# Ancestral lineages in the contact process: scaling and hitting properties

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Sebastian Steiber

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# Zusammenfassung

Diese Dissertation beschäftigt sich mit Ahnenlinien in einem zeitdiskreten, räumlichen Populationsmodell mit lokaler Regulierung, nämlich dem (zeitdiskreten) Kontaktprozess. Die Ahnenlinien können ebenfalls als gerichtete Irrfahrten in einer zufälligen, dynamischen Umgebung (RWDRE) interpretiert werden. Die Umgebung, die wir in dieser Arbeit betrachten, wird in der Literatur als "backbone" eines gerichteten Perkolationsclusters bezeichnet. In unserem Modell wählt ein Individuum seinen Elter in jedem diskreten Zeitschritt gleichverteilt unter allen Individuen, welche sich in der vorherigen Generation in seiner nächsten Nachbarschaft befinden. Die Wahl des Elters in jedem Zeitschritt geschieht dabei unabhängig von allem anderen. Im Jahr 2013 wurde dieses Modell von Birkner, Černý, Depperschmidt und Gantert analysiert, siehe [BČDG13]. Die Autoren haben bewiesen, dass die Irrfahrten, welche die Ahnenlinien modellieren, ein Gesetz der großen Zahl und einen "quenched" zentralen Grenzwertsatz erfüllen. Dieser Artikel ist im Zusammenhang mit weiteren, anschließenden Veröffentlichungen Grundlage für die in dieser Arbeit behandelten Fragestellungen und Probleme.
Im ersten Kapitel dieser Arbeit definieren wir das von uns betrachtete Modell und führen die im Weiteren verwendete Notation ein. Im zweiten Kapitel betrachten wir die gemeinsame Verteilung der Ahnenlinien aller Individuen der verschiedenen Generationen im eindimensionalen Fall. Der Ausdruck "eindimensionaler Fall" bezieht sich darauf, dass sich die Individuen in einem eindimensionalen Raum befinden. Es stellt sich heraus, dass die diffusiv reskalierte Sammlung aller Pfad schwach gegen das Brownsche Web konvergiert. Wir verifizieren hierzu die Konvergenzkriterien in [FINR04] und [Sun05]. Hauptaufgabe ist es, hierzu geeignete Abschätzungen für die Anzahl an Generationen bis zu einem Verschmelzen zweier Ahnenlinien zu finden. Es stellt sich heraus, dass der asymptotische Abfall für die Wahrscheinlichkeit, dass ein gemeinsamer Vorfahre erst nach $n$ Generationen gefunden wird, im eindimensionalen Fall von der Ordnung $\mathcal{O}(n^{-\frac{1}{2}})$ ist. Diese Abfallrate würde man auch für die Treffzeit zweier einfacher Irrfahrten erwarten. Man kann daher sagen, dass nicht besetzte Gebiete, welche die Verschmelzung der Ahnenlinien verhindern könnten, im eindimensionalen Fall die Wartezeit auf den ersten gemeinsamen Vorfahren nicht "wesentlich" verlängern.
Im dritten Kapitel beschäftigen wir uns mit Abschätzungen für die Differenz zwischen "annealed" und "quenched" Wahrscheinlichkeiten, Boxen unterschiedlicher Größe zu treffen. Das Finden solcher Abschätzungen ist durch einen aktuellen Artikel von Berger, Cohen und Rosenthal (siehe [BCR16]) motiviert, in welchem die Autoren Abschätzungen dieser Art verwenden, um einen "quenched" lokalen zentralen Grenzwertsatz für ballistische Irrfahrten in einer u.i.v. Umgebung zu beweisen. Hierbei ist es uns gelungen, ihre Beweisideen auf unser Modell bis auf das Treffen von Boxen der Seitenlänge $e^{\sqrt{\log(N)\log\log(N)}}$ zu übertragen. Für Raumdimensionen mindestens drei impliziert dieses Ergebnis bereits den "quenched" zentralen Grenzwertsatz (qCLT) von Birkner et al. und stellt eine wesentliche Verfeinerung der aus dem qCLT zu gewinnenden Abschätzungen zwischen "annealed" und "quenched" Wahrscheinlichkeiten dar.

## Abstract

This thesis deals with ancestral lineages in a time discrete spatial population model with local density regulation, namely the (discrete time) contact process. The ancestral lineages can be seen as directed random walks in a dynamic random environment (RWDRE), where at each discrete time step a particle chooses its parent uniformly among the particles in the previous generation, located at its nearest neighbourhood. The choice at each time step is independent of everything else in the model. In the literature the dynamic random environment we focus on is called the "backbone" of an oriented percolation cluster. In [BČDG13] Birkner, Černý, Depperschmidt and Gantert analysed this model and proved a law of large numbers and a quenched central limit theorem for the random walks that model the ancestral lineages. In this thesis we mainly focus on problems and questions that arise out of their work and which have been additionally inspired by subsequently published articles.

Within the first chapter we give a precise definition of the model and establish the notation that will be used within the rest of the thesis. Afterwards, in the second chapter we focus on the common distribution of the ancestral lineages of all individuals over all generations in the one-dimensional case. Talking about the "one-dimensional case", we mean that the dimension of the space in which the particles are located equals one. It turns out that the diffusively rescaled collection of all the ancestral paths converges weakly to the Brownian web. Checking the convergence criteria given in [FINR04] and extended by Sun in his PhD thesis (see [Sun05]), the main task is to find suitable bounds on the number of generations one has to wait, until the ancestral lineages of two individuals located within a fixed distance coalesce. We are able to prove that the tail bounds for the event of the coalescing time to be greater than $n$ are of order $\mathcal{O}(n^{-\frac{1}{2}})$ in the one-dimensional case. Therefore the tail bounds are of the same order one would expect from ordinary nearest neighbour simple random walks. Hence one could say that in the one-dimensional case unoccupied areas that might prevent a coalescing event do not substantially increase the time until a coalescing event occurs.

In the third chapter we prove estimates between quenched and annealed hitting probabilities of differently sized boxes. Investigation of this problem is motivated by a paper of Berger, Cohen and Rosenthal (see [BCR16]), in which the authors used this kind of estimates to prove a quenched local central limit theorem for a (ballistic) random walk in an i.i.d. environment. We are able to adapt their ideas to our model up to a comparison for boxes of side length $e^{\sqrt{\log(N)\log\log(N)}}$. This result already implies the quenched central limit theorem (qCLT) proved by Birkner et al., for space dimension at least three and provides a comparison between quenched and annealed hitting probabilities on a much finer scale than the comparison that follows out of the qCLT.

# Contents

# Introduction

In this thesis we focus on ancestral lineages in a spatial population model with local density regulation. In order to motivate our interest in models of this kind we quote a statement by Etheridge:

"The main purpose of theoretical population genetics is to understand the complex patterns of genetic variation that we observe in the world around us."[1]

The understanding of ancestral structures in a population model is one of the key elements to deduce information on genetic variation. Knowing the ancestral lineages of particles, we can answer questions on the type of individuals or identify the most recent common ancestor of an arbitrary subgroup of particles. The ancestral lineages within the population model we focus on can be seen as belonging to the huge class of random walks in random environment (RWRE). RWRE is a field that received considerable attention within the last decades. Models of RWRE can be helpful to understand physical, geological or biological problems such as the motion of electrons in crystals with impurities (see e.g. [BH91], [HK87, section 10] and [BG90b]), fluid flows in reservoirs consisting of a "mixture of good sandstones with high permeability (i.e. flow units) and poorer siltstones, mudstones and shales with low permeability"[2] (see e.g. [SZ11] ), and as already mentioned above, ancestral lineages in population models.

In general RWRE means that at each step the transition probabilities or transition rates of the random walk depend on the random configuration of the environment. The transition probabilities do not need to be determined by a local configuration of the random environment. In our case, for example, transition probabilities depend on the whole (possible) ancestry given by the environment. A random walk in random environment can basically be understood as a two-step random experiment. In the first step the environment is created according to some given probability measure. The outcome of the second step is a realisation of a random walk path whose dynamics depend on the environment. If the environment changes in time as well, we speak about random walk in dynamic random environment (RWDRE). Including the time-component as an additional dimension, we can, of course, think about RWDRE belonging to the class of RWRE but the special importance of the time component in many cases legitimates thinking about RWDRE as a different kind of model.

Recalling that we want to focus on ancestral lineages in a spatial population model, we are interested in the configuration of inhabited sites at different times in order to identify ancestors. Since the time

[1] [Eth11]

[2] see [SZ11], page 668, second paragraph

component has a "special role" within the model, it is obvious that our model belongs to the class of RWDRE. In this context of RWDRE one usually refers to the whole environment (including the time component) as a random space-time environment.

In the literature a huge variety of different random space-time environments is discussed. The first group of random space-time environments we want to mention are independent and identically distributed (i.i.d.) environments where at each space-time point transition probabilities of the random walk are chosen independently according to a identical distribution (see e.g. [RAS05]). A slight variation of this environment, where independence is only assumed between different time slices, can be found for example in [JRA11]. In this case one can think of the environment as being "refreshed at each time step". Random environments where individual sites evolve in time as independent Markov chains are for example analysed in [BMP08], [DL09] and [BZ06]. A more generalized case where uniform coupling conditions are imposed on the Markovian environment is discussed in [RV13]. At last we want to mention the group of environments generated by particle systems. For example [dHKS14] and [HdHdS$^+$15] focused on models of random walks on random walks. Random environments generated by an interacting particle system are for example considered by [AdHR11], [dHdSS13]. Classical examples are spin systems or exclusion processes. The environment we focus on is generated by a time-discrete version of the contact process.

In general the contact process $\eta := (\eta_t)_{t\in I}$ on $\mathbb{Z}^d$, where $I = [0,\infty)$ or $I = \mathbb{N}_0$, is a $\{0,1\}^{\mathbb{Z}^d}$ valued Markov process that can be seen as a model for the spread of an infection or the growth of a population. At some given time $t \in I$ we think of the sites $\{b \in \mathbb{Z}^d : \eta_t(x) = 1\}$ as being "infected", whereas the sites in $\{b \in \mathbb{Z}^d : \eta_t(x) = 0\}$ are considered to be "healthy". In the continuous time case a healthy site becomes infected at a rate proportional to the number of infected neighbours and an infected site becomes healthy at rate 1. Hence the flip rate $c(\eta, x)$ by which the state of $\eta_t$ at $x \in \mathbb{Z}^d$ is flipped from 0 to 1 or vice versa is given by

$$c(\eta,x) = \begin{cases} 1, & \text{if } \eta(x) = 1, \\ \lambda \cdot (\#\{y : \|x-y\|_1 = 1, \eta(y) = 1\}), & \text{if } \eta(x) = 0, \end{cases}$$

where $\lambda \geq 0$ is called the infection parameter. In this case the set $U(x) = \{y \in \mathbb{Z}^d : \|x-y\|_1 = 1\}$ is considered as the nearest neighbourhood of $x \in \mathbb{Z}^d$. In the discrete time case, a healthy site will be infected at the next time step with probability $p \in (0,1)$ if there exists an infected particle in its nearest neighbourhood and an infected site recovers with probability $(1-p)$. A precise definition of the discrete version of the contact process considered within this thesis will be given in section 1.2. Other articles dealing with random walks on discrete or continuous time versions of the contact process are [BČDG13], [BH15], [Mil16], [Bet16], [BČD16] and [BV16]. The list of examples on different environments given above is, of course, not complete and neither is the list of references. We just list some models in order to give an impression of how random environments could look like. When working on RWRE, one usually deals with the following two probability measures: The law of the random walk on a fixed environment is called the *quenched law*, whereas averaging over random environment and random walk is called the *annealed law* of the random walk.

The "groundwork" for this thesis is an article by Matthias Birkner, Jiří Černý, Andrej Depperschmidt

and Nina Gantert [BČDG13], which deals with the same RWDRE-model we are working on. In [BČDG13] the authors describe a regeneration construction for the random walk on the discrete time contact process and derive a law of large numbers (LLN) and a quenched central limit theorem (qCLT) from it. In [Mil16] the results by Birkner et al. are extended to a contact process with fluctuating population size. This is realized by supplementary “carrying capacities” fulfilling certain mixing conditions. In fact, the random walks are defined on a subgraph of an oriented (site) percolation cluster. The link between oriented percolation and the discrete time contact process is discussed in section 1.2.1 below. The restriction to the subgraph was necessary to avoid traps in which a directed random walk might get stuck.
In [BČDG13] Birkner et al. also proved that two random walks defined on the same oriented percolation cluster are “essentially independent” when they are far apart. Based on their observations the question arises how several (or infinitely many) random walks on the same oriented cluster behave.

In this thesis we prove that the diffusively rescaled system of coalescing random walks, starting from each space time point contained in the subgraph (traps are deleted) of an oriented percolation cluster of dimension $1+1$, converges weakly towards the *Brownian web* (BW). Systems of one-dimensional coalescing Brownian motions starting from $\mathbb{R} \times \{0\}$ have first been studied by Arratia [Arr79]. Later on Tóth and Werner [TW98] analyse “coalescing reflected-absorbed Brownian motions” which they need for construction and analysis of their “true self-repelling motion”. A new characterization of the BW is given by [FINR04], [FINR02]. They characterize the BW as a random variable taking values in a complete separable metric space, whose elements are compact sets of paths. Additionally, they give criteria for a whole system of rescaled coalescing random walks to converge in distribution towards the Brownian web. In [FINR04] convergence criteria for the case of non-crossing and crossing random walk paths have been developed. These results are extended by Sun in [Sun05] to the case of crossing random walks, whose increments fulfil a finite fifth moment condition. In the second chapter of this thesis we will prove that our model fulfils the generalized convergence criteria given by Sun. Some further articles dealing with properties of the BW and its dual are [NRS10] and [SSS14]. In [SS13] the authors prove that the centered and diffusively rescaled collection of the right most paths in an oriented percolation cluster converges weakly towards the Brownian web. In our case the situation is different, since one of the main problems we have to deal with is the crossing of “nearest neighbour paths” which does not occur, if always the rightmost path is chosen.

The second main problem we focus on is a comparison between quenched and annealed hitting probabilities of differently sized boxes. In [BCR16] Berger et al. use this kind of estimates to prove a quenched local central limit theorem (qLCLT) for (ballistic) random walks in an i.i.d. environment. Although we do not prove a qLCLT within this thesis, we are able to adapt some ideas in [BCR16] to our set-up and get a comparison between quenched and annealed probabilities for hitting boxes of side length $e^{\sqrt{\log(N)\log\log(N)}}$. This comparison together with the annealed central limit theorem implies the quenched central limit theorem (qCLT) proved by Birkner et al., for space dimension at least three. Additionally, it provides a comparison between quenched and annealed hitting probabilities on a much finer scale than the comparison that follows out of the qCLT. The most helpful tool for proving the estimates between the quenched and annealed probabilities is the environmental exposure procedure first invented by Bolthausen and Snitzman in [BS02]. A lot of the ideas we use can also be found in [Ber12].

# CHAPTER 1

## Description of the model: (Coalescing) Random walks on the backbone of an oriented percolation cluster

### 1.1. Primary notation

In this chapter we give a precise definition of the model we are working on. We want to point out that a very detailed description of the model can also be found in [BČDG13].

Let

$$\omega := \{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\} \tag{1.1}$$

be a family of i.i.d. Bernoulli($p$) random variables defined on some probability space

$$(\Omega, \mathcal{A}, \mathbb{P}). \tag{1.2}$$

Hence $\mathbb{P} \circ \omega^{-1}$ is a *Bernoulli product measure* on $\{0,1\}^{\mathbb{Z}^d \times \mathbb{Z}}$. Given a site $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$, we usually refer to the first component $x \in \mathbb{Z}^d$ as *space* and to the second component $n \in \mathbb{Z}$ as *time*. We call a (space-time) point $(x,n)$ to be *open* or *inhabitable* if $\omega(x,n) = 1$, and *closed* or *uninhabitable* if $\omega(x,n) = 0$. During the rest of the thesis $\|\cdot\|$ refers to the supremum-norm unless stated otherwise. An *open (directed) path* in $\omega$ that starts from $(y,m)$ and ends in $(x,n)$ for some $x, y \in \mathbb{Z}^d$ and $m, n \in \mathbb{Z}$ with $m \leq n$, is a sequence $x_m, x_{m+1}, ..., x_n \in \mathbb{Z}^d$ such that $x_m = y$, $x_n = x$, $\|x_k - x_{k-1}\| \leq 1$ for all $k = m+1, ..., n$ and $\omega(x_k, k) = 1$ for all $k = m, ..., n$. If there exists an open directed path in $\omega$ from $(y,m)$ to $(x,n)$ we will write $(y,m) \overset{\omega}{\to} (x,n)$. If for some $\omega$ for every $n \geq m$ there exists $x \in \mathbb{Z}^d$ such that $(y,m) \overset{\omega}{\to} (x,n)$, we will write $(x,n) \overset{\omega}{\to} \infty$.

**Remark 1.1.** In the proofs that follow, $C$ and $c$ denote some positive constants that are only allowed to depend on the success probability $p$ of $\omega(x,n)$, $(x,n) \in \mathbb{Z}^d$ in (1.1) and the space dimension $d$. If the explicit value of some constants is not important for the result, they will always be denoted by $C$ and $c$. The constants $C$ and $c$ may also vary within a chain of inequalities. If the value of a certain constant is important for a later step, we will add a subscript to it $C_1, c_1, C_2, c_2$....

### 1.2. The discrete time contact process

As already mentioned in the introduction we focus on random walks in random environment, where in our case the environment is generated by a discrete time version of the contact process. The (continuous time) contact process was first introduced by Harris in [Har74]. It is one of the classical interacting particle systems and serves as a model for the spread of an infection or the growth of a population. In the discrete time case, a healthy site will be infected at the next time step with probability $p$, if there exists an infected particle in its nearest neighbourhood and an infected site recovers with probability $(1-p)$. In this section we will give a precise definition of the discrete time contact process and the random environment we are interested in. Furthermore, some already known facts about the contact process will be listed. For the proof of these facts we will often refer to results about the continuous time contact process. Although in many cases a precise reference in the literature is missing, one agrees on these results to hold true for the discrete time contact process as well. We apologize for this inaccuracy, but giving all the proofs would go beyond the scope of this thesis.

**Definition 1.2.** We fix $m \in \mathbb{Z}$ and $A \subset \mathbb{Z}^d$. The *discrete time contact process* $\eta^{A,m} := (\eta_n^{A,m}(y))_{n \geq m}$ starting at time $m$ from the set $A$ is defined as

$$\eta_m^{A,m}(y) = \mathbb{1}_A(y), \; y \in \mathbb{Z}^d,$$

and for $n \geq m$

$$\eta_{n+1}^{A,m}(x) = \begin{cases} 1 & \text{if } \omega(x, n+1) = 1 \text{ and } \eta_n^{A,m}(y) = 1 \text{ for some } y \in \mathbb{Z}^d \text{ with } \|x-y\| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

With the convention that $\omega(x,m) = \mathbb{1}_A(x)$, whereas for $n > m$ the $\omega(x,n)$ are i.i.d. Bernoulli(p) as defined in (1.1), we have

$$\eta_n^{A,m}(x) = 1, \quad \text{if and only if} \quad (y,m) \xrightarrow{\omega} (x,n) \text{ for some } y \in A.$$

Sometimes we refer to $\eta^{A,m}$ as the contact process *driven by* $\omega$ or *defined on* $\omega$. For some given distribution $\mu$ on $\{0,1\}^{\mathbb{Z}^d}$ we write $\eta^{\mu,m} := (\eta_n^{\mu,m})_{n \geq m}$ for the discrete time contact process with initial configuration $\eta_m^{\mu,m}$ distributed according to $\mu$.

If $\eta_n^{A,m}(y) = 1$, we consider the particle in $y$ at time $n$ to be "infected" by some particle $x \in A$ at time $m$. Sometimes the random variable $\eta_n^{A,m}(y)$ also refers to the set of infected sites at time $n \geq m$. For $x \in \mathbb{Z}^d$, we often refer to the set

$$U(x) := \{y : \|x-y\| \leq 1\} \tag{1.4}$$

in (1.3) as the *nearest neighbourhood* of $x$ (with respect to $\|\cdot\|$). For $A \subset \mathbb{Z}^d$ the nearest neighbourhood of $A$ is defined as

$$U(A) := \bigcup_{x \in A} U(x). \tag{1.5}$$

If $A = \{x\}$ for some $x \in \mathbb{Z}^d$, we write $\eta_n^{(x,m)}$ instead of $\eta_n^{\{x\},m}$.

Define

$$\tau^A := \inf\{n \geq 0 : \eta_n^{A,0} = \emptyset\} \tag{1.6}$$

as the time that the contact process starting from $A \subset \mathbb{Z}^d$ at time 0 dies out. If $A = \{x\}$, $x \in \mathbb{Z}^d$ we write $\tau^x$ instead of $\tau^{\{x\}}$.

There exists a critical value $p_c \in (0,1)$ such that $\mathbb{P}(\tau^{\mathbf{0}} = \infty) = 0$ if $p \le p_c$ and $\mathbb{P}(\tau^{\mathbf{0}} = \infty) > 0$ if $p > p_c$ (see e.g. Theorem 1 in [GH02]), where $\mathbf{0} = (0, ..., 0) \in \mathbb{Z}^d$. During the rest of this thesis we assume $p > p_c$.

The discrete time contact process is a Markov process on $\{0,1\}^{\mathbb{Z}^d}$. Notice that on $\{0,1\}^{\mathbb{Z}^d}$, there exists a partial order of elements given by

$$\eta \le \eta', \text{ if } \eta(x) \le \eta'(x) \text{ for all } x \in \mathbb{Z}^d. \tag{1.7}$$

We always assume $\{0,1\}^{\mathbb{Z}^d}$ to be equipped with the product topology. A continuous function $f \in C(\{0,1\}^{\mathbb{Z}^d})$ is called *increasing* if

$$\eta \le \eta' \quad \text{implies} \quad f(\eta) \le f(\eta'). \tag{1.8}$$

We call two probability measures $\mu_1, \mu_2$ on $\{0,1\}^{\mathbb{Z}^d}$ to be *stochastically monotone*, denoted by $\mu_1 \le \mu_2$, iff

$$\int_{\{0,1\}^{\mathbb{Z}^d}} f d\mu_1 \le \int_{\{0,1\}^{\mathbb{Z}^d}} f d\mu_2 \quad \text{for all increasing } f \in C(\{0,1\}^{\mathbb{Z}^d}). \tag{1.9}$$

If $\mu_1 \le \mu_2$, we also say that $\mu_2$ *stochastically dominates* $\mu_1$.

Next we will focus on the weak limit of $\eta_0^{\mathbb{Z}^d,m}$ as $m$ tends to $-\infty$. Notice that the definition of the contact process defined on $\omega$ yields a monotone and additive coupling for arbitrary initial states $A, B \subset \mathbb{Z}^d$, which means that

$$A \subset B \ \Rightarrow \ \eta_n^{A,m} \subset \eta_n^{B,m}, \tag{1.10}$$

and

$$\eta_n^{A \cup B,m} = \eta_n^{A,m} \cup \eta_n^{B,m}. \tag{1.11}$$

Furthermore, for $A, B \subset \mathbb{Z}^d$ the following (self-)duality relation holds true

$$\mathbb{P}(\eta_n^{A,m} \cap U(B) \neq \emptyset) = \mathbb{P}(\eta_n^{B,m}(\omega) \cap U(A) \neq \emptyset) \tag{1.12}$$

for the definition of $U(A)$ see (1.5). The duality relation has to be written like this because of the convention that is made within the definition of $\eta_n^{A,m}$. The relation can be easily verified by reversing the directed paths in $\omega$ between $A \times \{m\}$ and $B \times \{n+1\}$.

Now we focus on the discrete time contact process with initial configuration $\mathbb{Z}^d$. Let $\mu_m$ be the distribution of $\eta_0^{\mathbb{Z}^d,m}$ and let $f \in C(\{0,1\}^{\mathbb{Z}^d})$ be an increasing function. Since $\eta_0^{\mathbb{Z}^d,m} \subset \mathbb{Z}^d$ a.s. for all $m \le 0$ the monotone coupling in (1.10) and the Markov property of the discrete time contact process imply

$$\int_{\{0,1\}^{\mathbb{Z}^d}} f d\mu_{m'} \le \int_{\{0,1\}^{\mathbb{Z}^d}} f d\mu_m \quad \text{for all } m' \le m \le 0. \tag{1.13}$$

which means that $\mu_{m'} \le \mu_m$ for all $m' \le m \le 0$. Hence compactness of the set of probability measures on $\{0,1\}^{\mathbb{Z}^d}$ implies the existence of a unique weak limit

$$\bar{\nu} \overset{w}{=} \lim_{m \to -\infty} \mu_m = \lim_{m \to -\infty} \mathcal{L}(\eta_0^{\mathbb{Z}^d,m}), \tag{1.14}$$

which is also non-trivial since we assume $p > p_c$. The measure $\bar{\nu}$ is called the *upper invariant measure* of the discrete time contact process. Hence, taking $m$ to $-\infty$, we obtain a stationary process $\eta := (\eta_n)_{n\in\mathbb{Z}} := (\eta_n^{\mathbb{Z}^d})_{n\in\mathbb{Z}}$, where for a given configuration $\omega \in \{0,1\}^{\mathbb{Z}^d\times\mathbb{Z}}$

$$\begin{aligned} &\eta_n(x) = 1, \quad \text{if and only if} \\ &\text{for every } m \le n \text{ there exists } y \in \mathbb{Z}^d \text{ such that } (y,m) \xrightarrow{\omega} (x,n). \end{aligned} \tag{1.15}$$

### 1.2.1. Link to oriented percolation

A very nice explanation of how oriented percolation is connected to the contact process can be found in [Lig99, page 13], yet for the sake of completeness we give a short explanation with notation adapted to our case.

We change our view on the contact process slightly. Let $A$ be a finite subset of $\mathbb{Z}^d$. If the contact process $(\eta_n^{A,m})_{n\ge m}$ is seen as a Markov process with state space given by the collection of finite subsets of $\mathbb{Z}^d$, its evolution in time can be described as follows:

Given the information on the process up to time $n$, the events $\{x \in \eta_{n+1}^{A,m}\}_{x\in\mathbb{Z}^d}$ are independent and

$$\mathbb{P}(\{x \in \eta_{n+1}^{A,m}\}|\eta_1^{A,m}, ..., \eta_n^{A,m}) = \begin{cases} \mathbb{P}(\omega(x,n+1) = 1) = p & \text{if } \eta_n^{A,m} \cap \{y : \|x-y\| \le 1\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

This description of the contact process is not usual but shows its connection to oriented (site) percolation and in fact exhibits it as a "probabilistic cellular automaton".

### 1.2.2. Some facts about the contact process and its upper invariant measure

In this section we will list some facts about the contact process, which will be needed for later results. As already mentioned the references often refer to the continuous time contact process, nevertheless, the statements also hold true for the discrete time case.

The first Lemma shows how the survival probability of the (discrete time) contact process is related to its initial configuration and gives an estimate on the probability that the contact process dies out after surviving for $n$ steps. Recall the definition of $\tau^A$ in (1.6).

**Lemma 1.3.** *There exist constants $C, c > 0$ such that for every $n \ge 0$ and $A \subset \mathbb{Z}^d$ we have*

$$\mathbb{P}(n \le \tau^{\mathbf{0}} < \infty) \le Ce^{-cn} \tag{1.17}$$

*and*

$$\mathbb{P}(\tau^A = \infty) \le Ce^{-c|A|} \tag{1.18}$$

**Proof:** For the proof of the continuous time case we refer to [Lig99, Theorem 2.30]. Furthermore, Birkner et al. gave a proof of (1.17) for the discrete time case in [BČDG13, Lemma A.1.].

□

We already mentioned that the upper invariant measure $\bar{\nu}$ is non-trivial. The next Lemma gives an easy example of a non-trivial measure that is stochastically dominated by $\bar{\nu}$.

**Lemma 1.4.** *The upper invariant measure of the contact process stochastically dominates a Bernoulli product measure* $\nu_{p'} := \mathrm{Ber}(p')^{\otimes \mathbb{Z}^d}$ *for some* $p' > 0$.

**Remark 1.5.** Recall the definition of stochastic domination from (1.9). The stochastic domination in Lemma 1.4 is equivalent to the existence of a probability measure $\mu$ on $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ with marginals $\nu_p$ and $\bar{\nu}$, which means

$$\nu_p(A) = \mu((\zeta, \zeta') \in \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d} : \zeta \in A),$$
$$\bar{\nu}(A) = \mu((\zeta, \zeta') \in \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d} : \zeta' \in A),$$

such that

$$\mu((\zeta, \zeta') : \zeta \leq \zeta') = 1$$

See also the remarks at the beginning of section two in [LS06] and [Lig85, page 72, Theorem 2.4].

**Proof of Lemma 1.4:** For the continuous time contact process, the proof of Lemma 1.4 is given in [LS06, Theorem 1.1].

□

The next Lemma is a discrete time analogue of the shape theorem given in [DG82, Theorem, equation (8)], see also [BG90a, Theorem (5)]. Before we are able to write it down, we need a little more notation. For $x, y \in \mathbb{Z}^d$ and $m \in \mathbb{Z}$ we define the stopping time

$$t^{(y,m)}(x) := \inf\{n \geq m : x \in \eta_n^{(y,m)}\},$$

with convention $\inf \emptyset := \infty$. The random time $t^{(y,m)}(x)$ is the first time at which a site $x \in \mathbb{Z}^d$ is infected by the discrete time contact process starting from $y \in \mathbb{Z}^d$ at time $m$. Furthermore, we define

$$H_n^{(y,m)} := \left\{ y' \in \mathbb{R}^d : \exists\, x \in \mathbb{Z}^d \text{ with } \left\| x - y' \right\| \leq \frac{1}{2} \text{ and } t^{(y,m)}(x) \leq n \right\}, \tag{1.19}$$

which is the set of all sites infected by the contact process $(\eta_n^{(y,m)})_n$ up to time $n$. Next we define the set of successful coupling between $\eta_n^{(y,m)}$ and $\eta_n^{\mathbb{Z}^d,m}$ by

$$K_n^{(y,m)} := \left\{ y \in \mathbb{R}^d : \exists\, x \in \mathbb{Z}^d \text{ with } \| x - y \| \leq \frac{1}{2} \text{ and } \eta_n^{(y,m)}(x) = \eta_n^{\mathbb{Z}^d,m}(x) \right\}. \tag{1.20}$$

Note that for each $x \in \mathbb{Z}^d$ that is contained in $H_n^{(y,m)}$ or $K_n^{(y,m)}$, the random sets also contain a cube of side length one and center $x$.

**Lemma 1.6. (shape theorem)** *There exists a convex subset* $U \subset \mathbb{R}^d$*, which is a neighbourhood of* $\mathbf{0} \in \mathbb{R}^d$*, such that for any* $\varepsilon > 0$

$$n \cdot (1 - \varepsilon) \cdot U \subset H_n^{(\mathbf{0},0)} \subset n \cdot (1 + \varepsilon) \cdot U \quad \textit{eventually},$$

*almost surely on the event* $\{\tau^{\mathbf{0}} = \infty\}$*. Additionally,*

$$n \cdot (1 - \varepsilon) \cdot U \subset H_n^{(\mathbf{0},0)} \cap K_n^{(\mathbf{0},0)} \subset n \cdot (1 + \varepsilon) \cdot U \quad \textit{eventually},$$

*almost surely on the event* $\{\tau^{\mathbf{0}} = \infty\}$.

**Proof:** For the continuous time contact process, this is proved in [DG82, Theorem, equation (8)], see also [BG90a, Theorem (5)].

□

There also exists a more quantitative version of the shape theorem which is literally formulated for the continuous time case:

**Lemma 1.7.** *There exist constants $C, c, \gamma > 0$ such that*

$$\mathbb{P}\left(\eta_n^{\mathbb{Z}^d,0}(x) \neq \eta_n^{\bar{\nu},0}(x)\right) \leq C^{-cn}, \quad x \in \mathbb{Z}^d, \tag{1.21}$$

$$\mathbb{P}\left(\eta_n^{(\mathbf{0},0)}(x) \neq \eta_n^{\mathbb{Z}^d,0}(x) \mid \tau^{\mathbf{0}} = \infty\right) \leq Ce^{-cn}, \quad \|x\| \leq \gamma n, \tag{1.22}$$

*and*

$$\mathbb{P}\left(t(x) > n \mid \tau^{\mathbf{0}} = \infty\right) \leq Ce^{-cn}, \quad \|x\| \leq \gamma n \tag{1.23}$$

**Proof:** The proof for the continuous time case can be found in [DG82]. See especially "Theorem" and Proposition 6, equation (33) and (34). Note that (1.21) follows immediately by (1.17), self duality of $\eta$ and the fact that

$$\begin{aligned}\mathbb{P}\left(\eta_n^{\mathbb{Z}^d,0}(x) \neq \eta_n^{\bar{\nu},0}(x)\right) &= \mathbb{P}\left(\eta_n^{\mathbb{Z}^d,0}(x) = 1\right) - \mathbb{P}\left(\eta_n^{\bar{\nu},0}(x) = 1\right) \\ &= \mathbb{P}\left(\tau^x > n\right) - \mathbb{P}\left(\tau^x = \infty\right).\end{aligned}$$

□

**Remark 1.8.** In [GM14] a shape theorem is proven for the continuous time contact process in random environment. The term "random environment" in the context of [GM14] refers to randomly chosen rates according to which an infected site infects its nearest neighbours.

## 1.3. The backbone of an oriented percolation cluster

In this section we give a new interpretation of $\eta$ and explain what is meant by the *backbone* of an oriented percolation cluster. The definition of $\eta$ can be found in (1.15). The term "*backbone of an oriented percolation cluster*" was used by Birkner et al. in [BČDG13]. A precise definition of the model can also be found therein.

For the rest of this thesis we want to think of $\eta$ as a population process. We call a site $x$ *inhabited* or *occupied* by an individual at time $n$, iff $\eta_n(x) = 1$. Recall the definition of $U$ in (1.4). By the way $\eta$ was constructed, $\eta_n(x) = 1$ for some $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ implies that at time $n-1$ there exists a particle in the nearest neighbourhood of $x$, more precisely $\eta_n(x) = 1$ implies $\eta_{n-1}(y) = 1$ for some $y \in U(x)$. We call the particle in $x$ at time $n$ to be an offspring of the particle in $y$ at time $n-1$. If there exists more than one particle in the nearest neighbourhood at time $n-1$, we choose the parent uniformly among the possible ones. This is also the place where local competition comes into play. In sparely crowded regions a particle has higher probability to leave an offspring for the next generation. Or from a different point of view, in sparely crowded areas a particle has an increased chance to be chosen as parent. Since we are interested in ancestral lineages, respectively random walks in random environment moving "backwards in time" (relative to the natural dynamics of $\eta$), we reverse the time direction to avoid negative signs. We denote the time reversal of $\eta$ by $\xi := (\xi_n)_{n\in\mathbb{Z}}$.

Recognize that for a given configuration $\omega \in \{0,1\}^{\mathbb{Z}^d\times\mathbb{Z}}$ the time-reversed process $\xi = (\xi_n)_{n\in\mathbb{Z}}$ can be characterized as

$$\xi_n(x) = \begin{cases} 1 & \text{if } (x,n) \xrightarrow{\omega} \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{1.24}$$

Since $\eta$ is a stationary Markov process its time reversal $\xi$ is a stationary Markov process as well. The invariant measure of $\xi$ is the upper invariant measure of the discrete time contact process. The random set

$$\mathcal{C} := \{(x,n) : \xi_n(x) = 1\} \tag{1.25}$$

is called the *backbone of an oriented percolation cluster*. If we want to emphasize the dependence of $\xi$ on $\omega$ or some special $\omega$ is chosen, we write $\xi_\omega := (\xi_\omega(x,n))_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}} := (\xi_n(x))_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}}$. Notice that $\xi$ is measurable with respect to $\sigma\big(\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\big)$.

## 1.4. Random walks on $\mathcal{C}$

In this section we give a precise definition of the random walks on $\mathcal{C}$ we are interested in. We fix some $z = (y,m) \in \mathbb{Z}^d \times \mathbb{Z}$ which will be the initial value of the random walk and restrict ourselves to the event

$$B_z := \{\xi_m(x) = 1\} = \{(x,m) \in \mathcal{C}\}. \tag{1.26}$$

By stationarity of $\eta$, it would have been enough to focus on the event $B_{(\mathbf{0},0)} := \{\xi_0(\mathbf{0}) = 1\}$. But the construction of the random walk for an arbitrarily chosen initial value is better to establish the notation that will be used later on. Note that $\mathbb{P}(B_z) = \mathbb{P}(B_{(\mathbf{0},0)}) > 0$.

We fix some configuration $\omega \in B_z$. As described before, given $\omega$, we want to construct a random walk starting from $y$ at time $m$ that chooses its next step (resp. its parent) uniformly among all possible states (resp. parents) in the next time-layer. We want the choice among the possible states to be independent of everything else in the model. Given some fixed environment the random walk is defined as follows:

**Definition 1.9. (Quenched law)** For some given $\omega \in B_z$ the *quenched law* of the random walk starting from $z$, usually denoted by $P_\omega^z$, is characterized by $P_\omega^z(X_m = y) = 1$ and transition probabilities

$$P_\omega^z(X_{n+1} = x' | X_n = x) := \frac{\xi_{n+1}(x')}{\sum_{\tilde{x}:\|\tilde{x}-x\|\le 1} \xi_{n+1}(\tilde{x})}, \tag{1.27}$$

which means that $X$ is a time-inhomogeneous Markov-chain under $P_\omega$.

**Remark 1.10.** By the definition of the quenched law above the random walk $X$ on the environment $\omega$ starting from space-time point $z = (y,m) \in \mathbb{Z}^d \times \mathbb{Z}$ remains undefined up to time $m$. This appears to be more natural since in fact we consider a $(d+1)$-dimensional directed random walk $(Y_n)_{n\ge 0} := (X_{n+m}, n+m)_{n\ge 0}$ and do not want to shift the space and time component against each other.

**Definition 1.11. (Annealed law)** Define $\mathsf{P}^z(\,\cdot\,) := \mathbb{P}(\,\cdot\,|B_z)$. The *annealed law* $\mathbb{P}^z$ of the random walk starting from the space time point $z$ is defined as

$$\mathbb{P}^z(\cdot) := \int P_w^z(\cdot)\,\mathsf{P}^z(d\omega) = \frac{1}{\mathbb{P}(B_z)}\int_{B_z} P_w^z(\cdot)\,\mathbb{P}(d\omega). \tag{1.28}$$

By a common abuse of notation, the measure $\mathbb{P}^z$ also refers to $\mathsf{P}^z \otimes P_w^z$, which is the joint law of the environment and the random walk. Note that $\mathsf{P}^z \otimes P_w^z$ is technically a semidirect product and not a product law.

We denote the expectations with respect to $P_\omega^z$, $\mathsf{P}^z$ and $\mathbb{P}^z$ by $E_\omega^z$, $\mathsf{E}^z$ and $\mathbb{E}^z$.

### 1.4.1. Definition of the regeneration structure

In this subsection we focus on a regeneration structure of the random walks on $\mathcal{C}$. This regeneration structure allows us to cut the random walks into independent and identically distributed (i.i.d.) increments. The lemmas and arguments within this subsection are given in much more detail at [BČDG13, section 2]. Birkner et al. adapted arguments from [Kuc89] and [Neu92] to prove exponential moments on the random walk increments between regeneration times. Cutting the random walk path into i.i.d. increments which have exponential and hence second moments, they immediately derived an annealed central limit theorem and a (strong) law of large numbers for the random walks. Also for their proof of the quenched central limit theorem, the regeneration structure was one of the main ingredients. For the sake of completeness we list some of the lemmas in [BČDG13], which will be needed later on and try to explain the ideas behind them as briefly as possible.

The first thing we want to explain is how a random walk path with dynamics given at (1.27) can be constructed using only "local" information on $\omega$. Instead of choosing some arbitrary space-time point the random walk starts from, we will locally construct a path of the random walk starting from $\mathbf{0}$ at time 0 on the event $B_{(\mathbf{0},0)}$ (see (1.26)). We start by defining some additional randomness on $(\Omega, \mathcal{A}, \mathbb{P})$ (see (1.2)). For each $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ let

$$\omega_{nb}(x,n) := (\omega_{nb}(x,n)[1], \dots, \omega_{nb}(x,n)[3^d]) \tag{1.29}$$

be a uniform (random) permutation of the nearest neighbours of $x$ that is independent of everything else in the model. Let $\gamma_k^{(\mathbf{0},0)} := (\gamma_k^{(\mathbf{0},0)}(n))_{n=0,\dots,k}$ be a directed open path of length $k \in \mathbb{N}$ starting from space-time point $(\mathbf{0},0) \in \mathbb{Z}^d \times \mathbb{Z}$, defined as follows:

$$\gamma_k^{(\mathbf{0},0)}(0) = \mathbf{0} \in \mathbb{Z}^d$$

and

$$\gamma_k^{(\mathbf{0},0)}(n+1) := \begin{cases} \left.\begin{array}{l}\text{the element of}\\ \{y \in U(x) : (y,n+1) \overset{\omega}{\to} \mathbb{Z}^d \times \{k-1\}\}\\ \text{that appears first in } \omega_{nb}(x,n)\end{array}\right\} & \text{if } \gamma_k^{(\mathbf{0},0)}(n) = x \text{ and } 1 \le n \le k-2, \\ \omega_{nb}(x,k-1)[1] & \text{if } \gamma_k^{(\mathbf{0},0)}(n) = x \text{ and } n = k-1. \end{cases}$$

We interpret $\gamma_k^{(\mathbf{0},0)}(k)$ as a "possible ancestor" of the individual at space-time point $(\mathbf{0},0)$. We want to point out that $\gamma_k^{(\mathbf{0},0)}$ is measurable with respect to

$$\mathcal{G}_0^k := \sigma(\omega(x,n),\omega_{nb}(x,n) : x\in\mathbb{Z}^d, 0\le n<k).$$

Furthermore, if $\gamma_k^{(\mathbf{0},0)}(n)\in\mathcal{C}$ for some $n\in 1,...,k$ it follows that $\gamma_m^{(\mathbf{0},0)}(n)=\gamma_k^{(\mathbf{0},0)}(n)$ for all $m\ge k$. On $B_{(\mathbf{0},0)}$ the limit

$$\gamma_\infty^{(\mathbf{0},0)}(j) := \lim_{k\to\infty}\gamma_k^{(\mathbf{0},0)}(j) \tag{1.30}$$

exists for all $j\in\mathbb{N}_0$. The properties of the local construction (see [BČDG13, Lemma2.1 and Remark 2.2]) yield a coupling on $B_{(\mathbf{0},0)}$ of the random variables $\omega,\omega_{nb}$ and the random walk $X$ starting from $\mathbf{0}$ at time 0 by

$$X_n := \lim_{k\to\infty}\gamma_k^{(\mathbf{0},0)}(n). \tag{1.31}$$

Using the local construction, we define a sequence $(T_n)_{n\ge0}$ of random times by

$$T_0 := 0 \quad\text{and}\quad T_j := \inf\left\{k>T_{j-1} : \xi_k\left(\gamma_k^{(\mathbf{0},0)}(k)\right)=1\right\}. \tag{1.32}$$

Notice that $\gamma_{T_j}^{(\mathbf{0},0)}(T_j)=\gamma_m^{(\mathbf{0},0)}(T_j)$ for all $m>T_j$, which by (1.31) and the definition of the local construction implies that $X_m=\gamma_{T_j}^{(\mathbf{0},0)}(m)$ for all $m\le T_j$. We interpret $(T_j)_{j\ge0}$ as the times at which the local construction discovers a "real ancestor" of the individual located at $(\mathbf{0},0)$ and call them *regeneration times*. For $i\ge1$, we define

$$\tau_i := T_i - T_{i-1} \quad\text{and}\quad Y_i := X_{T_i}-X_{T_{i-1}}. \tag{1.33}$$

According to [BČDG13, Lemma 2.5] $Y_i$ is symmetrically distributed, the sequence $(\tau_i,Y_i)_{i\ge1}$ is i.i.d and there exist constants $C,c>0$ such that

$$\mathbb{P}^{(\mathbf{0},0)}(\|Y_1\|>n)\le Ce^{-cn} \quad\text{and}\quad \mathbb{P}^{(\mathbf{0},0)}(\tau_1>n)\le Ce^{-cn}. \tag{1.34}$$

The proof of these statements is given in [BČDG13, section 2.3]. The main idea for the proof of (1.34) is to dominate the number of attempts before the local construction hits the cluster $\mathcal{C}$ by a geometrically distributed random variable with positive success probability. Furthermore, one needs to make use of the fact that for each failure one has to explore a "dead end" whose length has exponential tail bounds by (1.17).

### 1.4.2. Finite number of random walks starting from deterministic space-time points

In this section we introduce different models of several random walks defined on the same and independent copies of $\mathcal{C}$. The definitions within this subsection are essential for the rest of this thesis. If we consider several random walks, we slightly change our notation. Instead of considering the random walk $X$ with respect to the probability measures $\mathbb{P}^z$ and $P_\omega^z$ we rather add a subscript $z$ to $X$ in order to indicate the initial value. Since we extended the probability space $(\Omega,\mathcal{A},\mathbb{P})$ by some additional randomness $\omega_{nb}$, it is consistent with Definition 1.11 to refer to $\mathbb{P}(\,\cdot\,|B_z)$ as the annealed law of the random walk $X^{(z)}$ starting from the space-time point $z$. Recall the definition of $B_z$ in (1.26). Furthermore, for $z_1, ..., z_n \in \mathbb{Z}^d \times \mathbb{Z}$ we define

$$B_{z_1,...,z_n} := \bigcap_{k \leq n} B_{z_k}. \tag{1.35}$$

**(i) Random walks defined on a joint oriented percolation cluster**

At first we want to consider a model of $l \in \mathbb{N}$ random walks which for one given oriented percolation cluster move independently of each other on the same backbone. We refer to this model as the *joint* case, since several random walks are defined on a *joint* oriented percolation cluster, respectively its backbone. Notice that talking about "independent random walks" would be too much in this context since the cluster creates a dependence between them. If two random walks visit the same sparely crowded area, it is more likely that they choose the same ancestors. We fix $z_1 = (y_1, m_1), ..., z_l = (y_l, m_l) \in \mathbb{Z}^d \times \mathbb{Z}$ and extend the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (see (1.2)) by $l$ independent copies of $\omega_{nb}$, which will be denoted by $\omega_{nb}^{(1)}, ..., \omega_{nb}^{(l)}$. If the probability space is extended like this, we will add the subscript "joint" to it and denote it by $(\Omega_{joint}, \mathcal{A}_{joint}, \mathbb{P}_{joint})$. On the event $B_{z_1,...,z_n}$ we construct random walks $X^{(z_1)}, ..., X^{(z_l)}$ starting from $z_1, ..., z_l$ in the way we described in (1.31), but for each random walk the ancestral choice will be done according to some independent copy of $\omega_{nb}$. Since for a given configuration of $\omega$, for each random walk the choice of its parent at the next time step is independent of everything else in the model, we indeed get $l$ random walks which for a given oriented percolation cluster move independently of each other. Notice that

$$\mathbb{P}_{joint}(X_{n_1}^{(z_1)} \in A_1, ..., X_{n_l}^{(z_l)} \in A_l \mid B_{z_1,...,z_l}) = \frac{1}{\mathbb{P}(\cap_{k \leq l} B_{z_k})} \int_{\cap_{k \leq l} B_{z_k}} P_w^{z_1}(X_{n_1} \in A_1) \cdot ... \cdot P_w^{z_l}(X_{n_l} \in A_l) d\mathbb{P}$$

for $A_1, ..., A_l \in \mathcal{B}(\mathbb{R}),\ n_1 \geq m_1, ..., n_l \geq m_l$.

**(ii) Random walks defined on independent copies of the oriented percolation cluster**

Next we will define the model corresponding to what one would call "independent" random walks on oriented percolation clusters. Again we fix $z_1 = (y_1, m_1), ..., z_l = (y_l, m_l) \in \mathbb{Z}^d \times \mathbb{Z}$ and extend $(\Omega, \mathcal{A}, \mathbb{P})$ not only by $l$ independent copies $\omega_{nb}^{(1)}, ..., \omega_{nb}^{(l)}$ of $\omega_{nb}$, but also by $l$ independent copies $\omega^{(1)}, ..., \omega^{(l)}$ of $\omega$. The probability space extended like this will be denoted by $(\Omega_{ind}, \mathcal{A}_{ind}, \mathbb{P}_{ind})$. Now for each pair $(\omega^{(k)}, \omega_{nb}^{(k)})$ we construct a random walk $X^{(z_k)}$ starting from $z_k$, conditioned on the event $\{z_k \in \mathcal{C}(\omega^{(k)})\} = \{\xi_{\omega^{(k)}}(z_k) = 1\}$. What we obtain is a system of $l$ actually independent random walks defined on independent copies of $\mathcal{C}$, hence we refer to this model as the *independent* case. Notice that

$$\mathbb{P}_{ind}(X_{n_1}^{(z_1)} \in A_1, ..., X_{n_l}^{(z_l)} \in A_l \mid B_{z_1,...,z_l}^{ind}) = \prod_{k=1}^{l} \frac{1}{\mathbb{P}(B_{z_k})} \int_{B_{z_k}} P_w^{z_k}(X_{n_k} \in A_k) d\mathbb{P},$$

for $A_1, ..., A_l \in \mathcal{B}(\mathbb{R})$ $n_1 \geq m_1, ..., n_l \geq m_l$, where

$$B_{z_1,...,z_n}^{ind} := \{\xi_{\omega^{(1)}}(z_1) = ... = \xi_{\omega^{(l)}}(z_l) = 1\}. \tag{1.36}$$

**(iii) Coalescing random walks defined on a joint oriented percolation cluster**

At last we want to define a model of $l$ coalescing random walks moving on the same oriented percolation cluster. This model is basically the one that fits our interpretation of the random walks as ancestral lineages the most. Once a common ancestor of two particles is found, the ancestral lineages coalesce into one and we have to trace back the ancestral lineage of the common ancestor. As before we fix $z_1 = (y_1, m_1), ..., z_l = (y_l, m_l) \in \mathbb{Z}^d \times \mathbb{Z}$. On the event $\cap_{k \le l} B_{z_k}$ we construct random walks $X^{(z_1)}, ..., X^{(z_l)}$ starting from $z_1, ..., z_l$ in the way we described above, but for each random walk the ancestral choice will be done according to the same $\omega_{nb}$. This leads to $l$ coalescing random walks moving on the same oriented percolation cluster. Since for each space-time point $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ the permutation of the nearest neighbours $\omega_{nb}(x, n)$ is independent of everything else in the model, the random walks move independently until they hit a common space-time point and coalesce. We refer to this model as the *coalescing* case. The notation of the probability space will not be changed, since we already extended $(\Omega, \mathcal{A}, \mathbb{P})$ by a uniform (random) permutation of the nearest neighbourhood of each space-time point, see (1.29).

### 1.4.3. Definition of a joint regeneration structure

Considering several random walks on a joint oriented percolation cluster in analogy to subsection 1.4.1 one can define simultaneous regeneration times for the random walks (see [BČDG13, section 3]). We fix $z_1 = (y_1, m), ..., z_l = (y_l, m) \in \mathbb{Z}^d \times \mathbb{Z}$. Notice that we want the random walks to start at the same time $m \in \mathbb{Z}$. The individual regeneration times of the random walks $X^{(z_1)}, ..., X^{(z_l)}$ are given by

$$\begin{aligned} T_0^{(j)} &:= m, \\ T_{r+1}^{(j)} &:= \inf\left\{k > T_r^{(j)} : \xi_{m+k}\left(\gamma_k^{(z_j)}(k)\right) = 1\right\}, \end{aligned} \tag{1.37}$$

compare to (1.32). The simultaneous regeneration time for the random walks $X^{(z_1)}, ..., X^{(z_l)}$ are defined as

$$\begin{aligned} T_0^{sim} &:= m, \\ T_{k+1}^{sim} &:= \min \bigcap_{j=1}^{l} \{T_r^{(j)} : T_r^{(j)} > T_k^{sim}, r \geq 0\}. \end{aligned} \tag{1.38}$$

Extending the notation of (1.33) to the set-up of $l$ different random walks, we write

$$Y_k^{(z_j)} := X_{T_k^{(j)}}^{(z_j)} - X_{T_{k-1}^{(j)}}^{(z_j)}, \quad \tau_k^{(j)} := T_k^{(j)} - T_{k-1}^{(j)}. \tag{1.39}$$

and

$$\widehat{X}_k^{(z_j)} := X_{T_k^{sim}}^{(z_j)}. \tag{1.40}$$

The index of the individual regeneration time of $X^{(z_j)}$ at which a simultaneous regeneration occurs will be denoted by

$$\begin{aligned} J_0^{(j)} &:= 0, \\ J_{k+1}^{(j)} &:= \{r > J_k^{(j)} : T_r^{(j)} = T_{k+1}^{sim}\}. \end{aligned} \tag{1.41}$$

Furthermore, the pieces between simultaneous regenerations are defined as

$$\Xi_n := \left( (Y_k^{(z_1)}, \tau_k^{(1)})_{k=J_{n-1}^{(1)}+1}^{J_n^{(1)}}, ..., (Y_k^{(z_l)}, \tau_k^{(l)})_{k=J_{n-1}^{(l)}+1}^{J_n^{(l)}}, \widehat{X}_n^{(z_1)}, ..., \widehat{X}_n^{(z_l)} \right). \tag{1.42}$$

The random variable $\Xi_n$ takes values in $\times_{i=1}^{l} \mathbb{F} \times \times_{i=1}^{l} \mathbb{Z}^d$, where $\mathbb{F} := \bigcup_{n=1}^{\infty} (\mathbb{Z}^d \times \mathbb{N})^n$.
This construction is done in [BČDG13] for two random walks. Slightly adapting the proof of [BČDG13, Lemma 3.1] to an arbitrary number $l \in \mathbb{N}$ of random walks, we derive the following lemma.

