

C This entire section is conceptual and partly verbatim equivalent to subsection 2.5 of [173] with only minor modifications related to the non-trivial boundary while adapting the notation to the present work.

In this section we exploit the gauge fixing freedom to finalize the Hessian operator such that standard heat kernel techniques are applicable.

The outstanding task is to compute the RHS of the truncated FRGE, thereby retaining only the field monomials which are contained in the ansatz for the EAA. We want to utilize the known asymptotic expansions for the heat kernels of the operators involved, which is also well-studied in the case of ∂M . This would be particularly easy if those operators were functions of the Laplacian $\bar{D}^2$ only. A priori this is not the case, however. There are also operators such as $\bar{D}_\mu \bar{D}_\nu$ with uncontracted indices. One strategy to deal with them is to further decompose the fields $h_{\mu\nu}$, ξ^μ , and $\bar{\xi}_\mu$ into a sum of differentially constrained fields along the lines of the transverse-traceless (TT)- or York-decomposition. We already presented the extension to space-times with non-vanishing boundary where the fields satisfy Dirichlet conditions, see section 7.2. For further details and the complete story of the strategy we refer to ref. [172].

However, in this thesis we instead make use of the yet to be fixed parameters in the gauge fixing functional, a new technique introduced in ref. [173] that for the present truncation cancels all non-Laplacian differential operator terms. For reasons which will become apparent in a moment, we denote it the ‘ Ω deformed $\alpha = 1$ harmonic gauge’.

The properties

We have already noticed that the harmonic gauge condition $\varpi = 1/2$ reduces the operators occurring in the flow equation, in the ghost sector, to a function of the fully contracted Laplacian operator only. In the truncation ansatz, all operators of the type $\bar{D}^\mu \bar{D}^\nu$ are proportional to $(1 - 2\varpi)$ and thus drop out for $\varpi = 1/2$.

This choice, interpreted as a projection operator Π_ϖ also affects the ghost contributions on the RHS of the flow equation as can be seen from eq. (7.44b).

$$\Pi_\varpi \circ \Pi_{\text{m.s.}}((\widehat{\text{Hess}}_{\xi\bar{\xi}})^\mu_\alpha) = +2\sqrt{2}\Omega \cdot \bar{u}^{\text{gh}}(\mathfrak{g}_{\xi\bar{\xi}}^{-1})^\lambda_\alpha (\delta^\mu_\lambda (\bar{D}^2 + \frac{1}{d}C^2)) \quad (7.45a)$$

$$\Pi_\varpi \circ \Pi_{\text{m.s.}}((\widehat{\text{Hess}}_{\xi\bar{\xi}})^\alpha_\lambda) = -2\sqrt{2}\Omega \cdot \bar{u}^{\text{gh}}(\mathfrak{g}_{\xi\bar{\xi}}^{-1})^\alpha_\mu (\delta^\mu_\lambda (\bar{D}^2 + \frac{1}{d}C^2)) \quad (7.45b)$$

It is the only choice of ϖ that leads to a full cancellation of these terms in the ghost sector. Hence, from this point of view the harmonic gauge condition is necessary if the Hessian becomes of Laplacian type in the ghost sector.

The gravitational functional Γ_k^{grav} itself is free of uncontracted derivatives, but after a second variation they appear in the off-diagonal and the traceless-traceless component of the associated

Hessian operator. Quite remarkably, the harmonic gauge condition $\varpi = 1/2$ is also the best choice for the gravitational contributions. Employing $\varpi = 1/2$ all those terms that are proportional to some function of ϖ and contain uncontracted differential operators indeed vanish.

However, there is another source of troublesome covariant derivatives, namely those contained in $\mathcal{U}^{\mu\nu\rho\sigma}$ that in the trace-traceless decomposition contribute to the off-diagonal and the traceless part. Both are proportional to $[(1 + (d-6)\Omega)(\bar{u}_k^D) - \bar{u}_k^{gf}]$. Under the following choice of the gauge parameter these remaining terms disappear, leaving only Laplacian-type operators:

$$\bar{u}_k^{gf} = (1 + (d-6)\Omega)(\bar{u}_k^D) + \mathcal{O}(\Omega^2) \quad (7.46)$$

Before continuing note that $\bar{u}_k^{gf}$ has k dimension $(d-2)$ which in particular agrees with the one of $\bar{u}_k^D$. Introducing the *dimensionless* gauge fixing parameter $\alpha_k \equiv \bar{u}_k^D / \bar{u}_k^{gf}$, the preferred choice (7.46) corresponds to the value, up to $\mathcal{O}(\Omega^2)$ terms,

$$\alpha_k = 1 - (d-6)\Omega \equiv \alpha \quad (7.47)$$

Note that this α happens to be scale independent and in fact its full RG evolution is matched to the one of the dynamical Einstein-Hilbert functional.

Indeed, in all single-metric ('sm') calculations following ref. [122] the coefficient of the gauge fixing functional has been parametrized as

$$\bar{u}_k^{gf} = (\alpha^{sm})^{-1} \bar{u}_k^{sm} \quad (7.48)$$

and it was then the choice $\alpha_k^{sm} \equiv \alpha = 1$ that removed the uncontracted derivatives.

In order to achieve the same effect in the dynamical sector of the bi-metric setting we would have to generalize this choice in a Ω -dependent way, namely $\alpha = e^{(6-d)\Omega}$. Since only terms linear in Ω are relevant, we can think of

$$\alpha = 1 - (d-6)\Omega + \mathcal{O}(\Omega^2) \quad (7.49)$$

as being infinitesimally close to unity. It is this choice, eq. (7.49), which we refer to as the ' Ω deformed $\alpha = 1$ gauge'.

The justification

It should be clear that in general disposing of α as in (7.49), i.e. making it Ω dependent, is by no means legitimate from the conceptual point of view: The factor Ω represents the dynamical metric, $g = e^{2\Omega} \bar{g}$, it cannot appear in the final answer for the beta functions, but is rather a book keeping parameter we should expand in to disentangle terms with different powers of $h_{\mu\nu}$. The beta functions at levels-(0) and -(1), respectively, are obtained from the Taylor expansion of the traces:

$$\text{Tr}[\dots] = \text{Tr}[\dots]|_{\Omega=0} + \Omega \frac{d}{d\Omega} \text{Tr}[\dots]|_{\Omega=0} + \mathcal{O}(\Omega^2) \quad (7.50)$$

If we decide to eliminate the uncontracted covariant derivatives by giving the formally Ω -dependent value (7.49) to α we pay a price, namely we neglect a certain contribution to the linear term in (7.50), $\Omega \frac{d}{d\Omega} \text{Tr}[\dots]$, which arises through the Ω -dependence of $\alpha \equiv e^{(6-d)\Omega}$. By the chain rule, it has the form

$$(6-d)\Omega \frac{d}{d\alpha} \left\{ \text{Tr}[\dots]|_{\Omega=0} \right\} \Big|_{\alpha=1} \quad (7.51)$$

Now, the crucial observation is that the trace at $\Omega = 0$ whose α -derivative is taken here is to justify our choice, equals precisely the trace which appears in the single-metric computation.

Its α -dependence has been investigated in detail in ref. [135] and found to be rather small. In particular varying α over the interval from $\alpha = 0$ to $\alpha = 1$ causes changes in the flow properties which are smaller than those due to the cutoff scheme dependence and truncation error which, in fact, supplied the justification for setting α_k equal to a k -independent constant. This implies that, at the level of accuracy we may expect on the basis of the truncations already made, the term (7.51) is indeed negligible because $\text{Tr}[\dots]|_{\Omega=0}$ has no significant α -dependence near $\alpha = 1$. Thus, *in a truncation which neglects the running of α also the piece missed by the deformed gauge, (7.51), may be omitted.*

A second, independent justification

Formally using a Ω -dependent α -parameter requires a careful consideration of the resulting mixing of level-(0) with higher level contributions, which has to be kept as small as possible. Indeed, there is a way to justify the choice (7.49) from a different perspective. Assume we had started with a slightly different truncation ansatz in which the gauge fixing functional, Γ_k^{gf} , is substituted with

$$\Gamma_k^{(\gamma)}[h; \bar{g}] = \left(1 - \frac{1}{\gamma}\right) \frac{\bar{u}_k^{\text{gf}'}}{\bar{u}_k^{\text{gf}}} \Gamma_k^{\text{gf}}[h; \bar{g}] + \frac{1}{\gamma} \Gamma_k^{\text{subs}}[h; \bar{g}] \quad (7.52)$$

Here $\gamma \neq 0$ is a constant introduced for later convenience. Furthermore, $\Gamma_k^{\text{subs}}[h; \bar{g}]$ is a new functional which under the conformal projection ($g = e^{2\Omega} \bar{g}$), is required to obey

$$\begin{aligned} \Gamma_k^{\text{subs}}[h = 2\Omega \bar{g}; \bar{g}] &= 0 \quad \text{and} \\ \Pi_\Omega \widehat{\text{Hess}}_h [\Gamma_k^{\text{subs}}[h; \bar{g}]] &= e^{\gamma(d-6)\Omega} (\bar{u}_k^{\text{gf}'}/\bar{u}_k^{\text{gf}}) \Pi_\Omega \widehat{\text{Hess}}_h [\Gamma_k^{\text{gf}}[h; \bar{g}]] \end{aligned}$$

The first condition guarantees that the LHS, i.e. $\partial_t \Gamma_k$, remains unchanged, whereas the second requirement assures that the old and the new Hessians $\widehat{\text{Hess}}_\varphi [\Gamma_k[\varphi; \bar{\Phi}]]$ on the RHS agree at linear order in Ω for a certain choice of $\bar{u}_k^{\text{gf}'}$. Expanding the Hessian of (7.52) in powers of Ω ,

$$\Pi_\Omega \widehat{\text{Hess}}_h [\Gamma_k^{(\gamma)}[h; \bar{g}]] = (1 + (d-6)\Omega) (\bar{u}_k^{\text{gf}'}/\bar{u}_k^{\text{gf}}) \Pi_\Omega \widehat{\text{Hess}}_h [\Gamma_k^{\text{gf}}[h; \bar{g}]] + \mathcal{O}(\Omega^2)$$

we find it proportional to the ‘old’ gauge fixing contribution. The additional pre-factor can be absorbed by a redefinition of the gauge parameter as follows:

$$\bar{u}_k^{\text{gf}} \equiv (1 + (d-6)\Omega + \mathcal{O}(\Omega^2)) \bar{u}_k^{\text{gf}'} \quad (7.53)$$

The Hessians are equal then: $\Pi_\Omega \widehat{\text{Hess}}_h [\Gamma_k^{(\gamma)}[h; \bar{g}]] = \Pi_\Omega \widehat{\text{Hess}}_h [\Gamma_k^{\text{gf}}[h; \bar{g}]] + \mathcal{O}(\Omega^2)$. This in turn implies that the above steps for removing the uncontracted covariant derivatives work with all choices of $\Gamma_k^{(\gamma)}$, however now with the parameter

$$\bar{u}_k^{\text{gf}'} = \bar{u}_k^{\text{D}} \quad \Leftrightarrow \quad \alpha'_k = 1 \quad (7.54)$$

Within the generalized truncations (7.52), a Ω -independent choice for $\bar{u}_k^{\text{gf}'}$ is sufficient to account for the same simplification on the LHS of the flow equation that we used in the original ansatz with only Γ_k^{gf} containing the Ω -dependent coupling $\bar{u}_k^{\text{gf}}$. This Ω -dependence of a coupling in the truncation ansatz is thus merely an artifact of the conformal projection on the set of all possible invariants whereby different level- (p) monomials lead to the same Hessian. In this sense the Ω -dependence of $\bar{u}_k^{\text{gf}}$ can be thought of as a method for implicitly including additional field monomials in the truncation ansatz that lead to a removal of uncontracted covariant derivatives without introducing any new effects, otherwise.

There are several suitable candidates for $\Gamma_k^{\text{subs}}[h; \bar{g}]$ of which we consider here only those of the form

$$\Gamma_k^{\text{subs}}[h; \bar{g}] = \bar{u}_k^{\text{gf}'} \int \sqrt{\bar{g}} \bar{g}^{\mu\nu} \tilde{\mathcal{F}}_\mu^{(\gamma)}[g, \bar{g}](h) \tilde{\mathcal{F}}_\nu^{(\gamma)}[g, \bar{g}](h) \quad (7.55)$$

For the choice $\gamma = 2$ the pre-factor of the gauge fixing action in eq. (7.52) is $1/2$. The ‘missing’ half in the Hessian of the RHS of the flow equation now stems from the level-(2) contribution of $\Gamma_k^{\text{subs}}[h; \bar{g}]$. All higher level terms are new, additional field monomials with identical pre-factors fixed by $\bar{u}_k^{\text{gf}}$. Possible choices of the structure given in (7.55) with $\gamma = 2$ include

$$\tilde{\mathcal{F}}_\mu^{2;I}[g, \bar{g}] = \frac{\sqrt{\bar{g}}}{\sqrt{g}} g^{\alpha\lambda} g^{\beta\rho} g^{\kappa\sigma} \left[\bar{g}_{\mu\rho} \bar{g}_{\lambda\sigma} - \varpi \bar{g}_{\mu\sigma} \bar{g}_{\lambda\rho} \right] \bar{D}_\kappa \quad (7.56a)$$

$$\tilde{\mathcal{F}}_\mu^{2;II}[g, \bar{g}] = \frac{\sqrt{\bar{g}}}{\sqrt{g}} g^{\beta\lambda} \bar{g}_{\lambda\rho} \bar{g}_{\mu\sigma} \left[g^{\alpha\kappa} g^{\sigma\rho} - \varpi g^{\alpha\rho} g^{\kappa\sigma} \right] \bar{D}_\kappa \quad (7.56b)$$

It is straightforward to verify that all requirements on $\Gamma_k^{\text{subs}}[h; \bar{g}]$ are indeed fulfilled by these examples. (Notice that additional terms appearing in the Hessian associated to the action (7.55) are zero when evaluated for the conformally related metrics, $g = e^{2\Omega} \bar{g}$.)

Another interesting option is $\gamma = 1$. In this case one actually replaces the gauge fixing functional (6.26) with a new action containing the gauge fixing condition $\tilde{\mathcal{F}}_\mu^1[g, \bar{g}](h) = 0$. This choice requires however a new Faddeev-Popov operator $\tilde{\mathcal{M}}_\nu^\mu[g, \bar{g}]$ that corresponds to this class of gauge fixing condition. If we want to leave the RHS of the flow equation unchanged, we have to adopt an additional modification in the ghost sector.

First of all notice that with the choice (7.55) the requirements imposed on $\Gamma_k^{\text{subs}}[h; \bar{g}]$ can be transferred to requirements on $\tilde{\mathcal{F}}_\mu^\gamma[g, \bar{g}](h)$. We obtain $\Pi_\Omega \tilde{\mathcal{F}}_\mu^\gamma[g, \bar{g}](h) \stackrel{!}{=} 0$ from the vanishing of $\Gamma_k^{\text{subs}}[h; \bar{g}]$ under the conformal projection, and $(\partial_\nu \tilde{\mathcal{F}}_\mu^\gamma)[g, \bar{g}](h) = e^{\gamma(d-6)\Omega/2} \Pi_\Omega (\partial_\nu \mathcal{F}_\mu^\gamma)[g, \bar{g}](h)$ follows from the second criterion and by using $\Pi_\Omega (\tilde{\mathcal{F}}_\mu^\gamma[g, \bar{g}](h)) \stackrel{!}{=} 0$. Neglecting ghost contributions on the RHS of the flow equation allows us to rewrite the new Faddeev-Popov operator as being proportional to the original one, namely

$$\Pi_\Omega \tilde{\mathcal{M}}_\nu^\mu[g, \bar{g}] = e^{\gamma(d-6)\Omega/2} \mathcal{M}_\nu^\mu[g, \bar{g}] \quad (7.57)$$

This change of $\mathcal{M}$ could be compensated by a Ω -dependent ghost coupling $\bar{u}^{\text{gh}}$ in the original truncation. Thus, if there is a gauge fixing condition of the type

$$\tilde{\mathcal{F}}_\mu^{1;I}[g, \bar{g}](h) = \left(\frac{\sqrt{\bar{g}}}{\sqrt{g}} g^{\lambda\kappa} \bar{g}_{\lambda\kappa} \right)^{1/2} (\delta_\mu^\beta g^{\alpha\rho} \bar{D}_\rho - \varpi g^{\alpha\beta} \bar{D}_\mu) (g_{\alpha\beta} - \bar{g}_{\alpha\beta}) \quad (7.58)$$

for instance, it would correspond to the choice $\bar{u}_k^{\text{gf}} = \bar{u}^{\text{D}} [1 + (d-6)\Omega + \mathcal{O}(\Omega^2)]$ and $\bar{u}^{\text{gh}} = [1 + \frac{\gamma}{2}(d-6)\Omega] \bar{u}^{\text{gh}'}$ for the gauge parameter and the ghost coupling, respectively, in a truncation with the gauge fixing functional of eq. (11.56). We are not going to continue the analysis of the various possibilities and simply apply this prescribed technique to remove the undesirable uncontracted derivative terms in the Hessian operator.

The Hessian simplified

This particular choice of the gauge fixing sector simplifies the Hessian operator significantly. Thus, we are going to project the FRGE using the deformed $\alpha = 1$ harmonic gauge ($\varpi = 1/2$) with its associated projection $\Pi_{\alpha\Omega}$. Thereby, the RG running of the gauge coupling $\bar{\alpha}_k$ adjusts to the RG evolution of $\bar{u}_k^{\text{D}}$ in precisely the way that technical difficulties of uncontracted covariant derivatives gets eliminated. In the truncation ansatz, this corresponds to give a precise structure to the gauge fixing functional:

$$\Pi_{\alpha\Omega}(\Gamma_k^{\text{gf}}[h; \bar{g}]) = (1 + (d-6)\Omega) \bar{u}_k^{\text{D}} \int_{\text{M}} d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} \mathcal{F}_\mu[\bar{g}](h) \mathcal{F}_\nu[\bar{g}](h) \\ \text{with } \mathcal{F}_\mu[\bar{g}](h) \equiv \left(\bar{g}^{\alpha\rho} \bar{g}^{\beta\sigma} - \frac{1}{2} \bar{g}^{\alpha\beta} \bar{g}^{\rho\sigma} \right) \bar{g}_{\alpha\mu} \bar{D}_\beta h_{\rho\sigma}$$

When considering the bulk bi-metric beta-functions we shall provide another independent justification of this procedure by comparing our results to those of the calculation in ref. [172] where the uncontracted derivatives had been dealt with by a full transverse-traceless decomposition of $h_{\mu\nu}$.

Equation (7.45) already presents the final form of the Hessian operator in the ghost sector, since it is independent on $\bar{\alpha}_k$ and $\Pi_{\alpha\Omega}$ corresponds to setting $\varpi = 1/2$. These contributions basically consist of a simple differential operator of Laplace type with an additional constant summand: $(\bar{D}^2 + \frac{1}{d}C^2)$. In the gravitational sector, we experience an enormous reduction in particular because of the vanishing off-diagonal terms in the trace-traceless decomposition. Thus, what remains are the block diagonal contributions, both –the trace- and traceless-part– being functions of a single operator, $\bar{D}^2$:

$$\begin{aligned} \Pi_{\alpha\Omega} \circ \Pi_{\text{m.s.}} \circ \Pi_{\Omega}(\widehat{\text{Hess}}_{hh})_B^A &= +(\Pi_{\text{noTr}})_B^C (\widehat{\text{Hess}}_{hh}^{\text{noTr}})_C^D (\Pi_{\text{noTr}})_D^A \\ &\quad + (\Pi_{\text{Tr}})_B^C (\widehat{\text{Hess}}_{hh}^{\text{Tr}})_C^D (\Pi_{\text{Tr}})_D^A \end{aligned} \quad (7.59)$$

It is quite impressive in which way the choice of gauge fixing condition reduces the Hessian operators. For instance, the traceless-traceless constituent assumes the form:

$$\begin{aligned} \Pi_{\alpha\Omega} \circ \Pi_{\text{m.s.}} \circ \Pi_{\Omega}(\Pi_{\text{m.s.}}(\widehat{\text{Hess}}_{hh}^{\text{noTr}})_{\kappa\lambda}^{\mu\nu}) \\ = - [(\bar{u}_k^D \bar{\lambda}_k^D) \cdot (1 + (d-4)\Omega)] \cdot (\delta_\kappa^\nu \delta_\lambda^\mu) \\ + \frac{1}{2} [(1 + (d-6)\Omega)(\bar{u}_k^D)] \cdot \delta_\kappa^\nu \delta_\lambda^\mu \left\{ \left(\frac{d(d-3)+4}{d(d-1)} \right) C^2 - \bar{D}^2 \right\} \end{aligned} \quad (7.60)$$

Its trace-trace counterpart, which was already of Laplacian-type, also feels a strong reduction due to both, the harmonic gauge- and the $(\alpha = 1)$ -choice. In particular the global pre-factor disappears, since $(2-d)/(2-2d \cdot \varpi) = 1$ holds true in the harmonic case. Thus, the final result reads:

$$\begin{aligned} \Pi_{\alpha\Omega} \circ \Pi_{\text{m.s.}} \circ \Pi_{\Omega}(\Pi_{\text{m.s.}}(\widehat{\text{Hess}}_{hh}^{\text{Tr}})_{\kappa\lambda}^{\mu\nu}) \\ = - [(\bar{u}_k^D \bar{\lambda}_k^D) \cdot (1 + (d-4)\Omega)] \cdot \delta_\kappa^\nu \delta_\lambda^\mu \\ + \frac{1}{2} [(1 + (d-6)\Omega)(\bar{u}_k^D)] \cdot \left\{ \left(\frac{d-4}{d} \right) \cdot C^2 - \bar{D}^2 \right\} \delta_\kappa^\nu \delta_\lambda^\mu \end{aligned} \quad (7.61)$$

Notice that both operators have the trivial tensor structure $\delta_\kappa^\nu \delta_\lambda^\mu$.

Now all operators occurring under the traces on the RHS of the FRGE are functions of the Laplacian $\bar{D}^2$ alone, and we can easily apply the standard heat kernel techniques to compute the beta-functions for the six couplings of interest, $\bar{u}_k^D$, $\bar{\lambda}_k^D$, $\bar{u}_k^B$, $\bar{\lambda}_k^B$, and the boundary coefficients $(\bar{u}_k^{\partial B} + \bar{u}_k^{\partial D})$ as well as $(\bar{u}_k^{\partial B} \bar{\lambda}_k^{\partial B} + \bar{u}_k^{\partial D} \bar{\lambda}_k^{\partial D})$.

7.5 Heat kernel

Before we continue the evaluation of the functional trace on the RHS of the flow equation, it is instructive to summarize the so far obtained results. In order to extend the above projection techniques to affect the entire trace, notice that the cutoff operator $\mathfrak{R}_k[\bar{g}]$ is invariant under the projection $\Pi_{\text{oll}} := \Pi_{\alpha\Omega} \circ \Pi_{\text{m.s.}} \circ \Pi_{\Omega}$. In particular Π_{Ω} , as would be $\Pi_{\Omega(x)}$, leave $\mathfrak{R}_k[\bar{g}]$ unchanged. Hence, we can make use of Π_{oll} on the complete RHS with no further modifications.

In the last sections, we first demonstrated that under the action of $\Pi_{\xi=0=\xi}$ the ghost and

metric sector contribute separately, with independent traces given in eq. (7.20):

$$\begin{aligned} \text{STr}_{\text{F}} \left[\left(\widehat{\text{Hess}}_{\varphi} [\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \Big|_{\bar{\xi}=0=\xi} \\ = \text{STr}_{L^2}^{[h]} \left[\left(\widehat{\text{Hess}}_{hh} + \mathfrak{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{grav}} \right] \\ + \text{STr}_{L^2}^{[\xi]} \left[\left(\widehat{\text{Hess}}_{\xi\xi} + \mathfrak{R}_k^{\text{gh}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{gh}} \right] + \text{STr}_{L^2}^{[\xi]} \left[\left(\widehat{\text{Hess}}_{\xi\xi} + \mathfrak{R}_k^{\text{gh}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{gh}} \right] \end{aligned} \quad (7.62)$$

In the next steps, we mainly focused on the metric trace employing various projection techniques, Π_{all} , that ultimately lead to a trivial tensor structure of the Hessian with only Laplacian type operators. Therefore, we decomposed the identity $\mathbb{1}$ in its trace and traceless components and applied this projection on the Hessian operator. It turns out that for the harmonic gauge fixing ($\alpha = 1$)-choice $\widehat{\text{Hess}}_{hh}$ becomes diagonal w.r.t. Π_{Tr} and Π_{noTr} . Hence, since $\mathfrak{R}_k^{\text{grav}}[\bar{g}]$ is proportional to $\mathbb{1}$, the trace of the metric component decomposes in an orthogonal way⁴

$$\begin{aligned} \text{STr}_{L^2}^{[h]} \left[\left(\widehat{\text{Hess}}_{hh} + \mathfrak{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{grav}} \right] &= \text{STr}_{L^2}^{[h]} \left[\Pi_{\text{noTr}} \left(\widehat{\text{Hess}}_{hh}^{\text{noTr}} + \mathfrak{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{grav}} \right] \\ &+ \text{STr}_{L^2}^{[h]} \left[\Pi_{\text{Tr}} \left(\widehat{\text{Hess}}_{hh}^{\text{Tr}} + \mathfrak{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{grav}} \right] \end{aligned}$$

In total there are four traces over functions of the Laplacian operator. However, the structure of the traces is in all four cases quite similar, in particular the Hessian operators exhibit a common pattern that allows for a general treatment of the heat kernel expansion:

$$H(\ell, C; c) := -\bar{D}^2 - 2e^{2\Omega}\ell + \rho_p c C^2$$

Here, we introduced an x -independent constant c and an artificial bookkeeping parameter $\rho_p = 1$. The latter is inserted to study the impact of ‘paramagnetic’- or ‘diamagnetic’-like contributions to the RG evolution of the couplings [165]. Loosely speaking, ‘paramagnetic’ terms appear as non-minimal contribution in the propagator which here corresponds to the scalar curvature part, proportional to C^2 .

The four parts of the Hessian operator can now be related to $H(\ell, C; c)$, evaluated for particular values of c and ℓ . Substituting $(g_{\xi\xi}^{-1})_{\mu}^{\alpha} = \delta_{\mu}^{\alpha} = (g_{\xi\xi}^{-1})_{\mu}^{\alpha}$, we arrive at

$$\Pi_{\text{all}} \widehat{\text{Hess}}_{hh}^{\text{noTr}} \equiv \frac{1}{2} e^{(d-6)\Omega} \bar{u}_k^{\text{D}} \cdot H(\bar{\lambda}_k^{\text{D}}, C; c_{\text{T}}) \cdot \mathbb{1}_{[\text{noTr}]} \quad \text{with} \quad c_{\text{T}} \equiv \frac{d(d-3)+4}{d(d-1)} \quad (7.63a)$$

$$\Pi_{\text{all}} \widehat{\text{Hess}}_{hh}^{\text{Tr}} \equiv \frac{1}{2} e^{(d-6)\Omega} \bar{u}_k^{\text{D}} \cdot H(\bar{\lambda}_k^{\text{D}}, C; c_{\text{S}}) \cdot \mathbb{1}_{[\text{Tr}]} \quad \text{with} \quad c_{\text{S}} \equiv \frac{(d-4)}{d} \quad (7.63b)$$

$$\Pi_{\text{all}} \widehat{\text{Hess}}_{\xi\xi} \equiv -\sqrt{2} e^{2\Omega} \bar{u}^{\text{gh}} \cdot H(0, C; c_{\text{V}}) \cdot \mathbb{1}_{[\xi]} \quad \text{with} \quad c_{\text{V}} \equiv -\frac{1}{d} \quad (7.63c)$$

$$\Pi_{\text{all}} \widehat{\text{Hess}}_{\xi\xi}^{\text{gh}} \equiv \sqrt{2} e^{2\Omega} \bar{u}^{\text{gh}} \cdot H(0, C; c_{\text{V}}) \cdot \mathbb{1}_{[\xi]} \quad (7.63d)$$

Hereby, we assumed that all $\mathcal{O}(\Omega^2)$ terms are neglected.

The cutoff operator has the same (trivial) tensor structure than the above operators and furthermore it shares the coefficients with the kinetic operator $-\bar{D}^2$ in each field space component for $\Omega = 0$. In detail we have:

$$\begin{aligned} \mathfrak{R}_k^{\text{grav}}[\bar{g}] &\equiv +\frac{1}{2} \bar{u}_k^{\text{D}} \cdot R_k^{(0)}(-\bar{D}^2) \cdot \mathbb{1}_{[h]} \quad \text{and} \\ \mathfrak{R}_k^{\text{gh}}[\bar{g}] &\equiv -\sqrt{2} \bar{u}^{\text{gh}} \cdot R_k^{(0)}(-\bar{D}^2) \cdot \mathbb{1}_{[\xi]}, \quad \mathfrak{R}_k^{\text{gh}}[\bar{g}] \equiv +\sqrt{2} \bar{u}^{\text{gh}} \cdot R_k^{(0)}(-\bar{D}^2) \cdot \mathbb{1}_{[\xi]} \end{aligned}$$

⁴For brevity we omitted the additional projections, Π_{all} , which however are essentially to obtain the diagonal structure of the Hessian operator.

Hence, for the present truncation, the general form of the inverse operator, that appears on the RHS of the flow equation, can be written as follows:⁵

$$\begin{aligned} \left(\Pi_{\text{all}} \widehat{\text{Hess}}_{\bullet} + \mathfrak{R}_k^{\bullet}[\bar{g}] \right) &= \bar{u}_k^{\bullet} e^{\gamma \Omega} \left(-\bar{D}^2 - 2e^{2\Omega} \ell + c \rho_P C^2 \right) \mathbb{1}_{[\bullet]} + \bar{u}_k^{\bullet} R_k^{(0)}(-\bar{D}^2) \mathbb{1}_{[\bullet]} \\ &= +\bar{u}_k^{\bullet} \cdot \mathbb{1}_{[\bullet]} \cdot \left\{ \left(-\bar{D}^2 - 2\ell + R_k^{(0)}(-\bar{D}^2) \right) + c \rho_P C^2 \right\} \\ &\quad + \gamma \cdot \bar{u}_k^{\bullet} \cdot \mathbb{1}_{[\bullet]} \cdot \left\{ \left(-\bar{D}^2 - 2 \left(\frac{\gamma+2}{\gamma} \right) \ell \right) + c \rho_P C^2 \right\} \cdot \Omega + \mathbb{O}(\Omega^2) \end{aligned}$$

Here, γ denotes the conformal weight of the underlying invariant and in the present case is $\gamma^{\text{grav}} = (d-6)$ and $\gamma^{\text{gh}} = 2$ for the metric- and ghost-contribution, respectively.

Using the abbreviation $\mathcal{A}(\ell) \equiv (-\bar{D}^2 - 2\ell + R_k^{(0)}(-\bar{D}^2))$ for the diamagnetic $\Omega = 0$ contribution, we can expand the inverse of the above operator structure up to first order in Ω :

$$\begin{aligned} \left(\Pi_{\text{all}} \widehat{\text{Hess}}_{\bullet} + \mathfrak{R}_k^{\bullet}[\bar{g}] \right)^{-1} &= (\mathcal{A}(\ell) + c \rho_P C^2)^{-1} (\bar{u}_k^{\bullet})^{-1} \cdot \mathbb{1}_{[\bullet]} \\ &\quad - \Omega \cdot (\mathcal{A}(\ell) + c \rho_P C^2)^{-2} \left(-\bar{D}^2 - 2 \left(\frac{\gamma+2}{\gamma} \right) \ell + c \rho_P C^2 \right) (\bar{u}_k^{\bullet})^{-1} \gamma \cdot \mathbb{1}_{[\bullet]} \end{aligned} \quad (7.64)$$

This result is very useful, since all four traces are covered by its general form. In the end, we only have to insert the particular values of c , ℓ , and γ .

The operator of eq. (7.64) acts on $\partial_t \mathfrak{R}_k^{\bullet}$ and finally, we have to trace these terms using the heat kernel asymptotics. Hence, let us consider the general form of this missing operator:

$$\partial_t \mathfrak{R}_k^{\bullet}[\bar{g}] = \partial_t (\bar{u}_k^{\bullet} \cdot R_k^{(0)}(-\bar{D}^2)) \mathbb{1}_{[\bullet]}$$

Equation (7.64) implies that any k -independent scaling of the coefficient $\bar{u}_k^{\bullet}$ does not affect the RHS of the flow equation since it appears once in the denominator and as $\partial_t \bar{u}_k^{\bullet}$ in the numerator. In particular, the two traces in the ghost sector can be combined, since the Hessian L^2 -operators differed only by a factor of -1 which is compensated in combination with $\partial_t \mathfrak{R}_k^{\bullet}[\bar{g}]$. Hence, we only have to deal with three independent traces, namely

$$\begin{aligned} \Pi_{\text{all}} \text{STr}_F \left[\left(\widehat{\text{Hess}}_{\varphi}[\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \circ \partial_t \mathfrak{R}_k[\bar{\Phi}] \right] \\ = \Pi_{\text{all}} \text{STr}_{L^2}^{[h]} \left[\Pi_{\text{noTr}} \left(\widehat{\text{Hess}}_{hh}^{\text{noTr}} + \mathfrak{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{grav}} \right] \\ + \Pi_{\text{all}} \text{STr}_{L^2}^{[h]} \left[\Pi_{\text{Tr}} \left(\widehat{\text{Hess}}_{hh}^{\text{Tr}} + \mathfrak{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{grav}} \right] \\ + 2 \Pi_{\text{all}} \text{STr}_{L^2}^{[\xi]} \left[\left(\widehat{\text{Hess}}_{\xi\xi} + \mathfrak{R}_k^{\text{gh}}[\bar{g}] \right)^{-1} \partial_t \mathfrak{R}_k^{\text{gh}} \right] \end{aligned} \quad (7.65)$$

Let us now introduce one more functional $W_p(\Delta, \ell)$, with $\Delta = -\bar{D}^2$, in which the inverted operator and $\partial_t \mathfrak{R}_k^{\bullet}[\bar{g}]$ are combined:

$$\begin{aligned} W_p(\Delta; \ell, \bar{u}_k^{\bullet}) &:= \frac{1}{2} \mathcal{A}(\ell)^{-p} \cdot (\bar{u}_k^{\bullet})^{-1} \partial_t \mathfrak{R}_k^{\bullet} \\ &\equiv \frac{1}{2} \mathcal{A}(\ell)^{-p} \cdot \left(\partial_t R_k^{(0)}(\Delta) + R_k^{(0)}(\Delta) \partial_t \ln(\bar{u}_k^{\bullet}) \right) \cdot \mathbb{1}_{[\bullet]} \end{aligned} \quad (7.66)$$

In terms of the functional $W_p(\Delta, \ell)$ the Taylor expansion of the functional trace in the constant C assumes the form:

$$\begin{aligned} \frac{1}{2} \text{STr}_{L^2}^{[\bullet]}[\dots] &= \text{STr}_{L^2}^{[\bullet]}[W_1(\Delta; \ell, \bar{u}_k^{\bullet})] - \text{STr}_{L^2}^{[\bullet]} \left[\Omega \gamma \cdot \left(\Delta - 2 \left(\frac{\gamma+2}{\gamma} \right) \ell \right) \cdot W_2(\Delta; \ell, \bar{u}_k^{\bullet}) \right] \\ &\quad + \text{STr}_{L^2}^{[\bullet]} \left[C^2 \cdot (c \rho_P) \left\{ (\Omega \gamma - 1) W_2(\Delta; \ell, \bar{u}_k^{\bullet}) + 2 \Omega \gamma \cdot \left(\Delta - 2 \left(\frac{\gamma+2}{\gamma} \right) \ell \right) \cdot W_3(\Delta; \ell, \bar{u}_k^{\bullet}) \right\} \right] \end{aligned} \quad (7.67)$$

⁵For brevity, we include the pre-factors $\frac{1}{2}$ or $\pm\sqrt{2}$ in the k -dependent coefficient $\bar{u}_k^{\bullet}$.

Here, the neglected higher order terms are either of order $\mathcal{O}(\Omega^2)$ or $\mathcal{O}(C^4)$ and thus irrelevant for the present truncation ansatz.

In the final expansion based on the heat kernel asymptotics, we project the RHS on the basis invariants contained in our truncation ansatz. In total, there are 3 times 6 (−2) traces that have to be evaluated. The following provides a list of the sixteen components that ultimately constitute the beta-functions. We start with the terms of zeroth order in C :

$$\text{RHS}|_{C=0} = \text{STr}_{L^2}^{[\text{noTr}]} [W_1(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{noTr}]}] + \text{STr}_{L^2}^{[\text{Tr}]} [W_1(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{Tr}]}] \quad (7.68a)$$

$$+ \text{STr}_{L^2}^{[\text{gh}]} [W_1(\Delta; 0, \bar{u}^{\text{gh}}) \cdot \mathbb{1}_{[\text{gh}]}] \quad (7.68b)$$

$$- \text{STr}_{L^2}^{[\text{noTr}]} [(d-6)\Omega \Delta W_2(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{noTr}]}] \quad (7.68c)$$

$$- \text{STr}_{L^2}^{[\text{Tr}]} [(d-6)\Omega \Delta W_2(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{Tr}]}] \quad (7.68d)$$

$$- \text{STr}_{L^2}^{[\text{gh}]} [2(2\Omega) \Delta W_2(\Delta; 0, \bar{u}^{\text{gh}}) \cdot \mathbb{1}_{[\text{gh}]}] \quad (7.68e)$$

$$+ \text{STr}_{L^2}^{[\text{noTr}]} [2(d-4)\Omega \bar{\lambda}_k^D \cdot W_2(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{noTr}]}] \quad (7.68f)$$

$$+ \text{STr}_{L^2}^{[\text{Tr}]} [2(d-4)\Omega \bar{\lambda}_k^D \cdot W_2(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{Tr}]}] + \mathcal{O}(\Omega^2) \quad (7.68g)$$

The next order is already quadratic in C and thus does not induce any boundary contributions within our truncation (remember that $\bar{K}$ is of order C).

$$\text{RHS} = \text{RHS}|_{C=0} - \text{STr}_{L^2}^{[\text{noTr}]} [C^2 \cdot (c_T \rho_P) (1 + (d-6)\Omega) W_2(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{noTr}]}] \quad (7.69a)$$

$$- \text{STr}_{L^2}^{[\text{Tr}]} [C^2 \cdot (c_S \rho_P) (1 + (d-6)\Omega) W_2(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{Tr}]}] \quad (7.69b)$$

$$- \text{STr}_{L^2}^{[\text{gh}]} [2 \cdot C^2 \cdot (c_V \rho_P) (1 + 2\Omega) W_2(\Delta; 0, \bar{u}^{\text{gh}}) \cdot \mathbb{1}_{[\text{gh}]}] \quad (7.69c)$$

$$+ \text{STr}_{L^2}^{[\text{noTr}]} [2(d-6)\Omega C^2 \cdot (c_T \rho_P) \Delta W_3(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{noTr}]}] \quad (7.69d)$$

$$+ \text{STr}_{L^2}^{[\text{Tr}]} [2(d-6)\Omega C^2 \cdot (c_S \rho_P) \Delta W_3(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{Tr}]}] \quad (7.69e)$$

$$+ \text{STr}_{L^2}^{[\text{gh}]} [8\Omega (C^2 \cdot (c_V \rho_P) \Delta W_3(\Delta; 0, \bar{u}^{\text{gh}}) \cdot \mathbb{1}_{[\text{gh}]}] \quad (7.69f)$$

$$- \text{STr}_{L^2}^{[\text{noTr}]} [4\Omega C^2 \cdot (c_T \rho_P) (d-4) \bar{\lambda}_k^D \cdot W_3(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{noTr}]}] \quad (7.69g)$$

$$- \text{STr}_{L^2}^{[\text{Tr}]} [4\Omega C^2 \cdot (c_S \rho_P) (d-4) \bar{\lambda}_k^D \cdot W_3(\Delta; \bar{\lambda}_k^D, \bar{u}_k^D) \cdot \mathbb{1}_{[\text{Tr}]}] + \mathcal{O}(C^4, \Omega^2) \quad (7.69h)$$

Each function in this sum is either of type $\text{STr}_{L^2}^{[\text{noTr}]} [f W(\Delta) \cdot \mathbb{1}]$ or $\text{STr}_{L^2}^{[\text{noTr}]} [f \Delta W(\Delta) \cdot \mathbb{1}]$, with $f \equiv f(C, \Omega, d)$ an x -independent function.

R Notice that for the general conformal projection or in the case of an explicit h -expansion $f \equiv f(x)$ since $\Omega \equiv \Omega(x)$ becomes spacetime dependent.

The heat kernel

In the preceding steps we managed to reduce the Hessian operator to a function of the fully contracted Laplacian $\Delta \equiv -\bar{D}^2$ only. For the chosen Dirichlet boundary conditions and M being compact with non-vanishing boundary in general, $\Delta = -\bar{D}^2$ is not only elliptic but also strongly elliptic and self-adjoint with non-negative spectrum and a finite number of zero modes, $\dim(\ker(-\bar{D}^2)) < \infty$. For these operators, there exists a unique solution to the heat equation that can be used to expand the RHS of the flow equation.

Therefore, consider the operator $h_s \equiv e^{-s\Delta}$ which under the above conditions is related to a unique kernel, the heat kernel, defined by

$$(h_s f)(y) \equiv \int_M d^d x \sqrt{g} h_s(y, x) f(x) \quad \forall f \in L^2(M)$$

The heat kernel has important applications in physics, since it solves the heat equation, the ‘Wick-rotated’ analog of the Schrödinger differential equation:

$$(\Delta + \partial_s) h_s(y, x) = 0 \quad \text{with} \quad \lim_{s \rightarrow 0^+} \int_M d^d x \sqrt{g} h_s(y, x) f(x) = f(y) \quad \forall f \in L^2(M)$$

In the presence of (regular) boundaries we have to add a supplementary boundary condition to uniquely specify the kernel $h_s(y, x)$; for Dirichlet constraints this simply reads $h_s(y, x) = 0$ for all $y \in \partial M$.

Besides the symmetry $h_s(y, x) = h_s(x, y)$, the heat kernel inherits its homogeneity-, locality-, and translational invariance properties from the differential equation it satisfies. For non-empty boundaries the invariance under translations is broken. However, in the present context, its most important feature is given by the existence of a complete asymptotic expansion in the limits $s \rightarrow 0^+$ and $s \rightarrow +\infty$. The latter asymptotics results in the projection operator on the zero modes of Δ , which becomes apparent when expanding the heat kernel in eigenfunctions of Δ :

$$h_s(y, x) = \sum_j e^{-s\lambda_j} \phi_j(y) \phi_j(x) \quad \text{with} \quad \Delta \phi_j(x) = \lambda_j \phi_j(x) \quad (7.70)$$

If s approaches $+\infty$ the exponential starts dominating the eigenfunctions, which are locally confined due to the heat equation. What remains are the unsuppressed zero-modes, in particular we have

$$\lim_{s \rightarrow +\infty} h_s(x, x) = \dim \ker(-\bar{D}^2) < \infty$$

For our purpose more interesting is the opposite limit, where $h_s(x, y)$ stays close to the delta-distribution.

$$\lim_{s \rightarrow 0^+} h_s(x, x) = s^{-d/2} \sum_{j=0}^{\infty} \alpha_j(x) s^{j/2} \quad (7.71)$$

Hereby, the coefficients α_j are smooth functions and the equality holds uniformly, thus we can integrate over both sides and still retain the identity:

$$\begin{aligned} \lim_{s \rightarrow 0^+} \text{Tr}_{L^2} [h_s(f)] &= \lim_{s \rightarrow 0^+} \text{Tr}_{L^2} [e^{-s\Delta} f] = s^{-d/2} \sum_{j=0}^{\infty} a_j(f; \Delta) \sqrt{s^j} \quad \text{with} \\ a_j(f; \Delta) &= \int_M d^d x \sqrt{g} f(x) \alpha_j(x) + \sum_{r=0}^{j-1} \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} (n^\mu \bar{D}_\mu)^r f(x) \cdot \alpha_{j-r}(x) \end{aligned} \quad (7.72)$$

We introduced a smooth function f for later convenience that furthermore fulfills Dirichlet boundary conditions. In the present case $f(x)$ is constant, however in general it contains powers of $h(x)$ or the conformal factor $\Omega(x)$ when $\Pi_{\Omega(x)}$ is applied. Notice that whenever ∂M is empty, the sum only runs over even integers, since the odd ones represent boundary monomials. Explicitly, for Dirichlet conditions, the very general and universally valid expression of the few Seeley-DeWitt coefficients $a_j(f; \Delta)$, which were already known from the pioneering work of McKean and Singer [185], are given by⁶

$$a_0(f; \Delta) = +(4\pi)^{-\frac{d}{2}} \int_M d^d x \sqrt{g} f \quad (7.73a)$$

$$a_1(f; \Delta) = -(4\pi)^{-\frac{d}{2}} \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \frac{\sqrt{\pi}}{2} f \quad (7.73b)$$

$$a_2(f; \Delta) = +(4\pi)^{-\frac{d}{2}} \int_M d^d x \sqrt{g} \frac{1}{6} f \bar{R} \quad (7.73c)$$

$$+ (4\pi)^{-\frac{d}{2}} \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \frac{1}{6} (3n^\rho \bar{D}_\rho f + 2f \bar{K}) \quad (7.73d)$$

⁶Here $\text{str}(\mathbb{1})$ denotes the trace over the space on which Δ acts.

Thus, terminating the asymptotic series at order $s^{3/2}$ the terms retained are precisely those invariants contained in the truncation ansatz, i.e.

$$\begin{aligned} \text{Tr}_{L^2} \left[e^{s\bar{D}^2} f \cdot \mathbb{1} \right] &= \frac{\text{str}(\mathbb{1})}{(4\pi s)^{d/2}} \left\{ \int_M d^d x \sqrt{\bar{g}} f(x) - \frac{1}{4} \sqrt{4\pi s} \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} f(x) \right. \\ &\quad + \frac{1}{6} s \left(\int_M d^d x \sqrt{\bar{g}} \bar{R} f(x) + 2 \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \bar{K} f(x) \right) \\ &\quad \left. + \frac{1}{2} s \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} n^\rho \bar{D}_\rho f(x) + \mathcal{O}(s^{3/2}) \right\} \end{aligned} \quad (7.74)$$

The seemingly additional term containing the normal derivative $n^\rho \bar{D}_\rho f(x)$ will contribute to the dynamical curvature invariant of the boundary, where under conformal projection $f(x)$ is proportional to $\Omega(x)$. Extending the final truncation ansatz to include those derivative terms allows for deriving the beta-function of $\bar{u}_k^{\partial B}$ and $\bar{u}_k^{\partial D}$ independently.

R In the presence of boundary there appear Seeley-DeWitt coefficients $a_j(f; \Delta)$ of half-integer order [185, 186] that generate only boundary terms in the trace expansion. Furthermore, notice that in order $\mathcal{O}(s)$ the Einstein-Hilbert and Gibbons-Hawking-York terms appear in precisely the ‘preferred’ combination with a relative coefficient of $+2$, necessary for obtaining field equations in GR.

The great virtue for our present discussion is the special structure of the asymptotic expansion. By dimensional arguments it is already clear that each order in the expansion includes higher derivative operators, thus eq. (7.72) can really be understood as a series in basis invariants with increasing order in $\bar{D}$. This is especially useful to project the flow equation onto the truncation ansatz consisting of the zeroth, first and second order in $\bar{D}$. Now the trace over a general function $W(\Delta)$ of the Laplacian $\Delta = -\bar{D}^2$ can be expanded in this asymptotic series. Therefore, consider the Fourier transform of $W(\Delta)$ and substitute $s = -iv$:

$$\text{Tr}_{L^2} [W(\Delta) f \cdot \mathbb{1}] = \int_{-\infty}^{+\infty} dv \hat{W}(v) \text{Tr}_{L^2} [e^{iv\Delta} f \cdot \mathbb{1}] \equiv -i \int_{-\infty}^{+\infty} ds \hat{W}(is) \text{Tr}_{L^2} [h_s(f)] \text{str}(\mathbb{1})$$

In this way, the function specific information becomes separated from the geometrical data, which is represented by the trace over the heat kernel. Inserting the asymptotic expansion in the limit $s \rightarrow 0^+$ into the last equation, we arrive at a derivative expansion of $\text{Tr}_{L^2} [W(\Delta) f]$:

$$\begin{aligned} \text{Tr}_{L^2} [W(\Delta) f \cdot \mathbb{1}] &= -i \int_{-\infty}^{+\infty} ds \hat{W}(is) \left[\sum_{j \geq 0} \sqrt{s}^{j-d} a_j(f; \Delta) \right] \text{str}(\mathbb{1}) \\ &= + \sum_{j \geq 0} a_j(f; \Delta) \cdot \text{str}(\mathbb{1}) \cdot \int_{-\infty}^{+\infty} dv \hat{W}(v) \sqrt{-iv}^{j-d} \end{aligned} \quad (7.75)$$

Notice that the Fourier decomposition is compatible with the asymptotic expansion $s \rightarrow 0^+$, since the domain of integration has real part $\Re(s) = 0$. Hence, the above series precisely reflects a projection of the RHS of the flow equation onto the basis monomials that constitute our truncation ansatz. Furthermore, we have separated the geometric information contained in the Laplacian, from the specific shape of the function W .

7.5.1 Mellin transform

A feature of the heat kernel asymptotics is that the part containing the structure of the function $W(v)$ can be described by its Mellin transform, defined in [187]:

$$\mathcal{Q}_n[W] \equiv \int_{-\infty}^{+\infty} dv \hat{W}(v) (-iv)^{-n}. \quad (7.76)$$

This gives rise to the same integral transform that appears in the heat kernel expansion of W in eq. (7.75). Hence, the close relation between h_s and the Mellin transform is encoded in the following asymptotic series:

$$\mathrm{Tr}_{L^2} [W(\Delta) f \cdot \mathbb{1}] = \sum_{j \geq 0} a_j(f; \Delta) \cdot \mathrm{str}(\mathbb{1}) \cdot \mathcal{Q}_{(d-j)/2}[W] \quad (7.77)$$

The Mellin transform can be rewritten in a more elegant form that involves the function W rather than its Fourier transform $\widehat{W}$. Therefore, we first insert an identity based on the integral representation of the gamma function $\Gamma(n)$.

$$\begin{aligned} \mathcal{Q}_n[W] &= \int_{-\infty}^{+\infty} dv \widehat{W}(v) (-iv)^{-n} \cdot \frac{1}{\Gamma_n} \left[(-i)^n \int_0^{v \cdot \infty} dw w^{n-1} e^{i w} \right] \\ &= \frac{1}{\Gamma_n} \int_{-\infty}^{+\infty} dv \widehat{W}(v) \cdot \int_0^{v \cdot \infty} \frac{dw}{v} \left(\frac{w}{v} \right)^{n-1} e^{i w} \end{aligned}$$

In the last step we suggestively rearranged the terms in the second integral such that substituting $z = \frac{v}{u}$ for v in the measure yields

$$\mathcal{Q}_n[W] = \frac{1}{\Gamma_n} \int_{-\infty}^{+\infty} dv \widehat{W}(v) \int_0^{\infty} dz z^{n-1} e^{i v \cdot z} = \frac{1}{\Gamma_n} \int_0^{\infty} dz z^{n-1} \int_{-\infty}^{+\infty} du \widehat{W}(v) e^{i v \cdot z}$$

At this point notice that the v integration represents the inverse Fourier transformation. Hence, the Mellin transform assumes the form

$$\mathcal{Q}_n[W] = \frac{1}{\Gamma_n} \int_0^{\infty} dz z^{n-1} W(z) \quad (7.78)$$

Notice that for $n = 0$ and for functions W that have proper support on the positive real numbers, we obtain the limit $\mathcal{Q}_0[W] = W(0)$.

The Mellin transform of a function exhibits certain very useful properties that turn out to be relevant in studying the general properties of the later derived threshold-functions. As an integral transform the $\mathcal{Q}_n$ are linear in its argument. On the other hand a change of the degree n is related to a transformation of the argument. For instance the Mellin transforms of the functions $W(z)$ by $z \cdot W(z)$ are related as follows

$$\mathcal{Q}_n[z \cdot W(z)] = \frac{1}{\Gamma(n)} \int_0^{\infty} dz z^n W(z) = n \cdot \mathcal{Q}_{n+1}[W(z)] \quad (7.79)$$

This homogeneity property can be extended to integration- and differentiation operations of the argument in $\mathcal{Q}_n$. Again, it turns out that a simple change of the index n provides a mechanism to modified $W(z)$:

$$\begin{aligned} \mathcal{Q}_n[W] &= \frac{1}{\Gamma(n)} \int_0^{\infty} dz z^{n-1} W(z) = \frac{1}{\Gamma(n)} \int_0^{\infty} dz W(z) \partial_z \frac{z^n}{n} = -\frac{1}{\Gamma(n+1)} \int_0^{\infty} dz z^n W'(z) \\ &= -\mathcal{Q}_{n+1}[W'] \end{aligned} \quad (7.80)$$

In the second step we used integration by parts, assuming that for sufficient large values of z the function $W(z)$ drops off rapidly enough, thus $W(z)$ has sufficiently restricted support. This relation can be equivalently described as $\mathcal{Q}_n[W] = (-\partial_z)^{(-n)} W(0)$ for $n < 0$.⁷

⁷For half-integer negative n the fractional derivative has to be understood as being defined by the integral representation of eq. (7.80).

7.5.2 Heat kernel expansion

We can now apply the previous results to our truncation ansatz using the projection techniques described in the section 7.3, whereby in the end we have to insert the explicit form of W given by the combination of Hessian- and cutoff operator. The first reduction is obtained by projecting out all invariants in the heat kernel expansion that cannot be resolved within the present truncation, using $\Pi_{\text{Trunc}}(\bullet)$:

$$\begin{aligned}\Pi_{\text{Trunc}}(\text{Tr}_{L^2}[W(\Delta)f \cdot \mathbb{1}]) &= + \sum_{j \geq 0} \Pi_{\text{Trunc}}(a_j(f; \Delta) \cdot \text{str}(\mathbb{1}) \cdot Q_{(d-j)/2}[W]) \\ &= + \sum_{j \geq 0} \Pi_{\text{Trunc}}(a_j(f; \Delta)) \cdot \text{str}(\mathbb{1}) \cdot Q_{(d-j)/2}[W]\end{aligned}\quad (7.81)$$

In the second step we made use of the fact that the entire geometric information is contained in the Seeley-DeWitt coefficients $a_j(f; \Delta)$. In particular the basis invariants in the expansion are separated from the Mellin transformed W function and thus $\Pi_{\text{Trunc}}(\bullet)$ affects only the Seeley-DeWitt coefficients $a_j(f; \Delta)$. The great advantage of using the heat kernel series for the present truncation is that it shows a trivial projection in that $\Pi_{\text{Trunc}}(a_j(f; \Delta)) \equiv 0$ for all $j \geq 3$ and $\Pi_{\text{Trunc}}(a_j(f; \Delta)) \equiv a_j(f; \Delta)$ otherwise. Hence, we obtain

$$\begin{aligned}\Pi_{\text{Trunc}}(\text{Tr}_{L^2}[W(\Delta)f \cdot \mathbb{1}]) &= +a_0(f; \Delta) \cdot \text{str}(\mathbb{1}) \cdot Q_{d/2}[W] + a_1(f; \Delta) \cdot \text{str}(\mathbb{1}) \cdot Q_{(d-1)/2}[W] \\ &\quad + a_2(f; \Delta) \cdot \text{str}(\mathbb{1}) \cdot Q_{(d-2)/2}[W]\end{aligned}\quad (7.82)$$

Thus, $\Pi_{\text{Trunc}}(\bullet)$ terminates the series at order $\mathcal{O}(s^{3/2})$ and the trace on the RHS of the flow equation assumes the form:

$$\begin{aligned}\Pi_{\text{Trunc}}(\text{Tr}_{L^2}[W(\Delta)f \cdot \mathbb{1}]) &= \frac{\text{str}(\mathbb{1})}{(4\pi)^{d/2}} \left\{ + Q_{d/2}[W] \cdot \int_M d^d x \sqrt{\bar{g}} f(x) - \frac{\sqrt{\pi}}{2} Q_{(d-1)/2}[W] \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} f(x) \right. \\ &\quad + \frac{1}{6} Q_{(d-2)/2}[W] \cdot \left(\int_M d^d x \sqrt{\bar{g}} \bar{R} f(x) + 2 \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} \bar{K} f(x) \right) \\ &\quad \left. + \frac{1}{2} Q_{(d-2)/2}[W] \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} n^\rho \bar{D}_\rho f(x) \right\}\end{aligned}\quad (7.83)$$

The next two projection techniques, $\Pi_{\xi=0=\bar{\xi}}$ and Π_Ω actually do not affect the result of eq. (7.83), since the function W as well as f do not depend on the ghost fields or the fluctuation field $h_{\mu\nu}$ anymore. In addition, the simplified Hessian operator has no field dependence, except for the background metric $\bar{g}(x)$ and thus $f(x)$ is either constant in spacetime, depending only on the dimension d , the constant C and Ω , or it additionally inherits the x -dependence purely from the background metric $\bar{g}$. This has further implications on the expression of eq. (7.83), since the last contribution is proportional to the covariant derivative of $f(x)$, which thus vanishes. It is the consequence of the conformal projection technique, and thus also of its simplified version the single-metric projection, that the dynamical Gibbons-Hawking-York term disappears and cannot be resolved within the present truncation.

Furthermore, the trace-traceless decomposition acts as the identity on the outer part of the heat kernel expansion, since the tensor structure is encapsulated in the trace over the identity $\text{str}(\mathbb{1})$ and hence no mixing of different tensor subspaces takes place.

Under the final projection of the basis invariants onto maximally symmetric spaces, $\Pi_{\text{m.s.}}$

the scalar- and the extrinsic curvature get substituted with the constant C^2 and C , respectively.

$$\begin{aligned} \Pi_{\text{Trunc}}(\text{Tr}_{L^2}[W(\Delta)f \cdot \mathbb{1}]) & \quad (7.84) \\ &= \frac{\text{str}(\mathbb{1})}{(4\pi)^{d/2}} \left\{ \mathcal{Q}_{d/2}[W] \cdot \int_M d^d x \sqrt{\bar{g}} f(x) - \frac{\sqrt{\pi}}{2} \mathcal{Q}_{(d-1)/2}[W] \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} f(x) \right. \\ & \quad \left. + \frac{1}{6} \mathcal{Q}_{(d-2)/2}[W] \cdot \left(\int_M d^d x \sqrt{\bar{g}} C^2 f(x) + 2 \sqrt{\frac{(d-1)}{d}} \int_{\partial M} d^d x \sqrt{\bar{H}} C f(x) \right) \right\} \end{aligned}$$

Since $f(x)$ and the Mellin transform of W in principle contain additional factors of C and Ω what remains is an expansion in those constants up to second or first order, respectively. This final projection and the comparison of coefficients that leads to the beta-functions for our truncation ansatz concludes the second part of the thesis and is discussed in the next chapter. Before we turn to this final step, let us consider the remaining traces over the tensor spaces in the following subsection.

7.5.3 Tensor space traces

The heat kernel expansion takes care of the L^2 -operator trace. There is a remaining vector space trace $\text{str}(\mathbb{1})$ that factors each term in this series and relates to the tensor character of the underlying field. The general procedure to evaluate (infinite dimensional) endomorphisms was presented in subsection 1.4.2. However, in order to compute the remaining traces we exclusively deal with finite vector spaces, for which we simply have to ‘trace’ over all basis elements.

Though we have decomposed the metric field space into its trace- and traceless part, let us start with the identity element on the full space of symmetric 2-tensor fields, $(\mathbb{1}_s)_{\rho\sigma}^{\mu\nu} = \frac{1}{2}(\delta_\rho^\mu \delta_\sigma^\nu + \delta_\rho^\nu \delta_\sigma^\mu)$. Its trace corresponds to the dimension of the associated vector space and thus counts the number of independent components. It is given by

$$\text{str}(\mathbb{1}_s) = \text{tr}(\mathbb{1}_s) = (\mathbb{1}_s)_{\mu\nu}^{\mu\nu} = \frac{1}{2}(\delta_\mu^\mu \delta_\nu^\nu + \delta_\mu^\nu \delta_\nu^\mu) = \frac{d(d+1)}{2}$$

In $d = 4$ the space of symmetric 2-tensor fields is thus of dimension 10, with exactly the same number of free components. The same could be obtained by Young diagrams in a straightforward manner. Each index corresponds to a box and a generic 2-tensor splits into its symmetric- and anti-symmetric parts, i.e. $\square\square$ and $\square$ with dimension $\frac{d(d+1)}{2}$ or $\frac{d(d-1)}{2}$, respectively. Now, what about the traces of the orthogonal decomposition into trace- and traceless part? This is easily computed by projecting the identity $\mathbb{1}_s$ to the respective subspace using either Π_{Tr} for the trace component or Π_{noTr} for its traceless counterpart. Hence, we obtain:

$$\text{str}(\mathbb{1}_{\text{Tr}}) = \text{tr}(\Pi_{\text{Tr}} \mathbb{1}_s) = \text{tr}(\Pi_{\text{Tr}}) = (\Pi_{\text{Tr}})_{\mu\nu}^{\mu\nu} = \frac{1}{d} \bar{g}^{\mu\nu} \bar{g}_{\mu\nu} = 1 \quad (7.85)$$

$$\text{str}(\mathbb{1}_{\text{noTr}}) = \text{tr}(\Pi_{\text{noTr}} \mathbb{1}_s) = \text{tr}(\Pi_{\text{noTr}}) = \text{tr}(\Pi_s) - \text{tr}(\mathbb{1}_{\text{Tr}}) = \frac{d(d+1)-2}{2} = \frac{(d-1)(d+2)}{2} \quad (7.86)$$

In four dimensions, we thus have $10 = 9 \oplus 1$. Since the metric sector is described by ordinary tensor fields the supertrace str can be replaced by the usual one tr .

On the other hand, the ghost sector is of Grassmann type and therefore an additional minus sign occurs in the evaluation. Except of this subtlety, the remaining tensor structure is given by the tangent space which carries the same dimensionality as its underlying manifold, hence equals d or simply $\square$. Overall, the supertrace over the identity on $\theta \cdot \Gamma(\text{TM})$ reads:

$$\text{str}(\mathbb{1}_\xi) = -\text{tr}(\mathbb{1}_{\Gamma(\text{TM})}) = -d \quad (7.87)$$

In the following we write $\text{str}(\mathbb{1}_\xi) = -d \rho_{\text{gh}}$ with a bookkeeping parameter $\rho_{\text{gh}} = 1$ that allows to separate the ghost contributions from the metric traces.



8.1	Threshold functions
8.2	Beta-functions for the coefficients
8.3	Beta-functions for dimensionful couplings
8.4	Beta-functions for dimensionless couplings
8.5	The semiclassical approximation
8.6	Beta-functions in $d = 4$
8.A	Expanded beta-functions in $d = 2 + \varepsilon$
8.B	Beta-functions in $d = 3$

8. THE BETA-FUNCTIONS

This chapter is about the beta-functions of the truncated FRGE for metric gravity and it concludes the second part of this thesis. Hence, the results of the detailed calculation performed in the previous chapters are presented in the following sections, first in general dimensions and then for $d = 4$ in particular.

So far we have derived the Hessian operator for a bi-metric truncation ansatz that contains bulk and boundary invariants up to second and first order in the derivative $\bar{D}$, respectively. The dependence on a field space consisting of two (on the bulk) independent metrics and ghost fields which results in a theory space that at this order already consists of an infinite number of basis monomials. We focused on the Einstein-Hilbert and Gibbons-Hawking-York functionals that are either constructed by both, background- and dynamical metric, or purely by $\bar{g}$, respectively. This allows to study Background Independence in the context of the Asymptotic Safety scenario for QG as well as investigating the impact of spacetime topologies with boundaries on the RG evolution.

The last chapter presented a detail simplification of the Hessian operator and ultimately used the heat kernel expansion to project the RHS on the field monomials comprised in the truncation ansatz of eq. (5.15). In this chapter, we first of all give a summary of the so far obtained results, introduce the threshold functions and then compare coefficients of the LHS and RHS of the flow equation. We then briefly discuss the beta-functions for the dimensionful couplings that are more common in physics. Next, we convert those expressions into dimensionless quantities that define theory space and allow for the search of fixed points. The results are again presented for general spacetime dimension d and generic shape-functions. Since most of the numerical results of part III are obtained employing the optimized shape function, we append the results for the beta-functions for this special shape function. Finally, we focus on the spacetime dimension $d = 4$ which for practical reasons is the most relevant one. The discussions of the beta-functions in $d = 3$ and in the limit $d = 2 + \varepsilon$ are shifted to the appendix 8.A and 8.B.



Except for the boundary results, section 8.4 to 8.B follow closely ref. [173].

8.1 Threshold functions

The current status of the RHS of the flow equation involves a sum over L^2 -traces which either are of type $\text{STr}_{L^2}^{[\text{noTr}]}[f W_p(\Delta; \ell; \bar{u}_k^\bullet)]$ or $\text{STr}_{L^2}^{[\text{noTr}]}[f \Delta W_p(\Delta; \ell; \bar{u}_k^\bullet)]$, with $f \equiv f(C, \Omega, d)$ an x -independent function and

$$W_p(\Delta; \ell, \bar{u}_k^\bullet) \equiv \frac{1}{2} \mathcal{A}(\ell)^{-p} \cdot (\partial_t R_k^{(0)}(\Delta) + R_k^{(0)}(\Delta) \eta_k^\bullet) \cdot \mathbb{1}_{[\bullet]} \quad (8.1)$$

Whereby, $\mathcal{A}(\ell) \equiv (-\bar{D}^2 - 2\ell + R_k^{(0)}(-\bar{D}^2))$ denotes the denominator and $R_k^{(0)}(\Delta) \equiv k^2 R^{(0)}(\Delta/k^2)$ the shape function. Furthermore we defined the anomalous dimension $\eta_k^\bullet \equiv -\partial_t \ln(\bar{u}_k^\bullet)$ of the coupling $\bar{u}_k^\bullet$.

We start this section by evaluating these type of traces using the threshold-functions, $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$, introduced in [122]. Afterwards, we state some general properties of $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$.

8.1.1 The Mellin transform of $W_p(\Delta; \ell; \bar{u}_k^\bullet)$

In the heat kernel expansion the structure of W_p becomes separated from the geometric information encoded in the Laplacian operator, $-\bar{D}^2$, and from the function f . It thus suffices to consider the Mellin transform of $W_p(\Delta; \ell; \bar{u}_k^\bullet)$ independently, i.e. evaluate the following integral transformation

$$\begin{aligned} Q_n[W_p(\Delta; \ell; \bar{u}_k^\bullet)] &= \frac{\mathbb{1}_{[\bullet]}}{\Gamma(n)} \int_0^\infty dz z^{n-1} W_p(z; \ell; \bar{u}_k^\bullet) \\ &= \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{(1 - \frac{1}{2} \eta_k^\bullet) k^2 R^{(0)}(z/k^2) - z R^{(0)'}(z/k^2)}{(z - 2\ell + k^2 R^{(0)}(z/k^2))^p} \end{aligned} \quad (8.2)$$

The prime in $R^{(0)'}(z/k^2)$ denotes the derivative of the shape function w.r.t. its argument, i.e. $R^{(0)'}(z/k^2) \equiv \partial_s R^{(0)}(s)|_{s=z/k^2}$. This term originates from $\partial_t = k \partial_k$ acting on $R_k^{(0)}(\Delta)$ in the numerator, yielding

$$\partial_t (k^2 R^{(0)}(\Delta/k^2)) = k \partial_k (k^2 R^{(0)}(\Delta/k^2)) = 2k^2 \left(R^{(0)}(\Delta/k^2) - (z/k^2) R^{(0)'}(z/k^2) \right)$$

Introducing artificial factors of k^2 allows to rewrite the integrand in terms of the argument of the shape function, z/k^2 :

$$Q_n[W_p(\Delta; \ell; \bar{u}_k^\bullet)] = k^{2(n-p)} \frac{\mathbb{1}_{[\bullet]}}{\Gamma(n)} \int_0^\infty dz (z/k^2)^{n-1} \frac{(1 - \frac{1}{2} \eta_k^\bullet) R^{(0)}(z/k^2) - \frac{z}{k^2} R^{(0)'}(z/k^2)}{((z/k^2) - 2k^{-2}\ell + R^{(0)}(z/k^2))^p}$$

Now, we perform a change of integration variables, $y = z/k^2$ and $k^2 dy = dz$. For this transformation the integration range does not change, since $k \in \mathbb{R}^+$. We obtain

$$Q_n[W_p(\Delta; \ell; \bar{u}_k^\bullet)] = k^{2(n+1-p)} \frac{\mathbb{1}_{[\bullet]}}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{(1 - \frac{1}{2} \eta_k^\bullet) R^{(0)}(y) - y R^{(0)'}(y)}{(y - 2k^{-2}\ell + R^{(0)}(y))^p} \quad (8.3)$$

In this way, we can absorb the k -dependence of the Mellin transform in a global pre-factor $k^{2(n+1-p)}$ by simultaneously replacing ℓ by $k^{-2}\ell$. Equation (8.3) can be rewritten using the threshold function, a k -independent construction that contains the cutoff-specific information:

$$\Phi_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{R^{(0)}(y) - y R^{(0)'}(y)}{(y + w + R^{(0)}(y))^p} \quad (8.4a)$$

$$\tilde{\Phi}_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{R^{(0)}(y)}{(y + w + R^{(0)}(y))^p} \quad (8.4b)$$

In terms of this set of threshold functions, the Mellin transform of $W_p(\Delta; \ell; \bar{u}_k^\bullet)$ assumes the form:

$$\mathcal{Q}_n[W_p(\Delta; \ell; \bar{u}_k^\bullet)] = k^{2(n+1-p)} \left(\Phi_n^p(-2k^{-2}\ell) - \frac{1}{2} \eta_k^\bullet \tilde{\Phi}_n^p(-2k^{-2}\ell) \right) \cdot \mathbb{1}_{[\bullet]} \quad (8.5)$$

It is convenient to introduce $q_n^p(w, \eta_k^\bullet) := \Phi_n^p(w) - \frac{1}{2} \eta_k^\bullet \tilde{\Phi}_n^p(w)$ as abbreviation for this specific combination of threshold functions, since we will encounter those terms in several expressions for the beta-functions later on. Hence, eq. (8.5) can be rewritten as

$$\mathcal{Q}_n[W_p(\Delta; \ell; \bar{u}_k^\bullet)] = k^{2(n+1-p)} q_n^p(-2k^{-2}\ell, \eta_k^\bullet) \cdot \mathbb{1}_{[\bullet]} \quad (8.6)$$

8.1.2 The Mellin transform of $\Delta W_p(\Delta; \ell; \bar{u}_k^\bullet)$

Except for the traces of the above type, there are additional terms with an extra factor of $-\bar{D}^2$. However, from eq. (7.79) the Mellin transform can be directly related to the result of $W_p(\Delta; \ell; \bar{u}_k^\bullet)$ by means of the following identity

$$\mathcal{Q}_n[z \cdot W(z)] = \frac{1}{\Gamma(n)} \int_0^\infty dz z^n W(z) = n \cdot \mathcal{Q}_{n+1}[W(z)]$$

Hence, the Mellin transform of $\Delta W_p(\Delta; \ell; \bar{u}_k^\bullet)$ again gives rise to a combination of the threshold functions (8.4), which fully cover the contribution from the shape function times a k -dependent pre-factor:

$$\begin{aligned} \mathcal{Q}_n[\Delta W_p(\Delta; \ell; \bar{u}_k^\bullet)] &= n k^{2(n+2-p)} \left(\Phi_{n+1}^p(-2k^{-2}\ell) - \frac{1}{2} \eta_k^\bullet \tilde{\Phi}_{n+1}^p(-2k^{-2}\ell) \right) \cdot \mathbb{1}_{[\bullet]} \\ &\equiv n k^{2(n+2-p)} q_{n+1}^p(-2k^{-2}\ell, \eta_k^\bullet) \cdot \mathbb{1}_{[\bullet]} \end{aligned} \quad (8.7)$$

All terms in the heat kernel expansion (7.69) can be expressed using the results of eq. (8.6) and (8.7). What remains is the trace over the identity operator $\mathbb{1}_{[\bullet]}$ over the respective field subspace. Before we turn to this last ingredient, let us consider some of the general properties of the threshold functions.

8.1.3 Properties and estimates of the threshold functions

This subsection lists and proves a set of general properties the threshold functions as defined in eq. (8.4) exhibit. Furthermore, we state certain estimates that might prove useful when deducing general features of the beta-functions independent on any specific shape function. The results only depend on the following assumptions on the analytic properties of the shape function that is usually employed:

$$R^{(0)}(z=0) = 1, \quad \lim_{z \rightarrow \infty} R^{(0)}(z) = 0, \quad R^{(0)}(z) \geq 0 \quad \forall z \geq 0 \quad (8.8)$$

Under these assumptions it is possible to constraint the w dependence of the shape functions $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$, derive the limit $n \rightarrow 0+$ and present some general estimates of the p - and n -dependence.

We start with the following proposition for a general number of derivatives of $\Phi_n^p(w)$ w.r.t. w :

$$\partial_w^m \Phi_n^p(w) = (-1)^m \frac{\Gamma(p+m)}{\Gamma(p)} \Phi_n^{p+m}(w) \quad (8.9)$$

The proof is straightforward. Using the definition of $\Phi_n^p(w)$ in eq. (8.4) we obtain for the first differentiation

$$\begin{aligned}\partial_w \Phi_n^p(w) &= \partial_w \left(\frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{R^{(0)}(y) - yR^{(0)'}(y)}{(y+w+R^{(0)}(y))^p} \right) \\ &= (-p) \frac{1}{\Gamma(n)} \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{R^{(0)}(y) - yR^{(0)'}(y)}{(y+w+R^{(0)}(y))^{p+1}} = (-p) \cdot \Phi_n^{p+1}(w)\end{aligned}\quad (8.10)$$

Here we interchanged the order of integration and differentiation which is perfectly fine as long as the integrand is differentiable. Next, a successive application of this result yields

$$\partial_w^m \Phi_n^p(w) = (-1)^m \frac{\Gamma(p+m)}{\Gamma(p)} \Phi_n^{p+m}(w) \equiv (-1)^m \frac{(p-1+m)!}{(p-1)!} \Phi_n^{p+m}(w) \quad (8.11)$$

Whereby the latter equality holds for p and m positive integers.

A similar expression is found for the second type of threshold functions $\tilde{\Phi}_n^p(w)$, namely

$$\partial_w^m \tilde{\Phi}_n^p(w) = (-1)^m \frac{\Gamma(p+m)}{\Gamma(p)} \tilde{\Phi}_n^{p+m}(w) \quad (8.12)$$

Applying the derivative w.r.t. w on the integral representation of the threshold function, we obtain the same result as in the previous case:

$$\begin{aligned}\partial_w \tilde{\Phi}_n^p(w) &= \partial_w \left(\frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R(z)}{(z+R(z)+w)^p} \right) \\ &= (-p) \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R(z)}{(z+R(z)+w)^{p+1}} = (-p) \cdot \tilde{\Phi}_n^{p+1}(w)\end{aligned}$$

Again, an iterative application yields the above identity.

$$\partial_w^m \tilde{\Phi}_n^p(w) = (-1)^m \frac{\Gamma(p+m)}{\Gamma(p)} \tilde{\Phi}_n^{p+m}(w) \equiv (-1)^m \frac{(p-1+m)!}{(p-1)!} \tilde{\Phi}_n^{p+m}(w)$$

Notice that since both threshold functions behave the same under differentiation w.r.t. w , the q_n^p -function exhibit the equivalent property.

Assuming $p > 1$ and $n > 0$, the threshold functions can be expanded in a Taylor series for vanishing argument w , resulting in

$$\Phi_n^p(w) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{\Gamma(p+m)}{\Gamma(m+1)\Gamma(p)} \Phi_n^{p+m}(0) \right) \cdot w^m \quad (8.13)$$

If $\Phi_n^p(w)$ is continuous and $(j+1)$ -times differentiable on the interval $[w, 0]$, we can use Taylor's theorem to approximate the error when truncate the Taylor series at order m , say. The remainder is given by

$$R_m(w) = \frac{w^{m+1}}{(m+1)!} \partial_w^{m+1} \Phi_n^p(w) \Big|_{w=\xi_L \in [w, 0]} \quad (8.14)$$

whereby ξ_L is some real number in the interval $[w, 0]$. Substituting the maximum value the LHS can assume for any $\xi_L \in [w, 0]$ gives a bound on the contributions from the neglected terms. Along the same lines one can derive the Taylor expansion of $\tilde{\Phi}_n^p(w)$ and $q_n^p(w, \eta_k^\bullet)$. Thus, setting bounds on the coefficients of these Taylor series by deducing the properties of $\Phi_n^p(0)$, $\tilde{\Phi}_n^p(w)$, and $q_n^p(w, \eta_k^\bullet)$ may give some insight if these expansions converge.

Based on the above assumptions on the shape functions it is possible to rewrite $\Phi_n^p(w)$ into a more convenient form using the following substitution of integral variables:

$$v = R^{(0)}(y)y^{-1} \quad \Rightarrow \quad dv = -dy y^{-2} (R^{(0)}(y) - yR^{(0)'}(y))$$

At this point the positivity of $R^{(0)}(y)$ is important to retain the integral limits. Taking care of the extra minus sign by switching the integral limits, the standard threshold function $\Phi_n^p(w)$ can be cast into the following form:

$$\begin{aligned} \Phi_n^p(w) &\equiv \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} \frac{R^{(0)}(y) - yR^{(0)'}(y)}{(y+w+R^{(0)}(y))^p} \\ &\equiv \frac{1}{\Gamma(n)} \int_0^\infty dv \frac{R(v)^{n+1-p}}{(1+w/R(v)+v)^p} \end{aligned} \quad (8.15)$$

with $R(v) \equiv R^{(0)}(y(v))/v$. This turns out to be very useful when evaluating $\Phi_n^p(0)$ for vanishing arguments w .

For instance, let us consider the following estimate for the threshold functions, where $\alpha \geq 0$:

$$\frac{\Phi_{n+\alpha}^{p+\alpha}(0)}{\Phi_n^p(0)} \leq \frac{\Gamma(n)}{\Gamma(n+\alpha)} \leq 1 \quad (8.16)$$

This expression states that the coefficients in the Taylor expansion decrease and thus for suitable values of w the series converges. To prove this estimate, we make use of eq. (8.15) as follows

$$\begin{aligned} \Phi_{n+\alpha}^{p+\alpha}(0) &= \frac{1}{\Gamma(n+\alpha)} \int_0^\infty dv \frac{R(v)^{(n+\alpha)+1-(p+\alpha)}}{(1+v)^{p+\alpha}} \\ &= \frac{1}{\Gamma(n+\alpha)} \int_0^\infty dv \frac{R(v)^{n+1-p}}{(1+v)^p} \cdot (1+v)^{-\alpha} \end{aligned}$$

Since the integrand is a positive function and the integration interval extends only to positive values $v \geq 0$ in the integration interval, the additional factor is bounded as $(1+v)^{-\alpha} \leq 1$ for all $\alpha \geq 0$. Hence, we obtain the inequality

$$\Phi_{n+\alpha}^{p+\alpha}(0) \leq \frac{1}{\Gamma(n+\alpha)} \int_0^\infty dv \frac{R(v)^{n+1-p}}{(1+v)^p} = \left(\frac{\Gamma(n)}{\Gamma(n+\alpha)} \right) \frac{1}{\Gamma(n)} \int_0^\infty dv \frac{R(v)^{n+1-p}}{(1+v)^p}$$

The last term can be identified with the threshold function $\Phi_n^p(0)$ from which thus the above estimate follows

$$\Phi_{n+\alpha}^{p+\alpha}(0) \leq \frac{\Gamma(n)}{\Gamma(n+\alpha)} \Phi_n^p(0)$$

The same statement also holds for the second type of threshold functions $\tilde{\Phi}_n^p(0)$ and thus also for $q_n^p(0, \eta_k^\bullet)$:

$$\frac{\tilde{\Phi}_{n+\alpha}^{p+\alpha}(0)}{\tilde{\Phi}_n^p(0)} \leq \frac{\Gamma(n)}{\Gamma(n+\alpha)} \quad (8.17)$$

The proof of this estimate is quite similar to the previous one. We start by separating the extra power of α from the remaining integrand:

$$\begin{aligned} \tilde{\Phi}_{n+\alpha}^{p+\alpha}(0) &= \frac{1}{\Gamma(n+\alpha)} \int_0^\infty dz z^{n+\alpha-1} \frac{R(z)}{(z+R(z))^{p+\alpha}} \\ &= \left(\frac{\Gamma(n)}{\Gamma(n+\alpha)} \right) \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R(z)}{(z+R(z))^p} \left(\frac{z}{(z+R(z))} \right)^\alpha \end{aligned}$$

This time we use $R(z) \geq 0$ to set an upper bound of the additional factor. For $z \geq 0$, which corresponds to the integration range, we have

$$\frac{z}{(z+R(z))} = \frac{1}{(1+R(z)/z)} \leq 1$$

The case $z = 0$ is a null set of the integration and in fact also the integrand vanishes. Using this inequality, we arrive at the claimed estimate

$$\tilde{\Phi}_{n+\alpha}^{p+\alpha}(0) \leq \left(\frac{\Gamma(n)}{\Gamma(n+\alpha)} \right) \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R(z)}{(z+R(z))^p} = \left(\frac{\Gamma(n)}{\Gamma(n+\alpha)} \right) \tilde{\Phi}_n^p(0)$$

In conclusion we can truncate the Taylor series of the threshold functions at a certain order in the expansion provided that w is sufficiently small. For $|w| < 1$ the Taylor series converge, which is particular interesting for the semiclassical approximation.

Finally, the special case of $n = 0$ follows from (8.4) in the limit $n \rightarrow 0^+$ [122]:

$$\Phi_0^p(w) = \tilde{\Phi}_0^p(w) = (1+w)^{-p} \quad (8.18)$$

This concludes our general discussion on threshold functions. If not stated otherwise we later evaluate $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$ for the optimized cutoff shape function, for which the integrals can be explicitly computed.

8.2 Beta-functions for the coefficients

In this section we list the beta-functions for the coefficients that appear in the truncation ansatz of eq. (5.15) for general spacetime dimension d and arbitrary cutoff shape function.

8.2.1 Comparison of coefficients

As in any (finite) dimensional vector space, a full set of linearly independent basis elements is the basic ingredient to understand and study the whole space. While theory space in its complete form is surely not finite dimensional, the considered truncated subspace (5.15) is definitely finite, in fact a vector space of dimension 6. Therefore, the standard tools of algebra are directly applicable. In particular any vector of this truncated subspace is uniquely expressed in terms of a set of basis elements and as such, we can compare coefficients in any equality.

To this end, we have to project the set of traces given in eq. (7.69) onto the basis invariants of the truncation ansatz using the heat kernel expansion and the results of the previous section. While the actual expansion of the RHS is not illustrative, we focus here on some remarks only and present the results in the following subsections.

- A. From the eight contributions in $\text{RHS}|_{C=0}$ only the first three terms affect the level-(0) coefficients, since all others are linear in Ω .
- B. For eq. (7.69) we observe again that there are far more terms linear in Ω which thus only trigger the RG evolution of the dynamical invariants.
- C. Our truncation ansatz contains two basis invariants of order C^2 , the level-(0) and level-(1) Einstein-Hilbert functionals, and another one of order C for the boundary curvature term, thus

$$\text{LHS} \equiv \text{LHS}|_{C=0} + \left(\frac{d}{dC} \text{LHS} \right) \cdot C + \frac{1}{2} \left(\frac{d^2}{dC^2} \text{LHS} \right) \cdot C^2$$

On the other hand, the operators under the functional trace are given by

$$\text{RHS} \equiv \text{STr}[\dots \mathbb{G}(C^0)] + \text{STr}[\dots \mathbb{G}(C^2)] + \mathbb{G}(C^4)$$

In the heat kernel expansion these terms are multiplied by additional factors of C^0 , C , and C^2 . Hence, the heat kernel asymptotics of the C -quadratic terms in eq. (7.69) reduces to the bulk volume invariant $\int_M \sqrt{g}$ only. All other field monomials that appear in the expansion would combine to terms of at least order C^3 which is beyond the scope of the present truncation.

- D. The last remark in particular implies that all boundary effects stem from the terms contained in $\text{RHS}|_{C=0}$, while the Einstein-Hilbert coefficients obtains corrections also from the $\frac{1}{2}(\frac{d^2}{dC^2}\text{RHS}) \cdot C^2$. This asymmetry between boundary and bulk contributions makes a matching of $\bar{u}_k^{(0)}$ and $\bar{u}_k^{(1)}$ unlikely. However, for the present truncation we are unable to resolve $\bar{u}_k^{(0)}$. If the conformal projection technique is replaced by its x -dependent analog $\Omega(x)$ we will have additional terms that may compensate this imbalance.
- E. For the ghost contributions, later marked with the parameter $\rho_{\text{gh}} = 1$, the last argument of $W_p(\Delta; \ell; \bar{u}_k^{\text{gh}} \equiv \bar{u}^{\text{gh}})$ is in fact k -independent and thus its associated anomalous dimension vanishes, $\eta_k^{\text{gh}} \equiv -\partial_t \ln \bar{u}^{\text{gh}} = 0$. Hence, all q_n^p that occur contain the anomalous dimension of the dynamical Newton coupling $\eta^{\text{D}} \equiv -\partial_t \ln \bar{u}_k^{\text{D}} = \partial_t \ln G_k^{\text{D}}$:

$$q_n^p(w, \eta^{\text{D}}) \equiv \Phi_n^p(w) - \frac{1}{2} \eta_k^{\text{D}} \tilde{\Phi}_n^p(w) \quad (8.19)$$

In the sequel we list the beta-functions for the dimensionful coefficients that are obtained by comparison of coefficients for the six basis invariants. Therefore, we rewrite eq. (7.39) into the level-language, which leads to

$$\begin{aligned} \Pi_{\text{all}}(\partial_t \Gamma_k[\Phi, \bar{\Phi}]) &= +2 \cdot \int_M d^d x \sqrt{g} \left(\partial_t(\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)}) + d\Omega \cdot \partial_t(\bar{u}_k^{(1)} \bar{\lambda}_k^{(1)}) \right) \\ &\quad - \int_M d^d x \sqrt{g} C^2 \left(\partial_t \bar{u}_k^{(0)} + (d-2)\Omega \cdot \partial_t \bar{u}_k^{(1)} \right) \\ &\quad + 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{H} \partial_t(\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)}) \\ &\quad - 2 \cdot \int_{\partial M} d^{d-1} x \sqrt{H} \sqrt{\frac{(d-1)}{d}} C \partial_t \bar{u}_k^{(0)} + \mathcal{O}(\Omega^2) \end{aligned} \quad (8.20)$$

Each of the six basis invariants can be uniquely identified by its power in Ω and C as well as the integration range, which is either M or ∂M . While level-(1) contributions are linear in Ω , we can project onto level-(0) invariants by setting $\Omega = 0$ on both sides. Along similar lines the beta-functions for all coefficients then follow from the truncated FRGE:

$$\Pi_{\text{all}}(\partial_t \Gamma_k[g, \bar{g}]) = \frac{1}{2} \Pi_{\text{all}} \text{STr} \left[\left(\Pi_{\text{all}} \widehat{\text{Hess}}_{\varphi} [\Gamma_k[\Phi, \bar{\Phi}]] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \partial_t \mathfrak{R}_k[\bar{\Phi}] \right]$$

8.2.2 The Einstein-Hilbert invariants

Let us start with the Einstein-Hilbert invariants of the dynamical and background metric. They correspond to $\int_M d^d x \sqrt{g} C^2$ which are either linear in or independent of Ω . The coefficient for this invariant on the LHS reads

$$\Pi_{\text{EH}} \text{LHS} = -\partial_t \bar{u}_k^{(0)} - (d-2)\Omega \cdot \partial_t \bar{u}_k^{(1)}$$

The corresponding terms on the RHS can be easily projected using $\Pi_{\text{EH}}(\bullet) = \partial_{C^2}(\bullet)|_{C=0}$, since it is the only invariant of order $\mathcal{O}(C^2)$. The different levels, level-(0) and level-(1), are then obtained using $\Pi_{(0)}(\bullet) = (\bullet)|_{\Omega=0}$ or $\Pi_{(1)}(\bullet) = \partial_{\Omega}(\bullet)|_{\Omega=0}$, respectively. This results in two beta-functions

$$\partial_t \bar{u}_k^{(0)} \equiv \bar{\beta}_{\bar{u}}^{(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) \quad \text{and} \quad \partial_t \bar{u}_k^{(1)} \equiv \bar{\beta}_{\bar{u}}^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k)$$

Since level-(0) contribution do not enter the Hessian operator and thus the RHS of the flow equation, there is no back-reaction of $\bar{u}_k^{(0)}$ in the beta-function of $\bar{u}_k^{(1)}$. For level-(0) the explicit form of the beta-function is given by

$$\begin{aligned} \bar{\beta}_{\bar{u}}^{(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) = & -\frac{k^{d-2}d}{2(4\pi)^{\frac{d}{2}}} \left\{ \frac{(d+1)}{6} q_{(d-2)/2}^1(-2\lambda^{(1)}, \eta^D) \right. \\ & \left. - (d-1)\rho_P q_{d/2}^2(-2\lambda^{(1)}, \eta^D) \right\} \\ & + \frac{k^{d-2}}{(4\pi)^{\frac{d}{2}}} \rho_{\text{gh}} \left\{ \frac{d}{3} \Phi_{(d-2)/2}^1(0) + 2\rho_P \Phi_{d/2}^2(0) \right\} \end{aligned} \quad (8.21)$$

Here, we have introduced the dynamical dimensionless cosmological constant $k^2 \bar{\lambda}_k^{(1)} \equiv \lambda_k^D \equiv \lambda_k^{(1)} = \lambda_k^{(2)} = \dots$ and the dynamical anomalous dimension associated to Newton's coupling $\eta^D \equiv -\partial_t \ln \bar{u}_k^D \equiv -\partial_t \ln \bar{u}_k^{(1)}$. Notice that the beta-functions are indeed independent on the level-(0) coefficients. Again, we emphasize that $\rho_P = 1 = \rho_{\text{gh}}$ are only bookkeeping parameters that ultimately are set to unity. They are particularly convenient to understand how the different contributions affect the RG flow. We will make use of this technique in part III of this thesis.

The RG evolution for the higher level Newton-type coefficients is described by the following beta-function

$$\begin{aligned} \bar{\beta}_{\bar{u}}^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) = & \frac{k^{d-2}d}{4(4\pi)^{\frac{d}{2}}} \left\{ \frac{(d-6)(d+1)}{6} q_{d/2}^2(-2\lambda^{(1)}, \eta^D) \right. \\ & - \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^{(1)} q_{(d-2)/2}^2(-2\lambda^{(1)}, \eta^D) \\ & - 2 \frac{(d-6)(d-1)}{(d-2)} \rho_P [d q_{(d+2)/2}^3(-2\lambda^{(1)}, \eta^D) - q_{d/2}^2(-2\lambda^{(1)}, \eta^D)] \\ & \left. + 8 \frac{(d-4)(d-1)}{(d-2)} \rho_P \lambda^{(1)} q_{d/2}^3(-2\lambda^{(1)}, \eta^D) \right\} \\ & - \frac{k^{d-2}}{(4\pi)^{\frac{d}{2}}} \rho_{\text{gh}} \left\{ \left(\frac{d}{3} - \frac{4}{(d-2)} \rho_P \right) \Phi_{d/2}^2(0) + \frac{4d}{(d-2)} \rho_P \Phi_{(d+2)/2}^3(0) \right\} \end{aligned} \quad (8.22)$$

Notice that the entire dependence of $\bar{\beta}_{\bar{u}}^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k)$ on $\bar{u}_k^D$ arises from $q_n^p(-2\lambda_k^D, \eta^D)$ -functions. Thus, one could equivalently write $\bar{\beta}_{\bar{u}}^{(1)}(\eta^D, \bar{\lambda}_k^{(1)}; k)$. This in turn gives rise to an implicit equation for the anomalous dimension

$$\eta^D \equiv -\partial_t \ln \bar{u}_k^D = \bar{\beta}_{\bar{u}}^{(1)}(\eta^D, \bar{\lambda}_k^{(1)}; k)$$

that we are going to study later on in this chapter, after introducing the full set of dimensionless couplings.

In order to obtain the corresponding beta-functions in the D/B-language, one simply has to employ the following relations

$$\bar{u}_k^D \equiv \bar{u}_k^{(1)} \quad \text{and} \quad \bar{u}_k^B \equiv \bar{u}_k^{(0)} - \bar{u}_k^{(1)}$$

In the case of beta-functions for couplings we mostly make use of the D/B formulation.

8.2.3 The bulk volume invariant

In the next step, we consider the volume invariant on the bulk $\int_M d^d x \sqrt{g}$. It is associated to the running of the dynamical and background cosmological constant. In contrast to the previous

case of the Einstein-Hilbert functional, there is another basis invariant of the same degree in C , namely the volume counterpart of the boundary. Hence, the projection should look like $\Pi_{\text{vol}(M)}(\bullet) := (\bullet)|_{C=0; \partial M=\emptyset}$, for which the LHS assumes the form

$$\Pi_{\text{vol}(M)} \text{LHS} = 2\partial_t(\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)}) + 2d\Omega \cdot \partial_t(\bar{u}_k^{(1)} \bar{\lambda}_k^{(1)})$$

We then disentangle the different levels by means of $\Pi_{(0)}$ and $\Pi_{(1)}$, yielding the following two beta-functions for the coefficients:

$$\partial_t(\bar{u}_k^{(0)} \bar{\lambda}_k^{(0)}) \equiv \bar{\beta}_{\bar{u}\bar{\lambda}}^{(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) \quad \text{and} \quad \partial_t(\bar{u}_k^{(1)} \bar{\lambda}_k^{(1)}) \equiv \bar{\beta}_{\bar{u}\bar{\lambda}}^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k)$$

Applying $\Pi_{(0)}$ and $\Pi_{\text{vol}(M)}$ to the RHS of the flow equation results in

$$\bar{\beta}_{\bar{u}\bar{\lambda}}^{(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) = \frac{k^d d(d+1)}{4(4\pi)^{\frac{d}{2}}} q_{d/2}^1(-2\lambda^{(1)}, \eta^D) - \frac{k^d d}{(4\pi)^{\frac{d}{2}}} \rho_{\text{gh}} \Phi_{d/2}^1(0) \quad (8.23)$$

The level-(1) and therefore D-counterpart has a quite similar structure with an additional term proportional to $\lambda^{(1)}$ which vanishes in $d = 4$:

$$\begin{aligned} \bar{\beta}_{\bar{u}\bar{\lambda}}^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) &= \frac{k^d(d+1)}{4(4\pi)^{\frac{d}{2}}} \left\{ 2(d-4) \lambda^{(1)} q_{d/2}^2(-2\lambda^{(1)}, \eta^D) - \frac{(d-6)d}{2} q_{(d+2)/2}^2(-2\lambda^{(1)}, \eta^D) \right\} \\ &\quad + \frac{k^d d \rho_{\text{gh}}}{(4\pi)^{\frac{d}{2}}} \Phi_{(d+2)/2}^2(0) \end{aligned} \quad (8.24)$$

By construction, both beta-functions are purely diamagnetic in nature, since ρ_p multiplies terms of order $\mathcal{O}(C^2)$ and higher. Notice that eq. (8.22) and (8.24) give rise to a closed system of two coupled differential equations which can be solved independently of the other equations. Once the level-(1) or D solutions are obtained, one can insert them into the remaining beta-functions and solve for the level-(0) coefficients.

The transition to the D/B formulation of the beta-functions is straightforward. The conversion rules are given in eq. (5.18) and (5.17). They read:

$$\begin{aligned} \bar{u}_k^{(0)} \bar{\lambda}_k^{(0)} &= \bar{u}_k^D \bar{\lambda}_k^D + \bar{u}_k^B \bar{\lambda}_k^B & \bar{u}_k^{(0)} &= \bar{u}_k^D + \bar{u}_k^B \\ \bar{u}_k^{(p)} \bar{\lambda}_k^{(p)} &= \bar{u}_k^D \bar{\lambda}_k^D & \bar{u}_k^{(p)} &= \bar{u}_k^D \quad \forall p \geq 1 \end{aligned}$$

Along the same lines, one obtains the relations for the boundary coefficients we consider next.

8.2.4 The Gibbons-Hawking-York invariant

The Gibbons-Hawking-York basis invariant is the only one of order $\mathcal{O}(C)$. It is furthermore only of level-(0) since the next order vanishes under conformal projection. The corresponding LHS is given by

$$\Pi_{\text{GHY}} \text{LHS} = -2\sqrt{\frac{(d-1)}{d}} \partial_t \bar{u}_k^{\partial(0)}$$

On the RHS only the first three terms of eq. (7.68) contribute, for they are of order $\mathcal{O}(\Omega^0, C^0)$. Applying Π_{GHY} to the full flow equation, a comparison of coefficients leads to

$$\partial_t \bar{u}_k^{\partial(0)} \equiv \bar{\beta}_{\bar{u}}^{\partial(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k)$$

Hereby, the beta-functions of the level-(0) Gibbons-Hawking-York invariant reads

$$\bar{\beta}_{\bar{u}}^{\partial(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) = \frac{-k^{d-2}d}{12(4\pi)^{d/2}} \left\{ (d+1) q_{d/2-1}^1(-2\lambda^{(1)}, \eta^D) - 4\rho_{\text{gh}} \Phi_{d/2-1}^1(0) \right\} \quad (8.25)$$

Notice that it contains only diamagnetic contribution.

8.2.5 The boundary volume invariant

Finally, we consider the last basis invariant, the volume element on the boundary ∂M . In order to project the flow equation to this invariant, we have to set C to zero and ignore the bulk contribution. Under the corresponding projection $\Pi_{\text{vol}(\partial M)}(\bullet) := (\bullet)|_{C=0; \text{Int}(M)=\emptyset}$ the LHS assumes the following form:

$$\Pi_{\text{vol}(\partial M)} \text{LHS} = 2\partial_t(\bar{u}_k^{\partial(0)} \bar{\lambda}_k^{\partial(0)})$$

This time, Dirichlet boundary conditions for the metric fluctuations ensure that there are in fact no higher order terms. The k -dependence of the coefficient is then governed by the following equation:

$$\partial_t(\bar{u}_k^{\partial(0)} \bar{\lambda}_k^{\partial(0)}) \equiv \bar{\beta}_{\bar{u}\bar{\lambda}}^{\partial(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k)$$

Except for the global pre-factor, the explicit form of the beta-functions resembles eq. (8.25):

$$\bar{\beta}_{\bar{u}\bar{\lambda}}^{\partial(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}; k) = \frac{-k^{d-1}d}{16(4\pi)^{\frac{(d-1)}{2}}} \left\{ (d+1)q_{(d-1)/2}^1(-2\lambda, \eta^D) - 4\rho_{\text{gh}}\Phi_{(d-1)/2}^1(0) \right\} \quad (8.26)$$

This however, is not by chance but rather due to the fact that both beta-functions result from the same three traces and are evaluated only on the basis of the Seeley-DeWitt coefficient $a_1(f; \Delta)$ in eq. (7.73).

8.3 Beta-functions for dimensionful couplings

In physics language, the coefficients discussed in section 8.2 are usually replaced by the dimensionful couplings. For the considered truncation ansatz, this only affects the coefficients associated to the curvature invariants, which can be rewritten in terms of Newton-type couplings G_k :

$$\bar{u}_k^I \equiv \frac{1}{16\pi G_k^I} \quad \text{for all } I \in \{(p), \partial(p), B, \partial B, D, \partial D\} \quad (8.27)$$

Thus, the corresponding beta-functions of the Newton-type couplings can be read off from the results of section 8.2 using the following identity:

$$\partial_t G_k^I = -\frac{1}{16\pi}(\bar{u}_k^I)^{-2} \partial_t \bar{u}_k^I \equiv -16\pi(G_k^I)^2 \partial_t \bar{u}_k^I \equiv G_k^I \eta_k^I$$

In the final step we inserted the anomalous dimension $\eta_k^I \equiv -\partial_t \ln \bar{u}_k^I$ of $\bar{u}_k^I$. For the dimensionful G_k^I it assumes the form:

$$\eta_k^I \equiv \partial_t \ln G_k^I$$

The beta-functions $\bar{\beta}_G^I(\{G_k^I, \bar{\lambda}_k^D\}; k)$ of the dimensionful Newton couplings G_k^I are defined via $\partial_t G_k^I \equiv \bar{\beta}_G^I(\{G_k^I, \bar{\lambda}_k^D\}; k)$ and thus – except for a factor of G_k^I – their k -dependence is fully determined by the anomalous dimension:

$$\bar{\beta}_G^I(\{G_k^I, \bar{\lambda}_k^D\}; k) = G_k^I \eta_k^I(\{G_k^I, \bar{\lambda}_k^D\})$$

From these rules of conversion, we can deduce the anomalous dimensions of the three involved Newton-type couplings:

$$\begin{aligned} \eta_k^{(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}) &= \frac{2k^{d-2}d}{(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d+1)}{6} q_{(d-2)/2}^1(-2\lambda^{(1)}, \eta^D) \right. \\ &\quad \left. - (d-1)\rho_P q_{d/2}^2(-2\lambda^{(1)}, \eta^D) \right\} G_k^{(0)} \\ &\quad - \frac{4k^{d-2}}{(4\pi)^{\frac{d}{2}-1}} \rho_{\text{gh}} \left\{ \frac{d}{3} \Phi_{(d-2)/2}^1(0) + 2\rho_P \Phi_{d/2}^2(0) \right\} G_k^{(0)} \end{aligned} \quad (8.28a)$$

$$\begin{aligned} \eta_k^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}) &= \frac{k^{d-2}d}{(4\pi)^{\frac{d}{2}-1}} \left\{ -\frac{(d-6)(d+1)}{6} q_{d/2}^2(-2\lambda^{(1)}, \eta^D) \right. \\ &\quad + \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^{(1)} q_{(d-2)/2}^2(-2\lambda^{(1)}, \eta^D) \\ &\quad + 2 \frac{(d-6)(d-1)}{(d-2)} \rho_P [d q_{(d+2)/2}^3(-2\lambda^{(1)}, \eta^D) - q_{d/2}^2(-2\lambda^{(1)}, \eta^D)] \\ &\quad \left. - 8 \frac{(d-4)(d-1)}{(d-2)} \rho_P \lambda^{(1)} q_{d/2}^3(-2\lambda^{(1)}, \eta^D) \right\} G_k^{(1)} \\ &\quad + \frac{4k^{d-2}}{(4\pi)^{\frac{d}{2}-1}} \rho_{\text{gh}} \left\{ \left(\frac{d}{3} - \frac{4}{(d-2)} \rho_P \right) \Phi_{d/2}^2(0) + \frac{4d}{(d-2)} \rho_P \Phi_{(d+2)/2}^3(0) \right\} G_k^{(1)} \end{aligned} \quad (8.28b)$$

$$\eta_k^{\partial(0)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}) = \frac{G_k^{\partial(0)} \cdot k^{d-2}d}{3(4\pi)^{d/2-1}} \left\{ (d+1) q_{d/2-1}^1(-2\lambda^{(1)}, \eta^D) - 4\rho_{\text{gh}} \Phi_{d/2-1}^1(0) \right\} \quad (8.28c)$$

Notice that for the D/B-formulation we have the following identities

$$\eta_k^B/G_k^B \equiv \eta_k^{(0)}/G_k^{(0)} - \eta_k^{(1)}/G_k^{(1)} \quad \text{and} \quad \eta_k^D(\bar{u}_k^D, \bar{\lambda}_k^D) \equiv \eta_k^{(1)}(\bar{u}_k^{(1)}, \bar{\lambda}_k^{(1)}) \quad (8.29)$$

These equations will be extensively used in the following sections.

The remaining couplings are of the cosmological constant type. They multiply the basis invariants associated to volume elements. As coefficients they only occur in the combination $\bar{u}_k^I \bar{\lambda}_k^I$. In order to isolate their RG behavior, we use the following relation

$$\partial_t \bar{\lambda}_k^I = \eta_k^I \bar{\lambda}_k^I + (\bar{u}_k^I)^{-1} \partial_t (\bar{u}_k^I \bar{\lambda}_k^I) \equiv \eta_k^I \bar{\lambda}_k^I + 16\pi G_k^I \partial_t (\bar{u}_k^I \bar{\lambda}_k^I) \quad (8.30)$$

When applied to the beta-functions of section 8.2 we obtain

$$\partial_t \bar{\lambda}_k^{(0)} = \eta_k^{(0)} \bar{\lambda}_k^{(0)} + \frac{k^d d(d+1)}{(4\pi)^{\frac{d}{2}-1}} q_{d/2}^1(-2\lambda^{(1)}, \eta^D) G_k^{(0)} - \frac{4k^d d}{(4\pi)^{\frac{d}{2}-1}} \rho_{\text{gh}} \Phi_{d/2}^1(0) G_k^{(0)} \quad (8.31a)$$

$$\begin{aligned} \partial_t \bar{\lambda}_k^{(1)} &= \frac{k^d(d+1)}{(4\pi)^{\frac{d}{2}-1}} \left\{ 2(d-4) \lambda^{(1)} q_{d/2}^2(-2\lambda^{(1)}, \eta^D) - \frac{(d-6)d}{2} q_{(d+2)/2}^2(-2\lambda^{(1)}, \eta^D) \right\} G_k^{(1)} \\ &\quad + \frac{4k^d d \rho_{\text{gh}}}{(4\pi)^{\frac{d}{2}-1}} \Phi_{(d+2)/2}^2(0) G_k^{(1)} + \eta_k^{(1)} \bar{\lambda}_k^{(1)} \end{aligned} \quad (8.31b)$$

$$\partial_t \bar{\lambda}_k^{\partial(0)} = \eta_k^{\partial(0)} \bar{\lambda}_k^{\partial(0)} - \frac{k^{d-1} d}{4(4\pi)^{\frac{(d-3)}{2}}} \left\{ (d+1) q_{(d-1)/2}^1(-2\lambda, \eta^D) - 4\rho_{\text{gh}} \Phi_{(d-1)/2}^1(0) \right\} G_k^{\partial(0)} \quad (8.31c)$$

In the sequel of this chapter, we will consider dimensionless couplings, where the canonical scaling is subtracted and for which the topological features of theory can be studied, as for instance fixed points.

8.4 Beta-functions for dimensionless couplings

In this section, we present the beta-functions of all dimensionless couplings for general space-time dimension d . Therefore, it is necessary to subtract the canonical scaling dimension of the couplings and retain only their anomalous k -dependence. The conversion rules relating dimensionless couplings to dimensionful differ for boundary and bulk couplings only in case of the cosmological constant, where $\lambda_k^I = \bar{\lambda}_k^I k^{-2}$ holds for all $I \in \{D, B, (0), (1), (2), \dots\}$ and $\lambda_k^{\partial(0)} = \bar{\lambda}_k^{\partial(0)} k^{-1}$ for the boundary counterpart. On the other hand, the Newton-type couplings all share the same canonical dimension and thus can be converted to dimensionless quantities via $g_k^I = G_k^I k^{d-2}$ for all $I \in \{D, B, \partial(0), (0), (1), (2), \dots\}$. This amounts to the following relations between their scale derivatives:

$$\partial_t g_k^I = (d-2)k^{(d-2)} G_k^I + k^{d-2} \partial_t G_k^I = (d-2 + \eta_k^I) g_k^I \equiv \beta_g^I \quad \forall I \quad (8.32a)$$

Likewise for the cosmological constant type couplings, we obtain

$$\partial_t \lambda_k^I = k^{-2} \partial_t \bar{\lambda}_k^I - 2k^{-2} \bar{\lambda}_k^I = k^{-2} \partial_t \bar{\lambda}_k^I - 2\lambda_k^I \equiv \beta_\lambda^I \quad \forall I \neq \partial(0) \quad (8.32b)$$

$$\partial_t \lambda_k^{\partial(0)} = k^{-1} \partial_t \bar{\lambda}_k^{\partial(0)} - k^{-1} \bar{\lambda}_k^{\partial(0)} = k^{-1} \partial_t \bar{\lambda}_k^{\partial(0)} - \lambda_k^{\partial(0)} \equiv \beta_\lambda^{\partial(0)} \quad (8.32c)$$

In the following we will list the dimensionless beta-functions in their explicit, d -dependent form first in the $\{D, B\}$, then in the level-description. The threshold functions $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$ which they contain are defined in section 8.1 and are the same as in ref. [122]. Keep in mind that in order to identify the paramagnetic ghost and graviton contributions, respectively, the corresponding terms are multiplied by the factors ρ_p and ρ_{gh} which should be put to unity if this information is not needed.

In the sequel of this section we first study the beta-functions for arbitrary cutoff shape functions, while in the afterwards we specialize for the ‘optimized’ one. We have published the results for the bulk sector in ref. [173] and part of the following sections is carried over from this publication.

8.4.1 Arbitrary cutoff shape function

For practical applications, it is more convenient to use a specific choice of cutoff shape function that is consistent with the requirements we impose on the cutoff operator and at the same time simplifies the expression in a suitable way. However, in order to prove the cutoff scheme independence of certain results one either makes use of the general properties of $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$ discussed in section 8.1 or one tests the stability of the results for different shape functions. In this subsection, we present the beta-functions of the dimensionless couplings for arbitrary cutoff shape functions and in general spacetime dimensions d .

The anomalous dimension related to g^D

The additional term in β_g^D that adds to the canonical k -dependence in eq. (8.32a) is the anomalous dimension η^D that encapsulates all the non-trivial contributions generated by the RG flow.

In case of the anomalous dimension for ‘D’-Newton constant, we have to solve an implicit equation to obtain the corresponding η^D , since it also appears on the RHS of the flow equation. It yields

$$\eta^D(g^D, \lambda^D; d) = \frac{B_1^D(\lambda^D; d) g^D}{1 - B_2^D(\lambda^D; d) g^D} \quad (8.33)$$

Here, $B_1^D(\lambda^D; d)$ and $B_2^D(\lambda^D; d)$ are functions depending on the *dynamical* cosmological constant λ^D only, as well as parametrically on the spacetime dimension d . Their explicit forms are, for the function in the numerator,

$$\begin{aligned} B_1^D(\lambda^D; d) = & \frac{d}{(4\pi)^{\frac{d}{2}-1}} \left\{ -\frac{(d-6)(d+1)}{6} \Phi_{(d/2)}^2(-2\lambda^D) \right. \\ & + \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^D \Phi_{(d-2)/2}^2(-2\lambda^D) \\ & + 2 \frac{(d-6)(d-1)}{(d-2)} \rho_P [d \Phi_{(d+2)/2}^3(-2\lambda^D) - \Phi_{(d/2)}^2(-2\lambda^D)] \\ & - 8 \frac{(d-4)(d-1)}{(d-2)} \rho_P \lambda^D \Phi_{(d/2)}^3(-2\lambda^D) \left. \right\} \\ & + \frac{4}{(4\pi)^{\frac{d}{2}-1}} \rho_{gh} \left\{ \left(\frac{d}{3} - \frac{4}{(d-2)} \rho_P \right) \Phi_{(d/2)}^2(0) + \frac{4d}{(d-2)} \rho_P \Phi_{(d+2)/2}^3(0) \right\} \end{aligned} \quad (8.34)$$

Likewise the function in the denominator reads:

$$\begin{aligned} B_2^D(\lambda^D; d) = & \frac{d}{2(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d-6)(d+1)}{6} \tilde{\Phi}_{(d/2)}^2(-2\lambda^D) \right. \\ & - \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^D \tilde{\Phi}_{(d-2)/2}^2(-2\lambda^D) \\ & - 2 \frac{(d-6)(d-1)}{(d-2)} \rho_P [d \tilde{\Phi}_{(d+2)/2}^3(-2\lambda^D) - \tilde{\Phi}_{(d/2)}^2(-2\lambda^D)] \\ & + 8 \frac{(d-4)(d-1)}{(d-2)} \rho_P \lambda^D \tilde{\Phi}_{(d/2)}^3(-2\lambda^D) \left. \right\} \end{aligned} \quad (8.35)$$

Notice that in the present truncation the dimensionalities $d = 4$ and $d = 6$ play a special role: in these cases the graviton contributions to B_1^D and B_2^D are either all given by terms with an extra factor of λ^D multiplying the threshold functions, in $d = 6$, or precisely those terms are all absent, in $d = 4$. This pattern is not found in the ghost contributions (which can be identified by their ρ_{gh} factor).

The beta-function of λ^D

The running of the dynamical cosmological constant is governed by the beta-function

$$\begin{aligned} \beta_\lambda^D(g^D, \lambda^D; d) = & (\eta^D - 2) \lambda^D + g^D \frac{4d \rho_{gh}}{(4\pi)^{\frac{d}{2}-1}} \Phi_{(d+2)/2}^2(0) \\ & + g^D \frac{(d+1)}{(4\pi)^{\frac{d}{2}-1}} \left\{ 2(d-4) \lambda^D q_{(d/2)}^2(-2\lambda^D, \eta^D) - \frac{(d-6)d}{2} q_{(d+2)/2}^2(-2\lambda^D, \eta^D) \right\} \end{aligned} \quad (8.36)$$

Note that the non-canonical graviton contributions on the RHS of (8.36) show the same property as discussed above: they are proportional to *either* $(d-4)\lambda^D$ *or* $(d-6)$.

Together with eq. (8.35), eq. (8.36) gives rise to a closed system of two coupled differential equations which can be solved independently of the other equations.

The anomalous dimension related to g^B

The non-canonical k -dependence of the background Newton constant is given by the anomalous dimension

$$\begin{aligned} \eta^B(g^D, \lambda^D, g^B; d) = & \quad (8.37) \\ = & \frac{d}{(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d+1)}{3} q_{(d-2)/2}^1(-2\lambda^D, \eta^D) - \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^D q_{(d-2)/2}^2(-2\lambda^D, \eta^D) \right. \\ & + \left(\frac{(d-6)(d+1)}{6} - 8 \frac{(d-1)}{(d-2)} \rho_P \right) q_{(d/2)}^2(-2\lambda^D, \eta^D) \\ & + 8 \frac{(d-4)(d-1)}{(d-2)} \rho_P \lambda^D q_{(d/2)}^3(-2\lambda^D, \eta^D) \\ & \left. - 2 \frac{(d-6)(d-1)d}{(d-2)} \rho_P q_{(d+2)/2}^3(-2\lambda^D, \eta^D) \right\} g^B \\ & - \frac{4}{(4\pi)^{\frac{d}{2}-1}} \rho_{gh} \left\{ \frac{d}{3} \Phi_{(d-2)/2}^1(0) + \left(\frac{d}{3} + \frac{2(d-4)}{(d-2)} \rho_P \right) \Phi_{(d/2)}^2(0) + \frac{4d}{(d-2)} \rho_P \Phi_{(d+2)/2}^3(0) \right\} g^B \end{aligned}$$

The property found of η^D concerning its λ^D -dependence in $d = 4$ and $d = 6$ is only partially shared by η^B ; there are contributions from the graviton sector that neither drop out for $d = 4$ nor for $d = 6$.

The beta-function of λ^B

The last member in the hierarchy of the differential system of the bulk sector is the cosmological constant λ^B , which does not influence - but in turn shows a dependence on - all the other 3 bulk couplings. Its beta-function is given by

$$\begin{aligned} \beta_\lambda^B(g^D, \lambda^D, g^B, \lambda^B; d) = & (\eta^B - 2)\lambda^B - \frac{4d}{(4\pi)^{\frac{d}{2}-1}} \rho_{gh} \left\{ \Phi_{(d/2)}^1(0) + \Phi_{(d+2)/2}^2(0) \right\} g^B \\ & + \frac{(d+1)}{(4\pi)^{\frac{d}{2}-1}} \left\{ d q_{(d/2)}^1(-2\lambda^D, \eta^D) - 2(d-4) \lambda^D q_{(d/2)}^2(-2\lambda^D, \eta^D) \right. \\ & \left. + (d-6) \frac{d}{2} q_{(d+2)/2}^2(-2\lambda^D, \eta^D) \right\} g^B \end{aligned} \quad (8.38)$$

At this point all 4 RG equations of the D-B system are fully specified for the bulk sector. What remains are the boundary couplings that we are going to discuss in the level language. In the level description, the ‘D’ beta-functions provide the beta-functions at all higher levels $p = 1, 2, 3, \dots$

$$\eta^{(p)} = \eta^D \quad \text{and} \quad \beta_\lambda^{(p)} = \beta_\lambda^D \quad \text{for all } p \geq 1.$$

What is special are the level-(0) couplings. They are governed by combinations of the ‘B’ and ‘D’ beta-functions.

