Stochastic Foundations of Dynamic Trade and Labor Market Models

Dissertation

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vorgelegt von

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To my father

Contents

List of Abbreviations

CES	constant elasticity of substitution
CRRA	constant relative risk aversion
FPE	Fokker-Planck equation
i.i.d	independently and identically distributed
LHS	left-hand side
LID	London Institutional Database
ODE	ordinary differential equation
OECD	Organization for Economic Co-operation and Development
PDE	partial differential equation
PMF	probability mass function
PDF	probability density function
R&D	research and development
RHS	right-hand side
SDE	stochastic differential equation
UA	unemployment assistance
UI	unemployment insurance

Chapter 1

Introduction

This dissertation consists of three self-contained papers that are related to two main topics. In particular, the first and third studies focus on labor market modeling, whereas the second essay presents a dynamic international trade setup. In Chapter 2 of this dissertation, we investigate how expenses on labor market reforms change over time. In Chapter 3, we study how markup distributions adjust when a closed economy opens up. In Chapter 4, we examine effects of aggregate shocks on the distribution of the unemployment rates in the OECD member countries. In all three chapters we model systems that behave randomly and operate on stochastic processes. We therefore exploit stochastic calculus that establishes clear methodological links between the chapters.

Chapter 2 (”Expenses on Labor Market Reforms during Transitional Dynamics”) develops a dynamic model with heterogeneous employed and unemployed workers to analyze various effects of unemployment benefits system changes. The model exploits *stochastic calculus* to describe the transitional dynamics between two steady states of the economy that occurs after such changes. In so doing, the model estimates the duration of the reform, the dynamics of the unemployment rate and, most importantly, alteration of total unemployment benefit payments in countries with the two-tier unemployment benefits system.

The model predicts how expensive a potential reform might be and to which conditions in the labor market it might lead. As an example, the model foretells that the reduction of the length of unemployment insurance payments from 12 months to 6 months in Germany will cause a drop in the unemployment rate from 6.1% to 5.6% within 3 years. It moreover will lead to a decline in the government's expenses per unemployed individual by 10.7%.

Chapter 3 (”Endogenous Markup Distributions”, co-authored with Klaus Wälde) studies the effects of international trade on the markup distribution in an economy. In order to perform this analysis, we first present a closed-economy general-equilibrium industry dynamics model, where firms enter and exit markets. Within one market firms behave as Cournot competitors. This behavior yields an endogenous markup distribution in each market and across markets. Building on Fokker-Planck equations, the closed-economy model introduces the evolution of markup distributions over time from some initial distributions towards corresponding long-run stationary distributions.

We provide a simple estimation of the closed-economy model parameters to reach the average steady-state markup within the drugs and medicines market from U.S. data. We then present an open-economy model where two potentially asymmetric countries start trading. We thus analyze the evolution of markup distributions over time after the economies are opened up. As a result, we observe that opening up the economies leads to a decrease in markups in the mutual markets due to higher competition.

In Chapter 4 ("Unemployment in the OECD – Pure Chance or Institutions?", co-authored with Klaus Wälde), we portray the matching process in the labor market as a truly stochastic process that includes both *idiosyncratic* and *aggregate* components. The evolution of the distribution of unemployment rates follows on from this process. When transitions in the labor market are purely idiosyncratic, the mean and variance of the unemployment rates across OECD member countries accounted for by the standard textbook model are 5.71% and $1.6 \cdot 10^{-9}$ respectively. The mean value is plausible, whereas variance is insignificant.

Our model provides similar results with respect to the long-run average unemployment rate in the presence of aggregate shocks which is 5.86%. In contrast to the standard textbook model, it can also predict 1% of the observed long-run variation in unemployment rates. We therefore conclude that 99% of the long-run cross-country variation in the unemployment rates is due to *policy*. The model moreover foretells that the effect of aggregate shocks on the distribution of unemployment rates is even more significant in short-term dynamics. This means that the variation in the unemployment rates across OECD member countries just after aggregate shocks occur can be predominantly put down to *pure chance*.

Chapter 2

Expenses on Labor Market Reforms during Transitional Dynamics

by Alexey Cherepnev[1]

2.1 Introduction

Modern labor markets are characterized by a considerable number of fluctuations. Worker-employer matches get paired and destroyed, job seekers constantly look for new positions with varying intensity. Definitely, their job-search effort depends on how long they have been unemployed. On the other hand, job-search intensity also relies on the unemployment benefits system imposed in the economy.

It is commonly accepted that a decrease in the amount or duration of unemployment benefit payments reduces the unemployment rate. It is, however, extremely difficult to provide precise answers to questions like "By how much should unemployment benefits be changed in order to reduce the unemployment rate by, e.g., 0.1 percentage points?" or "How long does it take for a reform to exert an effect on the economy as a whole after the reform is pushed through?". Moreover, there is no clear answer to what the optimal unemployment benefits system is and how it can motivate job seekers to apply for new positions. However we must keep in mind that such an optimal system should hold unemployment payments relatively high in order to avoid poverty.

Thus, the objective of this study is to analyze various effects of a labor market reform on the unemployment benefits system. In particular, in this paper we estimate the duration of the reform, the dynamics of the unemployment rate and,

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most importantly, the alteration of total payments to job seekers in countries with the two-tier unemployment benefits system. The article also sheds light on the link between unemployment payments and the unemployment duration, i.e. on job-search intensity.

In order to address these topics, we present an elegant model with heterogeneous individuals that can be employed or unemployed over varying periods of time. Unemployed individuals are characterized by job-search intensity which, in turn, depends on how long individuals have remained unemployed (the unemployment spell), and on parameters of the unemployment benefits system, such as amount and duration of the payments. When unemployed, individuals influence the arrival rate of a new job offer by raising their job-search intensity. The increase in the job-search intensity, however, brings some disutility because unemployed individuals do not enjoy the process of looking for a new job.

The transitional dynamics, which occur when agents change their employment status, characterizes the economy between two steady states. The first steady state describes the economy before the reform is pushed through. The second equilibrium refers to a stable point in the economy afterwards. To detail such dynamics, the model exploits *stochastic calculus* and describes the status of agents at the individual level. For each agent we build a stochastic process that details the duration of employment and unemployment. Despite common characteristics, each process includes the optimal job-search effort of a corresponding individual given via an exit rate (a rate with which an individual exits from unemployment and becomes employed). This effort is a solution to the problem when the agent maximizes his expected future utility.

Every stochastic process in this model leads to a system of Fokker-Planck equations that characterizes the evolution of the probability density function of spells of unemployment and employment. Given this evolution and proper aggregation of all agents, we obtain the dynamics of the unemployment rate and the government expenses.

This study extends Launov and Wälde's (2013) equilibrium-matching model in order to analyze the dynamics of unemployment duration outside of the steady state. We also use the same modeling framework in the part of the endogenous effort formation. Furthermore, from the empirical prospective, the current paper utilizes the calibration parameters from that work.

Considering a two-tier unemployment benefits system, we take reduction in the duration of the unemployment insurance payments from 12 months to 6 months as an example of the labor market reform. Given this example, the model predicts a fall in the unemployment rate from 6.1% to 5.6% within 3 years. This fall appears due to the higher effort of individuals to find a new job caused by the shorter duration of unemployment insurance payments. After the calibration is done, the model provides other quantitative results as well. We, for instance, observe a decrease in the amount of government payments per job seeker by 10.7%.

This paper relates to various strands of literature. First, it builds on the theoretical literature which provides different benchmark models. This literature was pioneered by Mortensen (1982), Pissarides (1985), Blanchard and Diamond (1994), and Burdett and Mortensen (1998). These papers study how the labor market is affected by output shocks, R&D, unemployment, and wage changes. Second, the current work borrows heavily from the literature on the time dependent unemployment payments. This question was initially analyzed by Mortensen (1977) and then extended by, e.g., Acemoglu and Shimer (1999) and Albrecht and Vroman (2005).

Third, this paper also contributes to the literature on the link between unemployment compensation and unemployment duration (see Meyer, 1990, Roed and Zhang, 2003, Lalive, 2008, Farber and Valletta, 2013). In contrast to these papers which mainly focus on the steady state analysis, our study concentrates primarily on the transitional dynamics between steady states of the economy. In the same vein, this article provides support to numerous studies which have attempted to explain the link between time-dependent unemployment payments, endogenous job-search effort formation, and unemployment spells (see Cahuc and Lehmann (2000), Launov and Wälde (2013)). Indeed, optimal job-search effort of an individual to escape from unemployment relies not only on employment status, but also on how long the individual has been unemployed. Collectively, these studies outline a critical role for the duration of unemployment in explaining the dynamics of the unemployment rate and government expenses.

From the methodological perspective, we employ the Fokker-Planck equations (see Ross, 1996) which are solved by the method of characteristics (see for instance Courant and Hilbert, 1962, Evans, 1998, John, 1991, and Polyanin et al., 2001). In the economic literature, Fokker-Planck equations were also analyzed in various contexts, for example, by Klette and Kortum (2004) and Impullitti et al. (2013).

In this paper we continue with Section 2.2 where we provide the theoretical model that describes the transitional dynamics between states of the economy. Then, in Section 2.3, we obtain analytical results by deriving the Fokker-Planck equations and solving them. Using parameters from the paper by Launov and Wälde (2013) as a basis, we calibrate the model in Section 2.4. Finally, Section 2.5 concludes this paper with a summary.

2.2 The model

In this section we introduce the model. In doing so, we assume that the total number of agents representing the whole labor force is given by the positive finite discrete number N. The number of employed individuals varies over time and is denoted by $L(t) \in [0, N]$. The number of job seekers is $U(t) = N - L(t)$. There is a two-tier unemployment benefit scheme. The short-term unemployed agents receive high unemployment insurance (UI) compensation, which is comparable to their wage, whereas the long-term unemployed agents get low unemployment assistance (UA) payments. All agents are risk-averse, cannot save, and live for an infinitely long period of time. Time is continuous, the initial time point is $t = 0$.

Starting from the individual level, we introduce an endogenous job-search effort of an unemployed agent. This effort is based on the unemployment compensation which, in turn, depends on the unemployment spell. As an example of such dependence, we observe an increase in the job-search effort of an unemployed individual during the UI compensation period. The increase occurs because at the end of this period high UI payments are replaced by low UA payments. As a consequence of such replacement, unemployed agents try to avoid receiving these low payments by raising their effort to find a new position:

$$\underbrace{\text{unemployment spell} \rightarrow \text{search effort}}_{\text{individual level}} \Rightarrow$$
$$\Rightarrow \underbrace{\text{exit rate} \Rightarrow \text{aggregate unemployment rate}}_{\text{aggregate level}} \quad (2.1)$$

High individual effort is, in turn, not a guarantee that an unemployed individual finds a new job. It however increases the probability that the job match happens earlier. Mathematically speaking, high effort raises the rate to escape from unemployment (the exit rate). We therefore observe a link between the individual level and the aggregate level, since the aggregate unemployment level strongly depends on the exit rate.

On the aggregate level we study the out-of-the-steady-state evolution of the number of unemployment agents $U(t)$ and of the unemployment rate $u(t)$ at any future point in time $t > 0$. We furthermore expect that the variables $L(t)$, $U(t)$, and $u(t)$ will converge to their equilibrium values from any initial quantities at $t = 0$. Providing a link between the number of short-term and long-term unemployed agents and the two-tier unemployment compensation scheme, we moreover study the alteration of government expenses.

2.2.1 Individual level

We first focus on the employment state of an individual. Basically, individual i can be found in either of two states $x \equiv x_i(t) \in \{0, 1\}$ at each point in time t, where 1 stands for employment, and unemployment is denoted by 0. We accurately illustrate the transition dynamics between these two states and write these dynamics formally with the use of Poisson processes $q_\mu \equiv q_{i\mu_i}(t)$ and $q_\lambda \equiv q_{i\lambda}(t)$ via the following *stochastic differential equation* (SDE):[2]

$$dx = [1 - x]dq_\mu - xdq_\lambda. \tag{2.2}$$

In words, an event that an unemployed worker ($x = 0$) finds a new job appears with the individual-specific rate $\mu \equiv \mu_i(\cdot)$,[3] i.e. the change dq_μ equals 1 whenever q_μ jumps, i.e. when unemployed i finds a new job. This first part of the right-hand side becomes inactive when an agent becomes employed and $x = 1$, i.e. $1 - x = 0$.

It is a well-known fact that people adapt to a current situation and optimally choose their behavior. In this study we analyze individuals' incentives to find a new job when they are unemployed: an unemployed individual chooses the optimal effort to become employed given his current unemployment spell $s \equiv s_i(t)$ and thus earnings $b(s) \equiv b(s_i(t))$. It makes the use of individual shocks quite intuitive, i.e. agents are able to influence the arrival rate μ of a new job, e.g. increasing μ agents have a higher possibility of finding a new position faster. Individuals therefore find jobs through their own individual process q_μ.

Each worker in state $x = 1$ loses his job with rate λ which is constant for all individuals and is assumed to be exogenous. It happens when job destruction shock q_λ jumps, i.e. when $dq_\lambda = 1$. This assumption may reflect some policy changes or a labor market crisis; both are common to all individuals.[4]

Employed agents earn wage w and thus make no decisions about their future. The unemployment benefits, $b(s)$, paid to all job seekers, are normally lower than w and depend significantly on the length of the unemployment spell s. We therefore model behavior of individual spells s, given q_λ and q_μ, with the help of SDEs below. We do not allow for on-the-job searches.

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[2] We discuss in this section an individual i, but we suppress index i for the sake of notation simplicity.

[3] It reflects the fact that the exit rate depends on the effort of the individual (see scheme (2.1)). We show this connection below.

[4] The model described above is a continuous-time generalization of the model by Klette and Kortum (2004) where firms appear on the market with one product with rate λ and disappear from the market when the number of products they produce drops to zero.

Individual spells

Considering further one individual, we perceive that the unemployment spell s is positive, increases over time and drops to zero when the agent finds a job. We model this behavior by the component $dt - sdq_\mu$, where s is the unemployment spell of the individual measured in time units. The first component dt denotes an increment of unemployment duration over time. This means that if nothing else happens, the spell s increases between t_0 and t by $t - t_0$, where t_0 is a point in time when a worker has just lost the job (see Figure 2.1). The second component stands for an individual shock modeled by q_μ which matches person i to a new job. Note that process q_μ is exactly the same as in the SDE for the employment state x (see (2.2)).[5]

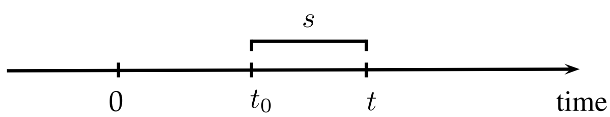


Figure 2.1: Relationship between t, t_0, and s, where t_0 is a point in time when the individual lost the job, t is some future point in time, and 0 is today.

The situation is similar when agent i is employed: The employment spell of agent i is given by negative values of variable s and described by the following equation $ds = -dt - sdq_\lambda$, where dt is again a trend component. The second term, sdq_λ, represents an idiosyncratic shock when the agent becomes unemployed again.[6]

We next combine two components of spell behavior into total dynamics of spell variable s and use status variable x from (2.2) to distinguish between employment and unemployment states:

$$ds = [1 - x][dt - sdq_\mu] - x[dt + sdq_\lambda]. \tag{2.3}$$

The first part is obviously functioning if the person is in the unemployment state, $x = 0$, representing the dynamics of unemployment spell. The second part, referring to the employment spell dynamics, is certainly inactive. The situation is opposite when the agent is employed, $x = 1$. Thus, as we emphasized above, positive values of s illustrate an unemployment spell which increases over time. Negative values of s, on the other hand, stand for a decreasing in time employment spell. Figure 2.2 reveals this dynamics.