**Lemma 1.12. (exponential tail bounds for simultaneous regeneration times)**
*There exist constants $C, c > 0$, such that*

$$\mathbb{P}_{joint}\left(T_1^{sim} \geq k \mid B_{z_1,...,z_n}\right) \leq Ce^{-ck}. \tag{1.43}$$

**Proof:** The lemma can be proven by a simple adaptation of the proof of Lemma 3.1 in [BČDG13].

□

The properties described in [BČDG13, Lemma 3.2 and Remark 3.3] still hold true. For the sake of completeness we list them in the following remark.

**Remark 1.13.** We fix $z_1 = (y_1, m), ..., z_l = (y_l, m) \in \mathbb{Z}^d \times \mathbb{Z}$. Let $(\Xi_k)_{k \in \mathbb{N}_0}$ denote the pieces between simultaneous regenerations of the random walks $X^{(z_1)}, ..., X^{(z_l)}$ as defined in (1.42), where $\Xi_0 := (\alpha^{(1)}, ..., \alpha^{(l)}, y_1, ..., y_l)$ for some arbitrarily chosen $\alpha^{(1)}, ..., \alpha^{(l)} \in \mathbb{F}$. Under $\mathbb{P}_{joint}(\,\cdot\,|B_{z_1,...,z_n})$ the stochastic process $(\Xi_k)_{k \in \mathbb{N}_0}$ is a discrete time Markov chain with transition probability function

$$\begin{aligned}
&\Psi_{joint}\left((\alpha^{(1)}, ..., \alpha^{(l)}, x_1, ..., x_l), (\beta^{(1)}, ..., \beta^{(l)}, y_1, ..., y_l)\right) \\
&\quad =: \Psi_{joint}\left((x_1, ..., x_l), (\beta^{(1)}, ..., \beta^{(l)}, y_1, ..., y_l)\right)
\end{aligned}$$

that has the following spatial-homogeneity property

$$\begin{aligned}
&\Psi_{joint}\left((x_1 + z, ..., x_l + z), (\beta^{(1)}, ..., \beta^{(l)}, y_1 + z, ..., y_l + z)\right) \\
&= \Psi_{joint}\left((x_1, ..., x_l), (\beta^{(1)}, ..., \beta^{(m)}, y_1, ..., y_l)\right)
\end{aligned} \tag{1.44}$$

for all $z \in \mathbb{Z}^d$. Note that the process $\left(\widehat{X}_k^{(z_1)}, ..., \widehat{X}_k^{(z_1)}\right)_{k \geq 0}$ is a Markov chain itself, with transition probabilities

$$\widehat{\Psi}_{joint}\left((x_1, ..., x_l), (y_1, ..., y_l)\right) := \Psi_{joint}\left((x_1, ..., x_l), \mathbb{F} \times ... \times \mathbb{F} \times \{(y_1, ..., y_l)\}\right). \tag{1.45}$$

The next lemma is quite essential for the proof of Proposition 2.1 in chapter 2. It basically tells us that random walks on a joint oriented percolation cluster behave similar to random walks defined on independent copies of the cluster as long as they are far apart of each other. The error that is made decays exponentially within the distance between the random walks.
Definitions (1.37)-(1.42) can be formulated analogously for random walks being defined on independent copies of the oriented percolation cluster. For the independent case the statements of Lemma 1.12 and

Remark 1.13 still hold true and appear even more obvious. Under $\mathbb{P}_{ind}(\,\cdot\,|B^{ind}_{z_1,\dots,z_n})$ the transition probabilities of $(\Xi_m)_m$ will be denoted by

$$\Psi_{ind}\Big((\alpha^{(1)},...,\alpha^{(l)},x_1,...,x_l),(\beta^{(1)},...,\beta^{(l)},y_1,...,y_l)\Big)$$
$$=: \Psi_{ind}\Big((x_1,...,x_l),(\beta^{(1)},...,\beta^{(l)},y_1,...,y_l)\Big).$$

Furthermore, let

$$\widehat{\Psi}_{ind}\left((x_1,...,x_l),(y_1,...,y_l)\right) := \Psi_{ind}\left((x_1,...,x_l),\mathbb{F}\times...\times\mathbb{F}\times\{(y_1,...,y_l)\}\right), \tag{1.46}$$

denote the transition probabilities of $\left(\widehat{X}^{(z_1)}_k,...,\widehat{X}^{(z_1)}_k\right)_{k\geq 0}$ with respect to $\mathbb{P}_{ind}(\,\cdot\,|B^{ind}_{z_1,\dots,z_n})$. Adapting the proof of [BČDG13, Lemma 3.4], we get the following:

**Lemma 1.14.** *There exist constants $C,c>0$, such that*

$$\|\Psi_{ind}((x_1,...,x_l),\cdot)-\Psi_{joint}((x_1,...,x_l),\cdot)\|_{TV}\leq Ce^{-c\min_{i\neq j}\|x_i-x_j\|}. \tag{1.47}$$

### 1.4.4. Construction of a coalescing stochastic flow

Now we want to define a model of infinitely many coalescing random walks $X^{(y,m)}=\big(X^{(y,m)}_n\big)_{n\geq m}$ starting (in principle) from any space-time point $(y,m)\in\mathbb{Z}^d\times\mathbb{Z}$, and moving on a joint oriented percolation cluster. Hence in comparison to the models introduced in section 1.4.2 we need to get rid of conditioning on the event that the space-time point the random walks start from is contained in the backbone of the oriented percolation cluster. This can be done by changing the transition probabilities in the following way:
Recall the definition of $\omega_{nb}$ in (1.29). Define

$$\Phi(x,n) := \begin{cases}\omega_{nb}(x,n)\big[\min\{i:(\omega_{nb}(x,n)[i],n+1)\in\mathcal{C}\}\big], & \text{if } \mathcal{C}\cap\big(U(x)\times\{n+1\}\big)\neq\emptyset,\\ \omega_{nb}(x,n)[1], & \text{otherwise.}\end{cases} \tag{1.48}$$

Note that for $(x,n)\in\mathcal{C}$, the first case occurs and $\Phi(x,n)$ gives back a uniformly chosen element of $\{y\in U(x):(y,n+1)\in\mathcal{C}\}$, which is the set of possible ancestors of $(x,n)$. For $(x,n)\notin\mathcal{C}$, $\Phi(x,n)$ gives back a uniformly chosen neighbour of $x$. We define

$$X^{(y,m)}_m := y, \qquad \text{and} \qquad X^{(y,m)}_{n+1} := \Phi(X^{(y,m)}_n,n), \quad n\geq m. \tag{1.49}$$

For fixed $(y,m)\in\mathbb{Z}^d\times\mathbb{Z}$, given $\omega$, $X^{(y,m)}$ is a time-inhomogeneous Markov chain with

$$P_\omega(X^{(y,m)}_{n+1}=x'\,|\,X^{(y,m)}_n=x) = \begin{cases}\frac{\xi_{n+1}(x')}{\sum_{\tilde{x}:\|\tilde{x}-x\|\leq 1}\xi_{n+1}(\tilde{x})}, & \text{if } \mathcal{C}\cap(U(x)\times\{n+1\})\neq\emptyset,\\ |U(x)|^{-1} & \text{if } \mathcal{C}\cap(U(x)\times\{n+1\})=\emptyset\end{cases} \tag{1.50}$$

and $P_\omega(X^{(y,m)}_m=y)=1$. In fact, (1.49) implements a coalescing stochastic flow with individual paths having transition probabilities given by (1.50). Note that for $z=(y,m)\in\mathbb{Z}^d\times\mathbb{Z}$ and $\omega\in B_z$ the transition probabilities (1.50) of the random walk $X^{(z)}$ coincide with the transition probabilities of the quenched law $P^z_\omega$ defined in (1.27). If $n=0$ is fixed, we will abbreviate $X^{(y)}:=X^{(y,0)}$.

## 1.5. (Neutral) multi-type contact process

In this section we extend the discrete time contact process driven by $\omega$ as defined in section 1.2 by assigning different types to the particles. Let $E$ be a set of possible types. We stipulate $0 \notin E$ (e.g. $E = \{1, 2\}$ or $E = (0, 1]$), since we want to interpret type 0 as “unoccupied site”. For simplicity we focus on the case $E = \{1, 2\}$. First we fix some $n_0 \in \mathbb{Z}$ and let $\eta_{n_0}^{\zeta, n_0} \equiv \zeta \in \{0, 1, 2\}^{\mathbb{Z}^d}$ be the initial configuration of “type 1”-and “type 2”-particles. The idea is to extend the contact process such that a “type 1”-particle can only give birth to “type 1”-particles and as before leave its offspring only at inhabitable sites. The same holds true for “type 2”-particles. In other words, we want each particle to inherit its type from its parent. Since we reversed time as mentioned at the end of section 1.3, the multi-type contact process evolves backwards with respect to the ancestral time. Remember the definition of $\Phi$ given at (1.48). For $n < n_0$ we define

$$\eta_{n-1}^{\zeta, n_0}(x) := \begin{cases} \eta_n^{\zeta, n_0}\big(\Phi(x, n-1)\big), & \text{if } (x, n-1) \in \mathcal{C}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.51}$$

since we think of the particles at time layer $n - 1$ to be offspring of the particles at time layer $n$, whereby the parental choice is determined by $\omega_{nb}$. See also the discussion in [BČDG13, p. 1–2].

By the definition of $\eta$ the following duality relation holds true

$$\eta_{n-1}^{\zeta, n_0}(x) := \begin{cases} \eta_m^{\zeta, n_0}(X_m^{(x,n)}), & \text{if } (x, n-1) \in \mathcal{C}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.52}$$

for $x \in \mathbb{Z}^d, n \le m \le n_0$. Here as in Definition 1.2 we use the convention that $\omega(x, n_0) = 1$ if $\zeta(x) > 0$, whereas the $\omega(x, n)$ are i.i.d. Bernoulli(p) at all other space-time points. This is the discrete-time analogue of the Harris construction. For $n \in \mathbb{Z}$ and pairwise different $x_1, \dots, x_l \in \mathbb{Z}^d$, $l \in \mathbb{N}$ write

$$B_{x_1,\dots,x_l;n} := \big\{(x_1, n), \dots, (x_l, n) \in \mathcal{C}\big\}. \tag{1.53}$$

In particular, for $C_1, \dots, C_l \subset E$ measurable and $l$ pairwise different points $x_1, \dots, x_l \in \mathbb{Z}^d$,

$$\mathbb{P}\big(\eta_n^{\zeta, n_0}(x_1) \in C_1, \dots, \eta_n^{\zeta, n_0}(x_l) \in C_l \mid B_{x_1,\dots,x_l;n}\big) = \mathbb{E}\Big[\prod_{i=1}^{l} \mathbb{1}_{C_i}\big(\eta_m^{\zeta, n_0}(X_m^{(x_i,n)})\big) \mid B_{x_1,\dots,x_l;n}\Big] \tag{1.54}$$

for all $m \in \{n, ..., n_0\}$. We expect that on sufficiently large space-time scales, any finite collection $X^{(x_0,t_0)}, X^{(x_1,t_1)}, \dots, X^{(x_n,t_n)}$ should look similar to (coalescing) random walks. By duality, this translates into a meta-theorem: “Everything” that is true for the (neutral) multi-type voter model should also be true for the (neutral) multi-type contact process. A first progress on this meta-theorem can be found in subsection 2.1.2 below.

# CHAPTER 2

## Brownian web scaling limit

In this chapter we will prove that the diffusively rescaled collection of random walk paths starting from every space time point contained in the backbone of an oriented percolation cluster of dimension $1+1$ converges in distribution to the Brownian web. This is done by verifying the convergence criteria Sun formulated in [Sun05]. The main ingredients are tail bounds on coalescing events, which will be proved in the first section of this chapter. The second section is dedicated to the characterization of the Brownian web. As mentioned in the introduction a nice characterization of the Brownian web can also be found in [FINR04]. A detailed proof that the convergence criteria given by Sun are fulfilled in our case is then given in the third section of this chapter.

## 2.1. Tail bounds on coalescing events

In the first section we focus on tail bounds for the hitting or meeting events of independent random walks on a "joint" oriented percolation cluster. A discussion on Proposition 2.1 can be found in Remark 2.3 and Remark 2.4 below. In order to prove the tail bounds we make use of the estimates on the transition probabilities given at (1.47) and estimates on return probabilities of supermartingales.

For $x_1, x_2 \in \mathbb{Z}^d$ define

$$T_{meet}^{(x_1,x_2)} := \inf\{n \geq 0 : X_n^{(x_1)} = X_n^{(x_2)}\}, \tag{2.1}$$

with $X^{(x_i)}$ as defined in (1.49) and the usual convention $\inf \emptyset = +\infty$. Note that both random walks start at time 0.

**Proposition 2.1.** *Let $x_1, x_2 \in \mathbb{Z}^d$, $x_1 \neq x_2$. We have*

$$\mathbb{P}_{joint}(T_{meet}^{(x_1,x_2)} < \infty \mid B_{x_1,x_2;0}) \begin{cases} = 1, & d \leq 2, \\ \in (0,1), & d \geq 3. \end{cases} \tag{2.2}$$

*In dimension $d = 1$ we have the following asymptotic behaviour*

$$\mathbb{P}_{joint}(T_{meet}^{(x_1,x_2)} > n \mid B_{x_1,x_2;0}) \asymp \frac{|x_1 - x_2|}{\sqrt{n}}, \tag{2.3}$$

*uniformly in $n$ and $x_1, x_2$ with $x_1 \neq x_2$ as $n$ tends to infinity. The definition of $B_{x_1,x_2;0}$ is given at (1.53).*

**Remark 2.2.** The asymptotic behaviour in (2.3) is needed to verify the convergence criteria given in section 2.3 below. Since we do not need the precise tail behaviour in the two-dimensional case, we do not further investigate this question here. We believe that this can be done analogously to the one-dimensional case, by showing that for

$$f(r) := C + \int_{r_0}^{r} C' \cdot \frac{r_0}{r} \cdot \exp\left(-ce^{-c'r}\right) dr, \quad \text{with } C, C', c, c', r_0 > 0 \text{ chosen properly,}$$

the stochastic process $f\left(\left\|\widehat{X}_n^{(x_1)} - \widehat{X}_n^{(x_2)}\right\|_2\right)$ is a supermartingale, up to the time that the random walks come closer than some fixed distance $K > 0$. If this holds true, the arguments given in section 2.1.4 below should be adaptable to the two-dimensional setting. Up to now we know that for every initial separation $x_0 = x_1 - x_2$ there exist constants $C, C' > 0$ and $M > 0$, such that

$$\frac{C}{\log(m)} \leq \mathbb{P}_{joint}\left(\inf\left\{k \geq 0 : \left\|\widehat{X}_n^{(x_1)} - \widehat{X}_n^{(x_2)}\right\|_2 \leq K\right\} \geq m \;\middle|\; B_{x_1,x_2;0}\right) \leq \frac{C'}{\log(m)} \tag{2.4}$$

for all $m > M$, where the dependency of $C$ and $C'$ on $x_0$ requires further investigation. The upper bound in (2.4) is a consequence of Remark 2.7 and [LPW09, Proposition 17.19], whereas the lower bound can be proved similar to the one-dimensional case.

### 2.1.1. Comments on Proposition 2.1

**Remark 2.3.** We are interested in collision events of two random walks $X^{(x_1)}, X^{(x_2)}$ moving on the space-time cluster $\mathcal{C}$. A collision event occurs, if the two walks are at the *same time* at the *same site*. Equation (2.2) tells us that a collision event between two random walks occurs almost surely in dimension d=1,2. This is not entirely obvious because the “holes” in the space-time cluster $\mathcal{C}$ might prevent collision events. Furthermore, in the one-dimensional case the tail bounds given in (2.3) of Proposition 2.1 coincide (at least up to a constant) with the tails bounds one would expect from ordinary random walks with i.i.d. increments, see e.g. Corollary 1.3 in [Uch11] or Theorem 8 in [Kes63]. Roughly speaking in the one-dimensional case the holes that occur in the space-time cluster have no substantial influence on the tail bounds for the probability of two random walks to meet after $n$ steps.

In order to prove (2.2), we have a look at the difference of two walks at their simultaneous regeneration times $(T_n^{sim})_n$, see (1.38). Recall the definition of $\widehat{X}_n^{(z)}$ in (1.40) and the definition of $\widehat{\Psi}_{joint}$ and $\widehat{\Psi}_{ind}$ in (1.45) and (1.46).

In the joint as well as in the independent case, the difference between the random walks $X^{(x_1)}$ and $X^{(x_2)}$ at their simultaneous regeneration times is a Markov chain with transition probabilities

$$\Psi_{joint}^{diff}(x, y) := \widehat{\Psi}_{joint}\left((x, 0), \bigcup_z \{(z + y, z)\}\right),$$

respectively

$$\Psi_{ind}^{diff}(x, y) := \widehat{\Psi}_{ind}\left((x, 0), \bigcup_z \{(z + y, z)\}\right),$$

see the remarks at the end of section 1.4.3. We want to point out that the transition probability function $\Psi^{diff}_{joint}$ is not (space-)homogeneous. This means that the transition probability for the distance between the random walks does not only depend on the increment but also on the initial distance between the two random walks itself. In the independent case the transition probability function of the difference is (space-)homogeneous, which means that the initial distance between two random walks is not important for calculating the probability of a certain increment. Hence we are allowed to define

$$\Psi^{diff}_{ind}(y-x) := \Psi^{diff}_{ind}(0, y-x) = \Psi^{diff}_{ind}(x,y).$$

Furthermore, the transition probability function in the independent case is symmetric. In both cases the transition probability functions have exponential tails by Lemma 1.12.
Notice that by the spatial homogeneity of $\Psi^{diff}_{ind}$, the difference between the two random walks is itself a $d$-dimensional random walk. By symmetry of the increments and the exponential tails, we know for example by the Chung-Fuchs-Theorem that the Markov chain $(\widehat{X}^{(x_1)}_k - \widehat{X}^{(x_2)}_k)_{k\geq 0}$ is recurrent under $\mathbb{P}_{ind}(\,\cdot\,|B^{ind}_{x_1,x_2;0})$ (in $d=1$). In the joint case we know at least that the Markov chain $(\widehat{X}^{(x_1)}_k - \widehat{X}^{(x_2)}_k)_{k\geq 0}$ is irreducible under $\mathbb{P}_{joint}(\,\cdot\,|B_{x_1,x_2;0})$.

**Remark 2.4.** Proposition 2.1 (see also Lemma 2.8 and Lemma 2.9 below) is in some sense a "trivial" instance of a so-called Lamperti problem: The difference $(X^{(x_1)}_{T_n} - X^{(x_2)}_{T_n})_n$ at simultaneous regeneration times is (under $\mathbb{P}_{joint}(\,\cdot\,|B_{x_1,x_2;0})$) a Markov chain that is a local perturbation of a symmetric random walk and the drift at $x$ vanishes exponentially fast in $\|x\|$, which is a consequence of Lemma 1.14. A very fine analysis in the nearest-neighbour case can be found in [Ale11], see also the references there for background.
Denis Denisov, Dima Korshunov and Vitali Wachtel (in contemporaneous work, see [DKW16]) have established a generalisation of Alexander's results to the non-nearest neighbour case which in particular refines the case $d=1$ in (2.3) to asymptotic equivalence. For the sake of completeness we present a short, rough proof of the coarser estimates that suffice for our purposes.

### 2.1.2. Consequences for the (neutral) multi-type contact process

In this subsection we discuss briefly consequences of Proposition 2.1 for the (neutral) multi-type contact process introduced in section 1.5.

**Proposition 2.5.**
*Let $\mu$ denote the Bernoulli product measure on $\{1,2\}^{\mathbb{Z}^d}$ with $\mu(\eta(x)=1) = 1-\mu(\eta(x)=2) = \alpha \in (0,1)$ for all $x \in \mathbb{Z}^d$. We write $\eta^{\mu,m} := (\eta^{\mu,m}_n)_{n\leq m}$ for the discrete time two-type contact process with initial configuration $\eta^{\mu,m}_m$ distributed according to $\mu$. Furthermore, we denote by $\bar{\nu}^{\mathbf{1}}$ and $\bar{\nu}^{\mathbf{2}}$ the weak limits*

$$\bar{\nu}^{\mathbf{1}} \overset{w}{=} \lim_{m\to\infty} \mathcal{L}(\eta^{\mathbf{1},m}_0),$$
$$\bar{\nu}^{\mathbf{2}} \overset{w}{=} \lim_{m\to\infty} \mathcal{L}(\eta^{\mathbf{2},m}_0),$$

*where $\eta^{\mathbf{1},m}$ and $\eta^{\mathbf{2},m}$ denote discrete time two-type contact processes starting at time-layer $m$ and evolving backwards in time, with initial configuration given by $\eta^{\mathbf{1},m}_m \equiv 1$ and $\eta^{\mathbf{2},m}_m \equiv 2$. In dimension*

*$d \le 2$ we have*

$$\mathcal{L}(\eta_0^{\mu,m}) \xrightarrow{w} \alpha\bar{\nu}^{\mathbf{1}} + (1-\alpha)\bar{\nu}^{\mathbf{2}}, \quad \textit{as } m \to \infty. \tag{2.5}$$

*If $d > 2$, then the weak limit*

$$\lim_{m\to\infty} \mathcal{L}(\eta_0^{\mu,m}) =: \nu^{\mu} \tag{2.6}$$

*exists and*

$$\nu^{\mu}\left(\zeta \in \{0,1,2\}^{\mathbb{Z}^d} : \zeta|_{A_1} \equiv 1, \zeta|_{A_1} \equiv 2\right) > 0$$

*for every finite $A_1, A_2 \subset \mathbb{Z}^d$, with $A_1 \cap A_2 = \emptyset$.*

**Proof:** In order to prove existence of the limits in (2.5) and (2.6), we need to show that for $n \in \mathbb{N}$, $x_1, ..., x_n \in \mathbb{Z}^d$ and $i_1, ..., i_n \in \{0,1,2\}$ chosen arbitrarily, the limit of

$$\mathbb{P}\left(\eta_0^{\mu,m}(x_1) = i_1, ..., \eta_0^{\mu,m}(x_n) = i_n\right) \tag{2.7}$$

exists as $m$ tends to infinity. We fix $n \in \mathbb{N}$, $x_1, ..., x_n \in \mathbb{Z}^d$ and $i_1, ..., i_n \in \{0,1,2\}$ and define $A_0 := \{x_k : i_k = 0\}$, $A_1 := \{x_k : i_k = 1\}$ and $A_2 := \{x_k : i_k = 2\}$. We can assume that $A_0$, $A_1$ and $A_2$ are disjoint, since the limit in (2.7) would be zero if $x_k = x_{k'}$ and $i_k \neq i_{k'}$ for some $k, k' \in \{1, ..., n\}$. Using the inclusion-exclusion formula we get that

$$\begin{aligned}
&\mathbb{P}\left(\eta_0^{\mu,m}|_{A_0} \equiv 0,\ \eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2\right) \\
&= \sum_{B \subset A_0} (-1)^{|B|} \sum_{B_1, B_2 \colon B_1 \dot\cup B_2 = B} \mathbb{P}\left(\eta_0^{\mu,m}|_{A_1 \cup B_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2 \cup B_1} \equiv 2\right).
\end{aligned}$$

Notice that we sum up over disjoint subsets $B_1, B_2 \subset B$. Hence it is enough to focus on the limit of

$$\mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2\right), \tag{2.8}$$

as $m$ tends to infinity. Define $A := A_1 \dot\cup A_2$ and note that

$$\begin{aligned}
&\mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2\right) \\
&= \mathbb{P}\left(\eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) \cdot \mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \;\middle|\; \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right)
\end{aligned} \tag{2.9}$$

We focus on the factors in (2.9) separately. The limit of the first factor can be characterized easily. Since $\mu$ is a measure on $\{1,2\}^{\mathbb{Z}^d}$ we know

$$\lim_{m\to\infty} \mathbb{P}\left(\eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) = \bar{\nu}\left(\zeta \in \{0,1\}^{\mathbb{Z}^d} : \zeta|_A \equiv 1\right),$$

where $\bar{\nu}$ is the upper invariant measure of the (single-type) contact process defined in (1.14). Hence we focus on the second factor. Recall the definition of $\xi$ in (1.24). First of all note that

$$\{\xi_0(x) = 1 \text{ for all } x \in A\} \subset \{\eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\},$$

and

$$\mathbb{P}\left(\eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) - \mathbb{P}\left(\xi_0(x) = 1 \text{ for all } x \in A\right) \le C|A|e^{-cm} \tag{2.10}$$

by (1.17). Since on the other hand the FKG-inequality yields

$$\mathbb{P}\left(\eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) \geq \mathbb{P}\left(\xi_0(x) = 1 \text{ for all } x \in A\right) \geq \left(\mathbb{P}(B_{(\mathbf{0},0)})\right)^{|A|},$$

we get that

$$\begin{aligned} &d_{TV}\Big(\mathbb{P}\left(\,\cdot\,\middle|\eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right), \mathbb{P}\left(\,\cdot\,\middle|\xi_0(x) = 1 \text{ for all } x \in A\right)\Big) \\ &= \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big), \quad \text{as } m \text{ tends to infinity,} \end{aligned}$$

by Lemma 3.2 proven below.
Therefore

$$\begin{aligned} &\mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \;\middle|\; \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) \\ &= \mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \;\middle|\; \xi_0(x) = 1 \text{ for all } x \in A\right) + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big) \\ &= \int \mathbb{P}\Big(\eta_0^{\zeta,m}|_{A_1} \equiv 1,\ \eta_0^{\zeta,m}|_{A_2} \equiv 2 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A\Big)\,\mu(d\zeta) + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big) \\ &= \int \mathbb{P}\Big(\zeta(X_m^{(x)}) = i \text{ for all } x \in A_i\,;\ i = 1,2 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A\Big)\,\mu(d\zeta) \qquad (2.11) \\ &\quad + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big). \qquad (2.12) \end{aligned}$$

By (2.2) of Proposition 2.1 this implies

$$\lim_{m\to\infty} \mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \;\middle|\; \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) = 0 \qquad (2.13)$$

if $A_1 \neq \emptyset$, $A_2 \neq \emptyset$ and $d \leq 2$. On the other hand if $d \leq 2$ and we assume without loss of generality that $A_2 = \emptyset$, we get

$$\begin{aligned} &\mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \;\middle|\; \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) \\ &= \mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1 \;\middle|\; \xi_0(x) = 1 \text{ for all } x \in A_1\right) + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A_1|}|A_1|e^{-cm}\big) \\ &= \int \mathbb{P}\Big(\zeta(X_m^{(x)}) = 1 \text{ for all } x \in A_1 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A_1\Big)\,\mu(d\zeta) + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A_1|}|A_1|e^{-cm}\big) \\ &\leq \mathbb{P}\Big(|\{X_m^{(z)}\ :\ z \in A_1\}| > 1 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A_1\Big) \\ &\quad + \sum_{y\in\mathbb{Z}^d} \mathbb{P}\Big(\{X_m^{(z)}\ :\ z \in A_1\} = \{y\} \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A_1\Big)\,\mu(\zeta(y) = 1) \\ &\quad + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A_1|}|A_1|e^{-cm}\big) \\ &\leq \alpha + \mathbb{P}\Big(|\{X_m^{(z)}\ :\ z \in A_1\}| > 1 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A_1\Big) \\ &\quad + \Big(\mathbb{P}\Big(|\{X_m^{(z)}\ :\ z \in A_1\}| = 1 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A_1\Big) - 1\Big)\,\alpha + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A_1|}|A_1|e^{-cm}\big) \end{aligned}$$

and hence

$$\begin{aligned} &\left|\mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1 \;\middle|\; \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A_1\right) - \alpha\right| \\ &\leq 2\mathbb{P}\Big(\Big|\{X_m^{(z)}\ :\ z \in A_1\}\Big| > 1 \;\Big|\; \xi_0(x) = 1 \text{ for all } x \in A_1\Big) + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A_1|}|A_1|e^{-cm}\big). \end{aligned}$$

The weak convergence in (2.5) of Proposition 2.5 then follows by (2.2) of Proposition 2.1. If $d \geq 3$ equation (2.11) yields

$$\begin{aligned}
&\mathbb{P}\left(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \,\middle|\, \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\right) \\
&= \int \mathbb{P}\left(\zeta(X_m^{(x)}) = i \text{ for all } x \in A_i\,;\ i = 1,2 \,\middle|\, \xi_0(x) = 1 \text{ for all } x \in A\right) \mu(d\zeta) \\
&\quad + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big) \\
&= \sum_{\substack{B_1 \subset \mathbb{Z}^d \\ |B_1| \leq |A_1|}} \sum_{\substack{B_2 \subset \mathbb{Z}^d \\ |B_2| \leq |A_2|}} \mathbb{P}\left(\{X_m^{(x)} : x \in A_1\} = B_1, \{X_m^{(x)} : x \in A_2\} = B_2 \,\middle|\, \xi_0(x) = 1 \text{ for all } x \in A\right) \\
&\qquad\qquad \cdot \mu(\zeta \in \{1,2\}^{\mathbb{Z}^d} : \zeta|_{B_1} \equiv 1, \zeta|_{B_2} \equiv 2) + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big) \\
&= \sum_{\substack{B_1 \subset \mathbb{Z}^d \\ |B_1| \leq |A_1|}} \sum_{\substack{B_2 \subset (\mathbb{Z}^d \setminus B_1) \\ |B_2| \leq |A_2|}} \mathbb{P}\left(\{X_m^{(x)} : x \in A_1\} = B_1, \{X_m^{(x)} : x \in A_2\} = B_2 \,\middle|\, \xi_0(x) = 1 \text{ for all } x \in A\right) \\
&\qquad\qquad \cdot \alpha^{|B_1|}(1-\alpha)^{|B_2|} + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big) \\
&= \mathbb{E}\left(\alpha^{|\{X_m^{(x)}:x \in A_1\}|} \cdot (1-\alpha)^{|\{X_m^{(x)}:x \in A_2\}|} \cdot \mathbb{1}_{\{\{X_m^{(x)}:x\in A_1\} \cap \{X_m^{(x)}:x \in A_2\} = \emptyset\}} \,\middle|\, \xi_0(x) = 1 \text{ for all } x \in A\right) \\
&\quad + \mathcal{O}\big([\mathbb{P}(B_{(\mathbf{0},0)})]^{-|A|}|A|e^{-cm}\big).
\end{aligned}$$

Since $\left|\left\{X_m^{(x)} \ : \ x \in A_i\right\}\right|$ on $\{\xi_0(x) = 1 \text{ for all } x \in A\}$ is non-increasing in $m$ and

$$\mathbb{P}\big(\{X_m^{(x)} : x \in A_1\} \cap \{X_m^{(x)} : x \in A_2\} = \emptyset \text{ for all } m > 0 \,\big|\, \xi_0(x) = 1 \text{ for all } x \in A\big) > 0$$

by Proposition 2.1, the last term converges by the monotone convergence theorem as $m$ tends to infinity and

$$\lim_{m\to\infty} \mathbb{P}\big(\eta_0^{\mu,m}|_{A_1} \equiv 1,\ \eta_0^{\mu,m}|_{A_2} \equiv 2 \,\big|\, \eta_0^{\mu,m}(x) > 0 \text{ for all } x \in A\big) > 0.$$

□

### 2.1.3. Proof of Proposition 2.1, equation (2.2)

**Proof of Proposition 2.1, equation (2.2) for $d \leq 2$**

In order to prove (2.2), we focus on the difference between the random walks $\widehat{X}^{(x_1)}$ and $\widehat{X}^{(x_2)}$. Notice that by translation invariance of $\mathbb{P}_{joint}$ and $\mathbb{P}_{ind}$ we have

$$\mathcal{L}(X^{(x)} - X^{(\mathbf{0})}|\mathbb{P}_{joint}(\,\cdot\,|B_{x,\mathbf{0};0})) = \mathcal{L}(X^{(x+y)} - X^{(y)}|\mathbb{P}_{joint}(\,\cdot\,|B_{x+y,y;0})),$$

$$\mathcal{L}(X^{(x)} - X^{(\mathbf{0})}|\mathbb{P}_{ind}(\,\cdot\,|B^{ind}_{x,\mathbf{0};0})) = \mathcal{L}(X^{(x+y)} - X^{(y)}|\mathbb{P}_{ind}(\,\cdot\,|B^{ind}_{x+y,y;0}))$$

for every $y \in \mathbb{Z}^d$. The same equalities hold true for $X$ replaced by $\widehat{X}$. This means that only the initial distance and not the exact configuration is important. Hence we define $\widehat{D}_n^{(x_1-x_2)} = \widehat{X}_n^{(x_1)} - \widehat{X}_n^{(x_2)}$ and $D_n^{(x_1-x_2)} = X_n^{(x_1)} - X_n^{(x_2)}$. Remember that $(\widehat{D}_n^{(x_1-x_2)})_n$ is a Markov chain with transition probability function $\Psi^{diff}_{joint}$ or $\Psi^{diff}_{ind}$ depending on whether the two random walks are defined on the same or on independent copies of the oriented percolation cluster. Since only the difference between the random walks will be considered, we omit the subscript $(x_1 - x_2)$ and indicate the initial distance as well as the fact that we condition on $(\mathbf{0}, 0)$ and $(x_1 - x_2, 0)$ to be contained in the backbone of the oriented percolation cluster by the family $\{\mathbb{P}^x_{joint}\}_{x\in\mathbb{Z}^d}$ (resp. $\{\mathbb{P}^x_{ind}\}_{x\in\mathbb{Z}^d}$) of probability measures, where $x$ takes the role of $x_1 - x_2$. This is done just to simplify notation within the proof of Proposition 2.1.

In dimension $d = 1, 2$, we can prove that there exists a function $s$ on $\mathbb{Z}^d$ such that $s(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$, which is superharmonic for $\Psi^{diff}_{joint}$ outside a finite subset of $\mathbb{Z}^d$. According to Proposition 5.3 in [Asm03], the existence of such a function implies recurrence for the Markov chain $(\widehat{D}_k)_{k\geq 0}$ in the joint case.

The key ingredient for the proof is the estimation on the total variation distance between $\Psi^{diff}_{joint}$ and $\Psi^{diff}_{ind}$ given in Lemma 1.14. By Lemma 1.14 we know that there exists $C, c > 0$ such that

$$|\Psi^{diff}_{joint}(x,y) - \Psi^{diff}_{ind}(x,y)| \leq 2\left\|\widehat{\Psi}_{joint}((x,0),\cdot) - \widehat{\Psi}_{ind}((x,0),\cdot)\right\|_{TV} \leq Ce^{-c\|x\|}, \tag{2.14}$$

for any $x, y \in \mathbb{Z}^d$. This means that the error term between $\Psi^{diff}_{joint}(x,y)$ and $\Psi^{diff}_{ind}(x,y)$ decays exponentially in the initial distance between the two random walks.

With the help of (2.14), we are able to prove that in dimension $d = 1$

$$\begin{aligned}\sum_y \Psi^{diff}_{joint}(x,y)|y|^\alpha &\leq |x|^\alpha - C|x|^{\alpha-2} + Ce^{-c\sqrt{|x|}} \\ &\leq |x|^\alpha \quad \text{for all } |x| \geq K_1. \end{aligned}\tag{2.15}$$

In dimension $d = 2$ let the covariance matrix of $\widehat{D}_1$ in the independent case be given by

$$\mathrm{Cov}_{ind}\left(\widehat{D}_1\right) = \begin{pmatrix} \bar{\sigma}^2 & \bar{\rho} \\ \bar{\rho} & \bar{\sigma}^2 \end{pmatrix},$$

where $|\bar{\rho}| < \bar{\sigma}^2$, since Birkner et al. proved in [BČDG13] that the limit law is not concentrated on a subspace. We can show that

$$\sum_y \Psi^{diff}_{joint}(x,y)\log^\alpha(\|Ay\|) \leq \log^\alpha(\|Ax\|) \quad \text{for all } \|x\| \geq K_2, \tag{2.16}$$

where $K_2$ is a large constant, $\alpha \in (0,1)$ and

$$A := \begin{pmatrix} \frac{1}{\sqrt{2(\bar{\sigma}^2+\bar{\rho})}} & \frac{1}{\sqrt{2(\bar{\sigma}^2+\bar{\rho})}} \\ \frac{1}{\sqrt{2(\bar{\sigma}^2-\bar{\rho})}} & -\frac{1}{\sqrt{2(\bar{\sigma}^2-\bar{\rho})}} \end{pmatrix}. \tag{2.17}$$

A detailed proof can be found in the appendix in section A.1. Notice that, inspired by the simple random walk case, $s(x) := |x|^\alpha$ is a natural candidate for a superharmonic function in the one-dimensional setting and $s(x) = \log^\alpha(\|x\|)$ is a natural candidate for a superharmonic function in the two-dimensional setting.

**Remark 2.6.** Let $d = 1$. If

$$\sum_{y\in\mathbb{Z}^d} \Psi^{diff}_{ind}(y-x)|y|^\alpha \leq |x|^\alpha \qquad \text{for all } x \geq K_1, \tag{2.18}$$

this means that $\left(\left|\widehat{D}_{n\wedge h(K_1)}\right|^\alpha\right)_n$ is a supermartingale with respect to the filtration

$$\mathbb{F}^1 := (\mathcal{F}^1_n)_n := (\sigma(\widehat{D}_m, m \leq n))_n, \tag{2.19}$$

where $h(K_1)$ is the first time the process $(\widehat{D}_n)_n$ enters the interval $[-K_1, K_1]$.

**Remark 2.7.** Let $d = 2$. If

$$\sum_y \Psi^{diff}_{joint}(x,y)\log^\alpha(\|Ay\|) \leq \log^\alpha(\|Ax\|) \quad \text{for all } \|x\| \geq K_2, \tag{2.20}$$

this means that $\left(\log^\alpha\left(\left\|A\widehat{D}_{n\wedge h(K_2)}\right\|_2 \vee 1\right)\right)_n$ is a supermartingale with respect to the filtration

$$\mathbb{F}^2 := (\mathcal{F}^2_n)_n := (\sigma(\widehat{D}_m, m \leq n))_n, \tag{2.21}$$

where $h(K_2)$ is the first time the process $(\widehat{D}_n)_n$ enters the ball of radius $K_2$ around zero.
Let $C_1, C_2$ be constants such that

$$C_1\|x\|_2 \leq \|Ax\| \leq C_2\|x\|_2 \tag{2.22}$$

and assume $K_2$ to be large enough such that $\frac{1}{C_1} \leq K_2$. Notice that above we truncated $\left\|A\widehat{D}_n\right\|_2$ from below to avoid difficulties. But since $\left\|A\widehat{D}_n\right\|_2 < 1$ implies $\left\|\widehat{D}_n\right\|_2 \leq \frac{1}{C_1} \leq K_2$, this is no problem because the process is stopped if $\left\|\widehat{D}_n\right\| \leq K_2$.

**Proof of Proposition 2.1, equation (2.2) for $d > 2$**

The proof of (2.2) for $d > 3$ is not very difficult. By Theorem 2(a) in [GH02], we already know that $\mathbb{P}_{joint}(T^{(x_1,x_2)}_{meet} < \infty \mid B_{x_1,x_2;0}) > 0$. Therefore it is left to prove that

$$\mathbb{P}_{joint}(T^{(x_1,x_2)}_{meet} = \infty \mid B_{x_1,x_2;0}) = \lim_{n\longrightarrow\infty} \mathbb{P}_{joint}(T^{(x_1,x_2)}_{meet} > n \mid B_{x_1,x_2;0}) > \varepsilon > 0.$$

This can be done by using similar arguments as in the proof of Lemma 2.9 below. Since Lemma 2.9 is more important for later use, we skip the proof at this place.

### 2.1.4. Proof of Proposition 2.1, equation (2.3)

In this section we focus our attention on "how fast" a collision event occurs. We will answer this question by finding upper and lower bounds for the (annealed) probability that a coalescing event happens after time $n$.
The structure of this section is as follows. At first we compute tails for the hitting times of a large but finite interval around zero. Since two random walks on the cluster are comparable to independent

random walks if the distance between them is large, this is much easier than the computation for the tails of hitting zero itself. At the end of the section we will prove that the hitting time of a large but finite neighbourhood of zero has the same tail behaviour as the hitting time of zero itself. As in [BČDG13, Lemma 3.6] we introduce for $r > 0$ the following stopping times

$$h(r) := \inf\{k \in \mathbb{N}_0 : |\widehat{D}_k| \le r\}, \tag{2.23}$$

$$H(r) := \inf\{k \in \mathbb{N}_0 : |\widehat{D}_k| \ge r\}, \tag{2.24}$$

additionally, we define

$$\widehat{T}_{meet} := \inf\{k \in \mathbb{N}_0 : |\widehat{D}_k| = 0\},$$

$$T_{meet} := \inf\{k \in \mathbb{N}_0 : |D_k| = 0\}.$$

Notice that $\widehat{T}_{meet}$ and $T_{meet}$ do not need to coincide, since $\widehat{T}_{meet}$ is the first time that the random walks meet at simultaneous regeneration times. Making use of the fact that the simultaneous regeneration times have exponential tail bounds (see Lemma 1.12), equation (2.3) of Proposition 2.1 holds true if it holds true for $T_{meet}$ replaced by $\widehat{T}_{meet}$ (see section A.1.4 in the appendix). Therefore it is enough to focus on calculating tail bounds for $\widehat{T}_{meet}$.

Before proving equation (2.3) of Proposition 2.1 we prove the following two lemmas.

**Lemma 2.8.** *Let $d = 1$. Consider two random walks defined on a joint oriented percolation cluster and let $\widehat{D}_n$ be the difference at their simultaneous regeneration times. There exist constants $C_3, K, M > 0$ such that for all $x_0 > K$*

$$\mathbb{P}^{x_0}_{joint}(H(\sqrt{m}) < h(K)) \le \frac{C_3 x_0}{\sqrt{m}} \quad \textit{for all} \quad m > M \quad \textit{and} \tag{2.25}$$

$$\mathbb{P}^{x_0}_{joint}(h(K) \ge m) \le \frac{C_3 x_0}{\sqrt{m}} \quad \textit{for all} \quad m > M. \tag{2.26}$$

**Lemma 2.9.** *Let $d = 1$. Consider two random walks defined on a joint oriented percolation cluster and let $\widehat{D}_n$ be the difference at their simultaneous regeneration times. There exist constants $C_4, K > 0$ such that for all $x_0 > K$ there exists $M := M(x_0) > 0$ such that*

$$\mathbb{P}^{x_0}_{joint}(H(\sqrt{m}) < h(K)) \ge \frac{C_4 x_0}{\sqrt{m}} \quad \textit{for all} \quad m > M \quad \textit{and} \tag{2.27}$$

$$\mathbb{P}^{x_0}_{joint}(h(K) \ge m) \ge \frac{C_4 x_0}{\sqrt{m}} \quad \textit{for all} \quad m > M. \tag{2.28}$$

**Proof of Lemma 2.8:** The proof will be divided into five steps.

Step 1: We fix some constants $K, L > 0$. Furthermore, for $j \in \mathbb{N}$ chosen arbitrarily we define

$$\tau_{jK,(j+1)K+L} := \inf\{n \ge 0 : \widehat{D}_n \ge (j+1)K + L \text{ or } \widehat{D}_n \le jK\}, \tag{2.29}$$

which is a stopping time with respect to the filtration defined in (2.19). Let $b_j := (j+1)K + L$ and $a_j := jK$ in order to shorten notation.

In the first step we will prove that there exist constants $C_5, M > 0$ such that for all $j \in \mathbb{N}$ and all $x_0 \in (jK, (j+1)K]$

$$\mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \geq b_j) \leq \frac{C_5}{L} \quad \text{for all } L > M, \tag{2.30}$$

if $K$ is chosen large enough and $L > M \gg K$.

First of all note that by (2.14) and Lemma 1.12 there exists a constant $c_1 > 0$ such that for all $x > K$

$$\begin{aligned} |\Psi_{ind}^{diff}(x,y) - \Psi_{ind}^{diff}(x,y)| &\leq e^{-c_1 \cdot x} \quad \text{for all } y \in \mathbb{Z}, \\ \mathbb{P}_{joint}^{x}\left(T_1^{sim} > \frac{y}{2}\right) + \mathbb{P}_{joint}^{x}\left(T_1^{sim} > \frac{y}{2}\right) &\leq e^{-c_1 \cdot y} \quad \text{for all } y > x, \end{aligned}$$

if $K$ is chosen large enough. Furthermore, we assume $K$ to be chosen so large that

$$\log(x) < \frac{c_1}{2} x \quad \text{for all } x > \tfrac{K}{2}.$$

We will show that

$$f(y) := \int_0^{|y|} \exp\left(e^{-\frac{c_1 \cdot s}{9}} - 1\right) ds$$

is a superharmonic function on $(K, \infty)$ with respect to $\psi_{joint}^{diff}$. Although this can be done similarly to the proof of (2.14) (see section A.1.1 in the appendix), we will give the main points at this place. Notice that

$$e^{-1} \cdot |y| \leq f(y) = \int_0^{|y|} \exp\left(e^{-\frac{c_1 \cdot s}{9}} - 1\right) ds \leq |y|$$

for all $y \in \mathbb{R}$. Hence

$$\begin{aligned} &\sum_y |\Psi_{ind}^{diff}(x,y) - \Psi_{joint}^{diff}(x,y)| f(y) \\ &\leq \sum_{y:|y-x| \leq \frac{x}{2}} |\Psi_{ind}^{diff}(x,y) - \Psi_{joint}^{diff}(x,y)||y| + \sum_{y:|y-x| > \frac{x}{2}} |\Psi_{ind}^{diff}(x,y) - \Psi_{joint}^{diff}(x,y)||y| \\ &\leq \frac{3}{2} x^2 e^{-c_1 x} + \sum_{y:|y-x| > \frac{x}{2}} (\mathbb{P}^{joint}(T_1^{sim} \geq \tfrac{|x-y|}{2}) + \mathbb{P}^{ind}(T_1^{sim} \geq \tfrac{|x-y|}{2}))|y| \\ &\leq \frac{3}{2} x^2 e^{-c_1 x} + 2 \sum_{y > \frac{x}{2}} e^{-c_1 y} y \leq \frac{3}{2} x^2 e^{-c_1 x} + \frac{4}{c_1} e^{\frac{-c_1 x}{4}}, \end{aligned}$$

where for the last inequality we made use of the fact that $\log(y) < \frac{c_1}{2} y$. Similarly we get that

$$\sum_{y:|y-x| > \frac{x}{2}} \Psi_{ind}^{diff}(x,y) f(y) \leq \sum_{y:|y-x| > \frac{x}{2}} \mathbb{P}^{ind}(T_1^{sim} \geq \tfrac{|x-y|}{2})|y| \leq \frac{4}{c_1} e^{\frac{-c_1 x}{4}}.$$

Furthermore,

$$\begin{aligned}\sum_{y:|y-x|\leq\frac{x}{2}}\Psi^{diff}_{ind}(x,y)f(y)&\leq\sum_{y:|y-x|\leq\frac{x}{2}}\Psi^{diff}_{ind}(x,y)(f(y)-f(x))+f(x)\\&\leq\sum_{y=1}^{\frac{x}{2}}\Psi^{diff}_{ind}(y)(f(x+y)-2f(x)+f(x-y))+f(x)\\&\leq\frac{x^2}{8}\cdot\sup_{s\in[-\frac{x}{2},\frac{3x}{2}]}\exp\left(e^{-\frac{c_1\cdot s}{9}}-1\right)\cdot\left(-\frac{c_1}{9}\right)e^{-\frac{c_1 s}{9}}+f(x)\\&\leq-\frac{c_1x^2}{72e}e^{-\frac{c_1x}{6}}+f(x),\end{aligned}$$

where in the second line one needs to replace $f(x+y)$ and $f(x-y)$ by their Taylor expansion to see that the third inequality holds true. Altogether we get

$$\begin{aligned}\sum_y\Psi^{diff}_{joint}(x,y)f(y)&\leq\sum_y|\Psi^{diff}_{joint}(x,y)-\Psi^{diff}_{ind}(x,y)|f(y)+\sum_y\Psi^{diff}_{ind}(x,y)f(y)\\&\leq f(x)-\frac{c_1x^2}{72e}e^{-\frac{c_1x}{6}}+\frac{3}{2}x^2e^{-c_1x}+\frac{8}{c_1}e^{\frac{-c_1x}{4}}\\&\leq f(x)\quad\text{for all }x\geq K,\end{aligned}$$

if $K$ is chosen large enough. Since by previous calculations $f\left(\widehat{D}_{n\wedge h(K)}\right)$ is a non-negative supermartingale (for the definition of $h(K)$ see (2.23)), we conclude

$$\begin{aligned}f(x_0)=\mathbb{E}^{x_0}_{joint}\left[f\left(\widehat{D}_0\right)\right]&\geq\mathbb{E}^{x_0}_{joint}\left[f\left(\widehat{D}_{\tau_{a_j,b_j}}\right)\right]\\&\geq f(b_j)\cdot\mathbb{P}^{x_0}_{joint}\left(\widehat{D}_{\tau_{a_j,b_j}}\geq b_j\right)+f(a_j)\cdot\mathbb{P}^{x_0}_{joint}(\widehat{D}_{\tau_{a_j,b_j}}\leq a_j)\\&\quad-\mathbb{E}^{x_0}_{joint}\left[f(a_j)-f\left(\widehat{D}_{\tau_{a_j,b_j}}\right)\mid\widehat{D}_{\tau_{a_j,b_j}}\leq a_j\right]\cdot\mathbb{P}^{x_0}_{joint}(\widehat{D}_{\tau_{a_j,b_j}}\leq a_j)\\&\geq f(b_j)\cdot\mathbb{P}^{x_0}_{joint}\left(\widehat{D}_{\tau_{a_j,b_j}}\geq b_j\right)+f(a_j)\cdot\mathbb{P}^{x_0}_{joint}(\widehat{D}_{\tau_{a_j,b_j}}\geq b_j)\\&\quad-\mathbb{E}^{x_0}_{joint}\left[|\widehat{D}_{\tau_{a_j,b_j}}-a_j|\mid\widehat{D}_{\tau_{a_j,b_j}}\leq a_j\right]\\&\geq f(b_j)\cdot\mathbb{P}^{x_0}_{joint}\left(\widehat{D}_{\tau_{a_j,b_j}}\geq b_j\right)+f(a_j)\cdot\mathbb{P}^{x_0}_{joint}(\widehat{D}_{\tau_{a_j,b_j}}\leq a_j)-C,\end{aligned}$$

where $C>0$ can be chosen independently of $j$. This yields

$$\mathbb{P}^{x_0}_{joint}\left(\widehat{D}_{\tau_{a_j,b_j}}\geq b_j\right)\leq\frac{f(x_0)-f(a_j)+C}{f(b_j)-f(a_j)}\leq\frac{x_0-a_j+C}{e^{-1}(b_j-a_j)}\leq\frac{K+C}{e^{-1}(K+L)},$$

if $x_0\in(jK,(j+1)K]$ which implies (2.30). Additionally, we define

$$\tilde{\tau}_{K,\sqrt{m}}:=\inf\{n\geq0:|\widehat{D}_n|\geq\sqrt{m}\text{ or }|\widehat{D}_n|\leq K\}.\tag{2.31}$$

The fact that $f\left(\widehat{D}_{n\wedge h(K)}\right)$ is a non-negative supermartingale also implies that

$$x_0\geq\mathbb{E}^{x_0}_{joint}\left[f\left(\widehat{D}_0\right)\right]\geq\mathbb{E}^{x_0}_{joint}\left[f\left(\widehat{D}_{\tilde{\tau}_{K,\sqrt{m}}}\right)\right]\geq f(\sqrt{m})\cdot\mathbb{P}^{x_0}_{joint}\left(|\widehat{D}_{\tilde{\tau}_{K,\sqrt{m}}}|\geq\sqrt{m}\right)$$

and (2.25) follows.