The anomalous dimension related to $g^{(0)}$

The anomalous dimension of the level-(0) Newton coupling fulfills

$$\frac{\eta^{(0)}}{g^{(0)}} = \frac{\eta^D}{g^D} + \frac{\eta^B}{g^B},$$

from which, using eqs. (8.35) and (8.37), it follows that

$$\begin{aligned} \eta^{(0)} = & + \frac{2d}{(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d+1)}{6} q_{(d-2)/2}^1(-2\lambda^D, \eta^D) - (d-1) \rho_P q_{(d/2)}^2(-2\lambda^D, \eta^D) \right\} g^{(0)} \\ & - \frac{4}{(4\pi)^{\frac{d}{2}-1}} \rho_{gh} \left\{ \frac{d}{3} \Phi_{(d-2)/2}^1(0) + 2 \rho_P \Phi_{(d/2)}^2(0) \right\} g^{(0)} \end{aligned} \quad (8.39)$$

For notational consistency, one should identify $\lambda^D = \lambda^{(1)} = \lambda^{(2)} = \dots$ on the RHS of this equation.

As a check on our calculation, we mention that if instead we identify $\eta^D = \eta^{(0)}$ and $\lambda^D = \lambda^{(0)}$ in the $q_n^p(-2\lambda^D, \eta^D)$ functions we re-obtain precisely the anomalous dimension of the single-metric truncation found in [122], $\eta^{(0)} = \eta^{\text{sm}}$, as it should be. (See also Section 10.3)

The beta-function of $\lambda^{(0)}$

For the cosmological constant at level-(0) one finds

$$\beta_\lambda^{(0)} = (\eta^{(0)} - 2)\lambda^{(0)} + g^{(0)} \frac{d(d+1)}{(4\pi)^{\frac{d}{2}-1}} q_{(d/2)}^1(-2\lambda^D, \eta^D) - g^{(0)} \frac{4d}{(4\pi)^{\frac{d}{2}-1}} \rho_{\text{gh}} \Phi_{(d/2)}^1(0) \quad (8.40)$$

where $\lambda^D = \lambda^{(1)} = \lambda^{(2)} = \dots$ in the present truncation. Again, it can be checked that $\beta_\lambda^{(0)}$ gives rise to precisely the single-metric beta-function $\beta_\lambda^{\text{sm}}$, obtained in [122] if instead we make the identification $\eta^D = \eta^{(0)}$ and $\lambda^D = \lambda^{(0)}$.

The anomalous dimension related to $g^{\partial(0)}$

For the present truncation, it is difficult to speak of D/B couplings in case of the boundary invariants. The reason is that level-(1) is not resolved in our final ansatz and thus it is purely speculation which contribution can be associated to the dynamical or background RG evolution. While for the cosmological constant, there is in fact no level-(1) counterpart, the direction in theory space associated to the Newton-type coupling $g_k^{\partial(1)}$ vanishes under conformal projection.

Thus, consider the anomalous dimension of $G_k^{\partial(0)}$ given in eq. (8.28c) and express it in terms of the dimensionless couplings.

$$\eta^{\partial(0)}(g^D, \lambda^D, g^{\partial(0)}) = \frac{g^{\partial(0)} d}{3(4\pi)^{d/2-1}} \left\{ (d+1) q_{d/2-1}^1(-2\lambda^{(1)}, \eta^D) - 4 \rho_{\text{gh}} \Phi_{d/2-1}^1(0) \right\} \quad (8.41)$$

For this anomalous dimension there is no obvious reduction for any spacetime dimension d , as observed for η^D in particular. However, for $\lambda^D = 0$ and $\eta^D = 0$, which corresponds to the semiclassical regime, there is a global factor of $(d-3)$. Notice that due to the general properties of the threshold functions the bracket is positive for $d \geq 3$. Thus non-trivial fixed point values for $g^{\partial(0)}$ will be negative.

The beta-function of $\lambda^{\partial(0)}$

The last ingredient describes the RG evolution of the boundary volume element.

$$\begin{aligned} \beta_\lambda^{\partial(0)} &= (\eta^{\partial(0)} - 1)\lambda^{\partial(0)} \\ &\quad - \frac{d}{4(4\pi)^{\frac{(d-3)}{2}}} \left\{ (d+1) q_{(d-1)/2}^1(-2\lambda, \eta^D) - 4 \rho_{\text{gh}} \Phi_{(d-1)/2}^1(0) \right\} g^{\partial(0)} \end{aligned} \quad (8.42)$$

The single-metric results obtained in [166] are structurally the same however with η^D and λ^D replaced by their single-metric analogs.

Notice that similar to the hierarchy $(g_k^{(1)}, \lambda_k^{(1)}) \rightarrow g_k^{(0)} \rightarrow \lambda_k^{(0)}$ on the bulk sector, we obtain a chain of dependencies for the bulk-boundary couplings:

$$(g_k^{(1)}, \lambda_k^{(1)}) \rightarrow g_k^{\partial(0)} \rightarrow \lambda_k^{\partial(0)}$$

This has some very practical consequences. For instance in deriving solutions for the full 6-dimensional truncated theory space, one first solves for the level-(1) bulk couplings and then insert the results into the remaining four differential equations to obtain a full RG trajectory.

8.4.2 Optimized cutoff shape function

The ‘optimized’ shape function [188] is given by $R^{(0)}(z) = (1-z)\theta(1-z)$ and allows for an explicit evaluation of the threshold-functions:

$$\Phi_n^p(w) = \frac{1}{\Gamma(n+1)}(1+w)^{-p}, \quad \text{and} \quad \tilde{\Phi}_n^p(w) = \frac{1}{\Gamma(n+2)}(1+w)^{-p} \quad (8.43)$$

In the following we list the beta-functions in general spacetime dimensions d within this optimized scheme.

The anomalous dimension related to g^D

The key ingredients for the anomalous dimension η^D in the D-sector, eq. (8.33), are the functions $B_1^D(\lambda^D; d)$ and $B_2^D(\lambda^D; d)$ given by (8.34) and (8.35), respectively. In the optimized scheme they are obtained upon inserting (8.43) into (8.34):

$$\begin{aligned} B_1^D(\lambda^D; d) = & \frac{(1-2\lambda^D)^{-2}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ -\frac{(d+1)}{3}((d-6) - 2\frac{(d-4)d}{d-2}\lambda^D) \right. \\ & \left. + 4\frac{(d-1)((6-d) + 2(d+2)\lambda^D)}{(d+2)(1-2\lambda^D)}\rho_P \right\} \\ & + \frac{8\left(\frac{1}{3} + \frac{4}{d(d+2)}\rho_P\right)}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)}\rho_{gh} \end{aligned} \quad (8.44a)$$

$$\begin{aligned} B_2^D(\lambda^D; d) = & \frac{d(1-2\lambda^D)^{-2}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+1)} \left\{ \frac{(d-6)(d+1)}{6(d+2)} - \frac{1}{3}\frac{(d-4)(d+1)}{(d-2)}\lambda^D + 2\frac{(d-6)(d-1)}{(d-2)(d+2)}\rho_P \right. \\ & \left. - \frac{4(d-1)}{(d-2)(d+2)}\frac{((d-6)d - 2(d-4)\lambda^D)}{(1-2\lambda^D)}\rho_P \right\} \end{aligned} \quad (8.44b)$$

If we are not interested in distinguishing the paramagnetic from diamagnetic terms, and ghost from metric contributions, we may set $\rho_{gh} = 1 = \rho_P$. This yields the following simplified expressions:

$$\begin{aligned} B_1^D(\lambda^D; d) = & \frac{(1-2\lambda^D)^{-3}}{3(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ -\frac{(d-7)(d-6)(d-2)}{(d+2)} + \frac{4(d^3 - 11d^2 + 18d - 6)}{(d-2)}\lambda^D \right. \\ & \left. - \frac{4d(d-4)(d+1)}{(d-2)}\lambda^{D2} \right\} + \frac{8\left(\frac{1}{3} + \frac{4}{d(d+2)}\right)}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \end{aligned} \quad (8.45a)$$

$$\begin{aligned} B_2^D(\lambda^D; d) = & \frac{d(1-2\lambda^D)^{-3}}{3(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+1)} \left\{ \frac{(d-6)(d^3 - 9d^2 + 54d - 56)}{2(d^2 - 4)(d+4)} \right. \\ & \left. - \frac{2(d^3 - 10d^2 + 15d - 10)}{(d^2 - 4)}\lambda^D + \frac{2(d-4)(d+1)}{(d-2)}\lambda^{D2} \right\} \end{aligned} \quad (8.45b)$$

The beta-function of λ^D

The beta-function of eq. (8.37) for the cosmological constant, in the optimized scheme, looks as follows:

$$\begin{aligned} \beta_\lambda^D(g^D, \lambda^D; d) = & (\eta^D - 2)\lambda^D + \frac{4d\rho_{gh}g^D}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+2)} \\ & + \frac{2(d+1)g^D}{(4\pi)^{\frac{d}{2}-1}d\Gamma(d/2)(1-2\lambda^D)^2} \left\{ 2(d-4)\lambda^D\left(1 - \frac{\eta^D}{(d+2)}\right) - \frac{(d-6)d}{(d+2)}\left(1 - \frac{\eta^D}{(d+4)}\right) \right\} \end{aligned} \quad (8.46)$$

The reduction to $\rho_{\text{gh}} = 1 = \rho_{\text{p}}$ is omitted here, since this expression contains no factor of ρ_{p} , and ρ_{gh} occurs only trivially in the second term on the RHS. So no further simplification can be achieved.

The anomalous dimension related to g^{B}

Eq. (8.38) for the choice (8.43) results in

$$\begin{aligned} \eta^{\text{B}}(g^{\text{D}}, \lambda^{\text{D}}, g^{\text{B}}; d) = & \quad (8.47) \\ = & \frac{(1 - 2\lambda^{\text{D}})^{-1}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ \frac{(d+1)}{3}(d - \eta^{\text{D}}) - \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^{\text{D}} \frac{(d - \eta^{\text{D}})}{(1 - 2\lambda^{\text{D}})} \right. \\ & + \left(\frac{(d-6)(d+1)}{3} - 16 \frac{(d-1)}{(d-2)} \rho_{\text{p}} \right) \frac{(1 - \eta^{\text{D}}/(d+2))}{(1 - 2\lambda^{\text{D}})} \\ & + 16 \frac{(d-4)(d-1)}{(d-2)} \rho_{\text{p}} \lambda^{\text{D}} \frac{(1 - \eta^{\text{D}}/(d+2))}{(1 - 2\lambda^{\text{D}})^2} \\ & \left. - 8 \frac{(d-6)(d-1)d}{(d^2-4)} \rho_{\text{p}} \frac{(1 - \eta^{\text{D}}/(d+4))}{(1 - 2\lambda^{\text{D}})^2} \right\} g^{\text{B}} \\ & - \frac{4}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \rho_{\text{gh}} \left\{ \frac{(d+2)}{3} + \frac{4(d+4)}{d(d+2)} \rho_{\text{p}} \right\} g^{\text{B}} \end{aligned}$$

This lengthy expression reduces slightly when we switch off the separation given by the ρ -parameters. For $\rho_{\text{gh}} = 1 = \rho_{\text{p}}$ we obtain

$$\begin{aligned} \eta^{\text{B}}(g^{\text{D}}, \lambda^{\text{D}}, g^{\text{B}}; d) = & \quad (8.48) \\ = & \frac{(1 - 2\lambda^{\text{D}})^{-1}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ \frac{(d+1)}{3}(d - \eta^{\text{D}}) - \frac{2}{3} \frac{(d-4)(d+1)}{(d-2)} \lambda^{\text{D}} \frac{(d - \eta^{\text{D}})}{(1 - 2\lambda^{\text{D}})} \right. \\ & + \left(\frac{(d-6)(d+1)(d-2) - 16(d-1)}{3(d-2)} \right) \frac{(1 - \eta^{\text{D}}/(d+2))}{(1 - 2\lambda^{\text{D}})} \\ & + 16 \frac{(d-4)(d-1)}{(d-2)} \lambda^{\text{D}} \frac{(1 - \eta^{\text{D}}/(d+2))}{(1 - 2\lambda^{\text{D}})^2} - 8 \frac{(d-6)(d-1)d}{(d^2-4)} \frac{(1 - \eta^{\text{D}}/(d+4))}{(1 - 2\lambda^{\text{D}})^2} \left. \right\} g^{\text{B}} \\ & - \frac{4}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left(\frac{d^3 + 4d^2 + 16d + 48}{3d(d+2)} \right) g^{\text{B}} \end{aligned}$$

The last term of (8.48) contains the ghost contributions.

The beta-function of λ^{B}

For the cosmological constant in the B-sector we only write down the result prior to setting $\rho_{\text{gh}} = 1$:

$$\begin{aligned} \beta_{\lambda}^{\text{B}}(g^{\text{D}}, \lambda^{\text{D}}, g^{\text{B}}, \lambda^{\text{B}}; d) = & (\eta^{\text{B}} - 2)\lambda^{\text{B}} - \frac{8(d+4)\rho_{\text{gh}}}{(4\pi)^{\frac{d}{2}-1}(d+2)\Gamma(d/2)} g^{\text{B}} \\ & + \frac{(d+1)(1 - 2\lambda^{\text{D}})^{-2}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+1)} \left\{ \frac{2d}{(d+2)} \left((d-2) - \frac{(d-1)}{(d+4)} \eta^{\text{D}} \right) \right. \\ & \left. - 4(d-2)(1 - \eta^{\text{D}}/(d+2))\lambda^{\text{D}} \right\} g^{\text{B}} \end{aligned} \quad (8.49)$$

This completes the list of the bulk beta-functions in the $\{\text{D}, \text{B}\}$ -description. As above, the boundary expressions are described in the level-language to which we turn next.

The anomalous dimension related to $g^{(0)}$

For the above made choice of the shape function the anomalous dimension for $g^{(0)}$ assumes the form

$$\eta^{(0)} = + \frac{2(1-2\lambda^D)^{-1}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ \frac{(d+1)}{6}(d-\eta^D) - 2\frac{(d-1)}{(d+2)} \frac{((d+2)-\eta^D)}{(1-2\lambda^D)} \rho_P \right\} g^{(0)} - \frac{4}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \rho_{gh} \left\{ \frac{d}{3} + \frac{4}{d} \rho_P \right\} g^{(0)} \quad (8.50)$$

Setting $\rho_{gh} = 1 = \rho_P$ this further simplifies to

$$\eta^{(0)} = \frac{4}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ \frac{(d+1)}{12} \frac{(d-\eta^D)}{(1-2\lambda^D)} - \frac{(d-1)}{(d+2)} \frac{((d+2)-\eta^D)}{(1-2\lambda^D)^2} - \left(\frac{d^2+12}{3d} \right) \right\} g^{(0)} \quad (8.51)$$

which can also be derived using the identity $\eta^{(0)}/g^{(0)} = \eta^B/g^B + \eta^D/g^D$ directly.

The beta-function of $\lambda^{(0)}$

For the scale-dependence of the cosmological constant $\lambda^{(0)}$ we have to consult eq. (8.40) which yields

$$\beta_\lambda^{(0)} = (\eta^{(0)} - 2)\lambda^{(0)} + g^{(0)} \frac{\frac{d}{2}(d+1)}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+2)} \frac{((d+2)-\eta^D)}{(1-2\lambda^D)} - g^{(0)} \frac{4d\rho_{gh}}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+1)} \quad (8.52)$$

Omitting the separation given by the ρ -parameters we find

$$\beta_\lambda^{(0)} = (\eta^{(0)} - 2)\lambda^{(0)} + g^{(0)} \frac{\frac{d}{2}((d-3+8\lambda^D)(d+2) - (d+1)\eta^D)}{(4\pi)^{\frac{d}{2}-1}\Gamma(d/2+2)(1-2\lambda^D)} \quad (8.53)$$

The higher level beta-functions are described by the D couplings given in eqs. (8.44a), (8.44b) and (8.46).

The anomalous dimension related to $g^{\partial(0)}$

Substituting the explicit expressions for the threshold functions in case of the optimized scheme, eq. (8.41) assumes the following form:

$$\eta^{\partial(0)} = \frac{g^{\partial(0)}}{3(4\pi)^{\frac{d}{2}-1}\Gamma(d/2)} \left\{ (d+1) \frac{(d-\eta^D)}{(1-2\lambda^D)} - 4d\rho_{gh} \right\} \quad (8.54)$$

This anomalous dimension defines the non-canonical scaling behavior of the boundary Newton-type coupling multiplying the Gibbons-Hawking-York functional.

The beta-function of $\lambda^{\partial(0)}$

Finally, we consider eq. (8.42) and insert the optimized shape functions. The differential equation for $\lambda_k^{\partial(0)}$ is then given by

$$\beta_\lambda^{\partial(0)} = (\eta^{\partial(0)} - 1)\lambda^{\partial(0)} + \frac{g^{\partial(0)} d}{4(4\pi)^{\frac{(d-3)}{2}}\Gamma(\frac{d+1}{2})} \left\{ \frac{(d+1-\eta^D)}{(1-2\lambda^D)} + 4\rho_{gh} \right\} \quad (8.55)$$

Identifying $\rho_{gh} = 1$ does not lead to any significant reduction of the formula, so we omit it here.

8.5 The semiclassical approximation

In this section we present an explicit solution to the RG equations which is valid approximately provided $g_k^I \ll 1$ and $\lambda_k^D \ll 1$. (The magnitude of λ_k^B plays no role for the validity.) We refer to it as the semiclassical approximation since in an (k/m_{Planck}) -expansion it describes the leading deviations from the strictly classical behavior with exactly constant dimensionful couplings $G_k^I \equiv G_0^I$ and $\bar{\lambda}_k^I \equiv \bar{\lambda}_0^I$, respectively.

Newton constants

The k -dependence of the Newton constants is covered by the corresponding anomalous dimension: $k\partial_k G_k^I = \eta^I G_k^I$ for $I \in \{D, B, (p), \partial(p)\}$. If $g_k^I \ll 1$ we may expand in g_k^I and retain the linear term only. Returning to dimensionful variables all anomalous dimensions have the following form then:

$$\eta^I = B_1^I(\bar{\lambda}_k^D/k^2) G_k^I \quad (8.56)$$

The solution to the ensuing RG equation with initial conditions posed at $k = 0$ is given by

$$\frac{1}{G_k^I} = \frac{1}{G_0^I} - \int_0^k dk' B_1^I(\bar{\lambda}_{k'}^D/k'^2) k'^{(d-1)} \quad (8.57)$$

The next step in the approximation consists in expanding the integrand of (8.57) in small values of $\lambda_k^D \equiv \bar{\lambda}_k^D/k^2$. In the lowest order we have $B_1^I(\bar{\lambda}_k^D/k^2) = B_1^I(0) + \mathcal{O}(\bar{\lambda}_k^D/k^2)$ for which the integration can be performed easily. Thus to leading order for $g_k^I \ll 1$ and $\lambda_k^D \ll 1$:

$$G_k^I = \frac{G_0^I}{1 + \omega_d^I G_0^I k^{d-2}} = G_0^I [1 - \omega_d^I G_0^I k^{d-2}] \quad (8.58)$$

Here we have defined the coefficients $\omega_d^I \equiv -B_1^I(0)/(d-2)$ which are analogous to those in [122] and its later generalizations.

For the D Newton coupling the coefficient ω_d^D follows from (8.34):

$$\begin{aligned} \omega_d^D &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d^4-7d^3+4d^2+12d)-8d(d-2)\rho_{\text{gh}}}{6(d-2)^2} \Phi_{(d/2)}^2(0) - \frac{2d((d-6)(d-1)d+8\rho_{\text{gh}})}{(d-2)^2} \rho_p \Phi_{(d+2)/2}^3(0) \right\}, \\ \omega_d^D &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d^3+7d^2-74d-48)}{6(d-2)} \Phi_{(d/2)}^2(0) - \frac{d(2d^2-10d-8)}{(d-2)} \Phi_{(d+2)/2}^3(0) \right\} \end{aligned} \quad (8.59)$$

The first equation contains the separation between contributions of different origin, the second relaxes this division and sets $\rho_{\text{gh}} = 1 = \rho_p$. For the B Newton constant, we find from eq. (8.37), with and without the ρ -factors, respectively:

$$\begin{aligned} \omega_d^B &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ -\frac{(d(d+1)-4d\rho_{\text{gh}})}{3(d-2)} \Phi_{(d-2)/2}^1(0) + \frac{2d((d-6)(d-1)d+8\rho_{\text{gh}})\rho_p}{(d-2)^2} \Phi_{(d+2)/2}^3(0) \right. \\ &\quad \left. - \frac{(d-2)d((d-5)d-6-8\rho_{\text{gh}})-48((d-1)+(d-4)\rho_{\text{gh}})\rho_p}{6(d-2)^2} \Phi_{(d/2)}^2(0) \right\} \quad (8.60) \\ \omega_d^B &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ -\frac{d(d-3)}{3(d-2)} \Phi_{(d-2)/2}^1(0) - \frac{(d^3-7d^2+74d+48)}{6(d-2)} \Phi_{(d/2)}^2(0) \right. \\ &\quad \left. + \frac{d(2d^2-10d-8)}{(d-2)} \Phi_{(d+2)/2}^3(0) \right\} \end{aligned}$$

In the level description the ω -coefficient is determined by eq. (8.39):

$$\begin{aligned}\omega_d^{(0)} &= \frac{2}{(4\pi)^{\frac{d}{2}-1}} \left\{ -\frac{d(d+1-4\rho_{\text{gh}})}{3(d-2)} \Phi_{(d-2)/2}^1(0) + \frac{2(d(d-1)+4\rho_{\text{gh}})}{(d-2)} \rho_{\text{p}} \Phi_{(d/2)}^2(0) \right\} \\ \omega_d^{(0)} &= \frac{2}{(4\pi)^{\frac{d}{2}-1}} \left\{ -\frac{d(d-3)}{6(d-2)} \Phi_{(d-2)/2}^1(0) + \frac{(d^2-d+4)}{(d-2)} \Phi_{(d/2)}^2(0) \right\}\end{aligned}\quad (8.61)$$

In the last equation we set again $\rho_{\text{gh}} = 1 = \rho_{\text{p}}$ to simplify the result.

Finally, the ω -coefficient for the boundary Newton-type coupling can be read off from eq. (8.41):

$$\omega_d^{\partial(0)} = -\frac{d}{3(4\pi)^{d/2-1}} \left\{ \frac{(d+1)}{(d-2)} \Phi_{d/2-1}^1(0) - \frac{4\rho_{\text{gh}}}{(d-2)} \Phi_{d/2-1}^1(0) \right\} \quad (8.62)$$

Omitting the separation of ghost and gravitational contributions by setting $\rho_{\text{gh}} = 1$, eq. (8.62) simplifies to

$$\omega_d^{\partial(0)} = -\frac{d(d-3)}{3(4\pi)^{d/2-1}} \left\{ \frac{1}{(d-2)} \Phi_{d/2-1}^1(0) \right\} \quad (8.63)$$

As already mentioned $\omega_d^{\partial(0)}$ vanishes in $d = 3$ and is negative (positive) for $d > 3$ ($d < 3$).

Cosmological constants

The RG equations for the bulk cosmological constants λ_k^I occurring in the present truncation have the structure

$$\partial_t \lambda_k^I = (\eta^I - 2) \lambda_k^I + A^I(\lambda_k^{\text{D}}, g_k^{\text{D}}) g_k^I \quad \text{for all } I \neq \partial(0) \quad (8.64)$$

and for the boundary counterpart

$$\partial_t \lambda_k^{\partial(0)} = (\eta^{\partial(0)} - 1) \lambda_k^{\partial(0)} + A^{\partial(0)}(\lambda_k^{\text{D}}, g_k^{\text{D}}) g_k^{\partial(0)} \quad (8.65)$$

whereby $A^I(\lambda_k^{\text{D}}, g_k^{\text{D}}) \equiv A_1^I(\lambda_k^I) - \frac{1}{2} \eta^{\text{D}} A_2^I(\lambda_k^I)$. Here we define $A_1^I \equiv A^I(\lambda_k^{\text{D}})|_{\eta^{\text{D}}=0}$ to be the contribution that remains after omitting η^{D} on the RHS of the flow equations. The dimensionful analog of eq. (8.64) and (8.65) can be solved formally to yield

$$\frac{\bar{\lambda}_k^I}{G_k^I} = \frac{\bar{\lambda}_0^I}{G_0^I} + \int_0^k dk' A^I(\lambda_{k'}^{\text{D}}, g_{k'}^{\text{D}}) k'^{d-1} \quad \text{or} \quad \frac{\bar{\lambda}_k^{\partial(0)}}{G_k^{\partial(0)}} = \frac{\bar{\lambda}_0^{\partial(0)}}{G_0^{\partial(0)}} + \int_0^k dk' A^{\partial(0)}(\lambda_{k'}^{\text{D}}, g_{k'}^{\text{D}}) k'^{d-2} \quad (8.66)$$

For $\lambda_k^{\text{D}} \ll 1$ we expand $A_1^I(\bar{\lambda}_k^{\text{D}}/k^2) = A_1^I(0) + \mathcal{O}(\bar{\lambda}_k^{\text{D}}/k^2)$ and solve the integral to leading order. Furthermore, we also insert the approximate result for Newtons coupling (8.58) into eq. (8.66). We finally obtain for the semiclassical approximation on the bulk

$$\bar{\lambda}_k^I = \frac{G_k^I}{G_0^I} \left(\bar{\lambda}_0^I + v_d^I G_0^I k^d + \mathcal{O}(\bar{\lambda}_0^{\text{D}}/k^2) \right) = \bar{\lambda}_0^I + v_d^I G_0^I k^d + \mathcal{O}(\bar{\lambda}_0^{\text{D}}/k^2, G_0^{I2} k^{d-2}) \quad (8.67)$$

whereby we defined the coefficients $v_d^I \equiv A_1^I(0)/d$, again following the conventions used in earlier publications [189, 190]. Likewise on the boundary we have the following semiclassical expansion

$$\begin{aligned}\bar{\lambda}_k^{\partial(0)} &= \frac{G_k^{\partial(0)}}{G_0^{\partial(0)}} \left(\bar{\lambda}_0^{\partial(0)} + v_d^{\partial(0)} G_0^{\partial(0)} k^{d-1} + \mathcal{O}(\bar{\lambda}_0^{\text{D}}/k^2) \right) \\ &= \bar{\lambda}_0^{\partial(0)} + v_d^{\partial(0)} G_0^{\partial(0)} k^{d-1} + \mathcal{O}(\bar{\lambda}_0^{\text{D}}/k^2, G_0^{I2} k^{d-2})\end{aligned}\quad (8.68)$$

this time however with $v_d^{\partial(0)} \equiv A_1^{\partial(0)}(0)/(d-1)$

For the D cosmological constant v_d^I can be deduced from eq. (8.36):

$$\begin{aligned} v_d^D &= -\frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ \frac{(d-6)}{2}(d+1) - 4\rho_{\text{gh}} \right\} \Phi_{(d+2)/2}^2(0), \\ v_d^D &= -\frac{(d-7)(d+2)}{2(4\pi)^{\frac{d}{2}-1}} \Phi_{(d+2)/2}^2(0) \end{aligned} \quad (8.69)$$

For the B cosmological coupling we find from eq. (8.38)

$$\begin{aligned} v_d^B &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ ((d+1) - 4\rho_{\text{gh}}) \Phi_{(d/2)}^1(0) - \left(\frac{(d-6)}{2}(d+1) - 4\rho_{\text{gh}} \right) \Phi_{(d+2)/2}^2(0) \right\}, \\ v_d^B &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ (d-3) \Phi_{(d/2)}^1(0) - \frac{(d-7)(d+2)}{2} \Phi_{(d+2)/2}^2(0) \right\} \end{aligned} \quad (8.70)$$

and in the level-description we obtain from eq. (8.40):

$$\begin{aligned} v_d^{(0)} &= \frac{1}{(4\pi)^{\frac{d}{2}-1}} \left\{ (d+1) - 4\rho_{\text{gh}} \right\} \Phi_{(d/2)}^1(0), \\ v_d^{(0)} &= \frac{(d-3)}{(4\pi)^{\frac{d}{2}-1}} \Phi_{(d/2)}^1(0) \end{aligned} \quad (8.71)$$

As always, the first equation of the above pairs retains arbitrary ρ 's, while the second is the true final result with $\rho_{\text{gh}} = 1 = \rho_P$.

Finally, we use eq. (8.68) and the result obtained in eq. (8.42) to derive the semiclassical expansion for $\tilde{\lambda}_k^{\partial(0)}$:

$$\begin{aligned} v_d^{\partial(0)} &= -\frac{d}{4(4\pi)^{\frac{(d-3)}{2}}} \left\{ \frac{(d+1) - 4\rho_{\text{gh}}}{(d-1)} \right\} \Phi_{(d-1)/2}^1(0), \\ v_d^{\partial(0)} &= -\frac{d(d-3)}{4(4\pi)^{\frac{(d-3)}{2}}(d-1)} \Phi_{(d/2)}^1(0) \end{aligned} \quad (8.72)$$

Notice that $v_d^{\partial(0)}$ has the opposite sign w.r.t. its bulk counterpart in eq. (8.71).

8.6 Beta-functions in $d = 4$

In this section we present the beta-functions for the dimensionless couplings derived in the last section for the particular interesting case of four spacetime dimensions, i.e. $d = 4$.

8.6.1 The D sector: beta-functions for g^D, λ^D

The RG equation for the dynamical Newton constant g_k^D is governed by the anomalous dimension η^D . The FRGE provides the following relation for it:

$$\begin{aligned} \eta^D(g^D, \lambda^D) &= +\frac{1}{\pi} \left\{ \frac{5}{3} q_2^2(-2\lambda^D, \eta^D) - 6\rho_P [4q_3^3(-2\lambda^D, \eta^D) - q_2^2(-2\lambda^D, \eta^D)] \right\} g^D \\ &\quad + \frac{2}{\pi} \rho_{\text{gh}} \left\{ \left(\frac{2}{3} - \rho_P \right) \Phi_2^2(0) + 4\rho_P \Phi_3^3(0) \right\} g^D \end{aligned} \quad (8.73)$$

It has a global factor of g_k^D and depends on λ_k^D through the threshold functions. In addition, $q_n^D(w, \eta^D)$ contains another term proportional to η^D . As a consequence, eq. (8.73) is an implicit equation for the anomalous dimension. It can be solved to yield the final result:

$$\eta^D(g^D, \lambda^D) = \frac{B_1^D(\lambda^D; 4) g^D}{1 - B_2^D(\lambda^D; 4) g^D} \quad (8.74)$$

We obtain a non-polynomial dependence of η^D on g^D , and $B_1^D(\lambda^D; 4)$ and $B_2^D(\lambda^D; 4)$ are functions of the cosmological constant, here specialized for $d = 4$. The first one contains graviton as well as ghost contributions,

$$B_1^D(\lambda^D; 4) = \frac{1}{\pi} \left\{ \frac{5}{3} \Phi_2^2(-2\lambda^D) - 6\rho_p [4\Phi_3^3(-2\lambda^D) - \Phi_2^2(-2\lambda^D)] \right\} + \frac{2}{\pi} \rho_{gh} \left\{ \left(\frac{2}{3} - \rho_p \right) \Phi_2^2(0) + 4\rho_p \Phi_3^3(0) \right\} \quad (8.75)$$

whereas the second one stems entirely from the graviton sector:

$$B_2^D(\lambda^D; 4) = -\frac{1}{2\pi} \left\{ \frac{5}{3} \tilde{\Phi}_2^2(-2\lambda^D) - 6\rho_p [4\tilde{\Phi}_3^3(-2\lambda^D) - \tilde{\Phi}_2^2(-2\lambda^D)] \right\} \quad (8.76)$$

Furthermore the running of the dynamical cosmological constant λ_k^D is described by

$$\partial_t \lambda_k^D = (\eta^D - 2) \lambda_k^D + g^D \frac{1}{\pi} [5q_3^2(-2\lambda_k^D, \eta^D) + 4\rho_{gh} \Phi_3^2(0)] \quad (8.77)$$

Remarks:

The beta-functions for the ‘D’ couplings exhibit certain properties that are reminiscent of the single-metric truncation [122, 191]: First, notice that the threshold functions $\Phi_n^D(-2\lambda^D)$ and $\tilde{\Phi}_n^D(-2\lambda^D)$ have singularities when the argument approaches -1 . This leads to a boundary of theory space at $\lambda^D = 1/2$ in both types of truncations. Second, in the single- and the bi-metric truncation we find a further divergence in η^D that can restrict the physically relevant part of theory space even stronger, namely a boundary caused by the singular curve $1 - B_2^{sm} g^{sm} = 0$ or $1 - B_2^D g^D = 0$, respectively. A certain class of RG trajectories, later referred to as type (IIIa)^D trajectories, terminate on these lines. We find that even though quantitatively the singularity properties change when moving from the single- to the bi-metric beta-functions, the qualitative picture remains the same.

8.6.2 The B-sector: beta-functions for g^B, λ^B

In the B-sector, the essential RG running is inherited from g_k^D and λ_k^D . The B-couplings themselves appear on the RHS of their differential equations only in the trivial canonical term since $\text{Tr}[\dots]$ did not depend on them. In particular, Newton’s constant g^B enters its own anomalous dimension η^B only as a global factor. Besides that it depends on the dynamical couplings only:

$$\eta^B(g^D, \lambda^D, g^B) = B_1^B(g^D, \lambda^D) g^B \quad (8.78a)$$

$$B_1^B(g^D, \lambda^D) = \frac{1}{\pi} \left\{ \frac{5}{3} q_1^1(-2\lambda^D, \eta^D) - \left(\frac{5}{3} + 12\rho_p \right) q_2^2(-2\lambda^D, \eta^D) + 24\rho_p q_3^3(-2\lambda^D, \eta^D) \right\} - \frac{4}{\pi} \rho_{gh} \left\{ \frac{1}{3} \Phi_1^1(0) + \frac{1}{3} \Phi_2^2(0) + 2\rho_p \Phi_3^3(0) \right\} \quad (8.78b)$$

Notice that eq. (8.78a) has no denominator analogous to its D counterpart (8.74). Note also that η^B , via the q_n^D -functions (8.19), indirectly depends on $\eta^D = \eta^D(g^D, \lambda^D)$.

The running of the cosmological constants in the B-sector is described by the differential equation

$$\partial_t \lambda_k^B = (\eta^B - 2) \lambda_k^B + A^B(\lambda_k^D, g_k^D) g_k^B \quad (8.79a)$$

Here the RG-effects that are not already covered by η^B are encoded in the function

$$A^B(\lambda_k^D, g_k^D) \equiv \frac{1}{\pi} \left\{ 5(q_2^1(-2\lambda_k^D, \eta^D) - q_3^2(-2\lambda_k^D, \eta^D)) - 4\rho_{gh}(\Phi_2^1(0) + \Phi_3^2(0)) \right\} \quad (8.79b)$$

Remarks:

The beta-functions of the ‘B’ (and also of the level-(0)) couplings are free from any additional singularities that would further reduce the physical part of theory space. In particular, as it is apparent from (8.78), the anomalous dimension η^B is well-defined for any value of g^B on which it depends linearly. Notice also that the threshold functions are evaluated at $-2\lambda_k^D$, and never at $-2\lambda_k^B$, so that there is *no restriction in the λ^B direction of theory space* that would be analogous to the $\lambda^D = 1/2$ boundary. The same holds true for the level-(0) plane.

We thus have all four beta-functions of the bulk sector at hand and can, at least in principle, solve the bulk system of differential equations. This will be carried out in part III of this thesis. Next, let us consider the level-language and the boundary differential equations.

8.6.3 The level-description: beta-functions for $g^{(p)}$, $\lambda^{(p)}$

The $\{B, D\}$ -language is equivalent to the one based upon the level number where the a priori different higher levels $p = 1, 2, \dots$ happens to be identical within the present truncation. The higher level Newton constants $g_k^{(p)}$ have the same running for all $p = 1, 2, 3, \dots$, for instance, and their beta-functions coincide in turn with that of g_k^D .

$$\begin{aligned} g_k^{(0)} &= \frac{g_k^D g_k^B}{g_k^D + g_k^B} & \text{and} & & g_k^{(p)} &= g_k^D & \text{for all } p \geq 1, \\ \lambda_k^{(0)} &= \frac{\lambda_k^D g_k^B + \lambda_k^B g_k^D}{g_k^D + g_k^B} & \text{and} & & \lambda_k^{(p)} &= \lambda_k^D & \text{for all } p \geq 1. \end{aligned} \quad (8.80)$$

Hence, for the anomalous dimensions, we obtain

$$\eta^{(1)} = \eta^{(2)} = \dots = \eta^D \quad (8.81)$$

with η^D given in equation (8.74). For the level-(0) anomalous dimension, i.e. the anomalous dimension corresponding to the running pre-factor of the $\sqrt{g}R$ -term $1/G_k^{(0)} = 1/G_k^B + 1/G_k^D$, we obtain instead:

$$\begin{aligned} \eta^{(0)}(g^D, \lambda^D, g^{(0)}) &= \frac{2}{\pi} \left\{ \frac{5}{6} q_1^1(-2\lambda^D, \eta^D) - 3\rho_P q_2^2(-2\lambda^D, \eta^D) - \rho_{gh} \left(\frac{2}{3} \Phi_1^1(0) \right. \right. \\ &\quad \left. \left. + \rho_P \Phi_2^2(0) \right) \right\} g^{(0)} \end{aligned} \quad (8.82)$$

Note that $\eta^{(0)}/g^{(0)}$ is a function of λ_k^D and g_k^D only.

For the cosmological constant at level-(0) we find

$$\partial_t \lambda_k^{(0)} = (\eta^{(0)} - 2) \lambda_k^{(0)} + g_k^{(0)} \frac{1}{\pi} \left\{ 5 q_2^1(-2\lambda_k^D, \eta^D) - 4\rho_{gh} \Phi_2^1(0) \right\}, \quad (8.83)$$

while for the higher cosmological constants

$$\lambda_k^{(1)} = \lambda_k^{(2)} = \dots = \lambda_k^D. \quad (8.84)$$

At all levels $p = 1, 2, 3, \dots$ their running is locked to that of λ_k^D , which in turn is governed by equation (8.77).

8.6.4 The boundary sector: beta-functions for $g^{\partial(0)}$, $\lambda^{\partial(0)}$

Not surprisingly, the structure of the anomalous scaling of the boundary couplings almost agrees with the one for $\lambda_k^{(0)}$ in eq. (8.83). That is, the anomalous dimension of $g^{\partial(0)}$ assumes the following form in $d = 4$:

$$\eta^{\partial(0)}(g^D, \lambda^D, g^{\partial(0)}) = \frac{1}{3\pi} \left\{ 5 q_1^1(-2\lambda^{(1)}, \eta^D) - 4\rho_{gh} \Phi_1^1(0) \right\} g^{\partial(0)} \quad (8.85)$$

The term in curly brackets is reminiscent of the one in eq. (8.83). Similarly, for the differential equation of the boundary cosmological constant we obtain in $d = 4$:

$$\partial_t \lambda_k^{\partial(0)} = (\eta^{\partial(0)} - 1) \lambda_k^{\partial(0)} - \frac{1}{2\sqrt{\pi}} \left\{ 5 q_{3/2}^1(-2\lambda_k, \eta^D) - 4 \rho_{\text{gh}} \Phi_{3/2}^1(0) \right\} g_k^{\partial(0)} \quad (8.86)$$

The reason for this structural similarity is that all three invariants obtain RG corrections to the canonical scaling from the same traces of eq. (7.69). All are of zeroth order in Ω and C and thus only the first three terms of eq. (7.68) contribute to the beta-functions of the associated couplings.

8.A Expanded beta-functions in $d = 2 + \varepsilon$

Besides the most relevant spacetime dimension $d = 4$, it is instructive to consider the case of $d = 2 + \varepsilon$. Here, we can study the dynamical effects that add to the trivial topological 2-dimensional theory by ‘turning on’ ε . The results we obtain give rise to certain universal terms that can be ascribed to the universal value $\Phi_n^{n+1}(0) = 1/\Gamma(n+1)$ which is obtained for any shape function $R^{(0)}$. In the sequel we display the beta-functions obtained by inserting $d = 2 + \varepsilon$ into the general results and expanding in ε , thereby discarding contributions of order ε and higher. If the limit $\varepsilon \rightarrow 0$ is finite, the result is a single term of order ε^0 , if not, there will be additional pole terms in $1/\varepsilon$.

The anomalous dimension related to g^D :

Instead of giving the full expression of the anomalous dimension η^D , it is more useful for further investigations to present the numerator and denominator functions $B_{1,2}^D$ expanded in ε separately. We obtain for the numerator in eq. (8.33):

$$\begin{aligned} B_1^D(\lambda^D) = & -\frac{8}{\varepsilon} \left\{ 2\Phi_2^3(-2\lambda^D) - 2\rho_P(\Phi_1^2(-2\lambda^D) + \lambda^D(\Phi_0^2(-2\lambda^D) - 2\Phi_1^3(-2\lambda^D))) \right\} \\ & -\frac{4}{3} \left\{ -2\rho_{\text{gh}}(1 + 3\rho_P) - \lambda^D(-2 + \log(64\pi^3))\Phi_0^2(-2\lambda^D) \right. \\ & + 3(-1 + \rho_P(-5 + \log(16\pi^2)))\Phi_1^2(-2\lambda^D) \\ & + 3[4\lambda^D\rho_P(-2 + \log(4\pi))\Phi_1^3(-2\lambda^D) + \lambda^D\Phi_0^{2(0,1,0)}(-2\lambda^D) \\ & - 2\rho_P((-7 + \log(16\pi^2))\Phi_2^3(-2\lambda^D) - \rho_{\text{gh}}\Phi_1^{2(0,1,0)}(0) \\ & + 2\lambda^D\Phi_1^{3(0,1,0)}(-2\lambda^D) + 2\rho_{\text{gh}}\Phi_2^{3(0,1,0)}(0) \\ & \left. + \Phi_1^{2(0,1,0)}(-2\lambda^D) - 2\Phi_2^{3(0,1,0)}(-2\lambda^D)) \right\} + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.87)$$

We see that B_1^D gives rise to a Laurent series with a singular part $\propto 1/\varepsilon$. The function that appears in the denominator of (8.33) assumes the form

$$\begin{aligned} B_2^D(\lambda^D) = & \frac{4}{\varepsilon} \left\{ 2\Phi_2^3(-2\lambda^D) - 2\rho_P(\Phi_1^2(-2\lambda^D) + \lambda^D(\Phi_0^2(-2\lambda^D) - 2\Phi_1^3(-2\lambda^D))) \right\} \\ & + \frac{2}{3} \left\{ -\lambda^D(-2 + \log(64\pi^3))\Phi_0^2(-2\lambda^D) \right. \\ & + 3(-1 + \rho_P(-5 + \log(16\pi^2)))\Phi_1^2(-2\lambda^D) \\ & + 3[4\lambda^D\rho_P(-2 + \log(4\pi))\Phi_1^3(-2\lambda^D) + \lambda^D\Phi_0^{2(0,1,0)}(-2\lambda^D) \\ & - 2\rho_P((-7 + \log(16\pi^2))\Phi_2^3(-2\lambda^D) + 2\lambda^D\Phi_1^{3(0,1,0)}(-2\lambda^D) \\ & \left. + \Phi_1^{2(0,1,0)}(-2\lambda^D) - 2\Phi_2^{3(0,1,0)}(-2\lambda^D)) \right\} + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.88)$$

Obviously B_2^D , too, is again divergent in ε . However, due to the relation $\eta^D = \frac{B_1^D(\lambda^D; 2+\varepsilon)g^D}{1-B_2^D(\lambda^D; 2+\varepsilon)g^D}$, the anomalous dimension is finite in the limit $\varepsilon \rightarrow 0$:

$$\begin{aligned} \eta^D(g^D, \lambda^D) &= \frac{\rho_P(\Phi_1^2(-2\lambda^D) - 2\Phi_2^3(-2\lambda^D) + 2\lambda^D\Phi_1^3(-2\lambda^D)) - 2\lambda^D\Phi_0^2(-2\lambda^D)}{\rho_P(\tilde{\Phi}_1^2(-2\lambda^D) - 2\tilde{\Phi}_2^3(-2\lambda^D) + 2\lambda^D\tilde{\Phi}_1^3(-2\lambda^D)) - 2\lambda^D\tilde{\Phi}_0^2(-2\lambda^D)} + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.89)$$

The beta-function of λ^D :

The running of the D cosmological constant expanded in terms of ε is given by

$$\begin{aligned} \beta_\lambda^D(g^D, \lambda^D) &= (\eta^D - 2)\lambda^D + 8\rho_{\text{gh}}\Phi_2^2(0)g^D \\ &\quad + 3(-4\lambda^D q_1^2(-2\lambda^D, \eta^D) + 4q_2^2(-2\lambda^D, \eta^D))g^D + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.90)$$

Notice that the anomalous dimension in the beta-function has to be understood as the expanded version of eq. (8.89).

The anomalous dimension related to g^B :

$$\begin{aligned} \eta^B(g^D, \lambda^D, g^B) &= \frac{8}{\varepsilon} \left\{ 4q_2^3(-2\lambda^D, \eta^D) - 2\rho_P\Phi_1^2(-2\lambda^D) + \lambda^D(q_0^2(-2\lambda^D, \eta^D) - 4q_1^3(-2\lambda^D, \eta^D)) \right\} g^B \\ &\quad - \frac{1}{3} \left\{ 4\lambda^D(-2 + \log(64\pi^3))q_0^2(-2\lambda^D, \eta^D) - 6q_0^1(-2\lambda^D, \eta^D) \right. \\ &\quad + 12(1 - 2\rho_P(-3 + \log(4\pi)))q_1^2(-2\lambda^D, \eta^D) \\ &\quad - 12 \left[4\lambda^D\rho_P(-2 + \log(4\pi))q_1^3(-2\lambda^D, \eta^D) + \lambda^D\Phi_0^{2(0,1,0)}(-2\lambda^D) \right. \\ &\quad - 2\rho_P((-7 + \log(16\pi^2))q_2^3(-2\lambda^D, \eta^D) + 2\lambda^Dq_1^{3(0,1,0)}(-2\lambda^D, \eta^D) \\ &\quad \left. \left. + q_1^{2(0,1,0)}(-2\lambda^D, \eta^D) - 2q_2^{3(0,1,0)}(-2\lambda^D, \eta^D) \right) \right] \\ &\quad \left. + 8 \left(2 + \rho_P(6 - 3q_1^{2(0,1,0)}(0, \eta^D) + 6q_2^{3(0,1,0)}(0, \eta^D)) \right) \rho_{\text{gh}} \right\} + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.91)$$

This series contains a divergent part $\propto 1/\varepsilon$ for the anomalous dimension η^B itself. It happens to vanish however for $\lambda^D = 0$.

The beta-function of λ^B :

$$\begin{aligned} \beta_\lambda^B(g^D, \lambda^D, g^B, \lambda^B) &= (\eta^B - 2)\lambda^B - 8\rho_{\text{gh}}(\Phi_1^1(0) + \Phi_2^2(0))g^B \\ &\quad + 3(2q_1^1(-2\lambda^D, \eta^D) + 4\lambda^Dq_1^2(-2\lambda^D, \eta^D) - 4q_2^2(-2\lambda^D, \eta^D))g^B + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.92)$$

Notice that while β_λ^B contains no explicit $1/\varepsilon$ -poles the anomalous dimension η^B to be used in (8.92) does have such poles.

The anomalous dimension related to $g^{(0)}$:

$$\begin{aligned} \eta^{(0)}(g^{(p)}, \lambda^{(p)}) &= 2(q_0^1(-2\lambda^D, \eta^D) - 2\rho_Pq_1^2(-2\lambda^D, \eta^D))g^{(0)} - 4\rho_{\text{gh}}\left(\frac{2}{3} + 2\rho_P\right)g^{(0)} + \mathcal{O}(\varepsilon) \end{aligned} \quad (8.93)$$

The beta-function of $\lambda^{(0)}$:

$$\beta_\lambda^{(0)}(g^{(p)}, \lambda^{(p)}) = (\eta^{(0)} - 2)\lambda^{(0)} + 6q_1^1(-2\lambda^D, \eta^D)g^{(0)} - 8\rho_{\text{gh}}\Phi_1^1(0)g^{(0)} + \mathcal{O}(\varepsilon) \quad (8.94)$$

Note that in the level-description neither $\eta^{(0)}$ nor $\beta_\lambda^{(0)}$ has explicit $1/\varepsilon$ -poles.

The anomalous dimension related to $g^{\partial(0)}$:

For the boundary Newton-type coupling the expansion in $d = 2 + \varepsilon$ yields

$$\begin{aligned} \eta^{\partial(0)}(g^D, \lambda^D, g^{\partial(0)}) &= g^{\partial(0)} \left\{ 2q_0^1(-2\lambda^{(1)}, \eta^D) - \frac{8}{3}\rho_{\text{gh}}\Phi_0^1(0) \right\} + \mathcal{O}(\varepsilon) \\ &= g^{\partial(0)} \left\{ \frac{(2 - \eta^D)}{(1 - 2\lambda^{(1)})} - \frac{8}{3}\rho_{\text{gh}} \right\} \end{aligned} \quad (8.95)$$

In the second step we used the property $\Phi_0^p(w) = \tilde{\Phi}_0^p(w) \equiv (1+w)^{-p}$. Since η^D is also free of $1/\varepsilon$ -poles, the anomalous dimension $\eta^{\partial(0)}$ is well defined in the limit $\varepsilon \rightarrow 0^+$.

The beta-function of $\lambda^{\partial(0)}$:

Finally, the beta-function for the cosmological constant $\lambda^{\partial(0)}$ is also found to be free of $1/\varepsilon$ -poles and its $\varepsilon \rightarrow 0^+$ -limit is given by

$$\beta_\lambda^{\partial(0)} = (\eta^{\partial(0)} - 1)\lambda^{\partial(0)} - \sqrt{\pi} \left\{ 3q_{1/2}^1(-2\lambda, \eta^D) - 4\rho_{\text{gh}}\Phi_{1/2}^1(0) \right\} g^{\partial(0)} + \mathcal{O}(\varepsilon) \quad (8.96)$$

8.B Beta-functions in $d = 3$

Next, we apply the results of section 8.4 to the special case of $d = 3$. We proceed in the usual manner, starting with the D couplings, then consider the B-sector, and finally take a look at the boundary couplings employing the language description of the beta-functions.

The anomalous dimension related to g^D :

The structure of the anomalous dimension follows eq. (8.33), with the numerator function

$$\begin{aligned} B_1^D(\lambda^D; 3) &= \frac{3}{2\sqrt{\pi}} \left\{ (2 + 12\rho_p)\Phi_{3/2}^2(-2\lambda^D) + 16\rho_p\lambda^D\Phi_{3/2}^3(-2\lambda^D) \right. \\ &\quad \left. - \frac{8}{3}\lambda^D\Phi_{1/2}^2(-2\lambda^D) - 36\rho_p\Phi_{5/2}^3(-2\lambda^D) \right\} \\ &\quad + \frac{2}{\sqrt{\pi}}\rho_{\text{gh}} \left\{ (1 - 4\rho_p)\Phi_{3/2}^2(0) + 12\rho_p\Phi_{5/2}^3(0) \right\} \end{aligned} \quad (8.97)$$

We see that B_1^D receives contributions of both gravitons and ghosts, of both dia- and para-magnetic nature. The denominator function B_2^D is unaffected by the ghosts:

$$\begin{aligned} B_2^D(\lambda^D; 3) &= \frac{3}{4\sqrt{\pi}} \left\{ (2 + 12\rho_p)\tilde{\Phi}_{3/2}^2(-2\lambda^D) + 16\rho_p\lambda^D\tilde{\Phi}_{3/2}^3(-2\lambda^D) \right. \\ &\quad \left. - \frac{8}{3}\lambda^D\tilde{\Phi}_{1/2}^2(-2\lambda^D) - 36\rho_p\tilde{\Phi}_{5/2}^3(-2\lambda^D) \right\} \end{aligned} \quad (8.98)$$

The beta-function of λ^D :

For the cosmological constant λ^D the beta-function reduces into the following form for $d = 3$:

$$\begin{aligned} \beta_\lambda^D(g^D, \lambda^D; 3) &= (\eta^D - 2)\lambda^D + g^D \frac{6}{\sqrt{\pi}}\rho_{\text{gh}}\Phi_{5/2}^2(0) \\ &\quad + g^D \frac{1}{\sqrt{\pi}} \left\{ -4\lambda^D q_{3/2}^2(-2\lambda^D, \eta^D) + 9q_{5/2}^2(-2\lambda^D, \eta^D) \right\} \end{aligned} \quad (8.99)$$

The anomalous dimension related to g^B :

Next, consider the anomalous dimension for the B-Newton coupling.

$$\begin{aligned} \eta^B(g^D, \lambda^D, g^B; 3) = & \frac{3}{2\sqrt{\pi}} \left\{ \frac{4}{3} q_{1/2}^1(-2\lambda^D, \eta^D) + \frac{8}{3} \lambda^D q_{1/2}^2(-2\lambda^D, \eta^D) - (2 + 16\rho_P) q_{3/2}^2(-2\lambda^D, \eta^D) \right. \\ & \left. - 16\rho_P \lambda^D q_{3/2}^3(-2\lambda^D, \eta^D) + 36\rho_P q_{5/2}^3(-2\lambda^D, \eta^D) \right\} g^B \\ & - \frac{2}{\sqrt{\pi}} \rho_{gh} \left\{ \Phi_{1/2}^1(0) + (1 - 2\rho_P) \Phi_{3/2}^2(0) + 12\rho_P \Phi_{5/2}^3(0) \right\} g^B \end{aligned} \quad (8.100)$$

The beta-function of λ^B :

Finally, we consider the beta-function of λ^B which represents the last ingredient in the hierarchy of the bulk couplings. In $d = 3$ it is given by

$$\begin{aligned} \beta_\lambda^B(g^D, \lambda^D, g^B, \lambda^B; 3) = & (\eta^B - 2)\lambda^B - \frac{6}{\sqrt{\pi}} \rho_{gh} \left\{ \Phi_{3/2}^1(0) + \Phi_{5/2}^2(0) \right\} g^B \\ & + \frac{1}{\sqrt{\pi}} \left\{ 6q_{3/2}^1(-2\lambda^D, \eta^D) + 4\lambda^D q_{3/2}^2(-2\lambda^D, \eta^D) - 9q_{5/2}^2(-2\lambda^D, \eta^D) \right\} g^B \end{aligned} \quad (8.101)$$

The anomalous dimension related to $g^{(0)}$:

Instead of using D/B it is sometimes more convenient to employ the level-language. The anomalous dimension associated to $g^{(0)}$ is given by

$$\begin{aligned} \eta^{(0)}(g^{(p)}, \lambda^{(p)}; 3) = & + \frac{3}{\sqrt{\pi}} \left\{ \frac{2}{3} q_{1/2}^1(-2\lambda^D, \eta^D) - 2\rho_P q_{3/2}^2(-2\lambda^D, \eta^D) \right\} g^{(0)} \\ & - \frac{2}{\sqrt{\pi}} \rho_{gh} \left\{ \Phi_{1/2}^1(0) + 2\rho_P \Phi_{3/2}^2(0) \right\} g^{(0)} \end{aligned} \quad (8.102)$$

The beta-function of $\lambda^{(0)}$:

Combining the D and B results, one can deduce the beta-function of $\lambda^{(0)}$, which reads in $d = 3$:

$$\beta_\lambda^{(0)}(g^{(p)}, \lambda^{(p)}; 3) = (\eta^{(0)} - 2)\lambda^{(0)} + g^{(0)} \frac{6}{\sqrt{\pi}} q_{3/2}^1(-2\lambda^D, \eta^D) - g^{(0)} \frac{6}{\sqrt{\pi}} \rho_{gh} \Phi_{3/2}^1(0) \quad (8.103)$$

In the level-description, the RG equations for the higher couplings ($p \geq 1$) involve $\eta^{(p)} = \eta^D$ and $\beta_\lambda^{(p)} = \beta_\lambda^D$.

The anomalous dimension related to $g^{\partial(0)}$:

The boundary Newton-type coupling has the following anomalous dimension in $d = 3$:

$$\eta^{\partial(0)}(g^D, \lambda^D, g^{\partial(0)}) = \frac{g^{\partial(0)}}{2\sqrt{\pi}} \left\{ 4q_{1/2}^1(-2\lambda^{(1)}, \eta^D) - 4\rho_{gh} \Phi_{1/2}^1(0) \right\} \quad (8.104)$$

The beta-function of $\lambda^{\partial(0)}$:

Finally, in $d = 3$, the beta-function for the cosmological constant describing the RG evolution of the boundary volume element is given by

$$\begin{aligned} \beta_\lambda^{\partial(0)} = & (\eta^{\partial(0)} - 1)\lambda^{\partial(0)} \\ & - \frac{3}{4} \left\{ 4q_1^1(-2\lambda, \eta^D) - 4\rho_{gh} \Phi_1^1(0) \right\} g^{\partial(0)} \end{aligned} \quad (8.105)$$



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One of the central questions we have to answer in this final part of the thesis, is whether or not the employed bi-metric-bulk – pure-background-boundary truncation studied in part II of this thesis supports the Asymptotic Safety conjecture for Quantum Gravity (QG). In the Effective Average Action (EAA) approach this corresponds to the existence of a non-trivial fixed point of the Renormalization Group (RG) flow with a finite dimensional ultraviolet (UV)-critical hypersurface, $\dim \mathcal{S}_{UV} < \infty$, rendering the underlying theory renormalizable in a non-perturbative sense. Today, there is strong evidence that theories of QG are indeed asymptotically safe [99, 100, 121, 123, 126, 130–135, 138, 139, 141, 142, 146–171]. There are also first observations that describe a reduction of infinite dimensional theory spaces to a finite set of relevant basis invariants [129]. While both aspects, a non-trivial fixed point and $\dim \mathcal{S}_{UV} < \infty$, are necessary conditions in order to obtain a fully predictive fundamental theory, most of the employed truncations are for technical reasons finite dimensional and thus extensions including an infinite number of basis invariants is very interesting to confirm the Asymptotic Safety conjecture. Indeed, recent studies of $f(R)$ -theories provide first promising results that find $\dim \mathcal{S}_{UV} < \infty$ even for infinite-dimensional truncations.

Nevertheless, a suitable theory of Quantum Gravity has to fulfill an even more fundamental requirement than renormalizability, i.e. Background Independence (BI). Since the geometry of spacetime is the dynamical object, there is a priori no background over which one can construct a Quantum Field theory (QFT). In the EAA approach one introduces an arbitrary background metric on intermediate steps in order to apply the standard techniques for the path integral while preserving the Ward Identities for split-symmetry (WISS) at least on-shell. The emergence from a suitable non-trivial fixed point in the UV is then just one criterion that a suitable theory of QG described by an RG trajectory of the corresponding Functional Renormalization Group Equation (FRGE) has to satisfy. In addition, in order to make the underlying theory manifestly background independent, one has to ensure that the same RG trajectory enters a regime of intact split-symmetry in the infrared (IR). One of our main objectives in this thesis, is to investigate the status of this global property by searching for RG solutions that are simultaneously compatible with the requirements of Background Independence and Asymptotic Safety. Thus, our truncation ansatz proceeds into a different direction than adding more and more basis invariants to existing calculations. We explore an up to now almost ‘dark’ region of the theory space by throw some light on the bi-metric nature of $\mathcal{T}$. While previous works almost exclusively relied on a severe assumption of fully intact split-symmetry in the gravitational sector along the entire RG evolution by invoking the so-called single-metric approximation, only recently [172] a first fully fledged bi-metric calculation was performed. Extrapolating the split-symmetry requirement from the limit $k \rightarrow 0$ up to the UV at $k \rightarrow \infty$ is a very strong assumption, since the gauge fixing functional and the cutoff action explicitly break split-symmetry. The technical difficulties which arise in studying this part of theory space prevented further progress in this direction and thus most of our today’s understanding is based on the single-metric results only.

In this thesis, we focus on exactly this almost undiscovered region of theory space by employing a new technique that significantly reduces the effort in evaluating bi-metric truncations. Furthermore, we allow spacetime to have a non-vanishing boundary, $\partial M \neq \emptyset$, and thus introduce another very important generalization to previous truncations. These geometries are in particular for cosmological applications very interesting.

This part of the thesis is structured as follows. In chapter 9 we study the UV properties of the RG flow by searching for suitable non-trivial fixed point in order to support the Asymptotic Safety conjecture also in case of $\partial M \neq \emptyset$ and for bi-metric calculations. Then, in chapter 10 we consider the IR regime of RG trajectories in order to understand whether or not Background Independence and Asymptotic Safety can coexist. Furthermore, we compare the results

derived in part II with the bi-metric calculation done in the ref. [172] which is based on a TT-decomposition. Both are used to study the reliability of their single-metric counterpart and define regions in theory space where the latter seems to be a good approximation to the general bi-metric RG flow.

After the UV and IR analysis, we discuss global aspects of the RG evolution, which is thus based on global solutions interconnecting the UV with the IR. In chapter 11 we propose a C -function like feature of exact solutions that reflects the ‘mode-counting’ property of Zamolodchikov’s C -function. We study its monotonicity for the two bi-metric truncations and the single-metric calculation and demonstrate its value in judging the reliability of truncations. Chapter 12 confronts the propagation of gravitons with the global results. Basically we distinguish three branches of the RG evolution, the UV, the semiclassical regime, and the IR. We present a possible interpretation of the observed results in terms of ‘dark matter’.

Except for chapter 9, the entire discussion so far is based on the beta-functions of the bulk-sector only. This is for the reason that the considered truncation has little to say about split-symmetry violation on the boundary. Thus, in chapter 13 we present some results of the boundary sector and apply their RG evolution to study black hole thermodynamics.

We conclude with a short summary of this part and present an outlook on future work along the direction presented in this thesis.



This part of the thesis relies mostly on already published work. Large parts of the following chapters follow closely ref. [166, 173, 192–194]. We reference the corresponding publication at the beginning of each chapter.



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	The classification of trajectories in the linearized region
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9.4	Conclusion
9.A	Critical exponents of the background-sector

9. INVESTIGATING ASYMPTOTIC SAFETY

 This chapter is based on [173]. It includes new results for the boundary couplings not presented in [173].

In this chapter we study the Asymptotic Safety conjecture for Quantum Gravity employing the beta-functions of the bi-metric-bulk – pure-background-boundary truncation derived in part II of this thesis.

The FRGE endows theory space, $\mathcal{T}$, with a vector field whose integral curves describe the RG evolution of theories. Each point in theory space, corresponding to an action functional respecting the symmetry constraints, thus generates a trajectory on $\mathcal{T}$ and thus interconnects a UV theory with its IR limit. From the general perspective all solutions of the FRGE are equally valid, even though they exhibit different properties. The Asymptotic Safety program sets two conditions for an RG trajectory to describe a well-behaved QFT. First, the underlying RG flow should emanate from a fixed point, thus its RG evolution should converge to a certain ‘stable’ theory which is regular in a sense and gives rise to a fundamental ‘bare’ prescription of the generating functional. Second, in order to have a full predictive formalism, the number of free parameters fixed by experiments should be finite dimensional. This translates into a finite dimensional UV-critical hypersurface, $\mathcal{S}_{UV}$, which is spanned by all relevant basis invariants associated to the considered fixed point. The family of RG trajectories which are asymptotically safe are parametrized by $\dim(\mathcal{S}_{UV}) < \infty$ free parameters and can be thought of as emanating from the corresponding fixed point in the UV and then evolve within the hypersurface $\mathcal{S}_{UV}$.

The vicinity of critical points of the flow equation can be described by linearizing $\mathcal{S}_{UV}$. The obtained structure defines the universality class of the underlying RG trajectories and is associated to the conformal (or in general scale-invariant) theory related to the fixed point functional. The characteristics of the universality class are best described by the critical exponents. Thus, it might be that conceptually very different theories reflecting possibly unrelated physical models exhibit the same critical behavior. While their description converge towards the fixed point their global evolution differs in general.

We start this chapter with section 9.1 making some general remarks on properties of the

RG flow for general spacetime dimensions d and for the special case of $d = 4$. In the following we then focus exclusively on the latter case of a four dimensional spacetime and search for fixed points of the RG flow, see section 9.2. The results are then further discussed by disentangling the contributions from the ghost and graviton sector as well as from paramagnetic and diamagnetic terms to understand their respective impact.

In section 9.3 we study the second condition for asymptotically safe theories, namely the UV critical hypersurface of the respective fixed points. With section 9.4 we conclude this chapter.

9.1 Structure

In this section we discuss some very general properties of the beta-functions derived in part II of this thesis and listed in chapter 8. Later on, we will study the RG equations only for the special case of $d = 4$. However, before we will make various remarks concerning their general structure which is valid for any $d > 2$. In particular, we investigate the hierarchy among the six beta-functions and take a first look at the semiclassical regime.

9.1.1 General remarks

Our truncation ansatz comprises a total of six independent couplings, three of Newton-type and three associated to the cosmological constant. Their respective impact on the RG evolution however is quite different. We will consider their general structure and certain topological properties in what follows.

Structure of the beta-functions

One can summarize the structure of the corresponding beta-functions by means of the anomalous dimensions $\eta^I(g_k^I, \lambda^\mathcal{D}, \eta^\mathcal{D})$ and a function $A^I(\lambda^\mathcal{D}, \eta^\mathcal{D})$. For all Newton-type couplings we have the following differential equation:

$$\partial_t g_k^I = (d - 2 + \eta^I(g_k^I, \lambda^\mathcal{D}, \eta^\mathcal{D})) g_k^I \quad \forall \quad I \in \{(p), \partial(p), B, \partial B, D, \partial D\} \quad (9.1)$$

While this structure is the same in the boundary and bulk sector, for the cosmological type couplings there is a difference related to their canonical scaling dimension. Thus, we have the two sets of differential equations:

$$\partial_t \lambda_k^I = (\eta^I(g_k^I, \lambda^\mathcal{D}, \eta^\mathcal{D}) - 2) \lambda_k^I + A^I(\lambda^\mathcal{D}, \eta^\mathcal{D}) g_k^I \quad \forall \quad I \in \{(p), B, D\} \quad (9.2a)$$

$$\partial_t \lambda_k^{\partial(0)} = (\eta^{\partial(0)}(g_k^{\partial(0)}, \lambda^\mathcal{D}, \eta^\mathcal{D}) - 1) \lambda_k^{\partial(0)} + A^{\partial(0)}(\lambda^\mathcal{D}, \eta^\mathcal{D}) g_k^{\partial(0)} \quad (9.2b)$$

On closer examination, it appears that the dynamical (or level- (p) for $p \geq 1$) couplings $\lambda^\mathcal{D}$ and $\eta^\mathcal{D}$ play a central role in defining the right-hand-side (RHS) of the differential equations. Except for this additional dependence on $\lambda^\mathcal{D}$ and $g^\mathcal{D}$, the pair (λ^I, g^I) decouples from the remaining equations for each I . This gives rise to a hierarchy among the couplings, which we discuss in section 9.1.2. The reason for this ‘imbalance’ between the dynamical sector and the remaining couplings is due to the structure of the FRGE,

$$\partial_t \Gamma_k[\varphi; \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\widehat{\text{Hess}}_\varphi[\Gamma_k[\varphi; \bar{\Phi}]] + \mathfrak{K}_k[\bar{\Phi}] \right) \partial_t \mathfrak{K}_k[\bar{\Phi}] \right]$$

Formally expanding the EAA in powers of the fluctuation fields φ , i.e. the level expansion $\Gamma_k[\varphi; \bar{\Phi}] = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma^{(n)}[\bar{\Phi}] \varphi^n$ whereby the couplings of level- (p) multiply terms of order $\mathcal{O}(\varphi^p)$, the flow equation reads

$$\sum_{n=0}^{\infty} \frac{1}{n!} \partial_t \Gamma^{(n)}[\bar{\Phi}] \varphi^n = \frac{1}{2} \text{STr} \left[\left(\sum_{m=2}^{\infty} \frac{1}{(m-2)!} \Gamma^{(m)}[\bar{\Phi}] + \mathfrak{K}_k[\bar{\Phi}] \right) \partial_t \mathfrak{K}_k[\bar{\Phi}] \right]$$

Since EAA only appears in its Hessian form on the RHS of the FRGE, all invariants of level-(0) and level-(1), along with their coefficients, disappear under the trace. Thus, for the truncation ansatz studied in this thesis, the level-(2) coefficients are trigger the non-trivial flow of the RG evolution; since here we identified $g_k^D \equiv g_k^{(1)} = g_k^{(2)} = \dots$ we can also substitute the level-(1) on the RHS of the flow equation. In fact, for a truncation ansatz that distinguishes between the various levels, the beta-functions of level-(p) can only dependent on other couplings of level-($p+2$) and higher.

For the beta-functions of chapter 8 this considerations imply that neither the bulk level-(0) nor the boundary level- $\partial(0)$ couplings affect the RHS of the flow equation. Their appearance in the beta-functions of the dimensionless couplings is purely due to the non-linear conversion rules relating them to the dimensionful coefficients. If we consider the (non-trivial) dependence of the beta-functions on λ^D and η^D for general spacetime dimensions d , as given in chapter 8, one observes that, quite remarkably, *all terms proportional to λ^D are also proportional to $(d-4)$* . Thus, for the special case of $d=4$ these contributions drop out, and the remaining λ^D -dependence is via the threshold functions, $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$, only, see section 8.1 for their definitions and properties. The threshold functions on the other hand only appear in two combinations. Either they are evaluated for zero argument, i.e. $\Phi_n^p(0)$, rendering the corresponding terms independent on λ^D and η^D , or they combine to expressions of type $q_n^p(-2\lambda^D, \eta^D)$ which are defined as

$$q_n^p(w, \eta^D) \equiv \Phi_n^p(w) - \frac{1}{2} \eta^D \tilde{\Phi}_n^p(w) \quad (9.3)$$

These latter contributions are associated to the graviton traces, while the former can be attributed to the ghost sector. In fact, keeping track of the bookkeeping parameter ρ_{gh} reveals this feature.

Note that it is always the anomalous dimension η^D that enters $q_n^p(-2\lambda^D, \eta^D)$ and there is no extra dependence of g^D on the RHS of the flow equation.

Separating para- and diamagnetic contributions

Similar to the pre-factor introduced for the traces of the ghost sector, ρ_{gh} , all formulae contain also a parameter $\rho_p \in \{0, 1\}$ which we also implemented as a book keeping device. Contributions proportional (not proportional) to ρ_p originate from the *paramagnetic* (*diamagnetic*) interaction terms¹ [165]. We refer to all terms that are *not* proportional to ρ_{gh} as ‘graviton’ contributions. In the nomenclature of ref. [165] those are either of ‘paramagnetic’ or ‘diamagnetic’ origin and correspondingly carry a factor of ρ_p or are missing a factor of ρ_p , respectively. In view of the physical picture of Asymptotic Safety developed in ref. [165] it will be instructive to keep the two types of contributions separately. Later on, when we are not interested in this distinction we ‘turn on’ all terms, putting $\rho_p = \rho_{\text{gh}} = 1$, unless stated otherwise.

Topological properties of the truncated RG flow

The general structure of the beta-functions for the six invariants, given in eq. (9.1) and (9.1), already reveals some topological features of the associated RG evolution. These are described by critical regions in the truncated theory space, where the beta-functions vanish. Values of the couplings where the entire set of equations becomes trivial define a critical point or in other terms a fixed point of the RG flow.

Focusing our attention on the Newton-type couplings their beta-functions vanish if the RHS of the differential equation (9.1) turns zero, i.e.

$$(d-2 + \eta^I(g^I, \lambda^D, \eta^D)) g^I \stackrel{!}{=} 0 \quad (9.4)$$

¹By definition [165, 195], and in accordance with the identical nomenclature in Yang-Mills theory, derivative (non-derivative) interaction terms coupling metric fluctuations $h_{\mu\nu}$ to the background $\bar{g}_{\mu\nu}$ are referred to as diamagnetic (paramagnetic), typical structures being $h\bar{D}^2 h$ ($h\bar{R}h$).

Assuming that η^I is regular at $g^I = 0$, there is a trivial and possibly a set of non-trivial solutions for this equation, namely

$$g_*^I = 0 \quad \text{and} \quad \eta^I(g_*^I, \lambda^D, \eta^D) = (2 - d) \quad (9.5)$$

Notice that the first, trivial solution is independent on any value of λ^D or η^D and thus the ‘line’ ($g^I = 0, \lambda^I, \lambda^D, g^D$) divides theory space into two portions, $g^I < 0$ and $g^I > 0$, for any I . No RG trajectory can cross this line and thus an initial positive (negative) Newton coupling will remain positive (negative) during for the entire RG flow.

R In fact, the anomalous dimension related to g^D is plagued by a singularity at $B_2^D(\lambda^D; d) g^D = 1$, in case of Einstein-Hilbert truncations. This affects the only type IIIa trajectories in the limit $k \rightarrow 0$.

On the other hand, solutions of $\eta^I(g_*^I, \lambda^D, \eta^D) = (2 - d)$ will depend on the explicit form of η^I and the point (λ^D, g^D) and thus defines running critical points of g^I . They give rise to a non-Gaussian fixed point (NGFP) and will be derived numerically in section 9.2 and further discusses in chapter 10.

The k -dependence of the cosmological-type couplings follows equation (9.2) and reads

$$\partial_t \lambda_k^I = \left(\eta^I(g_k^I, \lambda^D, \eta^D) - c_{M/\partial M}^I \right) \lambda_k^I + A^I(\lambda^D, \eta^D) g_k^I \quad (9.6)$$

Here, for simplicity we introduced $c_{M/\partial M}^I \equiv 1$ for $I = \partial(p)$ and $c_{M/\partial M}^I = 2$ otherwise. Together with its the dynamical sector and its associated Newton-type coupling it forms a complete differential system. The critical regions in the λ^I direction is less general, however in combination with eq. (9.5) we can spot the joined critical values (λ_*^I, g_*^I) .

First, we insert the trivial solution $g_*^I = 0$ which yields the critical values $(g_*^I = 0, \lambda_*^I)$ with λ_*^I solving

$$\left(\eta^I(0, \lambda^D, \eta^D) - c_{M/\partial M}^I \right) \lambda_*^I \stackrel{!}{=} 0 \quad (9.7)$$

Hence, $(g_*^I = 0, \lambda_*^I = 0)$ is a valid solution for any I independent on the values of λ^D and g^D . If all three pairs $I = (0)$, $I = (1)$, and $I = \partial(0)$ assume this trivial solution, the trajectory has entered the Gaussian fixed point (GFP). The second option of $\eta^I(0, \lambda^D, \eta^D) \equiv c_{M/\partial M}^I$ does not constraint the λ_*^I in general, but can be only satisfied for particular values of λ^D and g^D . This set of solutions, is only interesting for $I \equiv D$.

Second, the non-trivial solutions g_*^I satisfying $\eta^I(g_*^I, \lambda^D, \eta^D) = (2 - d)$ lead to an implicit equation for λ_*^I and g_*^I which also depends on the value of the D-couplings. Their properties can only be discussed making use of the explicit forms of the beta-functions.

In figure 9.1 we depict the phase diagram of the (λ^D, g^D) -plane for the bi-metric-bulk – pure-background-boundary truncation. In this plot the separation of RG trajectories at $g^D = 0$ becomes apparent. In the upper half plane there seems to be a fixed point with imaginary critical exponents, which can be inferred by the spirals in the vicinity of $\arctan(g^D) \approx 0.5$ and $\arctan(\lambda^D) \approx 0.2$.

9.1.2 The hierarchical structure of the truncated RG equations

As we have mentioned above the size of all physical RG-effects is controlled by the D-couplings only, in fact by couplings of at least level-(2) which, in the present case, are equal to G_k^D or $\bar{\lambda}^D$, respectively. The level-(0) and level-(1) couplings, and correspondingly the B-couplings, as

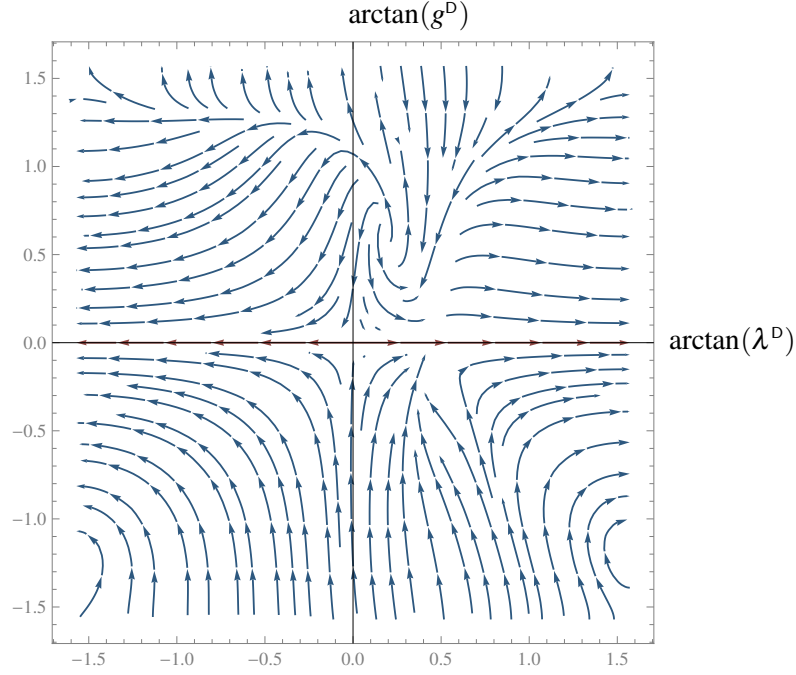


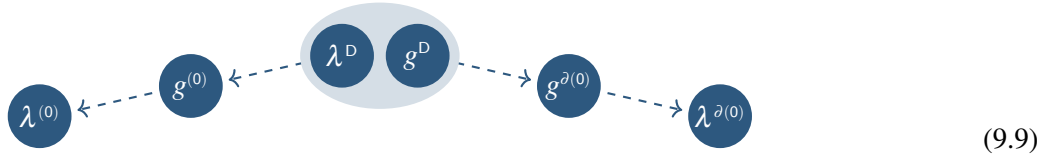
Figure 9.1: This plot shows the phase diagram in the (λ^D, g^D) -plane for the beta-functions of part II, using $\arctan(\cdots)$ to cover the full range $g^D, \lambda^D \in \{-\infty, \infty\}$. Apparently, the solution $g^D = 0$ separates the RG flow of negative and positive Newton coupling. Notice the signal of at least two fixed points, one situated at the origin, the other at positive g^D and λ^D .

well as their boundary analogs, cannot enter the RHS of the FRGE, for they are at most linear in the dynamical fields and hence drop out in the calculation of the Hessian operator.

No matter whether we employ the $\{B, D, \partial(0)\}$ or the $\{\partial(0), (0), (1), (2), \dots\}$ language, the truncated FRGE always comprises a coupled system of ordinary differential equation in six variables, $\{g_k^D, g_k^B, g_k^{\partial(0)}, \lambda_k^D, \lambda_k^B, \lambda_k^{\partial(0)}\}$ or $\{g_k^{\partial(0)}, g_k^{(0)}, g_k^{(p)} \equiv g_k^D, \lambda_k^{\partial(0)}, \lambda_k^{(0)}, \lambda_k^{(p)} \equiv \lambda_k^D, p \geq 1\}$, respectively. In either case *the dynamical couplings g_k^D and λ_k^D influence the beta-functions of the level-(0)- (or B-) and the boundary-couplings, but there is no back-reaction of $\{g_k^B, \lambda_k^B, g_k^{\partial(0)}, \lambda_k^{\partial(0)}\}$ or $\{g_k^{(0)}, \lambda_k^{(0)}, g_k^{\partial(0)}, \lambda_k^{\partial(0)}\}$ on the ‘D’ variables; furthermore $\lambda_k^{B/(0)/\partial(0)}$ does not back-react on $g_k^{B/(0)/\partial(0)}$* . The system of equations decomposes in the following hierarchical way:

$$\{g_k^D, \lambda_k^D\} \longrightarrow g_k^{B/(0)/\partial(0)} \longrightarrow \lambda_k^{B/(0)/\partial(0)}. \quad (9.8)$$

Notice that there is no mutual influence of $(0)/B$ and the boundary counterparts $\partial(0)$. An illustration of the hierarchy, which is formally written in terms of (9.8), is given as follows



Hereby, the level-(0) and $\partial(0)$ -couplings are seen to be mutually independent and the dependence propagates from top to bottom.

This observation fixes the strategy for the (mostly numerical) solution of the RG equations on which we embark in the following sections: First we solve the non-trivially coupled g_k^D - λ_k^D system, then we insert its solution into the single differential equation of the level-(0) (or alternatively, level- $\partial(0)$) Newton coupling, solve for their k -dependence, and finally use all

three solution functions obtained already to get the running of the corresponding cosmological constants of level-(0) (or its counterpart on the boundary level- $\partial(0)$).

The hierarchy (9.8) raises the following question: Is it really necessary for the couplings in the background sector (a) to assume fixed point values for $k \rightarrow \infty$, and (b) to restore split-symmetry at $k = 0$? After all, the B-couplings could even diverge in the UV without causing any problems for the D-couplings and the $h_{\mu\nu}$ -correlation functions given by $(\delta/\delta h)^n \Gamma_k[h; \bar{g}]|_{h=0}$. Or equivalently, should we insist that the level-(0) part of Γ_k , that is, $\Gamma_k[h; \bar{g}]|_{h=0}$, is well defined, too?

The answer to these questions is ‘yes’. In fact, the EAA at level zero is related to the partition function $Z_k[\bar{g}]$ whose $\bar{g}$ -dependence is indeed of physical interest, for instance, when it is used as a tool for ‘counting’ the field modes which are already integrated out at a given scale [166, 173]. This quantity is similar to a state sum and makes its appearance when one studies quantum gravity effects in black hole thermodynamics, for instance. (See ref. [166] or chapter 13 for further details.) Therefore we want also the level-(0) part of the EAA to be asymptotically safe, and to be linked to the higher levels in a split-symmetric way.

Moreover, this third part of the thesis is supposed to pave the way for future work on more complicated truncations which also allow lifting the degeneracy among all higher levels $p = 1, 2, \dots$ which is still assumed here. In this sense our analysis of the level-(0) /level-(1) interplay is supposed to have the character of a model for the general case.

9.1.3 No anti-screening in the semiclassical regime

In the next chapter we shall present a comprehensive analysis of the RG equations using the strategy of the previous subsection. Here, as a warm up, we present a simple analytic solution to these equation which is valid in the semiclassical regime, i.e. at scales in, and slightly above those of the classical regime where all dimensionful Newton and cosmological constants are strictly k -independent.

The technical details related to this approximation, over and above the approximation due to the truncation can be found in section 8.5. There we find that if $g_k^I \ll 1$ and $\lambda_k^D \ll 1$ all of the dimensionful quantities G_k^I and $\bar{\lambda}_k^I$, $I \in \{D, B, (p), \partial(p)\}$, behave as

$$G_k^I \approx G_0^I [1 - \omega_d^I G_0^I k^{d-2}] \quad (9.10a)$$

$$\bar{\lambda}_k^I \approx \bar{\lambda}_0^I + \nu_d^I G_0^I k^d \quad \text{or} \quad \bar{\lambda}_k^{\partial(0)} \approx \bar{\lambda}_0^{\partial(0)} + \nu_d^{\partial(0)} G_0^{\partial(0)} k^{d-1} \quad (9.10b)$$

The dimension-dependent constants ω_d^I and ν_d^I are tabulated in section 8.5. The general structure of the solution (9.10) is quite familiar; it also obtains in the single-metric (‘sm’) Einstein-Hilbert truncation. There it was found that the crucial ω -coefficient which governs the running of $G_k \equiv G_k^{\text{sm}}$ is *positive* in the most interesting case $d = 4$. With $\omega_4^{\text{sm}} > 0$ Newton’s constant decreases for increasing k , and this was interpreted as a kind of gravitational anti-screening [122].

Within the approximation (9.10), the anomalous dimension is given by

$$\eta^I = -(d-2)\omega_d^I g^I \quad (9.11)$$

so that $\omega_d^I > 0$, i.e. anti-screening, corresponds to a negative anomalous dimension $\eta^I < 0$. Therefore, the positive (negative) sign of ω_d^{sm} (η^{sm}) was highly welcome from the Asymptotic Safety point of view since at a non-trivial fixed point of the equation $\partial_t g_k^I = (d-2+\eta_k^I) g_k^I$ the anomalous dimension is negative, too:² $\eta_*^I = -(d-2) < 0$ whenever $g^I > 0$.

It is therefore somewhat disconcerting to discover that in the semi-classical regime the anti-screening sign is *not* re-obtained within the bi-metric truncation. In fact, in the bi-metric setting

²Here and in the following we always assume $d > 2$.

it is the dynamical Newton constant G_k^D that should be compared to G_k^{sm} . Therefore the hallmark of anti-screening is now $\eta^D < 0$, that is, $\omega_d^D > 0$ in the semiclassical approximation. However, using the explicit equations of section 8.5 for $d = 4$, we find that, with any cutoff, $\omega_4^D < 0$. Hence *the dynamical Newton constant shows a screening rather than anti-screening behavior in the semiclassical regime*: $\eta^D|_{\text{semiclass}} > 0$. Instead, the background Newton constant G_k^B runs in the opposite direction, $\eta^B|_{\text{semiclass}} < 0$, but this has no direct physical meaning.

Fortunately later on we shall also discover that in other parts of theory space η^D does become negative actually, and that even non-trivial fixed points form. Nevertheless, this simple example is a severe warning showing the limitations of the single-metric truncations and the range of validity of the semiclassical approximation.

For the anomalous dimension related to $g^{\partial(0)}$ the semiclassical approximation yields a negative $\omega_4^{\partial(0)} < 0$ for any cutoff shape function. In contrast to the dynamical bulk coupling, there is however no problem in having a negative $\omega_4^{\partial(p)}$. Therefore, notice that a negative $\omega_4^I < 0$ that remains negative beyond the semiclassical regime for $k \rightarrow \infty$, is still compatible with the existence of a non-trivial fixed point, defined by $\eta_*^I = -(d-2) < 0$. However, in this case the fixed point value of the associated Newton coupling g^I has to be negative, too. While this is perfectly fine for the boundary couplings, there is experimental evidence that the pre-factor of the Einstein-Hilbert action should be positive in the classical regime, when the theory resembles General Relativity. Since under the RG flow the sign of g_k^I is preserved, this in turn restricts $g_k^D > 0$ for all k , in order to recover the classical limit. On the other hand, except for case of enforcing the classical matching condition for the coefficients of Einstein-Hilbert- and Gibbons-Hawking-York-functional even on the quantum level, there is no preferred choice for the sign of $g^{\partial(p)}$.

9.2 Fixed points

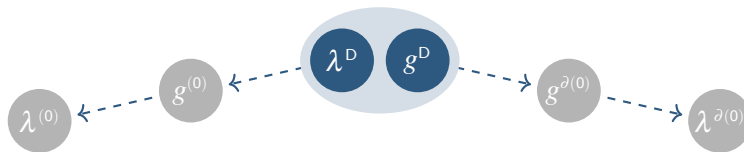
Based on the strategy outlined in subsection 9.1.2 we are going to solve the truncated flow equations in a stepwise manner, following the hierarchical order of dependencies by starting with the D-couplings and then proceed to the level-(0) counterparts. In this section, we are not relying on any approximation beyond the employed truncation techniques of part II and study the existence of special topological points, fixed points, of the RG flow.

Whenever numerical calculations are involved we employ the ‘optimized’ shape function [188] $R^{(0)}(z) = (1-z)\Theta(1-z)$ for which the threshold functions $\Phi_n^p(w)$ and $\tilde{\Phi}_n^p(w)$ [122] are given by a simple closed form, see eq. (8.43),

$$\Phi_n^p(w) = \frac{1}{\Gamma(n+1)}(1+w)^{-p}, \quad \text{and} \quad \tilde{\Phi}_n^p(w) = \frac{1}{\Gamma(n+2)}(1+w)^{-p}$$

However, while the explicit numerical values may differ for the various cutoff choices, the qualitative statements about the fixed points and their properties remain valid.

9.2.1 The dynamical sector



According to the hierarchy, we start with the dynamical couplings g^D, λ^D and try to find common zeros of their beta-functions: $\beta_g^D = 0 = \beta_\lambda^D$. In the following analysis we set $\rho_p = 1, \rho_{\text{gh}} = 1$, i.e.

we include all (dia- and paramagnetic, as well as ghost) contributions. Using the explicit form of the beta-functions given in chapter 8 we find three fixed points (g_*^D, λ_*^D) :

- ▶ A Gaußian fixed point at $\lambda_*^D = g_*^D = 0$, henceforth denoted G^D -FP
- ▶ Two non-Gaußian fixed point at which both g^D and λ^D are positive and negative, respectively; they will be denoted NG_+^D -FP and NG_-^D -FP in the following

For the optimized shape function the fixed point coordinates are given by

u_*	NG_+^D -FP	NG_-^D -FP	G^D -FP
$g_*^D =$	0.703	-3.54	0
$\lambda_*^D =$	0.207	-0.302	0

(9.12)

We recall that for the same shape function the well-known fixed point of the single-metric truncation [135, 191] is located at $(g_*^{\text{sm}} = 0.707, \lambda_*^{\text{sm}} = 0.193)$. These coordinates are remarkably close to those of the NG_+^D -FP.

An RG trajectory that initially starts at $g^D = 0$ will remain so for all $k \in [0, \infty)$. These class of trajectories are subject to the simple D-equations:

$$g_k^D = 0, \quad \lambda_k^D = \lambda_{k_0}^D (k_0/k)^2 \quad (0 \leq k < \infty) \quad (9.13)$$

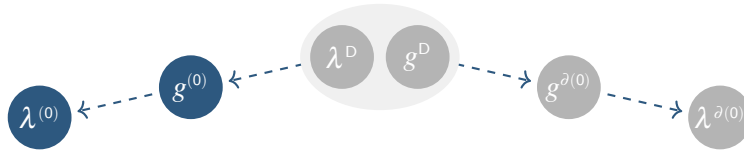
Here $\lambda_{k_0}^D$ denotes the value of the cosmological constant at $k = k_0$. This trajectory separates the positive $g^D > 0$ from the negative $g^D < 0$ regime and thus there is no RG trajectory which ever crosses the $g^D = 0$ plane.

As mentioned already, this is a very general feature of any Newton-type coupling. Thus, for any I there exist trajectories that separate the positive from the negative $g^{B/(0)}/\partial(0)$ regime:

$$g_k^{B/(0)/\partial(0)} = 0, \quad \lambda_k^{B/(0)} = \lambda_{k_0}^{B/(0)} (k_0/k)^2 \quad \text{or} \quad \lambda_k^{\partial(0)} = \lambda_{k_0}^{\partial(0)} (k_0/k) \quad (9.14)$$

The uniqueness of the integral curves ensures that there is no trajectory ever passes through $g^B = 0$, $g^{(0)} = 0$, or $g^{\partial(0)} = 0$. This, in particular implies that *there exists no crossover trajectory connecting NG_+^D -FP to NG_-^D -FP*. More generally, there is no trajectory along which any of the various Newton constants g^I would cross zero. Since, in the classical regime we observe a positive dynamical Newton coupling, we expect NG_+^D -FP to be more appropriate to describe real physics.

9.2.2 The level-(0) or B-sector



So far, we have seen that there are two non-Gaußian and one Gaußian fixed point solutions in the *dynamical* sector. Next, we insert their coordinates into the beta-functions of the *background* or similarly of the level-(0) quantities (g^B, λ^B) and $(g^{(0)}, \lambda^{(0)})$, respectively. We then search for zeros which would generalize the ‘D’ fixed points to the full four dimensional bulk part of theory space.

R Notice that the boundary couplings are unaffected by the level-(0) sector of the bulk and vice versa. Thus the fixed points can be computed independently.

The bulk fixed points in the D/B language

It turns out that for each of the fixed points in the D-sector there exists precisely one non-trivial solution of (g_*^B, λ_*^B) , i.e. which satisfies $(g_*^B, \lambda_*^B) \neq (0, 0)$. Those related to the three non-Gaussian D-FPs are located at

u_*	$\text{NG}_+^{\text{D-FP}}$	$\text{NG}_-^{\text{D-FP}}$	$\text{G}^{\text{D-FP}}$
$g_*^B =$	8.18	1.531	1.396
$\lambda_*^B =$	-0.008	-0.12	-0.111

(9.15)

Furthermore, for $g^B = 0 = \lambda^B$ the beta-functions for the B-sector vanish as well, for any value of the D-couplings. Thus, also a Gaussian fixed point $(g_*^B, \lambda_*^B) = (0, 0)$ exists in the B-sector, and it can be combined with each one of the D fixed point in (9.12).

In total we have six fixed points therefore: one which is purely Gaussian, having both $(g_*^D, \lambda_*^D) = 0$ and $(g_*^B, \lambda_*^B) = (0, 0)$, three mixed ones, and two are purely non-Gaussian ones. The table (9.16) gives a summary of all six combined fixed points and introduces the notation we shall use for them.

d=4	$\text{NG}_+^{\text{D-FP}}$	$\text{NG}_-^{\text{D-FP}}$	$\text{G}^{\text{D-FP}}$
	$(g_*^D=0.7, \lambda_*^D=0.2)$	$(g_*^D=-3.5, \lambda_*^D=-0.3)$	$(g_*^D=0, \lambda_*^D=0)$
$\text{NG}_+^{\text{B-FP}}$	$\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D-FP}}$	$\text{NG}_+^{\text{B}} \oplus \text{NG}_-^{\text{D-FP}}$	$\text{NG}_+^{\text{B}} \oplus \text{G}^{\text{D-FP}}$
	$(g_*^B=8.2, \lambda_*^B=-0.01)$	$(g_*^B=1.5, \lambda_*^B=-0.1)$	$(g_*^B=1.4, \lambda_*^B=-0.1)$
$\text{G}^{\text{B-FP}}$	$\text{G}^{\text{B}} \oplus \text{NG}_+^{\text{D-FP}}$	$\text{G}^{\text{B}} \oplus \text{NG}_-^{\text{D-FP}}$	$\text{G}^{\text{B}} \oplus \text{G}^{\text{D-FP}}$
	$(g_*^B=0, \lambda_*^B=0)$	$(g_*^B=0, \lambda_*^B=0)$	$(g_*^B=0, \lambda_*^B=0)$

(9.16)

An analogous table in the level language will be given below.

The bulk fixed points in the level-description

In the level-language the dynamical couplings, i.e. those at the higher levels $p = 1, 2, 3, \dots$, influence the beta-functions at level-(0), but not vice versa. The fixed points found in eq. (9.12) for the ‘D’ couplings entail $(g_*^{(p)}, \lambda_*^{(p)}) = (g_*^D, \lambda_*^D)$ for the levels $p \geq 1$.

Regarding $p = 0$, the fixed points listed in (9.16) give rise to three non-trivial ones for the level-(0) couplings. Two of them, namely $\text{NG}_+^{(0)} \oplus \text{NG}_+^{\text{D-FP}}$ and $\text{NG}_+^{(0)} \oplus \text{NG}_-^{\text{D-FP}}$ are easily computed using the conversion rules for the dimensionless couplings, i.e.

$$\begin{aligned}
 g_k^{(0)} &= \frac{g_k^D g_k^B}{g_k^D + g_k^B} & \text{and} & & g_k^{(p)} &= g_k^D & \text{for all } p \geq 1, \\
 \lambda_k^{(0)} &= \frac{\lambda_k^D g_k^B + \lambda_k^B g_k^D}{g_k^D + g_k^B} & \text{and} & & \lambda_k^{(p)} &= \lambda_k^D & \text{for all } p \geq 1.
 \end{aligned}
 \tag{9.17}$$

To obtain the third one, $\text{NG}_+^{(0)} \oplus \text{G}^{\text{D-FP}}$, related to the Gaussian fixed points of the ‘D’ sector, one must be careful: When directly using relation (9.17) any of the remaining 4 fixed points would seem to be located at $g_*^{(0)} = 0 = \lambda_*^{(0)}$. Because of a division by zero this is incorrect, however. From the coupled system (8.82)-(8.84) derived in part II, it turns out that the correct coordinates of $\text{NG}_+^{(0)} \oplus \text{G}^{\text{D-FP}}$ are actually non-zero:

$$\text{NG}_+^{(0)} \oplus \text{G}^{\text{D-FP}} : \quad g_*^{(0)} = 1.713, \quad \lambda_*^{(0)} = 0.068
 \tag{9.18}$$

They are obtained if we directly solve for zeros in the full $\{g_k^{(0)}, g_k^D, \lambda_k^{(0)}, \lambda_k^D\}$ system.

In total we thus have found the following three non-trivial fixed point values for the level-(0) couplings:

u_*	NG $_+^D$ -FP	NG $_-^D$ -FP	G ^{D} -FP
$g_*^{(0)} =$	0.647	2.697	1.713
$\lambda_*^{(0)} =$	0.190	0.0168	0.068

(9.19)

As a result, the list of all six combined fixed points looks as follows:

d=4	NG $_+^D$ -FP	NG $_-^D$ -FP	G ^{D} -FP
	$(g_*^D=0.7, \lambda_*^D=0.2)$	$(g_*^D=-3.5, \lambda_*^D=-0.3)$	$(g_*^D=0, \lambda_*^D=0)$
NG $_+^{(0)}$ -FP	NG $_+^{(0)} \oplus$ NG $_+^D$ -FP $(g_*^{(0)}=0.65, \lambda_*^{(0)}=0.2)$	NG $_+^{(0)} \oplus$ NG $_-^D$ -FP $(g_*^{(0)}=2.7, \lambda_*^{(0)}=0.02)$	NG $_+^{(0)} \oplus$ G ^{D} -FP $(g_*^{(0)}=1.7, \lambda_*^{(0)}=0.1)$
G $^{(0)}$ -FP	G $^{(0)} \oplus$ NG $_+^D$ -FP $(g_*^{(0)}=0, \lambda_*^{(0)}=0)$	G $^{(0)} \oplus$ NG $_-^D$ -FP $(g_*^{(0)}=0, \lambda_*^{(0)}=0)$	G $^{(0)} \oplus$ G ^{D} -FP $(g_*^{(0)}=0, \lambda_*^{(0)}=0)$

(9.20)

Summary

Each fixed point in the level description has an analog in the B - D system. Regardless of whether we consider the $\{B, D\}$ or the $\{(0), (1), (2), \dots\}$ language, we find a total of six fixed points on the bulk sector. Two of them lie in the negative g^D half-plane and will be not relevant for our further investigation. The remaining four are a doubly non-Gaussian fixed point NG $_+^B \oplus$ NG $_+^D$ -FP, a purely Gaussian one, and two mixed fixed points at which either the dynamical or the background couplings vanish. Their potential relevance to the Asymptotic Safety construction will be explored in the following chapters.

9.2.3 The boundary level- $\partial(0)$ -sector

Before we study to which extend the dia- and paramagnetic, as well as ghost and graviton contribution affect the presented results, let us consider the topological structure of the boundary sector, first.



This relates to the other branch of the hierarchy (9.9) and is completely independent of the fixed point structure of the level-(0) bulk couplings.

Different to the previous fixed point values of the level-bulk couplings, the qualitative picture of the semiclassical regime remains true for the non-Gaussian fixed points, leading to a negative Newton coupling as pre-factor of the Gibbons-Hawking-York term. Explicitly, we obtain again precisely one non-trivial solution $(g_*^{\partial(0)}, \lambda_*^{\partial(0)})$ for each NGFP of the D-sector, similar to the case of their bulk counterparts. For the optimized shape function, we find

u_*	NG $_+^D$ -FP	NG $_-^D$ -FP	G ^{D} -FP
$g_*^{\partial(0)} =$	-2.14	-27.9	-18.8
$\lambda_*^{\partial(0)} =$	1.204	0.72	1.33

(9.21)

If this picture of $g_*^{\partial(p)} < 0$ prevails for higher levels, the matching of the pre-factors of the Einstein-Hilbert and the Gibbons-Hawking-York functional would be inconsistent with the Asymptotic Safety requirement. Future work that disentangles the various level-contributions will be able to answer this question.

Apart from the non-trivial solutions for $(g_*^{\partial(0)}, \lambda_*^{\partial(0)})$, there exists a Gaussian type fixed point, independent on the values of the D-sector. Thus, we have again a total of six combined fixed points for the dynamical bulk level-(0) boundary sector. The complete list along with the naming conventions reads as follows:

d=4	NG ₊ ^D -FP	NG ₋ ^D -FP	G ^D -FP
	$(g_*^D=0.7, \lambda_*^D=0.2)$	$(g_*^D=-3.5, \lambda_*^D=-0.3)$	$(g_*^D=0, \lambda_*^D=0)$
NG ₋ ^{∂(0)} -FP	NG ₋ ^{∂(0)} ⊕ NG ₊ ^D -FP $(g_*^{\partial(0)}=-2.14, \lambda_*^{\partial(0)}=1.2)$	NG ₋ ^{∂(0)} ⊕ NG ₋ ^D -FP $(g_*^{\partial(0)}=-27.9, \lambda_*^{\partial(0)}=0.72)$	NG ₋ ^{∂(0)} ⊕ G ^D -FP $(g_*^{\partial(0)}=-18.8, \lambda_*^{\partial(0)}=1.3)$
G ^{∂(0)} -FP	G ^{∂(0)} ⊕ NG ₊ ^D -FP $(g_*^{\partial(0)}=0, \lambda_*^{\partial(0)}=0)$	G ^{∂(0)} ⊕ NG ₋ ^D -FP $(g_*^{\partial(0)}=0, \lambda_*^{\partial(0)}=0)$	G ^{∂(0)} ⊕ G ^D -FP $(g_*^{\partial(0)}=0, \lambda_*^{\partial(0)}=0)$

(9.22)

Summary

In the second branch in the hierarchy structure of the beta-functions, the boundary couplings appear. Similar to the previous results on the bulk sector we find a total of six fixed points for the D- $\partial(0)$ combinations. The important difference resides in the sign of the fixed point values of the boundary Newton-type coupling $g^{\partial(0)}$ which is negative for all non-trivial solutions. However, there is no physical argument why this coupling should non-negative and thus it is perfectly compatible with the Asymptotic Safety conjecture and experimental constraints.

9.2.4 Recovering the mechanism of paramagnetic dominance

In the single-metric analysis of [165] it was found that in $d = 4$ the NGFP owes its existence entirely to the paramagnetic interactions of the $h_{\mu\nu}$ fluctuations with the background metric; the diamagnetic interactions disfavor the formation of a fixed point instead. (Below $d = 3$ the situation changes, showing that the respective physical mechanism behind the NGFPs in $d = 2 + \varepsilon$ and $d = 4$ are quite different [165].) In this subsection we are study this picture in the bi-metric case making use of the parameter ρ_p to disentangle dia- and paramagnetic contributions.

In the g^D - λ^D -regime of interest, all qualitative properties of η^D in eq. (8.74) are well described by its linear approximation $\eta^D \approx B_1^D(\lambda^D)g^D$. The contribution of the denominator in (8.74), involving $B_2^D(\lambda^D)$, influences the resulting anomalous dimension only weakly. So let us focus on $B_1^D(\lambda^D)$. Assuming, as always, a positive dynamical Newton constant, the negative anomalous dimension which is indicative of anti-screening and is necessary for a NGFP, requires a negative B_1^D .

Now, what we find is that the paramagnetic interactions indeed drive B_1^D negative, but the diamagnetic ones have an antagonistic effect trying to make B_1^D and η^D positive. In fact, switching off the paramagnetic contributions yields, in the bi-metric setting,

$$B_1^D(\lambda^D)|_{\rho_p=0} > 0 \quad \text{for all values of } \lambda^D. \quad (9.23)$$

Instead including the paramagnetic parts we find that there is a crossover from negative to

positive values of the anomalous dimension at a certain critical value $\lambda_{\text{crit}}^{\text{D}}$, that is

$$B_1^{\text{D}}(\lambda^{\text{D}})|_{\rho_{\text{P}}=1} \begin{cases} > 0 & \forall \lambda^{\text{D}} < \lambda_{\text{crit}}^{\text{D}} \\ < 0 & \forall \lambda^{\text{D}} > \lambda_{\text{crit}}^{\text{D}} \end{cases} \quad (9.24)$$

The precise value of the critical cosmological constant, $\lambda_{\text{crit}}^{\text{D}}$, is cutoff scheme dependent, but in any scheme we find $0 < \lambda_{\text{crit}}^{\text{D}} < \lambda_*^{\text{D}}$.

The behavior (9.24) makes it explicit that, first, the absence of anti-screening in the semi-classical regime ($B_1^{\text{D}}(0) > 0$) can be reconciled with a non-trivial fixed point existing simultaneously ($B_1^{\text{D}}(\lambda_*^{\text{D}}) < 0$) and, second, *the NGFP can form only because in some part of theory space the paramagnetic interactions are stronger than the diamagnetic ones.*

Thus we essentially recover the single-metric picture according to which ‘paramagnetic dominance’ is the physical mechanism responsible for Asymptotic Safety. However, the bi-metric analysis suggests that this mechanism can occur only for a non-zero, positive cosmological constant.³ The implications of this result will be further discussed in chapter 12.

A similar effect of the diamagnetic terms rendering $B_1^{\text{I}} > 0$ can be observed for the anomalous dimension of the boundary Newton coupling $g^{\partial(0)}$. The associated beta-functions only contain diamagnetic contributions, since the corresponding basis invariant $\int_{\partial\text{M}} \sqrt{\bar{H}} \bar{K}$ does not feel the additional $\bar{R}$ -factors in the heat kernel expansion. In this case, the diamagnetic contributions are not compensated by counteracting paramagnetic terms and $B_1^{\partial(0)} > 0$ stays always positive requiring a negative fixed point value for $g^{\partial(0)}$.

9.2.5 Impact of the ghost contributions

It is instructive to look at the dependence of the fixed point data on the ghost contributions to the beta-functions. Since we multiplied each trace associated to the ghost functional with a factor of $\rho_{\text{gh}} = 1$, it is easy to assess the importance of ghosts relative to the gravitons by varying this parameter away from $\rho_{\text{gh}} = 1$, the value used in the previous subsections. This will give us an idea of the numerical accuracy one may expect within the present approximation which neglects the running of the ghosts’ wave function renormalization.

The fixed point coordinates in the dynamical sector depend on the ρ_{gh} non-trivially as shown in Fig. 9.2. While the trivial fixed point G^{D} -FP is independent of ρ_{gh} , the non-Gaussian ones

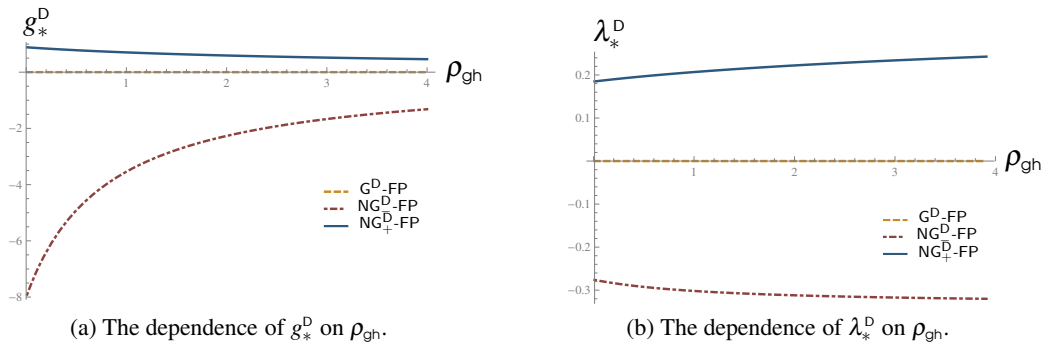


Figure 9.2: The dependence of g_*^{D} and λ_*^{D} on ρ_{gh} for the three types of fixed points, G^{D} -FP, NG_-^{D} -FP, and NG_+^{D} -FP. While the Gaussian one is insensitive to independent on ρ_{gh} the λ_*^{D} values of the others only slightly change when increasing ρ_{gh} . As for the Newton constant g_*^{D} , the fixed point in the lower half-plane ($g^{\text{D}} < 0$) is by far more sensitive to ρ_{gh} than the one in the upper, NG_+^{D} -FP.

³The same conclusion can be drawn from the beta-functions found in [172] with a bi-metric truncation too.

$\text{NG}_+^{\text{D}}\text{-FP}$ and $\text{NG}_-^{\text{D}}\text{-FP}$ are not. However, it turns out that g_*^{D} of $\text{NG}_-^{\text{D}}\text{-FP}$ is very sensitive to a change in ρ_{gh} , especially going from $\rho_{\text{gh}} = 0$ to $\rho_{\text{gh}} = 1$. This suggests that $\text{NG}_-^{\text{D}}\text{-FP}$ is likely to be a truncation artifact. On the other hand, the fixed point values of $\text{NG}_+^{\text{D}}\text{-FP}$ are quite stable when changing the strength of the ghost-contributions.

From the point of view of the Asymptotic Safety scenario both of these facts are encouraging: First, the fixed point at positive dynamical Newton constant, $\text{NG}_+^{\text{D}}\text{-FP}$, seems well described within the present approximation and hardly could be a truncation artifact. If we define a $k \rightarrow \infty$ limit there, since the running g_k^{D} never changes its sign, the resulting theory has a positive G_0^{D} , as it should be. Second, the fixed point at negative g^{D} , $\text{NG}_-^{\text{D}}\text{-FP}$, is much less stable and more likely to be a truncation artifact. If so, this would actually be welcome since then no UV limit could be taken there, and no asymptotically safe theory with $G_0^{\text{D}} < 0$ could be constructed, and the result of positive G_0^{D} implied by $\text{NG}_+^{\text{D}}\text{-FP}$ is a true prediction.

For the background couplings the picture is as follows. The Gaussian solution $g_*^{\text{B}} = 0 = \lambda_*^{\text{B}}$, leading to the three fixed points $\text{G}^{\text{B}} \oplus \text{G}^{\text{D}}\text{-FP}$, $\text{G}^{\text{B}} \oplus \text{NG}_-^{\text{D}}\text{-FP}$, and $\text{G}^{\text{B}} \oplus \text{NG}_+^{\text{D}}\text{-FP}$, is independent of ρ_{gh} , and so the same is true for their $(g^{\text{B}}, \lambda^{\text{B}})$ -coordinates. The fixed point that is situated

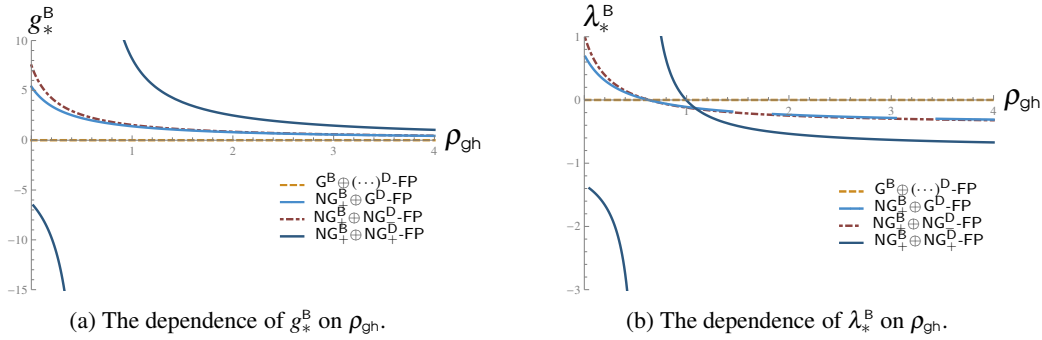


Figure 9.3: The dependence of λ_*^{B} and g_*^{B} on ρ_{gh} for the six fixed points. There are three Gaussian and three non-Gaussian fixed points in the B-sector, one for each D-fixed point. The Gaussian ones are independent on ρ_{gh} . In contrast to the D-part of these fixed points, the B fixed point values vary most strongly for $\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}\text{-FP}$. Especially for small ρ_{gh} it starts out negative, diverges and then turns positive for $\rho_{\text{gh}} < 1$.

in the negative g^{D} half-plane, $\text{NG}_+^{\text{B}} \oplus \text{NG}_-^{\text{D}}\text{-FP}$, is again very sensitive to the influence of the ghost sector. The background fixed point values exhibit a strong dependence on ρ_{gh} even for $\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}\text{-FP}$ which has the positive g_*^{D} . The B-fixed point values actually change the sign and turn negative below $\rho_{\text{gh}} < 0.6$, see Fig. 9.3.

We do not think that this apparent instability of $(g_*^{\text{B}}, \lambda_*^{\text{B}})$ sheds any doubt on the viability of the Asymptotic Safety construction at the doubly non-Gaussian $\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}\text{-FP}$. In fact, the B-couplings are *entirely unphysical* and owe their existence only to the extra $\bar{g}$ -dependence of Γ_k , hence to the split-symmetry violation in unobservable quantities. In fact, considering the level-(0) couplings in fig. 9.4 the instability of the B-sector is not visible for the ‘true’ coefficients multiplying the level-(0) invariants. The picture we obtain in this description is thus in full agreement with the D-results.

Finally, let us consider the ρ_{gh} -dependence of the boundary couplings, depicted in fig. 9.5. From this perspective it seems that the very large fixed point values associated to $\text{NG}_-^{\partial(0)} \oplus \text{NG}_-^{\text{D}}\text{-FP}$ and $\text{NG}_-^{\partial(0)} \oplus \text{G}^{\text{D}}\text{-FP}$ are caused by a ρ_{gh} -pole in the vicinity of the true value $\rho_{\text{gh}} = 1$. This further strengthens the assumption that those fixed points are artifacts of the truncation rather than stable solutions of the full RG flow. On the other hand, the D-fixed point of most interest, $\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^{\text{D}}\text{-FP}$, turns out to be almost insensitive on the ρ_{gh} parameter, especially in the

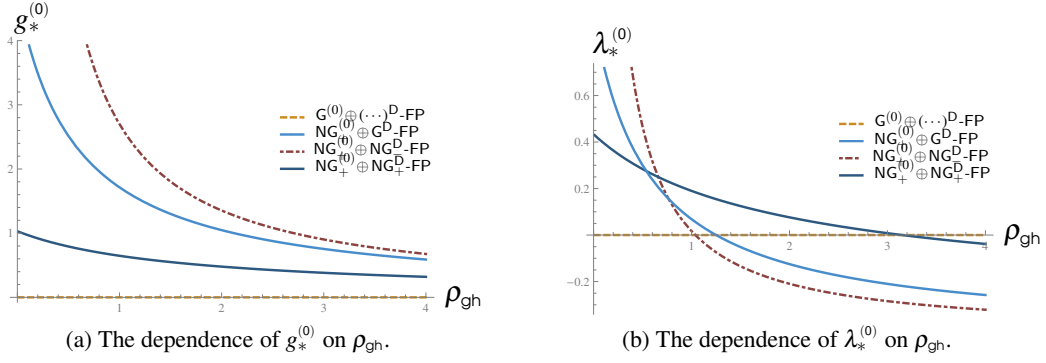


Figure 9.4: The dependence of $\lambda_*^{(0)}$ and $g_*^{(0)}$ on ρ_{gh} for the six fixed points. The general structure is quite similar to the D-sector, with a ρ_{gh} -insensitive Gaussian fixed point and the non-Gaussian one of the upper half-plane $g^{(0)} > 0$ being more stable than its negative counterpart. The divergence of the B-couplings does not affect the level-(0) couplings.

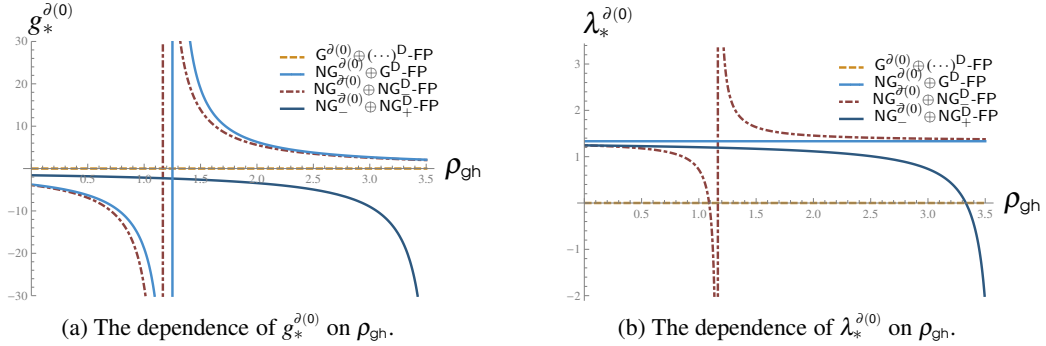


Figure 9.5: The dependence of $\lambda_*^{\partial(0)}$ and $g_*^{\partial(0)}$ on ρ_{gh} for the six fixed points. From very general arguments it follows that the Gaussian fixed point is independent on ρ_{gh} . In contrast, especially the non-Gaussian fixed point $\text{NG}_-^{\partial(0)} \oplus \text{NG}_-^{\text{D-FP}}$ is very sensitive to a change in ρ_{gh} . Indeed, in the vicinity of $\rho_{\text{gh}} = 1$ this fixed point value approaches a singularity and changes sign. On the other hand $\text{NG}_-^{\partial(0)} \oplus \text{G}^{\text{D-FP}}$ is almost constant in the ρ_{gh} -range of interest.

vicinity of $\rho_{\text{gh}} = 1$ which corresponds to including all ghost contributions in the way they appear on RHS of the flow equation.