[5]We could introduce the third component, an aggregated shock. Due to this type of shock, a certain fraction of employed workers gets fired simultaneously. It certainly becomes important in Chapter 4.

[6]We primarily analyze the unemployment state and therefore set the unemployment spell to be positive to simplify further calculations.

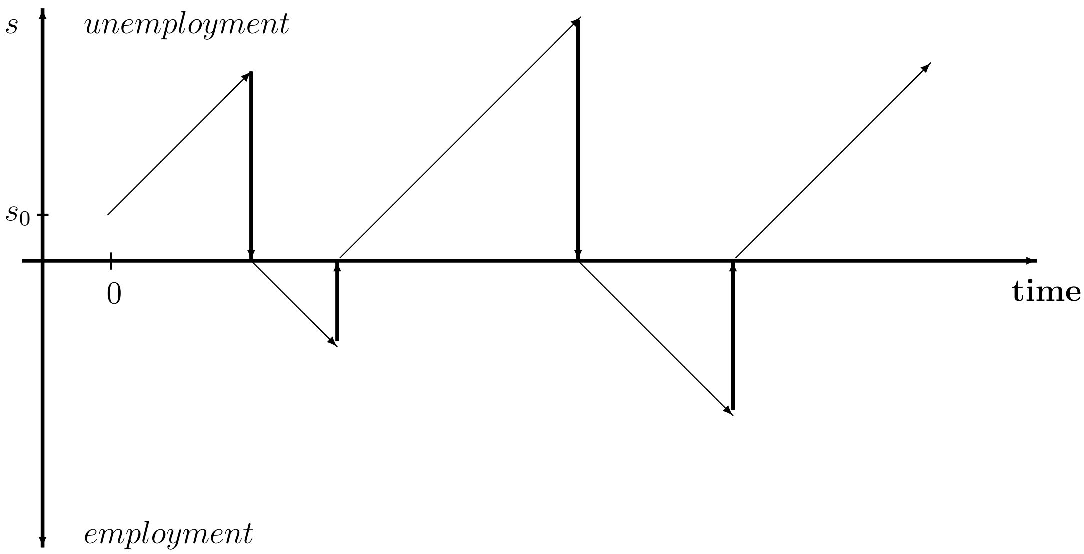


Figure 2.2: Spell dynamics according to the SDE (2.3).

Importantly, the introduced dynamics allows us to fully track an agent in both states and see how long the duration in these states is. It means that longer the individual i stays in state $x = 0$, the higher the unemployment spell s is. If the individual is employed, $x = 1$, the spell s decreases.

Endogenous effort formation

As we mentioned above, the unemployment spell of an individual defines the search effort. In this chapter we model this connection. In doing so, we consider the simplest adjustment dynamics of search effort based on the individual's intertemporal utility function:

$$U = E_0 \int_0^{+\infty} e^{-\rho t} u(c, \phi(s)) dt \tag{2.4}$$

and the time preference rate $\rho > 0$. The time point when we start the analysis is wisely normalized to 0 hereafter. The instantaneous utility depends on consumption $c \equiv c_i(t)$ and the job-search effort $\phi(s) \equiv \phi_i(s_i(t))$. Individuals dislike $\phi(s)$. We thus write the instantaneous utility function, which represents a CRRA utility in terms of consumption, as follows:

$$u(c, \phi(s)) = \frac{c^{1-\sigma} - 1}{1 - \sigma} - \phi(s) \tag{2.5}$$

Since on-the-job searching is not allowed in the model, employed individuals do not look for new jobs and their job-search effort therefore equals zero, i.e. $\phi(s) = 0$ when $s < 0$.

We assume that agents live infinitely long and have no capital income and no savings. They thus spend all earnings $z(s) \equiv z(s_i(t))$ on consumption c, i.e.

$$c = z(s). \tag{2.6}$$

We determine individual benefits $z(s)$ being in one of three states $\{w, b_{ui}, b_{ua}\}$, i.e. these benefits may be a constant wage w for an employed individual or unemployment payments $b(s) \equiv b(s_i(t)) \in \{b_{ui}, b_{ua}\}$ for a job seeker (b_{ui} as *unemployment insurance* or b_{ua} as *unemployment assistance*), or

$$z(s) = \begin{cases} b(s) = \begin{cases} b_{ui}, & \text{if } 0 < s < \bar{s}, \\ b_{ua}, & \text{if } s \geq \bar{s}, \end{cases} \\ w, & \text{if } s < 0, \end{cases}$$

where $0 < b_{ua} < b_{ui} < w$ and $\bar{s} > 0$ is the unemployment duration after which unemployment assistance is paid.

There is no technological growth in the economy. Hence, the labor income process $dz(s)$ can be described by an *SDE* with job matching and job destruction, already modeled by q_μ and q_λ respectively:

$$dz(s) = [1 - x][w - b(s)]dq_\mu - x[w - b_{ui}]dq_\lambda.$$

The first term plays a role when an agent is unemployed, i.e. $x = 0$. In this case the agent searches for a job with effort $\phi(s)$ that leads to the arrival rate of the job-finding event $\mu = \mu(\phi(s))$. When the individual finds a job, i.e. when q_μ jumps, the benefits $b(s)$ are replaced by wage w that the agent receives while being employed. It might also happen that the agent loses the job, i.e. when $x = 1$ and process q_λ jumps. At this moment benefits b_{ui} substitute for wage w, and the employed agent becomes unemployed, $x = 0$ (see (2.2)).

Value functions and Bellman equations. We form the maximization problem using the value function approach. The value function itself depends on state variables, i.e. on labor income $z(s)$ and (un)employment spell s. From (2.4) we thus write:

$$V(z(s), s) = \max_{\phi(s)} U = \max_{\phi(s)} E_0 \int_0^{+\infty} e^{-\rho t} u(c, \phi(s)) dt \tag{2.7}$$

Effort $\phi(s)$ is a *control* variable.

The general form of the Bellman equation from the time point $t = 0$ perspective is given by (see Cahuc and Zylberberg, 2004, Sennewald and Wälde, 2006, Sennewald, 2007, and Wälde, 2012):

$$\rho V(z(s), s) = \max_{\phi(s)} \left\{ u(c, \phi(s)) + \frac{1}{dt} E_0 dV(z(s), s) \right\} \tag{2.8}$$

with the budget constraint (2.6). Further derivations and additional information that lead to the Bellman equations in employment and unemployment states are provided in Appendix 2.A.

Employment. In cases when $z(s) = w$, the endogenous effort to find a new job is $\phi(s) = 0$ since the agent is employed and on-the-job searching is not allowed. Thus, Bellman equation (2.8) can be rewritten for the case $x = 1$ and $V(w) \equiv V(w, s)$ as following (similar to Launov and Wälde, 2013):

$$\rho V(w) = u(w, 0) - \lambda \left[V(w) - V\Big(b_{ui}, 0\Big) \right] \tag{2.9}$$

The instantaneous utility flow of being employed, $\rho V(w, s)$, is given as a combination of two components: The first part stands for the instantaneous utility when earning positive wage w and sparing no effort $\phi(s) = 0$. The second component denotes a random event of losing a job. When this event happens (with constant rate λ), an employed worker loses the difference between the value of being employed with wage w and the value of becoming unemployed with benefits b_{ui}.

Unemployment. For the case of unemployment, when $z = b(s)$, we obtain from (2.8):

$$\begin{aligned} \rho V(b(s), s) = \max_{\phi(s)} \Bigg\{ u(b(s), \phi(s)) + \frac{\partial V(b(s), s)}{\partial s} \\ + \mu(\phi(s)) \Big[V(w) - V(b(s), s) \Big] \Bigg\} \end{aligned} \tag{2.10}$$

The interpretation of this equation is similar to one for equation (2.9). We again have three components that build the instantaneous utility flow: The first component is the instantaneous utility of being unemployed, earning spell-dependent benefits $b(s)$ and applying some effort $\phi(s)$ to search for a new job. The second part defines a change in the value function when the unemployment spell changes over time. The plus sign appears because we assume a positive spell in this state. The third component describes a random event for an unemployed individual to find a new job. This event happens with the effort-dependent arrival rate, $\mu(\phi(s))$. An unemployed agent gains the difference between the value function of becoming employed and earning w and the value function of being unemployed with spell s and benefits $b(s)$.

The value function of being employed, $V(w)$, is unvarying given constant wage

w and rate λ. Intuitively an employed individual might become only short-term unemployed. As an effect of it, probabilities to this event at different time points and for different employment spells are identical (see (2.7)). Note that in this state an agent solves no maximization problem.

Similarly, $V(b_{ua}) \equiv V(b_{ua}, s)$ does not vary over time and spell given the optimal behavior of an individual and constant long-term benefits b_{ua}. It leads to the constant long-term effort $\phi(\bar{s})$ which must be higher than the short-term one $\phi(s)$, $0 < s < \bar{s}$.[7]

Obviously $V(b_{ua}) \leq V(b_{ui}, s) < V(w)\ \forall s$, which is also quite intuitive. If we focus on one period only, we write $u(b_{ua}, \phi(\bar{s})) \leq u(b_{ui}, \phi(s)) < u(w, 0)$, $0 < s \leq \bar{s}$, because the instantaneous utility $u(z(s), \phi(s))$ is an increasing function of the first argument, a decreasing function of the second argument, $b_{ua} < b_{ui} < w$ and $\phi(\bar{s}) \geq \phi(s) > 0$, $0 < s < \bar{s}$. The same holds for all periods.

Note that the right-hand side of the Bellman equation (2.10) of an unemployed individual is a one-dimensional differential equation with a unique fixed point defined by $\partial V(b(s), s)/\partial s = 0$. The derivative is negative to the left of the fixed point indicating that value function $V(b(s), s)$ decreases when the unemployment spell increases. Obviously, both $V(b_{ua})$ and $V(b_{ui}, s)$ are located in this area, $V(b_{ua}) < V(b_{ui}, s)$ and $V(b_{ui}, s)$ converges to $V(b_{ua})$ starting from $V(b_{ui}, 0)$ and following the dynamics of the system. For the demonstration purposes we assume that $V(w)$ lies in the area to the right from the fixed point, however it is not always the case.

Optimal effort. Given (2.5) and (2.16), we can solve the maximization problem in (2.10) that defines the optimal effort through the first-order condition (see derivations in Appendix 2.A):

$$\phi(s) = \left(\alpha\Big[V(w) - V(b(s), s)\Big]\right)^{\frac{1}{1-\alpha}}. \tag{2.11}$$

The optimal effort (2.11) differs for short-term and long-term unemployed individuals:

$$\phi(s) = \begin{cases} \left(\alpha\Big[V(w) - V(b_{ui}, s)\Big]\right)^{\frac{1}{1-\alpha}}, & s < \bar{s} \text{ and } b(s) = b_{ui}, \\ \left(\alpha\Big[V(w) - V(b_{ua})\Big]\right)^{\frac{1}{1-\alpha}}, & s \geq \bar{s} \text{ and } b(s) = b_{ua}. \end{cases} \tag{2.12}$$

Given $V(b_{ua}) < V(b_{ui}, s)$, the optimal effort of a short-term unemployed individual $0 < \phi(s) \leq \phi(\bar{s})$ and evolves over time (see Figure 2.3), whereas the optimal effort

[7] The time discounting rate ρ is also important here: for high values of ρ an individual does not assess future periods which makes the short-run value function $V(b_{ui}, s)$ constant, but $V(b_{ua}, s) < V(b_{ui}, s)$ also holds.

of a long-term unemployed individual $0 < \phi(s) = \phi(\bar{s})$ remains constant.

The maximized Bellman equation is derived when we plug the optimal effort from (2.11) into the Bellman equation (2.10) (see derivations in Appendix 2.A):

$$\rho V(b(s), s) = \frac{b(s)^{1-\sigma} - 1}{1-\sigma} + \frac{\partial V(b(s), s)}{\partial s} + \chi \Big[V(w) - V(b(s), s) \Big]^{\frac{1}{1-\alpha}}, \tag{2.13}$$

where $\chi = [\alpha\eta]^{\frac{\alpha}{1-\alpha}} [1 - \alpha\eta]$. The value function of being a long-term unemployed follows from this when we set $b(s) = b_{ua}$ and $\partial V(b(s), s)/\partial s = 0$:

$$\rho V(b_{ua}) = \frac{[b_{ua}]^{1-\sigma} - 1}{1-\sigma} + \chi \Big[V(w) - V(b_{ua}) \Big]^{\frac{1}{1-\alpha}} \tag{2.14}$$

in which $V(b_{ua})$ is expressed implicitly.

Solving for value function and optimal effort. In order to solve for the optimal effort (see (2.11)) we have to analyze two crucial points of the value function $V(z(s), s)$ which are $s = 0$ and $s = \bar{s}$. First, individuals do not know the time point when they lose their current positions. They moreover cannot influence this process, and it happens suddenly. Therefore their value function experiences a dramatic drop in its value when changing status from employed to short-term unemployed ($V(w) > V(b_{ui}, 0)$). Mathematically speaking, the value function $V(z(s), s)$ is discontinuous at $s = 0$ and therefore not differentiable at this point. Second, the point $\bar{s}$ reveals the time point when the individual becomes long-term unemployed and his benefits switch from unemployment insurance to unemployment assistance. An unemployed individual is able to predict this change in payments and adjust the job-search effort accordingly. This means that the value function is continuous at spell $\bar{s}$, but not differentiable, i.e. $V(b_{ui}, \bar{s}) = V(b_{ua})$ and $\partial V(b_{ui}, \bar{s})/\partial s \neq 0$.

In order to find the functional form of the optimal effort from (2.11) we exploit equations (2.9), (2.13), and (2.14) analyzing them jointly:[8]

$$\begin{cases} V(b_{ui}, 0) = \frac{(\rho+\lambda)}{\lambda} V(w) - \frac{1}{\lambda} \frac{w^{1-\sigma}-1}{1-\sigma} \\ \frac{\partial V(b_{ui}, s)}{\partial s} = \rho V(b_{ui}, s) - \frac{[b(s)]^{1-\sigma}-1}{1-\sigma} - \chi \Big[V(w) - V(b_{ui}, s) \Big]^{\frac{1}{1-\alpha}} \\ V(w) = V(b_{ua}) + \Big\{ \frac{\rho}{\chi} V(b_{ua}) - \frac{1}{\chi} \frac{[b_{ua}]^{1-\sigma}-1}{1-\sigma} \Big\}^{1-\alpha} \end{cases} \tag{2.15}$$

Starting from the last equation and assuming some value $V(b_{ua})$ we obtain the value of being employed $V(w)$. The latter value, in turn, specifies the condition at $s = 0$ of the short-term unemployment value $V(b_{ui}, 0)$ in the first equation of the system. The second equation can be transformed to an ordinary inhomogeneous

[8]We make straightforward algebra manipulations to obtain this system.

differential equation since the derivative with respect to only one argument s is present. This differential equation cannot be solved analytically. We therefore provide the numerical solution with calibrated parameters in Section 2.4.1.

We treat the latter system (2.15) to be solved when an assumed value of $V(b_{ua})$ fixes the boundary condition at $s = 0$ of the second differential equation, $V(b_{ui}, 0)$ (similarly to Launov and Wälde, 2013, Appendix A.3). Given this value the solution to the inhomogeneous differential equation must lead to $V(b_{ui}, \bar{s}) = V(b_{ua})$. If this is not the case, value $V(b_{ua})$ was inconsistently specified and should be adjusted accordingly.