Step 2: In the second step we will prove that if $K$ is chosen large enough there exist constants $C_6, M > 0$ such that for all $j \in \mathbb{N}$

$$\sup_{x \in (jK,(j+1)K]} \mathbb{P}^x_{joint}(\tau_{jK} \geq m) \leq \frac{C_6}{\sqrt{m}} \quad \text{for all} \quad m > M, \tag{2.32}$$

where

$$\tau_{jK} := \inf\{n \geq 0 : \widehat{D}_n \leq jK\}. \tag{2.33}$$

We fix some $x_0 \in (jK,(j+1)K] \cap \mathbb{Z}$ and define

$$M_n := \left(\widehat{D}_n - x_0\right)^2 - \sum_{k=0}^{k-1} d(\widehat{D}_k), \tag{2.34}$$

where

$$d_{x_0}(x) := \mathbb{E}^x_{joint}\left[\left(\widehat{D}_1 - x_0\right)^2 - (x - x_0)^2\right] \tag{2.35}$$

is the expected increment of $\left(\left(\widehat{D}_n - x_0\right)^2\right)_n$ after one step. Notice that by (2.14) we have

$$\begin{aligned}
d_{x_0}(x) &= \mathbb{E}^x_{joint}\left[\left(\widehat{D}_1 - x_0\right)^2 - (x - x_0)^2\right] \\
&= \mathbb{E}^x_{joint}\left[\left(\widehat{D}_1 - x\right)^2 + 2\left(\widehat{D}_1 - x\right)(x - x_0)\right] \\
&= \sum_y \psi^{diff}_{ind}(y) \cdot (y)^2 + \mathcal{O}(e^{-cx}) + 2(x - x_0)\sum_y \psi^{diff}_{ind}(y) \cdot (y) + (x - x_0)\mathcal{O}(e^{-cx}) \\
&= \tilde{\sigma}^2 + \mathcal{O}(e^{-cx}) + (x - x_0)\mathcal{O}(e^{-cx}) > 0,
\end{aligned}$$

where $\sum_y \psi^{diff}_{ind}(y) \cdot (y)^2 = \mathrm{Var}_{ind}(\widehat{D}_1) =: \tilde{\sigma}^2$. Since $|(x - x_0)| \leq K$ and $x \in (jK,(j+1)K]$ we can choose $K$ so large that for all $j \in \mathbb{N}$

$$|d_{x_0}(x) - \tilde{\sigma}^2| > \frac{\tilde{\sigma}^2}{2}$$

for all $x, x_0 \in (jK,(j+1)K] \cap \mathbb{Z}$. Since

$$\{\tau_{a_j} \geq m\} \subset \{\tau_{a_j,b_j} \geq m\} \cup \{\widehat{D}_{\tau_{a_j,b_j}} \geq b_j\}, \tag{2.36}$$

we can prove (2.32) by finding suitable bounds on the probability of the events $\{\tau_{a_j,b_j} \geq m\}$ and $\{\widehat{D}_{\tau_{a_j,b_j}} \geq b_j\}$. Note that $\mathbb{E}^x_{joint}[\tau_{a_j,b_j}] < \infty$. Furthermore,

$$\begin{aligned}
&\mathbb{E}^{x_0}_{joint}\left[\left(\widehat{D}_{\tau_{a_j,b_j}} - x_0\right)^2 \mid \widehat{D}_{\tau_{a_j,b_j}} \geq b_j\right] \\
&= (b_j - x_0)^2 + \mathbb{E}^{x_0}_{joint}\left[\left(\widehat{D}_{\tau_{a_j,b_j}} - x_0\right)^2 - (b_j - x_0)^2 \mid \widehat{D}_{\tau_{a_j,b_j}} \geq b_j\right] \\
&= (b_j - x_0)^2 + \mathbb{E}^{x_0}_{joint}\Big[\left(\widehat{D}_{\tau_{a_j,b_j}} - b_j\right)^2 + 2\underbrace{\left(\widehat{D}_{\tau_{a_j,b_j}} - b_j\right)(b_j - x_0)}_{>0 \text{ on } \{\widehat{D}_{\tau_{a_j,b_j}} \geq b_j\}} \mid \widehat{D}_{\tau_{a_j,b_j}} \geq b_j\Big] < \infty,
\end{aligned} \tag{2.37}$$

and

$$\mathbb{E}^{x_0}_{joint}\left[\left(\widehat{D}_{\tau_{a_j,b_j}} - x_0\right)^2 \mid \widehat{D}_{\tau_{a_j,b_j}} \le a_j\right]$$
$$= (a_j - x_0)^2 + \mathbb{E}^{x_0}_{joint}\Big[\left(a_j - \widehat{D}_{\tau_{a_j,b_j}}\right)^2 + 2\underbrace{\left(a_j - \widehat{D}_{\tau_{a_j,b_j}}\right)(x_0 - a_j)}_{>0 \text{ on } \{\widehat{D}_{\tau_{a_j,b_j}} \le a_j\}} \mid \widehat{D}_{\tau_{a_j,b_j}} \le a_j\Big] < \infty. \tag{2.38}$$

Since in (2.37) and (2.38) only differences between points inside the interval and the process occur, we can find $C > 0$ such that for all $j \in \mathbb{N}$

$$0 < \mathbb{E}^{x_0}_{joint}\Big[\left(\widehat{D}_{\tau_{a_j,b_j}} - b_j\right)^2 + 2\left(\widehat{D}_{\tau_{a_j,b_j}} - b_j\right)(b_j - x_0) \mid \widehat{D}_{\tau_{a_j,b_j}} \ge b_j\Big] < C,$$
$$0 < \mathbb{E}^{x_0}_{joint}\Big[\left(a_j - \widehat{D}_{\tau_{a_j,b_j}}\right)^2 + 2\left(a_j - \widehat{D}_{\tau_{a_j,b_j}}\right)(x_0 - a_j) \mid \widehat{D}_{\tau_{a_j,b_j}} \le a_j\Big] < C,$$

and this holds uniformly in $j \in \mathbb{N}$. By the previous calculation we obtain

$$0 = \mathbb{E}^{x_0}_{joint}[M_0] = \mathbb{E}^{x_0}_{joint}[M_{\tau_{a_j,b_j}}] = \mathbb{E}^{x_0}_{joint}\left[\left(\widehat{D}_{\tau_{a_j,b_j}} - x_0\right)^2\right] - \mathbb{E}^{x_0}_{joint}\left[\sum_{k=0}^{\tau_{a_j,b_j}-1} f(\widehat{D}_k)\right]$$
$$\le (b_j - x_0)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \ge b_j) + (a_j - x_0)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \le a_j)$$
$$+ \mathbb{E}^{x_0}_{joint}\Big[\left(\widehat{D}_{\tau_{a_j,b_j}} - b_j\right)^2 + 2\left(\widehat{D}_{\tau_{a_j,b_j}} - b_j\right)(b_j - x_0) \mid \widehat{D}_{\tau_{a_j,b_j}} \ge b_j\Big] \cdot \mathbb{P}^{x_0}_{joint}\left(\widehat{D}_{\tau_{a_j,b_j}} \ge b_j\right)$$
$$+ \mathbb{E}^{x_0}_{joint}\Big[\left(a_j - \widehat{D}_{\tau_{a_j,b_j}}\right)^2 + 2\left(a_j - \widehat{D}_{\tau_{a_j,b_j}}\right)(x_0 - a_j) \mid \widehat{D}_{\tau_{a_j,b_j}} \le a_j\Big] \cdot \mathbb{P}^{x_0}_{joint}\left(\widehat{D}_{\tau_{a_j,b_j}} \le a_j\right)$$
$$- \frac{\tilde{\sigma}}{2}\mathbb{E}^{x_0}_{joint}\left[\tau_{a_j,b_j}\right]$$
$$\le (b_j - x_0)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \ge b_j) + (a_j - x_0)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \le a_j) + C - \frac{\tilde{\sigma}}{2}\mathbb{E}^{x_0}_{joint}\left[\tau_{a_j,b_j}\right], \tag{2.39}$$

which yields

$$\mathbb{E}^{x_0}_{joint}[\tau_{a_j,b_j}] \le \frac{2}{\tilde{\sigma}^2}\Big((b_j - x_0)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \ge b_j) + (a_j - x_0)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \le a_j) + C\Big)$$
$$\le \frac{2((L+K)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \ge b_j) + (K)^2 + C)}{\tilde{\sigma}^2} \quad \text{for all } j \in \mathbb{N}.$$

Hence using the Markov inequality we conclude that

$$\mathbb{P}^{x}_{joint}(\tau_{a_j,b_j} \ge m) \le \frac{2((L+K)^2 \cdot \mathbb{P}^{x_0}(\widehat{D}_{\tau_{a_j,b_j}} \ge b_j) + (K)^2 + C)}{\tilde{\sigma}^2 m} \quad \text{for all } j \in \mathbb{N} \tag{2.40}$$

which implies (2.32) by (2.30) if $L = \sqrt{m}$.

Step 3: In the third step we will prove that there exist a constant $C_7 > 0$ and $\varepsilon > 0$ such that for all $j \in \mathbb{N}$

$$\mathbb{E}^{x}_{joint}\left[e^{-\lambda\tau_{jK}}\right] \ge 1 - C_7\sqrt{\lambda} \tag{2.41}$$

for all $x \in (-\infty, (j+1)K]$ and all $\lambda \in (0, \varepsilon)$.

First we fix $j \in \mathbb{N}$. Note that (2.41) holds true trivially for $x \in (-\infty, jK]$. Hence it is enough to focus on $x \in (jK,(j+1)K]$. For $x \in (jK,(j+1)K]$ chosen arbitrarily we define $G_x(m) := \mathbb{P}^x_{joint}(\tau_{jK} \geq m)$ in order to simplify notation. By (2.32) we know that for $a \in (M,\infty)$ chosen arbitrarily

$$\sup_{x\in(jK,(j+1)K]} G_x(ar) \leq \frac{C_6}{\sqrt{ar}} \quad \text{for all } r \in \left(\tfrac{M}{a},\infty\right). \tag{2.42}$$

Define $\alpha := a^{-1}$. If we multiply $G_x(ar)$ with the term $e^{-\lambda r}$ for some $\lambda > 0$ and integrate over $r$ we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda r}\, G_x(ar)\, dr &= \int_0^{M\alpha} e^{-\lambda r} dr + \int_{M\alpha}^\infty e^{-\lambda r} G_x(ar)\, dr \\ &\leq \frac{(1-e^{-\lambda M\alpha})}{\lambda} + C_6\sqrt{\alpha}\int_0^\infty e^{-\lambda r}\frac{1}{\sqrt{r}} dr \\ &= \frac{(1-e^{-\lambda M\alpha})}{\lambda} + \frac{C_6\sqrt{\alpha}\,\Gamma(\frac{1}{2})}{\sqrt{\lambda}},\end{aligned}$$

where $\Gamma(\,\cdot\,)$ is the $\Gamma$-function. Furthermore, notice that

$$\int_0^\infty e^{-\lambda r}\, G_x(ar)\, dr = \int_0^\infty e^{-(\lambda\alpha)(ar)}\, G_x(ar)\, dr = \alpha\, \widehat{G}_x\,(\lambda\alpha)\,, \tag{2.43}$$

where $\widehat{G}_x$ is the Laplace transform of $G_x$. Hence

$$\alpha\widehat{G}_x\,(\lambda\alpha) \leq \frac{(1-e^{-\lambda M\alpha})}{\lambda} + \frac{C_6\sqrt{\alpha}\,\Gamma(\frac{1}{2})}{\sqrt{\lambda}} \tag{2.44}$$

for all $\alpha^{-1} = a > M$, all $x \in (jK,(j+1)K]$ and $\lambda > 0$ chosen arbitrarily. If we evaluate this inequality at $\lambda = \alpha$, we obtain

$$\widehat{G}_x\left(\alpha^2\right) \leq \frac{(1-e^{-M\alpha^2})}{\alpha^2} + \frac{C_6\;\Gamma(\frac{1}{2})}{\alpha}.$$

Taking into account that $\widehat{G}_x(\alpha) = \frac{1-\mathbb{E}^{x_o}_{joint}[e^{-\alpha\tau_{jK}}]}{\alpha}$ we conclude that

$$1-\mathbb{E}^x_{joint}[e^{-\alpha^2\tau_{jK}}] \leq \frac{(1-e^{-M\alpha^2})}{\alpha} + C_6\;\Gamma(\tfrac{1}{2})\cdot\alpha$$

for all $\alpha = a^{-1} \leq M^{-1}$ and all $x \in (jK,(j+1)K]$, which implies (2.41).

Step 4: In the fourth step we will prove that there exist $\varepsilon > 0$ and a constant $C_8 > 0$ such that for all $x > K$

$$\mathbb{E}^x_{joint}\left[e^{-\lambda\tau_K}\right] \geq 1 - C_8\cdot x\cdot\sqrt{\lambda} \quad \text{for all } \lambda \in (0,\varepsilon). \tag{2.45}$$

We fix some $x > K$ and choose $j \in \mathbb{N}$ such that $x \in (jK,(j+1)K]$. Hence by (2.41)

$$\begin{aligned}\mathbb{E}^x_{joint}[e^{-\lambda\tau_K}] &= \sum_y \mathbb{E}^x_{joint}[e^{-\lambda\tau_{2K}}\mathbb{1}_{\{\widehat{D}_{\tau_{2K}}=y\}}e^{-\lambda(\tau_K-\tau_{2K})}] \\ &= \sum_y \mathbb{E}^x_{joint}[e^{-\lambda\tau_{2K}}\mathbb{1}_{\{\widehat{D}_{\tau_{2K}}=y\}}\mathbb{E}^x_{joint}[e^{-\lambda(\tau_K-\tau_{2K})} \mid \mathcal{F}^1_{\tau_{2K}}]] \\ &= \sum_y \mathbb{E}^x_{joint}[e^{-\lambda\tau_{2K}}\mathbb{1}_{\{\widehat{D}_{\tau_{2K}}=y\}}\mathbb{E}^y_{joint}[e^{-\lambda(\tau_K)}]] \\ &\geq \sum_y \mathbb{E}^x_{joint}[e^{-\lambda\tau_{2K}}\mathbb{1}_{\{\widehat{D}_{\tau_{2K}}=y\}}(1-C_7\sqrt{\lambda})],\end{aligned}$$

if $\lambda < \varepsilon$, where $\mathbb{F}^1 := (\mathcal{F}_n^1)_n := (\sigma(\widehat{D}_m, m \leq n))_n$ (see (2.19)). A repetition of this argument leads to

$$\mathbb{E}^x_{joint}[e^{-\lambda \tau_K}] \geq (1 - C_7\sqrt{\lambda})^j \geq (1 - C_7\sqrt{\lambda})^{\frac{x}{K}} \geq 1 - \frac{C_7}{K} x\sqrt{\lambda},$$

where the last inequality holds true by Bernoulli's inequality, if $\lambda < \frac{1}{C_7^2} \wedge \varepsilon$ small enough. Hence (2.45) follows.

Step 5: Since every time the interval $[-K, K]$ is crossed by the process $(D_n)_n$ we have a positive probability to regenerate inside the interval. The number of times that the process $(\widehat{D}_n)_n$ jumps over the interval $[-K, K]$ before it finally lands inside can be bounded by a geometrically distributed random variable. Hence there exists a constant $C_9 > 0$ and $\varepsilon > 0$ such that

$$\mathbb{E}^x_{joint}[e^{-\lambda h(K)}] \geq 1 - C_9 \cdot x \cdot \sqrt{\lambda} \quad \text{for all } \lambda \in (0, \varepsilon) \tag{2.46}$$

for all $x > K$. Note that a detailed proof of a similar statement is given at (2.52) below. We skip the proof at this place. Out of (2.46) we conclude that

$$(1 - e^{-\lambda m})\mathbb{P}^x_{joint}(h(K) \geq m) \leq \mathbb{E}^x_{joint}[1 - e^{-\lambda h(K)}],$$

which for $\lambda = m^{-1}$ yields

$$\mathbb{P}^x_{joint}(h(K) \geq m) \leq \frac{C_9 \cdot x}{\sqrt{m}(1 - e^{-1})}.$$

Hence we conclude that for all $x > K$

$$\mathbb{P}^x_{joint}(h(K) \geq m) \leq \frac{C_3 \cdot x}{\sqrt{m}} \quad \text{for all } m > M, \tag{2.47}$$

where $C_3 := \frac{C_9}{(1-e^{-1})}$ and $M := \varepsilon^{-1}$.

□

**Proof of Lemma 2.9:** Let $K$ be a large constant. We choose $x_0 > K$ and assume that $m \geq M$ for some $M = M(x_0) > 0$. We prove (2.27) first. The idea is to show that

$$\mathbb{P}^{2^j K}_{joint}(H(2^{j+1}K) < h(K)) \approx \frac{1}{2}.$$

Hence using the Markov property we get

$$\mathbb{P}^K_{joint}(H(\sqrt{m}) < h(K)) \approx \prod_{j=0}^{\frac{\log(mK^{-2})}{2\log(2)}-1} \mathbb{P}^{2^j K}_{joint}(H(2^{j+1}K) < h(K)) \approx \left(\frac{1}{2}\right)^{\frac{\log(mK^{-2})}{2\log(2)}} = \frac{C \cdot K}{\sqrt{m}}.$$

Because of technical reasons we assume that $2^5K > x_0$. We treat the case $2^5K < x_0$ at the end of the proof. Notice that the event, that distance $2^5K$ is reached before the distance between the two random walks becomes less than $K$, has positive probability and can be bounded away from zero. In order to get a suitable bound for this event which is independent of $x_0$ we construct "corridors" as described in the proof of [BČDG13, proof of Lemma 3.8] and force the random walks to walk along these corridors. The probability for the random walks to increase their distance by at least 1 and regenerate simultaneously within the next step can be bounded from below by some $\delta_1 > 0$. Hence using the Markov property at simultaneous regeneration times we get that

$$\mathbb{P}^{x_0}_{joint}(H(\sqrt{m}) < h(K)) \geq (\delta_1)^{2^5 K} \cdot \mathbb{P}^{2^5 K}_{joint}(H(\sqrt{m}) < h(K)),$$

where $\delta_1$ is independent of $x_0$.
Now we choose $j \in \left[5, \frac{\log(mK^{-2})}{2\log(2)}\right] \cap \mathbb{Z}$ arbitrarily. The following estimation between the "joint" and "independent" law holds true by the coupling argument given in Lemma 1.14:

$$\begin{aligned}\mathbb{P}^{2^j K}_{joint}(H(2^{j+1}K) < h(jK)) &\geq \mathbb{P}^{2^j K}_{joint}(H(2^{j+1}K) < h(jK) \wedge (2^{j+1}K)^3)\\ &\geq \mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) < h(jK) \wedge (2^{j+1}K)^3) - C(2^{j+1}K)^3 e^{-cjK}.\end{aligned}$$

In the independent case the process $(\widehat{D}_n)_n$ is a sum of independent increments (with zero mean) and therefore a martingale. We define $\tau_j := \inf\{k \geq 1 : |\widehat{D}_k| \leq jK \text{ or } |\widehat{D}_k| \geq 2^{j+1}K\}$. Using the martingale property and the fact that the simultaneous regeneration times have exponential tail bounds we get that

$$\begin{aligned}2^j K = \mathbb{E}^{2^j K}_{ind}[\widehat{D}_{\tau_j}] &= \mathbb{E}^{2^j K}_{ind}[\widehat{D}_{\tau_j} \mathbb{1}_{\{|\widehat{D}_{\tau_j} - \widehat{D}_{\tau_j - 1}| < jK\}}] + \mathbb{E}^{2^j K}_{ind}[\widehat{D}_{\tau_j} \mathbb{1}_{\{|\widehat{D}_{\tau_j} - \widehat{D}_{\tau_j - 1}| \geq jK\}}]\\ &\leq jK \cdot (1 - \mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) < h(jK)))\\ &\quad + (2^{j+1}K + jK) \cdot \mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) < h(jK))\\ &\quad + C 2^{j+1} K e^{-cjK},\end{aligned}$$

where $C2^{j+1}Ke^{-cjK} < K$ if $K$ is large. Therefore we get

$$\mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) < h(jK)) \geq \frac{1}{2} - \frac{j+1}{2^{j+1}}.$$

The following estimation looks strange but will give us good control over all the error terms that occur, see (2.48). Because $j \geq 5$, we have

$$\frac{1}{2} = \frac{2^j}{2^{j+1}} \geq \frac{2^j - 5(j-1)}{2^{j+1} - 5j} + \frac{2j-5}{2^{j+1}}.$$

Since

$$\begin{aligned}\mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) < h(jK) \wedge (2^{j+1}K)^3) \geq \mathbb{P}^{2^j K}_{ind}&(H(2^{j+1}K) < h(jK))\\ &- \mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) \geq (2^{j+1}K)^3),\end{aligned}$$

we need to find bounds for the second term on the right side. A simple application of Donsker's invariance principle yields

$$\max_{x:|x| \leq 2^{j+1}K} \mathbb{P}^x_{ind}(H(2^{j+1}K) > (2^{j+1}K)^2) \leq (1 - \varepsilon),$$

for some $\varepsilon \in (0,1)$ if $K$ is chosen large enough. Hence making use of the Markov property, we get that

$$\mathbb{P}^{2^j K}_{ind}(H(2^{j+1}K) \geq (2^{j+1}K)^3) \leq (1 - \varepsilon)^{2^j K}.$$

Putting all the error terms together we conclude

$$\mathbb{P}^{2^jK}_{joint}(H(2^{j+1}K) < h(jK)) \geq \frac{2^j - 5(j-1)}{2^{j+1} - 5j},$$

if $K$ is chosen sufficiently large. If we consider the product over these factors we obtain

$$\begin{aligned}\mathbb{P}^{2^5K}_{joint}(H(\sqrt{m}) \leq h(K)) &\geq \exp\left(\sum_{j=5}^{\frac{\log(mK^{-2})}{2\log 2}-1} \log(2^j - 5(j-1)) - \log(2^{j+1} - 5j)\right) \\ &\geq C \cdot \frac{x_0}{\sqrt{m}}, \end{aligned} \tag{2.48}$$

if $m \geq M$ and $M = M(x_0)$ is chosen sufficiently large.
The invariance principle yields that after having reached distance $\sim \sqrt{m}$, the probability that the distance between the two random walks remains greater than $\frac{1}{2}\sqrt{m}$ during the next $m$ steps is bounded from below by a small constant $\delta_2 > 0$. Therefore

$$\mathbb{P}^{x_0}_{joint}(h(K) \geq m) \geq \frac{Cx_0}{\sqrt{m}}.$$

If $2^5K \leq x_0$, we define $K' := \lceil 2^{-5}x_0 \rceil \geq K$ and hence

$$\mathbb{P}^{x_0}_{joint}(H(\sqrt{m}) < h(K)) \geq (\delta_1)^{2^5K}\mathbb{P}^{2^5K'}_{joint}(H(\sqrt{m}) < h(K')).$$

Adapting the proof of the previous case we get that

$$\mathbb{P}^{2^5K'}_{joint}(H(\varepsilon\sqrt{m}) < h(K')) \geq C \cdot \frac{2^5K'}{\sqrt{m}} \geq C \cdot \frac{x_0}{\sqrt{m}},$$

as before.

□

Now that we finished the proof of Lemma 2.8 and Lemma 2.9, we turn to the proof of Proposition 2.1, equation (2.3):

**Proof of Proposition 2.1, equation (2.3):**
Remember that $\widehat{T}_{meet}$ was defined as the time $(\widehat{D}_n)_n$ hits zero. As mentioned at the beginning of the section it is enough to prove that there exist constants $C$, $M > 0$ such that for all $x_0 > 0$

$$\mathbb{P}^{x_0}_{joint}(\widehat{T}_{meet} \geq m) \leq \frac{C \cdot x_0}{\sqrt{m}} \quad \text{for all } m > M. \tag{2.49}$$

Some comments on the lower bound can be found in section A.1.4 in the appendix. Assume for the moment that there exist $C, \varepsilon > 0$ such that

$$1 - \mathbb{E}^{x_0}_{joint}[e^{-\lambda \widehat{T}_{meet}}] \leq C \cdot \sqrt{\lambda} \cdot x_0 \tag{2.50}$$

for all $\lambda \in (0, \varepsilon)$. By the simple estimation

$$(1 - e^{-\lambda m})\mathbb{P}^{x_0}_{joint}(\widehat{T}_{meet} \geq m) \leq \mathbb{E}^{x_0}_{joint}[1 - e^{-\lambda \widehat{T}_{meet}}],$$

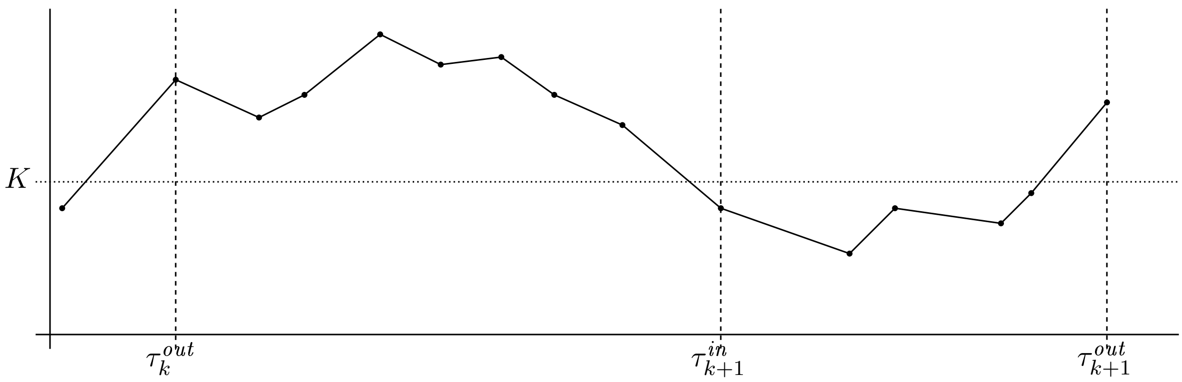
we conclude that

$$\mathbb{P}^{x_0}_{joint}(\widehat{T}_{meet} \geq m) \leq \frac{C \cdot \sqrt{\lambda} \cdot x_0}{(1 - e^{-\lambda m})},$$

which for $\lambda = \frac{1}{m}$ implies (2.49).

Hence it is enough to prove (2.50). We define the following stopping times

$$\begin{aligned}
\tau_0^{out} &:= 0, \\
\tau_{k+1}^{in} &:= \inf\{n \geq \tau_k^{out} : |\widehat{D}_k| \leq K\}, \\
\tau_{k+1}^{out} &:= \inf\{n \geq \tau_{k+1}^{in} : |\widehat{D}_k| > K\}.
\end{aligned}$$




We define

$$p_1 := \min_{x \,\in\, [-K,K]} \mathbb{P}^{x}_{joint}(|\widehat{D}_{m_0}| > K) > 0$$

and

$$p_2 := \min_{x \,\in\, [-K,K]} \mathbb{P}^{x}_{joint}(\widehat{D}_{m_0} = 0) > 0,$$

where $m_0 = m_0(K)$ is suitably chosen. Recognize that $\left(\tau_{k+1}^{in} - \tau_k^{out}, \tau_{k+1}^{out} - \tau_{k+1}^{in}, \widehat{D}_{\tau_{k+1}^{out}}\right)_{k \geq 0}$ is a Markov process with state space $\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}$ and adapted to the filtration

$$(\widetilde{\mathcal{F}}_n)_n := \left(\sigma\left(\widehat{D}_{\tau_{k+1}^{out}}, k \leq n\right)\right)_n, \tag{2.51}$$

where

$$\mathbb{P}^{K_0}_{joint}\left(\tau_{k+1}^{in} - \tau_k^{out} \geq m \mid \widehat{D}_{\tau_1^{out}} = y_1 \ldots \widehat{D}_{\tau_k^{out}} = y_k\right) \leq \mathbb{P}^{y_k}_{joint}(h(K) \geq m) \leq C_3 \cdot \frac{y_k}{\sqrt{m}}$$

for all $m > M$ (see (2.26)) and

$$\mathbb{P}^{K_0}_{joint}\left(\tau_{k+1}^{out} - \tau_{k+1}^{in} \geq m_0 \cdot m \mid \widehat{D}_{\tau_1^{out}} = y_1 \ldots \widehat{D}_{\tau_k^{out}} = y_k\right) \leq (1 - p_1)^{m-1}.$$

We assume without loss of generality that the process $(\widehat{D}_n)_n$ always exits the interval $[-K, K]$ to the positive side, since $\Psi^{diff}_{joint}(-x, -y) = \Psi^{diff}_{joint}(x, y)$ for all $x, y > 0$. For the first time leaving the interval $[-K, K]$ the following bound holds true:

Let the initial difference between the random walks be given by $y \in [-K, K]$, hence $\tau_0^{out} \equiv \tau_1^{in} \equiv 0$ and

$$\mathbb{E}^{y}_{joint}\left[e^{-\lambda(\tau_1^{out} - \tau_1^{in})}\right] = \mathbb{E}^{y}_{joint}\left[e^{-\lambda \tau_1^{out}}\right] \geq 1 - \lambda \cdot \mathbb{E}^{y}_{joint}\left[\tau_1^{out}\right] \geq 1 - C_{10} \cdot \lambda,$$

where $C_{10} := \max_{y\in[-K,K]} \mathbb{E}^{y}_{joint}\left[\tau_1^{out}\right] < \infty$, since the time inside the interval $[-K, K]$ has exponential tail bounds. Choose $x \geq K$. By (2.46) there exists $\varepsilon > 0$ such that for the whole period of entering and exiting the interval $[-K, K]$ we get

$$\begin{aligned}
&\mathbb{E}^{x}_{joint}\left[e^{-\lambda(\tau_1^{out}-\tau_0^{out})}\right] = \mathbb{E}^{x}_{joint}\left[e^{-\lambda\tau_1^{out}}\right] \\
&= \sum_{y=-K}^{K}\sum_{m=1}^{\infty} \mathbb{E}^{x}_{joint}\left[e^{-\lambda\left(\tau_1^{in}+(\tau_1^{out}-\tau_1^{in})\right)}\mathbb{1}_{\{\tau_1^{in}=m,\widehat{D}_m=y\}}\right] \\
&= \sum_{y=-K}^{K}\sum_{m=1}^{\infty} \mathbb{E}^{x}_{joint}\left[e^{-\lambda m}\mathbb{1}_{\{\tau_1^{in}=m,\widehat{D}_m=y\}}\mathbb{E}^{x}_{joint}\left[e^{-\lambda(\tau_1^{out}-\tau_1^{in})} \mid \tau_1^{in}, \widehat{D}_1, ..., \widehat{D}_m\right]\right] \\
&= \sum_{y=-K}^{K}\sum_{m=1}^{\infty} \mathbb{E}^{x}_{joint}\left[e^{-\lambda m}\mathbb{1}_{\{\tau_1^{in}=m,\widehat{D}_m=y\}}\mathbb{E}^{y}_{joint}\left[e^{-\lambda\tau_1^{out}}\right]\right] \\
&\geq \mathbb{E}^{x}_{joint}\left[e^{-\lambda\tau_1^{in}}\right](1 - C_{10}\lambda) \\
&\geq \left(1 - C_9\sqrt{\lambda}\cdot x\right)(1 - C_{10}\lambda) \\
&\geq 1 - C_9\sqrt{\lambda}\cdot x - C_{10}\lambda
\end{aligned}$$

for all $\lambda \in (0, \varepsilon)$ and all $x > K$. Next, we split up the whole path up to the first hitting time of zero into pieces inside and outside the interval $[-K, K]$. Let $x_0 > K$ be fixed. Let $N$ be a geometrically distributed random variable with success probability $p_2$, which is independent of everything else. $N$ gives us an upper bound on the number of times the random walks enters the interval $[-K, K]$ before it hits zero. We get that

$$\begin{aligned}
\mathbb{E}^{x_0}_{joint}[e^{-\lambda\widehat{T}_{meet}}] &\geq \mathbb{E}^{x_0}_{joint}\left[\exp\left(-\lambda\sum_{k=1}^{N}(\tau_k^{in}-\tau_{k-1}^{out}) + (\tau_k^{out}-\tau_k^{in})\right)\right] \\
&= \sum_{n=1}^{\infty} p_2(1-p_2)^{n-1}\mathbb{E}^{x_0}_{joint}\left[\exp\left(-\lambda\sum_{k=1}^{n}(\tau_k^{in}-\tau_{k-1}^{out}) + (\tau_k^{out}-\tau_k^{in})\right)\right],
\end{aligned} \tag{2.52}$$

where

$$\begin{aligned}
&\mathbb{E}^{x_0}_{joint}\left[\exp\left(-\lambda\sum_{k=1}^{n}(\tau_k^{in}-\tau_{k-1}^{out}) + (\tau_k^{out}-\tau_k^{in})\right)\right] \\
&= 2\sum_{x_{n-1}=1+K}^{\infty} \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\mathbb{1}_{\{\widehat{D}_{\tau_{n-1}^{out}}=x_{n-1}\}}\cdot e^{-\lambda((\tau_n^{in}-\tau_{n-1}^{out})+(\tau_n^{out}-\tau_n^{in}))}\right] \\
&= 2\sum_{x_{n-1}=1+K}^{\infty} \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\cdot\mathbb{1}_{\{\widehat{D}_{\tau_{n-1}^{out}}=x_{n-1}\}}\cdot\mathbb{E}^{x_0}_{joint}\left[e^{-\lambda((\tau_n^{in}-\tau_{n-1}^{out})+(\tau_n^{out}-\tau_n^{in}))}\middle|\widetilde{\mathcal{F}}_{n-1}\right]\right] \\
&= 2\sum_{x_{n-1}=1+K}^{\infty} \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\cdot\mathbb{1}_{\{\widehat{D}_{\tau_{n-1}^{out}}=x_{n-1}\}}\cdot\mathbb{E}^{x_{n-1}}_{joint}\left[e^{-\lambda\tau_1^{out}}\right]\right] \\
&\geq 2\sum_{x_{n-1}=1+K}^{\infty} \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\cdot\mathbb{1}_{\{\widehat{D}_{\tau_{n-1}^{out}}=x_{n-1}\}}\left(1 - C_9\sqrt{\lambda}\cdot x_{n-1} - C_{10}\lambda\right)\right] \\
&= \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\cdot\left(1 - C_9\sqrt{\lambda}\cdot\widehat{D}_{\tau_{n-1}^{out}} - C_{10}\lambda\right)\right], \\
&= \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\right] - \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau_{n-1}^{out}}\cdot\left(C_9\sqrt{\lambda}\cdot\widehat{D}_{\tau_{n-1}^{out}} + C_{10}\lambda\right)\right].
\end{aligned}$$

For the definition of $(\widetilde{\mathcal{F}}_n)_n$ see (2.51). We claim that there exists a constant $C > 0$ such that

$$\mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau^{out}_{n-1}} \cdot \left(C_9\sqrt{\lambda} \cdot \widehat{D}_{\tau^{out}_{n-1}} + C_{10}\lambda\right)\right] \leq \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau^{out}_{n-1}}\right] \cdot C\sqrt{\lambda} \tag{2.53}$$

for all $\lambda \in (0, \varepsilon)$. If we assume this claim to be true, we get by a repetition of the arguments above that

$$\mathbb{E}^{joint}_{x_0}[e^{-\lambda\tau^{out}_n}] \geq \left(1 - C\sqrt{\lambda}\right)^n \geq 1 - Cn\sqrt{\lambda},$$

and hence

$$\mathbb{E}^{x_0}_{joint}[e^{-\lambda\widehat{T}_{meet}}] = \sum_{n=1}^{\infty} p_2(1-p_2)^{n-1}(1 - nC\sqrt{\lambda}) \geq 1 - \frac{C}{p_2}\sqrt{\lambda},$$

which implies (2.50) and therefore (2.3). So all that is left to do is to prove (2.53). For ease of notation we define the following event

$$A_n(k, y) := \{\tau^{in}_{n-1} < k,\ |\widehat{D}_r| < K \text{ for all } \tau^{in}_{n-1} \leq r < k,\ \widehat{D}_{k-1} = y\} \in \sigma(\widehat{D}_0, ..., \widehat{D}_{k-1}),$$

which is the event that the process $(\widehat{D}_n)_n$ visits the interval $[-K, K]$ for the $(n-1)$-th time before time $k$, that it stays within the interval from $\tau^{in}_{n-1}$ to time $k-1$ and that at time $k-1$ the process is at $y \in [-K, K]$. Hence

$$\begin{aligned}
&\mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau^{out}_{n-1}} \cdot \left(C_9\sqrt{\lambda} \cdot \widehat{D}_{\tau^{out}_{n-1}} + C_{10}\lambda\right)\right] \\
&\leq \sum_{|y|\leq K}\sum_{k\in\mathbb{N}} \mathbb{E}^{x_0}_{joint}\left[\mathbb{1}_{A_n(k,y)}\mathbb{1}_{\{|\widehat{D}_k|>K\}}e^{-\lambda k} \cdot \left(C_9\sqrt{\lambda} \cdot \widehat{D}_{\tau^{out}_{n-1}} + C_{10}\lambda\right)\right] \\
&\leq \sum_{|y|\leq K}\sum_{k\in\mathbb{N}} \mathbb{E}^{x_0}_{joint}\left[\mathbb{1}_{A_n(k,y)}e^{-\lambda k} \cdot \mathbb{E}^{x_0}_{joint}\left[\mathbb{1}_{\{\widehat{D}_k>K\}}\left(C_9\sqrt{\lambda} \cdot \widehat{D}_{\tau^{out}_{n-1}} + C_{10}\lambda\right) \,\Big|\, \widehat{D}_0, ..., \widehat{D}_{k-1}\right]\right] \\
&\leq \sum_{|y|\leq K}\sum_{k\in\mathbb{N}} \mathbb{E}^{x_0}_{joint}\left[\mathbb{1}_{A_n(k,y)}e^{-\lambda k} \cdot \mathbb{E}^{joint}_{y}\left[\mathbb{1}_{\{\widehat{D}_1>K\}}\left(C_9\sqrt{\lambda} \cdot \widehat{D}_{\tau^{out}_1} + C_{10}\lambda\right)\right]\right] \\
&\leq \sum_{|y|\leq K}\sum_{k\in\mathbb{N}} \mathbb{E}^{x_0}_{joint}\left[\mathbb{1}_{A_n(k,y)}e^{-\lambda k}\right] \cdot \sum_{x>K}\left(C_9\sqrt{\lambda} \cdot x + C_{10}\lambda\right)\Psi^{diff}_{joint}(y, x) \\
&\leq (C\sqrt{\lambda}) \cdot \sum_{|y|\leq K}\sum_{k\in\mathbb{N}} \mathbb{E}^{x_0}_{joint}\left[\mathbb{1}_{A_n(k,y)}e^{-\lambda k}\right] \cdot \sum_{x>K}\Psi^{diff}_{joint}(y, x)x \\
&\leq (C\sqrt{\lambda}) \cdot \mathbb{E}^{x_0}_{joint}\left[e^{-\lambda\tau^{out}_{n-1}}\right],
\end{aligned}$$

where the fifth inequality holds true since

$$0 \leq \sum_{x>K}(C_9 \cdot x + C_{10})\,\Psi^{diff}_{joint}(y, x) < \infty,$$

which together with the exponential tail bounds for $\Psi^{diff}_{joint}$ (compare Lemma 1.12 and Lemma 1.14) implies the existence of a constant $C > 0$, such that

$$\sum_{x>K}\left(C_9\sqrt{\lambda} \cdot x + C_{10}\lambda\right)\Psi^{diff}_{joint}(y, x) \leq C\sqrt{\lambda}\sum_{x>K}\Psi^{diff}_{joint}(y, x).$$

□

## 2.2. Characterization of the Brownian web

In this section give a short introduction into the topic of the Brownian web and formulate the convergence theorem precisely. The characterization of the Brownian web given below, can also be found for example in [FINR04] or [Sun05].
We define a metric on $\mathbb{R}^2$ by

$$\rho((x_1,t_1),(x_2,t_2)) := |\tanh(t_1)-\tanh(t_2)| \vee \left|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\right|.$$

Let $R_c^2$ be the completion of $\mathbb{R}^2$ under $\rho$. We can think of $R_c^2$ as the image of $[-\infty,\infty]\times[-\infty,\infty]$ under the mapping

$$(x,t) \mapsto (\Phi(x,t),\Psi(t)) := \left(\frac{\tanh(x)}{1+|t|}, \tanh(t)\right) \in R_c^2.$$

This means that $R_c^2$ can be identified with the square $[-1,1]\times[-1,1]$ where the line $[-1,1]\times 1$ and the line $[-1,1]\times -1$ are squeezed to two single points which will be denoted by $(*,\infty)$ and $(*,-\infty)$. We define $\Pi$ to be the set of functions $f:[\sigma,\infty]\longrightarrow[-\infty,\infty]$ with "starting time" $\sigma\in[-\infty,\infty]$, such that the mapping $t\mapsto(f(\sigma\vee t),t)$ from $(\mathbb{R},|\cdot|)$ to $(R_c^2,\rho)$ is continuous. We consider the elements in $\Pi$ as a tuple of the function $f$ and its starting time $\sigma$. If we identify the elements in $\Pi$ with their paths $(f(\sigma\vee t),t)_{t\in\mathbb{R}}$ in $R_c^2$, the set $\Pi$ together with the metric

$$d((f,\sigma),(g,\sigma')) := |\tanh(\sigma)-\tanh(\sigma')| \vee \sup_{t\in\mathbb{R}}\left|\frac{\tanh(f(t\vee\sigma))}{1+|t|} - \frac{\tanh(g(t\vee\sigma'))}{1+|t|}\right|,$$

becomes a complete separable metric space. Let $\mathcal{H}$ be the set of compact subsets of $(\Pi,d)$. Together with the Hausdorff metric

$$d_{\mathcal{H}}(K_1,K_2) := \sup_{(f,\sigma)\in K_1}\inf_{(g,\sigma')\in K_2} d((f,\sigma),(g,\sigma')) \vee \sup_{(g,\sigma')\in K_2}\inf_{(f,\sigma)\in K_1} d((f,\sigma),(g,\sigma')),$$

$\mathcal{H}$ becomes a complete separable metric space. Let $\mathcal{B}_{\mathcal{H}}$ be the Borel $\sigma$-algebra associated with the metric $d_{\mathcal{H}}$. We can characterize the Brownian web (BW) as follows:

**Definition 2.10. (Brownian web)** The Brownian web is a $(\mathcal{H},\mathcal{B}_{\mathcal{H}})$-valued random variable $\mathcal{W}$, defined on a probability space $(\Omega,\mathcal{A},P)$, whose distribution is uniquely determined by the following properties:

(i) For any deterministic point $(x,t)\in\mathbb{R}^2$, there is almost surely a unique path $W^{(x,t)}$ starting from $(x,t)$.

(ii) For any deterministic $z_1,...,z_k\in\mathbb{R}^2$, the joint distribution of $W^{(z_1)},...,W^{(z_k)}$ is that of coalescing Brownian motions (with unit diffusion constant).

(iii) For any deterministic, countable dense subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure of $\{W^{(z)}:z\in\mathcal{D}\}$ in $(\Pi,d)$.

## 2.3. Verification of convergence criteria

Now we give a precise definition of the system of coalescing random walks starting from each point contained in the oriented percolation cluster, which we expect to converge to the Brownian web. Remember that as said at the beginning of this chapter, the space dimension $d$ equals one.
Let $\mathcal{C} \subset \mathbb{Z} \times \mathbb{Z}$ be the set of all space-time points connected to infinity, as defined in (1.25). If a point $z = (x, n) \in \mathbb{Z} \times \mathbb{Z}$ is in $\mathcal{C}$ let $\pi^z$ be the linearly interpolated path of the random walk $X^{(z)}$ starting from $z$ defined in (1.49). If a point $z \in \mathbb{Z} \times \mathbb{Z}$ is not in $\mathcal{C}$, we choose the next point left to $z$ that is connected to infinity and define $\pi^z$ as the linearly interpolated copy of the random walk path starting there. In particular, if $z = (x, n) \notin \mathcal{C}$ we define

$$c((x,n)) := \max\{y < x : (y,n) \in \mathcal{C}\} \text{ and } (\pi^z(t))_{t \geq n} := (\pi^{(c(z),n)}(t))_{t \geq n}. \tag{2.54}$$

We formulate our result in a similar way Sarkar and Sun did in [SS13]. Let $\mathbf{\Gamma}$ be the collection of all paths $\mathbf{\Gamma} := \{\pi^z \in \mathbb{Z} \times \mathbb{Z}\} = \{\pi^z : z \in \mathcal{C}\}$. Since all paths in $\mathcal{C}$ are equicontinuous the closure of $\mathbf{\Gamma}$, which we also denote by $\mathbf{\Gamma}$, is a random variable taking values in $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$.
There will be situations in which we need to consider the collection of the piecewise constant paths instead of the linearly interpolated ones. We denote the piecewise constant paths by $\kappa^z$ and the collection of these paths by $\mathbf{K} := \{\kappa^z : z \in \mathbb{Z} \times \mathbb{Z}\} = \{\kappa^z : z \in \mathcal{C}\}$.
In order to formulate the convergence theorem precisely we define

$$S_{b,\delta} := (S^1_{b,\delta}, S^2_{b,\delta}) : (R^2_c, d) \longrightarrow (R^2_c, d),$$

where

$$S_{b,\delta}(x,t) := (S^1_{b,\delta}(x,t), S^2_{b,\delta}(t)) := \begin{cases} (\frac{x\delta}{b}, \delta^2 t), & \text{if } (x,t) \in \mathbb{R}^2, \\ (\pm\infty, \delta^2 t), & \text{if } (x,t) = (\pm\infty, t), t \in \mathbb{R}, \\ (*, \pm\infty), & \text{if } (x,t) = (*, \pm\infty). \end{cases}$$

In the literature, the mapping $S_{b,\delta}$ is called the *diffusive scaling map*. The mapping $S_{b,\delta}$ can be extended to $(\Pi, d)$, where $S_{b,\delta}((\pi, t_0))$ is the path whose graph equals the image of $(\pi(t), t)_{t \in [t_0, \infty]}$ under $S_{b,\delta}$. If $K$ is a subset of $\Pi$ we define $S_{b,\delta}K := \{S_{b,\delta}((\pi, t)) : (\pi, t) \in K\}$. For $K \in \mathcal{H}$ we have $S_{b,\delta}K \in \mathcal{H}$.

**Theorem 2.11.** *The sequence of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variables $(S_{\sigma,\delta}\mathbf{\Gamma})_\delta$, where*

$$\sigma^2 := \frac{\mathbb{E}[Y_1^2]}{\mathbb{E}[\tau_1]} \quad \textit{(compare [BČDG13, Remark 1.2])},$$

*converges in distribution to the Brownian web.*

$$\mu_\delta := \mathcal{L}(S_{\sigma,\delta}\mathbf{\Gamma}|\mathbb{P}) \xrightarrow{w} \mathcal{L}(\mathcal{W}|P) =: \mu, \quad \textit{for } \delta \longrightarrow 0. \tag{2.55}$$

**Notation 2.12.** First we introduce a little more notation which will be needed to formulate Sun's convergence criteria. Most of the ideas Sun used to verify the convergence criteria for his model are adaptable to our case. Therefore the work done in this chapter is very similar to the work done by

Sun in [Sun05].
We define $\Lambda_{L,T} := [-L, L] \times [-T, T] \subset \mathbb{R}^2$. For some $x_0, t_0 \in \mathbb{R}$ and $u, t > 0$ let $R(x_0, t_0, u, t)$ be the rectangle $[x_0 - u, x_0 + u] \times [t_0, t_0 + t] \subset \mathbb{R}^2$ and define $A_{u,t}(x_0, t_0)$ to be the event that $K \in \mathcal{H}$ contains a path that touches both the rectangle $R(x_0, t_0, u, t)$ and the left or right boundary of the bigger rectangle $R(x_0, t_0, 20u, 2t)$, see Figure 2.1 below.
For $a, b, t_0, t \in \mathbb{R}, a < b, 0 < t$ and $K \in \mathcal{H}$ we define the number of distinct points in $\mathbb{R} \times \{t_0 + t\}$ which are touched by some path in $K \in \mathcal{H}$ that also touches $[a, b] \times \{t_0\}$ by

$$\begin{aligned}&\eta_K(t_0, t; a, b) := \\ &\#\{y \in \mathbb{R} : \exists x \in [a, b] \text{ and a path in } K \text{ which touches both } (x, t_0) \text{ and } (y, t_0 + t)\}.\end{aligned}$$

If $\mathcal{X}$ is a $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable, we define $\mathcal{X}^{s^-}$ to be the subset of paths in $\mathcal{X}$ which start before or at time $s$. We restrict the paths in $\mathcal{X}^{s^-}$ to $[t, \infty]$ and define $\mathcal{X}^{s^-, t_T}$ as the collection of these restricted paths. Note that $\mathcal{X}^{s^-, t_T}$ is also an $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable. If $s = t$, we will simply write $\mathcal{X}^{s_T}$.

Sun shows [Sun05, Theorem 1.3.2, Lemma 3.4.1] that a family $\{\mathcal{X}_n\}_n$ of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variables with distribution $\{\mu_n\}$ converges in distribution to the standard Brownian web $\mathcal{W}$, if it satisfies the following conditions:

$(I_1)$ There exist single path valued random variables $\theta_n^{(y)} \in \mathcal{X}_n$, for $y \in \mathbb{R}^2$, satisfying: For $\mathcal{D}$ a deterministic countable dense subset of $\mathbb{R}^2$, for any deterministic $z_1, ..., z_m \in \mathcal{D}$, $\theta_n^{(z_1)}, ..., \theta_n^{(z_m)}$ converge jointly in distribution to coalescing Brownian motions (with unit diffusion constant) starting from $z_1, ..., z_m$, as $n$ tends to infinity.

$(T_1)$ For every $u, L, T \in (0, \infty)$

$$\widetilde{g}(t, u; L, T) \equiv t^{-1} \limsup_{n \longrightarrow \infty} \sup_{(x_0, t_0) \in \Lambda_{L,T}} \mu_n(A_{t,u}(x_0, t_0)) \to 0 \text{ as } t \longrightarrow 0^+,$$

which is a sufficient condition for the family $\{\mathcal{X}_n\}_n$ to be tight.

$(B_1')$ For all $\beta > 0$

$$\limsup_{n \longrightarrow \infty} \sup_{t > \beta} \sup_{t_0, a \in \mathbb{R}} \mu_n(\{K \in \mathcal{H} : \eta_K(t_0, t; a - \varepsilon, a + \varepsilon) > 1\}) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0^+.$$

$(E_1')$ Fix $t_0 \in \mathbb{R}$. If $\mathcal{Z}_{t_0}$ is any subsequential limit of $\{\mathcal{X}_n^{t_0^-}\}_n$, defined on some probability space $(\Omega, \mathcal{A}, P)$, then for all $t, a, b \in \mathbb{R}$, with $t > 0$ and $a < b$,

$$E_P[\eta_{\mathcal{Z}_{t_0}}(t_0, t; a, b)] \leq E_P[\eta_{\mathcal{W}}(t_0, t; a, b)] = \frac{b - a}{\sqrt{\pi t}}.$$

**Remark 2.13.** Instead of $\mathcal{X}_n$, we usually write $\mathcal{X}_\delta$ to denote the $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable $S_{\sigma,\delta}\mathbf{\Gamma}$ (compare (2.55)). If we want to consider the weak limit of $(\mathcal{X}_\delta)_{\delta > 0}$ along a certain subsequence $(\delta_n)_n$, where $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$, we denote the random variables $S_{\sigma,\delta_n}\mathbf{\Gamma}$ by $\mathcal{X}_{\delta_n}$. The probability measure $\mathbb{P} \circ (S_{\sigma,\delta_n}\mathbf{\Gamma})^{-1}$ on $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ is denoted by $\mu_{\delta_n}$.

### Checking condition $(I_1)$

First of all let $\mathcal{D}$ be a dense countable subset of $\mathbb{R}^2$ and choose $y_1 = (x_1, t_1), ..., y_m = (x_m, t_m) \in \mathcal{D}$. Define $y_{\delta,i} := (\tilde{x}_i, \tilde{t}_i) := (\lfloor x_i \sigma \delta^{-1} \rfloor, \lfloor t_i \delta^{-2} \rfloor)$ and let $(\pi_\delta^i, t_\delta^i)$ denote the function in $(\Pi, d)$ whose graph equals the image of the graph of $(\pi^{(y_{\delta,i})}, \tilde{t}_i)$ under $S_{\sigma,\delta}$. In order to shorten notation, we will suppress the starting time in the following calculations.

Step 1: The first thing to show is that for every $i \in \{1, ..., m\}$ the linearly interpolated and diffusively rescaled random walk $\pi_\delta^i$, converges weakly under $\mathbb{P}$ to a Brownian motion starting from $y_i$. Using [BČDG13, Theorem 1.1, Remark 1.5] we know that for $(x,n) \in \mathbb{Z} \times \mathbb{Z}$ the diffusively rescaled random walk $\pi_\delta^{(x,n)}$ converges weakly under $\mathbb{P}(\,\cdot\,|B_{(x,n)})$ to a Brownian motion, where $B_{(x,n)}$ is the event that $(x,n)$ is connected to infinity. Define $G_{(x,n)}$ to be the event that the quenched functional central limit theorem holds for a path starting in $(x,n)$. From [BČDG13, Theorem 1.1, Theorem 1.4] we get that $\mathbb{P}(G_{(x,n)}|B_{(x,n)}) = 1$, which means that $\mathbb{P}((G_{(x,n)})^c \cap B_{(x,n)}) = 0$. Define

$$G := \bigcap_{(x,n)\in\mathbb{Z}^2} G_{(x,n)} \cup (B_{(x,n)})^c = \left( \bigcup_{(x,n)\in\mathbb{Z}^2} (G_{(x,n)})^c \cap B_{(x,n)} \right)^c .$$

Note that $\mathbb{P}(G) = 1$ since the complement is a countable union of null sets. Hence up to a $\mathbb{P}$-null set either $(x,n) \in \mathbb{Z} \times \mathbb{Z}$ is not connected to infinity or the functional central limit theorem holds in $(x,n)$. Keeping this in mind, the only thing left to do in order to prove the claim of step 1, is to show that

$$\frac{c(y_{\delta_n,i})\delta_n}{\sigma} \underset{n\to\infty}{\longrightarrow} x_i,$$

in probability, for any $(\delta_n)_n$ with $\delta_n \downarrow 0$, where $c((x,n))$ was defined in (2.54). We fix some null sequence $(\delta_n)_n$. According to [Dur84, Section 10] we know that there exist $K, C > 0$ such that

$$\mathbb{P}\left(|x - c((x,m))| \geq K \log(1/\delta_n)\right) \leq C\delta_n^2 \quad \text{for all} \quad (x,m) \in \mathbb{Z} \times \mathbb{Z}, \tag{2.56}$$

where $\{|x - c((x,m))| \geq K\}$ is the event that the next point left to $x$, which is contained in the oriented percolation cluster, is more than distance $K$ apart from $x$. Or, said differently, (2.56) is an estimate on the probability of holes of order $\sim \log(1/\delta_n)$ to occur. Hence

$$\mathbb{P}\left( \left| x_i - \frac{c(y_{\delta_n,i})\delta_n}{\sigma} \right| > \varepsilon \right) = \mathbb{P}\left( \left| x_i\sigma\delta_n^{-1} - c(y_{\delta_n,i}) \right| > \frac{\varepsilon\sigma}{\delta_n} \right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

from which the desired convergence follows.