9.2.6 Summary

In this section, we investigated the fixed point structure of the beta-functions in the bi-metric-bulk – pure-background-boundary truncation. Therefore, we made use on a hierarchical order in the differential system, by first solving the fixed point condition for the D-couplings, yielding one Gaussian and two non-Gaussian fixed points. The latter two are separated by the trajectories satisfying $g_k^{\text{D}} = 0$ and lay in the upper (lower) half-plane of Newton's coupling g^{D} and are denoted $\text{NG}_+^{\text{D-FP}}$ and $\text{NG}_-^{\text{D-FP}}$, respectively. For asymptotically safe RG trajectories describing a QFT that enters the familiar classical regime of General Relativity there is the additional constraint $g_*^{\text{D}} > 0$. Thus $\text{NG}_+^{\text{D-FP}}$ seems to be favored by experimental observations. In fact, it turns out that $\text{NG}_-^{\text{D-FP}}$ is very sensitive to changes of the ghost sector indicating that this non-trivial solution might be only an artifact of the truncation, while the dependence of $\text{NG}_+^{\text{D-FP}}$ on ρ_{gh} is almost negligible. If this picture confirms for extended truncations, then the requirement of Asymptotic Safety yields a unique fundamental theory in the D-sector. Inserting the solution separately into both branches of the hierarchy model, we found for each of the three fixed

points (λ_*^D, g_*^D) a trivial and a non-trivial solution for the zeros of the bulk and boundary level-(0) beta-functions, respectively. This gives rise to a total of $3_D \times 2_{(0)} + 3_D \times 2_{\partial(0)} = 12$ fixed points in the six dimensional truncated theory space of the bi-metric-bulk – pure-background-boundary truncation. From the physical and – as the stabilization checks indicate – also from the mathematical point of view the most interesting fixed points are combinations as $(\dots) \oplus \text{NG}_+^D$ -FP, in particular the ‘double’-NGFPs $\text{NG}_+^{(0)} \oplus \text{NG}_+^D$ -FP and $\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^D$ -FP. Notice that bulk and boundary level-(0) couplings are independent of each other and thus the complete form of, say $\text{NG}_+^{(0)} \oplus \text{NG}_+^D$ -FP, is given by $G^{\partial(0)} \oplus \text{NG}_+^{(0)} \oplus \text{NG}_+^D$ -FP or $\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^{(0)} \oplus \text{NG}_+^D$ -FP. In the next section, we study the UV critical hypersurfaces associated to the twelve fixed points.

9.3 Predictivity

In investigating AS conjecture for QG in the bi-metric-bulk – pure-background-boundary truncation, we identify the topological landscape of theory space by searching for trivial points of the RG flow. The obtained structure characterizes the underlying differential system and is of uttermost importance to define non-perturbatively renormalizable theories in the AS sense. In section 9.2, we have found a total of twelve fixed points within the six dimensional truncated theory space. In the sequel we use their linearized critical hypersurfaces $\mathcal{S}_{UV}$ to get a first understanding of their properties and the associated scale invariant theory. As mentioned above, the corresponding critical exponents classify the universality class of the fixed point theory and can in fact be the same for fundamentally very different physical models. Thus, while the global properties is hidden in the non-linear parts of $\mathcal{S}_{UV}$, its UV structure is fully described by its universality class. For the AS conjecture the most important information is the dimensionality of $\mathcal{S}_{UV}$, which in order to describe fully predictive theories has to be finite dimensional. However, due to the complexity of the FRGE on usually projects the RG flow of the full infinite dimensional theory space $\mathcal{T}$ to a subspace by means of truncations. For finite dimensional truncations the UV-critical hypersurface is always finite and thus there is little to say about this aspect of Asymptotic Safety. Nevertheless, by increasing the number of basis invariants one observes a promising trend, suggesting that in the UV asymptotically safe trajectory can be fixed by only a finite set of experiments, which is also confirmed by first infinite dimensional truncations of $f(R)$ -type, where the UV-critical hypersurface satisfies $\dim(\mathcal{S}_{UV}) < \infty$ [129].

We start this section by recapitulating the classification of RG trajectories in the vicinity of critical points and introduce the notion of stability matrix, relevant and irrelevant directions, and critical exponents. Afterwards, we follow the lines of the previous section and study the two orthogonal subspaces $(g^D, \lambda^D, g^{(0)}, \lambda^{(0)})$ and $(g^D, \lambda^D, g^{\partial(0)}, \lambda^{\partial(0)})$, separately, which is justified by the hierarchical structure of the differential system (9.9).

9.3.1 The classification of trajectories in the linearized region

In this subsection, we go back to the general formulation, denoting $u^{(n)}(k)$ to be a generic dimensionful coupling and $\beta^{(n)}(\{u^{(m)}(k)\})$ its associated beta-function. Furthermore, let N_u be the number of couplings of the possibly truncated theory space. If the beta-functions are sufficiently smooth, there is a neighborhood of every point in theory space $u_0 \equiv (u_0^{(m)})$ which is well described by the linearized RG flow, obtained by expanding the beta-functions up to linear order in the couplings, i.e.

$$\beta^{(n)}(\{u^{(m)}(k)\}) = \beta^{(n)}(\{u_0^{(m)}\}) + \sum_r \frac{\partial \beta^{(n)}(\{u^{(m)}\})}{\partial u^{(r)}} \Big|_{u^{(m)} \equiv u_0^{(m)}} \left(u^{(r)}(k) - u_0^{(r)} \right) + \dots$$

The coefficient of the linear term defines the stability matrix $\mathcal{B}$ at u_0 and contains all essential information of the RG flow in the vicinity of u_0 . Its components are thus given by

$$\mathcal{B}_{nr}(u_0) = \left. \frac{\partial \beta^{(n)}(\{u^{(m)}\})}{\partial u^{(r)}} \right|_{u^{(m)} \equiv u_0^{(m)}} \quad (9.25)$$

We thus arrive at a linear differential system of N_u variables, that can be solved independently if the stability matrix $\mathcal{B}$ is diagonalizable and thus the system can be decoupled. In the following we assume that there is a basis of (right) eigenvectors of the stability matrix, satisfying $\sum_n^{N_u} \mathcal{B}_{rn}(u_0) v^{(n)} = \lambda_r v^{(r)}$, with⁴

$$v^{(r)}(k) \equiv \sum_m^{N_u} S_{rm}^{-1} u^{(m)}(k), \quad u^{(m)}(k) = \sum_r^{N_u} S_{mr} v^{(r)}(k) \quad (9.26)$$

Here S denotes an invertible linear map such that $S^{-1} \mathcal{B} S$ is diagonal. Notice that the eigenvalues λ_r are in general complex for $\mathcal{B}$ usually fails to be symmetric.

If we substitute (9.26) into the linearized differential equations, the system indeed decouples, yielding N_u isolated ordinary differential equations of the following form

$$k \partial_k v^{(r)}(k) = \lambda_r \cdot v^{(r)}(k) + \sum_n^{N_u} S_{rn}^{-1} \left[\beta^{(n)}(u^0) - \sum_m^{N_u} \mathcal{B}_{nm}(u_0) u_0^{(m)} \right] \quad (9.27)$$

The second term is a constant, independent of k and $v^{(r)}$, which we abbreviate in the sequel by $b^{(r)}(u_0) \equiv \sum_n^{N_u} S_{rn}^{-1} [\beta^{(n)}(u^0) - \sum_m^{N_u} \mathcal{B}_{nm}(u_0) u_0^{(m)}]$. In case of $\lambda_r = 0$, the RHS of eq. (9.27) is constant and the k -dependence of the associated eigenvectors becomes logarithmic:

$$v^{(r)}(k) = b^{(r)}(u_0) \cdot \ln \left(\frac{k}{k_0} \right) + v^{(r)}(k_0) \quad (9.28)$$

For the more interesting case of non-vanishing eigenvalues, $\lambda_r \neq 0$, eq. (9.27) is linear in $t = \ln(k/k_0)$. Using an exponential ansatz, the eigenvectors of the stability matrix associated to $\lambda_r \neq 0$ shows the following k -dependence

$$v^{(r)}(k) = \left(v^{(r)}(k_0) - \frac{1}{\theta_r} b^{(r)}(u_0) \right) \cdot \left(\frac{k}{k_0} \right)^{-\theta_r} + \frac{b^{(r)}(u_0)}{\theta_r} \quad \text{for } \lambda_r \neq 0 \quad (9.29)$$

In the last two equations we have introduced $v^{(r)}(k_0)$ as the value of the eigenvector at k_0 and $\theta_r := -\lambda_r$, the negative of the associated eigenvalue, which is referred to as critical exponent if $u_0 \equiv u_*$ describes a fixed point. The sign of its real part, i.e. of $\Re(\theta_r)$, indicates if a direction that is generated by a local eigenvector diverges from the chosen coordinate u_0 or converges when lowering the scale k . We thus classify all eigenvectors in the linear regime of an arbitrary point u_0 in theory space as relevant, marginal or irrelevant direction depending on the real part of the negative eigenvalue θ_r that might be either positive, zero or negative, respectively, see table 9.1. In the marginal case, in which the eigenvalue vanishes, the linear approximation is insufficient to deduce a converging or diverging behavior of the solutions along this direction. Therefore higher order contributions have to be considered.

Now, we apply this machinery to very special points in theory space, fixed points. The previous discussion simplifies for critical points u_* , since the beta-function vanishes there and thus the zeroth order in the Taylor expansion of $\beta^{(n)}$ drops out, implying in particular

$$b^{(r)}(u_*) \equiv - \sum_{n,m}^{N_u} S_{rn}^{-1} \mathcal{B}_{nm}^* u_*^{(m)} \equiv \theta_r \sum_m^{N_u} S_{rm}^{-1} u_*^{(m)} \equiv \theta_r v_*^{(r)} \quad (9.30)$$

⁴ In this context, the eigenvalues of $\mathcal{B}$ denoted λ_r should not be confused with the cosmological constant.

Classification of trajectories		
Critical exponent ($= -\lambda_r$)	Behavior	Classification
$\Re(\theta_r) > 0$	$k^{- \theta_r }$	relevant
$\Re(\theta_r) = 0$	not considered	marginal
$\Re(\theta_r) < 0$	$k^{+ \theta_r }$	irrelevant

Table 9.1: Classification of trajectories in the linearized region of a point u_0

Here we wrote $\mathcal{B}_{nm}^* \equiv \mathcal{B}_{nm}(u_*)$ and noticed that $\mathcal{B}_{nm}^*$ diagonalizes w.r.t. S . We then used eq. (9.26) to identify the remaining sum as the fixed point coordinate in the basis of the eigenvectors $v^{(m)}$, i.e. $v_*^{(r)} \equiv \sum_m S_{rm}^{-1} u_*^{(m)}$. The resulting expression for the k -dependence of the (right) eigenvectors (9.29) assumes the following very illuminating form:

$$v^{(r)}(k) = \left(v^{(r)}(k_0) - v_*^{(r)} \right) \cdot \left(\frac{k}{k_0} \right)^{-\theta_r} + v_*^{(r)} \quad \text{for } \theta_r \neq 0 \quad (9.31)$$

Discount marginal direction for a moment, let us consider the RG evolution in the vicinity of v_* for a generic trajectory $k \mapsto v(k)$ with initial condition $v(k_0) \equiv v_0 \neq v_*$. The first term in eq. (9.31) describes the deviation from the fixed point coordinate. To be explicit, we focus on the UV flow of the trajectory, hence assuming $k > k_0$ and thus $k/k_0 > 1$. For positive (negative) real part of θ_r the initial separation of $v_0^{(r)}$ from the fixed point coordinate $v_*^{(r)}$ shrinks (blows up) while increasing k . This has the consequence that if the RG trajectory is asymptotically safe, i.e. it converges to v_* for $k \rightarrow \infty$, the coordinate in all ‘irrelevant’ directions, $\Re(\theta_r) < 0$, has to coincide with the fixed point value, i.e.

$$k \mapsto v(k) \quad \text{asymptotically safe} \quad \implies \quad v_0^{(r)} \equiv v_*^{(r)} \quad \forall r \in N_u \text{ with } \Re(\theta_r) < 0$$

Otherwise, any initial discrepancy potentiates pulling the trajectory away from the fixed point. On the other hand, if $\Re(\theta_r) > 0$ the trajectory is pulled into the fixed point for $k \rightarrow \infty$.

Thus, the critical exponent indeed characterize the asymptotic behavior of the corresponding couplings. Depending on their sign small disturbances from the fixed point value in the respective direction of theory space either leads to an attractive or repulsive evolution. In com-

Classification of fixed point trajectories			
Critical exponent ($= -\lambda_r$)	Classification	Attractive	Repulsive
$\Re(\theta_r) < 0$	Attractive	IR ($k \rightarrow 0$)	UV ($k \rightarrow \infty$)
$\Re(\theta_r) > 0$	Repulsive	UV ($k \rightarrow \infty$)	IR ($k \rightarrow 0$)

Table 9.2: Classification of trajectories in the linearized region of a fixed point u^* .

bination with the AS requirement, all trajectories which are attracted towards the fixed point for increasing k , are suitable candidates for a renormalizable quantum theory of gravity. On the other hand, if we are in search for a safe IR theory the arguments have to be reverted. Hence, any repulsive direction for increasing k , say, describes an attractive direction for decreasing k . In the Wilsonian renormalization group approach, the flow evolves from the UV towards the IR and therefore we may classify the trajectories that enter into an IR fixed point as attractive and

those which depart as repulsive. Table 9.2 presents an incomplete overview of the classification scheme, where we omitted the marginal directions since the vanishing of their eigenvalues require a separate analysis.

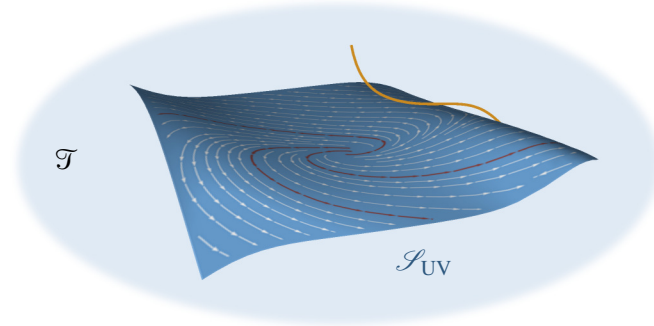


Figure 9.6: This plot illustrates the vicinity of a non-Gaussian fixed point embedded in theory space $\mathcal{T}$. Its UV-critical hypersurface, $\mathcal{S}_{UV}$, is highlighted in blue along with a set of asymptotically safe RG trajectories (marked in white and red). Here, as usually, arrows point from $k \rightarrow \infty$ to $k = 0$. The case of RG solutions which do not respect the AS condition is presented by the (orange) line that separates from $\mathcal{S}_{UV}$ under the inverse flow. Corresponding theories are plagued by UV divergences and are therefore not renormalizable.

In the sequel, we focus on the UV interpretation of the fixed point regimes, since we are in search for a suitable UV completion of Quantum Gravity. In this case, the relevant eigenvectors span the subspace of UV attractive trajectories which defines the linear part of the UV-critical hypersurface $\mathcal{S}_{UV}$. While each irrelevant direction, $\Re(\theta_r) < 0$, is constraint by the Asymptotic Safety program, the relevant counterparts are unspecified. Hence, the dimensionality of $\mathcal{S}_{UV}$, which equals the number of relevant directions, is related to the predictivity of the system. Any trajectory within $\mathcal{S}_{UV}$ gives rise to the same fundamental UV theory, for it converges to the fixed point for $k \rightarrow \infty$, see fig. 9.6. Thus, we need $s_{UV} := \dim(\mathcal{S}_{UV})$ experiments to fix the remaining freedom of the differential system in order to obtain a unique renormalizable theory of Quantum Gravity. From this perspective it is very natural to impose the condition of $\dim(\mathcal{S}_{UV}) < \infty$ being finite to define a generalized criterion of renormalizability [168].

R Remember that any initial parameter of a theory has to be fixed by some additional constraint, either experimentally or by theoretical consistency. Asymptotic Safety is a very strong constraint of the latter case, where usually infinitely many free parameters are allocated values specified by the mathematical formalism.

9.3.2 Critical exponents and scaling fields

Following the analysis of the previous subsection, we are now going to study the critical behavior of the twelve fixed points for the bi-metric-bulk – pure-background-boundary truncation. This discussion presents the level-language, first for the bulk and then independently for the boundary couplings. The corresponding results for the B-sector are found in appendix 9.A of this chapter.

The bulk-sector

In the previous section we found a total of twelve fixed points that represents trivial, though very interesting parts of the RG flow. We now extend the list of properties, which so far consists of their coordinates, by describing their associated universality classes characterized by the critical exponents θ_r .

Making again use of the hierarchy among the beta-functions, we start with the bulk sector and consider all bulk projections of fixed points associated to the dynamical (or from level-(1) upwards) Gaußian fixed point for which $(g_*^D, \lambda_*^D) \equiv (0, 0)$ holds true. Besides a full Gaußian one, there is mixed combination with non-canonical scaling properties of the level-(0) couplings, i.e.

$G^{(0)} \oplus G^D\text{-FP } (g_*^{(0)}, \lambda_*^{(0)}) = (0, 0)$		$NG_+^{(0)} \oplus G^D\text{-FP } (g_*^{(0)}, \lambda_*^{(0)}) = (1.7, 0.07)$	
θ_r	eigenvectors $v^{(r)}$	θ_r	eigenvectors $v^{(r)}$
−2	$\frac{8\pi}{3}\hat{e}_g^D + \hat{e}_\lambda^D$	−2	$0.97\hat{e}_g^D + 0.12\hat{e}_\lambda^D - 0.23\hat{e}_g^{(0)} + 0.03\hat{e}_\lambda^{(0)}$
2	$+\hat{e}_\lambda^D$	2	$5 \times 10^{-17}\hat{e}_\lambda^D + 1.00\hat{e}_g^{(0)} + 0.38\hat{e}_\lambda^{(0)}$
−2	$8\pi\hat{e}_g^{(0)} + \hat{e}_\lambda^{(0)}$	2	$1.00\hat{e}_g^{(0)} + 0.04\hat{e}_\lambda^{(0)}$
2	$\hat{e}_\lambda^{(0)}$	4	$\hat{e}_\lambda^{(0)}$

(9.32)

In both tables, we list the critical exponents along with the ‘scaling fields’, the eigenvectors $v^{(r)}$ of the associated stability matrix. Here the $\hat{e}$ ’s are Cartesian unit vectors; $\hat{e}_g^D \equiv \hat{e}_g^{(1)} = \hat{e}_g^{(2)} = \dots$ points in the direction of the dynamical (or level-(p), $p \geq 1$) Newton constant, etc. For the directions emanating from the Gaußian part of the fixed points we observe the canonical scaling behavior with critical exponent -2 and 2 for Newton’s and cosmological coupling, respectively. Notice that the Gaußian fixed point is situated on the separating line $g^I = 0$ that prevents zero-crossing of RG trajectories. It is UV repulsive (attractive) in the direction of the cosmological (Newton) coupling and thus there is a unique RG solution for $g^D > 0$ entering the point $(0, 0)$. This is also reflected in the dimensionality of its UV critical manifold, which is $s_{UV} := \dim(\mathcal{S}_{UV}) = 2$, as given by the canonical scaling behavior of the couplings. For the non-trivial fixed point combination of D- and level-(0) sector, $NG_+^{(0)} \oplus G^D\text{-FP}$, there appears an additional relevant direction along $\lambda^{(0)}$ yielding a three dimensional subspace of asymptotically safe trajectories, i.e. $s_{UV} = 3$.

Next, we consider the fixed points associated to $NG_-^D\text{-FP}$. Its sensitivity on the ghost sector contributions has raised the question, if its existence is rather an artifact of the truncation than a true property of the exact flow. Nevertheless, it is instructive to discuss its properties listed in the following table

$G^{(0)} \oplus NG_-^D\text{-FP } (g_*^{(0)}, \lambda_*^{(0)}) = (0, 0)$		$NG_+^{(0)} \oplus NG_-^D\text{-FP } (g_*^{(0)}, \lambda_*^{(0)}) = (2.7, 0.02)$	
θ_r	eigenvectors $v^{(r)}$	θ_r	eigenvectors $v^{(r)}$
2.21	$1.0\hat{e}_g^D + 0.1\hat{e}_\lambda^D$	2.21	$-0.50\hat{e}_g^D - 0.03\hat{e}_\lambda^D - 0.88\hat{e}_g^{(0)} - 0.01\hat{e}_\lambda^{(0)}$
5.12	$-0.6\hat{e}_g^D + 0.8\hat{e}_\lambda^D$	5.12	$0.2\hat{e}_g^D - 0.3\hat{e}_\lambda^D - 0.5\hat{e}_g^{(0)} + 0.8\hat{e}_\lambda^{(0)}$
−2	$1.00\hat{e}_g^{(0)} + 0.01\hat{e}_\lambda^{(0)}$	2	$1.00\hat{e}_g^{(0)} + 0.01\hat{e}_\lambda^{(0)}$
2	$\hat{e}_\lambda^{(0)}$	4	$\hat{e}_\lambda^{(0)}$

(9.33)

The same structure as discussed previously appears for the mixed combination of non-trivial and Gaußian fixed points, whereby in the present case the level-(0) part obeys the canonical scaling relation, yielding $G^{(0)} \oplus NG_-^D\text{-FP}$. This fixed point has again a three dimensional UV-critical manifold, $s_{UV} = 3$, with qualitatively interchanged roles of $g^{(0)}$ and g^D . On the other hand, the doubly non-Gaußian fixed point $NG_+^{(0)} \oplus NG_-^D\text{-FP}$ is UV-attractive in all four bulk directions, and thus each coefficient of the truncation ansatz has to be experimentally fixed by an independent measurement. Notice that the hierarchy structure is also visible on the level of critical exponents, which remain the same in the D-sector for different values of the level-(0) couplings.

We now turn to the most promising candidate for an asymptotically safe quantum theory of gravity, $NG_+^D\text{-FP}$. This fixed point seems to be stable under severe deformations and extensions

of theory space and exhibits also very appealing properties concerning its dependence on the ghost contributions, for instance. Due to the existence of the separating line at $g^D = 0$, it is so to say also situated in the half-plane $g^D > 0$ consistent with a classical IR limit reproducing the positive sign of Newton's coupling in GR. In combination with the level-(0) results for the fixed point condition, we obtain two further critical points, characterized as follows:

$G^{(0)} \oplus NG_+^D\text{-FP } (g_*^{(0)}, \lambda_*^{(0)}) = (0, 0)$	
θ_r	eigenvectors $V^{(j)}$
$3.6 + 4.3i$	$-0.99\hat{e}_g^D - (0.04 + 0.15i)\hat{e}_\lambda^D$
$3.6 - 4.3i$	$-0.99\hat{e}_g^D - (0.04 - 0.15i)\hat{e}_\lambda^D$
-2	$0.96\hat{e}_g^{(0)} + 0.28\hat{e}_\lambda^{(0)}$
2	$\hat{e}_\lambda^{(0)}$

$NG_+^{(0)} \oplus NG_+^D\text{-FP } (g_*^{(0)}, \lambda_*^{(0)}) = (0.65, 0.19)$	
θ_r	eigenvectors $v^{(r)}$
$3.6 + 4.3i$	$0.86\hat{e}_g^D + (0.04 + 0.13i)\hat{e}_\lambda^D + (0.40 - 0.03i)\hat{e}_g^{(0)} - (0.2 - 0.2i)\hat{e}_\lambda^{(0)}$
$3.6 - 4.3i$	$0.86\hat{e}_g^D + (0.04 - 0.13i)\hat{e}_\lambda^D + (0.40 + 0.03i)\hat{e}_g^{(0)} - (0.2 + 0.2i)\hat{e}_\lambda^{(0)}$
2	$0.96\hat{e}_g^{(0)} + 0.28\hat{e}_\lambda^{(0)}$
4	$\hat{e}_\lambda^{(0)}$

(9.34)

According to eq. 9.34, the fixed points which stem from $NG_+^D\text{-FP}$, located in the physically relevant $g^D > 0$ half-space, are special in that some of their critical exponents are complex. This leads to an oscillating part in the solution of eq. (9.31) which further gives rise to the characteristic spirals of the RG trajectories approaching the FP for $k \rightarrow \infty$. This feature was already observed in the single-metric Einstein-Hilbert truncation [135, 191] and in fact, the complex conjugate pair $\theta_{1/2}$ of eq. 9.34 is reminiscent of the single-metric results. This close numerical similarity lends further credit to the conjecture that *the single-metric fixed point should correspond to one of the two fixed points of the full system which are related to the $NG_+^D\text{-FP}$* . In chapter 10 we investigate the cause of this coincidence. In contrast to the complex, non-integer values of the critical exponents in the D-sector their level-(0) counterparts are found to be always of integer type. In fact, from all previous results we see that they either appear with the canonical scaling -2 and 2 for the Gaussian solution related to $G^{(0)}\text{-FP}$, or with 2 and 4 in case of $NG_+^{(0)}\text{-FP}$. For the dimensionality of the UV critical hypersurface we read off $s_{UV} = 3$ for the mixed fixed point $G^{(0)} \oplus NG_+^D\text{-FP}$ and $s_{UV} = 4$ for the ‘doubly non-Gaussian’ fixed points, $NG_+^{(0)} \oplus NG_-^D\text{-FP}$.

For the present truncation, we observe that each Gaussian part of the fixed point is associated with one relevant and one irrelevant direction, while a non-Gaussian component introduces two relevant dimensions. In particular for the ‘doubly’ NGFPs we thus have a four dimensional UV-critical hypersurface, implying that four independent experimental results are needed in order to impose sufficient initial condition to obtain the unique trajectory realized in Nature. Furthermore, whereas the critical exponents related to the level-(0) couplings are always real integers, the D-sector is of non-integer type usually and in the most interesting case of $NG_+^D\text{-FP}$ we find a pair of complex numbers for $\theta_{1/2}$, resulting in a spiral behavior of the RG trajectories.

The UV-critical manifold for the boundary sector

We conclude this section by presenting the critical behavior of the fixed points in the boundary sector. We are not going to repeat the discussion for the D-sector, since the results remain the

same and are unaffected by $g^{\partial(0)}$ and $\lambda^{\partial(0)}$. In fact, we will only focus on the difference of the level-(0) bulk and boundary results and keep the discussion short on those aspects, which turn out to be the same in both cases.

However, the first distinction already appears for the Gaußian fixed point in the boundary sector. Since it described the canonical scaling of the free theory, the critical exponents differ for $\lambda^{(0)}$ and $\lambda^{\partial(0)}$, since for the latter the dimensionful coupling scales as k^1 rather than k^2 in the canonical case. Hence, for G^D -FP, we have the following characterization of the critical regimes when combined with the Gaußian and non-trivial fixed point solutions of the boundary sector:⁵

$G^{\partial(0)} \oplus G^D\text{-FP } (g_*^{\partial(0)}, \lambda_*^{\partial(0)}) = (0,0)$		$NG_-^{\partial(0)} \oplus G^D\text{-FP } (g_*^{\partial(0)}, \lambda_*^{\partial(0)}) = (-18.8, 4/3)$	
θ_r	eigenvectors $v^{(r)}$	θ_r	eigenvectors $v^{(r)}$
-2	$\frac{8\pi}{3}\hat{e}_g^D + \hat{e}_\lambda^D$	-2	$0.12\hat{e}_g^D + 0.01\hat{e}_\lambda^D + 0.99\hat{e}_g^{\partial(0)} + 0.02\hat{e}_\lambda^{\partial(0)}$
2	$+\hat{e}_\lambda^D$	2	$0.0\hat{e}_g^D + 0.0\hat{e}_\lambda^D + 0.0\hat{e}_g^{\partial(0)} + 0.0\hat{e}_\lambda^{\partial(0)}$
-2	$8\pi\hat{e}_g^{\partial(0)} - \hat{e}_\lambda^{\partial(0)}$	2	$1.00\hat{e}_g^{\partial(0)} - 0.07\hat{e}_\lambda^{\partial(0)}$
1	$\hat{e}_\lambda^{\partial(0)}$	3	$g^{\partial(0)}$

(9.35)

We obtain the same picture as for the bulk sector: each Gaußian part of the fixed points comes along with one irrelevant direction, while for the non-Gaußian component both directions turn out to be relevant and thus have to be fixed by additional constraints. The only apparent differences compared to the bulk contribution are the negative sign multiplying the $\lambda^{\partial(0)}$ -part of the eigenvector and the decremented critical exponent associated to $g^{\partial(0)}$.

Next, consider the vicinity of the NG_-^D -FP fixed point and its combination with the zeros of the beta-functions for the boundary couplings. The resulting critical exponents and the scaling fields read:

$G^{\partial(0)} \oplus NG_-^D\text{-FP } (g_*^{\partial(0)}, \lambda_*^{\partial(0)}) = (0,0)$		$NG_-^{\partial(0)} \oplus NG_-^D\text{-FP } (g_*^{\partial(0)}, \lambda_*^{\partial(0)}) = (-27.9, 0.72)$	
θ_r	eigenvectors $v^{(r)}$	θ_r	eigenvectors $v^{(r)}$
2.21	$1.0\hat{e}_g^D + 0.1\hat{e}_\lambda^D$	2.21	$0.02\hat{e}_g^D + 0.001\hat{e}_\lambda^D + 1.0\hat{e}_g^{\partial(0)} - 0.03\hat{e}_\lambda^{\partial(0)}$
5.12	$0.6\hat{e}_g^D - 0.8\hat{e}_\lambda^D$	5.12	$0.04\hat{e}_g^D - 0.05\hat{e}_\lambda^D + 9.97\hat{e}_g^{\partial(0)} + 0.7\hat{e}_\lambda^{\partial(0)}$
-2	$1.00\hat{e}_g^{\partial(0)} - 0.03\hat{e}_\lambda^{\partial(0)}$	2	$1.00\hat{e}_g^{\partial(0)} - 0.03\hat{e}_\lambda^{\partial(0)}$
1	$\hat{e}_\lambda^{\partial(0)}$	3	$\hat{e}_\lambda^{\partial(0)}$

(9.36)

Except for the differences that already appeared on for the Gaußian fixed point, there is no further peculiarity. In fact, the results resembles the properties of the bulk sector exhibiting integer eigenvalues in the descending boundary part of the differential system.

Finally, let us consider the most interesting fixed point NG_+^D -FP. In combination with the trivial solution for the boundary couplings, we obtain the familiar picture of a three dimensional UV-critical hypersurface with real, integer critical exponents on the boundary and a complex

⁵For technical reasons, we could not resolve the eigenvector associated to θ_2 of $NG_-^{\partial(0)} \oplus G^D$ -FP. Whether this is a numerical problem or if it is related to the large negative or the fractional value of $g^{\partial(0)}$ and $\lambda^{\partial(0)}$, respectively, remains unclear.

pair in the D-sector. The details of $G^{\partial(0)} \oplus \text{NG}_+^{\text{D}}\text{-FP}$ read:

$G^{\partial(0)} \oplus \text{NG}_+^{\text{D}}\text{-FP}_{(g_*^{\partial(0)}, \lambda_*^{\partial(0)})=(0,0)}$	
θ_r	eigenvectors $V^{(j)}$
$3.6 + 4.3i$	$-0.99\hat{e}_g^{\text{D}} - (0.04 + 0.15i)\hat{e}_\lambda^{\text{D}}$
$3.6 - 4.3i$	$-0.99\hat{e}_g^{\text{D}} - (0.04 - 0.15i)\hat{e}_\lambda^{\text{D}}$
-2	$0.87\hat{e}_g^{\partial(0)} - 0.49\hat{e}_\lambda^{\partial(0)}$
1	$\hat{e}_\lambda^{\partial(0)}$

(9.37)

The second combination results in the doubly non-Gaussian fixed point. As any non-trivial zero of the beta-functions of the boundary part, the corresponding fixed point value of the Newton coupling is negative, $g_*^{\partial(0)}$. Thus, asymptotically safe trajectories evolve in the lower half-plane $g^{\partial(0)} < 0$ which is opposite to the bulk level-(0) as well as D-solutions. Apart from this subtlety, the general structure of $\mathcal{S}_{\text{UV}}$ is very similar compared to (9.34):

$\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^{\text{D}}\text{-FP}_{(g_*^{\partial(0)}, \lambda_*^{\partial(0)})=(-2.14, 1.20)}$	
θ_r	eigenvectors $v^{(r)}$
$3.6 + 4.3i$	$(0.3 + 0.1i)\hat{e}_g^{\text{D}} + (0.003 + 0.01i)\hat{e}_\lambda^{\text{D}} - 0.9\hat{e}_g^{\partial(0)} - (0.2 - 0.2i)\hat{e}_\lambda^{\partial(0)}$
$3.6 - 4.3i$	$(0.3 - 0.1i)\hat{e}_g^{\text{D}} + (0.003 - 0.01i)\hat{e}_\lambda^{\text{D}} - 0.9\hat{e}_g^{\partial(0)} - (0.2 + 0.2i)\hat{e}_\lambda^{\partial(0)}$
2	$0.87\hat{e}_g^{\partial(0)} - 0.49\hat{e}_\lambda^{\partial(0)}$
3	$\hat{e}_\lambda^{\partial(0)}$

(9.38)

While the D-sector forces the trajectories to spiral in the vicinity of $\text{NG}_+^{\text{D}}\text{-FP}$, the real exponents in the boundary part implies a linear translation along $\hat{e}_\lambda^{\partial(0)}$ and $\hat{e}_g^{\partial(0)}$, respectively. This, as well as $s_{\text{UV}} =$, resembles the findings of the previous subsection where the level-(0) boundary couplings are replaced by their bulk counterparts.

Hence, besides the canonical difference that also affects the non-Gaussian solutions, the critical behavior of bulk and boundary sector on level-(0) are quite similar. Nevertheless, we already discovered a very fundamental distinction between both cases, namely the half-spaces of the Newton-couplings asymptotically safe RG trajectories live in. Furthermore, there are subtle differences in the eigenvectors, where certain directions are combined with a relative opposite sign compared to its bulk analog.

9.4 Conclusion

So far, the Asymptotic Safety conjecture for Quantum Gravity has undergone several, all successful tests each revealing a non-trivial fixed point compatible with the classical theory of General Relativity. Besides increasing the number of basis invariants to cover more and more regions of theory space, one also studies the stability of the previous findings by deforming the underlying assumptions or by employing conceptual extensions to the standard program. In this thesis, we study two such conceptual different directions in theory space, one which modifies the topological class of the underlying spacetime, allowing for manifolds with boundary, the other addressing the intrinsically bi-metric character of the Effective Average Action. While for technical reasons, most of our today's understanding is based on so-called single-metric calculations, which impose an a priori unjustified split-symmetry along the entire RG evolution, recently there has been first evidence for the continued existence of the NGFP in a bi-metric setting [172].

Remarkably, employing a new technique that should encourage further work beyond the single-metric approximation, we recover the familiar NGFP reminiscent of the single-metric Einstein-Hilbert results [122]. Searching for extensions including the boundary invariants, each fixed point in the dynamical sector gives rise to a trivial and non-trivial solution for the boundary level-(0) couplings. In combination with the level-(0) bulk couplings we obtain twelve critical points of the RG flow, out of which those associated to NG_+^{D} -FP are the most interesting ones. In order to understand the topological properties of asymptotically safe trajectories, we first studied the general properties of the differential equations, revealing a separating line at $g^I = 0$ which implies that for each RG solution the sign of Newton's coupling is the same for all k . We then utilized the hierarchical structure of the beta-functions obtained the fixed point values for the optimized shape function and studied the possible combination of individual solutions to a full fledged critical point in the six-dimensional truncated theory space.

For the boundary Newton coupling we observed that only negative non-trivial solutions can occur, which is in contrast to the bulk sector where in fact g^{D} is bounded to be positive by experimental data. While numerically and also qualitatively different situated, the respective directions of the UV-critical hypersurfaces are very similar for level-(0) on bulk and boundary. In both directions only real integer critical exponents occur, while in the D-sector the especially important NG_+^{D} -FP fixed point is related to complex scaling exponents resulting in the spiral effect observed in fig. 9.1. Concerning the predictivity of the underlying theory, within this truncation we revealed a correspondence between the number of Gaußian components in the fixed point and the number of its irrelevant directions. While each non-trivial solution for a pair (g^I, λ^I) also requires two initial conditions, thus gives rise to positive real parts of θ only, a trivial contribution $(g^I, \lambda^I) = (0, 0)$ is UV-attractive (-repulsive) along $\hat{e}_g^I$ (or $\hat{e}_\lambda^I$). For the very important case of the threefold non-Gaußian fixed point $\text{NG}_-^{\partial(0)}$ - $\text{NG}_+^{(0)}$ - NG_+^{D} -FP this results in a six-dimensional UV-critical manifold, being restricted only to the respective half-planes of the Newton couplings. Hence, we may conclude that we found further evidence for the Asymptotic Safety conjecture by severely deforming the standard truncations, exploring different topological spacetimes and the study the effect of split-symmetry breaking on the existence of the non-Gaußian fixed point. Its stability seems quite promising indicating that the so far observed topological property might be in fact a feature of the exact flow equation. For the bi-metric-bulk – pure-background-boundary truncation the Asymptotic Safety condition imposes the following requirement on the solutions of FRGE to be asymptotically safe:

$$\mathcal{S}_{\text{UV}}^{\text{EH-GHY}} \equiv \{k \rightarrow A_k[\varphi; \bar{\Phi}; \{g^I, \lambda^I\}] \in \mathcal{T}^{\text{EH-GHY}} \mid g^{\text{D}}, g^{(0)} > 0 \wedge g^{\partial(0)} < 0\}$$

Therefore, we indeed obtain $\dim(\mathcal{S}_{\text{UV}}^{\text{EH-GHY}}) = 6$ and the present truncation includes four more relevant invariants, compared to the single-metric Einstein-Hilbert truncation. In the next chapter, we will combine the requirements of Asymptotic Safety and Background Independence to form a reduction of $\mathcal{S}_{\text{UV}}^{\text{EH-GHY}}$ to the physical relevant part, including only those theories which respect the fundamental requirement of Background Independence.

9.A Critical exponents of the background-sector

This appendix list the characterizing features of the UV-critical hypersurface for the six fixed points in the background sector. It includes the critical exponents as well as the corresponding

scaling fields.

$G^B \oplus G^D$ -FP $(g_*^B, \lambda_*^B) = (0, 0)$		$NG_+^B \oplus G^D$ -FP $(g_*^B, \lambda_*^B) = (1.40, -0.11)$		(9.39)
θ_r	eigenvectors $v^{(r)}$	θ_r	eigenvectors $v^{(r)}$	
-2	$\frac{8\pi}{3}\hat{e}_g^D + \hat{e}_\lambda^D$	-2	$\frac{8\pi}{3}\hat{e}_g^D + \hat{e}_\lambda^D$	
2	$+\hat{e}_\lambda^D$	2	$+\hat{e}_\lambda^D$	
-2	$-4\pi\hat{e}_g^B + \hat{e}_\lambda^B$	-2	$-4\pi\hat{e}_g^B + \hat{e}_\lambda^B$	
2	$\hat{e}_\lambda^B$	2	$\hat{e}_\lambda^B$	

$G^B \oplus NG_-^D$ -FP $(g_*^B, \lambda_*^B) = (0, 0)$		$NG_+^B \oplus NG_-^D$ -FP $(g_*^B, \lambda_*^B) = (1.53, -0.12)$		(9.40)
θ_r	eigenvectors $v^{(r)}$	θ_r	eigenvectors $v^{(r)}$	
2.21	$1.0\hat{e}_g^D + 0.1\hat{e}_\lambda^D$	2.21	$-0.93\hat{e}_g^D - 0.06\hat{e}_\lambda^D - 0.35\hat{e}_g^B + 0.04\hat{e}_\lambda^B$	
5.12	$-0.6\hat{e}_g^D + 0.8\hat{e}_\lambda^D$	5.12	$0.4\hat{e}_g^D - 0.6\hat{e}_\lambda^D - 0.4\hat{e}_g^B + 0.5\hat{e}_\lambda^B$	
-2	$1.0\hat{e}_g^B - 0.1\hat{e}_\lambda^B$	2	$1.00\hat{e}_g^B - 0.08\hat{e}_\lambda^B$	
2	$\hat{e}_\lambda^B$	4	$\hat{e}_\lambda^B$	

$NG_+^B \oplus NG_+^D$ -FP $(g_*^B, \lambda_*^B) = (8.18, -0.01)$		(9.41)
θ_r	eigenvectors $v^{(r)}$	
$3.6 + 4.3i$	$-(1.6 - 0.2i)10^{-2}\hat{e}_g^D - (0.9 + 2.4i)10^{-2}\hat{e}_\lambda^D + 1.0\hat{e}_g^B + (1.7 - 2.3i)10^{-2}\hat{e}_\lambda^B$	
$3.6 - 4.3i$	$-(1.6 + 0.2i)10^{-2}\hat{e}_g^D - (0.9 - 2.4i)10^{-2}\hat{e}_\lambda^D + 1.0\hat{e}_g^B + (1.7 + 2.3i)10^{-2}\hat{e}_\lambda^B$	
2	$\hat{e}_g^B - 9 \times 10^{-4}\hat{e}_\lambda^B$	
4	$\hat{e}_\lambda^B$	

$G^B \oplus NG_+^D$ -FP $(g_*^B, \lambda_*^B) = (0, 0)$		(9.41)
θ_r	eigenvectors $v^{(r)}$	
$3.6 + 4.3i$	$-0.99\hat{e}_g^D - (0.04 + 0.15i)\hat{e}_\lambda^D$	
$3.6 - 4.3i$	$-0.99\hat{e}_g^D - (0.04 - 0.15i)\hat{e}_\lambda^D$	
2	$\hat{e}_g^B - 9 \times 10^{-4}\hat{e}_\lambda^B$	
2	$\hat{e}_\lambda^B$	



10.1	Introduction
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10. BACKGROUND INDEPENDENCE

 This chapter follows closely section 1 and 5-9 of [173].

The most momentous requirement a quantum theory of gravity must satisfy is Background Independence, necessitating in particular an ab initio derivation of the arena all non-gravitational physics takes place in, namely spacetime. Using the background field technique, this requirement translates into the condition of an unbroken split-symmetry connecting the (quantized) metric fluctuations to the (classical) background metric. If the regularization scheme used violates split-symmetry during the quantization process it is mandatory to restore it in the end at the level of observable physics. In this chapter we present a detailed investigation of split-symmetry breaking and restoration within the EAA approach to Quantum Einstein Gravity (QEG) with a special emphasis on the Asymptotic Safety conjecture, based on the results of part II for the bulk sector. In ref. [173] we demonstrate for the first time in a non-trivial setting that the two key requirements of Background Independence and Asymptotic Safety can be satisfied simultaneously. This important result is shown by the existence of a subset of RG trajectories which are both asymptotically safe and split-symmetry restoring: In the ultraviolet they emanate from a non-Gaussian fixed point, and in the infrared they lose all symmetry violating contributions inflicted on them by the non-invariant functional RG equation. As an application, we compute the scale dependent spectral dimension which governs the fractal properties of the effective QEG spacetimes at the bi-metric level. Earlier tests of the Asymptotic Safety conjecture almost exclusively employed ‘single-metric truncations’ which are blind towards the difference between quantum and background fields. We explore in detail under which conditions they can be reliable, and we discuss how the single-metric based picture of Asymptotic Safety needs to be revised in the light of the new results. We shall conclude that the next generation of truncations for quantitatively precise predictions (of critical exponents, for instance) is bound to be of the bi-metric type.

10.1 Introduction

One of the key requirements every candidate for a quantum theory of the gravitational interaction and spacetime geometry should satisfy is *Background Independence*. The theory's basic kinematical rules and dynamical laws should be formulated without reference to any distinguished spacetime such as Minkowski space, for instance. Rather, the possible states of a 'quantum spacetime' should be a prediction of the theory. In addition, it must provide us with a set of special observables which, by means of their expectation values in a given state, 'interpret' this state in terms of classical geometry, or a generalized notion thereof. Among them the expectation value of the metric would play a significant role. If non-degenerate, smooth and approximately flat, on large length scales at least, the underlying state might appear like a classical spacetime macroscopically, possibly similar to the real Universe we live in. We can then try to match the predictions against concrete measurements and observations.[72, 73, 75, 76, 93, 196–198]

However, in general one would also expect states without any interpretation in terms of concepts from classical General Relativity, Riemannian geometry in particular. A simple example are situations in which the metric has an expectation value which is degenerate, identically vanishing, for instance. While the gravitational physics implied by such states is certainly very different from the one we know, they might realize a 'symmetric phase' of gravity which arguably is easier to understand than the broken phase we live in. In the latter, diffeomorphism symmetry is broken down to the stability group of the metric expectation value, the Poincaré group in the flat case.

There exist two fundamentally different approaches to deal with the requirement of Background Independence. They differ in particular in the way they deal with the rather severe conceptual and technical difficulties which are caused by this requirement and are of a kind never encountered in conventional matter field theories on Minkowski space:

- A. The most obvious strategy is to literally employ no background geometry at all in setting up the foundational structures of the theory, then work out its quantum dynamical consequences, and try to find states on which appropriate geometric operators (metric etc.) signal the existence of almost classical spacetimes at the expectation value level. Examples of such literally background independent settings include statistical field theory models like the Causal Dynamical Triangulations (CDT) [94–98], and Loop Quantum Gravity (LQG) [72, 75, 93]. Following this route, there are (at least) two characteristic difficulties one has to cope with. First, since most of the traditional tools of quantum (field) theory presuppose a rigid background metric, considerable conceptual problems must be overcome. The proposed solutions are usually outside the realm of quantum field theory. Often they encode the fundamental degrees of freedom in new types of variables which replace the quantum fields, triangulated spaces or spin-foams being typical examples. Second, at the computational level the difficulty arises that an *ab initio* explanation of a (non-degenerate, smooth, approximately flat) macroscopic metric has to bridge an enormous gap of scales if one starts from a microscopic theory governing the 'atoms of geometry'. For Planck sized building blocks of spacetime, say, even the typical scales of particle physics are about 20 orders of magnitude away. This calls for an application of Wilson's renormalization group but, again, most of the existing tools are inapplicable in the background-free context [199, 200].
- B. Rather than using literally no background at all, 'Background Independence' can also be achieved in the diametrically opposite way, namely by actually taking advantage of background structures in formulating and evaluating the theory, but making sure that *all possible backgrounds are treated on a completely equal footing*. This latter requirement implies in particular that the background geometry may appear only in intermediate steps

of the calculations while predictions for observables may never depend on it [27].

This second approach is realized for instance in the present case employing the Effective Average Action to study theories of quantum gravity [122]. In the simplest case of metric gravity we use the background-quantum field split of the dynamical metric $\hat{g}_{\mu\nu}$, the integration variable in the underlying functional integral, in the form $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$. Here $\bar{g}_{\mu\nu}$ is the arbitrary classical background metric, and $\hat{h}_{\mu\nu}$ a nonlinear fluctuation which plays the role of the quantum field, i.e. it is functionally integrated over (or promoted to an operator, in the canonical formulation). Thanks to the presence of the arbitrary but, by assumption, non-degenerate background metric $\bar{g}_{\mu\nu}$ the original problem appears in a new, somewhat different guise now: it consists in the conceptually easier task of quantizing the, now matter-like field $\hat{h}_{\mu\nu}$ ‘living’ on the classical background geometry given by $\bar{g}_{\mu\nu}$.

At this point the technical challenge resides in the fact that this background is completely generic and enjoys no special symmetries in particular. However, the availability of $\bar{g}_{\mu\nu}$ opens the door for the application of a considerable arsenal of quantum field theory techniques, namely basically all those that have been developed for matter systems on Minkowski space or on curved classical backgrounds. The price we pay for this enormous advantage is that *in the $\hat{h}_{\mu\nu}$ -theory we must have complete control over the $\bar{g}_{\mu\nu}$ -dependence of all expectation values*, the n -point functions of $\hat{h}_{\mu\nu}$ in particular. Their IPI version, with an IR cutoff at the scale k , is generated by the Effective Average Action $\Gamma_k[h_{\mu\nu}; \bar{g}_{\mu\nu}]$. As usual this functional depends on the expectation value field $h_{\mu\nu} \equiv \langle \hat{h}_{\mu\nu} \rangle$, but also the background metric, $\bar{g}_{\mu\nu}$, which here acquires the status of an indispensable second argument of Γ_k . The expectation value of the full metric is

$$g_{\mu\nu} \equiv \langle \hat{g}_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + \langle \hat{h}_{\mu\nu} \rangle \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Sometimes it is more natural to consider $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, rather than the pair $(h_{\mu\nu}, \bar{g}_{\mu\nu})$ as the independent variables on which the EAA depends, and to set

$$\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}] \equiv \Gamma_k[h_{\mu\nu}; \bar{g}_{\mu\nu}]|_{h=g-\bar{g}}$$

In this formulation the intrinsic *bi-metric character* of the EAA becomes manifest. We refer to the second argument of $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}]$ as its *extra $\bar{g}_{\mu\nu}$ -dependence* since, contrary to the $\bar{g}$ -dependence within $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$, this ‘extra’ dependence does not combine with $h_{\mu\nu}$ to form a full dynamical metric.

The Functional Renormalization Group Equation governs the k -dependence of Γ_k , [110–114], and in principle the functional integral over $\hat{g}_{\mu\nu}$ can be evaluated indirectly by solving the FRGE instead [121]. For this to be possible, and the Functional Renormalization Group Equation to exist in the first place, it is unavoidable to employ a running action functional which admits an arbitrary extra $\bar{g}_{\mu\nu}$ -dependence and, in fact, depends also on Faddeev-Popov ghost fields ξ^μ and $\bar{\xi}_\mu$, respectively: $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu]$. (See ref. [122] for further details.)

The approaches **A.** and **B.** have complementary advantages and disadvantages. In **A.** the strategy of literally avoiding any background, and starting from a ‘vacuum’ state with no space-time interpretation at all, it is comparatively easy to describe a possible phase of unbroken diffeomorphism invariance. But the corresponding broken phase is a very hard problem since it is due to the cooperative effect of a huge number of ‘atoms of spacetime’. Conversely, in **B.**, the broken phase is the easier one since, at least when the background is chosen self-consistently so that $g_{\mu\nu} = \bar{g}_{\mu\nu}$ holds true, the quantum fluctuations can be relatively weak, with vanishing expectation value $h_{\mu\nu} \equiv \langle \hat{h}_{\mu\nu} \rangle = 0$. The symmetric phase, on the other hand, requires huge quantum effects as $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \langle \hat{h}_{\mu\nu} \rangle$ is supposed to vanish identically even though $\bar{g}_{\mu\nu}$ stays always non-degenerate.

In the present context we realize Background Independence in the second way, i.e., loosely speaking, by quantizing the fluctuations of the metric in all possible background spacetimes at a time. We are going to explore the RG evolution of the EAA with a particular emphasis on how the amount of ‘extra’ $\bar{g}$ -dependence the functional Γ_k suffers from depends on the RG scale k . The extra $\bar{g}$ -dependence is an entirely unphysical artifact present only at intermediate computational steps. To establish Background Independence it must be turned to zero at the end. The extra $\bar{g}$ -dependence is most conveniently discussed in terms of the *background quantum split-symmetry*, its breaking, and its restoration. This symmetry transformation changes $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ according to

$$\delta_\epsilon^{\text{split}} h_{\mu\nu} = \epsilon_{\mu\nu}(x), \quad \delta_\epsilon^{\text{split}} \bar{g}_{\mu\nu} = -\epsilon_{\mu\nu}(x) \quad (10.1)$$

and it has an obvious action on the functionals¹ $\Gamma_k[h; \bar{g}] \equiv \Gamma_k[g, \bar{g}]$. Here $\epsilon_{\mu\nu}(x)$ is an arbitrary symmetric tensor field. Clearly, the full dynamical metric $g = \bar{g} + h$ is invariant under $\delta_\epsilon^{\text{split}}$, while $\bar{g}$ is not. As a consequence, Γ_k is invariant under split transformations, $\delta_\epsilon^{\text{split}} \Gamma_k = 0$, precisely when the two-metrics functional $\Gamma_k[g, \bar{g}]$ is actually independent of its second argument $\frac{\delta}{\delta \bar{g}} \Gamma_k[g, \bar{g}] = 0$, or equivalently, when $\Gamma_k[h; \bar{g}]$ happens to be a functional of the sum of its arguments only, viz. $\bar{g} + h \equiv g$.

In order to understand where the unavoidable extra $\bar{g}$ -dependence of the EAA comes from recall that the EAA derives from a functional integral which, after the split $\hat{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$, has the form [122]

$$\int \mathcal{D}\hat{h}_{\mu\nu} \int \mathcal{D}\xi^\mu \mathcal{D}\bar{\xi}_\mu \exp \left(-S[\bar{g} + \hat{h}] - S_{\text{gf}}[\hat{h}; \bar{g}] - S_{\text{gh}}[\hat{h}, \xi, \bar{\xi}; \bar{g}] - \Delta S_k[\hat{h}, \xi, \bar{\xi}; \bar{g}] \right) \quad (10.2)$$

While the (arbitrary) bare action $S[\bar{g} + \hat{h}]$ is indeed $\delta_\epsilon^{\text{split}}$ -invariant, the same is *not* true for the gauge fixing term S_{gf} , the ghost action S_{gh} , and in particular the cutoff action ΔS_k ; it implements the IR cutoff in the familiar way by mode suppression terms which are quadratic in the quantum fields [110–114]. This shows that the very concept of an EAA, namely the idea to ‘coarse grain’ (that is, smoothly regularize in the infrared) a gauge-fixed functional integral hinges in a crucial way on the availability of $\bar{g}_{\mu\nu}$: neither the gauge fixing, nor the ‘cutting out’ of the IR modes necessary to derive a functional RG equation could be implemented without promoting the background metric to an independent entity, different from $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ in general. That this is indeed necessary is easily traced back to the gravitational field’s special status among the carriers of the fundamental interactions, namely its very relation to the geometry of spacetime.²

We also recall that the EAA employs a gauge fixing condition which belongs to the special class of *background gauges*, as a result of which $S_{\text{gf}} + S_{\text{gh}}$ is invariant under the so-called background gauge transformations, δ_v^{B} , not to be confused with the ‘quantum gauge transformations’ to be fixed, of course. Furthermore, ΔS_k is constructed so as to enjoy the same invariance property. For a diffeomorphism generated by any vector field v , the background gauge transformation δ_v^{B} acts as the Lie derivative $\mathcal{L}_v$ on both the dynamical fields $(\hat{g}, \hat{\eta}, \hat{\bar{\eta}})$ or $(\hat{h}, C, \bar{C})$, and on $\bar{g}_{\mu\nu}$. Ultimately this leads to an EAA which is an invariant functional of its arguments, $\delta_v^{\text{B}} \Gamma_k[h, \xi, \bar{\xi}; \bar{g}] = 0$, and its RG flow preserves this property.

In the limit $k \rightarrow 0$, where the IR regulator is removed, $\mathcal{R}_k$ and ΔS_k vanish by construction, and as a result $\Gamma \equiv \lim_{k \rightarrow 0} \Gamma_k$ coincides with the ordinary effective action, for the specific (background-type) gauge chosen.

¹We suppress the ghosts when they are inessential for the discussion. On them, $\delta_\epsilon^{\text{split}} \xi^\mu = 0$ and $\delta_\epsilon^{\text{split}} \bar{\xi}_\mu = 0$.

²Note that there exists no analogous ‘Background Independence’ issue in Yang-Mills gauge theories on Minkowski space. There, an ‘extra $\bar{A}_\mu$ -dependence’ can always be avoided in a trivial way, namely by simply not using a gauge fixing condition involving a background gauge field, whereupon no such field will appear anywhere. Not so in gravity: even if we were to give up background gauge invariance and use a $\bar{g}_{\mu\nu}$ -free gauge fixing condition, a mode suppression term ΔS_k with the necessary properties could not even be written down without having a second metric at our disposal [122].

While Γ , like Γ_k at any $k > 0$, is perfectly δ_v^B -invariant, it is still not $\delta_\epsilon^{\text{split}}$ -invariant: While one source of split-symmetry violation, the one due to ΔS_k , disappears at $k = 0$, the other, the gauge fixing and ghost sector, still precludes complete $\delta_\epsilon^{\text{split}}$ -invariance. However, in a sense, this is a very weak violation since it concerns the gauge modes only, and should disappear, too, upon going on-shell [68, 201–205].

Nevertheless, at intermediate scale $k > 0$ the RG flow generates in principle all possible, generically $\delta_\epsilon^{\text{split}}$ -violating functionals $\Gamma_k[g, \bar{g}, \xi, \bar{\xi}]$ of four independent arguments. They are constrained only by their built-in δ_v^B -invariance, and proper approximation schemes for solving the functional flow equation such as truncations of theory space must take account of this fact.

To date, almost all available RG studies of the Asymptotic Safety scenario still involve the same type of approximation to the exact EAA, the so-called *single-metric truncation*, which was used very early on as the first testing ground for the gravitational Functional Renormalization Group Equation [122],

$$\partial_t \Gamma_k[g, \bar{g}, \xi, \bar{\xi}] = \frac{1}{2} \text{STr} \left[\left(\Gamma^{(2)} + \mathfrak{R}_k \right)^{-1} \partial_t \mathfrak{R}_k \right], \quad (10.3)$$

Truncations of this general class project the RG flow implied by the (exact!) equation (10.3) on the subspace spanned by functionals of the form

$$\Gamma_k[g, \bar{g}, \xi, \bar{\xi}] = \Gamma_k^{\text{sm}}[g] + (S_{\text{gf}} + S_{\text{gh}})[g, \bar{g}, \xi, \bar{\xi}] \quad (10.4)$$

While Γ_k^{sm} is a generic (diffeomorphism invariant) functional of the dynamical metric only, S_{gf} and S_{gh} are the classical gauge fixing and ghost actions³, depending on the expectation value fields now. An example of (10.4) is the Einstein-Hilbert truncation [122, 135, 191] in which $\Gamma_k^{\text{sm}}[g]$ is specialized further to contain the two invariants $\int \sqrt{g} R$ and $\int \sqrt{g}$, only, with k -dependent pre-factors involving a running Newton constant G_k and cosmological constant $\bar{\lambda}_k$, respectively.

Because of their quite substantial technical complexity, the work on more general truncations that would go beyond (10.4) started only recently. In [136] a first ‘bi-metric’ truncation with separate $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ -dependence was analyzed in conformally reduced gravity and, in [137], for matter induced gravity (in the large N -limit). The first bi-metric investigation of fully fledged Quantum Einstein Gravity employed a truncation ansatz with two separate Einstein-Hilbert terms for the dynamical (‘D’) and the background (‘B’) metric [172]. It consists of the following ‘graviton’ (‘grav’) part, added to the classical gauge fixing and ghost terms whose RG evolution is still neglected :

$$\Gamma_k^{\text{grav}}[g, \bar{g}] = -\frac{1}{16\pi G_k^{\text{D}}} \int d^d x \sqrt{g} (R(g) - 2\bar{\lambda}_k^{\text{D}}) - \frac{1}{16\pi G_k^{\text{B}}} \int d^d x \sqrt{\bar{g}} (R(\bar{g}) - 2\bar{\lambda}_k^{\text{B}}) \quad (10.5)$$

(Generalizing the truncation in a different direction, single-metric ansätze with a running ghost sector were studied in [132–134].)

This chapter is devoted to a detailed investigation of the RG flow related to the bi-metric Einstein-Hilbert truncation using the RG equations for the bulk invariants derived in the previous part and thus based on the (bulk) beta-functions listed in chapter 8. The analysis will be far more comprehensive than the preliminary one in ref. [172]. We shall be specifically interested in *global* properties of the flow, in particular the question as to whether Asymptotic Safety can coexist with Background Independence, that is, the restoration of split-symmetry at the physical level. This question is not easy to answer as it requires control over the fully extended RG trajectories, their limits $k \rightarrow \infty$ and $k \rightarrow 0$ in particular. Coexistence of Asymptotic Safety

³With the running field renormalization of $h_{\mu\nu}$ as implied by G_k included usually.

with Background Independence implies that there exists at least one RG trajectory $k \mapsto \Gamma_k[g, \bar{g}]$ which is non-singular for all $k \in [0, \infty)$, and which approaches a non-Gaussian fixed point at its upper end, i.e. for $k \rightarrow \infty$, while at the lower end, in the ‘physical’ limit $k \rightarrow 0$ where the EAA equals the ordinary effective action, it is split-symmetry-restoring, i.e. there Γ_k loses its extra $\bar{g}$ -dependence.⁴ In particular on a truncated theory space it is by no means obvious that such trajectories do indeed exist and both requirements can be met simultaneously.

One of our main results will be that within the bi-Einstein-Hilbert truncation Asymptotic Safety and Background Independence can indeed coexist: *Some*, but not all RG trajectories which emanate from a non-Gaussian fixed point in the UV also restore split-symmetry in the IR. It will be instructive to uncover the very elegant mechanism of how the concrete RG differential equations bring about this symmetry restoration; we shall see it involves a *moving attractor* in the background part of theory space. We shall see that from a 4-parameter family of asymptotically safe trajectories a 2-parameter subset is symmetry restoring.

The existence of this subset of symmetry restoring trajectories is good news for the Asymptotic Safety program for two independent reasons: First of all, it shows that at least in this truncation Asymptotic Safety and Background Independence are not mutually exclusive. Secondly, since a physically meaningful theory can only be based on a RG trajectory from this subset it follows that the predictive power of the Asymptotic Safety scenario is actually *higher* than what one would expect by just counting relevant perturbations at the fixed point; the indispensable subsidiary condition of Background Independence narrows down the possibilities of constructing physically inequivalent theories at a given fixed point.

Recently the crucial importance of split-symmetry has also been demonstrated in a very impressive way within a 3-dimensional scalar toy model [206]: being careless about its restoration, one can even destroy the Wilson-Fisher fixed point!

Most of this chapter is devoted to an analysis of the beta-functions in $d = 4$ elaborately derived in part II. In section 10.2 we first present a detailed analysis of the global properties of the RG flow. In particular we demonstrate that Asymptotic Safety and Background Independence are indeed compatible by explicitly constructing RG trajectories which restore split-symmetry in the IR. We explain how the special properties of these trajectories come about and we uncover the role played by a moving attractor point in theory space.

Section 10.3 contains a very detailed comparison of the bi-metric Einstein-Hilbert truncation with its single-metric approximation. The goal of this section is to find out under what conditions the latter is sufficiently reliable, and under what circumstances it is mandatory to employ the former. We shall find the somewhat ‘miraculous’ result, unexplained by any general principles, that the single-metric approximation seems to perform best near a non-Gaussian fixed point. Since section 10.3 is rather technical, and mostly intended to establish the lessons one can learn from the calculation of part II and which can help to design future truncations optimally, this section can be skipped by the reader who is mostly interested in the new results.

Next, we consider a first application of the bi-metric RG equation in section 10.4: we compute the scale dependent spectral dimension of the effectively fractal spacetimes QEG gives rise to.

In section 10.5 we briefly comment on the bi-metric RG flow in $d = 2 + \varepsilon$ and $d = 3$ spacetime dimension. Near 2 dimensions, where all Newton constants become dimensionless, their beta-functions contain a universal leading term which we compute. The resulting *universal* discrepancies between the single- and the bi-metric results are rather striking. In this setting it becomes particularly obvious that single-metric approximations can be very misleading even at the qualitative level. Subsequently, we conclude this chapter by giving a short summary and discussion of the presented results.

⁴Except the one due to the gauge modes, to be precise.

10.2 Can split-symmetry coexists with Asymptotic Safety?

We are now going to study the fully fledged RG flow on the four dimensional theory space of the bulk sector by analyzing the following system of coupled differential equations with both analytical and numerical methods:

$$\partial_t g_k^D = \beta_g^D(g_k^D, \lambda_k^D) \equiv [d - 2 + \eta^D(g_k^D, \lambda_k^D)] g_k^D \quad (10.6a)$$

$$\partial_t \lambda_k^D = \beta_\lambda^D(g_k^D, \lambda_k^D) \quad (10.6b)$$

$$\partial_t g_k^B = \beta_g^B(g_k^D, \lambda_k^D, g_k^B) \equiv [d - 2 + \eta^B(g_k^D, \lambda_k^D, g_k^B)] g_k^B \quad (10.6c)$$

$$\partial_t \lambda_k^B = \beta_\lambda^B(g_k^D, \lambda_k^D, g_k^B, \lambda_k^B) \quad (10.6d)$$

In this section we are mostly interested in the global properties of the RG flow. In particular we investigate to what extent split-symmetry at $k = 0$ can be realized by a judicious choice of the trajectory's 'initial' conditions. In practice they are imposed at an intermediate scale k_0 , and the differential equations are solved then both in the upward and downward direction. The all decisive question we try to answer is: **Do there exist RG trajectories for which split-symmetry, i.e. Background Independence for $k \rightarrow 0$ coexists with Asymptotic Safety?** What makes this question hard is its global nature: Background Independence and Asymptotic Safety concern exactly the opposite ends of the RG trajectories, the limits of very small and very large scales, respectively.

Since the D-couplings are not affected by the B-sector but, conversely, enter the B-beta-functions, we first solve the eqs. (10.6a) and (10.6b) for g_k^D and λ_k^D , and then substitute the solutions into the beta-functions of g_k^B and λ_k^B in eqs. (10.6c) and (10.6d) to obtain the 'B'-components of the trajectory. We describe the results of the two steps in turn.

10.2.1 Trajectories of the 'D' sub-system

The set of solutions for the B-independent (g_k^D, λ_k^D) -system (10.6a), (10.6b) decomposes into a subset with positive g_k^D for all k , one with $g_k^D < 0$ always, and a single trajectory with $g_k^D = 0 \forall k$ that separates the two regions. The sign of the Newton coupling never changes along any trajectory. From chapter 9 we know already that the D-system allows for three fixed points: the Gaussian one, G^D -FP, which is located on the separating trajectory, and two non-Gaussian ones, NG_-^D -FP and NG_+^D -FP, that lie below ($g^D < 0$) or above it ($g^D > 0$), respectively. We will focus in the sequel on the positive g^D domain.

Fig. 10.1 shows the phase portrait on the g^D - λ^D plane which we obtained numerically.⁵ We see that it is impressively similar to the well-known phase portrait of the single-metric Einstein-Hilbert truncation [191].

Following ref. [191] the trajectories in the upper half-plane in Fig. 10.1 can be classified in the same way as their single-metric relatives, namely as of type (Ia)^D, type (IIa)^D, and type (IIIa)^D respectively, depending on whether the cosmological constant λ_k^D approaches $-\infty$, 0, or $+\infty$ in the far IR, i.e. for $k \rightarrow 0$. The type (IIa)^D solution separates the two regimes and is, henceforth, also called separatrix. (In Fig. 10.1 it is represented by a dashed line, and we have also highlighted a representative of the other types, a (Ia)^D and a (IIIa)^D trajectory.) The separatrix 'crosses over' from the NG_+^D -FP in the UV to the G^D -FP in the IR. Notice that NG_+^D -FP is UV-attractive in both directions, and thus all trajectories are pulled into this fixed point when $k \rightarrow \infty$. Due to the imaginary part of the critical exponents they form spirals. Further details on the fixed point structure can be found in chapter 9.

⁵Here and in all similar diagrams the arrows always point from the UV towards the IR.

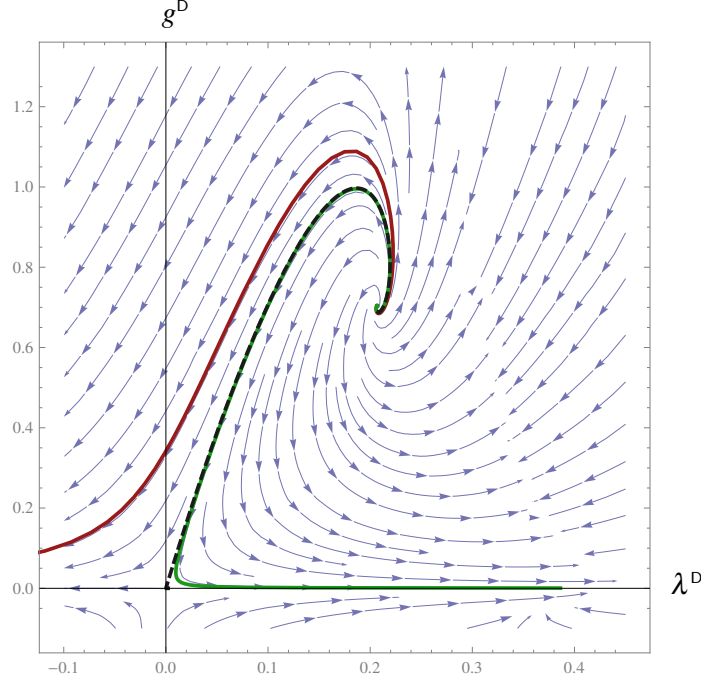


Figure 10.1: The phase portrait of the bi-metric ‘D’-sector. The vertical (horizontal) axis corresponds to g^D (λ^D). Note the remarkable similarity with the phase portrait of the single-metric Einstein-Hilbert truncation [191].

10.2.2 Solving the non-autonomous ‘B’ system

Each one of the D trajectories obtained above can now be substituted into the two RG equations of the B- or level-(0) couplings. After fixing initial conditions they can be solved to give the k -dependence of the two remaining coordinates of the 4-dimensional trajectories, namely $g_k^{B/(0)}$ and $\lambda_k^{B/(0)}$.

Let us start by investigating their qualitative behavior for arbitrary initial conditions in the B-sector. Once a solution (g_k^D, λ_k^D) is picked, the beta-functions in eqs. (10.6c), (10.6d) for the B-couplings are polynomials with known, but *scale dependent* coefficients $A^B(k) \equiv A^B(g_k^D, \lambda_k^D)$ and $B_1^B(k) \equiv B_1^B(g_k^D, \lambda_k^D)$:

$$\partial_t g_k^B = \beta_g^B(g_k^B, \lambda_k^B; k) = 2g_k^B + B_1^B(k)(g_k^B)^2 \quad (10.7a)$$

$$\partial_t \lambda_k^B = \beta_\lambda^B(g_k^B, \lambda_k^B; k) = -2\lambda_k^B + A^B(k)g_k^B + B_1^B(k)g_k^B \lambda_k^B \quad (10.7b)$$

It is important to appreciate that when the functions $A^B(k)$ and $B_1^B(k)$ are given, the eqs. (10.7) form a closed coupled system for the two remaining (g_k^B, λ_k^B) which, however, contrary to that for (g_k^D, λ_k^D) , is *not autonomous*. The beta-functions on the RHS of eqs. (10.7a) and (10.7b) possess an *explicit* dependence on k . Hence the vector field on the g^B - λ^B -plane they give rise to and, as a consequence, the entire phase portrait on this plane are RG-time dependent. This will complicate the analysis considerably.

The other fact to be appreciated is that the beta-functions of (10.7) are *polynomial* in the unknowns g^B and λ^B . This is in sharp contradistinction to the dynamical sector which involves complicated threshold functions $\Phi_n^D(-2\lambda^D)$. For later use we also mention that, in terms of the dimensionful quantities $1/G^B$ and $\tilde{\lambda}^B/G^B$, the system (10.7) has the following general solution:

$$\frac{1}{G_k^B} = \frac{1}{G_{k_0}^B} - \int_{k_0}^k dk' k' B_1^B(k') \quad (10.8a)$$

$$\frac{\bar{\lambda}_k^B}{G_k^B} = \frac{\bar{\lambda}_{k_0}^B}{G_{k_0}^B} + \int_{k_0}^k dk' k'^3 A^B(k') \quad (10.8b)$$

Here, as always, k_0 denotes an arbitrary (initial, or intermediate) scale somewhere along the trajectory.

The ‘B’ Newton constant. Returning to dimensionless quantities the hierarchy among the two equations (10.7) allows to first solve (10.7a) for g_k^B , and then determine the k -dependence of λ_k^B from (10.7b). The differential equation (10.7a) is of Bernoulli type. It determines g_k^B and is independent of λ_k^B . Besides the trivial solution $g_k^B = 0$ there exists a non-trivial one, namely

$$g_k^B = \frac{k^2 g_{k_0}^B}{k_0^2 - g_{k_0}^B \int_{k_0}^k k' B_1^B(k') dk'} \quad (10.9)$$

Let us demonstrate that *the solution (10.9) is regular everywhere*. The denominator in eq. (10.9) does not vanish at any k . For this to happen the k' -integral in (10.9) would have to be positive. It turns out however that $B_1^B(k) = B_1^B(g_k^D, \lambda_k^D)$ in the integrand is positive in the small interval $\lambda^D \in [0.213, 0.5]$ only; furthermore the integral $\int_{k_0}^k k' B_1^B(k') dk'$ in (10.9) remains negative even in the corresponding possibly ‘dangerous’ regimes, as for example the IR branch for the type (IIIa)^D trajectories or the spirals in the vicinity of NG₊^D-FP. Fig. 10.2 shows the decrease of $\int_{k_0}^k k' B_1^B(k') dk'$ as a function of k towards the UV, and the fact that it is negative in the far IR, for the three different types of trajectories in the D-sector. In fact, in the UV – when

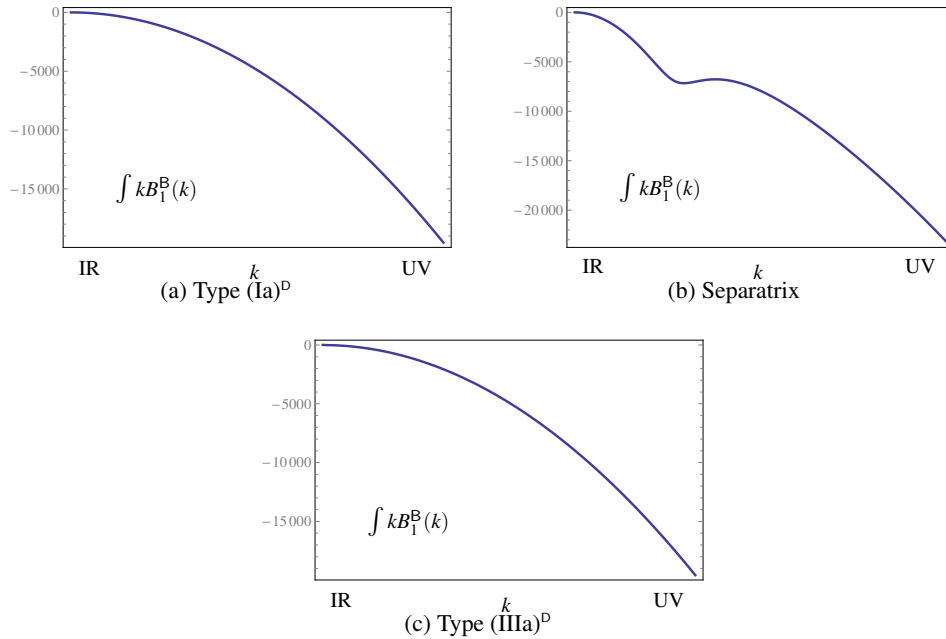


Figure 10.2: The function $k \mapsto \int_{k_0}^k k' B_1^B(k') dk'$ for representative examples of the three types of trajectories. The function is seen to be negative throughout, demonstrating that the solution (10.9) is regular everywhere.

the D-trajectories spiral into NG_+^{D} -FP – the denominator in (10.9) becomes very large so that the k_0^2 term can be neglected. Then (10.9) approaches

$$g_k^{\text{B}} \stackrel{\text{UV}}{=} \frac{k^2}{-\int_{k_0}^k k' B_1^{\text{B}}(k') dk'} \xrightarrow{k \rightarrow \infty} 8.18 \equiv g_*^{\text{B}} \quad (10.10)$$

This limit equals precisely the NG_+^{B} -FP-value of g_k^{B} for any initial value $g_{k_0}^{\text{B}}$. This shows that in the UV, and for any trajectory,

$$\lim_{k' \rightarrow \infty} B_1^{\text{B}}(k') = C \quad \text{with } |C| < \infty \text{ and } C < 0 \quad (10.11)$$

should hold true, where C is some finite constant.

The ‘B’ cosmological constant. Upon inserting the solutions of the D-sector and g_k^{B} we obtain from (10.7b) a single linear, inhomogeneous ODE with scale-dependent coefficients which determines λ_k^{B} :

$$\partial_t \lambda_k^{\text{B}} = A^{\text{B}}(k) g_k^{\text{B}} + [B_1^{\text{B}}(k) g_k^{\text{B}} - 2] \lambda_k^{\text{B}} \quad (10.12)$$

The coefficient functions $A^{\text{B}}(k)$ and $B_1^{\text{B}}(k)$ are fixed once initial conditions are imposed on g_k^{D} and λ_k^{D} . The solution to eq. (10.12) reads then

$$\lambda_k^{\text{B}} = \frac{g_k^{\text{B}}}{g_{k_0}^{\text{B}}} \left(\frac{k_0}{k} \right)^4 \left[\lambda_{k_0}^{\text{B}} + \frac{g_{k_0}^{\text{B}}}{k_0^4} \int_{k_0}^k dk' k'^3 A^{\text{B}}(k') \right] \quad (10.13)$$

where for g_k^{B} the expression (10.9) is to be inserted. Inside the square brackets on the RHS of (10.13) we can distinguish two contributions, of first and of zeroth order in $\lambda_{k_0}^{\text{B}}$, respectively. The moment the denominator in (10.9) starts increasing rapidly, g_k^{B} becomes almost independent of the initial value $g_{k_0}^{\text{B}}$. This renders the term in (10.13) which is of zeroth order in $\lambda_{k_0}^{\text{B}}$ completely independent of the initial data $(\lambda_{k_0}^{\text{B}}, g_{k_0}^{\text{B}})$.

In Fig. 10.3 we display the k -dependence of both contributions separately. From the diagrams we conclude that the function $k \mapsto \lambda_k^{\text{B}}$ is indeed independent of the initial value $g_{k_0}^{\text{B}}$ over a large range of scales. For those k -values the constant contribution, in the approximation of eq. (10.10) given by the fraction $-\frac{\int k^3 A^{\text{B}}(k)}{k^2 \int k B_1^{\text{B}}(k)}$, starts dominating over the $\lambda_{k_0}^{\text{B}}$ -linear term already at small scales k and finally approaches the $\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}$ -FP value of λ^{B} in the UV.

Summary. Recalling that above the same qualitative property was found for g_k^{B} also, we see that *under upward evolution, all solutions $k \mapsto (g_k^{\text{B}}, \lambda_k^{\text{B}})$ of the B-sector ‘forget’ their values at $k = k_0$ and converge to a single trajectory ultimately hitting $\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}$ -FP when moving towards the UV.*

This behavior is related to the observation that for $k \rightarrow \infty$ all solutions in the D-sector spiral into NG_+^{D} -FP (assuming, as always, $g_{k_0}^{\text{D}} > 0$). This fact is a global, and nonlinear extension of the above linear analysis, yielding 2 attractive directions at NG_+^{D} -FP. Since all such solutions share the same fate in the UV, the differential equations for the B-couplings, too, become independent of $(g_{k_0}^{\text{D}}, \lambda_{k_0}^{\text{D}})$ in the UV. In principle the B-couplings could still depend on their own initial values, $(g_{k_0}^{\text{B}}, \lambda_{k_0}^{\text{B}})$. However, we saw that the influence of $g_{k_0}^{\text{B}}$ and $\lambda_{k_0}^{\text{B}}$ is significant only in the IR. Hence, we discover that the UV attractivity of $\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}$ -FP in all 4 directions which previously was established on the basis of the critical exponents at the linearized level only, possesses a nonlinear extension: *The RG flow on the 4-dimensional theory space has*

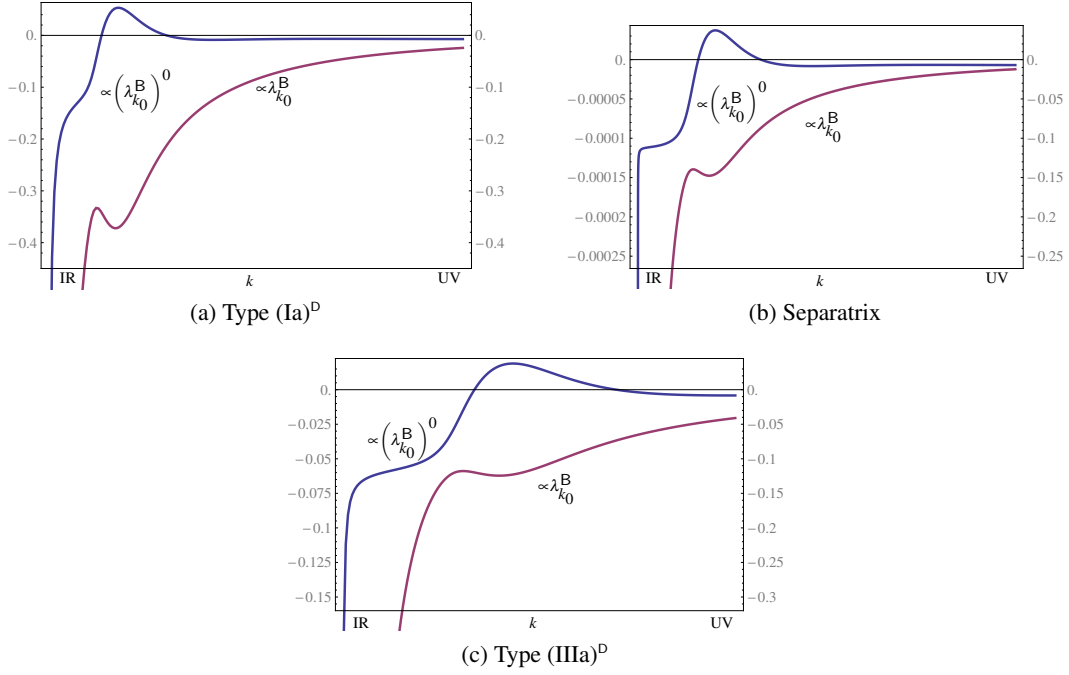


Figure 10.3: The two contributions to the function λ_k^B which are constant and linear in $\lambda_{k_0}^B$, respectively, see eq. (10.13). Shown is the regime of scales when the denominator in (10.9) becomes large. The vertical scales on the LHS (RHS) of the diagrams correspond to the linear (constant) contribution.

the global property that all trajectories approach the doubly non-Gaussian fixed point under upward evolution:

$$(g_k^D, \lambda_k^D, g_k^B, \lambda_k^B) \xrightarrow{k \rightarrow \infty} \text{NG}_+^B \oplus \text{NG}_+^D\text{-FP} \quad \text{for all } (g_{k_0}^D > 0, \lambda_{k_0}^D > 0, g_{k_0}^B > 0, \lambda_{k_0}^B)$$

To be precise, the fixed point's 'basin of attraction' consists of all points with positive dynamical and background Newton constants.

10.2.3 The running UV attractor in the B-sector

We saw that in the D-sector the differential equations are autonomous: The beta-functions are invariant under translations in the RG-time t , and we obtain a time independent phase portrait in the D-sector, see Fig. 10.1. However, after choosing initial values $(g_{k_0}^D, \lambda_{k_0}^D)$ and inserting the resulting Dyn-solution into the 'B'-equations, their translation invariance gets broken and the B-system becomes non-autonomous, depending explicitly on k . As a result, the 2 dimensional phase portrait on the g^B - λ^B plane is explicitly 'time' dependent.

We shall nevertheless be able to deduce the essential qualitative properties of the phase portrait by analytical methods. Its structure is essentially determined by the k -dependent analogue of fixed points in the B-system.

The 'moving fixed point'. We consider the two equations (10.7) for g_k^B and λ_k^B with given, externally prescribed coefficient functions $A^B(k)$ and $B_1^B(k)$ and search for k -dependent points $(g_\bullet^B, \lambda_\bullet^B)$ on the g^B - λ^B -plane at which both beta-functions vanish simultaneously:

$$\begin{aligned} \beta_g^B(g_\bullet^B(k), \lambda_\bullet^B(k); k) &= 0 \\ \beta_\lambda^B(g_\bullet^B(k), \lambda_\bullet^B(k); k) &= 0 \end{aligned} \tag{10.14}$$

As long as $B_1^B(k) \neq 0$, there exist two solutions to these equations.

- A. The first one is trivial, $g_\bullet^B = 0 = \lambda_\bullet^B$, and happens to be independent of k . As a consequence, we expect a 4-dimensional Gaussian fixed point $(g_*^D, \lambda_*^D, g_*^B, \lambda_*^B) = 0$ to be present in all phase diagrams.
- B. The second solution is non-trivial and explicitly k -dependent:

$$g_\bullet^B(k) = -2/B_1^B(k) \quad \text{and} \quad \lambda_\bullet^B(k) = -\frac{1}{2}A^B(k)/B_1^B(k) \quad (10.15)$$

Its k -dependence is shown in Fig. 10.4 for three typical D-trajectories, one of each type, which determine $A^B(k)$ and $B_1^B(k)$. Thus, for $k \rightarrow \infty$ the ‘running fixed point’ $(g_\bullet^B(k), \lambda_\bullet^B(k))$ approaches a true one, namely $\text{NG}_+^B\text{-FP}$.

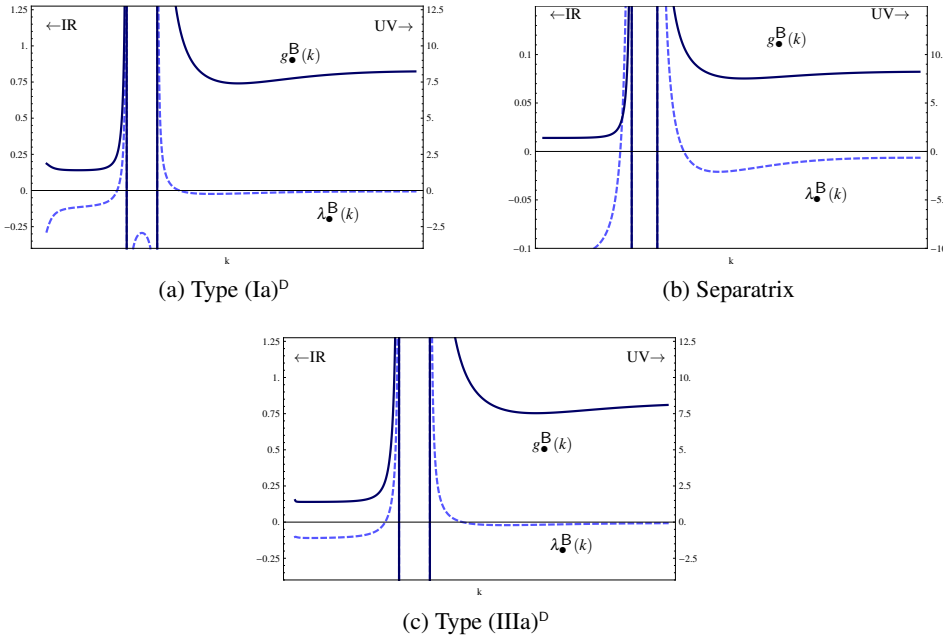


Figure 10.4: The k -dependence of the running UV attractor $(g_\bullet^B(k), \lambda_\bullet^B(k))$ for three representative D trajectories. The left (right) scale corresponds to $\lambda_\bullet^B$ ($g_\bullet^B$). The running attractor $(g_\bullet^B(k), \lambda_\bullet^B(k))$ approaches the $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP coordinates in the UV and finite, non-vanishing values in the IR. In between the running UV attractor touches the boundary of theory space as can be seen from the divergent coordinates. Notice that $(g_\bullet^B(k), \lambda_\bullet^B(k))$ is the position of the ‘sink’ the inverse RG flow in the B-sector is pointing to, but not a solution to the RG equations.

The UV behavior of $(g_\bullet^B(k), \lambda_\bullet^B(k))$ is seen to be the same for all underlying D trajectories (g_k^D, λ_k^D) , namely

$$(g_\bullet^B(k), \lambda_\bullet^B(k)) \xrightarrow{k \rightarrow \infty} \text{NG}_+^B\text{-FP} \quad \text{for all } (g_{k_0}^D > 0, \lambda_{k_0}^D, g_{k_0}^B > 0, \lambda_{k_0}^B) \quad (10.16)$$

Let us consider the embedding of the running fixed point into the 4-dimensional theory space. It moves along a curve parametrized by

$$u_\bullet(k) \equiv (g_k^D, \lambda_k^D, g_\bullet^B(k), \lambda_\bullet^B(k)) \quad (10.17)$$

Note that the curve $k \mapsto u_\bullet(k)$ is not an RG trajectory. Since all D trajectories approach NG_+^D -FP for $k \rightarrow \infty$, it is clear that $u_\bullet(k)$ approaches $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP in the UV. However, its global properties depend significantly on the type of the D-trajectory:

A. The type (IIa)^D-trajectory, the separatrix, for instance, describes a cross-over:

$$G^D\text{-FP} \xleftarrow{k \rightarrow 0} (g_k^D, \lambda_k^D) \xrightarrow{k \rightarrow \infty} NG_+^D\text{-FP} \quad (10.18)$$

Hence the resulting curve (10.17) connects two fixed points, namely the ‘doubly non-Gaussian’ one in the UV, and the mixed $NG_+^B \oplus G^D\text{-FP}$ in the IR:

$$NG_+^B \oplus G^D\text{-FP} \xleftarrow{k \rightarrow 0} u_\bullet(k) \xrightarrow{k \rightarrow \infty} NG_+^B \oplus NG_+^D\text{-FP} \quad (10.19)$$

B. For type (Ia)^D and (IIIa)^D trajectories, where λ^D for $k \rightarrow 0$ goes to $-\infty$ and $+\infty$ ⁶, respectively, the point (10.15) approaches, in the IR,

$$(g_\bullet^B, \lambda_\bullet^B) \xrightarrow{\lambda^D \rightarrow \pm\infty} \left(\frac{3\pi}{5}, -\frac{2}{5} \right) \quad (10.20)$$

At a certain intermediate scale ($0 < k < \infty$), the running attractor touches the boundary of theory space, i.e. it is pulled to infinity⁷, then returns to the interior of theory space, and finally moves towards $NG_+^B\text{-FP}$, under upward evolution. This behavior is clearly seen in the diagrams of Fig. 10.4.

Stability of the ‘moving fixed point’. Let us linearize the B-system (10.7) about $(g_\bullet^B, \lambda_\bullet^B)$ and deduce a k -dependent analogue of a stability matrix, $\mathcal{B}(g_\bullet^B, \lambda_\bullet^B; k)$. This matrix turns out to have two k -dependent eigenvectors $V_\bullet^{(1/2)}(k)$, associated to the ‘would-be critical exponents’, i.e. its negative eigenvalues $\theta_\bullet^{(1)} = 2$ and $\theta_\bullet^{(2)} = 4$, respectively:

$$\mathcal{B}(g_\bullet^B, \lambda_\bullet^B; k) = \begin{pmatrix} -2 & 0 \\ \frac{1}{2}A^B(k) & -4 \end{pmatrix}, \quad V_\bullet^{(1)}(k) = 4\hat{e}_g^B + A^B(k)\hat{e}_\lambda^B, \quad V_\bullet^{(2)}(k) = \hat{e}_\lambda^B \quad (10.21)$$

Here, $\hat{e}_g^B$ and $\hat{e}_\lambda^B$ are unit vectors in the g^B - and λ^B -direction, respectively. As both $\theta_\bullet^{(1)}$ and $\theta_\bullet^{(2)}$ are found to be positive we may conclude that *at all scales the point $(g_\bullet^B, \lambda_\bullet^B)$ is UV-attractive in both B-directions*. No matter which initial conditions we choose for the B-couplings, under upward evolution the B-trajectories $k \mapsto (g_k^B, \lambda_k^B)$ are always pulled towards this point for $k \rightarrow \infty$. Therefore we shall refer to it as the *running UV attractor* and denote it Attr^B or $\text{Attr}^B(k)$ in the following.

Global structure of the B-flow. For $k \rightarrow 0$ the attractor property Attr^B implies that all B-trajectories converge to the values (10.20) in the (Ia)^D and (IIIa)^D cases, and to $NG_+^B \oplus G^D\text{-FP}$ in the case of the separatrix (IIa)^D.

Under upward evolution, when $\text{Attr}^B(k)$ moves due to an increasing k , the B-trajectories try to follow its motion until they all end up in the doubly non-Gaussian fixed point $NG_+^B \oplus NG_+^D\text{-FP}$ for $k \rightarrow \infty$. This behavior is shown in the phase portraits of Figs. 10.5 - 10.7 for representative type (Ia)^D, (IIa)^D, and (IIIa)^D trajectories, respectively.

Each one of the ‘snapshots’ displayed in Figs. 10.5 - 10.7 is structured as follows. The ‘current RG-time’ can be inferred from the position of the five-pointed star on the underlying D-trajectory; it is sketched inside the small box on the right of the phase portrait. The arrows in this larger diagram represent the 2-component vector field $\vec{\beta}_B \equiv (\beta_g^B, \beta_\lambda^B)$ on the g^B - λ^B -plane at

⁶For a moment we ignore the singularity in the beta-functions at $\lambda^D = 1/2$.

⁷Note that there is nothing wrong with diverging values of $g_\bullet^B$ and $\lambda_\bullet^B$ at some k since, as we stressed already, the curve followed by the ‘running fixed point’ is not an RG trajectory. A divergent $g_\bullet^B$ and/or $\lambda_\bullet^B$ simply means that at this special value of k there exists no such fixed point.

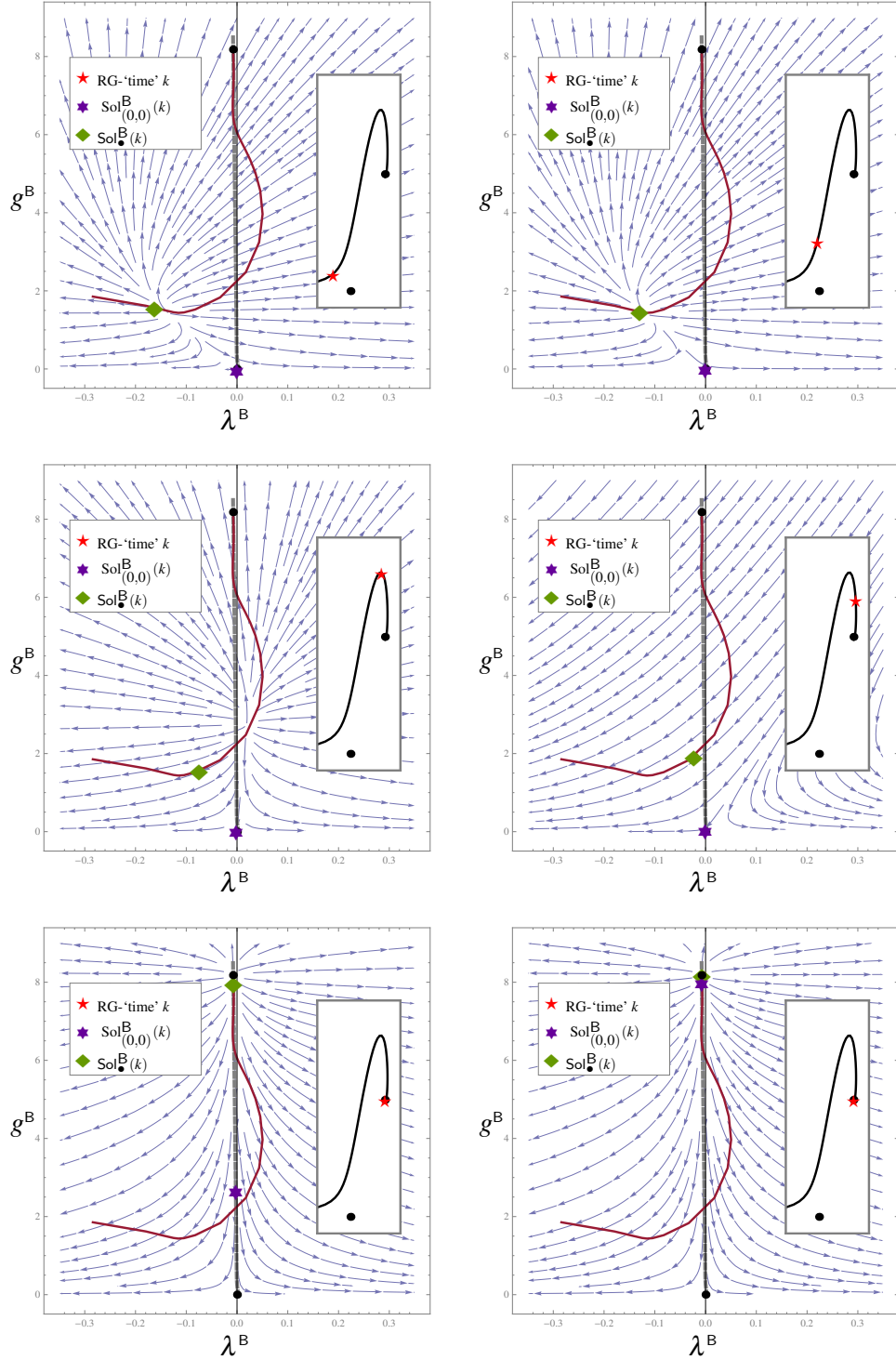


Figure 10.5: The B-phase portraits at increasing scales k . The underlying type $(Ia)^D$ trajectory in the D-sector is shown in the inset on the right, and the current RG time is marked with a star therein. The arrows point towards the IR and picture the instantaneous vector field in the B-sector. The (red) solid and the (gray) dashed curve highlight two important solutions in the B-sector, namely $\text{Sol}^B_{\bullet}(k)$ and $\text{Sol}^B_{(0,0)}(k)$, respectively. Their current position is indicated by the (green) diamond and the (violet) six-pointed star, respectively.

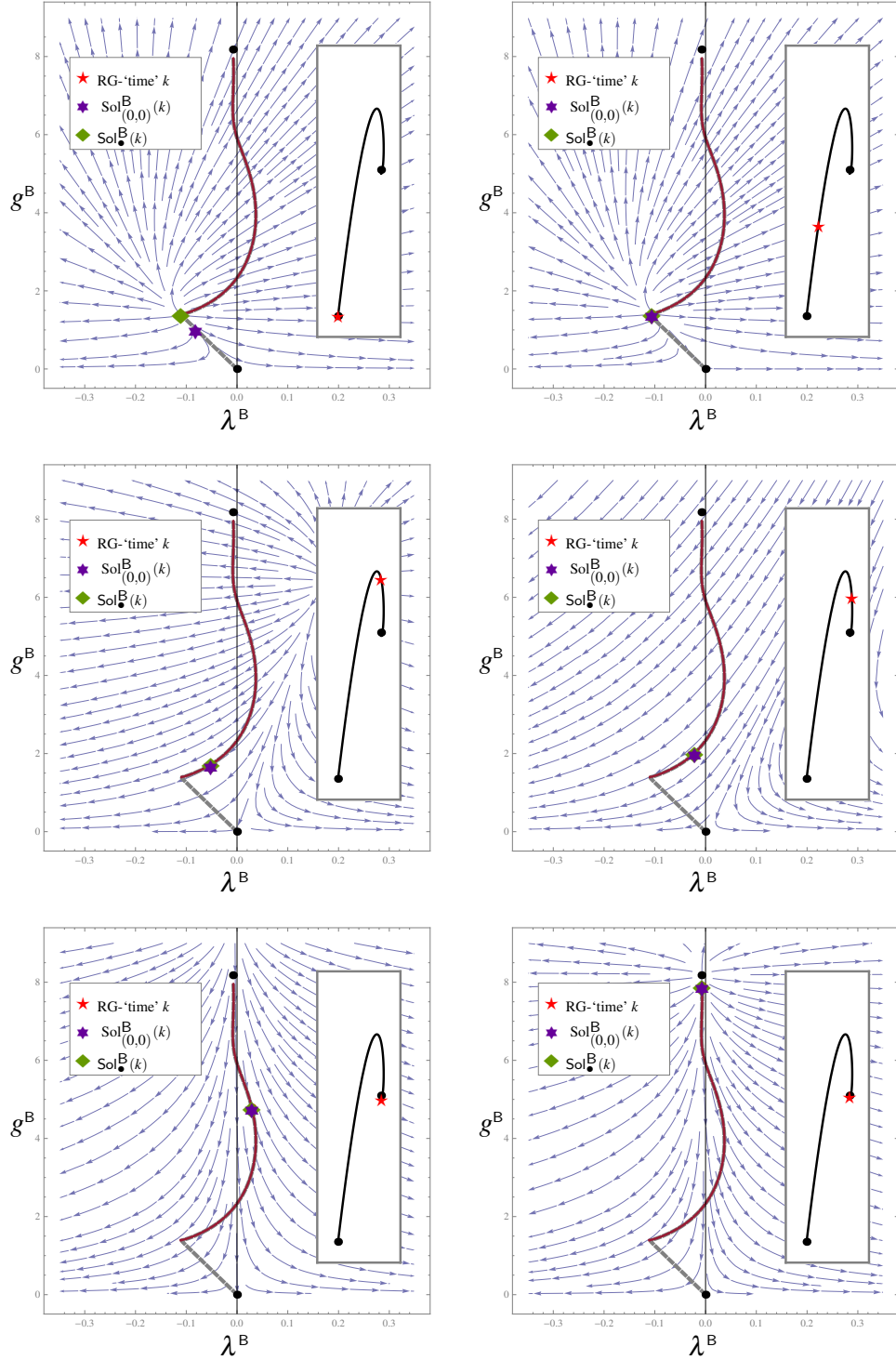


Figure 10.6: The B-phase portraits at increasing scales k . The underlying separatrix in the D-sector is shown in the inset on the right, and the current RG time is marked with a star therein. The arrows point towards the IR and picture the instantaneous vector field in the B-sector. The (red) solid and the (gray) dashed curve highlight two important solutions in the B-sector, namely $\text{Sol}_\bullet^B(k)$ and $\text{Sol}_{(0,0)}^B(k)$, respectively. Their current position is indicated by the (green) diamond and the (violet) six-pointed star, respectively.

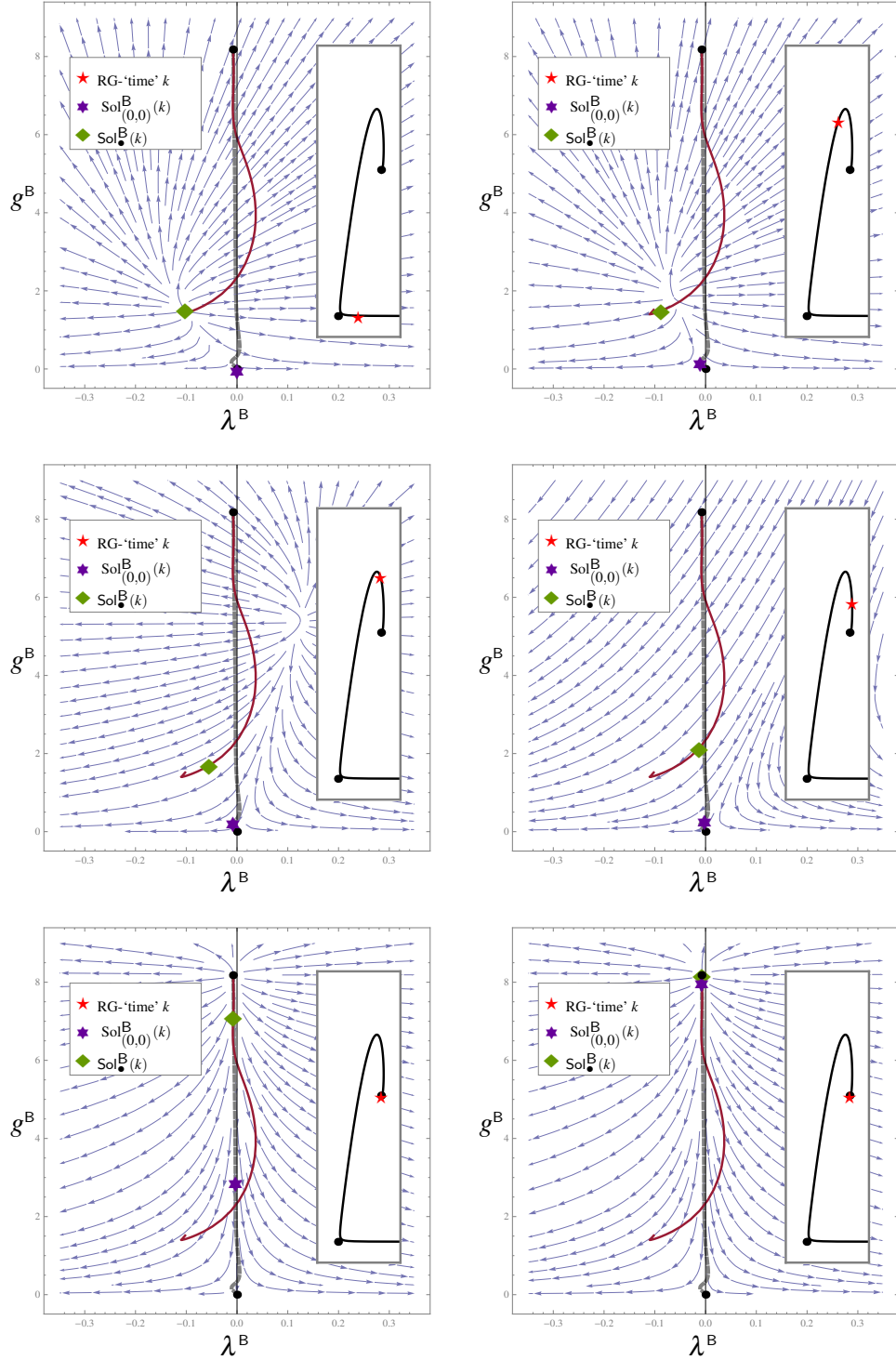


Figure 10.7: The B-phase portraits at increasing scales k . The underlying type $(\text{IIIa})^D$ trajectory in the D-sector is shown in the inset on the right, and the current RG time is marked with a star therein. The arrows point towards the IR and picture the instantaneous vector field in the B-sector. The (red) solid and the (gray) dashed curve highlight two important solutions in the B-sector, namely $\text{Sol}^B_{\bullet}(k)$ and $\text{Sol}^B_{(0,0)}(k)$, respectively. Their current position is indicated by the (green) diamond and the (violet) six-pointed star, respectively.

this particular instant of time. The instantaneous integral curves of this vector field are shown, too; because of its time dependence, those integral curves are no RG trajectories, however.

Furthermore, information about a single, especially interesting RG trajectory is provided by the time dependent location of the six-pointed star which can be seen in all snap shots. It indicates the current position of the solution (g_k^B, λ_k^B) of the RG equations which is fixed by the ‘final condition’ $\lim_{k \rightarrow 0} (g_k^B, \lambda_k^B) = (0, 0)$. This solution is denoted $\text{Sol}_{(0,0)}^B(k)$ in the diagrams. The dashed curve, for clarity shown in the plots at any time, is the set of points visited by $\text{Sol}_{(0,0)}^B(k)$ for $0 \leq k < \infty$. It is a true RG trajectory, i.e. a solution to the eqs. (10.7).

Similarly, we included in all phase portraits a (red) curve that shows another genuine RG trajectory, denoted $\text{Sol}_\bullet^B(k)$. It is the solution picked by the ‘final condition’ $\lim_{k \rightarrow 0} (g_k^B, \lambda_k^B) = \lim_{k \rightarrow 0} (g_\bullet^B(k), \lambda_\bullet^B(k))$. In other words, under upward evolution it grows out of the UV attractor, with which it coincides at $k = 0$.

As time elapses from $k = 0$ to $k = \infty$, we expect $\text{Sol}_\bullet^B(k)$ and $\text{Attr}^B(k)$ to follow different paths, since the former curve is an RG trajectory, while the latter is not. This characteristic feature can be clearly observed in the diagrams. The trajectory $\text{Sol}_\bullet^B(k)$ ultimately gets pulled into the doubly non-Gaussian fixed point for $k \rightarrow \infty$:

$$\text{Attr}^B(0) \oplus (\dots)^D \xleftarrow{k \rightarrow 0} \text{Sol}_\bullet^B(k) \xrightarrow{k \rightarrow \infty} \text{NG}_+^B \oplus \text{NG}_+^D\text{-FP}$$

Here $(\dots)^D$ stands for the various possible IR regimes of the underlying dynamical trajectory. The RG-trajectory $\text{Sol}_\bullet^B(k)$ is an especially important one since *all trajectories converge towards $\text{Sol}_\bullet^B(k)$ when k is increased*.

In the snapshots of Fig. 10.6, obtained from the D-separatrix, this convergence is clearly seen to occur for the trajectory which, under upward evolution, begins in the doubly Gaussian fixed point $G^B \oplus G^D\text{-FP}$ at $k = 0$. This solution coincides with the attractor’s position already at a rather low RG-scale where $(g_\bullet^B, \lambda_\bullet^B)$ has not yet moved much and seems to be k -independent. Only after both trajectories have merged, $(g_\bullet^B, \lambda_\bullet^B)$ actually starts running rapidly, and the remaining evolution towards $k \rightarrow \infty$ can be entirely described by the red curve that sits on top of the dashed one.

In the plots the attraction towards $(g_\bullet^B, \lambda_\bullet^B)$ is well visible since the running UV attractor is first heading for infinity in the vertical direction, then returns, moves to $\lambda^D \approx 0$, and lowers its g^D value until it ultimately reaches $\text{NG}_+^B\text{-FP}$. This motion of the running attractor reflects itself in the bow of the trajectories.

In Fig. 10.5, pertaining to a $(\text{Ia})^D$ trajectory, this feature is less pronounced: The $\text{Sol}_\bullet^B$ -solution remains close to the Gaussian fixed point value for a long period of RG-time, and the confluence of both curves takes place in the far UV only.

Finally, Fig. 10.7 for the $(\text{IIIa})^D$ case looks very similar to the type $(\text{Ia})^D$ result and most of the significant running in the B-sector takes place in the UV only. In the IR we went down only to RG-scales which are such that λ_k^D is still well separated from the singular boundary of theory space at $\lambda^D = 1/2$.

Summary of the attractor mechanism. In Fig. 10.8 we give a schematic description of the attractor mechanism which we uncovered in the ‘snapshots’. It is sufficient to focus on the (upward!) evolution of $\text{Attr}^B(k)$ and its most interesting ‘follower’ the RG trajectory $\text{Sol}_\bullet^B(k)$. The dashed (black) and solid (red) curves describe the footprints of $\text{Attr}^B(k)$ and $\text{Sol}_\bullet^B(k)$, respectively, during their full evolution from $k = 0$ to $k \rightarrow \infty$. In the upper right part of the plot the (orange) dotted curve adumbrates the boundary of (g^B, λ^B) -space where the B-couplings diverge. The clocks mark equal-time positions on both curves; their black filling indicates the RG time elapsed since they left their common initial point, $\text{Attr}^B(k = 0)$. The attraction of $\text{Sol}_\bullet^B(k')$ to the current position of $\text{Attr}^B(k')$ is indicated by the dashed (blue) arrows.

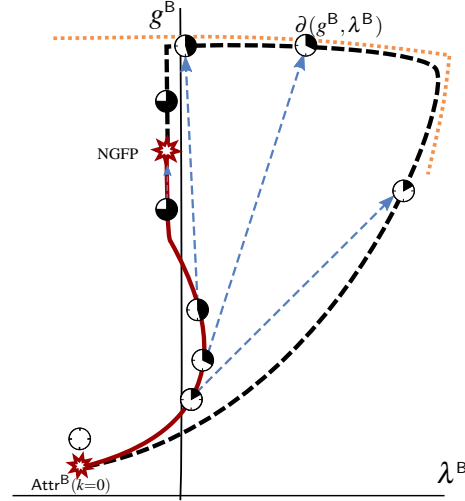


Figure 10.8: The dashed (black) and solid (red) curve show the k -dependent positions of $\text{Attr}^B(k)$ and $\text{Sol}^B(k)$ on the g^B - λ^B -plane. Recall that $\text{Sol}^B(k)$ is an RG trajectory while $\text{Attr}^B(k)$ is not. The (orange) dotted curve indicates the boundary ' $\partial(g^B, \lambda^B)$ ' where the B-couplings diverge. The clocks mark equal-time positions on the curves; the black filling indicates the elapsed RG time for upward evolution. The dashed (blue) arrows indicate the direction in which $\text{Sol}^B(k')$ is pulled at 'time' k' , namely the position of the attractor at this instant of time, $\text{Attr}^B(k')$. While initially, at $k = 0$, the trajectory $\text{Sol}^B(0)$ coincides with $\text{Attr}^B(0)$ the curves depart due to their different velocities. They meet again at $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP for $k \rightarrow \infty$. However, the motion of $\text{Attr}^B(k)$ at intermediate scales is encoded in the indentation of the $\text{Sol}^B(k)$ curve before it approaches $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP.

Notice that the two tracks meet only in the IR, at $\text{Attr}^B(0)$ (red star on the bottom), and at $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP in the UV. This is due to the following fact. While $\text{Sol}^B(k')$ is an RG trajectory whose velocity is determined by the beta-functions, $\text{Attr}^B(k)$ is a k -dependent solution to a 'non-evolution' equation. The latter moves to the boundary of (g^B, λ^B) -space for some intermediate scales, but rapidly turns back to finite values of g^B and λ^B , and then slowly approaches $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP from above. On the other hand, the RG trajectories, in particular $\text{Sol}^B(k)$, have a smaller velocity and thus only see the 'tailamp' of $\text{Attr}^B(k)$ with which they try to catch up. The journey of $\text{Attr}^B(k)$ to the boundary and back is reflected in the indentation of the $\text{Sol}^B(k)$ curve before it approaches $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP.

Summary: We have seen that the RG-evolution in the B-sector is crucially determined by the scale dependent UV attractor $\text{Attr}^B(k)$. Whereas, in the UV, the RG-trajectories have no other choice but ultimately run into $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP for $k \rightarrow \infty$, they differ strongly in their IR behavior, in particular in the way they approach the physical point $k = 0$. Following the trajectory backward, i.e. for increasing k , the dependence on their 'initial' point $(g_{k_0}^{D/B}, \lambda_{k_0}^{D/B})$ reduces the more the closer (g_k^B, λ_k^B) gets to the running UV attractor $(g_\bullet^B(k), \lambda_\bullet^B(k))$.

10.2.4 Split-symmetry restoration in the physical limit $k \rightarrow 0$

We will now search for RG trajectories which comply with the requirement of split-symmetry restoration in the IR, i.e. when k is lowered towards the physical point $k = 0$.

Recall that split-symmetry is a property of idealized solutions of the FRGE where $\Gamma_k[h; \bar{g}]$ reduces to a functional of a single field, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. If fully intact in the truncation ansatz (11.52), its $G_k^{(p)}$'s and $\bar{\lambda}_k^{(p)}$'s are the same then at all levels, $p = 0, 1, 2, \dots$. In B-D language, this

is tantamount to saying that $\Gamma_k^{\text{grav}}[g, \bar{g}]$ loses its ‘extra $\bar{g}$ -dependence’ so that, for $p = 1, 2, 3, \dots$:

$$\frac{1}{G_k^{(p)}} \equiv \frac{1}{G_k^D} \stackrel{!}{=} \frac{1}{G_k^{(0)}} \equiv \frac{1}{G_k^D} + \frac{1}{G_k^B} \Leftrightarrow \frac{1}{G_k^B} \equiv \frac{k^2}{g_k^B} \stackrel{!}{=} 0 \quad (10.22a)$$

$$\frac{\bar{\lambda}^{(p)}}{G_k^{(p)}} \equiv \frac{\bar{\lambda}^D}{G_k^D} \stackrel{!}{=} \frac{\bar{\lambda}_k^{(0)}}{G_k^{(0)}} \equiv \frac{\bar{\lambda}^D}{G_k^D} + \frac{\bar{\lambda}_k^B}{G_k^B} \Leftrightarrow \frac{\bar{\lambda}_k^B}{G_k^B} \equiv k^4 \frac{\lambda_k^B}{g_k^B} \stackrel{!}{=} 0 \quad (10.22b)$$

Clearly we cannot expect those conditions to hold everywhere along a trajectory, at best, and only approximately, in a restricted regime of scales. After all, split-symmetry is broken explicitly both by the gauge fixing and the cutoff term. It is desirable, however, to base the construction of QEG on an asymptotically safe trajectory which reinstalls split-symmetry as exactly as possible⁸ for $k \rightarrow 0$: In this limit the EAA approaches the standard effective action whose n -point functions, taken on-shell, are related to observable S -matrix elements.