2.2.2 Aggregate level

After obtaining the results for the optimal effort on the individual level, we are now able to describe the aggregated state, in particular, the job-matching rate and the aggregate unemployment rate. Note that the *job-destruction* rate, λ, is assumed to be constant and common to all individuals.

Exit rate

An *endogenous effort* of a worker of finding a new job, denoted by $\phi(s)$, certainly influences the probability to find a new position. We model this effect via what is generally called *the job-finding rate*, *hazard rate* or *rate to exit from unemployment* of the form:

$$\mu(\phi(s)) = \eta[\phi(s)]^{\alpha} \tag{2.16}$$

with $0 < \alpha < 1$. We discuss the scale parameter η, which reflects agent's search ability, below.

All agents are characterized by the unemployment or employment duration s. In addition, there are other kinds of heterogeneity. In particular, we cannot ignore unobserved heterogeneity of agents' job-search productivity type, which becomes important if we build an aggregate exit rate and compare it to the real data.

We therefore introduce two groups of agents with different unobserved ability to search for a job: N_1 is the number of people with a *high* search productivity type. The number of agents with of a *low* type is $N_0 = N - N_1$. To include search ability into the individual exit rate $\mu(\phi(s))$, we let the scale parameter η denote an individual's unobserved ability in the following way:

$$\eta = \begin{cases} \eta_1, & \text{with probability } \pi = N_1/N, \\ \eta_0, & \text{with probability } 1 - \pi = N_0/N. \end{cases} \tag{2.17}$$

All individuals are otherwise identical. This fact is modeled via the common optimal effort (2.11), i.e. all individuals within one ability group being in the same state x and with an identical spell s behave in exactly the same way.

Aggregate unemployment rate

We suggest a trivial aggregation mechanism. However, it includes unknown probabilities of being employed and unemployed, as well as the probability of observing an individual with certain duration s at each point in time. Mathematically speaking, the key ingredient of the model, the state of an individual i, $x_i(t)$, is unfortunately not certain at every time point $t > 0$: The agents can be observed in any one of two states, $x_i \in \{0, 1\}$, with certain probabilities which strongly depend on the individual job matching rate $\mu_i(\phi_i(s))$ (see *SDEs* (2.2) and (2.3), and (2.16)).

Suppose for each agent i we know these probabilities, or, more precisely, probability density functions. In this case we are able to compute the expected value of variable $x_i(t)$ as well as the total number of employed agents according to

$$L(t) = \sum_{i=1}^{N} x_i(t).$$

The number of unemployed agents is then defined by

$$U(t) = N - L(t).$$

The unemployment rate is then given by

$$u(t) = \frac{U(t)}{N} = \frac{N - L(t)}{N}.$$

We detail the aggregation algorithm in Section 2.3.3, where we cover the aggregate dynamics and obtain a missing evolution of individual probabilities of being employed and unemployed. The analysis of expenses on labor market reforms is discussed thereafter.

2.3 Analytical results

It is essential to derive the evolution of the key ingredient of the model, the unconditional probability density function $p_\eta(t, x, s)$.[9] Its integral over certain spell and time intervals gives the probability for the random duration s of an agent of taking on a particular value within this spell interval at a time point inside the time interval. One important feature of the *pdf* $p_\eta(t, x, s)$ is that it splits the state prob-

[9]This problem can be also solved via simulations (see Lewis and Shedler, 1979 and Ross, 2002).

abilities for being employed $P_\eta(t,1)$ and unemployed $P_\eta(t,0)$ into spell components:

$$P_\eta(t,1) = \int_{-\infty}^{0} p_\eta(t,1,s)ds \quad \text{and} \quad P_\eta(t,0) = \int_{0}^{\infty} p_\eta(t,0,s)ds. \tag{2.18}$$

It shows that the probability of the employment or unemployment state, $P_\eta(t,x)$, can be easily derived from the probability density function $p_\eta(t,x,s)$ where $x \in \{0,1\}$.

We also note that the *pdf* depends on the search productivity type η. However, the dynamics of *pdf* for each individual with the common type is identical. Therefore, in most cases, we write the general η meaning that it holds for both η_1 and η_0.

2.3.1 Fokker-Planck equations with state dependent arrival rates

The unknown evolution of *pdf* $p_\eta(t,x,s)$ over both time and spell components makes our analysis fairly complicated. However, in general, the evolution the *pdf* can be fully described by a system of Fokker-Planck equations. To find this system we first combine the given above stochastic differential equations (2.2) and (2.3) into a system:

$$\begin{cases} dx &= [1-x]dq_\mu - xdq_\lambda \\ ds &= [1-2x]dt - [1-x]sdq_\mu - xsdq_\lambda \end{cases} \tag{2.19}$$

where $\mu = \mu(\phi(s))$ and λ is constant. We derive the system of Fokker-Planck equations in Appendix 2.B and write the result as follows:

$$\begin{cases} p_\eta(t,0,0) = \lambda P_\eta(t,1) \\ p_\eta(t,1,0) = \int_0^\infty \mu(\phi(s))p_\eta(t,0,s)ds \\ \frac{\partial p_\eta(t,0,s)}{\partial s} + \frac{\partial p_\eta(t,0,s)}{\partial t} = -\mu(\phi(s))p_\eta(t,0,s) \\ \frac{\partial p_\eta(t,1,s)}{\partial s} - \frac{\partial p_\eta(t,1,s)}{\partial t} = \lambda p_\eta(t,1,s). \end{cases} \tag{2.20}$$

This is a partial differential equation system of the probability density function $p_\eta(t,x,s)$ since it contains partial derivatives with respect to two variables, time t and spell s.

The system includes four equations: The first equation describes the evolution of the probability density function at the point in time t of an individual who just become unemployed, $p_\eta(t,0,0)$, i.e. it forms the boundary conditions at $s = 0$. It means that an employed agent, when losing the job, immediately becomes unemployed with the spell $s = 0$, and this process is independent of the spell s of an employed worker. The second equation of the system (2.20) is a generalization of the first one. It describes the opposite situation, i.e. when an unemployed individual finds a new job. The unemployment spell becomes then $s = 0$ and starts falling from the zero level. However, the exit rate $\mu(\phi(s))$ depends on the job-search effort $\phi(s)$ and, therefore, on the unemployment spell s. In particular, according to (2.11) and Figure 2.3, the high unemployment spell implies the high job-search effort of an unemployed individual and the high job-matching rate. The latter, in turn, leads to high probability of finding a new job. If job-matching rate μ was independent of spell s, the second equation would appear to be similar to the first equation of the system given (2.18).

The partial derivatives with respect to t in the third and fourth equations label the density evolution over time for a given spell s. The spell derivatives on the other hand describes how the probability density function changes when the time point t is fixed. As we can see, the density at each point in time and for each spell level depends on the arrival rates $\mu(\phi(s))$ and λ. These rates together with initial conditions on $p_\eta(t,0,s)$ and $p_\eta(t,1,s)$, $p_\eta(0,0,s)$ and $p_\eta(0,1,s)$ respectively determine a unique solution that we discuss below.

2.3.2 Unemployment and employment. Solution to the system of equations

We obtain the solution to the system of Fokker-Planck equations (2.20) through a characteristic system of equations (see Polyanin et al., 2001, Chapter 5). The evolution of unemployment spell distribution is given as a solution to the first and third equations of the system (2.20) with the positive spell s (see Appendix 2.C):

$$p_\eta(t,0,s) = p_\eta(0,0,s-t)e^{-\int_{s-t}^{s}\mu(\phi(\xi))d\xi} + \mathbf{1}_{0\leq s\leq t}\lambda P_\eta(t-s,1)e^{-\int_0^s \mu(\phi(\xi))d\xi} \tag{2.21}$$

where $s \geq 0$, $p_\eta(0,0,s)$ is an initial *pdf* of an unemployed individual at time point $t=0$, $P_\eta(t,1)$ - probability of being employed at t.

The solution (2.21) consists of two summands: The first one describes the evolution of the probability density function of an unemployed agent who never finds a job. This *pdf* diminishes over time due to the positive probability of finding a new job which is linked to $\mu(\phi(s))$ (see the term $e^{-\int_{s-t}^{s}\mu(\phi(\xi))d\xi}$ depreciating over time). In the first summand we also observe that the unemployment spell s grows over time when no movements happen. It is described in detail by the initial condition $p_\eta(0,0,s-t)$ that moves over time t. As a representation of such dynamics of the first summand, we discuss the solution in Section 2.4.2, where Figure 2.7a shows diminishing initial distribution that shifts each time point towards the positive values of the unemployment duration s.

The second part of (2.21) represents a situation when a worker loses his job and becomes unemployed. Since, when it happens, the unemployment spell $s = 0$, the *pdf* is given by $\lambda P_\eta(t, 1)$. This term includes the probability $P_\eta(t, 1)$ that an agent is employed with any employment spell s and the spell-independent job-separation rate λ. After becoming unemployed the spell starts growing with time t. However there is still a positive probability of getting a new job with rate $\mu(\phi(s))$. Therefore the diminishing term $e^{-\int_0^s \mu(\phi(\xi))d\xi}$ appears in the second summand as well. The indicator function $\mathbf{1}_{0\leq s\leq t}$ keeps track of the fact that for any individual who switched from employed to unemployed the unemployment spell, s, cannot exceed the time interval t. Figure 2.7a qualitatively shows that only this part of the solution survives in the long run $\forall s \in [0, \infty)$.

The long-run solution consists of only the second part (see proof in Appendix 2.C):

$$p^*_\eta(\infty, 0, s) = \lambda P^*_\eta(\infty, 1)e^{-\int_0^s \mu(\phi(\xi))d\xi},$$

where $s \geq 0$ and $0 < s << t$.

To obtain the probability that an agent is unemployed, we integrate over the positive values of spell s, i.e.

$$P_\eta(t, 0) = \int_0^{+\infty} p_\eta(t, 0, s)ds \tag{2.22}$$

Employment is denoted by negative values of the spell s. Similar to the case of unemployment, we can obtain the evolution of the *pdf* in the employment state, $p_\eta(t, 1, s)$, and interpret it (see Appendix 2.C). However we are not interested in the employment state in such detail. We obtain a sufficient description of it using the property of complement events:

$$P_\eta(t, 1) = 1 - P_\eta(t, 0). \tag{2.23}$$

We refer to (2.21), (2.22), and (2.23) below. Thus we combine them into the system

$$\begin{cases} p_\eta(t, 0, s) & = p_\eta(0, 0, s-t)e^{-\int_{s-t}^s \mu(\phi(\xi))d\xi} + \mathbf{1}_{0\leq s\leq t}\lambda P_\eta(t-s, 1)e^{-\int_0^s \mu(\phi(\xi))d\xi} \\ P_\eta(t, 0) & = \int_0^{+\infty} p_\eta(t, 0, s)ds \\ P_\eta(t, 1) & = 1 - P_\eta(t, 0). \end{cases} \tag{2.24}$$

2.3.3 Aggregate dynamics

We build the aggregate dynamics in a simple manner. Starting from the case of one individual (see Section 2.2.1), we extrapolate our analysis to the example with two agents. Finally, we take all N individuals into detailed consideration and replace them by two representative agents who denote each search productivity type $\eta \in \{\eta_0, \eta_1\}$.

Constructing the aggregate dynamics, we inspect the probability of selecting an agent with a certain spell. More precisely, considering the economy between time points t and $t + \Delta t$, it is important to define the probability of choosing one individual out of N with an unemployment spell within a certain duration interval $[s, s+\Delta s]$. Obviously, this probability in general identifies the state of the economy overall: if the great part of the distribution locates, e.g., on the unemployment spell values $s > 0$, it describes a bad state of the economy with a considerable fraction of unemployed agents. In addition, this probability is defined by a corresponding probability density function. We name this *pdf* the *aggregate* probability density function and discuss it in detail below.

Two representative agents

An individual can be observed in either of two states, unemployment or employment, of a certain duration. To understand the overall dynamics, let us now study an economy that consists of two agents with the same search productivity type η as an example. Suppose agent 1 is employed with spell $s_1 < 0$ and agent 2 is unemployed with spell $s_2 > 0$ at time point t. Then the probability of picking one agent out of those two with spell s_1 or with spell s_2 is simply $\frac{1}{2}$ at t and 0 otherwise. Moreover, as we show above, the individual *pdfs* follow (2.24) with their own initial conditions.

The identical situation might be then described by a degenerate probability density function, s_1 with probability $\frac{1}{2}$ and s_2 with probability $\frac{1}{2}$, which we name as a *joint* probability density function of the simplistic economy with two agents. Importantly, the joint *pdf* is given as a sum of the individual *pdfs* with weights $\frac{1}{2}$. Thus, the evolution of the joint *pdf* is also defined by (2.24), similarly to the individual *pdfs*. However, the initial *pdf* must be combined with weights $\frac{1}{2}$ from initial individual *pdfs*. Note that such a replacement is valid only because all agents of the same type being in the same state with the identical duration s would behave equally (see Section 2.2.1).

We simplify the aggregation in the case of N heterogeneous agents with different job-search productivity type $\eta \in \{\eta_0, \eta_1\}$ as a next step. The straight-forward, but fairly complicated approach would be to compute the evolutions of all N individual *pdfs* and aggregate them afterwards. However, we obtain the same results by considering an economy with only two representative agents, the first – of type η_0 and the second – of type η_1, whose initial joint *pdfs*, $p_{\eta_1}(0, x, 1)$ and $p_{\eta_0}(0, x, 1)$, are combinations of initial distributions of agents with the corresponding type.[10]

The joint *pdfs* simplify our analysis for three reasons. First, the dynamics of the joint *pdfs* over time t fully describes the economy as it would be done by the combination of individual probability density functions. Second, the joint *pdfs* follow the same rules of motion already studied in Section 2.3.1 (see (2.24)), i.e. the qualitative analysis in Section 2.3.1 and 2.3.2 is valid.[11] Finally, within any future time interval $[t, t + \Delta t]$, the probability of picking one individual with an unemployment spell within a certain interval $[s, s + \Delta s]$ is fully determined by a combination of *pdfs* of two representative agents, $p_{\eta_1}(t, x, s)$ and $p_{\eta_0}(t, x, s)$, in the following manner:

$$p(t, x, s) = \pi p_{\eta_1}(t, x, s) + (1 - \pi)p_{\eta_0}(t, x, s). \tag{2.25}$$

We call this mixture the *aggregate* probability density function.

Evaluation of labor market reforms

To evaluate a potential labor market reform, it is sufficient to study the aggregate *pdf* $p(t, x, s)$ as defined by (2.25). Bearing this in mind, we are able to address many more questions than just finding the unemployment rate, which is:

$$u(t) = \int_0^{+\infty} p(t, 0, s)ds \tag{2.26}$$

For instance, the predicted number of employed agents is

$$L(t) = NP(t, 1),$$

whereas the number of unemployed individuals is described by:

$$U(t) = N - L(t) = N[1 - P(t, 1)] = NP(t, 0).$$

Interestingly, we now can obtain, e.g., the average unemployment duration at each time point t:

$$E_0[s] = \frac{\int_0^{+\infty} sp(t, 0, s)ds}{P(t, 0)}$$

[10] A mixture of initial distributions becomes a smooth function when the number of agents is large (see e.g. Figure 2.7).

[11] To obtain the initial values $p_\eta(0, 0, s)$ and $p_\eta(0, 1, s)$, the observed initial distribution might be required to be decomposed on two distributions, for each search productivity type.