Step 2: Before formulating the claim we want to prove in step 2, we require a little more notation. Consider the tuple $(\pi_\delta^1, ..., \pi_\delta^m)$ (the starting time is suppressed) as a random variable on the product space $(\Pi^m, d^{*m})$, where

$$d^{*m}\left[ ((f_1,\sigma_1), ..., (f_m,\sigma_m)), ((g_1,\sigma_1')..., (g_m,\sigma_m')) \right] = \max_{1\leq i\leq m} d\left((f_i,\sigma_i),(g_i,\sigma_i')\right).$$

Denote by $\overline{\Pi}$ the space of càdlàg paths and define

$$\bar{d}((f,\sigma),(g,\sigma')) := |\sigma - \sigma'| \vee \sup_{t\in\mathbb{R}} |f(t\vee\sigma) - g(t\vee\sigma')|.$$

The metric $\bar{d}^{*m}$ on the product space $\overline{\Pi}^m$ is defined analogously. Next we will define two coalescing rules, which can be considered as mappings from $(\overline{\Pi}^m, \bar{d}^{*m})$ to $(\overline{\Pi}^m, \bar{d}^{*m})$.
The first coalescing rule $\Gamma_\alpha$ is defined as follows:
Let $((f_1,t_1),...,(f_m,t_m))$ be an element of $\overline{\Pi}^m$. Define $T_\alpha^{i,j}$ as the first time that the paths $f_i, f_j$ coincide or change their relative order after time $t_i \vee t_j$

$$T_\alpha^{i,j} := \inf\{t > t_i \vee t_j : (f_i(t_i \vee t_j) - f_j(t_i \vee t_j))\,(f_i(t) - f_j(t)) \leq 0\}.$$

Start with the equivalence relation $i \sim i$, $i \not\sim j$ for all $i \neq j$ on $\{1,...,m\}$. Define $\Gamma_\alpha$ on $((f_1,t_1),...,(f_m,t_m))$ by

$$\tau_\alpha := \min_{1\leq i,j\leq m, i\not\sim j} T_\alpha^{i,j}, \qquad \text{with } \min\emptyset = \infty,$$

and

$$\Gamma_\alpha(f_i(t)) := \begin{cases} f_i(t), & \text{if } t < \tau_\alpha, \\ f_{i^*}(t), & \text{if } t \geq \tau_\alpha, \end{cases}$$

where $i^* = \min\{j | (j \sim i) \text{ or } (j \not\sim i \text{ and } T_\alpha^{i,j} = \tau_\alpha)\}$. Enhance the equivalence relation by $i \sim i^*$. Iterating this procedure, we get the desired structure of coalescing random walks. We label the successive times $\tau_\alpha$ by $\tau_\alpha^1, ..., \tau_\alpha^k$, where $k \in 1,...,m$ is chosen such that $\tau_\alpha^k = \infty$.
The second coalescing rule $\Gamma_\beta$ is defined very similarly, but $T_\alpha^{i,j}$ is replaced by

$$T_\beta^{i,j} := \inf\{t \geq t_i \vee t_j : f_i(t) = f_j(t)\},$$

the time when two paths coincide. In order to simplify notation we add subscripts $\alpha$ or $\beta$ to $(f_1,...,f_m)$ if $\Gamma_\alpha$ or $\Gamma_\beta$ is applied to it.
Let $\kappa_\delta^i$ denote the piecewise constant càdlàg versions of $\pi_\delta^i$. The claim of step 2 is that for all $\varepsilon > 0$

$$\mathbb{P}\left(d^{*m}\left[\left(\kappa_{\delta,\alpha}^1, ..., \kappa_{\delta,\alpha}^m\right), \left(\kappa_{\delta,\beta}^1, ..., \kappa_{\delta,\beta}^m\right)\right] \geq \varepsilon\right) \longrightarrow 0, \tag{2.57}$$

as $\delta \downarrow 0$.
Notice that $d((f_1,t_1),(f_2,t_2)) \leq \bar{d}((f_1,t_1),(f_2,t_2))$ for all $(f_1,t_1),(f_2,t_2) \in \Pi$, since $\tanh(\cdot)$ is Lipschitz continuous with Lipschitz constant one. Hence, in order to prove (2.57), it is enough to show that

$$\mathbb{P}\left(\bar{d}^{*m}\left[\left(\kappa_{1,\alpha}^1, ..., \kappa_{1,\alpha}^m\right), \left(\kappa_{1,\beta}^1, ..., \kappa_{1,\beta}^m\right)\right] \geq \frac{\sigma\varepsilon}{\delta}\right) \longrightarrow 0, \tag{2.58}$$

as $\delta \downarrow 0$, where $\kappa_1^i$ denotes the piecewise constant càdlàg version of $\pi_\delta^i$ in the case $\delta = 1$. If $\delta = 1$ we sometimes denote $\kappa_\delta^i$ by $\kappa^i$. We prove the claim by induction over $m$. Let $m = 2$. Since $\kappa_{1,\alpha}^1 = \kappa_{1,\beta}^1 = \kappa^1$ we get that

$$\bar{d}^{*m}\left[\left(\kappa_{1,\alpha}^1, \kappa_{1,\alpha}^2\right), \left(\kappa_{1,\beta}^1, \kappa_{1,\beta}^2\right)\right] = \bar{d}[\kappa_{1,\alpha}^2, \kappa_{1,\beta}^2].$$

In Proposition 2.1 (see also Lemma A.1 and discussion there) we proved that two random walks on a joint oriented percolation cluster coalesce almost surely. Choose $\varepsilon' > 0$ arbitrarily and let $N, K$ be large enough such that $\mathbb{P}(T_\alpha^{1,2} > N) < \frac{\varepsilon'}{2}$ and

$$\mathbb{P}\left(\max_{n,|x-y_2|<N} T_{meet}^{(x,n),(x+1,n)} > K\right) < \frac{\varepsilon'}{2(2N+1)(N+1)}.$$

Furthermore, choose $\delta$ such that $\frac{\varepsilon\sigma}{\delta} > 2K$. We get that

$$\begin{aligned}
&\mathbb{P}\left(\bar{d}(\kappa^2_{1,\alpha}, \kappa^2_{1,\beta}) > \frac{\varepsilon\sigma}{\delta}\right) \\
&\leq \frac{\varepsilon'}{2} + \sum_{0\leq |x|,n\leq N} \mathbb{P}\left(\sup_{s<T^{(x,n),(x+1,n)}_{meet}} |\kappa^{(x,n)}(s) - \kappa^{(x+1,n)}(s)| > \frac{\varepsilon\sigma}{\delta},\ \kappa^2(T^{1,2}_\alpha) = x,\ T^{1,2}_\alpha = n\right) \\
&\leq \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} \leq \varepsilon',
\end{aligned} \tag{2.59}$$

where the first inequality follows from the fact that when two paths cross, they can miss each other by at most "one step". Since $\varepsilon' > 0$ is chosen arbitrarily, the statement in (2.57) is proved for the case $m = 2$.

Now let $m > 2$. There are two possibilities for the event in (2.58) to occur.

The first possibility is that a "wrong ($\alpha-$)coalescing event" occurs, which means that for some $k$ and $i < j$ a path $\kappa^l_\delta$, $l < i$ coalesces or changes its relative order with $\kappa^i_\delta$ after time $\tau^k_{\delta,\alpha} = T^{i,j}_{\delta,\alpha}$ and before time $T^{i,j}_{\delta,\beta}$, where there is no need for $\kappa^l_\delta$ and $\kappa^j_\delta$ to coalesce "soon", since their paths never crossed. Let us consider this case. First notice that $(\pi^1_\delta, ..., \pi^m_\delta)$ converges weakly in $(\Pi^m, d^{*m})$ to $m$ independent Brownian motions $(\mathcal{B}^1, ..., \mathcal{B}^m)$ starting from $(y_1, ..., y_m)$ by [BČDG13, Theorem 1.3, Remark 1.5, Remark 3.11] and step 1. Hence Skorohod's representation theorem yields that we can choose a joint probability space on which

$$d^{*m}\left[\left(\pi^1_\delta, ..., \pi^m_\delta\right), \left(\mathcal{B}^1, ..., \mathcal{B}^m\right)\right] \longrightarrow 0 \quad \text{almost surely}, \quad \text{as } \delta \to 0. \tag{2.60}$$

Since $\kappa^i_\delta$ is the piecewise constant càdlàg version of $\pi^i_\delta$, we know that also

$$d^{*m}\left[\left(\kappa^1_\delta, ..., \kappa^m_\delta\right), \left(\mathcal{B}^1, ..., \mathcal{B}^m\right)\right] \longrightarrow 0 \quad \text{almost surely}, \quad \text{as } \delta \to 0. \tag{2.61}$$

We denote the crossing and coalescing times of $(\kappa^1_\delta, ..., \kappa^m_\delta)$ by $\{T^{i,j}_{\delta,\alpha}\}_{1\leq i,j\leq m}$. Since $\{T^{i,j}_{\delta,\alpha}\}_{1\leq i,j\leq m}$ are continuous mappings from $(\overline{\Pi}^m, d^{*m})$ to $\mathbb{R}$ we get that

$$T^{i,j}_{\delta,\alpha} \longrightarrow \tau^{i,j} \quad \text{almost surely}, \quad \text{as } \delta \to 0, \tag{2.62}$$

where $\tau^{i,j}$ is the first time that the paths of $\mathcal{B}_i$ and $\mathcal{B}_j$ cross. Since all these times are distinct a. s. and hence

$$|T^{l,i}_{\delta,\alpha} - T^{i,j}_{\delta,\alpha}| > 0 \quad \text{almost surely}, \quad \text{as } \delta \to 0, \tag{2.63}$$

whereas

$$\sup_{1\leq i<j\leq m} |T^{i,j}_\alpha - T^{i,j}_\beta| \longrightarrow 0 \quad \text{in probability}, \tag{2.64}$$

by arguments similar to the ones we used to prove (2.59). Hence the probability of a "wrong" ($\alpha-$)coalescing event to occur tends to 0.

The second possibility for the event in (2.58) to occur is that there is "too much" time between the crossing and the coalescence of two paths, without any other interactions with additional paths. "Too much" time means there is a positive probability that two random walks need more than $\sim \delta^{-2}\varepsilon^2$ steps to coalesce after their paths crossed, for some $\varepsilon > 0$. Therefore the random walks have positive probability to reach distance $\delta^{-1}\varepsilon$ before they finally coincide. According to the case $m = 2$ the probability of this event to occur tends to zero as $\delta$ tends to zero. This proves (2.58) for $m > 2$.

Step 3: Verification of $(I_1)$

This step is just putting together the previous steps:
Since $(\kappa^1_{\delta,\alpha}, ..., \kappa^m_{\delta,\alpha})$ converges in distribution to Brownian motions $(\mathcal{B}^1_\alpha, ..., \mathcal{B}^m_\alpha)$ starting from $y_1, ..., y_m$ by step 1 and the distance between $((\kappa^1_{\delta,\alpha}, y_{\delta,1}), ..., (\kappa^m_{\delta,\alpha}, y_{\delta,m}))$ and $(\kappa^1_{\delta,\beta}, ..., \kappa^m_{\delta,\beta})$ converges to zero in probability by step 2, we get that $(\kappa^1_{\delta,\beta}, ..., \kappa^m_{\delta,\beta})$ and therefore $(\pi^1_{\delta,\beta}, ..., \pi^m_{\delta,\beta})$ converges in distribution to $(\mathcal{B}^1_\alpha, ..., \mathcal{B}^m_\alpha)$.

### Checking condition $(T_1)$

Let $A^+_{t,u}(x_0, t_0)$ be the event that $K \in \mathcal{H}$ contains a path touching both $R(x_0, t_0, u, t)$ and the right boundary of the bigger rectangle $R(x_0, t_0, 20u, 2t)$. Similarly, we define $A^-_{t,u}(x_0, t_0)$ as the event that the path hits the left boundary of the bigger rectangle. If a variable is diffusively scaled we will add "$\sim$" to it, where $\tilde{t} = t\delta^{-2}$ if $t$ is a time variable and $\tilde{x} = \sigma x \delta^{-1}$ if $x$ is a space-variable. In order to verify condition $(T_1)$ it is enough to show that for every $u \in (0, \infty)$

$$t^{-1} \limsup_{\delta \to 0} \mu_1(A^+_{\tilde{t},\tilde{u}}(0,0)) \longrightarrow 0 \text{ as } t \downarrow 0, \tag{$T_1^+$}$$

where in comparison to condition $(T_1)$ we omitted the supremum because of the spatial invariance of $\mu_1 := \mathbb{P} \circ (S_{\sigma,1}\mathbf{\Gamma})^{-1}$. Condition $(T_1^-)$ is defined analogously with the event $A^+$ replaced by $A^-$. It is enough to show $(T_1^+)$ since condition $(T_1^-)$ can be verified similarly and $(T_1)$ is true if condition $(T_1^+)$ and $(T_1^-)$ hold, since $A_{\tilde{t},\tilde{u}}(x_0, t_0) = A^+_{\tilde{t},\tilde{u}}(x_0, t_0) \cup A^-_{\tilde{t},\tilde{u}}(x_0, t_0)$.
We will show that for every $u > 0$ fixed, $\limsup_{\delta \longrightarrow 0} \mu_1(A^+_{\tilde{t},\tilde{u}}(0,0))$ is in $\mathbf{o}(t)$. Let $u > 0$ and define $x_{1,\delta} := \lfloor 3\tilde{u} \rfloor, x_{2,\delta} := \lfloor 8\tilde{u} \rfloor, x_{3,\delta} := \lfloor 13\tilde{u} \rfloor$ and $x_{4,\delta} := \lfloor 18\tilde{u} \rfloor$. We are interested in the paths $\pi^{x_{i,\delta}} := \pi^{(x_{i,\delta},0)}$, $i = 1,2,3,4$.
We denote by $B_i$ the event that $\pi^{x_{i,\delta}}$ stays within distance $\tilde{u}$ from $x_{i,\delta}$ up to time $2\tilde{t}$. For a fixed $(x, m) \in R(\tilde{u}, \tilde{t}) := R(0, 0, \tilde{u}, \tilde{t})$ denote the times (stopping times) when the random walker $\pi^{(x,m)}$ first exceeds $5\tilde{u}$, $10\tilde{u}$, $15\tilde{u}$ and $20\tilde{u}$ by $\tau_1^{(x,m)}$, $\tau_2^{(x,m)}$, $\tau_3^{(x,m)}$ and $\tau_4^{(x,m)}$. Furthermore, define $\tau_0^{(x,m)} = 0$ and $\tau_5^{(x,m)} = 2\tilde{t}$. Denote by $C_i(x, m)$ the event that $\pi^{(x,m)}$ does not coalesce with $\pi^{x_{i,\delta}}$ before time $2\tilde{t}$. We assume that $\tilde{t} \in \mathbb{Z}$, if not we replace $\tilde{t}$ by $\lceil \tilde{t} \rceil$. We estimate the probability in $(T_1^+)$ in the following way

$$\mu_1\left(A^+_{\tilde{t},\tilde{u}}(0,0)\right) \leq \mu_1\left(\bigcup_{i=1}^4 B_i^c\right) \tag{$*$}$$

$$+ \mu_1\left(\bigcap_{i=1}^4 B_i\, , \bigcup_{(x,m)\in R(\tilde{u},\tilde{t})} \left(\bigcap_{i=1}^4 C_i(x,m)\right) \cap \{\tau_4^{(x,m)} < 2\tilde{t}\}\right). \tag{$**$}$$

We estimate the terms $(*)$ and $(**)$ separately. First note that

$$\mu_1\left(\bigcup_{i=1}^4 B_i^c\right) \leq \mu_1(B_1^c) + \mu_1(B_2^c) + \mu_1(B_3^c) + \mu_1(B_4^c),$$

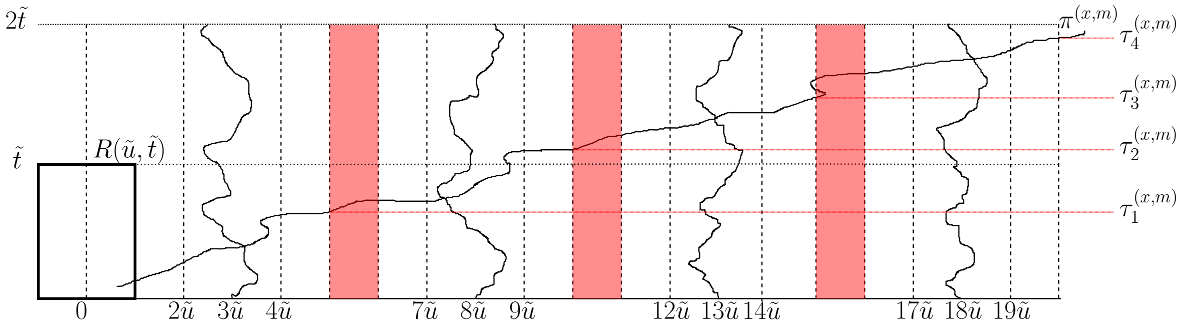



Figure 2.1.: Illustration of the event $A^{+}_{t,u}(x_0,t_0)$.

and that

$$\lim_{\delta\downarrow 0}\mu_1(B_1^c) = P\left(\sup_{s\in[0,t]}|B_s| > u\right) < 4e^{-\frac{u^2}{2t}} \in \mathbf{o}(t) \text{ as } t\downarrow 0,$$

where $B$ is a standard Brownian motion on a probability space $(\Omega,\mathcal{A},P)$. The second term $(**)$ can be estimated by

$$(**) \leq \sum_{\substack{x\in[-\tilde{u},\tilde{u}]\cap\mathbb{Z}\\ m\in[0,\tilde{t}]\cap\mathbb{Z}}} \mu_1\left(\bigcap_{i=1}^{4} B_i, \bigcap_{i=1}^{4} C_i(x,m), \{\tau_4^{(x,m)} < 2\tilde{t}\}\right).$$

From now on we come back to the underlying Markovian structure of the random walks $X^{(x,m)}$ and $X^{(x_{i,\delta},0)}; i=1,..,4$ and focus on their simultaneous regeneration times. Note that the estimates in Lemma 1.12 and Lemma 1.14 hold true for any finite number of random walks. We fix $x\in[-\tilde{u},\tilde{u}]\cap\mathbb{Z}$ and $m\in\left[0,\tilde{t}\right]\cap\mathbb{Z}$. We denote by $\theta_i$ the first regeneration time that $\pi^{(x,m)}(n)-\pi^{(x_{i,\delta})}(n) > 0$. Recognize that on the event in $(**)$ we have $\theta_i < 2\tilde{t}$. Furthermore, let $\widehat{B}_i$ be the event that $\pi^{(x_{i,\delta})}$ stays within distance $\tilde{u}$ of $x_{i,\delta}$ at simultaneous regeneration times up to time $2\tilde{t}$ and denote by $\widehat{C}_i(x,m)$ the event that $\pi^{(x,m)}$ does not coincide with $\pi^{(x_{i,\delta})}$ at simultaneous regeneration times before time $2\tilde{t}$. In analogy to the previous notation let $\hat{\tau}_i^{(x,m)}$ be the first time that a simultaneous regeneration event occurs after the random walk path $\pi^{(x,m)}$ exceeds $(5\cdot i)\tilde{u}$. Only considering the random walks at simultaneous regeneration times, we can estimate a single summand of the sum above by

$$\begin{aligned}
&\mu_1\left(\bigcap_{i=1}^{4} B_i, \bigcap_{i=1}^{4} C_i(x,m), \{\tau_4^{(x,m)} < 2\tilde{t}\}\right)\\
&\leq \mu_1\left(\bigcap_{i=1}^{4} \widehat{B}_i, \bigcap_{i=1}^{4} \widehat{C}_i(x,m), \{\hat{\tau}_4^{(x,m)} < (2+\varepsilon)\tilde{t}\}\right) \qquad (2.65)\\
&\quad + \text{correction term, that sim. reg. after } \tau_4^{(x,m)} \text{ takes “too long”}\\
&\leq \mu_1\left(\bigcap_{i=1}^{4} \widehat{B}_i, \bigcap_{i=1}^{4} \widehat{C}_i(x,m), \{\hat{\tau}_4^{(x,m)} < (2+\varepsilon)\tilde{t}\}, \{T^{sim}_{\theta_4} - T^{sim}_{\theta_4-1} < C\log(\tfrac{1}{\delta})\}\right) + \delta^C\\
&\leq \mu_1\left(\bigcap_{i=1}^{3} \widehat{B}_i, \bigcap_{i=1}^{3} \widehat{C}_i(x,m), \{\hat{\tau}_3^{(x,m)} < (2+\varepsilon)\tilde{t}\}\right)\mathbb{P}_{joint}^{C\log(\frac{1}{\delta})}\left(H(\varepsilon\tilde{u}) < \widehat{T}_{meet}\right) + (2\tilde{t})Ce^{-c\tilde{u}} + \delta^C, \qquad (2.66)
\end{aligned}$$

where $C, c > 0$ are positive constants. The last inequality holds true since by the regeneration structure, every information we gained up to time $T^{sim}_{\theta_4}$ about the "future" of the cluster is that every random walk is placed at a space-time point that is connected to infinity, therefore the "future" of the cluster after time $T^{sim}_{\theta_4}$ can be replaced by some identical copy in which all the points the random walks sit in are connected to infinity. By the coupling argument alluded to before Lemma 1.14 the cluster right to the third red bar can be chosen independently of what is left to the third red bar.
Note that by (2.25), we know (with suitably controlled error term) that for every $0 < \alpha' < \alpha < 1$

$$\begin{aligned}
&\mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(H(\varepsilon\tilde{u}) < \widehat{T}_{meet}\right)\\
&\leq \mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(H(\varepsilon\tilde{u}) < \widehat{T}_{meet}, \widehat{T}_{meet} \geq 2\tilde{t}^{\alpha}\right) + \mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(H(\varepsilon\tilde{u}) < \widehat{T}_{meet}, \widehat{T}_{meet} < 2\tilde{t}^{\alpha}\right)\\
&\leq \mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(\widehat{T}_{meet} \geq 2\tilde{t}^{\alpha}\right) + \mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(H(\varepsilon\tilde{u}) < \widehat{T}_{meet}, \widehat{T}_{meet} < 2\tilde{t}^{\alpha}\right)\\
&\leq \frac{C\delta^{\alpha'}}{\sqrt{t^{\alpha}}} + \mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(H(\varepsilon\tilde{u}) < 2\tilde{t}^{\alpha}\right).
\end{aligned}$$

We define

$$R^{sim}_{\frac{1}{\delta}} := \{T^{sim}_{k} - T^{sim}_{k-1} \leq \log^2(\delta^{-1}) \text{ for all } k \leq \tfrac{2t}{\delta^2}\}.$$

Notice that

$$\begin{aligned}
\mathbb{P}^{C\log(\frac{1}{\delta})}_{joint}\left(H(\varepsilon\tilde{u}) \leq 2\tilde{t}^{\alpha}\right) &\leq \mathbb{P}^{\frac{\varepsilon}{2}\tilde{u}}_{joint}\left(H(\varepsilon\tilde{u}) \leq 2\tilde{t}^{\alpha}\right) \leq \mathbb{P}^{\frac{\varepsilon}{2}\tilde{u}}_{ind}\left(H(\varepsilon\tilde{u}) \leq 2\tilde{t}^{\alpha}\right) + C(2\tilde{t}^{\alpha})e^{-c\delta^{-1}}\\
&\leq \sum_{n=1}^{\lfloor 2\tilde{t}^{\alpha}\rfloor} \mathbb{P}^{\frac{\varepsilon}{2}\tilde{u}}_{ind}\left(\widehat{D}_n > \varepsilon\tilde{u}\right) + Ce^{-\frac{c}{\delta}}\\
&\leq \sum_{n=1}^{\lfloor 2\tilde{t}^{\alpha}\rfloor} \mathbb{P}^{\frac{\varepsilon}{2}\tilde{u}}_{ind}\left(|\widehat{D}_n - \tfrac{\varepsilon\tilde{u}}{2}| > \tfrac{\varepsilon\tilde{u}}{2} \mid R^{sim}_{\frac{1}{\delta}}\right) + Ce^{-c\log^2(\delta)} + Ce^{-c\delta^{-\varepsilon}}\\
&\leq C\sum_{k=1}^{2\tilde{t}^{\alpha}} \exp\left(-\frac{(\varepsilon\tilde{u})^2}{Ck\log^2(\frac{1}{\delta})}\right) + Ce^{-c\log^2(\delta)}\\
&\leq Ct^{\alpha}\delta^{-2\alpha}\exp\left(-\frac{(\varepsilon u)^2\delta^{-2+2\alpha}}{Ct^{2\alpha}\log^2(\frac{1}{\delta})}\right) + Ce^{-c\log^2(\delta)} \in \mathbf{o}(\delta^{\alpha})
\end{aligned}$$

where the second inequality holds true by (1.14) for some $\varepsilon > 0$ and in the fifth inequality we made use of Azuma's inequality, since under $\mathbb{P}^{x}_{ind}$ the process $(\widehat{D}_n)_n$ is a martingale. Repetition of the arguments given in (2.65) leads to

$$\mu_1\left(\bigcap_{i=1}^{4} B_i, \bigcap_{i=1}^{4} C_i(x,m), \tau_4^{(x,m)} < 2\tilde{t}\right) \leq (C\delta^{\alpha'})^4,$$

for some $\alpha' \in (0,1)$, where $\alpha'$ can be chosen close to one. Using this estimation, the term in $(**)$ can be bounded from above by

$$\begin{aligned}
&\mu_1\left(\bigcap_{i=1}^{4} B_i, \exists (x,m) \in R(\tilde{u},\tilde{t}) \text{ s.t.} \bigcap_{i=1}^{4} C_i(x,m) \text{ and } \tau_4^{(x,m)} < 2\tilde{t}\right)\\
&\leq \sum_{x\in[-\tilde{u},\tilde{u}]\cap\mathbb{Z}} \sum_{m\in[0,\tilde{t}]\cap\mathbb{Z}} (C\delta^{\alpha'})^4\\
&\leq (C\delta^{4\alpha'}) \cdot 2\tilde{u}\tilde{t} \leq C(u)t\delta^{\varepsilon} \text{ for some } \varepsilon > 0,
\end{aligned}$$

since $\alpha'$ can be chosen close to one. Hence condition $(T_1^+)$ is satisfied.

### Checking condition $(B_1')$

We fix $t > \beta > 0$ and $t_0, a \in \mathbb{R}$. We want to show that for each $\varepsilon' > 0$ there exists $\varepsilon > 0$ independent of $t, t_0$ and $a$, such that

$$\mu_\delta(\eta(t_0, t; a - \varepsilon, a + \varepsilon) > 1) = \mu_1(\eta(\tilde{t}_0, \tilde{t}; \tilde{a} - \tilde{\varepsilon}, \tilde{a} + \tilde{\varepsilon}) > 1) < \varepsilon'$$

for all $\delta > 0$ sufficiently small. First we assume that $\tilde{t}_0 = n_0 \in \mathbb{Z}$. In this case only paths that start from the interval $[\tilde{a} - \tilde{\varepsilon}, \tilde{a} + \tilde{\varepsilon}] \cap \mathbb{Z}$ at time $n_0$ are counted by $\eta$. Therefore

$$\begin{aligned}
&\mu_1(\eta(n_0, \tilde{t}; \tilde{a} - \tilde{\varepsilon}, \tilde{a} + \tilde{\varepsilon}) > 1) \\
&\leq \sum_{\{x, x+1\} \subset [\tilde{a} - \tilde{\varepsilon}, \tilde{a} + \tilde{\varepsilon}] \cap \mathbb{Z}} \mathbb{P}(\pi^{(x,n_0)}(k) \neq \pi^{(x+1,n_0)}(k) \text{ for all } k \in [n_0, n_0 + \lfloor \tilde{t} \rfloor]).
\end{aligned}$$

By (2.3) of Proposition 2.1 and Lemma A.1 in the appendix, which is the extension of (2.3) for random starting points as defined in (2.54), we get that

$$\mathbb{P}\left(\pi^{(x,n_0)}(k) \neq \pi^{(x+1,n_0)}(k) \text{ for all } k \in [n_0, n_0 + \lfloor \tilde{t} \rfloor]\right) \leq \frac{C}{\sqrt{\tilde{t}}},$$

for some $C > 0$. Hence

$$\mu_1(\eta(n_0, \tilde{t}; \tilde{a} - \tilde{\varepsilon}, \tilde{a} + \tilde{\varepsilon}) > 1) \leq \frac{2\tilde{\varepsilon}C}{\sqrt{\tilde{t}}} \leq \frac{2\sigma\varepsilon C}{\sqrt{t}} \leq \frac{2\sigma\varepsilon C}{\sqrt{\beta}},$$

which is smaller than $\varepsilon'$ if $\varepsilon < \frac{\varepsilon'\sqrt{\beta}}{2\sigma C}$.

If $\tilde{t}_0 \in (n_0, n_0 + 1)$ for some $n_0 \in \mathbb{N}$, it is enough to show that $\mu_1(\eta(n_0, \tilde{t}; \tilde{a} - 2\tilde{\varepsilon}, \tilde{a} + 2\tilde{\varepsilon}) > 1) < \varepsilon'$, which is true by similar estimates as above.

### Checking condition $(E_1')$

In order to verify condition $(E_1')$ we need to prove a statement similar to Lemma 3.5.2 in [Sun05] which is formulated in Lemma 2.15 below. This can be done by adapting Lemma 2.0.7 in [Sun05] to our case (see Lemma 2.14 below). The rest of the proof follows by more general results, proved by Sun and does not need any adaptation.

**Lemma 2.14.** *Remember that* $\mathbf{K}$ *was defined as the collection of piecewise constant random walk paths. For* $A \subset \mathbb{Z}$ *and* $m, n \in \mathbb{N}$, $m > n$, *we define*

$$\mathbf{K}_m^{A,n} := \{\kappa^{(x,n)}(m) : x \in A\}.$$

*If* $n = 0$ *we simply write* $\mathbf{K}_m^A := \mathbf{K}_m^{A,0}$. *Then*

$$p_m := \mathbb{P}\left(0 \in \mathbf{K}_m^{\mathbb{Z}}\right) \leq \frac{C}{\sqrt{m}},$$

*for some constant* $C$ *independent of* $m$.

**Proof:** Let $B_M := [0, M-1] \cap \mathbb{Z}$ and in order to simplify notation define

$$\mathbf{K}_m^A(x) := \begin{cases} 1, & \text{if } (x,m) \in \mathbf{K}_m^A \\ 0, & \text{otherwise,} \end{cases}$$

for some $A \subset \mathbb{Z}$. Using translation invariance of $\mathbb{P}$ we obtain

$$e_m(B_M) := \mathbb{E}[|\mathbf{K}_m^{\mathbb{Z}} \cap B_M|] = \mathbb{E}\left[\sum_{x \in B_M} \mathbf{K}_m^{\mathbb{Z}}(x)\right] = \sum_{x \in B_M} \mathbb{E}\left[\mathbf{K}_m^{\mathbb{Z}}(x)\right] = p_m \cdot M,$$

where $e_m(B)$ can be estimated by

$$e_m(B_M) \leq \sum_{k \in \mathbb{Z}} \mathbb{E}[|\mathbf{K}_m^{B_M + kM} \cap B_M|] = \sum_{k \in \mathbb{Z}} \mathbb{E}[|\mathbf{K}_m^{B_M} \cap (B_M + kM)|] = \mathbb{E}[|\mathbf{K}_m^{B_M}|].$$

Recognize that the difference $M - |\mathbf{K}_m^{B_M}|$ is larger than the number of nearest neighbour pairs that coalesced before time $m$. Using translation invariance of $\mathbb{P}$ we get that

$$\begin{aligned} \mathbb{E}[M - |\mathbf{K}_m^{B_M}|] &\geq \sum_{x=0}^{M-2} \mathbb{E}[\mathbb{1}_{\{\kappa^{(x,0)}(t) = \kappa^{(x+1,0)}(t) \text{ for some } t \leq m\}}] \\ &= (M-1)\mathbb{P}[\kappa^{(0,0)}(t) = \kappa^{(1,0)}(t) \text{ for some } t \leq m]. \end{aligned}$$

By (2.3) of Proposition 2.1 and Lemma A.1 in the appendix, we obtain

$$\begin{aligned} \mathbb{E}[|\mathbf{K}_m^{B_M}|] &\leq M - (M-1)\mathbb{P}[\kappa^{(0,0)}(t) = \kappa^{(1,0)}(t) \text{ for some } t \leq m] \\ &\leq M - (M-1)\left(1 - \frac{C}{\sqrt{m}}\right) \\ &< 1 + M\frac{C}{\sqrt{m}}, \end{aligned}$$

and therefore

$$p_m < \frac{1}{M} + \frac{C}{\sqrt{m}}.$$

Using the fact that $M$ can be chosen arbitrarily large, we get that

$$p_m \leq \frac{C}{\sqrt{m}}.$$

□

Now we are ready to prove our analogue of [Sun05, Lemma 3.5.2]. Recall the definition of $\mathcal{X}_\delta^{t_0^-}$ within the comments after Theorem 2.11. Also recall that $(T_1)$ is a sufficient condition for the family $(\{\mathcal{X}_\delta\}_{\delta>0})_\delta$ to be tight, hence let $\mathcal{Z}_{t_0}$ be a subsequential limit of $\mathcal{X}_\delta^{t_0^-}$, defined some probability space $(\Omega, \mathcal{A}, P)$, where $\mathcal{X}_\delta := S_{\sigma,\delta}\mathbf{\Gamma}$.

**Lemma 2.15.** *The intersection of paths in $\mathcal{Z}_{t_0}$ with the line $\{t_0+\varepsilon\} \times \mathbb{R}$ is almost surely locally finite.*

**Proof** Let $\mathcal{Z}_{t_0}$ be the weak limit of a sequence $(\mathcal{X}_{\delta_n}^{t_0^-})_n$. We denote the diffusively scaled piecewise constant paths that start before or at time $t_0$ by $\mathcal{Y}_{\delta_n}^{t_0^-}$, where $\mathcal{Y}_\delta := S_{\sigma,\delta}\mathbf{K}$. Remember that $\mathbf{K}$ is the collection of all piecewise constant paths.

If we denote the space of compact subsets of $(R_c^2, \rho)$ by $(\mathcal{P}, \rho_{\mathcal{P}})$, where $\rho_{\mathcal{P}}$ is the induced Hausdorff metric and consider $\mathcal{Y}_{\delta_n}^{t_0^-}(t_0+\varepsilon)$ and $\mathcal{X}_{\delta_n}^{t_0^-}(t_0+\varepsilon)$ as $(\mathcal{P}, \rho_{\mathcal{P}})$-valued random variables, we get that

$$\rho_{\mathcal{P}}\left(\mathcal{X}_{\delta_n}^{t_0^-}(t_0+\varepsilon), \mathcal{Y}_{\delta_n}^{t_0^-}(t_0+\varepsilon)\right) \longrightarrow 0 \quad \text{as} \quad n \to \infty \tag{2.67}$$

in probability. Therefore $\mathcal{Y}_{\delta_n}^{t_0^-}(t_0+\varepsilon)$ also converges weakly to $\mathcal{Z}_{t_0}(t_0+\varepsilon)$ as $(\mathcal{P}, \rho_{\mathcal{P}})$ valued random variables. Since for all $a, b \in \mathbb{R},\ a < b$ the set

$$\{K \in (\mathcal{P}, \rho_{\mathcal{P}}) : |K \cap (a,b) \times \mathbb{R}| \geq k\}$$

is an open set in $(\mathcal{P}, \rho_{\mathcal{P}})$, we get that

$$\begin{aligned}
E_P[|\mathcal{Z}_{t_0}(t_0+\varepsilon) \cap (a,b) \times \mathbb{R}|] &= \sum_{k=1}^{\infty} P[|\mathcal{Z}_{t_0}(t_0+\varepsilon) \cap (a,b) \times \mathbb{R}| \geq k] \\
&\leq \sum_{k=1}^{\infty} \liminf_{n \longrightarrow \infty} \mathbb{P}[|\mathcal{Y}_{\delta_n}^{t_0^-}(t_0+\varepsilon) \cap (a,b) \times \mathbb{R}| \geq k] \\
&\leq \liminf_{n \longrightarrow \infty} \mathbb{E}[|\mathcal{Y}_{\delta_n}^{t_0^-}(t_0+\varepsilon) \cap (a,b) \times \mathbb{R}|] \\
&\leq \frac{C(b-a)}{\sqrt{\varepsilon}},
\end{aligned}$$

where the last inequality holds true by Lemma 2.14, since

$$\begin{aligned}
&\mathbb{E}[|\mathcal{Y}_{\delta}^{t_0^-}(t_0+\varepsilon) \cap (a,b) \times \mathbb{R}|] \\
&\leq \mathbb{E}\Bigg[\sum_{x \in (\tilde{a}, \tilde{b}) \cap \mathbb{Z}} \mathbf{K}_{\tilde{\varepsilon}}^{\mathbb{Z}}(x)\Bigg] \leq \sum_{x \in (\tilde{a}, \tilde{b}) \cap \mathbb{Z}} \mathbb{E}[\mathbf{K}_{t_0+\tilde{\varepsilon}}^{\mathbb{Z}}(x)] \leq \frac{C(\tilde{b}-\tilde{a})}{\sqrt{\tilde{\varepsilon}}} \leq \frac{C(b-a)}{\sqrt{\varepsilon}}.
\end{aligned}$$

□

Condition $(E_1')$ then follows by Lemma 2.15 and [Sun05, Lemma 3.5.3]. For completeness' sake [Sun05, Lemma 3.5.3] is given below. Recall the definition of $\mathcal{X}^{s_T}$ in Notation 2.12.

**Lemma 2.16. (siehe [Sun05, Lemma 3.5.3])**
*For any $\varepsilon > 0$, $\mathcal{Z}_{t_0}^{(t_0+\varepsilon)_T}$, the set of paths in $\mathcal{Z}_{t_0}$ truncated before time $t_0+\varepsilon$, is distributed as $\mathcal{B}^{\mathcal{Z}_{t_0}(t_0+\varepsilon)}$, i.e., coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0+\varepsilon) \subset \mathbb{R}^2$.*

**Verification of $(E_1')$:**
Notice that $\mu' := \mathcal{L}(\mathcal{B}^{\mathcal{Z}_{t_0}(t_0+\varepsilon)}|P) \leq \mathcal{L}(\mathcal{W}|P) =: \mu$ which means that for every bounded measurable function $f$ with

$$f(K) \leq f(K'), \quad \text{for every } K, K' \in \mathcal{H}, \text{ with } K \subset K'$$

we have

$$\int f d\mu' \leq \int f d\mu.$$

Hence we conclude

$$\begin{aligned}
E[\eta_{\mathcal{Z}_{t_0}}(t_0, t; a, b)] &= E[\eta_{\mathcal{Z}_{t_0}^{(t_0+\varepsilon)_T}}(t_0+\varepsilon, t-e; a, b)] \\
&= E[\eta_{\mathcal{B}^{\mathcal{Z}_{t_0}(t_0+\varepsilon)}}(t_0+\varepsilon, t-e; a, b)] \\
&\leq E[\eta_{\mathcal{W}}(t_0+\varepsilon, t-e; a, b)] = \frac{b-a}{\sqrt{\pi(t-\varepsilon)}},
\end{aligned}$$

for every $\varepsilon \in (0, t)$ which implies $(E_1')$.

# CHAPTER 3

## Comparison between annealed and quenched hitting probabilities

In this chapter we focus on the difference between quenched and annealed probabilities of hitting boxes with different side length. By the quenched central limit theorem given by Birkner et al. in [BČDG13, Theorem 1.1], we know that

$$\begin{aligned}
&\left|E_\omega^z\left[f\left(X_n/\sqrt{n}\right)\right] - \mathbb{E}^z\left[f\left(X_n/\sqrt{n}\right)\right]\right| \\
&\leq \left|E_\omega^z\left[f\left(X_n/\sqrt{n}\right)\right] - \Phi(f)\right| + \left|\Phi(f) - \mathbb{E}^z\left[f\left(X_n/\sqrt{n}\right)\right]\right| \\
&\longrightarrow 0, \quad \text{as} \quad n \to \infty, \quad \text{for } \mathsf{P}^z\text{-almost all } \omega,
\end{aligned} \tag{3.1}$$

where $f \in C_b(\mathbb{R}^d)$ and $\Phi(f) := \int f(x)\Phi(dx)$ with $\Phi$ a non-trivial, centered isotropic $d$-dimensional normal law.

If we choose $f$ to be a smooth approximation of an indicator function, the quenched CLT tells us that the error between quenched and annealed hitting probability of boxes of side length $\sqrt{n}$ vanishes. Within this chapter we will refine the estimates in (3.1) down to boxes of sub-algebraic side length $e^{\sqrt{\log(N)\log\log(N)}}$, which gives a comparison between quenched and annealed hitting probabilities on a much finer scale than proven by Birkner et al. in [BČDG13]. One of the key ingredients is, as it was in the previous chapter, the regeneration structure of the random walks. The techniques within the proofs below have been used by Berger, Cohen and Rosenthal in [BCR16] to prove a quenched local central limit theorem (qLCLT) for random walks in an i.i.d. and uniformly elliptic environment of dimension $d \geq 4$. Uniform ellipticity means that there exists a uniform positive lower bound on the transition probabilities of nearest neighbour jumps. This condition is violated in our case. We have been able to work out the proofs for $d \geq 3$, since we focus on a directed random walk in a dynamic random environment and hence we have an "additional" time component. The next step towards proving a qLCLT would be an estimate on hitting probabilities of constant box size which would give us a coupling between quenched and annealed probability measures. This coupling can be used to prove the existence of a probability measure on the set of environments, which is invariant with respect to the point of view of the particle and also absolutely continuous with respect to the original environmental measure $\mathsf{P}^z(\,\cdot\,)$ (see Definition 1.11). In Remark 3.29 we will point out some problems that arise within the proof of Theorem 3.28 and which prevent us from getting an analogue of (3.1) for constant box size. Up to now, we have no idea how to solve them.

One of the main tools that we use within this chapter is the environmental exposure procedure originally developed by Bolthausen and Snitzman in [BS02] for i.i.d. environments and which we need to adapt to our case. We also want to point to Corollary 3.25 which shows how a qCLT for dimension $d \geq 3$ can be derived from the estimates between quenched and annealed hitting probabilities given below. We start with a section on useful notation and Lemmas, whereas the rest of the chapter is dedicated to decrease the box size within the estimates on the hitting probabilities.

## 3.1. Useful notation and general results

**Definition 3.1.**
Define an order relation "$\prec$" on

$$\mathcal{P}(N) := \left(\mathbb{Z}^d \times \mathbb{Z}\right) \cap \left([-N\log^3(N), N\log^3(N)]^d \times [0, N^2 + \log^3(N)]\right),$$

by ordering the sites in $\mathcal{P}(N)$ increasing in time and then lexicographically in each time-layer. Let $(z_k)_{k\geq 1} := \left(z_k^{(N)}\right)_{k\geq 1}$ be an "increasing" enumeration of all sites in $\mathcal{P}(N)$, which means that $z_k \prec z_{k+1}$ for all $k$. We usually denote by $y_k$ the space component and by $m_k$ the time component of $z_k$.
Furthermore, define

$$\widetilde{\mathcal{P}}(N) := \left(\mathbb{Z}^d \times \mathbb{Z}\right) \cap \left([-\tfrac{1}{3}N\log^3(N), \tfrac{1}{3}N\log^3(N)]^d \times [0, \tfrac{1}{3}N^2]\right), \tag{3.2}$$

$$\mathbf{I}_k := \mathbf{I}_k(N) := \{z_n : n \leq k\}, \tag{3.3}$$

$$\mathbf{O}_k := \mathbf{O}_k(N) := \left(\mathbb{Z}^d \times \mathbb{Z}\right) \setminus \mathbf{I}_k, \tag{3.4}$$

$$\mathbf{L}_k := \mathbf{L}_k(N) := \{(y,m) : \ m < m_k - \log^2(N)\} \cap \mathcal{P}(N), \tag{3.5}$$

$$\partial^+\mathbf{I}_k := \partial^+\mathbf{I}_k(N) := \{(y,m) \in \mathbf{I}_k : m = \max\{n : (y,n) \in \mathbf{I}_k\}\}, \tag{3.6}$$

$$\mathcal{F}_k := \mathcal{F}_k^{\mathbf{i}}(N) := \sigma(\omega(z_n) : n \leq k), \quad \mathcal{F}_0 := \{\Omega, \emptyset\}, \tag{3.7}$$

$$\mathcal{F}_k^{\mathbf{o}} := \mathcal{F}_k^{\mathbf{o}}(N) := \sigma(\omega(z) : z \in \mathbf{O}_k), \quad \mathcal{F}_0^{\mathbf{o}} := \{\Omega, \emptyset\}, \tag{3.8}$$

$$A_k(\omega|_{\mathbf{I}_k}) := A_k(\omega, N) := \{\omega' : \omega'|_{\mathbf{I}_k} = \omega|_{\mathbf{I}_k}\}. \tag{3.9}$$

These definitions are needed to simplify notation within the proofs below, see Figure 3.1. We will write $(\omega|_{\mathbf{I}_k}, \omega'|_{\mathbf{O}_k})$ for an element $\tilde{w} \in \Omega$ such that $\tilde{\omega}|_{\mathbf{I}_k} = \omega|_{\mathbf{I}_k}$ and $\tilde{\omega}|_{\mathbf{O}_k} = \omega'|_{\mathbf{O}_k}$. The notation $(\omega|_{\mathbf{I}_{k-1}}, 1, \omega'|_{\mathbf{O}_k})$ should also be clear from that point of view.

**Lemma 3.2.** *Let $\mu$ be a probability measure on a measurable space $(\Omega, \mathcal{A})$. Choose $\varepsilon > 0$ and assume there exists $A \in \mathcal{A}$ such that $\mu(A) > 1 - \varepsilon$. Then*

$$d_{TV}(\mu, \mu(\,\cdot\,|A)) \leq \mathcal{O}(\varepsilon).$$

**Proof:**

$$\begin{aligned} d_{TV}(\mu, \mu(\,\cdot\,|A)) &= \sup_{B\in\mathcal{A}} |\mu(B) - \mu(B|A)| = \sup_{B\in\mathcal{A}} \left|\mu(B) - \frac{\mu(B\cap A)}{\mu(A)}\right| \\ &\leq \sup_{B\in\mathcal{A}} \left|\mu(B) - \frac{\mu(B)}{\mu(A)}\right| + \frac{\mu(A^c)}{\mu(A)} \leq \sup_{B\in\mathcal{A}} \left|\frac{\varepsilon\mu(B)}{\mu(A)}\right| + \frac{\varepsilon}{1-\varepsilon} \\ &\leq \frac{2\varepsilon}{1-\varepsilon} = \mathcal{O}(\varepsilon). \end{aligned}$$

□

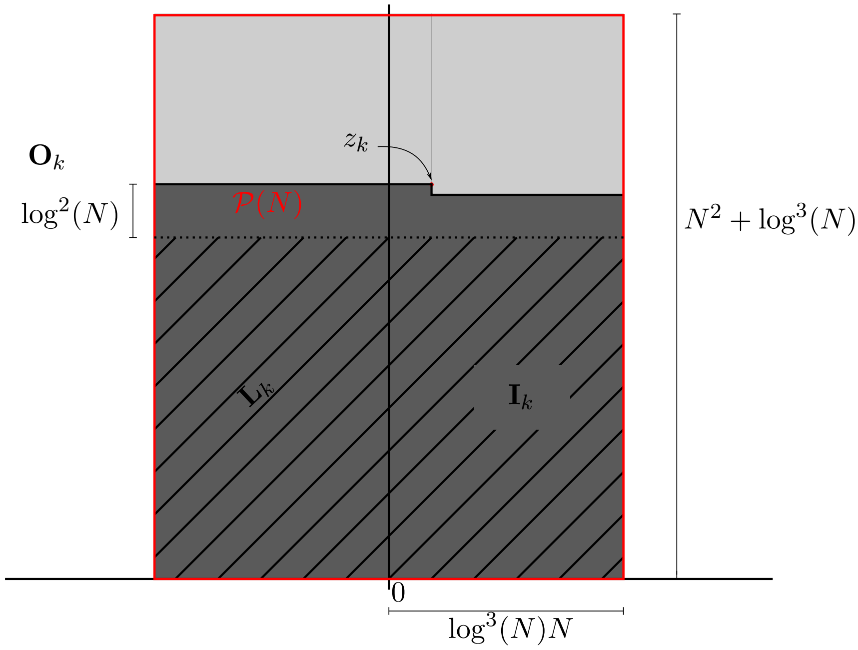



Figure 3.1.: Figure shows the subsets of $\mathbb{Z}^d \times \mathbb{Z}$ defined above. $\mathcal{P}(N)$ is surrounded by a red frame. $\mathbf{O}_k$ is the union of the light grey and white area. The area of $\mathbf{L}_k$ is hatched. $\mathbf{I}_k$ is painted in dark grey.

In this chapter the positive constants $C, c > 0$ will be used in the way described in Remark 1.1. They are only allowed to depend on the space dimension $d$ and the success probability $p$ of the Bernoulli random variables $(\omega(x,n))_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}}$.

**Lemma 3.3.** *Let $z = (y,m) \in \mathcal{P}(N)$ and $(\eta_n^z)_{n\geq m}$ be the time discrete version of the contact process defined in Definition 1.2. There exists $C, c, \rho > 0$ such that*

$$\mathbb{P}\left((B_z)^c \cup \left(B_z \cap \{|\eta_n^z| \geq \rho \log^2(N)\}\right)\right) \geq 1 - CN^{-c\log(N)}, \tag{3.10}$$

*for $n$ chosen arbitrarily such that $n - m \geq \log^2(N)$ and $N$ chosen large enough.*

**Proof of Lemma 3.3:** We choose $z = (y,m) \in \mathbb{Z}^d \times \mathbb{Z}$ arbitrarily. Remember the definition of $(H_n^{(y,m)})_{n\geq m}$ and $(K_n^{(y,m)})_{n\geq m}$ in (1.19) and (1.20). By a discrete time version of the shape theorem given in Lemma 1.6 and Lemma 1.7, we know that there exists a convex subset $U \subset \mathbb{R}^d$ and constants $\varepsilon' = \varepsilon'(U), C, c > 0$, such that

$$\mathsf{P}^{(y,m)}\left((y + (1-\varepsilon')(n-m)\cdot U) \subset (H_n^{(y,m)} \cap K_n^{(y,m)})\right) \geq 1 - CN^{-c\log(N)}$$

for all $n \geq m + \log^2(N)$, where $N$ is chosen sufficiently large. Lemma 1.4 together with (lower) large deviation estimates for the particle density in the upper invariant measure of the discrete time contact process yield the existence of $C, c, \rho > 0$ such that

$$\mathsf{P}^{(y,m)}\left(|\eta_n^{(y,m)}| \geq \rho\log^2(N)\right) \geq 1 - CN^{-c\log(N)}$$

for all $n \geq m + \log^2(N)$. We conclude that

$$\begin{aligned}\mathbb{P}\left(B_{(y,m)} \cap \{\eta_n^{(y,m)} \geq \rho\log^2(N)\}\right) &\geq \mathbb{P}(B_{(y,m)}) - CN^{-c\log(N)}\mathbb{P}(B_{(y,m)})\\ &\geq \mathbb{P}(B_{(y,m)}) - CN^{-c\log(N)}\mathbb{P}(B_0)\\ &\geq \mathbb{P}(B_{(y,m)}) - CN^{-c\log(N)},\end{aligned}$$

by translation invariance of $\mathbb{P}$.

□

**Corollary 3.4.** *Lemma 3.3 together with* (1.17) *implies the existence of* $c, C, \rho > 0$ *such that*

$$\mathbb{P}\left(\bigcap_{(y,m)\in\mathcal{P}(N)} \left((B_{(y,m)})^c \cap \{l(y,m) \leq \log^2(N) - 2\}\right) \cup \left(B_{(y,m)} \cap \{|\eta^{(y,m)}_{m+\log^2(N)}| \geq \rho\log^2(N)\}\right)\right)$$
$$\geq 1 - CN^{-c\log(N)},$$

*where $l(y,m)$ denotes the length of the longest open path starting from $(y,m)$. In the following we refer to the set above as $D(N)$.*

**Remark 3.5.** Recall the definition of $(T_n)_{n\geq 0}$ in (1.32). Define

$$R_N := R_N(X) := \{T_k - T_{k-1} \leq \log^2(N) \text{ for all } k \leq N^2\}. \tag{3.11}$$

By (1.34) we know that

$$\mathbb{P}^z(R_N) \geq 1 - CN^{-c\log(N)}.$$

**Lemma 3.6.** *Let $z = (y,m) \in \widetilde{\mathcal{P}}(N)$ and $\frac{N^2}{2} \leq n \leq N^2$. There exist constants $C, c > 0$ such that*

$$\mathbb{P}^z\left(\|X_n - y\| \geq \sqrt{n}\log^3(N)\right) \leq CN^{-c\log(N)}. \tag{3.12}$$

*Additionally, let $Q(z,N)$ be the event that*

$$P^z_\omega\left(\|X_n - y\| \geq \sqrt{n}\log^3(N)\right) \leq CN^{-\frac{c}{2}\log(N)}, \tag{3.13}$$

*then $\mathbb{P}^z(Q(z,N)) \geq 1 - CN^{-\frac{c}{2}\log(N)}$.*

**Remark 3.7.** Notice that by Lemma 3.6 a random walk starting from $z = (y,m) \in \widetilde{\mathcal{P}}(N)$ stays within $\mathcal{P}(N)$ up to time $N^2$ with *high probability*. An event is said to occur with "high probability", if the probability of the complement decays super-algebraically in $N$.