By eqs. (10.22), approximate split-symmetry demands G_k^B to be very ‘large’, and $\bar{\lambda}^B$ to be very ‘small’, in an appropriate sense. Note that these are conditions on the B-couplings only, the D ones are left unconstrained.

If the relations (10.22) indeed hold true for all k in some interval (k_1, k_2) , the k -differentiated relations are satisfied, too. They require the beta-functions of the dimensionful B-couplings to vanish; from (10.8):

$$\partial_t(1/G_k^B) = -k^2 B_1^B(\lambda_k^D, g_k^D) \stackrel{!}{=} 0 \quad (10.23a)$$

$$\partial_t(\bar{\lambda}_k^B/G_k^B) = k^4 A^B(\lambda_k^D, g_k^D) \stackrel{!}{=} 0 \quad (10.23b)$$

The conditions (10.23) guarantee the stability of (10.22) under the RG-evolution. Notice that, by them, the running of the B-couplings is not explicitly restricted, rather they put constraints on the D quantities λ_k^D and g_k^D . This is just opposite as above.

When we try to solve (10.23) by finding simultaneous zeros of $B_1^B \stackrel{!}{=} 0$ and $A^B \stackrel{!}{=} 0$ we find that (on the $g^D > 0$ half-plane) there exists *only one* such zero, namely the point $(g_{\text{zero}}^D, \lambda_{\text{zero}}^D) \approx (0.708, 0.207)$. This shows clearly that we have to abandon the idea of finding a *full* trajectory that preserves split-symmetry, but rather look for RG-trajectories that restore split-symmetry in the physical limit $k \rightarrow 0$ at least, which is perfectly sufficient.

Nevertheless, it is quite remarkable that the point $(g_{\text{zero}}^D, \lambda_{\text{zero}}^D)$ is strikingly close to the NG_+^D -FP fixed point which we located at $(g_*^D, \lambda_*^D) \approx (0.703, 0.207)$. There is no obvious general reason for this ‘miracle’ to happen.

Split-symmetric ‘final conditions’ in the B-sector

From now on we shall be modest and try to establish split-symmetry at $k = 0$ only. In order to explore the implications of (10.22) for this case let us assume we have solved the differential equations of the D-sector, found all trajectories $k \mapsto (g_k^D, \lambda_k^D)$, and labeled them by their position $(g_{k_0}^D, \lambda_{k_0}^D)$ at some intermediate scale, $0 < k_0 < \infty$. Then, inserting the D-trajectories into the flow eqs. (10.8b) for g_k^B and λ_k^B , our task is to identify those initial, or more appropriately, final conditions for the B couplings that lead to intact split-symmetry in the physical limit $k \rightarrow 0$.

⁸Which is not to say, *fully*. The gauge fixing dependent contents of Γ_k which never makes its way into observables may remain split-symmetry violating. Note that in the present truncation this ‘gauge fixing dependent contents’ is the $\Gamma^{\text{gf}} + \Gamma^{\text{gh}}$ part of the EAA which never becomes split-symmetric, of course. Our discussion concerns only the Γ^{grav} -part of the EAA ansatz.

Taking advantage of the explicit solution to the two B-equations given in (10.9) and (10.13) the requirement of split-symmetry, at some k which is still arbitrary, assumes the form

$$\frac{1}{G_k^B} = k^2 \frac{1}{g_k^B} = \frac{k_0^2}{g_{k_0}^B} - \int_{k_0}^k k' B_1^B(k') dk' \stackrel{!}{=} 0 \quad (10.24a)$$

$$\frac{\bar{\lambda}^B}{G_k^B} = k^4 \frac{\lambda_k^B}{g_k^B} = k_0^4 \frac{\lambda_{k_0}^B}{g_{k_0}^B} + \int_{k_0}^k dk' k'^3 A^B(k') \stackrel{!}{=} 0 \quad (10.24b)$$

We want split-symmetry to be intact at $k = 0$, so we now let $k \rightarrow 0$ in the conditions (10.24), while keeping k_0 strictly nonzero throughout. Then these conditions *uniquely* fix ‘initial’ values $(G_{k_0}^B, \bar{\lambda}_{k_0}^B)$ for the dimensionful⁹ background couplings at $k = k_0$:

$$G_{k_0}^B = g_{k_0}^B / k_0^2 = - \left(\int_0^{k_0} k' B_1^B(k') dk' \right)^{-1} \quad (10.25a)$$

$$\bar{\lambda}_{k_0}^B = \lambda_{k_0}^B k_0^2 = G_{k_0}^B \int_0^{k_0} dk' k'^3 A^B(k') \quad (10.25b)$$

Here $B_1^B(k) \equiv B_1^B(g_k^D, \lambda_k^D)$ and $A^B(k) \equiv A^B(g_k^D, \lambda_k^D)$ depend manifestly on the D trajectory under consideration.

This result is good news for the Asymptotic Safety program in a twofold way: First, there does indeed exist a trajectory in the B-sector which complies with the requirement of split-symmetry at $k = 0$, but second, there is *only one* such trajectory; as a consequence, when solving the RG equations for the B-couplings there are no constants of integration that could be chosen freely, and this increases the predictivity of the theory.

Another remarkable feature of the initial conditions (10.25) is their close relationship to the running attractor $\text{Attr}^B(k)$, to which we turn next.

The $k \rightarrow 0$ asymptotics of the D trajectories

As a necessary preparation for the exploration of the split-symmetry restoration of the full 4-dimensional system and to demonstrate the role played by the running attractor we first summarize the IR behavior of (g_k^D, λ_k^D) along trajectories of the three types, (Ia)^D, (IIa)^D, and (IIIa)^D, respectively.

Type (Ia)^D: This type of trajectories has the defining property that $\lambda_k^D \xrightarrow{k \rightarrow 0} -\infty$. If we consider the D-system for very large negative (positive) values of λ_k^D we obtain the following asymptotic solutions (for $g_{k_0}^D > 0$):

$$g_k^D = \frac{2\pi g_{k_0}^D k^2}{g_{k_0}^D (k_0^2 - k^2) + 2\pi k_0^2} \xrightarrow{k \rightarrow 0} 0 \quad (10.26)$$

$$\lambda_k^D = \frac{g_{k_0}^D (k_0^4 - k^4) - 6\pi k_0^4 \lambda_{k_0}^D}{3k^2 (g_{k_0}^D (k^2 - k_0^2) - 2\pi k_0^2)} \xrightarrow{k \rightarrow 0} \pm\infty \quad (10.27)$$

Here the sign for the limit of λ_0^D agrees with that of $\lambda_{k_0}^D \text{sign}(6\pi\lambda_{k_0}^D - g_{k_0}^D)$. For type (Ia)^D trajectories the initial point $(g_{k_0}^D, \lambda_{k_0}^D)$ is chosen such that the IR-limit of λ_k^D is negative. (The

⁹Note that because of the explicit factors of k^2 and k^4 , respectively, which relate dimensionless and dimensionful quantities in eqs. (10.24) the implications of split-symmetry in the limit $k \rightarrow 0$ are discussed most easily in *dimensionful* terms.

positive sign in (10.27) applies to the type (IIIa)^D we will discuss below.) The corresponding *dimensionful* D-couplings have the following asymptotic behavior:

$$\frac{1}{G_k^D} = \frac{g_{k_0}^D (k_0^2 - k^2) + 2\pi k_0^2}{2\pi g_{k_0}^D} \xrightarrow{k \rightarrow 0} \frac{1}{G_{k_0}^D} + \frac{k_0^2}{2\pi} \quad (10.28a)$$

$$\frac{\bar{\lambda}_k^D}{G_k^D} = k_0^4 \left[\lambda_{k_0}^D / g_{k_0}^D - \frac{1}{6\pi} (1 - (k/k_0)^4) \right] \xrightarrow{k \rightarrow 0} \frac{\bar{\lambda}_{k_0}^D}{G_{k_0}^D} - \frac{k_0^4}{6\pi} \quad (10.28b)$$

The IR-limit of G_k^D in eq. (10.28a) will later on be used to define the Planck mass for the class of type (Ia)^D trajectories.

Type (IIa)^D: Along the separatrix the dimensionless cosmological and Newton's constant approach zero in the IR: $\lambda_k^D \xrightarrow{k \rightarrow 0} 0$ and $g_k^D \xrightarrow{k \rightarrow 0} 0$. Linearizing around the values $g^D = 0 = \lambda^D$ we find

$$g_k^D = g_{k_0}^D (k/k_0)^2 \xrightarrow{k \rightarrow 0} 0 \quad 1/G_k^D = \frac{k_0^2}{g_{k_0}^D} \xrightarrow{k \rightarrow 0} \frac{k_0^2}{g_{k_0}^D} \quad (10.29a)$$

$$\lambda_k^D = \frac{3}{8\pi} g_{k_0}^D (k/k_0)^2 \xrightarrow{k \rightarrow 0} \pm 0 \quad \bar{\lambda}_k^D / G_k^D = \frac{3}{8\pi} k^4 \xrightarrow{k \rightarrow 0} 0 \quad (10.29b)$$

The initial values $\lambda_{k_0}^D$ and $g_{k_0}^D$, imposed near the Gaussian fixed point, are constrained by the 'separatrix condition' $\lambda_{k_0}^D = (3/8\pi) g_{k_0}^D$.

Type (IIIa)^D: The type (IIIa)^D trajectories suffer from the well known problem that they run into a divergence of the beta-functions and terminate at some low but nonzero scale k_{term} when λ_k^D reaches the value 1/2: Neither within the single- nor the bi-metric Einstein-Hilbert truncation their full extension to $k = 0$ can be computed. Nevertheless, the situation is fairly clear for the trajectories of interest, namely those that before hitting the singularity enjoy a long classical regime [207–209] in which $G_k^D \equiv G_{\text{cl.reg.}}^D$ and $\bar{\lambda}^D \equiv \bar{\lambda}_{\text{cl.reg.}}^D$ are approximately constant. Their dimensionless counterparts are then given by, for $k \gg k_{\text{term}}$ in the classical regime ('cl.reg.'),

$$g_k^D = G_{\text{cl.reg.}}^D k^2 \quad (10.30a)$$

$$\lambda_k^D = \Lambda_{\text{cl.reg.}}^D / k^2 \quad (10.30b)$$

For the purposes of the present paper we hypothesize that in reality the classical regime even extends to scales $k \rightarrow 0$. The running (10.30) corresponds to that found in (10.27) then.

The attractor mechanism of split-symmetry restoration

Next we are going to insert the various types of asymptotic (for $k \rightarrow 0$) solutions $k \mapsto (g_k^D, \lambda_k^D)$ into $B_1^B(k) \equiv B_1^B(g_k^D, \lambda_k^D)$ and $A^B(k) \equiv A^B(g_k^D, \lambda_k^D)$. Then we use the resulting functions in eqs. (10.25) for the initial values $(G_{k_0}^B, \bar{\lambda}_{k_0}^B)$ whereby we can perform the two k' -integrals analytically. For the asymptotic formulae to be sufficiently good approximations in the integrands we must choose k_0 very close to zero (more precisely, much smaller than the Planck scale, $k_0 \ll (G_0^D)^{-1/2}$, see below). To convince yourself that the use of the asymptotic solutions is indeed permissible then, and to see what it entails, notice also the following facts:

- A. Since g_k^D approaches zero in the IR for all three classes of D trajectories, contributions to B_1^B and A^B containing the anomalous dimension $\eta^D \propto g^D$ produce only terms of subleading order in the asymptotic expansion, which we may neglect. This implies in particular $q_n^D(-2\lambda_k^D) \approx \Phi_n^D(-2\lambda_k^D)$ for $g_k^D \rightarrow 0$.

- B.** For the large values of λ_k^D which occur in IR of type (Ia)^D and (IIIa)^D trajectories we may exploit that $\lim_{\lambda_k^D \rightarrow \pm\infty} \Phi_n^D(-2\lambda_k^D) = 0$ for $n \geq 1$. Only the ghost terms contribute to A^B and B_1^B . For the separatrix we instead use $\lim_{\lambda_k^D \rightarrow 0} \Phi_n^D(-2\lambda_k^D) = \Phi_n^D(0)$ in lowest order.
- C.** The ghost contributions to both B_1^B and A^B are unaffected by any approximation in the D-sector, simply because they do not depend on the D couplings at all and are thus k -independent and independent of the D initial conditions.

Going through the explicit formulae for B_1^B and A^B it is now easy to check that, as a consequence of these three simplifying properties, we are entitled to perform the integrals (10.25) in the far IR by substituting *constant* functions $B_1^B(k) \approx B_1^B(0)$ and $A^B(k) \approx A^B(0)$. In this manner we find that the initial values of the B couplings that lead to a fully intact split-symmetry at $k = 0$, are given by

$$G_{k_0}^B = (-2/B_1^B(0))/k_0^2 = g_\bullet^B(0)/k_0^2 \quad (10.31a)$$

$$\bar{\lambda}_{k_0}^B = (-\frac{1}{2}A^B(0)/B_1^B(0))k_0^2 = \lambda_\bullet^B(0)k_0^2 \quad (10.31b)$$

This is an important result, and various comments are in order here:

- A.** By comparing the initial values (10.31) to the coordinates (10.15) of $\text{Attr}^B(k)$ in the limit $\lambda^D \rightarrow \pm\infty$ or $\lambda^D \rightarrow 0$, respectively, we find that *the split-symmetry restoring initial point $(g_{k_0}^B, \lambda_{k_0}^B)$ coincides exactly with the location of the running UV attractor $(g_\bullet^B, \lambda_\bullet^B)$ for $k = 0$.*
- B.** In the first place, the result demonstrates that *there exists indeed a fully extended RG trajectory, well behaved at all scales between zero and infinity.* It defines an asymptotically safe theory, hitting a NGFP for $k \rightarrow \infty$, and at the same time restores split-symmetry in the physical limit $k \rightarrow 0$ when all fluctuations are integrated out. At least numerically, we can compute this trajectory for all $k \in [0, \infty)$. In this way we have verified that the trajectory lies indeed on the UV critical hypersurface of the doubly non-Gaussian fixed point, $\text{NG}_+^B \oplus \text{NG}_+^D\text{-FP}$.
- C.** *There exists one, and only one set of initial values in the B-sector, $(g_{k_0}^B, \lambda_{k_0}^B)$, that leads to split-symmetry at $k = 0$.* (The uniqueness follows from the fact that the running UV attractor is IR repulsive in all directions.) This has the positive side effect that the number of free parameters that characterize the asymptotically safe quantum theories one can construct does not increase when we generalize the 2-parameter single-metric Einstein-Hilbert truncation to the 4-parameter bi-metric one. In fact, the newly introduced parameters immediately get ‘eaten up’ by the necessity to turn the split-symmetry violations to zero at $k = 0$.
- D.** For type (Ia)^D and (IIIa)^D trajectories the presence of the ghost contributions ($\rho_{\text{gh}} = 1$) is found to be essential for the existence of the symmetry restoring point. Making ρ_{gh} explicit, the corresponding IR solutions assume the form

$$\frac{1}{G_k^B} \xrightarrow{k \rightarrow 0} \frac{1}{G_{k_0}^B} - \frac{5\rho_{\text{gh}}}{3\pi} k_0^2 \quad (10.32a)$$

$$\frac{\bar{\lambda}_k^B}{G_k^B} \xrightarrow{k \rightarrow 0} \frac{\bar{\lambda}_{k_0}^B}{G_{k_0}^B} + \frac{2\rho_{\text{gh}}}{3\pi} k_0^4 \quad (10.32b)$$

From here the split-symmetry restoring initial values $G_{k_0}^B = 3\pi/(5k_0^2\rho_{\text{gh}})$ and $\bar{\lambda}_{k_0}^B = -2/5$ can be deduced. Omitting the ghost terms ($\rho_{\text{gh}} = 0$) split-symmetry restoration in the IR would require $G_{k_0}^B \rightarrow \infty$ at a nonzero k_0 ! Note also that the corresponding dimensionless initial data, for $\rho_{\text{gh}} = 1$, are $(g_{k_0}^B, \lambda_{k_0}^B) = (3\pi/5, -2/5)$ which, by (10.21), is exactly the IR position of the attractor, $\text{Attr}^B(k \rightarrow 0)$.

- E.** For the separatrix there are additional graviton contributions shifting the initial data towards smaller values:

$$\frac{1}{G_k^B} \xrightarrow{k \rightarrow 0} \frac{1}{G_{k_0}^B} - \frac{(7 + 20\rho_{\text{gh}})}{12\pi} k_0^2 \quad (10.33a)$$

$$\frac{\bar{\lambda}^B}{G_k^B} \xrightarrow{k \rightarrow 0} \frac{\bar{\lambda}_{k_0}^B}{G_{k_0}^B} - \frac{(5 - 8\rho_{\text{gh}})}{12\pi} k_0^4 \quad (10.33b)$$

‘Switching on’ the ghosts ($\rho_{\text{gh}} = 1$) the split-symmetry restoration happens at $(g_{k_0}^B, \lambda_{k_0}^B) = (4\pi/9, -1/9)$. These are precisely the B-coordinates of the fixed point $\text{NG}_+^B \oplus \text{G}^D$ -FP, which in turn equals the $k \rightarrow 0$ limit of $\text{Attr}^B(k)$ for the type (IIa)^D trajectory.

Summary: For every RG trajectory of the D-sector there exists precisely one associated trajectory of the B couplings which restores split-symmetry for $k \rightarrow 0$. In the IR, the B trajectory approaches the (UV attractive, i.e., IR repulsive) running attractor $\text{Attr}^B(k)$. For $k \rightarrow \infty$, the combined 4-dimensional trajectory is asymptotically safe and runs into the fixed point $\text{NG}_+^B \oplus \text{NG}_+^D$ -FP.

10.2.5 All classes of split-symmetry restoring trajectories

In this subsection we provide a survey of all classes of RG trajectories with restored split-symmetry in the IR, and we present explicit examples. In the D-sector, we do have the freedom of choosing initial data $g_{k_0}^D$ and $\lambda_{k_0}^D$ at some $k = k_0$, and so we will select a typical representative of each type (Ia)^D, (IIa)^D, and (IIIa)^D, respectively. Once such a solution is picked we have no further freedom: the values of $g_{k_0}^B$ and $\lambda_{k_0}^B$ are then uniquely determined by the requirement of split-symmetry at $k = 0$, and so there is a unique ‘lift’ of the 2-dimensional D trajectory to the 4-dimensional theory space. The resulting trajectories will be referred to as of type (Ia)^D-Attr^B, (IIa)^D-Attr^B, and (IIIa)^D-Attr^B, respectively. Clearly the last type is the most interesting one since it is closest to real Nature, presumably.

In the following we shall always express the RG-scale k and the dimensionful couplings in units of the Planck mass defined by the dynamical Newton constant: $m_{\text{Pl}} = [G_{k_{\text{IR}}}^D]^{-1/2}$. For the infrared normalization scale k_{IR} we choose $k_{\text{IR}} = 0$ for the type (Ia)^D and (IIa)^D trajectories. In the (IIIa)^D case where we cannot follow the evolution down to $k = 0$ we choose for k_{IR} a value in the (semi-)classical regime where G_k^D and $\bar{\lambda}_k^D$ are approximately constant (lower horizontal branch of the trajectory). In principle the notion of a ‘Planck mass’ depends on the level of the Newton constant used to define it. In our case there is no ambiguity since we enforce split-symmetry in the IR. Hence all $G_k^{(p)}$ give rise to the same Planck mass:

$$m_{\text{Pl}} = \lim_{k \rightarrow 0} (1/\sqrt{G_k^{(p)}}) \equiv 1/\sqrt{G_0^{(0)}} \equiv 1/\sqrt{G_0^D}$$

This definition depends on the chosen trajectory, however.

Next we present the k -dependence of all running couplings, both in dimensionless and dimensionful form, for one representative numerical solution in each class of trajectories.¹⁰ For comparison we include the single-metric result for the Einstein-Hilbert truncation [122] using the same initial data as for the D couplings. The diagrams in Figs. 10.9-10.11, 10.12-10.14, and 10.15-10.17, respectively, are devoted to the (Ia)^D-, (IIa)^D-, and (IIIa)^D-Attr^Btrajectories.

¹⁰For the ‘initial’ conditions at the scale $k_0 = 1$ we set $(g_{k_0}^D, \lambda_{k_0}^D) = (0.3, -0.01)$, $(g_{k_0}^D, \lambda_{k_0}^D) = (\frac{3\pi}{8} 10^{-4}, 10^{-4})$, and $(g_{k_0}^D, \lambda_{k_0}^D) = (0.05, 0.01)$ for the representative of type (Ia)^D, (IIa)^D, and (IIIa)^D, respectively.

R In all plots we employ the following color / style-coding to distinguish the couplings g^I , G^I , λ^I , and $\bar{\lambda}^I$, for $I \in \{D, B, (0), \text{sm}\}$:

dashed (red):		$I = \text{sm}$ (single-metric)
solid (dark-blue):		$I = D \equiv (p)$ for $p \geq 1$
solid (light-blue):		$I = (0)$
dot-dashed (blue):		$I = B$

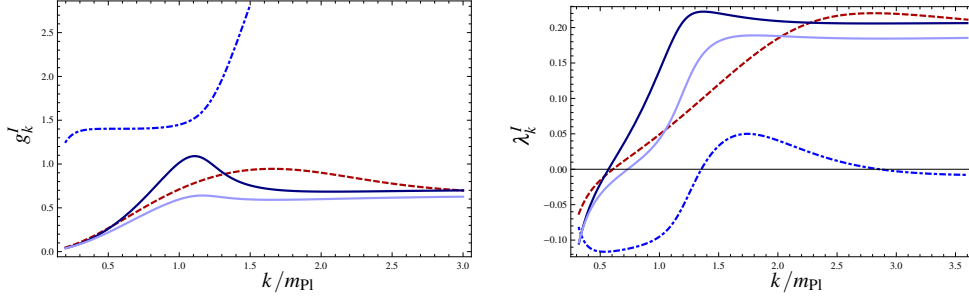


Figure 10.9: Type (Ia)^D-Attr^B trajectory: dimensionless couplings.

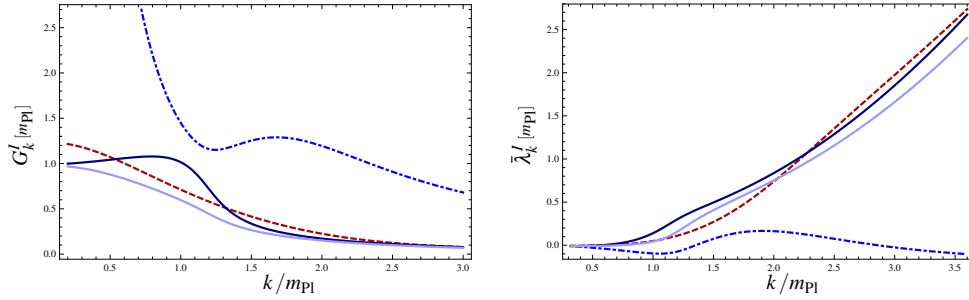


Figure 10.10: Type (Ia)^D-Attr^B trajectory: dimensionful couplings.

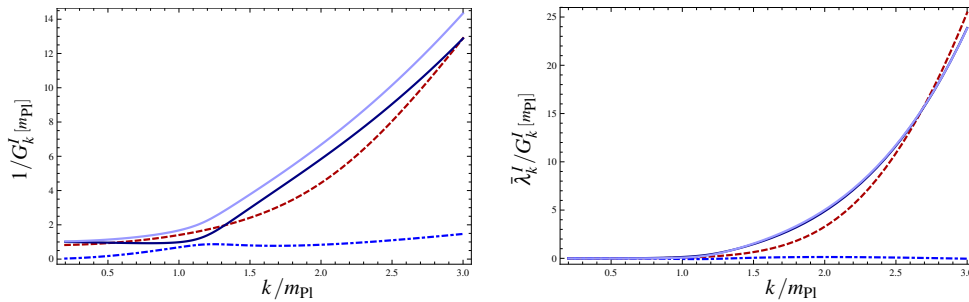
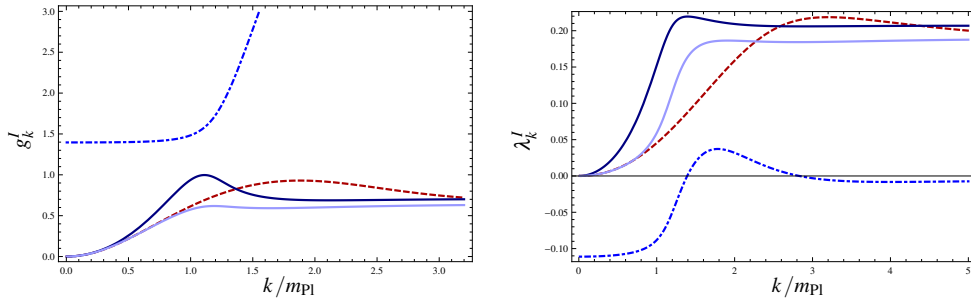
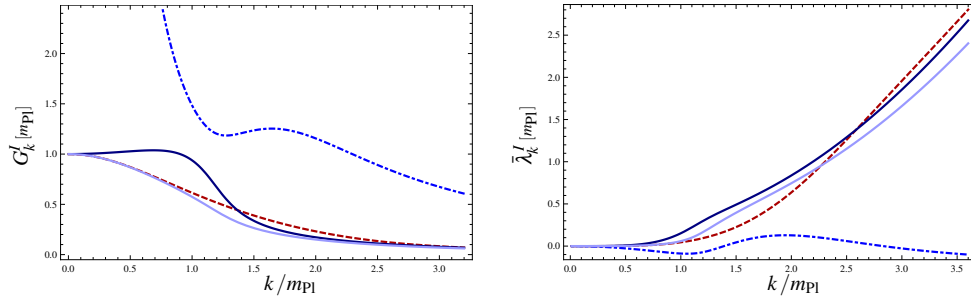
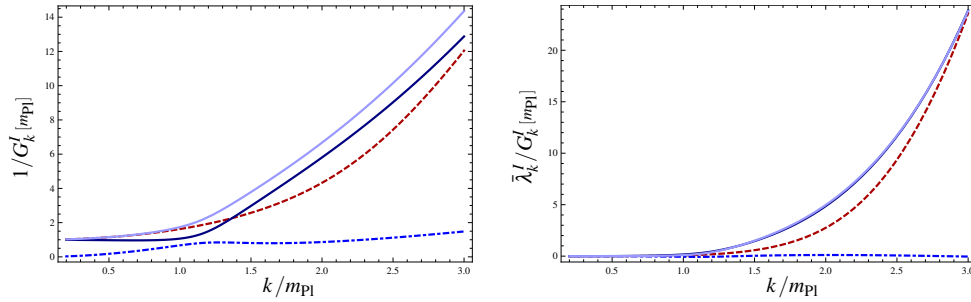
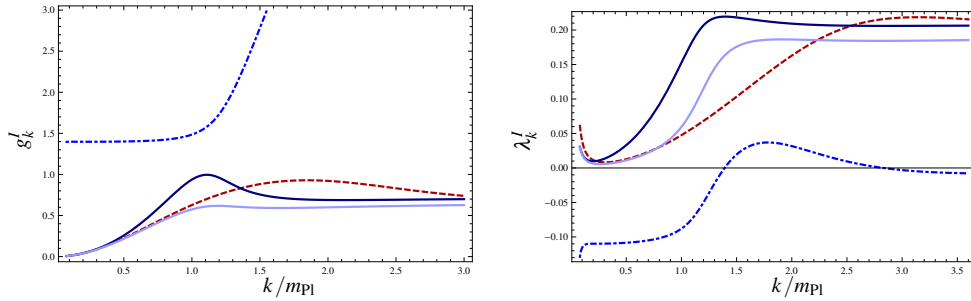


Figure 10.11: Type (Ia)^D-Attr^B trajectory: the coefficients as they appear in the EAA. Note the perfect split-symmetry restoration in the IR: $1/G_k^B$ and $\bar{\lambda}^B/G_k^B$ vanish for $k \rightarrow 0$, implying that Γ_k^{grav} loses its extra $\bar{g}_{\mu\nu}$ dependence.

Dimensionless couplings. Let us first consider the running *dimensionless* couplings shown in Figs. 10.9, 10.12, and 10.15. The following features are shared by all three types of trajectory-

Figure 10.12: Type (IIa)^D-Attr^B trajectory: dimensionless couplings.Figure 10.13: Type (IIa)^D-Attr^B trajectory: dimensionful couplings.Figure 10.14: Type (IIa)^D-Attr^B trajectory: the coefficients as they appear in the EAA. Note the split-symmetry restoration for $k \rightarrow 0$.Figure 10.15: Type (IIIa)^D-Attr^B trajectory: dimensionless couplings.

ries:

The solutions for the single-metric, the level-(0) and the D couplings do not agree in any ap-

proximate sense, but differ quite significantly for most k . In the IR, we imposed the requirement of split-symmetry and this is clearly seen even in the results for the dimensionless quantities: For $k \rightarrow 0$, in the classical regimes, the $p = 0$ and $p \geq 1$ curves overlap basically. Likewise, in the UV, we observe that, consistent with the analysis in Section 10.3, there is a remarkable similarity of the single- and bi-metric curves in the vicinity of their non-Gaussian fixed points. All plots confirm this numerical ‘miracle’ which, as we emphasized already, is not due to any general principle. (But it is highly welcome of course.) At intermediate scales the single-metric and the bi-metric solutions are found to be rather different, even qualitatively. This is precisely the symptom of the broken split-symmetry.

In conclusion we can say in comparison with the bi-metric truncation, the single-metric treatment seems to be a good approximation in the far IR and UV, but at the quantitative level it does not account for what happens in between. It must be said that the single-metric results convey the correct qualitative picture, nevertheless.

Dimensionful couplings. Next we take a look at the *dimensionful* couplings, expressed in units of the Planck mass. Figs. 10.10, 10.13, and 10.16 show the results for the Newton and cosmological constants for the three classes. As for the asymptotic k -dependence of the Newton constants G_k^I , $I \in \{D, B, (0), \text{sm}\}$ we observe that all of them *vanish* for $k \rightarrow \infty$.

Thus we recover gravitational anti-screening in the bi-metric setting, but only at a high (Planckian) scale. In fact, it is quite impressive to see that, for k below the Planck scale, the dynamical Newton constant G_k^D actually *increases* with k then, assumes a maximum near $k \approx m_{\text{Pl}}$, and finally decreases for $k \gtrsim m_{\text{Pl}}$. On the other hand, the single-metric Newton constant G_k^{sm} decreases with k at all scales $k \geq 0$. Clearly this behavior of G_k^D is a consequence of the sign-flip of $B_1^D(\lambda^D)$, and therefore η^D , at $\lambda^D = \lambda_{\text{crit}}^D$.

Coefficients. For $k \rightarrow 0$ the background Newton constant G_k^B diverges, and $\bar{\lambda}_k^B$ vanishes, exactly as it should be in order to make $1/G_k^B$ vanish in this limit, which is necessary for split-symmetry. This is best seen in Figs. 10.11, 10.14, and 10.17. There the dependence of the

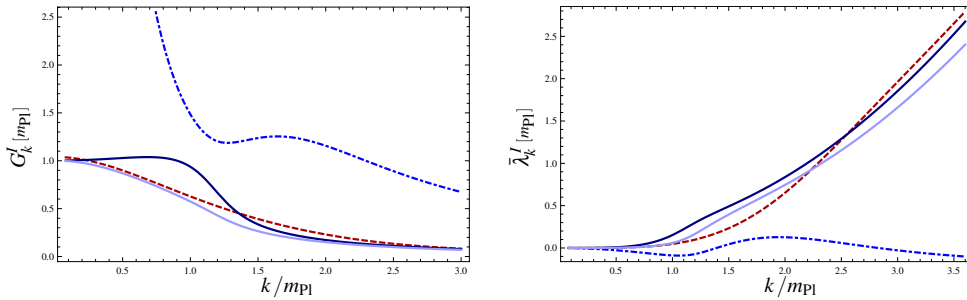


Figure 10.16: Type (IIIa)^D-Attr^Btrajectory: dimensionful couplings.

pre-factors of the B-type invariants in the truncation ansatz on k is shown, namely $1/G_k^B$ and $\bar{\lambda}_k^B/G_k^B$, respectively. Remember that the (blue) dot-dashed line is related to the B-sector, it is very impressive to see how close to zero it stays in the IR for all three types of trajectories. These plots confirm that we were indeed successful in combining Background Independence with Asymptotic Safety.

Notice that for moderately large values of k the B-pre-factors increase. Their deviation from zero is relatively small when compared with the D- or level-(0) sector, and this implies that split-symmetry is intact at least approximately. For intermediate scales we again find considerable violation of split-symmetry, which manifests itself by B-coefficients which are now of the same

order as the D- and level-(0) ones. Therefore the single-metric (red, dashed line) only converges to the bi-metric curves for $k \rightarrow 0$ and $k \rightarrow \infty$.

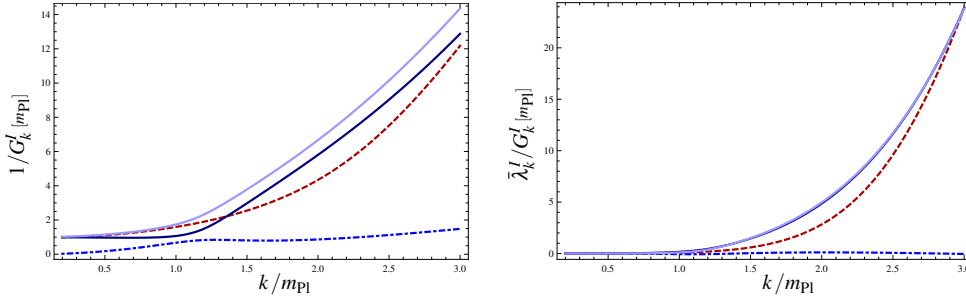


Figure 10.17: Type (IIIa)^D-Attr^B trajectory: the coefficients as they appear in the EAA. Note again the vanishing $1/G_k^B$ and $\bar{\lambda}_k^B/G_k^B$, indicative of split-symmetry restoration in the limit $k \rightarrow 0$.

From the differences between the $p = 0$ and $p \geq 1$ curves, too, we see again that for all three trajectory types split-symmetry is apparently well restored in the IR and the UV, but in between it suffers from a considerable breaking; the D couplings show a more pronounced k -dependence than the level-(0) couplings, whereas the single-metric functions are monotone.

10.3 Single-metric vs. bi-metric truncation: a confrontation

In this section we perform an in-depth analysis of the differences between the bi-metric Einstein-Hilbert truncation and its single-metric approximation. We shall describe how precisely their results are interrelated on general grounds, and what can be learned from the numerical comparison about the validity of the single-metric truncation. As we shall see, its degree of reliability varies considerably over the theory space. For future work it will be important to know of course where, and to what extent it can be trusted. In particular we shall also understand why in the past it has always been notoriously difficult to obtain accurate and stable results for the critical exponents.

In this section, in subsection 10.3.7, we shall also critically examine how our new method based on the ‘deformed $\alpha = 1$ gauge’ compares to the bi-metric calculation in ref. [172] which employed the transverse-traceless approach.

This section is of a somewhat technical nature, and can be skipped by the reader who is mostly interested in the results.

10.3.1 Collapsed level hierarchies

Let us write the level-expanded EAA symbolically as $\Gamma_k[h; \bar{g}] = \sum_{p=0}^{\infty} F_k^{(p)}[\bar{g}] (h_{\mu\nu})^p$ where the $F_k^{(p)}$ ’s depend on the background metric only. When we insert this expansion into the FRGE and project on a fixed level p we see that $\partial_t F_k^{(p)}$ which appears on its left-hand-side (LHS) gets equated to an expression exclusively involving $\{F_k^{(q)} | q = 2, \dots, p+2\}$. Hence the scale derivative of all (dimensionful) level-(p) couplings is given by a beta-function depending on the level-(q) couplings, with $q = 2, \dots, p+2$, only.¹¹ This is the generic situation when no special restrictions on the form of the EAA are assumed: the FRGE amounts to an infinite hierarchy of equations $\partial_t F_k^{(p)} = \dots$ for $p = 0, 1, 2, 3, \dots$ which does not terminate at any finite level and couples all levels therefore. Only if it was possible to realize split-symmetry exactly this tower of equations collapses to a single equation that governs all levels.

¹¹The *dimensionless* couplings also contain trivial canonical terms in their beta-functions, of course. They play no role in this discussion and are ignored here.

The bi-Einstein-Hilbert truncation used in the present paper involves the assumption that split-symmetry is broken only weakly, and that differences among the ‘higher’ levels $p = 1, 2, 3, \dots$ are sufficiently small so that they may be ignored. The lowest level, $p = 0$, however is dealt with separately and is allowed to show a RG behavior different from $p \geq 1$. As always, the $p = 0$ couplings have beta-functions which depend on the $p = 2$ couplings. For the present truncation the latter happen to be equal to those at all non-zero levels $p \geq 1$.

Stated more abstractly, what reduced the infinite hierarchy of RG equations to just 2 equations was an *additional hypothesis about the RG flow*, namely that the split-symmetry breaking is such that it lifts only the degeneracy between levels with $p = 0$ and $p > 0$, while those with $p > 0$ remain degenerate among themselves.

The logical status of the familiar single-metric truncations can be characterized analogously. Here the additional hypothesis invoked is even stronger: one pretends that the solutions to the FRGE exhibit *exact* split-symmetry so that *all* levels $p = 0, 1, 2, 3, \dots$ undergo an equivalent RG evolution. It is sufficient then to retain the RG equations for the lowest level at $p = 0$ to fix the k -dependence of all running couplings.

10.3.2 Relating single- and bi-metric beta-functions

Let us return to the concrete example of the bi-Einstein-Hilbert truncation with its 4 independent couplings $\{g^{(0)}, \lambda^{(0)}, g^{(1)}, \lambda^{(1)}\}$ and let us see in which way precisely its beta-functions are related to those of the standard single-metric Einstein-Hilbert truncation with only 2 running couplings. Recall that after the conformal projection $g_{\mu\nu} = e^{2\Omega} \bar{g}_{\mu\nu}$ the EAA of the former equals that of the latter for $\Omega = 0$. In the bi-metric case, the flow equation is expanded in powers of Ω , whereby the zeroth and first orders in Ω correspond to the level-(0) and level-(1) couplings, respectively. Structurally the single-metric beta-functions β_g^{sm} and $\beta_\lambda^{\text{sm}}$ thus coincide with the beta-functions of the level-(0) couplings, however only *after we have identified all couplings of different orders*.

Even though this sounds trivial it changes the form of the beta-functions quite significantly so that the new differential equations are of a rather different type, with qualitatively new properties. For example, the anomalous dimension η^D that (contrary to η^B !) can appear on the RHS of the bi-metric flow equation is no longer related to an independent coupling $g^{(2)}$ then, but in fact to the η^B -related $g^{(0)}$. This transition changes the simple, Bernoulli-type differential equation (10.7) into a much more complicated non-polynomial (but autonomous) one.

In detail, we have to apply the following identifications:

$$\beta_{g/\lambda}^{\text{sm}}(g^{\text{sm}}, \lambda^{\text{sm}}; d) \equiv \beta_{g/\lambda}^{(0)}(\{g^{(q)} = g^{\text{sm}}, \lambda^{(q)} = \lambda^{\text{sm}}\}; d) \Big|_{\eta^{(p)} \equiv \eta^{\text{sm}}} \quad p \in \{0, 1, 2, \dots\} \quad (10.34a)$$

$$g^{(0)} = g^{(1)} = \dots = g^{\text{sm}}, \quad \lambda^{(0)} = \lambda^{(1)} = \dots = \lambda^{\text{sm}}. \quad (10.34b)$$

The running of all couplings at higher levels is pretended to be described by the two beta-functions from level-(0).

10.3.3 Conditions for the reliability of a single-metric calculation

To check whether the single-metric truncation is a good approximation to the bi-metric one, we must study the beta-functions of the higher levels, the conditions (10.34), and their implications:

$$g_k^{\text{sm}} \stackrel{!}{=} g_k^{(0)} = g_k^{(1)} = \dots, \quad \lambda_k^{\text{sm}} \stackrel{!}{=} \lambda_k^{(0)} = \lambda_k^{(1)} = \dots, \quad (10.35a)$$

$$\beta_{g/\lambda}^{\text{sm}}(g^{\text{sm}}, \lambda^{\text{sm}}; d) \stackrel{!}{=} \beta_{g/\lambda}^{(p)}(\{g^{(q)} = g^{\text{sm}}, \lambda^{(q)} = \lambda^{\text{sm}}\}; d) \quad p, q \in \{0, 1, 2, \dots\} \quad (10.35b)$$

Notice that contrary to split-symmetry requirement where it was more natural to consider *dimensionful* couplings, the requirements (10.35) are constraints on the *dimensionless* couplings. But of course as long as we are not taking the $k \rightarrow 0$ or $k \rightarrow \infty$ limit we can simply strip off the explicit k -factors from the split-symmetry condition and end up with the conditions (10.35):

$$G_k^{\text{sm}} = k^{-(d-2)} g_k^{\text{sm}} \stackrel{!}{=} k^{-(d-2)} g_k^{(0)} = k^{-(d-2)} g_k^{(1)} = \dots = G_k^{(p)} \quad (10.36a)$$

$$\bar{\lambda}_k^{\text{sm}} = k^2 \lambda_k^{\text{sm}} \stackrel{!}{=} k^2 \lambda_k^{(0)} = k^2 \lambda_k^{(1)} = \dots = \bar{\lambda}_k^{(p)} \quad (10.36b)$$

Though in principle there is the possibility that the explicit k -dependence of the dimensionful couplings gives rise to split-symmetry for $k = 0$ or $k \rightarrow \infty$ only, we never relied on this possibility, and we shall never do it in what follows. This puts the respective requirements for intact split-symmetry and a valid single-metric approximation on an equal footing.

10.3.4 The anomalous dimensions: structural differences

In the remainder of this section we restrict the discussion to the Newton couplings. This covers already all subtleties that arise in concretely working out the conditions (10.35) and their generalization to the full 4-dimensional theory space.

The beta-functions of the Newton constants in the level-description were found in eqs. (8.73) and (8.82) of chapter 8, respectively:

$$\beta_g^{(0)}(g^{\text{D}}, \lambda^{\text{D}}, g^{(0)}; d) = \left[d - 2 + (B_1^{(0)}(\lambda^{\text{D}}; d) + \eta^{\text{D}} B_2^{(0)}(\lambda^{\text{D}}; d)) g^{(0)} \right] g^{(0)} \quad (10.37a)$$

$$\beta_g^{\text{D}}(g^{\text{D}}, \lambda^{\text{D}}; d) = \left[d - 2 + (B_1^{\text{D}}(\lambda^{\text{D}}; d) + \eta^{\text{D}} B_2^{\text{D}}(\lambda^{\text{D}}; d)) g^{\text{D}} \right] g^{\text{D}} \quad (10.37b)$$

Due to the back-reaction of the D-couplings, i.e. those with $p \geq 1$, especially via η^{D} , eqs. (10.37a) and (10.37b) amount to structurally quite different expressions for the anomalous dimensions:

$$\eta^{(0)}(g^{\text{D}}, \lambda^{\text{D}}, g^{(0)}; d) = [B_1^{(0)}(\lambda^{\text{D}}; d) + \eta^{\text{D}} B_2^{(0)}(\lambda^{\text{D}}; d)] g^{(0)} \quad (10.38a)$$

$$\eta^{\text{D}}(g^{\text{D}}, \lambda^{\text{D}}; d) = \frac{B_1^{\text{D}}(\lambda^{\text{D}}; d) g^{\text{D}}}{1 - B_2^{\text{D}}(\lambda^{\text{D}}; d) g^{\text{D}}} \quad (10.38b)$$

The ‘confusion’ of different levels by the single-metric truncation yields a formula for its anomalous dimension η^{sm} that combines the non-polynomial g^{D} dependence (10.38b) at the levels $p \geq 1$, with the dependence on the cosmological constant from level-(0), the latter given by $B_{1/2}^{(0)}(\lambda; d)$. Explicitly, we can extract the single-metric beta-functions for Newton’s coupling by applying (10.34) to eq. (10.37):

$$\begin{aligned} \beta_g^{\text{sm}}(g^{\text{D}}, \lambda^{\text{D}}, g^{(0)}; d) &= \beta_g^{(0)}(g^{\text{sm}}, \lambda^{\text{sm}}, g^{\text{sm}}; d) \big|_{\eta^{\text{D}} = \eta^{\text{sm}}} \\ &= \left[d - 2 + (B_1^{(0)}(\lambda^{\text{sm}}; d) + \eta^{\text{sm}} B_2^{(0)}(\lambda^{\text{sm}}; d)) g^{\text{sm}} \right] g^{\text{sm}} \end{aligned} \quad (10.39)$$

The identification of η^{D} with η^{sm} leads to an implicit equation for η^{sm} from which we obtain

$$\eta^{\text{sm}}(g^{\text{sm}}, \lambda^{\text{sm}}; d) = \frac{B_1^{(0)}(\lambda^{\text{sm}}; d) g^{\text{sm}}}{1 - B_2^{(0)}(\lambda^{\text{sm}}; d) g^{\text{sm}}} \quad (10.40)$$

Eq. (10.40) highlights the limitations of the single-metric formula in approximating the full bi-metric RG flow: Even though η^{sm} inherits the dynamical g -dependence and thus reproduces all findings based on its non-polynomial form, it completely loses any information on the dynamical λ -dependence. It is thus rather non-trivial that our bi-metric results, even at the numerical level, in many cases stayed very close to the single-metric ones.

10.3.5 (Un-)Reliable portions of the single-metric theory space

Since intact split-symmetry is closely related to the reliability of the single-metric truncation, we expect the IR regime of the trajectories — where we explicitly restored split-symmetry — to be well approximated by the trajectories of g^{sm} and λ^{sm} . The plots presented in section 10.2.5 actually confirm this expectation, see Figs. 10.9, 10.12, and 10.15.

Nevertheless, we already pointed out that there exists no fully extended solution $\{\Gamma_k, k \in [0, \infty)\}$ along which split-symmetry would be intact everywhere. As can be seen in the diagrams of section 10.2.5, only in the extreme IR and UV, that is *only when* $k/m_{\text{Planck}} \ll 1$ or $k/m_{\text{Planck}} \gg 1$ *the dimensionless bi-metric couplings are satisfactorily approximated by the single-metric ones, in between the curves disagree even qualitatively.*

Instead of focusing on a single trajectory only, we next investigate the validity of the single-metric approximation in an extended region of the bi-metric theory space. For this purpose we determine the set $\mathfrak{R}$ of all pairs (g, λ) with $g > 0$ such that at the points of theory space associated to them via $(g^{(0)}, \lambda^{(0)}, g^{\text{D}}, \lambda^{\text{D}}) = (g, \lambda, g, \lambda) \in \mathcal{T}$ the ‘single-metric condition’ (10.35) is satisfied. Those points form a subset of the 4-dimensional theory space, denoted

$$\mathcal{T}_{\mathfrak{R}} = \{(g, \lambda, g, \lambda) \mid (g, \lambda) \in \mathfrak{R}\} \subset \mathcal{T}$$

We may think of this submanifold as set of initial points for RG trajectories at which the single-metric approximation is exact. The crucial question is to what extent this submanifold is invariant under the 4-dimensional RG flow.

If we start from a point with $(g^{(0)}, \lambda^{(0)}) = (g, \lambda) = (g^{\text{D}}, \lambda^{\text{D}})$, and insist that this equality is preserved under an infinitesimal RG transformation, we must require that the corresponding anomalous dimensions agree: $\eta^{(0)}(g, \lambda, g; d) \stackrel{!}{=} \eta^{\text{D}}(g, \lambda; d)$. Using (10.38) this entails a constraint for the pair (g, λ) which involves the $B_{1/2}$ -functions:

$$0 \stackrel{!}{=} g \left[B_1^{(0)}(\lambda; d) - B_1^{\text{D}}(\lambda; d) \right] + g^2 \left[B_1^{\text{D}}(\lambda; d) B_2^{(0)}(\lambda; d) - B_1^{(0)}(\lambda; d) B_2^{\text{D}}(\lambda; d) \right] \quad (10.41)$$

This equation is a necessary condition for $(g, \lambda) \in \mathfrak{R}$.

Besides the requirement $\eta^{(0)} \stackrel{!}{=} \eta^{\text{D}}$, leading to (10.41), the ‘single-metric validity conditions’ (10.35) also include the constraint arising from $\eta^{\text{sm}}(g, \lambda; d) \stackrel{!}{=} \eta^{\text{D}}(g, \lambda; d)$. Expressed in terms of the $B_{1/2}$ -functions it is found to be exactly identical with eq. (10.41).¹² As a result, *equation (10.41) is a sufficient condition for the validity of the single-metric truncation within $\mathfrak{R}$.*

Let us now solve eq. (10.41) explicitly to determine the pairs $(g, \lambda) \in \mathfrak{R}$.

Solution class $(g, \lambda_{\mathfrak{R}})$. A first class of solutions consists of points $(g, \lambda_{\mathfrak{R}})$ with g arbitrary and $\lambda_{\mathfrak{R}}$ satisfying both $B_1^{(0)}(\lambda; d) = B_1^{\text{D}}(\lambda; d)$ and $B_2^{(0)}(\lambda; d) = B_2^{\text{D}}(\lambda; d)$.

Are there λ -values for which these conditions are satisfied, or satisfied approximately at least? The explicit comparison of $B_{1/2}^{(0)}$ and $B_{1/2}^{\text{D}}$ with respect to their λ -dependence in $d = 4$ is depicted in Fig. 10.18 and Fig. 10.19, respectively. The dashed vertical line marks the fixed point value of the dynamical cosmological constant, λ_*^{D} . The dark line gives the dependence of $B_{1/2}^{\text{D}}$ on λ , whereas the light one shows the functions $B_{1/2}^{(0)}$. In the inset pictures we plotted the difference between the D- and level-(0) functions.

We observe that while $B_{1/2}^{(0)}(\lambda)$ and $B_{1/2}^{\text{D}}(\lambda)$ have a qualitatively similar λ -dependence, the exact equality $B_{1/2}^{(0)}(\lambda) = B_{1/2}^{\text{D}}(\lambda)$ holds only at a single value of λ . However, what comes as a real surprise is that this distinguished λ -value at which the single-metric truncation performs

¹²The coincidence of the $\eta^{(0)} = \eta^{\text{D}}$ and $\eta^{\text{sm}} = \eta^{\text{D}}$ conditions is owed to the fact that $B_{1/2}^{\text{sm}} \equiv B_{1/2}^{(0)}$, which reflects the general relationship between the single-metric and the level-(0) conformally projected bi-metric results.

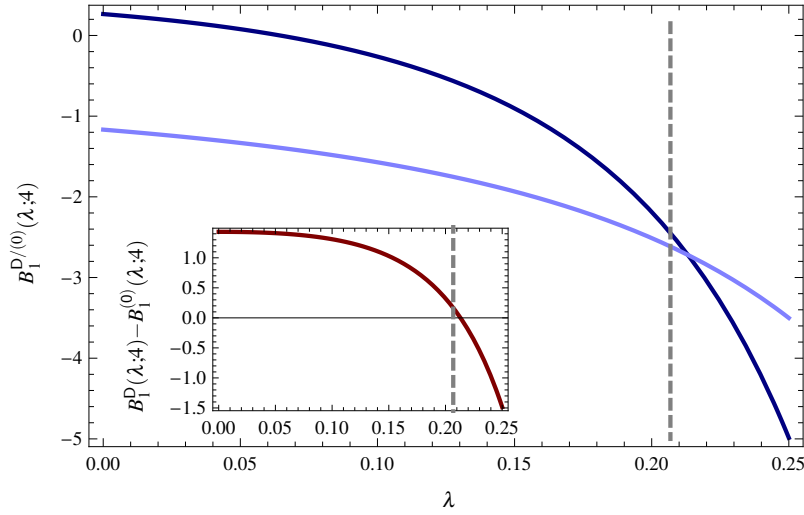


Figure 10.18: This figure depicts the dependence of $B_1^D(\lambda;4)$ and $B_1^{(0)}(\lambda;4)$ on their argument λ by the dark, respectively light, blue line. The difference of the functions is shown as the (red) curve inserted in the lower left corner. In both plots the fixed point value λ_*^D is marked with the dashed vertical line. The smaller the difference of $B_1^D(\lambda;4)$ and $B_1^{(0)}(\lambda;4)$ the better is the single-metric approximation to the bi-metric truncation. While in the vicinity of the Gaußian fixed point there is a large discrepancy, the point of best approximation is seen to be (miraculously) close to the NGFP value λ_*^D .

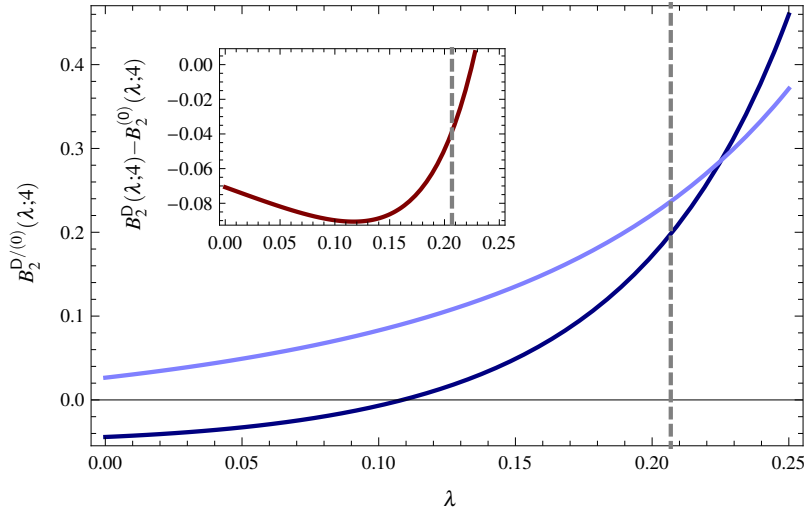


Figure 10.19: The second condition for the fulfillment of eq. (10.41) is the agreement of $B_2^D(\lambda;4)$ with $B_2^{(0)}(\lambda;4)$. Their dependence on λ is given by the dark, respectively light, blue line. The difference of the functions is shown in the upper left corner. The fixed point value λ_*^D is marked with the dashed vertical line. Notice that the agreement of $B_2^D(\lambda;4)$ with $B_2^{(0)}(\lambda;4)$ is best in the vicinity of the NGFP. There, the single-metric truncation is a good approximation to the bi-metric one. Away from the NGFP its quality deteriorates considerably.

best is impressively close to the λ -coordinate of NG $_+$ -FP, λ_*^D . This ‘miracle’ is not explained by any general principle. Its implication is clear though: *The single-metric approximation to the bi-Einstein-Hilbert truncation is most reliable precisely in that region of theory space where the non-Gaußian fixed point is located.* This discovery may be seen as an a posteriori justification

of the single-metric investigations of the Asymptotic Safety conjecture.

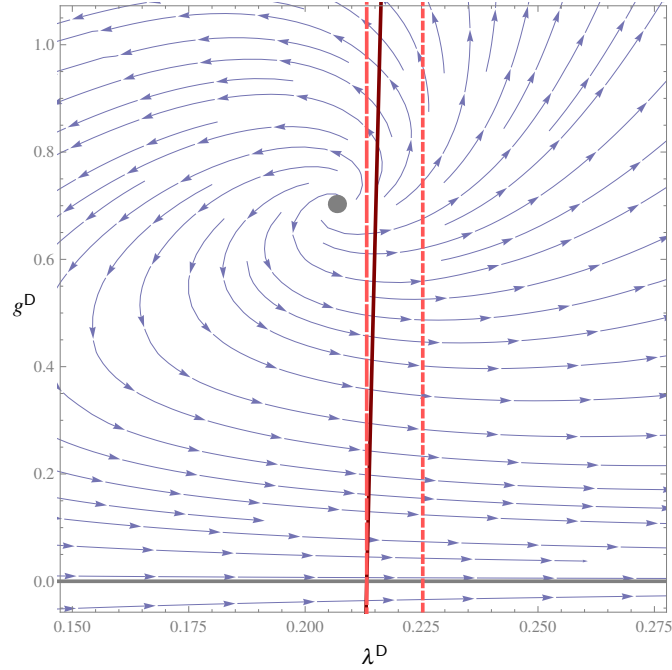


Figure 10.20: The two solutions to the split-symmetry condition (10.41), given by pairs $(g, \lambda_{\mathfrak{R}})$ and $(g_{\mathfrak{R}}(\lambda), \lambda)$, respectively, are shown by the two light, and the dark (red) curve. They are superimposed on the phase portrait of the ‘D’ sector. In the vicinity of these curves split-symmetry is approximately intact. The ‘miraculous’ result is that both of them, the almost linear (dark red) curve, and the (light red) exactly vertical line, pass very close to the NGFP (gray disc). This justifies the use of the single-metric truncation in the vicinity of the NGFP. On the other hand, it is also apparent that every RG trajectories stays only briefly within a neighborhood of $(g, \lambda_{\mathfrak{R}})$ or $(g_{\mathfrak{R}}(\lambda), \lambda)$. Away from these regions the single-metric truncation might become problematic.

Solution class $(g_{\mathfrak{R}}(\lambda), \lambda)$. Up to now we solved the condition (10.41) by setting to zero the coefficients of g and g^2 separately. There is a second type of solutions consisting of pairs $(g_{\mathfrak{R}}(\lambda), \lambda)$ where

$$g_{\mathfrak{R}}(\lambda) \equiv \frac{B_1^{(0)}(\lambda; d) - B_1^D(\lambda; d)}{B_1^{(0)}(\lambda; d)B_2^D(\lambda; d) - B_1^D(\lambda; d)B_2^{(0)}(\lambda; d)} \quad (10.42)$$

In Fig. 10.20 the function $g_{\mathfrak{R}}(\lambda)$, along with the above $(g, \lambda_{\mathfrak{R}})$ solution, is superimposed on the phase portrait of the ‘D’ sector which, as we know, is qualitatively similar to the ‘sm’ one.

The good news we learn from this plot is that *the NGFP is situated very close to the curve $(g_{\mathfrak{R}}(\lambda), \lambda)$* . This is a second ‘miracle’, again unexplained by any general argument, and independent of the first one. So the single-metric approximation seems indeed most reliable when it comes to locating the NGFP and exploring its properties.

The bad news is that there does not exist a single RG trajectory that would stay on, or close to the $(g_{\mathfrak{R}}(\lambda), \lambda)$ -line for *all* scales; the trajectories intersect it at most once or twice. The consequence is that every ‘sm’ trajectory unavoidably contains segments where it differs substantially from its bi-metric, i.e. ‘D’ analogue.

Summary: The general picture which emerges is as follows. In the extreme UV, all RG trajectories start out from initial points which are infinitesimally close to the non-Gaussian fixed point which is at the heart of the Asymptotic Safety construction. In this region, the bi-metric description is well approximated by the single-metric truncation. Once the trajectories escape from the non-Gaussian fixed point regime and enter an intermediate region where $\lambda \lesssim 0.15$, say, split-symmetry is significantly violated, and the single-metric approximation becomes unreliable. This can be seen in the Figs. 10.18, 10.19, and the explicit RG trajectories given in the diagrams of section 10.2.5.

10.3.6 The (un-)reliability of the critical exponent calculations

What remains to be investigated is the precise relation between the single- and bi-metric results concerning the non-Gaussian fixed point, in particular its location and critical exponents. Figs. 10.18 and 10.19 already shed some light on this question since ‘miraculously’ the best agreement of single-metric and bi-metric results was found to be close to λ_*^D . In this subsection we concentrate on the implications of this observation and, most importantly, try to understand why the predictions for the critical exponents differ so much in the two truncations.

- A.** Consider the single-metric and the bi-metric non-Gaussian fixed point regimes. Exactly at the NGFP all anomalous dimensions are $\eta_*^I = -(d-2)$. We first turn our attention to the bi-metric Newton couplings at the fixed point:

$$g_*^D = \frac{(d-2)}{B_2^D(\lambda_*^D; d)(d-2) - B_1^D(\lambda_*^D; d)} \quad (10.43a)$$

$$g_*^{(0)} = \frac{(d-2)}{B_2^{(0)}(\lambda_*^D; d)(d-2) - B_1^{(0)}(\lambda_*^D; d)} \quad (10.43b)$$

It is obvious that the better $B_{1/2}^{(0)}(\lambda_*^D; d) \approx B_{1/2}^D(\lambda_*^D; d)$ is satisfied the closer are $g_*^{(0)}$ and $g_*^D = g_*^{(p)}$, $p \geq 1$, and the better is the split-symmetry. At the NG_+^D -FP the deviation is small but non-zero; we expect $g_*^{(0)}$ to be at best approximately equal to g_*^D , which in fact was found in section 9.2. The main reason for the differing fixed point values is the difference in the $B_1^{(0)/D}$ -functions. Both, $B_1^D(\lambda; d)$ and $B_1^{(0)}(\lambda; d)$ decrease for increasing λ , and so does their difference. For $\lambda < \lambda_*^D$ the difference $B_1^D(\lambda; d) - B_1^{(0)}(\lambda; d)$ is positive, and thus $g_*^{(0)} < g_*^D$.

- B.** The relation of $g_*^{(0)}$ and g_*^D to their single-metric cousin g_*^{sm} is more involved:

$$g_*^{\text{sm}} = \frac{(d-2)}{B_2^{(0)}(\lambda_*^{\text{sm}}; d)(d-2) - B_1^{(0)}(\lambda_*^{\text{sm}}; d)} \quad (10.44)$$

The difference of g_*^{sm} and $g_*^{(0)}$ is a consequence of the differing fixed point values of the cosmological constants, in particular we find $\lambda_*^{\text{sm}} < \lambda_*^D$. This discrepancy has a balancing effect such that $g_*^D \approx g_*^{\text{sm}}$ while $g_*^{(0)} < g_*^{\text{sm}}$ owed to the fact that $B_1^{(0)}(\lambda_*^{\text{sm}}; d) > B_1^{(0)}(\lambda_*^D; d)$ for $\lambda_*^{\text{sm}} < \lambda_*^D$.

- C.** Moving away from the NGFPs the first question that arises is whether the respective linearized flows in their vicinity are similar. The solution for the Newton constants in the bi-metric truncation reads

$$g^D(k) = g_*^D + 2c_1 V_1^{(1)} \left(\frac{k_0}{k} \right)^{\theta'} \cos(\vartheta_c + \vartheta^{(1)} + \theta'' \ln(k_0/k)) \quad (10.45a)$$

$$g^{(0)}(k) = g_*^{(0)} + 2c_1 V_3^{(1)} \left(\frac{k_0}{k} \right)^{\theta'} \cos(\vartheta_c + \vartheta^{(1)} + \theta'' \ln(k_0/k)) + c_3 V_3^{(3)} \left(\frac{k_0}{k} \right)^{\theta_3} \quad (10.45b)$$

Here, c_j , ϑ_c are real constants of integration, $V_r^{(j)}$ is the r^{th} component of the j^{th} (real) eigenvector, and $\vartheta^{(j)}$ is its phase. The set of critical exponents consists of the complex pair $\theta_{1/2} = \theta' \pm i\theta''$, where $\theta' > 0$, and the purely real, positive critical exponent θ_3 . In addition to the spiral motion into the non-Gaussian fixed point present for the dynamical coupling, there is an additional k^2 -term ($\theta_3 = -2$) that contributes to the running of $g^{(0)}(k)$.

In the single-metric truncation the linearization yields instead

$$g^{\text{sm}}(k) = g_*^{\text{sm}} + 2c_1^{\text{sm}} V_1^{(1;\text{sm})} \left(\frac{k_0}{k} \right)^{\theta'_{\text{sm}}} \cos(\vartheta_c^{\text{sm}} + \vartheta^{(1;\text{sm})} + \theta''_{\text{sm}} \ln(k_0/k)) \quad (10.46)$$

We see that qualitatively $g^{\text{sm}}(k)$ has the same kind of running as $g^{\text{D}}(k)$. Again, there is a pair of complex conjugate critical exponents with a positive real part θ'_{sm} and a non-vanishing imaginary part θ''_{sm} that produces spirals. From a quantitative perspective the results differ considerably, however, and it is instructive to understand why.

- D.** Despite the small difference of the respective fixed point coordinates, **the critical exponents disagree substantially** in the single- and bi-metric truncation, as we discuss next. Considering the single-metric and the dynamical bi-metric sector the two critical exponents are given by

$$\theta_{1/2} = -\frac{1}{2}(\mathcal{B}_{11} + \mathcal{B}_{22}) \mp \frac{1}{2} \sqrt{4\mathcal{B}_{12}\mathcal{B}_{21} + (\mathcal{B}_{11} - \mathcal{B}_{22})^2} \quad (10.47)$$

where $\mathcal{B}_{rs}$ is the r - s entry of the respective stability matrix:

$$\mathcal{B} = \begin{pmatrix} \partial_g \beta_g & \partial_\lambda \beta_g \\ \partial_g \beta_\lambda & \partial_\lambda \beta_\lambda \end{pmatrix} \Rightarrow \mathcal{B}^{\text{D}} = \begin{pmatrix} -2.32 & -27.8 \\ 0.72 & -4.9 \end{pmatrix}, \quad \mathcal{B}^{\text{sm}} = \begin{pmatrix} -2.34 & -10.4 \\ 0.96 & -0.61 \end{pmatrix} \quad (10.48)$$

Comparing B_1^{D} to B_1^{B} we observe that their first columns agree quite well, but the second columns are rather different. The main difference between the matrices $\mathcal{B}^{\text{D}}$ and $\mathcal{B}^{\text{sm}}$ is the way they depend on the cosmological constant. Whereas a change in the Newton couplings has, even quantitatively, a similar impact on B_1^{D} and B^{sm} , a change in the cosmological constant affects the bi-metric system much more strongly than the ‘sm’ one. Since in both cases the product $4\mathcal{B}_{12}\mathcal{B}_{21}$ is negative and much larger than $(\mathcal{B}_{11} - \mathcal{B}_{22})^2$, the square root in (10.47) is purely imaginary, yielding

$$\theta' = -\frac{1}{2}(\mathcal{B}_{11} + \mathcal{B}_{22}), \quad \theta'' = \frac{1}{2} \sqrt{-4\mathcal{B}_{12}\mathcal{B}_{21} - (\mathcal{B}_{11} - \mathcal{B}_{22})^2} \quad (10.49)$$

The differences of the single-metric and dynamical bi-metric critical exponents are therefore approximately

$$\theta' - \theta'_{\text{sm}} \approx -\frac{1}{2}(\mathcal{B}_{22}^{\text{D}} - \mathcal{B}_{22}^{\text{sm}}) = 2.2, \quad (10.50a)$$

$$\theta'' - \theta''_{\text{sm}} \approx \sqrt{-\mathcal{B}_{12}^{\text{D}}\mathcal{B}_{21}^{\text{D}}} - \sqrt{-\mathcal{B}_{12}^{\text{sm}}\mathcal{B}_{21}^{\text{sm}}} = 1.3. \quad (10.50b)$$

So, numerically the error due to the single-metric approximation is indeed considerable, more than two units (one unit) in the real (imaginary) part.

Taking the explicit structure of the beta-functions into account it thus becomes clear that it is the *slope* of the functions $B_{1/2}(\lambda; d)$ and $A_{1/2}(\lambda; d)$ at $\lambda = \lambda_*$ that is mainly responsible for the differences. It is apparent from Figs. 10.18 and 10.19 that these slopes are indeed quite different for the ‘D’ and the ‘sm’ case, even at the intersection points where the functions themselves agree. These λ -derivatives are the main reason for the quantitative differences

Summary: In conclusion we can say that the vicinity of the non-Gaussian fixed point is sufficiently well described within the single-metric approximation if we are satisfied with ‘semi-quantitative’ results. It correctly captures all qualitative properties of the flow. Our experience with the present truncation suggests however that it will hardly be possible to perform precision calculations of critical exponents in a single-metric truncation, not even with a very general ansatz for Γ_k .

10.3.7 Comparison with the TT-based approach of ref. [172]

At this point it is worthwhile to check how our present bi-metric results compare to those obtained in ref. [172] where a similar truncation ansatz including two Einstein-Hilbert actions for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ was used, and the same running couplings were investigated. Both calculations rely on the conformal projection technique, actually first employed in [172], however with two main differences:

- A.** The gauge choice: While in ref. [172] the ‘anharmonic gauge’, $\varpi = 1/d$, with gauge parameter $\alpha \rightarrow 0$ was chosen, the present calculation uses the harmonic gauge fixing condition, $\varpi = 1/2$, with gauge parameter $\alpha = 1 - (d-6)\Omega + \mathcal{O}(\Omega^2)$. The dependence of the ‘sm’ results on the gauge fixing parameter had already been investigated in ref. [123, 135, 147], and the changes between $\alpha = 0$ and $\alpha = 1$ were found to be of the order of a few percent only, so we expect the choice of α to be of minor importance.
- B.** Uncontracted derivatives: In the present calculation the uncontracted derivative terms cancel due to the gauge choice. Therefore the heat kernel expansion becomes straightforward. Instead, in ref. [172], one had to deal with contracted as well as uncontracted derivative terms, and it was necessary to apply a TT-decomposition to project the traces in the FRGE onto the truncated theory space. This led to a set of new field variables (irreducible component fields) and made the heat kernel expansion by far more involved and lengthy.

Recall that a truncation of theory space is not specified by the ansatz for the EAA alone, but in addition by a prescription for the projection on the field monomials. Variation of either ingredient may alter the results. The difference of our present calculation to [172] can be understood as stemming from the Γ_k -ansatz only, namely in the gauge fixing part of the EAA. Furthermore, the cutoff action ΔS_k is slightly different in the two cases, since in [172] it is formulated in terms of the irreducible (TT) components of $h_{\mu\nu}$, while in our new approach simply in terms of the undecomposed $h_{\mu\nu}$. The comparison can help distinguishing between artifacts of the specific truncation and robust, truncation-independent, results.

In what follows we focus on the properties of the doubly non-Gaussian fixed point in $d = 4$, and compare the results of the TT-based bi-metric calculation in [172] with our present one, as well as with the single-metric approximation.

Existence and location of non-Gaussian fixed points

In ref. [172] the same number of fixed points, in the same six categories as in our present approach has been found, namely three different D fixed points, and for each of them two B fixed points. The most important one for Asymptotic Safety is NG_+^{D} -FP, having $g_*^{\text{D}} > 0$. In Tab. 10.1 we summarize its properties. Notice that the new and the old bi-metric results in the D sector and their single-metric counterparts share the same qualitative as well as semi-quantitative features, the B-sector differs in that the two approaches lead to λ_*^{B} and g_*^{B} values, with the opposite signs even. However, the B-couplings describe only differences between level-(0) and the D or $p \geq 1$ couplings and thus their precise values and signs are not meaningful as such; they are an indication for the degree of split-symmetry, however. (The negative g_*^{B} found in [172] only tells us that in this case $g_*^{(0)} > g_*^{\text{D}}$.)

Bi-metric [172]	Bi-metric (present)	Single-metric [122]
$\text{NG}_-^{\text{B}} \oplus \text{NG}_+^{\text{D}}\text{-FP}$	$\text{NG}_+^{\text{B}} \oplus \text{NG}_+^{\text{D}}\text{-FP}$	NG-FP
$(g_*^{\text{D}} = 1.05, \lambda_*^{\text{D}} = 0.22)$ $(g_*^{\text{B}} = -41.6, \lambda_*^{\text{B}} = 0.58)$	$(g_*^{\text{D}} = 0.70, \lambda_*^{\text{D}} = 0.21)$ $(g_*^{\text{B}} = 8.2, \lambda_*^{\text{B}} = -0.01)$	$(g_*^{\text{sm}} = 0.71, \lambda_*^{\text{sm}} = 0.19)$
$\text{NG}_+^{(0)} \oplus \text{NG}_+^{\text{D}}\text{-FP}$	$\text{NG}_+^{(0)} \oplus \text{NG}_+^{\text{D}}\text{-FP}$	NG-FP
$(g_*^{\text{D}} = 1.05, \lambda_*^{\text{D}} = 0.22)$ $(g_*^{(0)} = 1.08, \lambda_*^{(0)} = 0.21)$	$(g_*^{\text{D}} = 0.70, \lambda_*^{\text{D}} = 0.21)$ $(g_*^{(0)} = 0.65, \lambda_*^{(0)} = 0.19)$	$(g_*^{\text{sm}} = 0.71, \lambda_*^{\text{sm}} = 0.19)$
$\theta_{\pm} = 4.5 \pm 4.2i$	$\theta_{\pm} = 3.6 \pm 4.3i$	$\theta_{\pm} = 1.5 \pm 3.0i$
$s_{\text{UV}} = 2_{\text{D}} + 2_{(0)} = 4$	$s_{\text{UV}} = 2_{\text{D}} + 2_{(0)} = 4$	$s_{\text{UV}} = 2$

Table 10.1: The properties of the doubly non-Gaussian fixed points are listed for the bi-metric calculation in [172], for the present one, and for the single-metric approximation [122]. The dynamical fixed point properties obtained by the two bi-metric approaches are seen to be qualitatively equivalent, and also well approximated by the single-metric truncation. In the background sector, the results for g_*^{B} and λ_*^{B} differ by sign, but only their combination with the ‘D’ parameters yielding the level-(0) couplings is numerically meaningful, and those are indeed qualitatively similar. In addition the critical exponents for the dynamical sector are given. They are complex, with a positive real part.

Impact of α and ϖ on the NGFPs

The impact of changing the functional form of the gauge fixing condition, concretely the parameter ϖ , and the parameter α in its pre-factor can be observed in the fixed point coordinates. The single-metric, and the present bi-metric truncation employ the same harmonic gauge fixing and $\alpha = 1$ choices; in Table 10.1 we see that their fixed point values almost coincide. The TT-based bi-metric calculation [172] which uses the ‘anharmonic’ choice for ϖ together with $\alpha = 0$ leads to a somewhat different fixed point value $g_*^{\text{D}} > 1$, with about the same λ_*^{D} . This is a small, but visible effect due to the different gauge choices.

For the design of future, more advanced truncations it is also instructive to monitor how the gauge choice influences the ghost sector and the beta-functions. In Fig. 10.21 we therefore plot the ρ_{gh} -dependence of the fixed point coordinates obtained with the old approach [172]. For comparison our findings from Fig. 9.2 are indicated as light gray lines. The diagrams show basically overlapping curves for most coordinates. If $\rho_{\text{gh}} \lesssim 2$ the positive g_*^{D} values, though qualitatively displaying the same increasing behavior for decreasing ρ_{gh} , differ quantitatively, but by much less than one order of magnitude. The standard choice $\rho_{\text{gh}} = 1$ is within this regime and thus reflects the small, but visible difference in the fixed point values of Tab. 10.1. At large ρ_{gh} , the influence of ϖ in the ghost sector becomes completely negligible, at least in the non-Gaussian fixed point regime.

We can get an indication for the quality of the split-symmetry in the vicinity of the NGFP by the size of its g_*^{B} value, or more appropriately, by its inverse. The smaller $1/g_*^{\text{B}}$, the better is the coincidence of $g_*^{(0)}$ and g_*^{D} . In Fig. 10.22 the dependence of $1/g_*^{\text{B}}$ on ρ_{gh} is shown for both bi-metric calculations, that is [172] and the present one. As already pointed out, they yield different signs for g_*^{B} , but this is irrelevant.

The good news we learn from Fig. 10.22 is that $1/g_*^{\text{B}}$ possesses a zero, and that this zero occurs in both calculations very close to $\rho_{\text{gh}} = 1$, that is, to the actually implemented value of the ghost normalization! There, the magnitude of g_*^{B} diverges, and this in turn forces $g_*^{(0)}$ and

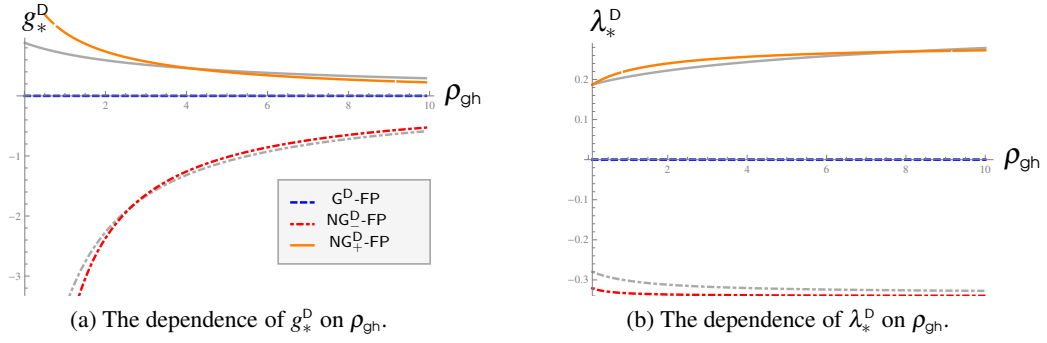


Figure 10.21: The dependence of g_*^D and λ_*^D on ρ_{gh} for the three fixed points, G^D -FP, NG_-^D -FP, and NG_+^D -FP according to the approach of [172] (dark lines). The gray lines indicate the corresponding results of the present calculation. The qualitative agreement of these results for all fixed points is obvious. Except for very small values of ρ_{gh} , even quantitatively the results are seen to be almost equal, indicating that the impact of ϖ on the ghost sector is relatively small, and is probably negligible compared to the other sources of uncertainty.

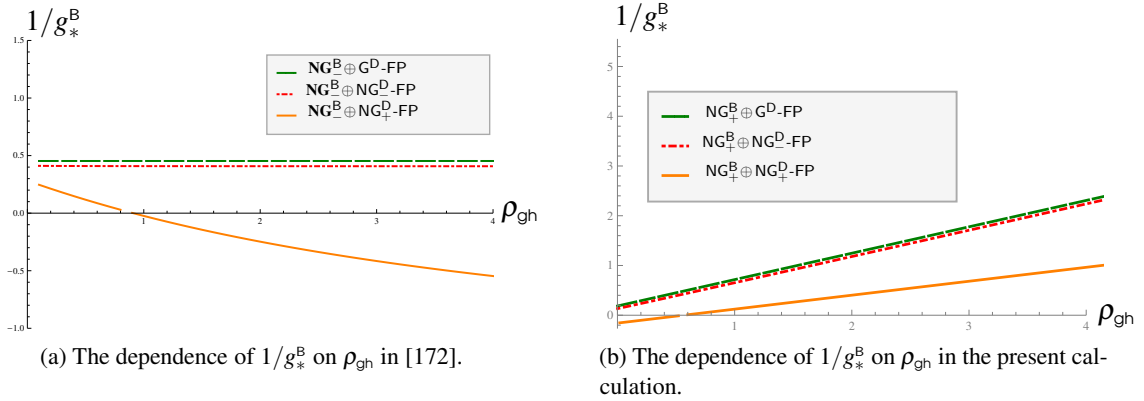


Figure 10.22: The dependence of $1/g_*^B$ on ρ_{gh} is shown for the three ‘B’ non-Gaussian fixed points. The smaller $1/g_*^B$, the better split-symmetry is realized at the respective non-Gaussian fixed point. We see that the curves for the physically most relevant one, based upon NG_+^D -FP, displays a zero which in both calculations is located very close to $\rho_{gh} = 1$.

$g_*^D = g_*^{(p)}$, $p \geq 1$ to be equal. The privileged status of a choice near $\rho_p = 1$ seems to be at least one of the reasons for the ‘miraculously’ good agreement of the single- and bi-metric truncation in the vicinity of the NGFP.

Critical exponents and UV-critical hypersurface

Turning next to the linearized flow near the (doubly) non-Gaussian fixed point, Tab. 10.1 shows that the critical exponents for the two bi-metric calculations are more similar among themselves than in comparison to the single-metric results. In fact, for the spiral motion they predict almost the same frequency.

Still there is a difference between the bi-metric truncations, the origin of which can again be traced back to the λ^D -dependence of the beta-functions. The stability matrix implied by the differential equations in [172] reads:

$$\mathcal{B} = \begin{pmatrix} -2.37 & -46.0 \\ 0.49 & -6.6 \end{pmatrix} \quad (\text{TT-approach}) \quad (10.51)$$

While the first column, corresponding to the derivatives with respect to g^D , shows only relatively small deviations from B_1^D in eq. (10.48), the source of the quantitative change in the critical exponents again originates in the λ^D -derivatives in the second column, which in turn are governed by the λ^D -dependence of $B_1^D(\lambda^D)$ in the vicinity of the non-Gaussian fixed point.

The agreement of the dimensionality of the critical hypersurface as predicted by both bi-metric calculations is an additional point in support of the following general picture: The results of the present, new bi-metric approach and the earlier TT-based one in [172] are in close analogy, at least in the vicinity of the non-Gaussian fixed point. Deviations in the UV are of a minor numerical kind, but most importantly all qualitative results for physically essential quantities agree among the two approaches.

10.4 An application: the running spectral dimension

It has been observed very early on that the effective spacetimes described by the EAA, in particular those along asymptotically safe RG trajectories display self-similar properties reminiscent of fractals, with a k -dependent effective dimensionality $d_{\text{eff}} \equiv 4 + \eta_N$ which interpolates between $d_{\text{eff}} = 4$ macroscopically and $d_{\text{eff}} = 2$ microscopically [123, 135]. This observation gave rise to the development of a general scale dependent analog of Riemannian geometry for those spacetimes [210, 211], and the discovery of a dynamically generated minimum length, a notion that turned out surprisingly subtle [210].

Computing the spectral dimension $\mathcal{D}_s$ of those spacetimes [99] revealed the same crossover from 4 dimensions in the IR to 2 in the UV which was observed on the basis of d_{eff} . It is an exact, truncation independent prediction of asymptotically safe gravity, with or without matter.

More recently [100] the scale dependence of $\mathcal{D}_s$ was reconsidered under the more restrictive assumptions of (i) pure gravity, (ii) the validity of the (single-metric) Einstein-Hilbert truncation, and (iii) the choice of an RG trajectory which admits a long classical regime [209, 212]. Under these conditions, the EAA predicts in addition an extended semiclassical regime at intermediate scales in which the spectral dimension assumes the rational value $\mathcal{D}_s = 4/3$. It has been argued [100] and substantiated by a detailed comparison with Monte Carlo data that the dimensional reduction observed in numerical CDT simulations [213–215] actually originates in this semiclassical regime rather than the asymptotic scaling region of a fixed point. (See [100, 168, 216] for further details, and [125, 217] for extensions.)

The scale dependent spectral dimension $\mathcal{D}_s(k) \equiv \mathcal{D}_s(g_k, \lambda_k)$ derived in [100], under the above conditions, reads (for $d = 4$)

$$\mathcal{D}_s(g, \lambda) = \frac{8}{4 + \lambda^{-1} \beta_\lambda(g, \lambda)} \quad (10.52)$$

where λ was the dimensionless cosmological constant of the single-metric truncation, $\lambda^{\text{sm}} \equiv \bar{\lambda}^{\text{sm}}/k^2$. From the EAA-based derivation of eq. (10.52) it is obvious [99] that λ_k enters this formula via the (contracted) Einstein-equation $R(\langle g \rangle_k) = 4\bar{\lambda}_k^{\text{sm}} = 4k^2 \lambda_k^{\text{sm}}$ which describes how the effective metric responds to the scale dependence of the cosmological constant.

When we now go over from the single- to the bi-metric Einstein-Hilbert truncation we interpret $\langle g \rangle_k$ as a self-consistent background metric. It is given by the tadpole equation¹³ $(\delta\Gamma_k/\delta h)|_{h=0} = 0$ which, for the present truncation, has again the same structure as the classical vacuum Einstein equation, but now containing the running *level-(1) cosmological constant*: $\bar{G}_{\mu\nu} = -\bar{\lambda}_k^{(1)} \bar{g}_{\mu\nu}$. Going through its derivation [100] it is therefore easy to see that the above formula for the spectral dimension remains correct for the bi-metric truncation of the present

¹³See Section 4 of ref. [137] for a detailed discussion.

paper provided we set $\lambda \equiv \lambda^{(1)}$ in eq. (10.52), and interpret β_λ as the beta-function of $\lambda^{(1)}$, depending on the level-(1) couplings:¹⁴

$$\mathcal{D}_s(g^{(1)}, \lambda^{(1)}) = \frac{8}{4 + (\lambda^{(1)})^{-1} \beta_\lambda^{(1)}(g^{(1)}, \lambda^{(1)})} \quad (10.53)$$

Note that $\mathcal{D}_s$ is a scalar function on the $g^{(1)}$ - $\lambda^{(1)}$ theory space: it depends only on the value of the couplings, but not the scale at which the RG trajectory passes there. It is well-defined if the denominator on the RHS of (10.53) is always positive. This is indeed the case for the separatrix and all type (IIIa)^D trajectories, but not for those of type (Ia)^D. They pass through a point where the assumptions behind (10.52) and (10.53) do not apply [122] and $\mathcal{D}_s$ diverges.

It is instructive to insert solutions of the RG equations, $k \mapsto (g_k^{(1)}, \lambda_k^{(1)}) = (g_k^D, \lambda_k^D)$, into eq. (10.53) and determine the resulting scale-dependence of $\mathcal{D}_s$. For each class of trajectories we take one representative and depict the result in the following. While $\mathcal{D}_s$ is actually independent of the B-couplings, it is most natural to think of these trajectories as 4-dimensional ones whose B-sector is chosen in the split symmetry-restoring way.

Type (Ia)^D trajectories.

For this class of trajectories the cosmological constant turns negative in the IR. This entails a

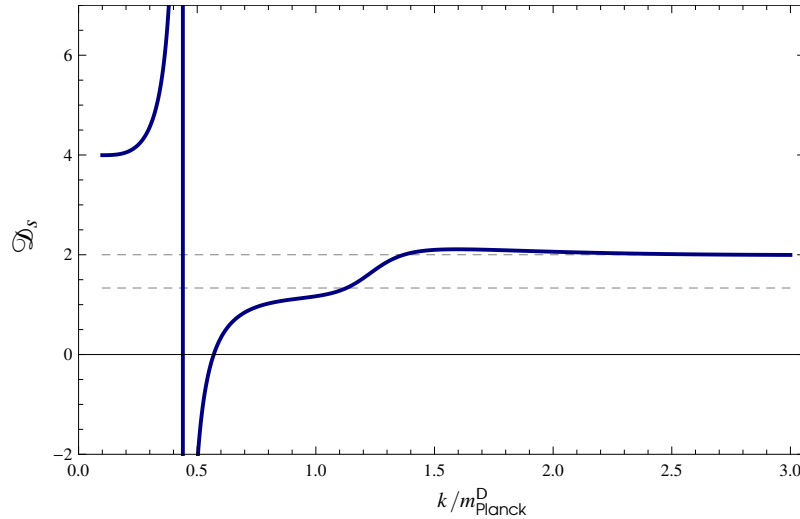


Figure 10.23: The spectral dimension along a typical (Ia)^D trajectory. The interpretation as a spectral dimension is lost near the singularity where the cosmological constant turns negative. Besides a short semiclassical plateau at $\mathcal{D}_s = 4/3$, disturbed by the divergence, the IR and the UV limits at 4 and 2, respectively, are well visible.

divergence of $\mathcal{D}_s$ at a certain scale where the denominator of eq. (10.52) vanishes. In the vicinity of this scale, the function $\mathcal{D}_s$ admits no physical interpretation as a ‘ k -dependent spectral dimension’, since this would require a very slow (‘adiabatic’) dependence on k . Apart from that, we find in the corresponding Fig. 10.23 qualitatively precisely the same k -dependence of $\mathcal{D}_s$, which had been obtained within the single-metric approximation [100, 125]: In the IR the running spectral dimension approaches $\mathcal{D}_s = 4$, there exists a semiclassical plateau at $\mathcal{D}_s = 4/3$, here relatively short because it is disturbed by the divergence, and in the UV the running $\mathcal{D}_s$ converges to its fixed point value, 2.

¹⁴In a truncation that distinguishes between $g^{(p)}$ for $p \geq 1$ the spectral dimension inherits an implicit dependence on higher level couplings since those appear in $\beta_\lambda^{(1)}$.

Type (IIa)^D trajectory.

The separatrix in the D-sector gives rise to the scale-dependent $\mathcal{D}_s$ shown in Fig. 10.24. The

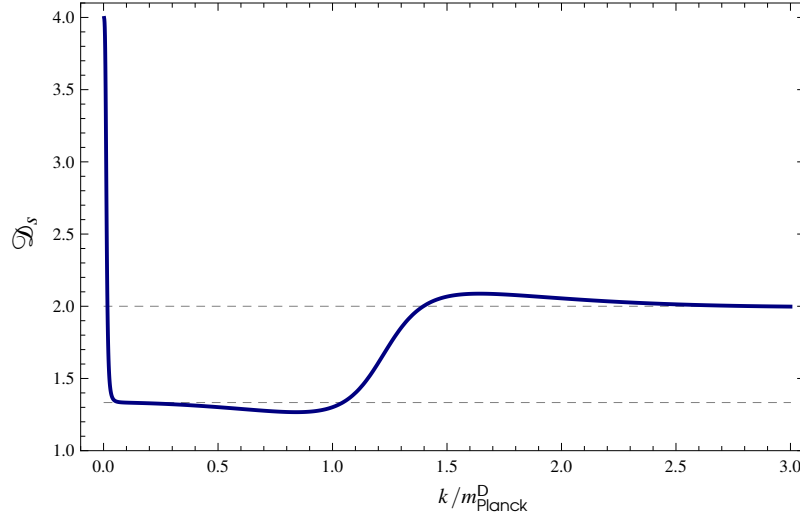


Figure 10.24: Spectral dimension along the (IIa)^D trajectory, the separatrix in the dynamical sector. The jump at $k = 0$ is a computational artifact; in reality the semiclassical regime with $\mathcal{D}_s = 4/3$ extends down to $k = 0$.

semiclassical plateau at $\mathcal{D}_s = 4/3$ is quite pronounced in this case, and it seems to terminate in a sudden jump to $\mathcal{D}_s = 4$ for $k \rightarrow 0$. Actually this jump is a computational artifact since in a numerical calculation one is never able to find the separatrix exactly; rather, almost always the computer will generate a type (Ia) or (IIIa) trajectory. Hence it is ultimately pushed away from the GFP along the λ -direction, at some very low scale $k/m_{\text{Planck}} \ll 1$, and this is exactly what caused the apparent jump to $\mathcal{D}_s = 4$ in Fig. 10.24. In reality, for the perfect separatrix solution (and in absence of matter!) the semiclassical regime with $\mathcal{D}_s = 4/3$ extends down to $k = 0$; there exists no genuinely classical regime with $\mathcal{D}_s = 4$ [100]. Towards the UV, we find a smooth cross over of the semiclassical plateau to $\mathcal{D}_s = 2$, as expected in the NGFP regime.

Type (IIIa)^D trajectories.

For this type of trajectories, Fig. 10.25 shows the running spectral dimension along a typical example. All three plateaus are well visible here, with $\mathcal{D}_s = 2$ in the UV, $\mathcal{D}_s = 4$ in the IR, and an intermediate plateau, well below the Planck scale, with the semiclassical value $\mathcal{D}_s = 4/3$. This, again, is in accord with the results obtained in [100] by means of a single-metric truncation.

Summarizing this subsection we can say that as far as the running spectral dimension is concerned, the single-metric Einstein-Hilbert truncation is a fully reliable approximation to its bi-metric generalization. By its very definition, a k -dependent spectral dimension makes sense only if $\mathcal{D}_s(k)$ changes with k at most ‘adiabatically’ [100]. Basically $\mathcal{D}_s$ can be interpreted meaningfully only when it develops a plateau. Yet, for all three types of trajectories, the single- and bi-metric truncations agree on the respective plateau structures, and on the values which $\mathcal{D}_s$ assumes there.

10.5 A brief look at $d = 2 + \varepsilon$ and $d = 3$

Gravity in, or near two dimensions has always been an important theoretical laboratory for quantum gravity. In particular, the Asymptotic Safety scenario was first proposed in the $2 + \varepsilon$

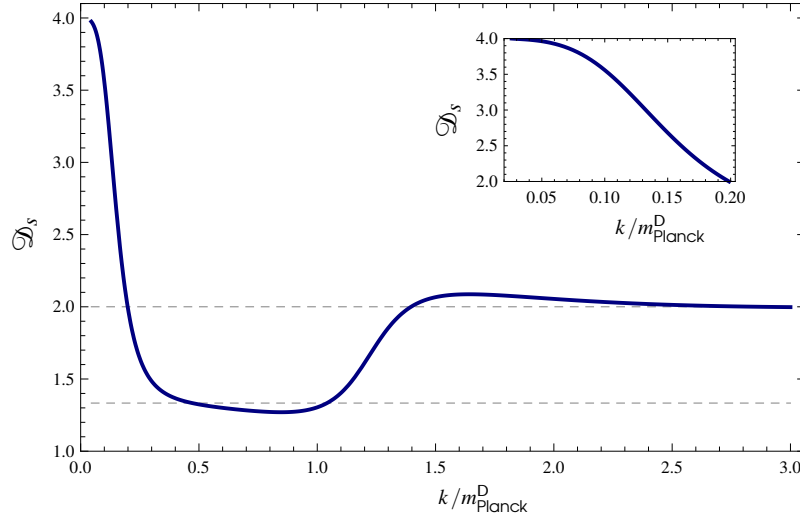


Figure 10.25: The spectral dimension along a type (IIIa)^D trajectory. All three plateaus are well developed: we find $\mathcal{D}_s = 2$ in the UV, $\mathcal{D}_s = 4$ in the IR, and in between a semiclassical regime with $\mathcal{D}_s = 4/3$. The insert shows the classical regime at $k \ll m_{\text{Planck}}^D$.

dimensional Einstein-Hilbert theory. In this section we re-analyze this theory in the bi-metric setting. Because of its universality properties, absent in higher dimensions, our findings about the relation between the single- and bi-metric treatment are particularly clearcut, and in fact quite striking.

Also three dimensions are of special interest since in $d = 3$ the metric and ghost fluctuations compensate exactly *on shell*. For the EAA which is a typical off shell object this implies by no means that there is no RG flow in $d = 3$ (as is often believed wrongly). Rather, while certain characteristic terms indeed disappear from the beta-functions, there is still a non-trivial RG running which needs to be taken seriously, for instance, when one uses the EAA to construct the continuum limit of a regularized functional integral for gravity. In $d = 3$ we have the advantage that this characteristic ‘off-shell running’ can be studied *in isolation*.

In the following two subsections we discuss $d = 2 + \varepsilon$ and $d = 3$ in turn; a complete list of the pertinent beta-functions can be found in chapter 8.

10.5.1 Near dimension two

The case of $d = 2 + \varepsilon$ dimensions is special in that all Newton constants become dimensionless for $\varepsilon \rightarrow 0$. In the lowest nontrivial order in ε , and for vanishing (dynamical) cosmological constant, all anomalous dimensions have the structure

$$\eta^I = -b^I g^I + \mathcal{O}(\varepsilon), \quad I \in \{D, B, (p)\} \quad (10.54)$$

with certain constants b^I . For $\varepsilon \searrow 0$, the leading term in η^I is of order ε^0 . Within our approximation of retaining in all formulae only the lowest nontrivial order the condition for a non-Gaussian fixed point ($\varepsilon + \eta^I = 0$) has a solution which is linear in ε , namely $g_*^I = \varepsilon/b^I$. Note also that in the approximation (10.54) the general relationship (10.22b) connecting the level- to the D-B-language boils down to the simple statement

$$b^{(0)} = b^B + b^D \quad \text{and} \quad b^{(p)} = b^D \quad \text{for } p \geq 1. \quad (10.55)$$

In the single-metric Einstein-Hilbert truncation [122], too, the anomalous dimension is well known to have the structure (10.54). In its dia- vs. para-magnetic decomposed form, the perti-

ment coefficient $b^I \equiv b^{\text{sm}}$ was found to be [165]

$$b^{\text{sm}} = \frac{2}{3} \left[(-3)_{\text{dia}} + (4)_{\text{gh-dia}} + (6)_{\text{para}} + (12)_{\text{gh-para}} \right] = \frac{2}{3} \times [+19] = \frac{38}{3} \quad (10.56)$$

All four contributions to b^{sm} are separately universal, i.e. independent of the cutoff shape function $R_k^{(0)}$.

Guided by our experience from 4 dimensions we might expect that the result of the single-metric calculation approximately agrees at some level of accuracy with that in the D sector of the bi-metric computation. In particular b^{sm} of (10.56) should not be very different from b^{D} , the crucial coefficient in the anomalous dimension η^{D} of the dynamical Newton constant G_k^{D} . What we actually find reads as follows:

$$\begin{aligned} b^{\text{D}} &= \frac{2}{3} \left[-2(3 + 2\rho_{\text{gh}} + 15\rho_{\text{p}}) \Phi_1^2(0) + 12(7 - 2\rho_{\text{gh}}) \rho_{\text{p}} \Phi_2^3(0) \right] \\ &= \frac{2}{3} \left[-6 - 4\rho_{\text{gh}} + 12\rho_{\text{p}} - 12\rho_{\text{gh}}\rho_{\text{p}} \right] \end{aligned} \quad (10.57)$$

The diamagnetic contributions are obtained from this expression by setting $\rho_{\text{p}} = 0$, and the graviton part by letting $\rho_{\text{gh}} = 0$; furthermore, the paramagnetic contributions are those proportional to ρ_{p} , and the ghost part is proportional to ρ_{gh} . Hence, when written in the style of the single-metric result, eq. (10.57) reads

$$b^{\text{D}} = \frac{2}{3} \left[(-6)_{\text{dia}} + (-4)_{\text{gh-dia}} + (12)_{\text{para}} + (-12)_{\text{gh-para}} \right] = \frac{2}{3} \times [-10] = -\frac{20}{3} \quad (10.58)$$

Obviously b^{sm} and b^{D} are quite different, not even their signs are in agreement so that screening and anti-screening behavior get interchanged. While the single-metric calculation predicts $b^{\text{sm}} > 0$, hence a NGFP at $g_*^{\text{sm}} > 0$ (for $\varepsilon > 0$), the bi-metric analogue has $b^{\text{D}} < 0$ with a corresponding fixed point at a *negative* Newton constant, something one normally considers unphysical. This clash is particularly striking since, like in b^{sm} , *all 4 separate contributions appearing in the dia/para, metric/ghost decomposition of b^{D} are separately scheme independent, but none of them agrees with its single-metric analogue.* Indeed, in the second line of (10.57) we exploited that all threshold functions of the type $\Phi_n^{n+1}(0)$, for vanishing argument, assume the universal, i.e. $R_k^{(0)}$ -independent values $\Phi_n^{n+1}(0) = 1/\Gamma(n+1)$, $n \geq 0$. In this sense, both the single- and the bi-metric results can be considered particularly robust and ‘clean’.

Turning to the anomalous dimension of the B-sector we find likewise

$$\begin{aligned} b^{\text{B}} &= \frac{2}{3} \left[(4\rho_{\text{gh}} - 3) \Phi_0^1(0) + 2(3 + 2\rho_{\text{gh}} + 6(3 + \rho_{\text{gh}})\rho_{\text{p}}) \Phi_1^2(0) - 12(7 - 2\rho_{\text{gh}}) \rho_{\text{p}} \Phi_2^3(0) \right] \\ &= \frac{2}{3} \left[3 + 8\rho_{\text{gh}} - 6\rho_{\text{p}} + 24\rho_{\text{gh}}\rho_{\text{p}} \right]. \end{aligned} \quad (10.59)$$

This expression, again, contains only universal values of the threshold functions. Casting (10.59) in a more instructive way, we obtain

$$b^{\text{B}} = \frac{2}{3} \left[(3)_{\text{dia}} + (8)_{\text{gh-dia}} + (-6)_{\text{para}} + (24)_{\text{gh-para}} \right] \equiv \frac{2}{3} \times [+29] = \frac{58}{3} \quad (10.60)$$

Notice that in total b^{B} is *positive* while b^{D} was negative.

What makes these findings particularly alarming, or at least puzzling at first sight is that they show a considerable degree of internal consistency. To see this, let us add up the b -coefficients of the background and the dynamical sector, thereby maintaining the dia/para, metric/ghost decomposition:

$$b^{\text{B}} + b^{\text{D}} = \frac{2}{3} \left[(3 - 6)_{\text{dia}} + (8 - 4)_{\text{gh-dia}} + (-6 + 12)_{\text{para}} + (24 - 12)_{\text{gh-para}} \right] = b^{\text{sm}} \quad (10.61)$$

Remarkably enough, not only does the sum $b^B + b^D$ exactly agree with the old single-metric result, even all 4 terms of the decomposition separately do so. The ‘miracle’ behind (10.61) finds its explanation when we evaluate the level-(0) anomalous dimension, the corresponding coefficient $b^{(0)}$ being

$$\begin{aligned} b^{(0)} &= \frac{2}{3} \left[(4\rho_{\text{gh}} - 3)\Phi_0^1(0) + 6(1 + 2\rho_{\text{gh}})\rho_P\Phi_1^2(0) \right] \\ &= \frac{2}{3} \left[-3 + 4\rho_{\text{gh}} + 6\rho_P + 12\rho_{\text{gh}}\rho_P \right] \end{aligned} \quad (10.62)$$

This result coincides, not only as a sum but even term by term, exactly with the single-metric result (10.56):

$$b^{(0)} = \frac{2}{3} \left[(-3)_{\text{dia}} + (4)_{\text{gh-dia}} + (6)_{\text{para}} + (12)_{\text{gh-para}} \right] = b^{\text{sm}} \quad (10.63)$$

From our discussion in Section 10.3 of the relation between single- and bi-metric beta-functions the equality $b^{(0)} = b^{\text{sm}}$ was indeed to be expected; there we demonstrated that quite generally the single-metric RG equations are closely related to the level-(0) ones in the bi-metric computation. It is reassuring to see this rule at work here, despite potential subtleties related to the limit $\varepsilon \rightarrow 0$.

On the other hand, from (10.55) we know that at the level of the b -coefficients the translation rule from the D-B to the level description is simply $b^{(0)} = b^B + b^D$. Combining this with $b^{(0)} = b^{\text{sm}}$ from (10.63) we obtain $b^B + b^D = b^{\text{sm}}$, and this is precisely what we found in eq. (10.61)!

At this point several important remarks are in order:

- A.** The main message conveyed by the above universal numbers is that *the leading order in ε is suffering from a significant violation of split-symmetry*, as is testified by the values $b^{(0)} = b^{\text{sm}} = +\frac{38}{3}$ at level zero, and $b^{(1,2,3,\dots)} = b^D = -\frac{20}{3}$ at the higher levels. Equivalent to that, there is a corresponding disagreement between the single- and bi-metric results, even at the level of signs.
- B.** In order to judge to what extent these somewhat disconcerting findings might carry over to higher dimensions, $d = 4$ in particular, the (perhaps) worried reader should recall the following facts.
 - B.1.** In general dimensions, $\eta^D(g^D, \lambda^D)$ is given by eq. (8.33), and it involves two functions of the dynamical cosmological constant, B_1^D and B_2^D . Expanding this anomalous dimension in g^D , and retaining the leading, i.e. linear term only, it reads

$$\eta^D(g^D, \lambda^D; d) = B_1^D(\lambda^D; d)g^D + \mathcal{O}(g^{D^2}) \quad (10.64)$$

Upon an additional expansion of B_1^D for small λ^D the approximation for η^D boils down to

$$\eta^D(g^D, \lambda^D; d) = -(d-2)\omega_d^D g^D + \mathcal{O}(g^{D^2}) + \mathcal{O}(\lambda^D) \quad (10.65)$$

It is characterized by a single coefficient only, $\omega_d^D \equiv -B_1^D(0; d)/(d-2)$.

- B.2.** If we set $d \equiv 2 + \varepsilon$ and then expand η^D of eq. (8.33) in powers of ε we obtain $\eta^D = -b^D g^D + \mathcal{O}(\varepsilon)$ for the lowest non-trivial order in ε , with $b^D = \lim_{\varepsilon \rightarrow 0} (\varepsilon \omega_{2+\varepsilon}^D)$ a well defined, and non-zero number. Notice that this simple equation for η^D has the same structure as the doubly expanded (in g^D and λ^D) general result (10.65) valid for all d . Here, however, it results from nothing more than the $\varepsilon \rightarrow 0$ limit alone [122]; no separate assumption about the smallness of the couplings and, based on that, expansions with respect to g^D , λ^D are invoked. Stated differently, if we retain only at the lowest order in ε , it is unavoidable that η^D becomes linear in g^D and, more importantly, that the coefficient function $B_1^D(\lambda^D)$ automatically always gets

evaluated at the point $\lambda^D = 0$ only. So, the essential conclusion for the present discussion is that *in the limit $\varepsilon \rightarrow 0$, to leading order in ε , the RG flow completely ‘forgets’ about how a non-zero value of λ^D affects the RG running of g^D .*

B.3. In accord with this last remark, even the NGFP at $g_*^D = \varepsilon/b^D$ is fully determined by b^D , i.e. by $B_1^D(\lambda^D = 0)$. For $\varepsilon \searrow 0$, the very same coefficient which for general d controls the semiclassical regime near the GFP only also decides about whether or not there exists the desired NGFP on the $g^D > 0$ half-space. We saw that this led to a clash between the single- and bi-metric calculation, since $b^{\text{sm}} > 0$, but $b^D < 0$. However, in subsection 9.2.4 we also saw already that in $d = 4$ the situation is different in a crucial way. There $B_1^D(\lambda^D)$ changes its sign between $\lambda^D = 0$ and $\lambda^D = \lambda_*^D$; as a result, the semiclassical regime on the $g^D > 0$ half-space exhibits gravitational screening ($\eta^D > 0$), as in $d = 2 + \varepsilon$, while for larger values of λ^D the sign of η^D turns negative, ultimately approaches $\eta_*^D = -2$, and a NGFP forms at a *positive* value of Newton’s constant, exactly where we would like it to be.

C. Summarizing the above remarks, we can say that *the most problematic property of the $(2 + \varepsilon)$ -dimensional theory, the dynamical Newton constant having a negative fixed point value¹⁵, is an artifact of the ε -expansion.* In $d = 4$, instead, screening at small λ^D can coexist with anti-screening and a NGFP at larger values of the cosmological constant thanks to the λ^D -dependence of η^D , a property the leading order of the ε -expansion is completely insensitive to.

10.5.2 Three dimensions

Next, let us turn our attention to spacetimes of dimension $d = 3$. The full set of results is given in section 8.B of chapter 8. First we focus on the anomalous dimensions in the semiclassical regime which, for all d , read $\eta^I = -(d-2)\omega_d^I g^I + \mathcal{O}(g^{I^2}, \lambda^D)$. In $d = 2 + \varepsilon$ the coefficients ω_d^I were universal, in $d = 3$ they explicitly depend on the cutoff $R_k^{(0)}$ instead. To make our main point as clear as possible, it is instructive to write down the dia/para and metric/ghost decomposed form of the coefficients appearing in $\eta^I = -\omega_3^I g^I + \mathcal{O}(g^{I^2}, \lambda^D)$ for $I = \{D, B, (0)\}$. In the D and B sector we find, respectively:

$$\begin{aligned}\omega_3^D &= -\frac{1}{\sqrt{\pi}} \left[(3 + 2\rho_{\text{gh}} + 2\rho_P(9 - 4\rho_{\text{gh}})) \Phi_{3/2}^2(0) - 6\rho_P(9 - 4\rho_{\text{gh}}) \Phi_{5/2}^3(0) \right] \\ \omega_3^B &= \frac{1}{\sqrt{\pi}} \left[2(\rho_{\text{gh}} - 1) \Phi_{1/2}^1(0) + (3 + 2\rho_{\text{gh}} + 4\rho_P(6 - \rho_{\text{gh}})) \Phi_{3/2}^2(0) \right. \\ &\quad \left. - 6\rho_P(9 - 4\rho_{\text{gh}}) \Phi_{5/2}^3(0) \right]\end{aligned}\quad (10.66)$$

Notice that ω_3^D and ω_3^B are perfectly generic, in the sense that they receive contributions of both paramagnetic and diamagnetic origin, from both ghost and metric fluctuations.

The expected ‘magic’ happens only in the case of the level-(0) coefficient: From the general identity

$$\frac{\eta^{(0)}}{g_k^{(0)}} = \frac{\eta^D}{g_k^D} + \frac{\eta^B}{g_k^B} \quad \text{and} \quad \frac{\eta^{(p)}}{g_k^{(p)}} = \frac{\eta^D}{g_k^D} \quad \text{for all } p \geq 1. \quad (10.67)$$

we infer $\omega_3^{(0)} = \omega_3^D + \omega_3^B$ in the present approximation, yielding

$$\omega_3^{(0)} = \frac{2}{\sqrt{\pi}} \left[(\rho_{\text{gh}} - 1) \Phi_{1/2}^1(0) + \rho_P(3 + 2\rho_{\text{gh}}) \Phi_{3/2}^2(0) \right] \quad (10.68)$$

¹⁵We assume $\varepsilon > 0$ here, as always.

Upon ‘switching on’ the ghosts by setting $\rho_{\text{gh}} = 1$, the entire coefficient $\omega_3^{(0)}$ is seen to be proportional to ρ_p . Hence *the level-(0) anomalous dimension $\eta^{(0)} \propto \omega_3^{(0)}$ is of purely paramagnetic origin.*

The non-zero diamagnetic contributions from ω_3^D and ω_3^B have canceled precisely. At the level of the bi-metric EAA, for generic arguments (‘off shell’ in particular), this cancellation is the only immediate reflection of the fact that classical Einstein-Hilbert gravity has no propagating modes in $d = 3$. There is no comparable compensation of metric and ghost modes at higher orders.

The observed perfect cancellation of all diamagnetic terms in $\omega_3^{(0)}$ is fully consistent with earlier results on the single-metric case [165] where the same cancellation was found to occur in the analogous coefficient ω_3^{sm} . Indeed, it can be verified again that $\omega_3^{(0)} = \omega_3^{\text{sm}}$, as it should be for general reasons.

Finally, let us leave the semiclassical regime and consider the possibility of non-Gaussian fixed points in 3 dimensions. Using the full fledged beta-functions of section 8.B we do indeed find such fixed points. The results can be summarized as follows:

d=3	NG ₊ ^D -FP	NG ₋ ^D -FP	G ^D -FP
	$(g_*^D=0.13, \lambda_*^D=0.2)$	$(g_*^D=-0.38, \lambda_*^D=-0.25)$	$(g_*^D=0, \lambda_*^D=0)$
NG ₊ ^B -FP	NG ₊ ^B ⊕NG ₊ ^D -FP	NG ₊ ^B ⊕NG ₋ ^D -FP	NG ₊ ^B ⊕G ^D -FP
	$(g_*^B=1.3, \lambda_*^B=-0.97)$	$(g_*^B=0.17, \lambda_*^B=-0.14)$	$(g_*^B=0.18, \lambda_*^B=-0.15)$
G ^B -FP	G ^B ⊕NG ₊ ^D -FP	G ^B ⊕NG ₋ ^D -FP	G ^B ⊕G ^D -FP
	$(g_*^B=0, \lambda_*^B=0)$	$(g_*^B=0, \lambda_*^B=0)$	$(g_*^B=0, \lambda_*^B=0)$

(10.69)

We find, as in $d = 4$, a total of six fixed points, five of which are non-Gaussian. Remarkably enough, the qualitative picture is exactly the same as in the four dimensional case, displaying a non-Gaussian fixed point both in the upper and lower half-plane of g^D , as well as a Gaussian one at $g_*^D = 0$. Instead, the background coordinate value g_*^B (λ_*^B) is found to be positive (negative). Furthermore, a relatively large hierarchy $g_*^D \approx 10 g_*^B$ for the doubly non-Gaussian NG₊^B⊕NG₊^D-FP, indicating approximate split-symmetry, is found also in $d = 3$.

10.6 Summary and conclusion

In this section we start with a summary of this chapter by means of a brief and concise list displaying the main results. Therefore, it is convenient to also recapitulate the findings of chapter 9 that includes the fixed points of the RG flow and its properties, thus addressing the question of Asymptotic Safety within the present truncation. We then close with a number of general conclusions and the essential lessons for future investigations, which we learned here.

10.6.1 Summary of the main results

A. On the technical side, we employed and tested a new method for dealing with operator traces involving uncontracted covariant derivatives which is much simpler than those used before. We found that, within the expected truncation uncertainty and cutoff dependence, the resulting RG flow matches the results obtained with the conventional method based on a York decomposition, even though, mathematically, the pertinent beta-functions are quite different. We hope that similar methods will be helpful also in future bi-metric calculations.

B. On the 4-dimensional theory space $\mathcal{T} \equiv \{(g^D, \lambda^D, g^B, \lambda^B)\}$ the RG flow was found to decompose hierarchically according to $(g^D, \lambda^D) \rightarrow g^B \rightarrow \lambda^B$. This allows us to compute the flow on the dynamical subspace $\mathcal{T}_D \equiv \{(g^D, \lambda^D)\}$ without reference to the background couplings.

- C.** The main characteristics of the 2-dimensional flow on $\mathcal{T}_D$ are as follows:
- C.1.** There exist two non-Gaussian and one Gaussian fixed point on $\mathcal{T}_D$, namely $\text{NG}_-^D\text{-FP}$, $\text{NG}_+^D\text{-FP}$, and $G^D\text{-FP}$. They are located at a negative, positive, and vanishing coordinate g_*^D , respectively.
 - C.2.** Reliability analyses reveal that $\text{NG}_+^D\text{-FP}$ is very robust, while $\text{NG}_-^D\text{-FP}$ is likely to be a truncation artifact.
 - C.3.** The critical exponents of $\text{NG}_+^D\text{-FP}$ are given by a complex conjugate pair, with a non-zero imaginary part leading to spiral-shaped trajectories. The fixed point is UV attractive in both directions.
 - C.4.** There are no RG trajectories crossing the $g^D = 0$ line, in particular there exists no cross-over trajectory connecting $\text{NG}_-^D\text{-FP}$ to $\text{NG}_+^D\text{-FP}$. We may therefore restrict $\mathcal{T}_D$ to the half-plane with $g^D > 0$ which is invariant under the flow.
 - C.5.** The phase portrait on $\mathcal{T}_D$ is very similar to the one obtained with the single-metric truncation. In particular the properties of $\text{NG}_+^D\text{-FP}$ are numerically similar to those of the single-metric fixed point, and the RG trajectories admit exactly the same ('type Ia, IIa, IIIa') classification that has been introduced for the single-metric case.
 - C.6.** In contrast to all single-metric based predictions, gravitational anti-screening is lost near the Gaussian fixed point $G^D\text{-FP}$.
- D.** Each RG trajectory on $\mathcal{T}_D$, together with initial conditions for g^B and λ^B , gives rise to a 4-dimensional trajectory. We analyzed the corresponding flow on the complete theory space $\mathcal{T}$.
- D.1.** On $\mathcal{T}$, there exists a total of 6 fixed points, each one of the three D-fixed points above can be combined with both a Gaussian and a non-Gaussian fixed point of the background couplings.
 - D.2.** The doubly non-Gaussian fixed point on $\mathcal{T}$, $\text{NG}_+^B \oplus \text{NG}_+^D\text{-FP}$, is the natural candidate for the construction of an asymptotically safe infinite cutoff limit. It possesses 4 relevant directions, so the UV critical hypersurface associated to it, $\mathcal{S}_{UV}$, has maximum dimension within the present truncation.
- E.** In order to define a quantum field theory, we need an RG trajectory which, at ' $k = \infty$ ', starts out infinitesimally close to the fixed point, always runs on $\mathcal{S}_{UV}$, and thereby gradually becomes split-symmetric, at the very least in the physical limit when k approaches zero. This is the crucial requirement of Background Independence. Within the bi-Einstein-Hilbert truncation it requires Γ_k^{grav} to loose its 'extra $\bar{g}$ -dependence', that is, $1/G_k^B$ and $\bar{\lambda}^B/G_k^B$ must vanish in the IR limit of low scales k approaching zero.
- E.1.** For the first time, it was possible to demonstrate explicitly there do indeed exist trajectories which meet both of the two key requirements, Asymptotic Safety and Background Independence, simultaneously. Those trajectories are labeled by only *two* free parameters. Thus the theory's predictivity is actually higher than expected on the basis of a 4-dimensional $\mathcal{S}_{UV}$.
 - E.2.** The RG trajectories which restore split-symmetry in the IR (at $k = k_{\text{IR}}$) were found to be precisely those which merge with the 'running UV attractor' at low scales, $\text{Attr}^B(k)$.
- F.** The relevance of the running UV attractor $\text{Attr}^B(k)$ to the problem of split-symmetry restoration was uncovered by the following sequence of results and observations.
- F.1.** For every fixed trajectory U^D on $\mathcal{T}_D$, i.e. $k \mapsto (g_k^D, \lambda_k^D) \equiv U^D(k)$ the background couplings g_k^B and λ_k^B are governed by a non-autonomous, i.e. explicitly RG-time dependent system of differential equations on the 2-dimensional subspace $\mathcal{T}_B \equiv \{(g^B, \lambda^B)\}$. It can be visualized as a time dependent vector field $\vec{\beta}_B(k)$ on $\mathcal{T}_B$, which also depends on the D-trajectory chosen.
 - F.2.** For every trajectory U^D , and at every fixed RG time k , the vector field $\vec{\beta}_B(k)$ has an (instantaneous) zero at $(g_\bullet^B(k), \lambda_\bullet^B(k)) \in \mathcal{T}_B$. The (likewise instantaneous) stability anal-

ysis reveals that it is UV attractive in both B-directions, hence the curve $k \mapsto \text{Attr}^B(k) \equiv (g_k^D, \lambda_k^D, g_\bullet^B(k), \lambda_\bullet^B(k))$ acts as a ‘running UV attractor’. It is *not* an RG trajectory by itself.