For further analysis however it is important to define the conditional *pdf*:

$$p(t, s|0) = \frac{p(t, 0, s)}{\int_0^{+\infty} p(t, 0, s)ds} = \frac{p(t, 0, s)}{P(t, 0)}.$$

The total government expenses per unemployed individual paid at each time point are then given by

$$\begin{aligned} C(t) &= \frac{\int_0^{+\infty} b(s)p(t, 0, s)ds}{P(t, 0)} \\ &= \frac{b_{ui} \int_0^{\bar{s}} p(t, 0, s)ds + b_{ua} \int_{\bar{s}}^{+\infty} p(t, 0, s)ds}{P(t, 0)} \\ &= b_{ui} \int_0^{\bar{s}} p(t, s|0)ds + b_{ua} \int_{\bar{s}}^{+\infty} p(t, s|0)ds, \end{aligned} \tag{2.27}$$

where the first summand represents the amount paid to one short-term unemployed agent, i.e. unemployment insurance per unemployed individual. The long-term unemployed agents receive unemployment assistance given by the second summand. The integral $\int_0^{\bar{s}} p(t, s|0)ds$ gives the fraction of the short-term unemployed, whereas $\int_{\bar{s}}^{+\infty} p(t, s|0)ds$ represents the fraction of the long-term unemployed.

2.3.4 Definition of equilibrium

Each steady state is characterized by the set of parameters of the model. If one or more parameters is changed, the economy transits from the old equilibrium to a new one. Our setup explains in this case how fast this transition is completed and what is a new steady state with a new set of parameters.

Equilibrium in this model follows the simple procedure:

1. For each search productivity type $\eta = \{\eta_1, \eta_0\}$, we do the next steps:

   (a) Given the set of parameters in the model $\{\rho, \sigma, \lambda, \alpha, w, b_{ui}, b_{ua}, \bar{s}\}$, the system (2.15) is solved and value functions $V(b_{ui}, s)$, $V(b_{ua})$, and $V(w)$ are obtained.

   (b) The value function computed in the previous step defines the optimal effort via (2.11).

   (c) The optimal effort defines the exit rate $\mu(s)$ according to (2.16).

   (d) The exit rate in turn enters the evolution of the probability density functions (2.24). Assuming an initial distribution and solving these equations, we obtain the evolution of the probability density function in the unemployment state.

2. We next combine pdfs for each search productivity type with weights π and $1-\pi$ to get the evolution of the aggregate probability density function $p(t,x,s)$ (see (2.25)).

3. From the dynamics of the aggregate probability density function we obtain all necessary values to evaluate the potential reform, i.e. the evolution of the unemployment rate and of government expenses, given by (2.26) and (2.27) respectively.

To solve the model the total set of parameters of the model therefore includes

$$\left\{\rho, \sigma, \lambda, \alpha, w, b_{ui}, b_{ua}, \bar{s}, \pi, \eta_1, \eta_0\right\}. \tag{2.28}$$

The sufficient set of equations is $\left\{(2.11), (2.15), (2.16), (2.24), (2.25), (2.26), (2.27)\right\}$.

2.4 Calibration

We further perform a predictive analysis of changes during a labor market reform, which, e.g., is to reduce the unemployment rate. In doing so, the government might change any subset of parameter set (2.28). As an example of such a reform, we study how the reduction of the length of unemployment insurance payments from $\bar{s}_{old} = 12$ to $\bar{s}_{new} = 6$ months influences the unemployment rate and the government expenses. For our analysis we choose Germany as a typical OECD country with a two-tier unemployment benefits system.

In particular we calibrate the model using parameters obtained by Launov and Wälde (2013) (see Table 2.1). Following the sequence provided in Section 2.3.4, we first estimate the value functions, optimal effort, exit rates. Second, we compute the evolution of the probability density function. The last step is to obtain the dynamics of the unemployment rate and alteration of government expenses.

ρ	σ	λ	α	w	b_{ui}	b_{ua}	π	η_1	η_0
0.024	0.7639	0.01	0.406	1166.26	727.46	350	0.91	0.0911	0.0167

Table 2.1: Parameters of the model from Launov and Wälde (2013).

We perform our analysis twice: We complete the first round with $\bar{s}_{old}$ with any initial *pdf* to obtain the long-run *pdf*. Second, we set this long-run *pdf* to be an initial *pdf* for the case with $\bar{s}_{new}$ and run the same procedure again. After doing so, we obtain the transitional dynamics of the system from the old steady state characterized by $\bar{s}_{old}$ to the new equilibrium with $\bar{s}_{new}$.

2.4.1 Value functions, optimal effort, job matching rates

The optimal behavior of an unemployed individuals suggests that the search effort $\phi(s)$ must increase when the unemployment duration s rises towards $\bar{s}$ (see Figure 2.3).[12] It happens because the individual observes the end of unemployment insurance payments. The unemployed agent therefore wants to avoid the situation when low unemployment assistance are received and tries more intensively to find a job and earn wage w instead. If it is not successful, the agent gets unemployment assistance and the effort becomes constant because from the UA state only a new job can be found, i.e. no further uncertainty.

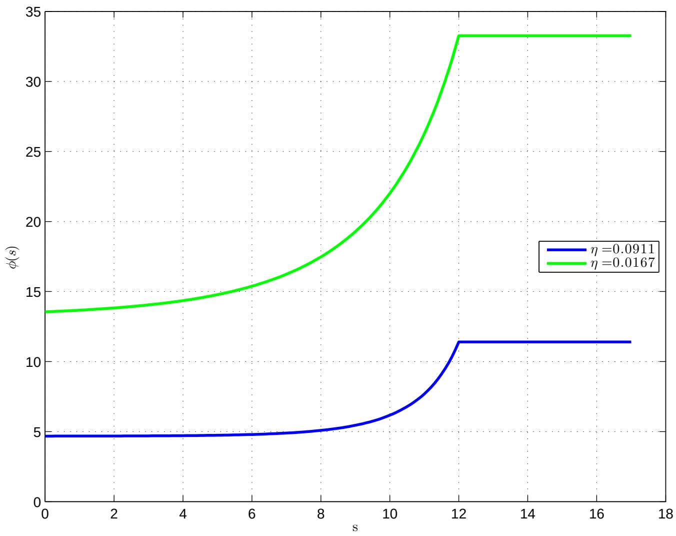


Figure 2.3: Optimal effort $\phi(s)$.

The low search productivity type unemployed obviously must look for a job more intensively. Figure 2.3 illustrates this fact: The effort of the η_0-type agent is higher than of the η_1-type agent for each value of unemployment duration s, i.e. $\phi_{\eta_1}(s) < \phi_{\eta_1}(s)\ \forall s > 0$.

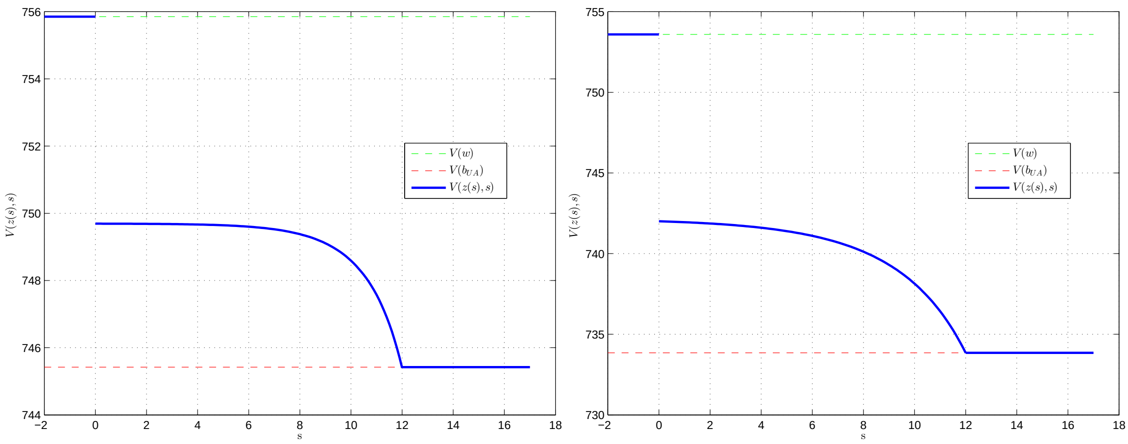
In order to find the alteration to the optimal effort and value function, we solve the system (2.15) numerically. The key ingredient of this system is the constant value function of an unemployed agent, V_{ua}, which is given in Table 2.2 for each agent's type η and duration of UI payments, $\bar{s}$.

	$\eta_1 = 0.0911$	$\eta_0 = 0.0167$
$\bar{s}_{old} = 12$	$V_{ua} = 745.42$	$V_{ua} = 733.87$
$\bar{s}_{new} = 6$	$V_{ua} = 745.39$	$V_{ua} = 733.50$

Table 2.2: The value function of the long-term job seeker, V_{ua}, in the old and new steady states for two search productivity types.

[12]We draw all figures in this section with $\bar{s}_{old} = 12$. The same analysis with value $\bar{s}_{new} = 6$ would produce similar results.

The value function of a short-term unemployed individual, $V(b_{ui}, s)$, obviously lies between constant levels of the value functions of being employed, $V(w)$, and of being long-term unemployed, $V(b_{ua})$. Moreover, since the job destruction happens unexpectedly, the agent is not able to gradually adjust his preferences right after this time point. This fact explains the drop in the value function from $V(w)$ to $V(b_{ui}, 0)$ at $s = 0$ (see Figures 2.4a, 2.4b). We further observe that the individual with lower search productivity type η_0 loses more in terms of the value function when the job destruction occurs. It happens because the new conditions demand higher search effort for this type.



(a) $\eta_1 = 0.0911$ (b) $\eta_0 = 0.0167$

Figure 2.4: Evolution of the value function $V(z(s), s)$ with $\bar{s}_{old} = 12$.

With rising effort the value function $V(b_{ui}, s)$ gradually falls until $V(b_{ua})$ when spell s increases towards $\bar{s}$ (see Figures 2.4a, 2.4b). At the unemployment sell $\bar{s}$ the value function of a short-term unemployed reaches its boundary condition, $V(b_{ui}, \bar{s}) = V(b_{ua})$, and remains constant afterwards together with invariable effort.

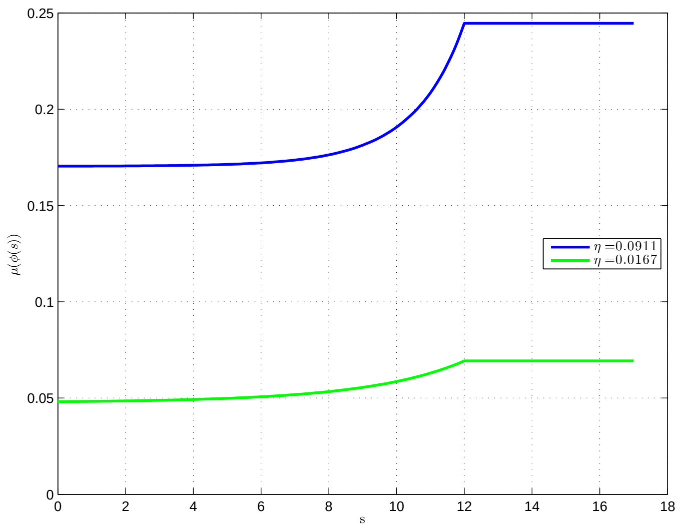


Figure 2.5: The job matching rate $\mu(\phi(s))$.

After estimating the optimal effort and the value functions, we calibrate the job matching rate $\mu(\phi(s))$ with scale parameters $\eta_1 = 0.0911$ and $\eta_0 = 0.0167$ according to (2.16) (see Figure 2.5). Importantly, according to (2.16) the high search productivity type is characterized by high ability to convert the effort into the job matching rate. We model this fact choosing appropriate scale parameters $\eta_1 > \eta_0$. Figure 2.5 illustrates that the exit rate of the high-type unemployed always lies above the exit rate of a low-type agent, $\mu(\phi_{\eta_1}(s)) > \mu(\phi_{\eta_0}(s))\ \forall s > 0$. The job-matching rates moreover rise with the unemployment spell s, when s increases from 0 to $\bar{s}$. It happens due to an increase in the effort $\phi_\eta(s)$. In addition, for any $s > \bar{s}$ the exit rates becomes flat together with the effort (compare Figure 2.3 and 2.5).

2.4.2 Unemployment rate and probability density functions

We obtain the numerical representation of the evolution the *pdfs* for each search productivity type η_1 and η_0. This evolution relies on the system of equations (2.24). In order to aggregate the results of two types, we use (2.25) to derive the dynamics of aggregate *pdf*, $p(t, x, s)$. These results are already sufficient to compute the evolution of the unemployment rate.

Unemployment rate

The labor market reform (the drop in the UI payments duration from $\bar{s}_{old}$ to $\bar{s}_{new}$) leads to a decrease in the unemployment rate from 6.1% to 5.6% within approximately 3 years (30-40 months) (see Figure 2.6). We obtain this result using (2.26).

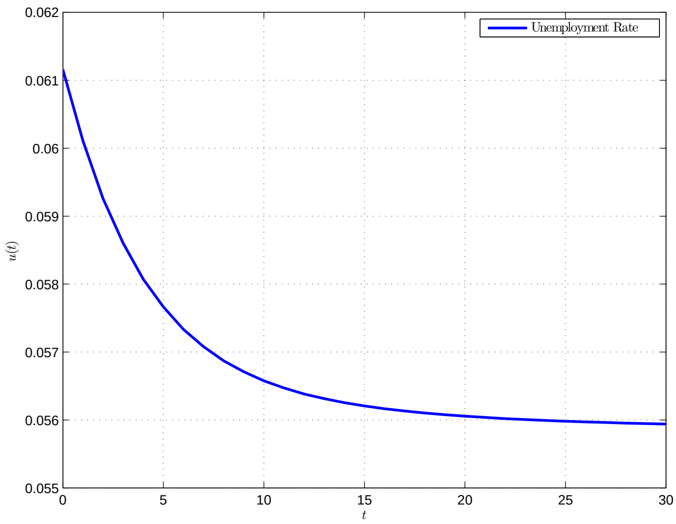


Figure 2.6: Evolution of the unemployment rate $u(t)$ over time after the labor market reform.

A fraction of job seekers find jobs due to the higher effort for both search productivity types imposed by new labor market conditions. The unemployment rate, observed initially at 6.1%, therefore falls sharply within the first year after the reform and then decreases gradually later on.

Evolution of probability density functions

We are mainly interested in the part of the aggregate unconditional *pdf* that reflects the unemployment state, $p(t, 0, s)$. We further convert this part of the aggregate *pdf* into the conditional *pdf* to provide an intuitive explanation of the government expenses dynamics. Given the textbook definition, we write the *pdf* of an agent *conditional on* being unemployed as follows:

$$p(t, s|0) = \frac{p(t, 0, s)}{P(t, 0)} = \frac{p(t, 0, s)}{\int_0^\infty p(t, 0, s)ds} \tag{2.29}$$

with

$$\int_0^\infty p(t, s|0)ds = 1.$$

Figure 2.7 presents the entire behavior of the conditional *pdf* from (2.29) where part 2.7a describes the overall picture, whereas the initial and terminal distributions are given in part 2.7b.