**Proof of Lemma 3.6:** We prove the Lemma for $z = (\mathbf{0}, 0)$. Conditioned on the event $R_N$, that the time between two regenerations up to time $N^2$ is at most $\log^2(N)$, we get that

$$\begin{aligned}\mathbb{P}^{(\mathbf{0},0)}(\|X_n\| \geq \sqrt{n}\log^3(N)) &\leq \mathbb{P}^{(\mathbf{0},0)}(\|X_n\| \geq \sqrt{n}\log^3(N)|R_N) + CN^{-c\log(N)}\\ &\leq \mathbb{P}^{(\mathbf{0},0)}(\exists k \leq n : \|X_{T_k}\| \geq \frac{1}{2}\sqrt{n}\log^3(N)|R_N) + CN^{-c\log(N)}\\ &\leq \sum_{k=1}^n \mathbb{P}^{(\mathbf{0},0)}(\|X_{T_k}\| \geq \frac{1}{2}\sqrt{n}\log^3(N)|R_N) + CN^{-c\log(N)}\\ &\leq d\sum_{k=1}^n \exp\left(-\frac{Cn\log^6(N)}{4k\log^4(N)}\right) + CN^{-c\log(N)}\\ &\leq CN^{-c\log(N)},\end{aligned}$$

where the fourth inequality holds true by Azuma's inequality applied to each coordinate.
For the second inequality note that on $R_N$ it is impossible for the random walk to leave the box $[-\sqrt{n}\log^3(N), \sqrt{n}\log^3(N)]^d$ between two regeneration times and then be inside $[-\frac{1}{2}\sqrt{n}\log^3(N), \frac{1}{2}\sqrt{n}\log^3(N)]^d$ when the next regeneration occurs.
We turn to the proof of (3.13). By (3.12) we know that $\mathbb{P}^{(\mathbf{0},0)}\left(\|X_n\| \geq \sqrt{n}\log^3(N)\right) \leq CN^{-c\log(N)}$. Hence the Markov inequality yields

$$
\begin{aligned}
&\mathsf{P}^{(\mathbf{0},0)}\left(\left\{\omega \in \Omega : P_\omega^{(\mathbf{0},0)}\left(\|X_n\| \geq \sqrt{n}\log^3(N)\right) \geq \sqrt{CN^{-c\log(N)}}\right\}\right) \\
&\leq \frac{\mathsf{E}^{(\mathbf{0},0)}\left[P_\omega^{(\mathbf{0},0)}\left(\|X_n\| \geq \sqrt{n}\log^3(N)\right)\right]}{\sqrt{CN^{-c\log(N)}}} \\
&\leq CN^{-\frac{c}{2}\log(N)},
\end{aligned}
$$

which proves (3.13).

□

As required within the next Lemma, let $z = (y, m) \in \widetilde{\mathcal{P}}(N)$ and $z_k = (y_k, m_k) \in \mathcal{P}(N)$, $k \in \mathbb{N}$. Lemma 3.8 basically tells us that the law of $\omega$ on $\mathbf{O}_k$ under $\mathsf{P}^z$ is "similar" in total variation distance to some Bernoulli product measure, as long as $(m_k - m) > \log^2(N)$. Recall definitions (3.3)-(3.9).

**Lemma 3.8.** *Fix $z = (y, m) \in \widetilde{\mathcal{P}}(N)$ and choose $k \in \mathbb{N}$ such that $z \prec z_k$. For $\omega$ fixed we define*

$$
V_k^{(z)}(\omega) := \{(x, n) \in \partial^+\mathbf{I}_k : z \xrightarrow{\omega} (x, n)\}.
$$

*Note that $V_k^{(z)}$ is measurable with respect to $\mathcal{F}_k$. Choose $z_k = (y_k, m_k)$ and $\omega_z \in B_z$, then*

$$
\mathsf{P}^z(\omega|_{\mathbf{O}_k} \in \ \cdot\ |\mathcal{F}_k)(\omega_z) =: \kappa_k^z(\omega_z|_{\mathbf{I}_k}, \ \cdot\ ) = \mathbb{P}(\omega|_{\mathbf{O}_k} \in \ \cdot\ |V_k^{(z)}(\omega_z) \xrightarrow{\omega} \infty).
$$

*If $(m_k - m) > \log^2(N)$ there exist constants $C, c > 0$ such that*

$$
\mathsf{P}^z\left(d_{\mathrm{TV}}(\kappa_k^z(\omega|_{\mathbf{I}_k}, \ \cdot\ ), \mathrm{Ber}^{\otimes \mathbf{O}_k}) \leq CN^{-c\log(N)}\right) \geq 1 - CN^{-c\log(N)}.
$$

**Proof:** Note that $\mathbb{P} \circ (\omega|_{\mathbf{O}_k})^{-1}$ is a Bernoulli product measure on $\{0,1\}^{\mathbf{O}_k}$. As required, we fix $z = (y, m) \in \widetilde{\mathcal{P}}(N)$ and $\omega_z \in B_z$. Furthermore, we choose

$$
\widetilde{A} \in \mathcal{F}_k^{\mathbf{O}} \subset \sigma(\omega(x, n) : (x, n) \in \mathbb{Z}^d \times \mathbb{Z})
$$

arbitrarily and define $A := \{\omega|_{\mathbf{O}_k} : \omega \in \widetilde{A}\}$. We get

$$
\begin{aligned}
\kappa_k^z(\omega_z|_{\mathbf{I}_k}, A) &= \mathsf{P}^z(\omega|_{\mathbf{O}_k} \in A\ |\mathcal{F}_k)(\omega_z|_{\mathbf{I}_k}) \\
&= \frac{1}{\mathsf{P}^z(A_k(\omega_z|_{\mathbf{I}_k}))} \int_{A_k(\omega_z|_{\mathbf{I}_k})} \mathbb{1}_A(\omega|_{\mathbf{O}_k})\ \mathsf{P}^z(d\omega) \\
&= \frac{1}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k}) \cap B_z)} \int_{A_k(\omega_z|_{\mathbf{I}_k}) \cap B_z} \mathbb{1}_A(\omega|_{\mathbf{O}_k})\ \mathbb{P}(d\omega) \\
&= \frac{1}{\mathrm{Ber}^{\otimes \mathbf{O}_k}(V_k^{(z)}(\omega_z) \xrightarrow{\vartheta} \infty)} \int_{\{V_k^{(z)}(\omega|_{\mathbf{I}_k}) \xrightarrow{\vartheta} \infty\}} \mathbb{1}_A(\vartheta)\ \mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta).
\end{aligned}
$$

The definition of $A_k(\omega|_{\mathbf{I}_k})$ is given in (3.9). Recall that $\omega_z \in B_z$ is fixed, hence $V_k^{(z)}(\omega_z)$ is fixed subset of $\partial^+\mathbf{I}_k$.
If $z_k = (y_k, m_k),\ k \in \mathbb{N}$ is chosen such that $(m_k - m) > \log^2(N)$, then by Lemma 3.3 there exists $C, c, \rho > 0$ such that

$$\mathsf{P}^z\left(\{\omega : \#V_k^{(z)}(\omega) \geq \rho \log^2(N)\}\right) \geq 1 - CN^{-c\log(N)}. \tag{3.14}$$

Note that for all $\omega_z \in B_z$ with $\#V_k^{(z)}(\omega_z) \geq \rho\log^2(N)$ we have

$$\mathrm{Ber}^{\otimes \mathbf{O}_k}\left(\{\vartheta : V_k^{(z)}(\omega_z) \overset{\vartheta}{\to} \infty\}\right) \geq 1 - CN^{-c\log(N)},$$

since the probability that $\sim \log^2(N)$ points are not connected to infinity is of order $CN^{-c\log(N)}$ by (1.18). The Theorem then follows by (3.14) and Lemma 3.2.

□

**Lemma 3.9. (Annealed derivative estimates)**
*We fix $z = (y, m) \in \widetilde{\mathcal{P}}(N)$. There exists a constant $C > 0$ such that*

*i) for every $x \in [-\log^3(N)N, \log^3(N)N]$ and every $M$ such that $\frac{2}{5}N^2 \leq M \leq N^2$*

$$\mathbb{P}^z(X_M = x) < CN^{-d}$$

*ii) for every $x \in [-\log^3(N)N, \log^3(N)N]$, every $\frac{2}{5}N^2 \leq M \leq N^2$ and every $1 \leq j \leq d$*

$$|\mathbb{P}^{(y,m)}(X_M = x) - \mathbb{P}^{(y+e_j,m)}(X_M = x)| < CN^{-(d+1)}$$

*iii) for every $x \in [-\log^3(N)N, \log^3(N)N]$ and every $\frac{2}{5}N^2 \leq M \leq N^2$*

$$|\mathbb{P}^{(y,m)}(X_M = x) - \mathbb{P}^{(y,m+1)}(X_M = x)| < CN^{-(d+1)}.$$

**Proof:** See Appendix A.2.1.

□

The following Theorem belongs to the class of Azuma type inequalities and is proven by McDiarmid in 1998 (see [HMRAR98, Theorem 3.14]). Some comments on the usefulness of this theorem for proving the estimates on the hitting probabilities are given in Remark 3.14 below, after some further notation is established.

**Theorem 3.10. (McDiarmid (1998))** *Let $\{M_k\}_{k\geq 0}$ be a martingale with respect to some probability measure $P$ and some filtration $(\mathcal{F}_k)_{k\geq 0}$, given by*

$$M_k := \mathbb{E}_P[X|\mathcal{F}_k], \quad \textit{where} \quad M_0 = E_P[X] \quad \textit{and} \quad |X| \leq C \quad P-a.s..$$

*For $1 \leq k \leq n$ define*

$$U_k := \operatorname{esssup}\left(|M_k - M_{k-1}|\middle|\mathcal{F}_{k-1}\right) \quad \textit{and} \quad U := \sum_{k=1}^n U_k^2.$$

*Then*

$$P(|M_n - M_0| \geq \alpha, U \leq c) \leq 2e^{-\frac{\alpha^2}{2c}}.$$

**Proof:** See [HMRAR98, Theorem 3.14] and [BCR16, Theorem 2.13] .

□

## 3.2. Estimates on hitting probabilities for “large” boxes

This section is dedicated to the proof of Proposition 3.11. Some important comments on Proposition 3.11 can be found in Remark 3.12 below. Notice that in Proposition 3.11 the side length of the boxes is of order $N^\theta$ and $\frac{d}{d+1} < \theta \le 1$, whereas $M$ is of order $N^2$. The proof of Proposition 3.11 is split up into two cases treated separately by Lemma 3.15 and Lemma 3.18. The distinction of cases is also illustrated in Figure 3.2. The uniformity of Proposition 3.11 in $M$ and $\Delta$ is necessary for further improvements on the box size, done within the next sections. Recall the definition of $D(N)$ in Corollary 3.4.

**Proposition 3.11.** *Let $d \ge 3$ and $\frac{d}{d+1} < \theta \le 1$. In addition we fix some starting point $z \in \widetilde{\mathcal{P}}(N)$. Let $G_1(z, \theta, N) \subset B_z \cap D(N)$ be the event that for every $\frac{2}{5}N^2 \le M \le N^2$ and every (d-dimensional) cube $\Delta \subset [-N\log^3(N), N\log^3(N)]^d$ of side length $N^\theta$ we have*

$$|P_\omega^z(X_M \in \Delta) - \mathbb{P}^z(X_M \in \Delta)| \le N^{d(\theta-1)}. \tag{3.15}$$

*Then for every $\frac{d}{d+1} < \theta \le 1$ there exist constants $C, c > 0$, independent of $z$, such that*

$$\mathsf{P}^z(G_1(z, \theta, N)) = 1 - CN^{-c\log(N)}, \tag{3.16}$$

*and hence*

$$\begin{aligned} \mathbb{P}\left(\bigcap_{z \in \widetilde{\mathcal{P}}(N)} G_1(z, \theta, N) \cup (B_z)^c\right) &\ge 1 - \sum_{z \in \widetilde{\mathcal{P}}(N)} \mathbb{P}\left(G_1(z, \theta, N))^c \cap B_z\right) \\ &\ge 1 - CN^{-c\log(N)}. \end{aligned}$$

**Remark 3.12.** Proposition 3.11 is not an improvement of the quenched central limit theorem given by Birkner et al. in the sense that Proposition 3.11 already implies the qCLT. It should be considered as a quenched analogue of Lemma 3.9 i) instead, since

$$P_\omega^z(X_M \in \Delta) - \mathbb{P}^z(X_M \in \Delta) \le |P_\omega^z(X_M \in \Delta) - \mathbb{P}^z(X_M \in \Delta)| \le N^{d(\theta-1)}, \tag{3.17}$$

and hence

$$P_\omega^z(X_M \in \Delta) \le CN^{d(\theta-1)} \quad \text{on} \quad G_1(z, \theta, N), \tag{3.18}$$

by Lemma 3.9 i), where parameters are chosen as in Proposition 3.11. This is exactly the sense in which the proposition will be used later on.

As already mentioned within the introduction of this section, we will prepare the proof of Proposition 3.11 by first proving some lemmas. For the rest of the section we fix $z \in \widetilde{\mathcal{P}}(N)$, $\frac{d}{d+1} < \theta \le 1$, $\omega_z \in D(N) \cap B_z$ and $z_k = (m_k, y_k) \in \mathcal{P}(N)$. We define the set $\mathbf{M}_k$ depending on $z_k$ and $N$ as

$$\mathbf{M}_k := \mathbf{M}_k(z_k, N) := \{(y, m) \in \mathcal{P}(N) : 0 \le \|y - y_k\| \le m_k - m \le \log^2(N)\}.$$

See also Figure 3.2. The “bottom” of $\mathbf{M}_k$ is defined as

$$\partial^- \mathbf{M}_k := \{(y, m) \in \mathbf{M}_k : m = \min\{n : (y, n) \in \mathbf{M}_k\}\}.$$

Furthermore, we fix $\frac{2}{5}N^2 \leq M \leq N^2$, $\frac{d}{d+1} < \theta' < \theta$, $v \in \mathbb{Z}^d$ and define $V := \left\lfloor N^{2\theta'} \right\rfloor$. During the rest of the section, we focus on the quantity

$$\begin{aligned}
U_k(\omega_z) &= \operatorname{esssup}\Big( \left| \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k-1}\right] - \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]\right| \Big| \mathcal{F}_{k-1}\Big)(\omega_z) \\
&\leq \operatorname{esssup}\Big( \Big| \mathsf{E}^z\left[ P^z_\omega\left(\{X_{M+V}=v\} \cap \{\mathbf{M}_k \text{ is not visited}\}\right) \middle| \mathcal{F}_{k-1}\right] \\
&\qquad - \mathsf{E}^z\left[ P^z_\omega\left(\{X_{M+V}=v\} \cap \{\mathbf{M}_k \text{ is not visited}\}\right) \middle| \mathcal{F}_{k}\right] \Big| \;\Big|\; \mathcal{F}_{k-1}\Big)(\omega_z) \qquad (*) \\
&\quad + \operatorname{esssup}\Big( \Big| \mathsf{E}^z\left[ P^z_\omega\left(\{X_{M+V}=v\} \cap \{\mathbf{M}_k \text{ is visited}\}\right) \middle| \mathcal{F}_{k-1}\right] \\
&\qquad - \mathsf{E}^z\left[ P^z_\omega\left(\{X_{M+V}=v\} \cap \{\mathbf{M}_k \text{ is visited}\}\right) \middle| \mathcal{F}_{k}\right] \Big| \;\Big|\; \mathcal{F}_{k-1}\Big)(\omega_z). \qquad (**)
\end{aligned}$$

The event we are interested in is illustrated in Figure 3.2. The meaning of the differently coloured areas is the following:
Notice that $\omega_z \in D(N) \cap B_z$ is fixed whereas "$\mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]$" has to be read as the conditional expectation of the random variable $(P^z_\cdot\left(X_{M+V}=v\right))(\omega) = P^z_\omega\left(X_{M+V}=v\right)$. Due to the conditional expectation, one should think of $\omega$ being fixed or *exposed* within the grey area, whereas we average over $\omega$ within the white area conditioned on the fact that $z$ is connected to infinity. Hence "$\mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]$" is in some sense a mixture of quenched and annealed laws. If we focus on a specific realization $U_k(\omega_z)$ of $U_k$, as it is done above, the configuration of $\omega$ within the fixed area coincides with $\omega_z$. Computing the conditional essential supremum of

$$\left| \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k-1}\right] - \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]\right| \tag{3.19}$$

we get an estimate on how the "quenched" hitting probability of $v$ changes if in addition the space-time point $z_k$ of the environment is fixed or exposed. One way in which the process $(U_k)_k$ could also be interpreted path-wise is that at each time $k$ the random real number $U_k(\omega_z)$ equals the distance between $\mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]$ and $\mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k-1}\right]$ in $L^\infty\left(\mathsf{P}^z\left(\,\cdot\,| A_{k-1}(\omega_z|_{\mathbf{I}_{k-1}})\right)\right)$. In detail

$$\begin{aligned}
U_k(\omega_z) &= \operatorname{esssup}\Big( \Big| \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k-1}\right] - \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]\Big| \;\Big| \mathcal{F}_{k-1}\Big)(\omega_z) \\
&= \Big\| \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k-1}\right] - \mathsf{E}^z\left[ P^z_\omega\left(X_{M+V}=v\right) \middle| \mathcal{F}_{k}\right]\Big\|_{L^\infty\left(\mathsf{P}^z(\,\cdot\,|\, A_{k-1}(\omega_z|_{\mathbf{I}_{k-1}}))\right)}.
\end{aligned}$$

In order to prove Proposition 3.11, we expose the environment step by step up to time $M + \log^2(N)$. Hence one should think of $k$ being chosen such that $z_k = (y_k, m_k) \in \mathcal{P}(N)$ where $m_k \leq M + \log^2(N)$.

**Remark 3.13.** Notice that by Lemma 3.6 and mixing properties of the environment (see Lemma 1.14 and Lemma 3.8), the configuration of the environment on

$$\left(\mathbb{Z}^d \times \{x \in \mathbb{Z} : x \in [0, N^2 + \log^3(N)]\}\right) \setminus \mathcal{P}(N) \tag{3.20}$$

has negligible influence on the probability $P^z_\omega\left(X_{M+V}=v\right)$ and therefore does not need to be exposed. "Negligible" means that the set of environments for which $P^z_\omega\left(X_{M+V}=v\right)$ is influenced by a change of the environment on (3.20) has small probability. To be more precise the probability of the set of those environments decays exponentially in $N$ and can therefore be hidden within the complement of $G_1(z, \theta, N)$ (see (3.16)). Further comments on mixing properties of the environment can be found in [Mil16, section 2].

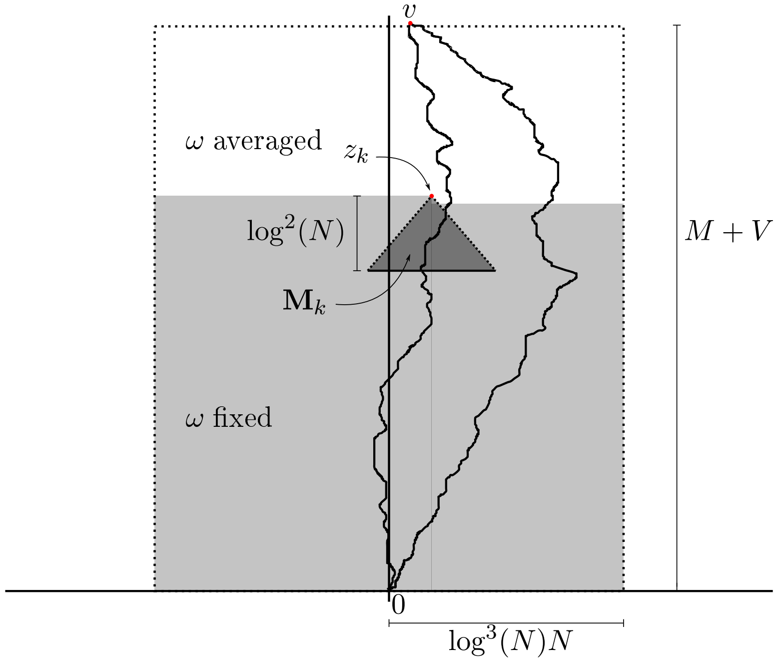



Figure 3.2.: Hitting $v$ at time $M+V$ with and without visiting $\mathbf{M}_k$

**Remark 3.14.** First we want to give a short overview on how the estimates of the difference between the quenched and annealed hitting probabilities will be proved. Roughly speaking, the squared "errors" we make exposing the environment step by step up to a certain time layer need to be summable and "well" bounded. The errors mentioned in the previous sentence are exactly the random variables $U_k$ defined above. The bound on $U = \sum_k U_k^2$ will of course be growing in $N$. This is the point where McDiarmid's inequality (see Theorem 3.10) comes into play. The martingale considered in Theorem 3.10 will be defined as $M_k := \mathsf{E}^z\left[P_\omega^z\left(X_{M+V}=v\right)|\mathcal{F}_k\right]$. Conditioning on $\mathcal{F}_0 = \{\Omega, \emptyset\}$ means "nothing is exposed" and $M_0 = \mathsf{E}^z\left[P_\omega^z\left(X_{M+V}=v\right)|\mathcal{F}_0\right] = \mathbb{P}^z\left(X_{M+V}=v\right)$ equals the annealed law. If on the other hand the environment up to time-layer $M+\log^2(N)$ is "exposed", with high probability $\mathsf{P}^z\left[P_\omega^z\left(X_M \in \,\cdot\,\right)|\mathcal{F}_{k_0}\right] = P_\omega^z\left(X_M \in \,\cdot\,\right)$ by Remark 3.13. In the previous sentence $k_0$ is chosen such that the configuration of the environment in $\mathcal{P}(N)$ up to time to time-layer $M+\log^2(N)$ is measurable with respect to $\mathcal{F}_{k_0}$. Making use of annealed estimates for the last $\sim V$ steps, by McDiarmid's inequality the difference between quenched and annealed hitting probabilities fulfils (3.15) with high probability, if the bound on the sum of squared errors is of the "right" order.

**Lemma 3.15. (The term $(*)$ - $\mathbf{M}_k$ is not visited)**
*If we choose the parameters as described above, there exists constants $C, c > 0$ such that*

$$\begin{aligned} \operatorname{esssup}\Big(\Big|&\mathsf{E}^z\left[P_\omega^z\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is not visited}\}\right)|\mathcal{F}_{k-1}\right] \\ &- \mathsf{E}^z\left[P_\omega^z\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is not visited}\}\right)|\mathcal{F}_k\right]\Big| \,\Big|\mathcal{F}_{k-1}\Big)(\omega_z) \le CN^{-c\log(N)}. \qquad (3.21) \end{aligned}$$

**Proof:** Let $z$, $\omega_z$ and $k$ be as required. It is obvious that (3.21) holds true if $z_k \prec z$ or $z = z_k$, therefore we assume that $z \prec z_k$. We distinguish the cases $m_k - m > \log^2(N)$ and $m_k - m \le \log^2(N)$.

First we consider the case $m_k - m > \log^2(N)$. For $\omega \in B_z$, we define

$$f_k^{\mathrm{nv}}(\omega) := P^z_\omega\left(\{X_{M+V} = v\} \cap \{\mathbf{M}_k \text{ is not visited}\}\right).$$

To be more precise, we can define $f_k^{\mathrm{nv}}$ as

$$\begin{aligned} &f_k^{\mathrm{nv}}\left(\xi_\omega|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right) := P_{\tilde\omega}\left(\{X_{M+V} = v\} \cap \{\mathbf{M}_k \text{ is not visited}\}\right), \\ &\text{for some } \tilde\omega \in \left\{\omega' : \xi_{\omega'}|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k} = \xi_\omega|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right\}, \end{aligned}$$

which means that the quenched probability of hitting the space-time point $(v, M+V)$ and not visiting $\mathbf{M}_k$ in fact only depends on the values of $\xi$ on $(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k$. For the definition of $\xi$ see (1.24). Hence we get that

$$\begin{aligned} &\Big(\mathsf{E}^z\left[P^z_\omega\left(\{X_{M+V} = v\} \cap \{\mathbf{M}_k \text{ is not visited}\}\right)|\mathcal{F}_{k-1}\right] \\ &\quad - \mathsf{E}^z\left[P^z_\omega\left(\{X_{M+V} = v\} \cap \{\mathbf{M}_k \text{ is not visited}\}\right)|\mathcal{F}_k\right]\Big)(\omega_z) \\ &= \int f_k^{\mathrm{nv}}\left(\xi_{(\omega_z|_{\mathbf{I}_{k-1}},\,\vartheta)}\big|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right)\ \kappa^z_{k-1}(\omega_z|_{\mathbf{I}_{k-1}}, d\vartheta) - \int f_k^{\mathrm{nv}}\left(\xi_{(\omega_z|_{\mathbf{I}_k},\,\vartheta')}\big|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right)\ \kappa^z_k(\omega_z|_{\mathbf{I}_k}, d\vartheta') \\ &\le \int f_k^{\mathrm{nv}}\left(\xi_{(\omega_z|_{\mathbf{I}_{k-1}},\,\vartheta)}\big|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right)\ \mathrm{Ber}^{\otimes\mathbf{O}_{k-1}}(d\vartheta) - \int f_k^{\mathrm{nv}}\left(\xi_{(\omega_z|_{\mathbf{I}_k},\,\vartheta')}\big|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right)\ \mathrm{Ber}^{\otimes\mathbf{O}_k}(d\vartheta') \\ &\quad + CN^{-c\log(N)} \\ &\le C\cdot\mathrm{Ber}^{\otimes\mathbf{O}_k}\left(\left\{\vartheta : \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}\big|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k} \ne \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}\big|_{(\mathbb{Z}^d\times\mathbb{Z})\setminus\mathbf{M}_k}\right\}\right) + CN^{-c\log(N)}, \end{aligned}$$

where the first inequality holds true by Lemma 3.8. It is obvious by the way in which $\xi$ is defined, that

$$\xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}\big|_{\{(y,m):\,(y,m)\notin\mathbf{M}_k,\,m\ge m_k-\log^2(N)\}} = \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}\big|_{\{(y,m):\,(y,m)\notin\mathbf{M}_k,\,m\ge m_k-\log^2(N)\}}.$$

Therefore it is enough to bound

$$\begin{aligned} &\mathrm{Ber}^{\otimes\mathbf{O}_k}\left(\left\{\vartheta : \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}\big|_{\{(y,m):\,m<m_k-\log^2(N)\}} \ne \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}\big|_{\{(y,m):\,m<m_k-\log^2(N)\}}\right\}\right) \\ &\quad = \mathrm{Ber}^{\otimes\mathbf{O}_k}\left(\left\{\vartheta :\ \exists(y,m) : m < m_k - \log^2(N),\ \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}(y,m) \ne \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}(y,m)\right\}\right), \end{aligned}$$

Remember the definition of $V_k^{(z)}(\omega_z|_{\mathbf{I}_k})$ in Lemma 3.8. By Remark 3.13 it is enough to focus on

$$\{(y,m) :\ m < m_k - \log^2(N)\} \cap \mathcal{P}(N) = \mathbf{L}_k, \quad \text{see (3.5)}.$$

Since $\omega_z \in D(N)$ (see Corollary 3.4), we know that for all $(y,m) \in \mathbf{L}_k$ for which $\xi_{\omega_z}(y,m) = 1$ we have $\omega_z \in \left\{\eta^{(y,m)}_{m+\log^2(N)} \ge \rho\log^2(N)\right\}$. Hence

$$\begin{aligned} &\mathrm{Ber}^{\otimes\mathbf{O}_k}\left(\left\{\vartheta :\ \exists(y,m) : m < m_k - \log^2(N),\ \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}(y,m) \ne \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}(y,m)\right\}\right) \\ &\le \sum_{(y,m)\in\mathbf{L}_k} \mathrm{Ber}^{\otimes\mathbf{O}_k}\left(\left\{\vartheta : \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}(y,m) \ne \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}(y,m)\right\}\right) \\ &\le \sum_{(y,m)\in\mathbf{L}_k} \mathrm{Ber}^{\otimes\mathbf{O}_k}\Big(\Big\{\vartheta : \xi_{(\omega_z|_{\mathbf{I}_{k-1}},0,\,\vartheta)}(y',m') = 0 \text{ for all } (y',m') \in \eta^{(y,m)}_{m+\log^2(N)} \\ &\qquad\qquad \text{and } \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}(y,m) = \xi_{(\omega_z|_{\mathbf{I}_{k-1}},1,\,\vartheta)}(z_k) = 1\Big\}\Big) \\ &\le \sum_{(y,m)\in\mathbf{L}_k} CN^{-c\log(N)} \\ &\le CN^{-c\log(N)}. \end{aligned} \tag{3.22}$$

This completes the proof for $m_k - m > \log^2(N)$. If $m_k - m \leq \log^2(N)$ the function $f_k^{\mathrm{nv}}(\cdot)$ is positive iff $z \notin \mathbf{M}_k$. But then the value of $f_k^{\mathrm{nv}}(\omega)$ is not changed by changing the value of $\omega$ at $z_k$, which implies that also in this case the Lemma holds true.

$\square$

**Lemma 3.16. (special regeneration point)**
*We fix some $z = (y, m) \in \mathbb{Z}^d$ and define a "special" regeneration point $R$ as*

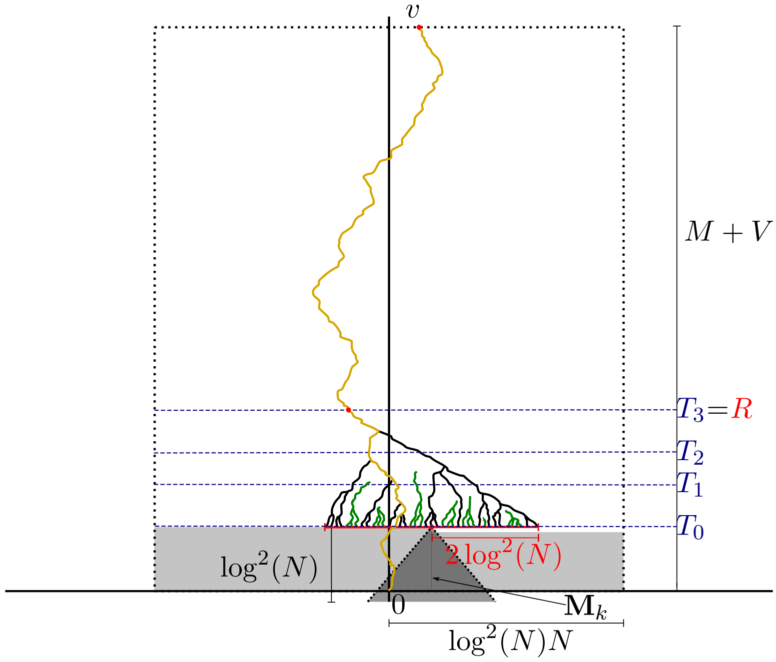
$$\begin{aligned} R(z) := R(y,m,N) := \inf\{T_n : \textit{for all } x \in [y - 2\log^2(N), y + 2\log^2(N)]^d \cap \mathbb{Z}^d \textit{ we have} \\ (x,m) \xrightarrow{\omega} (X_{T_n}, T_n) \quad \textit{if } \xi_\omega(x,m) = 1, \\ l(x,m) < T_n \qquad \textit{if } \xi_\omega(x,m) = 0\}, \end{aligned}$$

*where $(T_k)_{k\geq 0}$ denote the regeneration times of $X$ with respect to $\mathbb{P}^{(x_0,m)}$ for some $x_0 \in [y - 2\log^2(N), y + 2\log^2(N)]^d \cap \mathbb{Z}^d$. As before, $l(x,m)$ denotes the length of the longest directed open path starting from $(x,m)$.*
*Fix $z_k \in \mathcal{P}(N)$. There exist constants $C, c > 0$, independent of $z_k$, such that*

$$\mathbb{P}^{(x_0,m_k)}\Big(R(z_k) \geq m_k + 2\log^{6d+9}(N) \;\big|\; \{\xi(y_k + r, m_k) = \zeta(r)\ \forall r\}\Big) \leq C N^{-c\log(N)} \tag{3.23}$$

*for all $x_0 \in [y_k - 2\log^2(N), y_k + 2\log^2(N)]^d \cap \mathbb{Z}^d$ and $\zeta \in \{0,1\}^{[-2\log^2(N), 2\log^2(N)]^d \cap \mathbb{Z}^d}$ chosen such that $\zeta(x_0 - y_k) = 1$.*

Note that in (3.23) the variable "$r$" ranges over all elements of $[-2\log^2(N), +2\log^2(N)]^d \cap \mathbb{Z}^d$. The regeneration point $R(z_k)$ is illustrated in the following picture:



Before proving Lemma 3.16, we need to prove the following lemma:

**Lemma 3.17.** *As before, we denote by $(\eta_n^{(x,m)})_{n\geq m}$ the discrete time version of the contact process as defined in Definition 1.2. We fix $\zeta \in \{0,1\}^{[-2\log^2(N),2\log^2(N)]^d\cap\mathbb{Z}^d}$, $\zeta \not\equiv 0$ and $z=(y,m)\in\mathcal{P}(N)$ and define*

$$\mathbf{C}(\zeta,z) := \bigcap_{r:\zeta(r)=1} \eta^{(y+r,m)}_{m+\log^{2d+2}(N)}.$$

*Then there exist $C,c,\varepsilon>0$ such that*

$$\mathbb{P}\left(\#\mathbf{C}(\zeta,z)\geq \varepsilon\log^{2d+2}(N)\;\middle|\;\{\xi(y+r,m)=\zeta(r)\;\forall r\}\right)\geq 1-CN^{-c\log(N)}. \tag{3.24}$$

*Moreover, there exist $C,c>0$ such that for any $x_0\in[y-2\log^2(N),y+2\log^2(N)]^d\cap\mathbb{Z}^d$ and any $\zeta\in\{0,1\}^{[-2\log^2(N),2\log^2(N)]^d\cap\mathbb{Z}^d}$ with $\zeta(x_0-y)=1$, we have*

$$\mathbb{P}^{(x_0,m)}\left(\mathbf{C}(\zeta,z)\times\{m+\log^{2d+2}(N)\}\overset{\omega}{\longrightarrow}(X_{m+\log^{6d+9}(N)},m+\log^{6d+9}(N))\;\middle|\;\{\xi(y+r,m)=\zeta(r)\;\forall r\}\right)$$
$$\geq 1-CN^{-c\log(N)}.$$

**Proof:** Fix some $\zeta\in\{0,1\}^{[-2\log^2(N),2\log^2(N)]^d\cap\mathbb{Z}^d}$ and let $x_0\in[y-2\log^2(N),y+2\log^2(N)]^d\cap\mathbb{Z}^d$ be such that $\zeta(x_0-y)=1$. We define

$$J(\zeta):=J(N,\zeta):=\left\{\xi(y_k+r,m_k)=\zeta(r)\;\forall r\in[-2\log^2(N),2\log^2(N)]^d\cap\mathbb{Z}^d\right\}. \tag{3.25}$$

The proof of the Lemma will be separated into three steps.
Step 0: Since we condition on the event $J(\zeta)$, we need to bound the probability for this event to occur from below. Recognize that

$$\begin{aligned}\mathbb{P}(J(\zeta)) &\geq \mathbb{P}(\{\xi(y+r,m+1)=1\;\forall r\in[y-2\log^2(N)+1,y+2\log^2(N)+1]^d\cap\mathbb{Z}^d\})\\ &\quad\cdot\mathbb{P}(\{\omega(y+r,m)=\zeta(r)\;\forall r\in[y-2\log^2(N),y+2\log^2(N)]^d\cap\mathbb{Z}^d\})\\ &\geq \mathbb{P}(B_0)^{(4\log^2(N)+2)^d}\cdot(p(1-p))^{(4\log^2(N)+2)^d}\\ &\geq (p(1-p)\mathbb{P}(B_0))^{\log^{2d+1}(N)},\end{aligned} \tag{3.26}$$

where the second inequality holds true by the FKG-inequality.
Step 1: Let $(x_l)_{l\geq1}$ be an enumeration of the elements

$$\{x\in[y-2\log^2(N),y+2\log^2(N)]^d\cap\mathbb{Z}^d:\zeta(x-y)=1\}.$$

Remember the definition of $(H_n^{(y,m)})_{n\geq m}$ and $(K_n^{(y,m)})_{n\geq m}$ in (1.19) and (1.20). By the discrete time version of the shape theorem given in Lemma 1.6 and Lemma 1.7, we know that there exists a convex subset $U\subset\mathbb{R}^d$ and constants $\varepsilon'=\varepsilon'(U),C,c>0$, such that

$$\mathsf{P}^{(y,m)}\left(\left\{(x_l+(1-\varepsilon')(n-m)\cdot U)\subset(H_n^{(x_l,m)}\cap K_n^{(x_l,m)})\right\}\cap J(\zeta)\right)\geq\mathsf{P}^{(y,m)}(J(\zeta))-Ce^{-c(n-m)}$$

for all $l$ and all $n$ chosen sufficiently large. Additionally choose $\tilde\varepsilon=\tilde\varepsilon(U)>0$ such that

$$\tilde\varepsilon\log^{2d+2}(N)\leq\left|\bigcap_l\left(x_l+(1-\varepsilon')\log^{2d+2}(N)\cdot U\right)\right|$$

for all $N$ sufficiently large. Since by Lemma 1.7 the upper invariant measure of the discrete time contact process has positive density, there exist $C, c > 0$ and $\varepsilon > 0$ such that

$$\mathbb{P}\left(\{\#\mathbf{C}(\zeta, z) \geq \varepsilon \log^{2d+2}(N)\} \cap J(\zeta)\right) \geq \mathbb{P}\left(J(\zeta)\right) - Ce^{-c\log^{2d+2}(N)}.$$

Hence inequality (3.24) follows from (3.26).

Step 2: Making use of (1.18) we get that

$$\begin{aligned}
&\mathbb{P}\left(\{\exists (x,n) \in \mathbf{C}(\zeta, z) :\ \xi(x,n) = 1\} \cap \{\#\mathbf{C}(\zeta, z) \geq \varepsilon \log^{2d+2}(N)\} \cap J(\zeta)\right) \\
&= \mathbb{P}\left(\{\#\mathbf{C}(\zeta, z) \geq \varepsilon \log^{2d+2}(N)\} \cap J(\zeta)\right) \\
&\quad - \mathbb{P}\left(\{\xi(x,n) = 0\ \forall (x,n) \in \mathbf{C}(\zeta, z)\ \} \cap \{\#\mathbf{C}(\zeta, z) \geq \varepsilon \log^{2d+2}(N)\} \cap J(\zeta)\right) \\
&\geq \mathbb{P}\left(J(\zeta)\right) - Ce^{-c\log^{2d+2}(N)} - Ce^{-c\log^{2d+2}(N)}
\end{aligned}$$

for some constants $C, c > 0$. On the event

$$G_1(q) := \{\xi(q, \log^{2d+2}(N) + m) = 1\} \cap \{q \in \mathbf{C}(\zeta, z)\},$$

for some $q \in \mathbb{Z}^d$, the discrete time contact process $\eta^{(q, m+\log^{2d+2}(N))}$ fulfils a shape theorem as described in step 1. Hence we know that there exists a convex subset $U \subset \mathbb{R}^d$ and constants $\varepsilon' = \varepsilon'(U), C, c > 0$, such that

$$\begin{aligned}
&\mathbb{P}\left(\left\{(q + (1-\varepsilon')(n - m - \log^{2d+2}(N)) \cdot U) \subset (H_n^{(q,m+\log^{2d+2}(N))} \cap K_n^{(q,m+\log^{2d+2}(N))})\right\} \cap G_1(q)\right) \\
&\geq \mathbb{P}(G_1(q)) - Ce^{-c\log^{2d+2}(N)}
\end{aligned}$$

for all $n \geq m + 2\log^{2d+2}(N)$, where $N$ is chosen sufficiently large. Define

$$U_n(q) := (q + (1-\varepsilon')n \cdot U).$$

In analogy to (3.11) we define

$$\widetilde{R}_N := \{T_{k+1} - T_k \leq \log^{2d+2}(N) \text{ for all } k < N^2\}. \tag{3.27}$$

Calculations similar to the proof Lemma 3.6 yield

$$\begin{aligned}
&\mathbb{P}^{(x_0,m)}\left(\left\|X_{m+\log^{6d+9}(N)} - x_0\right\| > \log^{6d+8}(N)\right) \\
&\leq \mathbb{P}^{(x_0,m)}\left(\left\|X_{m+\log^{6d+9}(N)} - x_0\right\| > \log^{6d+8}(N) \middle| \widetilde{R}_N\right) + Ce^{-c\log^{2d+2}(N)} \\
&\leq \mathbb{P}^{(x_0,m)}\left(\exists k \leq \log^{6d+9}(N) : \|X_{T_k} - x_0\| > \frac{1}{2}\log^{6d+8}(N) \middle| \widetilde{R}_N\right) + Ce^{-c\log^{2d+2}(N)} \\
&\leq \sum_{k=1}^{\log^{6d+9}(N)} \mathbb{P}^{(x_0,m)}\left(\|X_{T_k} - x_0\| > \frac{1}{2}\log^{6d+8}(N) \middle| \widetilde{R}_N\right) + Ce^{-c\log^{2d+2}(N)} \\
&\leq d \sum_{k=1}^{\log^{6d+9}(N)} \exp\left(-\frac{C\log^{12d+16}(N)}{k\log^{4d+4}(N)}\right) + Ce^{-c\log^{2d+2}(N)} \\
&\leq Ce^{-c\log^{2d+2}(N)},
\end{aligned}$$

which, using the triangle inequality, implies that

$$\mathbb{P}^{(x_0,m)}\left(\left\|X_{m+\log^{6d+9}(N)}-y\right\|>2\log^{6d+8}(N)\right)\leq Ce^{-c\log^{2d+2}(N)}.$$

We focus on the event

$$G_2(q'):=\{X_{m+\log^{6d+9}(N)}=q'\},$$

for some $q'$ such that $\|q'-y\|<2\log^{6d+8}(N)$.
We define $\widetilde{\eta}^{(q',m+\log^{6d+9}(N))}$ as the dual discrete time contact process starting from a single infection $(q',m+\log^{6d+9}(N))$ and evolving backwards in time. Since there exists a "backwards"-path of length $\log^{6d+9}(N)$, the dual process survives with probability greater than $1-Ce^{-c\log^{6d+9}(N)}$, for some $C,c>0$. On the event that $\widetilde{\eta}^{(q',m+\log^{6d+9}(N))}$ survives (which happens with high probability) the process fulfils a shape theorem backwards in time. Let $\widetilde{H}$ and $\widetilde{K}$ be the "backward" analogues of $H$ and $K$ in (1.19) and (1.20). There exist $\widetilde{U}\subset\mathbb{R}^d$ and $\varepsilon=\varepsilon(U),C,c>0$ such that

$$\begin{aligned}&\mathbb{P}\left(\left\{(q'+(1-\varepsilon)(m+\log^{6d+9}(N)-n)\cdot\widetilde{U})\subset(\widetilde{H}_n^{(q',\log^{6d+9}(N)+m)}\cap\widetilde{K}_n^{(q',\log^{6d+9}(N)+m)})\right\}\cap G_2(q')\right)\\&\geq\mathbb{P}(G_2(q'))-Ce^{-c\log^{2d+2}(N)}\end{aligned}$$

for all $n\leq m+\log^{6d+9}(N)-\log^{2d+2}(N)$, where $N$ is chosen sufficiently large. Define

$$\widetilde{U}_n(q'):=(q'+(m+\log^{6d+9}(N)-n)\cdot\widetilde{U})).$$

For $n_0:=(\log^{6d+9}(N)-\log^{2d+2}(N))/2$ and since $\|q'-q\|\leq\|q'-y\|+\|q-y\|<C\log^{6d+8}(N)$, we know that

$$q'\in U_{2n_0}(q).$$

Since convergence towards the upper invariant measure of the contact process happens exponentially fast we know that with probability greater than $1-Ce^{-cn_0}$ the intersection between $\eta_{n_0}^{(q,m+\log^{2d+2}(N))}$ and $\widetilde{\eta}_{n_0}^{(q',m+\log^{6d+9}(N))}$ is non-trivial. Since all the error terms are elements of $\mathbf{o}(N^{-c\log(N)})$ after dividing by $\mathbb{P}(J(\zeta))$ the desired result follows on $G_1(q)\cap G_2(q')$. The union over "typical" $q$ and $q'$ then gives the result.

□

**Proof of Lemma 3.16:** We fix $z_k\in\mathcal{P}(N)$. By Lemma 3.17 we know that for any configuration $\zeta$ with high probability the set $\mathbf{C}(\zeta,z_k)\times\{m+\log^{2d+2}(N)\}$ is connected with the random walk path at time $m_k+\log^{6d+9}(N)$. In order to shorten notation, we define

$$L(N,\zeta):=\{l(y_k+r,m_k)<\log^{2d+2}(N)\ \text{ if }\zeta(r)=0\},\tag{3.28}$$

where $l(y,m)$ denotes the length of the longest open path starting from $(y,m)$,

$$I(N,\zeta):=\left\{\mathbf{C}(\zeta,z_k)\times\{m_k+\log^{2d+2}(N)\}\xrightarrow{\omega}\left(X_{m_k+\log^{6d+9}(N)},m_k+\log^{6d+9}(N)\right)\right\}\tag{3.29}$$

Also recall the definition of $J(N,\zeta)$ and $\widetilde{R}_N$ in (3.25) and (3.27). Note that there exist constants $C,c>0$ such that

$$\mathbb{P}^{(x_0,m_k)}\left(\widetilde{R}_N\right)\geq1-Ce^{-c\log^{2d+2}(N)},$$

(see (1.34)) and

$$\mathbb{P}^{(x_0,m_k)}\left(L(N,\zeta)\cap I(N,\zeta)\cap J(N,\zeta)\right)\geq \mathbb{P}^{(x_0,m_k)}\left(J(N,\zeta)\right)-Ce^{-c\log^{2d+2}(N)}$$

for all $x_0,\zeta$ such that $\zeta(x_0-y_k)=1$.
Hence

$$\begin{aligned}
&\mathbb{P}^{(x_0,m_k)}(L(N,\zeta)\cap I(N,\zeta)\cap J(N,\zeta))\\
&\leq \mathbb{P}^{(x_0,m_k)}(L(N,\zeta)\cap I(N,\zeta)\cap J(N,\zeta)\cap \widetilde{R}_N)+Ce^{-c\log^{2d+2}(N)}\\
&\leq \mathbb{P}^{(x_0,m_k)}(\{R(z_k)<m_k+2\log^{6d+9}(N)\}\cap J(N,\zeta))+Ce^{-c\log^{2d+2}(N)},
\end{aligned}$$

together with (3.26) implies the desired result.

□

**Lemma 3.18. (The term (**) - $\mathbf{M}_k$ is visited)**
*Let the parameters be as described at the beginning of the section. There exist constants $C,c>0$ such that*

$$\begin{aligned}
&\operatorname{esssup}\Big(\Big|\mathsf{E}^z\left[P^z_\omega\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_{k-1}\right]\\
&\qquad -\mathsf{E}^z\left[P^z_\omega\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_{k}\right]\Big|\ \Big|\ \mathcal{F}_{k-1}\Big)(\omega_z)\\
&\leq C\left(\log^{6d+9}(N)\right)P^z_{\omega_z}(\mathbf{M}_k \text{ is visited})V^{-\frac{d+1}{2}}+\mathcal{O}(N^{-c\log(N)}).
\end{aligned}$$

**Proof:** If $z_k\prec z$ the random walk starting from $z$ has no possibility to visit $\mathbf{M}_k$, therefore we assume that $z\prec z_k$ or $z=z_k$. As in the proof of Lemma 3.15 we distinguish between the cases $m_k-m\leq\log^2(N)$ and $m_k-m>\log^2(N)$. We first consider the case $m_k-m\leq\log^2(N)$.
Recognize that $m_k-m\leq\log^2(N)$ and the fact that $\mathbf{M}_k$ is visited, implies $z\in\mathbf{M}_k$. Remember the definition of $J(\zeta):=J(N,\zeta)$ in (3.25), where $\zeta\in\{0,1\}^{[-2\log^2(N),2\log^2(N)]^d\cap\mathbb{Z}^d}$. Additionally, we define

$$f(x,\omega_z|_{\mathbf{I}_k},\zeta):=\begin{cases}\mathsf{E}^z\left[P^z_\omega(X_{m_k}=x)\ \middle|\ A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta)\right], & \text{if } \mathsf{P}^z\left(A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta)\right)>0,\\ 0, & \text{if } \mathsf{P}^z\left(A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta)\right)=0.\end{cases}$$

Note that on $A_k(\omega_z|_{\mathbf{I}_k})\cap B_z\cap J(\zeta)$ the random variable $P^z_\cdot(X_{m_k}=x)(\omega)$ is almost surely constant (since we assumed $m\geq m_k-\log^2(N)$) and for $\omega_z$ and $\zeta$ fixed

$$\sum_{x:\zeta(x-y_k)=1}f(x,\omega_z|_{\mathbf{I}_k},\zeta)=1.$$

Choose some $\omega\in A_k(\omega_z|_{\mathbf{I}_k})\cap B_z$. By making use of the Markov property of the quenched law, we get that

$$\begin{aligned}
&\mathbb{1}_{J(\zeta)}(\omega)P^z_\omega\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\\
&=\sum_{x:\zeta(x-y_k)=1}\mathbb{1}_{J(\zeta)}(\omega)P^z_\omega(X_{m_k}=x)P^{(x,m_k)}_\omega\left(X_{M+V}=v\right)\\
&=\sum_{x:\zeta(x-y_k)=1}\mathbb{1}_{J(\zeta)}(\omega)f(x,\omega_z|_{\mathbf{I}_k},\zeta)P^{(x,m_k)}_\omega\left(X_{M+V}=v\right).
\end{aligned}$$

Hence for $\zeta \in \{0,1\}^{[-2\log^2(N),2\log^2(N)]^d\cap\mathbb{Z}^d}$ and $x$ such that $f(x,\omega_z|_{\mathbf{I}_k},\zeta)>0$, it is enough to focus on the conditional expectation of the term $\mathbb{1}_{J(\zeta)}(\omega)P^{(x,m_k)}_\omega(X_{M+V}=v)$ in the sum above.

$$
\begin{aligned}
&\mathsf{E}^z\left[\mathbb{1}_{J(\zeta)}(\omega)P^{(x,m_k)}_\omega(X_{M+V}=v)\,\middle|\mathcal{F}_k\right](\omega_z)\\
&=\frac{1}{\mathsf{P}^z(A_k(\omega_z|_{\mathbf{I}_k}))}\int_{A_k(\omega_z|_{\mathbf{I}_k})}\mathbb{1}_{J(\zeta)}(\omega)P^{(x,m_k)}_\omega(X_{M+V}=v)\,\mathsf{P}^z(d\omega)\\
&=\frac{1}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap B_z)}\int\mathbb{1}_{A_k(\omega_z|_{\mathbf{I}_k})\cap B_z}(\omega)\mathbb{1}_{J(\zeta)}(\omega)P^{(x,m_k)}_\omega(X_{M+V}=v)\,\mathbb{P}(d\omega)\\
&=\frac{1}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap B_z)}\int\mathbb{1}_{A_k(\omega_z|_{\mathbf{I}_k})\cap B_{(x,m_k)}}(\omega)\mathbb{1}_{J(\zeta)}(\omega)P^{(x,m_k)}_\omega(X_{M+V}=v)\,\mathbb{P}(d\omega)\\
&=\frac{\mathbb{P}(B_{(x,m_k)})}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap B_z)}\int\mathbb{1}_{A_k(\omega_z|_{\mathbf{I}_k})}(\omega)\mathbb{1}_{J(\zeta)}(\omega)P^{(x,m_k)}_\omega(X_{M+V}=v)\,\mathsf{P}^{(x,m_k)}(d\omega)\\
&=\frac{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta))}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap B_z)}\int P^{(x,m_k)}_\omega(X_{M+V}=v)\,\mathsf{P}^{(x,m_k)}(d\omega\mid A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta))
\end{aligned}
$$

(The previous equality holds true since by construction $A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta)\subset B_{(x,m_k)}$.)

$$
=\frac{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta))}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap B_z)}\mathbb{P}^{(x,m_k)}(X_{M+V}=v\mid A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta)),
$$

where

$$
\begin{aligned}
&\mathbb{P}^{(x,m_k)}(X_{M+V}=v\mid A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta))\\
&=\mathbb{P}^{(x,m_k)}(X_{M+V}=v\mid J(\zeta))\\
&=\mathcal{O}(N^{-c\log(N)})+\sum_{(w,l):\|w-x\|<l-m_k<2\log^{6d+9}(N)}\mathbb{P}^{(x,m_k)}\left(X_{R(z_k)}=w,R(z_k)=l\mid J(\zeta)\right)\\
&\qquad\cdot\mathbb{P}^{(w,l)}(X_{M+V}=v,).
\end{aligned}
$$

The random variable $R$ denotes the special regeneration point defined in Lemma 3.16.
Also note that

$$
\sum_\zeta\frac{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap J(\zeta))}{\mathbb{P}(A_k(\omega_z|_{\mathbf{I}_k})\cap B_z)}=1.
$$

Altogether we get that

$$
\begin{aligned}
&\left|\mathsf{E}^z\left[P_\omega(\{X_{M+V}=v\}\cap\{\mathbf{M}_k\text{ is visited}\})|\mathcal{F}_k\right]-\mathsf{E}^z\left[P_\omega(\{X_{M+V}=v\}\cap\{\mathbf{M}_k\text{ is visited}\})|\mathcal{F}_{k+1}\right]\right|(\omega_z)\\
&\le\sup_{(x_1,n_1),(x_2,n_2)\in\mathbf{G}(z_k,N)}\left|\mathbb{P}^{(x_1,n_1)}(X_{M+V}=v)-\mathbb{P}^{(x_2,n_2)}(X_{M+V}=v)\right|+CN^{-c\log(N)}
\end{aligned}
\tag{3.30}
$$

where $\mathbf{G}(z_k,N):=\{(x,n):\|y_k-x\|<3\log^{6d+9}(N),0\le n-m_k<2\log^{6d+9}(N)\}$.
Hence Lemma 3.9 yields

$$
\begin{aligned}
&\left|\mathsf{E}^z\left[P_\omega(\{X_{M+V}=v\}\cap\{\mathbf{M}_k\text{ is visited}\})|\mathcal{F}_k\right]-\mathsf{E}^z\left[P_\omega(\{X_{M+V}=v\}\cap\{\mathbf{M}_k\text{ is visited}\})|\mathcal{F}_{k+1}\right]\right|(\omega_z)\\
&\le C\left(\log^{6d+9}(N)\right)V^{-\frac{d+1}{2}},
\end{aligned}
$$

where the last inequality holds true uniformly in $k$ since we expose the environment only up to time $M+\log^2(N)$. This proves the Lemma for $m_k-m\le\log^2(N)$, since in this case $P^z_{\omega_z}(\mathbf{M}_k\text{ is visited})=1$.