F.3. The UV limit of the running attractor is precisely the doubly non-Gaussian fixed point: $\lim_{k \rightarrow \infty} \text{Attr}^B(k) = \text{NG}_+^B \oplus \text{NG}_+^D\text{-FP}$.

F.4. Picking a D-trajectory U^D we are equipped with a k -dependent vector field $\vec{\beta}_B$ on $\mathcal{T}_B$. Integral curves of $\vec{\beta}_B(k')$ at any fixed moment of ‘time’, k' , are *not* projections onto $\mathcal{T}_B$ of an RG trajectory on the full theory space in general; they correspond to ‘snapshots’ of the phase portrait on $\mathcal{T}_B$ at this very moment.

F.5. For every ‘initial’ point $(g_{\text{in}}^B, \lambda_{\text{in}}^B)$ specified on $\mathcal{T}_B$ at the time k_{in} there exists a unique solution of the RG equations through this point, $k \mapsto U^B(k)$, with $U^B(k_{\text{in}}) = (g_{\text{in}}^B, \lambda_{\text{in}}^B)$. It is obtained by integrating the (now explicitly k dependent) equation $\partial_t U^B(k) = \vec{\beta}_B(U^B(k); k)$ both upward and downward. Making all input data explicit, we denote this solution as $U^B\{U^D; k_{\text{in}}; g_{\text{in}}^B, \lambda_{\text{in}}^B\}(k)$. It gives rise to an RG trajectory on the full theory space: $U(k) = (U^D(k), U^B(k))$.

F.6. We find that in the limit $k \gg k_{\text{in}}$ of a long upward evolution, $U^B\{U^D; k_{\text{in}}; g_{\text{in}}^B, \lambda_{\text{in}}^B\}$ loses its memory of the initial point $(g_{\text{in}}^B, \lambda_{\text{in}}^B)$, and a universal limit curve is obtained:

$$\lim_{k \gg k_{\text{in}}} U^B\{U^D; k_{\text{in}}; g_{\text{in}}^B, \lambda_{\text{in}}^B\}(k) = U_\bullet^B(k)$$

F.7. Every D trajectory U^D implies a specific limit curve $U_\bullet^B(k)$, and together they define an RG trajectory on the full theory space: $k \mapsto U_\bullet(k) = (U^D(k), U_\bullet^B(k))$. This trajectory is precisely the one which, in the IR, ends on the running attractor $\text{Attr}^B(k)$.¹⁶ It is determined by the ‘final condition’ $(g_{k_{\text{IR}}}^B, \lambda_{k_{\text{IR}}}^B) = (g_\bullet^B(k_{\text{IR}}), \lambda_\bullet^B(k_{\text{IR}}))$ with $k_{\text{IR}} \rightarrow 0$. In short, $U_\bullet(k_{\text{IR}}) = \text{Attr}^B(k_{\text{IR}})$, while in the opposite limit $U_\bullet(k)$ approaches the doubly non-Gaussian fixed point: $U_\bullet(k \rightarrow \infty) = \text{NG}_+^B \oplus \text{NG}_+^D\text{-FP}$.

F.8. We found that the class of all trajectories $U_\bullet(k)$, obtained by varying the underlying $U^D(k)$, are of special importance: The trajectories in this class, and only those, restore split-symmetry in the IR, being at the same time asymptotically safe. This explains the relevance of the running UV attractor to the problem of split-symmetry restoration.

G. Performing a detailed comparison of the single- and the bi-metric Einstein-Hilbert truncation we found that, quite unexpectedly, the former is a rather precise approximation to the latter in the vicinity of the non-Gaussian fixed point. In the far IR the split-symmetry restoring trajectories, by construction, give rise to another regime in which the two truncations agree well. However, in between there are strong qualitative differences, for instance with respect to the sign of the dynamical anomalous dimension η^D . Furthermore, the quantitative differences of the critical exponents in both settings are quite significant, despite the ‘miraculous’ precision of the single-metric truncation near the NGFP. They clearly show the limitations of the single-metric approximation. Results from an independent calculation using the TT-decomposition techniques [172] support these findings.

H. As a concrete application of the bi-metric flow in $d = 4$ we computed the running spectral dimension $\mathcal{D}_s(k)$ of the emergent fractal spacetimes according to the definition proposed in [100]. We found that the bi- and single-metric results agree on all universal predictions this definition can give rise to, namely the formation of plateaus on which $\mathcal{D}_s(k)$ assumes the values $\mathcal{D}_s = 4$, $\mathcal{D}_s = 4/3$, and $\mathcal{D}_s = 2$, respectively.

I. We saw that in $d = 3$ the bi-metric flow is very similar to the one in $d = 4$, even though the classical Einstein-Hilbert theory in $d = 3$ has no propagating modes. This makes it particularly

¹⁶Note that numerically constructing the trajectory reaching $\text{Attr}^B(k_{\text{IR}})$ involves fine-tuning, in the sense that the desired final point is IR repulsive in both B-directions. From this perspective the UV attractor is better called an ‘IR repeller’.

clear that, in any dimension, due to the off-shell nature of the EAA not only the ‘radiative’ field modes but also the ‘coulombic’ ones play an important role in driving the RG evolution. This is in accord with the physical picture of ‘paramagnetic dominance’ put forward in [165, 195].

J. The most striking and even qualitatively essential discrepancies between the single- and the bi-metric predictions we encountered in the leading order of the ε -expansion about two dimension ($d = 2 + \varepsilon$, $\varepsilon > 0$). Disentangling metric /ghost and dia- /paramagnetic contributions, both the anomalous dimension η^D and its single-metric approximation η^{sm} are characterized by 4 *separately universal* coefficients. All 4 of them were found to *disagree* between η^D and η^{sm} . Thus, contrary to the single-metric prediction, the bi-metric RG flow possesses *no NGFP at positive Newton constant*. We were able to show that this discrepancy and the related strong breaking of split-symmetry are an artifact of the leading order ε -expansion, which does not generalize to higher dimensions.

10.6.2 Conclusions and outlook

The main message of the present investigations for future work on the Asymptotic Safety program is quite clear: From a certain degree of precession onward it seems to make little sense to keep including further invariants in the truncation ansatz that are built from the dynamical metric alone. Rather, to the same extent we allow the effective average action to depend on $g_{\mu\nu}$ in a more complicated way, also its dependence on the background metric $\bar{g}_{\mu\nu}$ must be generalized. While the picture conveyed by the single-metric truncations is qualitatively correct often, they are certainly insufficient for quantitatively precise calculations, the determination of critical exponents for example.

Taking the bi-metric structure of the EAA seriously we find ourselves in a situation which is very widespread in quantum field theory: One starts from a classical field theory with a certain symmetry, tries to quantize it, thereby discovers that some sort of regularization is needed, then picks a regulator which spoils the symmetry, perhaps because there exists no invariant regularization, and finally after the quantization one tries to re-establish this symmetry at the observable level, or close to it. A well known example of this situation is the quantization of the electromagnetic field in a scheme which violates gauge invariance. Computing radiative corrections one then might encounter a non-zero photon mass at the intermediate steps of the calculation. At the end, however, it must always be possible to choose the bare parameters in such a way that the renormalized, physical photon mass turns out exactly zero.

In quantum gravity, the background quantum split-symmetry should be seen as analogous to gauge invariance in this example. Split-symmetry is a formal way to express the arbitrariness of the background on which we quantize the metric fluctuations, and unbroken split-symmetry is tantamount to Background Independence. The effective average action is not invariant per se, but the space of RG trajectories contains special solutions which re-establish split-symmetry at the physical level.

In this paper we showed explicitly that the restoration of split-symmetry is indeed possible, and how it can be achieved in practice. In future bi-metric analyses of Quantum Einstein Gravity this will always be a central and important step. The analogy with the photon mass makes it very clear that the implementation of split-symmetry deserves considerable attention since otherwise we never can be sure to deal with the right ‘universality class’. After all, comparing QEG to the familiar matter field theories on flat space, its most momentous distinguishing feature is Background Independence; it implies in particular the necessity of an ab initio derivation of the arena all non-gravitational physics takes place in, namely spacetime. Clearly this has much more profound consequences for the general structure of the theory than its notorious perturbative non-renormalizability, for example, which shows up at a secondary technical level only.



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11. A GENERALIZED $\mathcal{C}$ -THEOREM

 This chapter follows closely ref. [193].

We develop a generally applicable method for constructing functions, $\mathcal{C}$, which have properties similar to Zamolodchikov’s C -function, and are geometrically natural objects related to the theory space explored by non-perturbative functional Renormalization Group (RG) equations. Employing the Euclidean framework of the Effective Average Action (EAA), we propose a $\mathcal{C}$ -function which can be defined for arbitrary systems of gravitational, Yang-Mills, ghost, and bosonic matter fields, and in any number of spacetime dimensions. It becomes stationary both at critical points and in classical regimes, and decreases monotonically along RG trajectories provided the breaking of the split-symmetry which relates background and quantum fields is sufficiently weak. Within the Asymptotic Safety approach we test the proposal for Quantum Einstein Gravity in $d > 2$ dimensions, performing detailed numerical investigations in $d = 4$. We find that the bi-metric Einstein-Hilbert truncation of theory space, here discussed for the results of II and for those of ref. [172], is general enough to yield perfect monotonicity along the RG trajectories, while its more familiar single-metric analog fails to achieve this behavior which we expect on general grounds. Investigating generalized crossover trajectories connecting a fixed point in the ultraviolet to a classical regime with positive cosmological constant in the infrared, the $\mathcal{C}$ -function is shown to depend on the choice of the gravitational instanton which constitutes the background spacetime. For de Sitter space in 4 dimensions, the Bekenstein-Hawking entropy is found to play a role analogous to the central charge in conformal field theory. We also comment on the idea of a ‘ $\bar{\lambda}$ - N connection’ and the ‘ N -bound’ discussed earlier.

11.1 Introduction

One of the most remarkable results in 2-dimensional conformal field theory is Zamolodchikov’s c -theorem [218, 219]. It states that every 2D Euclidean Quantum Field theory with reflection positivity, rotational invariance, and a conserved energy momentum tensor possesses a function

C of its coupling constants, which is non-increasing along the renormalization group trajectories and is stationary at fixed points where it equals the central charge of the corresponding conformal field theory.

After the advent of this theorem many authors tried to find a generalization that would be valid also in dimensions greater than two [220–229]. This includes, for instance, suggestions by Cardy [220] to integrate the trace anomaly of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$ over a 4-sphere of unit radius, $C \propto \int_{S^4} d^4x \sqrt{g} \langle T_{\mu}^{\mu} \rangle$, the work of Osborn [221, 222], and ideas based on the similarity of C to the thermodynamical free energy [223], leading to a conjectural ‘ F -theorem’ which states that, under certain conditions, the finite part of the free energy of 3-dimensional field theories on S^3 decreases along RG trajectories and is stationary at criticality [224]. Cappelli, Friedan and Latorre [225] proposed to define a C -function on the basis of the spectral representation of the 2-point function of the energy-momentum tensor.

While these investigations led to many important insights into the expected structure of the hypothetical higher-dimensional C -function, the search was successful only recently [230, 231] with the proof of the ‘ a -theorem’ [220, 229]. According to the a -theorem, the coefficient of the Euler form term in the induced gravity action of a 4D theory in a curved, but classical, background spacetime is non-increasing along RG-trajectories.

Clearly theorems of this type are extremely valuable as they provide non-perturbative information about quantum field theories or statistical systems in the strong coupling domain. They constrain the structure of possible RG flows on theory space, and they rule out exotic behavior such as limit cycles, for instance (at least for a suitable class of beta-functions [232]).

In this paper we describe and test a broadly applicable search strategy by means of which generalized ‘ C -like’ functions could be identified under rather general conditions, in particular in cases where the known c - and the a -theorems do not apply. Our main motivation is in fact theories which include quantized gravity, in particular those based upon the Asymptotic Safety construction [122, 145, 168–171, 233].

In a first step, we try to generalize only one specific feature of Zamolodchikov’s C -function for a generic field theory in any number of dimensions, namely its ‘counting property’: the sought-after function should roughly be equal to (or at least in a known way be related to) the number of degrees of freedom that are integrated out along the RG trajectory when the scale is lowered from the ultraviolet (UV) towards the infrared (IR). Technically, we shall do this by introducing a higher-derivative mode-suppression factor in the underlying functional integral which acts as an IR cutoff. We can then take advantage of the well established framework of the Effective Average Action (EAA) to control the scale dependence [110–114].

In a generic theory comprising a certain set of dynamical fields, Φ , and corresponding background fields, $\bar{\Phi}$, the EAA is a ‘running’ functional $\Gamma_k[\Phi, \bar{\Phi}]$ similar to the standard effective action, but with a built-in IR cutoff at a variable mass scale k . Its k -dependence is governed by an exact functional RG equation the FRGE we discussed in detail in chapter 4.

The specific property of the EAA which will play a crucial role in our approach is the following: For a broad class of theories, those that are ‘positive’ in a sense we shall explain, the very structure of the FRGE implies that the EAA is a monotonically increasing function of the IR cutoff:

$$\partial_k \Gamma_k[\Phi, \bar{\Phi}] \geq 0 \quad \forall (\Phi, \bar{\Phi}) \quad (11.1)$$

We shall refer to this property as the *pointwise monotonicity* of the EAA since it applies at all points $(\Phi, \bar{\Phi})$ of field space independently. Thus the EAA provides us with even infinitely many monotone functions of the RG scale k , one for each field configuration $(\Phi, \bar{\Phi})$. So one might wonder if those functions, or a combination thereof, deserve being considered a generalization of Zamolodchikov’s C -function.

Unfortunately the answer is negative, for the following reason. The pointwise monotonicity property refers to Γ_k when it is evaluated at *dimensionful* field arguments, and is parametrized by dimensionful running coupling constants. However, the c -theorem and its generalizations apply to the RG flow on *theory space*, $\mathcal{T}$, a manifold which is coordinatized by the *dimensionless* couplings. The latter differ from the dimensionful ones by explicit powers of k fixed by the canonical scaling dimensions. As a consequence, when rewritten in terms of dimensionless fields and couplings, the property (11.1) does not precisely translate into a monotonicity statement about the de-dimensionalized theory space analog of Γ_k , henceforth denoted $\mathcal{A}_k$. Rather, when the derivative ∂_k hits the explicit powers of k , additional canonical scaling terms arise which prevent us from concluding simply ‘by inspection’ that $\mathcal{A}_k$ is monotone along RG trajectories.

Nevertheless, our main strategy will be to take maximum advantage of the pointwise monotonicity of Γ_k as the primary input, and then try to get a handle on the monotonicity violations that occur in going from Γ_k to $\mathcal{A}_k$. We shall see that by evaluating the action functionals on special field configurations this difficulty can be reduced to a far more tractable level.

A related issue is that Γ_k , while having attractive monotonicity properties, does not in addition also become stationary at fixed points, as a sensible generalization of the C -function in 2D should. But, again for the above reason, this is not necessarily a drawback since at an RG fixed point it is anyhow not the dimensionful couplings but rather the dimensionless coordinates of theory space that are supposed to assume stationary values, i.e. actually it is $\mathcal{A}_k$ that approaches a ‘fixed functional’, $\mathcal{A}_*$.

When we look at Γ_k and $\mathcal{A}_k$ pointwise, or equivalently, at the infinitely many k -dependent couplings they parametrize them independently, then there is a clash between stationarity at fixed points and monotonicity along the RG flow: $\mathcal{A}_k$ is stationary at fixed points, but not monotone, while for Γ_k the situation is the other way around. However, by adopting the pointwise perspective we are expecting by far too much, namely that all dimensionless couplings individually behave like a C -function. Presumably we can hope to find at best a single, or perhaps a few, real valued quantities with all desired properties. We shall denote such a hypothetical function by $\mathcal{C}_k$ in the following. Assuming it exists, the quantity $\mathcal{C}_k$ is *one* function depending on *infinitely many* running couplings along the RG trajectory, so the transition from Γ_k to $\mathcal{C}_k$ amounts to a tremendous ‘data reduction’.

Thus, within the EAA framework, the central question is whether there exists a kind of ‘essentially universal’ map from k -dependent functionals Γ_k to a function $\mathcal{C}_k$ that is monotone along the flow and stationary at fixed points. Here the term ‘universal’ is to indicate that we would require only a few general properties to be satisfied, comparable to reflection positivity, rotational invariance, etc. in the case of Zamolodchikov’s theorem. The reason why we believe that there should exist such a map is that the respective monotonicity properties of Γ_k and the C -function in 2D have essentially the same simple origin. They both ‘count’ in a certain way the degrees of freedom (more precisely: fluctuation modes) that are integrated out at a given RG scale intermediate between the UV and the IR.

We begin by considering a class of $\mathcal{C}_k$ -candidates which are obtained by evaluating $\Gamma_k[\Phi, \bar{\Phi}]$ at a particularly chosen pair of arguments $(\Phi, \bar{\Phi})$ that will have an explicit dependence on k in general. In this chapter we propose to use *self-consistent background field configurations* for this purpose. We evaluate the EAA at a scale dependent point in field space, namely $\Phi = \bar{\Phi} \equiv \bar{\Phi}_k^{\text{s.c.}}$. By definition, a background field $\bar{\Phi} \equiv \bar{\Phi}_k^{\text{s.c.}}$ is said to be self-consistent (‘s.c.’) if the equation of motion for the dynamical field Φ that is implied by Γ_k admits the solution $\Phi = \bar{\Phi}$. With other words, if the system is put in a background which is self-consistent, the fluctuations of the dynamical field, $\varphi \equiv \Phi - \bar{\Phi}$, have zero expectation value and, in this sense, do not modify this special background. As we shall demonstrate in detail in section 11.2, the

proposal $\mathcal{C}_k = \Gamma_k[\bar{\Phi}_k^{s.c.}, \bar{\Phi}_k^{s.c.}]$ is indeed a quite promising candidate for a generalized C -function. It is stationary at fixed points and it is ‘close to’ being monotonically decreasing along the flow.

The phrase ‘close to’ requires an explanation. Especially in quantum gravity, Background Independence is a central requirement [73, 234]. While in the causal dynamical triangulation approach [94–98] or in loop quantum gravity [72, 93], for instance, this requirement is met by strictly not using a background spacetime structure at all, the EAA framework uses the background field technique [27]. At the intermediate steps of the quantization one does introduce a background spacetime, equipped with a non-degenerate background metric in particular, but at the same time one makes sure that no observable prediction will depend on it.¹ This can be done by means of the Ward identities pertaining to the split-symmetry [136, 235, 236] which governs the interrelation between φ and $\bar{\Phi}$. This symmetry, if intact, ensures that the physical contents of a theory is independent of the chosen background structures. Usually, at the ‘off-shell’ level of Γ_k , in particular when $k > 0$, the symmetry is broken by the gauge fixing and cutoff terms in the bare action. Insisting on unbroken split-symmetry in the physical sector restricts the admissible RG trajectories the EAA may follow, see chapter 10 for more details.

The wording ‘close to’ used above has the precise meaning that *in the idealized situation of negligible split-symmetry violation the monotonicity of $\mathcal{C}_k$ is manifest for the entire class of theories with pointwise monotonicity of the EAA*. This can indeed be seen without embarking on any complicated analysis, whose outcome could then possibly depend on the type of theory under consideration. Instead, in reality where the breaking of split-symmetry often is an issue, such an analysis is indeed necessary in order to check whether or not the split-symmetry violation is strong enough to destroy the monotone behavior of $\mathcal{C}_k$. We believe that under weak conditions whose precise form needs to be found, the monotonicity is not destroyed so that the proposed $\mathcal{C}_k$ indeed complies both with the stationarity and the monotonicity requirement.

In this chapter, rather than attempting a general proof we investigate a concrete system, asymptotically safe gravity in d dimensions, $d = 4$ in particular, determine its RG flow, and explore the properties of the resulting function $\mathcal{C}_k$. Clearly, for practical reasons we can study the flow only on a truncation of the a priori infinite dimensional theory space. Nevertheless, we shall be able to demonstrate that, provided the truncation is sufficiently refined, the corresponding approximation to the exact $\mathcal{C}_k$ is indeed a non-decreasing function of k . It will be quite impressive to see how non-trivially the various components of the truncation ansatz for Γ_k must, and actually do conspire in order to produce this result.

Concretely we shall explore Quantum Einstein Gravity (QEG) both within the familiar (‘single-metric’) Einstein-Hilbert-truncation [122] in which the background metric appears only in the gauge fixing part of the action, as well as within a more refined ‘bi-metric truncation’ [136, 137] where the EAA can depend on it via a second Einstein-Hilbert term, over and above the one for the dynamical metric. The beta-functions for this ‘bi-metric Einstein-Hilbert truncation’ have been derived in [172] using TT-decomposition techniques and in part II utilizing a Ω -deformed gauge sector. Those results form the basis of the present analysis.

We emphasize that the goal of the present investigation is not, or at least not primarily, to reanalyze or reformulate the known c - and a -theorems within the EAA approach. (Work along these lines has been reported in [237, 238].) Rather, we want to investigate the properties of a candidate for a generalized C -function which is distinguished and natural in its own right, namely from the perspective of the EAA. As such it can be tentatively defined under conditions that are far more general than those leading to the a - and c -theorems (with respect to dimensionality, field contents, symmetries, etc.). We expect that after restricting this generality appropriately $\mathcal{C}_k$ can be given properties similar to a C -function. The long-term objective of

¹The construction in section 11.2 will force us to deal with a background metric and include it into the set $\bar{\Phi}$ even when analyzing pure matter theories on a classical (for instance, flat) spacetime.

this research program is to find out which restrictions precisely lead to interesting properties of $\mathcal{C}_k$. Here we provide a first example where this strategy can be seen to actually work, namely (truncated) QEG in 4 dimensions.

The structure of this chapter is as follows. In section 11.2 we develop the general theory for using the EAA and $\mathcal{C}_k$ as a counting device², after recalling some necessary background material. Then, in section 11.3 we apply the resulting framework to the particularly important case of asymptotically safe Quantum Einstein Gravity. Our setting will apply to an arbitrary dimensionality of spacetime; the numerical calculations needed to verify the claimed properties of $\mathcal{C}_k$ are performed for the most interesting case of 4 dimensions though. We conclude this chapter in section 11.4 by discussing related topics, as the N -bound or the Bekenstein-Hawking entropy, and present an outlook for future work in this direction. This chapter contains a small appendix, including the special status of Faddeev-Popov ghost and some remark on the results for different classes of RG trajectories.

11.2 The effective average action as a ‘C-function’

In this section we develop a generally applicable framework for constructing functions $\mathcal{C}_k$ which have properties similar to a C-function, and at the same time are ‘geometrically natural’ objects from the perspective of the theory space explored by the EAA. To prepare the ground, and to fix various notations, it is unavoidable to embark on some special aspects of the EAA technique first.

11.2.1 Counting field modes

Eigenfunction representation of the path integral. We consider a general quantum field theory on a d dimensional Euclidean spacetime, either rigid or fluctuating, that is governed by a functional integral $Z = \int [d\hat{\Phi}] e^{-S[\hat{\Phi}, \bar{\Phi}]}$. The bare action S depends on a set of commuting and anticommuting dynamical fields, $\hat{\Phi}$, and on a corresponding set of background fields, $\bar{\Phi}$. (Here and in the following we use a compact notation, leaving implicit all field indices and possible sign factors depending on the Grassmann parity of the field components.) We assume that the functional integral is regularized in the UV in some way; we shall be more precise about this point in a moment.

In the case of a Yang-Mills theory, $\hat{\Phi}$ contains both the gauge field and the Faddeev-Popov ghosts, and S is understood to include gauge fixing and ghost terms. Furthermore, the corresponding background fields are part of $\bar{\Phi}$. As a rule, the *fluctuation field* $\hat{\phi} \equiv \hat{\Phi} - \bar{\Phi}$ is always required to gauge-transform homogeneously, i.e. like a matter field. Henceforth we regard $\hat{\phi}$ rather than $\hat{\Phi}$ as the true dynamical variable and interpret Z as an integral over the fluctuation variables: $Z = \int [d\hat{\phi}] \exp(-S[\hat{\phi}, \bar{\Phi}])$.

For conceptual reasons that will become apparent below, the set of background fields, $\bar{\Phi}$, always contains a classical spacetime metric $\bar{g}_{\mu\nu}$. In typical particle physics applications on a rigid (flat, say) spacetime for instance one would not be interested in how Z depends on the background metric and one might set $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ throughout. But in quantum gravity, when Background Independence is an issue, one wants to know $Z \equiv Z[\bar{g}_{\mu\nu}]$ for *any* background. In fact, employing the background field technique to implement Background Independence [27] one represents the dynamical metric as $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$ and requires split-symmetry at the level of observable quantities, see chapter 10. When the spacetime is dynamical, $\hat{g}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ are special components of $\hat{\Phi}$ and $\hat{\phi}$, respectively.

²A brief discussion and an application of the method to the example of black hole physics appeared already in [166] and will be revised in chapter 13. (See also ref. [239].)

Next we pick a basis in field space, $\{\varphi_\omega\}$, and expand the fields that are integrated over. Then, symbolically, $\hat{\varphi}(x) = \sum_\omega a_\omega \varphi_\omega(x)$ where $\sum_\omega$ stands for a summation and/or integration over all labels carried by the basis elements, and $\int[d\hat{\Phi}]$ is now interpreted as the integration over all possible values that can be assumed by the expansion coefficients $a \equiv \{a_\omega\}$. Thus, $Z = \prod_\omega \int_{-\infty}^{\infty} da_\omega \exp(-S[a; \bar{\Phi}])$.

To be more specific, let us assume that the basis $\{\varphi_\omega\}$ is constituted by the eigenfunctions of a certain differential operator, $\mathcal{L}$, which may depend on the background fields $\bar{\Phi}$, and which has properties similar to the negative Laplace-Beltrami operator, $-\bar{D}^2$, appropriately generalized for the types of (tensor, spinor, $\dots$) fields present in $\hat{\varphi}$. We suppose that $\mathcal{L}$ is built from covariant derivatives involving $\bar{g}_{\mu\nu}$ and the background Yang-Mills fields, if any, so that it is covariant under spacetime diffeomorphism and gauge-transformations. We assume an eigenvalue equation $\mathcal{L} \varphi_\omega = \Omega_\omega^2 \varphi_\omega$ with positive spectral values $\Omega_\omega^2 > 0$. The precise choice of $\mathcal{L}$ is arbitrary to a large extent.

The only property of $\mathcal{L}$ we shall need is that it should associate small (large) distances on the rigid spacetime equipped with the metric $\bar{g}_{\mu\nu}$ to large (small) values of Ω_ω^2 . A first (but for us not the essential) consequence is that we can now easily install a UV cutoff by restricting the ill-defined infinite product $\prod_\omega$ to only those ω 's which satisfy $\Omega_\omega < \Omega_{\max}$. This implements a UV cutoff at the mass scale $\Omega_{\max}$.

More importantly for our purposes, we also introduce a smooth IR cutoff at a variable scale $k \leq \Omega_{\max}$ into the integral, replacing it with

$$Z_k = \prod'_\omega \int_{-\infty}^{\infty} da_\omega e^{-S[a; \bar{\Phi}]} e^{-\Delta S_k} \quad (11.2)$$

where the prime indicates the presence of the UV cutoff, and

$$\Delta S_k \equiv \frac{1}{2} \sum_\omega \mathfrak{R}_k(\Omega_\omega^2) a_\omega^2 \quad (11.3)$$

implements the IR cutoff. The extra piece in the bare action, ΔS_k , is designed in such a way that those φ_ω -modes which have eigenvalues $\Omega_\omega^2 \ll k^2$ get suppressed by a small factor $e^{-\Delta S_k} \ll 1$ in eq. (11.2), while $e^{-\Delta S_k} = 1$ for the others. The function $\mathfrak{R}_k$ is essentially arbitrary, except for its interpolating behavior between $\mathfrak{R}_k(\Omega_\omega^2) \sim k^2$ if $\Omega_\omega \ll k$ and $\mathfrak{R}_k(\Omega_\omega^2) = 0$ if $\Omega_\omega \gg k$.

The operator $\mathcal{L}$ defines the precise notion of ‘coarse graining’ field configurations. We regard the φ_ω 's with $\Omega_\omega > k$ as the ‘short wavelength’ modes, to be integrated out first, and those with small eigenvalues $\Omega_\omega < k$ as the ‘long wavelength’ ones whose impact on the fluctuation's dynamics is not yet taken into account. This amounts to a diffeomorphism and gauge covariant generalization of the standard Wilsonian renormalization group, based on standard Fourier analysis on $\mathbb{R}^d$, to situations with arbitrary background fields $\bar{\Phi} = (\bar{g}_{\mu\nu}, \bar{A}_\mu, \dots)$.

While helpful for the interpretation, for most practical purposes it is often unnecessary to perform the expansion of $\hat{\varphi}(x)$ in terms of the $\mathcal{L}$ -eigenfunctions explicitly. Rather, one thinks of (11.2) as a ‘basis independent’ functional integral

$$Z_k = \int [d'\hat{\varphi}] e^{-S[\hat{\varphi}; \bar{\Phi}]} e^{-\Delta S_k[\hat{\varphi}; \bar{\Phi}]} \quad (11.4)$$

for which the $\mathcal{L}$ -eigenbasis plays no special role, while the operator $\mathcal{L}$ as such does so, of course. In particular the cutoff action ΔS_k is now rewritten with Ω_ω^2 replaced by $\mathcal{L}$ in the argument of R_k :

$$\Delta S_k[\hat{\varphi}; \bar{\Phi}] = \frac{1}{2} \int d^d x \sqrt{\bar{g}} \hat{\varphi}(x) \mathfrak{R}_k(\mathcal{L}) \hat{\varphi}(x) \quad (11.5)$$

Note that at least when $k > 0$ the modified partition function Z_k depends on the respective choices for $\mathcal{L}$ and $\bar{\Phi}$ separately.

The generality of the framework. In this chapter we propose to use the one-parameter family of partition function $k \mapsto Z_k$, for $k \in [0, \infty)$, as a diagnostic tool to investigate the RG flow between different quantum field theories. This is a quite general and flexible framework. There is considerable freedom in choosing the cutoff operator, and even when $\mathcal{L} \equiv \mathcal{L}(\bar{\Phi})$ is fixed we may still choose $\bar{\Phi}$ in a large variety of different ways so as to 'project out' different information from the partition function. However, as we shall discuss in the next subsection, there exists a natural, almost 'canonical' choice of background configurations $\bar{\Phi}$.

Monotonicity of Z_k Our discussion in the following sections is based upon the key observation that Z_k enjoys a simple property which is strikingly reminiscent of the C-theorem in 2 dimensions.

Let us assume for simplicity that all component fields constituting $\hat{\phi}$ are commuting, and that $\bar{\Phi}$ has been chosen k -independent. Then (11.4) is a (regularized, and convergent for appropriate S) purely bosonic integral with a positive integrand which, thanks to the suppression factor $e^{-\Delta S_k}$, decreases with increasing k . Therefore, Z_k and the 'entropy' $\ln Z_k$, are monotonically decreasing functions of the scale:

$$\partial_k \ln Z_k < 0 \quad (11.6)$$

The interpretation of (11.6) is clear: Proceeding from the UV to the IR by lowering the infrared cutoff scale, an increasing number of field modes get un-suppressed, thus contribute to the functional integral, and as a consequence the value of the partition function increases. Thus, in a not too literal sense of the word, $\ln Z_k$ 'counts' the number of field modes that have been integrated out already.

Before we can make this intuitive argument more precise and explicit we must introduce a number of technical tools in the following subsections.

11.2.2 Running actions and self-consistent backgrounds

The generating functional. Introducing a source term for the fluctuation fields turns the partition functions into the generating functional

$$Z_k[J; \bar{\Phi}] \equiv e^{W_k[J; \bar{\Phi}]} = \int \mathcal{D}' \hat{\phi} \exp \left(-S[\hat{\phi}; \bar{\Phi}] - \Delta S_k[\hat{\phi}; \bar{\Phi}] + \int d^d x \sqrt{\bar{g}} J(x) \hat{\phi}(x) \right) \quad (11.7)$$

Hence the $\bar{\Phi}$ - and k -dependent expectation value $\langle \hat{\phi} \rangle \equiv \varphi$ reads

$$\varphi(x) \equiv \langle \hat{\phi}(x) \rangle = \frac{1}{\sqrt{\bar{g}(x)}} \frac{\delta W_k[J; \bar{\Phi}]}{\delta J(x)} \quad (11.8)$$

If we can solve this relation for J as a functional of $\bar{\Phi}$, the definition of the Effective Average Action (EAA), essentially the Legendre transform of W_k , may be written as

$$\Gamma_k[\varphi; \bar{\Phi}] = \int d^d x \sqrt{\bar{g}} \varphi(x) J(x) - W_k[J; \bar{\Phi}] - \Delta S_k[\varphi; \bar{\Phi}] \quad (11.9)$$

with the solution to (11.8) inserted, $J \equiv J_k[\varphi; \bar{\Phi}]$. (In the general case, Γ_k is the Legendre-Fenchel transform of W_k , with ΔS_k subtracted.)

The EAA gives rise to a source-field relationship which includes an explicit cutoff term linear in the fluctuation field:

$$\frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma_k[\varphi; \bar{\Phi}]}{\delta \varphi(x)} + \mathfrak{R}_k[\bar{\Phi}] \varphi(x) = J(x) \quad (11.10)$$

Here and in the following we write $\mathfrak{R}_k \equiv \mathfrak{R}_k(\mathcal{L})$, and the notation $\mathfrak{R}_k[\bar{\Phi}]$ is used occasionally to emphasize that the cutoff operator may depend on the background fields. The solution to (11.10), and more generally all fluctuation correlators $\langle \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \rangle$ obtained by multiple differentiation of Γ_k , are functionally dependent on the background, e.g. $\varphi(x) \equiv \varphi_k[J; \bar{\Phi}](x)$.

The bi-field variant. For the expectation value of the full, i.e. un-decomposed field $\hat{\Phi} = \bar{\Phi} + \hat{\phi}$ we employ the notation

$$\Phi = \bar{\Phi} + \varphi \quad \text{with } \Phi \equiv \langle \hat{\Phi} \rangle \text{ and } \varphi \equiv \langle \hat{\phi} \rangle. \quad (11.11)$$

Using the complete field Φ instead of φ as the second independent variable, accompanying $\bar{\Phi}$, entails the ‘bi-field’ variant of the EAA,

$$\Gamma_k[\Phi, \bar{\Phi}] \equiv \Gamma_k[\varphi; \bar{\Phi}]|_{\varphi=\Phi-\bar{\Phi}} \quad (11.12)$$

which, in particular, is always ‘bi-metric’: $\Gamma_k[g_{\mu\nu}, \dots, \bar{g}_{\mu\nu}, \dots]$.

The level representation. Later on it will often be helpful to organize the terms contributing to $\Gamma_k[\varphi; \bar{\Phi}]$ according to their *level* which, by definition, is their degree of homogeneity in the φ ’s. The underlying assumption is that the EAA admits a *level expansion* of the form

$$\Gamma_k[\varphi; \bar{\Phi}] = \sum_{p=0}^{\infty} \check{\Gamma}_k^p[\varphi; \bar{\Phi}] \quad (11.13)$$

where $\check{\Gamma}_k^p[c\varphi; \bar{\Phi}] = c^p \check{\Gamma}_k^p[\varphi; \bar{\Phi}]$ for $c > 0$. If $\Gamma_k[\varphi; \bar{\Phi}]$ admits a Taylor expansion in φ about $\varphi = 0$, this expansion exists, of course, with the level- (p) contribution $\check{\Gamma}_k^p$ being its p -derivative term, but this is not guaranteed in general.

The self-consistent backgrounds. We are interested in how the dynamics of the fluctuations $\hat{\phi}$ depends on the environment they are placed in, the background metric $\bar{g}_{\mu\nu}$, for instance, and the other classical fields collected in $\bar{\Phi}$. It would be instructive to know if there exist special backgrounds in which the fluctuations are particularly ‘tame’ such that, for vanishing external source, they amount to only small oscillations about a stable equilibrium, with a vanishing mean: $\varphi \equiv \langle \hat{\phi} \rangle = 0$. Such distinguished backgrounds $\bar{\Phi} \equiv \bar{\Phi}^{\text{s.c.}}$ are referred to as *self-consistent* (s.c.) since, if we pick one of those, the expectation value of the field $\langle \hat{\Phi} \rangle = \Phi = \bar{\Phi}$ does not get changed by any violent $\hat{\phi}$ -excitations that, generically, can shift the point of equilibrium.

From eq. (11.10) we obtain the following condition $\bar{\Phi}^{\text{s.c.}}$ must satisfy (since $J = 0$ by assumption):

$$\frac{\delta}{\delta \varphi(x)} \Gamma_k[\varphi; \bar{\Phi}]|_{\varphi=0, \bar{\Phi}=\bar{\Phi}_k^{\text{s.c.}}} = 0 \quad (11.14)$$

This is the *tadpole equation* from which we can compute the self-consistent background configurations, if any. In general $\bar{\Phi}^{\text{s.c.}} \equiv \bar{\Phi}_k^{\text{s.c.}}$ will have an explicit dependence on k . A technically convenient feature of (11.14) is that it no longer contains the somewhat disturbing $\mathfrak{R}_k \varphi$ -term that was present in the general field equation (11.10). Self-consistent backgrounds are equivalently characterized by eq. (11.8),

$$\frac{\delta}{\delta J(x)} W_k[J; \bar{\Phi}]|_{J=0, \bar{\Phi}=\bar{\Phi}_k^{\text{s.c.}}} = 0 \quad (11.15)$$

which again expresses the vanishing of the fluctuation’s one-point function.

Note that provided the level expansion (11.13) exists we may replace (11.14) with

$$\frac{\delta}{\delta\varphi(x)}\check{\Gamma}_k^1[\varphi;\bar{\Phi}]\Big|_{\varphi=0,\bar{\Phi}=\bar{\Phi}_k^{s.c.}}=0 \quad (11.16)$$

which involves only the level-(1) functional $\check{\Gamma}_k^1$. Later on in the applications this trivial observation has the important consequence that *self-consistent background field configurations* $\bar{\Phi}_k^{s.c.}(x)$ can contain only running coupling constants of level $p=1$, that is, the couplings parameterizing the functional $\check{\Gamma}_k^1$ which is linear in φ .³

The level-dependence of EAA for self-consistent backgrounds. In our later discussions the value of the EAA at $\varphi=0$ will be of special interest. While it is still a rather complicated functional for a generic background where $\Gamma_k[0;\bar{\Phi}]=-W_k[J_k[0;\bar{\Phi}];\bar{\Phi}]$, the source which is necessary to achieve $\varphi=0$ for self-consistent backgrounds is precisely $J=0$, implying

$$\Gamma_k[0;\bar{\Phi}_k^{s.c.}]\equiv\check{\Gamma}_k^0[0;\bar{\Phi}_k^{s.c.}]=-W_k[0;\bar{\Phi}_k^{s.c.}] \quad (11.17)$$

Here we also indicated that in a level expansion only the $p=0$ term of Γ_k survives putting $\varphi=0$.

So we can summarize saying that *the value of $\Gamma_k[0;\bar{\Phi}_k^{s.c.}]$ can contain only the running couplings of the levels $p=0$ and $p=1$, respectively, the former entering via $\check{\Gamma}_k^0$, the latter via $\bar{\Phi}_k^{s.c.}$.*

11.2.3 FIDE, FRGE, and WISS

The EAA satisfies a number of important exact functional equations which include a Functional Integro-Differential Equation (FIDE), the Functional Renormalization Group Equation (FRGE), the Ward Identities for split-symmetry (WISS), and the Becchi, Rouet, Stora and Tyutin (BRST)-Ward identity.

Functional Integro-Differential Equation

The FIDE is obtained by substituting (11.9) in (11.7), using (11.10), and reads

$$e^{-\Gamma_k[\varphi;\bar{\Phi}]}=\int[d'\hat{\varphi}]\exp\left(-S[\hat{\varphi};\bar{\Phi}]-\Delta S_k[\hat{\varphi};\bar{\Phi}]+\int d^dx\hat{\varphi}(x)\frac{\delta\Gamma_k}{\delta\varphi(x)}[\varphi;\bar{\Phi}]\right) \quad (11.18)$$

Here, as always, summation over field components and their tensor, spinor, internal symmetry, etc. indices is understood. For the purposes of the present investigation, the most important property of (11.18) is that the last term on its RHS, the one linear in $\hat{\varphi}$, vanishes if the background happens to be self-consistent, and at the same time the argument $\varphi=0$ is inserted on both sides of the FIDE:

$$\exp(-\Gamma_k[0;\bar{\Phi}_k^{s.c.}])=\int[d'\hat{\varphi}]\exp(-S[\hat{\varphi};\bar{\Phi}_k^{s.c.}]-\Delta S_k[\hat{\varphi};\bar{\Phi}_k^{s.c.}]) \quad (11.19)$$

We shall come back to this identity soon.

Functional Renormalization Group Equation

Another important exact relation satisfied by the EAA is the Functional Renormalization Group Equation (FRGE)),

$$k\partial_k\Gamma_k[\varphi;\bar{\Phi}]=\frac{1}{2}\text{STr}\left[\left(\Gamma_k^{(2)}[\varphi;\bar{\Phi}]+\mathfrak{R}_k[\bar{\Phi}]\right)^{-1}k\partial_k\mathfrak{R}_k[\bar{\Phi}]\right] \quad (11.20)$$

³Notice that the k -derivative of $\bar{\Phi}_k^{s.c.}(x)$ is in general governed also by higher level couplings due to their appearance in the beta-functions of the level $p=1$ couplings.

with the Hessian matrix of the fluctuation derivatives $\Gamma_k^{(2)} \equiv \delta^2 \Gamma_k / \delta \varphi^2$. The supertrace ‘STr’ in (11.20) provides the additional minus sign which is necessary for the φ -components with odd Grassmann parity, Faddeev-Popov ghosts and fermions.

Ward Identities for split-symmetry

The EAA, written as $\Gamma_k[\Phi, \bar{\Phi}]$, satisfies the following exact functional equation which governs the ‘extra’ background dependence it has over and above the one which combines with the fluctuations to form the full field Φ :

$$\frac{\delta}{\delta \bar{\Phi}(x)} \Gamma_k[\Phi, \bar{\Phi}] = \frac{1}{2} \text{STr} \left[\left(\Gamma_k^{(2)}[\Phi, \bar{\Phi}] + \mathfrak{R}_k[\bar{\Phi}] \right)^{-1} \frac{\delta}{\delta \bar{\Phi}(x)} S_{\text{tot}}^{(2)}[\Phi, \bar{\Phi}] \right] \quad (11.21)$$

Here $S_{\text{tot}}^{(2)}$ is the Hessian of $S_{\text{tot}} = S + \Delta S_k$ with respect to Φ , where S includes gauge fixing and ghost terms. The equation (11.21) is the Ward identity induced by the split-symmetry transformations $\delta \varphi = \varepsilon$, $\delta \bar{\Phi} = -\varepsilon$, hence the abbreviation WISS. It was first obtained in [235] in the context of Yang-Mills theory. In quantum gravity, extensive use has been made of (11.21) in ref. [136] where it served as a tool to assess the degree of split-symmetry breaking and, related to that, the reliability of certain truncations of QEG.

For a discussion of the modified BRST-Ward identity enjoyed by the EAA we refer to [235] and [122].

11.2.4 Pointwise monotonicity

Our search for a generalized C -type counting function which depends monotonically on k along the RG trajectories will be based upon the following structural property of the FRGE (11.20). From the very definition of the EAA by a Legendre transform it follows that for all $\bar{\Phi}$ the sum $\Gamma_k + \Delta S_k$ is a convex functional of φ , and that $\Gamma_k^{(2)} + \mathfrak{R}_k$ is a strictly positive definite operator therefore which can be inverted at all scales $k \in (0, \infty)$. Now let us suppose that the theory under consideration contains Grassmann-even fields only. Then the supertrace in (11.20) amounts to the ordinary, and convergent trace of a positive operator so that the FRGE implies

$$k \partial_k \Gamma_k[\varphi; \bar{\Phi}] \geq 0 \quad \text{at all fixed } \varphi, \bar{\Phi}. \quad (11.22)$$

Thus, at least in a class of distinguished theories the EAA, evaluated at any fixed pair of arguments φ and $\bar{\Phi}$, is a monotonically increasing function of k . With other words, *lowering k from the UV towards the IR the value of $\Gamma_k[\varphi; \bar{\Phi}]$ decreases monotonically.*

We refer to this property as *pointwise monotonicity* in order to emphasize that it applies at all points of field space, $(\varphi, \bar{\Phi})$, separately. In particular this means that the argument of $\Gamma_k[\varphi, \bar{\Phi}]$ is assumed to have no k -dependence of its own here.

In presence of fields with odd Grassmann parity, fermions and Faddeev-Popov ghosts, the RHS of the FRGE is no longer obviously non-negative. However, if the only Grassmann-odd fields are ghosts the pointwise monotonicity (11.22) can still be made a general property of the EAA, the reason being as follows. At least when one implements the gauge fixing condition strictly, it cuts-out a certain subspace of the space of fields $\hat{\Phi}$ to be integrated over, namely the gauge orbit space. Hereby the integral over the ghosts represents the measure on this subspace, the Faddeev-Popov determinant. The subspace and its geometrical structures are invariant under the RG flow, however. Hence the EAA pertaining to the manifestly Grassmann-even integral *over the subspace* is of the kind considered above, and the argument implying (11.22) should therefore be valid again. For a more explicit version of this reasoning we refer to appendix 11.A.

11.2.5 Monotonicity vs. stationarity

The EAA evaluated at fixed arguments shares the monotonicity property with a C -function. One of the problems is however that $\Gamma_k[\varphi; \bar{\Phi}]$ is not stationary at fixed points of the RG flow. In order to see why, and how to improve the situation, some care is needed concerning the interplay of dimensionful and dimensionless variables, to which we turn next.

The FRGE in component form. Let us assume that the space constituted by the functionals of φ and $\bar{\Phi}$ admits a basis $\{I_\alpha\}$ so that we can expand the EAA as

$$\Gamma_k[\varphi; \bar{\Phi}] = \sum_{\alpha} \bar{u}_{\alpha}(k) I_{\alpha}[\varphi; \bar{\Phi}] \quad (11.23)$$

with dimensionful running coupling constants $\bar{u} \equiv (\bar{u}_{\alpha})$. They obey a FRGE in component form, $k \partial_k \bar{u}_{\alpha}(k) = \bar{b}_{\alpha}(\bar{u}(k); k)$, whereby the functions $\bar{b}_{\alpha}$ are defined by the expansion of $\text{STr}[\dots]$ with respect to the basis:

$$\frac{1}{2} \text{STr} \left[\left(\sum_{\alpha} \bar{u}_{\alpha}(k) I_{\alpha}^{(2)}[\varphi; \bar{\Phi}] + \mathfrak{R}_k \right)^{-1} k \partial_k \mathfrak{R}_k \right] =: \sum_{\alpha} \bar{b}_{\alpha}(\bar{u}(k); k) I_{\alpha}[\varphi; \bar{\Phi}] \quad (11.24)$$

Note that the statement of monotonicity in (11.22), when it holds true, translates into the point-wise positivity of the sum $\sum_{\alpha} \bar{b}_{\alpha} I_{\alpha}$.

The dimensionless fields. Denoting the canonical mass dimension⁴ of the running couplings by $[\bar{u}_{\alpha}] \equiv d_{\alpha}$, their dimensionless counterparts are defined by $u_{\alpha} \equiv k^{-d_{\alpha}} \bar{u}_{\alpha}$. In terms of the dimensionless couplings the expansion of Γ_k reads

$$\Gamma_k[\varphi; \bar{\Phi}] = \sum_{\alpha} u_{\alpha}(k) k^{d_{\alpha}} I_{\alpha}[\varphi; \bar{\Phi}] \quad (11.25)$$

Now observe that since Γ_k is dimensionless the basis elements have dimensions $[I_{\alpha}[\varphi; \bar{\Phi}]] = -d_{\alpha}$. Purely by dimensional analysis, this implies that⁵

$$I_{\alpha}[c^{[\varphi]} \varphi; c^{[\bar{\Phi}]} \bar{\Phi}] = c^{-d_{\alpha}} I_{\alpha}[\varphi; \bar{\Phi}] \quad \text{for any constant } c > 0. \quad (11.26)$$

This relation expresses the fact that the nontrivial dimension of I_{α} is entirely due to that of its field arguments; there are simply no other dimensionful quantities available after the k -dependence has been separated off. Using (11.26) for $c = k^{-1}$ yields

$$k^{d_{\alpha}} I_{\alpha}[\varphi; \bar{\Phi}] = I_{\alpha}[k^{-[\varphi]} \varphi; k^{-[\bar{\Phi}]} \bar{\Phi}] \equiv I_{\alpha}[\tilde{\varphi}; \tilde{\bar{\Phi}}] \quad (11.27)$$

Here we introduced the sets of dimensionless fields,

$$\tilde{\varphi}(x) \equiv k^{-[\varphi]} \varphi(x), \quad \tilde{\bar{\Phi}}(x) \equiv k^{-[\bar{\Phi}]} \bar{\Phi}(x) \quad (11.28)$$

which include, for instance, the dimensionless metric and its fluctuations:

$$\tilde{h}_{\mu\nu}(x) \equiv k^2 h_{\mu\nu}(x), \quad \tilde{g}_{\mu\nu}(x) \equiv k^2 \bar{g}_{\mu\nu}(x) \quad (11.29)$$

⁴Our conventions are as follows. We use dimensionless coordinates, $[x^{\mu}] = 0$. Then $[ds^2] = -2$ implies that all components of the various metrics have $[\hat{g}_{\mu\nu}] = [\bar{g}_{\mu\nu}] = [g_{\mu\nu}] = -2$, and likewise for the fluctuations: $[\hat{h}_{\mu\nu}] = [h_{\mu\nu}] = -2$.

⁵We use the notation $c^{[\varphi]} \varphi \equiv \{c^{[\varphi]} \varphi_i\}$ for the set in which each field is rescaled according to its individual canonical dimension.

Note that the quantity (11.27), in whatever way we write it, is dimensionless. When we insert the dimensionless fields rather than φ and $\tilde{\Phi}$ into the basis functionals, the latter lose their nonzero dimension: $[I_\alpha[\tilde{\varphi}; \tilde{\Phi}]] = 0$.

Exploiting (11.27) in (11.25) we obtain the following representation of the EAA which is entirely in terms of dimensionless quantities⁶

$$\Gamma_k[\varphi; \tilde{\Phi}] = \sum_{\alpha} u_{\alpha}(k) I_{\alpha}[\tilde{\varphi}; \tilde{\Phi}] \equiv \mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}] \quad (11.30)$$

Alternatively, one might wish to make its k -dependence explicit, writing,

$$\Gamma_k[\varphi; \tilde{\Phi}] = \sum_{\alpha} u_{\alpha}(k) I_{\alpha}[k^{-[\varphi]} \varphi; k^{-[\tilde{\Phi}]} \tilde{\Phi}] \quad (11.31)$$

In the second equality of (11.30) we introduced the new functional $\mathcal{A}_k$ which, by definition, is numerically equal to Γ_k , but its independent variables (arguments) are the dimensionless fields $\tilde{\varphi}$ and $\tilde{\Phi}$. Hence the k -derivative of $\mathcal{A}_k[\tilde{\varphi}, \tilde{\Phi}]$ is to be performed at fixed $(\tilde{\varphi}, \tilde{\Phi})$, while the analogous derivative of $\Gamma_k[\varphi; \tilde{\Phi}]$ refers to fixed dimensionful arguments. This leads to the following two trivial but momentous equations:

$$k \partial_k \mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}] = \sum_{\alpha} k \partial_k u_{\alpha}(k) I_{\alpha}[\tilde{\varphi}; \tilde{\Phi}] \quad (11.32a)$$

$$k \partial_k \Gamma_k[\varphi; \tilde{\Phi}] = \sum_{\alpha} \{k \partial_k u_{\alpha}(k) + d_{\alpha} u_{\alpha}(k)\} k^{d_{\alpha}} I_{\alpha}[\varphi; \tilde{\Phi}] \quad (11.32b)$$

The extra term $\propto d_{\alpha} u_{\alpha}(k)$ in eq. (11.32b) arises by differentiating the factor $k^{d_{\alpha}}$ in (11.25). It leads to the well-known canonical scaling term in the β -functions of the dimensionless couplings.

The (dimensionless) theory space. For the following it is crucial to recall that it is the *dimensionless* couplings $u \equiv (u_{\alpha})$ that serve as local coordinates on theory space, henceforth denoted $\mathcal{T}$. Its points are functionals $\mathcal{A}$ which depend on dimensionless arguments: $\mathcal{A}[\tilde{\varphi}; \tilde{\Phi}] = \sum_{\alpha} u_{\alpha} I_{\alpha}[\tilde{\varphi}; \tilde{\Phi}]$. The RG trajectories are curves $k \mapsto \mathcal{A}_k = \sum_{\alpha} u_{\alpha}(k) I_{\alpha} \in \mathcal{T}$ that are everywhere tangent to

$$k \partial_k \mathcal{A}_k = \sum_{\alpha} \beta_{\alpha}(u(k)) I_{\alpha} \quad (11.33)$$

The functions β_{α} , components of a vector field $\vec{\beta}$ on $\mathcal{T}$, are obtained by translating $k \partial_k \bar{u}_{\alpha}(k) = \bar{b}_{\alpha}(\bar{u}(k); k)$ into the dimensionless language. This leads to the autonomous system of differential equations

$$k \partial_k u_{\alpha}(k) \equiv \beta_{\alpha}(u(k)) = -d_{\alpha} u_{\alpha}(k) + b_{\alpha}(u(k)) \quad (11.34)$$

Here b_{α} , contrary to its dimensionful precursor $\bar{b}_{\alpha}$, has no explicit k -dependence, thus defining an RG-time independent vector field, the ‘RG flow’ $(\mathcal{T}, \vec{\beta})$.

If it has a fixed point at some u^* then $\beta_{\alpha}(u^*) = 0$, and the ‘velocity’ of any trajectory passing this point vanishes there⁷, $k \partial_k u_{\alpha} = 0$. Hence by (11.33) the redefined functional $\mathcal{A}_k$ becomes stationary there, that is, its scale derivative vanishes pointwise,

$$k \partial_k \mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}] = 0 \quad \text{for all fixed } \tilde{\varphi}, \tilde{\Phi}. \quad (11.35)$$

⁶Here one should also switch from k to the manifestly dimensionless ‘RG time’ $t \equiv \ln(k) + \text{const}$, but we shall not indicate this notationally.

⁷To keep the notation simple, we assume here that among the u_{α} ’s there are no ‘inessential’, aka ‘redundant’, couplings that would not have to approach fixed point values.

So the entire functional $\mathcal{A}_k$ approaches a limit, $\mathcal{A}_* = \sum_{\alpha} u_{\alpha}^* I_{\alpha}$. The standard EAA instead keeps running in the fixed point regime:

$$\Gamma_k[\varphi; \tilde{\Phi}] = \sum_{\alpha} u_{\alpha}^* k^{d_{\alpha}} I_{\alpha}[\varphi; \tilde{\Phi}] \quad \text{when} \quad u_{\alpha}(k) = u_{\alpha}^*. \quad (11.36)$$

The monotonicity of $\mathcal{A}_k$ This brings us back to the 'defect' of Γ_k we wanted to repair: While $\Gamma_k[\varphi; \tilde{\Phi}]$ was explicitly seen to decrease monotonically along RG trajectories, it does not come to a halt at fixed points in general. The redefined functional $\mathcal{A}_k$, instead, approaches a finite limit $\mathcal{A}_*$ at fixed points, but can we argue that it is monotone along trajectories?

Unfortunately this is not the case, and the culprit is quite obvious, namely the $d_{\alpha} u_{\alpha}$ -terms present in the scale derivative of Γ_k , but absent for $\mathcal{A}_k$: The positivity of the RHS of eq. (11.32b) does not imply the positivity of the RHS of eq. (11.32a), and *there is no obvious structural reason for $k \partial_k \mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}] \geq 0$ at fixed $\tilde{\varphi}, \tilde{\Phi}$* . The best we can get is the bound

$$k \partial_k \mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}] \geq - \sum_{\alpha} d_{\alpha} u_{\alpha}(k) I_{\alpha}[\tilde{\varphi}; \tilde{\Phi}] \quad (11.37)$$

which follows by subtracting the two eqs. (11.32) and making use of (11.27).

11.2.6 The proposal

The complementary virtues of $\mathcal{A}_k$ and Γ_k with respect to monotonicity along trajectories and stationarity at critical points suggest the following strategy for finding a C-type function with better properties: Rather than considering the functionals pointwise, i.e. with fixed configurations of either the dimensionless or dimensionful fields inserted, one should evaluate them at *explicitly scale dependent arguments*: $\mathcal{C}_k \stackrel{?}{=} \Gamma_k[\varphi_k; \tilde{\Phi}_k] \equiv \mathcal{A}_k[\tilde{\varphi}_k; \tilde{\Phi}_k]$.

The hope is that the respective arguments $\varphi_k \equiv k^{[\varphi]} \tilde{\varphi}_k$, and $\tilde{\Phi}_k \equiv k^{[\tilde{\Phi}]} \tilde{\Phi}_k$ can be given a k -dependence which is intermediate between the two extreme cases $(\varphi, \tilde{\Phi}) = \text{const}$ and $(\tilde{\varphi}, \tilde{\Phi}) = \text{const}$, respectively, so as to preserve as much as possible of the monotonicity properties of Γ_k , while rendering $\mathcal{C}_k$ stationary at fixed points of the RG flow.

The most promising candidate of this kind which we could find is

$$\mathcal{C}_k = \Gamma_k[0; \tilde{\Phi}_k^{\text{s.c.}}] = \mathcal{A}_k[0; \tilde{\Phi}_k^{\text{s.c.}}] \quad (11.38)$$

Here the fluctuation argument is set to zero, $\varphi_k \equiv 0$, and for the background we choose a self-consistent one, $\tilde{\Phi}_k^{\text{s.c.}}$, a solution to the tadpole equation (11.14), or equivalently its dimensionless variant

$$\frac{\delta}{\delta \tilde{\varphi}(x)} \mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}] \Big|_{\tilde{\varphi}=0, \tilde{\Phi}=\tilde{\Phi}_k^{\text{s.c.}}} = 0 \quad (11.39)$$

The function $k \mapsto \mathcal{C}_k$ defined by eq. (11.38) has a number of interesting properties to which we turn next.

Stationarity at critical points

When the RG trajectory approaches a fixed point, $\mathcal{A}_k[\tilde{\varphi}; \tilde{\Phi}]$ approaches $\mathcal{A}_*[\tilde{\varphi}; \tilde{\Phi}]$ pointwise. Furthermore, the tadpole equation (11.39) becomes $(\delta \mathcal{A}_* / \delta \tilde{\varphi})[0; \tilde{\Phi}_*] = 0$. It is completely k -independent, and so is its solution, $\tilde{\Phi}_*$. Thus $\mathcal{C}_k$ approaches a well defined, finite constant:

$$\mathcal{C}_k \xrightarrow{\text{FP}} \mathcal{C}_* = \mathcal{A}_*[0; \tilde{\Phi}_*] \quad (11.40)$$

Of course we can write this number also as $\mathcal{C}_* = \Gamma_k[0; k^{[\tilde{\Phi}]} \tilde{\Phi}_*]$ wherein the explicit and the implicit scale dependence of the EAA cancel exactly when a fixed point is approached.

Stationarity at classicality

In a classical regime ('CR'), by definition, $\bar{b}_\alpha \rightarrow 0$, so that it is now the *dimensionful* couplings whose running stops: $\bar{u}_\alpha(k) \rightarrow \bar{u}_\alpha^{\text{CR}} = \text{const.}$ Thus, by (11.23), Γ_k approaches $\Gamma_{\text{CR}} = \sum_\alpha \bar{u}_\alpha^{\text{CR}} I_\alpha$ pointwise. Hence the dimensionful version of the tadpole equation, (11.14), becomes k -independent, and the same is true for its solution, $\bar{\Phi}_{\text{CR}}^{\text{s.c.}}$. So, when the RG trajectory approaches a classical regime, $\mathcal{C}_k$ loses its k -dependences and approaches a constant:

$$\mathcal{C}_k \xrightarrow{\text{CR}} \mathcal{C}_{\text{CR}} = \Gamma_{\text{CR}}[0; \bar{\Phi}_{\text{CR}}^{\text{s.c.}}] \quad (11.41)$$

Alternatively we can write $\mathcal{C}_{\text{CR}} = \mathcal{A}_k[0; k^{-[\bar{\Phi}]} \bar{\Phi}_{\text{CR}}^{\text{s.c.}}]$ where it is now the explicit and implicit k -dependence of $\mathcal{A}_k$ which cancel mutually.

We observe that there is a certain analogy between 'criticality' and 'classicality', in the sense that dimensionful and dimensionless couplings exchange their roles. The difference is that the former situation is related to special *points* of theory space, while the latter concerns extended *regions* in $\mathcal{T}$. In those regions, $\mathcal{A}_k$ keeps moving as $\mathcal{A}_k[\cdot] = \sum_\alpha \bar{u}_\alpha^{\text{CR}} k^{-d_\alpha} I_\alpha[\cdot]$. Nevertheless it is thus plausible, and of particular interest in quantum gravity, to apply a (putative) C -function not only to crossover trajectories in the usual sense which connect two fixed points, but also to *generalized crossover transitions* where one of the fixed points, or even both, get replaced by a classical regime.

Monotonicity at exact split-symmetry

If split-symmetry is exact in the sense that $\Gamma_k[\varphi; \bar{\Phi}]$ depends on the single independent field variable $\bar{\Phi} + \varphi \equiv \Phi$ only, and the theory is such that pointwise monotonicity (11.22) holds true, then $k \mapsto \mathcal{C}_k$ is a monotonically increasing function of k . In fact, differentiating (11.38) and using the chain rule yields

$$\partial_k \mathcal{C}_k = (\partial_k \Gamma_k)[0; \bar{\Phi}_k^{\text{s.c.}}] + \int d^d x (\partial_k \bar{\Phi}_k^{\text{s.c.}}(x)) \left(\frac{\delta}{\delta \bar{\Phi}(x)} - \frac{\delta}{\delta \varphi(x)} \right) \Gamma_k[\varphi; \bar{\Phi}] \Big|_{(0, \bar{\Phi}_k^{\text{s.c.}})} \quad (11.42)$$

In the first term on the RHS of (11.42) the derivative ∂_k hits only the explicit k -dependence of the EAA. By eq. (11.22) we know that this contribution is non-negative. The last term, the $\delta/\delta\varphi$ -derivative, is actually zero by the tadpole equation (11.14). Including it here it becomes manifest that the integral term in (11.42) vanishes when Γ_k depends on φ and $\bar{\Phi}$ only via the combination $\varphi + \bar{\Phi}$. Thus we have shown that

$$\partial_k \mathcal{C}_k \geq 0 \quad \text{at exact split-symmetry} \quad (11.43)$$

Note that the result for $\partial_k \mathcal{C}_k$ in (11.42) is much closer to what one needs to prove in order to rightfully call $\mathcal{C}_k$ a ' C -function' than the inequality (11.37). In theories that require no breaking of split-symmetry, for instance, the integral term in (11.42) is identically zero and we know that $\partial_k \mathcal{C}_k \geq 0$ holds true.

The degree of split-symmetry violation varies over theory space in general. Split-symmetry is unbroken at points $u = (u_\alpha)$ where at most those coordinates u_α are non-zero that belong to basis functionals $I_\alpha[\varphi; \bar{\Phi}]$ which happen to depend on $\bar{\Phi} + \varphi$ only.

The breaking of split-symmetry is best discussed in terms of the functional $\Gamma_k[\Phi, \bar{\Phi}] \equiv \Gamma_k[\Phi - \bar{\Phi}; \bar{\Phi}]$ for which perfect symmetry amounts to independence of the second argument: $\frac{\delta}{\delta \bar{\Phi}} \Gamma_k[\Phi, \bar{\Phi}] = 0$. In this language, $\mathcal{C}_k$ is written as $\mathcal{C}_k = \Gamma_k[\bar{\Phi}_k^{\text{s.c.}}, \bar{\Phi}_k^{\text{s.c.}}]$, and its scale derivative assumes the form

$$\partial_k \mathcal{C}_k = (\partial_k \Gamma_k)[\bar{\Phi}_k^{\text{s.c.}}, \bar{\Phi}_k^{\text{s.c.}}] + \int d^d x (\partial_k \bar{\Phi}_k^{\text{s.c.}}(x)) \left. \frac{\delta \Gamma_k[\Phi, \bar{\Phi}]}{\delta \bar{\Phi}(x)} \right|_{\Phi = \bar{\Phi} = \bar{\Phi}_k^{\text{s.c.}}} \quad (11.44)$$

Whether or not $\partial_k \mathcal{C}_k$ is always non-negative depends on the size of the split-symmetry breaking the EAA suffers from. To prove monotonicity of $\mathcal{C}_k$ one would have to show on a case-by-case basis that the second term on the RHS of (11.44) never can override the first one, known to be non-negative, so as to render their sum negative.

In fact, for the derivative $\delta\Gamma_k/\delta\bar{\Phi}$ in (11.44) we have an exact formal identity at our disposal, the WISS of eq. (11.21). The equations (11.44) and (11.21) together with the FRGE for the $(\partial_k\Gamma_k)$ -term could be the starting point of future work on exact estimates.

In the next section of the present paper we shall investigate the monotonicity properties of $\mathcal{C}_k$ in certain truncations of pure Quantum Einstein Gravity using a different strategy. In this case it is easier to work directly with the definition of $\mathcal{C}_k$, eq. (11.38), rather than using the WISS.

11.2.7 The mode counting property revisited

Equipped with the EAA machinery, we now return to the heuristic argument about the mode 'counting' property of Z_k that was presented at the end of subsection 11.2.1. Trying to make it more precise, we shall now demonstrate that our candidate $\mathcal{C}_k \equiv \Gamma_k[0; \bar{\Phi}_k^{\text{s.c.}}]$ is really a measure for the 'number' of field modes, since it is closely related to a spectral density.

In fact, after our preparations in the previous subsections this should not be too much of a surprise. By virtue of the general identity (11.19), a special case of the FIDE satisfied by the EAA, the exponential $e^{-\Gamma_k[0; \bar{\Phi}_k^{\text{s.c.}}]}$ equals precisely the partition function considered Z_k in subsection 11.2.1, provided the latter is specialized for a self-consistent background.

In order to present a picture which is as clear as possible let us make a number of specializations and approximations. In particular we consider purely bosonic theories again, and invoke the idealization of perfect split-symmetry. Then, by eq. (11.42), we have $\partial_k \mathcal{C}_k = (\partial_k \Gamma_k)[0; \bar{\Phi}_k^{\text{s.c.}}]$, and upon expressing $(\partial_k \Gamma_k)$ via the FRGE we arrive at⁸

$$k\partial_k \mathcal{C}_k = \frac{1}{2} \text{Tr} \left[\left(\Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}] + R_k(\mathcal{L}) \right)^{-1} k\partial_k \mathfrak{R}_k(\mathcal{L}) \right] \quad (11.45)$$

Now we consider a situation where the Hessian appearing in this equation itself qualifies as a cutoff operator. When we choose $\mathcal{L} = \Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}]$ we obtain

$$k\partial_k \mathcal{C}_k = \frac{1}{2} \text{Tr} \left[(\mathcal{L} + \mathfrak{R}_k(\mathcal{L}))^{-1} k\partial_k \mathfrak{R}_k(\mathcal{L}) \right] = \frac{1}{2} \sum_{\Omega^2} \frac{k\partial_k \mathfrak{R}_k(\Omega^2)}{\Omega^2 + \mathfrak{R}_k(\Omega^2)} \quad (11.46)$$

Here Ω^2 denotes the eigenvalues of $\mathcal{L}$, and $\sum$ indicates the summation and/or integration over its spectrum, leaving the corresponding spectral density implicit. To proceed, we opt for a particularly convenient cutoff function $\mathfrak{R}_k$, namely the sharp cutoff⁹:

$$\mathfrak{R}_k(\Omega^2) = \lim_{\hat{\mathfrak{R}} \rightarrow \infty} \hat{\mathfrak{R}} \Theta(k^2 - \Omega^2) \quad (11.47)$$

The limit $\hat{\mathfrak{R}} \rightarrow \infty$ in (11.47) is to be understood in the distributional sense. It should be taken only after the integration over Ω^2 has been performed. If we formally use (11.47) in the equation (11.46) this leads us to $k\partial_k \mathcal{C}_k = 2 \int_{\Omega^2} \delta(1 - \Omega^2/k^2) = 2k^2 \text{Tr} [\delta(k^2 - \mathcal{L})]$, or equivalently,

$$\frac{d}{dk^2} \mathcal{C}_k = \text{Tr} \left[\delta \left(k^2 - \Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}] \right) \right] \geq 0 \quad (11.48)$$

⁸Note that the equation (11.45) is of course insufficient to *determine* the function $k \mapsto \mathcal{C}_k$. We rather use it to *interpret* a given $\mathcal{C}_k$ which was derived from a known solution to the full-fledged FRGE.

⁹See ref. [191] for a detailed discussion of the sharp cutoff. It is often used in quantum gravity since it allows for an easy closed-form evaluation of the threshold functions Φ_n^p and $\bar{\Phi}_n^p$ that frequently appear in QEG beta-functions [122].

The equation (11.48) is quite remarkable and sheds some light on the interpretation of $\mathcal{C}_k$: Its derivative equals exactly the spectral density of the Hessian operator evaluated at the s.c.-background field configuration and for vanishing fluctuations, $\Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}]$. It is a manifestly non-decreasing function of k therefore.

Let us integrate (11.48) over k^2 . Provided the RG effects are weak and Γ_k runs only very slowly, the k -dependence of the resulting field $\bar{\Phi}_k^{\text{s.c.}}$ is weak, too, so that it may be a sensible approximation to neglect the k -dependence of $\Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}]$ in the δ -function of (11.48) relative to the explicit k^2 . Under these special circumstances, the integrated version of (11.48) reads:

$$\mathcal{C}_k = \text{Tr} \left[\Theta \left(k^2 - \Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}] \right) \right] + \text{const} \quad (11.49)$$

Thus, our conclusion is that, at least under the conditions described, the function $\mathcal{C}_k$ indeed counts field modes, in the almost literal sense of the word, namely the eigenfunctions of the Hessian operator which have eigenvalues not exceeding k^2 .

Regardless of the present approximation we define in general

$$\mathcal{N}_{k_1, k_2} \equiv \mathcal{C}_{k_2} - \mathcal{C}_{k_1} \quad (11.50)$$

Then, in the cases when the above assumptions apply and (11.49) is valid, $\mathcal{N}_{k_1, k_2}$ has a simple interpretation: it equals the number of eigenvalues between k_1^2 and $k_2^2 > k_1^2$ of the Hessian operator $\Gamma_k^{(2)}[0; \bar{\Phi}_k^{\text{s.c.}}]$, when the spectrum is discrete. When the assumptions leading to (11.49) are not satisfied, the interpretation of $\mathcal{N}_{k_1, k_2}$, and $\mathcal{C}_k$ in the first place, is less intuitive, but these functions are well defined nevertheless.

As an aside let us also mention that the function (11.49) is closely related to the Chamseddine-Connes spectral action [240–243] in Noncommutative Geometry, where the squared Dirac operator plays the same role as the Hessian operator above.

11.3 Asymptotically safe quantum gravity

In this section we make the above ideas concrete and apply them to an appropriately truncated form of Quantum Einstein Gravity (QEG) which is asymptotically safe, that is, all physically relevant RG trajectories start out in the UV, for $k \rightarrow \infty$, at a point infinitesimally close to a non-Gaussian fixed point (NGFP). When k is lowered they run towards the IR, always staying within the fixed point's UV critical manifold, and ultimately approach the (dimensionless) ordinary effective action [99, 100, 121, 123, 126, 130–135, 138, 139, 141, 142, 146–171].

11.3.1 The single- and bi-metric Einstein-Hilbert truncations

This subsection presents a short summary of the general truncation setting and the differences between the three employed calculations. For more details on the explicit evaluation steps of the FRGE we refer to part II.

Truncated theory space. In the following we study the C -function properties of $\mathcal{C}_k$ in pure, metric-based quantum gravity in an arbitrary spacetime dimension. We rely on results obtained with the so called single- and bi-metric Einstein-Hilbert truncations where the considered subspace of theory space is spanned by the invariants $\int \sqrt{g}$ and $\int \sqrt{g} R$ only, with $g_{\mu\nu}$ - and $\bar{g}_{\mu\nu}$ -contributions disentangled in the bi-metric case.

In either case the ansatz for the EAA in this subspace is given by

$$\Gamma_k[g, \xi, \bar{\xi}, \bar{g}] = \Gamma_k^{\text{grav}}[g, \bar{g}] + \Gamma_k^{\text{gf}}[g, \bar{g}] + \Gamma_k^{\text{gh}}[g, \xi, \bar{\xi}, \bar{g}]. \quad (11.51)$$

It consists of a purely gravitational part, $\Gamma_k^{\text{grav}}[g, \bar{g}]$, and an essentially classical gauge sector¹⁰ based on the coordinate condition $(\delta_\mu^\beta \bar{g}^{\alpha\gamma} \bar{D}_\gamma - \bar{\omega} \bar{g}^{\alpha\beta} \bar{D}_\mu) h_{\mu\nu} = 0$ from which the gauge fixing term $\Gamma_k^{\text{gf}}[g, \bar{g}]$ and the corresponding ghost action $\Gamma_k^{\text{gh}}[g, \xi, \bar{\xi}, \bar{g}] \propto \int \bar{\xi} \mathcal{M}[g, \bar{g}] \xi$ are derived. Here ξ^μ and $\bar{\xi}_\mu$ denote the diffeomorphism ghosts, and $\mathcal{M}[g, \bar{g}]$ is the Faddeev-Popov operator [122].

We will mostly focus in the following on the Einstein-Hilbert truncation in a bi-metric setting. In this case, Γ_k^{grav} comprises two separate Einstein-Hilbert terms built from the dynamical metric $g_{\mu\nu}$ and its background analog, $\bar{g}_{\mu\nu}$, respectively (cf. part II):

$$\begin{aligned} \Gamma_k^{\text{grav}}[g, \bar{g}] = & -\frac{1}{16\pi G_k^{\text{D}}} \int d^d x \sqrt{g} (R(g) - 2\bar{\lambda}^{\text{D}}) \\ & -\frac{1}{16\pi G_k^{\text{B}}} \int d^d x \sqrt{\bar{g}} (R(\bar{g}) - 2\bar{\lambda}^{\text{B}}) \end{aligned} \quad (11.52)$$

The couplings, G_k^{D} , $\bar{\lambda}^{\text{D}}$ and G_k^{B} , $\bar{\lambda}^{\text{B}}$ represent k -dependent generalizations of the classical Newton or cosmological constant in the dynamical ('D') and the background ('B') sector, respectively. Expanding eq. (11.52) in terms of the fluctuation field $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ yields the level-expansion of the EAA:

$$\begin{aligned} \Gamma_k^{\text{grav}}[h; \bar{g}] = & -\frac{1}{16\pi G_k^{(0)}} \int d^d x \sqrt{\bar{g}} (R(\bar{g}) - 2\bar{\lambda}_k^{(0)}) \\ & -\frac{1}{16\pi G_k^{(1)}} \int d^d x \sqrt{\bar{g}} \left[-\bar{G}^{\mu\nu} - \bar{\lambda}_k^{(1)} \bar{g}^{\mu\nu} \right] h_{\mu\nu} \\ & -\frac{1}{2} \int d^d x \sqrt{\bar{g}} h^{\mu\nu} \Gamma_k^{\text{grav}(2)}[\bar{g}, \bar{g}]_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \mathcal{O}(h^3) \end{aligned} \quad (11.53)$$

In the level-description, the background and dynamical couplings appear in certain combinations in front of invariants that have a definite level, i.e. order in $h_{\mu\nu}$. The two sets of coupling constants are related by

$$\frac{1}{G_k^{(0)}} = \frac{1}{G_k^{\text{B}}} + \frac{1}{G_k^{\text{D}}}, \quad \frac{\bar{\lambda}_k^{(0)}}{G_k^{(0)}} = \frac{\bar{\lambda}_k^{\text{B}}}{G_k^{\text{B}}} + \frac{\bar{\lambda}_k^{\text{D}}}{G_k^{\text{D}}}, \quad (11.54a)$$

$$\frac{1}{G_k^{(p)}} = \frac{1}{G_k^{\text{D}}} \text{ for } p \geq 1, \quad \frac{\bar{\lambda}_k^{(p)}}{G_k^{(p)}} = \frac{\bar{\lambda}_k^{\text{D}}}{G_k^{\text{D}}} \text{ for } p \geq 1. \quad (11.54b)$$

Notice that the level-(0) couplings, $G_k^{(0)}$ and $\bar{\lambda}_k^{(0)}$, multiply pure background invariants and thus do not contribute to the dynamical field equations. They are, however, relevant to the statistical mechanics of black holes, for instance [166] or chapter 13 of this thesis.

In the present ansatz all couplings of higher level, $p \geq 1$, are identical and agree with the dynamical ('D') ones. However, the level-(1) Newton and cosmological constants, $G_k^{(1)} \equiv G_k^{\text{D}}$ and $\bar{\lambda}_k^{(1)} \equiv \bar{\lambda}_k^{\text{D}}$, which enter the effective field equations and the tadpole equation, differ in general from the level-(0) couplings.

Single-metric approximation. When the distinction of the different levels is artificially suppressed in the truncation ansatz by hypothesizing perfect split-symmetry along the entire RG trajectory, i.e. if we set $G_k^{(0)} = G_k^{(p)} \equiv G_k^{\text{sm}}$ and $\bar{\lambda}_k^{(0)} = \bar{\lambda}_k^{(p)} \equiv \bar{\lambda}_k^{\text{sm}}$ for all p and k , then the gravitational action $\Gamma_k^{\text{grav}}[g, \bar{g}]$ reduces to a functional of a single metric:

$$\Gamma_k^{\text{grav}}[g, \bar{g}] = -\frac{1}{16\pi G_k^{\text{sm}}} \int d^d x \sqrt{g} (R(g) - 2\bar{\lambda}_k^{\text{sm}}) \quad (11.55)$$

¹⁰For a *single-metric* extension of the ghost sector, see refs. [132–134].

The second argument of the action functional, $\bar{g}$, has actually disappeared from the RHS of (11.55). This approximation, for obvious reasons, is referred to as the single-metric Einstein-Hilbert truncation. In general split-symmetry is violated during the RG evolution, and a detailed comparison with the more advanced bi-metric ansatz (11.53) has revealed that there are even qualitative differences in the respective flows, especially in the crossover regime, see chapter 10.

Gauge sector. In the sequel we will analyze the RG flows obtained in three different RG studies. One is based upon the single-metric ansatz (11.55), while the other two are bi-metric calculations which employ the same, more general 4-parameter ansatz (11.53), but differ in various details of the computational setting, the gauge choice in particular.

All three calculations use a gauge fixing action of the form

$$\Gamma_k^{\text{gf}}[g, \bar{g}] = \frac{1}{32\pi\alpha G_k^{\text{D/sm}}} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} \left[\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] (g_{\alpha\beta} - \bar{g}_{\alpha\beta}) \right] \left[F S g f_v^{\rho\sigma}[\bar{g}] (g_{\rho\sigma} - \bar{g}_{\rho\sigma}) \right] \quad (11.56)$$

It depends on the gauge parameter α and the coefficient ϖ occurring in the gauge condition $\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] h_{\mu\nu} \equiv (\delta_\mu^\beta \bar{g}^{\alpha\gamma} \bar{D}_\gamma - \varpi \bar{g}^{\alpha\beta} \bar{D}_\mu) h_{\mu\nu}$. Both α and ϖ are k -independent by assumption. The single-metric results obtained in [122] are based on the choice ($\varpi = 1/2$, $\alpha = 1$), whereas the bi-metric calculations performed in [172], henceforth denoted [I], and in part II of this thesis, in the following referred to as [II], use ($\varpi = 1/d$, $\alpha \rightarrow 0$), and ($\varpi = 1/2$, $\alpha = 1$), respectively.

In this chapter, the investigation of $\mathcal{C}_k$ will be based upon the beta-functions derived in refs. [122], [I], and [II], respectively.

RG flow and phase diagram. The beta-functions describing the flow on theory space pertain to the dimensionless couplings $g_k^I \equiv k^{d-2} G_k^I$ and $\lambda_k^I \equiv k^{-2} \bar{\lambda}_k^I$ for $I \in \{\text{B}, \text{D}, \text{sm}, (0), (1), \dots\}$. In the bi-metric case, the 4 independent RG differential equations are partially decoupled, displaying the hierarchical structure [173]:

$$(g_k^{\text{D/(p)}}, \lambda_k^{\text{D/(p)}}) \rightarrow g_k^{\text{B/(0)}} \rightarrow \lambda_k^{\text{B/(0)}}, \quad \text{for } p \geq 1 \quad (11.57)$$

In order to solve the system of differential equations one starts by finding solutions of the D or $p \geq 1$ -sector, and then substitutes them successively into the decoupled RG equations of the B- or level-(0) couplings, depending on which ‘language’ one uses.

In [II] it was shown that the beta-functions obtained for the different gauge choices in [I] and [II] yield the same qualitative results, with only minor numerical differences. In particular, a UV-fixed point was found in both cases with remarkably stable properties under the change of gauge. Quite surprisingly, its ‘D’-coordinates agree quite well with the results from the single-metric approximation; in fact the latter turned out to be unusually reliable within this regime.

Since the RG flows in the single- and bi-metric truncations are qualitatively similar their (projected) phase portraits in the $g^{\text{D}}\text{-}\lambda^{\text{D}}$ and $g^{\text{sm}}\text{-}\lambda^{\text{sm}}$ plane, respectively, share the same overall structure, as depicted in Fig. 11.1. The integral curves in the upper half plane ($g^{\text{D/sm}} > 0$) are classified as type Ia, type IIa, or type IIIa trajectories, depending on whether the cosmological constant $\lambda^{\text{D/sm}}$ approaches $-\infty$, 0, or $+\infty$ in the IR,¹¹ respectively [146]. The type IIa trajectory

¹¹The Einstein-Hilbert truncation is known to be inapplicable to type IIIa trajectories when $\lambda^{\text{D/sm}}$ approaches values of order unity. Here, we assume that their classical regime (CR) (having $\lambda^{\text{D/sm}} \ll 1$) represents their true $k \rightarrow 0$ limit. Even if ultimately this should turn out not to be the case, our treatment of the NGFP $\rightarrow$ CR crossover will remain valid.

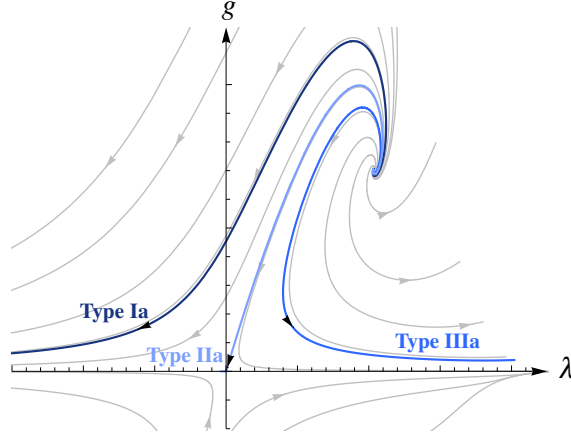


Figure 11.1: The schematic structure of the phase portrait on $g^{\text{sm}}-\lambda^{\text{sm}}$ or the projected $g^{\text{D}}-\lambda^{\text{D}}$ plane as predicted by all three truncations considered. Here and in the following the arrows always point in the direction of decreasing k .

is a separatrix: it separates solutions with an ultimately positive cosmological constant from those with a negative one at $k = 0$. Likewise the trajectory $g^{\text{D/sm}} = 0$ separates the upper and lower half plane, indicating that once the Newton coupling is chosen positive, it remains so on all scales.

The type IIIa trajectories display a generalized crossover of the kind mentioned in section 11.2.6. It connects a fixed point in the UV to a classical regime in the IR. The latter is located on its lower, almost horizontal branch where $g, \lambda \ll 1$ [209, 212].

11.3.2 Gravitational instantons

General self-consistent background solutions. Let us now set up the tadpole equations which result from the truncation ansatz $\Gamma_k = \Gamma_k^{\text{grav}} + \Gamma_k^{\text{gf}} + \Gamma_k^{\text{gh}}$. To be consistent with the conventions in eq. (11.14) we must introduce background fields also for the ghosts, at least for a moment. We decompose them as $\xi^\mu = \Xi^\mu + \eta^\mu$ and $\bar{\xi}_\mu = \bar{\Xi}_\mu + \bar{\eta}_\mu$ where $(\Xi, \bar{\Xi})$ and $(\eta, \bar{\eta})$ denote their backgrounds and fluctuations, respectively. Then the tadpole condition (11.14) amounts to the following three coupled equations for $\varphi \in \{h, \eta, \bar{\eta}\}$:

$$0 = \left. \frac{\delta (\Gamma_k^{\text{grav}} + \Gamma_k^{\text{gf}} + \Gamma_k^{\text{gh}})}{\delta \varphi(x)} \right|_{h=0, \eta=0, \bar{\eta}=0; \bar{g}=\bar{g}_k^{\text{s.c.}}, \Xi=\Xi_k^{\text{s.c.}}, \bar{\Xi}=\bar{\Xi}_k^{\text{s.c.}}} \quad (11.58)$$

If we begin by solving the equations involving $\delta/\delta\eta$ and $\delta/\delta\bar{\eta}$ and take advantage of the fact that $\Gamma_k^{\text{gh}} \propto \int (\bar{\Xi} + \bar{\eta}) \mathcal{M}(\Xi + \eta)$ is bilinear in the ghosts we conclude immediately that the only self-consistent background they admit for a non-degenerate $\mathcal{M}$ is the trivial one, $\Xi_k^{\text{s.c.}} = 0 = \bar{\Xi}_k^{\text{s.c.}}$. As a consequence, the third equation, $\delta\Gamma_k/\delta h|_{\dots} = 0$, receives no contribution from Γ_k^{gh} since its h -derivative vanishes upon inserting the vanishing background ghosts and $\eta = 0 = \bar{\eta}$. Furthermore, the gauge fixing action, too, does not contribute since $\Gamma_k^{\text{gf}} \propto \int (\mathcal{F}h)^2$, being bilinear in h , has a vanishing derivative at $h = 0$.

As a result, for every truncation ansatz of the above form, that is, for any choice of the ‘gravitational’ piece Γ_k^{grav} , the tadpole condition for $\bar{\Phi}_k^{\text{s.c.}} \equiv (\bar{g}_k^{\text{s.c.}}, \Xi_k^{\text{s.c.}}, \bar{\Xi}_k^{\text{s.c.}}) = (\bar{g}_k^{\text{s.c.}}, 0, 0)$ boils down to a single non-trivial equation, namely

$$\left. \frac{\delta}{\delta h_{\mu\nu}(x)} \Gamma_k^{\text{grav}}[h; \bar{g}] \right|_{h=0; \bar{g}=\bar{g}_k^{\text{s.c.}}} = 0 \quad (11.59)$$

This equation determines the self-consistent metrics which can ‘live’ on a given spacetime manifold, M , without being modified by the agitation of the quantum fluctuations.

For pure metric gravity, in truncations of the type $\Gamma_k = \Gamma_k^{\text{grav}} + \Gamma_k^{\text{gf}} + \Gamma_k^{\text{gh}}$, the C -function candidate $\mathcal{C}_k \equiv \Gamma_k[\varphi = 0; \bar{\Phi}_k^{\text{s.c.}}]$ which we motivated above for a generic theory now becomes concretely $\mathcal{C}_k = \Gamma_k|_{h=\eta=\bar{\eta}=\Xi=\bar{\Xi}=0, \bar{g}=\bar{g}_k^{\text{s.c.}}}$. It involves only the ‘grav’-part of the ansatz:

$$\mathcal{C}_k = \Gamma_k^{\text{grav}}[h = 0; \bar{g}_k^{\text{s.c.}}] \quad (11.60)$$

The Einstein-Hilbert case. In the special case of the Einstein Hilbert truncation the tadpole equation (11.59) happens to have the same mathematical structure as the classical vacuum Einstein equation in presence of a cosmological constant. The $h_{\mu\nu}$ -derivative of the bi-metric ansatz (11.53) for Γ_k^{grav} yields at $h_{\mu\nu} = 0$:

$$G_{\mu\nu}(\bar{g}_k^{\text{s.c.}}) = -\bar{\lambda}_k^{(1)} \bar{g}_{k\mu\nu}^{\text{s.c.}}, \quad \text{or} \quad R_{\mu\nu}(\bar{g}_k^{\text{s.c.}}) = \frac{2}{d-2} \bar{\lambda}_k^{(1)} \bar{g}_{k\mu\nu}^{\text{s.c.}} \quad (11.61)$$

In the single-metric approximation the tadpole equation is the same, except that $\bar{\lambda}_k^{(1)}$ is replaced with $\bar{\lambda}_k^{\text{sm}}$ then. Thus $\bar{g}_k^{\text{s.c.}}$ is always an Einstein metric, and upon contraction we get from eq. (11.61):

$$R(\bar{g}_k^{\text{s.c.}}) = \frac{2d}{(d-2)} \bar{\lambda}_k^{(1)} \quad (11.62)$$

Inserting this expression for the curvature scalar into Γ_k^{grav} yields the following representation of the EAA, evaluated for $h_{\mu\nu} = 0$ and a self-consistent background geometry:

$$\begin{aligned} \mathcal{C}_k = \Gamma_k^{\text{grav}}[0; \bar{g}_k^{\text{s.c.}}] &= -\frac{1}{16\pi G_k^{(0)}} \int_M d^d x \sqrt{\bar{g}} \left\{ R(\bar{g}) - 2\bar{\lambda}_k^{(0)} \right\} \Big|_{\bar{g}=\bar{g}_k^{\text{s.c.}}} \\ &= -\frac{1}{8\pi G_k^{(0)}} \left[\left(\frac{d}{d-2} \right) \bar{\lambda}_k^{(1)} - \bar{\lambda}_k^{(0)} \right] \text{Vol}(M, \bar{g}_k^{\text{s.c.}}) \end{aligned} \quad (11.63)$$

Here $\text{Vol}(M, g) \equiv \int_M d^d x \sqrt{g}$ denotes the Euclidean volume of the manifold M measured with the metric written in the argument, $g_{\mu\nu}$.

Note that Γ_k^{grav} evaluated at $(h; \bar{g}) = (0; \bar{g}_k^{\text{s.c.}})$ depends on both the level-(0) and the level-(1) couplings in a non-trivial way: the former enter via the action $\Gamma_k|_{h=0}$ which has a level-(0) component only, the latter via the tadpole equation which is entirely ‘level-(1)’.

Assuming the running dimensionful couplings are regular at the scale k considered, eq. (11.63) shows that $\mathcal{C}_k$ is finite if, and only if, the spacetime manifold has finite volume. The self-consistent background being an Einstein metric, its curvature structure and other details play no role for the value of the action, it is only the volume that matters.

Instanton solutions. Trying to find solutions to (11.61) that exist for all scales from ‘ $k = \infty$ ’ down to $k = 0$ the simplest situation arises when all metrics $\bar{g}_k^{\text{s.c.}}$, $k \in [0, \infty)$ can be put on *the same smooth manifold* M , leading in particular to the same spacetime topology at all scales, thus avoiding the delicate issue of a topological change. This situation is realized, for example, if the level-(1) cosmological constant is positive on all scales, which is indeed the case along the type (IIIa) trajectories: $\bar{\lambda}_k^{(1)} > 0$, $k \in [0, \infty)$.

In the following we focus on precisely this situation. The requirement of a finite action is then met by a well studied class of Einstein spaces which exist for an arbitrary positive value of the cosmological constant, namely certain 4-dimensional gravitational instantons [244–246], see Table 11.1 for some examples.

Metric	M	$\mathcal{V}$
Eucl. de Sitter	S^4	3π
Page	$P_2 + \bar{P}_2$	1.8π
$S^2 \times S^2$	$S^2 \times S^2$	2π
Fubini-Study	$P_2(\mathbb{C})$	$9\pi/4$

Table 11.1: Various 4-dimensional gravitational instantons and the related normalized volumes $\mathcal{V}(M, \mathring{g})$. (See [244] for a detailed account.)

Let $\mathring{g}_{\mu\nu}$ be the metric of one such instanton, corresponding to a fixed reference value of the cosmological constant, $\mathring{\lambda}$, say. Then the tadpole equation (11.61), at any k , is solved by the following rescaled metric [210, 211]:

$$\bar{g}_{k\mu\nu}^{\text{s.c.}} = \frac{\mathring{\lambda}}{\bar{\lambda}_k^{(1)}} \mathring{g}_{\mu\nu} \quad (11.64)$$

As a result, the k -dependence of the total volume behaves as, for arbitrary d ,

$$\text{Vol}(M, \bar{g}_k^{\text{s.c.}}) = 8\pi [\bar{\lambda}_k^{(1)}]^{-d/2} \cdot \mathcal{V}(M, \mathring{g}) \quad (11.65)$$

Here we introduced the dimensionless constant

$$\mathcal{V}(M, \mathring{g}) \equiv \frac{1}{8\pi} \mathring{\lambda}^{d/2} \text{Vol}(M, \mathring{g}) \quad (11.66)$$

which is characteristic of the instanton under consideration.¹² The number $\mathcal{V}$ is manifestly independent of k , and it is easy to see that it is also *independent of $\mathring{\lambda}$* . The reason is that $\mathring{g}$ depends on $\mathring{\lambda}$ via the equation $R_{\mu\nu}(\mathring{g}) = \frac{2}{d-2} \mathring{\lambda} \mathring{g}_{\mu\nu}$. This implies that upon rescaling $\mathring{\lambda}$ by a constant factor, $\mathring{\lambda} \rightarrow c^2 \mathring{\lambda}$, the metric responds according to $\mathring{g}_{\mu\nu} \rightarrow c^{-2} \mathring{g}_{\mu\nu}$, and so the volume behaves as $\text{Vol}(M, \mathring{g}) \rightarrow c^{-d} \text{Vol}(M, \mathring{g})$. In the definition of $\mathcal{V}$, eq. (11.66), the factor c^{-d} coming from the volume is therefore precisely canceled by a corresponding factor c^{+d} which is produced by its pre-factor, $\mathring{\lambda}^{d/2} \rightarrow c^d \mathring{\lambda}^{d/2}$.

Thus the value of $\mathcal{V}$ is a universal number which depends only on the type of the instanton considered¹³. For the round metric on S^d we find, for instance,

$$\mathcal{V}(S^d) = \pi^{\frac{(d-1)}{2}} \frac{[(d-1)(d-2)]^{d/2}}{2^{(d+4)/2} \Gamma(\frac{d+1}{2})} \quad (11.67)$$

Table 11.1 contains the corresponding $\mathcal{V}$ values for some more examples in $d = 4$.

The $\mathcal{C}_k$ -function for the Einstein-Hilbert truncation. Using (11.65) in (11.63) we obtain the following two equivalent representations of $\mathcal{C}_k$:

$$\begin{aligned} \mathcal{C}_k &= -\frac{1}{G_k^{(0)} [\bar{\lambda}_k^{(1)}]^{d/2}} \left[\left(\frac{d}{d-2} \right) \bar{\lambda}_k^{(1)} - \bar{\lambda}_k^{(0)} \right] \mathcal{V}(M, \mathring{g}) \\ &= -\frac{1}{g_k^{(0)} [\lambda_k^{(1)}]^{d/2}} \left[\left(\frac{d}{d-2} \right) \lambda_k^{(1)} - \lambda_k^{(0)} \right] \mathcal{V}(M, \mathring{g}) \end{aligned} \quad (11.68)$$

¹²It is closely related to the normalized volume $\bar{v}(M, g)$ defined in the mathematical literature [245, 247, 248].

¹³For a discussion of the topological properties of the normalized volume see [245, 247, 248].

In the second line of (11.68) we eliminated the dimensionful quantities $G_k^{(p)}$ and $\bar{\lambda}_k^{(p)}$ in favor of their dimensionless analogs whereby all explicit factors of k dropped out.

We observe that the result (11.68) for the function $k \mapsto \mathcal{C}_k$ has the general structure

$$\mathcal{C}_k \equiv \mathcal{C}(g_k^{(0)}, \lambda_k^{(0)}, \lambda_k^{(1)}) = Y(g_k^{(0)}, \lambda_k^{(0)}, \lambda_k^{(1)}) \mathcal{V}(M, \hat{g}) \quad (11.69)$$

Here $Y(\cdot) \equiv \mathcal{C}(\cdot)/\mathcal{V}$ stands for the following function over theory space:

$$Y(g^{(0)}, \lambda^{(0)}, \lambda^{(1)}) = - \frac{\left[\left(\frac{d}{d-2}\right) \lambda^{(1)} - \lambda^{(0)}\right]}{g^{(0)} [\lambda^{(1)}]^{d/2}} \quad (11.70)$$

Equation (11.70) represents our main result. We shall study its properties below. In 4 dimensions we have in particular

$$Y(g^{(0)}, \lambda^{(0)}, \lambda^{(1)}) = - \frac{2\lambda^{(1)} - \lambda^{(0)}}{g^{(0)} (\lambda^{(1)})^2} \quad (d=4) \quad (11.71)$$

Several comments are in order here.

- A.** The function $\mathcal{C}$ depends on both the RG trajectory and on the solution to the running self-consistency condition, along this very trajectory, that has been picked. In eq. (11.69) those two dependencies factorize: the former enters via the function Y , the latter via the constant factor $\mathcal{V}(M, \hat{g})$ that characterizes the gravitational instanton.
- B.** The dependence on the RG trajectory, parametrized as $k \mapsto (g_k^{(0,1)}, \lambda_k^{(0,1)})$, is obtained by evaluating a *scalar function on theory space* along this curve, namely $Y: \mathcal{T} \rightarrow \mathbb{R}$, $(g^{(0,1)}, \lambda^{(0,1)}) \mapsto Y(g^{(0)}, \lambda^{(0)}, \lambda^{(1)})$. It is defined at all points of $\mathcal{T}$ where $g^{(0)} \neq 0$ and $\lambda^{(1)} \neq 0$, and turns out to be actually independent of $g^{(1)}$.
- C.** We shall refer to $Y_k \equiv Y(g_k^{(0)}, \lambda_k^{(0)}, \lambda_k^{(1)}) \equiv \mathcal{C}_k/\mathcal{V}(M, \hat{g})$ and $Y(\cdot) \equiv \mathcal{C}(\cdot)/\mathcal{V}(M, \hat{g})$ as the *reduced $\mathcal{C}_k$ and $\mathcal{C}(\cdot)$ functions*, respectively.
- D.** Denoting the collection of running couplings by $u(k)$, we may write the scale derivative of $\mathcal{C}_k$ as $k\partial_k \mathcal{C}_k = \mathcal{V}(M, \hat{g}) \left(\beta_\alpha \frac{\partial}{\partial u_\alpha} Y \right) (u(k))$ which involves the directional derivative $\vec{\beta} \cdot \vec{\nabla}$ acting upon scalar functions on theory space. This motivates defining the subset $\mathcal{T}_+$ of $\mathcal{T}$ on which $Y: \mathcal{T} \rightarrow \mathbb{R}$ has a positive directional derivative in the direction of $\vec{\beta}$:

$$\mathcal{T}_+ \equiv \left\{ u \in \mathcal{T} \mid \beta_\alpha(u) \frac{\partial}{\partial u_\alpha} Y(u) > 0 \right\}. \quad (11.72)$$

The interpretation is that $\mathcal{C}_k$ increases monotonically with k along those (parts of) RG trajectories that lie entirely inside $\mathcal{T}_+$, i.e. $k\partial_k \mathcal{C}_k > 0$ at all points of $\mathcal{T}_+$.

- E.** Invoking the idealization of exact split-symmetry, i.e. assuming that the Newton and cosmological constants of different levels are all equal ($g_k^{(p)} \equiv g_k^{\text{sm}}, \lambda_k^{(p)} \equiv \lambda_k^{\text{sm}}, p=0,1,2,\dots$), we obtain $\mathcal{C}_k$ in the single-metric approximation.

It reads $\mathcal{C}_k = \mathcal{C}(g_k^{\text{sm}}, \lambda_k^{\text{sm}}) = Y^{\text{sm}}(g_k^{\text{sm}}, \lambda_k^{\text{sm}}) \cdot \mathcal{V}(M, \hat{g})$ with

$$Y^{\text{sm}}(g^{\text{sm}}, \lambda^{\text{sm}}) = - \left(\frac{2}{d-2} \right) \frac{1}{g^{\text{sm}} (\lambda^{\text{sm}})^{d/2-1}} \quad (11.73)$$

Note that it depends only on the dimensionless combination $G_k^{\text{sm}} (\bar{\lambda}_k^{\text{sm}})^{d/2-1} = g_k^{\text{sm}} (\lambda_k^{\text{sm}})^{d/2-1}$ whose robustness properties under changes of the cutoff and the gauge fixing has often been used to check the reliability of single-metric truncations [123, 135, 151, 191].

- F.** In general, the beta-functions which govern the RG evolution of Γ_k may depend on the topology of M , see [236] for an example. Within the truncation considered here this is not the case, however, the reason being the universality of the heat-kernel asymptotics which is exploited in the computation of the beta-functions, see section 7.5 of part II of this thesis and ref. [172].

- G.** Switching from the level language, which employs the couplings $g^{(p)}$ and $\lambda^{(p)}$, to the B-D language, based upon the couplings $\{g^D, \lambda^D, g^B, \lambda^B\}$, with

$$\frac{1}{g^{(0)}} = \frac{1}{g^B} + \frac{1}{g^D}, \quad \frac{\lambda^{(0)}}{g^{(0)}} = \frac{\lambda^B}{g^B} + \frac{\lambda^D}{g^D} \quad \text{and} \quad g^{(1)} = g^D, \quad \lambda^{(1)} = \lambda^D,$$

the function $Y : \mathcal{T} \rightarrow \mathbb{R}$ assumes the following form, for $d = 4$,

$$Y(g^D, \lambda^D, g^B, \lambda^B) = -\frac{1}{g^D \lambda^D} - \frac{1}{g^B \lambda^D} \left[2 - \frac{\lambda^B}{\lambda^D} \right] \quad (11.74)$$

This representation is particularly convenient for part of the numerical analyses to which we turn in the next section.

11.3.3 Numerical results

Above we investigated the general properties of $\mathcal{C}_k$ and we derived its explicit form for the Einstein-Hilbert truncation. In this subsection we will study the monotonicity properties of $\mathcal{C}_k$ in 4 spacetime dimensions for solutions of the FRGE in both the single-metric [122] and the two bi-metric approaches [I], [II]. We will focus on type IIIa trajectories (see Fig.11.1) which exhibit the aforementioned NGFP $\rightarrow$ CR crossover. Possibly those solutions are relevant to real Nature even [209, 212]. For the corresponding discussion of type Ia and type IIa trajectories we refer to appendix 11.B.

- A.** Returning to $\mathcal{C}_k$ for the bi-metric truncation given in eq. (11.69), the entire information on the RG trajectory is contained in the factor

$$Y_k \equiv Y(g_k^{(0)}, \lambda_k^{(0)}, \lambda_k^{(1)}) = -\frac{2\lambda_k^{(1)} - \lambda_k^{(0)}}{g_k^{(0)} [\lambda_k^{(1)}]^2} \quad (11.75)$$

We are going to evaluate this function of k for a number of RG trajectories on the 4-dimensional theory space which we generate numerically.

- B.** Likewise we compute the reduced $\mathcal{C}_k$ -function predicted by the single-metric truncation, that is, we evaluate

$$Y_k^{\text{sm}} \equiv Y^{\text{sm}}(g_k^{\text{sm}}, \lambda_k^{\text{sm}}) = -\frac{1}{g_k^{\text{sm}} \lambda_k^{\text{sm}}} \quad (11.76)$$

for trajectories on the corresponding 2-dimensional theory space. We obtain them numerically by solving a system consisting of 2 differential equations only.

- C.** We are also going to perform a *hybrid calculation* which is intermediate between the 2- and 4-dimensional treatment, in the following sense.

As a rule, the single-metric approximation to a bi-metric truncation is a valid description of the flow if split-symmetry is only weakly broken, i.e. there is no significant difference between couplings at different levels: $u_\alpha^{(0)} = u_\alpha^{(1)} = u_\alpha^{(2)} = \dots$. If we make the corresponding identifications $\lambda_k^{(0)} = \lambda_k^{(1)} = \dots \equiv \lambda_k$ and $g_k^{(0)} = g_k^{(1)} = \dots \equiv g_k$ in (11.75) we obtain

$$Y_k^{\text{split-sym}} \equiv Y(g_k, \lambda_k, \lambda_k) = -\frac{1}{g_k \lambda_k} \quad (11.77)$$

Here (g_k, λ_k) stands for $(g_k^{(0)}, \lambda_k^{(0)})$ or, what should be the same, $(g_k^{(1)}, \lambda_k^{(1)}) \equiv (g_k^D, \lambda_k^D)$ as obtained from the bi-metric RG equations. If the trajectory respects split-symmetry it does not matter from which level we take the couplings. If split-symmetry is not perfect, it

does however matter from which level they come, and we obtain two, in general different functions:

$$Y_k^{\text{split-sym},(0)} \equiv -\frac{1}{g_k^{(0)} \lambda_k^{(0)}}, \quad Y_k^{\text{split-sym},(1)} \equiv -\frac{1}{g_k^{(1)} \lambda_k^{(1)}} \quad (11.78)$$

It will be instructive to compare the two functions (11.78) for various representative trajectories on 4-dimensional theory space. This will provide us with some insights about what is more important in an approximate calculation of $\mathcal{C}_k$: good control over the details of the underlying RG trajectory, or precise (analytic) knowledge about how $\Gamma_k[0; \bar{\Phi}_k^{\text{s.c.}}]$ depends on the couplings from the various levels when split-symmetry is broken.

Note that while (11.77) and (11.78) have the same *structure* as the single-metric result (11.76), there is a crucial difference: the former Y_k -functions involve running couplings obtained from the 4-dimensional bi-metric system of RG equations, whereas the latter, Y_k^{sm} , has the solutions to the 2-dimensional single-metric flow equations as its input.

Note also that by virtue of (11.74) the full-fledged bi-metric Y_k may be written as

$$Y_k = Y_k^{\text{split-sym},(1)} + \Delta Y_k \quad \text{with} \quad \Delta Y_k \equiv -\frac{1}{g_k^{\text{B}} \lambda_k^{\text{D}}} \left[2 - \frac{\lambda_k^{\text{B}}}{\lambda_k^{\text{D}}} \right] \quad (11.79)$$

The magnitude of the ΔY_k -term is a measure for the degree of split-symmetry violation as $\Delta Y_k = 0$ when the symmetry is exact¹⁴ and $Y_k \equiv Y_k^{\text{split-sym},(p)}$ for all $p = 0, 1, 2, \dots$.

Single-metric truncation

We begin the computation of the reduced single-metric $\mathcal{C}_k$ -function by numerically calculating a number of type IIIa trajectories on the 2-dimensional $g^{\text{sm}}\text{-}\lambda^{\text{sm}}$ theory space, and then evaluate Y_k^{sm} for them. We find that the reduced $\mathcal{C}_k$ -functions thus obtained always have the same qualitative properties: they become stationary (approach plateaus) for $k \rightarrow \infty$ and $k \rightarrow 0$, but *they are not on all scales monotonically increasing with k* . For all trajectories there exists a regime of scales where $\partial_k \mathcal{C}_k < 0$. This negative derivative typically occurs while the trajectory crosses over from the NGFP to its turning point close to the GFP.

In Fig. 11.2 we display a representative single-metric example. For reasons of a clearer presentation we plot here, and in all analogous diagrams that will follow, *the inverse* of the reduced $\mathcal{C}_k$ function, along with its derivative. Furthermore, here and in the following, the scale k is always measured in units of the Planck mass defined by the classical regime, $m_{\text{Planck}} \equiv 1/\sqrt{G_{\text{CR}}}$. The example of Fig. 11.2 shows the ‘wrong sign’ of the scale derivative ($\partial_k Y_k^{\text{sm}} < 0$) for k in an interval between about 3 and 5 Planck masses, which is the typical order of magnitude.

Bi-metric Einstein-Hilbert truncation

Turning now to the bi-metric Einstein-Hilbert truncation with its 4-dimensional theory space we employ the two sets of RG equations from [I] and [II], respectively, and compare the results they imply.

Furthermore, we must distinguish two fundamentally different cases with respect to the RG trajectories, namely trajectories which restore split-symmetry in the IR, and trajectories which do not.

In either case we begin by numerically computing a type IIIa trajectory of the decoupled $g^{\text{D}}\text{-}\lambda^{\text{D}}$ subsystem. Then, when we ‘lift’ this 2D trajectory to a 4D one, we must pick initial

¹⁴Of course, this can also be seen directly. Reinstating dimensionful couplings, the two contributions to $\Delta Y_k = -\frac{1}{G_k^{\text{B}} \lambda_k^{\text{D}}} \left[2 - \frac{\lambda_k^{\text{B}}}{\lambda_k^{\text{D}}} \right]$ are proportional to $1/G_k^{\text{B}}$ and $\bar{\lambda}_k^{\text{B}}/G_k^{\text{B}}$, respectively. Those quantities are the pre-factors of the monomials responsible for the *extra* background dependence of Γ_k , and so they must vanish to achieve split-symmetry.

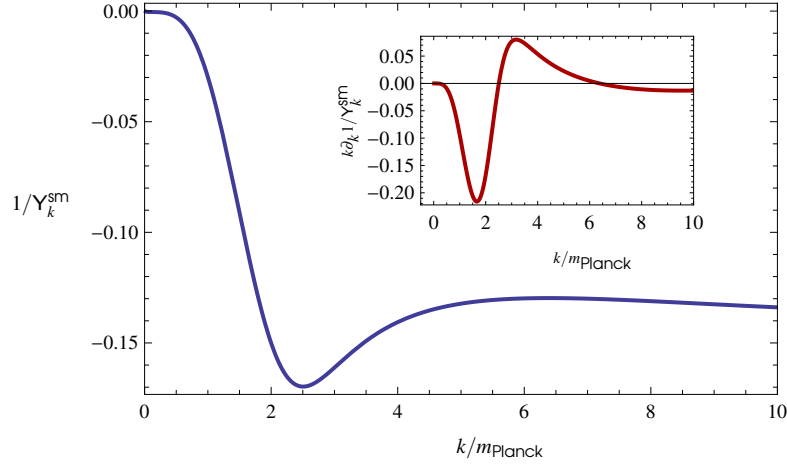


Figure 11.2: The inverse of Y_k^{sm} for a typical single-metric type IIIa trajectory. The inset shows its k -derivative whose positive values indicate a violation of monotonicity.

conditions for g^{B} and λ^{B} , and it is at this point that we must decide about restoring, or not restoring the symmetry. As we showed in detail in ref. [173], the requirement of split-symmetry implies uniquely fixed values for the couplings g_k^{B} and λ_k^{B} in the limit $k \searrow 0$, namely precisely the coordinates $(g_{\bullet}^{\text{B}}(k), \lambda_{\bullet}^{\text{B}}(k))$ of the running UV-attractor. We now discuss the cases with and without symmetry restoration in turn.

Split-symmetry restoring trajectories. Opting for the symmetry restoring IR values of the B-couplings, what remains free to vary is the underlying type IIIa trajectory in the D sector. We find that the qualitative properties of the resulting functions Y_k are the same for all trajectories of this type, and that these properties do not depend on whether we use the RG equations from [I] or from [II]. The picture is always the following: The reduced $\mathcal{C}_k$ -function Y_k ap-

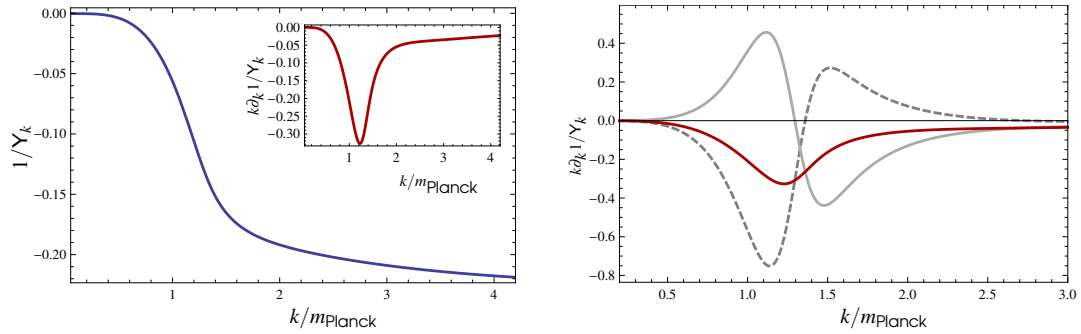


Figure 11.3: The left plot shows $1/Y_k$ and its scale derivative for a typical bi-metric type IIIa trajectory that restores split-symmetry in the IR. It is based on the RG equations of [I]. For these trajectories, $\mathcal{C}_k$ is always found to be perfectly monotone. The inset in the left plot shows $k\partial_k 1/Y_k$, which is decomposed in the right plot into the derivative of the split-symmetric component $1/Y_k^{\text{split-sym},(1)}$ (dashed, gray curve) and of $\Delta(1/Y_k)$ (solid, gray curve). Neither of the two contributions is negative definite separately, but their sum is (solid, dark red curve).

proaches plateau values in the fixed point and in the classical regime, i.e. it becomes stationary there, and it is a strictly monotone function of k on all scales in between, $k\partial_k Y_k > 0$.

Clearly the latter property is in marked contrast with the single-metric results. In Fig. 11.3

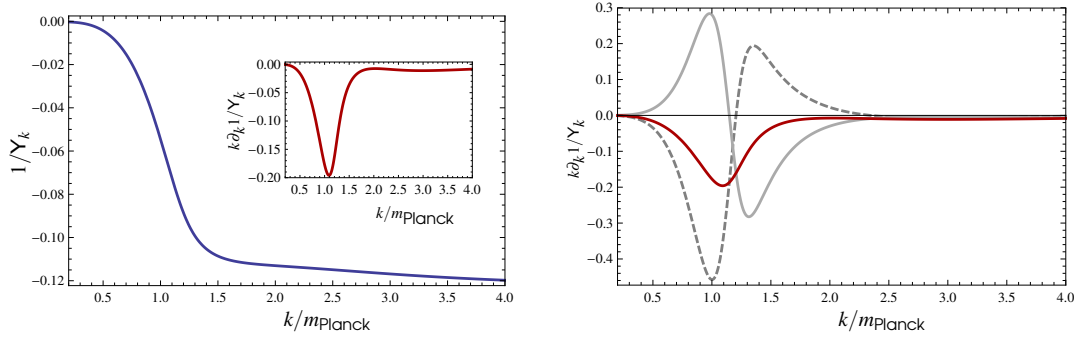


Figure 11.4: The function $1/Y_k$ as in Fig. 11.3, but now based on the RG equations of [II].

we show Y_k for a representative IIIa-trajectory. The plots were obtained with the RG equations derived in [I]. Their analogs based on the equations from [II] are displayed in Fig. 11.4. It is gratifying to see that there is hardly any difference between the results from the two calculational schemes.

In eq. (11.79) we decomposed Y_k as $Y_k = Y_k^{\text{split-sym},(1)} + \Delta Y_k$ in order to make its split-symmetry violating part explicit. For $1/Y_k$ we have correspondingly $1/Y_k = 1/Y_k^{\text{split-sym},(1)} + \Delta(1/Y_k)$ with $\Delta(1/Y_k) = -\Delta Y_k / (Y_k^{\text{split-sym},(1)} Y_k)$ and exact split-symmetry ($\Delta Y_k = 0$) amounts to $\Delta(1/Y_k) = 0$, of course.