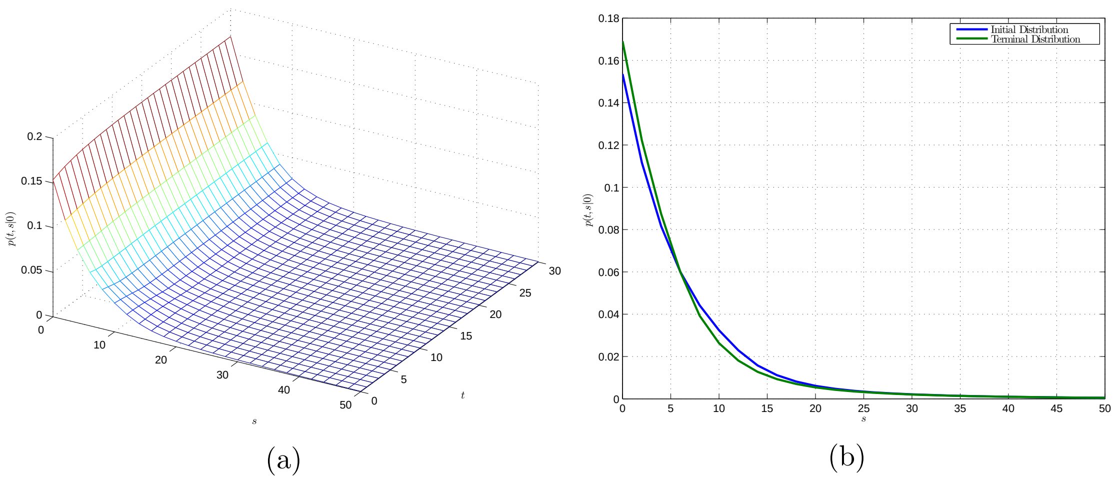
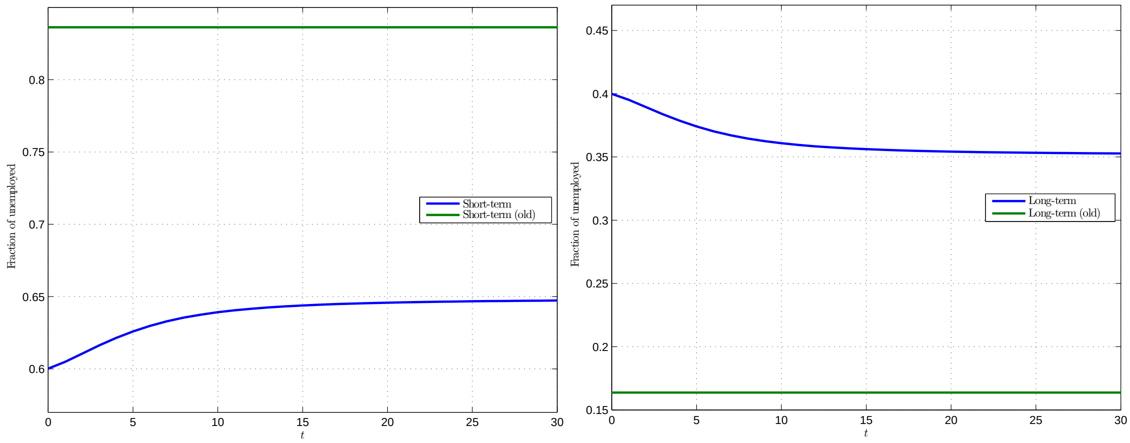


Figure 2.7: Evolution of conditional probability density function $p(t, s|0)$ (a) and its initial and limiting cases (b).

To sum up, there are two sources of the redistribution of the unemployment duration which we observe in Figure 2.7b. First, unemployed agents become employed due to the higher effort for both search productivity types. Second, a number of long-term unemployed individuals become short-term unemployed. After the labor market reform is applied, the fraction of short-term unemployed, defined in (2.27), obviously drops. It happens because all short-term unemployed individuals with the unemployment spell between $\bar{s}_{new}$ and $\bar{s}_{old}$ suddenly become the long-term unemployed. After this drop the fraction of short-term unemployed increases due to the redistribution of job seekers from the long-term unemployment via the employment state.



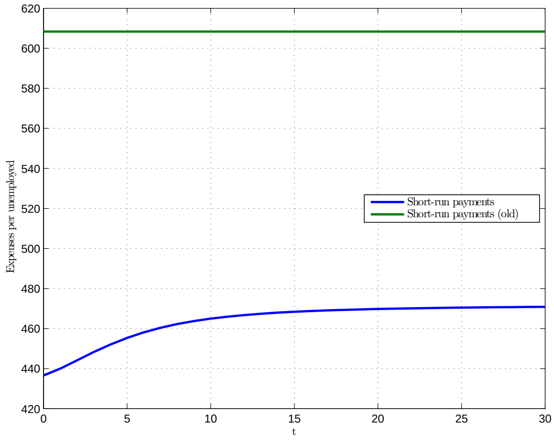
(a) (b)

Figure 2.8: Evolution of the fraction of the short-term (a) and the long-term (b) unemployed agents.

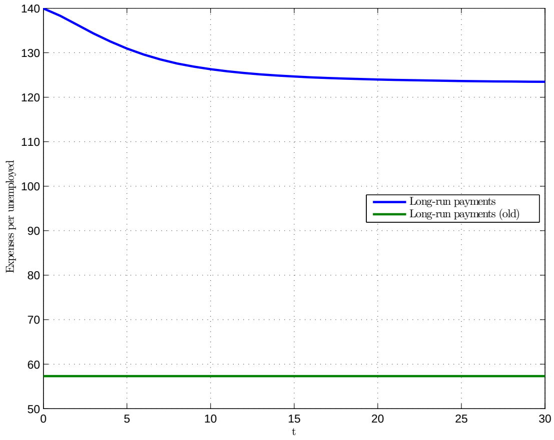
Figure 2.8 shows that the fraction of short-term unemployed initially falls from 87% to 60% but then gradually increases to the 65% level. The exact opposite happens with the fraction of long-term job seekers. The latter fraction sharply rises from 13% to 40% and then slowly decreases back down to 35% level.

2.4.3 Government expenses

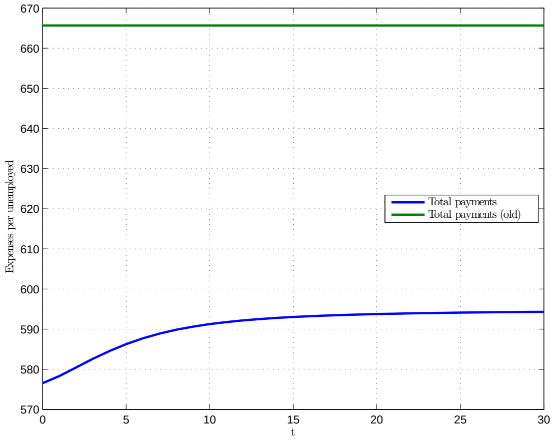
The model predicts future government spendings to support the unemployment system under new circumstances. Together with the evolution of the fraction of the short- and long-term unemployed workers (see Figures 2.8), we compute the government expenditure on both short- and long-term unemployed agents using (2.27). The model therefore predicts decrease in the total government expenses per unemployed individual by 10.7%. Figure 2.9 reveals the same trends in government expenses as in the fraction of unemployed agents (see Figure 2.8).



(a)



(b)



(c)

Figure 2.9: Short-term (a), long-term (b) and total government expenses per unemployed individual (c).

2.5 Conclusion

In this paper we analyze various effects of unemployment benefits system changes. In doing so, we estimate the duration of the reform, the dynamics of the unemployment rate, and alteration of total payments to job seekers in countries with the two-tier unemployment benefits system. We also show that job-search intensity links the unemployment payments and the unemployment duration.

To perform such analysis, we develop the dynamic model with heterogeneous employed and unemployed workers. Exploiting *stochastic calculus* the model describes the transitional dynamics between two steady states of the economy that occurs after the reduction of the length of unemployment insurance compensation duration. Taking into account these dynamics, the model predicts the decrease in the value function and the increase in the optimal effort of an unemployed worker. This alteration raises, in turn, the job-matching rate. Since the probability of escaping from the state of unemployment becomes higher, the unemployment rate falls.

The dynamics of the probability density function of the unemployment duration reveals the change in the fraction of the short-term and the long-term job seekers. The fraction of the short-term unemployed individuals initially drops because a considerable number of them automatically become long-term unemployed after the reform. This fraction then rises due to the redistribution of the long-run job seekers until its new steady state. The fraction of the long-term unemployed individuals reveals the opposite dynamics. The total government expenses on both groups changes in the same way as the fraction of the short-term job seekers.

For the quantitative analysis we take the reduction of the unemployment insurance benefits payments from 12 months to 6 months as an example. The model predicts that this reduction causes a decrease in the unemployment rate from 6.1% to 5.6% within 3 years. The fraction of the short-term unemployed workers falls from .84 to .65, whereas the fraction of the long-term ones rises from .16 to .35. The reform moreover leads to a decline in the government expenses per unemployed individual by 10.7%.

The approach presented in this study seems to be a highly promising benchmark for future empirical work. Analyzing labor market reforms outside Germany would, however, require another careful calibration of the model parameters. In addition, the model works well with multiple-tier unemployment benefits system. It can also perform similar analysis of the employment duration with employment spell dependent wage.

Appendix

2.A Bellman equation

We formally write the differential of the value function following Wälde (2012) for two states $x \in \{0, 1\}$.

Employment

An employed worker is in state $x = 1$ and receives wage $z(s) = w$ and therefore the differential of value function reads:

$$dV(w,s) = -\frac{\partial V(w,s)}{\partial s}dt + \left[V\Big(w,s\Big) - V(w,s)\right]dq_\mu \\ + \left[V\Big(b_{ui},0\Big) - V(w,s)\right]dq_\lambda \quad (2.30)$$

In the employment state the value function does not depend on the employment spell. Thus, $\frac{\partial V(w,s)}{\partial s} = 0$ and

$$\frac{E_0 dV(w,s)}{dt} = \lambda\left[V\Big(b_{ui},0\Big) - V(w,s)\right]$$

The Bellman equation is then given by:

$$\rho V(w,s) = u(w,0) + \lambda\left[V\Big(b_{ui},0\Big) - V(w,s)\right]$$

Note that an employed spares no effort in this state. The maximization problem therefore vanishes.

Unemployment

An unemployed finds himself in state $x = 0$ with positive spell $s > 0$ and benefits $b(s) \in \{b_{ui}, b_{ua}\}$. Thus,

$$dV(z(s),s) = \frac{\partial V(z(s),s)}{\partial s}dt \\ + \{V(b(s-s) + w - b(s-s), s-s) - V(b(s),s)\}\, dq_\mu \\ + \{V(b(s),s) - V(b(s),s)\}\, dq_\lambda \quad (2.31)$$

It leads to

$$dV(z(s),s) = \frac{\partial V(z(s),s)}{\partial s} dt + \{V(w,0) - V(b(s),s)\}\, dq_\mu \tag{2.32}$$

The positive sign of the first term reflects the positive sign of unemployment spell s.

The job-matching rate, $\mu(\phi(s))$, depends on time component t since $s = t - t_0$, where t_0 is a time point of the previous jump. It makes q_μ to be an *inhomogeneous* Poisson process. Forming expectations of dq_μ, we exploit the definition of this type of processes (see Ross, 1996, Chapter 2.4), write the number of jumps at t as $N(t)$ and write:

$$P\{dq_\mu = 1\} = P\{N(t+dt) - N(t) = 1\} = \mu(\phi(s))dt + o(dt)$$

$$P\{dq_\mu > 1\} = P\{N(t+dt) - N(t) > 1\} = o(dt)$$

$$\begin{aligned} P\{dq_\mu = 0\} &= P\{N(t+dt) - N(t) = 0\} = 1 - \mu(\phi(s))dt - 2o(dt) \\ &= 1 - \mu(\phi(s))dt - o(dt) \end{aligned}$$

Thus,

$$\begin{aligned} E_0[dq_\mu] &= 0 \cdot [1 - \mu(\phi(s))dt - o(dt)] + 1 \cdot [\mu(\phi(s))dt + o(dt)] + o(dt)\sum_{n=2}^{\infty} n \\ &= \mu(\phi(s))dt + o(dt) \end{aligned}$$

If we neglect $o(dt)$, we obtain a similar to homogeneous process formula:

$$E_0[dq_\mu] = \mu(\phi(s))dt$$

Thus, taking expectations from both sides of (2.32), we write:

$$\frac{E_0 dV(b(s),s)}{dt} = \frac{\partial V(b(s),s)}{\partial s} + \mu(\phi(s))\Big[V\Big(w,0\Big) - V(b(s),s)\Big]$$

The Bellman equation is then given by:

$$\begin{aligned} \rho V(b(s),s) = \max_{\phi(s)} \Big\{ u(b(s),\phi(s)) + \frac{\partial V(b(s),s)}{\partial s} \\ + \mu(\phi(s))\Big[V\Big(w,0\Big) - V(b(s),s)\Big]\Big\} \end{aligned} \tag{2.33}$$

Given (2.5) and (2.16), we can solve the latter maximization problem also given by

(2.10) that defines the optimal effort through the first-order condition

$$-u'_{\phi}(b(s),\phi(s)) = \mu'_{\phi}(\phi(s))\Big[V\Big(w,0\Big) - V(b(s),s)\Big]$$

Unemployed individuals choose search effort optimally such that after any increase in effort the utility loss is equal to the expected utility gain from it. Given (2.16), we write

$$-1 + \alpha\eta[\phi(s)]^{\alpha-1}\Big[V\Big(w,0\Big) - V(b(s),s)\Big] = 0$$

from where we derive the optimal effort:

$$\phi(s) = \Big(\alpha\eta\Big[V(w) - V(b(s),s)\Big]\Big)^{\frac{1}{1-\alpha}}$$

The maximized Bellman equation is derived when we plug the optimal effort from above or (2.11) into the Bellman equation (2.10):

$$\rho V(b(s),s) = \frac{b(s)^{1-\sigma}-1}{1-\sigma} - \Big\{\alpha\eta\Big[V(w,0) - V(b(s),s)\Big]\Big\}^{\frac{1}{1-\alpha}} + \frac{\partial V(b(s),s)}{\partial s}$$
$$+ \Big\{\alpha\eta\Big[V(w,0) - V(b(s),s)\Big]\Big\}^{\frac{\alpha}{1-\alpha}}\Big[V\Big(w,0\Big) - V(b(s),s)\Big] \quad (2.34)$$

$$\rho V(b(s),s) = \frac{b(s)^{1-\sigma}-1}{1-\sigma} - \Big\{\alpha\eta\Big[V(w,0) - V(b(s),s)\Big]\Big\}^{\frac{1}{1-\alpha}} + \frac{\partial V(b(s),s)}{\partial s}$$
$$+ [\alpha\eta]^{\frac{\alpha}{1-\alpha}}\Big[V(w,0) - V(b(s),s)\Big]^{\frac{1}{1-\alpha}} \quad (2.35)$$

$$\rho V(b(s),s) = \frac{b(s)^{1-\sigma}-1}{1-\sigma} + \frac{\partial V(b(s),s)}{\partial s}$$
$$+ \Big[(\alpha\eta)^{\frac{\alpha}{1-\alpha}} - (\alpha\eta)^{\frac{1}{1-\alpha}}\Big]\Big[V(w,0) - V(b(s),s)\Big]^{\frac{1}{1-\alpha}} \quad (2.36)$$

2.B Derivation of the Fokker-Planck equations. 5-steps procedure

In order to derive the Fokker-Planck equations we follow the 5-step procedure developed by Bayer and Wälde (2013). We also simplify notation denoting the job-matching rate by $\mu(s) \equiv \mu(\phi(s))$. In addition, we do not write the subscript η of the probability density function $p_\eta(t,x,s)$ and use this place to denote the partial derivatives of the *pdf*.