Now let $m_k - m > \log^2(N)$. Recall the definition of $\mathbf{L}_k$ in (3.5). Furthermore, by Lemma 3.8 there exists a subset

$$\widetilde{G}_1(z,N) \subset D(N) \cap B_z \cap \left\{\omega : d_{\mathrm{TV}}(\kappa_k^z(\omega|_{\mathbf{I}_k},\ \cdot\ ), \mathrm{Ber}^{\otimes \mathbf{O}_k}) \leq CN^{-c\log(N)}\right\},$$

with $\mathbb{P}^z(\widetilde{G}_1(z,N)) \geq 1 - CN^{-c\log(N)}$ such that for all $\omega_z \in \widetilde{G}_1(z,N)$ we have

$$\begin{aligned}
&\mathsf{E}^z\left[P_\omega^z\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_k\right](\omega_z)\\
&= \int P^z_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\kappa_k^z(\omega_z|_{\mathbf{I}_k}, d\vartheta)\\
&= \int P^z_{(\omega|_{\mathbf{I}_k},\vartheta)}\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)})\\
&= \int \mathbb{1}_{\{\xi_{(\omega_z|_{\mathbf{I}_k},\vartheta)}|_{\mathbf{L}_k}=\xi_{\omega_z}|_{\mathbf{L}_k}\}} P^z_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta)\\
&\quad + \int \mathbb{1}_{\{\xi_{(\omega_z|_{\mathbf{I}_k},\vartheta)}|_{\mathbf{L}_k}\neq\xi_{\omega}|_{\mathbf{L}_k}\}} P^z_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)}).
\end{aligned}$$

By an argumentation similar to (3.22) we get that

$$\begin{aligned}
&\mathsf{E}^z\left[P_\omega^z\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_k\right](\omega_z)\\
&= \int \mathbb{1}_{\{\xi_{(\omega_z|_{\mathbf{I}_k},\vartheta)}|\mathbf{L}_k=\xi_{\omega_z}|\mathbf{L}_k\}} P^z_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\big(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\big)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)}).
\end{aligned}$$

This implies

$$\begin{aligned}
&\mathsf{E}^z\left[P_\omega^z\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_k\right](\omega_z)\\
&= \int \mathbb{1}_{\{\xi_{(\omega_z|_{\mathbf{I}_k},\vartheta)}|_{\mathbf{L}_k}=\xi_{\omega_z}|_{\mathbf{L}_k}\}} P^z_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)})\\
&= \int \mathbb{1}_{\{\xi_{(\omega_z|_{\mathbf{I}_k},\vartheta)}|_{\mathbf{L}_k}=\xi_{\omega_z}|_{\mathbf{L}_k}\}}\\
&\qquad \cdot \sum_{x\in\partial^-\mathbf{M}_k} P^z_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(X_{m_k-\log^2(N)}=x\right) P^{(x,m_k-\log^2(N))}_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(X_{M+V}=v\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)})\\
&= \sum_{x\in\partial^-\mathbf{M}_k} P^z_{\omega_z}\left(X_{m_k-\log^2(N)}=x\right)\\
&\qquad\qquad \cdot \int \mathbb{1}_{\{\xi_{(\omega_z|_{\mathbf{I}_k},\vartheta)}|_{\mathbf{L}_k}=\xi_{\omega_z}|_{\mathbf{L}_k}\}} P^{(x,m_k-\log^2(N))}_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(X_{M+V}=v\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)})\\
&= \sum_{x\in\partial^-\mathbf{M}_k} P^z_{\omega_z}\left(X_{m_k-\log^2(N)}=x\right)\\
&\qquad\qquad \cdot \int P^{(x,m_k-\log^2(N))}_{(\omega_z|_{\mathbf{I}_k},\vartheta)}\left(X_{M+V}=v\right)\mathrm{Ber}^{\otimes \mathbf{O}_k}(d\vartheta) + \mathcal{O}(N^{-c\log(N)})\\
&= \sum_{x\in\partial^-\mathbf{M}_k} P^z_{\omega_z}\left(X_{m_k-\log^2(N)}=x\right)\\
&\qquad\qquad \cdot \mathsf{E}^{(x,m_k-\log^2(N))}\left[P_\omega^{(x,m_k-\log^2(N))}\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)\Big|\mathcal{F}_k\right](\omega_z) + \mathcal{O}(N^{-c\log(N)}),
\end{aligned}$$

where the conditional expectation in the last line fits exactly the case we discussed before. Hence altogether we get

$$\begin{aligned}
&\Big|\mathsf{E}^z\left[P_\omega\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_k\right]-\mathsf{E}^z\left[P_\omega\left(\{X_{M+V}=v\}\cap\{\mathbf{M}_k \text{ is visited}\}\right)|\mathcal{F}_{k+1}\right]\Big|(\omega_z)\\
&\leq CP^z_{\omega_z}\left(\mathbf{M}_k \text{ is visited}\right)\left(\log^{6d+9}(N)\right)V^{-\frac{d+1}{2}}+\mathcal{O}(N^{-c\log(N)}). \qquad (3.31)
\end{aligned}$$

□

**Lemma 3.19.** *Let $X^{(1)}$ and $X^{(2)}$ be two independent random walks defined on the same oriented percolation cluster. As in the previous chapter we denote the difference between the random walks by $(D_m)_{m\geq 0}:=(X^{(1)}_m-X^{(2)}_m)_{m\geq 0}$. We define*

$$R_N(X^{(i)}):=\{T^{(i)}_k-T^{(i)}_{k-1}\leq\log^2(N) \textit{ for all } k\leq N^2\}$$

*and*

$$R^{sim}_N:=\{T^{sim}_k-T^{sim}_{k-1}\leq\log^2(N) \textit{ for all } k\leq N^2\},$$

*where $(T^{sim}_n)_n$ are the times at which both random walks regenerate. For the definition of $(T^{(i)}_n)_n$ and $(T^{sim}_n)_n$ see* (1.37) *and* (1.38)*. If $d\geq 3$, there exist constants $C,c>0$ such that for every $n$*

$$\begin{aligned}
&\mathbb{P}^{\mathbf{0}}_{joint}\left(\left\{\#\left\{m<N^2:\|D_m\|<\log^2(N)\right\}>n\log^8(N)\right\}\cap R^{sim}_N\right)\\
&\leq\left(1-\frac{C}{\log^2(N)}\right)^n+CN^{-c\log(N)}.
\end{aligned}$$

The probability measure $\mathbb{P}^x_{joint}$ is defined within the comments at the beginning of section 2.1.3.

**Proof:** We know that

$$\mathbb{P}^{\mathbf{0}}_{joint}\left(R^{sim}_N\right)\geq 1-CN^{-c\log(N)}.$$

We focus on the difference between the random walks at their simultaneous regeneration times. As in chapter 2 let $\widehat{D}_n=\widehat{X}^{(1)}_n-\widehat{X}^{(2)}_n=X^{(1)}_{T^{sim}_n}-X^{(2)}_{T^{sim}_n}$. Furthermore, we define the following sequence of stopping times

$$\begin{aligned}
\tau^{in}_{N,0}&:=0,\\
\tau^{out}_{N,k+1}&:=\inf\{n\geq\tau^{in}_{N,k}:\left\|\widehat{D}_n\right\|>\log^2(N)\},\\
\tau^{in}_{N,k+1}&:=\inf\{n\geq\tau^{out}_{N,k+1}:\left\|\widehat{D}_n\right\|\leq\log^2(N)\},
\end{aligned}$$

with the convention $\inf\emptyset:=\infty$.
Notice that

$$\mathbb{P}^{\mathbf{0}}_{joint}(\exists\, k \text{ such that } \tau^{in}_{N,k}=\infty)=1,$$

since in dimension $d\geq 3$ the process $(\widehat{D}_n)_{n\geq 0}$ is transient. Denote by

$$p:=p(N):=\min_{x:\|x\|\leq\log^2(N)}\mathbb{P}^x_{joint}\left(\left\|\widehat{D}_{\log^4(N)}\right\|\geq\log^2(N)\right)>\varepsilon,$$

where $\varepsilon>0$ can be chosen independently of $N$. Notice that

$$\mathbb{P}^{\mathbf{0}}_{joint}\left(\tau^{out}_{N,k+1}-\tau^{in}_{N,k}\geq\log^6(N)\mid\widehat{D}_{\tau^{in}_{N,k}}=x,\ \tau^{in}_{N,k}<\infty\right)\leq(1-\varepsilon)^{\log^2(N)}\leq CN^{-c\log(N)}.$$

Therefore the probability of the event

$$\begin{aligned}\widehat{R}_N &:= R_N^{sim} \cap \{\tau_{N,k}^{out} - \tau_{N,k}^{in} < \log^6(N) \text{ for all } k \le N^2 : \tau_{N,k}^{in} < \infty\} \\ &= R_N^{sim} \cap \left( \bigcap_{k \le N^2} \left[ (\{\tau_{N,k}^{out} - \tau_{N,k}^{in} < \log^6(N)\} \cap \{\tau_{N,k}^{in} < \infty\}) \cup \{\tau_{N,k}^{in} = \infty\} \right] \right)\end{aligned}$$

is bounded from below by

$$\mathbb{P}^{joint}(\widehat{R}_N) \ge 1 - CN^{-c\log(N)}.$$

On the event $\widehat{R}_N$ the number of times that the distance between the random walks becomes less than $\log^2(N)$ can be bounded by

$$\#\left\{n < N^2 : \|D_n\| < \log^2(N)\right\} < \log^8(N) \cdot (\inf\{k \le N^2 : \tau_{N,k}^{in} - \tau_{N,k-1}^{in} > N^2\} \wedge N^2),$$

with the convention $\inf \emptyset := \infty$, since on $\widehat{R}_N$ the difference $\tau_{N,k}^{out} - \tau_{N,k}^{in}$ is bounded by $\log^6(N)$ regeneration steps, each of which is bounded by $\log^2(N)$. Next we will prove that

$$\mathbb{P}_{joint}^{\mathbf{0}}(\tau_{N,k}^{in} - \tau_{N,k-1}^{in} > N^2 \mid \tau_{N,k-1}^{in} < \infty) \ge \frac{1}{\log^2(N)} > 0, \tag{3.32}$$

which allows us to bound $\inf\{k : \tau_{N,k}^{in} - \tau_{N,k-1}^{in} > N^2\}$ by a geometrical random variable with success probability $e(N) := \frac{1}{\log^2(N)} > 0$. Using the Markov property it is enough to prove that

$$\mathbb{P}_{joint}^{y}(\tau_{N,1}^{in} > N^2) \ge e(N) > 0, \tag{3.33}$$

for $y \in \mathbb{Z}^d, \|y\| \le \log^2(N)$ chosen arbitrarily. By the strong Markov property, we factorize (3.33) as follows

$$\mathbb{P}_{joint}^{y}(\tau_{N,1}^{in} > N^2) = \sum_{x : \|x\| \ge \log^2(N)} \mathbb{P}_{joint}^{y}(D_{\tau_{N,1}^{out}} = x) \mathbb{P}_{joint}^{x}(\tau_{N,1}^{in} > N^2).$$

We focus on $\mathbb{P}_{joint}^{x}(\tau_{N,1}^{in} > N^2)$ for some $x \in \mathbb{Z}^d$ with $\|x\| > \log^2(N)$ and assume without loss of generality that the first component of $x$, denoted by $x_1$, satisfies $|x_1| > \log^2(N)$. By Lemma 1.14 we can couple the joint and independent measures until $\tau_{N,1}^{in}$. The probability that the coupling breaks within the next $N^2$ steps is bounded by $CN^2 e^{-c\log^2(N)}$. The estimation of independent random walks (see [BČDG13, Lemma 3.6]) gives us

$$\begin{aligned}
&\mathbb{P}_{ind}^{x}(\tau_{N,1}^{in} > N^2) \\
&\ge \mathbb{P}_{ind}^{x}(\tau_{N,1}^{in} = \infty) \\
&\ge \sum_{y \in \mathbb{Z}^d, n \in \mathbb{N}} \mathbb{P}_{ind}^{x}\left(H_1(K\log^2(N)) = n < h_1(\log^2(N)), D_n = y\right) \mathbb{P}_{ind}^{y}\left(h(\log^2(N)) = \infty\right) \\
&\ge (1-\varepsilon) \sum_{y \in \mathbb{Z}^d, n \in \mathbb{N}} \mathbb{P}_{ind}^{x}\left(H_1(K\log^2(N)) = n < h_1(\log^2(N)), D_n = y\right) \frac{(\log^2(N))^{2-d} - \|y\|_2^{2-d}}{(\log^2(N))^{2-d}} \\
&\ge (1-\varepsilon) \sum_{y \in \mathbb{Z}^d, n \in \mathbb{N}} \mathbb{P}_{ind}^{x}\left(H_1(K\log^2(N)) = n < h_1(\log^2(N)), D_n = y\right) (1 - 1/C) \\
&\ge (1-\varepsilon) \frac{|x_1| - \log^2(N)}{K\log^2(N) - \log^2(N)} (1 - 1/C) \\
&\ge \frac{C}{\log^2(N)},
\end{aligned}$$

where $K > 1$ is a large constant, $C > 0$ and

$$h_1(r) := \inf\{n \geq 0 : (D_n)_1 \leq r\} \quad \text{and} \quad H_1(r) := \inf\{n \geq 0 : (D_n)_1 \geq r\}.$$

The last estimate yields

$$\begin{aligned}
&\mathbb{P}^{\mathbf{0}}_{joint}\left(\{\#\left\{n < \mathbb{N}^2 : \|D_n\| < \log^2(N)\right\} > n\log^8(N)\} \cap R_N^{sim}\right) \\
&\leq \mathbb{P}^{\mathbf{0}}_{joint}\left(\{\#\left\{n < \mathbb{N}^2 : \|D_n\| < \log^2(N)\right\} > n\log^8(N)\} \cap \widehat{R}_N\right) + CN^{-c\log(N)} \\
&\leq \left(1 - \frac{C}{\log^2(N)}\right)^n + CN^{-c\log(N)}. 
\end{aligned} \tag{3.34}$$

□

**Corollary 3.20.** *Let $X^{(1)}$ and $X^{(2)}$ be two independent random walks defined on the same oriented percolation cluster. If $d \geq 3$, then*

$$\mathbb{P}^{\mathbf{0}}_{joint}\left[\left\{\#\left\{n < N^2 : \left\|X_n^{(1)} - X_n^{(2)}\right\| < \log^2(N)\right\} \leq \log^{12}(N)\right\} \cap R_N^{sim}\right] \geq 1 - CN^{-c\log(N)}.$$

**Proof:**

$$\begin{aligned}
&\mathbb{P}^{\mathbf{0}}_{joint}\left(\left\{\#\left\{n < \mathbb{N}^2 : \left\|X_n^{(1)} - X_n^{(2)}\right\| < \log^2(N)\right\} > \log^4(N)\log^8(N)\right\} \cap R_N^{sim}\right) \\
&< \left(1 - \tfrac{C}{\log^2(N)}\right)^{\log^4(N)} + CN^{-c\log(N)} \\
&< \left(\left(1 - \tfrac{C}{\log^2(N)}\right)^{\log^2(N)}\right)^{\log^2(N)} + CN^{-c\log(N)} \\
&\leq CN^{-c\log(N)}.
\end{aligned}$$

□

**Remark 3.21.** Note that Corollary 3.20 implies

$$\begin{aligned}
&\mathsf{P}^z\left((E_\omega^z \otimes E_\omega^z)\left[\#\left\{n < N^2 : \left\|X_n^{(1)} - X_n^{(2)}\right\| < \log^2(N)\right\}\mathbb{1}_{R_N^{sim}}\right] \geq C\log^{12}(N)\right) \\
&\leq \mathsf{P}^z\left((E_\omega^z \otimes E_\omega^z)\left[1 + N^2\mathbb{1}_{\{\#\{n<N^2:\|X_n^{(1)}-X_n^{(2)}\|<\log^2(N)\}\geq\log^{12}(N)\}}\mathbb{1}_{R_N^{sim}}\right] \geq C\right) \\
&\leq \mathsf{P}^z\left((P_\omega^z \otimes P_\omega^z)\left[\{\#\{n < N^2 : \|X_n^{(1)} - X_n^{(2)}\| < \log^2(N)\} \geq \log^{12}(N)\} \cap R_N^{sim}\right] \geq \frac{C-1}{N^2}\right) \\
&\leq CN^{-c\log(N)},
\end{aligned}$$

for some constant $C > 1$.

Now we turn to the proof of Proposition 3.11.

**Proof of Proposition 3.11:** By Lemma 3.15 and Lemma 3.18, we know that there exists a subset $\widetilde{G}_1(z,N)$ of $D(N) \cap B_z$ with $\mathbb{P}^z(\widetilde{G}_1(z,N)) \geq 1 - CN^{-c\log(N)}$ such that on $\widetilde{G}_1(z,N)$

$$\begin{aligned}
U_k &= \operatorname{esssup}\left(\left|\mathsf{E}^z\left[P_\omega^z\left(X_{M+V} = v\right)|\mathcal{F}_{k-1}\right] - \mathsf{E}^z\left[P_\omega^z\left(X_{M+V} = v\right)|\mathcal{F}_k\right]\right| \,\middle|\, \mathcal{F}_{k-1}\right) \\
&\leq C\left(\log^{6d+9}(N)\right)P_\cdot^z\left(\mathbf{M}_k \text{ is visited}\right)V^{-\frac{d+1}{2}} + CN^{-c\log(N)}.
\end{aligned}$$

We define

$$\left\{\omega \in B_z : (E^z_\omega \otimes E^z_\omega)\left[\#\left\{n < N^2 : \left\|X^{(1)}_n - X^{(2)}_n\right\| < \log^2(N)\right\} \mathbb{1}_{R^{sim}_N}\right] < C\log^{12}(N)\right\}$$
$$=: W(z,N) \subset B_z \tag{3.35}$$

and prove that for $U := \sum_k U_k^2$ we have

$$U \leq C \cdot l(N) \cdot V^{-d-1}$$

on

$$\widetilde{G}_1(z,N) \cap W(z,N), \tag{3.36}$$

where $l(N)$ is a slowly varying function. The sum $\sum_k U_k^2$ is taken up to time-layer $M + \log^2(N)$.
In order to simplify notation we define $\tilde{l}(N) := \left(\log^{6d+9}(N)\right)$, which is also a slowly varying function.
For $\omega \in \widetilde{G}_1(z,N) \cap W(z,N)$ we have

$$\begin{aligned}
&\sum_k U_k^2 \\
&\leq C\tilde{l}^2(N)V^{-d-1}\sum_k \left(P^z_\omega\left(\mathbf{M}_k \text{ is visited}\right)\right)^2 + CN^{-c\log(N)} \\
&\leq C\tilde{l}^2(N)V^{-d-1}\sum_k \left(\sum_{x_1 \in \partial^-\mathbf{M}_k} P^z_\omega\left(X_{m_k-\log^2(N)} = x_1\right)\right)\left(\sum_{x_2 \in \partial^-\mathbf{M}_k} P^z_\omega\left(X_{m_k-\log^2(N)} = x_2\right)\right) \\
&\quad + CN^{-c\log(N)} \\
&\leq C\tilde{l}^2(N)V^{-d-1}\sum_k \left(\sum_{x_1,x_2 \in \partial^-\mathbf{M}_k} P^z_\omega\left(X_{m_k-\log^2(N)} = x_1\right)P^z_\omega\left(X_{m_k-\log^2(N)} = x_2\right)\right) + CN^{-c\log(N)} \\
&\leq C(\tilde{l}^2(N)V^{-d-1}) \cdot \\
&\qquad \sum_k \left(\sum_{x_1 \in \partial^-\mathbf{M}_k} (P^z_\omega \otimes P^z_\omega)\left(\left\|X^{(1)}_{m_k-\log^2(N)} - X^{(2)}_{m_k-\log^2(N)}\right\| < 2\log^2(N), X_{m_k-\log^2(N)} = x_1\right)\right) \\
&\quad + CN^{-c\log(N)} \\
&\leq C\tilde{l}^2(N)V^{-d-1}2\log^2(N)\sum_{k=1}^{M+\log^2(N)} (P^z_\omega \otimes P^z_\omega)\left(\left\|X^{(1)}_k - X^{(2)}_k\right\| < 2\log^2(N)\right) + CN^{-c\log(N)} \\
&\leq C\tilde{l}^2(N)V^{-d-1}2\log^2(N)\log^{12}(N) + CN^{-c\log(N)},
\end{aligned} \tag{3.37}$$

where the last inequality holds true by the definition of $W(z,n)$ in (3.35).
With $l(N) = 2\tilde{l}^2(N)\log^2(N)\log^{12}(N)$ the desired result follows.
Let $k_0$ be such that $m_{k_0} = M + \log^2(N)$ and $(y_k, m_k) = z_k \prec z_{k_0} = (y_{k_0}, m_{k_0})$ for all $z_k$ such that $m_k \leq M + \log^2(N)$. We denote the $\sigma$-Algebra $\mathcal{F}_{k_0}$ by $\mathcal{G}$. Using McDiarmids inequality (see Theorem 3.10) we get that

$$\begin{aligned}
&\mathsf{P}^z\left(\left|\mathsf{E}^z\left[P^z_{\tilde{\omega}}\left(X_{M+V} = v\right)|\mathcal{G}\right](\omega) - \mathbb{P}^z_\omega\left(X_{M+V} = v\right)\right| > \frac{1}{2}N^{-d}\right) \\
&\leq \mathsf{P}^z\left(D(N)^c \cup W(z,N)^c\right) + 2\exp\left(-\frac{CN^{-2d}}{l(N)V^{-d-1}}\right).
\end{aligned}$$

Remember that $V := \left\lfloor N^{2\theta'} \right\rfloor$ and $\frac{d}{d+1} < \theta' < \theta$, hence

$$\exp\left(-\frac{CN^{-2d}}{l(N)V^{-d-1}}\right) = \exp\left(-\frac{C}{l(N)}N^{-2d}N^{2\theta'(d+1)}\right) = \exp\left(-\frac{C}{l(N)}N^{-2(d-\theta'(d+1))}\right) \le CN^{-c\log(N)}.$$

For $z \in \widetilde{\mathcal{P}}(N)$ chosen arbitrarily, let $\widetilde{G}_2(z,N) \subset B_z$ be the event that

$$\left|\mathsf{E}^z\left[P^z_\omega\left(X_{M+V}=v\right)|\mathcal{G}\right](\omega) - \mathbb{P}^z_\omega\left(X_{M+V}=v\right)\right| \le \frac{1}{2}N^{-d}$$

for every $\frac{2}{5}N^2 \le M \le N^2$ and every $v \in \mathbb{Z}^d$ with $\|v\| \le N\log^3(N)$. By the previous calculations we know that $\mathbb{P}^z(\widetilde{G}_2(z,N)) \ge 1 - CN^{-c\log(N)}$. Now we fix $\omega \in \widetilde{G}_2(z,N)$, $\frac{d}{d+1} < \theta \le 1$ and a cube $\Delta_x \subset Z^d$ of side length $N^\theta$ and center $x \in \mathbb{Z}^d$. We are interested in estimates on the following quantity

$$\left|P^z_\omega(X_M = \Delta_x) - \mathbb{P}^z(X_M \in \Delta_x)\right|.$$

We denote by $\Delta_x^{(1)}$ a cube with center $x$ and side length $\frac{9}{10}N^\theta$ that is slightly smaller than $\Delta_x$ and by $\Delta_x^{(2)}$ a cube with center $x$ and side length $\frac{11}{10}N^\theta$ that is slightly bigger than $\Delta_x$. There exist $C, c > 0$ such that

$$\mathbb{P}^z(X_{M+V} \in \Delta_x^{(1)}) < \mathbb{P}^z(X_M \in \Delta_x) + CN^{-c\log(N)}, \tag{3.38}$$

$$\mathbb{P}^z(X_{M+V} \in \Delta_x^{(2)}) > \mathbb{P}^z(X_M \in \Delta_x) - CN^{-c\log(N)}. \tag{3.39}$$

Furthermore, there exists a subset $\widetilde{G}_3(z,N) \subset B_z$ such that for all $\omega_z \in \widetilde{G}_3(z,N)$

$$\mathsf{E}^z\left[P^z_\omega\left(X_{M+V} \in \Delta_x^{(1)}\right)\middle|\mathcal{G}\right](\omega_z) < P^z_{\omega_z}(X_M \in \Delta) + CN^{-c\log(N)}, \tag{3.40}$$

$$\mathsf{E}^z\left[P^z_\omega\left(X_{M+V} \in \Delta_x^{(2)}\right)\middle|\mathcal{G}\right](\omega_z) > P^z_{\omega_z}(X_M \in \Delta) - CN^{-c\log(N)}, \tag{3.41}$$

and $P^z(\widetilde{G}_3(z,N)) \ge 1 - CN^{-c\log(N)}$. The proof of (3.38)-(3.41) can be found in section A.2.2 in the appendix. Since $\widetilde{G}_2(z,N) \cap \widetilde{G}_3(z,N) \subset G_1(z,\theta,N)$ the proof of Proposition 3.11 is complete.

□

## 3.3. Estimates on hitting probabilities for "small" boxes

As already mentioned in Remark 3.12, the next thing we want to do is to use Proposition 3.11 in order to improve our bounds on the term

$$\sum_k \left(P^z_\omega\left(\mathbf{M}_k \text{ is visited}\right)\right)^2, \tag{3.42}$$

that appears in (3.37). The new bounds on (3.42) we get out of Lemma 3.22 below, will then be used to prove Lemma 3.23 which leads to an improved version of Proposition 3.11 (see Theorem 3.24). In Corollary 3.25 we discuss how the quenched central limit theorem given by Birkner et al. in [BČDG13] can be derived from Theorem 3.24 for dimension $d \ge 3$.

**Lemma 3.22.** *Assume that $d \geq 3$. For every $0 < \theta \leq 1$ and $z \in \widetilde{\mathcal{P}}(N)$ let $G_2(z,\theta,h,N) \subset B_z$ be the event that for every $\frac{2}{5}N^2 \leq M \leq N^2$ and every cube $\Delta$ of side length $N^\theta$ that is contained in $[-N\log^3(N), N\log^3(N)]^d$*

$$P_\omega^z(X_M \in \Delta) \leq \log^h(N) N^{-d(1-\theta)}.$$

*Then for every $0 < \theta \leq 1$, $z \in \widetilde{\mathcal{P}}(N)$ there exist $C, c > 0$ and $h = h(\theta) \geq 0$, independent of $z$, such that*

$$\mathsf{P}^z(G_2(z,\theta,h,N)) \geq 1 - CN^{-c\log(N)}.$$

*Hence*

$$\begin{aligned}\mathbb{P}\left(\bigcap_{z\in\widetilde{\mathcal{P}}(N)} G_2(z,\theta,h,N) \cup (B_z)^c\right) &\geq 1 - \sum_{z\in\widetilde{\mathcal{P}}(N)} \mathbb{P}\left((G_2(z,\theta,h,N))^c \cap B_z\right) \\ &\geq 1 - CN^{-c\log(N)}.\end{aligned}$$

**Proof:** Let $(\theta_n)_{n\geq 0}$ be a decreasing sequence of real numbers with

$$\theta_0 \in (\tfrac{d}{d+1}, 1) \quad \text{and} \quad \tfrac{d}{d+1}\theta_n < \theta_{n+1} < \theta_n.$$

We prove the lemma by induction over $n$ for the sequence $(\theta_n)_{n\geq 0}$. For $\theta_0$ the Lemma holds true by Proposition 3.11 and Lemma 3.9 *i*). For the induction step we fix $n \geq 0$ and assume that the statement holds true for $\theta_n$. We define $\rho := \frac{\theta_{n+1}}{\theta_n} > \frac{d}{d+1}$. By Proposition 3.11 and Lemma 3.9 *i*) there exists $h = h(\rho)$ such that by translation invariance of $\mathbb{P}$, we have

$$\mathbb{P}\left(\bigcap_{z\in\widetilde{\mathcal{P}}(N)} G_2(z,\rho,h,N) \cup (B_z)^c\right) \geq 1 - CN^{-c\log(N)}.$$

Since the statement holds true by induction hypothesis for $\theta_n$ there also exists $h' = h'(\theta_n)$ such that

$$\mathbb{P}\left(\bigcap_{z\in\widetilde{\mathcal{P}}(N)} G_2(z,\theta_n,h',\lfloor N^\rho\rfloor) \cup (B_z)^c\right) \geq 1 - C\lfloor N^\rho\rfloor^{-c\log(\lfloor N^\rho\rfloor)}.$$

We define

$$\begin{aligned}\bar{R}(\rho,h,N) &:= \bigcap_{z\in\widetilde{\mathcal{P}}(N)} \left(G_2(z,\rho,h,N) \cup (B_z)^c\right), \\ \bar{R}(\theta_n,h',\lfloor N^\rho\rfloor) &:= \bigcap_{z\in\widetilde{\mathcal{P}}(N)} \left(G_2(z,\theta_n,h',\lfloor N^\rho\rfloor) \cup (B_z)^c\right),\end{aligned}$$

and

$$L := \bar{R}(\rho,h,N) \cap \bigcap_{x\in\mathcal{P}(2N)} \sigma_x(\bar{R}(\theta_n,h',\lfloor N^\rho\rfloor)),$$

where $\sigma_x$ denotes the shift of the environment in direction $x$. In detail for $x = (v,t) \in \mathbb{Z}^d \times \mathbb{Z}$ the shift operator $\sigma_x$ on $\Omega$ is defined as

$$\sigma_x(\omega) := \omega(\,\cdot\, + v,\ \cdot\, + t).$$

Note that $\mathbb{P}(L) \geq 1 - CN^{-c\log(N)}$ by translation invariance of $\mathbb{P}$.
Now we choose $z = (y,m) \in \widetilde{\mathcal{P}}(N)$, $\frac{2}{5}N^2 \leq M \leq N^2$ and a cube $\Delta_x$ of side length $N^{\theta_{n+1}} = (N^\rho)^{\theta_n}$ and center $x$ that is contained in $[-N\log^3(N), N\log^3(N)]^d$ arbitrarily. Let $V = \lfloor N^{2\rho} \rfloor$.
Denote by $Q(N)$ the event that

$$\sum_{v:\|v-x\|>2N^\rho \log^3(N)} \mathbb{1}_{\{\xi(v,M-V)=1\}}(\omega) P^{(v,M-V)}_\omega (X_M \in \Delta_x) \leq CN^{-c\log(N)},$$

where $C, c > 0$. By (3.13) we know that there exist $\widetilde{C}, \tilde{c} > 0$ such that $\mathbb{P}(Q(N)) \geq 1 - \widetilde{C}N^{-\tilde{c}\log(N)}$. We will prove the lemma by showing that

$$L \cap Q(N) \subset G_2(z, \theta_{n+1}, h, N) \cup (B_z)^c.$$

We fix $\omega \in L \cap Q(N)$. If $\xi_\omega(z) = 0$ we have $\omega \in (B_z)^c \subset G_2(z,\theta,h,N) \cup (B_z)^c$, hence we focus on the case that $z$ is chosen such that $\xi_\omega(z) = 1$. By the Markov property of the quenched measure we have

$$P^{(y,m)}_\omega (X_M \in \Delta_x) = \sum_{v:\|v-x\|\leq 2N^\rho \log^3(N)} P^{(y,m)}_\omega (X_{M-V} = v) P^{(v,M-V)}_\omega (X_M \in \Delta_x) + CN^{-c\log(N)}.$$

If $\xi_\omega(v, M-V) = 0$ we have $P^{(y,m)}_\omega (X_{M-V} = v) = 0$. On the other hand if $\xi_\omega(v, M-V) = 1$, the fact that $\omega \in \bar{R}(\rho, h, N)$ implies that for all cubes $\Delta'$ of side length $N^\rho$, we have

$$P^{(y,m)}_\omega (X_{M-V} \in \Delta') \leq \log^h(N)(N)^{-d(1-\rho)}.$$

Additionally, $\omega \in \bigcap_{x\in\mathcal{P}(2N)} \sigma_x(\bar{R}(\theta_n, h', \lfloor N^\rho \rfloor))$ and the fact that $N^{\theta_{n+1}} = (N^\rho)^{\theta_n}$ implies that

$$P^{(v,M-V)}_\omega (X_M \in \Delta_x) \leq \log^{h'}(N^\rho)(N^\rho)^{-d(1-\theta_n)} \leq \log^{h'+1}(N)(N)^{-d(\rho-\theta_{n+1})}.$$

Since $[-N^\rho \log^3(N^\rho), N^\rho \log^3(N^\rho)]^d$ is the union of at most $C\log^{3d}(N) \leq \log^{3d+1}(N)$ cubes of side length $N^\rho$, we get that

$$\begin{aligned} P^{(y,m)}_\omega (X_M \in \Delta_x) &\leq C\log^{3d+1}(N)N^{-d(1-\rho)} \cdot \log^{h'+1}(N)N^{-d(\rho-\theta_{n+1})}, \\ &\leq C\log^{3d+h'+2}(N)N^{-d(1-\theta_{n+1})}. \end{aligned}$$

For $h(\theta_{n+1}) := 2d + h' + 2$ the statement holds true for $\theta_{n+1}$.

□

**Lemma 3.23.** *Let $d \geq 3$. Similar to the proof of Proposition 3.11 let $\mathcal{G}$ denote the $\sigma$-Algebra $\mathcal{F}_{k_0}$, where $k_0$ is such that $m_{k_0} = \lfloor N^2 + \log^2(N) \rfloor$ and $(y_k, m_k) = z_k \prec z_{k_0} = (y_{k_0}, m_{k_0})$ for all $z_k = (y_k, m_k)$ with $m_k \leq N^2 + \log^2(N)$. Let $\eta > 0$, $V = \lfloor N^\eta \rfloor$ and define $G_3(z, V, N)$ as the event that for every $v \in \mathbb{Z}^d$*

$$|\mathsf{E}^z \left[P^z_\omega (X_{N^2+V} = v) | \mathcal{G}\right] - \mathbb{P}^z(X_{N^2+V} = v)| \leq N^{-d}V^{-\frac{d}{5}}.$$

*Then there exist $C, c > 0$ such that*

$$\mathbb{P}^z(G_3(z, V, N)) \geq 1 - CN^{-c\log(N)}.$$

*Hence*

$$\mathbb{P}\left(\bigcap_{z\in\widetilde{\mathcal{P}}(N)} G_3(z,V,N)\cup(B_z)^c\right)\geq 1-\sum_{z\in\widetilde{\mathcal{P}}(N)}\mathbb{P}\left((G_3(z,V,N))^c\cap B_z\right)$$
$$\geq 1-CN^{-c\log(N)}.$$

**Proof:** Fix $v\in\mathbb{Z}^d$, $\theta>0$ and let $\eta>0$ be such that $\theta<\frac{1}{20}\eta$. Furthermore, let $L:=L(N)$ be a large integer such that $2^{-(L+1)}N^2\leq V-\log^2(N)<2^{-L}N^2$. For $1\leq l<L$ we define

$$\mathcal{P}^{(l)}:=\mathcal{P}(N)\cap\left\{(x,n):x\in\mathbb{Z}^d,2^{-l-1}N^2\leq N^2-n<2^{-l}N^2\right\}.$$

Additionally, we define

$$\begin{aligned}
\mathcal{P}^{(0)}&:=\mathcal{P}(N)\cap\left\{(x,n):x\in\mathbb{Z}^d,0\leq n\leq\frac{N^2}{2}\right\},\\
\mathcal{P}^{(L)}&:=\mathcal{P}(N)\cap\left\{(x,n):x\in\mathbb{Z}^d,0\leq N^2-n<2^{-L}N^2\right\},\\
F(v)&:=\left\{(x,n)\in\mathcal{P}(N):\|x-v\|\leq\sqrt{N^2+V-n}\log^3(N)\right\},
\end{aligned}$$

and

$$\mathcal{P}^{(l)}(v):=\mathcal{P}^{(l)}\cap F(v),\quad\widehat{\mathcal{P}}^{(l)}(v):=\{y:\exists z\in\mathcal{P}^{(l)}(v)\text{ such that }\|y-z\|\leq\log^3(N)\}.$$

First we want to improve the estimates on the term

$$\sum_k\left(P_\omega^z\left(\mathbf{M}_k\text{ is visited}\right)\right)^2,$$

which appears in (3.37), where $\omega$ is chosen out of $G_2(z,\theta,h,N)$ (see Lemma 3.22). We define

$$V(l):=\sum_{k:z_k\in\mathcal{P}^{(l)}(v)}\left(P_\omega^z\left(\mathbf{M}_k\text{ is visited}\right)\right)^2.$$

First of all recall the definition of $W(z,N)$ in (3.35), and note that on $W(z,N)$ we have

$$V(0)\leq\left(E_\omega^z\otimes E_\omega^z\right)\left[\#\left\{n<N^2:\left\|X_n^{(1)}-X_n^{(2)}\right\|<\log^2(N)\right\}\mathbb{1}_{R_N^{sim}}\right]\leq\log^{12}(N).$$

Remember that we usually denote by $y_k$ the space component and by $m_k$ the time component of $z_k$ According to Lemma 3.22 we can bound $V(l)$, $l\geq 1$, for $\omega\in G_2(\theta,z,h,N)$ by

$$\begin{aligned}
V(l)&=\sum_{k:(x_k,m_k-\log^2(N))\in\mathcal{P}^{(l)}(v)}\left(\sum_{x\in\partial^-\mathbf{M}_k}P_\omega^z\left(X_{m_k-\log^2(N)}=x\right)\right)^2\\
&\leq C\sum_{k:(x_k,m_k-\log^2(N))\in\mathcal{P}^{(l)}(v)}\log^{2d}(N)\sum_{x\in\partial^-\mathbf{M}_k}\left(P_\omega^z\left(X_{m_k-\log^2(N)}=x\right)\right)^2\\
&\leq C\log^{2d+2}(N)\sum_{z'\in\widehat{\mathcal{P}}^{(l)}(v)}\left(P_\omega^z\left(X_{m_k-\log^2(N)}=z'\right)\right)^2\\
&\leq C\log^{2d+2}(N)\sum_{z'\in\widehat{\mathcal{P}}^{(l)}(v)}\log^{2h}(N)N^{2(\theta-1)d}\\
&\leq Cl'(N)N^{2(\theta-1)d}N^{d+2}2^{-l\frac{d+2}{2}}.
\end{aligned}\tag{3.43}$$

For the first estimate we made use of the Cauchy-Schwarz inequality. The second inequality is due to the fact that each point is counted at most $\log^2(N)$ times. The third inequality follows by Lemma 3.22 and for the last inequality we estimated the number of points in $\widehat{\mathcal{P}}^{(l)}(v)$. The function $l'(N)$ is a slowly varying function. We consider the process

$$U_k = \operatorname{esssup}\left(\left|\mathsf{E}^z\left[P_\omega^z\left(X_{N^2+V}=v\right)|\mathcal{F}_{k-1}\right] - \mathsf{E}^z\left[P_\omega^z\left(X_{N^2+V}=v\right)|\mathcal{F}_k\right]\right|\Big|\mathcal{F}_{k-1}\right),$$

on $G_2(\theta, z, h, N)$. Remember that the calculations in the proof of Lemma 3.15 and Lemma 3.18 have lead us to

$$\begin{aligned} U_k(\omega) \le\ & l(N)\cdot P_\omega^z\left(\mathbf{M}_k \text{ is visited}\right) \\ & \cdot \sup_{(x_i,n_i)\in\mathbf{G}(z_k,N)}\left|\mathbb{P}^{(x_1,n_1)}\left(X_{N^2+V}=v\right) - \mathbb{P}^{(x_2,n_2)}\left(X_{N^2+V}=v\right)\right| + CN^{-c\log(N)}, \end{aligned}$$

where $l(N)$ is a slowly varying function and

$$\mathbf{G}(z_k, N) := \{(x,n) : \|y_k - x\| < 3\log^{6d+9}(N), 0 \le n - m_k < 2\log^{6d+9}(N)\}.$$

For $k$ such that $z_k \in \mathcal{P}^{(l)}(v)$ we have

$$\sup_{(x_i,n_i)\in\mathbf{G}(z_k,N)}\left|\mathbb{P}^{(x_1,n_1)}\left(X_{N^2+V}=v\right) - \mathbb{P}^{(x_2,n_2)}\left(X_{N^2+V}=v\right)\right| \le C(2^{-l}N^2)^{-\frac{d+1}{2}}.$$

Now making use of the more precise estimates given in Lemma 3.22 which yield to (3.43), we get

$$\begin{aligned} U &\le C\sum_{l=0}^{L}(2^{-l}N^2)^{-(d+1)}V(l) + CN^{-c\log(N)} \\ &\le CN^{-2(d+1)}V(0) + CN^{-2(d+1)}N^{2(\theta-1)d}N^{d+2}\sum_{k=1}^{L}2^{l(d+1)}2^{-l\frac{d+2}{2}} \\ &\le CN^{-2(d+1)}V(0) + CN^{-3d+2\theta d}2^{L\frac{d}{2}} \\ &\le CN^{-2(d+1)}V(0) + CN^{-3d+2\theta d}N^dV^{-\frac{d}{2}} \\ &\le CN^{-2d+2\theta d}V^{-\frac{d}{2}}. \end{aligned}$$

Hence, using McDiarmid's inequality, we get $\mathbb{P}^z(G_3(z,V,N)) \ge 1 - CN^{-c\log(N)}$.

□

**Theorem 3.24.** *Let $d \ge 3$. For every $0 < \theta \le 1$ and $z \in \widetilde{\mathcal{P}}(N)$ let $G_4(z,N)$ denote the event that for every cube $\Delta$ of side length $N^\theta$ we have*

$$|P_\omega^z(X_{N^2} \in \Delta) - \mathbb{P}^z(X_{N^2} \in \Delta)| \le CN^{-d(1-\theta)-\frac{1}{3}\theta}.$$

*Then there exist $C, c > 0$ such that*

$$\mathbb{P}^z(G_4(z,N)) \ge 1 - CN^{-c\log(N)}.$$

*Hence*

$$\begin{aligned} \mathbb{P}\left(\bigcap_{z\in\widetilde{\mathcal{P}}(N)} G_4(z,N)\cup(B_z)^c\right) &\ge 1 - \sum_{z\in\widetilde{\mathcal{P}}(N)}\mathbb{P}\left((G_4(z,N))^c\cap B_z\right) \\ &\ge 1 - CN^{-c\log(N)}. \end{aligned}$$

**Proof:** Fix $0 < \theta \leq 1$, $z \in \widetilde{\mathcal{P}}(N)$ as required. We choose $\frac{3}{4}\theta < \theta' < \theta$ and $V = \left\lfloor N^{\frac{4\theta'}{d}} \right\rfloor$. We know by Lemma 3.23 that there exist constants $C, c > 0$ such that the event $G_3(z, V, N)$, that

$$|\mathsf{E}^z\left[P_\omega^z(X_{N^2+V} = v)|\mathcal{G}\right] - \mathbb{P}^z(X_{N^2+V} = v)| \leq N^{-d}V^{-\frac{d}{5}}$$

for all $v \in \mathbb{Z}^d$, has probability $\mathbb{P}^z(G_3(z, V, N)) \geq 1 - CN^{-c\log(N)}$. Let $\Delta$ be a cube of side length $N^\theta$ and center $c(\Delta) = x$ that is contained in $[-N\log^3(N), N\log^3(N)]^d$. Let $\Delta^{(1)}$ and $\Delta^{(2)}$ be cubes of side length $N^\theta - \log^3(N)\sqrt{V}$ and $N^\theta + \log^3(N)\sqrt{V}$ and center $c(\Delta^{(1)}) = c(\Delta^{(2)}) = c(\Delta) = x$. Then on $G_3(z, V, N)$ we have

$$\left|\mathsf{E}^z\left[P_\omega^z(X_{N^2+V} \in \Delta^{(i)})|\mathcal{G}\right] - \mathbb{P}^z(X_{N^2+V} = \Delta^{(i)})\right| \leq |\Delta^{(i)}|N^{-d}V^{-\frac{d}{5}}.$$

As in the proof of Proposition 3.11 we know that

$$\begin{aligned}
&\mathbb{P}^z(X_{N^2+V} \in \Delta^{(1)}) < \mathbb{P}^z(X_{N^2} \in \Delta) + CN^{-c\log(N)},\\
&\mathbb{P}^z(X_{N^2+V} \in \Delta^{(2)}) > \mathbb{P}^z(X_{N^2} \in \Delta) - CN^{-c\log(N)}.
\end{aligned}$$

Additionally, there exists $\widetilde{G}_4(z, N) \subset B_z$ such that for all $\omega_z \in \widetilde{G}_4(z, N)$ we have

$$\begin{aligned}
&\mathsf{E}^z\left[P_\omega^z\left(X_{N^2+V} \in \Delta^{(1)}\right)|\mathcal{G}\right](\omega_z) < P_{\omega_z}^z(X_{N^2} \in \Delta) + CN^{-c\log(N)},\\
&\mathsf{E}^z\left[P_\omega^z\left(X_{N^2+V} \in \Delta^{(2)}\right)|\mathcal{G}\right](\omega_z) > P_{\omega_z}^z(X_{N^2} \in \Delta) - CN^{-c\log(N)},
\end{aligned}$$

and $\mathbb{P}^z(\widetilde{G}_4(z, N)) \geq 1 - CN^{-c\log(N)}$ for some $C, c > 0$. Making use of Lemma 3.23 and “standard” bounds on the annealed transition kernel we obtain

$$\begin{aligned}
&P_\omega^z(X_{N^2} \in \Delta) - \mathbb{P}^z(X_{N^2} \in \Delta)\\
&\leq \mathsf{E}^z\left[P_{\tilde{\omega}}^z\left(X_{N^2+V} \in \Delta^{(2)}\right)\middle|\mathcal{G}\right] - \mathbb{P}^z\left(X_{N^2+V} \in \Delta^{(1)}\right) + CN^{-c\log(N)}\\
&\leq \mathsf{E}^z\left[P_{\tilde{\omega}}^z\left(X_{N^2+V} \in \Delta^{(1)}\right)\middle|\mathcal{G}\right] - \mathbb{P}^z\left(X_{N^2+V} \in \Delta^{(1)}\right) + \mathsf{E}^z\left[P_{\tilde{\omega}}^z\left(X_{N^2+V} \in \Delta^{(2)} \setminus \Delta^{(1)}\right)|\mathcal{G}\right]\\
&\quad + CN^{-c\log(N)}\\
&\leq \left|\mathsf{E}^z\left[P_{\tilde{\omega}}^z\left(X_{N^2+V} \in \Delta^{(1)}\right)\middle|\mathcal{G}\right] - \mathbb{P}^z\left(X_{N^2+V} \in \Delta^{(1)}\right)\right| + \mathsf{E}^z\left[P_{\tilde{\omega}}^z\left(X_{N^2+V} \in \Delta^{(2)} \setminus \Delta^{(1)}\right)\middle|\mathcal{G}\right]\\
&\quad + CN^{-c\log(N)}\\
&\leq |\Delta^{(1)}|N^{-d}V^{-\frac{d}{5}} + N^{-d}\left|\Delta^{(2)} \setminus \Delta^{(1)}\right| + CN^{-c\log(N)}\\
&\leq CN^{-d(1-\theta)}N^{-\frac{3\theta}{5}} + C\log^3(N)N^{-d}N^{\theta(d-1)}N^{\frac{4\theta'}{2d}} + CN^{-c\log(N)}\\
&\leq CN^{-d(1-\theta)}N^{-\frac{\theta}{3}}.
\end{aligned}$$

By similar estimates on $\mathsf{E}^z\left[P_{\tilde{\omega}}^z\left(X_{N^2+V} \in \Delta^{(1)}\right)\middle|\mathcal{G}\right] - \mathbb{P}^z\left(X_{N^2+V} \in \Delta^{(2)}\right)$ the theorem follows. □

**Corollary 3.25. (Quenched CLT)** *Let $d \geq 3$. For any continuous and bounded function $f \in C_b(\mathbb{R}^d)$*

$$|E_\omega^z\left[f\left(X_{N^2}/N\right)\right] - \Phi(f)| \longrightarrow 0, \quad \text{as} \quad N \to \infty, \quad \text{for } \mathsf{P}^z\text{-almost all } \omega, \tag{3.44}$$

*where $\Phi(f) := \int f(x)\Phi(dx)$ with $\Phi$ a non-trivial, centered isotropic $d$-dimensional normal law.*

In [BČDG13] Birkner et al. proved a quenched CLT for all $d \geq 1$.

**Notation 3.26.** For any real number $M \in \mathbb{R}$ we define $\Pi_M$ as a partition of $\mathbb{Z}^d$ into boxes of side length $M$. For $\Delta \in \Pi_M$ we denote the center of $\Delta$ by $c(\Delta)$. If the center of the box or the side length is important, we sometimes denote the box $\Delta$ with $x = c(\Delta)$ and side length $M$ by $\Delta_x$ or $\Delta_x^M$. We define $\mathcal{I} := \mathcal{I}(\Pi_M) := \{c(\Delta) : \Delta \in \Pi_M\}$ as the set of all centers of boxes in $\Pi_M$. The partition $\Pi_M$ is sometimes also denoted by $(\Delta_x)_{x\in\mathcal{I}}$.
For some $z \in \widetilde{\mathcal{P}}(N)$, $n \leq \mathbb{N}$ and $\omega \in \Omega$ we say that a box $\Delta \in \Pi_M$ is $z$-$n$-reachable in $\omega$ if there exists $x \in \Delta$ such that $z \overset{\omega}{\to} (x, n) \overset{\omega}{\to} \infty$.

**Proof of Corollary 3.25:** By the Portmanteau theorem it is enough to prove (3.44) for all bounded and uniformly continuous functions $f \in C_u(\mathbb{R}^d)$.
We fix $\theta \in (0,1)$, $f \in C_u(\mathbb{R}^d)$ and $\varepsilon > 0$. We choose $\delta > 0$ such that for all $x, y \in \mathbb{R}^d$ with $\|x - y\| \leq \delta$ we have $|f(x) - f(y)| < \frac{\varepsilon}{3}$. Let $(\Delta_x)_{x\in\mathcal{I}}$ be a partition of $\mathbb{Z}^d$ into boxes of side length $N^\theta$. Furthermore, we assume $N$ to be large enough such that $N^{(\theta-1)} < \delta$. We prove (3.44) for $z = (\mathbf{0}, 0) \in \mathbb{Z}^d \times \mathbb{Z}$ but omit the superscript "$(\mathbf{0}, 0)$". By Theorem 3.24 we get

$$
\begin{aligned}
E_\omega\left[f\left(X_{N^2}/N\right)\right] &= \sum_{x\in\mathbb{Z}^d} f\left(\frac{x}{N}\right) P_\omega\left(X_{N^2} = x\right) \\
&= \sum_{y\in\mathcal{I}} \sum_{x\in\Delta_y} f\left(\frac{x}{N}\right) P_\omega\left(X_{N^2} = x\right) \\
&\leq \sum_{y\in\mathcal{I}} f\left(\frac{y}{N}\right) P_\omega\left(X_{N^2} \in \Delta_y\right) + \frac{\varepsilon}{3} \\
&\leq \sum_{y\in\mathcal{I}} f\left(\frac{y}{N}\right) \mathbb{P}\left(X_{N^2} \in \Delta_y\right) + C \sum_{y\in\widetilde{\mathcal{I}}} N^{-d(1-\theta)-\frac{1}{3}\theta} + CN^{-c\log(N)} + \frac{\varepsilon}{3} \\
&\leq \sum_{x\in\mathbb{Z}^d} f\left(\frac{x}{N}\right) \mathbb{P}\left(X_{N^2} = x\right) + C\left(\frac{N\log^3(N)}{N^\theta}\right)^d N^{-d(1-\theta)-\frac{1}{3}\theta} + CN^{-c\log(N)} + \frac{2\varepsilon}{3} \\
&\leq \mathbb{E}\left(f(X_{N^2}/N)\right) + \varepsilon,
\end{aligned}
$$

where $\widetilde{\mathcal{I}} := \{x \in \mathcal{I} : \|x\| \leq N\log^3(N)\}$ and $N$ is chosen large enough. Corollary 3.25 follows by the annealed CLT.