In the Figs. 11.3 and 11.4 we show how the scale derivative of $1/Y_k$ decomposes into the derivative of $1/Y_k^{\text{split-sym},(1)}$ and of the symmetry violation term $\Delta(1/Y_k)$. For all trajectories of the class considered, and with the RG equations from both [I] and [II], we always find that the scale derivative of neither $1/Y_k^{\text{split-sym},(1)}$, nor of $\Delta(1/Y_k)$ is negative definite separately, but their sum is!

When split-symmetry is intact, $\Delta Y_k = 0$, (violation of) monotonicity for $1/Y_k^{\text{split-sym},(1)}$ is equivalent to a (non-) monotone function $1/Y_k$. Now, for *generic* RG trajectories from the bi-metric calculations [I] and [II] this condition is known to be approximately satisfied only for $k \rightarrow \infty$, i.e. in the vicinity of the NGFP. The trajectories considered in the present paragraph are fine-tuned to fulfill the requirement of split-symmetry restoration in the IR, so ΔY_k vanishes also there. But on all intermediate scales split-symmetry is broken, the monotonicity of $\mathcal{C}_k$ is not guaranteed by any general argument, and in general $Y_k \neq Y_k^{\text{split-sym},(1)}$. And indeed Figs. 11.3 and 11.4 show a strong violation of this equality. In fact, the part $Y_k^{\text{split-sym},(1)}$ is seen to be non-monotone exactly in the regime where the split-symmetry of the RG trajectory is known to be significantly broken.

The status of split-symmetry violation displayed by an RG trajectory is thus also reflected by the deviation of the pertinent Y_k from its ‘split-symmetry enforced’ version $Y_k^{\text{split-sym},(1)}$. Quite remarkably, for the present class of trajectories *a perfect compensation of the split-symmetry violation the RG trajectories suffer from, and a nonzero correction term ΔY_k takes place*. Miraculously, the term ΔY_k modifies the non-monotone $Y_k^{\text{split-sym},(1)}$ in precisely such a way that the total Y_k is monotone.

This perfect compensation *for all eligible trajectories* strongly supports our hope that the candidate $\mathcal{C}_k$ -function really qualifies as a ‘C-function’ since the structure of ΔY_k , i.e. the way how it depends on the couplings, is a direct consequence of having set $\mathcal{C}_k = \Gamma_k[0; \bar{\Phi}_k^{\text{s.c.}}]$. It is indeed surprising to see that a function as simple as the ΔY_k of eq. (11.79) can do the job of rendering $\mathcal{C}_k$ monotone for all physically relevant trajectories at once.

Split-symmetry violating trajectories. We continue to use the bi-metric RG equations from [I] and [II], but now we deliberately break split-symmetry by selecting a generic trajectory in the g^B - λ^B -subspace, one that would *not* hit the running UV-attractor for $k \searrow 0$. After having generated solutions $k \mapsto (g_k^D, \lambda_k^D)$ of the two decoupled D-equations, again corresponding to a type IIIa trajectory, we solve the resulting B-equations with initial values for (g_k^B, λ_k^B) that explicitly break split-symmetry even in the IR.

In Figs. 11.5 and 11.6 the numerical results for $1/Y_k$ are displayed for the RG equations of [I] and [II], respectively. They show the same qualitative behavior: While the function Y_k becomes stationary towards the NGFP-regime in the UV, the second plateau in the IR, which we had found for trajectories restoring split-symmetry, is now destroyed by the appearance of extrema in the function Y_k , rendering it non-monotone. In the right panels of Figs. 11.5 and 11.6 this is reflected by the changing sign of the derivative $k\partial_k Y_k$ plotted there.

In order to visualize how sign flips of $k\partial_k Y_k$ can come about it is helpful to define, and to determine numerically, the following subset of the g^B - λ^B -plane:

$$\mathcal{T}_+^B(k) \equiv \{(g^B, \lambda^B) \in \mathbb{R}^2 \mid (g_k^D, \lambda_k^D, g^B, \lambda^B) \in \mathcal{T}_+ \subset \mathcal{T}\} \quad (11.80)$$

The RG time-dependent set $\mathcal{T}_+^B(k)$ consists of all those points of the 4D theory space at which the directional derivative is positive, $\beta_\alpha \partial Y / \partial u_\alpha > 0$, and which have (g^D, λ^D) -coordinates that agree with the current position of the selected ‘D’ trajectory at time k , i.e. (g_k^D, λ_k^D) .

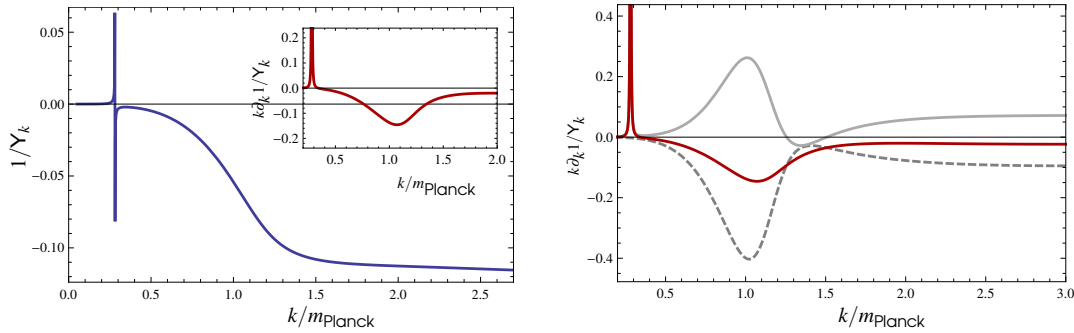


Figure 11.5: The left plot shows the function $1/Y_k$ for a bi-metric trajectory of type IIIa that does *not* restore split-symmetry in the IR. It is based on the RG equations of [I]. We observe a sign-change of $k\partial_k(1/Y_k)$ at moderate values of k , indicating a violation of monotonicity. In the decomposed form of $k\partial_k(1/Y_k)$, shown in the right plot, we see that the contribution $1/Y_k^{\text{split-sym},(1)}$ is in fact monotone, but the correction term $\Delta(1/Y_k)$ is not, and neither is their sum. Not restoring split-symmetry in the IR results in a violation of the monotonicity of $\mathcal{C}_k$ along the trajectory considered.

Using the same typical IIIa trajectory in the dynamical sector as above in the split-symmetry restoring case, we now study the RG evolution in the g^B - λ^B -plane, see Fig. 11.7. To this end, we subdivide this plane into $\mathcal{T}_+^B(k)$, the shaded regions in the diagrams of Fig. 11.7, and its complement, the white regions in Fig. 11.7. This subdivision is different at each instant of RG time. We are particularly interested in those 4D trajectories $k \mapsto (g_k^D, \lambda_k^D, g_k^B, \lambda_k^B)$ that give rise to a monotonically increasing $\mathcal{C}_k$ -function, or in other words, in trajectories whose projection on the B-plane is such that $(g_k^B, \lambda_k^B) \in \mathcal{T}_+^B(k)$ holds true for all k .

From Fig. 11.7 it is now clear why the trajectory restoring split-symmetry in the IR, starting at the UV-attractor located at P_2 , is so special: As the scale k changes, so does the region defined by $\mathcal{T}_+^B(k)$. In the IR, the domain $\mathcal{T}_+^B(k)$ defines a narrow band around the running UV-attractor (P_2), and this results in a monotonically increasing $\mathcal{C}_k$. While the distinguished split-symmetry

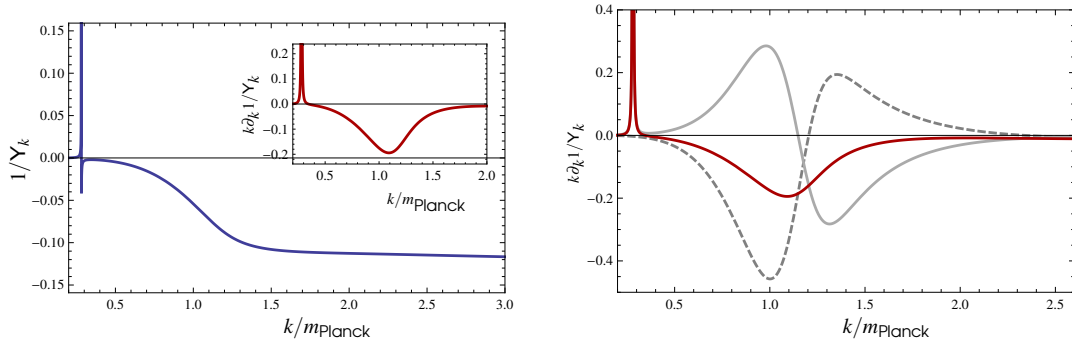


Figure 11.6: The function $1/Y_k$ as in Fig. 11.5, but now based on the RG equations of [II].

restoring trajectory is safely within this band, most of the split-symmetry breaking trajectories lie well outside $\mathcal{T}_+^B(k)$ at low k . Increasing k , we move towards the UV, and $\mathcal{T}_+^B(k)$ extends especially to regions with negative λ^B , while its boundary approaches, and ultimately touches the asymptotic position of the NGFP.

The crucial fact to notice is the following. At all scales, the symmetry restoring trajectory is seen to stay within $\mathcal{T}_+^B(k)$, and this is in agreement with the results obtained in paragraph (A). Since when k is increased sooner or later all trajectories converge to this particular one¹⁵, they are necessarily all pulled towards a regime $\mathcal{T}_+^B(k)$, if they are not yet inside already. This can be observed in Fig. 11.7 by following the trajectory that passes through the point P_1 at some low scale. As P_1 lies outside $\mathcal{T}_+^B(k)$ this implies that $\partial_k \mathcal{C}_k < 0$ in the IR. Increasing k the trajectory is pulled towards the running UV-attractor and between the first and second snapshot of Fig. 11.7 it crosses the boundary of $\mathcal{T}_+^B(k)$. At this moment the derivative of $\mathcal{C}_k$ crosses zero and from this point onward we have $\partial_k \mathcal{C}_k > 0$. In the third snapshot the trajectory is already well inside $\mathcal{T}_+^B$ and it approaches the symmetry-restoring one. Once close to this ‘guiding trajectory’ it remains in its vicinity and together, for $k \rightarrow \infty$, they approach the boundary of $\mathcal{T}_+^B(k)$ from its interior. This is as it should be since we know that $k\partial_k \mathcal{C}_k = 0$ at the NGFP.

The hybrid calculation. In the hybrid calculation, we retain only the $Y_k^{\text{split-sym}}$ -part of Y_k , omitting the correction term ΔY_k . In Fig. 11.8 we show (the inverse of) its two variants $Y_k^{\text{split-sym},(0)}$ and $Y_k^{\text{split-sym},(1)}$ which are obtained by extracting the couplings from, respectively, the 0th and the 1st level of a bi-metric type IIIa trajectory. This particular trajectory restores split-symmetry in the IR. Fig. 11.8 shows that the graphs of the resulting functions $Y_k^{\text{split-sym},(0)}$ and $Y_k^{\text{split-sym},(1)}$ are quite different, the former function is monotone, the latter is not.

This observation once more tells us that the correction term ΔY_k is needed in order to compensate for the split-symmetry violation that goes into $\mathcal{C}_k$ via the trajectories. In fact, at intermediate scales, *all* trajectories suffer from this disease, both the symmetry restoring and the non-restoring ones; the unmistakable symptoms are the substantial differences among the levels.

This confirms our earlier findings: The correction term ΔY_k is indispensable. It is needed in order to protect the sum $Y_k^{\text{split-sym}} + \Delta Y_k$ against the otherwise unavoidable infection with the symmetry violation the trajectories must live with. This protection is successful, i.e. $\mathcal{C}_k = (Y_k^{\text{split-sym}} + \Delta Y_k)\mathcal{V}$ has a monotone dependence on k , provided we do not break split-symmetry by hand, that is, by selecting inappropriate initial conditions for the background couplings.

¹⁵See ref. [173] for a detailed demonstration of this behavior.

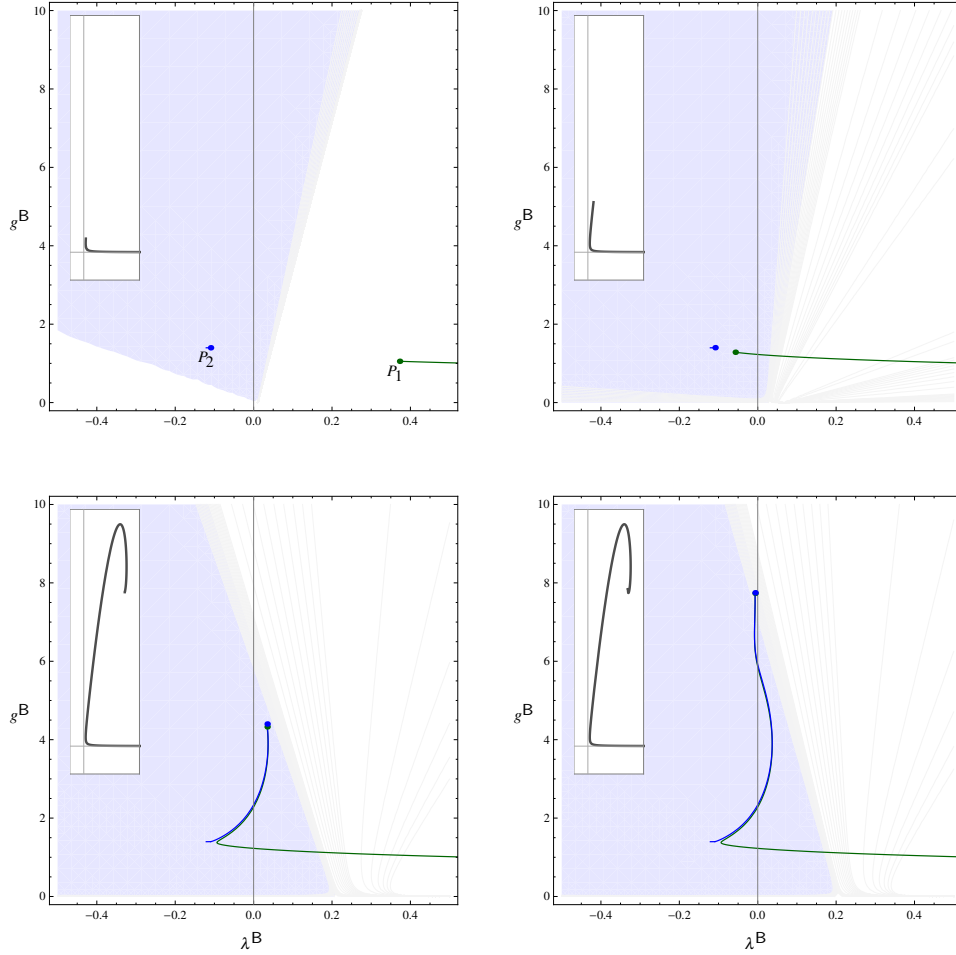


Figure 11.7: This series of snapshots represents the g^B - λ^B -plane at four RG times which increase from the upper left to the lower right diagram. They are given by the maximum k -value of the incomplete dynamical trajectory $k \mapsto (g_k^D, \lambda_k^D)$ shown in the respective inset. The shaded regions correspond to $\mathcal{T}_+^B(k)$ at that particular time; hence every trajectory in the shaded (white) region will give rise to a positive (negative) value of $k\partial_k\mathcal{C}_k$ at the instant of time k . Furthermore, two different B-trajectories that are evolved upward (towards increasing scales k) are shown at the corresponding moments. The one passing the point P_1 (P_2) is split-symmetry violating (restoring). The symmetry restoring trajectory starts its upward evolution close to P_2 , the position of the running UV attractor [173]; we see that this trajectory never leaves the shaded area, and thus its $\mathcal{C}_k$ -function is strictly monotone. This is different for the trajectory through P_1 : Attracted by the running UV-attractor, it is pulled into the shaded region, thus unavoidably crossing the boundary of $\mathcal{T}_+^B(k)$, which causes a sign flip of $\partial_k\mathcal{C}_k$, rendering $\mathcal{C}_k$ non-monotone.

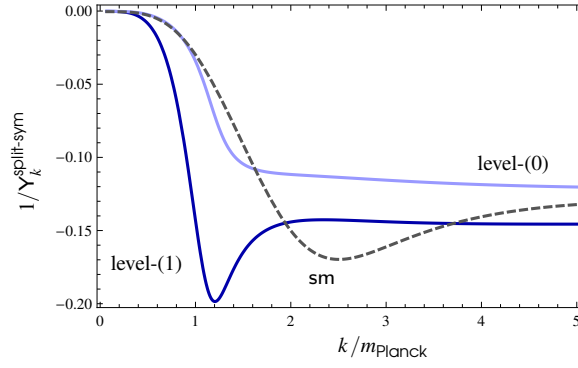


Figure 11.8: The inverse of $Y_k^{\text{split-sym},(0)}$ and $Y_k^{\text{split-sym},(1)}$ is shown for a generic bi-metric type IIIa trajectory that restores split-symmetry in the IR. On intermediate scales, the RG trajectory is not split-symmetric, as is evident from the different graphs of the level-(1) (dark, solid) and the level-(0) (light, solid) variants of $Y_k^{\text{split-sym}}$. For comparison, the single-metric result Y_k^{sm} is also included (dashed curve).

Testing pointwise monotonicity. We have seen in eq. (11.42) that for *exact* RG trajectories $k \mapsto \Gamma_k$ the only source of obstructions for $\mathcal{C}_k$ to become a monotone function is the second term on its RHS, which measures to what extent Γ_k breaks split-symmetry. In the case of exact RG solutions, we know that the first term on the RHS is positive, $(\partial_k \Gamma_k)[0; \bar{\Phi}_k^{s.c.}] \geq 0$, since this is a special case of the pointwise monotonicity, $(\partial_k \Gamma_k)[\varphi; \bar{\Phi}] \geq 0 \forall (\varphi, \bar{\Phi}), \forall k$. However, the latter property might not always be true for approximate solutions to the flow equation, those obtained by using truncations, for instance. Testing pointwise monotonicity, $(\partial_k \Gamma_k)[\varphi; \bar{\Phi}] \geq 0$, may therefore serve as a device to judge the validity of a truncation. In future work we will come back to this method for arbitrary arguments $(\varphi, \bar{\Phi})$. We focus here only on $(\partial_k \Gamma_k)$ evaluated at the special arguments $(\varphi, \bar{\Phi}) = (0, \bar{\Phi}_k^{s.c.})$.

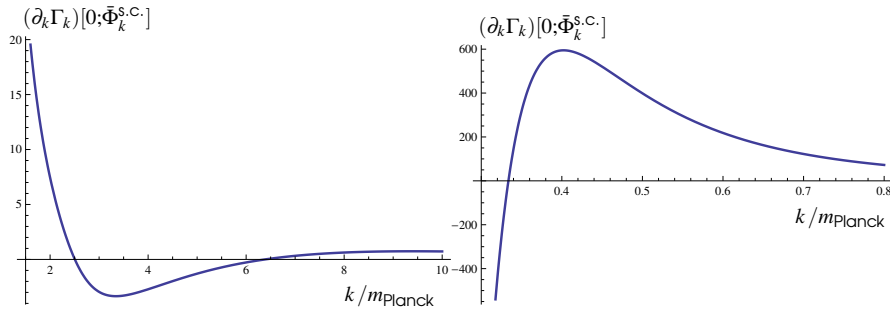


Figure 11.9: The first term on the RHS of eq. (11.42), i.e. $(\partial_k \Gamma_k)[0; \bar{\Phi}_k^{s.c.}]$ is evaluated for a typical single-metric (left) and split-symmetry violating bi-metric (right) trajectory. In both cases, it is seen to be negative for certain scales. This indicates a severe failure of the underlying approximation since, at the exact level, $(\partial_k \Gamma_k)$ is known to be positive at all field arguments and for any k .

It turns out that the single-metric and the ‘unphysical’ bi-metric RG trajectories (those without split-symmetry restoration) actually fail this pointwise monotonicity test. As shown in Fig. 11.9, there are k -intervals on which $(\partial_k \Gamma_k)[0; \bar{\Phi}_k^{s.c.}]$ is *negative*. On the other hand, for the split-symmetry restoring bi-metric trajectories this quantity is positive throughout, as it is at the exact level, see Fig. 11.10. These findings make it very clear that the non-monotonicity displayed by our $\mathcal{C}_k$ -function candidate, when applied to single-metric and symmetry violating

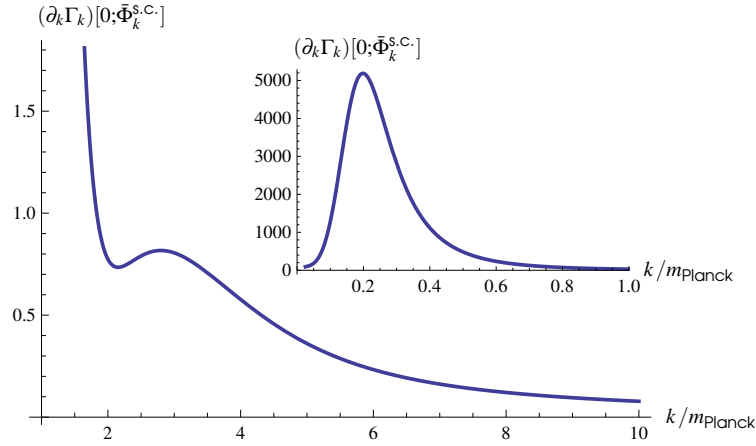


Figure 11.10: The quantity $(\partial_k \Gamma_k)[0; \bar{\Phi}_k^{s.c.}]$ is now evaluated for a split-symmetry restoring bi-metric trajectory. It always stays non-negative, even in those regimes where the single-metric or the split-symmetry violating bi-metric (see inset) trajectories fail the pointwise monotonicity test.

bi-metric truncations, is *not due to a defect of the proposed form of $\mathcal{C}_k$* but rather originates in insufficient approximations. Only the symmetry-restoring, bi-metric trajectories are close enough to the exact ones to render both $(\partial_k \Gamma_k)[0; \bar{\Phi}_k^{s.c.}]$ and the full $\partial_k \mathcal{C}_k$ positive.

Summary and Conclusion

To sum it up we can say that the expected monotonicity of $\mathcal{C}_k$ arises under the following conditions: First, the bi-metric version of the Einstein-Hilbert truncation is used, and second, the underlying RG trajectory is split-symmetry restoring. Violating either of these conditions may destroy the monotonically increasing behavior of $\mathcal{C}_k$. We saw that there are cases where the split-symmetry violation of Γ_k is sufficiently small to leave the monotonicity of $\mathcal{C}_k$ intact, the main example being the bi-metric type IIIa trajectories approaching the UV attractor for $k \rightarrow 0$.

We have seen that down-grading the bi-metric truncation ansatz to the level of a single-metric approximation is paid by loosing the monotonicity property of $\mathcal{C}_k$. As for its dependence on the running couplings, the reduced $\mathcal{C}_k$ -function in the bi-metric truncation, Y_k , differs from its single-metric counterpart Y_k^{sm} by the correction term ΔY_k , which vanishes when Γ_k^{grav} is exactly split-symmetric. (In the single-metric approximation, this is always the case, by decree.) We found that there is a numerically highly non-trivially conspiracy and compensation between ΔY_k and those properties of the RG trajectories which stem from the split-symmetry violation in the flow equation, and which could easily destroy the monotonicity of $\mathcal{C}_k$. The fact that this does not happen for any of the physically relevant trajectories is directly linked to the specific properties of our candidate function, $\mathcal{C}_k = \Gamma_k[0; \bar{\Phi}_k^{s.c.}]$, its scale dependent argument in particular, since it determines the structure of ΔY_k .

Taken together these findings strongly support the following conjecture: *In the full theory, QEG in 4 dimensions, or in a sufficiently general truncation thereof, the proposed candidate for a generalized C-function is a monotonically increasing function of k along all RG trajectories that restore split-symmetry in the IR and thus comply with the fundamental requirement of Background Independence.*

If the conjecture can be established we will have a particularly easy to apply diagnostic tool for testing the reliability of truncations. Since then solutions to the untruncated flow equation for sure have a monotone $\mathcal{C}_k$, any truncation that violates the monotonicity misses qualitatively important features of the RG flow and would therefore be judged an insufficient approximation

to the full flow. In this light we provisionally conclude that the single-metric approximation is not fully reliable, while the bi-metric Einstein-Hilbert truncation is superior as it keeps the monotonicity of $\mathcal{C}_k$ intact at least. Of course this conclusion is fully consistent with all other results available on the bi-metric Einstein-Hilbert truncation, ref. [172] as well as the chapters 10 and 12 of this thesis.

As split-symmetry is essential in this context, it might be helpful to recall its physical contents. Split-symmetry and the corresponding Ward Identities for split-symmetry (WISS) are the technical device by means of which Background Independence in the physical sector ('on-shell') is imposed on the effective action and similar 'off-shell' quantities¹⁶. The prototypical example of a Background Independent theory is classical General Relativity [234]. Now, even though we describe it by an effective action $\Gamma[g, \bar{g}]$, we would like QEG to enjoy Background Independence exactly at the same level as General Relativity.

Let us contrast QEG with *genuine bi-metric theories*, in the original sense of the word, that is, extensions of General Relativity employing two *physically distinct* metrics, $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$, say. Depending on the structure of their action $S[g_{\mu\nu}^{(1)}, g_{\mu\nu}^{(2)}, \dots]$ they could differ, for instance, in their coupling to matter, or their propagation properties. In an appropriate limit, matter particles of a certain species could, for example, follow the geodesics of $g_{\mu\nu}^{(1)}$, or of $g_{\mu\nu}^{(2)}$; but it also can happen that trajectories are no geodesics at all and have no geometric interpretation. In a genuine bi-metric theory, these different cases are *experimentally distinguishable*. The metrics have equal status in that both of them, independently, can make their way into observables. In canonical quantization both $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$ are turned into operators. This is fundamentally different when one applies the background field technique to the quantization of a system with a bare action $S[\hat{g}]$ depending on one metric only, and introduces $\bar{g}_{\mu\nu}$ only as a technical convenience, for coarse-graining and gauge-fixing purposes in particular. Setting $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$ we transfer the physical degrees of freedom entirely from $\hat{g}_{\mu\nu}$ to $\hat{h}_{\mu\nu}$ which is made the new dynamical quantum field by replacing $\int[d\hat{g}_{\mu\nu}]$ with $\int[d\hat{h}_{\mu\nu}]$. From the functional perspective, $\bar{g}_{\mu\nu}$ is merely an arbitrary shift on which no observable consequence of the theory may depend. In canonical quantization, $\hat{g}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ are operators, while $\bar{g}_{\mu\nu}$ continues to be a classical *c*-number field. The logical dissimilarity between dynamical and background metric gets slightly obscured at the level of the expectation values $h_{\mu\nu} \equiv \langle \hat{h}_{\mu\nu} \rangle$ and $g_{\mu\nu} \equiv \langle \hat{g}_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + h_{\mu\nu}$ since $\Gamma_k[h; \bar{g}] \equiv \Gamma_k[g, \bar{g}]$ depends on two independent fields, two metrics in fact, if one uses the EAA in the 'comma notation', $\Gamma_k[g, \bar{g}]$. Now, the role of the split-symmetry as encoded in the WISS of eq. (11.21), is to express the requirement that *there is only one physical metric* and that no observable quantity may depend on how $\bar{g}_{\mu\nu}$ was chosen. Setting for example $k = 0$ and ignoring gauge fixing issues for a moment, invariance of the bare action under $\{\delta \hat{h}_{\mu\nu} = \varepsilon_{\mu\nu}, \delta \bar{g}_{\mu\nu} = -\varepsilon_{\mu\nu}\} \iff \{\delta \hat{g}_{\mu\nu} = 0, \delta \bar{g}_{\mu\nu} = -\varepsilon_{\mu\nu}\}$ implies that $\Gamma_0[h; \bar{g}]$ can depend on the sum $\bar{g} + h \equiv g$ only, while $\Gamma_0[g, \bar{g}] \equiv \Gamma_0[g]$ simply does not depend on its second argument. In reality, because we use a 'background-type' gauge fixing condition, $\Gamma_0[g, \bar{g}]$ does have a certain $\bar{g}$ -dependence, again dictated by the WISS, but it disappears upon going on-shell and cannot be seen in any experiment therefore.

11.3.4 Crossover trajectories and their mode count

Stationarity of $\mathcal{C}_k$. In subsection 11.2.6 we proved that the exact $\mathcal{C}_k$ is stationary at fixed points as well as in classical regimes. The explicit $\mathcal{C}_k$ functions obtained from both the single- and the bi-metric truncation indeed display this behavior. In fact, looking at the two alternative formulas for $\mathcal{C}_k$ in (11.68) it is obvious that $\mathcal{C}_k$ becomes stationary when the *dimensionless* couplings are at a fixed point of the flow, and when the *dimensionful* ones become scale independent; this is the case in a classical regime ('CR') where by definition no physical RG effects

¹⁶For them, 'Background Independence' is not naively 'independence of $\bar{g}_{\mu\nu}$ '.

occur. If $\bar{\lambda}_{\text{CR}}^I$ and G_{CR}^I are the constant values of the cosmological and Newton constants there, this regime amounts to the trivial canonical scaling $\lambda_k^I = k^{-2}\bar{\lambda}_{\text{CR}}^I$ and $g_k^I = k^{d-2}G_{\text{CR}}^I$.

Generalized crossovers and limits of $\mathcal{C}_k$. In subsection 11.2.6 we mentioned already the possibility of generalized crossover transitions, not only in the standard way from one fixed point to another, but rather from a fixed point to a classical regime or vice versa. Thereby $\mathcal{C}_k$ will always approach well defined stationary values $\mathcal{C}_*$ and $\mathcal{C}_{\text{CR}}$ in the respective fixed point or classical regime. (See Fig. 11.11 for a schematic sketch.)

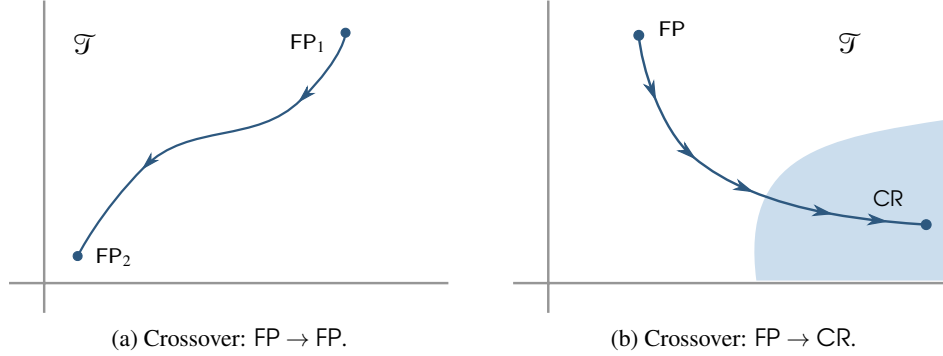


Figure 11.11: Crossover trajectories in theory space: from one fixed point to another, (a), and from a UV fixed point to a classical regime, (b).

In the case of an asymptotically safe RG trajectory, the initial point in the UV is a non-Gaussian fixed point, by definition. For the corresponding limit $\mathcal{C}^{\text{UV}} \equiv \lim_{k \rightarrow \infty} \mathcal{C}_k$ the bi-metric calculation yields $\mathcal{C}^{\text{UV}} = \mathcal{C}_*$, with

$$\mathcal{C}_* = - \frac{\left(\frac{d}{d-2}\right) \lambda_*^{(1)} - \lambda_*^{(0)}}{g_*^{(0)} [\lambda_*^{(1)}]^{d/2}} \mathcal{V}(M, \hat{g}) \quad (11.81)$$

This result simplifies to

$$\mathcal{C}_*^{\text{sm}} = - \left(\frac{2}{d-2} \right) \frac{\mathcal{V}(M, \hat{g})}{g_*^{\text{sm}} [\lambda_*^{\text{sm}}]^{d/2-1}} \quad (11.82)$$

in the single-metric approximation. Note that $\mathcal{C}_*$ diverges at the trivial (‘Gaussian’) fixed point at which all dimensionless couplings vanish.

If the trajectory ends in an IR fixed point the corresponding limit $\mathcal{C}^{\text{IR}} \equiv \lim_{k \rightarrow 0} \mathcal{C}_k$, if it exists, is again given by the formula (11.81), $\mathcal{C}^{\text{IR}} \equiv \mathcal{C}_*$, but for different fixed point coordinates. If the trajectory is instead destined to enter a classical regime and to approach $k = 0$ by an infinitely ‘long’, and boring, since purely canonical running on $\mathcal{T}$, then the value at the end point equals $\mathcal{C}^{\text{IR}} = \mathcal{C}_{\text{CR}}$ where

$$\mathcal{C}_{\text{CR}} = - \left(\frac{2}{d-2} \right) \frac{\mathcal{V}(M, \hat{g})}{G_{\text{CR}} [\bar{\lambda}_{\text{CR}}]^{d/2-1}} \quad (11.83)$$

In writing down eq. (11.83) we assumed that the same values of G_{CR} and $\bar{\lambda}_{\text{CR}}$ apply at all levels, as required by split-symmetry.

Thus, for any of the above crossover types we expect a finite value of

$$\mathcal{N} \equiv \mathcal{N}_{0,\infty} \equiv \mathcal{C}^{\text{UV}} - \mathcal{C}^{\text{IR}} \quad (11.84)$$

Physical interpretation of $\mathcal{C}_k$. Beside monotonicity and stationarity, $\mathcal{C}_k$ has another essential property in common with a C-function: The limiting value $\mathcal{C}_*$ has a genuine inherent interpretation at the fixed point itself. It is a number characteristic of the NGFP which does not depend on the direction it is approached, and in this role it is analogous to the central charge. The interpretation of $\mathcal{C}_*$ is best known for $d = 4$ and the single-metric approximation where, apart from inessential constants, it is precisely the inverse of the dimensionless combination $g_*\lambda_* = G_{k \rightarrow \infty} \bar{\lambda}_{k \rightarrow \infty}$. Its physical interpretation is that of an ‘intrinsic’ measure for the size of the cosmological constant at the fixed point, namely the limit of the running cosmological constant in units of the running Planck mass ($G_k^{-1/2}$). In numerous single-metric studies the product $g_*\lambda_*$ has been investigated, and it was always found that $g_*\lambda_*$ is a *universal quantity*, i.e. it is independent of the cutoff scheme and the gauge fixing, within the accuracy permitted by the approximation. In fact, typically the universality properties of $g_*\lambda_*$ were even much better than those of the critical exponents. Completely analogous remarks apply to $d \neq 4$, and to the quantity (11.81) in the bi-metric generalization.

We interpret this number, with all due care, as a measure for the total ‘number of field modes’ integrated out along the entire trajectory. Clearly $\mathcal{C}^{\text{UV}}$ and $\mathcal{C}^{\text{IR}}$ play a role analogous to the central charges of the 2D conformal field theories sitting at the end points of the trajectory in the case of Zamolodchikov’s theorem.

Within the Einstein-Hilbert truncation, our numerical results for $\mathcal{C}_*$ at the NGFP in $d = 4$ are as follows for the three calculations we compared:

$$\mathcal{C}_* = -\mathcal{V}(M, g) \times \begin{cases} 7.3 & \text{single-metric} \\ 4.3 & \text{bi-metric I} \\ 8.1 & \text{bi-metric II} \end{cases} \quad (11.85)$$

Obviously, in all calculations $\mathcal{C}_*$ is a negative number of order unity.

Let us now focus on the case depicted in Fig. 11.11b, which can be seen as a simple caricature of the real Universe. We consider a family of de Sitter spaces along a type IIIa trajectory which is known to possess a classical regime with $\bar{\lambda}_{\text{CR}} > 0$. Assuming that this regime represents the true final state of the evolution, we obtain

$$\mathcal{C}^{\text{IR}} = -\frac{3\pi}{G_{\text{CR}} \bar{\lambda}_{\text{CR}}} \quad (11.86)$$

Note that this $\mathcal{C}^{\text{IR}}$ is negative, too, and that $-\mathcal{C}^{\text{IR}}$ equals precisely the well known semi-classical Bekenstein-Hawking entropy of de Sitter space [79].

Thus, combining (11.85) and (11.86) for $\mathcal{C}^{\text{UV}}$ and $\mathcal{C}^{\text{IR}}$, respectively, we arrive at the following important conclusion: In an asymptotically safe theory of quantum gravity which is built upon a generalized crossover trajectory from criticality (the NGFP) to classicality *the total number of modes integrated out*, $\mathcal{N} = \mathcal{C}^{\text{UV}} - \mathcal{C}^{\text{IR}}$, is *finite* according to the natural counting device provided by the EAA itself.

In the special situation when $G_{\text{CR}} \bar{\lambda}_{\text{CR}} \ll 1$, like in the real world, we have $|\mathcal{C}^{\text{IR}}| \gg 1$, while $|\mathcal{C}^{\text{UV}}| = \mathcal{O}(1)$ according to the NGFP data of (11.85). As a consequence, the number $\mathcal{N}$ is completely dominated by the IR part of the trajectory, $\mathcal{N} = \mathcal{C}^{\text{UV}} - \mathcal{C}^{\text{IR}} \approx -\mathcal{C}^{\text{IR}}$, and so we obtain

$$\mathcal{N} \approx +\frac{3\pi}{G_{\text{CR}} \bar{\lambda}_{\text{CR}}} \gg 1 \quad (11.87)$$

It is tempting to identify $\bar{\lambda}_{\text{CR}}$ and G_{CR} with the corresponding values measured in the real Universe. Because of their extremely tiny product $G_{\text{CR}} \bar{\lambda}_{\text{CR}}$ we find a tremendous number of modes then: $\mathcal{N} \approx 10^{120}$. Nevertheless, in sharp contradistinction to what standard perturbative field theory would predict, this number is *finite*.

Different types of trajectories. Concerning the finiteness of $\mathcal{N}$, the situation changes if we try to define the function $k \mapsto \mathcal{C}_k$ along trajectories of the type Ia, those heading for a negative cosmological constant λ^D after leaving the NGFP regime, and of type IIa, the single trajectory which crosses over from the NGFP to the Gaussian fixed point (GFP) at which all 4 couplings vanish.

In the **type IIa** case, eq. (11.81) yields ‘ $\mathcal{C}_* = -\infty$ ’ at the IR end point of the trajectory so that the total number of modes diverges, ‘ $\mathcal{N} = +\infty$ ’. Clearly this behavior can be seen as the limit $\bar{\lambda}_{\text{CR}} \rightarrow 0$ of eq. (11.87) since the GFP has a vanishing cosmological constant. The divergent value of $\mathcal{N}$ is the signal of a ‘topology change’ that occurs at $\bar{\lambda}^{(1)} = 0$: While the self-consistent backgrounds (of maximal symmetry, say) are spheres S^d for $\bar{\lambda}^{(1)} > 0$, it is flat space (R^d) if $\bar{\lambda}^{(1)} = 0$. The Euclidean volume of the former is always finite, but that of R^d is infinite.

While along the type IIa trajectory the divergence of $\mathcal{C}_k$ occurs only at the very end of the RG evolution, i.e. in the limit $k \rightarrow 0$, for **type Ia** trajectories $\mathcal{C}_k$ becomes singular already at a *finite* scale $k = k_{\text{sing}} > 0$. All trajectories of this type cross the hyperplane $\lambda^D = 0$ at a nonzero scale, k_{sing} . However, as eq. (11.74) shows, $Y(\cdot)$ and $\mathcal{C}(\cdot)$ are singular on this plane¹⁷, so that $\mathcal{C}_k$ diverges in the limit $k \searrow k_{\text{sing}}$. The number of modes, $\mathcal{N}_{k_{\text{sing}}, \infty}$ is infinite then, which however by no means implies that *all* modes have been integrated out already. In fact, there is a non-trivial RG evolution also between k_{sing} and $k = 0$.

Along a type Ia trajectory, the tadpole equation has qualitatively different solutions for $k > k_{\text{sing}}$, $k = k_{\text{sing}}$, and $k < k_{\text{sing}}$, namely spherical, flat, and hyperbolic spaces, respectively (S^d , R^d , and H^d , say). This topology change prevents us from smoothly continuing the mode count across the $\lambda^D = 0$ plane. This is the reason why in this paper we mostly focused on type IIIa trajectories. Some further details for the type Ia and IIa cases can be found in appendix 11.B, however.

11.4 Discussion and outlook

The proposed $\mathcal{C}_k$ -function

The effective average action is a variant of the standard effective action which has an IR cutoff built-in at a sliding scale k . As such, it possesses a natural ‘mode counting’ and monotonicity property which is strongly reminiscent of Zamolodchikov’s C -function in 2 dimensions, at least at a heuristic level. For a broad class of systems, this property (‘pointwise monotonicity’) is easy to demonstrate, the essential input being that in every system with a well-defined RG flow the action $\Gamma_k + \Delta S_k$ is a strictly convex functional on all scales, that is, the Hessian operator satisfies the positivity constraint $\Gamma_k^{(2)} + \mathfrak{R}_k > 0$, $k \in (0, \infty)$. Motivated by this observation, and taking advantage of the structures and tools that are naturally provided by the manifestly non-perturbative EAA framework, we tried to find a map from the functional $\Gamma_k[\Phi, \bar{\Phi}]$ to a single real valued function $\mathcal{C}_k$ that shares two main properties with the C -function in 2 dimensions, namely monotonicity along RG trajectories and stationarity at RG fixed points.

We do not expect such a map to exist in full generality. In fact, an essential part of the research program we are proposing consists in finding suitable restrictions on, or specializations of the *admissible trajectories* (restoring split-, or other symmetries, etc.), the *theory space* (with respect to field contents and symmetries), the underlying *space of fields* (boundary conditions, regularity requirements, etc.), and the *coarse graining methodology* (choice of cutoff, treatment of gauge modes, etc.) that will guarantee its existence.

¹⁷One might be worried about the hyperplanes $g^D = 0$ and $g^B = 0$ on which $Y(\cdot)$ is singular too, see eq. (11.74). However, within the truncation considered there exist no trajectories that would ever cross or touch those planes.

In this chapter we motivated and analyzed a specific candidate for a map of this kind, namely $\mathcal{C}_k = \Gamma_k[\bar{\Phi}_k^{\text{s.c.}}, \bar{\Phi}_k^{\text{s.c.}}]$ where $\bar{\Phi}_k^{\text{s.c.}}$ is a running self-consistent background, a solution to the tadpole equation implied by Γ_k . We showed that the function $\mathcal{C}_k$ is stationary at fixed points, and a non-decreasing function of k when the breaking of the split-symmetry which relates fluctuation fields and backgrounds is sufficiently weak. Thus, for a concrete system the task is to identify the precise conditions under which the split-symmetry violation does not destroy the monotonicity property of $\mathcal{C}_k$, and to give a corresponding proof then.

It would be interesting to work out the properties of this $\mathcal{C}_k$ -function for further concrete examples, but also to explore and test structurally different maps from Γ_k to $\mathcal{C}_k$. A new kind of map should in particular be devised if one wants to count the modes of fermionic fields with the same weight as those of bosons. Because of the sign factors produced by the super-trace in the flow equation, $\mathcal{C}_k = \Gamma_k[\bar{\Phi}_k^{\text{s.c.}}, \bar{\Phi}_k^{\text{s.c.}}]$ rather counts bosons and fermions with opposite signs, and so $\mathcal{N}$ equals the total number of bosonic modes integrated out minus the number of fermionic ones. (The different count for bosons and fermion is reminiscent of, but not precisely identical to the functional integral representing the Witten index instead of the partition function. There, fermionic states of fermion number F contribute with the weight $(-1)^F$.) While the ‘boson minus fermion’ counting is at variance with the standard c -theorem, the properties and potential applications of the present $\mathcal{C}_k$ with fermions should be explored in more detail before dismissing it prematurely. As we stressed already, rather than reproducing known results, our main goal consists in finding maps $\Gamma_k \rightarrow \mathcal{C}_k$ that are simple and ‘geometrically natural’ in the theory space and functional RG context.

Einstein-Hilbert truncations

By means of a particularly relevant example, QEG in $d > 2$ dimensions, we demonstrated that our approach is viable in principle and can indeed lead to interesting candidates for ‘ C -functions’ under conditions which are not covered by the known c - and a -theorems. As exact proofs are not within reach for the time being, the practical problem is of course the same as in all non-perturbative functional RG studies, namely the necessity to truncate the theory space. Here, for asymptotically safe quantum gravity we computed $\mathcal{C}_k$ directly from the RG trajectories obtained with both the single- and bi-metric Einstein-Hilbert truncation, respectively.

It is one of our main results that in the bi-metric truncation the function $\mathcal{C}_k$ has the desired properties of monotonicity and stationarity, while the single-metric truncation is too poor an approximation to correctly reproduce the monotonicity which we expect at the exact (un-truncated) level. We demonstrated explicitly that the monotonicity property obtains only for the RG trajectories which are physically meaningful, that is, those which lead to a restoration of split-symmetry once all field modes are integrated out.

We studied generalized crossover trajectories from a fixed point in the UV to a classical regime in the IR, in which by definition the dimensionful cosmological and Newton constants loose their k -dependence. In many ways they are analogous to a standard (fixed point $\rightarrow$ fixed point) crossover. In quantum gravity they are of special importance as one of the main challenges consists in explaining the emergence of a classical spacetime from the quantum regime.

For the trajectories with positive cosmological constant (‘type IIIa’) the self-consistent background configurations needed are gravitational instantons. The resulting $\mathcal{C}_k$ depends on the instanton type via the normalized volume, a quantity of topological significance. For the example of Euclidean de Sitter space, the sphere S^4 , for instance, we obtained the ‘integrated $\mathcal{C}$ -theorem’

$$\mathcal{N} = \mathcal{C}^{\text{UV}} - \mathcal{C}^{\text{IR}} \approx 3\pi/G_{\text{CR}}\bar{\lambda}_{\text{CR}} \quad (11.88)$$

This result is intriguing for several reasons. First of all, $\mathcal{N}$, and also the values of $\mathcal{C}^{\text{UV}}$ and $\mathcal{C}^{\text{IR}}$ separately, are well defined finite numbers, in marked contrast to expectations based on the

counting in perturbative field theory. The quantity $\mathcal{N}$ can be interpreted as a measure for the ‘number of modes’ which are integrated out while the cutoff is decreased from $k \rightarrow \infty$ to $k = 0$.

Hereby the notion of ‘counting’ and the precise meaning of a ‘number of field modes’ is defined by the EAA itself, namely via the identification $\mathcal{C}_k = \Gamma_k[\bar{\Phi}_k^{s.c.}, \bar{\Phi}_k^{s.c.}]$. Under special conditions it reduces to a literal counting of the $\Gamma_k^{(2)}$ -eigenvalues in a given interval. Generically we are dealing with a non-trivial generalization thereof which, strictly speaking, amounts to a *definition* of ‘counting’. As such it is the most natural one from the EAA perspective, however.

The approximate equality $\mathcal{N} \approx 3\pi/G_{\text{CR}}\bar{\lambda}_{\text{CR}}$ is valid if $G_{\text{CR}}\bar{\lambda}_{\text{CR}} \ll 1$ in the classical regime. In this limiting case, $\mathcal{N} \approx |\mathcal{C}^{\text{IR}}|$ equals exactly the Bekenstein-Hawking entropy of de Sitter space, and the contribution from the UV fixed point is negligible, $|\mathcal{C}^{\text{UV}}| \ll |\mathcal{C}^{\text{IR}}|$. Asymptotic Safety is crucial for this result, making $\mathcal{C}^{\text{UV}}$ finite.

So we are led to the following interpretation of the entropy of de Sitter space: it equals the number $\mathcal{N}$ of metric and ghost fluctuation modes that are integrated out between the NGFP in the UV and the classical regime in the IR. Asymptotic Safety is ‘taming’ the ultraviolet and renders this number perfectly finite. (In the real Universe, $\mathcal{N} \approx 10^{120}$.)

Outlook

Future work along these lines will be in various different directions. Clearly one of the goals will be to corroborate our result based on the bi-metric Einstein-Hilbert truncation on larger theory spaces. This should also lead to a better understanding and to a physics interpretation of the stationary values $\mathcal{C}_*$ and $\mathcal{C}_{\text{CR}}$ which replace the central charge of 2D conformal field theory. We found that $\mathcal{C}_{\text{CR}}$ coincides with the familiar Bekenstein-Hawking entropy¹⁸, but this is likely to change beyond the Einstein-Hilbert truncation.

Ultimately one could hope to establish, perhaps even at some level of rigor, the existence of a C -function in 4D asymptotically safe Quantum Einstein Gravity by using a combination of WISS and FRGE in order to derive bounds for the ‘disturbing’ term in the equation (11.44) for $\partial_k \mathcal{C}_k$.

Besides its obvious relevance to the global structure of the RG flow, this will also allow us to use the monotonicity of $\mathcal{C}_k$ as a powerful criterion and easy to apply practical test for assessing the reliability of truncations or other approximations.

Furthermore, it would be interesting to find further examples, based on other theory spaces which admit a simple map from Γ_k to some $\mathcal{C}_k$. It also remains to clarify the precise relationship between the existing c - and a -theorems on the one side, and the present framework on the other. We will address this question in future work.

In the existing work on the known (generalized) c -theorems in 2, 3, and 4 dimensions, usually no reference is made to a (bare) action and fields it depends on; only the existence of a local energy momentum tensor is assumed. In the present approach the emphasis is instead on (effective) action functionals depending on a set of fields that is fixed from the start. However, recalling the discussion (of the ‘reconstruction problem’) in ref. [121] it becomes clear that this clash is much less profound than it seems: In the EAA approach to Asymptotic Safety, $\Gamma_k[\cdot]$ should primarily be seen as a generating function (or functional) for a set of n -point functions. Hereby the field arguments of the EAA serve a purely technical purpose, and in general *the relationship between those field arguments and the fundamental physical degrees of freedom (DOF) whose quantization would result in a given RG trajectory is at best a highly indirect one.*

The main reason is that in the case of an asymptotically safe theory $\Gamma_{k \rightarrow \infty}$, i.e. the fixed point action is extremely complicated, highly non-linear, contains higher derivatives, and is nonlocal

¹⁸It is nevertheless intriguing that the Bekenstein-Hawking entropy appears here in a role analogous to the central charge in conformal field theory. In fact the thermodynamics of 2-dimensional black holes, or correspondingly dimensionally reduced ones, is closely related to the Virasoro algebra and its central charge [249].

probably. Furthermore, $\Gamma_{k \rightarrow \infty}$ is a gauge-fixed action, but it will *not* be of the familiar form $S + S_{\text{gr}} + S_{\text{gh}}$ in general, with some invariant action S and a quadratic, second derivative ghost action S_{gh} , as it is the case when one applies the Faddeev-Popov trick. This is another issue that complicates the identification of the physical contents of the fixed point theory. In perturbation theory, $\Gamma_{k \rightarrow \infty}$ which is essentially the same as the bare action S , contains only a few relevant field monomials and only second derivative terms. As a result, there is basically a one-to-one relation between degrees of freedom and fields. In Asymptotic Safety, rather than an ad hoc input, the theory's bare action, or what comes closest to it, $\Gamma_{k \rightarrow \infty}$, is the result of a complicated non-perturbative evaluation of the fixed point condition. As the structure of propagating modes is crucially affected by higher derivative and nonlocal terms, it is the fixed point condition that decides about the nature of the underlying DOF's. To identify them, a phase-space functional integral of the form $\int dx \int d\pi \exp(i \int \pi_j \dot{x}_j - H[\pi, x])$ must be found which reproduces the RG trajectory obtained from the FRGE. We can then read off canonically conjugate pairs x_j , π_j and the (local) Hamiltonian $H[\pi, x]$ which governs their bare dynamics. To bring the original functional integral [121] $\int [d\hat{\Phi}] e^{-\Gamma_{k \rightarrow \infty}}$ to this form, field redefinitions and the introduction of further, or different fields will be necessary in order to remove nonlocal and higher-derivative terms. It is quite conceivable that there is more than one set $\{\pi_j, x_j\}$ and Hamiltonian H that reproduces a given RG trajectory. In this case we would say that the quantum theory defined by the latter has 'dual' descriptions employing different (bare) actions and fields. As yet, not much work has been devoted to this 'reconstruction problem', see however ref. [121] for a first step.

Relation of $\mathcal{C}_k$ to the N -bound

To close with, we mention another intriguing aspect of the integrated $\mathcal{C}$ -theorem (11.88) which deserves being investigated further, namely its connection to the hypothesis of the ' **N -bound**' which is due to Banks [250] and, in a stronger form, to Bousso [251]. In Bousso's formulation, the claim is that in any universe with a positive cosmological constant, containing arbitrary matter that even may dominate at all times, the observable entropy S_{obs} is bounded by $S_{\text{obs}} \leq 3\pi/G\tilde{\lambda} \equiv N$.

Here S_{obs} includes both matter and horizon entropy, but excludes entropy that cannot be observed in a causal experiment. As for the notion of an 'observable entropy', it is identified [251] with the entropy contained in the causal diamond of an observer, i.e. the spacetime region which can be both influenced and seen by the observer. It is bounded by the past and future light cones based at the endpoints of the observer's world line.

Remarkably, while the number N equals the Bekenstein-Hawking entropy of *empty* de Sitter space, the bound is believed to apply in presence of arbitrary matter, and for arbitrary spacetimes with $\tilde{\lambda} > 0$, which not even asymptotically need to be de Sitter.¹⁹

Given the methods developed in this chapter the intriguing possibility arises to check whether the N -bound holds in asymptotically safe field theories and to tentatively identify N with $\mathcal{N} \equiv \mathcal{C}^{\text{UV}} - \mathcal{C}^{\text{IR}}$. In principle we have all tools available for a fully non-perturbative test that treats gravity at a level well beyond the semi-classical approximation. We would have to add matter fields to the truncation ansatz [130, 152, 153] and include for all types of fields the corresponding (Gibbons-Hawking-York, etc.) surface terms that are needed on spacetimes with a non-empty boundary [166, 252, 253].

Originally the N -bound grew out of string theory based arguments which hinted at the possibility of a ' **$\tilde{\lambda}$ - N -connection**' [250, 251]. It would be such that all universes with a *positive* cosmological constant are described by a fundamental quantum theory which has only a *finite* number of degrees of freedom, and that this number is determined by $\tilde{\lambda}$.²⁰

¹⁹In [251] the original requirement [250] of spacetimes that are asymptotically de Sitter has been dropped.

²⁰For a similar discussion in Loop Quantum Gravity see ref. [254].

Is there a corresponding ‘ $\bar{\lambda}$ - $\mathcal{N}$ -connection’ in asymptotically safe field theory? For pure gravity we can answer this question in the affirmative already now: The fundamental quantum field theory is defined by the Asymptotic Safety construction with an RG trajectory of the type IIIa, we get the required positive cosmological constant in the IR, $\bar{\lambda}_{\text{CR}}$, which in turn fixes the number of degrees of freedom, here to be interpreted as $\mathcal{C}^{\text{UV}} - \mathcal{C}^{\text{IR}}$, by $\mathcal{N} = 3\pi/G_{\text{CR}}\bar{\lambda}_{\text{CR}} < \infty$. Recalling our discussion of the Ia and IIa trajectories in section 11.3.3 and appendix 11.B we can now easily understand what is special about a strictly *positive* $\bar{\lambda}$, and why the connection fails for a negative or vanishing classical cosmological constant: in the latter cases, we found that $\mathcal{N}$ is *not* finite.

11.A The special status of Faddeev-Popov ghosts

In subsection 11.2.4 we argued that Faddeev-Popov ghosts, even though they contribute with a negative sign to the supertrace on the RHS of the flow equation, do not destroy the pointwise monotonicity of the EAA when they are the only fields present with odd Grassmann parity. The reason was that the ghosts are merely a way of representing the Faddeev-Popov determinant, $\det(\mathcal{M})$, the functional integral actually being $Z = \int [dg] \det(\mathcal{M}) e^{-S_{\text{gf}}} e^{-S}$, wherein the gauge fixing term and the determinant effectively restrict the integration over all metrics to an integral over the gauge orbit space of metrics modulo diffeomorphisms. If we had parametrized the latter directly we were dealing with a purely Grassmann-even integral [62], which when modified by an IR cutoff, obviously leads to a pointwise monotone EAA as the gauge orbit space is independent of k . In this appendix, we briefly indicate how this general argument can be made concrete.

We start out from the functional integral that has been gauge-fixed à la Faddeev-Popov, but without IR cutoff yet. Then, after the usual background split, we perform a partial²¹ transverse-traceless (TT)-decomposition of the fluctuation field,

$$h_{\mu\nu} = h_{\mu\nu}^T + \left(\bar{D}_\mu V_\nu + \bar{D}_\nu V_\mu - \frac{2}{d} \bar{g}_{\mu\nu} \bar{D}^\alpha V_\alpha \right) + \frac{1}{d} \bar{g}_{\mu\nu} h \quad (11.89)$$

with $\bar{D}^\mu h_{\mu\nu}^T = 0$, $\bar{g}^{\mu\nu} h_{\mu\nu}^T = 0$, $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$. Henceforth we interpret $[dg_{\mu\nu}]$ as $[dh_{\mu\nu}^T][dh][dV_\mu]$. Furthermore, we write the Faddeev-Popov determinant as $\det(\mathcal{M}[g, \bar{g}]) \equiv \det(\mathcal{M}) e^{-S_1}$ with $\mathcal{M} \equiv \mathcal{M}[\bar{g}, \bar{g}]$. Diagrammatically speaking the action S_1 contains the ghost-antighost-graviton vertices and $\mathcal{M}^{-1}$ is the ‘free’ ghost propagator. Thus

$$Z[\bar{g}] = \int [dh_{\mu\nu}^T][dh][dV_\mu] \det(\mathcal{M}) e^{-S_{\text{gf}}} e^{-\tilde{S}} \quad (11.90)$$

with $\tilde{S} \equiv S + S_1$ and the representation $\det(\mathcal{M}) = \int [d\xi][d\bar{\xi}] \exp \int \bar{\xi} \mathcal{M} \xi$.

Next, consider the family of gauge fixing functions

$$\begin{aligned} \mathcal{F}_\mu[\bar{g}](h) &= \bar{D}^\nu h_{\mu\nu} - \varpi \bar{D}_\mu h \\ &= \bar{D}^2 V_\mu + \left(1 - \frac{2}{d}\right) \bar{D}_\mu \bar{D}^\nu V_\nu + \bar{R}_\mu^\nu V_\nu + \left(\frac{1}{d} - \varpi\right) \bar{D}_\mu h \end{aligned} \quad (11.91)$$

As for the parameter ϖ , we choose the ‘un-harmonic’ gauge [124, 255, 256], setting $\varpi = 1/d$. (The harmonic gauge has $\varpi = 1/2$ instead.) This gauge leads to a remarkable conspiracy of the gauge fixing action, $S_{\text{gf}} = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} \mathcal{F}_\mu[\bar{g}](h) \mathcal{F}_\nu[\bar{g}](h)$, and the ghost action at $h_{\mu\nu} = 0$,

²¹It is ‘partial’ in that the vector field V_μ is not decomposed further here as this is usually done, setting $V_\mu = V_\mu^T + \bar{D}_\mu \sigma$ with $\bar{g}^{\mu\nu} \bar{D}_\mu V_\nu^T = 0$.

$S_{\text{gf}} = \int d^d x \sqrt{\bar{g}} \bar{\xi}_\mu \mathcal{M}^\mu_\nu \xi^\nu$. One finds that S_{gf} is a bilinear form in V_μ (and only V_μ !) whose kernel is precisely the *square of the inverse ghost propagator*²²:

$$S_{\text{gf}} = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}} V_\mu \mathcal{M}^\mu_\alpha \mathcal{M}^\alpha_\nu V^\nu \quad (11.92)$$

In this gauge, $\mathcal{M}^\mu_\nu \equiv \mathcal{M}[\bar{g}, \bar{g}]^\mu_\nu = \bar{D}^2 \delta^\mu_\nu - (1 - 2/d) \bar{D}^\mu \bar{D}_\nu - \bar{R}^\mu_\nu$. As a consequence, the integral over the ghosts, producing the determinant $\det(\mathcal{M})$, when combined with $e^{-S_{\text{gf}}}$, yields a Dirac δ -functional in the limit $\alpha \rightarrow 0$:

$$\det(\mathcal{M}) e^{-\frac{1}{2\alpha} \int V \mathcal{M}^2 V} \rightarrow \delta[V] \quad (11.93)$$

It satisfies $\int [dV] \delta[V] \rightarrow 1$. That $\delta[V]$ is indeed correctly normalized, up to a constant, follows from $\det(\mathcal{M}) \int [dV] e^{-\frac{1}{2\alpha} \int V \mathcal{M}^2 V} = \det(\mathcal{M}) \det^{-1/2}(\mathcal{M}^2) = \det(\mathcal{M}) \det^{-1}(\mathcal{M}) = 1$.

As a result, the limit $\alpha \rightarrow 0$ simplifies the integral for Z quite considerably: after integrating over V_μ and using (11.93) we are left with

$$Z = \int [dh^T_{\mu\nu}] \mathcal{D}h \exp \left(-\tilde{S}[h^T_{\mu\nu} + d^{-1} \bar{g}_{\mu\nu} h; \bar{g}_{\mu\nu}] \right) \quad (11.94)$$

This functional integral is manifestly over fields of even Grassmann parity only. So when we go through the usual procedure and define the associated EAA, the derivation of $\partial_k \Gamma_k \geq 0$ in the main part of this paper applies to it, provided the above exact compensation of the ghost and V_μ contributions persists in presence of an IR cutoff. While this is not the case for a generic cutoff, it has been shown [126] that if the cutoff operators $\mathfrak{R}_k$ of the ghost and metric fluctuations, respectively, are appropriately related, which always can be achieved, the compensation does indeed persist. For further details the reader is referred to [126].

Thus we have shown that (at the very least) when the ghosts are the only Grassmann-odd fields it is in principle always possible to set up the gauge fixing and ghost sector of the EAA and its FRGE in such a way that $\partial_k \Gamma_k \geq 0$ holds true pointwise.

The various sets of beta-functions studied in this chapter were *not* obtained using this very special set-up for the gauge-fixing and ghost sector. However, as the truncations considered here anyhow neglect all RG effects in this sector we have the freedom to use any gauge at this level of accuracy since this should not lead to an extra error. A similar remark applies to the choice of the cutoff operators.

11.B The trajectory types Ia and IIa

In this appendix we evaluate $\mathcal{C}_k = \Gamma_k^{\text{grav}}[0; \bar{g}_k^{\text{s.c.}}]$ along type Ia and IIa trajectories in $d = 4$ for the bi-metric Einstein-Hilbert truncation. The analysis parallels to some extent the one in subsection 11.3.3 for the type IIIa case. Since in the Ia and IIa cases the cosmological constant $\lambda^{(1)}$ turns zero or even negative at some scale the corresponding self-consistent background undergoes a topological change, from S^4 to flat Euclidean space R^4 and to H^4 , respectively. While for the separatrix, the type IIa trajectory, the change from S^4 to R^4 happens in the limit $k \rightarrow 0$ only, the type Ia solutions have a negative $\lambda^{(1)}$ at finite scales $k < k_{\text{sing}}$ already. At the transition point $k = k_{\text{sing}}$ equation (11.75) is no longer valid and Y_k diverges: $\lim_{k \searrow k_{\text{sing}}} Y_k = \infty$. For $k < k_{\text{sing}}$ a different formula for $\mathcal{C}_k$, employing a new background configuration, could be derived. We shall not do this here and rather restrict our attention to the subspace of $\mathcal{T}$ with $\lambda^{(1)} > 0$.

The Figs. 11.12 and 11.13 depict the explicit k -dependence of $1/Y_k$ for a representative type Ia trajectory and the unique IIa trajectory, respectively, both for the case of restored split-symmetry in the IR. The plots are based on the RG equations of [II]; the results with those

²²This ‘magic’ property has been discovered by F. Saueressig et al. [126, 257].

from [I] are quite similar. For the type Ia trajectory in Fig. 11.12 the singularity of Y_k occurs at about $k_{\text{sing}} \approx 0.5m_{\text{Planck}}$, and Y_k is seen to be perfectly monotone above this scale.

Along this trajectory, we integrated out $\mathcal{N}_{k_{\text{sing}},\infty} = \infty$ modes already before the end of the trajectory. But as there is a nontrivial RG evolution also below k_{sing} , there are still further modes left to be integrated out. Using a hyperbolic background we could count how many there are in some interval $[k_1, k_2]$ with $k_1 < k_2 < k_{\text{sing}}$. But clearly there is no meaningful way of associating a finite number $\mathcal{N}_{0,\infty}$ to the complete trajectory as this was possible in the IIIa case.

For the symmetry-restoring bi-metric separatrix, the plot of $1/Y_k$ is very similar to the case of the IIIa-trajectories discussed in the main part of the paper, see Fig. 11.13. The only difference is that $1/Y_k$ vanishes exactly at $k = 0$, while $1/Y_k$ was always nonzero for the type IIIa solutions. So the separatrix is the marginal case where $\mathcal{N} = \infty$ is reached precisely at the IR-end point of the trajectory.

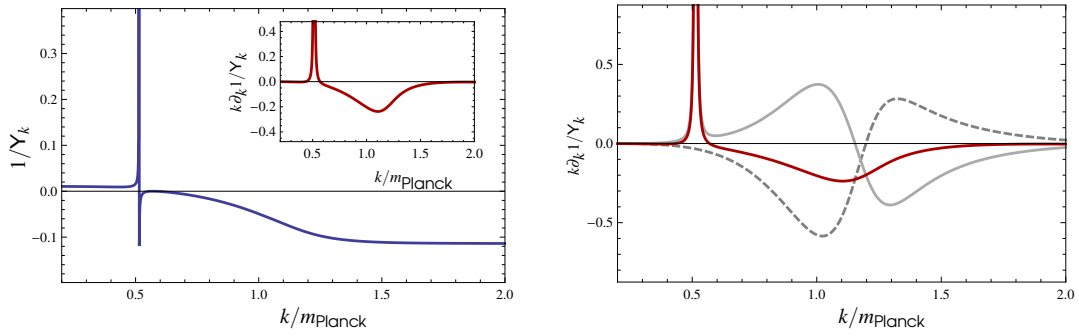


Figure 11.12: The function $1/Y_k$ and its derivative are shown for a typical bi-metric type Ia trajectory which restores split-symmetry in the IR. Due to the sign flip of $\lambda_k^{(1)}$ near $k_{\text{sing}} \approx 0.5m_{\text{Planck}}$, the function $1/Y_k$ has a zero there and $\mathcal{E}_k$ diverges. As long as (11.75) is still valid, to $k > k_{\text{sing}}$, the function Y_k is seen to be monotone.

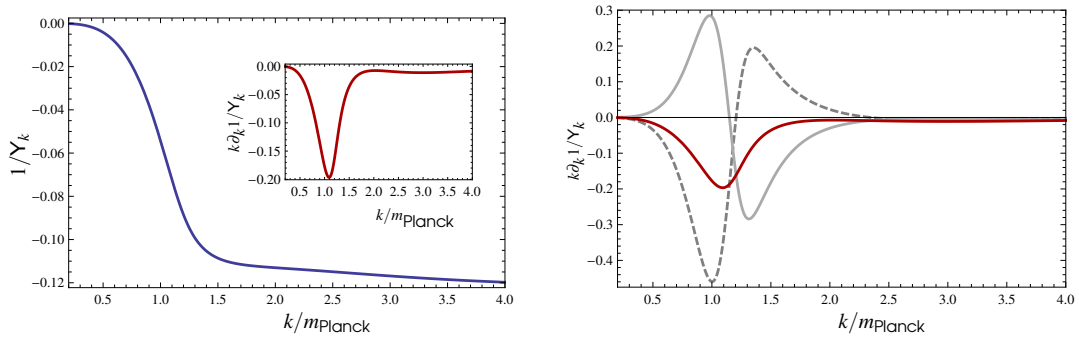


Figure 11.13: The function $1/Y_k$ for the bi-metric type Ia trajectory (separatrix) and its derivative. Its properties are similar to the type IIIa results obtained in subsection 11.3.3. In particular Y_k is seen to be monotone. The asymptotic topology change of the self-consistent background in the limit $k \rightarrow 0$ is not directly visible in these plots. It can be checked though that Y_k diverges for $k \rightarrow 0$ (and that it does not in the IIIa case).

In a series of snapshots, Figs. 11.14 and 11.15 show the evolution of the RG trajectories in the background sector, from the IR to the UV, on the basis of typical Ia and IIa dynamical trajectories, respectively. The shaded (white) regions partition the $g^{\text{B}}\text{-}\lambda^{\text{B}}$ -plane into subsets of positive (negative) slope $k\partial_k Y_k$. A crossing of the corresponding boundary indicates a violation

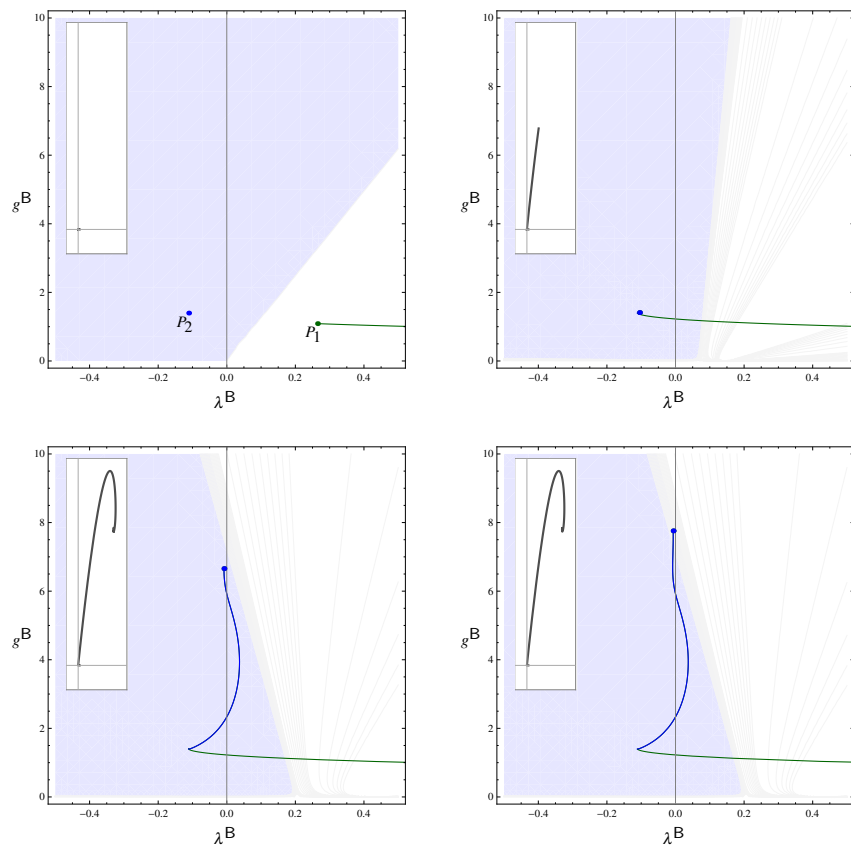


Figure 11.15: A series of snapshots as in Fig. 11.14, but for the *type IIa* trajectory. The results are similar to those in subsection 11.3.3 for the type IIIa trajectories.

of the monotonicity of $\mathcal{C}_k$. While in the case of the separatrix the requirement of split-symmetry restoration in the IR is sufficient to assure this condition, a more careful study is needed for type Ia trajectories. There $\lambda^{(1)}$ turns negative at finite scales k , and thus makes eq. (11.75) inapplicable. In any case, trajectories that break split-symmetry at $k = 0$ are more vulnerable to monotonicity violation than those restoring it.



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	Single-metric truncation
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12. PROPAGATING MODES IN QUANTUM GRAVITY

 This chapter follows closely ref. [192].

Within the Asymptotic Safety scenario, we discuss whether Quantum Einstein Gravity (QEG) can give rise to a semi-classical regime of propagating physical gravitons (gravitational waves) governed by an effective theory which complies with the standard rules of local quantum field theory. According to earlier investigations based on single-metric truncations there is a tension between this requirement and the condition of Asymptotic Safety since the former (latter) requires a positive (negative) anomalous dimension of Newton's constant. In this chapter we show that the problem disappears using the bi-metric renormalization group flows that became available recently: They admit an asymptotically safe UV limit and, at the same time, a genuine semi-classical regime with a positive anomalous dimension. This brings the gravitons of QEG on a par with arbitrary (standard model, etc.) particles which exist as asymptotic states. We also argue that metric perturbations on almost Planckian scales might not be propagating, and we propose an interpretation as a form of 'dark matter'.

12.1 Introduction

One of the indispensable requirements an acceptable fundamental quantum gravity theory must satisfy is the emergence of a classical regime where in particular small perturbations, i.e. gravitational waves, can propagate on an almost flat background spacetime. This regime should be well described by classical General Relativity or, if one pushes its boundary towards the quantum domain a bit further, by the effective quantum field theory approach pioneered by Donoghue [258, 259].

In the present context we consider the scenario where the ultraviolet (UV) completion of quantized gravity is described by an asymptotically safe quantum field theory [145]. In a formulation based upon the gravitational average action [122], this quantum field theory is defined by a specific Renormalization Group (RG) trajectory $k \mapsto \Gamma_k$ which lies entirely within the UV-critical hypersurface of a non-Gaussian fixed point (NGFP). Here $\Gamma_k \equiv \Gamma_k[h_{\mu\nu}; \bar{g}_{\alpha\beta}]$ denotes

the Effective Average Action, a ‘running’ action functional which, besides the scale k , depends on the (expectation value of the) metric fluctuations, $h_{\mu\nu}$, and the metric of the background spacetime on which they are quantized, $\bar{g}_{\alpha\beta}$, see chapter 4 and part II of this thesis.

To recover classical General Relativity in this setting it would be most natural if the asymptotically safe RG trajectory of the fundamental theory, emanating from the NGFP in the UV ($k \rightarrow \infty$), contains a segment in the low energy domain ($k \rightarrow 0$) where the full fledged description in terms of the effective average action, valid for all scales and all backgrounds, smoothly goes over into the effective field theory of spin-2 quanta propagating on a rigid background Minkowski spacetime. The simplest picture would then be that the approximating low energy theory which is implied by the fundamental asymptotically safe one is ‘standard’ in the sense that it complies with the usual axiomatics of local quantum field theory on Minkowski space which underlies all of particle physics, for instance.

However, almost all existing RG studies of the Asymptotic Safety scenario, using functional RG methods, indicate that there is a severe tension, if not a clash, between their predictions and the picture of a *conventional Minkowski space theory* describing propagating gravitons or gravitational waves at low energies [99, 100, 121, 123, 126, 130–135, 138, 139, 141, 142, 146–171].

In the following we try to describe this tension as precisely as possible. It is necessary to distinguish the real question of (non-)existing propagating gravitational waves in the classical regime from certain objections against Asymptotic Safety in general that were raised occasionally but were based on misconceptions and are unfounded therefore. One of these misconceptions is the believe that the anomalous dimensions of quantum fields must be *positive, always*.

In fact, for asymptotically safe Quantum Einstein Gravity (QEG) it is crucial that the anomalous dimension of the metric fluctuations, η_N , is *negative*, at least in the vicinity of the NGFP. There, by the very construction of the theory’s UV completion, it assumes the value $\eta_N^* = -(d-2)$, in d spacetime dimensions.¹ And indeed, the RG equations obtained within the special class of non-perturbative approximations that have been considered in the past almost exclusively, the so called ‘single-metric’ truncations of theory space, had always given rise to a negative anomalous dimension [168–171]. Moreover, $\eta_N < 0$ was found not only near the NGFP but even *everywhere* on the truncated theory space considered.

In these truncations the ansatz for the Effective Average Action (EAA) always included a term $\propto G_k^{-1} \int d^d x \sqrt{\bar{g}} R(g)$ from which η_N was obtained as the scale derivative of the running Newton constant: $\eta_N = k \partial_k \ln G_k$. Since in this term the metric $g_{\mu\nu}$ is to be interpreted as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the running Newton constant fixes the normalization of the fluctuation field, $h_{\mu\nu}$. While extremely tiny in magnitude, η_N turned out negative with this entire class of truncations even in the ‘classical regime’ displayed by the special (Einstein-Hilbert truncated, Type IIIa) trajectory which matches the observed values of Newton’s constant and the cosmological constant [191, 209, 212].

To see why the sign of the anomalous dimension is important let us consider an arbitrary field in d spacetime dimensions with an inverse propagator $\propto Z(k^2)p^2$ which depends on an RG scale k . In absence of other relevant scales we may identify $k^2 = p^2$, obtaining the dressed propagator $\tilde{\mathcal{G}}(p) \propto [Z(p^2)p^2]^{-1}$. For example in a regime where $Z(k^2) \propto k^{-\eta}$ with a constant exponent η we have, in momentum space, $\tilde{\mathcal{G}}(p) \propto 1/(p^2)^{1-\eta/2}$. If this propagator pertains to an Euclidean field theory on flat space it is natural to perform a Fourier transformation with respect to all d coordinates, whence

$$\mathcal{G}_E(x-y) \propto \frac{1}{|x-y|^{d+\eta-2}} \quad (12.1)$$

¹We assume $d > 2$ throughout.

For field theories on Minkowski space the static limit of the propagator is particularly interesting; setting the time component of p_μ to zero and taking the $(d-1)$ dimensional Fourier transform of $\tilde{\mathcal{G}}(p)$ we get, with $x \equiv (x^0, \vec{x})$ and $y \equiv (x^0, \vec{y})$ at equal times,

$$\mathcal{G}_M(0, \vec{x} - \vec{y}) \propto \frac{1}{|\vec{x} - \vec{y}|^{d+\eta-3}} \quad (12.2)$$

Eqs. (12.1) and (12.2) confirm that the exponent η which comes into play via the scale dependent field normalization $Z(k^2) \propto k^{-\eta}$ indeed deserves the name of an ‘anomalous dimension’: the renormalization effects changed the effective dimensionality of spacetime, which manifests itself by the fall-off behavior of the 2-point function, from d to $d + \eta$. In $d = 3 + 1$, for instance, we obtain the modified Coulomb potential

$$\mathcal{G}_M(0, \vec{x} - \vec{y}) \propto \frac{1}{|\vec{x} - \vec{y}|^{1+\eta}} \quad (12.3)$$

The point to be noted here is that, *as compared to the classical Coulomb Green’s function, a positive value of the anomalous dimensions renders the propagator more short ranged, while it becomes more long ranged when η is negative.*

Thus we conclude that the anomalous dimension $\eta_N < 0$ found by the single-metric truncations of QEG corresponds to a graviton propagator on flat space which falls off for increasing distance *more slowly* than $1/|\vec{x}|$. Also notice that, strictly speaking, eq. (12.1) holds only when $d + \eta - 2 \neq 0$. If $d + \eta - 2 = 0$ one has an increasing behavior even, $\mathcal{G}_E(x - y) \propto \ln(x - y)^2$. This is precisely the case relevant at the NGFP of quantum gravity where $\eta_N^* = -(d - 2)$. In the fixed point regime the momentum dependence is $\tilde{\mathcal{G}}(p) \propto 1/p^d$. Note that at the NGFP the function (12.2) becomes *linear*: $\mathcal{G}_M(0, \vec{x} - \vec{y}) \propto |\vec{x} - \vec{y}|$.

The fall-off properties of the propagator have occasionally been adduced as a difficulty for the Asymptotic Safety idea. We emphasize that in reality there is no such difficulty. It is nevertheless instructive to go through the argument, and to see where it fails. For this purpose, consider an arbitrary bosonic quantum field Φ on 4D Minkowski space. Under very weak conditions one can derive a Källén-Lehmann spectral representation [260, 261] for its dressed propagator:

$$\Delta'_F(x - y) = \int_0^\infty d\mu^2 \rho(\mu^2) \Delta_F(x - y; \mu^2) \quad (12.4)$$

Here

$$\Delta_F(x - y; \mu^2) = - \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 - \mu^2 + i\epsilon} \quad (12.5)$$

is the free Feynman propagator (with possible tensorial structures suppressed), and the spectral weight function

$$\rho(q^2) = (2\pi)^3 \sum_\alpha \delta^4(p_\alpha - q) |\langle 0 | \Phi(0) | \alpha \rangle|^2 \quad (12.6)$$

contains a sum over all states $|\alpha\rangle$ with momenta p_α where $p_\alpha^2 \geq 0$, $p_{\alpha 0} \geq 0$ (the one-particle contribution included). It is assumed that the states are elements of a vector space which is equipped with a positive-definite inner product. Therefore it follows directly from its definition (12.6) that $\rho(\mu^2)$ is a *non-negative* function. The Källén-Lehmann representation itself follows from only a few, very basic additional assumptions: (a) completeness of the momentum eigenstates, in particular completeness of the asymptotic states, (b) the spectral condition $p^2 \geq 0$, $p_0 \geq 0$ for the states, (c) Poincaré covariance, in particular invariance of the vacuum state.

If a dressed propagator Δ'_F possess a Källén-Lehmann representation it follows that its Fourier transform behaves as $1/p^2$ for $p^2 \rightarrow \infty$ limit, exactly as for the free one, Δ_F . Conversely, for $|\vec{x} - \vec{y}| \rightarrow \infty$ at equal times, Δ'_F cannot decay more slowly than $\propto 1/|\vec{x} - \vec{y}|$. Indeed, the free massive Feynman propagator behaves as $\Delta_F(0, \vec{x} - \vec{y}; \mu^2) \propto \exp(-\mu |\vec{x} - \vec{y}|)$ in this limit, so that the μ^2 -integral in (12.4) amounts to a superposition of decaying exponentials with non-negative weight, since $\rho(\mu^2) \geq 0$. The best that can happen is that $\rho(\mu^2)$ has support at $\mu^2 = 0$, in which case the free propagator behaves Coulomb-like $\propto 1/|\vec{x} - \vec{y}|$, and, as a consequence, the dressed one as well, $\Delta'_F(0, \vec{x} - \vec{y}) \propto 1/|\vec{x} - \vec{y}|$. Obviously this is the behavior corresponding to an anomalous dimension $\eta = 0$. If a Källén-Lehmann representation exists, Δ'_F may fall off faster, so $\eta > 0$ is possible, but not more slowly.

As a consequence, under the conditions implying the existence of a Källén-Lehmann representation negative anomalous dimensions $\eta < 0$ cannot occur. This entails that, conversely, whenever an anomalous dimension is found to be negative one or several of those conditions must be violated.

In the case of asymptotically safe gravity, described by the EAA, we can easily identify at least one of the above necessary conditions which is *not* satisfied: The functional integral related to $\Gamma_k[h_{\mu\nu}, \xi_\mu, \bar{\xi}^\mu; \bar{g}_{\mu\nu}]$ is a modified version (containing an IR regulator term) of the standard Faddeev-Popov gauge-fixed and BRST invariant functional integral which quantizes $h_{\mu\nu}$ in some background gauge, usually the de Donder-Weyl gauge [122]. However, the operatorial reformulation of this quantization scheme is well-known to involve a state space with an *indefinite metric* [262]. Therefore, $\rho(q^2)$ has no reason to be positive, and the short distance behavior of the dressed $h_{\mu\nu}$ propagator may well be different from $1/p^2$ in momentum space. In fact, Asymptotic Safety makes essential use of this possibility: For $p^2 \rightarrow \infty$, and in $d = 4$, the propagator must be proportional to $1/p^4$ as a consequence of the UV fixed point.

A well-known example with similar properties is the Lorentz-covariant quantization of Yang-Mills theories on flat space, Quantum chromodynamics (QCD), for instance. Here the anomalous dimension related to the gluon, $\eta \equiv \eta_F$, is negative too, and its negative sign is precisely the one responsible for asymptotic freedom. Analogous to the computation done for the Newton constant, one can obtain η_F in the EAA approach by using a (covariant) background type gauge and reading off η_F from the term $\frac{1}{4g_k^2} \int F_{\mu\nu}^2$ in Γ_k as the logarithmic scale derivative of the gauge coupling g_k , see ref. [110–114] for details. A long ranged gluon propagator due to $\eta < 0$ could be indicative of gluon confinement, at least in certain gauges. Again the pertinent state space is not positive-definite, and so even propagators increasing with distance are not excluded by general principles.

It is actually quite intriguing that a linear confinement potential $\propto |\vec{x} - \vec{y}|$ for static color charges, corresponding to a $1/p^4$ behavior in the IR, is precisely what in gravity is realized in the UV. While the fixed point regime of QEG is realized at small rather than large distances, the graviton carries a *large negative anomalous dimension* there.

Up to now we exploited only a rather technical, non-dynamical property of the quantization scheme used, namely the indefinite metric on state space, in order to reject the implications of a Källén-Lehmann representation with a positive spectral density. This was sufficient to demonstrate that within the setting of the (background gauge invariant) gravitational EAA of ref. [122] *the exact anomalous dimension derived from the running Newton constant is not bound to be positive for any general reason*. Therefore there is nothing obviously wrong with the negative η_N 's that were found in concrete QEG calculations on truncated theory spaces, and a similar statement is true for Yang-Mills theory.

However, the previous argument has not yet much to do with the *dynamical* properties of the respective theory. Taking QCD as an example again, we can solve the BRST cohomology problem which underlies its perturbative quantization, and in this way we learn how to reduce

the indefinite-metric state space to a subspace of ‘physical’ states which carries a positive definite inner product. One finds that, in this sense, transverse gluons and quarks are ‘physical’, while longitudinal and temporal gluons, as well as Faddeev-Popov ghosts are ‘unphysical’.

Now, it is a highly non-trivial question whether the dynamics of the ‘physical’ states is such that the above requirements (a), (b), (c) are satisfied so that a Källén-Lehmann representation of the transverse gluon propagator could exist. The general belief is that the answer is negative since gluons, being confined, do not form a complete system of asymptotic states. So here we have a deep dynamical rather than merely kinematical reason to reject the implications of the Källén-Lehmann representation concerning the propagator’s fall-off behavior. This opens the door for a gluon propagator which might even increase with distance, like, for instance, the ‘IR enhanced’ propagator proportional to $1/p^4$ for $p^2 \rightarrow 0$.²

Even though the gluon propagator is gauge dependent there is a direct connection to the gauge invariant confinement criterion of an area law for Wilson loops. It has been shown [267] that if the gluon propagator possesses the singular $1/p^4$ behavior for $p^2 \rightarrow 0$ in just *one* gauge then QCD is confining in the Wilson loop sense; in any other gauge it need not show this singular behavior. In covariantly gauge fixed QCD, it is of interest to know the properties of the gluon, ghost, and quark propagators also because they contain information about the non-perturbative dynamical mechanism by means of which the theory cuts down the indefinite state space to a positive-definite subspace, containing ‘physical’ states only.

In gravity, the analogous question concerns the status of the transverse gravitons, that is, the $h_{\mu\nu}$ modes which are not ‘pure gauge’ but rather ‘physical’ in the BRST sense. Let us envisage a universe which, on all its vastly different scales, from the Planck regime to cosmological distances, is governed by QEG, and let us ask whether a transverse graviton which it may contain is more similar to a photon (unconfined, freely propagating, exists as an asymptotic state³) or to a gluon (confined, no asymptotic state, no Källén-Lehmann representation with positive ρ)?

In its full generality this is a very hard question. The attempt at an answer on the basis of existing single-metric computations would be that the graviton is more similar to the gluon than to the photon, a claim that might appear surprising, in particular if one thinks of astrophysical gravitational waves.

In quantum gravity, where Background Independence adds to the standard principles of quantum field theory, a particular convenient way to ensure a covariant formalism in presence of a gauge fixing and a cutoff term is the background field method. Thereby one introduces a generic background metric $\bar{g}_{\mu\nu}$ at intermediate stages of the quantization in addition to the usual dynamical metric $g_{\mu\nu}$. Consequently, the most general ansatz for Γ_k (possibly including matter fields) has to be of ‘bi-metric’ type and thus contains all possible field monomials that can be constructed from $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$, and the matter fields which respect all relevant symmetries (diffeomorphisms, (gauge-) symmetries in the matter sector, etc.) [136, 137]. On the one hand, this affects for instance the mathematical description of the UV-completion of the theory, where a suitable UV fixed point defines the relation between the various invariants of background and physical metric. However, on the other hand, the background is a purely technical artifice that does not affect the observables. Those are obtained from the physical sector of $\Gamma_{k=0} \equiv \Gamma$ that must respect Background Independence, i.e. which is independent of the auxiliary background field in the IR limit. Hence, $\bar{g}_{\mu\nu}$ – even though crucial in the construction of an exact RG flow

²Please note that by no means we are saying here that this behavior *must* occur, rather only that it *can* occur without violating any of the general principles discussed. In fact, detailed analyses of the IR properties of QCD, employing various independent non-perturbative techniques, indicate that in reality the picture is far more complex. For recent results concerning the gluon propagator, the properties of the spectral densities and the positivity properties of Yang-Mills theory we must refer to the literature [263–266].

³to the extent this can make sense as an approximate notion in curved spacetime

at all intermediate scales k – becomes redundant when invoking fully intact split-symmetry in the IR ($k = 0$). A technical simplification in the RG computations consists of neglecting the background invariants in the ansatz for the *non-gauge* part of the EAA, giving rise to the aforementioned single-metric approximation. Since the gauge fixing and the cutoff action still rely explicitly on $\bar{g}_{\mu\nu}$, the RHS of the FRGE generates invariants depending on the background field and thus remain unresolved. Hence, the reliability of the single-metric approximation has to be tested by comparison of the class of solutions, in particular the UV completion with its more general bi-metric counterparts. Thus, from a physical perspective, the ultimate theory (physical sector of $\Gamma_{k=0}$, on-shell S-matrix elements, for example) is *not* ‘bi-metric’ in the sense that two independent metric tensors would play a role individually. There is only one physical metric; the background metric at intermediate stages of the quantization is only a technical artifice. Fully intact split-symmetry in the physical sector, for vanishing IR cutoff, is precisely the statement that $\bar{g}_{\mu\nu}$ has become redundant and no observable depends on it.

The purpose of this chapter is to go beyond the single-metric approximation and investigate the crucial sign of the anomalous dimension η_N using differently truncated functional RG flows of asymptotically safe metric gravity, i.e. QEG. In particular we explore the corresponding predictions of two ‘bi-metric truncations’ of theory space. They have been studied in ref. [172], henceforth denoted [I], and in ref. [173] or part II of this thesis (with boundary terms omitted) which in the sequel is referred to as [II], respectively. Both employ a similar truncation ansatz for $\Gamma_k[g, \bar{g}]$, namely two *separate* Einstein-Hilbert terms for the dynamical and the background metric $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively. The calculations in [I] and [II] differ, however, with respect to the gauge fixing-conditions and -parameters they use, as well as the field parameterization they employ. In [I] the ‘geometric’ or ‘anharmonic’ gauge fixing [62, 124, 172, 255, 256] is used, with gauge fixing parameter $\alpha = 0$, while [II] relies on the harmonic gauge and $\alpha = 1$. Furthermore, in [I], the functional flow equation and in particular its mode suppression operator $\mathfrak{R}_k$ was formulated in terms of a transverse-traceless (TT) decomposed field basis for $h_{\mu\nu}$, while in part II we have seen that no such decomposition is necessary to obtain the results of [II]. It is to be expected that these differences of the coarse graining schemes employed should have only a minor impact on the RG flow and leave its essential qualitative features unchanged.

This chapter is organized as follows. In section 12.2 we present a detailed analysis of the two bi-metric calculations [I], [II] and a comparison of their respective RG flows with the well-known one based on the single-metric Einstein-Hilbert truncation. We demonstrate that the former imply a positive anomalous dimension, hence a ‘photon-like’ behavior of gravitons in the semi-classical regime. There is no obvious physical reason or qualitative argument that would explain the sign flip of η in going from the single- to the bi-metric truncation. Therefore, detailed quantitatively precise calculations are particularly important here.

Section 12.3 is devoted to metric fluctuations outside this regime. Their precise propagation properties near, but close to the Planck scale remain unknown for the time being. We argue that, in this range of covariant momenta, they behave as a form of gravitating, but non-propagating ‘dark matter’. Possible implications for the early Universe are also discussed. Finally section 12.4 contains a brief summary.

12.2 Anomalous dimension in single- and bi-metric truncations

As a short reminder, our approach to the quantization of gravity assumes that the fundamental degrees of freedom mediating the gravitational interaction are carried by the spacetime metric. It heavily relies upon the Effective Average Action (EAA), a k -dependent functional $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu]$ which, in the case of QEG, depends on the dynamical metric $g_{\mu\nu}$, the background metric $\bar{g}_{\mu\nu}$, and the diffeomorphism ghost ξ^μ and anti-ghost $\bar{\xi}_\mu$, respectively. We employ the background field method to deal with the key requirement of Background Indepen-

dence, and are thus led to the task of quantizing the metric fluctuations $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ in all fixed but arbitrary backgrounds simultaneously.⁴

For all truncations of theory space studied in this chapter the corresponding ansatz for the EAA has the same general structure, namely

$$\Gamma_k[g, \bar{g}, \xi, \bar{\xi}] = \Gamma_k^{\text{grav}}[g, \bar{g}] + \Gamma_k^{\text{gf}}[g, \bar{g}] + \Gamma_k^{\text{gh}}[g, \bar{g}, \xi, \bar{\xi}] \quad (12.7)$$

Concretely we consider the Einstein-Hilbert truncation, both in its familiar single-metric form [122, 191] and a more advanced bi-metric variant thereof [172, 173]. In the single-metric truncation the gravitational (‘grav’) part of the ansatz has the form

$$\Gamma_k^{\text{grav}}[g, \bar{g}] = -\frac{1}{16\pi G_k^{\text{sm}}} \int d^d x \sqrt{\bar{g}} \left(R(g) - 2\bar{\lambda}^{\text{sm}} \right) \quad (12.8)$$

It contains two running coupling constants, Newton’s constant G_k^{sm} and the cosmological constant $\bar{\lambda}_k^{\text{sm}}$. (The superscript ‘sm’ stands for single-metric.)

For the most general bi-metric refinement of this truncation one should in principle include the infinitely many invariants which one can construct from the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ that reduce to (12.8) when both metrics are identified, $g = \bar{g}$. Here, we follow [I] and [II] retaining for technical simplicity only four such invariants, namely two independent Einstein-Hilbert actions for g and $\bar{g}$, respectively:

$$\begin{aligned} \Gamma_k^{\text{grav}}[g, \bar{g}] = & -\frac{1}{16\pi G_k^{\text{D}}} \int d^d x \sqrt{g} \left(R(g) - 2\bar{\lambda}^{\text{D}} \right) \\ & -\frac{1}{16\pi G_k^{\text{B}}} \int d^d x \sqrt{\bar{g}} \left(R(\bar{g}) - 2\bar{\lambda}^{\text{B}} \right) \end{aligned} \quad (12.9)$$

This family of actions comprises 4 running coupling constants, the dynamical (‘D’) Newton and cosmological constants as well as their background (‘B’) counterparts.

An equivalent and sometimes more useful description of the action (12.9) is obtained by expanding $\Gamma_k^{\text{grav}}[g, \bar{g}]$ in powers of the fluctuation field $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$. We have, up to terms of second order in $h_{\mu\nu}$:

$$\begin{aligned} \Gamma_k^{\text{grav}}[h, \bar{g}] = & -\frac{1}{16\pi G_k^{(0)}} \int d^d x \sqrt{\bar{g}} \left(R(\bar{g}) - 2\bar{\lambda}_k^{(0)} \right) \\ & -\frac{1}{16\pi G_k^{(1)}} \int d^d x \sqrt{\bar{g}} \left[-\bar{G}^{\mu\nu} - \bar{\lambda}_k^{(1)} \bar{g}^{\mu\nu} \right] h_{\mu\nu} \\ & -\frac{1}{2} \int d^d x \sqrt{\bar{g}} h^{\mu\nu} \Gamma_k^{\text{grav}(2)}[\bar{g}, \bar{g}] h_{\rho\sigma} + \mathcal{O}(h^3) \end{aligned} \quad (12.10)$$

This expansion in powers of $h_{\mu\nu}$ is referred to as the ‘level representation’ of the EAA, and a term is said to belong to level- (p) if it contains p factors of $h_{\mu\nu}$, for $p = 0, 1, 2, \dots$. The level- (p) couplings $G_k^{(p)}, \bar{\lambda}_k^{(p)}$, by definition, correspond to invariants that are of order $(h_{\mu\nu})^p$. Their relation to the ‘D’ and ‘B’ couplings that were used in eq. (12.9) is given by, for $p = 0$,

$$\frac{1}{G_k^{(0)}} = \frac{1}{G_k^{\text{B}}} + \frac{1}{G_k^{\text{D}}}, \quad \frac{\bar{\lambda}_k^{(0)}}{G_k^{(0)}} = \frac{\bar{\lambda}_k^{\text{B}}}{G_k^{\text{B}}} + \frac{\bar{\lambda}_k^{\text{D}}}{G_k^{\text{D}}}, \quad (12.11)$$

⁴In this chapter we are dealing with pure gravity. If one includes matter fields a general truncation ansatz for Γ_k contains all possible field monomials that can be constructed from $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$, and the matter fields that respect the full set of imposed symmetries. At the fundamental level ($k \rightarrow \infty$) the fixed point condition will fix the precise combination in which $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ occur; in the final theory ($k \rightarrow 0$) instead it is, again, split-symmetry that forces one of the two metrics to become irrelevant or more precisely, ‘invisible’ by the physical observables.

and $G_k^{(p)} = G_k^D$, $\bar{\lambda}_k^{(p)} = \bar{\lambda}_k^D$ at all higher levels $p \geq 1$.

Note that the couplings at level-(1) are precisely those which enter the field equation for self-consistent backgrounds, $\delta\Gamma_k/\delta h_{\mu\nu}|_{h=0} = 0$, while those at level-(2) and levels-(3, 4, ...) determine the propagator and the vertices of the $h_{\mu\nu}$ -self-interactions, respectively. In the present truncation the latter roles are played by the same coupling namely $G_k^{(1)} = G_k^{(2)} = \dots \equiv G_k^D$, and likewise $\bar{\lambda}_k^{(1)} = \bar{\lambda}_k^{(2)} = \dots \equiv \bar{\lambda}_k^D$. However, it goes beyond a single-metric truncation as it resolves the differences between level-(0) and level-(1).⁵ Single-metric calculations retain only terms of order $(h_{\mu\nu})^0$, i.e. of level-(0), and then postulate that the RG running of the couplings at the higher levels is well approximated by that at level-(0). (See section 10.3 for a detailed discussion.)

The gauge fixing and the ghost terms Γ_k^{gf} and Γ_k^{gh} in (12.7) are determined by the gauge fixing function

$$\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] h_{\mu\nu} \equiv (\delta_\mu^\beta \bar{g}^{\alpha\gamma} \bar{D}_\gamma - \bar{\omega} \bar{g}^{\alpha\beta} \bar{D}_\mu) h_{\mu\nu} \quad (12.12)$$

which involves a free parameter, $\bar{\omega}$, whose RG running is neglected here. Special cases include the harmonic gauge ($\bar{\omega} = 1/2$) and the geometric, or ‘anharmonic’ gauge ($\bar{\omega} = 1/d$). In addition there appears the gauge parameter α in the gauge fixing action whose k -dependence will be neglected as well:

$$\Gamma_k^{\text{gf}}[g, \bar{g}] = \frac{1}{32\pi\alpha G_k^{D/\text{sm}}} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} \left[\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] (g_{\alpha\beta} - \bar{g}_{\alpha\beta}) \right] \left[\mathcal{F}_\nu^{\rho\sigma}[\bar{g}] (g_{\rho\sigma} - \bar{g}_{\rho\sigma}) \right] \quad (12.13)$$

Specifically, the two gauge fixing parameters were chosen as $(\bar{\omega} = 1/2, \alpha = 1)$, $(\bar{\omega} = 1/d, \alpha \rightarrow 0)$, and $(\bar{\omega} = 1/2, \alpha = 1)$ in the single-metric truncation of [122], the ‘TT-decomposed’⁶ bi-metric calculation of [I], and the ‘ Ω -deformed’⁷ bi-metric analysis in [II], respectively.

When the full ansatz is inserted into the Functional Renormalization Group Equation (FRGE) we obtain a coupled system of RG differential equations which, when expressed in terms of dimensionless couplings⁸, has the following structure:

$$\partial_t g_k^{D/\text{sm}} = [d - 2 + \eta^{D/\text{sm}}(g_k^{D/\text{sm}}, \lambda_k^{D/\text{sm}})] g_k^{D/\text{sm}} \quad (12.14a)$$

$$\partial_t \lambda_k^{D/\text{sm}} = \beta_\lambda^{D/\text{sm}}(g_k^{D/\text{sm}}, \lambda_k^{D/\text{sm}}) \quad (12.14b)$$

$$\partial_t g_k^{(0)} = [d - 2 + \eta^{(0)}(g_k^D, \lambda_k^D, g_k^{(0)})] g_k^{(0)} \quad (12.14c)$$

$$\partial_t \lambda_k^{(0)} = \beta_\lambda^{(0)}(g_k^D, \lambda_k^D, g_k^{(0)}, \lambda_k^{(0)}) \quad (12.14d)$$

The two equations (12.14a) and eq. (12.14b) constitute the single-metric system, while the bi-metric system is described by the full set of all 4 differential equations.

Since the above equations are partially decoupled, solutions $k \mapsto (g_k^D, \lambda_k^D, g_k^{(0)}, \lambda_k^{(0)})$ can be obtained from (12.14) in a hierarchical fashion: $(g_k^D, \lambda_k^D) \Rightarrow g_k^{(0)} \Rightarrow \lambda_k^{(0)}$. Notice that the explicit

⁵For structurally different calculations disentangling background and fluctuation fields see [130, 206, 268–270].

⁶The Hessian of Γ_k in the Einstein-Hilbert truncation contains uncontracted derivative operators such as $\bar{D}_\mu \bar{D}_\nu$. In [I] a transverse-traceless (TT) decomposition of the fluctuation field $h_{\mu\nu}$ was employed to deal with this complication. The problematic operators act on the component fields as fully contracted Laplacian $\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$ then, and heat kernel methods can be applied to evaluate the functional traces due to the various irreducible fields.

⁷In [II], Ω denotes a conformal parameter introduced as a tool to distinguish between dynamical and background contributions. The freedom in choosing a gauge parameter α was exploited to reduce the functional trace on the RHS of the FRGE to a function of the Laplacian $\bar{D}^2$ alone, which then could be computed using standard heat kernels again.

⁸The dimensionless couplings, g_k^I and λ_k^I , are related to the dimensionful ones, G_k^I and $\bar{\lambda}_k^I$, appearing in the truncation ansatz, by $G_k^I = k^{2-d} g_k^I$ and $\bar{\lambda}_k^I = k^2 \lambda_k^I$, respectively.

form of the beta-functions to be used is different for the three truncations we are going to consider here; they can be found in [122], [I], and [II], respectively.

In the sequel, we mostly focus on the Newton couplings G_k^I and their non-canonical RG running which is described by the respective anomalous dimension $k\partial_k \ln G_k^I \equiv \eta^I$. In all truncations considered here its general structure is

$$\eta^I = \frac{B_1^I(\lambda) g^I}{1 - B_2^I(\lambda) g^I} \quad \text{for } I \in \{D, B, (0), \text{sm}\} \quad (12.15)$$

The level- and background- η^I 's are related by $\eta^{(0)}/g^{(0)} = \eta^B/g^B + \eta^D/g^D$.

In the sequel we employ the language of levels and always present the couplings of the $h_{\mu\nu}$ -independent invariants, denoted by a superscript (0), together with the higher level couplings which are collectively denoted by 'D', standing for (p) , $p \geq 1$. (The 'B' couplings could be obtained from (12.11) if needed.)

In the following subsections we analyze the anomalous dimensions related to the various versions of Newton's constant. We begin with the single-metric case and then proceed to the two bi-metric calculations [I] and [II].

Unless stated otherwise, we always assume 4 spacetime dimensions ($d = 4$) in the rest of this paper, and we employ the optimized cutoff shape function [188].

12.2.1 Single-metric truncation

In the single-metric Einstein-Hilbert truncation the RG running of Newton's constant is governed by

$$\eta^{\text{sm}}(g^{\text{sm}}, \lambda^{\text{sm}}) = \frac{B_1^{\text{sm}}(\lambda^{\text{sm}}) g^{\text{sm}}}{1 - B_2^{\text{sm}}(\lambda^{\text{sm}}) g^{\text{sm}}} \quad (12.16)$$

The function $B_1^{\text{sm}}(\lambda^{\text{sm}})$ in the numerator of eq. (12.16) is given by

$$B_1^{\text{sm}}(\lambda^{\text{sm}}) = -\frac{1}{3\pi} \left\{ -5\Phi_1^1(-2\lambda^{\text{sm}}) + 18\Phi_2^2(-2\lambda^{\text{sm}}) + 4\Phi_1^1(0) + 6\Phi_2^2(0) \right\} \quad (12.17)$$

and $B_2^{\text{sm}}(\lambda^{\text{sm}})$ in the denominator reads

$$B_2^{\text{sm}}(\lambda^{\text{sm}}) = \frac{1}{6\pi} \left\{ -5\tilde{\Phi}_1^1(-2\lambda^{\text{sm}}) + 18\tilde{\Phi}_2^2(-2\lambda^{\text{sm}}) \right\} \quad (12.18)$$

Here Φ and $\tilde{\Phi}$ are the standard threshold functions introduced in [122] which depend on the details of the cutoff scheme, its 'shape function' $R_k^{(0)}$ in particular.

We are interested in the sign of η^{sm} in dependence on g^{sm} and λ^{sm} , the two coordinates on theory space. As can be seen from the plot in Fig. 12.1a, in the single-metric truncation, *the anomalous dimension η^{sm} is negative in the entire physically relevant region of the g^{sm} - λ^{sm} theory space.* This is a well-known fact, already mentioned in the Introduction, and has been confirmed also by all single-metric truncations with more than the $\int \sqrt{g}$ and $\int \sqrt{g}R$ terms in the ansatz that were analyzed so far [123–126, 128, 129, 147, 236, 255, 256, 271–276].

In the semi-classical regime⁹ where $0 < g^{\text{sm}}, \lambda^{\text{sm}} \ll 1$ the term $B_2^{\text{sm}}(\lambda^{\text{sm}})g^{\text{sm}}$ in the denominator on the RHS of (12.16) is negligible, hence the negative sign of η^{sm} is entirely due to the negative sign of $B_1^{\text{sm}}(\lambda^{\text{sm}})$ that occurs for small arguments $\lambda^{\text{sm}} \ll 1$. Here it is a reliable approximation to set $\eta^{\text{sm}} \approx B_1^{\text{sm}}(\lambda^{\text{sm}})g^{\text{sm}}$.

⁹To be precise, we consider a 'type IIIa' trajectory here, which, by definition, has a *positive* cosmological constant in the IR, see [191].

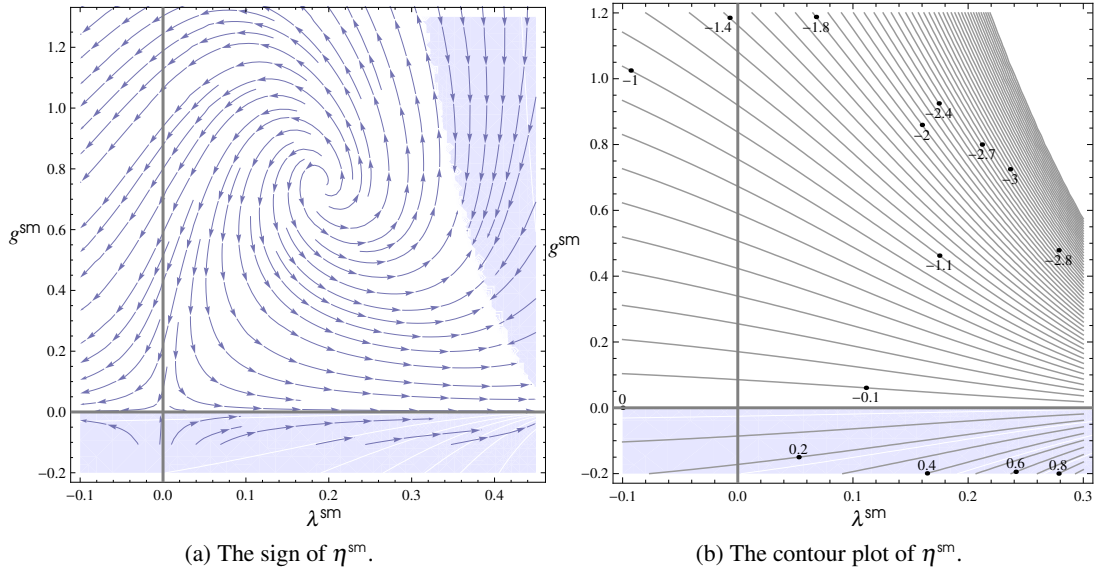


Figure 12.1: The phase-portrait of the single-metric Einstein-Hilbert truncation. The shaded areas in the left diagram indicate regions in the $(g^{\text{sm}}, \lambda^{\text{sm}})$ -plane of positive anomalous dimension η^{sm} . In the single-metric approximation, η^{sm} is seen to be negative everywhere on the physically accessible part of theory space. The contour plot of the right diagram shows the lines of constant η^{sm} values (‘iso- η ’ lines).

It is instructive to expand the function B_1^{sm} for small values of the (dimensionless) cosmological constant:

$$B_1^{\text{sm}}(\lambda^{\text{sm}}) = \frac{1}{3\pi} \left[\Phi_1^1(0) - 24\Phi_2^2(0) \right] - \frac{26}{3\pi} \lambda^{\text{sm}} + \mathcal{O}((\lambda^{\text{sm}})^2) \quad (12.19)$$

This linear approximation confirms the negative values of B_1^{sm} in the semi-classical regime: Its λ^{sm} -independent term $B_1^{\text{sm}}(0)$ is known to be negative for any admissible cutoff [122], and the term linear in the cosmological constant is negative, too, when $\lambda^{\text{sm}} > 0$.

Notice that the slope of the linear function (12.19) is *universal*, i.e. cutoff scheme independent. Every choice of the shape function $R_k^{(0)}$ used in the threshold functions Φ and $\tilde{\Phi}$ yields the same slope, $-26/3\pi$, which is negative and thus favors an anomalous dimension which is negative, too. The constant term in (12.19) is cutoff scheme dependent, however its negative sign is not. Hence, starting from $B_1^{\text{sm}}(0) < 0$, the function $B_1^{\text{sm}}(\lambda^{\text{sm}})$, and therefore also $\eta^{\text{sm}}(g^{\text{sm}}, \lambda^{\text{sm}})$, decreases with increasing values of λ^{sm} , and in fact stays negative throughout the relevant part of theory space ($\lambda^{\text{sm}} < 1/2$).

12.2.2 The (TT-based) bi-metric calculation (I)

Turning to truncations of bi-metric type now, let us consider the approach followed in [I] first. In the dynamical sector the dependence of the corresponding anomalous dimension

$$\eta^{\text{D}}(g^{\text{D}}, \lambda^{\text{D}}) = \frac{B_1^{\text{D}}(\lambda^{\text{D}}) g^{\text{D}}}{1 - B_2^{\text{D}}(\lambda^{\text{D}}) g^{\text{D}}} \quad (12.20)$$

on the cosmological constant λ^{D} is described by the numerator function,

$$B_1^{\text{D}}(\lambda^{\text{D}}) = \frac{1}{6\pi} \left\{ 25\Phi_2^2(-2\lambda^{\text{D}}) + \Phi_2^2\left(-\frac{4}{3}\lambda^{\text{D}}\right) - 80\Phi_3^3(-2\lambda^{\text{D}}) \right. \\ \left. - 3\Phi_2^2(0) + 28\Phi_3^3(0) + 72\Phi_4^4(0) \right\} \quad (12.21)$$

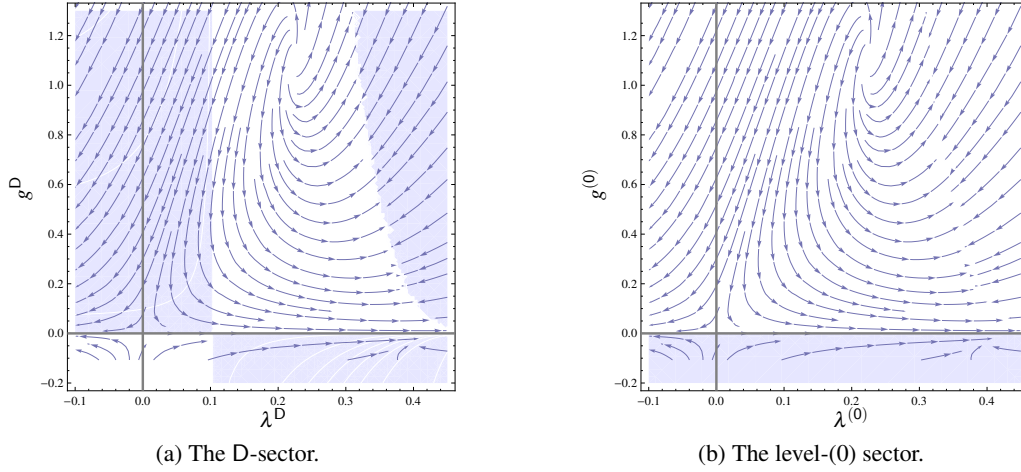


Figure 12.2: The phase portraits on the (g^D, λ^D) - and the $(g^{(0)}, \lambda^{(0)})$ -plane, respectively, obtained using the bi-metric results of [I]. The shaded (white) areas are regions of positive (negative) anomalous dimension η^D and $\eta^{(0)}$, respectively. While the dynamical anomalous dimension η^D exhibits regions of physical interest where it is positive, $\eta^{(0)}$ does not.

while the denominator contribution in eq. (12.20) contains

$$B_2^D(\lambda^D) = -\frac{1}{12\pi} \left\{ 25 \tilde{\Phi}_2^2(-2\lambda^D) + \tilde{\Phi}_2^2\left(-\frac{4}{3}\lambda^D\right) - 80 \tilde{\Phi}_3^3(-2\lambda^D) \right\} \quad (12.22)$$

The beta-functions of the level-(0) and the background-sector are sensitive to the dynamical couplings as well. In particular the sign of the anomalous dimension $\eta^{(0)}$, pertaining to the *level-(0)* Newton constant $g^{(0)}$, is strongly dependent on the *dynamical* cosmological constant, λ^D . Explicitly,

$$\eta^{(0)}(g^D, \lambda^D, g^{(0)}) = \frac{1}{12\pi} \left[10 q_1^1(-2\lambda^D) + 2 q_1^1\left(-\frac{4}{3}\lambda^D\right) - 40 q_2^2(-2\lambda^D) - 8 \Phi_1^1(0) - 13 \Phi_2^2(0) \right] g^{(0)} \quad (12.23)$$

where $q_n^p(w) \equiv \Phi_n^p(w) - \frac{\eta^D}{2} \tilde{\Phi}_n^p(w)$ involves $\eta^D \equiv \eta^D(g^D, \lambda^D)$.

In Figs. 12.2a and 12.2b we display the (g^D, λ^D) and $(g^{(0)}, \lambda^{(0)})$ phase portraits for the dynamical and the level-(0) couplings, respectively. For the latter, the overall picture is essentially the same as for the single-metric truncations: The anomalous dimension $\eta^{(0)}$ is negative everywhere on theory space (where $g^{(0)} > 0$), in particular in the semi-classical regime. However, the dynamical (g^D, λ^D) -flow reveals a novel aspect of the bi-metric truncation: the anomalous dimension η^D is *positive* for λ^D smaller than a certain critical value $\lambda_{\text{crit}}^D > 0$, and turns negative only when $\lambda^D > \lambda_{\text{crit}}^D$.

While this conclusion is drawn on the basis of the complete formula (12.15) including the denominator, the sign of η^D coincides with the sign of B_1^D since $B_2^D g^D \ll 1$ in the entire region of interest, so that $\eta^D \approx B_1^D(\lambda^D) g^D$ is a good approximation. As a consequence, the domains where $\eta^D > 0$ and $\eta^D < 0$, respectively, are separated by a straight line on the (g^D, λ^D) -plane located at $\lambda^D = \lambda_{\text{crit}}^D$ with $B_1^D(\lambda_{\text{crit}}^D) = 0$.