Step I. The expected change of an auxiliary function

We consider an auxiliary function $f(x, s)$ of two random variables, state x and spell s, with two-dimensional bounded support $\mathcal{S}$. We then write its differential:

$$\begin{aligned} df(x,s) =& f_s(x,s)[1-2x]dt + \{f(x+1-x, s-(1-x)s) - f(x,s)\}\, dq_\mu \\ &+ \{f(x-x, s-xs) - f(x,s)\}\, dq_\lambda \end{aligned}$$

$$\begin{aligned} E[df(x,s)] =& f_s(x,s)[1-2x]dt + \{f(x+1-x, s-(1-x)s) - f(x,s)\}\, E[dq_\mu] \\ &+ \{f(x-x, s-xs) - f(x,s)\}\, E[dq_\lambda] \end{aligned}$$

Note that by the definition of inhomogeneous Poisson processes $E[dq_\mu] = \mu(s)dt$ (see Ross (1996), Chapter 2.4). Process q_λ is homogeneous and therefore $E[dq_\lambda] = \lambda dt$. Thus

$$\begin{aligned} \frac{E[df(x,s)]}{dt} =& f_s(x,s)[1-2x] + \{f(1, s-(1-x)s) - f(x,s)\}\, \mu(s) \\ &+ \{f(0, s-xs) - f(x,s)\}\, \lambda, \end{aligned}$$

where the subscript s denotes the partial derivative with respect to s.

Step II. Dynkin's formula and its right-hand side

Given the differential version of Dynkin's formula,

$$\frac{\partial E[f(x,s)]}{\partial t} = E\left[\frac{E[df(x,s)]}{dt}\right], \tag{2.37}$$

we write its right-hand side as follows:

$$\begin{aligned} E\left[\frac{E[df(x,s)]}{dt}\right] =& P(t,0)\int_0^\infty \Bigg\{\Bigg[f_s(0,s) + \{f(1,0) - f(0,s)\}\,\mu(s) \\ &+ \{f(0,s) - f(0,s)\}\,\lambda\Bigg] p(t,s|0)\Bigg\} ds \\ &+ P(t,1)\int_{-\infty}^0 \Bigg\{\Bigg[-f_s(1,s) + \{f(1,s) - f(1,s)\}\,\mu(s) \\ &+ \{f(0,0) - f(1,s)\}\,\lambda\Bigg] p(t,s|1)\Bigg\} ds, \end{aligned}$$

where $P(t, x)$ is the probability of the state $x \in \{0, 1\}$, $p(t, s|x)$ is the conditional probability density function to observe spell s at time point t given the state $x\{0, 1\}$. Remember that state $x = 0$ stands for unemployment, $x = 1$ – for employment.

Note also that one of the terms under both integrals disappears, to the extend that we simplify the latter equation and write:

$$E\left[\frac{E[df(x,s)]}{dt}\right] = P(t,0)\int_0^{\infty}[f_s(0,s) + \{f(1,0) - f(0,s)\}\,\mu(s)]p(t,s|0)ds \\ + P(t,1)\int_{-\infty}^{0}[-f_s(1,s) + \{f(0,0) - f(1,s)\}\,\lambda]p(t,s|1)ds.$$

In terms of unconditional probabilities $p(t,x,s) = P(t,x)\cdot p(t,s|x)$ we rewrite the expression above as follows:

$$E\left[\frac{E[df(x,s)]}{dt}\right] = \int_0^{\infty}[f_s(0,s) + \{f(1,0) - f(0,s)\}\,\mu(s)]p(t,0,s)ds \\ + \int_{-\infty}^{0}[-f_s(1,s) + \{f(0,0) - f(1,s)\}\,\lambda]p(t,1,s)ds$$

Step III. Integration by parts

Exploiting integration by parts, we simplify spell derivatives of the auxiliary function $f_s(x,s)$:

$$\int_0^{\infty} f_s(0,s)p(t,0,s)ds = f(0,s)p(t,0,s)\Big|_0^{\infty} - \int_0^{\infty} f(0,s)p_s(t,0,s)ds \\ = -f(0,0)p(t,0,0) - \int_0^{\infty} f(0,s)p_s(t,0,s)ds$$

and

$$-\int_{-\infty}^{0} f_s(1,s)p(t,1,s)ds = -f(1,s)p(t,1,s)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} f(1,s)p_s(t,1,s)ds \\ = -f(1,0)p(t,1,0) + \int_{-\infty}^{0} f(1,s)p_s(t,1,s)ds$$

where $f(0,\infty)p(t,0,\infty) = 0$ and $f(1,-\infty)p(t,1,-\infty) = 0$ because $f(x,s)$ has bounded support $\mathcal{S}$ outside of which $f(x,s) = 0$, i.e. $f(0,\infty) = 0$ and $f(1,$-$infty) = 0$. Moreover it is reasonable to assume that there are no agents in the economy with the infinitely high spell $s \to \pm\infty$, i.e. $p(t,0,\pm\infty) = 0$. Therefore the right-hand side of Dynkin's formula (2.37) reads:

$$\begin{aligned} E\left[\frac{E[df(x,s)]}{dt}\right] =& -f(0,0)p(t,0,0) \\ &+ \int_0^\infty \Big\{-f(0,s)p_s(t,0,s) + \{f(1,0)-f(0,s)\}\,\mu(s)p(t,0,s)\Big\}ds \\ &- f(1,0)p(t,1,0) \\ &+ \int_{-\infty}^0 \Big\{f(1,s)p_s(t,1,s) + \{f(0,0)-f(1,s)\}\,\lambda p(t,1,s)\Big\}ds \end{aligned} \tag{2.38}$$

Step IV. Derivative of the expected value

On the other hand,

$$E[f(x,s)] = \int_0^\infty p(t,0,s)f(0,s)ds + \int_{-\infty}^0 p(t,1,s)f(1,s)ds$$

Thus, the left-hand side of Dynkin's formula is given by:

$$\frac{\partial E[f(x,s)]}{\partial t} = \int_0^\infty p_t(t,0,s)f(0,s)ds + \int_{-\infty}^0 p_t(t,1,s)f(1,s)ds \tag{2.39}$$

Step V. Collecting all terms

Given the differential version of Dynkin's formula (2.37), we further combine all terms of (2.38) and (2.39) with:

$$\begin{aligned} f(0,0) &: -p(t,0,0) + \int_{-\infty}^0 \lambda p(t,1,s)ds = 0 \\ f(1,0) &: -p(t,1,0) + \int_0^\infty \mu(s)p(t,0,s)ds = 0 \\ f(0,s) &: -p_s(t,0,s) - \mu(s)p(t,0,s) = p_t(t,0,s) \\ f(1,s) &: \quad p_s(t,1,s) - \lambda p(t,1,s) = p_t(t,1,s) \end{aligned} \tag{2.40}$$

We note that the first equation includes the integral of the unconditional pdf $p(t,1,s) = P(t,1)p(t,s|1)$ and, given constant job separation rate λ, can be simplified using $\int_{-\infty}^0 p(t,s|1) = 1$. Thus,

$$\int_{-\infty}^0 \lambda p(t,1,s)ds = \lambda \int_{-\infty}^0 P(t,1)p(t,s|1)ds = \lambda P(t,1)\int_{-\infty}^0 p(t,s|1)ds = \lambda P(t,1).$$

We next group all four equations from (2.40) into the system:

$$\begin{cases} p(t,0,0) = \lambda P(t,1) \\ p(t,1,0) = \int_0^\infty \mu(s)p(t,0,s)ds \\ p_s(t,0,s) + p_t(t,0,s) = -\mu(s)p(t,0,s) \\ p_s(t,1,s) - p_t(t,1,s) = \lambda p(t,1,s) \end{cases}$$

2.C Solution to the system of differential equations

In this section we derive the solution to the system of the first and third FPEs from (2.20). These two equations describe the evolution of the probability density function of being unemployed $p(t,0,s)$. Note that the probability density function in the employment case $p(t,1,s)$ from the second and fourth equations of the system (2.20) can be derived in the similar way and, thus, is not presented here. We also simplify notation denoting the job-matching rate by $\mu(s) \equiv \mu(\phi(s))$. In addition, we do not write the subscript η of the probability density function $p_\eta(t,x,s)$ and use this place to denote the partial derivatives of the *pdf*.

Solution to the unemployment part

Unemployment is denoted by positive spell s. We therefore look at equation:

$$p_s(t,0,s) + p_t(t,0,s) = -\mu(s)p(t,0,s) \tag{2.41}$$

with $s \geq 0$ and two boundary conditions: the initial condition for $t = 0$

$$p(0,0,s) \tag{2.42}$$

and the zero-spell condition which comes from the first equation of (2.20)

$$p(t,0,0) = \lambda P(t,1). \tag{2.43}$$

We next look at equations (2.41) and (2.42). The method of solution is based on solving the characteristic system of equations (see Polyanin et al. (2001), Chapter 5):

$$\frac{ds}{1} = \frac{dt}{1} = -\frac{dp(t,0,s)}{\mu(s)p(t,0,s)}$$

and it has two independent integrals:

$$s - t = C_1$$

and

$$p(t, 0, s)e^{\int_0^s \mu(\xi)d\xi} = C_2,$$

such that the general solution of equation (2.41) is given by $C_2 = \Phi(C_1)$, where $\Phi(\cdot)$ is an arbitrary function.[13] We thus obtain the following results:

$$p(t, 0, s) = \Phi(s - t)e^{-\int_0^s \mu(\xi)d\xi} \tag{2.44}$$

Step I. Initial condition

When $t = 0$, given the initial condition (2.42), from (2.44) we find:

$$p(0, 0, s) = \Phi(s)e^{-\int_0^s \mu(\xi)d\xi}$$

from where

$$\Phi(s) = p(0, 0, s)e^{\int_0^s \mu(\xi)d\xi}$$

or, changing the variable $s \to s - t$,

$$\Phi(s - t) = p(0, 0, s - t)e^{\int_0^{s-t} \mu(\xi)d\xi}.$$

Plugging the latter expression in the solution (2.44) gives:

$$p(t, 0, s) = p(0, 0, s - t)e^{-\int_0^s \mu(\xi)d\xi}e^{\int_0^{s-t} \mu(\xi)d\xi} = p(0, 0, s - t)e^{-\int_{s-t}^s \mu(\xi)d\xi} \tag{2.45}$$

which is obviously valid only on $s \geq t$ because the initial condition $p(0, 0, s - t) = 0$ $\forall s < t$. That is why we do not need to add an indicator function $\mathbf{1}_{s \geq t}$ into the solution.

Proposition 1 *The solution* (2.45) *fades away in the long run.*

Proof. Due to $s \geq t$ $s \to \infty$ in the long run. It in turn makes $\lim_{t\to\infty} \mu(s)$ to be constant because of constant long-run effort (see (2.12) and Fig.2.3). From that it follows that the integral $\lim_{t\to\infty} \int_{s-t}^s \mu(\xi)d\xi = \mu t$ and $\lim_{t\to\infty} e^{-\mu t} \to 0$. Given that $p(0, 0, s - t)$ is finite $\forall s, t$, the solution (2.45) equals to zero in the long run. ■

[13] In general the solution is given by $\Omega(C_1, C_2) = 0$ or $C_2 = \Phi(C_1)$ since every constant can be expressed as a function of another constant.

Step II. Boundary condition

The second part describes solution of (2.41) on $0 \leq s \leq t$. We therefore look at equation (2.44) with the independent from spell s boundary condition (2.43). For $s = 0$ (2.44) reads:

$$p(t, 0, 0) = \lambda P(t, 1) = \Phi(-t)$$

from where

$$\Phi(t) = \lambda P(-t, 1)$$

or, changing the variable $t \to s - t$,

$$\Phi(s - t) = \lambda P(t - s, 1)$$

Then (2.44) reads

$$p(t, 0, s) = \mathbf{1}_{0 \leq s \leq t} \lambda P(t - s, 1) e^{-\int_0^s \mu(\xi) d\xi}.$$

In this solution we have to explicitly add the indicator function, $\mathbf{1}_{0 \leq s \leq t}$, because $P(t - s, 1) e^{-\int_0^s \mu(\xi) d\xi} \neq 0$ for $s \notin [0, t]$.

Step III. Combination

Combining two solutions into one, we obtain:

$$p(t, 0, s) = p(0, 0, s - t) e^{-\int_{s-t}^s \mu(\xi) d\xi} + \mathbf{1}_{0 \leq s \leq t} \lambda P(t - s, 1) e^{-\int_0^s \mu(\xi) d\xi} \tag{2.46}$$

where $s \geq 0$.

In the long-run only the second part persists, such that for relatively small nonnegative spell values $0 < s << t$ we write:

$$p^*(\infty, 0, s) = \lambda P^*(\infty, 1) e^{-\int_0^s \mu(\xi) d\xi}.$$

where $P^*(\infty, 1)$ is the long run probability to observe an individual in the employment state $x = 1$.

Solution to the employment part

The solution in case of employment, where spell $s \leq 0$, can be derived similarly and is given by:

$$p(t, 1, s) = p(0, 1, t + s) e^{-\lambda t} + \mathbf{1}_{\{|s| \leq t\}} e^{\lambda s} \int_0^{+\infty} \mu(s) p(t + s, 0, s) ds \tag{2.47}$$

Combination of employment and unemployment parts

The difficulty we observe when solving (2.46) and (2.47) (or (2.21), (2.22), and (2.23)) jointly is that $p(t,0,0)$ and $p(t,1,0)$ are unknown and depend on each other. All other values $p(t,0,s)$ and $p(t,1,s)$ for $s \neq 0$ can be found from the previous periods.

Given that $P(t,0)+P(t,1) = 1$ and $p(t,0,0) = \lambda P(t,1)$ we see that the unknown value p(t,0,0) can be found through

$$p(t,0,0) = \lambda(1-P(t,0)) = \lambda\left(1 - \int_0^\infty p(t,0,s)ds\right) = \lambda\left(1 - \lim_{x\to 0^+}\int_x^\infty p(t,0,s)ds\right)$$

where $p(t,0,s)$ is known for all $s > 0$.

Constant arrival rates

If we assumed constant arrival rate μ, equations (2.46) and (2.47) would become:

$$p(t,0,s) = p(0,0,s-t)e^{-\mu t} + \mathbf{1}_{0\leq s\leq t}\lambda P(t-s,1)e^{-\mu s}$$

and

$$p(t,1,s) = p(0,1,t+s)e^{-\lambda t} + \mathbf{1}_{\{|s|\leq t\}}\mu P(t+s,0)e^{\lambda s}$$

respectively.

Chapter 3

Endogenous Markup Distributions

BY ALEXEY CHEREPNEV[1] AND KLAUS WÄLDE[2]

3.1 Introduction

In recent years many studies have emphasized the importance of international trade effects on economic stability. At the country level, the stationarity of markups and prices has been thought of as a key factor of such stability. In contrast, various firm-level fluctuations have considerably influenced prices of goods. Significant changes in prices have subsequently lowered living standards, affecting overall stability. Such price fluctuations have been primarily caused by changes in firms' productivity, market competition (i.e. by changes in the number of firms in a market) and, consequently, by changes in firms' markups.

While focusing on the demand-supply relationship, a considerable amount of macroeconomic literature disregards the impact of firm-level fluctuations on the markup and price distributions. Moreover, relatively little has been done on studying international trade effects on the price formation just after trade barriers fall. Thus, we formulate the aim of this study by posting the following questions: "What are the main drivers of markup fluctuations?", "In which cases does the price in a market grow or fall?", "What are the transitional dynamics of markup and price distributions when a closed economy opens up?".

We observe that static economic models are limited. They are able to only partially explain the real world dynamics. In this respect, the modern tendency towards analysis performed outside of the steady state shows untapped potential. Recent studies show a little progress towards modeling various dynamics between two steady states (see, e.g., Gustafsson and Segerstrom, 2010, and Burstein and

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Melitz, 2011). We would also like to perform such analysis to shed more light on the evolution of the markup and price distributions.

In this paper we argue that movements at the firm level are fairly complicated, but also crucial for understanding both markup and price fluctuations. These movements become especially important for an open-economy analysis due to major difficulty in firm aggregation. This work thus attempts to fill the gaps in the literature by presenting the dynamics of the markup distribution between two steady states in both closed and open economies.