□

## 3.4. Estimates on hitting probabilities for "sub-algebraic boxes"

Decreasing the box size within the estimates between the quenched and annealed hitting probabilities down to a constant real number would probably be the next step towards proving a quenched local central limit theorem (qLCLT). At least this is the next step within the proof of the qLCLT for ballistic random walks in an uniformly elliptic, i.i.d. environment published by Berger et al. (see [BCR16, Theorem 5.1]). In this section we give a proof of decreasing the box size in Theorem 3.24 down to boxes of side length $e^{\sqrt{\log(N)\log\log(N)}}$ using techniques similar to the proof of [BCR16, Theorem 5.1]. If only boxes of sub-algebraic side length greater than $\log^2(N)$ will be considered, this guarantees that all

boxes in $[-\sqrt{N}\log^3(N), \sqrt{N}\log^3(N)]$ are $(\mathbf{0},0)$-$n$-reachable (see Notation 3.26) with high probability, at least if $n$ is of order $N$. This statement is proved in Lemma 3.27 below.
Adapting the proof of [BCR16, Theorem 5.1] to our case, some problems arise which we have not been able to solve up to now, and which prevent us from decreasing the box size down to constant side length independent of $N$. These problems also seem to appear within the proof of [BCR16, Theorem 5.1] itself. It is not quite clear to us how the authors overcome these difficulties and if there exists a suitable solution for our case as well. The analogue of [BCR16, Theorem 5.1] for boxes of side length $e^{\sqrt{\log(N)\log\log(N)}}$ is formulated in Theorem 3.28 below. Nevertheless, Theorem 3.28 provides a comparison between quenched and annealed hitting probabilities on a finer scale than Theorem 3.24 which is already finer than the comparison that follows from the qCLT by Birkner at al. in [BČDG13, Theorem 1.1] (see (3.1)).
See also Remark 3.29, where the problem of proving a version of Theorem 3.28 for constant box size is discussed.

**Lemma 3.27.** *Let $z=(y,m)\in\widetilde{\mathcal{P}}(\sqrt{N})$ and $\Pi_M=(\Delta_x)_{x\in\mathcal{I}}$ be a partition of $\mathbb{Z}^d$ into boxes of side length $M\geq\log^2(N)$. For every $\|x\|\leq\sqrt{n}\log^3(N)$, where $\frac{N}{2}\leq n\leq N$, we have*

$$\mathbb{P}^z\left(\Delta_x \text{ is } z\text{-}n\text{-reachable}\right)\geq 1-CN^{-c\log(N)}. \tag{3.45}$$

**Proof:** The proof of Lemma 3.27 is very similar to the proof of Lemma 3.8. Using analogous arguments we can show that there exists $\rho>0$ such that

$$\mathsf{P}^z\left(|\Delta_x\cap\eta_n^z|\geq\rho\log^2(N)\right)\geq 1-CN^{-c\log(N)},$$

if $N$ is sufficiently large and hence

$$\mathsf{P}^z\left(\Delta_x\cap\eta_n^z\xrightarrow{\omega}\infty\right)\geq 1-CN^{-c\log(N)},$$

where $(\eta_n^z)_{n\geq m}$ denotes the discrete time contact process starting at time $m$ with only one infected particle at site $y$.

□

**Theorem 3.28.** *Let $d\geq 3$ and define $\Pi_{sub\text{-}alg.}$ as a partition of $\mathbb{Z}^d$ into "sub-algebraic boxes" of side length $e^{\sqrt{\log(N)\log\log(N)}}$. For $N\in\mathbb{N}$ denote by $G_5(N):=G_5(N,C',c')\subset B_{(\mathbf{0},0)}$ the set of environments such that*

$$\sum_{\Delta\in\Pi_{sub\text{-}alg.}}|P_\omega^{(\mathbf{0},0)}(X_N\in\Delta)-\mathbb{P}^{(\mathbf{0},0)}(X_N\in\Delta)|\leq C'e^{-c'\sqrt{\log(N)\log\log(N)}}. \tag{3.46}$$

*Then for an appropriate choice of $C',c'>0$ there exist constants $C,c>0$ such that*

$$\mathbb{P}^{(\mathbf{0},0)}(G_5(N))\geq 1-CN^{-c\log(\log(N))}.$$

Note that there is a small "notation break" compared to Theorem 3.24: Now, time runs up to $N$ and not up to $N^2$.

**Proof:** Let $\theta > 0$ be some small constant. We define $N_j := \lfloor N^{\frac{1}{2^j}} \rfloor$ and

$$r(N) := \left\lfloor \log_2 \left( \frac{\theta \log(N)}{2\sqrt{\log(N) \log\log(N)}} \right) \right\rfloor,$$

such that $r(N)$ is the minimal integer for which $N_{r(N)}^{\frac{\theta}{2}} \leq e^{\sqrt{\log(N)\log\log(N)}}$. In addition we define $n_0 := N - \sum_{j=1}^{r(N)} N_j$ and $n_k := \sum_{j=1}^{k} N_j + n_0$, where $1 \leq k \leq r(N)$. For $0 \leq k \leq r(N)$ we define $\Pi_k := \Pi_{\lfloor N_k^{\frac{\theta}{2}} \rfloor}$ and

$$\lambda_k := \sum_{\Delta \in \Pi_k} \left| P_\omega^{(\mathbf{0},0)}(X_{n_k} \in \Delta) - \mathbb{P}^{(\mathbf{0},0)}(X_{n_k} \in \Delta) \right|.$$

The definition of $\Pi_M$ for some arbitrary real number $M \in \mathbb{R}$ is given in Notation 3.26. We will prove that

$$\lambda_k \leq \lambda_{k-1} + C N_k^{-\alpha}, \quad \text{for some } \alpha \in (0,1), \tag{3.47}$$

and hence

$$\lambda_{r(N)} \leq \lambda_1 + C \sum_{k=1}^{r(N)} N_k^{-\alpha},$$

where $\lambda_1 \leq CN^{-c}$ for some $C, c > 0$ by Theorem 3.24 and the second term on the right side is bounded by $C' e^{-c'\sqrt{\log(N)\log\log(N)}}$ for some $C', c' > 0$. Note that

$$\begin{aligned}
\lambda_{r(N)} &= \sum_{\Delta \in \Pi_{r(N)}} \left| P_\omega^{(\mathbf{0},0)}(X_{n_{r(N)}} \in \Delta) - \mathbb{P}^{(\mathbf{0},0)}(X_{n_{r(N)}} \in \Delta) \right| \\
&= \sum_{\Delta \in \Pi_{r(N)}} |P_\omega^{(\mathbf{0},0)}(X_N \in \Delta) - \mathbb{P}^{(\mathbf{0},0)}(X_N \in \Delta)|,
\end{aligned}$$

is the total variation distance between quenched and annealed hitting probabilities of boxes of side length $N_{r(N)}^{\frac{\theta}{2}} \leq e^{\sqrt{\log(N)\log\log(N)}}$ we are interested in. To be more precise, within the last iteration step in (3.47) one should replace $N_{r(N)}^{\frac{\theta}{2}}$ by $e^{\sqrt{\log(N)\log\log(N)}}$. However the arguments given below hold true in both cases.

Let $k \geq 2$. Define

$$J_{n_k} := \{\Delta_x \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reachable for all } x \text{ with } \|x\| < \sqrt{N}\log^3(N)\}.$$

Note that by (3.45) we have $\mathbb{P}^{(\mathbf{0},0)}(\cap_{k=1}^{r(N)} J_{n_k}) \geq 1 - CN^{-c\log(N)}$. To shorten notation, we define

$$\widetilde{\Pi}_k := \{\Delta_x \in \Pi_k : \|x\| \leq \sqrt{n_k}\log^3(N)\}. \tag{3.48}$$

By Lemma 3.6 we have

$$\begin{aligned}\lambda_k &\leq \sum_{\Delta\in\widetilde{\Pi}_k}\left|P_\omega^{(\mathbf{0},0)}(X_{n_k}\in\Delta)-\mathbb{P}^{(\mathbf{0},0)}(X_{n_k}\in\Delta)\right|\mathbb{1}_{\{\Delta_x \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}}(\omega)+CN^{-c\log(N)}\\ &\leq \sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\left|P_\omega^{(\mathbf{0},0)}(X_{n_k}\in\Delta,X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_k}\in\Delta,X_{n_{k-1}}\in\Delta')\right|\\ &\qquad\cdot\mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}}(\omega)\mathbb{1}_{\{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)+CN^{-c\log(N)},\end{aligned}$$

on $J_N := \cap_{k=1}^{r(N)} J_{n_k}\cap Q((\mathbf{0},0),N)$. For the definition of $Q((\mathbf{0},0),N)$ see Lemma 3.6. The triangle inequality and the Markov property of $P_\omega$ then yield

$$\begin{aligned}&\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\left|P_\omega^{(\mathbf{0},0)}(X_{n_k}\in\Delta,X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_k}\in\Delta,X_{n_{k-1}}\in\Delta')\right|\\ &\qquad\cdot\mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}}(\omega)\mathbb{1}_{\{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\ &\leq \sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}P_\omega^{(u,n_{k-1})}(X_{n_k}\in\Delta)\\ &\qquad\cdot\left|P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u)-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\right|\\ &\qquad\cdot\mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega) \qquad (3.49)\\ &+\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\\ &\qquad\cdot\left|P_\omega^{(u,n_{k-1})}(X_{n_k}\in\Delta)-\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta)\right|\\ &\qquad\cdot\mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega) \qquad (3.50)\\ &+\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\Bigg|\sum_{u\in\Delta'}\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta)\\ &\qquad\cdot\Big(\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}=u)\Big)\Bigg|\\ &\qquad\cdot\mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega) \qquad (3.51)\\ &+\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\left|\sum_{u\in\Delta'}\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta)\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}=u)-\mathbb{P}^{(\mathbf{0},0)}(X_{n_k}\in\Delta,X_{n_{k-1}}\in\Delta')\right|\\ &\qquad\cdot\mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega). \qquad (3.52)\end{aligned}$$

We estimate (3.49)-(3.52) separately. We start with (3.49).

$$\begin{aligned}
&\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'} P_\omega^{(u,n_{k-1})}(X_{n_k}\in\Delta)\\
&\qquad\cdot\left|P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u)-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta\text{ is }(\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
\le&\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\left|P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u)-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
=&\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\left|P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
=&\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\left|P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')\right|\mathbb{1}_{\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
\le&\sum_{\Delta'\in\Pi_{k-1}}\left|P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')\right|=\lambda_{k-1}.
\end{aligned}$$

Now we turn to the second term (3.50). First of all note that

$$\begin{aligned}
&\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\\
&\qquad\cdot\left|P_\omega^{(u,n_{k-1})}(X_{n_k}\in\Delta)-\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta)\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta\text{ is }(\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
=&\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\\
&\qquad\cdot\sum_{\Delta\in\widetilde{\Pi}_k}\left|P_\omega^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta)-\mathbb{P}^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta)\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta\text{ is }(\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega).
\end{aligned}$$

As in the proof of [BCR16, Theorem 5.1] we call a cube $\Delta'\in\widetilde{\Pi}_{k-1}$ to be "good" if for every $u\in\Delta'$ with $\xi(u,n_{k-1})=1$ and every $\Delta\in\widetilde{\Pi}_k$

$$\left|P_\omega^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta)-\mathbb{P}^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta)\right|\le CN_k^{-\frac{d}{2}(1-\theta)-\frac{1}{6}\theta},\tag{3.53}$$

otherwise we call $\Delta'$ to be "bad".

Additionally, we call a cube $\Delta'\in\widetilde{\Pi}_{k-1}$ to be "well connected" if for every $u\in\Delta'$ with $\xi(u,n_{k-1})=1$

$$\sum_{\substack{\Delta_x\in\widetilde{\Pi}_k:\\ \|x-u\|\le\sqrt{N_k}\log^3(N_k)}}P_\omega^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta_x)\ge 1-N_k^{-\log(N_k)},\tag{3.54}$$

otherwise we call $\Delta'$ to be "badly connected".
Recall the definition of $\widetilde{\Pi}_{k-1}$ in (3.48) and note that the number of boxes $\Delta' \in \widetilde{\Pi}_{k-1}$ is bounded by

$$\left( C \frac{\sqrt{n_{k-1}} \log^3(n_{k-1})}{N_{k-1}^{\frac{\theta}{2}}} \right)^d .$$

By Lemma 3.6 there exists a set $Q((u, n_{k-1}), N_k) \subset B_{(u,n_{k-1})}$, with

$$\mathbb{P}\left( \left(B_{(u,n_{k-1})}\right)^c \cup Q((u, n_{k-1}), N_k) \right) \geq 1 - C N_k^{-c\log(N_k)},$$

such that on $Q((u, n_{k-1}), N_k)$ the bound given in (3.54) holds true. Furthermore, Theorem 3.24 yields that for every $u \in \Delta' (\in \widetilde{\Pi}_{k-1})$ there exists a set $G_4((u, n_{k-1}), N_k)$ with

$$\mathbb{P}\left( \left(B_{(u,n_{k-1})}\right)^c \cup G_4((u, n_{k-1}), N_k) \right) \geq 1 - C \left(N_k\right)^{-c\log(N_k)},$$

such that on $G_4((u, n_{k-1}), N_k)$ for every $\Delta (\in \Pi_k)$ the bound given in (3.53) holds true. Hence for "good" and "well connected" boxes $\Delta' \in \widetilde{\Pi}_{k-1}$ we get that

$$\begin{aligned}
&\sum_{u \in \Delta'} \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta') P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \sum_{\Delta \in \widetilde{\Pi}_k} \left| P_\omega^{(u,n_{k-1})}(X_{n_k} \in \Delta) - \mathbb{P}^{(u,n_{k-1})}(X_{n_k} \in \Delta) \right| \\
&\quad \cdot \mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\} \cap \{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega) \\
&\leq C \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta') \left( \left( \sqrt{N_k} \log^3(N_k) N_k^{-\frac{\theta}{2}} \right)^d N_k^{-\frac{d}{2}(1-\theta) - \frac{1}{6}\theta} + C \left(N_k\right)^{-c\log(N_k)} \right) \\
&\leq C \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta') \log^{3d}(N_k) N_k^{-\frac{1}{6}\theta}.
\end{aligned}$$

Since basically $\left( C \frac{\sqrt{n_{k-1}} \log^3(n_{k-1})}{N_{k-1}^{\frac{\theta}{2}}} \right)^d$ cubes $\Delta' \in \widetilde{\Pi}_{k-1}$ need to be considered where the probability for each cube $\Delta'$ to be "good" and "well connected" is of order

$$\begin{aligned}
&\mathbb{P}\left( \bigcap_{u \in \Delta'} \left[ \left(B_{(u,n_{k-1})}\right)^c \cup \left( Q((u, n_{k-1}), N_k) \cap G_4((u, n_{k-1}), N_k) \right) \right] \right) \\
&\geq 1 - \sum_{u \in \Delta'} \left( \mathbb{P}\left( B_{(u,n_{k-1})} \cap (Q((u, n_{k-1}), N_k))^c \right) + \mathbb{P}\left( B_{(u,n_{k-1})} \cap (G_4((u, n_{k-1}), N_k))^c \right) \right) \\
&\geq 1 - N^{-c\log\log(N)} \quad \text{for all } k \leq r(N),
\end{aligned}$$

we get that the probability of the event

$$G_N := \{\text{all the cubes in } \widetilde{\Pi}_{k-1} \text{ are "good" and "well connected"}\} \tag{3.55}$$

is bounded from below by

$$\begin{aligned}
\mathbb{P}(G_N) &\geq 1 - \sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \mathbb{P}\left( \{\Delta' \text{ is "bad" and "badly connected"}\} \right) \\
&\geq 1 - \left( C \frac{\sqrt{n_{k-1}} \log^3(n_{k-1})}{N_{k-1}^{\frac{\theta}{2}}} \right)^d N^{-c\log\log(N)} \\
&\geq 1 - C N^{-c\log\log(N)}.
\end{aligned} \tag{3.56}$$

Hence on $G_N$

$$\begin{aligned}
&\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')\\
&\qquad\cdot\sum_{\Delta\in\widetilde{\Pi}_k}\left|P_\omega^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta)-\mathbb{P}^{(u,n_{k-1})}(X_{n_{k-1}+N_k}\in\Delta)\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta\text{ is }(\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
&\le\sum_{\substack{\Delta'\in\widetilde{\Pi}_{k-1}:\\ \text{“good” and “well connected”}}}C\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')\log^3(N_k)N_k^{-\frac{1}{6}\theta}+\sum_{\substack{\Delta'\in\widetilde{\Pi}_{k-1}:\\ \text{“bad” or “badly connected”}}}C\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')\\
&\le CN_k^{-\frac{1}{6}\theta}.
\end{aligned}$$

Next we focus on (3.51). We have

$$\begin{aligned}
&\sum_{\Delta\in\widetilde{\Pi}_k}\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\Bigg|\sum_{u\in\Delta'}\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta)\\
&\qquad\cdot\left[\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')P_\omega^{(\mathbf{0},0)}(X_{n_{k-1}}=u|X_{n_{k-1}}\in\Delta')-\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}=u)\right]\Bigg|\\
&\qquad\cdot\mathbb{1}_{\{\Delta\text{ is }(\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
&\le\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{\substack{\Delta\in\widetilde{\Pi}_k:\text{dist}(\Delta',c(\Delta))\\ \le\sqrt{N_k}\log^3(N_k)}}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')\left|\max_{u\in\Delta'}\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta_x)-\min_{u\in\Delta'}\mathbb{P}^{(u,n_{k-1})}(X_{n_k}\in\Delta_x)\right|\\
&\quad+C\,(N_k)^{-c\log(N_k)},
\end{aligned}$$

on $G_N$, defined in the previous step. Making use of the annealed derivative estimates we get that the last term is bounded by

$$\sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \sum_{\substack{\Delta_x \in \widetilde{\Pi}_k : \mathrm{dist}(\Delta', x) \\ \leq \sqrt{N_k} \log^3(N_k)}} \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta')$$

$$\cdot \left| \max_{u \in \Delta'} \mathbb{P}^{(u,n_{k-1})}(X_{n_{k-1}+N_k} \in \Delta_x) - \min_{u \in \Delta'} \mathbb{P}^{(u,n_{k-1})}(X_{n_{k-1}+N_k} \in \Delta_x) \right|$$

$$+ C\,(N_k)^{-c \log(N_k)}$$

$$\leq \sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta') \sum_{\substack{\Delta_x \in \widetilde{\Pi}_k : \mathrm{dist}(\Delta', x) \\ \leq \sqrt{N_k} \log^3(N_k)}} \frac{C N_k^{\frac{d\theta}{2}} N_{k-1}^{\frac{\theta}{2}}}{N_k^{\frac{d+1}{2}}} + C\,(N_k)^{-c \log(N_k)}$$

$$\leq \sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta') \cdot \left( \frac{N_{k-1}^{\frac{\theta}{2}} + 2\sqrt{N_k} \log^3(N_k)}{N_k^{\frac{\theta}{2}}} \right)^d \cdot \frac{C N_k^{\frac{d\theta}{2}} N_{k-1}^{\frac{\theta}{2}}}{N_k^{\frac{d+1}{2}}} + C\,(N_k)^{-c \log(N_k)}$$

$$\leq \sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} \in \Delta') \cdot \frac{C \left(\sqrt{N_k}\right)^d \log^{3d}(N_k)}{N_k^{\frac{d\theta}{2}}} \cdot \frac{C N_k^{\frac{d\theta}{2}} N_{k-1}^{\frac{\theta}{2}}}{N_k^{\frac{d+1}{2}}} + C\,(N_k)^{-c \log(N_k)}$$

$$\leq C N_k^{-\frac{1}{2}+\theta} \log^{3d}(N_k).$$

Finally we consider the last term (3.52).

$$\sum_{\Delta \in \widetilde{\Pi}_k} \sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(u,n_{k-1})}(X_{n_k} \in \Delta) \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} = u) - \mathbb{P}^{(\mathbf{0},0)}(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') \right|$$

$$\cdot \mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\} \cap \{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)$$

$$\leq \sum_{\Delta' \in \widetilde{\Pi}_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}} = u)$$

$$\cdot \sum_{\Delta \in \widetilde{\Pi}_k} \left| \mathbb{P}^{(u,n_{k-1})}(X_{n_k} \in \Delta) - \mathbb{P}^{(\mathbf{0},0)}(X_{n_k} \in \Delta | X_{n_{k-1}} = u) \right|$$

$$\cdot \mathbb{1}_{\{\Delta \text{ is } (\mathbf{0},0)\text{-}n_k\text{-reach.}\} \cap \{\Delta' \text{ is } (\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega).$$

On $R_N$ (see (3.11)) the first regeneration after time $n_k - 1$ occurs with probability greater than $1 - n_k^{-\log(n_k)}$ before time $n_k + \log^2(n_k)$. Similar arguments hold true for the annealed walk starting in $u$. Hence in fact we have to deal with the difference of two annealed laws whose starting points differ in space and time at most by $2\log^2(N)$. Hence the annealed derivative estimates yield

$$\left| \mathbb{P}^{(u,n_{k-1})}(X_{n_k} \in \Delta) - \mathbb{P}^{(\mathbf{0},0)}(X_{n_k} \in \Delta | X_{n_{k-1}} = u) \right| \leq \frac{C \log^2(N_k) N_k^{\frac{d\theta}{2}}}{(N_k - \log^{3d}(N_k))^{\frac{d+1}{2}}} \leq \frac{C \log^2(N_k) N_k^{\frac{d\theta}{2}}}{(N_k)^{\frac{d+1}{2}}}.$$

Since we only need to consider boxes $\Delta$ whose center is distance $\sqrt{N_k} \log^3(N_k)$ apart from $u$, we get that

$$\begin{aligned}
&\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}=u)\\
&\qquad\cdot\sum_{\Delta\in\widetilde{\Pi}_k}\left|\mathbb{P}^{(u,n_{k-1})}(X_{n_k-n_{k-1}}\in\Delta)-\mathbb{P}^{(\mathbf{0},0)}(X_{n_k}\in\Delta|X_{n_{k-1}}=u)\right|\\
&\qquad\cdot\mathbb{1}_{\{\Delta\text{ is }(\mathbf{0},0)\text{-}n_k\text{-reach.}\}\cap\{\Delta'\text{ is }(\mathbf{0},0)\text{-}n_{k-1}\text{-reach.}\}}(\omega)\\
&\le\sum_{\Delta'\in\widetilde{\Pi}_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}=u)\cdot\left(\frac{\sqrt{N_k}\log^3(N_k)}{N_k^{\frac{\theta}{2}}}\right)^d\cdot\frac{C\log^{3d}(N_k)N_k^{\frac{d\theta}{2}}}{(N_k)^{\frac{d+1}{2}}}\\
&\le C\log^{6d}(N_k)N_k^{-\frac{1}{2}}.
\end{aligned}$$

Hence all together we get that on the event $R_N\cap G_N$ with

$$\mathbb{P}^{(\mathbf{0},0)}(R_N\cap G_N)\ge 1-CN^{-c\log\log(N)}$$

for every $k\in\{1,..,r(N)\}$ we have

$$\lambda_k\le\lambda_{k-1}+CN_k^{-\frac{1}{3}\theta}+CN_k^{-\frac{1}{2}+\theta}\log^{3d}(N_k)+C\log^{6d}(N_k)N_k^{-\frac{1}{2}}\le\lambda_{k-1}+CN_k^{-\frac{\theta}{3}},$$

if $N$ is sufficiently large. This yields

$$\begin{aligned}
\lambda_{r(N)}&\le\lambda_1+C\sum_{k=1}^{r(N)}N_k^{-\frac{\theta}{3}}\le\lambda_1+C\sum_{k=1}^{r(N)}N^{-\frac{\theta}{3\cdot 2^k}}\\
&\le\lambda_1+C\int_1^{r(N)+1}e^{-\log(N)\frac{\theta}{3\cdot 2^s}}ds\\
&\overset{u=\log(N)\frac{\theta}{3\cdot 2^s}}{=}\lambda_1+\int_{\alpha_N}^{\beta_N}\frac{e^{-u}}{-\log(2)u}du,
\end{aligned}$$

where $\alpha_N=\frac{\theta}{6}\log(N)$ and $\beta_N=\frac{\theta}{3\cdot 2^{r(N)+1}}\log(N)\ge\frac{\sqrt{\log(N)\log\log(N)}}{6}$. Hence

$$\begin{aligned}
\lambda_{r(N)}&\le\lambda_1+\int_{\alpha_N}^{\beta_N}\frac{e^{-u}}{-\log(2)u}ds\le\lambda_1+\int_{\alpha_N}^{\beta_N}-e^{-u}du\\
&\le\lambda_1+\left[e^{-u}\right]_{\alpha_N}^{\beta_N}\le\lambda_1+e^{-\beta_N}\le\lambda_1+e^{-c\sqrt{\log(N)\log\log(N)}},
\end{aligned}$$

for some $c>0$. The proof is complete since by Theorem 3.24 we have $\lambda_1\le CN^{-\frac{\theta}{3}}$.

□

**Remark 3.29.** Recall the definition of $G_N$ in (3.55). Decreasing the box size of the partition of $\mathbb{Z}^d$ down to a constant real number, we need to find suitable bounds on the number of "bad" or "badly connected" boxes in which we have not succeed up to now. "Suitable" means that

$$\begin{aligned}
\sum_{\substack{\Delta'\in\widetilde{\Pi}_{k-1}:\\ \text{"bad" or "badly connected"}}}C\mathbb{P}^{(\mathbf{0},0)}(X_{n_{k-1}}\in\Delta')&\le\sum_{\substack{\Delta'\in\widetilde{\Pi}_{k-1}:\\ \text{"bad" or "badly connected"}}}C(n_{k-1})^{-\frac{d}{2}}N_{k-1}^{\frac{d\theta}{2}}\\
&\le CN^{-\alpha},\quad\text{for some }\alpha>0.
\end{aligned}$$

Hence the number of "bad" or "badly connected" cubes should be of order $N^\beta$ for some $\beta<\frac{d}{2}$ with high probability. Further investigation in decreasing box size down to a constant real number would probably be the next step towards proving a quenched local central limit theorem.

# Bibliography


[AdHR11] L. Avena, F. den Hollander, and F. Redig, *Law of large numbers for a class of random walks in dynamic random environments*, Electron. J. Probab. **16** (2011), no. 21, 587–617. MR 2786643

[AG89] R. Arratia and L. Gordon, *Tutorial on large deviations for the binomial distribution*, Bull. Math. Biol. **51** (1989), no. 1, 125–131. MR 978907

[Ale11] Kenneth S. Alexander, *Excursions and local limit theorems for Bessel-like random walks*, Electron. J. Probab. **16** (2011), no. 1, 1–44. MR 2749771

[Arr79] Richard Alejandro Arratia, *COALESCING BROWNIAN MOTIONS ON THE LINE*, ProQuest LLC, Ann Arbor, MI, 1979, Thesis (Ph.D.)–The University of Wisconsin - Madison. MR 2630231

[Asm03] Søren Asmussen, *Applied probability and queues*, second ed., Applications of Mathematics (New York), vol. 51, Springer-Verlag, New York, 2003, Stochastic Modelling and Applied Probability. MR 1978607 (2004f:60001)

[BČD16] Matthias Birkner, Jiří Černý, and Andrej Depperschmidt, *Random walks in dynamic random environments and ancestry under local population regulation*, Electron. J. Probab. **21** (2016), Paper No. 38, 43. MR 3508685

[BČDG13] Matthias Birkner, Jiří Černý, Andrej Depperschmidt, and Nina Gantert, *Directed random walk on the backbone of an oriented percolation cluster*, Electron. J. Probab. **18** (2013), no. 80, 35. MR 3101646

[BCR16] Noam Berger, Moran Cohen, and Ron Rosenthal, *Local limit theorem and equivalence of dynamic and static points of view for certain ballistic random walks in i.i.d. environments*, Ann. Probab. **44** (2016), no. 4, 2889–2979. MR 3531683

[Ber12] Noam Berger, *Slowdown estimates for ballistic random walk in random environment*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 1, 127–174. MR 2862036

[Bet16] Stein Andreas Bethuelsen, *The contact process as seen from a random walk*, arXiv preprint arXiv:1607.03410 (2016).

[BG90a] Carol Bezuidenhout and Geoffrey Grimmett, *The critical contact process dies out*, Ann. Probab. **18** (1990), no. 4, 1462–1482. MR 1071804

[BG90b] Jean-Philippe Bouchaud and Antoine Georges, *Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications*, Phys. Rep. **195** (1990), no. 4-5, 127–293. MR 1081295

[BH91] E. Buffet and P. Hannigan, *Directed random walks in random environments*, J. Statist. Phys. **65** (1991), no. 3-4, 645–672. MR 1137427

[BH15] Stein Andreas Bethuelsen and Markus Heydenreich, *Law of large numbers for random walks on attractive spin-flip dynamics*, arXiv preprint arXiv:1411.3581v2 (2015).

[BMP08] C Boldrighini, RA Minlos, and A Pellegrinotti, *Random walks in random environment with markov dependence on time*, Condensed Matter Physics **11** (2008), no. 2, 209–221.

[BS02] Erwin Bolthausen and Alain-Sol Sznitman, *On the static and dynamic points of view for certain random walks in random environment*, Methods Appl. Anal. **9** (2002), no. 3, 345–375, Special issue dedicated to Daniel W. Stroock and Srinivasa S. R. Varadhan on the occasion of their 60th birthday. MR 2023130

[BV16] Stein Andreas Bethuelsen and Florian Völlering, *Absolute continuity and weak uniform mixing of random walk in dynamic random environment*, arXiv preprint arXiv:1601.07710 (2016).

[BZ06] Antar Bandyopadhyay and Ofer Zeitouni, *Random walk in dynamic Markovian random environment*, ALEA Lat. Am. J. Probab. Math. Stat. **1** (2006), 205–224. MR 2249655

[DG82] Richard Durrett and David Griffeath, *Contact processes in several dimensions*, Z. Wahrsch. Verw. Gebiete **59** (1982), no. 4, 535–552. MR 656515

[dHdSS13] F. den Hollander, R. dos Santos, and V. Sidoravicius, *Law of large numbers for non-elliptic random walks in dynamic random environments*, Stochastic Process. Appl. **123** (2013), no. 1, 156–190. MR 2988114

[dHKS14] Frank den Hollander, Harry Kesten, and Vladas Sidoravicius, *Random walk in a high density dynamic random environment*, Indag. Math. (N.S.) **25** (2014), no. 4, 785–799. MR 3217035

[DKW16] Denis Denisov, Dmitry Korshunov, and Vitali Wachtel, *At the edge of criticality: Markov chains with asymptotically zero drift*, arXiv preprint arXiv:1612.01592 (2016).

[DL09] Dmitry Dolgopyat and Carlangelo Liverani, *Non-perturbative approach to random walk in Markovian environment*, Electron. Commun. Probab. **14** (2009), 245–251. MR 2507753

[Dur84] Richard Durrett, *Oriented percolation in two dimensions*, Ann. Probab. **12** (1984), no. 4, 999–1040. MR 757768 (86g:60117)

[Eth11] Alison Etheridge, *Some mathematical models from population genetics*, Lecture Notes in Mathematics, vol. 2012, Springer, Heidelberg, 2011, Lectures from the 39th Probability Summer School held in Saint-Flour, 2009. MR 2759587

[Fel71] William Feller, *An introduction to probability theory and its applications. Vol. II.*, Second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971. MR 0270403

[FINR02] L. R. G. Fontes, M. Isopi, C. M. Newman, and K. Ravishankar, *The Brownian web*, Proc. Natl. Acad. Sci. USA **99** (2002), no. 25, 15888–15893 (electronic). MR 1944976

[FINR04] ———, *The Brownian web: characterization and convergence*, Ann. Probab. **32** (2004), no. 4, 2857–2883. MR 2094432

[GH02] Geoffrey Grimmett and Philipp Hiemer, *Directed percolation and random walk*, In and out of equilibrium (Mambucaba, 2000), Progr. Probab., vol. 51, Birkhäuser Boston, Boston, MA, 2002, pp. 273–297. MR 1901958 (2003b:60159)

[GM14] Olivier Garet and Régine Marchand, *Large deviations for the contact process in random environment*, Ann. Probab. **42** (2014), no. 4, 1438–1479. MR 3262483

[Gri10] Geoffrey Grimmett, *Probability on graphs*, Institute of Mathematical Statistics Textbooks, vol. 1, Cambridge University Press, Cambridge, 2010, Random processes on graphs and lattices. MR 2723356

[Har74] T. E. Harris, *Contact interactions on a lattice*, Ann. Probability **2** (1974), 969–988. MR 0356292

[HdHdS+15] M. R. Hilário, F. den Hollander, R. S. dos Santos, V. Sidoravicius, and A. Teixeira, *Random walk on random walks*, Electron. J. Probab. **20** (2015), no. 95, 35. MR 3399831

[HK87] Joseph W Haus and Klaus W Kehr, *Diffusion in regular and disordered lattices*, Physics Reports **150** (1987), no. 5-6, 263–406.

[HMRAR98] M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, and B. Reed (eds.), *Probabilistic methods for algorithmic discrete mathematics*, Algorithms and Combinatorics, vol. 16, Springer-Verlag, Berlin, 1998. MR 1678554

[JRA11] Mathew Joseph and Firas Rassoul-Agha, *Almost sure invariance principle for continuous-space random walk in dynamic random environment*, ALEA Lat. Am. J. Probab. Math. Stat. **8** (2011), 43–57. MR 2748407

[Kes63] Harry Kesten, *Ratio theorems for random walks. II*, J. Analyse Math. **11** (1963), 323–379. MR 0163365

[Kuc89] Thomas Kuczek, *The central limit theorem for the right edge of supercritical oriented percolation*, Ann. Probab. **17** (1989), no. 4, 1322–1332. MR 1048929

[Lig85] Thomas M. Liggett, *Interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 276, Springer-Verlag, New York, 1985. MR 776231

[Lig99] ———, *Stochastic interacting systems: contact, voter and exclusion processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 324, Springer-Verlag, Berlin, 1999. MR 1717346 (2001g:60247)

[LL10] Gregory F. Lawler and Vlada Limic, *Random walk: a modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010. MR 2677157

[LPW09] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer, *Markov chains and mixing times*, American Mathematical Society, Providence, RI, 2009, With a chapter by James G. Propp and David B. Wilson. MR 2466937 (2010c:60209)

[LS06] Thomas M. Liggett and Jeffrey E. Steif, *Stochastic domination: the contact process, Ising models and FKG measures*, Ann. Inst. H. Poincaré Probab. Statist. **42** (2006), no. 2, 223–243. MR 2199800

[Mil16] Katja Miller, *Random walks on weighted, oriented percolation clusters*, ALEA, Lat. Am. J. Probab. Math. Stat. **13** (2016), 53–77.

[Neu92] Claudia Neuhauser, *Ergodic theorems for the multitype contact process*, Probab. Theory Related Fields **91** (1992), no. 3-4, 467–506. MR 1151806 (93c:60162)

[NRS10] C. M. Newman, K. Ravishankar, and E. Schertzer, *Marking* $(1,2)$ *points of the Brownian web and applications*, Ann. Inst. Henri Poincaré Probab. Stat. **46** (2010), no. 2, 537–574. MR 2667709

[RAS05] Firas Rassoul-Agha and Timo Seppäläinen, *An almost sure invariance principle for random walks in a space-time random environment*, Probab. Theory Related Fields **133** (2005), no. 3, 299–314. MR 2198014

[RV13] Frank Redig and Florian Völlering, *Random walks in dynamic random environments: a transference principle*, Ann. Probab. **41** (2013), no. 5, 3157–3180. MR 3127878

[Saw76] Stanley Sawyer, *Results for the stepping stone model for migration in population genetics*, Ann. Probability **4** (1976), no. 5, 699–728. MR 0682605 (58 #33157)

[Spi76] Frank Spitzer, *Principles of random walk*, second ed., Springer-Verlag, New York-Heidelberg, 1976, Graduate Texts in Mathematics, Vol. 34. MR 0388547 (52 #9383)

[SS13] Anish Sarkar and Rongfeng Sun, *Brownian web in the scaling limit of supercritical oriented percolation in dimension* $1+1$, Electron. J. Probab. **18** (2013), no. 21, 23. MR 3035749

[SSS14] Emmanuel Schertzer, Rongfeng Sun, and Jan M. Swart, *Stochastic flows in the Brownian web and net*, Mem. Amer. Math. Soc. **227** (2014), no. 1065, vi+160. MR 3155782

[Sun05] Rongfeng Sun, *Convergence of coalescing nonsimple random walks to the Brownian Web*, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–New York University. MR 2706830

[SZ11] M Shegeftfard and SM Zamanzadeh, *Utilization of percolation theory to estimate water injection breakthrough time*, Energy Exploration & Exploitation **29** (2011), no. 5, 667–678.

[TW98] Bálint Tóth and Wendelin Werner, *The true self-repelling motion*, Probab. Theory Related Fields **111** (1998), no. 3, 375–452. MR 1640799

[Uch11] Kôhei Uchiyama, *The first hitting time of a single point for random walks*, Electron. J. Probab. **16** (2011), no. 71, 1960–2000. MR 2851052

[WK65] George H. Weiss and Motoo Kimura, *A mathematical analysis of the stepping stone model of genetic correlation*, J. Appl. Probability **2** (1965), 129–149. MR 0177827 (31 #2089)

# CHAPTER A

# Appendix

## A.1. Proofs of chapter 1

### A.1.1. Proof of Proposition 2.1; equation (2.2) for $d = 1$

We separate the proof into 4 steps:

Step 1: We prove that there exists a large constant $K_1$ such that

$$\sum_y |\Psi^{diff}_{ind}(x,y) - \Psi^{diff}_{joint}(x,y)||y|^\alpha \leq Ce^{-c\sqrt{|x|}} \quad \text{for all } |x| > K_1.$$

We choose $K_1$ sufficiently large, split the sum in two parts and estimate them separately.

$$\begin{aligned}\sum_y |\Psi^{diff}_{ind}(x,y) - \Psi^{diff}_{joint}(x,y)||y|^\alpha = &\sum_{y=x-\sqrt{|x|}}^{y=x+\sqrt{|x|}} |\Psi^{diff}_{ind}(x,y) - \Psi^{diff}_{joint}(x,y)||y|^\alpha \\ &+ \sum_{y:|y-x|>\sqrt{|x|}} |\Psi^{diff}_{ind}(x,y) - \Psi^{diff}_{joint}(x,y)||y|^\alpha.\end{aligned}$$

By (2.14) the first sum can be bounded by

$$\sum_{y=x-\sqrt{|x|}}^{y=x+\sqrt{|x|}} |\Psi^{diff}_{joint}(x,y) - \Psi^{diff}_{ind}(x,y)||y|^\alpha \leq 2\sqrt{|x|}(|x| + \sqrt{|x|})^\alpha Ce^{-c|x|} \leq Ce^{-c|x|}$$

for all $|x| \geq K_1$, if $K_1$ is chosen large enough. The second sum can be estimated as

$$\begin{aligned}&\sum_{y:|y-x|>\sqrt{|x|}} |\Psi^{diff}_{joint}(x,y) - \Psi^{diff}_{ind}(x,y)||y|^\alpha \\ &\leq \sum_{y:|y-x|>\sqrt{|x|}} (\mathbb{P}^{joint}(T^{sim}_1 \geq |x-y|) + \mathbb{P}^{ind}(T^{sim}_1 \geq |x-y|))|y|^\alpha \\ &\leq Ce^{-c\sqrt{|x|}}.\end{aligned}$$

Thus step 1 is proved.

Step 2: By using similar estimations as in step 1, we see that

$$\sum_{y:|y-x|>\sqrt{|x|}} \Psi^{diff}_{ind}(x,y)|y|^\alpha \leq Ce^{-c\sqrt{|x|}} \quad \text{for all } |x| \geq K_1,$$

if $K_1$ is chosen large enough.

Step 3: We prove

$$\sum_{y=x-\sqrt{x}}^{y=x+\sqrt{x}} \Psi^{diff}_{ind}(x,y)|y|^\alpha \leq |x|^\alpha - C|x|^{\alpha-2} \quad \text{for all } |x| > K_1.$$

Using Taylor's theorem we know that $(1+z)^\alpha = 1 + \alpha z + \frac{1}{2}\alpha(\alpha-1)z^2(1+\xi)^{\alpha-2}$ for some $\xi \in (-|z|,|z|)$. Therefore we get

$$\sum_{y=x-\sqrt{|x|}}^{y=x+\sqrt{|x|}} \Psi_{ind}^{diff}(x,y)|y|^\alpha = |x|^\alpha \sum_{y=-\sqrt{|x|}}^{y=\sqrt{|x|}} \Psi_{ind}^{diff}(y)(1+\tfrac{y}{x})^\alpha \le |x|^\alpha - \tfrac{C}{8}\alpha|1-\alpha||x|^{\alpha-2} \le |x|^\alpha - C|x|^{\alpha-2},$$

where the linear term vanishes because of the symmetry of $\Psi_{ind}^{diff}$.

Step 4: Putting together the results of step 1-3 we obtain

$$\begin{aligned}\sum_y \Psi_{joint}^{diff}(x,y)|y|^\alpha &\le \sum_y |\Psi_{joint}^{diff}(x,y) - \Psi_{ind}^{diff}(x,y)||y|^\alpha + \sum_{y:|y-x|>\sqrt{|x|}} \Psi_{ind}^{diff}(x,y)|y|^\alpha + \sum_{y=x-\sqrt{x}}^{y=x+\sqrt{x}} \Psi_{ind}^{diff}(x,y)|y|^\alpha \\ &\le |x|^\alpha - C|x|^{\alpha-2} + Ce^{-c\sqrt{|x|}} \\ &\le |x|^\alpha \quad \text{for all } |x| \ge K_1,\end{aligned}$$

which therefore proves the recurrence of $(\widehat{X}_k^{(x_1)} - \widehat{X}_k^{(x_2)})_{k\ge 0}$ in the joint case.

□

### A.1.2. Proof of Proposition 2.1, equation (2.2) for $d = 2$

In order to simplify notation we define

$$\widehat{D}_n^{(x_1-x_2)} := (\widehat{D}_{1,n}^{(x_1-x_2)}, \widehat{D}_{2,n}^{(x_1-x_2)}) := \widehat{X}_n^{(x_1)} - \widehat{X}_n^{(x_2)},$$

where $\mathbb{P}_{joint}(\widehat{D}_0^{(x_1-x_2)} = x_1 - x_2 | B_{x_1-x_2,\mathbf{0};0}) = 1$. First we assume the covariance for $\widehat{D}_{1,1}^{(x_1-x_2)}$ and $\widehat{D}_{2,1}^{(x_1-x_2)}$ under $\mathbb{P}_{ind}(\,\cdot\,|B_{x_1-x_2,\mathbf{0};0})$ to be zero. The proof in the two-dimensional case is very similar to the one-dimensional case. In the two-dimensional case a natural candidate for the superharmonic function with the desired properties is $h(x) := \log^\alpha(\|x\|_2)$ for some $\alpha \in (0,1)$. Within the following calculations $\|\cdot\| := \|\cdot\|_2$. We divide the proof into similar steps:

Step 1-2: The estimations of step 1 and step 2 in the one-dimensional case can be adapted to the two-dimensional setting. Therefore similar results hold true:

There exists a large constant $K_2$ such that

$$\sum_y |\Psi_{ind}^{diff}(x,y) - \Psi_{joint}^{diff}(x,y)| \log^\alpha(\|y\|) \le Ce^{-c\sqrt[4]{\|x\|}} \quad \text{for all } \|x\| > K_2,$$

and

$$\sum_{y:\|y-x\|>\sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(x,y)\log^\alpha(\|y\|) \le Ce^{-c\sqrt[4]{\|x\|}} \quad \text{for all } \|x\| \ge K_2.$$

The fourth root is needed for technical reasons in order to get suitable bounds on the remainder of the Taylor expansion in step 3.
Step 3: As in the one-dimensional case we use a Taylor expansion to prove that there exists a positive function $F$ which decays polynomially as $\|x\|$ tends to infinity, such that

$$\sum_{y:\|y-x\|\le\sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(x,y)\log^\alpha(\|y\|) \le \log(\|x\|)^\alpha - F(x) \quad \text{for all } \|x\| > K_2.$$

We define $f_x(h_1,h_2) := \log^\alpha\left(\|x\|^2\left[(\frac{x_1}{\|x\|}+h_1)^2 + (\frac{x_2}{\|x\|}+h_2)^2\right]\right)$ and write the term above as

$$\sum_{y:\|y-x\|\le\sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(x,y)\log^\alpha(\|y\|) = \frac{1}{2^\alpha} \sum_{y:\|y\|\le\sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(y)\cdot f_x(\tfrac{y_1}{\|x\|}, \tfrac{y_2}{\|x\|}). \tag{A.1}$$

Using Taylor's theorem, the function $f_x(h_1,h_2)$ can be written as

$$f_x(h_1,h_2) = \log^\alpha(\|x\|^2) \; + P_x(h_1,h_2) + R_x(h_1,h_2),$$

where $P_x(h_1,h_2) := T_x^3(h_1,h_2) - \log^\alpha(\|x\|^2)$ and $T_x^3$ is the third Taylor polynomial. Notice that $P_x(0,0)=0$. At first we focus on the term $\sum_{y:\|y\|\le \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(y)\cdot P_x(\frac{y_1}{\|x\|},\frac{y_2}{\|x\|})$. The linear and cubic terms in

$$\sum_{y:\|y\|\le \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(y)\cdot P_x(\tfrac{y_1}{\|x\|},\tfrac{y_2}{\|x\|})$$

vanish because of the symmetry of $\Psi_{ind}^{diff}$. Since we assumed the covariance of $\widehat{D}_1^{(x_1-x_2)}$ and $\widehat{D}_1^{(x_1-x_2)}$ to be zero, we know that

$$\sum_{y:\|y\|\le \sqrt[4]{\|x\|}} \partial_{h_1}\partial_{h_2} f_x(0,0)\Psi_{ind}^{diff}(y)\tfrac{y_1y_2}{\|x\|^2} \le Ce^{-c\sqrt[4]{\|x\|}}.$$

For the quadratic terms the following bound holds true

$$\begin{aligned}
&\sum_{y:\|y\|\le \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(y)\partial_{h_1}^2 f_x(0,0)\frac{y_1^2}{\|x\|^2} + \sum_{y:\|y\|\le \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(y)\partial_{h_2}^2 f_x(0,0)\frac{y_2^2}{\|x\|^2}\\
&\le -4\alpha(1-\alpha)\sigma^2\frac{\log^{(\alpha-2)}(\|x\|^2)}{\|x\|^2} + Ce^{-c\sqrt[4]{\|x\|}}.
\end{aligned}$$

The remainder $R_x(h_1,h_2)$ can be bounded by $\frac{C}{\|x\|^{9/4}}$.
Altogether we get that

$$\sum_{y:\|y-x\|\le \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(x,y)\log^\alpha(\|y\|) \le \log^\alpha(\|x\|) - \frac{C}{\log^{(2-\alpha)}(\|x\|^2)\,\|x\|^2} + \frac{C}{\|x\|^{9/4}} + Ce^{-c\sqrt[4]{\|x\|}}.$$

<u>Step 4:</u>
Using the results of step 1-3 we get that

$$\begin{aligned}
&\sum_y \Psi_{joint}^{diff}(x,y)\log^\alpha(\|y\|)\\
&\le \sum_y |\Psi_{joint}^{diff}(x,y) - \Psi_{ind}^{diff}(x,y)|\log^\alpha(\|y\|) + \sum_{y:\|y-x\|\le \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(x,y)\log^\alpha(\|y\|)\\
&\quad + \sum_{y:\|y-x\|> \sqrt[4]{\|x\|}} \Psi_{ind}^{diff}(x,y)\log^\alpha(\|y\|)\\
&\le \log^\alpha(\|x\|) - \frac{C}{\log^{(2-\alpha)}(\|x\|^2)\,\|x\|^2} + \frac{C}{\|x\|^{9/4}} + Ce^{-c\sqrt[4]{\|x\|}}\\
&\le \log^\alpha(\|x\|) \quad \text{for all } \; \|x\| \ge K_2,
\end{aligned}$$

which proves the recurrence of $(\widehat{X}_k^{(x_1)} - \widehat{X}_k^{(x_2)})_{k\ge 0}$ under $\mathbb{P}_{ind}(\;\cdot\;|B_{x_1-x_2,\mathbf{0};0})$ in dimension two, if the covariance of $\widehat{D}_{1,1}^{(x_1-x_2)}$ and $\widehat{D}_{2,1}^{(x_1-x_2)}$ is zero.
Now we assume that

$$\mathrm{Cov}^{ind}\left(\widehat{D}_1^{(x_1-x_2)}\right) := \begin{pmatrix}\bar\sigma^2 & \bar\rho\\ \bar\rho & \bar\sigma^2\end{pmatrix},$$

where $|\bar\rho| < \bar\sigma^2$, since Birkner et al. proved in [BČDG13] that the limit law is not concentrated on a subspace. If we define $A := \begin{pmatrix}\frac{1}{\sqrt{2(\sigma^2+\rho)}} & \frac{1}{\sqrt{2(\sigma^2+\rho)}}\\ \frac{1}{\sqrt{2(\sigma^2-\rho)}} & -\frac{1}{\sqrt{2(\sigma^2-\rho)}}\end{pmatrix}$, we know that $\mathrm{Cov}\left(A\widehat{D}_1^{(x_1-x_2)}\right) := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

There exist constants $C_1, C_2$, depending only on $\rho$ and $\sigma$ such that $C_1\|x\| \le \|Ax\| \le C_2\|x\|$ for all $x\in\mathbb{R}^2$. We choose the function $h$ to be $h(x) := \log^\alpha(\|Ax\|)$ and define $\tilde{x} := Ax$. From the inequalities above, we know that $\|\tilde{x}\|$ is large if

$\|x\|$ is large.

If $K_2$ is chosen large enough, similar estimates as in step $1-2$ hold true.

$$\sum_{y} |\Psi^{diff}_{ind}(x,y) - \Psi^{diff}_{joint}(x,y)| \log^\alpha(\|Ay\|) \le Ce^{-c\sqrt[4]{\|x\|}} \quad \text{for all } \|x\| > K_2,$$
$$\sum_{y:\|y-x\|>\sqrt[4]{\|\tilde{x}\|}} \Psi^{diff}_{ind}(x,y) \log^\alpha(\|Ay\|) \le Ce^{-c\sqrt[4]{\|x\|}} \quad \text{for all } \|x\| \ge K_2.$$

In order to get a result similar to step 3 we introduce the function $f_{\tilde{x}}(h_1,h_2) := \log^\alpha\left(\|\tilde{x}\|^2\left[(\frac{\tilde{x}_1}{\|\tilde{x}\|}+h_1)^2+(\frac{\tilde{x}_2}{\|\tilde{x}\|}+h_2)^2\right]\right)$ and write the term

$$\sum_{y:\|y-x\|\le\sqrt[4]{\|\tilde{x}\|}} \Psi^{diff}_{ind}(x,y) \log^\alpha(\|Ay\|)$$

on which we focus in step 3 in the following way

$$\sum_{y:\|y-x\|\le\sqrt[4]{\|\tilde{x}\|}} \Psi^{diff}_{ind}(x,y) \log^\alpha(\|Ay\|) = \frac{1}{2^\alpha} \sum_{y:\|y\|\le\sqrt[4]{\|\tilde{x}\|}} \Psi^{diff}_{ind}(y) \cdot f_{\tilde{x}}\left(\frac{(Ay)_1}{\|\tilde{x}\|}, \frac{(Ay)_2}{\|\tilde{x}\|}\right).$$

We know that

$$\frac{\|Ay\|}{\|\tilde{x}\|} \le \frac{C_2\,\|y\|}{\|Ax\|} \le \frac{C}{\|x\|^{3/4}},$$

which means that $\frac{(Ay)_1}{\|\tilde{x}\|}$ and $\frac{(Ay)_2}{\|\tilde{x}\|}$ are small if $K_2$ is large.

By using similar arguments as in the uncorrelated case we get that

$$\sum_{y} \Psi^{diff}_{joint}(x,y) \log^\alpha(\|Ay\|) \le \log^\alpha(\|Ax\|) \quad \text{for all } \|x\| \ge K_2,$$

which proves the result in the general case. □

### A.1.3. Proposition 2.1 equation (2.3) for random initial values

In Proposition 2.1 we prove tail bounds conditioned on the event that the initial points are connected to infinity. In this section we give a proof for the tail bounds of dimension $d = 1$ to hold true, if we do not condition on the event that the starting points are connected to infinity. We use the convention introduced at the beginning of section 2.3 where the random walks start from the next point left to the given site that is connected to infinity. See especially (2.54).