The sign flip of B_1^D can be demonstrated analytically by expanding B_1^D in powers of the cosmological constant:

$$B_1^D(\lambda^D) = \frac{1}{6\pi} \left[23 \Phi_2^2(0) + 72 \Phi_4^4(0) - 72 \Phi_3^3(0) \right] - \frac{43}{9\pi} \lambda^D + \mathcal{O}((\lambda^D)^2) \quad (12.24)$$

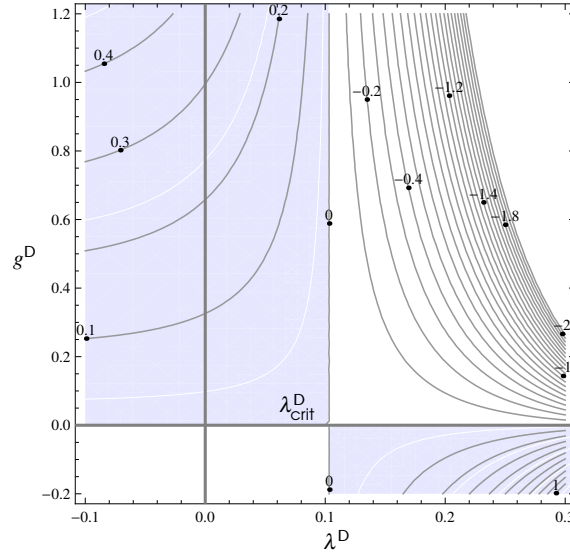


Figure 12.3: The contour plot shows the lines of constant anomalous dimension η^D implied by the bi-metric calculation [I]. The shaded regions correspond to $\eta^D > 0$. Note the sign flip of η^D at a non-zero critical value of the cosmological constant, $\lambda_{\text{crit}}^D > 0$.

Again, the slope of this linear function is found to be both universal and negative, $-43/9\pi$ in this case. The difference in comparison with the single-metric truncation lies in the constant term $B_1^D(0)$: according to the bi-metric calculation it is *positive* for all plausible cutoff schemes, in sharp contradistinction to $B_1^{\text{sm}}(0) < 0$ in the single-metric case.

For the example of the optimized shape function we have $B_1^D(0) = +35/36\pi$, yielding the critical cosmological constant $\lambda_{\text{crit}}^D|_{\text{opt.cutoff}} \approx 0.2$. The quadratic terms $\propto (\lambda^D)^2$ in B_1^D correct this result to about $\lambda_{\text{crit}}^D|_{\text{opt.cutoff}} \approx 0.1$ which is then stable under the addition of still higher orders and coincides with the exact value.

So we may conclude that *the dynamical anomalous dimension η^D is positive in the semi-classical regime where $0 < g^D, \lambda^D \ll 1$ and becomes negative for $\lambda^D > \lambda_{\text{crit}}^D \approx 0.1$* . This is also seen in Fig. 12.3 which shows the lines of constant η^D values on the (g^D, λ^D) plane. Recall that the NGFP, for instance, is located on the curve with $\eta^D = -2$.

To verify that the novel feature of a positive η^D in the semi-classical regime is independent of the cutoff-scheme chosen, we have checked the corresponding condition $B_1^D(0) > 0$ for the one-parameter family of exponential shape functions $R_k^{(0)}(y) = sy [\exp(sy) - 1]^{-1}$, for example [123, 135, 191, 277]. Its threshold functions at vanishing argument can be evaluated exactly:

$$B_1^D(0) = \frac{1}{6\pi(s-1)^3} \left[-(7s-3)(s-1) + (9 + s(14s-19)) \ln(s) \right] \quad (12.25)$$

This result for the s -dependence, plotted in Fig. 12.4, is indeed reassuring: even though the value of $B_1^D(0)$ decreases for increasing ‘shape parameter’ s , it stays always positive. This yields a critical value λ_{crit}^D which is positive, too. Thus, the bi-metric calculation [I], very robustly, predicts a semi-classical regime with a positive value of the dynamical anomalous dimension η^D .

12.2.3 The (‘ Ω -deformed’) bi-metric calculation (II)

A different bi-metric approach that is more closely related to the single-metric computation in [122] was developed in [II] recently. While it employs the same truncation ansatz, namely two separate Einstein-Hilbert actions for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, the gauge fixing and the field parametrization

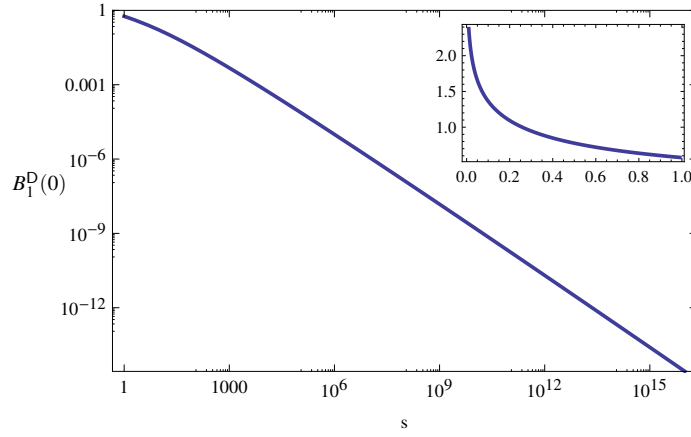


Figure 12.4: The constant $B_1^D(0)$ as obtained in [I] for different values of the shape parameter s characterizing the exponential shape functions. While cutoff scheme dependent, its sign is always positive, implying the existence of a critical cosmological constant $\lambda_{\text{crit}}^D > 0$ such that $\eta^D > 0$ for $\lambda^D < \lambda_{\text{crit}}^D$.

chosen are different from the calculation [I]. In order to explore whether the novel properties displayed by [I] are actually due to its bi-metric character, and to what extent gauge fixing and field parametrization issues play a role possibly, we shall now repeat the analysis of the previous subsection, this time using the beta-functions obtained in [II].

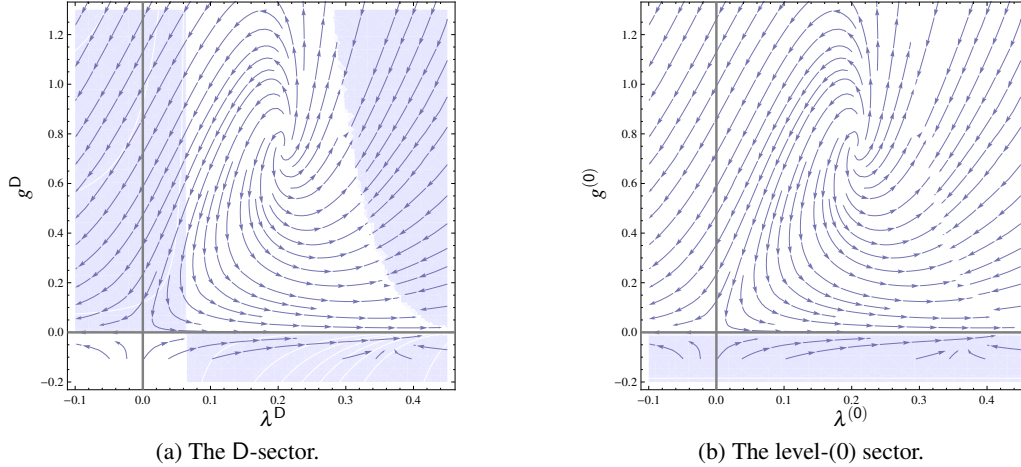


Figure 12.5: The phase portraits for the dynamical and the level-(0) sector according to the beta-functions of [II]. As can be seen from the shading the dynamical RG flow exhibits a transition from negative to positive anomalous dimensions at $\lambda_{\text{crit}}^D > 0$, while its level-(0) counterpart is negative everywhere in the physically relevant part of theory space.

The property at stake is the λ^D -dependence of η^D . It has the structure (12.20) again, but with seemingly rather different functions in the numerator and denominator:

$$B_1^D(\lambda^D) = \frac{1}{\pi} \left\{ \frac{23}{3} \Phi_2^2(-2\lambda^D) - 24 \Phi_3^3(-2\lambda^D) - \frac{2}{3} \Phi_2^2(0) + 8 \Phi_3^3(0) \right\} \quad (12.26a)$$

$$B_2^D(\lambda^D) = -\frac{1}{2\pi} \left\{ \frac{23}{3} \tilde{\Phi}_2^2(-2\lambda^D) - 24 \tilde{\Phi}_3^3(-2\lambda^D) \right\} \quad (12.26b)$$

The level-(0) sector is governed by the following anomalous dimension $\eta^{(0)}$:

$$\eta^{(0)}(g^D, \lambda^D, g^{(0)}) = \frac{2}{\pi} \left[\frac{5}{6} q_1^1(-2\lambda^D) - 3 q_2^2(-2\lambda^D) - \frac{2}{3} \Phi_1^1(0) - \Phi_2^2(0) \right] g^{(0)} \quad (12.27)$$

The resulting phase-portraits for the dynamical and level-(0) sectors are depicted in Fig. 12.5. Only in case of the dynamical anomalous dimension η^D do the shaded areas, indicating regions of positive anomalous dimension, appear in the physically relevant part of the phase diagram. For the level-(0) sector we obtain a negative value of $\eta^{(0)}$ everywhere. The contour plot over the (g^D, λ^D) plane showing the lines of constant η^D is displayed in Fig. 12.6.

Comparing the diagrams in Figs. 12.5 and 12.6 to their analogs of the calculation [I], in Figs. 12.2 and 12.3, we find perfect agreement at the qualitative level between the two bi-metric approaches [I] and [II], respectively. However, the results differ significantly from their single-metric counterparts in Fig. 12.1.

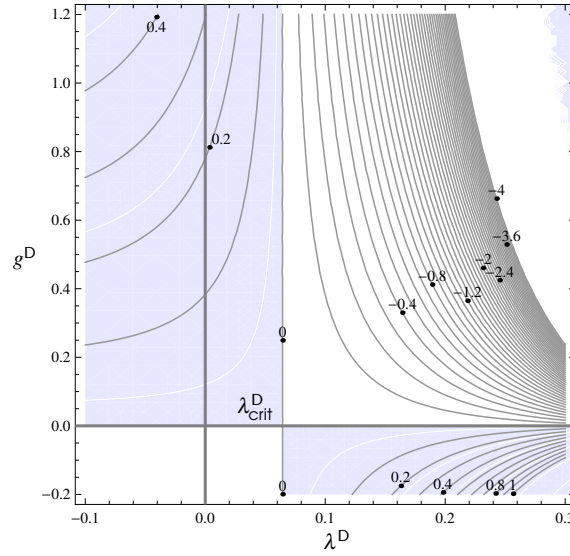


Figure 12.6: The contour plot shows the lines of constant anomalous dimension η^D implied by the bi-metric calculation [II]. The shaded regions correspond to $\eta^D > 0$. Consistent with the calculation in [I], η^D is found to change its sign on a straight line $\lambda^D = \lambda_{\text{crit}}^D$ with $\lambda_{\text{crit}}^D > 0$. In the semi-classical regime, η^D is seen to be positive.

As the semi-classical regime is of special importance let us expand $B_1^D(\lambda^D)$ in the vicinity of $\lambda^D = 0$ again:

$$B_1^D(\lambda^D) = \frac{1}{\pi} \left[7\Phi_2^2(0) - 16\Phi_3^3(0) \right] - \frac{26}{3\pi} \lambda^D + \mathcal{O}((\lambda^D)^2) \quad (12.28)$$

Eq. (12.28) confirms the picture implied by the previous bi-metric calculation [I]: a universal, negative slope (which in this case happens to coincide with the single-metric value, $-26/3\pi$) along with a universally *positive* constant term, $B_1^D(0)$. Together they give rise to a region in which $\eta^D > 0$.

The cutoff-scheme dependence of $B_1^D(0)$ was again checked by evaluating $B_1^D(0)$ for the optimized and the s -family of exponential shape-functions, for instance. The optimized cutoff yields $B_1^D(0) = 5/6\pi$. The linear approximation (12.28) corresponds to the critical value $\lambda_{\text{crit}}^D \approx 0.096$ in this case; it coincides almost perfectly with the corresponding exact value from the full non-linear equation: $\lambda_{\text{crit}}^D \approx 0.064$. For the exponential shape-functions the constant term in

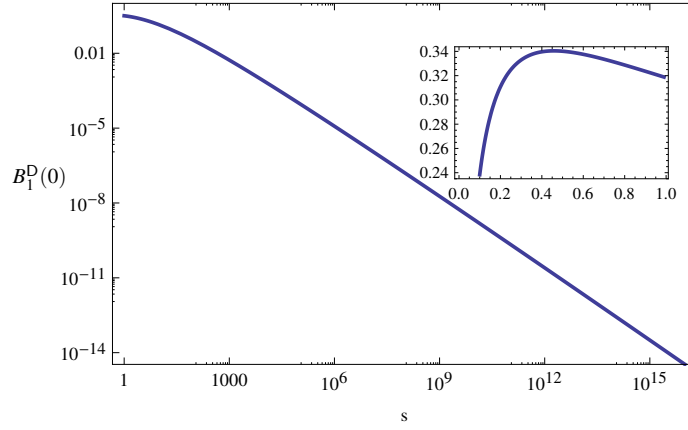


Figure 12.7: The value of $B_1^D(0)$ according to the bi-metric calculation [II] is shown for different choices of the family parameter s characterizing the exponential cutoff shape functions.

(12.28) evaluates to $B_1^D(0) = \frac{4-4s+\ln(s)+3s\ln(s)}{\pi(s-1)^2}$ which is positive for all admissible values¹⁰ of s , as shown in Fig. 12.7.

Thus, the second set of bi-metric results fully confirms all conclusions drawn in the previous subsection on the basis of the RG equations obtained in [I].

12.2.4 Summary: significance of the cosmological constant

We investigated the possibility of a positive anomalous dimension (η^D or η^{sm}) in the semi-classical regime of three different truncations. In section 12.1 we discussed already that while at negative η^I near the NGFP is the very hallmark of Asymptotic Safety, there is no general reason that would forbid η^I to be positive in other parts of theory space, the semi-classical regime in particular. While a transition to a positive η^I was not observed in any single-metric truncation, we found that both bi-metric calculations which we analyzed do indeed show that η^D is actually positive on a large portion of theory space, namely the half plane $-\infty < \lambda^D < \lambda_{crit}^D$. Here λ_{crit}^D is a strictly positive critical cosmological constant, necessarily smaller than the NGFP coordinate λ_* .

The region in theory space with a negative η^D , which is indispensable for a non-Gaussian fixed point and the non-perturbative renormalizability of QEG, crucially owes its existence to the *negative, universal slope* of $B_1^I(\lambda)$ at $\lambda = 0$. It occurs in all three truncations, including the single-metric one, and indicates an *anti-screening* component in the beta-function of g^D . In the ‘sm’ case the intercept $B_1^{sm}(0)$ is negative as well, and so $B_1^{sm}(\lambda)$ is negative for all λ . In both bi-metric truncations $B_1^D(0)$ is positive, however, and this gives rise to a window $\lambda^D \in (-\infty, \lambda_{crit}^D)$ with a certain $\lambda_{crit}^D > 0$ in which $B_1^D(\lambda)$ is positive.

In the semi-classical regime, the linear (in g^D) relationship $\eta^D \approx B_1^D(\lambda)g^D$ always turned out to be an excellent approximation. Hence, for a positive Newton constant (which we always assume) the anomalous dimension is positive in the window $\lambda^D \in (-\infty, \lambda_{crit}^D)$. The precise value of λ_{crit}^D depends on the cutoff shape function; generically it is of the order 10^{-1} or 10^{-2} , say.

The main message is summarized in Fig. 12.8 which depicts the exact (i.e., all-order) λ -dependence of B_1^I . The single- and bi-metric functions all decrease with increasing λ . But while the ‘sm’ function B_1^{sm} is negative everywhere, both of the dynamical bi-metric functions are non-negative in the vicinity of $\lambda = 0$, implying a positive dynamical anomalous dimension

¹⁰It is known that for quantitative reliability the shape parameter $s \in (0, \infty)$ should not be chosen too small, $s \gtrsim 0.5$, say [123, 135, 191].

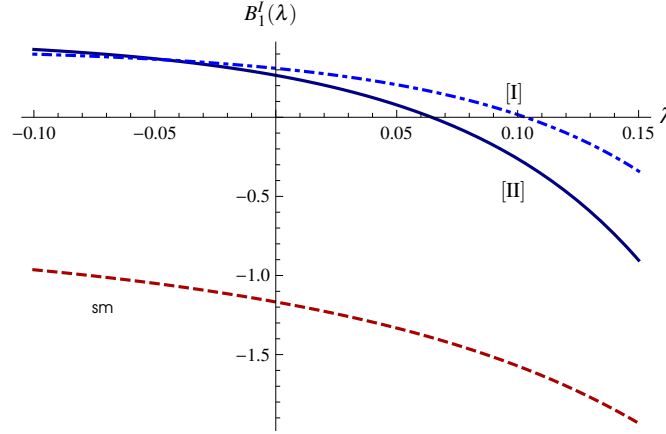


Figure 12.8: The λ -dependence of $B_1^I(\lambda)$ for the two bi-metric truncations [I] and [II], as well as the single-metric approximation (sm). Notice that the latter has not only a negative slope but also a negative intercept $B_1^{sm}(0) < 0$, while both bi-metric functions are positive in the semi-classical regime of not too large dimensionless cosmological constant.

there: $\eta^D(g^D, \lambda^D) > 0$ for all $g^D > 0$ and $-\infty < \lambda^D < \lambda_{crit}^D$.

12.2.5 From anti-screening to screening and back

Recalling the definition $\eta^D \equiv k \partial_k \ln G_k^D$, it follows from the above that along every RG trajectory running on the half space $\lambda^D < \lambda_{crit}^D$ the dynamical Newton constant G_k^D *increases* with increasing scale k . Stated differently, the gravitational interaction shows a *screening* behavior there. This is in stark contrast to its anti-screening character in the NGFP regime.

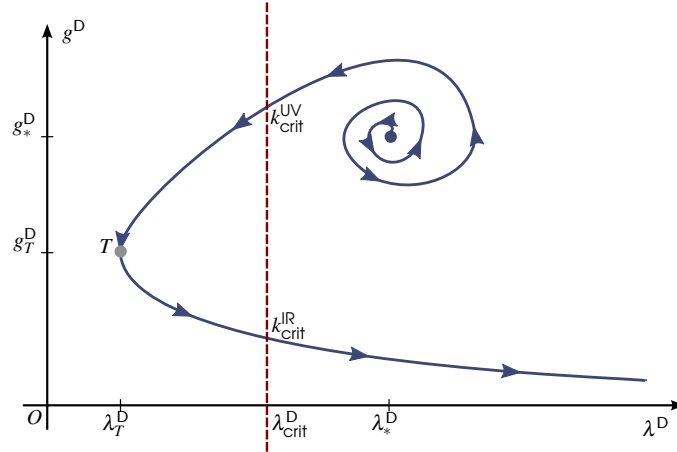


Figure 12.9: Schematic behavior of a bi-metric type IIIa trajectory on the (g^D, λ^D) -projection of theory space. The dashed line separates the half spaces with $\eta^D > 0$ and $\eta^D < 0$, respectively. The part of the trajectory located above (below) the turning point T is referred to as the trajectory's UV (IR) branch.

For the example of a bi-metric trajectory which is of type IIIa in the ‘D’ projection [173] the situation is depicted schematically in Fig. 12.9. The trajectory $k \mapsto (g_k^D, \lambda_k^D)$ emanates from the NGFP at ‘ $k = \infty$ ’, then leaves the asymptotic scaling regime for $k \approx m_{\text{planck}}$, but stays in the half-space with $\eta^D > 0$ as long as k is larger than a certain critical scale k_{crit}^{UV} at which the running cosmological constant λ_k^D drops below λ_{crit}^D . As k decreases further below k_{crit}^{UV} , the cosmological constant continues to decrease until the turning point T is reached, beyond which

the (dimensionless!) λ_k^D now increases for decreasing k . Ultimately, it will re-enter the half-space with $\eta^D < 0$, namely at a second critical scale, $k_{\text{crit}}^{\text{IR}}$. So, by definition,

$$\lambda_k^D|_{k=k_{\text{crit}}^{\text{IR}}} = \lambda_{\text{crit}}^D = \lambda_k^D|_{k=k_{\text{crit}}^{\text{UV}}} \quad \text{with } k_{\text{crit}}^{\text{IR}} < k_{\text{crit}}^{\text{UV}}. \quad (12.29)$$

As it is already well-known for the type IIIa trajectories in the single-metric truncation [191, 209, 212], the bi-metric trajectories of this type, too, can have a long classical regime where the (dimensionful!) Newton- and cosmological constant are approximately constant. This requires tuning the turning point T very close to the Gaußian fixed point, the origin $(0,0)$ in Fig. 12.9. The point T is passed at $k = k_T$ with $k_{\text{crit}}^{\text{IR}} \ll k_T \ll k_{\text{crit}}^{\text{UV}}$ where the two critical scales are far apart then.

For example, the ‘RG trajectory realized in Nature’, that is, the specific single-metric $(g_k^{\text{sm}}, \lambda_k^{\text{sm}})$ - or bi-metric (g_k^D, λ_k^D) -trajectory whose parameters are matched against the measured values of G and $\bar{\lambda}$ [209, 212] is well-known to be highly fine-tuned, with turning point coordinates as tiny as $g_T \approx \lambda_T \approx 10^{-60}$. Following the discussion in [209, 212] it is easy to see that, for this trajectory, and for a λ_{crit}^D value of, say, 10^{-2} , the UV critical scale is about $k_{\text{crit}}^{\text{UV}} \approx m_{\text{Planck}}/10$, while the one in the IR is slightly above the present Hubble parameter, $k_{\text{crit}}^{\text{IR}} \approx 10H_0$. Newton’s constant reaches its maximum at $k = k_{\text{crit}}^{\text{UV}}$; it is about 2% larger there than at laboratory scales.

12.3 Interpretation and Applications

The ‘dynamical’ anomalous dimension η^D governs the running of that particular version of Newton’s constant which controls the strength of the gravitational self-interaction and the coupling of gravity to matter. We found gravitational screening (rather than anti-screening, as predicted by the single-metric truncations) in the semi-classical regime, that is, G_k^D grows with k as long as $\lambda_k^D < \lambda_{\text{crit}}^D$. The strong renormalization effects associated with Asymptotic Safety, the formation of a fixed point, anti-screening, and large negative values of η^D , are confined to the half-space with $\lambda^D > \lambda_{\text{crit}}^D$ instead.

In the following two subsections we discuss a number of possible implications of these findings. In subsection 12.3.1 we interpret the sign change of η^D in terms of a dark matter description, and in subsection 12.3.2 we briefly comment on an application in cosmology.

12.3.1 The dark matter interpretation

Physical significance of the dimensionless cosmological constant

For the interpretation of the above results it is helpful to recall that, upon going on-shell, the value of the *dimensionful* cosmological constant $\bar{\lambda}^D \equiv k^2 \lambda_k^D$ determines the curvature of space-time when it is explored with an experiment, or a ‘microscope’ of resolving power¹¹ $\ell \propto 1/k$. The radius of curvature of spacetime is of the order $r_c \propto (\bar{\lambda}^D)^{-1/2}$ then, and the *dimensionless* cosmological constant is approximately the (squared) ratio of the two distance scales involved:

$$\lambda_k^D \approx \left(\frac{\ell}{r_c} \right)^2 \quad (12.30)$$

Thus we see that *the sign-flip of η^D is controlled by the background curvature*: on self-consistent backgrounds [166] which are only weakly curved on the scale of the microscope, $\ell \ll r_c$, we have $\lambda_k^D \ll 1$, therefore $\eta^D > 0$, and so we observe a screening behavior of the gravitational interaction. Conversely, when the spacetime is strongly curved on the scale of the microscope (i.e. the scale set by the modes just being integrated out at this k) the ratio ℓ/r_c approaches unity, implying $\lambda_k^D > \lambda_{\text{crit}}^D$ and, as a result, strong anti-screening effects.

¹¹This estimate could be made more precise using the method of the ‘cutoff modes’, see refs. [210, 211].

Propagating gravitons in the semi-classical regime

The positive η^D in the semi-classical regime resolves the puzzle raised in the Introduction: On a nearly flat background spacetime the dynamics of the $h_{\mu\nu}$ fluctuations is such that the interactions get weaker at large distance, and the corresponding Green's function is short ranged. The positive η^D causes no conflict with the existence of a Källén-Lehmann representation with a positive spectral density, and the EAA may be seen as describing an effective field theory very similar to those on Minkowski space. It describes weakly interacting gravitons and, in the classical limit, gravitational waves. In the opposite extreme when the curvature is large on the scale set by k there is no description of the $h_{\mu\nu}$ -dynamics in terms of a Minkowski space-like effective field theory. The propagator $\propto 1/(-\bar{D}^2)^{1-\eta^D/2}$ is very different from the one on flat space then, both because of the background curvature and of the large negative η^D which renders it long ranged. In this regime the $h_{\mu\nu}$ -dynamics is anti-screening and results in the formation of a non-trivial RG fixed point.

This general picture points in a similar direction as the mechanism of the ‘paramagnetic dominance’ found in [165] which likewise emphasizes the importance of the background curvature for Asymptotic Safety.

The positive sign of η^D near the Gaußian fixed point is furthermore consistent with the perturbative calculations on a flat background¹² performed by Bjerrum-Bohr, Donoghue, and Holstein [278].

The screening behavior in the semi-classical regime is also consistent with the first analyses of the ‘lines of constant physics’ [279, 280] found by numerical simulations within the CDT approach [98].

Strong curvature regime: ‘physical’, gravitating, and (non-)propagating $h_{\mu\nu}$ modes

An important issue about which we can only speculate at this point is the properties of the metric fluctuations in the regime where $\lambda^D \gtrsim \lambda_{\text{crit}}^D$. There, the field $h_{\mu\nu}$ still carries ‘physical’, in the sense of ‘non-gauge’ excitations which, however, admit no description as ‘particles’ approximately governed by an effective field theory similar to those on Minkowski space. This would not be surprising from an *on-shell* perspective as now the background is curved on a scale comparable to the physics considered. However, it is not completely trivial that the quantum fluctuations driving the RG flow¹³ reflect this transition, too, since those are far off-shell in general.

All we can say about the $h_{\mu\nu}$ quantum field in this regime is that it is likely to carry ‘physical’ excitations which, due to the non-linearity of the theory, interact gravitationally. We do not know the precise propagation properties of those excitations, however. They might, or might not behave like a curved space version of the graviton, as propagating little ripples on a strongly curved background.

What comes to mind here is the analogy to transverse gluons in QCD, at the transition from the asymptotic freedom to the confinement regime. In either regime they are ‘physical’, i.e. ‘non-gauge’ excitations, but only in the former regime they behave similar to propagating particles, while they are confined in the latter.

Also the *unparticles* which were proposed by Georgi in a different context [281, 282] are examples of such perfectly ‘physical’ field excitations which admit no particle interpretation, not even on flat space.

¹²Provided the latter are restricted to the vacuum-polarization diagrams, i.e. those related to η^D .

¹³By contributing to the functional trace on the RHS of the FRGE.

The $h_{\mu\nu}$ propagator by RG improvement

The physics of the $h_{\mu\nu}$ excitations in the strong curvature regime could be explored by computing their n -point functions $\delta^n \Gamma_0[h; \bar{g}] / \delta h^n|_{h=0}$ from the standard effective action $\Gamma_0 = \lim_{k \rightarrow 0} \Gamma_k$ on a self-consistent, in general curved background $\bar{g} \equiv \bar{g}^{\text{s.c.}}$. Particularly important is the inverse propagator $\mathcal{G}^{-1} \propto \delta^2 \Gamma_0[h; \bar{g}^{\text{s.c.}}] / \delta h^2|_{h=0}$. It describes the properties of both the ‘radiative’ modes carried by $h_{\mu\nu}$, and the ‘Coulombic’ modes. The latter determine in particular the response of the $h_{\mu\nu}$ field to an externally prescribed (static) source $\mathcal{T}_{\mu\nu}$, the source-field relationship having the symbolic structure $\mathcal{G}^{-1} h = \mathcal{T}$.

The calculation of $\mathcal{G}$ is a very hard problem, not only because of the much more general truncation ansatz it requires, but also because we do not yet know any realistic candidate for a consistent background $\bar{g}_k^{\text{s.c.}}$ in the domain of interest [283, 284]. Clearly a technically simple background like $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ is excluded here since a flat background is far from consistent when λ^D is large.

Despite these difficulties we can try to get a rough first impression of this domain if we restrict our attention to the $h_{\mu\nu}$ propagator in a regime of covariant momenta in which $\eta^D \equiv \eta$ is approximately k -independent. Then, by a standard argument [285], RG improvement of the 2-point function suggests that the inverse propagator in Γ_0 equals $\mathcal{G}^{-1} \propto (-\bar{D}^2)^{1-\eta/2}$. In general this is a complicated operator with a non-local integral kernel. Let us consider the corresponding source-field relation,

$$L^{-\eta} (-\bar{D}^2)^{1-\eta/2} \phi = -4\pi G \rho \quad (12.31)$$

with a now scale-independent Newton constant G , and a length parameter L included for dimensional reasons. Here we suppress the tensor structure and employ a notation reminiscent of the Newtonian limit which we shall take later on only; the following argument is fully relativistic still.

Non-locality mimics dark matter

For a generic real, i.e. non-integer value of η the LHS of eq. (12.31) involves a highly non-local operator acting on ϕ . In order to understand how the solutions of this equation differ from the classical ones, let us act with the operator $(-L^2 \bar{D}^2)^{\eta/2}$ on both sides of (12.31). Leaving domain issues aside this yields an equation similar to (12.31), but now with $\eta = 0$ and a modified source instead:

$$\bar{D}^2 \phi = 4\pi G \tilde{\rho} \quad (12.32a)$$

$$\tilde{\rho} \equiv (-L^2 \bar{D}^2)^{\eta/2} \rho \quad (12.32b)$$

We see that the modifications caused by a non-zero anomalous dimension can be shifted from the differential operator acting on the gravitational field to the source function. In the Newtonian limit, for instance, eq. (12.32a) has the interpretation of the classical Poisson equation for the gravitational potential ϕ generated by the mass density $\tilde{\rho}$. However, the density function $\tilde{\rho}$ does not coincide with the mass distribution that has actually been externally prescribed, namely ρ . The RG effects are encoded in the way the ‘bare’ mass distribution ρ gets ‘dressed’ by quantum effects which turn it into the ‘renormalized’ $\tilde{\rho}$.

Being more explicit, the operator application in (12.32b) amounts to the convolution of ρ

with a non-local integral kernel:¹⁴

$$\tilde{\rho}(x) = \int d^d x' \sqrt{\bar{g}(x')} K_\eta(x, x') \rho(x') \quad (12.33a)$$

$$K_\eta(x, x') \equiv \langle x | (-L^2 \bar{D}^2)^{\eta/2} | x' \rangle \quad (12.33b)$$

Note that the kernel K_η , and therefore $\tilde{\rho}$, still depend on the background $\bar{g}_{\mu\nu}$.

While in general x and x' are 4-dimensional coordinates they reduce to 3D space coordinates if we invoke the Newtonian limit where ρ , $\tilde{\rho}$, and ϕ are time independent. In fact, to gain a rough, but qualitatively correct intuition for the ‘dressing’ $\rho \mapsto \tilde{\rho}$, it suffices to consider the Newtonian limit, an approximately flat background in particular, but to maintain a non-zero value of η . Then, with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, eq. (12.32a) boils down to the time independent Poisson equation $\nabla^2 \phi = 4\pi G \tilde{\rho}$, and the kernel $K_\eta(\vec{x}, \vec{x}') \equiv K_\eta(|\vec{x} - \vec{x}'|)$ is easily evaluated in the plane wave eigenbasis of the Laplacian on flat space, ∇^2 :

$$K_\eta(r) = \int \frac{d^3 p}{(2\pi)^3} (L^2 \vec{p}^2)^{\eta/2} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')}, \quad r \equiv |\vec{x} - \vec{x}'| \quad (12.34)$$

Focusing on the simplest case, $\eta \in [-2, -1]$, this integral yields¹⁵, at $r \neq 0$,

$$K_\eta(r) = -[4\pi\Gamma(-1-\eta)\cos(\frac{\pi}{2}\eta)]^{-1} \frac{L^\eta}{r^{3+\eta}} \quad (12.35)$$

Now, even if the ‘bare’ $\rho(\vec{x})$ is due to a point mass, for example, $\rho(\vec{x}) = M\delta(\vec{x})$, the ‘renormalized’ or ‘dressed’ mass distribution amounts to an extended, smeared out cloud with a density profile $\tilde{\rho}(\vec{x}) = MK_\eta(|\vec{x}|)$. If (12.35) applies, $\tilde{\rho}$ has support also away from $\vec{x} = 0$, falling off according to the power law

$$\tilde{\rho}(r) \propto 1/r^{3+\eta} \quad (r > 0) \quad (12.36)$$

If η is negative, the $\tilde{\rho}$ distribution is the more extended the larger is $|\eta|$.

While strictly speaking (12.36) is valid only for $\eta \in [-2, -1]$, it highlights the main impact a negative η has on gravity, also beyond the Newtonian limit: If one sticks to the classical form of the field equation (here: Poisson’s equation) the gravitational field is sourced not only by the energy momentum tensor of the true matter (here: ρ) but in addition by a fictitious energy-momentum-, and in particular mass-distribution ($\tilde{\rho}$) which is obtained by a non-local integral transformation applied to the true, or ‘bare’, source.

In the simplest case the integral transformation is linear and assumes the form (12.33a). Where it applies, the ‘fictitious’ matter traces the ‘genuine’ one, the latter sources the former. Hence it seems indeed appropriate to regard the transition from ρ to $\tilde{\rho}$ as due to the ‘dressing’ of the bare source by quantum effects, similar to the dressing of electrons in Quantum Electrodynamics (QED) by clouds of virtual particles surrounding them. It is quite clear then, in particular in a massless theory, that the dressing of point sources results in spatially extended, non-local structures.

¹⁴If needed, the non-integer power of $\bar{D}^2$ can be expressed by an appropriate integral representation. For a general discussion of fractional powers of the Laplacian and d’Alembertian and their Green’s functions, see [286, 287].

¹⁵For other values of η we must introduce explicit distance or momentum cutoffs into the integral (12.34) in order to take account of the fact that the approximation $\mathcal{G}^{-1} \propto (\bar{D}^2)^{1-\eta/2}$ with a constant value of η is valid only in a restricted regime. Being interested in qualitative effects only we shall not do this here. One also has to be careful about delta-function singularities at the origin; in particular we have $K_0(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$, as it should be.

Modified gravity in astrophysics: a digression

Applying this discussion to the realm of astrophysics, to galaxies or clusters of galaxies, one is tempted to interpret the fictitious matter contained in $\tilde{\rho}$, over and above the true one, as the long sought-for dark matter, and to identify ρ with the actually observed ‘luminous’ matter.

To avoid any misunderstanding we emphasize that the presently available RG flows do not (yet?) reliably predict large negative anomalous dimension ($\eta \approx -1$, say) on astrophysical scales¹⁶. All we can say for the time being is that the mathematical structure of the field equations we encounter here is *potentially* relevant to the astrophysical dark matter problem, but clearly much more work will be needed to settle the issue.

The much more direct reason why the mechanism of non-local gravity mimicking dark matter is relevant to Asymptotic Safety is that *on a type IIIa trajectory large negative η ’s occur in two regimes*: not only at astrophysical or cosmological scales, $k \lesssim k_{\text{crit}}^{\text{IR}}$, but also near the Planck regime, $k \gtrsim k_{\text{crit}}^{\text{UV}}$.

As it is shown schematically in Fig. 12.9, the trajectories of type IIIa, like the one that could perhaps apply to the real Universe, have two sections with a sufficiently large λ^{D} to make η^{D} negative, one on the UV-, the other on the IR-branch. The main difference between the branches is their typical value of g^{D} : it is much smaller on the IR-branch than on the UV-branch. As a result, on the IR-branch $|\eta^{\text{D}}| = |B_1^{\text{D}}(\lambda^{\text{D}})g^{\text{D}}|$ assumes values of order unity, say, only when λ^{D} is increased much further beyond $\lambda_{\text{crit}}^{\text{D}}$ than this would be necessary on the UV-branch. This distinction is best seen in the contour plots (‘iso- η -lines’) of Figs. 12.3 and 12.6. Since the Einstein-Hilbert truncation becomes unreliable near $\lambda^{\text{D}} = 1/2$, it can deal with the large negative η ’s on the UV-branch only.

After the above precautionary remark it is nevertheless interesting to note that on the astrophysical side an integral transform like (12.33a), connecting luminous to dark matter in real galaxies, has indeed been proposed long ago on a purely phenomenological basis: It is at the heart of the Tohline-Kuhn modified-gravity approach [288–290]. Recently this approach has attracted attention also because it was found to emerge naturally from a certain classical, fully relativistic, and *non-local* extension of General Relativity [291–293].

Above we saw that quantum gravity effects can modify Einstein’s equations in precisely the Tohline-Kuhn style. The similarity between the two theories becomes most explicit for $\eta = -1$ which leads to the integral kernel

$$K_{-1}(\vec{x}, \vec{x}') = \frac{1}{2\pi^2 L} \frac{1}{|\vec{x} - \vec{x}'|^2} \quad (12.37)$$

This is exactly the one which appears also in the Tohline-Kuhn framework.

Using this kernel in eq. (12.33a), a point mass with $\rho(\vec{x}) = M\delta(\vec{x})$ is seen to surround itself with a spherical ‘dark matter halo’ whose radial density profile is given by $\tilde{\rho}(r) = M/(2\pi^2 L r^2)$. By virtue of $\nabla^2 \phi = 4\pi G \tilde{\rho}$, this dark matter distribution generates the logarithmic potential $\phi(r) = (2GM/\pi L) \ln(r)$. In the Newtonian limit, it is well known to yield a perfectly flat rotation curve, that is, a test particle on a circular orbit has a velocity which is independent of its radius¹⁷, $v^2 = 2GM/\pi L$.

Non-local constitutive relations as a QEG vacuum effect

Recently the Tohline-Kuhn framework turned out to describe the Newtonian limit of a fully relativistic generalization of General Relativity which allows the incorporation of non-locality

¹⁶See, however, ref. [212].

¹⁷From the idealized case of a point particle where $v^2 \propto M$ it might appear that this approach has difficulties reproducing the Tully-Fisher law for spiral galaxies [294] which favors $v^4 \propto M$. However, the detailed studies on the basis of fits to realistic galaxy data reported in [295, 296] seem not to encounter such difficulties.

at a phenomenological, purely classical level [291–293, 297]. This theory, proposed by Hehl and Mashhoon, relies on the observation that the teleparallel equivalent of General Relativity, a special gauge theory of the translation group, is amenable to generalization through the introduction of a non-trivial ‘constitutive relation’ similar to the constitutive relations between $(\vec{E}, \vec{B})$ and $(\vec{D}, \vec{H})$ in electrodynamics.

Because of memory effects, such relations are non-local typically. They make their appearance both in the classical electrodynamics of matter, and in vacuum Quantum Electrodynamics where loop effects are well known to give rise to a complicated relationship between $\vec{E}$ and $\vec{D}$, say, which is both non-linear and non-local [298]. As for quantum gravity, it was pointed out [165, 195] that QEG, like QED, has a non-trivial vacuum structure with a *non-linear* relationship between the gravitational analogs of the $\vec{E}$ and $\vec{D}$ fields. From this perspective it is quite natural that the source-field relation of quantum gravity, in a regime with large negative η , turns out not only non-linear, but also *non-local*.

In this sense, a phenomenological theory like the one in [291–293], as far as its general structure is concerned, may well be regarded as an effective field theory description of the QEG vacuum in the large- η regime.

In fact, in QEG and the theory of ref. [291–293] the size of the new effects is determined by essentially the same control parameter. In [291–293] the degree of non-locality is governed by the ratio $\rho \equiv L_{\text{acc}}/L_{\text{phen}}$, where L_{phen} denotes the length scale of the phenomenon under consideration, and L_{acc} is the acceleration length of the observer. Interestingly, $\rho^{-2} \equiv (L_{\text{phen}}/L_{\text{acc}})^2$ is basically the same as the dimensionless cosmological constant $\lambda_k = \bar{\lambda}_k/k^2$ which controls the size of η and the non-local effects in QEG. There, $\ell \approx k^{-1}$ characterizes the length scale of the physical process under consideration and so it takes the place of L_{phen} , while the radius of curvature, r_c , may be identified with L_{acc} ¹⁸.

Planck scale non-locality as ‘dark matter’

At this point of the discussion we switch back from the IR to the UV regime. As we emphasized already the beta-functions considered in the present chapter, where they are reliable, yield only tiny values for η on astrophysical scales. So here we focus on the dark matter interpretation which applies to the UV *branch* of the ‘RG trajectory realized in Nature’, see Fig 12.9. Of course, the UV-branch exists not only for the trajectories of type IIIa but for all asymptotically safe ones. Along any of them, for k near the Planck scale, but still above $k_{\text{crit}}^{\text{UV}}$, the anomalous dimension is large and negative since the trajectory just left the NGFP regime where $\eta^{\text{D}} \approx \eta_*^{\text{D}} = -2$.

It is thus plausible to re-apply the above discussion of astrophysical dark matter which is mimicked by non-locality in the ultraviolet. The situation would then be as follows. When we approach the UV regime, above a certain scale $k_{\text{crit}}^{\text{UV}}$ located about one or two orders of magnitude below the Planck scale, non-local effects start becoming essential. Now, the regime in question, $k_{\text{crit}}^{\text{UV}} \lesssim k \lesssim m_{\text{Planck}}$, is exactly the one for which we concluded already that the $h_{\mu\nu}$ excitations cannot be described there by an effective field theory of the conventional local form; in particular their propagation properties are not easily established, and we conjectured that there are indeed no propagating gravitons above $k_{\text{crit}}^{\text{UV}}$.

Assuming this picture is correct it suggests the interpretation of the physical, but non-propagating $h_{\mu\nu}$ modes as a type of Planckian dark matter that admits an effective description in terms of a (fully relativistic!) Hehl-Mashhoon-type theory [291–293]. In this scenario the modes of the metric fluctuations with covariant momenta above $k_{\text{crit}}^{\text{UV}}$ do not propagate, but are still physical (in the sense of ‘non-gauge’). They interact gravitationally with matter and among themselves, they can condense to form spatially extended structures, and they dress ordinary

¹⁸In cosmology, for instance, one has indeed $L_{\text{acc}} \sim H^{-1} \sim \bar{\lambda}^{-1/2} \sim r_c$.

localized energy-momentum distributions by ‘dark matter halos’ which are approximately described by a Tohline-Kuhn-type integral transform.

This reflects an antagonism between gravitons and dark matter: The semi-classical modes of the fluctuation field have a particle interpretation, describe massless gravitons or essentially classical gravitational waves, while those with larger momenta are equally physical, gravitate, but do not propagate presumably.

To visualize this situation it helps to recall the example of the transverse gluon modes in QCD: Those with momenta well above the confinement scale propagate approximately particle-like, the others are confined, and they form the homogeneous gluon condensate characteristic of the QCD vacuum state.

12.3.2 Primordial density perturbations from the NGFP regime

The conjectured absence of propagating gravitons in a certain range of momenta can also be relevant to cosmology presumably, for example in the context of the Cosmic Microwave Background (CMB). In refs. [189, 190, 209] an Asymptotic Safety-based alternative to the standard inflationary paradigm has been proposed in which the source of the primordial density perturbations, responsible for later structure formation, are the quantum fluctuations of geometry itself which occur during the Planck epoch.¹⁹ Within QEG the fluctuations in this regime are governed by the NGFP, and so they could provide a perfect window to the very physics of Asymptotic Safety.

It has been argued that when the Universe was in the Planck-, or NGFP-regime the scale-free form of the $h_{\mu\nu}$ -propagator $\propto 1/D^4$ gave rise to a kind of cosmic ‘critical phenomenon’ which displays metric fluctuations on all length scales [189, 190, 209, 216]. The scale-free nature of all physics at the fixed point renders the fluctuation spectrum scale-free automatically. Towards the end of the Planck era, the RG trajectory leaves the asymptotic scaling regime of the NGFP, the fluctuations ‘freeze out’, and thus prepare the initial state for the subsequent classical evolution. They lead to a Harrison-Zeldovich like CMB spectrum with a spectral index of $n_s = 1$ plus small corrections [189, 190, 209, 216].

Here the absence of propagating gravitational waves at high scales could come into play as follows. At the end of the Planck epoch the geometry fluctuations get imprinted on the (by then essentially classical) spacetime metric and the matter fields. The imprints then evolve classically, and ultimately, at decoupling, get encoded in the CMB. Now, a priori the frozen-in geometry perturbations present at the end of the Planck era ($k \approx m_{\text{Planck}}$) would affect the scalar and the radiative (‘tensor’) parts of the metric alike. If, however, there do not yet exist physical radiative excitations at this scale, or they are suppressed, then one has a natural reason to expect that in real Nature *the CMB tensor-to-scalar-ratio should be smaller than unity*. The power in the tensor modes is suppressed relative to the scalar ones since by the time the Universe leaves the fixed point regime gravitational waves cannot propagate yet, the relevant scales being in the range $m_{\text{Planck}} > k > k_{\text{crit}}^{\text{UV}}$.

12.4 Summary

Since the early investigations of the Einstein-Hilbert truncation it was clear that a subset of its RG trajectories contain a long classical regime at low scales in which G_k and $\bar{\lambda}_k$ are constant to a very good approximation; from these single-metric calculations it appeared, however, that in the adjacent semi-classical regime at slightly larger scales the Newton constant decreases immediately, thus rendering the anomalous dimension $\eta \equiv k \partial_k \ln G_k$ negative. Even though at the endpoint of the separatrix, for example, we have $\bar{\lambda} = 0$ and so the effective field equations

¹⁹For a different approach to asymptotically safe inflation see [299] and [233].

admit Minkowski space as a solution, the quantized metric fluctuations on this background, the gravitons, would have unexpected properties, being more similar to gluons than to photons. However, in this chapter we provided evidence from two independent bi-metric analyses, one based on part II, which indicate that this is actually not the case. Between the strictly classical ($\eta = 0$) and the fixed point regime ($\eta < 0$) there exists an intermediate interval of scales with a positive anomalous dimension. Those RG trajectories which have a positive cosmological constant in the classical domain possess two regimes displaying a negative anomalous dimension, one at Planckian, and the other on cosmological scales. At least in the former the existence of propagating gravitons seems questionable, and we proposed a natural interpretation of the pertinent physical, non-propagating, but gravitating $h_{\mu\nu}$ excitations as a form of Planckian ‘dark matter’.



13.1	Introduction
13.2	Global properties of the boundary sector
	The running UV attractor in the $\partial(0)$ -sector
	The UV behavior of the boundary sector
	The IR behavior of the boundary sector
	The semiclassical regime
13.3	Variational problems
	Effective field equations and stationary points
	Variational principle in presence of a boundary
13.4	Counting field modes
	Split-symmetry and $\partial M = \emptyset$
	Euclidean space with non-vanishing boundary
	Thermodynamics of the Schwarzschild black hole
13.5	Conclusion

13. THE GIBBONS-HAWKING-YORK BOUNDARY TERM

 Section 13.4 is a condensed version of subsection 4.3 in [166]. Parts of section 13.4 and 13.5 can be found in [166].

Previous explorations of the Asymptotic Safety scenario in Quantum Einstein Gravity (QEG) by means of the effective average action and its associated functional Renormalization Group (RG) equation assumed spacetime manifolds which have no boundaries. In ref. [166] we took a first step towards a generalization to non-trivial boundaries, restricting ourselves to action functionals which are at most of second order in the derivatives acting on the metric. The beta-functions considered in [166] were derived in a single-metric calculation as well as for a bi-metric matter induced truncation in which the gravitational RG evolution is generated by a coupled scalar matter sector.

In this chapter we are going to analyze truncated actions with running boundary terms for full fledged QEG within the bi-metric Einstein-Hilbert truncation augmented by a scale dependent Gibbons-Hawking-York surface term of the background metric, see part II of this thesis.

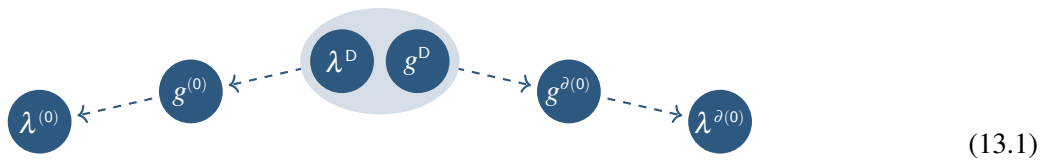
Since in the current setting the RG evolution of the boundary terms is purely induced by the bulk contributions, the single-metric and bi-metric beta-functions associated to the boundary invariants structurally agree, however with g_k^{sm} and λ_k^{sm} replaced by their dynamical counterparts g_k^{D} and λ_k^{D} . It turns out that

We find that the bulk and the boundary Newton constant, pertaining to the Einstein-Hilbert and Gibbons-Hawking-York term, respectively, show opposite RG running; proposing a scale dependent variant of the Arnowitt-Deser-Misner (ADM) mass we argue that the running of both couplings is consistent with gravitational anti-screening. We discuss the status of the ‘bulk-boundary matching’ usually considered necessary for a well defined variational principle within the functional RG framework, and we explain a number of conceptual issues related to the ‘zoo’ of (Newton-type, for instance) coupling constants, for the bulk and the boundary, which result from the bi-metric character of the gravitational average action. In particular we describe a simple device for counting the number of field modes integrated out between the infrared cutoff scale and the ultraviolet. This method makes it manifest that, in an asymptotically safe theory,

there are effectively no field modes integrated out while the RG trajectory stays in the scaling regime of the underlying fixed point. As an application, we investigate how the semiclassical theory of Black Hole Thermodynamics gets modified by quantum gravity effects and compare the new picture to older work on ‘RG-improved black holes’ which incorporated the running of the bulk Newton constant only. We find, for instance, that the black hole’s entropy vanishes and its specific heat capacity turns to zero at Planckian scales, indicating a stabilization of the thermodynamical properties when approaching the ultraviolet.

13.1 Introduction

In this chapter we conclude the analysis of the beta-functions obtained in part II of this thesis, by focusing on the boundary sector of theory space. One of the ideas to study a bi-metric full fledged truncation of Quantum Einstein Gravity (QEG) including surface terms, is the importance of spacetime boundaries in cosmology, in particular the evolution of the Universe [86]. We shall be thus interested in the Renormalization Group (RG) evolution of scale dependent gravitational actions which include surface terms and extend our previous single-metric results by the bi-metric structure of the bulk sector. Concretely we will use the results of part II relying on the effective average action to formulate a diffeomorphism invariant and, most importantly, background independent coarse graining flow on the ‘theory space’ of action functionals for the metric [122]. While in the past this approach was limited to the quantization of gravity on spacetimes M without a boundary¹ the bi-metric-bulk – pure-background-boundary truncation ansatz considered in this thesis comprises a total of six independent basis invariants of which four are defined on the bulk and two on the boundary. From the corresponding ‘running’ coefficients, only the dynamical couplings of the bulk sector, i.e. the D-coefficients, enter the RHS of the FRGE. Since it is the Hessian operator of the EAA that appears under the functional trace on the RHS, all coefficients associated to basis invariants of less than second order in the fluctuation fields do not affect the RG evolution at all. In the level-language in which $\Gamma_k[\varphi; \bar{\Phi}]$ is expanded in powers of φ , this property translates into the vanishing of all level-(0) and level-(1) couplings in the non-canonical part of the RG flow. This has a very practical implication resulting in a differential system that exhibits a hierarchy among the couplings. For the present truncation, the dynamical sector, g^D and λ^D , completely decouples and significantly influences the RG behavior of the remaining four coefficients related to level-(0) bulk and boundary invariants. In detail, we have the following hierarchy structure for the six beta-functions:



For the present investigation another consequence of eq. (13.1) is the mutual independence of the $(g^{(0)}, \lambda^{(0)})$ and $(g^{\partial(0)}, \lambda^{\partial(0)})$. Thus any result derived in either the (0)- or $\partial(0)$ -sector is not affected by the value and properties of the other one. In the previous chapters of this thesis, we exploited this fact to study bi-metric results independently of the boundary sector. While in general, it is very interesting to investigate split-symmetry, Background Independence, and related issues also for the boundary terms, the present truncation is unable to disentangle the $\partial(0)$ -, $\partial(1)$ -, $\partial(2)$ -, ..., -sectors. In future work we want to incorporate this separation extending the present truncation by a dynamical Gibbons-Hawking-York functional. Therefore, in part II of this thesis we already paved the ground for this future project by keeping the

¹ See however [300] for an early perturbative calculation of the induced surface term in $2 + \varepsilon$ dimensional gravity.

discussion and derivation very general. It will be very interesting to compare the results from the previous chapters of the bulk sector with its boundary counterpart.

Our focus in this chapter is thus on the conceptual part of boundary terms within the FRGE, which from the cosmological point of view is a very important generalization of spacetime manifolds. Hence, we now follow the second branch of eq. (13.1) and consider the $D\text{-}\partial(0)$ part of the differential system, comprising the RG evolution of the four couplings g^D , λ^D , $g^{\partial(0)}$, and $\lambda^{\partial(0)}$. For the same reasons the previous sections ignored the boundary couplings, we will now omit the discussion on level-(0) bulk couplings, which is treated extensively in chapter 10.

We start this chapter by investigating the general (global) properties of the RG flow in the boundary sector. We detect a similar running UV attractor for $\partial(0)$ that gives rise to the non-Gaussian fixed point in the ultraviolet which is relevant for the Asymptotic Safety conjecture. For more details and its properties, we refer to chapter 9.

In the next step, we consider the variational problem in the context of the EAA. While the Gibbons-Hawking-York term was initially introduced to recover Einstein's field equations in the presence of spacetime boundaries, from the Functional Renormalization Group (FRG) perspective there is nothing special about this invariant. In particular the classical matching technique is now subject to the RG evolution and will in general be spoiled.

Next, we evaluate the $\mathcal{C}_k$ -function of chapter 11 for the average actions containing boundary contributions. In particular we study the Euclidean Schwarzschild solution and derive its thermodynamical quantities, which now become k -dependent.

Finally, we conclude this chapter by summarizing the results in section 13.5.

13.2 Global properties of the boundary sector

The beta-functions for the D-couplings are very involved, containing non-polynomial dependencies of g^D and λ^D due to their appearance in the Hessian operator under the functional trace. On the other hand, the differential equations for the boundary couplings are quite trivial, once some values for g^D and λ^D are inserted. Any RG solution obeys the following two relations

$$\partial_t g_k^{\partial(0)} = \beta_g^{\partial(0)}(g_k^{\partial(0)}, \lambda_k^{\partial(0)}; k) = 2g_k^{\partial(0)} + B_1^{\partial(0)}(g_k^D, \lambda_k^D)(g_k^{\partial(0)})^2 \quad (13.2a)$$

$$\partial_t \lambda_k^{\partial(0)} = \beta_\lambda^{\partial(0)}(g_k^{\partial(0)}, \lambda_k^{\partial(0)}; k) = -3\lambda_k^{\partial(0)} + A^{\partial(0)}(g_k^D, \lambda_k^D)g_k^{\partial(0)} + B_1^{\partial(0)}(g_k^D, \lambda_k^D)g_k^{\partial(0)}\lambda_k^{\partial(0)} \quad (13.2b)$$

The explicit form of the functions $B_1^{\partial(0)}(k) \equiv B_1^{\partial(0)}(g_k^D, \lambda_k^D)$ and $A^{\partial(0)}(k) \equiv A^{\partial(0)}(g_k^D, \lambda_k^D)$ can be deduced from the beta-functions in chapter 8. For our purpose, it suffices to observe that these functions are independent on the boundary couplings.

The remaining structure of the beta-functions is quite trivial, containing at most quadratic orders in the couplings. Further notice, that in agreement with the hierarchy (13.1), the beta-function of $g^{\partial(0)}$ does not depend on $\lambda^{\partial(0)}$.

In the following we draw some conclusions about the global picture of the boundary sector, very similar to those obtained for the background couplings in chapter 10. This is due to the fact, that except for the explicit form of $A^{\partial(0)}$, $B_1^{\partial(0)}$ or A^B , B_1^B , both sectors satisfy the same class of differential equations. We focus on $d = 4$, even though most of these properties remain valid beyond.

13.2.1 The running UV attractor in the $\partial(0)$ -sector

Once we have chosen a solution $k \mapsto (g_k^D, \lambda_k^D)$ in the dynamical sector, the differential equations for $g_k^{\partial(0)}$ and $\lambda_k^{\partial(0)}$ become non-autonomous, with explicit k -dependences introduced by $B_1^{\partial(0)}(k)$ and $A^{\partial(0)}(k)$. Hence, the invariance of the beta-functions under translations in the RG-time

$t \equiv \ln(k/k_0)$ gets broken and the quantitative and possibly qualitative structure of the phase diagram will change with k .

Nevertheless, considering deformation invariant properties of the RG flow, we will be able to deduce the general picture of the boundary sector, again by means of *k-dependent analogues of fixed points*. To search for these trivial points of the RG flow is actually a very common technique in mathematics to understand the underlying differential system. Thus, let us consider the set of solutions for which the beta-functions vanish:

$$\beta_g^{\partial(0)}(g_{\bullet}^{\partial(0)}(k), \lambda_{\bullet}^{\partial(0)}(k); k) = 0 \quad \text{and} \quad \beta_{\lambda}^{\partial(0)}(g_{\bullet}^{\partial(0)}(k), \lambda_{\bullet}^{\partial(0)}(k); k) = 0 \quad (13.3)$$

Here, we treated the dynamical sector as k -dependent parameters and thus solutions to eq. (13.3) inherit a k -dependence. In quite general terms, if $B_1^{\partial(0)}(k) \neq 0$, the conditions of eq. (13.3) are satisfied by the following two types of solutions

- A. From the structure of the beta-functions it is apparent that the trivial solution, $g_{\bullet}^{\partial(0)} = 0 = \lambda_{\bullet}^{\partial(0)}$, always exists, independent of the dynamical sector and thus of k . This k -stable solution represents a special point on the separating line $g^{\partial(0)} = 0$, which divides the RG flow in the boundary sector into two classes with either positive and negative $g^{\partial(0)}$ for all k .
- B. Furthermore there is a second, this time non-trivial solution which is explicitly k -dependent:²

$$g_{\bullet}^{\partial(0)}(k) = -2/B_1^{\partial(0)}(k) \quad \text{and} \quad \lambda_{\bullet}^{\partial(0)}(k) = -\frac{2}{3}A^{\partial(0)}(k)/B_1^{\partial(0)}(k) \quad (13.4)$$

Since the underlying equations are of quadratic order in the couplings, this solution can be associated to the non-Gaussian fixed point for asymptotically safe RG trajectories in the limit $k \rightarrow \infty$. We denote this non-trivial solution $\text{Attr}^{\partial(0)}(k) \equiv (g_{\bullet}^{\partial(0)}, \lambda_{\bullet}^{\partial(0)})$ as the running UV-attractor of the boundary sector, for reasons which become clear in a moment.

The non-trivial solution $\text{Attr}^{\partial(0)} \equiv \text{Attr}^{\partial(0)}(k)$ is very interesting, for it describes the evolution of the later non-Gaussian fixed point and in addition decisively influences the RG flow of the entire boundary sector. In fig. 13.1 we depict its k -dependent position for a typical IIIa-trajectory of the D-sector corresponding to the class of solutions which are assumed to be realized in Nature. The structure we observe for the UV part of this particular solution extends to all asymptotically safe trajectory without any significant modification, since the D-couplings approach the same coordinate, the fixed point values of NG_+^{D} -FP. In contrast to the bulk couplings, notice that asymptotically safe trajectories evolve in the *negative* half-plane of Newton's coupling of the boundary sector, i.e. $g^{\partial(0)} < 0$. This is reflected by the negative fixed point value $g_{\bullet}^{\partial(0)}$ for $\text{NG}_+^{\partial(0)} \oplus \text{NG}_+^{\text{D}}$ -FP and is also seen in fig. 13.1.

In the following, we comment on the general properties of $\text{Attr}^{\partial(0)}(k)$, valid as long as $B_1^{\partial(0)}(k) \neq 0$.

- A. Since $\text{Attr}^{\partial(0)}$ is not subject to the FRGE and does not correspond to an RG solution, the running attractor can in principle disappear, change sign in its $g^{\partial(0)}$ -coordinate, or violate any other condition which exact solutions of the FRGE have to satisfy.
- B. Fig. 13.2 shows that the considered D-trajectory evolves in a very extended region of theory space where the sign of the $\text{Attr}^{\partial(0)}$ -coordinates is preserved. The situation of $k \rightarrow 0$ is difficult to answer for type IIIa trajectories, for the Einstein-Hilbert truncation in the linear field parametrization is known to be plagued by a singularity of η^{D} for $\lambda^{\text{D}} \rightarrow 1/2$. We will postpone the discussion on the IR behavior of $\text{Attr}^{\partial(0)}$ to subsection 13.2.3. However, notice that fig. 13.2 also indicates regions where $\text{Attr}^{\partial(0)}$ is situated in the upper $g^{\partial(0)}$ -half-plane of the boundary sector. RG trajectories that emanate in the ultraviolet

²Notice that in comparison with the background result, $\lambda_{\bullet}^{\partial(0)}(k)$ and $\lambda_{\bullet}^{\text{B}}(k)$ have different pre-factors of $-(d-2)/(d-1)|_{d=4} = -2/3$ and $-(d-2)/d|_{d=4} = -1/2$, respectively. This is a result of their different canonical scaling.

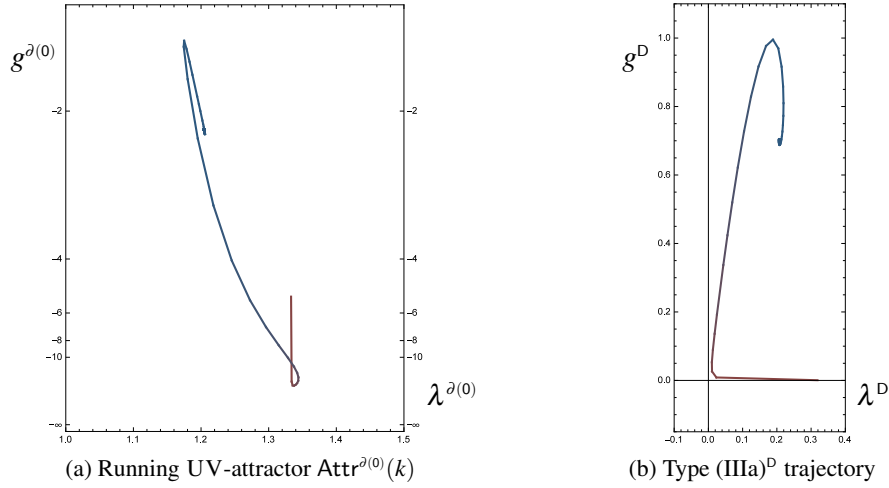


Figure 13.1: The position of the running UV attractor of the boundary sector, $\text{Attr}^{\partial(0)}(k)$ is shown in the left panel for typical type IIIa trajectory $k \mapsto (g^D, \lambda^D)$ as given in the right panel. Though the $g^{\partial(0)}$ coordinate assumes very large values on intermediate scales $0 < k < \infty$, it always stays finite. For $k \rightarrow \infty$ (the direction of increasing RG-time $t = \ln(k/k_0)$ is indicated by the transition from red to blue) $\text{Attr}^{\partial(0)}(k)$ approaches the $\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^D$ -FP coordinates. Hence, the existence of the UV-limit for $\text{Attr}^{\partial(0)}(k)$ strengthens the evidence for the Asymptotic Safety conjecture.

from NG_+^D -FP, where $g_{\bullet}^{\partial(0)}$ is negative, and enter a region of positive $g_{\bullet}^{\partial(0)}$ at finite $0 < k < \infty$, experience a significant change of the topological landscape in the boundary-sector. This is in particular the case for asymptotically safe type Ia trajectories being trapped in the lower $g^{\partial(0)}$ -half-plane for all k , while in the IR they cross the boundary of a region for which $g_{\bullet}^{\partial(0)}$ turns positive.

- C. While the $g_{\bullet}^{\partial(0)}$ -coordinate assumes almost any negative value, the corresponding $\lambda_{\bullet}^{\partial(0)}$ -coordinate changes only within a very small range around its fixed point value $\lambda_{*}^{\partial(0)} \approx 1.204$. However, compared to its background counterpart Attr^B , it never touches the boundary of theory space and is always well-defined for a very large class of type IIa and IIIa trajectories.
- D. The name of a running UV-attractor is in fact justified. For each k , $\text{Attr}^{\partial(0)}(k)$ attracts the RG flow in the boundary sector towards its current position, at least for all trajectories in the same $g^{\partial(0)}$ -half plane. This can be seen by linearizing the boundary-system (13.2) about $(g_{\bullet}^{\partial(0)}, \lambda_{\bullet}^{\partial(0)})$ and deduce the k -dependent analogue of a stability matrix, $\mathcal{B}(g_{\bullet}^{\partial(0)}, \lambda_{\bullet}^{\partial(0)}; k)$:

$$\mathcal{B}(g_{\bullet}^{\partial(0)}, \lambda_{\bullet}^{\partial(0)}; k) = \begin{pmatrix} -2 & 0 \\ \frac{2}{3}A^{\partial(0)}(k) & -3 \end{pmatrix} \quad (13.5)$$

Since for each k both eigenvalues of $\mathcal{B}$, $\theta_{1;\partial\bullet}$ and $\theta_{2;\partial\bullet}$, are found to be positive we may conclude that *at all scales the point $(g_{\bullet}^{\partial(0)}, \lambda_{\bullet}^{\partial(0)})$ is UV-attractive in both $\partial(0)$ -directions*. Independent on the imposed initial conditions for the $\partial(0)$ -couplings, RG trajectories considered in the boundary sector of theory space, i.e. solutions $k \mapsto (g_k^{\partial(0)}, \lambda_k^{\partial(0)})$, are always pulled towards $\text{Attr}^{\partial(0)}$ for $k \rightarrow \infty$.

Next, we consider the limits $k \rightarrow \infty$ and $k \rightarrow 0$ of $\text{Attr}^{\partial(0)}(k)$ and thus the topological structure of the boundary level-(0) phase diagrams for the ultraviolet and infrared, respectively.

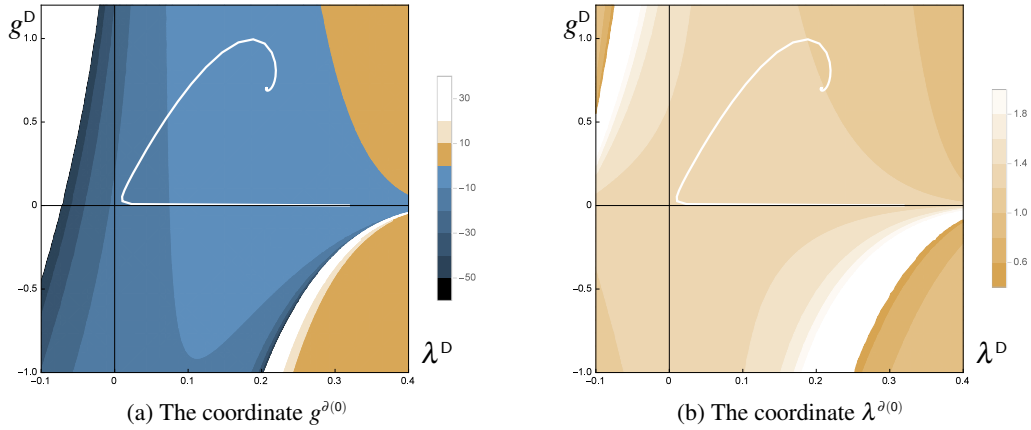


Figure 13.2: In the left (right) plot, we compute the coordinate value of Newton's (cosmological) coupling of $\text{Attr}^{\partial(0)}(g^D, \lambda^D)$ for several points of the D-couplings. Negative (positive) values of $g_{\bullet}^{\partial(0)}$ or $\lambda_{\bullet}^{\partial(0)}$ are indicated by dark blue (light orange) regions. In the most interesting part of theory space, associated to asymptotically safe RG trajectories of type-IIIa (as e.g. the white superimposed curve with $g^D, \lambda^D > 0$), the running UV-attractor is trapped in the region $g^{\partial(0)} < 0$ and $\lambda^{\partial(0)} > 0$. For type Ia with a negative λ^D at finite k , the position $\text{Attr}^{\partial(0)}$ approaches the boundary of theory space and in fact its $g_{\bullet}^{\partial(0)}$ coordinate changes sign. On the left of each plot, the sharp jump in the position of $\text{Attr}^{\partial(0)}$ is a truncation artifact associated to a pole in the beta-functions of the D-couplings.

13.2.2 The UV behavior of the boundary sector

From the Asymptotic Safety point view, the most interesting part of the RG evolution is encapsulated in the UV regime where asymptotically safe RG trajectories have to enter the realm of a suitable fixed point. In previous exploration of the Asymptotic Safety program studying space-time manifolds without boundary, a very promising candidate in the shape of a non-Gaussian fixed point was discovered [122, 123, 135, 147, 191]. This candidate corresponds to $\text{NG}_+^D\text{-FP}$ which from the discussion in chapter 9 is also the most stable one w.r.t. the ghost sector dependence. The results obtained in chapter 9 further showed that $\text{NG}_+^D\text{-FP}$ extends to the bi-metric subspace of $\mathcal{T}$ studied in this thesis which contains the lowest order of boundary invariants $\int_{\partial M} \sqrt{H}$ and $\int_{\partial M} \sqrt{H} \bar{K}$.

In the last subsection, we discovered a running UV-attractor in the boundary-sector, $\text{Attr}^{\partial(0)}$, that determines the RG evolution of the coefficients multiplying the two boundary invariants. Its properties are essential to understand the global flow within the boundary part of theory space and its UV limit for asymptotically safe RG trajectories coincides with the non-Gaussian fixed point $\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^D\text{-FP}$. That is, for all RG solutions $k \mapsto (g_k^D, \lambda_k^D, g_k^{\partial(0)}, \lambda_k^{\partial(0)})$ which are compatible with the Asymptotic Safety requirement, we have

$$\lim_{k \rightarrow \infty} \text{Attr}^{\partial(0)}(g_k^D, \lambda_k^D) = \text{Attr}^{\partial(0)}(g_*^D, \lambda_*^D) \equiv (g_*^{\partial(0)}, \lambda_*^{\partial(0)}) \quad (\text{asymptotically safe})$$

While an extensive discussion on the full set of fixed points and their UV-critical hypersurfaces can be found in chapter 9, we here focus only on the results related to $G^D\text{-FP}$ and $\text{NG}_+^D\text{-FP}$ in $d = 4$. The latter is – as mentioned above – the most interesting fixed point from the Asymptotic

Safety point of view.³

$d = 4$	g_*^D	λ_*^D	$g_*^{\partial(0)}$	$\lambda_*^{\partial(0)}$
$\text{NG}_-^{\partial(0)} \oplus \text{G}^D\text{-FP}$	0	0	-18.8	1.33
$\text{G}^{\partial(0)} \oplus \text{G}^D\text{-FP}$	0	0	0	0
$\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^D\text{-FP}$	0.70	0.21	-2.14	1.2
$\text{G}^{\partial(0)} \oplus \text{NG}_+^D\text{-FP}$	0.70	0.21	0	0

(13.6)

The most important observation is the negative sign of $g_*^{\partial(0)}$ that we already mentioned above. Conceptually, there is nothing wrong with $g^{\partial(0)} < 0$, which is different for its g^D counterpart. Nevertheless, it has some severe consequence for the classical matching condition, where $g^{(1)} \stackrel{!}{=} g^{\partial(1)}$ is chosen in order to have a well-posed variational principle that leads to the Einstein field equations of General Relativity. While for the present truncation we only resolve $g^{\partial(0)}$ which in general can differ from $g^{\partial(1)}$, in particular, a negative $g^{\partial(0)}$ would imply that split-symmetry together with Asymptotic Safety cannot be reconciled with the classical matching condition. We come back to this issue in the next section.

Now, let us linearize the GR flow about the latter two fixed points of eq. (13.6). We find the following set of their associated critical exponents:

$$\text{G}^{\partial(0)} \oplus \text{NG}_+^D\text{-FP:} \quad \theta_{1,2} = 3.60 \pm 4.28i, \quad \theta_3 = -2, \quad \theta_4 = 1 \quad (13.7a)$$

$$\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^D\text{-FP:} \quad \theta_{1,2} = 3.60 \pm 4.28i, \quad \theta_3 = 3, \quad \theta_4 = 2 \quad (13.7b)$$

Whereas in the case of the $\text{G}^{\partial(0)} \oplus \text{G}^D\text{-FP}$ the $g^{\partial(0)}$ -direction is classified to be irrelevant, for the doubly non-Gaussian fixed point, $\text{NG}_-^{\partial(0)} \oplus \text{NG}_+^D\text{-FP}$, all four scaling fields are relevant, i.e. the corresponding values of $\Re(\theta)$ are all positive. Thus, the dimensionalities of the respective UV critical manifolds are $\dim \mathcal{S}_{\text{UV}}(\text{G}^{\partial(0)} \oplus \text{G}^D\text{-FP}) = 3$ and $\dim \mathcal{S}_{\text{UV}}(\text{NG}_-^{\partial(0)} \oplus \text{G}^D\text{-FP}) = 4$, demanding a total of 3 or 4 additional initial conditions, respectively, in order to uniquely specify an RG solution. Notice that the critical scaling behavior of the boundary part is found to be the same for the present bi-metric and the corresponding single-metric calculation [166].

13.2.3 The IR behavior of the boundary sector

Now, let us briefly comment on the $k \rightarrow 0$ limit of the running UV attractor of the boundary sector. Since we shown that it is indeed UV-attractive in the $g^{\partial(0)}$ - and $\lambda^{\partial(0)}$ -direction, there is at most a single RG solution which starts at $\text{Attr}^{\partial(0)}(0)$ for $k = 0$. Following the flow in the direction of decreasing k , i.e. towards the infrared, $\text{Attr}^{\partial(0)}(k)$ has to be understood as a running IR ‘repeller’. Still, it determines the non-trivial structure of the phase diagram, however now RG solutions depart from the current position of $\text{Attr}^{\partial(0)}$.

In order to have a clear as possible picture of the global behavior of RG solutions in the boundary sector, we have to consider the influence of $\text{Attr}^{\partial(0)}$ on asymptotically safe trajectories also in the IR. The D-sector, which drives the k -dependence of $\text{Attr}^{\partial(0)}(k)$, can be classified into three types of asymptotically safe RG trajectories which differ in the IR behavior of the cosmological constant λ^D , see fig. 13.3. The position of $\text{Attr}^{\partial(0)}(k)$ for these three cases is listed in the following.

³Here, and in what follows all numerical computations, if not stated otherwise, are based on the optimized shape function.

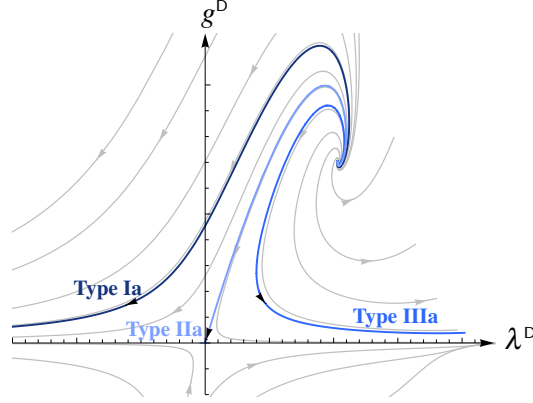


Figure 13.3: The schematic structure of the phase portrait on g^D - λ^D including the three types of asymptotically safe RG trajectories. As always, the arrows point in the direction of decreasing k . Thus, classification is based on the IR behavior of λ^D which either becomes negative, zero, or positive corresponding to type Ia, IIa (separatrix), or IIIa solutions, respectively.

Ia The defining property for this type of RG solutions is $\lambda_k^D \xrightarrow{k \rightarrow 0} -\infty$. If we consider the D-system for very large negative values of λ_k^D we obtain the following asymptotic solutions (for $g_{k_0}^D > 0$):

$$g_k^D = \frac{2\pi g_{k_0}^D k^2}{g_{k_0}^D (k_0^2 - k^2) + 2\pi k_0^2} \xrightarrow{k \rightarrow 0} 0 \quad (13.8a)$$

$$\lambda_k^D = \frac{g_{k_0}^D (k_0^4 - k^4) - 6\pi k_0^4 \lambda_{k_0}^D}{3k^2 (g_{k_0}^D (k^2 - k_0^2) - 2\pi k_0^2)} \xrightarrow{k \rightarrow 0} \pm\infty \quad (13.8b)$$

Notice that this result is also valid for solutions of $\lambda_k^D \xrightarrow{k \rightarrow 0} +\infty$ (the type-IIIa trajectories discussed below). The IR behavior of the underlying trajectory is specified by the sign of λ_0^D . For type (Ia)^D trajectories we insert solution (13.8) for negative $\lambda_{k_0}^D$ and obtain the corresponding IR position of $\text{Attr}^{\partial(0)}(k)$:

$$\lim_{k \rightarrow 0} \text{Attr}^{\partial(0)}(k) = \left(\frac{3\pi}{2}, \frac{4}{3} \right) \quad (\text{type Ia}) \quad (13.9)$$

Notice that opposite to the ultraviolet part of the trajectories $\text{Attr}^{\partial(0)}(k)$ now resides on the upper $g^{\partial(0)}$ -half plane and thus has no impact on the IR evolution of asymptotically safe type Ia trajectories.

IIa Due to the negative critical exponent in the g^D direction, there is a unique asymptotically safe RG solution (up to rescaling) that interconnects NG_+^D -FP with the Gaußian fixed point G^D -FP. It defines the dividing line between the regions of type Ia and IIIa trajectories and is therefore also denoted separatrix. In the vicinity of G^D -FP we can linearize the RG flow and determine the separatrix solution of the D-couplings:

$$g_k^D = g_{k_0}^D (k/k^2) \quad \text{and} \quad \lambda_k^D = \frac{3}{8\pi} g_{k_0}^D (k/k^2)$$

When this solution is inserted into the differential equations of the boundary-sector, we can solve for the zeros of the corresponding beta-functions yielding the IR limit of $\text{Attr}^{\partial(0)}(k)$:

$$\lim_{k \rightarrow 0} \text{Attr}^{\partial(0)}(k) = \left(-6\pi, \frac{4}{3} \right) \quad (\text{separatrix}) \quad (13.10)$$

In this case, as is also apparent from fig. 13.2, $g_{\bullet}^{\partial(0)}$ remains negative.

IIIa The problem in studying the IR behavior of type IIIa trajectories for the Einstein-Hilbert truncations in the linear field parametrization is the occurrence of a singularity in the beta-functions of the D-sector for λ^D close to $1/2$. Extensions to non-local truncations and exponential type field parametrization suggests that this pole is in fact an artifact of truncating theory space and disappears for exact solutions [146, 175]. Thus, for the present discussion we ignore this singularity and assume we can prolong the type IIIa trajectories up to $\lambda^D \rightarrow +\infty$. Then, we can make use of the IR solution (13.8) and derive the $\text{Attr}^{\partial(0)}$ position for $k = 0$:

$$\lim_{k \rightarrow 0} \text{Attr}^{\partial(0)}(k) = \left(\frac{3\pi}{2}, \frac{4}{3} \right) \quad (\text{type IIIa}) \quad (13.11)$$

It turns out that independent on the sign of λ^D in the limit $\lambda^D \rightarrow \pm\infty$, the running UV-attractor of the boundary sector approaches the same position in theory space.

Notice that whenever g^D approaches zero, which is the case for the IR limits of the RG trajectories and in particular for the Gaußian fixed point solutions, the value of $\lambda_{\bullet}^{\partial(0)}$ assumes $4/3$, independent on λ^D .

13.2.4 The semiclassical regime

In between the infrared and ultraviolet limits we have just discussed, the dynamical RG solutions pass a region in theory space which is best described by the semiclassical approximation. Hence, for some intermediate scales $0 < k < \infty$, we can easily solve the D-sector and study the corresponding boundary equations within this semiclassical regime. As already mentioned, ‘semiclassical’ refers to an approximation in which the RG equations are reduced to their one loop form. Thereby, the ‘improvement’ terms proportional to η^D and $\eta^{\partial(0)}$ in the beta-functions are neglected and evaluating all threshold functions are evaluated at zero cosmological constant λ^D . Thus, approximating the anomalous dimensions η in this way, we obtain

$$\eta^D = -(d-2) \omega_d g^D, \quad \eta^{\partial(0)} = -(d-2) \omega_d^{\partial(0)} g^{\partial(0)} \quad (13.12)$$

Here, ω_d^D and $\omega_d^{\partial(0)}$ denote numerical constants, independent on k , that classify the respective semiclassical behavior. Using $\eta \equiv \partial_t \ln G_k$ the differential equations of the dimensionful Newton couplings admit the following exact solutions:

$$G_k^I = \frac{G_0^I}{1 + \omega_d^I G_0^I k^{d-2}} \quad (13.13)$$

If the dimensionless Newton coupling $g_0^I = G_0^I k^{d-2}$ is sufficiently small in the semiclassical regime, we can expand eq. (13.13) for small $G_0^I k^{d-2}$, which yields:

$$G_k = G_0 \left[1 - \omega_d G_0 k^{d-2} + \dots \right] \quad (13.14a)$$

$$G_k^{\partial(0)} = G_0^{\partial(0)} \left[1 - \omega_d^{\partial(0)} G_0^{\partial(0)} k^{d-2} + \dots \right] \quad (13.14b)$$

Along the same lines we next solve for the dimensionful running cosmological constants in the same regime. Since $\bar{\lambda}$ and $\bar{\lambda}^{\partial(0)}$ show different canonical scaling, the respective semiclassical approximations develop different k -dependencies:

$$\bar{\lambda}_k = \bar{\lambda}_0 + v_d G_0 k^d + \dots \quad (13.14c)$$

$$\bar{\lambda}_k^{\partial(0)} = \bar{\lambda}_0^{\partial(0)} + v_d^{\partial(0)} G_0^{\partial(0)} k^{d-1} + \dots \quad (13.14d)$$

Here G_0 , $G_0^{\partial(0)}$, $\bar{\lambda}_0$, and $\bar{\lambda}_0^{\partial(0)}$ are free constants of integration, and the dots represent higher orders in $G_0 k^{d-2}$ and $G_{\partial(0)} k^{d-2}$, respectively.

In the following we consider the case of $d = 4$ and employ the ‘optimized’ threshold functions. For a more detailed discussion and results for general spacetime dimensions d , we refer to chapter 8 of this thesis. We will focus on the qualitative structure of the semiclassical results and consider the signs of ω_d^I and ν_d^I coefficients, which in fact agree for any suitable shape function. In detail, we obtain

$$\omega_4^D = -\frac{5}{12\pi} > 0 \quad \text{and} \quad \nu_4^D = +\frac{3}{8\pi} > 0 \quad (13.15a)$$

$$\omega_4^{(0)} = +\frac{11}{6\pi} > 0 \quad \text{and} \quad \nu_4^{(0)} = +\frac{1}{8\pi} > 0 \quad (13.15b)$$

$$\omega_4^{\partial(0)} = -\frac{1}{6\pi} < 0 \quad \text{and} \quad \nu_4^{\partial(0)} = -\frac{2}{9\pi} < 0 \quad (13.15c)$$

Here, we included also the level-(0) bulk couplings, for it is very instructive to compare the relative signs of the full sector. In fact, as discussed in chapter 10 the semiclassical results of the level-(0) couplings agrees with the ones obtained by a single-metric calculation. Hence, knowing $\omega_4^{(0)} = \omega_4^{\text{sm}}$ and $\nu_4^{(0)} = \nu_4^{\text{sm}}$ holds true, we can directly compare the present results of the bi-metric truncation with the previous one derived in ref. [166]. Based on eq. (13.15) we make the following observations:

- A.** Within the approximation (13.14), the bulk Newton constant G_k^D and the boundary counterpart $G_k^{\partial(0)}$ have both negative coefficient ω_4 . Hence, in contrast to the single-metric calculation and the level-(0) results, G_k^D *increases* for increasing k . Nevertheless, away from the semiclassical regime this behavior changes when leaving the vicinity of the Gaussian fixed points and we re-obtain the familiar picture of gravitational anti-screening [122], see chapter 10 and 12. On the other hand, $\omega_4^{\partial(0)} < 0$ does not generally imply that $G_k^{\partial(0)}$ also increases towards the UV. In the boundary sector, there is a priori no experimental observation that would favor a positive $G_k^{\partial(0)}$ and in fact the condition of Asymptotic Safety requires $G_k^{\partial(0)} < 0$. Thus, albeit the different signs of $\omega_4^{(0)/\partial(0)}$, the semiclassical behavior of asymptotically safe trajectories agrees on level-(0).
- B.** It can be checked that the negative signs of ω_4^D and $\omega_4^{\partial(0)}$ in the semiclassical regime, are robust with respect to changes of the cutoff shape function.
- C.** In fact, while there is sign change of the anomalous dimension in the dynamical sector, the results on the boundary, $\eta^{\partial(0)}$, extends beyond the semiclassical approximation. In $d = 4$ and for the ‘optimized’ threshold functions (8.4) we obtain

$$\eta^{\partial(0)} = \frac{g^{\partial(0)}}{3\pi} \frac{1 - \frac{5}{4}\eta^D + 8\lambda^D}{(1 - 2\lambda^D)} \quad (13.16)$$

This result is exact in the sense that the improvement term $\propto \eta^D$ on the RHS of the FRGE was retained and λ^D has not been set to zero in the arguments of the Φ ’s. It is obvious that the expression (13.16) is always negative in the regime of interest, $g^{\partial(0)} < 0$, $\eta^D < 0$, and $\lambda^D \in [0, 1/2)$.

- D.** As a result of the previous remarks, we can conclude that within this truncation we find that the equality $G_k^{(0)} = G_k^{\partial(0)}$ is inconsistent with Asymptotic Safety as long as split-symmetry is preserved. Hence, at least on the basis of the here derived beta-functions we can say that different to the previous single-metric results there is no point in the asymptotically space branch of theory space for which the average action Γ_k has the desired well posed variational principle by simply including a Gibbons-Hawking-York boundary term. This also includes the ‘physical point’ $k = 0$. We are going to discuss this mismatch

in the next section, but mention only that there are general arguments indicating that the technique introduced by Gibbons, Hawking, and York is in general insufficient and has to be replaced by different surface corrections, see for instance [301, 302].

The well known *decrease* of the bulk Newton constant in the UV has been interpreted as an indication for the *anti-screening* character of QEG at short distances [122]. In the bi-metric setting, we have already seen in chapter 10 that the dynamical sector enters a regime of positive η^D and thus has a screening property in the semiclassical regime. While this changes beyond this region, the level-(0) boundary Newton's coupling has the same sign for all k similar to its bulk counterpart.

The analysis of the global properties for the RG flow in the boundary sector revealed that opposite to the bulk part, Asymptotic Safety requires a negative $g^{\partial(0)}$. Since once we start with an initial negative Newton coupling $g^{\partial(0)}(k_0) < 0$ there is no solution in theory space that crosses the $g^{\partial(0)} = 0$ and thus would lead to a positive $g^{\partial(0)}$ on some intermediate scale. From this perspective the Gibbons-Hawking-York technique is incompatible with the two requirements of Asymptotic Safety and Background Independence, at least on the results obtained within the present truncation. In the next section, we briefly comment on the classical idea of introducing boundary terms to obtain Einstein's field equations from a well-posed variational principle.

13.3 Variational problems

Classical General Relativity is defined by Einstein's field equation which can also be derived from the requirement that the Einstein-Hilbert functional,

$$S_{\text{EH}}[g_{\mu\nu}] = -\frac{1}{16\pi G} \int_M d^d x \sqrt{g} R \quad (13.17)$$

becomes stationary, *provided* the spacetime manifold, M , has no boundary. In order to generalize the variational principle to very interesting case of spacetimes with a non-empty boundary ∂M the problem arises that S_{EH} responds to a change $\delta g_{\mu\nu} = g + v_{\mu\nu}$ of the metric, with $v_{\mu\nu}$ vanishing on ∂M , by producing a certain surface term, over and above the desired 'bulk' term containing the Einstein tensor $G^{\mu\nu}$:

$$\partial_\nu S_{\text{EH}} = \frac{1}{16\pi G} \left(\int_M d^d x \sqrt{g} G^{\mu\nu} v_{\mu\nu} + \int_{\partial M} d^{d-1} x \sqrt{H} H^{\alpha\beta} n^\mu \partial_\mu v_{\alpha\beta} \right) \quad (13.18)$$

In eq. (13.18) we have introduced the normal vector field n^μ related to the embedding of the boundary in M and the tensor $H_{\alpha\beta}$ which is related to the induced metric on the boundary. Under Dirichlet conditions variations of the metric do not affect its boundary value; hence, by assumption, $v_{\mu\nu}$ vanishes on ∂M . This implies that the derivatives of $v_{\mu\nu}$ in directions tangential to ∂M will also vanish. On the other hand, its normal derivative $n^\mu \partial_\mu v_{\alpha\beta}|_{\partial M}$ is in general unrestricted and thus non-zero. For this reason the surface term in (13.18) remains for general $v_{\mu\nu}$ and the condition of stationarity $\partial_\nu S_{\text{EH}} = 0$ is not equivalent to Einstein's (vacuum) field equation $G^{\mu\nu} = 0$.

Notice that these kind of difficulties are not uncommon in Lagrangian or Hamiltonian systems. In order to re-establish the standard variational problem, one includes an additional boundary contribution to the action whose variation exactly cancels the unwanted surface term originating from the bulk part.

In General Relativity, the most popular proposal⁴ for a surface correction with this property is the Gibbons-Hawking-York term [79],

$$S_{\text{GHY}} = \frac{1}{16\pi G} \int_{\partial M} d^{d-1} x \sqrt{H} (-2K) \quad (13.19)$$

⁴For recent proposals of different surface corrections see [301, 302]; for a general discussion see [76, 197, 198].

where K denotes the trace of the extrinsic curvature of the embedded boundary.⁵ If one applies the variation on the sum $S_{\text{EH}} + S_{\text{GHY}}$ we find that the obstructing surface term on the RHS of (13.18) is precisely compensated by the variation of the Gibbons-Hawking-York action of eq. (13.19). Hence, the stationary points of the total action are exactly those metrics satisfying Einstein's (vacuum) field equation: $\partial_\nu (S_{\text{EH}} + S_{\text{GHY}}) = 0 \Leftrightarrow G^{\mu\nu} v_{\mu\nu} = 0$ for all $v_{\mu\nu}$.

This generalization of GR to spacetimes with boundary has lead to interesting applications in cosmology. For instance, the Gibbons-Hawking-York term has played an important role in the Euclidean functional integral approach to black hole thermodynamics [79, 303]. In the leading order of the semiclassical expansion the black hole's free energy is given by the 'on-shell' value of the classical action functional. Since, for vacuum solutions, $R_{\mu\nu} = 0$ implies a vanishing contribution from the bulk term, the free energy, and hence all derived thermodynamical quantities such as the entropy, for instance, stem entirely from the surface term. It is important to ask how this picture presents itself in full fledged quantum gravity [73]. There exist already detailed investigations within Loop Quantum Gravity [72, 75, 93], for instance [304].

At this point however, one has to be careful in translating these classical techniques into the framework of Quantum Gravity. Fixing the relative size of coefficients of two independent basis invariants, as done by the matching of boundary and bulk couplings, will in general be spoiled by quantum effects. Only if certain symmetry principles or other fundamental algebraic constraints require the precise matching of the classical conditions, there is a reasonable chance that we encounter this tuning also on a quantum level. For the classically preferred ratio of boundary and bulk coefficient there is actually a weak geometric property, namely it turns out that the heat kernel expansion preserves the relative factor of 2. Whether or not this is sufficient cannot be answered within the present truncation, for which details of the level-(1) boundary coefficients are needed. However, already in the classical description, there are severe constraints on the reliability of the Gibbons-Hawking-York implementation, especially when spacetime is non-compact, see ref. [301, 302].

13.3.1 Effective field equations and stationary points

From the FRG point of view, the motivation to study truncations which include the Gibbons-Hawking-York term are not primary related to the classical variational problem. Rather, when allowing for spacetimes with non-empty boundary, $\partial M \neq \emptyset$, which is very interesting to study quantum effects in cosmological models, theory space gets extended by an infinite number of basis invariants defined on ∂M . The FRGE on this enlarged space in general only closes if we include all field monomials on ∂M which are compatible with the symmetry principle and the field content, the latter being equipped with a set of suitable boundary conditions. Thus, the relevance of the Gibbons-Hawking-York term in the EAA approach to QEG is better understood from a technical perspective, in that it makes its appearance in Γ_k when employing a systematic derivative expansion up to second order in D or $\bar{D}$. It is thus the natural candidate to start exploring directions in theory spaces associated to $\partial M \neq \emptyset$. For the exact solutions, we have to include all further boundary terms will also comprises the proposed surface terms in [301, 302].

In the Effective Average Action approach based on the FRGE classical field equations get replaced by the *running stationarity condition* and the *running tadpole equation*, which in absence of split-symmetry are not equivalent [166]. In following we present a short summary of both concepts for vanishing spacetime boundaries, $\partial M = \emptyset$.

⁵In the literature, S_{GHY} is usually normalized by replacing $K \rightarrow K - K_0$ in (13.19), with K_0 the trace of the extrinsic curvature tensor appropriate for an embedding of the boundary manifold in flat space. While subtracting K_0 does not change $\partial_\nu S_{\text{GHY}}$, we shall refrain from it here for reasons to be discussed below.

Running stationarity condition

For the first equation, notice that it is $\tilde{\Gamma}_k \equiv \Gamma_k + \Delta_k S$ rather than the Effective Average Action, Γ_k , for which the k -dependent field-source relationship between sources and expectation fields $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle$, $h_{\mu\nu} \equiv \langle h_{\mu\nu} \rangle$ is established. On an arbitrary theory space involving fluctuation fields, φ , coupled to sources J_r , the corresponding stationary condition evaluated at background fields $\bar{\Phi}$ reads

$$\frac{\delta \tilde{\Gamma}_k[\varphi; \bar{\Phi}]}{\delta \varphi_r} \equiv \frac{\delta \Gamma_k[\varphi; \bar{\Phi}]}{\delta \varphi_r} + \frac{\delta \Delta_k S[\varphi; \bar{\Phi}]}{\delta \varphi_r} = J_r \quad (13.20)$$

This equation can be interpreted as a first notion of a scale dependent ‘effective field equation’: the running stationary point condition of $\tilde{\Gamma}_k$. The complicate character of this equation resides in its dependence on both the fluctuations, φ , and the background fields $\bar{\Phi}$.

In case of full-fledged Quantum Einstein Gravity, here evaluated for vanishing sources, eq. (13.20) therefore assume the following form

$$\frac{\delta \Gamma_k[h; \bar{g}]}{\delta h^{\mu\nu}(x)} + \sqrt{\bar{g}} \mathfrak{R}_k h_{\mu\nu}(x) = 0 \quad (13.21)$$

This coupled system yields the solutions $h_{\mu\nu}(x) \equiv h_{\mu\nu}[\bar{g}](x)$ as functionals of the fixed but arbitrary background metric $\bar{g}$.

Running tadpole equations

A very special, and interesting class of solutions is obtained by evaluating (13.21) for $h_{\mu\nu} = 0$. This gives rise to the definition of ‘self-consistent’ backgrounds, which satisfy the *running tadpole equation*:

$$\left. \frac{\delta}{\delta h_{\mu\nu}(x)} \Gamma_k[h; \bar{g}_k^{\text{s.c.}}] \right|_{h=0} = 0 \quad (13.22)$$

Notice that the $\mathfrak{R}_k h_{\mu\nu}$ term vanishes for $h_{\mu\nu} = 0$. While the functional variation addresses the fluctuation field h , the solutions are obtained for another field, $\bar{g}_{\mu\nu}$. This implies that in general *the tadpole equation cannot be written as the $\bar{g}$ -derivative of any diffeomorphism invariant functional $F[\bar{g}]$* . Furthermore, the integrability of this system is not automatic, but under special circumstances solutions can exist, see ref. [137] for further details.

R Even though self-consistent background metrics, $\bar{g}_k^{\text{s.c.}}$, define k -dependent solutions of the tadpole equation for $h_{\mu\nu} = 0$, this does not correspond to the idea of small fluctuations in the perturbative sense. It rather corresponds to restricting the class of theories μ for which the expectation value $h_{\mu\nu} = \langle \hat{h}_{\mu\nu} \rangle_\mu$ vanishes.

We have thus found another generalization of the classical field equations: a scale dependent version of the tadpole condition. Though it derives from the running stationary point condition, it has the significant difference of depending only on the background metric $\bar{g}_{\mu\nu}$ and defines the so-called self-consistent background solutions.

Summary

Both the *running stationarity condition* and the *running tadpole equation* generalize the field equations of classical gravity theory in a way which goes far beyond replacing the Einstein-Hilbert action $S_{\text{EH}}[g]$ with some other diffeomorphism invariant single-metric functional $S[g]$. In the first case one deals with field equations involving two metrics $\bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu} + h_{\mu\nu} \equiv g_{\mu\nu}$; their structure is constrained only by the requirement that they must be representable as the h -derivative of some invariant functional. In the second case the effective field equations contain

only one metric, namely $\bar{g}$, but there is no longer the requirement to be the $\bar{g}$ -derivative of an invariant action. Thus the gravitational average action allows for a great variety of potential modifications of the classical Einstein equations. A purely phenomenological analysis of such bi-metric actions should therefore be a worthwhile complement to the flow equation studies.

Thus the gravitational average action allows for a great variety of potential modifications of the classical Einstein equations. A purely phenomenological analysis of such bi-metric actions should therefore be a worthwhile complement to the flow equation studies.

13.3.2 Variational principle in presence of a boundary

We now come back to the case of manifolds with boundaries, $\partial M \neq \emptyset$ and address the problem of giving a consistent meaning of the variational principle in the presence of boundary terms. Therefore, consider the truncation ansatz of eq. (5.15) with a bi-metric gravitational effective action comprising the Einstein-Hilbert term of dynamical and background field as well as a Gibbons-Hawking-York term of $\bar{g}$ and further volume contributions.

For the time being, we extend Γ_k by incorporating a second, dynamical Gibbons-Hawking-York term and allow for some matter field content, A , in form of an energy-momentum tensor $T^{\mu\nu}[A; \bar{g}]$. Thus, the EAA assumes the form (whereby for simplicity we consider only terms up to quadratic order in h):

$$\begin{aligned} \Gamma_k[h; \bar{g}] = & \Gamma_k[A; \bar{g}] + \frac{1}{16\pi G_k^{(1)}} \cdot \int_M d^d x \sqrt{\bar{g}} \left\{ \bar{G}^{\mu\nu} + \bar{\lambda}_k^{(1)} \bar{g}^{\mu\nu} - 8\pi G_k^{(1)} T^{\mu\nu}[A; \bar{g}] \right\} h_{\mu\nu} \\ & + \frac{1}{2} \int_M d^d x \sqrt{\bar{g}} h^{\rho\sigma} \widehat{\text{Hess}}_h [\Gamma_k[A, h; \bar{g}]]_{\rho\sigma}^{\mu\nu} h_{\mu\nu} \\ & - \frac{1}{16\pi} \left(\frac{1}{G_k^{(1)}} - \frac{1}{G_k^{\partial(1)}} \right) \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} (\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) h_{\mu\nu} \end{aligned} \quad (13.23)$$

If the ∂M -integral in (13.23) is absent, the running stationary condition, eq. (13.21), reads:

$$(8\pi G_k^{(1)})^{-1} (\bar{G}^{\mu\nu} + \bar{\lambda}_k^{(1)} \bar{g}^{\mu\nu} - 8\pi G_k^{(1)} T^{\mu\nu}[A; \bar{g}]) = -\widehat{\text{Hess}}_h [\Gamma_k[A, h; \bar{g}]]_{\rho\sigma}^{\mu\nu} h^{\rho\sigma} - \Re_k h^{\mu\nu} \quad (13.24)$$

The term on the RHS is reminiscent of Einstein's field equation, however with k -dependent coefficients associated to level-(1). The Hessian operator introduces further terms of at least level-(2) and contributions of the fluctuation field h . Apparently, this equation depends on both h and $\bar{g}$ and its solution gives rise to fields of the form $h_k[A; \bar{g}]$.

In order to obtain the running tadpole equation we have to evaluate (13.24) at $h_{\mu\nu} = 0$, yielding a k -dependent analog of the Einstein field equations:

$$\bar{G}^{\mu\nu}[\bar{g}^{\text{s.c.}}] + \bar{\lambda}_k^{(1)} (\bar{g}^{\text{s.c.}})^{\mu\nu} = 8\pi G_k^{(1)} T^{\mu\nu}[A; \bar{g}^{\text{s.c.}}] \quad (13.25)$$

By construction, this equation involves only level-(1) couplings and defines the class of self-consistent background metrics. Obviously, eq. (13.24) and (13.25) are in general not equivalent. However, this familiar structure of the effective field equations can only be obtained if the boundary term in eq. (13.23) vanishes.

Let us consider the running tadpole equation for simplicity. A functional variation of eq. (13.23) and subsequently setting $h = 0$, results in the following condition for self-consistent background solutions:

$$\begin{aligned} 0 = & \frac{1}{16\pi G_k^{(1)}} \cdot \int_M d^d x \sqrt{\bar{g}} \left\{ \bar{G}^{\mu\nu} + \bar{\lambda}_k^{(1)} \bar{g}^{\mu\nu} - 8\pi G_k^{(1)} T^{\mu\nu}[A; \bar{g}] \right\} v_{\mu\nu} \\ & - \frac{1}{16\pi} \left(\frac{1}{G_k^{(1)}} - \frac{1}{G_k^{\partial(1)}} \right) \cdot \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} (\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) v_{\mu\nu} \end{aligned} \quad (13.26)$$

This condition has to hold for all tangent vectors $v_{\mu\nu}$ of $h_{\mu\nu}$. In order to get rid of the ∂M -integral, there are basically two possibilities. Either one imposes the matching condition $G_k^{\partial(1)} = G_k^{(1)}$ or one modifies the space of expectation fields, $h_{\mu\nu} \equiv \langle \hat{h}_{\mu\nu} \rangle$, such that the integral vanishes.⁶ We comment on both ideas in the following:

- A.** Already in a first single-metric and bi-metric matter induced truncation comprising a Gibbons-Hawking-York boundary term, we observed that the classical matching condition can only be satisfied at a single value of k [166]. The reason for this is founded in the opposite sign in the respective anomalous dimensions.
- B.** In order to fully resolve the issue of the matching condition, we have to resolve the level-(1) boundary and bulk coefficients. While the present truncation makes a first step towards this end by lifting the degeneracy in the bulk sector, more involved techniques are necessary to obtain the same results on the boundary. Nevertheless, if the level-(0) bulk and boundary results are mainly reliable there is the following constraint on a possible matching. We have seen that asymptotically safe trajectories are described by the family of RG solutions with $g^{\partial(0)} < 0$ and $g^D \equiv g^{(1)} > 0$. Furthermore, the requirement of Background Independence translates to split-symmetry in the limit $k \rightarrow 0$. Hence, combining these fundamental constraints with the result of the present truncation, we obtain the subset of theory space containing all candidates of a theory of Quantum Gravity, namely $\mathcal{T}_{\text{AS\&BI}}^{\text{trunc}} \equiv \{(g^{(p)}, g^{\partial(p)}, \lambda^{(p)}, \lambda^{\partial(p)}) | g^{(p)} > 0 \text{ and } g^{\partial(p)} < 0\}$. That the observation $g^{\partial(0)} < 0$ restricts all higher level boundary Newton couplings to be also negative, is a consequence of split-symmetry at $k = 0$ and the topological property of theory space that there is no transition from the negative, $g^I < 0$, to the positive half-plane, $g^I > 0$. From this it follows that if the present level-(0) results qualitatively remain valid even in the exact case, the matching condition for the coefficients of the boundary term in eq. (13.26) would be inconsistent with the joined assumption of Asymptotic Safety and Background Independence. This would then completely rule out the classical matching technique.
- C.** The second option can be understood as restricting the class of possible action functionals, $S[g]$, and thus theories μ , to only those which yield expectation values of the fluctuation fields $h_{\mu\nu} \equiv \langle \hat{h}_{\mu\nu} \rangle_\mu$ which simultaneously satisfy $(\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) h_{\mu\nu}|_{\partial M} = 0$ and Dirichlet constraints. The former condition cancels the boundary term in eq. (13.26) independent of its coefficient. It is reminiscent of the gauge condition, $\mathcal{F}[\bar{g}]_\mu^{\rho\sigma} h_{\rho\sigma} = (\bar{D}^\rho \bar{D}^\sigma - \bar{\omega} \bar{g}^{\rho\sigma} \bar{D}^2) h_{\rho\sigma}$ (similar for $v_{\mu\nu}$) however with $\bar{\omega} = -1$ instead of the harmonic choice $\bar{\omega} = 1/2$ utilized in this work. In fact rewriting the surface term into a bulk contribution, we obtain:

$$\begin{aligned} \int_{\partial M} d^{d-1} x \sqrt{\bar{H}} (\bar{n}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{n}^\rho \bar{D}_\rho) v_{\mu\nu} &= \int_M d^d x \sqrt{\bar{g}} (\bar{D}^\rho \bar{D}^\sigma + \bar{g}^{\rho\sigma} \bar{D}^2) v_{\rho\sigma} \\ &= \int_M d^d x \sqrt{\bar{g}} (\bar{D}^\mu \mathcal{F}[\bar{g}]_\mu^{\rho\sigma} + \frac{3}{2} \bar{g}^{\rho\sigma} \bar{D}^2) v_{\rho\sigma} \end{aligned}$$

Thus, part of the boundary term can be absorbed in the gauge sector and what remains is proportional to $n^\rho \bar{D}_\rho v_{\mu\nu}$. From this we can deduce that the difficulty of the ‘disturbing’ surface term concerns only one of several irreducible components of the metric fluctuation. If one performs a transverse-traceless decomposition of $h_{\mu\nu}$, ref. [123, 135, 147] or section 7.2, it is only its trace part $h^{\text{Tr}} \bar{g}_{\mu\nu}/d$, with $h^{\text{Tr}} \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$, which is affected by the surface term in eq. (13.26) which assumes the form $\int_{\partial M} d^{d-1} x \sqrt{\bar{H}} n^\alpha \partial_\alpha \phi$. Obviously the other (i.e. TT, TL, and LL) parts of the York decomposition do not contribute so that those irreducible components, at fixed h^{Tr} , enjoy a standard variational principle. Note that h^{Tr}

⁶Though we fully appreciate the possibility of obtaining a well-posed variational problem by means of different surface terms, we here focus only on the Gibbons-Hawking-York approach.

amounts to a fluctuation of the conformal factor of $g_{\mu\nu}$. Thus, we can recover Einstein's field equations for all tensor components for Dirichlet constraints, if we restrict the metric fluctuation to have a vanishing derivative of the trace part on the boundary, or likewise if the conformal factor and its normal derivative are simultaneously fixed on ∂M .

- D. If we generalize the previous restriction of the conformal factor to all fluctuation fields, we consider a smaller function space of expectation fields, which satisfy Dirichlet conditions and also have a vanishing normal derivative $\bar{D}_n$ on the boundary:

$$v_{\mu\nu} \in T_{\Phi}F' \equiv \{s_{\mu\nu} \text{ tensor on } M; \quad s_{\mu\nu} = 0 \text{ and } \bar{D}_n s_{\mu\nu} = 0 \text{ on } \partial M\} \quad (13.27)$$

For an EAA defined over F' the surface terms in the first functional variation disappear *always*, and the effective field equation is unambiguously given by (13.25).

At first glance, the conventional point of view on this reduction is rather reserved, in particular when the underlying functional integral is intended to represent a transition amplitude between 3-geometries; M is the portion of spacetime between 'initial' and 'final' time slices, making up ∂M , then. If the dynamical field and its normal derivative are fixed on the initial and final slice, the underlying equation is in general overdetermined since we imposed twice too many boundary conditions for a second order field equation. Hence, there may be no solutions that also satisfy the additional constraints and while having preserved the form of the field equations, we have lost all of its information.

However, following the lines of [166], we argue that the FRG perspective on this restriction is far more optimistic. A generic (second order in the derivatives) ansatz of the EAA, $\Gamma_k[h; \bar{g}]$, depends on two independent metrics, a result of the implementation of Background Independence and the explicit breaking of split-symmetry by cutoff and gauge fixing action. The running tadpole equation then yields a special class of stationary points, the self-consistent backgrounds evaluated for $h_{\mu\nu} = 0$. In this context, $\partial_v \Gamma_k[h; \bar{g}] = 0$ has to have a meaning, not for all h , but only near $h_{\mu\nu} \equiv 0$, where $\bar{g}$ becomes self-consistent. The essential observation is that the self-consistent solution $h_{\mu\nu} \equiv 0$ on all of M , if it exists, is *not* lost when we restrict field space to F' , the trivial reason being that the zero solution has vanishing derivatives everywhere on M and, by continuity, a vanishing normal derivative on ∂M .

In this section we are far from exhaust the investigation of the variational problem in the context of the Asymptotic Safety program to Quantum Gravity. Even though we have some first indication of a possibly global 'mismatch' of the bulk-boundary coefficients, it still remains an open issue if a full resolution of the different levels for the Newton couplings will change this first impression or if the classical matching technique has to be replaced. Most likely, we therefore have to fully appreciate that Γ_k is not a classical but an effective action containing arbitrarily high derivatives acting on the metric and correspondingly complicated surface terms. This will require a major structural generalization of the FRGE, for the following reason.

For the present, and although in ref. [166], the boundary contributions vanish under second variation and Dirichlet conditions and thus its couplings do not enter the Hessian $\widehat{\text{Hess}}_h[\Gamma_k]$ on the RHS of the flow equation. This implies that they cannot back-react on the RG evolution which rather is fully determined by the dynamical bulk couplings. In more complicated truncations, and at the exact level this situation will change; $\widehat{\text{Hess}}_h[\Gamma_k]$ will consist of bulk-bulk, bulk-boundary, and boundary-boundary blocks, and also the cutoff operator $\mathfrak{R}_k$ has an analogous block structure. At this point one must take a decision about how to coarse-grain fields living on the boundary. A priori there is a considerable freedom in choosing a cutoff for them, and clearly this choice will be crucially important for the bulk-boundary matching.

Returning to the more restricted scope of the present paper we shall consider it legitimate to extract effective field equations from the bulk action at level-(1) and assume that no surface

terms interfere with that. This is justified by either invoking the restriction from $\mathcal{F}$ to $\mathcal{F}'$ or, very conservatively, by narrowing down the truncation to the ‘perfect’ one of (B) above; this will not take anything away from the non-trivial results of the next section, in particular on black hole thermodynamics.

13.4 Counting field modes



This section, as well as part of the conclusion, follows closely the single-metric and matter induced bi-metric discussion published in [166].

In this subsection we present a number of examples which illustrate the ‘state counting’ property of $\ln \mathbb{Z}_k := -\Gamma_k[0; \bar{g}_k^{\text{s.c.}}]$, see chapter 11 for more details. Here we focus on the relevance of surface terms. For simplicity we fix $d = 4$ in this section.

13.4.1 Split-symmetry and $\partial M = \emptyset$

In a first example, let us assume $\partial M = \emptyset$ for a moment. With vanishing ghosts, self-consistent backgrounds are solutions of the (conventional looking, but ‘running’) Einstein equation

$$G_{\mu\nu}(\bar{g}_k^{\text{s.c.}}) = -\bar{\lambda}_k^{(1)}(\bar{g}_k^{\text{s.c.}})_{\mu\nu} \quad (13.28)$$

Notice that only level-(1) couplings, here identified with all higher levels and subsumed in the D-sector, enter the running tadpole equation. For every given solution to (13.28), the bulk part of eq. (13.23) leads to the running on-shell -action

$$\Gamma_k^{\text{grav}}[0; \bar{g}_k^{\text{s.c.}}] = -\frac{\bar{\lambda}_k^{(0)}}{8\pi G_k^{(0)}} \int_M d^4x \sqrt{g} \Big|_{g=\bar{g}_k^{\text{s.c.}}} \quad (13.29)$$

This quantity is strictly negative, and $\ln \mathbb{Z}_k$ positive therefore. (We assume $\bar{\lambda}_k^{(0)}$ and $G_k^{(0)}$ positive here.)

As an example, consider the maximally symmetric solution to (13.28) for $\bar{\lambda}_k^{(1)} > 0$, namely the 4-sphere $S^4(L)$ with radius $L_k = (3/\bar{\lambda}_k^{(1)})^{1/2}$. It has scalar curvature $R = 12/L_k^2 = 4\bar{\lambda}_k^{(1)}$ and the volume⁷ $\int d^4x \sqrt{g} = s_4 L_k^4 = 9s_4/(\bar{\lambda}_k^{(1)})^2$. Hence

$$\ln \mathbb{Z}_k = \frac{9s_4}{8\pi} \frac{\bar{\lambda}_k^{(0)}}{G_k^{(0)}(\bar{\lambda}_k^{(1)})^2} = \frac{9s_4}{8\pi} \frac{\bar{\lambda}_k^{(0)}}{g_k^{(0)}(\lambda_k^{(1)})^2} \quad (13.30)$$

In case of split-symmetry, $\lambda_k \equiv \lambda_k^{(0)} \equiv \lambda_k^{(1)} = \dots$ and $g_k \equiv g_k^{(0)} \equiv g_k^{(1)} = \dots$, this further reduces to $\ln \mathbb{Z}_k = \frac{9s_4}{8\pi} \frac{1}{g_k \lambda_k}$. This is a very intriguing and important result. It shows that the weighted number of modes integrated out between infinity and the IR cutoff k depends only on the properties of the dimensionless combination of couplings, which for the split-symmetric case is given by $G_k \bar{\lambda}_k = g_k \lambda_k$.

Along a RG trajectory of type IIIa, for instance [191], this product decreases from its fixed point value $\lim_{k \rightarrow \infty} g_k \lambda_k = g_* \lambda_* = \mathcal{O}(1)$ to the infrared value $G_{\text{obs}} \bar{\lambda}_{\text{obs}}$ which is observed at low scales; in real Nature it is of the order 10^{-120} .

It is known [190, 209] that for all type IIIa trajectories admitting a long classical regime there is a huge hierarchy $G_{\text{obs}} \bar{\lambda}_{\text{obs}} \ll g_* \lambda_*$. Hence, as expected from the monotonicity results of chapter 11

$$\ln \mathbb{Z}_{k \rightarrow 0} \gg \ln \mathbb{Z}_{k \rightarrow \infty} \quad (13.31)$$

⁷Here and in the following we write $s_n \equiv \text{Vol } S^n(1) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ and $b_n \equiv \text{Vol } B^n(1) = \pi^{n/2}/\Gamma(n/2 + 1)$ for the volume of the unit n -sphere and n -ball, respectively.

Along the hypothetical trajectory realized in Nature [209, 212] the Boltzmann weighted number of modes integrated out when the cutoff approaches zero is about $\ln \mathbb{Z}_{k \rightarrow 0} \approx 10^{120}$, while the NGFP value $\ln \mathbb{Z}_{k \rightarrow \infty}$ is basically zero.

13.4.2 Euclidean space with non-vanishing boundary

As a second example we consider a four dimensional Euclidean spacetime with a non-empty boundary. We use the associated tadpole equation (13.26) to explore its contents employing the results of the bi-metric-bulk – pure-background-boundary truncation. For simplicity we specialize for a regime of the underlying RG trajectory in which the cosmological constant in this equation, the level-(1) coupling $\bar{\lambda}_k^{(1)}$, is negligible, yielding

$$\bar{R}_{\mu\nu} = 0 \quad \text{with} \quad \bar{\lambda}_k^{(1)} = 0 \quad (13.32)$$

Clearly, the simplest Ricci flat solution is flat space. So let us assume M is a subset of R^4 , with $\partial M \neq \emptyset$, and equipped with a flat metric. Inserting the configuration

$$\bar{g}_{\mu\nu}^{\text{s.c.}} = \delta_{\mu\nu} \quad \text{with} \quad \bar{\lambda}_k^{(1)} = 0, \quad (13.33)$$

into $\Gamma_k[h=0; \bar{g}_{\mu\nu}]$, the ‘mode counting’ device assumes the following form

$$\begin{aligned} -\ln \mathbb{Z}_k = \Gamma_k[0; \delta_{\mu\nu}] &= \frac{\bar{\lambda}_k^{(0)}}{8\pi G_k^{(0)}} \text{Vol}(M) + \frac{\bar{\lambda}_k^{\partial(0)}}{8\pi G_k^{\partial(0)}} \text{Vol}(\partial M) \\ &\quad - \frac{1}{8\pi G_k^{\partial(0)}} \int_{\partial M} d^3x \sqrt{\bar{H}} \bar{K} \end{aligned} \quad (13.34)$$

A perhaps surprising property of this equation is that it involves a non-zero (bulk) cosmological constant term even though it applies to flat space. However, the condition for flat space to be a self-consistent background is that the cosmological constant *at level-(1)* is negligible. Its counterpart at level-(0), the one appearing in (13.34) may have any value. Only when split-symmetry happens to be intact we have $\bar{\lambda}_k^{(0)} = \bar{\lambda}_k^{(1)}$ so that the bulk term on the RHS of (13.34) indeed vanishes. This results in a kind of ‘holographic’ property of the function $\mathbb{Z}_k$ which then is expressed by surface terms only. While the Gibbons-Hawking-York term is the most important contribution, the condition $\bar{\lambda}_k^{(0)} = \bar{\lambda}_k^{(1)} = 0$ still leaves room for a non-zero boundary cosmological constant $\bar{\lambda}_k^{\partial(0)}$, leading to a term proportional to $\text{Vol}(\partial M)$.