This study relates to various strands of literature. First, this paper relies on a large group of closed-economy models that includes entries and exits of firms. Such models are examined by Jovanovic (1982), Hopenhayn (1992), Klette and Kortum (2004), Luttmer (2007) and Lentz and Mortensen (2008) among others. Many of them incorporate stochastic elements into the models in order to explain various firm dynamics.

Second, we borrow the productivity dynamics from the international trade model presented by Melitz (2003). In doing so, we assume that after market entrance each firm draws its productivity level from a common distribution. This level then remains constant. In addition, we structure our paper in a similar way first presenting the closed-economy model and then extending it to the open-economy case when trade barriers fall. In contrast to the Melitz model, we employ a continuous-time generalization of the Dixit-Stiglitz model with a discrete finite number of goods (see Dixit and Stiglitz, 1977).

Third, this paper also contributes to the literature on international trade. There are many international trade papers where authors use various trade liberalization scenarios (see, e.g., Melitz, 2003, Gustafsson and Segerstrom, 2010, Burstein and Melitz, 2011, Impullitti et al., 2013). In these papers the international trade liberalization is considered to be partial (e.g. reduction of the entrance and iceberg costs) or complete (e.g. the fall of the trade barriers). We focus on the latter case only.

Fourth, our methodology is related to models which are built on the stochastic foundation and study transitional dynamics. For instance, one of the first attempts to describe a complete picture of transition dynamics of firms' productivity and output is performed by Burstein and Melitz (2011). Their paper also refers to the work by Atkeson and Burstein (2010), where a two-side iteration procedure is suggested to compute the transition between two states of the modeled economy. The general overview of the stochastic calculus is given by, e.g.,Ross (1996) and Wälde (2012).

Finally, to describe dynamics of probability distributions we employ the Kolmogorov (Fokker-Planck) equations which are widely used in mathematics and physics, but rarely presented in economic literature. The limited number of examples in economic literature includes Merton (1975), Lo (1988), Klette and Kortum (2004), Moscarini (2005), Koeniger and Prat (2007), Prat (2007), Bayer and Wälde (2010a,b, 2013).

In this paper we therefore present an alternative continuous-time model to study firm reallocations across available markets. Our model includes discrete and finite quantities of firms, goods and markets. We basically concentrate on dynamics at the firm level, i.e. we decompose all movements of a firm into entries and exits. When a firm is an outsider, it might start supplying each market with a certain probability related to the market *attractiveness*. The latter depends on the number of firms already producing in this market. If this number is large, it is less rewarding for a potential producer to enter this market due to strong competition and low markup. Within one market firms behave as Cournot competitors.

Such dynamics give us a distinguishing feature that we can describe the whole market at any future point in time knowing only the initial conditions of the market, the market attractiveness and the total number of entrepreneurs in the whole economy. These dynamics are fully determined by the Kolmogorov equations that clarify how distribution of prices, markups and the number of firms in a market change over time and converge to their new steady states. Importantly, this relatively simple setup allows us to discuss firm dynamics out of steady states in both closed-economy and open-economy models.

Although the dynamics of the probability distributions are interesting in themselves, this study also produces results which corroborate the findings of the previous works in all fields mentioned above. First, our paper adds to the literature on market entry and exit in a closed-economy model. For example, we observe that if a market is occupied by productive firms with low marginal costs, it will lower the price in this market. Moreover, a high number of firms producing in a market, also set lower prices. In the latter case, the price approaches average marginal costs as in the market with intense competition. Another important finding of our paper is that the markup distribution within a market is a result of the distribution of the number of firms only. We also provide a simple estimation of the closed-economy model parameters to reach the average steady-state markup within the drugs and medicines market equal to 1.04 (the U.S. data).

The results observed in this study also mirror those of the previous studies that have examined the effect of trade liberalization. As an example, we model the fall of the trade barriers in a two-country world and observe that, when two closed economies open up, the number of manufacturers in a mutual market increases as a result of the summation of a number of manufacturers in the first and the second countries. This leads to a decline in the market attractiveness in the mutual market due to stronger competition, which, in turn, forces some firms to exit this market.

The remainder of this paper is organized as follows. In Section 3.2, we describe the closed-economy model with stochastic entries and exits. In the same section we look at consumption of the representative agent and at production of a firm. We then study the equilibrium distributions and look at the cross-section analysis in Section 3.3. Section 3.4 defines the equilibrium in the closed-economy world together with the long-run distributions and their properties. This section also includes some qualitative analysis of the evolution of the distribution of the number of firms and the markups distribution. In Section 3.5 we assess a simplified trade scenario when impassable trade barriers between two asymmetric closed-economy countries fall. Section 3.6 provides both summary and concluding remarks.

3.2 The closed-economy model

3.2.1 The basic structure

There is a fixed discrete number I of markets i. Each market is characterized by a single good denoted also by i. There is the Dixit-Stiglitz structure of demand for goods $i = 1, 2, \ldots, I$ (see Dixit and Stiglitz (1977)). Each good i is produced by an endogenous discrete finite number $n_i(t)$ of firms z that operate in market i and behave like Cournot oligopolists. The total number of incumbents in all I markets is therefore endogenously given by

$$N(t) = \sum_{i=1}^{I} n_i(t).$$

The number of firms out of the market and the number of incumbents sum up to a fixed discrete finite number Z. One can think of Z as the number of entrepreneurs in the whole economy with entrepreneurs being in or out of the market. We therefore count all producing firms and outsiders from 1 to Z.

We build our model such that few firms operate in few markets.[3] To become a producer, outsiders simultaneously search for technologies to enter each market i. A technology for market i arrives at an endogenous rate β_i. This rate forms an aggregate rate of $\beta_i[Z - N(t)]$ with which outsiders find technologies and therefore enter into market i.

The technology level of a firm z in a market i at t is denoted by its productivity $\varphi_{zi}(t)$ which remains constant throughout the whole period of time when the firm is present in this market, i.e. we consider no R&D process performed by a firm.[4]

[3]One could think of $I = 100$ markets and $Z = 1000$ potential entrepreneurs as an example.

[4]The productivity changes when a firm leaves a market and enters another market afterwards.

Once a technology to enter a market i is found, firms first check if this technology level is high enough to overcome production costs. If yes, firms start producing in a market i competing with other producers. Due to competition or other effects, incumbents leave a market i at an endogenous death rate ξ_i. This implies that the aggregate rate of inflow from all markets into the pool of outsiders is $\sum_{i=1}^{I} \xi_i n_i(t)$. The I markets, the pool of outsiders Z, arrival rates and the overall dynamics are illustrated in Figure 3.1.

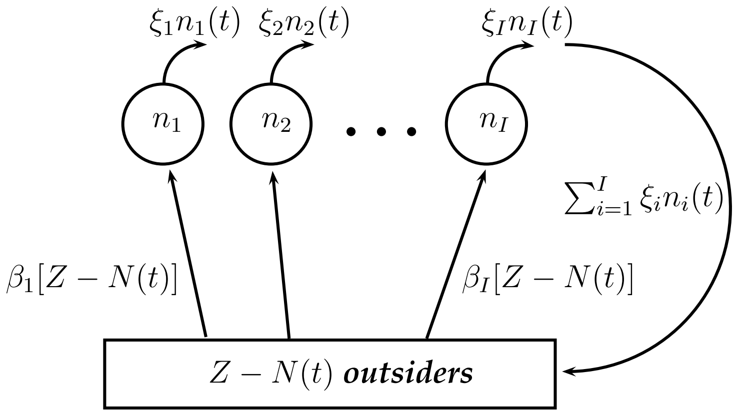

Figure 3.1: Markets and flows of firms.

We are fundamentally interested in firm dynamics (the rest, like markups, follows from this). A firm z is uniquely described at each point in time t by the market i in which firm z produces and by a productivity φ_{zi} firm z possesses within this market i. It forms two state variables of a firm:

$$\boldsymbol{s_z}(t) \equiv [s_{z1}, s_{z2}, \ldots, s_{zi}, \ldots, s_{zI}], \text{ and } \varphi_{zi}(t), \tag{3.1}$$

where $s_{zi}(t) \in \{0, 1\}$ and $\varphi_{zi}(t) \in G_i(\varphi)$. The state variable $s_{zi}(t)$ equals 1 and $\varphi_{zi}(t)$ is some positive value from the market specific distribution $G_i(\varphi)$ if firm z produces in market i at time point t. Variables $s_{zi}(t) = 0\ \forall i$ and $\varphi_{zi}(t) = 0$ when the firm z is an outsider.

When the realization of the two state variables in given, all other endogenous variables in this economy are derived from them. If firm z produces in market i, only $s_{zi}(t) = 1$ and all other values of the vector-valued variable $\boldsymbol{s_z}(t)$ are zeros, i.e. $s_{zj} = 0\ \forall j \neq i$. It means that the market in which the firm z produces at t is defined as[5]

$$m_z(t) = \begin{cases} i, & \text{when } s_{zi}(t) = 1, s_{zj} = 0\ \forall j \neq i, \\ 0, & \text{when } s_{zi}(t) = 0\ \forall i. \end{cases}$$

We discuss complete dynamics in Appendix 3.E which are considered as an extension to the current model.

[5]The variable $m_z(t)$ is redundant, not used in derivations, and therefore used for descriptive purposes.

We will capture dynamics of entry and exit processes formally by using two (sets of) Poisson processes. One set of processes, denoted by q_{zi} with the arrival (entry) rate β_i, describes how a firm z makes the transition into market i. The processes x_{zi} describe how a firm z leaves market i. The corresponding arrival (exit) rate is ξ_i.

All arrival rates related to the market i are functions of the number of firms producing in this market, $n_i(t)$. Hence, we allow for

$$\begin{aligned} \beta_i &= \beta_i(n_i(t)), \\ \xi_i &= \xi_i(n_i(t)). \end{aligned} \tag{3.2}$$

First, if there are many firms in market i, it is more difficult (or economically less rewarding) to enter this market for any new incumbent. This suggests both $\beta_i = \beta_i(n_i(t))$ and $\partial\beta_i/\partial n_i < 0$. Second, the exit rate, $\xi_i(n_i(t))$, is increasing when $n_i(t)$ is high, i.e. $\partial\xi_i/\partial n_i > 0$. The interpretation behind this fact is also straightforward: the high number of firms producing in the market i imposes *stronger* competition within this market which, in turn, forces some firms to leave. We will also specify the arrival rates in more detail below.[6]

In summary, one firm is now described via a set of variables $s_{zi}(t)\ \forall i$ and $\varphi_{zi}(t)$, i.e. $I+1$ variables in total. The number of realizations is different. If we assume that the distribution $G_i(\varphi)$ is discrete and defined on $[1, \bar{\varphi}]$, where finite $\bar{\varphi}$ defines the highest possible level of productivity, then in each market firm z can reveal φ productivity levels. This brings $\bar{\varphi} \cdot I$ states (realizations) if firm is producing and 1 realization for being out of all markets. We obtain $\bar{\varphi} \cdot I + 1$ realizations in total for one firm z. Later in Section 3.3 we present probability for each of $\bar{\varphi} \cdot I + 1$ states. Given the number of firms Z, the whole economy is described by $Z(\bar{\varphi} \cdot I + 1)$ states. We also summarize all abbreviations in the following table.

[6] One simplified version of this paper includes constant arrival rates. As a generalization one can consider β_i and ξ_i as functions of state variables (3.1) of the economy, i.e. $\boldsymbol{s_z}(t)$ and $\varphi_{zi}(t)$.

Elements of the model	Abbreviations
markets, goods	$i = 1, 2, \ldots, I$
firms, potential entrepreneurs	$z = 1, 2, \ldots, Z$
total number of workers	L
state of firm i in market z	$s_{zi}(t) \in \{0, 1\}$
number of firms in market i	$n_i(t) = \sum_{z=1}^{Z} s_{zi}(t)$
number of firms out of all markets	$Z - \sum_{i=1}^{I} n_i(t)$
individual entry rate	$\beta(n_i(t))$
individual exit rate	$\xi(n_i(t))$
probability that firm z in market i at t	$P^s_{zi}(t)$
productivity of firm z in market i at t	$\varphi_{zi}(t) \in G_i(\varphi)$

Table 3.1: Notation of the model.

Below we consider consumers and production of firms. After it, in order to describe movements in the economy, we form a stochastic differential equation system which has a Markov nature, i.e. where all arrival rates (and therefore all future distributions) may depend on current state variables only. We therefore apply the Fokker-Planck logic and compute joint distributions by the steps worked out in Bayer and Wälde (2013).

3.2.2 Consumers

Production needs to be good for firms, i.e. produced goods are consumed and firms make profits. We therefore specify consumers that value goods $i = 1, 2, \ldots, I$. Agents essentially choose quantity of goods $1, 2, \ldots, I$ that they desire to consume, $Y_1(t)$, $Y_2(t)$, …, $Y_I(t)$ according to the Dixit-Stiglitz instantaneous utility function:

$$u(Y_1(t), Y_2(t), \ldots, Y_I(t)) = \left(\sum_{i=1}^{I} [Y_i(t)]^\rho \right)^{1/\rho}$$

given $\rho \in (0, 1)$, the constant elasticity of substitution (CES) between goods, $\sigma = 1/(1-\rho) > 1$, and the budget constraint:

$$w = \sum_{i=1}^{I} p_i Y_i(t),$$

where w is wage, and p_i is the price of good i.

After solving the corresponding maximization problem (see Appendix 3.A), we obtain the following a well-known demand function for each good i:

$$Y_i = w \left[\frac{p_i}{P} \right]^{-\sigma}. \tag{3.3}$$

given the price index $P = \left(\sum_{i=1}^{I} p_i^{1-\sigma}\right)^{-\sigma}$ (see, e.g., Dixit and Stiglitz, 1977, and Melitz, 2003).

3.2.3 Production and market structure

We allow many firms to be active in market i at the same time. We assume that they interact as Cournot oligopolists inside each market i. A firm z produces with technology

$$y_{zi}(t) = \varphi_{zi}(t) l_z(t) - f_{di}, \tag{3.4}$$

where $y_{zi}(t)$ is output of firm z in market i at t and $l_z(t)$ is labor force size is required for this production given the current productivity level, $\varphi_{zi}(t)$. The last term, f_{di}, represents the market specific costs of production. Wage of one labor unit is fixed and denoted by w. We also define the price elasticity of market demand by $\varepsilon_i \equiv \frac{dY_i/dp_i(Y_i)}{Y_i/p_i(Y_i)}$, where $Y_i(t) = \sum_{z=1}^{Z} y_{zi}(t)$ is aggregate output in market i and $p(Y_i)$ is the price in this market.

We link the elasticity of substitution, σ, and the elasticity of market demand, ε_i, using (3.3) (see derivations in Appendix 3.A):

$$\varepsilon_i = -\sigma. \tag{3.5}$$

Economically it means that the price elasticity of market demand is identical in all markets. We therefore define $\varepsilon \equiv \varepsilon_i$. Since $\sigma > 1$, we conclude that $\varepsilon_i < -1$. If goods can be easily substituted (high σ), negative ε_i implies a dramatic fall of demand of good i as a response to a small change in the price of good i, and vise versa.