**Lemma A.1.** *Let $d = 1$. There exist constants $C, M > 0$ such that*

$$\mathbb{P}(T^{c((-1,0)),c((0,0))}_{meet} > m) \le \frac{C}{\sqrt{m}} \quad \textit{for all} \quad m > M. \tag{A.2}$$

**Proof:** Since both random walks start at time 0, we suppress the time component in $c(\,\cdot\,)$ and write $B_x$ instead of $B_{(x,0)}$, $x \in \mathbb{Z}$, see (1.26). First notice that $c(-1) = c(0)$ on $(B_0)^c$, hence $T^{(c(-1),c(0))}_{meet} = 0$. Therefore

$$\mathbb{P}(T^{(c(-1),c(0))}_{meet} > m) = \mathbb{P}(T^{(c(-1),c(0))}_{meet} > m, B_0).$$

We get

$$\begin{aligned}\mathbb{P}\left(T^{(c(-1),c(0))}_{meet} > m\right) &= \sum_{k=1}^{\infty} \mathbb{P}\left(T^{(c(-1),c(0))}_{meet} > m, B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right)\\ &\le \sum_{k=1}^{K\log(m)} \mathbb{P}\left(T^{(-k,0)}_{meet} > m, B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) + Ce^{-cK\log(m)}\\ &\le \sum_{k=K'}^{K\log(m)} \mathbb{P}\left(T^{(-k,0)}_{meet} > m, B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) + \frac{C}{\sqrt{m}},\end{aligned}$$

where $K$ is chosen such that $K \cdot c > 2$ and $K' > 0$ is a constant. The first inequality holds true by (2.56), see also [Dur84, Section 10], whereas the last inequality follows by (2.3) of Proposition 2.1. Hence we focus on

$$\mathbb{P}\left(T^{(-k,0)}_{meet} > m, B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right), \quad \text{for } k \in \{K', ..., \lfloor K\log(m)\rfloor\}. \tag{A.3}$$

Conditioned on the event that there exists a hole of length $k-1$ left to 0 we get

$$\begin{aligned}
&\mathbb{P}\left(T_{meet}^{(-k,0)} > m, B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&= \mathbb{P}\left(T_{meet}^{(-k,0)} > m \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \cdot \mathbb{P}\left(B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \mathbb{P}\left(T_{meet}^{(-k,0)} > m \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \cdot Ce^{-ck},
\end{aligned}$$

since the probability of holes of length $k-1$ to occur, decays exponentially in $k$. Define

$$R_k := \inf\{T_n^{sim} > 0 : l(-r,0) + 1 < T_n \text{ for all } 0 < r < k\},$$

where $l(y,m)$ denotes the length of the longest open path starting from $(y,m)$. Then

$$\begin{aligned}
&\mathbb{P}\left(T_{meet}^{(-k,0)} > m \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&= \sum_{r>0} \sum_{\substack{x_1 : |x_1+k|\} \leq r \\ x_2 : |x_2| \leq r}} \mathbb{P}\left(T_{meet}^{(-k,0)} > m, R_k = r, X_r^{(-k)} = x_1, X_r^{(0)} = x_2 \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \sum_{r>0} \sum_{\substack{x_1 : |x_1+k| \leq r \\ x_2 : |x_2| \leq r}} \left(\frac{C(k+2r)}{\sqrt{m-r}} \wedge 1\right) \mathbb{P}\left(R_k = r, X_r^{(-k)} = x_1, X_r^{(0)} = x_2 \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&= \sum_{r>0} \left(\frac{C(k+2r)}{\sqrt{m-r}} \wedge 1\right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right)
\end{aligned} \tag{A.4}$$

where the second inequality holds true by Proposition 2.1 with fixed starting points $x_1$ and $x_2$ with $|x_1 - x_2| \leq k + 2r$. Note that

$$\begin{aligned}
&\mathbb{P}\left(B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\geq \mathbb{P}\left(\omega(-k,0) = 1, \omega(-k+1,0) = 0, ..., \omega(-1,0) = 0, \omega(0,0) = 1, B_{(-k,1)}, B_{(0,1)}\right) \\
&= p^2(1-p)^{k-1}\mathbb{P}\left(B_{-k}, B_0\right) \\
&\geq e^{-ck}\mathbb{P}\left(B_{-k}, B_0\right).
\end{aligned} \tag{A.5}$$

For the event $\{R_k > \frac{m}{2}\}$ conditioned on $\{B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\}$, we get that

$$\begin{aligned}
&\mathbb{P}\left(R_k > \frac{m}{2} \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \mathbb{P}\left(R_k > \frac{m}{2}, (B_{-k+1})^c, ..., (B_{-1})^c \mid B_{-k}, B_0\right) \cdot e^{ck} \\
&\leq \left(C \sum_{0<j<k} \mathbb{P}\left(l(-j,0) > \frac{m}{4}, (B_{-j})^c\right) + C \sum_{0 \leq j \leq m} \mathbb{P}_{joint}\left(T_{j+1}^{sim} - T_j^{sim} > \frac{m}{4}\right)\right) \cdot e^{ck} \\
&\leq \left(C\log(m)e^{-cm} + Cme^{-cm}\right) \cdot e^{c\log(m)} \\
&\leq Ce^{-cm},
\end{aligned} \tag{A.6}$$

where the first inequality holds true by (A.5) and the third inequality holds true since $k \leq K\log(m)$, by (A.3). Together with (A.4), this yields

$$\begin{aligned}
&\mathbb{P}\left(T_{meet}^{(-k,0)} > m \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \sum_{r>0} \left(\frac{C(k+2r)}{\sqrt{m-r}} \wedge 1\right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \sum_{r=0}^{\frac{m}{2}} \left(\frac{C(k+2r)}{\sqrt{m-r}} \wedge 1\right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\quad + \sum_{r=\frac{m}{2}+1}^{\infty} \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \sum_{r=0}^{\frac{m}{2}} \left(\frac{C(k+2r)}{\sqrt{m}} \wedge 1\right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\quad + Ce^{-cm}.
\end{aligned}$$

Hence we focus on

$$\sum_{r=0}^{\frac{m}{2}} \left( \frac{C(k+2r)}{\sqrt{m}} \wedge 1 \right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right).$$

Calculations similar to (A.6) yield

$$\begin{aligned}
&\mathbb{P}\left(R_k \geq \frac{k^2+\tilde{r}}{2} \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \mathbb{P}\left(R_k \geq \frac{k^2+\tilde{r}}{2}, (B_{-k+1})^c, ..., (B_{-1})^c \mid B_{-k}, B_0\right) \cdot e^{ck} \\
&\leq \left( C \sum_{0<j<k} \mathbb{P}\left(l(-j,0) \geq \frac{k^2+\tilde{r}}{4}, (B_{-j})^c\right) + C \sum_{0 \leq j \leq k^2+\tilde{r}} \mathbb{P}_{joint}\left(T^{sim}_{j+1} - T^{sim}_j \geq \frac{k^2+\tilde{r}}{4}\right)\right) \cdot e^{ck} \\
&\leq \left(Cke^{-c(k^2+\tilde{r})} + C(k^2+\tilde{r})e^{-c(k^2+\tilde{r})}\right) \cdot e^{ck} \\
&\leq Ce^{-c(k^2+\tilde{r})},
\end{aligned} \tag{A.7}$$

if $k > K'$, $\tilde{r} > 0$. Hence

$$\begin{aligned}
&\sum_{r=0}^{\frac{m}{2}} \left( \frac{C(k+2r)}{\sqrt{m}} \wedge 1 \right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \frac{Ck^2}{\sqrt{m}} + \sum_{r=k^2}^{\infty} \frac{C(k+2r)}{\sqrt{m}} \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \\
&\leq \frac{Ck^2}{\sqrt{m}} + C\sum_{r=1}^{\infty} \frac{(k^2+r)}{\sqrt{m}} e^{-c(k^2+r)} \\
&\leq \frac{Ck^2}{\sqrt{m}},
\end{aligned}$$

if $k > K'$. Altogether we get that

$$\begin{aligned}
&\mathbb{P}\left(T^{(c(-1),c(0))}_{meet} > m\right) \\
&\leq \sum_{k=K'}^{K\log(m)} \mathbb{P}\left(T^{(-k,0)}_{meet} > m, B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) + \frac{C}{\sqrt{m}} \\
&\leq \sum_{k=K'}^{K\log(m)} \mathbb{P}\left(T^{(-k,0)}_{meet} > m \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) \cdot Ce^{-ck} + \frac{C}{\sqrt{m}} \\
&\leq \sum_{k=K'}^{K\log(m)} \left( \sum_{r=0}^{\frac{m}{2}} \left( \frac{C(k+2r)}{\sqrt{m}} \wedge 1 \right) \mathbb{P}\left(R_k = r \mid B_{-k}, (B_{-k+1})^c, ..., (B_{-1})^c, B_0\right) + Ce^{-cm} \right) \cdot Ce^{-ck} + \frac{C}{\sqrt{m}} \\
&\leq \sum_{k=K'}^{K\log(m)} \left( \frac{Ck^2}{\sqrt{m}} + Ce^{-cm} \right) \cdot Ce^{-ck} + \frac{C}{\sqrt{m}},
\end{aligned}$$

which proves (A.2).

□

### A.1.4. From regeneration times to real time

Let the parameters be as defined in section 2.1.4. We show that (2.3) of Proposition 2.1 holds true if it holds true for $T_{meet}$ replaced by $\widehat{T}_{meet}$.

We fix some $x \in \mathbb{Z}^d$. Let $M > 0$ be some large constant. We choose $K > 0$ large enough, such that

$$\mathbb{P}^x_{joint}(T^{sim}_m \geq Km) \leq e^{-cm} \tag{A.8}$$

for all $m > M$ and some $c > 0$. Inequality (A.8) holds true by Lemma 1.12 and standard large deviation estimates. By Lemma 2.8 we obtain

$$\begin{aligned}\mathbb{P}^x_{joint}\left(T_{meet} > Km\right) &\leq \mathbb{P}^x_{joint}\left(T_{meet} > Km, T^{sim}_m < Km\right) + e^{-cm}\\ &\leq \mathbb{P}^x_{joint}\left(\widehat{T}_{meet} > m\right) + e^{-cm}\\ &\leq \frac{\sqrt{K}C_3}{\sqrt{Km}} + e^{-cm}\\ &\leq \frac{C}{\sqrt{Km}},\end{aligned}$$

which proves the upper bound on $\mathbb{P}^x_{joint}\left(T_{meet} > m\right)$. On the other hand notice that within the proof of (2.28) we could also have conditioned on the event that the time between two regeneration is smaller than $\frac{jK}{2}$ within the next $(2^{j+1}K)^3$ steps if the initial value of $(\widehat{D}_n)_n$ lies inside the interval $[2^jK, 2^{j+1}K)$, which would have give us a lower bound on

$$\mathbb{P}^x_{joint}\left(T_{meet} > m, \left(-\tfrac{K}{2}, \tfrac{K}{2}\right) \text{ is never visited between two regeneration times }\right).$$

Since

$$\begin{aligned}&\mathbb{P}^x_{joint}\left(T_{meet} > m\right)\\ &\geq \mathbb{P}^x_{joint}\left(T_{meet} > m, \left(-\tfrac{K}{2}, \tfrac{K}{2}\right) \text{ is never visited between two regeneration times }\right)\end{aligned}$$

the lower bound on $\mathbb{P}^x_{joint}\left(T_{meet} > m\right)$ follows.

## A.2. Annealed estimates

### A.2.1. ADE

The proof of Lemma 3.9 is very similar to the proof of Lemma 2.14 in [BCR16]. Nevertheless, we give the proofs here for the sake of completeness and because the requirements in [BCR16] such as "uniform ellipticity (UE)" are not satisfied in our case. First we need to prove two useful Lemmas.

**Lemma A.2.** *Let $\{Y_i\}_{i=1}^\infty$ and $\{Z_i\}_{i=1}^\infty$ be a sequence of $d$-dimensional random variables and a sequence of $1$-dimensional non-negative integer valued random variables, such that $\{(Y_i, Z_i)\}_{i=1}^\infty$ are independent and identically distributed with respect to some probability measure $P$. Assume in addition that there exists $v \in \mathbb{Z}^d$, $k \in \mathbb{N}$ such that $P((Y_1, Z_1) = (v, k)) > 0$ and $P((Y_1, Z_1) = (w, k+1)) > 0$ for every $w$ with $\|w - v\| \leq 1$. Let $S_n = \sum_{i=1}^n Y_i$ and $T_n = \sum_{i=1}^n Z_i$. Then there exists $C < \infty$ which is determined by $P$ such that for every $n, m \in \mathbb{N}$, every $x, y \in \mathbb{Z}^d$ with $\|x - y\| = 1$*

$$P((S_n, T_n) = (x, m)) < Cn^{-\frac{d+1}{2}}, \tag{A.9}$$

$$|P((S_n, T_n) = (x, m)) - P((S_n, T_n) = (x, m+1))| < Cn^{-\frac{d+2}{2}}, \tag{A.10}$$

*and*

$$|P((S_n, T_n) = (x, m)) - P((S_n, T_n) = (y, m))| < Cn^{-\frac{d+2}{2}}. \tag{A.11}$$

**Remark A.3.** The Lemma above is similar to Claim A.2 in [BCR16]. But since in our case the space-time random walk $(X_n, T_n)$ does not always have the same parity, we adapted the Lemma slightly.

*Proof Lemma A.2:* Let $\chi$ be the characteristic function of $(Y_1, Z_1)$. Note that the characteristic function $\chi$ is periodic, as $(Y_1, Z_1)$ is concentrated on a lattice. The existence of $(v, k) \in \mathbb{Z}^d \times \mathbb{N}$ as described in the requirements of Lemma A.2, implies that the period is $2\pi$. By Lemma 2.3.2 in [LL10] there exist $D, \delta > 0$ such that

i) $|\chi(\theta, s)| < 1 - D$ for every $(\theta, s) \in [-\pi, \pi]^{d+1}$ such that $\|(\theta, s)\|_1 \geq \delta$,

ii) $|\chi(\theta, s)| < 1 - D\,\|(\theta, s)\|_1^2$ for every $(\theta, s) \in [-\pi, \pi]^{d+1}$ such that $\|(\theta, s)\|_1 < \delta$.

Inequality (A.9) follows by i) and ii) since

$$\begin{aligned}
P\left(\sum_{i=1}^{n}(Y_i,Z_i)=(x,m)\right) &= \frac{1}{(2\pi)^{d+1}}\int_{[-\pi,\pi]^{d+1}} e^{-i\langle\theta,x\rangle-i(s\cdot m)}\chi^n(\theta,s)d\theta ds\\
&\leq \int_{[-\pi,\pi]^{d+1}}|\chi^n(\theta,s)|d\theta ds\\
&\leq \int_{\|(\theta,s)\|_1\geq\delta}|\chi^n(\theta,s)|d\theta ds+\int_{\|(\theta,s)\|_1<\delta}|\chi^n(\theta,s)|d\theta ds\\
&\leq (2\pi)^{d+1}(1-D)^n+\int_{\|(\theta,s)\|_1<\delta}(1-D\|(\theta,s)\|_1^2)^n d\theta ds\\
&\leq (2\pi)^{d+1}(1-D)^n+\int_{\|(\theta,s)\|_1<\delta}e^{-nD\|(\theta,s)\|_1^2}d\theta ds\\
&\overset{(*)}{\leq} (2\pi)^{d+1}(1-D)^n+\frac{1}{(\sqrt{n})^{d+1}}\int_{\|(\vartheta,t)\|_1<\sqrt{n}\delta}e^{-D\|(\vartheta,t)\|_1^2}d\vartheta dt\\
&\leq \frac{C}{n^{\frac{d+1}{2}}},
\end{aligned}$$

where $(*)$ can be obtained by substituting $\vartheta=\sqrt{n}\theta$, $t=s\sqrt{n}$. Inequality (A.10) follows by i) and ii) since

$$\begin{aligned}
&\left|P\left(\sum_{i=1}^{n}(Y_i,Z_i)=(x,m)\right)-P\left(\sum_{i=1}^{n}(Y_i,Z_i)=(x,m+1)\right)\right|\\
&=\frac{1}{(2\pi)^{d+1}}\left|\int_{[-\pi,\pi]^{d+1}}e^{-i\langle\theta,x\rangle-i(s\cdot m)}\chi^n(\theta,s)d\theta ds-\int_{[-\pi,\pi]^{d+1}}e^{-i\langle\theta,x\rangle-i(s\cdot(m+1))}\chi^n(\theta,s)d\theta ds\right|\\
&\leq\left(\int_{[-\pi,\pi]^{d+1}}\left|e^{-i\langle\theta,x\rangle-i(s\cdot m)}-e^{-i\langle\theta,x\rangle-i(s\cdot(m+1))}\right||\chi(\theta,s)|^n d\theta ds\right)\\
&\leq\left(\int_{[-\pi,\pi]^{d+1}}\left|e^{-i(s\cdot m)}-e^{-i(s\cdot(m+1))}\right||\chi(\theta,s)|^n d\theta ds\right)\\
&\leq\left(\int_{[-\pi,\pi]^{d+1}}\left|1-e^{-is}\right||\chi(\theta,s)|^n d\theta ds\right)\\
&\leq\left(\int_{[-\pi,\pi]^{d+1}}|s||\chi(\theta,s)|^n d\theta ds\right)\\
&\leq C(1-D)^n+\int_{\|(\theta,s)\|_1<\delta}|s|e^{-nD\|(\theta,s)\|_1^2}d\theta ds\\
&\leq\frac{C}{n^{\frac{d+2}{2}}}.
\end{aligned}$$

Inequality (A.11) follows by i) and ii) in the same way. □

**Lemma A.4.** *Let $d\geq 3$. Fix $\frac{2}{5}N^2\leq M\leq N^2$ and some starting point $z=(y,m)\in\widetilde{\mathcal{P}}(N)$. Note that $\mathbb{P}^z(T_0=m)=1$. We define the events $Z(l):=\cup_k\{T_k-T_0=l\}$ and*

$$\widehat{Z}_{M-k}(l):=Z(l)\cap\bigcap_{j=l+1}^{M-k}(Z(j))^c,\quad \text{where } m\leq k\leq M-l-1.$$

*Then the following holds:*

*i) For every $l\leq M-m$ and $x\in\mathbb{Z}^d$*

$$\mathbb{P}^z\left(X_{m+l}=x\mid\widehat{Z}_{M-m}(l)\right)\leq Cl^{-\frac{d}{2}}. \tag{A.12}$$

*ii) For every $l\leq M-m$ and $x,y\in\mathbb{Z}^d$ such that $\|x-y\|=1$*

$$\left|\mathbb{P}^{(y,m)}\left(X_{m+l}=x\mid\widehat{Z}_{M-m}(l)\right)-\mathbb{P}^{(y+e_j,m)}\left(X_{m+l}=x\mid\widehat{Z}_{M-m}(l)\right)\right|<Cl^{-\frac{d+1}{2}}. \tag{A.13}$$

*iii) For every $l\leq M-m$ and $x\in\mathbb{Z}^d$*

$$\left|\mathbb{P}^{(y,m)}\left(X_{m+l}=x\mid\widehat{Z}_{M-m}(l)\right)-\mathbb{P}^{(y,m+1)}\left(X_{m+l}=x\mid\widehat{Z}_{M-m-1}(l-1)\right)\right|<Cl^{-\frac{d+1}{2}}. \tag{A.14}$$

**Remark A.5.** The Lemma above is almost the same as Lemma A.4 in [BCR16], we give the proof in order to convince the reader that the Lemma also holds true in our case with conditions such as "(UE)" and "i.i.d. environment" being violated.

**Proof:** We first prove part $i)$. Since on $Z(l)$ the event $\{X_{m+l} = x\}$ is independent of $\bigcap_{j=l+1}^{M-m} (Z(j))^c$ we get that

$$\begin{aligned}
\mathbb{P}^z(X_{m+l} = x \big| \widehat{Z}_{M-m}(l)) &= \mathbb{P}^z(X_{m+l} = x \big| Z(l)) \\
&= \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^{\infty} \mathbb{P}^z((X_{T_k}, T_k) = (x, m+l)) \\
&= \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^{L} \mathbb{P}^z((X_{T_k}, T_k) = (x, m+l), T_{\lceil \frac{k}{2} \rceil} \geq \tfrac{l}{2} + m) \\
&\quad + \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^{L} \mathbb{P}^z((X_{T_k}, T_k) = (x, m+l), T_k - T_{\lceil \frac{k}{2} \rceil} > \tfrac{l}{2}) \\
&\quad + \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=L+1}^{\infty} \mathbb{P}^z((X_{T_k}, T_k) = (x, m+l), T_{\lceil \frac{k}{2} \rceil} \leq \tfrac{l}{2} + m) \\
&\quad + \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=L+1}^{\infty} \mathbb{P}^z((X_{T_k}, T_k) = (x, m+l), T_k - T_{\lceil \frac{k}{2} \rceil} < \tfrac{l}{2}). \qquad \text{(A.15)}
\end{aligned}$$

Lemma A.2 yields

$$\begin{aligned}
&\mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l), T_{\lceil \frac{k}{2} \rceil} \leq \tfrac{l}{2} + m\right) \\
&= \sum_{s=1}^{\frac{l}{2}} \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l) \;\Big|\; (X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (w, m+s)\right) \mathbb{P}^z\left((X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (w, m+s)\right) \\
&\leq Ck^{-\frac{d+1}{2}} \sum_{s=1}^{\frac{l}{2}} \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z\left((X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (w, m+s)\right) \\
&\leq Ck^{-\frac{d+1}{2}} \mathbb{P}^z\left(T_{\lceil \frac{k}{2} \rceil} \leq \tfrac{l}{2} + m\right),
\end{aligned}$$

and using similar arguments and translation invariance of $\mathbb{P}^z$

$$\begin{aligned}
&\mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l), T_k - T_{\lceil \frac{k}{2} \rceil} < \tfrac{l}{2}\right) \\
&\leq \sum_{s=\frac{l}{2}}^{l-1} \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l) \;\Big|\; (X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (w, m+s)\right) \mathbb{P}^z\left((X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (w, m+s)\right) \\
&\leq Ck^{-\frac{d+1}{2}} \sum_{s=\frac{l}{2}}^{l-1} \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l) \;\Big|\; (X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (y+w, m+s)\right) \\
&\leq Ck^{-\frac{d+1}{2}} \sum_{s=\frac{l}{2}}^{l-1} \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z\left((X_{T_k}, T_k) = (x-w, m+l) \;\Big|\; (X_{T_{\lceil \frac{k}{2} \rceil}}, T_{\lceil \frac{k}{2} \rceil}) = (y, m+s)\right) \\
&\leq Ck^{-\frac{d+1}{2}} \mathbb{P}^z\left(T_{\lceil \frac{k}{2} \rceil} \leq \tfrac{l}{2} + m\right).
\end{aligned}$$

Additionally, we get that

$$\begin{aligned}
\mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l), T_{\lceil \frac{k}{2} \rceil} \geq \tfrac{l}{2} + m\right) &\leq Ck^{-\frac{d+1}{2}} \mathbb{P}^z\left(T_{\lceil \frac{k}{2} \rceil} \geq \tfrac{l}{2} + m\right), \\
\mathbb{P}^z\left((X_{T_k}, T_k) = (x, m+l), T_k - T_{\lceil \frac{k}{2} \rceil} > \tfrac{l}{2}\right) &\leq Ck^{-\frac{d+1}{2}} \mathbb{P}^z\left(T_{\lceil \frac{k}{2} \rceil} \geq \tfrac{l}{2} + m\right).
\end{aligned}$$

Coming back to (A.15) we have

$$\mathbb{P}^z(X_{m+l} = x \big| Z(l)) \leq C\left(\sum_{k=1}^{L} k^{-\frac{d+1}{2}} \mathbb{P}^z\left(T_{\lceil \frac{k}{2} \rceil} \geq \tfrac{l}{2} + m\right) + \sum_{k=L+1}^{\infty} k^{-\frac{d+1}{2}} \mathbb{P}^z\left(T_{\lceil \frac{k}{2} \rceil} \leq \tfrac{l}{2} + m\right)\right).$$

We choose

$$L := \frac{l}{\mathbb{E}^z[T_1 - T_0]} = \mathcal{O}(l),$$

and since

$$\begin{aligned}
\mathbb{P}^z\left(T_{\lceil \frac{k}{2}\rceil} \geq \tfrac{l}{2} + m\right) &= \mathbb{P}^z\left(\sum_{i=1}^{\frac{k}{2}}((T_i - T_{i-1}) - \mathbb{E}^z[T_1 - T_0]) \geq \tfrac{l}{2} - \tfrac{k}{2}\mathbb{E}^z[T_1 - T_0]\right) \\
&\leq \mathbb{P}^z\left(\left|\sum_{i=1}^{\frac{k}{2}}((T_i - T_{i-1}) - \mathbb{E}^z[T_1 - T_0])\right| \geq \left|\tfrac{(L-k)\mathbb{E}^z[T_1 - T_0]}{2}\right|\right) \\
&\leq \frac{C}{(L-k)^{2d}}\mathbb{E}^z\left[\left(\sum_{i=1}^{\frac{k}{2}}((T_i - T_{i-1}) - \mathbb{E}^z[T_1 - T_0])\right)^{2d}\right] \\
&\leq \frac{C}{(L-k)^{2d}}\mathbb{E}^z\left[\left(\sqrt{\frac{2}{k}}\sum_{i=1}^{\frac{k}{2}}((T_i - T_{i-1}) - \mathbb{E}^z[T_1 - T_0])\right)^{2d}\right]k^d \\
&\leq \frac{Ck^d}{(L-k)^{2d}},
\end{aligned}$$

we get that

$$\mathbb{P}^z\left(T_{\lceil \frac{k}{2}\rceil} \geq \tfrac{l}{2} + m\right), \mathbb{P}^z\left(T_{\lceil \frac{k}{2}\rceil} \leq \tfrac{l}{2} + m\right) \leq \min\left\{1, \frac{Ck^d}{(L-k)^{2d}}\right\}.$$

Hence

$$\mathbb{P}^z(X_{m+l} = x \big| Z(l)) \leq C\sum_{k=1}^{\infty} k^{-\frac{d+1}{2}} \min\left\{1, \frac{k^d}{(L-k)^{2d}}\right\}.$$

As in [BCR16] we split up the sum into four parts. For $k \in [1, \frac{L}{2}]$ we have that $\frac{k^d}{(L-k)^{2d}} < 1$ and hence

$$\sum_{k=1}^{\frac{L}{2}} k^{-\frac{d+1}{2}} \min\left\{1, \frac{k^d}{(L-k)^{2d}}\right\} = \sum_{k=1}^{\frac{L}{2}} \frac{k^{\frac{d-1}{2}}}{(L-k)^{2d}} \leq \sum_{k=1}^{\frac{L}{2}} Ck^{\frac{d-1}{2}} L^{-2d} \leq CL^{-d} = Cl^{-d}.$$

For $k \in [\frac{L}{2}, L - \sqrt{L}]$ we also have $\frac{k^d}{(L-k)^{2d}} \leq 1$ and hence

$$\begin{aligned}
\sum_{k=\frac{L}{2}}^{L-\sqrt{L}} k^{-\frac{d+1}{2}} \min\left\{1, \frac{k^d}{(L-k)^{2d}}\right\} &= \sum_{k=\frac{L}{2}}^{L-\sqrt{L}} \frac{k^{\frac{d-1}{2}}}{(L-k)^{2d}} \leq C\int_{\frac{L}{2}}^{L-\sqrt{L}} \frac{s^{\frac{d-1}{2}}}{(L-s)^{2d}} ds \\
&= C\int_{\sqrt{L}}^{\frac{L}{2}} (L-t)^{\frac{d-1}{2}} t^{-2d} dt \leq CL^{\frac{d-1}{2}} \int_{\sqrt{L}}^{\frac{L}{2}} t^{-2d} dt \\
&\leq CL^{\frac{d-1}{2}} \left(\sqrt{L}\right)^{-2d+1} \leq CL^{-\frac{d}{2}} \leq Cl^{-\frac{d}{2}}.
\end{aligned}$$

For $k \in [L - \sqrt{L}, L + \sqrt{L}]$ we have

$$\sum_{k=L-\sqrt{L}}^{L+\sqrt{L}} k^{-\frac{d+1}{2}} \min\left\{1, \frac{k^d}{(L-k)^{2d}}\right\} \leq \sum_{k=L-\sqrt{L}}^{L+\sqrt{L}} k^{\frac{-d-1}{2}} \leq CL^{\frac{-d-1}{2}}\sqrt{L} \leq CL^{-\frac{d}{2}} \leq Cl^{-\frac{d}{2}}.$$

At last for $k \in [L+\sqrt{L},\infty]$, the proof is very similar to the second case and we get that

$$\begin{aligned}
\sum_{k=L+\sqrt{L}}^{\infty} k^{-\frac{d+1}{2}} \min\left\{1, \frac{k^d}{(L-k)^{2d}}\right\} &\leq \sum_{k=L+\sqrt{L}}^{\infty} \frac{k^{\frac{d-1}{2}}}{(L-k)^{2d}} \\
&\leq C\int_{L+\sqrt{L}}^{2L} \frac{s^{\frac{d-1}{2}}}{(s-L)^{2d}} ds + C\int_{k=2L}^{\infty} \frac{s^{\frac{d-1}{2}}}{(s-L)^{2d}} ds \\
&\leq C\int_{\sqrt{L}}^{L} \frac{(L+t)^{\frac{d-1}{2}}}{t^{2d}} dt + C\int_{k=L}^{\infty} \frac{(L+t)^{\frac{d-1}{2}}}{t^{2d}} dt \\
&\leq CL^{\frac{d-1}{2}} \int_{\sqrt{L}}^{L} t^{-2d} dt + C\int_{k=L}^{\infty} t^{\frac{d-1}{2}} t^{-2d} dt \\
&\leq CL^{-\frac{d}{2}} \leq Cl^{-\frac{d}{2}}.
\end{aligned}$$

The last estimates yield

$$\mathbb{P}^z(X_{m+l} = x \big| Z(l)) \leq Cl^{-\frac{d}{2}}.$$

Next we prove part *ii*). As in the previous part we have

$$\begin{aligned}
&\left|\mathbb{P}^{(y,m)}\left(X_{m+l} = x \mid \widehat{Z}_{M-m}(l)\right) - \mathbb{P}^{(y+e_j,m)}\left(X_{m+l} = x \mid \widehat{Z}_{M-m}(l)\right)\right| \\
&= \left|\mathbb{P}^{(y,m)}\left(X_{m+l} = x \mid Z(l)\right) - \mathbb{P}^{(y+e_j,m)}\left(X_{m+l} = x \mid Z(l)\right)\right| \\
&= \left|\mathbb{P}^{(y,m)}\left(X_{m+l} = x \mid Z(l)\right) - \mathbb{P}^{(y,m)}\left(X_{m+l} = x - e_j \mid Z(l)\right)\right| \\
&\leq \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^{\infty} \left|\mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x, m+l)) - \mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x-e_j, m+l))\right| \\
&\leq \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^{L} \left|\mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x, m+l), T_{\lceil\frac{k}{2}\rceil} \geq \tfrac{l}{2}+m) - \mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x-e_j, m+l), T_{\lceil\frac{k}{2}\rceil} \geq \tfrac{l}{2}+m)\right| \\
&+ \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^{L} \left|\mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x, m+l), T_{\lceil\frac{k}{2}\rceil} - T_k \geq \tfrac{l}{2}) - \mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x-e_j, m+l), T_{\lceil\frac{k}{2}\rceil} - T_k \geq \tfrac{l}{2})\right| \\
&+ \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=L+1}^{\infty} \left|\mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x, m+l), T_{\lceil\frac{k}{2}\rceil} \leq \tfrac{l}{2}+m) - \mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x-e_j, m+l), T_{\lceil\frac{k}{2}\rceil} \leq \tfrac{l}{2}+m)\right| \\
&+ \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=L+1}^{\infty} \left|\mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x, m+l), T_{\lceil\frac{k}{2}\rceil} - T_k \leq \tfrac{l}{2}) - \mathbb{P}^{(y,m)}((X_{T_k}, T_k) = (x-e_j, m+l), T_{\lceil\frac{k}{2}\rceil} - T_k \leq \tfrac{l}{2})\right|
\end{aligned}$$

By applying (A.10) to

$$\begin{aligned}
&\left|\mathbb{P}^{(y,m)}\left((X_{T_k}, T_k) = (x, m+l), T_{\lceil\frac{k}{2}\rceil} \leq \tfrac{l}{2}+m\right) - \mathbb{P}^{(y,m)}\left((X_{T_k}, T_k) = (x-e_j, m+l), T_{\lceil\frac{k}{2}\rceil} \leq \tfrac{l}{2}+m\right)\right| \\
&\leq \sum_{s=1}^{\lceil\frac{k}{2}\rceil} \sum_{w\in\mathbb{Z}^d} \Bigg|\mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right) \mathbb{P}^{(y,m)}\left((X_{T_k}, T_k) = (x, m+l) \;\Big|\; (X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right) \\
&\qquad - \mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right) \mathbb{P}^{(y,m)}\left((X_{T_k}, T_k) = (x-e_j, m+l) \;\Big|\; (X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right)\Bigg| \\
&\leq \sum_{s=1}^{\lceil\frac{k}{2}\rceil} \sum_{w\in\mathbb{Z}^d} \mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right) \\
&\qquad \cdot \Bigg|\mathbb{P}^{(y,m)}\left((X_{T_k}, T_k) = (x, m+l) \;\Big|\; (X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right) \\
&\qquad - \mathbb{P}^{(y,m)}\left((X_{T_k}, T_k) = (x-e_j, m+l) \;\Big|\; (X_{T_{\lceil\frac{k}{2}\rceil}}, T_{\lceil\frac{k}{2}\rceil}) = (w, m+s)\right)\Bigg|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbb{P}^{(y,m)}\left((X_{T_k},T_k)=(x,m+l),T_{\lceil\frac{k}{2}\rceil}>\tfrac{l}{2}+m\right)-\mathbb{P}^{(y,m)}\left((X_{T_k},T_k)=(x-e_j,m+l),T_{\lceil\frac{k}{2}\rceil}>\tfrac{l}{2}+m\right)\right|\\
&\le\sum_{s=\lceil\frac{k}{2}\rceil+1}^{l-1}\sum_{w\in\mathbb{Z}^d}\Bigg|\mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w,m+s)\right)\mathbb{P}^{(y,m)}\left((X_{T_k},T_k)=(x,m+l)\;\middle|\;(X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w,m+s)\right)\\
&-\mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w-e_j,m+s)\right)\mathbb{P}^{(y,m)}\left((X_{T_k},T_k)=(x-e_j,m+l)\;\middle|\;(X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w-e_j,m+s)\right)\Bigg|\\
&\le\sum_{s=\lceil\frac{k}{2}\rceil+1}^{l-1}\sum_{w\in\mathbb{Z}^d}\left|\mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w,m+s)\right)-\mathbb{P}^{(y,m)}\left((X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w-e_j,m+s)\right)\right|\\
&\qquad\cdot\mathbb{P}^{(y,m)}\left((X_{T_k},T_k)=(x,m+l)\;\middle|\;(X_{T_{\lceil\frac{k}{2}\rceil}},T_{\lceil\frac{k}{2}\rceil})=(w,m+s)\right)
\end{aligned}$$

we get, by similar bounds on the other terms in (A.15), that

$$\begin{aligned}
&\left|\mathbb{P}^{(y,m)}\left(X_{m+l}=x\;\middle|\;\widehat{Z}_{M-m}(l)\right)-\mathbb{P}^{(y+e_j,m)}\left(X_{m+l}=x\;\middle|\;\widehat{Z}_{M-m}(l)\right)\right|\\
&\le C\left(\sum_{k=1}^{L}k^{-\frac{d+2}{2}}\mathbb{P}^z\left(T_{\lceil\frac{k}{2}\rceil}\ge\tfrac{l}{2}+m\right)+\sum_{k=L+1}^{\infty}k^{-\frac{d+2}{2}}\mathbb{P}^z\left(T_{\lceil\frac{k}{2}\rceil}\le\tfrac{l}{2}+m\right)\right)\\
&\le C\sum_{k=1}^{\infty}k^{-\frac{d+2}{2}}\min\left\{1,\frac{k^d}{(L-k)^{2d}}\right\}
\end{aligned}$$

and hence

$$\left|\mathbb{P}^{(y,m)}\left(X_{m+l}=x\;\middle|\;\widehat{Z}_{M-m}(l)\right)-\mathbb{P}^{(y+e_j,m)}\left(X_{m+l}=x\;\middle|\;\widehat{Z}_{M-m}(l)\right)\right|\le Cl^{-\frac{d+1}{2}}.$$

The proof of *iii)* requires slightly different arguments. We start in a similar fashion as in the previous case.

$$\begin{aligned}
&\left|\mathbb{P}^{(y,m)}\left(X_{m+l}=x\;\middle|\;\widehat{Z}_{M-m}(l)\right)-\mathbb{P}^{(y,m+1)}\left(X_{m+l}=x\;\middle|\;\widehat{Z}_{M-m-1}(l-1)\right)\right|\\
&=\left|\mathbb{P}^{(y,m)}\left(X_{m+l}=x\mid Z(l)\right)-\mathbb{P}^{(y,m+1)}\left(X_{m+l}=x\mid Z(l-1)\right)\right|\\
&=\left|\mathbb{P}^{(y,m+1)}\left(X_{m+l+1}=x\mid Z(l)\right)-\mathbb{P}^{(y,m+1)}\left(X_{m+l}=x\mid Z(l-1)\right)\right|\\
&=\Bigg|\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l))}\sum_{k=1}^{\infty}\mathbb{P}^{(y,m+1)}\left((X_{T_k},T_k)=(x,m+l+1)\right)\\
&\quad-\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l-1))}\sum_{k=1}^{\infty}\mathbb{P}^{(y,m+1)}\left((X_{T_k},T_k)=(x,m+l)\right)\Bigg|\\
&\le\left|\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l))}-\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l-1))}\right|\sum_{k=1}^{\infty}\mathbb{P}^{(y,m+1)}\left((X_{T_k},T_k)=(x,m+l+1)\right)\\
&\quad+\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l-1))}\sum_{k=1}^{\infty}\left|\mathbb{P}^{(y,m+1)}\left((X_{T_k},T_k)=(x,m+l+1)\right)-\mathbb{P}^{(y,m+1)}\left((X_{T_k},T_k)=(x,m+l)\right)\right|,
\end{aligned}$$

where the last summand can be estimated analogously to part *ii)*. The first summand can be estimated in the following way

$$\begin{aligned}
&\left|\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l))}-\frac{1}{\mathbb{P}^{(y,m+1)}(Z(l-1))}\right|\sum_{k=1}^{\infty}\mathbb{P}^{(y,m+1)}\left((X_{T_k},T_k)=(x,m+l+1)\right)\\
&=\frac{|\mathbb{P}^{(y,m+1)}(Z(l-1))-\mathbb{P}^{(y,m+1)}(Z(l))|}{\mathbb{P}^{(y,m+1)}(Z(l-1))}\mathbb{P}^{(y,m+1)}(X_{m+l+1}=x|Z(l))\\
&\le C|\mathbb{P}^{(y,m+1)}(Z(l-1))-\mathbb{P}^{(y,m+1)}(Z(l))|l^{-\frac{d}{2}}.
\end{aligned}$$

Therefore we need to prove that

$$|\mathbb{P}^{(y,m+1)}(Z(l-1))-\mathbb{P}^{(y,m+1)}(Z(l))|\le Cl^{-\frac{1}{2}}. \tag{A.16}$$

Since

$$|\mathbb{P}^{(y,m+1)}(Z(l-1)) - \mathbb{P}^{(y,m+1)}(Z(l))| = |\sum_{k=1}^{\infty}\mathbb{P}^{(y,m+1)}(T_k = m+l) - \sum_{k=1}^{\infty}\mathbb{P}^{(y,m+1)}(T_k = m+l+1)|$$
$$\leq \sum_{k=1}^{\infty}|\mathbb{P}^{(y,m)}(T_k = m+l) - \mathbb{P}^{(y,m)}(T_k = m+l+1)|,$$

which can be treated similarly to the differences we had a look at before. Hence the result follows by standard Fourier analysis similar to Lemma A.2.

□

**Proof of Lemma 3.9:** First we prove part $i$). We have

$$\begin{aligned}
&\mathbb{P}^z(X_M = x)\\
&= \sum_{l\leq M-m}\mathbb{P}^z\left(\widehat{Z}_{M-m}(l)\right)\sum_{w\in\mathbb{Z}^d}\mathbb{P}^z\left(X_{m+l} = w\middle|\widehat{Z}_{M-m}(l)\right)\mathbb{P}^z\left(X_M = x\middle|\widehat{Z}_{M-m}(l),\ X_{m+l} = w\right)\\
&\leq \sum_{l\leq M-m}\mathbb{P}^z\left(\widehat{Z}_{M-m}(l)\right)\sum_{w\in\mathbb{Z}^d}Cl^{-\frac{d}{2}}\mathbb{P}^z\left(X_M = x\middle|\widehat{Z}_{M-m}(l),\ X_{m+l} = w\right)\\
&\leq \sum_{l\leq M-m}\mathbb{P}^z\left(\widehat{Z}_{M-m}(l)\right)Cl^{-\frac{d}{2}}\\
&\leq \sum_{l\leq M-m}Ce^{-c(M-m-l)}l^{-\frac{d}{2}}\\
&\leq C\left(\sum_{l=1}^{\frac{M-m}{2}}e^{-c(M-m-l)}l^{-\frac{d}{2}} + \sum_{l=\frac{M-m}{2}}^{M-m}e^{-c(M-m-l)}l^{-\frac{d}{2}}\right)\\
&\leq Ce^{-c\frac{M-m}{2}} + C(M-m)^{-\frac{d}{2}}\sum_{l=\frac{M-m}{2}}^{M-m}e^{-c(M-m-l)}\\
&\leq C(M-m)^{-\frac{d}{2}},
\end{aligned}$$

where for the first inequality we made use of the translation invariance of $\mathbb{P}$.
For part $ii$) we have

$$\begin{aligned}
&|\mathbb{P}^{(y,m)}(X_M = x) - \mathbb{P}^{(y+e_j,m)}(X_M = x)|\\
&= |\mathbb{P}^{(y,m)}(X_M = x) - \mathbb{P}^{(y,m)}(X_M = x-e_j)|\\
&= \Big|\sum_{l\leq M-m}\mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l))\sum_{w\in\mathbb{Z}^d}\mathbb{P}^{(y,m)}(X_{m+l} = w|\widehat{Z}_{M-m}(l))\mathbb{P}^{(y,m)}(X_M = x|X_{m+l} = w, \widehat{Z}_{M-m}(l))\\
&\quad - \sum_{l\leq M-m}\mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l))\sum_{w\in\mathbb{Z}^d}\mathbb{P}^{(y,m)}(X_{m+l} = w-e_j|\widehat{Z}_{M-m}(l))\mathbb{P}^{(y,m)}(X_M = x-e_j|X_{m+l} = w-e_j, \widehat{Z}_{M-m}(l))\Big|\\
&\leq \sum_{l\leq M-m}\mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l))\\
&\qquad\cdot\sum_{w\in\mathbb{Z}^d}\Big|\mathbb{P}^{(y,m)}(X_{m+l} = w|\widehat{Z}_{M-m}(l)) - \mathbb{P}^{(y,m)}(X_{m+l} = w-e_j|\widehat{Z}_{M-m}(l))\Big|\mathbb{P}^{(y,m)}(X_M = x|X_{m+l} = w, \widehat{Z}_{M-m}(l))\\
&\leq \sum_{l\leq M-m}\mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l))\sum_{w\in\mathbb{Z}^d}Cl^{-\frac{d+1}{2}}\mathbb{P}^{(y,m)}(X_M = x|X_{m+l} = w, \widehat{Z}_{M-m}(l))\\
&\leq \sum_{l\leq M-m}Ce^{-c(M-m-l)}l^{-\frac{d+1}{2}}\\
&\leq C(M-m)^{-\frac{d+1}{2}}.
\end{aligned}$$

As before the proof of *iii)* requires slightly different arguments.

$$
\begin{aligned}
&|\mathbb{P}^{(y,m)}(X_M = x) - \mathbb{P}^{(y,m+1)}(X_M = x)| \\
&= \Big| \sum_{l \le M-m} \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m)}(X_{m+l} = w | \widehat{Z}_{M-m}(l)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \\
&- \sum_{l \le M-m} \mathbb{P}^{(y,m+1)}(\widehat{Z}_{M-m-1}(l-1)) \\
&\qquad \cdot \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \mathbb{P}^{(y,m+1)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m-1}(l-1)) \Big| \\
&= \Big| \sum_{l \le M-m} \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m)}(X_{m+l} = w | \widehat{Z}_{M-m}(l)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \\
&- \sum_{l \le M-m} \mathbb{P}^{(y,m+1)}(\widehat{Z}_{M-m-1}(l-1)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \Big| \\
&\le \Big| \sum_{l \le M-m} \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m)}(X_{m+l} = w | \widehat{Z}_{M-m}(l)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \\
&\quad - \sum_{l \le M-m} \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \Big| \\
&+ \Big| \sum_{l \le M-m} \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \\
&\quad - \sum_{l \le M-m} \mathbb{P}^{(y,m+1)}(\widehat{Z}_{M-m-1}(l-1)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \Big| \\
&\le \sum_{l \le M-m} \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) \sum_{w \in \mathbb{Z}^d} \Big| \mathbb{P}^{(y,m)}(X_{m+l} = w | \widehat{Z}_{M-m}(l)) - \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \Big| \\
&\cdot \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l)) \\
&+ \sum_{l \le M-m} \Big| \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) - \mathbb{P}^{(y,m+1)}(\widehat{Z}_{M-m-1}(l-1)) \Big| \\
&\cdot \sum_{w \in \mathbb{Z}^d} \mathbb{P}^{(y,m+1)}(X_{m+l} = w | \widehat{Z}_{M-m-1}(l-1)) \mathbb{P}^{(y,m)}(X_M = x | X_{m+l} = w, \widehat{Z}_{M-m}(l))
\end{aligned}
$$

The first summand can be estimated similarly to part *ii)*. For the last summand note that

$$
\begin{aligned}
&\Big| \mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l)) - \mathbb{P}^{(y,m+1)}(\widehat{Z}_{M-m-1}(l-1)) \Big| \\
&= \left| \mathbb{P}^{(y,m)} \left( Z(l) \cap \bigcap_{j=l+1}^{M-m} (Z(j))^c \right) - \mathbb{P}^{(y,m+1)} \left( Z(l-1) \cap \bigcap_{j=l}^{M-m-1} (Z(j))^c \right) \right| \\
&= \left| \mathbb{P}^{(y,m)}(Z(l)) \mathbb{P}^{(y,m)} \left( \bigcap_{j=l+1}^{M-m} (Z(j))^c \;\middle|\; Z(l) \right) - \mathbb{P}^{(y,m+1)}(Z(l-1)) \mathbb{P}^{(y,m+1)} \left( \bigcap_{j=l}^{M-m-1} (Z(j))^c \;\middle|\; Z(l-1) \right) \right| \\
&= \Big| \mathbb{P}^{(y,m)}(Z(l)) - \mathbb{P}^{(y,m+1)}(Z(l-1)) \Big| \mathbb{P}^{(y,m)} \left( \bigcap_{j=l+1}^{M-m} (Z(j))^c \;\middle|\; Z(l) \right) \\
&\le C \Big| \mathbb{P}^{(y,m)}(Z(l)) - \mathbb{P}^{(y,m)}(Z(l-1)) \Big| e^{-c(M-m-l)} \\
&\le C l^{-\frac{1}{2}} e^{-c(M-m-l)},
\end{aligned}
$$

where the last inequality follows by (A.16). Hence

$$\begin{aligned}
&\sum_{l\leq M-m}\left|\mathbb{P}^{(y,m)}(\widehat{Z}_{M-m}(l))-\mathbb{P}^{(y,m)}(\widehat{Z}_{M-m-1}(l-1))\right|\\
&\cdot\sum_{w\in\mathbb{Z}^d}\mathbb{P}^{(y,m+1)}(X_{m+l}=w|\widehat{Z}_{M-m-1}(l-1))\mathbb{P}^{(y,m)}(X_M=x|X_{m+l}=w,\widehat{Z}_{M-m}(l))\\
&\leq\sum_{l\leq M-m}Cl^{-\frac{1}{2}}e^{-c(M-m-l)}\sum_{w\in\mathbb{Z}^d}\mathbb{P}^{(y,m+1)}(X_{m+l}=w|\widehat{Z}_{M-m-1}(l-1))\mathbb{P}^{(y,m)}(X_M=x|X_{m+l}=w,\widehat{Z}_{M-m}(l))\\
&\leq\sum_{l\leq M-m}Cl^{-\frac{1}{2}}e^{-c(M-m-l)}\sum_{w\in\mathbb{Z}^d}l^{-\frac{d}{2}}\mathbb{P}^{(y,m+1)}(X_M=x|X_{m+l}=w,\widehat{Z}_{M-m}(l))\\
&\leq\sum_{l\leq M-m}Ce^{-c(M-m-l)}l^{-\frac{d+1}{2}}\\
&\leq C(M-m)^{-\frac{d+1}{2}}.
\end{aligned}$$

Hence part *iii*) follows. ☐

### A.2.2. Further annealed estimates

In this section we prove estimates (3.38)-(3.41). As required let $\frac{2}{5}\leq M\leq N^2$, $\frac{d}{d+1}<\theta\leq 1$, $\frac{d}{d+1}<\theta'<\theta$ and $V:=\left\lfloor N^{2\theta'}\right\rfloor$. Furthermore, we fix some cube $\Delta_x\subset\mathbb{Z}^d$ of side length $N^\theta$ and center $x\in\mathbb{Z}^d$. We denote by $\Delta_x^{(1)}$ a cube with center $x$ and side length $\frac{9}{10}N^\theta$ that is slightly smaller than $\Delta_x$ and by $\Delta_x^{(2)}$ a cube with center $x$ and side length $\frac{11}{10}N^\theta$ that is slightly bigger than $\Delta_x$. Hence

$$\begin{aligned}
&\mathbb{P}^{(\mathbf{0},0)}(X_{M+V}\in\Delta_x^{(1)})\\
&=\mathbb{P}^{(\mathbf{0},0)}(X_{M+V}\in\Delta_x^{(1)},X_M\in\Delta_x)+\mathbb{P}^{(\mathbf{0},0)}(X_{M+V}\in\Delta_x^{(1)},X_M\notin\Delta_x)\\
&\leq\mathbb{P}^{(\mathbf{0},0)}(X_M\in\Delta_x)+\mathbb{P}^{(\mathbf{0},0)}(X_{M+V}\in\Delta_x^{(1)},X_M\notin\Delta_x,R_N)+CN^{-c\log(N)}\\
&\leq\mathbb{P}^{(\mathbf{0},0)}(X_M\in\Delta_x)+\sum_{k=1}^{M}\sum_{z'}\mathbb{P}^{(\mathbf{0},0)}(X_{M+V}\in\Delta_x^{(1)},X_M\notin\Delta_x,R_N,T_{k-1}<M\leq T_k,(X_{T_k},T_k)=z')+CN^{-c\log(N)}\\
&\overset{z'=(x',n')}{\leq}\mathbb{P}^{(\mathbf{0},0)}(X_M\in\Delta_x)+M\sum_{\substack{n'\leq M+\log^2(N)\\ \|x-x'\|>N^\theta-\log^2(N)}}\mathbb{P}^{(x',n')}(X_{M+V}\in\Delta_x^{(1)})+CN^{-c\log(N)}\\
&\leq\mathbb{P}^{(\mathbf{0},0)}(X_M\in\Delta_x)+M\sum_{\substack{M\leq n'\leq M+\log^2(N)\\ \|x-x'\|>N^\theta-\log^2(N)}}\mathbb{P}^{(\mathbf{0},0)}(X_{M+V-n'}\in\Delta_{x-x'}^{(1)})+CN^{-c\log(N)}\\
&\leq\mathbb{P}^{(\mathbf{0},0)}(X_M\in\Delta_x)+M\sum_{M\leq n'\leq M+\log^2(N)}\mathbb{P}^{(\mathbf{0},0)}(\|X_{M+V-n'}\|\geq\frac{1}{10}N^\theta)+CN^{-c\log(N)}\\
&\leq\mathbb{P}^{(\mathbf{0},0)}(X_M\in\Delta_x)+CN^{-c\log(N)},
\end{aligned}$$

where the last inequality holds true by Lemma 3.6. Inequality (3.39) follows by similar arguments. Hence we turn to the proof of (3.40). An argumentation similar to (3.22) (see also Remark 3.13) leads to

$$\mathsf{E}^z\left[P_\omega^z(X_M\in\Delta)|\mathcal{G}\right](\omega_z)<P_{\omega_z}^z(X_M\in\Delta)+CN^{-c\log(N)},$$

hence it is enough to prove that

$$\mathsf{E}^z\left[P_\omega^z\left(X_{M+V}\in\Delta_x^{(1)}\right)-P_\omega^z(X_M\in\Delta)\Big|\mathcal{G}\right]<CN^{-c\log(N)}.$$

Notice that

$$\mathsf{E}^z\left[P_\omega^z\left(X_{M+V}\in\Delta_x^{(1)}\right)-P_\omega^z(X_M\in\Delta)\Big|\mathcal{G}\right]\leq\mathsf{E}^z\left[P_\omega^z\left(X_{M+V}\in\Delta_x^{(1)},X_M\notin\Delta_x\right)\Big|\mathcal{G}\right].$$

Arguing as in Lemma 3.18, we can compare the term above to the annealed probability of hitting $\Delta_x^{(1)}$ at time $M+V$ without hitting $\Delta_x$ at time $M+l(N)$, where $l(N)$ is a slowly varying function. Hence (3.39) follows by the annealed estimates we proved above.