In the language of thermodynamics equation (13.34) defines a certain ‘free energy’. The terms proportional to $\bar{\lambda}_k^{(0)} \text{Vol}(M)$ and $\bar{\lambda}_k^{\partial(0)} \text{Vol}(\partial M)$ amount to a homogeneously distributed bulk and surface energy density reminiscent of the volume and surface energy of a liquid droplet. In this picture the last term in (13.34), the Gibbons-Hawking-York contribution, is equally natural and describes how the droplet gains or loses energy by developing a curved or crumpled surface.

What is the statistical mechanics, and what are the pertinent degrees of freedom which underlie this thermodynamics at the microscopic level?

In the context of the effective average action the answer is clear: It is the statistical mechanics of the matter and geometry *fluctuations* about their respective backgrounds. The generalized harmonic modes of those fluctuations are ‘counted’ by the partition function $\mathbb{Z}_k$ when the IR cutoff k is lowered from infinity to zero. The various running coupling constants contained in Γ_k and $\ln \mathbb{Z}_k$ parametrize how the number of fluctuation modes, contributing to the functional integral and weighted with the ‘Boltzmann factor’ e^{-S} , decreases when we ‘zoom’ deeper and deeper into the microscopic structure of spacetime by increasing k .

Another remark is in order at this point. We stress that the occurrence of surface terms in the partition function $\mathbb{Z}_k$ on empty flat space is both *unavoidable* and *natural* from the physics point

of view. It is unavoidable because the RG flow generates such terms when $\partial M \neq \emptyset$. Therefore, contrary to the classical action underlying the variational principle of General Relativity we may not subtract any terms on an ad hoc basis ‘by hand’ from Γ_k . The surface terms are also natural because the number of field modes counted by $\mathbb{Z}_k$ will depend on the shape of ∂M in general, and this dependence can lead to observable effects, the most famous example being the Casimir effect.

In classical relativity, in order to obtain a finite action for asymptotically flat spacetimes, one usually replaces $K \rightarrow K - K_0$ in the Gibbons-Hawking-York action S_{GH} . Here K_0 is the extrinsic curvature of ∂M when embedded into a *flat* spacetime. From the above remarks it should be clear that within the average action approach this procedure would not only be unmotivated but wrong since we might lose essential physics.

Closed Euclidean 4-ball

To be more concrete about M let us take M to be a 4-ball, $M = B^4(L)$, with arbitrary radius L . Embedding the boundary $\partial M = S^3(L)$ into R^4 its extrinsic curvature equals $\bar{K} = 3/L$. As a result,

$$-\ln \mathbb{Z}_k = \frac{b_4}{8\pi} \frac{\bar{\lambda}_k^{(0)} L^4}{G_k^{(0)}} + \frac{s_3}{8\pi} \frac{\bar{\lambda}_k^{\partial(0)} L^3}{G_k^{\partial(0)}} - \frac{3s_3}{8\pi} \frac{L^2}{G_k^{\partial(0)}} \quad (13.35)$$

Here it is particularly obvious that the asymptotic series for the heat kernel gives rise to a systematic expansion in powers of $1/L$.

‘Cylindrical’ foliation

As a second example consider a simple Euclidean caricature of the foliated cylinder type spacetimes one considers in relation with the initial or boundary value problem of Lorentzian gravity. Again we embed M into $(R^4, \delta_{\mu\nu})$. We fix a foliation of R^4 in terms of flat 3-dimensional hypersurfaces labeled by a parameter t referred to as ‘Euclidean time’. This gives rise to a corresponding foliation on M . We consider M foliated by hypersurfaces Σ_t , $t \in [t_1, t_2]$ which are bounded by closed 2-surfaces S_t . Thus ∂M consists of the hypersurfaces Σ_{t_1} , Σ_{t_2} , and the union of all $S_t = \partial \Sigma_t$.

For simplicity we take M to be the direct product of a 3-ball of radius ℓ , $B^3(\ell)$, with the time interval. Thus $\Sigma_t = B^3(\ell)$ and $S_t = S^2(\ell)$ for any $t \in [t_1, t_2]$ so that we have $\text{Vol}(M) = b_3 \ell^3 (t_2 - t_1)$ and $\text{Vol}(\partial M) = 2b_3 \ell^3 + s_2 \ell^2 (t_2 - t_1)$. The extrinsic curvature of the flat $t = \text{const}$ surfaces in R^4 vanishes and so Σ_{t_1} and Σ_{t_2} do not contribute to the surface integral over $\bar{K}$. The only contribution comes from the union of all 2-spheres S_t . Since the trace of the extrinsic curvature of $S^2(\ell)$ embedded in flat R^3 is given by $K = 2/\ell$ we therefore get $\int d^3x \sqrt{H} \bar{K} = (t_2 - t_1) \cdot (2/\ell) \cdot \text{Vol} S^2(\ell) = 2s_2 (t_2 - t_1) \ell$.

This brings us to the final result

$$-\ln \mathbb{Z}_k = \left[\frac{\bar{\lambda}_k^{(0)}}{6G_k^{(0)}} \ell^3 + \frac{\bar{\lambda}_k^{\partial(0)}}{2G_k^{\partial(0)}} \ell^2 - \frac{\ell}{G_k^{\partial(0)}} \right] (t_2 - t_1) + \frac{\bar{\lambda}_k^{\partial(0)}}{3G_k^{\partial(0)}} \ell^3 \quad (13.36)$$

The coefficient of $(t_2 - t_1)$ on the RHS of (13.36) may be thought of as a certain energy associated to the empty flat 3-dimensional space (not ‘spacetime’) interior to a 2-sphere of radius ℓ .

13.4.3 Thermodynamics of the Schwarzschild black hole

We continue to restrict ourselves to the self-consistent backgrounds of the type (13.32), i.e. Ricci-flat metrics, $R_{\mu\nu}(\bar{g}) = 0$. Perhaps the most prominent representative of this class is the

Euclidean Schwarzschild solution $ds^2 = f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2$ with

$$f(r) = 1 - \frac{R_s}{r} \quad (13.37)$$

Here $r \in (R_s, \infty)$, $t \in [0, \beta]$, and time is now compactified to a circle of circumference $\beta \equiv 4\pi R_s$.

Note that the Schwarzschild radius R_s has the status of a free constant of integration with the dimension of a *length*. It is usually re-expressed in terms of a mass, M , so as to recover Newtonian gravity asymptotically. Then $R_s = 2GM$ where G is the Newton constant of the classical theory. Since in quantum gravity it is not a priori obvious which constant $G_k^{(0)}$, $G_k^{(1)}$, $G_k^{(2)}$, $\dots$ should be used to convert R_s to a mass we shall refrain from doing this and continue to label the family of Schwarzschild metrics $g_{\mu\nu}^{\text{Schw}}$ by the length parameter R_s .

The running on-shell action

We consider the Euclidean Schwarzschild manifold M foliated by 3D hypersurfaces Σ_t of constant t . They carry the metric $ds_{\Sigma_t}^2 = f(r)^{-1}dr^2 + r^2d\Omega^2$. With the time compactified and the period chosen as⁸ $\beta = 4\pi R_s$, the only boundary of M is the union of the asymptotic 2-spheres $\partial\Sigma_t = S^2$ on which $r = \text{const} \equiv \hat{r}$, $\hat{r} \rightarrow \infty$.

Let us insert this background into the truncation ansatz (13.23) with $A = 0$ where, by assumption, $\bar{\lambda}_k^{(1)} = 0$. All that needs to be evaluated is $-\ln \mathbb{Z}_k = \Gamma_k[h = 0; \bar{g} = g^{\text{Schw}}]$. Since all terms involving $\bar{R}$, will vanish we are left only with the bulk and boundary cosmological constant terms, respectively, together with the extrinsic curvature term:

$$-\frac{2}{16\pi G_k^{\partial(0)}} \int_{\partial M} d^3x \sqrt{\bar{H}} (\bar{K} - \bar{K}_0) - \frac{2}{16\pi G_k^{\partial(0)}} \int_{\partial M} d^3x \sqrt{\bar{H}} \bar{K}_0 \quad (13.38)$$

Here we rewrote the extrinsic curvature of M in ∂M by adding and subtracting $\bar{K}_0$, the curvature of ∂M embedded in flat space, $\bar{K} \equiv (\bar{K} - \bar{K}_0) + \bar{K}_0$. As a result, the first integral of (13.38) is the usual subtracted Gibbons-Hawking-York term, while the second, to leading order in $R_s/r \rightarrow 0$, becomes independent of the spacetime curvature caused by the black hole. For very large $\hat{r}$ it corresponds to the surface term of flat spacetime considered in Section 13.4.2. When added to the bulk and boundary cosmological constant terms it yields the free energy of flat spacetime. Thus the on-shell average action boils down to

$$-\ln \mathbb{Z}_k = -\frac{1}{8\pi G_k^{\partial(0)}} \int_{\partial M} d^3x \sqrt{\bar{H}} (\bar{K} - \bar{K}_0) + \dots \quad (13.39)$$

where the dots stand for the contributions of flat space. The evaluation of the integral in (13.39) is standard; it yields $\int d^3x \sqrt{\bar{H}} (\bar{K} - \bar{K}_0) = -2\pi R_s \beta$ when $\hat{r} \rightarrow \infty$. For the partition function this leads us to

$$-\ln \mathbb{Z}_k = \frac{\beta R_s}{4G_k^{\partial(0)}} + \dots = \frac{A}{4G_k^{\partial(0)}} + \dots \quad (13.40)$$

where $A \equiv 4\pi R_s^2$ denotes the area of the event horizon (in the Lorentzian interpretation).

Eq. (13.40) is a very instructive formula. Structurally it coincides with the familiar semi-classical Gibbons-Hawking-York result [79]. However, the (unique and truly constant) classical Newton constant appearing there got replaced by a specific member of the various infinite families of running Newton-type couplings which parametrize a general bi-metric average action. Thus we see that *the scale dependence of the partition function $\mathbb{Z}_k$ and the derived thermodynamical quantities are governed by the boundary Newton constant at level zero.*

⁸If $\beta \neq 4\pi R_s$ there is another boundary at the horizon, however. We shall not consider this generalization here.

Thermodynamics at finite scale

Note that the Bekenstein-Hawking temperature is scale independent,

$$T = \frac{1}{\beta} = \frac{1}{4\pi R_s}, \quad (13.41)$$

while the free energy $F_k \equiv -\beta^{-1} \ln \mathbb{Z}_k$ inherits its k -dependence from $G_k^{\partial(0)}$:

$$F_k = \frac{R_s}{4G_k^{\partial(0)}} = \frac{1}{16\pi G_k^{\partial(0)}} \frac{1}{T} \quad (13.42)$$

If we apply the standard relations $U = -T^2 \frac{\partial}{\partial T} (F/T)$ and $S = -\partial F / \partial T$ at every *fixed value of* k we obtain for the internal energy and entropy, respectively:

$$U_k = \frac{R_s}{2G_k^{\partial(0)}} = \frac{1}{8\pi G_k^{\partial(0)}} \frac{1}{T} = 2F_k \quad (13.43)$$

$$S_k = \pi \frac{R_s^2}{G_k^{\partial(0)}} = \frac{A}{4G_k^{\partial(0)}} = \frac{1}{16\pi G_k^{\partial(0)}} \frac{1}{T^2} \quad (13.44)$$

Likewise $C = \partial U / \partial T$ yields the specific heat capacity

$$C_k = -2\pi \frac{R_s^2}{G_k^{\partial(0)}} = -\frac{1}{8\pi G_k^{\partial(0)}} \frac{1}{T^2} \quad (13.45)$$

Obviously the RG running of all thermodynamical functions of interest is governed by a single running coupling, namely $1/G_k^{\partial(0)}$.

Running ADM mass

Up to now we never ascribed any mass to the Schwarzschild spacetime. As we mentioned already, it is more natural to characterize it by a length such as R_s . Its conversion to a mass is a matter of convention, strictly speaking, which in quantum gravity becomes particularly ambiguous. Nevertheless, our result (13.40) suggests that a natural way of relating a mass to a black hole with a given parameter R_s is by means of $G_k^{\partial(0)}$:

$$M_k \equiv \frac{R_s}{2G_k^{\partial(0)}} \quad (13.46)$$

We emphasize again that R_s has no k -dependence; it labels different solutions of the truncated tadpole equation, $\bar{R}_{\mu\nu} = 0$, which happens to be independent of any running coupling. So the k -dependence of the running mass M_k is entirely due to $G_k^{\partial(0)}$. It can be seen as a *scale dependent generalization of the classical ADM mass*.⁹ The definition (13.46) is motivated by observing that all relations of semiclassical black hole thermodynamics retain their form when quantum gravity effects are included via the average action *provided we replace the classical mass by the running mass* M_k . The mass (13.46) controls the partition function of a single black hole,

$$-\ln \mathbb{Z}_k = 4\pi G_k^{\partial(0)} M_k^2 = \frac{1}{2} \beta M_k, \quad (13.47)$$

and the ensuing thermodynamical relations

$$F_k = \frac{1}{2} M_k \quad U_k = M_k, \quad S_k = 4\pi G_k^{\partial(0)} M_k^2, \quad (13.48)$$

look like their semiclassical counterparts with the replacement $M \rightarrow M_k$. In its other roles, the classical mass M might possibly get replaced by running masses with a different k -dependence.

⁹In more general backgrounds it becomes $M_k = -(8\pi G_k^{\partial(0)})^{-1} \oint (K - K_0)$ where the integral is over an asymptotic sphere.

The range of validity of self-consistent Schwarzschild solutions

In order to understand the range of validity for the present discussion, let us go back to the running tadpole equation that defines the k -dependent ‘self-consistent’ background solutions:

$$G^{\mu\nu}(\bar{g}_k^{\text{s.c.}}) = -\bar{\lambda}_k^{(1)} \bar{g}_k^{\text{s.c.}\mu\nu} \iff R(\bar{g}_k^{\text{s.c.}}) = 4\bar{\lambda}_k^{(1)} \quad (13.49)$$

This defines $\bar{g}_k^{\text{s.c.}}$ to be an Einstein-metric for all k associated to some constant, but k -dependent curvature $R(\bar{g}_k^{\text{s.c.}}) = 4\bar{\lambda}_k^{(1)}$. For the Schwarzschild solution, we impose certain homogeneity and isotropy constraints on the spatial part of the metric and consider the case of $\bar{\lambda}_k^{(1)} \approx 0$. This leads to $\bar{g}_k^{\text{s.c.}} \equiv \bar{g}^{\text{Schw}}$. However, the assumption $\bar{\lambda}_k^{(1)} \approx 0$ is only justified for some range of scales $k \in (k_{\text{IR}}^{\text{Schw}}, k_{\text{UV}}^{\text{Schw}})$, in particular there exists no RG trajectory for which $\bar{\lambda}_k^{(1)} \equiv k^2 \lambda_k^{(1)} \approx 0$ for all k . Figure 13.4 shows the k -dependence of the cosmological bulk coupling, $\bar{\lambda}_k^{(1)}$, for a typical type IIIa trajectory. While towards decreasing values of k , the approximation $\bar{\lambda}_k^{(1)} \approx 0$ is well justified,

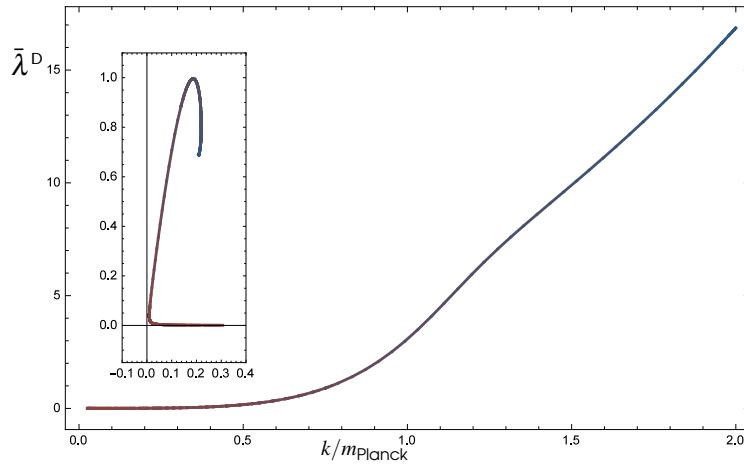


Figure 13.4: The k -dependence of the dimensionful dynamical cosmological constant on the bulk, $\bar{\lambda}_k^{(1)} \equiv k^2 \lambda_k^{(1)}$, evaluated for a typical IIIa trajectory (inset) in units of Planck mass. From this result we see that the range in which the self-consistent background can be described by a Euclidean Schwarzschild metric with vanishing $\bar{\lambda}_k^{(1)}$, extends to arbitrary small values of k , i.e. $k_{\text{IR}}^{\text{Schw}} \rightarrow 0$. Towards the ultraviolet, leaving the semiclassical regime, $\bar{\lambda}_k^{(1)}$ increases significantly and cannot be neglected anymore. Thus, from this scale $k_{\text{UV}}^{\text{Schw}}$ onwards, the k -dependence of $\bar{g}_k^{\text{s.c.}}$ becomes important.

implying in particular $k_{\text{IR}}^{\text{Schw}} = 0$, we see that when the trajectory leaves the semiclassical regime towards the UV fixed point the dimensionful cosmological constant increases and becomes non-negligible, hence $k_{\text{UV}}^{\text{Schw}} < \infty$. This implies that solutions of the running tadpole equations (13.49) turn k -dependent above $k_{\text{UV}}^{\text{Schw}}$ and Ricci-flatness has to be replaced by the constant curvature condition, i.e.

$$\bar{g}_k^{\text{s.c.}} = \bar{g}^{\text{Schw}} \quad \forall k \in (0, k_{\text{UV}}^{\text{Schw}} \approx 0.7 m_{\text{Planck}}) \quad \text{else} \quad R(\bar{g}_k^{\text{s.c.}}) = 4\bar{\lambda}_k^{(1)} \quad \bar{\lambda}_k^{(1)} \neq 0 \quad (13.50)$$

In the following we make use of the observation that as long as the background curvature effects are small, i.e. $R(\bar{g}_k^{\text{s.c.}}) \approx 0$, the self-consistent solution is k -independent. Since the range of validity includes the classical and semiclassical regime, we will be most interested in the results at the edge, just below $k_{\text{UV}}^{\text{Schw}}$, where the semiclassical approximation is still valid.

Explicit k -dependence of the boundary Newton constant

Let us see now what we obtain for M_k from the running of $G_k^{\partial(0)}$. Its k -dependence is governed by the anomalous dimension $\eta^{\partial(0)}$ which depends on the bulk couplings $g^{(1)} \equiv g^{(2)} \equiv \dots$ and

$\lambda^{(1)} \equiv \lambda^{(2)} \equiv \dots$. In the range of the Schwarzschild solution $k \in (0, k_{\text{UV}}^{\text{Schw}} \approx 0.7 m_{\text{Planck}})$ we can use the semiclassical approximation that yields

$$\eta^{\partial(0)} = -(d-2) \omega_d^{\partial(0)} g^{\partial(0)} \quad (13.51)$$

with the crucial coefficient obtained in eq. (8.63)

$$\omega_d^{\partial(0)} = -\frac{d(d-3)}{3(4\pi)^{d/2-1}} \frac{1}{(d-2)} \Phi_{d/2-1}^1(0) < 0 \quad (13.52)$$

The following semiclassical solution for the RG equation $\partial_t (1/G_k^{\partial(0)}) = -\eta^{\partial(0)} (1/G_k^{\partial(0)})$ can thus be obtained, which reads in 4 dimensions:

$$\frac{1}{G_k^{\partial(0)}} = \frac{1}{G_0^{\partial(0)}} + \omega_4^{\partial(0)} k^2 \quad (13.53)$$

Notice, that this is the same result as obtained with the full-fledged single-metric- and the bi-metric matter induced-truncation in ref. [166].

Explicit k -dependence of the ADM mass

Using (13.53) in (13.46) we associate the following running mass to the black hole with Schwarzschild radius R_s :

$$M_k = [1 + \omega_4^{\partial(0)} G_0^{\partial(0)} k^2] M_0 \quad \text{where} \quad M_0 \equiv \frac{R_s}{2G_0^{\partial(0)}} \quad (13.54)$$

Though, from the present truncation it seems incompatible with the Asymptotic Safety requirement, let us nevertheless match the surface and bulk Newton constants at $k=0$ and identify this quantity with the standard Newton constant, $G_0^{\partial(0)} = G_0^{(0)} \equiv G \equiv m_{\text{Planck}}^{-2}$. Then we recover the ordinary relationship $M_0 = R_s/(2G)$ in the extreme infrared (at $k=0$), but at higher scales the mass associated to the very same geometry is *smaller* than M_0 :

$$M_k = \left[1 - |\omega_4^{\partial(0)}| \left(\frac{k}{m_{\text{Planck}}} \right)^2 \right] M_0 \quad (13.55)$$

In writing down (13.55) we made it manifest that the coefficient $\omega_4^{\partial(0)}$ turned out negative. As a consequence, M_k *decreases for increasing k , reaches zero at a scale near $k = m_{\text{Planck}}$, and becomes negative for even larger k -values.*

An attempt at interpreting this behavior could be as follows. It is known that in QEG the bulk Newton constant G_k decreases for increasing k , and this was interpreted as an indication of gravitational anti-screening due to the energy and momentum of the virtual particles surrounding every massive body; because of the attractivity of gravity, they are pulled towards this body, adding positively to its bare mass, whence the virtual cloud leads to an effective mass that increases with increasing distance [122].

Now we have seen that the boundary Newton constant $G_k^{\partial(0)}$ *increases* for increasing k at least in the upper half plane. Interestingly enough, what at first sight might seem to contradict the picture of gravitational anti-screening, in view of $M_k = \frac{1}{2} R_s / G_k^{\partial(0)}$, at least heuristically, actually *confirms* it: According to this definition of ‘mass’, *the running mass of any material body decreases with increasing k , or decreasing distance.* Somewhere near $k = m_{\text{Planck}}$ it even seems to vanish, indicating probably that in this regime a more elaborate treatment is necessary.

Running thermodynamic quantities

The running free and internal energy, the entropy and the specific heat capacity are governed by the same function of k as M_k in (13.55): $F_k, U_k, S_k, C_k \propto [1 - |\omega_4^{\partial(0)}|(k/m_{\text{Planck}})^2]$. Note, however, that the specific heat has a negative IR value, $C_0 = -2\pi R_S^2/G_0^{\partial(0)}$, and tends to zero in the Planck regime, see Fig. 13.5. At this point, when the underlying RG trajectory leaves the semiclassical regime and approaches the vicinity of the non-Gaussian fixed point, we expect further effects due to the k -dependence of the self-consistent background solution.

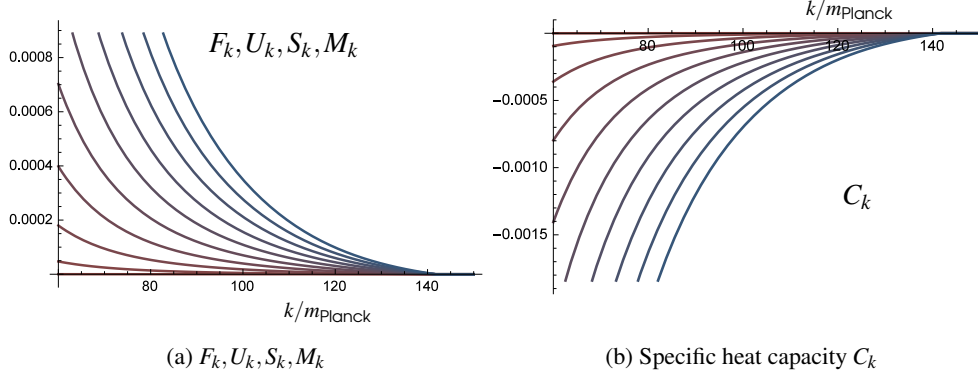


Figure 13.5: The k -dependence of the thermodynamical quantities.

A natural cutoff identification.

Even though the shortcut to extracting physical information from the running couplings in Γ_k by identifying k with some physical scale ('RG improvement') is notoriously ambiguous in general, it is clear that the black hole spacetime specified by a given R_S has a distinguished intrinsic mass scale associated to it that does not rely on any artificial conversion factor, namely $1/R_S$, or the temperature $T = (4\pi R_S)^{-1}$.

If we tentatively adopt the cutoff identification $k \approx 1/R_S$, and go from macroscopic astrophysical black holes to microscopic ones with a Planckian Schwarzschild radius, we find that M_k decreases monotonically, heading for $M_k = 0$ near $R_S = \ell_{\text{Planck}}$. Remarkably, by eqs. (13.47), the thermodynamical quantities, the entropy in particular, all vanish in this limit: $F_k \rightarrow 0$, $U_k \rightarrow 0$, $S_k \rightarrow 0$ for $R_S \searrow \ell_{\text{Planck}}$.

It is particularly intriguing that *the specific heat capacity C_k exhibits the same behavior however starting from negative values*. This suggests that near ℓ_{Planck} the notorious instability of classical gravity possibly gets tamed in a dynamical way: The system no longer can lower its energy by accreting further mass and the gravitational collapse might come to a halt.

This picture based on the boundary Newton constant is surprisingly similar to what we found in [305–309] by a rather different reasoning, namely the 'RG improvement' of the *bulk* Newton constant in the classical formula $f(r) = 1 - 2GM/r$. Keeping M fixed we replaced $G \rightarrow G_k$ and identified $1/k$ with the radial proper distance.

Future work will have to clarify the precise relationship between the two treatments, in particular whether they are different pictures of the same phenomenon or should be superimposed rather. In addition, the disentanglement of the boundary level couplings turns out very important, in particular to understand if the level-(0) Newton coupling is indeed trapped in the lower half plane for negative Newton couplings. In this case the previous results have to be evaluated for negative $G_k^{\partial(0)}$ which then implies that we need further surface invariants that compensate this sign and lead to a positive entropy and ADM mass.

13.5 Conclusion

In this chapter we studied the results of the functional RG flows of Quantum Einstein Gravity on spacetime manifolds with boundary. This extends our previous work [166] to a bi-metric truncation where the running of the boundary coefficients is induced by a dynamical Newton and cosmological coupling, now being disentangled from their level-(0) counterparts. Most of the single-metric and matter induced bi-metric results of the boundary sector could be recovered. All these truncations have in common that they contained various surface terms such as the Gibbons-Hawking-York term for instance. While the motivation to incorporate these boundary invariants into the EAA ansatz exceeds far beyond the classical matching condition that was established to recover Einstein's field equation from a variational principle.

We studied the global RG properties of the boundary sector, which due to their structural similarity exhibits certain global features reminiscent of the bulk level-(0) part of theory space. In particular, we discovered a running UV-attractor that however is mainly situated in the lower half plane of negative $g^{\partial(0)} < 0$ and coincides with the non-Gaussian fixed point for asymptotically safe trajectories. Though the present truncation is only a first step towards understanding the boundary sector, if the results remain valid while including higher level invariants, it would suggest that Asymptotic Safety together with Background Independence exclude the possibility of a bulk-boundary matching as proposed by Gibbons, Hawking, and York. From the FRG perspective, this is actually not a real drawback, but supposed to happen, if not for the Gibbons-Hawking-York functional than for other boundary invariants. Leaving the structural generalization of the FRGE which is needed for arbitrary (untruncated) action functionals to future work, we justified the variational procedure for the second derivative actions considered here. For a proper interpretation of the surface terms and the variational principle it was crucial to take the bi-metric character of gravitational average action into account. In an expansion with respect to $h_{\mu\nu}$ a generic functional Γ_k contains 'towers' of bulk Newton constants $G_k^{(p)}$, boundary Newton constants $G_k^{\partial(0)}$, and many more similar couplings whereby the 'level' $p = 0, 1, 2, \dots$ is indicative of the $h_{\mu\nu}$ -power in the corresponding field monomial. Since the background-quantum split-symmetry is broken by the IR cutoff, the different levels evolve independently under the RG flow.

A key observation in this context is the following. The partial differential equation which determines self-consistent backgrounds, that is, backgrounds which once prepared by external means are not modified by the intrinsic quantum fluctuations, involve only the couplings of *level-(1)*. The (thermodynamical, etc.) properties of the backgrounds they imply are determined by the *level-(0)* couplings in addition. In the example considered, in fact *only* those of level-(0) happened to be relevant. It is therefore possible to have a bulk-boundary matching of Newton's constant at level-(1), $G_k^{(1)} = G_k^{(1,\partial)}$, hence a standard variational principle, but nevertheless a mismatch at level zero: $G_k^{(0)} \neq G_k^{(0,\partial)}$. This mismatch can encode important information relevant to the effective field theory description of physics at finite scales k ; black hole thermodynamics turned out to be a prime example, however special care has to be exercised concerning the sign of $G^{\partial(0)}$.

In (semi-)classical black hole physics there exists only a single mass parameter, M , and this parameter plays various conceptually rather different roles. It controls, for instance, the semiclassical partition function of a single black hole but it also describes the strength of the Newtonian force between two black holes at large distances, say. In quantum gravity, in the context of the average action, there does not exist a single running mass which serves all these purposes at a time. Likewise, there is not a single, but actually quite many different running Newton constants. Each of them takes over one specific role played by the standard Newton constant, or the classical concept of mass, respectively, and depending on this role its k -dependence

is different in general. We saw that the partition function and the related thermodynamics of an isolated black hole is governed by $M_k \propto 1/G_k^{(0,\partial)}$, while the effective Einstein equation involve the level-(1) bulk couplings only.

Based upon the concept of self-consistent backgrounds, solutions of the running tadpole equation, we employed the proposed $\mathcal{C}_k$ -function of chapter 11, to count the number of field modes integrated out by the average action. The associated partition function related to a given RG trajectory and a background metric, $\bar{g}$, describes the statistical mechanics of the metric fluctuations relative to this background. While this mode count is of interest also on spacetimes without boundary, here it led us to an intriguing scale dependent generalization of black hole thermodynamics which represents the physical basis and motivation for the specific definition of M_k .

A more phenomenological analysis of the corresponding quantum effects in black hole spacetimes is an interesting playground to test and understand the implications of the RG results. This seems to be a very promising path to follow in future work.



14.1 Summary

Asymptotic Safety
Background Independence vs. Asymptotic Safety
Towards a C -function in quantum gravity
Positivity vs. Asymptotic Safety
Asymptotic Safety on manifolds with boundary

14.2 Outlook

14. CONCLUSION

In this chapter we give a brief summary of the results discussed in this part of the thesis and then conclude with possible future projects in these directions.

14.1 Summary

On the path towards a deep understanding of Nature physicists are confronted with two theoretical concepts, the Standard Model of particle physics and General Relativity, each outstanding in its predictivity and precision. In quest of a unified theory that intertwine the quantum nature of matter with the geometrical construction of spacetime, the search still continues. Many decades of research have brought deep insights in both concepts, have revealed their deficiencies and their mathematical foundations, and many physicists nowadays believe that a solid theory of Quantum Gravity may cure most, or even all, of it. Promoting General Relativity to a quantum level is a very sophisticated task, since most of the methods available in the construction of previous quantum field theories turned out to be inapplicable.

One of the reasons may be that, considered as a gauge theory, gravity is special because it relates the geometry of spacetime (its base space) with the structure of its gauge group and thus renders geometry as the mediating gauge field. Hence, Background Independence, the requirement that the formulation should be independent on any – possibly prescribed – background geometry, is central to all constructions of quantum theories of gravity.

On the other hand, Quantum Gravity seem to manifest as a nontrivial theory at a fundamental level, i.e. on very small scales (in the ultraviolet, so to say), thus that perturbative approaches may only get a glimpse of the full beauty of gravity well below the Planck energy. In addition, it turned out that from the perspective of conventional perturbative approaches quantum gravity seems to be non-renormalizable.

A variety of different approaches trying to circumvent these difficulties brought to light hidden generalized concepts that only become relevant when gravity enters the game. A conservative approach to Quantum Gravity and thus to a complete, unified theory is Asymptotic Safety. Based on the ideas of Weinberg [145] there is a (non-)perturbative notion of renormalizability as a (non-)trivial fixed point of the Renormalization Group (RG) flow. Employing the

Effective Average Action (EAA) formulation, Wetterich [112] succeeded in deriving an exact functional differential equation, the FRGE, which is UV- and IR-regulated. This non-linear equation defines RG trajectories on theory space, a manifold containing all possible invariants compatible with the required symmetries and field content, from which only those that emanate from a suitable UV fixed point are physical relevant. The position of this fixed point defines the fundamental theory, which is thus a prediction of the formalism instead of a mere input.

However, the non-linearity of the FRGE forces to employ, still non-perturbative, approximations, so called truncations, that consider only a subspace of all possible invariants. Imposing different symmetries, varying the field contents, or changing the underlying topology has provided strong evidence for the existence of a suitable non-trivial UV fixed point, thus supporting the Asymptotic Safety conjecture of Quantum Gravity. Besides increasing the complexity of truncations by including further invariants, studying the general properties of the FRGE for gravity and test its consistency with the requirement a theory of Quantum Gravity should fulfill, is at least equally important.

In this thesis, we studied action functionals of bi-metric Einstein-Hilbert type augmented with a boundary Gibbons-Hawking-York surface term, resolving the RG evolution of six basis invariants. The derivation of the corresponding beta-functions was carried out in great detail in part II. Therefore, we introduced a new technique that simplifies bi-metric calculations by exploiting the properties of the employed truncation on the basis of the gauge sector.

In part III of this thesis we studied the derived beta-functions with respect to various central questions of the Asymptotic Safety program to QG. We here present only a very brief summary of the results and refer to the conclusions of the respective chapters for more details.

14.1.1 Asymptotic Safety

In chapter 9 the dimensionless beta-functions were analyzed for (non-trivial) fixed points of the RG flow in four spacetime dimensions. We revealed a hierarchy among the beta-functions dominated by the dynamical couplings which allows to study their influence on bulk- and boundary-sector separately. For the dynamical part of theory space we found three fixed points, one Gaussian and two non-Gaussian solutions. In particular we recovered a fixed point reminiscent of the previous results in the Asymptotic Safety program to Quantum Gravity, with a positive Newton and cosmological constant. It was found that for each dynamical solution both, the background and the bulk level-(0) sector, give rise to a trivial and non-trivial solution, which yields a total of 12 fixed points.

We then discussed their critical behavior in detail showing that in the present case each (non-)trivial component in the fixed point comes along with (two) one relevant and (zero) one irrelevant direction. Thus dimensionality of their UV-critical hypersurfaces assumes values between 3 and 6, whereby the lower limit represents a triple-GFP and the upper limit corresponds to a triple-NGFP.

14.1.2 Background Independence vs. Asymptotic Safety

Background Independence is a fundamental requirement every quantum theory of gravity has to take care of. The principle that geometry is the dynamical object itself is found to be implemented by invoking no background at all (e.g. LQG, CDT, ...), or by introducing a background geometry at intermediate states of the quantization while assuring that observations do not depend on this particular choice. In the EAA approach the background field method realizes the latter option at the expense of a now bi-field (in QEG bi-metric) formulation of the FRGE. Background Independence translates into a reduction of the EAA to a single-field dependence at the physical scale where it agrees with the standard effective action Γ and is known as split-symmetry. In chapter 10 we investigated in which way the conditions of Background In-

dependence and non-perturbative renormalizability can be fulfilled simultaneously. The results uncover a topological feature of the Renormalization Group flow, the running UV-attractor, that guarantees Asymptotic Safety on the one hand and assures Background Independence on the other hand, thereby increasing the predictivity of the setting. Thus, there is a first evidence for the coexistence of Background Independence and Asymptotic Safety. Furthermore, we studied the conceptual and quantitative differences of bi-metric calculations and their single-metric approximation, revealing only small portions of theory space where the latter seems to be reliable. Miraculously, the regime of the non-Gaussian fixed point is one part of theory space where the single-metric approximation qualitatively agrees with its bi-metric counterparts, even though especially the critical exponents differ in both cases.

14.1.3 Towards a C -function in quantum gravity

In chapter 11 we studied a C -function like feature of the Effective Average Action under Renormalization Group evolution. The conjectured function on theory space acts as a mode counting device and at the same time should fulfill the monotonicity condition along suitable Renormalization Group trajectories. To test the violation of this monotonicity condition for a particular truncation would then be a very powerful tool to draw conclusions about the reliability and the error of the corresponding results. The properties of the proposed candidate of a C -function could be established for the bi-metric Einstein-Hilbert truncation for all RG solutions compatible with Background Independence. In sharp contrast, a large class of split-symmetry violating and the full set of single-metric RG trajectories fail to reproduce a monotonicity. The clash between the proposed C -function and the non-monotonicity for those trajectories was shown to be not a failure of the C -function but an indication for the unreliability of those solutions. While our proposal consists of two components, the first which is manifestly monotone for exact solutions and the second which ‘measures’ the split-symmetry violation along the RG trajectory, it turns out to be the first term which generates the non-monotone behavior of the C -function in case of the single-metric or split-symmetry violating trajectories. Thus in confirmation with the observations of chapter 10, these solutions are only approximately valid in parts of theory space, while the split-symmetry restoring bi-metric trajectories are in accordance with the properties of the exact solutions.

14.1.4 Positivity vs. Asymptotic Safety

Bi-metric calculations, even though rare, were already studied several years ago. However, the perspective we adopted in chapters 10 and 11 brought new insights into the finer structures underlying the bi-metric nature of the gravitational FRGE. It turned out that the Einstein-Hilbert single-metric approximation that was used occasionally for Renormalization Group-based predictions in cosmology for instance, come close to their bi-metric analogs only in the classical and the UV regime. Especially on intermediate scales, where the single-metric approximation was found unreliable, the full bi-metric picture revealed completely new, so far uncovered features, one of them being the sign change in the anomalous dimension of Newton’s coupling, η . In chapter 12 we studied the effect a positive anomalous dimension has on the propagation of gravitational modes and the particle interpretation of the fields. It is argued that when approaching the nontrivial fixed point towards the UV, the anomalous dimension turns negative, indicating a break down of a Källén-Lehmann – and thus particle – representation. At the same time the equation of motion predict a strongly curved spacetime where again the particle concept loses its meaning for independent reasons. The appearance of dark matter could be due to a cloaking of the actual sources by these ‘unparticle’ formation, as was pointed out in section 12.3. A transition from a negative anomalous dimension, crucial for the existence of a nontrivial fixed point, to a positive η had never been observed in the single-metric trunca-

tions. Being negative in the UV and positive in the (semi-)classical regime, the η obtained by bi-metric truncations was opening the possibility to have a coexistence of Asymptotic Safety at the fundamental level with the standard Wightman axioms of flat spacetime QFT in the flat space on-shell limit.

14.1.5 Asymptotic Safety on manifolds with boundary

In chapter 13 we then studied the effect of adding a Gibbons-Hawking-York like boundary term to the groundbreaking Einstein-Hilbert truncation, [122] in the FRG setting. We explored its Renormalization Group evolution, in particular the interplay of boundary and bulk terms. Embedding the Asymptotic Safety program on manifolds having a non-vanishing boundary finally made for instance black hole spacetimes accessible for a detailed study within this approach without Renormalization Group improvement which suffers from the need for ambiguous cut-off identification. Even under for this completely different setting the necessary conditions for non-perturbative renormalizability in the shape of a nontrivial fixed point could be established, extending the results we obtained in a previous single-metric and matter induced bi-metric study [166]. Applying the Renormalization Group results to the Euclidean Schwarzschild geometry provided an indication for a stabilization of the thermodynamical properties due to quantum effects.

14.2 Outlook

A satisfactory theory of quantum gravity is one longterm objective in theoretical physics. In order to achieve this ambitious goal, while lacking experimental data, it will be very important to find parallels among different approaches to quantum gravity. Each of these theories acknowledge the fact that gravity is somewhat special and at least on a mathematical level more involved, revealing new features and open issues.

Following the path of the Asymptotic Safety program within the Effective Average Action approach the difficulty resides in the functional non-linear differential equation, the FRGE, which requires a completely new catalog of mathematical techniques, yet to be discovered. Thus physicists invent (non-perturbative) approximation schemes, known as truncations, which allow to stepwise uncover possible candidates for a theory of Quantum Gravity which are renormalizable in a non-perturbative sense. From this perspective there are basically two ways future work will extend our understanding of the underlying Renormalization Group flow on theory space, the manifold containing all possible action functionals. The first way is formulated on the level of truncations, where the previous results are tested for its stability and extended by including more and more basis invariants. The second way addresses the general properties of the FRGE on the level of exact solutions, revealing possibly hidden features of the RG flow that can be utilized to judge the reliability of truncations. These additional information may be found very instructive to construct new truncations in a systematic way.

In this thesis we studied partially vaguely explored and completely untouched regions in theory space. Both extensions, the bi-metric ansatz which is actually mandatory and the inclusion of boundary invariants, are very interesting aspects of Quantum Gravity and it had been only for the technical difficulty that previous investigations almost exclusively omitted their influence. Encouraged by the results obtained in this thesis, further bi-metric truncations are needed to test the reliability of the single-metric results and addresses fundamental questions as Background Independence within the Asymptotic Safety program to Quantum Gravity. The introduced method to reduce the complexity of bi-metric studies may be helpful in this direction.

Furthermore, we kept the assumptions and notation as general as possible, especially in view of the boundary sector. Only at the very end, in order to obtain definite results, we specified the

gauge fixing condition, the boundary constraints the fields satisfy, and the field parametrization related to the field space connection. This has the great advantage that future works can continue the investigation of bi-metric and/ or boundary type truncations from this point onwards.

In particular, we have paved the ground to study truncations that contain a dynamical Gibbons-Hawking-York term that would require to substitute the ordinary conformal projection technique by an x -dependent analog and then employ the TT-decomposition to evaluate the functional trace by means of the heat kernel expansion. We have mentioned the necessary steps in detail. Further studies in this direction can be used to study non-perturbative results to black hole physics or cosmology either with or without boundary contributions. Thereby, dissolving the tied bound of different invariants on the classical level already revealed hidden features on the quantum level and will in future help understanding to which extent those results may effect observations on our scales.

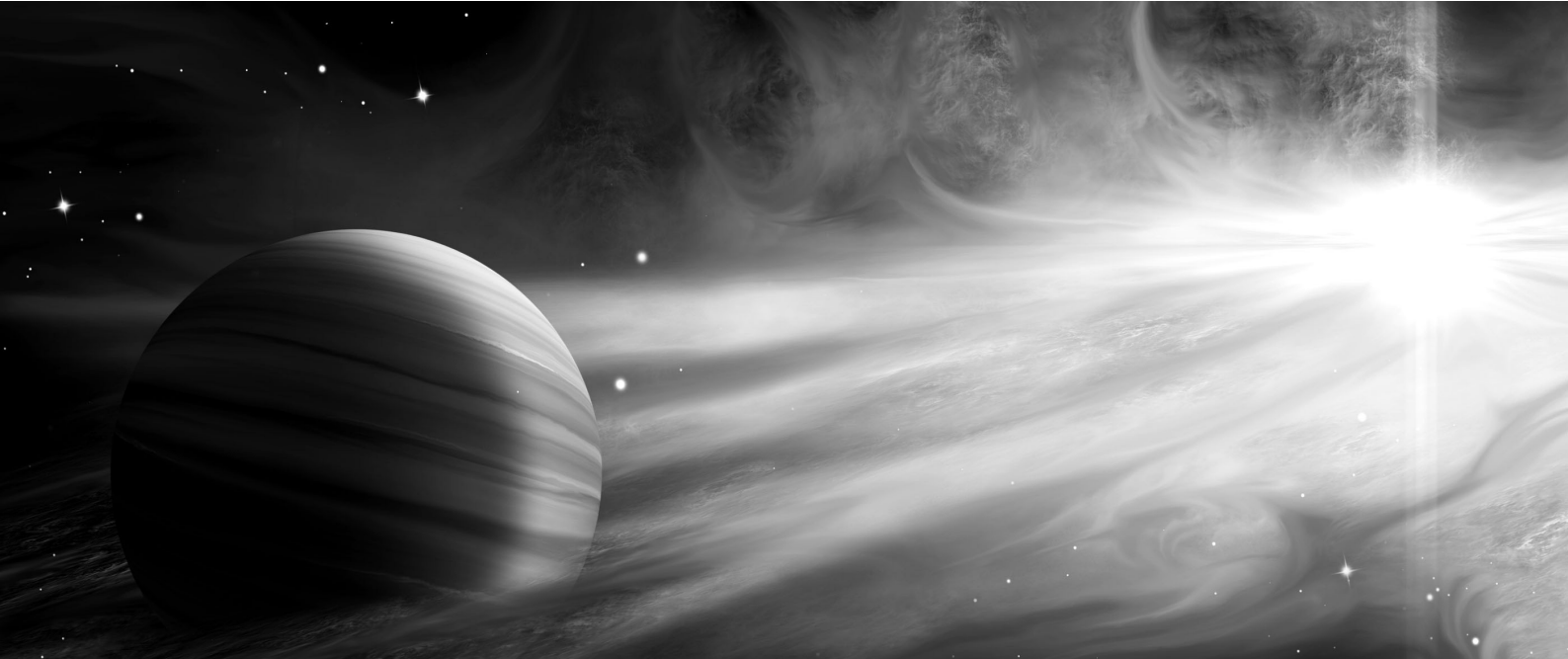
Another, very promising aspect studied in this thesis relates to the proposed C -function property of the Effective Average Action. Establishing a suitable C -function can be an excellent guiding principle from the theoretical perspective that would give rise to many diverse applications also outside Quantum Gravity. One possible way can be to employ the Ward Identities for split-symmetry and use its properties to prove or adjust our proposal, ultimately leading to a non-perturbative constraint on the RG evolution.

Thus, there are many ways to continue the journey towards a better understanding of Nature.



Appendix

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BIBLIOGRAPHY

Books

- [1] John M. Lee. *Introduction to Topological Manifolds (Graduate Texts in Mathematics)*. Springer, 2013 (cited on pages 2, 8).
- [2] John M. Lee. *Introduction to Smooth Manifolds (Graduate Texts in Mathematics)*. Springer, 2002 (cited on pages 2, 9, 12 sq., 38).
- [3] M. Nakahara. *Geometry, topology and physics*. 2003 (cited on pages 2, 38, 66).
- [4] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry*. Volume I & II. New York Chichester: John Wiley, 1996 (cited on pages 2, 38, 45, 52, 56).
- [11] Morton Hamermesh. *Group Theory and Its Application to Physical Problems (Dover Books on Physics)*. Dover Publications, 1989 (cited on pages 14, 34, 37).
- [12] William Fulton and Joe Harris. *Representation Theory: A First Course (Graduate Texts in Mathematics / Readings in Mathematics)*. Springer, 1999 (cited on pages 14, 29 sq., 32, 34).
- [13] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry (London Mathematical Society Student Texts)*. Cambridge University Press, 1996 (cited on pages 14, 34).
- [14] H.F Jones. *Groups, Representations and Physics*. CRC Press, 1998 (cited on pages 14, 30, 34).
- [15] Stefan Scherer. *Symmetrien und Gruppen in der Teilchenphysik*. in press. Heidelberg: Springer Spektrum (cited on pages 14, 34).
- [21] Richard P. Stanley. *Enumerative Combinatorics, Volume 2*. Cambridge University Press, 2001 (cited on page 33).
- [24] Gregory L. Naber. *Topology, Geometry and Gauge fields: Foundations*. Springer New York, 2011 (cited on pages 38, 54, 103).

-
- [25] Gregory L. Naber. *Topology, Geometry and Gauge fields: Interactions*. Springer New York, 2011 (cited on pages 38, 54, 103).
 - [26] G. Esposito, A.Yu. Kamenshchik, and G. Pollifrone. *Euclidean Quantum Gravity on Manifolds with Boundary (Fundamental Theories of Physics)*. Springer, 2012 (cited on pages 38, 66, 119, 122, 131).
 - [27] Bryce Dewitt. *Global approach to quantum field theory*. Oxford University Press, 2014 (cited on pages 38, 307, 356 sq.).
 - [28] James Glimm and Arthur Jaffe. *Quantum Physics*. Springer New York, 1987 (cited on pages 38, 66, 68, 76, 84, 86, 103, 118 sq.).
 - [33] Stephen Bruce Sontz. *Principal Bundles: The Classical Case*. Springer, 2015 (cited on page 53).
 - [42] Charles Nash. *Differential Topology and Quantum Field Theory*. Academic Press, 1992 (cited on page 66).
 - [43] Stephen A. Fulling. *Aspects of Quantum Field Theory in Curved Spacetime (London Mathematical Society Student Texts)*. Cambridge University Press, 1989 (cited on page 66).
 - [45] N. Bourbaki. *Algebra I: Chapters 1-3*. Springer, 1998 (cited on page 70).
 - [47] Donald L. Cohn. *Measure Theory*. Birkhäuser Boston, 1980 (cited on pages 76 sq., 81, 86).
 - [48] Giuseppe Da Prato. *An Introduction to Infinite-Dimensional Analysis*. Springer Berlin Heidelberg, 2006 (cited on pages 84, 86).
 - [49] Xia Dao-Zing and E.J Brody. *Measure and Integration Theory on Infinite-dimensional Spaces. Abstract Harmonic Analysis*. Amsterdam, Boston: Academic Press, 1972 (cited on page 86).
 - [65] R. F. Streater and A. S. Wightman. *PCT, spin and statistics, and all that*. 1989 (cited on page 118).
 - [71] Abhay Ashtekar. *General relativity and gravitation : a centennial perspective*. New York: Cambridge University Press, 2015 (cited on page 122).
 - [72] Abhay Ashtekar. *Lectures on non-perturbative canonical gravity*. Singapore River Edge, N.J: World Scientific, 1991 (cited on pages 122, 133, 306, 356, 432).
 - [73] Claus Kiefer. *Quantum gravity*. Oxford: Oxford University Press, 2012 (cited on pages 122, 306, 356, 432).
 - [75] Thomas Thiemann. *Modern canonical quantum general relativity*. Cambridge: Cambridge University Press, 2007 (cited on pages 122, 306, 432).
 - [76] T Padmanabhan. *Gravitation : foundations and frontiers*. Cambridge, UK New York: Cambridge University Press, 2010 (cited on pages 122, 306, 431).
 - [89] Michael B. Green, J. H. Schwarz, and Edward Witten. *SUPERSTRING THEORY. VOL. 1: INTRODUCTION*. 1988 (cited on page 132).
 - [90] Michael B. Green, J. H. Schwarz, and Edward Witten. *SUPERSTRING THEORY. VOL. 2: LOOP AMPLITUDES, ANOMALIES AND PHENOMENOLOGY*. 1988 (cited on page 132).
 - [91] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge University Press, 2007 (cited on page 132).

- [92] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond*. Cambridge University Press, 2007 (cited on page 132).
- [93] Carlo Rovelli. *Quantum gravity*. Cambridge, UK New York: Cambridge University Press, 2004 (cited on pages 133, 306, 356, 432).
- [196] H. W. Hamber. *Quantum gravitation the Feynman path integral approach*. Berlin: Springer, 2009 (cited on page 306).
- [201] Manfred Böhm. *Gauge theories of the strong and electroweak interaction*. Stuttgart: B.G. Teubner, 2001 (cited on page 309).
- [245] Arthur L. Besse. *Einstein Manifolds*. Berlin: Springer, 1987 (cited on pages 372 sq.).
- [297] Milutin Blagojevic and Friedrich W. Hehl. *Gauge Theories of Gravitation*. London: Imperial College Press, 2013. arXiv: 1210.3775 [gr-qc] (cited on page 418).
- [298] W. Dittrich and M. Reuter. *Effective Lagrangians in Quantum Electrodynamics*. Berlin: Springer, 1985 (cited on page 418).
- [303] G. W. Gibbons and S. W. Hawking. *Euclidean quantum gravity*. Singapore River Edge, NJ: World Scientific, 1993 (cited on page 432).

Articles

- [5] Kurt Gödel. “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I”. German. In: *Monatshefte für Mathematik und Physik* 38.1 (1931), pages 173–198 (cited on page 2).
- [6] E. Zermelo. “Untersuchungen über die Grundlagen der Mengenlehre. I”. German. In: *Mathematische Annalen* 65.2 (1908), pages 261–281 (cited on page 2).
- [7] A. A. Fraenkel. “Über die Zermelosche Begründung der Mengenlehre (abstract)”. In: *Jahresber. Deutsch. Math. Verein.* 30.II (1921) (cited on page 2).
- [8] Edwin E. Moise. “Geometric Topology in Dimensions 2 and 3”. In: *Graduate Texts in Mathematics* (1977) (cited on page 12).
- [9] James Munkres. “Obstructions to the smoothing of piecewise-differentiable homeomorphisms”. In: *Bull. Amer. Math. Soc.* 65.5 (Jan. 1959), pages 332–335 (cited on page 12).
- [10] Giampiero Esposito. “From spinor geometry to complex general relativity”. In: *Int. J. Geom. Meth. Mod. Phys.* 2 (2005), pages 675–731. arXiv: hep-th/0504089 [hep-th] (cited on page 14).
- [16] Issai Schur. “Neue Begründung der Theorie der Gruppencharaktere”. In: *Sitzungsber. Preuss. Akad. Wiss. Berlin* Part 1 (1905), pages 406–432 (cited on page 21).
- [17] Heinrich Maschke. “Ueber den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen”. In: *Mathematische Annalen* 50.4 (Dec. 1898), pages 492–498 (cited on page 23).
- [18] Heinrich Maschke. “Beweis des Satzes, dass diejenigen endlichen linearen Substitutionsgruppen, in welchen einige durchgehends verschwindende Coefficienten auftreten, intransitiv sind”. In: *Mathematische Annalen* 52.2-3 (June 1899), pages 363–368 (cited on page 23).
- [19] N. D. Mermin. “The topological theory of defects in ordered media”. In: *Rev. Mod. Phys.* 51 (3 July 1979), pages 591–648 (cited on page 23).
- [20] J. Milnor. “Remarks On Infinite Dimensional Lie Groups”. In: *In Les Houches 1983, Proceedings, Relativity, Groups and Topology* (1985), pages 1007–1057 (cited on page 27).

-
- [22] A. M. Vershik, I. M. Gelfand, and M. I. Graev. “Representations of the group of diffeomorphisms”. In: *Russian Mathematical Surveys* 30.6 (1975), page 1 (cited on page 38).
 - [23] Takeshi Hirai. “Irreducible unitary representations of the group of diffeomorphisms of a non-compact manifold”. In: *J. Math. Kyoto Univ.* 33.3 (1993), pages 827–864 (cited on page 38).
 - [29] L. Mangiarotti and G. Sardanashvily. “Connections in Classical and Quantum Field Theory”. In: (2000) (cited on pages 45, 56).
 - [30] B. G. Schmidt. “Conditions on a connection to be a metric connection”. In: *Comm. Math. Phys.* 29.1 (1973), pages 55–59 (cited on page 51).
 - [31] H. Hopf and W. Rinow. “Ueber den Begriff der vollständigen differentialgeometrischen Fläche”. In: *Commentarii Mathematici Helvetici* 3.1 (Dec. 1931), pages 209–225 (cited on page 52).
 - [32] C. J. Atkin. “The Hopf-Rinow Theorem is false in infinite Dimensions”. In: *Bulletin of the London Mathematical Society* 7.3 (Nov. 1975), pages 261–266 (cited on page 52).
 - [34] O. Babelon and C.-M. Viallet. “The Riemannian geometry of the configuration space of gauge theories”. In: *Comm. Math. Phys.* 81.4 (1981), pages 515–525 (cited on pages 59, 61).
 - [35] O. Babelon and C.-M. Viallet. “The geometrical interpretation of the Faddeev-Popov determinant”. In: *Physics Letters B* 85.2-3 (Aug. 1979), pages 246–248 (cited on page 59).
 - [36] I. M. Singer. “Some remarks on the Gribov ambiguity”. In: *Comm. Math. Phys.* 60.1 (1978), pages 7–12 (cited on pages 59 sq., 165).
 - [37] Andreas Kriegl and Peter W Michor. “Aspects of the theory of infinite dimensional manifolds”. In: *Differential Geometry and its Applications* 1.2 (Sept. 1991), pages 159–176 (cited on pages 60, 64).
 - [38] Peter W. Michor. “Gauge Theory for Diffeomorphism Groups”. In: *Differential Geometrical Methods in Theoretical Physics* (1988), pages 345–371 (cited on page 60).
 - [39] Bryce S. DeWitt. “Quantum Theory of Gravity. I. The Canonical Theory”. In: *Phys. Rev.* 160 (5 Aug. 1967), pages 1113–1148 (cited on pages 64, 121 sq., 163, 195).
 - [40] David G. Ebin. “On the space of Riemannian metrics”. In: *Bull. Amer. Math. Soc.* 74.5 (Jan. 1968), pages 1001–1004 (cited on pages 64, 128).
 - [41] David G Ebin. “The manifold of Riemannian metrics”. In: *Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968)* (1970), pages 11–40 (cited on page 64).
 - [44] M. F. Atiyah and I. M. Singer. “The index of elliptic operators on compact manifolds”. In: *Bull. Amer. Math. Soc.* 69.3 (May 1963), pages 422–434 (cited on page 70).
 - [46] S.A. Fulling et al. “Normal forms for tensor polynomials. 1: The Riemann tensor”. In: *Class.Quant.Grav.* 9 (1992), pages 1151–1197 (cited on pages 73 sq., 139).
 - [50] Konrad Osterwalder and Robert Schrader. “Axioms for Euclidean Green’s functions”. In: *Comm. Math. Phys.* 31.2 (1973), pages 83–112 (cited on pages 87, 118).
 - [51] Konrad Osterwalder and Robert Schrader. “Axioms for Euclidean Green’s functions. II”. In: *Comm. Math. Phys.* 42.3 (1975), pages 281–305 (cited on pages 87, 118).
 - [52] Walter Rudin. “Fourier Analysis on Groups”. In: (Jan. 1990) (cited on page 87).
 - [53] Serge Lang. “Differential and Riemannian Manifolds”. In: *Graduate Texts in Mathematics* (1995) (cited on page 89).

-
- [54] V. N. Gribov. “Quantization of Nonabelian Gauge Theories”. In: *Nucl. Phys.* B139 (1978), page 1 (cited on pages 90, 165).
 - [55] C Becchi, A Rouet, and R Stora. “Renormalization of gauge theories”. In: *Annals of Physics* 98.2 (June 1976), pages 287–321 (cited on page 91).
 - [56] Claude Chevalley and Samuel Eilenberg. “Cohomology theory of Lie groups and Lie algebras”. In: *Trans. Amer. Math. Soc.* 63.1 (Jan. 1948), pages 85–85 (cited on page 91).
 - [57] Rudolf Schmid. “BRST Cohomology and Anomalies”. In: *Geometrical and Algebraic Aspects of Nonlinear Field Theory* (1989), pages 159–172 (cited on page 91).
 - [58] Rudolf Schmid. “Local cohomology in gauge theories, BRST transformations and anomalies”. In: *Differential Geometry and its Applications* 4.2 (June 1994), pages 107–116 (cited on page 91).
 - [59] D. Binosi and A. Quadri. “Anti-BRST symmetry and background field method”. In: *Phys. Rev. D* 88.8 (Oct. 2013) (cited on page 93).
 - [60] L.D. Faddeev and V.N. Popov. “Feynman diagrams for the Yang-Mills field”. In: *Physics Letters B* 25.1 (July 1967), pages 29–30 (cited on page 93).
 - [61] Dmitri V. Vassilevich. “The Faddeev-Popov trick in the presence of boundaries”. In: *Physics Letters B* 421.1-4 (Mar. 1998), pages 93–98 (cited on pages 94, 196).
 - [62] Emil Mottola. “Functional integration over geometries”. In: *J.Math.Phys.* 36 (1995), pages 2470–2511. arXiv: hep-th/9502109 [hep-th] (cited on pages 106, 391, 402).
 - [63] Barry R. Holstein and John F. Donoghue. “Classical physics and quantum loops”. In: *Phys. Rev. Lett.* 93 (2004), page 201602. arXiv: hep-th/0405239 [hep-th] (cited on page 115).
 - [64] Stanley J. Brodsky and Paul Hoyer. “The $\hbar$ Expansion in Quantum Field Theory”. In: *Phys. Rev. D* 83 (2011), page 045026. arXiv: 1009.2313 [hep-ph] (cited on page 115).
 - [67] Stefan Hollands and Robert M. Wald. “Axiomatic quantum field theory in curved space-time”. In: *Commun. Math. Phys.* 293 (2010), pages 85–125. arXiv: 0803.2003 [gr-qc] (cited on page 119).
 - [68] Bryce S. DeWitt. “Quantum Theory of Gravity. II. The Manifestly Covariant Theory”. In: *Phys. Rev.* 162 (5 Oct. 1967), pages 1195–1239 (cited on pages 121 sq., 129, 195, 309).
 - [69] Bryce S. DeWitt. “Quantum Theory of Gravity. III. Applications of the Covariant Theory”. In: *Phys. Rev.* 162 (5 Oct. 1967), pages 1239–1256 (cited on pages 121 sq., 195).
 - [70] Marc H. Goroff and Augusto Sagnotti. “The Ultraviolet Behavior of Einstein Gravity”. In: *Nucl. Phys.* B266 (1986), page 709 (cited on pages 121, 131).
 - [74] Carlo Rovelli. “Quantum Gravity: third edition”. In: *Class. Quantum Grav.* 29.17 (Aug. 2012), page 179001 (cited on page 122).
 - [77] Albert Einstein. “On the General Theory of Relativity”. In: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1915 (1915). [Addendum: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1915, 799 (1915)], pages 778–786 (cited on page 123).
 - [78] James W. York. “Role of Conformal Three-Geometry in the Dynamics of Gravitation”. In: *Physical Review Letters* 28.16 (Apr. 1972), pages 1082–1085 (cited on page 125).
 - [79] G. W. Gibbons and S. W. Hawking. “Action integrals and partition functions in quantum gravity”. In: *Phys. Rev. D* 15.10 (May 1977), pages 2752–2756 (cited on pages 125, 198, 386, 431 sq., 440).

-
- [80] Stephen Hawking and G. F. R. Ellis. “Singularities in homogeneous world models”. In: *Phys. Lett.* 17 (1965), pages 246–247 (cited on page 127).
 - [81] S. W. Hawking and R. Penrose. “The Singularities of gravitational collapse and cosmology”. In: *Proc. Roy. Soc. Lond.* A314 (1970), pages 529–548 (cited on page 127).
 - [82] S. W. Hawking and R. Penrose. “The Nature of space and time”. In: *Sci. Am.* 275 (1996), pages 44–49 (cited on page 127).
 - [83] David E. Blair. “Chapter 2: Spaces of metrics and curvature functionals”. In: *Handbook of Differential Geometry* (2000), pages 153–185 (cited on pages 127 sq.).
 - [84] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. “RG flows of Quantum Einstein Gravity in the linear-geometric approximation”. In: *Annals Phys.* 359 (2015), pages 141–165. arXiv: 1412.7207 [hep-th] (cited on pages 128, 195).
 - [85] Liliana Maxim-Răileanu. “The Manifold of Riemannian Metric on a Compact Manifold with Boundary”. In: *An. St. Univ. Al. I. Cuza Iasi* 34 (1998), pages 67–72 (cited on page 129).
 - [86] J. B. Hartle and S. W. Hawking. “Wave function of the Universe”. In: *Phys. Rev. D* 28.12 (Dec. 1983), pages 2960–2975 (cited on pages 130 sq., 422).
 - [87] Gerard ’t Hooft and M. J. G. Veltman. “One loop divergencies in the theory of gravitation”. In: *Annales Poincare Phys. Theor.* A20 (1974), pages 69–94 (cited on page 131).
 - [88] A. E. M. van de Ven. “Two loop quantum gravity”. In: *Nucl. Phys.* B378 (1992), pages 309–366 (cited on page 131).
 - [94] J. Ambjørn, J. Jurkiewicz, and R. Loll. “Dynamically triangulating Lorentzian quantum gravity”. In: *Nuclear Physics B* 610.1-2 (Sept. 2001), pages 347–382 (cited on pages 133, 306, 356).
 - [95] J. Ambjørn, J. Jurkiewicz, and R. Loll. “Reconstructing the Universe”. In: *Phys. Rev. D* 72.6 (Sept. 2005) (cited on pages 133, 306, 356).
 - [96] J. Ambjørn et al. “Nonperturbative quantum de Sitter universe”. In: *Phys. Rev. D* 78.6 (Sept. 2008) (cited on pages 133, 306, 356).
 - [97] J. Ambjørn et al. “Planckian Birth of a Quantum de Sitter Universe”. In: *Physical Review Letters* 100.9 (Mar. 2008) (cited on pages 133, 306, 356).
 - [98] J. Ambjørn et al. “Nonperturbative quantum gravity”. In: *Physics Reports* 519.4-5 (Oct. 2012), pages 127–210 (cited on pages 133, 306, 356, 414).
 - [99] O. Lauscher and M. Reuter. “Fractal spacetime structure in asymptotically safe gravity”. In: *JHEP* 0510 (2005), page 050. arXiv: hep-th/0508202 [hep-th] (cited on pages 133, 158, 278, 342, 368, 398).
 - [100] Martin Reuter and Frank Saueressig. “Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data”. In: *JHEP* 1112 (2011), page 012. arXiv: 1110.5224 [hep-th] (cited on pages 133, 158, 278, 342 sqq., 351, 368, 398).
 - [101] Domenico J. W. Giulini, Claus Kiefer, and Claus Lammerzahl. “Proceedings, 271st WE-Heraeus Seminar on Aspects of Quantum Gravity: From Theory to Experiment Search”. In: *Lect. Notes Phys.* 631 (2003), pp.1–242 (cited on page 134).
 - [102] Claus Kiefer and Manuel Kraemer. “Quantum Gravitational Contributions to the CMB Anisotropy Spectrum”. In: *Phys. Rev. Lett.* 108 (2012), page 021301. arXiv: 1103.4967 [gr-qc] (cited on page 134).

-
- [103] Claus Kiefer and Manuel Kraemer. “Can effects of quantum gravity be observed in the cosmic microwave background?” In: *Int. J. Mod. Phys. D* 21 (2012), page 1241001. arXiv: 1205.5161 [gr-qc] (cited on page 134).
 - [105] L. P. Kadanoff. “Scaling laws for Ising models near $T(c)$ ”. In: *Physics* 2 (1966), pages 263–272 (cited on page 137).
 - [106] K. G. Wilson. “Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture”. In: *Phys.Rev.* B4 (1971), pages 3174–3183 (cited on page 137).
 - [107] K.G. Wilson and John B. Kogut. “The Renormalization group and the epsilon expansion”. In: *Phys.Rept.* 12 (1974), pages 75–200 (cited on page 137).
 - [108] Kenneth G. Wilson. “The renormalization group: Critical phenomena and the Kondo problem”. In: *Rev. Mod. Phys.* 47.4 (Oct. 1975), pages 773–840 (cited on page 137).
 - [109] Joseph Polchinski. “Renormalization and effective lagrangians”. In: *Nuclear Physics B* 231.2 (Jan. 1984), pages 269–295 (cited on page 137).
 - [110] M. Reuter and C. Wetterich. “Average action for the Higgs model with Abelian gauge symmetry”. In: *Nucl.Phys.* B391 (1993), pages 147–175 (cited on pages 137, 149, 307 sq., 354, 400).
 - [111] M. Reuter and C. Wetterich. “Running gauge coupling in three-dimensions and the electroweak phase transition”. In: *Nucl.Phys.* B408 (1993), pages 91–132 (cited on pages 137, 149, 307 sq., 354, 400).
 - [112] Christof Wetterich. “Exact evolution equation for the effective potential”. In: *Phys.Lett.* B301 (1993), pages 90–94 (cited on pages 137, 145, 149, 307 sq., 354, 400, 448).
 - [113] M. Reuter and C. Wetterich. “Effective average action for gauge theories and exact evolution equations”. In: *Nucl.Phys.* B417 (1994), pages 181–214 (cited on pages 137, 149, 307 sq., 354, 400).
 - [114] M. Reuter and C. Wetterich. “Exact evolution equation for scalar electrodynamics”. In: *Nucl.Phys.* B427 (1994), pages 291–324 (cited on pages 137, 149, 307 sq., 354, 400).
 - [115] Oliver J. Rosten. “Fundamentals of the Exact Renormalization Group”. In: *Phys. Rept.* 511 (2012), pages 177–272. arXiv: 1003.1366 [hep-th] (cited on page 138).
 - [116] Juergen Berges, Nikolaos Tetradis, and Christof Wetterich. “Non-perturbative renormalization flow in quantum field theory and statistical physics”. In: *Phys. Rept.* 363 (2002), pages 223–386. arXiv: hep-ph/0005122 (cited on page 145).
 - [117] Bertrand Delamotte. “An introduction to the nonperturbative renormalization group”. In: (2007). arXiv: cond-mat/0702365 (cited on page 145).
 - [118] Janos Polonyi. “Lectures on the functional renormalization group method”. In: *Central Eur. J. Phys.* 1 (2003), pages 1–71. arXiv: hep-th/0110026 (cited on page 145).
 - [119] D M McAvity and H Osborn. “A DeWitt expansion of the heat kernel for manifolds with a boundary”. In: *Classical and Quantum Gravity* 8.4 (1991), page 603 (cited on pages 148, 187).
 - [120] D M McAvity and H Osborn. “Asymptotic expansion of the heat kernel for generalized boundary conditions”. In: *Classical and Quantum Gravity* 8.8 (1991), page 1445 (cited on pages 148, 187).
 - [121] Elisa Manrique and Martin Reuter. “Bare Action and Regularized Functional Integral of Asymptotically Safe Quantum Gravity”. In: *Phys.Rev.* D79 (2009), page 025008. arXiv: 0811.3888 [hep-th] (cited on pages 150, 158, 278, 307, 368, 389 sq., 398).

-
- [122] M. Reuter. “Nonperturbative evolution equation for quantum gravity”. In: *Phys.Rev.* D57 (1998), pages 971–985. arXiv: hep-th/9605030 [hep-th] (cited on pages 153, 158, 172, 236, 250, 254, 260, 263, 267, 270, 286 sq., 303, 307 sq., 327, 340, 343, 345, 347, 354, 356, 362, 367, 369 sq., 375, 397, 400, 403–406, 408, 422, 426, 430 sq., 443, 450).
 - [123] O. Lauscher and M. Reuter. “Flow equation of quantum Einstein gravity in a higher derivative truncation”. In: *Phys.Rev.* D66 (2002), page 025026. arXiv: hep-th/0205062 [hep-th] (cited on pages 153, 158, 278, 339, 342, 368, 374, 398, 405, 408, 411, 426, 435).
 - [124] Alessandro Codello, Roberto Percacci, and Christoph Rahmede. “Ultraviolet properties of $f(R)$ -gravity”. In: *Int. J. Mod. Phys.* A23 (2008), pages 143–150. arXiv: 0705.1769 [hep-th] (cited on pages 153, 391, 402, 405).
 - [125] Stefan Rechenberger and Frank Saueressig. “The R^2 phase-diagram of QEG and its spectral dimension”. In: *Phys. Rev.* D86 (2012), page 024018. arXiv: 1206.0657 (cited on pages 153, 342 sq., 405).
 - [126] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. “RG flows of Quantum Einstein Gravity on maximally symmetric spaces”. In: *JHEP* 06 (2014), page 026. arXiv: 1401.5495 [hep-th] (cited on pages 153, 158, 278, 368, 392, 398, 405).
 - [127] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. “Fixed Functionals in Asymptotically Safe Gravity”. In: (2015), pages 2227–2229. arXiv: 1302.1312 [hep-th] (cited on page 153).
 - [128] Dario Benedetti, Pedro F. Machado, and Frank Saueressig. “Asymptotic safety in higher-derivative gravity”. In: *Mod. Phys. Lett.* A24 (2009), pages 2233–2241. arXiv: 0901.2984 [hep-th] (cited on pages 153, 405).
 - [129] Dario Benedetti. “On the number of relevant operators in asymptotically safe gravity”. In: *Europhys.Lett.* 102 (2013), page 20007. arXiv: 1301.4422 [hep-th] (cited on pages 153, 278, 295, 405).
 - [130] Pietro Donà, Astrid Eichhorn, and Roberto Percacci. “Matter matters in asymptotically safe quantum gravity”. In: *Phys. Rev.* D89.8 (2014), page 084035. arXiv: 1311.2898 [hep-th] (cited on pages 153, 158, 278, 368, 390, 398, 404).
 - [131] Djamel Dou and Roberto Percacci. “The running gravitational couplings”. In: *Class. Quant. Grav.* 15 (1998), pages 3449–3468. arXiv: hep-th/9707239 [hep-th] (cited on pages 153, 159, 278, 368, 398).
 - [132] Kai Groh and Frank Saueressig. “Ghost wavefunction renormalization in asymptotically safe quantum gravity”. In: *J. Phys. A: Math. Theor.* 43.36 (July 2010), page 365403 (cited on pages 153, 159, 166, 221 sq., 278, 309, 368 sq., 398).
 - [133] Astrid Eichhorn, Holger Gies, and Michael M. Scherer. “Asymptotically free scalar curvature-ghost coupling in quantum Einstein gravity”. In: *Phys. Rev. D* 80.10 (Nov. 2009) (cited on pages 153, 159, 166, 221 sq., 278, 309, 368 sq., 398).
 - [134] Astrid Eichhorn and Holger Gies. “Ghost anomalous dimension in asymptotically safe quantum gravity”. In: *Phys. Rev. D* 81.10 (May 2010) (cited on pages 153, 159, 166, 221 sq., 278, 309, 368 sq., 398).
 - [135] O. Lauscher and M. Reuter. “Ultraviolet fixed point and generalized flow equation of quantum gravity”. In: *Phys.Rev.* D65 (2002), page 025013. arXiv: hep-th/0108040 [hep-th] (cited on pages 153, 159, 166, 237, 278, 288, 300, 309, 339, 342, 368, 374, 398, 408, 411, 426, 435).

-
- [136] Elisa Manrique and Martin Reuter. “Bimetric Truncations for Quantum Einstein Gravity and Asymptotic Safety”. In: *Annals Phys.* 325 (2010), pages 785–815. arXiv: 0907.2617 [gr-qc] (cited on pages 153, 159, 309, 356, 362, 401).
 - [137] Elisa Manrique, Martin Reuter, and Frank Saueressig. “Matter Induced Bimetric Actions for Gravity”. In: *Annals Phys.* 326 (2011), pages 440–462. arXiv: 1003.5129 [hep-th] (cited on pages 153, 159, 309, 342, 356, 401, 433).
 - [138] Elisa Manrique, Stefan Rechenberger, and Frank Saueressig. “Asymptotically Safe Lorentzian Gravity”. In: *Physical Review Letters* 106.25 (June 2011) (cited on pages 153, 159, 278, 368, 398).
 - [139] Stefan Rechenberger and Frank Saueressig. “A functional renormalization group equation for foliated spacetimes”. In: *J. High Energ. Phys.* 2013.3 (Mar. 2013) (cited on pages 153, 159, 278, 368, 398).
 - [140] Jan-Eric Daum and Martin Reuter. “Running Immirzi Parameter and Asymptotic Safety”. In: *PoS(CNCFG2010)* (2010), page 003. arXiv: 1111.1000 [hep-th] (cited on page 153).
 - [141] Jan-Eric Daum and M. Reuter. “Renormalization Group Flow of the Holst Action”. In: *Phys. Lett. B* 710 (2012), pages 215–218. arXiv: 1012.4280 [hep-th] (cited on pages 153, 159, 162, 278, 368, 398).
 - [142] J.E. Daum and M. Reuter. “Einstein-Cartan gravity, Asymptotic Safety, and the running Immirzi parameter”. In: *Annals Phys.* 334 (2013), pages 351–419. arXiv: 1301.5135 [hep-th] (cited on pages 153, 159, 162, 278, 368, 398).
 - [143] Ulrich Harst and Martin Reuter. “A new functional flow equation for Einstein–Cartan quantum gravity”. In: *Annals Phys.* 354 (2015), pages 637–704. arXiv: 1410.7003 [hep-th] (cited on pages 153, 162).
 - [144] Ulrich Harst and Martin Reuter. “The ‘Tetrad only’ theory space: Nonperturbative renormalization flow and Asymptotic Safety”. In: *JHEP* 05 (2012), page 005. arXiv: 1203.2158 [hep-th] (cited on pages 153, 162).
 - [146] M. Reuter and Frank Saueressig. “A Class of nonlocal truncations in quantum Einstein gravity and its renormalization group behavior”. In: *Phys.Rev.* D66 (2002), page 125001. arXiv: hep-th/0206145 [hep-th] (cited on pages 159, 278, 368, 370, 398, 429).
 - [147] O. Lauscher and M. Reuter. “Is quantum Einstein gravity nonperturbatively renormalizable?” In: *Class.Quant.Grav.* 19 (2002), pages 483–492. arXiv: hep-th/0110021 [hep-th] (cited on pages 159, 278, 339, 368, 398, 405, 426, 435).
 - [148] O. Lauscher and M. Reuter. “Towards nonperturbative renormalizability of quantum Einstein gravity”. In: *Int.J.Mod.Phys.* A17 (2002), pages 993–1002. arXiv: hep-th/0112089 [hep-th] (cited on pages 159, 278, 368, 398).
 - [149] Roberto Percacci and Daniele Perini. “Constraints on matter from asymptotic safety”. In: *Phys. Rev.* D67 (2003), page 081503. arXiv: hep-th/0207033 [hep-th] (cited on pages 159, 278, 368, 398).
 - [150] Roberto Percacci and Daniele Perini. “Asymptotic safety of gravity coupled to matter”. In: *Phys. Rev.* D68 (2003), page 044018. arXiv: hep-th/0304222 [hep-th] (cited on pages 159, 278, 368, 398).
 - [151] A Bonanno and M Reuter. “Proper Time Flow Equation for Gravity”. In: *Journal of High Energy Physics* 2005.02 (Feb. 2005), page 035 (cited on pages 159, 278, 368, 374, 398).

-
- [152] G. P. Vacca and O. Zanusso. “Asymptotic Safety in Einstein Gravity and Scalar-Fermion Matter”. In: *Physical Review Letters* 105.23 (Nov. 2010) (cited on pages 159, 278, 368, 390, 398).
 - [153] O. Zanusso et al. “Gravitational corrections to Yukawa systems”. In: *Physics Letters B* 689.2-3 (May 2010), pages 90–94 (cited on pages 159, 278, 368, 390, 398).
 - [154] Max R. Niedermaier. “Gravitational Fixed Points from Perturbation Theory”. In: *Physical Review Letters* 103.10 (Sept. 2009) (cited on pages 159, 278, 368, 398).
 - [155] M. Niedermaier. “Gravitational fixed points and asymptotic safety from perturbation theory”. In: *Nuclear Physics B* 833.3 (July 2010), pages 226–270 (cited on pages 159, 278, 368, 398).
 - [156] Martin Reuter and Holger Weyer. “Background Independence and Asymptotic Safety in Conformally Reduced Gravity”. In: *Phys.Rev. D* 79 (2009), page 105005. arXiv: 0801.3287 [hep-th] (cited on pages 159, 278, 368, 398).
 - [157] Martin Reuter and Holger Weyer. “The Role of Background Independence for Asymptotic Safety in Quantum Einstein Gravity”. In: *Gen.Rel.Grav.* 41 (2009), pages 983–1011. arXiv: 0903.2971 [hep-th] (cited on pages 159, 278, 368, 398).
 - [159] Carlo Pagani and Roberto Percacci. “Quantization and fixed points of non-integrable Weyl theory”. In: *Class. Quant. Grav.* 31 (2014), page 115005. arXiv: 1312.7767 [hep-th] (cited on pages 159, 278, 368, 398).
 - [160] Alessandro Codello, Roberto Percacci, and Christoph Rahmede. “Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation”. In: *Annals Phys.* 324 (2009), pages 414–469. arXiv: 0805.2909 [hep-th] (cited on pages 159, 278, 368, 398).
 - [161] Jan-Eric Daum and Martin Reuter. “Effective Potential of the Conformal Factor: Gravitational Average Action and Dynamical Triangulations”. In: *Adv. Sci. Lett.* 2 (2009), pages 255–260. arXiv: 0806.3907 [hep-th] (cited on pages 159, 278, 368, 398).
 - [162] Jan-Eric Daum, Ulrich Harst, and Martin Reuter. “Running Gauge Coupling in Asymptotically Safe Quantum Gravity”. In: *JHEP* 01 (2010), page 084. arXiv: 0910.4938 [hep-th] (cited on pages 159, 162, 278, 368, 398).
 - [163] Jan-Eric Daum and Martin Reuter. “The Effective Potential of the Conformal Factor in Quantum Einstein Gravity”. In: *PoS CLAQG08* (2011), page 013. arXiv: 0910.5401 [hep-th] (cited on pages 159, 162, 278, 368, 398).
 - [164] U. Harst and M. Reuter. “QED coupled to QEG”. In: *JHEP* 05 (2011), page 119. arXiv: 1101.6007 [hep-th] (cited on pages 159, 278, 368, 398).
 - [165] Andreas Nink and Martin Reuter. “On the physical mechanism underlying Asymptotic Safety”. In: *JHEP* 01 (2013), page 062. arXiv: 1208.0031 [hep-th] (cited on pages 159, 240, 278, 283, 291, 346, 349, 352, 368, 398, 414, 418).
 - [166] Daniel Becker and Martin Reuter. “Running boundary actions, Asymptotic Safety, and black hole thermodynamics”. In: *JHEP* 1207 (2012), page 172. arXiv: 1205.3583 [hep-th] (cited on pages 159, 263, 278 sq., 286, 357, 368 sq., 390, 398, 413, 421, 427, 430, 432, 435 sqq., 443, 445, 450).
 - [167] D. Becker and M. Reuter. “Asymptotic Safety and Black Hole Thermodynamics”. In: *Proceedings - C14-09-29.2* (2015), pages 2230–2232. arXiv: 1212.4274 [hep-th] (cited on pages 159, 278, 368, 398).

-
- [168] Martin Reuter and Frank Saueressig. “Quantum Einstein Gravity”. In: *New J.Phys.* 14 (2012), page 055022. arXiv: 1202.2274 [hep-th] (cited on pages 159, 278, 298, 342, 354, 368, 398).
 - [169] Max Niedermaier and Martin Reuter. “The Asymptotic Safety Scenario in Quantum Gravity”. In: *Living Rev.Rel.* 9 (2006), pages 5–173 (cited on pages 159, 278, 354, 368, 398).
 - [172] Elisa Manrique, Martin Reuter, and Frank Saueressig. “Bimetric Renormalization Group Flows in Quantum Einstein Gravity”. In: *Annals Phys.* 326 (2011), pages 463–485. arXiv: 1006.0099 [hep-th] (cited on pages 159, 168, 207, 224, 235, 239, 278 sq., 292, 302, 309, 331, 339–342, 351, 353, 356, 370, 374, 384, 402 sq.).
 - [173] Daniel Becker and Martin Reuter. “En route to Background Independence: Broken split-symmetry, and how to restore it with bi-metric average actions”. In: *Annals Phys.* 350 (2014), pages 225–301. arXiv: 1404.4537 [hep-th] (cited on pages 159, 224, 235, 249, 260, 279, 281, 286, 305, 370, 377, 380 sq., 402 sq., 412).
 - [175] Nobuyoshi Ohta and Roberto Percacci. “Ultraviolet Fixed Points in Conformal Gravity and General Quadratic Theories”. In: (2015). arXiv: 1506.05526 [hep-th] (cited on pages 166, 429).
 - [176] M. Berger and D. Ebin. “Some decompositions of the space of symmetric tensors on a Riemannian manifold”. In: *J. Differential Geom.* 3.3-4 (1969), pages 379–392 (cited on pages 213 sq.).
 - [177] James W. Jr. York. “Covariant decompositions of symmetric tensors in the theory of gravitation”. eng. In: *Annales de l’institut Henri Poincaré (A) Physique théorique* 21.4 (1974), pages 319–332 (cited on page 213).
 - [178] Qusay S.A. Al-Zamil and James Montaldi. “Witten-Hodge theory for manifolds with boundary and equivariant cohomology”. In: *Differential Geometry and its Applications* 30.2 (Apr. 2012), pages 179–194 (cited on page 213).
 - [179] Sylvain Cappell et al. “Cohomology of harmonic forms on Riemannian manifolds with boundary”. In: *Forum Mathematicum* 18.6 (Jan. 2006) (cited on page 213).
 - [180] O. Gil-Medrano and A. Montesinos Amilibia. “About a decomposition of the space of symmetric tensors of compact support on a Riemann manifold.” eng. In: *The New York Journal of Mathematics [electronic only]* 1 (1994), pages 10–25 (cited on page 213).
 - [181] Tsuan Wu Ting. “Problem of compatibility and orthogonal decomposition of second-order symmetric tensors in a compact Riemannian manifold with boundary”. In: *Arch. Rational Mech. Anal.* 64.3 (Sept. 1977), pages 221–243 (cited on pages 213 sq.).
 - [182] Harald P. Pfeiffer and James W. York. “Extrinsic curvature and the Einstein constraints”. In: *Phys. Rev. D* 67.4 (Feb. 2003) (cited on page 213).
 - [183] Kunihiko Kodaira. “Harmonic Fields in Riemannian Manifolds (Generalized Potential Theory)”. In: *The Annals of Mathematics* 50.3 (July 1949), page 587 (cited on page 215).
 - [184] Matthias Blau. “Lecture Notes on General Relativity”. In: <http://www.blau.itp.unibe.ch/newlecturesGR.pdf> (Oct. 2014), pages 346–350 (cited on page 233).
 - [185] H. P. McKean and I. M. Singer. “Curvature and eigenvalues of the Laplacian”. In: *J. Diff. Geom.* 1 (1967), pages 43–69 (cited on pages 243 sq.).
 - [186] D. V. Vassilevich. “Heat kernel expansion: User’s manual”. In: *Phys. Rept.* 388 (2003), pages 279–360. arXiv: hep-th/0306138 [hep-th] (cited on page 244).

-
- [187] H. Mellin. “Über die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma- und der hypergeometrischen Funktionen”. In: *Acta Soc. Sci. Fennicae* 1 (1896), page 21 (cited on page 244).
 - [188] Daniel F. Litim. “Fixed points of quantum gravity”. In: *Phys. Rev. Lett.* 92 (2004), page 201301. arXiv: hep-th/0312114 [hep-th] (cited on pages 264, 287, 405).
 - [189] A. Bonanno and M. Reuter. “Cosmology of the Planck era from a renormalization group for quantum gravity”. In: *Phys. Rev. D* 65 (2002), page 043508. arXiv: hep-th/0106133 [hep-th] (cited on pages 268, 419).
 - [190] M. Reuter and Frank Saueressig. “From big bang to asymptotic de Sitter: Complete cosmologies in a quantum gravity framework”. In: *JCAP* 0509 (2005), page 012. arXiv: hep-th/0507167 [hep-th] (cited on pages 268, 419, 437).
 - [191] M. Reuter and Frank Saueressig. “Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation”. In: *Phys. Rev. D* 65 (2002), page 065016. arXiv: hep-th/0110054 [hep-th] (cited on pages 270, 288, 300, 309, 311 sq., 367, 374, 398, 403, 405, 408, 411, 413, 426, 437).
 - [192] Daniel Becker and Martin Reuter. “Propagating gravitons vs. ‘dark matter’ in asymptotically safe quantum gravity”. In: *JHEP* 1412 (2014), page 025. arXiv: 1407.5848 [hep-th] (cited on pages 279, 397).
 - [193] Daniel Becker and Martin Reuter. “Towards a C -function in 4D quantum gravity”. In: *JHEP* 1503 (2015), page 065. arXiv: 1412.0468 [hep-th] (cited on pages 279, 353).
 - [195] Andreas Nink and Martin Reuter. “On quantum gravity, Asymptotic Safety, and paramagnetic dominance”. In: *Int. J. Mod. Phys. D* 22 (2013), pages 138–157. arXiv: 1212.4325 [hep-th] (cited on pages 283, 352, 418).
 - [197] T. Padmanabhan. “Holographic gravity and the surface term in the Einstein-Hilbert action”. In: *Braz. J. Phys.* 35.2a (2005), pages 362–372 (cited on pages 306, 431).
 - [198] T. Padmanabhan. “Gravity and the thermodynamics of horizons”. In: *Physics Reports* 406.2 (Jan. 2005), pages 49–125 (cited on pages 306, 431).
 - [199] Bianca Dittrich, Mercedes Martin-Benito, and Erik Schnetter. “Coarse graining of spin net models: dynamics of intertwiners”. In: *New Journal of Physics* 15.10 (Oct. 2013), page 103004 (cited on page 306).
 - [200] Bianca Dittrich, Mercedes Martin-Benito, and Sebastian Steinhaus. “Quantum group spin nets: Refinement limit and relation to spin foams”. In: *Phys. Rev. D* 90.2 (July 2014) (cited on page 306).
 - [202] L. F. Abbott. “The Background Field Method Beyond One Loop”. In: *Nucl. Phys.* B185 (1981), page 189 (cited on page 309).
 - [203] Marcus T. Grisaru, P. van Nieuwenhuizen, and C. C. Wu. “Background Field Method Versus Normal Field Theory in Explicit Examples: One Loop Divergences in S Matrix and Green’s Functions for Yang-Mills and Gravitational Fields”. In: *Phys. Rev. D* 12 (1975), page 3203 (cited on page 309).
 - [204] D. M. Capper, J. J. Dulwich, and M. Ramon Medrano. “The Background Field Method for Quantum Gravity at Two Loops”. In: *Nucl. Phys.* B254 (1985), page 737 (cited on page 309).
 - [205] Stephen L. Adler. “Einstein Gravity as a Symmetry Breaking Effect in Quantum Field Theory”. In: *Rev. Mod. Phys.* 54 (1982). [Erratum: *Rev. Mod. Phys.* 55, 837 (1983)], page 729 (cited on page 309).

-
- [206] I. Hamzaan Bridle, Juergen A. Dietz, and Tim R. Morris. “The local potential approximation in the background field formalism”. In: *JHEP* 03 (2014), page 093. arXiv: 1312.2846 [hep-th] (cited on pages 310, 404).
 - [207] M. Reuter and H. Weyer. “Renormalization group improved gravitational actions: A Brans-Dicke approach”. In: *Phys.Rev.* D69 (2004), page 104022. arXiv: hep-th/0311196 [hep-th] (cited on page 325).
 - [208] M. Reuter and H. Weyer. “Running Newton constant, improved gravitational actions, and galaxy rotation curves”. In: *Phys.Rev.* D70 (2004), page 124028. arXiv: hep-th/0410117 [hep-th] (cited on page 325).
 - [209] Alfio Bonanno and Martin Reuter. “Entropy signature of the running cosmological constant”. In: *JCAP* 0708 (2007), page 024. arXiv: 0706.0174 [hep-th] (cited on pages 325, 342, 371, 375, 398, 413, 419, 437 sq.).
 - [210] Martin Reuter and Jan-Markus Schwindt. “A Minimal length from the cutoff modes in asymptotically safe quantum gravity”. In: *JHEP* 01 (2006), page 070. arXiv: hep-th/0511021 [hep-th] (cited on pages 342, 373, 413).
 - [211] Martin Reuter and Jan-Markus Schwindt. “Scale-dependent metric and causal structures in Quantum Einstein Gravity”. In: *JHEP* 01 (2007), page 049. arXiv: hep-th/0611294 [hep-th] (cited on pages 342, 373, 413).
 - [212] M. Reuter and H. Weyer. “Quantum gravity at astrophysical distances?” In: *JCAP* 0412 (2004), page 001. arXiv: hep-th/0410119 [hep-th] (cited on pages 342, 371, 375, 398, 413, 417, 438).
 - [213] J. Ambjørn, J. Jurkiewicz, and R. Loll. “Spectral dimension of the universe”. In: *Phys. Rev. Lett.* 95 (2005), page 171301. arXiv: hep-th/0505113 [hep-th] (cited on page 342).
 - [214] J. Ambjørn, J. Jurkiewicz, and R. Loll. “Reconstructing the universe”. In: *Phys. Rev.* D72 (2005), page 064014. arXiv: hep-th/0505154 [hep-th] (cited on page 342).
 - [215] Dario Benedetti and Joe Henson. “Spectral geometry as a probe of quantum space-time”. In: *Phys. Rev.* D80 (2009), page 124036. arXiv: 0911.0401 [hep-th] (cited on page 342).
 - [216] Gianluca Calcagni et al. “Proceedings, 6th Aegean Summer School on Quantum gravity and quantum cosmology”. In: *Lect.Notes Phys.* 863 (2013). Edited by Gianluca Calcagni, pp.1–399 (cited on pages 342, 419).
 - [217] Gianluca Calcagni, Astrid Eichhorn, and Frank Saueressig. “Probing the quantum nature of spacetime by diffusion”. In: *Phys.Rev.* D87.12 (2013), page 124028. arXiv: 1304.7247 [hep-th] (cited on page 342).
 - [218] A.B. Zamolodchikov. “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory”. In: *JETP Lett.* 43 (1986), pages 730–732 (cited on page 353).
 - [219] A.B. Zamolodchikov. “Renormalization Group and Perturbation Theory Near Fixed Points in Two-Dimensional Field Theory”. In: *Sov.J.Nucl.Phys.* 46 (1987), page 1090 (cited on page 353).
 - [220] John L. Cardy. “Is There a c Theorem in Four-Dimensions?” In: *Phys.Lett.* B215 (1988), pages 749–752 (cited on page 354).
 - [221] H. Osborn. “Derivation of a Four-dimensional c Theorem”. In: *Phys.Lett.* B222 (1989), page 97 (cited on page 354).

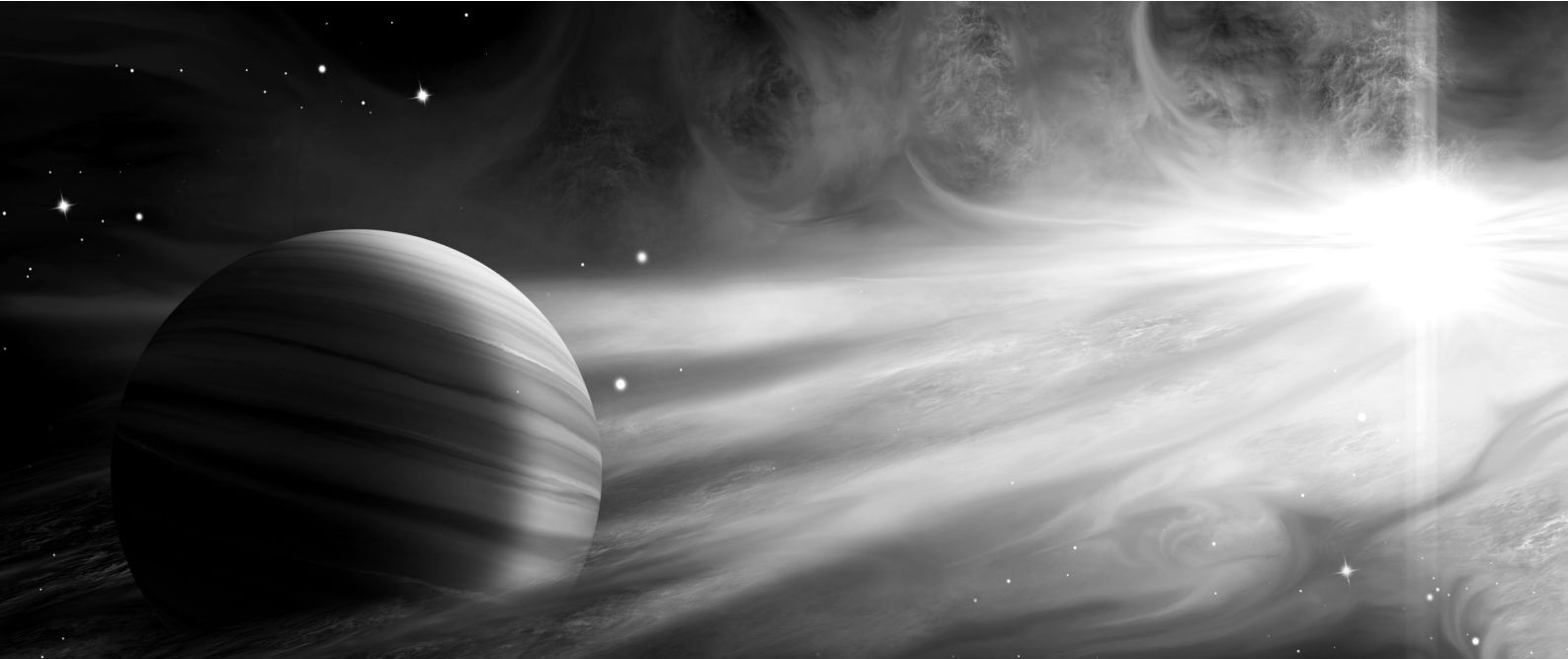
-
- [222] I. Jack and H. Osborn. “Analogues for the c Theorem for Four-dimensional Renormalizable Field Theories”. In: *Nucl.Phys.* B343 (1990), pages 647–688 (cited on page 354).
 - [223] A.H. Castro Neto and Eduardo H. Fradkin. “The Thermodynamics of quantum systems and generalizations of Zamolodchikov’s C theorem”. In: *Nucl.Phys.* B400 (1993), pages 525–546. arXiv: cond-mat/9301009 [cond-mat] (cited on page 354).
 - [224] Igor R. Klebanov, Silviu S. Pufu, and Benjamin R. Safdi. “F-Theorem without Supersymmetry”. In: *JHEP* 1110 (2011), page 038. arXiv: 1105.4598 [hep-th] (cited on page 354).
 - [225] Andrea Cappelli, Daniel Friedan, and Jose I. Latorre. “ C theorem and spectral representation”. In: *Nucl.Phys.* B352 (1991), pages 616–670 (cited on page 354).
 - [226] G.M. Shore. “A new c theorem in four-dimensions”. In: *Phys.Lett.* B253 (1991), pages 380–390 (cited on page 354).
 - [227] G.M. Shore. “The $C(F)$ theorem”. In: *Phys.Lett.* B256 (1991), pages 407–415 (cited on page 354).
 - [228] F. Bastianelli. “Tests for c theorems in 4-D”. In: *Phys.Lett.* B369 (1996), pages 249–254. arXiv: hep-th/9511065 [hep-th] (cited on page 354).
 - [229] D. Anselmi et al. “Nonperturbative formulas for central functions of supersymmetric gauge theories”. In: *Nucl.Phys.* B526 (1998), pages 543–571. arXiv: hep-th/9708042 [hep-th] (cited on page 354).
 - [230] Zohar Komargodski and Adam Schwimmer. “On Renormalization Group Flows in Four Dimensions”. In: *JHEP* 1112 (2011), page 099. arXiv: 1107.3987 [hep-th] (cited on page 354).
 - [231] Markus A. Luty, Joseph Polchinski, and Riccardo Rattazzi. “The a -theorem and the Asymptotics of 4D Quantum Field Theory”. In: *JHEP* 1301 (2013), page 152. arXiv: 1204.5221 [hep-th] (cited on page 354).
 - [232] Thomas L. Curtright, Xiang Jin, and Cosmas K. Zachos. “RG flows, cycles, and c -theorem folklore”. In: *Phys. Rev. Lett.* 108 (2012), page 131601. arXiv: 1111.2649 [hep-th] (cited on page 354).
 - [233] Alfio Bonanno. “An effective action for asymptotically safe gravity”. In: *Phys. Rev.* D85 (2012), page 081503. arXiv: 1203.1962 [hep-th] (cited on pages 354, 419).
 - [234] Abhay Ashtekar, Martin Reuter, and Carlo Rovelli. “From General Relativity to Quantum Gravity”. In: (2014). arXiv: 1408.4336 [gr-qc] (cited on pages 356, 384).
 - [235] M. Reuter and C. Wetterich. “Gluon condensation in nonperturbative flow equations”. In: *Phys.Rev.* D56 (1997), pages 7893–7916. arXiv: hep-th/9708051 [hep-th] (cited on pages 356, 362).
 - [236] Martin Reuter and Holger Weyer. “Conformal sector of Quantum Einstein Gravity in the local potential approximation: Non-Gaussian fixed point and a phase of unbroken diffeomorphism invariance”. In: *Phys.Rev.* D80 (2009), page 025001. arXiv: 0804.1475 [hep-th] (cited on pages 356, 374, 405).
 - [237] M. Reuter and C. Wetterich. “Quantum Liouville field theory as solution of a flow equation”. In: *Nucl.Phys.* B506 (1997), pages 483–520. arXiv: hep-th/9605039 [hep-th] (cited on page 356).
 - [238] Alessandro Codello, Giulio D’Odorico, and Carlo Pagani. “A functional RG equation for the c -function”. In: *JHEP* 07 (2014), page 040. arXiv: 1312.7097 [hep-th] (cited on page 356).

-
- [239] Benjamin Koch and Frank Saueressig. “Structural aspects of asymptotically safe black holes”. In: *Class.Quant.Grav.* 31 (2014), page 015006. arXiv: 1306.1546 [hep-th] (cited on page 357).
 - [240] Ali H. Chamseddine and A. Connes. “Universal formula for noncommutative geometry actions: Unification of gravity and the standard model”. In: *Phys. Rev. Lett.* 77 (1996), pages 4868–4871 (cited on page 368).
 - [241] Ali H. Chamseddine and Alain Connes. “The Spectral action principle”. In: *Commun. Math. Phys.* 186 (1997), pages 731–750. arXiv: hep-th/9606001 [hep-th] (cited on page 368).
 - [242] Ali H. Chamseddine and Alain Connes. “Quantum Gravity Boundary Terms from Spectral Action”. In: *Phys. Rev. Lett.* 99 (2007), page 071302. arXiv: 0705.1786 [hep-th] (cited on page 368).
 - [243] Natalia Alkofer, Frank Saueressig, and Omar Zanusso. “Spectral dimensions from the spectral action”. In: *Phys. Rev. D* 91.2 (2015), page 025025. arXiv: 1410.7999 [hep-th] (cited on page 368).
 - [244] Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson. “Gravitation, Gauge Theories and Differential Geometry”. In: *Phys.Rept.* 66 (1980), page 213 (cited on pages 372 sq.).
 - [246] Don Page. “A compact rotating gravitational instanton”. In: *Physics Letters B* 79.3 (Nov. 1978), pages 235–238 (cited on page 372).
 - [247] Steven Carlip. “Space-time foam and the cosmological constant”. In: *Phys. Rev. Lett.* 79 (1997), pages 4071–4074. arXiv: gr-qc/9708026 [gr-qc] (cited on page 373).
 - [248] Steven Carlip. “Dominant topologies in Euclidean quantum gravity”. In: *Class. Quant. Grav.* 15 (1998), pages 2629–2638. arXiv: gr-qc/9710114 [gr-qc] (cited on page 373).
 - [249] M. Reuter. “Hawking Radiation and the Virasoro Algebra”. In: *Class. Quant. Grav.* 6 (1989), page 1149 (cited on page 389).
 - [250] T. Banks. “Cosmological Breaking of Supersymmetry?” In: *Int. J. Mod. Phys. A* 16.05 (Feb. 2001), pages 910–921 (cited on page 390).
 - [251] Raphael Bousso. “Positive vacuum energy and the N bound”. In: *JHEP* 11 (2000), page 038. arXiv: hep-th/0010252 [hep-th] (cited on page 390).
 - [252] Ted Jacobson and Alejandro Satz. “On the renormalization of the Gibbons-Hawking boundary term”. In: *Phys.Rev. D* 89.6 (2014), page 064034. arXiv: 1308.2746 [gr-qc] (cited on page 390).
 - [253] Alejandro Satz and Ted Jacobson. “Black hole entropy and the renormalization group”. In: (2015), pages 1413–1415. arXiv: 1301.3171 [hep-th] (cited on page 390).
 - [254] Lee Smolin. “Quantum gravity with a positive cosmological constant”. In: (2002). arXiv: hep-th/0209079 [hep-th] (cited on page 390).
 - [255] Alessandro Codello and Roberto Percacci. “Fixed points of higher derivative gravity”. In: *Phys. Rev. Lett.* 97 (2006), page 221301. arXiv: hep-th/0607128 [hep-th] (cited on pages 391, 402, 405).
 - [256] Pedro F. Machado and Frank Saueressig. “On the renormalization group flow of f(R)-gravity”. In: *Phys. Rev. D* 77 (2008), page 124045. arXiv: 0712.0445 [hep-th] (cited on pages 391, 402, 405).
 - [257] Dario Benedetti et al. “The Universal RG Machine”. In: *JHEP* 06 (2011), page 079. arXiv: 1012.3081 [hep-th] (cited on page 392).

-
- [258] John F. Donoghue. “Leading quantum correction to the Newtonian potential”. In: *Phys. Rev. Lett.* 72 (1994), pages 2996–2999. arXiv: gr - qc / 9310024 [gr-qc] (cited on page 397).
 - [259] John F. Donoghue. “General relativity as an effective field theory: The leading quantum corrections”. In: *Phys. Rev. D* 50 (1994), pages 3874–3888. arXiv: gr - qc / 9405057 [gr-qc] (cited on page 397).
 - [260] Gunnar Källén. “On the definition of the Renormalization Constants in Quantum Electrodynamics”. In: *Helv. Phys. Acta* 25.4 (1952). [,509(1952)], page 417 (cited on page 399).
 - [261] H. Lehmann. “On the Properties of propagation functions and renormalization constants of quantized fields”. In: *Nuovo Cim.* 11 (1954), pages 342–357 (cited on page 399).
 - [262] N. Nakanishi and I. Ojima. “Covariant operator formalism of gauge theories and quantum gravity”. In: *World Sci. Lect. Notes Phys.* 27 (1990), pages 1–434 (cited on page 400).
 - [263] Axel Maas. “Describing gauge bosons at zero and finite temperature”. In: *Phys. Rept.* 524 (2013), pages 203–300. arXiv: 1106.3942 [hep-ph] (cited on page 401).
 - [264] Leonard Fister and Jan M. Pawłowski. “Confinement from Correlation Functions”. In: *Phys. Rev. D* 88 (2013), page 045010. arXiv: 1301.4163 [hep-ph] (cited on page 401).
 - [265] Stefan Strauss, Christian S. Fischer, and Christian Kellermann. “Analytic structure of the Landau gauge gluon propagator”. In: *Phys. Rev. Lett.* 109 (2012), page 252001. arXiv: 1208.6239 [hep-ph] (cited on page 401).
 - [266] Valentin Mader et al. “Infrared Saturation and Phases of Gauge Theories with BRST Symmetry”. In: *Eur. Phys. J. C* 74 (2014), page 2881. arXiv: 1309.0497 [hep-th] (cited on page 401).
 - [267] Geoffrey B. West. “Confinement, the Wilson Loop and the Gluon Propagator”. In: *Phys. Lett.* B115 (1982), page 468 (cited on page 401).
 - [268] Ivan Donkin and Jan M. Pawłowski. “The phase diagram of quantum gravity from diffeomorphism-invariant RG-flows”. In: (2012). arXiv: 1203.4207 [hep-th] (cited on page 404).
 - [269] Nicolai Christiansen et al. “Global Flows in Quantum Gravity”. In: (2014). arXiv: 1403.1232 [hep-th] (cited on page 404).
 - [270] Alessandro Codello, Giulio D’Odorico, and Carlo Pagani. “Consistent closure of renormalization group flow equations in quantum gravity”. In: *Phys. Rev. D* 89.8 (2014), page 081701. arXiv: 1304.4777 [gr-qc] (cited on page 404).
 - [271] Kai Groh et al. “Higher Derivative Gravity from the Universal Renormalization Group Machine”. In: *PoS EPS-HEP2011* (2011), page 124. arXiv: 1111.1743 [hep-th] (cited on page 405).
 - [272] Dario Benedetti, Pedro F. Machado, and Frank Saueressig. “Taming perturbative divergences in asymptotically safe gravity”. In: *Nucl. Phys.* B824 (2010), pages 168–191. arXiv: 0902.4630 [hep-th] (cited on page 405).
 - [273] Dario Benedetti, Pedro F. Machado, and Frank Saueressig. “Four-derivative interactions in asymptotically safe gravity”. In: (2009). [AIP Conf. Proc.1196,44(2009)]. arXiv: 0909.3265 [hep-th] (cited on page 405).
 - [274] Pedro F. Machado and R. Percacci. “Conformally reduced quantum gravity revisited”. In: *Phys. Rev. D* 80 (2009), page 024020. arXiv: 0904.2510 [hep-th] (cited on page 405).

-
- [275] Juergen A. Dietz and Tim R. Morris. “Asymptotic safety in the $f(R)$ approximation”. In: *JHEP* 01 (2013), page 108. arXiv: 1211.0955 [hep-th] (cited on page 405).
 - [276] Nobuyoshi Ohta and Roberto Percacci. “Higher Derivative Gravity and Asymptotic Safety in Diverse Dimensions”. In: *Class.Quant.Grav.* 31 (2014), page 015024. arXiv: 1308.3398 [hep-th] (cited on page 405).
 - [277] Wataru Souma. “Nontrivial ultraviolet fixed point in quantum gravity”. In: *Prog. Theor. Phys.* 102 (1999), pages 181–195. arXiv: hep-th/9907027 [hep-th] (cited on page 408).
 - [278] N. E. J Bjerrum-Bohr, John F. Donoghue, and Barry R. Holstein. “Quantum gravitational corrections to the nonrelativistic scattering potential of two masses”. In: *Phys. Rev. D* 67 (2003). [Erratum: *Phys. Rev. D* 71, 069903(2005)], page 084033. arXiv: hep-th/0211072 [hep-th] (cited on page 414).
 - [280] J. Ambjørn et al. “Renormalization Group Flow in CDT”. In: *Class. Quant. Grav.* 31 (2014), page 165003. arXiv: 1405.4585 [hep-th] (cited on page 414).
 - [281] Howard Georgi. “Unparticle physics”. In: *Phys. Rev. Lett.* 98 (2007), page 221601. arXiv: hep-ph/0703260 [hep-ph] (cited on page 414).
 - [282] Howard Georgi. “Another odd thing about unparticle physics”. In: *Phys. Lett. B* 650 (2007), pages 275–278. arXiv: 0704.2457 [hep-ph] (cited on page 414).
 - [283] O. Lauscher, M. Reuter, and C. Wetterich. “Rotation symmetry breaking condensate in a scalar theory”. In: *Phys.Rev. D* 62 (2000), page 125021. arXiv: hep-th/0006099 [hep-th] (cited on page 415).
 - [284] Alfio Bonanno and Martin Reuter. “Modulated Ground State of Gravity Theories with Stabilized Conformal Factor”. In: *Phys.Rev. D* 87.8 (2013), page 084019. arXiv: 1302.2928 [hep-th] (cited on page 415).
 - [285] Christof Wetterich. “The Average action for scalar fields near phase transitions”. In: *Z. Phys.* C57 (1993), pages 451–470 (cited on page 415).
 - [286] C. G. Bollini and J. J. Giambiagi. “Arbitrary powers of D’Alembertians and the Huygens’ principle”. In: *J. Math. Phys.* 34 (1993), pages 610–621 (cited on page 416).
 - [287] D. Lopez Nacir and F. D. Mazzitelli. “Running of Newton’s constant and non integer powers of the d’Alembertian”. In: *Phys. Rev. D* 75 (2007), page 024003. arXiv: hep-th/0610031 [hep-th] (cited on page 416).
 - [289] J. R. Kuhn and L. Kruglyak. “Non-Newtonian forces and the invisible mass problem”. In: *The Astrophysical Journal* 313 (Feb. 1987), page 1 (cited on page 417).
 - [291] Hans-Joachim Blome et al. “Nonlocal Modification of Newtonian Gravity”. In: *Phys. Rev. D* 81 (2010), page 065020. arXiv: 1002.1425 [gr-qc] (cited on pages 417 sq.).
 - [292] Friedrich W. Hehl and Bahram Mashhoon. “Nonlocal Gravity Simulates Dark Matter”. In: *Phys. Lett. B* 673 (2009), pages 279–282. arXiv: 0812.1059 [gr-qc] (cited on pages 417 sq.).
 - [293] Friedrich W. Hehl and Bahram Mashhoon. “A Formal framework for a nonlocal generalization of Einstein’s theory of gravitation”. In: *Phys. Rev. D* 79 (2009), page 064028. arXiv: 0902.0560 [gr-qc] (cited on pages 417 sq.).
 - [294] R. B. Tully and J. R. Fisher. “A New method of determining distances to galaxies”. In: *Astron. Astrophys.* 54 (1977), pages 661–673 (cited on page 417).
 - [295] C. Chicone and B. Mashhoon. “Nonlocal Gravity: Modified Poisson’s Equation”. In: *J. Math. Phys.* 53 (2012), page 042501. arXiv: 1111.4702 [gr-qc] (cited on page 417).

- [296] S. Rahvar and B. Mashhoon. “Observational Tests of Nonlocal Gravity: Galaxy Rotation Curves and Clusters of Galaxies”. In: *Phys. Rev. D* 89 (2014), page 104011. arXiv: 1401.4819 [astro-ph.GA] (cited on page 417).
- [299] Steven Weinberg. “Asymptotically Safe Inflation”. In: *Phys. Rev. D* 81 (2010), page 083535. arXiv: 0911.3165 [hep-th] (cited on page 419).
- [300] S. M. Christensen and M. J. Duff. “Quantum Gravity in Two + ϵ Dimensions”. In: *Phys. Lett. B* 79 (1978), page 213 (cited on page 422).
- [301] Abhay Ashtekar, Jonathan Engle, and David Sloan. “Asymptotics and Hamiltonians in a First order formalism”. In: *Class. Quant. Grav.* 25 (2008), page 095020. arXiv: 0802.2527 [gr-qc] (cited on pages 431 sq.).
- [302] Robert B. Mann et al. “On the Stress Tensor for Asymptotically Flat Gravity”. In: *Class. Quant. Grav.* 25 (2008), page 225019. arXiv: 0804.2079 [hep-th] (cited on pages 431 sq.).
- [304] A. Ashtekar, John C. Baez, and Kirill Krasnov. “Quantum geometry of isolated horizons and black hole entropy”. In: *Adv. Theor. Math. Phys.* 4 (2000), pages 1–94. arXiv: gr-qc/0005126 [gr-qc] (cited on page 432).
- [305] Alfio Bonanno and Martin Reuter. “Renormalization group improved black hole spacetimes”. In: *Phys. Rev. D* 62 (2000), page 043008. arXiv: hep-th/0002196 [hep-th] (cited on page 444).
- [306] A. Bonanno and M. Reuter. “Spacetime structure of an evaporating black hole in quantum gravity”. In: *Phys. Rev. D* 73 (2006), page 083005. arXiv: hep-th/0602159 [hep-th] (cited on page 444).
- [307] Alfio Bonanno and Martin Reuter. “Quantum gravity effects near the null black hole singularity”. In: *Phys. Rev. D* 60 (1999), page 084011. arXiv: gr-qc/9811026 [gr-qc] (cited on page 444).
- [308] M. Reuter and E. Tuiran. “Quantum Gravity Effects in the Kerr Spacetime”. In: *Phys. Rev. D* 83 (2011), page 044041. arXiv: 1009.3528 [hep-th] (cited on page 444).



ABBREVIATIONS

Acronyms

AC	axiom of choice
ADM	Arnowitt-Deser-Misner
AS	Asymptotic Safety
BH	Black Hole
BI	Background Independence
BRST	Becchi, Rouet, Stora and Tyutin
CDT	Causal Dynamical Triangulations
CMB	Cosmic Microwave Background
DOF	degrees of freedom
EAA	Effective Average Action
EQFT	Euclidean Quantum Field theory
ERGE	Exact Renormalization Group Equation
FLRW	Friedmann–Lemaître–Robertson–Walker
FP	fixed point
FRG	Functional Renormalization Group
FRGE	Functional Renormalization Group Equation
GFP	Gaussian fixed point
GR	General Relativity
IR	infrared
LHC	Large Hadron Collider
LHS	left-hand-side
LQG	Loop Quantum Gravity
NGFP	non-Gaussian fixed point
QCD	Quantum chromodynamics
QED	Quantum Electrodynamics
QEG	Quantum Einstein Gravity
QFT	Quantum Field theory

- QG** Quantum Gravity
- RG** Renormalization Group
- RHS** right-hand-side
- SM** Standard Model of particle physics
- SR** Special Relativity
- TT** transverse-traceless
- UV** ultraviolet
- WISS** Ward Identities for split-symmetry
- ZF** Zermelo–Fraenkel
- ZFC** Zermelo–Fraenkel set theory with the axiom of choice



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