All firms producing in market i, differ in their productivities, i.e. marginal costs $c_{zi}(t) = w/\varphi_{zi}(t)$. The optimal price $p_i = p(Y_i)$ is then given by a solution to the profit maximization problem of a firm (see Varian, 1992, or Appendix 3.B):

$$p_i = \frac{c_{zi}(t)}{1 + \frac{y_{zi}(t)}{Y_i(t)\varepsilon}} \tag{3.6}$$

Defining further $\Theta_{zi} = y_{zi}(t)/Y_i(t)$ as the market share of firm z in market i, we write the price in market i as

$$p_i = \frac{c_{zi}(t)}{1 + \frac{\Theta_{zi}}{\varepsilon}} \tag{3.7}$$

In this equality, price p_i links firm specific terms, c_{zi} and Θ_{zi}. It means that if firm z is highly productive in market i (it produces with lower costs), it gets higher market share Θ_{zi} within this market.

3.2.4 Labor market

The labor market fixes the wage w and makes sure that exogenous labor supply L equals labor demand. Demand comes from employment $l_z(t)$ in firm z at t. As firms can be out of the market, labor demand for a given firm z will be zero at certain points in time. Labor market equilibrium therefore reads

$$\sum_{z=1}^{Z} l_z(t) = L. \tag{3.8}$$

3.2.5 Market entry and exit

When we think about a market i, we need to know two things: how many firms are in this market and which are they? We can follow individual firms through markets by looking at the set of variables $\{s_{zi}(t)\}_{i=1}^{I}$. Each variable $s_{zi}(t)$ describes the state of firm z in market i at t. This variable can take only two values, $s_{zi}(t) \in \{0, 1\}$, where $s_{zi}(t) = 1$ if the firm z produces in market i at t. Variable $s_{zi}(t)$ is set to 0 when firm z is not producing in the market i at t, i.e. the firm either produces on another market $j \neq i$, $s_{zj} = 1$, or the firm is an outsider, $s_{zi}(t) = 0 \; \forall i = 1, 2, \ldots, I$. We formulate this logic employing the following stochastic differential equation for $s_{zi}(t)$,

$$ds_{zi}(t) = \left(1 - \sum_{j=1}^{I} s_{zj}(t)\right) dq_{zi} - s_{zi}(t)dx_{zi}. \tag{3.9}$$

If z is an outsider, it can start business in any of the markets $i = 1, 2, \ldots, I$ depending on which Poisson process q_{zi}, $i = 1, 2, \ldots, I$, jumps first. Obviously, market entry can only take place when a firm is not yet in a market. If the producer z is currently in the market i, it cannot jump to any other market $j \neq i$, but must first leave the market i. This exit process is governed by x_{zi}. The process described by (3.9) can therefore be easily understood: When the firm z is out of the market, all $s_{zi} = 0 \; \forall i$. Obviously, only the first term plays a role then, whereas the second term vanishes. When a firm makes a transition to market i, i.e. when $dq_{zi}(t) = 1$, state $s_{zi} = 1$, and other states $s_{zj} = 0 \; j \neq i$, the second term is non-zero while the first term does not play any role anymore, it equals zero. When the firm exits the market i as $dx_{zi}(t) = 1$, s_{zi} again becomes 0 among other states s_{zj}, $j \neq i$.

The arrival rate of the market entry process q_{zi} is denoted by $\beta_i(n_i)$. The argument $n_i(t)$ explicitly shows that the arrival rate depends on the number of firms present in the market i at t. This rate is thus market specific, but firm-independent. This means that all outsiders have identical probability to enter the market i. This probability is equal for all potential producers because all of them observe the same situation in all markets an instant before they might join one of them. Mathematically, the arrival rate $\beta_i(n_i)$ is the same for all outsiders since $n_i(t)$ is simultaneously observed by all potential producers at the time point t whereas the entrance may only happen an instant after t. The arrival rate $\beta_i(n_i)$ is also independent of the state of the firm: the process q_{zi} constantly jumps over time, but influences variable s_{zi} only if firm z is an outsider, i.e. when $\sum_{j=1}^{I} s_{zj} = 0$. The index i of $\beta_{\boldsymbol{i}}(n_i)$ captures all features of market i that has an impact on market entry such as entry barriers, congestion in the market and the like.

The arrival rate of the market exit process x_{zi}, $\xi_i(n_i)$, reveals similar characteristics as $\beta_i(n_i)$. This rate is also market specific, but firm-independent, i.e. it depends on aggregate characteristics of the market. It similarly allows process x_{zi} constantly to jump. However, x_{zi} changes s_{zi} exclusively when $s_{zi} = 1$, i.e. firm z is producing in market i at the moment t. In this case, the first term of (3.9) vanishes.

Coefficients $1-\sum_{j=1}^{I} s_{zj}$ and s_{zi} switch the effects of processes q_{zi} and x_{zi} respectively depending on the current states of firm z in all markets, s_{zi}, $i = 1, 2, \ldots, I$. It allows us to build the process (3.9) to be self-consistent, i.e. state s_{zi} does not depend on any other characteristic of the system and thus can be solved independently (see Section 3.3.1).

Other characteristics of the model apparently depend on the states s_{zi}. Let us now understand how many firms $n_i(t)$ there are in market i. This number simply reads:

$$n_i(t) = \sum_{z=1}^{Z} s_{zi}(t). \tag{3.10}$$

Term $n_i(t)$ is given via the sum of states $s_{zi} \in \{0, 1\}$ of both active and inactive firms in market i. This sum however captures the increase in $n_i(t)$ driven by active in market i firms only.

The total number of producing firms is obtained via the sum of the number of firms, $n_i(t)$, over all markets:

$$N(t) = \sum_{i=1}^{I} n_i(t) = \sum_{i=1}^{I} \sum_{z=1}^{Z} s_{zi}(t). \tag{3.11}$$

3.2.6 Productivity of a firm

Another important attribute of the model is firm z's productivity. We define $\varphi_{zi}(t)$ to denote the productivity of firm z in market i at point in time t. When a firm z is out of the market at t, her productivity is obviously zero in all markets, $\varphi_{zi}(t) = 0$ $\forall i = 1, 2, \ldots, I$. However, the firm can find a technology to enter one market i as the arrival rate for finding a technology is positive, $\beta_i(n_i) > 0$. The process q_{zi} captures this transition of entrepreneur z into market i. As a firm simultaneously searches for technologies in all markets, transition into the production state is captured by the sum over all processes q_{zi}. As no two Poisson processes ever jump at the same point in time,[7] a firm always ends up in one well-defined market i.

The entry is therefore affected in two ways. First, in order to enter a market i, a firm z must have a chance to choose the technology level appropriate to this market i (governed by the process q_{zi} with rate β_i). Second, the technology level when chosen from distribution $G_i(\varphi)$ must exceed some reservation level and therefore must yield a positive profit (see Appendix 3.C).

3.3 Equilibrium distributions

Equilibrium is described in $t = 0$ by the optimality conditions in industries defined in Sections 3.2.2 and 3.2.3 plus, for $t > 0$, by an evolution of probability distributions over time. In this section we will focus on $t > 0$ describing the evolution of the probability mass function (*PMF*) of firm's state $s_{zi}(t)$. Knowing this *PMF* allows us to carry out further analysis of distributions of the number of firms and of markups.

We must distinguish a probability density function (*PDF*) from a probability mass function (*PMF*). A *PDF* is associated with continuous random variables, whereas a *PMF* is linked with discrete random variables. Since the key variable in our model, the state variable, $s_{zi}(t) \in \{0, 1\}$ is a discrete random variable, we perform further analysis employing a *PMF.* In this respect, we employ the term "probability distribution" meaning a discrete probability distribution defined by a probability mass function.

We start by looking at individual distributions. By this we imply probability distributions for *one* firm. Since the closed-economy model presented in Section 3.2 reveals the Markov property, knowing the initial conditions at $t = 0$ predicts the distributions for any future point in time. The entire system for one firm consists of I equations for where a firm is currently supplying as described by (3.9). The number of firms in a market i is given by (3.10). After describing distributions of the firm's state, distributions of the number of firms and distributions of markups, we will discuss cross-section distributions.

3.3.1 The fundamental distributions

We call fundamental distributions those that determine the state s of a firm. All other distributions are derived from these distributions. If we consider one firm z, it can be an outsider or produces in one of the markets $i = 1, 2, \cdots, I$. A

[7] Poisson processes jump at random points in time. Assume that one jump takes place at some t. The probability that another jump takes place at this same point in time is zero, given that time is continuous.

firm z is out of all markets if $s_{zi} = 0$ $\forall i = 1, 2, \ldots, I$. We denote this situation as if the firm z produces in market 0, $s_{z0} = 1$. On the other hand, if one of $s_{zi} = 1$, a firm z is producing in market i and $s_{zj} = 0$ $\forall j \neq i$. These two situations describe the whole state space, meaning that the state of a firm is denoted by vector $\boldsymbol{s_z}(t) \equiv [s_{z1}(t), s_{z2}(t), \ldots, s_{zI}(t)]$.

After the state space is discussed, we detail the probability mass functions defined on this state space. To be precise with notation from the outset, $P^s_{zi}(t)$ denotes the probability that a firm z is in a market i at t. Because the time is continuous, the probability mass function $P^s_{zi}(t)$ continuously evolves over time. In line with the vector-valued variable $\boldsymbol{s_z}(t)$, vector $\boldsymbol{P^s_z}(t) = [P^s_{z0}(t), P^s_{z1}(t), \ldots, P^s_{zI}(t)]$ includes only one element different from zero.

We describe the evolution of the probability mass functions of a firm z to be in a market i at t by a system of differential equations (see detailed derivations in Appendix 3.D). It reads for markets 1 to I

$$\begin{cases} \frac{\partial P^s_{z1}(t)}{\partial t} &= \beta_1(n_1)P^s_{z0}(t) - \xi_1(n_1)P^s_{z1}(t) \\ &\vdots \\ \frac{\partial P^s_{zI}(t)}{\partial t} &= \beta_I(n_I)P^s_{z0}(t) - \xi_I(n_I)P^s_{zI}(t) \end{cases} \tag{3.12}$$

which is a system of I equations with $I+1$ unknowns $\{P^s_{z0}(t), P^s_{z1}(t), \ldots, P^s_{zI}(t)\}$. Each equation includes the partial derivatives with respect to time t on its left-hand side that label evolution of the probability mass function over time t for a given market i. As we can see on the right-hand side of these equations, the evolution of the probability mass function at each point in time depends on the inflow and the outflow of the firms which happens with the arrival rates $\beta_i(n_i)$ and $\xi_i(n_i)$ respectively. These rates together with initial conditions $\{P^s_{z0}(0), P^s_{z1}(0), \ldots, P^s_{zI}(0)\}$ determine a unique solution.

We find an additional $(I+1)$th equation using the property of the complement events:

$$P^s_{z0}(t) + \sum_{i=1}^{I} P^s_{zi}(t) = 1.$$

It says that for a firm there are no states other than producing in one of the market or being an outsider. Probability to observe a firm i in market 0, i.e. out of all markets, $P^s_{z0}(t)$, as well as its time derivative, $\frac{\partial P^s_{z0}(t)}{\partial t}$, follows from the latter equation.

The closed system of equations that fully describes the model is fairly complicated. Since all firms act simultaneously, the system must consist of $Z \times I$ differential equations with the arrival rates $\beta_i(n_i)$ and exit rates $\xi_i(n_i)$ dependent on the number of firms in markets, n_i.

Another characteristic of a firm is its productivity $\varphi_{zi}(t)$ when this firm is active in market i at t. The productivity can take values from $G_i(\varphi)$ that exceeds the cutoff productivity (see Appendix 3.C).

3.3.2 Derived distributions

Before we present the derived distributions, we must clearly indicate that all conclusions in the rest of Section 3.3 and Section 3.4 require constant market entry and exit rates.

We denote the first important probability, the probability that there are n firms in the market i at t, by $P_i^n(t,n)$. Then $P_i^m(t,m)$ is the probability that markup in market i at t equals m. Both these probability distributions are derived from the fundamental distributions $\boldsymbol{P_z^s}(t)$.

The number of firms in a market

Assuming that states $s_{zi}(0)$ $\forall z, i$ are initially drawn from the same distribution $P_{zi}(0)$, all firms enter a certain market i with the same probability,

$$P_{1i}^s(t) = P_{2i}^s(t) = \ldots = P_{Zi}^s(t) \equiv p_i(t), \tag{3.13}$$

where $P_{zi}^s(t)$ is the solution of (3.12). Despite the fact that each firm has her own entry and exit processes, these processes are independent of each other. It happens because the arrival rates β_i and ξ_i are assumed to be constant. Then the variables $s_{1i}(t)$, $s_{2i}(t)$, $\ldots$, $s_{Zi}(t)$, indicating whether firms z are in the market i at time point t, are independent and identically distributed *Bernoulli* random variables with parameter $p_i(t) \in [0,1]$.

After all possible jumps, the number $n_i(t)$ of active firms in the market i at t is given by the sum from (3.10). As this sum of *Bernoulli* random variables is a *binomial* random variable with parameters Z and $p_i(t)$, the probability that exactly n firms produce in market i at t is therefore given by

$$P_i^n(t,n) = \left(\begin{array}{c} Z \\ n \end{array} \right) p_i(t)^n (1 - p_i(t))^{Z-n}, \tag{3.14}$$

which obviously increases in $\beta_i(n_i(t^-))$ and decreases in $\xi_i(n_i(t^-))$.

The finding that $n_i(t)$ is *Bernoulli* distributed implies that the mean and variance are[8]

$$E[n_i(t)] = Zp_i(t), \tag{3.15}$$

$$\text{var}\,[n_i(t)] = Zp_i(t)\,[1 - p_i(t)]\,. \tag{3.16}$$

When we are interested in the expected *share* of firms in market i, the mean and variance are

$$E\left[\frac{n_i(t)}{Z}\right] = p_i(t), \tag{3.17}$$

$$\text{var}\left[\frac{n_i(t)}{Z}\right] = \frac{p_i(t)\,[1 - p_i(t)]}{Z}. \tag{3.18}$$

Interestingly, as long as Z is large enough, the variance of the share is basically zero and the expected share (3.17) gives the realized share which we can compare with the real data.

The markup distribution in a market

There is a distribution of prices within a market and between markets. The distribution within a market i results from heterogeneous productivities and entry and exit rates of this market. This distribution directly follows from (3.7). On the other hand, the distribution of prices across goods i depends on the whole set of arrival rates, $\{\beta_i\}$ and $\{\xi_i\}$, and the average productivities in all markets.

Defining $\mu_i^c(t)$ as average marginal costs in market i

$$\mu_i^c(t) = \frac{1}{n_i(t)} \sum_{z=1}^{Z} c_{zi}(t),$$

we obtain from (3.7) the optimal price in the market i as a function of average quantities (see Appendix 3.B),

$$p_i(t) = \frac{\mu_i^c(t)}{1 + \frac{1}{n_i(t)\varepsilon}}. \tag{3.19}$$

We therefore define the equilibrium price using the market specific variables, the average marginal costs, $\mu_i^c(t)$, and the average market share, $1/n_i(t)$. It is also fairly easy to interpret this result: If market i is occupied by productive firms with low marginal costs, it will lower the price p_i. Moreover, given that $\varepsilon < -1$, a high number of firms, n_i, producing in market i also set lower prices. In the latter case, the price p_i approaches average marginal costs, μ_i^c, as in the market with intense

[8]Unless otherwise noted, E denotes the expectations operator with respect to the current point in time which we normalize to zero.

competition, $n_i \to \infty$.

We define the (unweighted) average markup $m_i(n_i)$ using (3.19) as

$$m_i(n_i) \equiv \frac{1}{1 + \frac{1}{n_i(t)\varepsilon}}. \tag{3.20}$$

Similarly to the equilibrium price, markups are distributed within a market and between markets. The distribution within a market is a result of the distribution of the number of firms $n_i(t)$ only. Besides, the distribution of markups across markets can be compared to empirical measures by looking at the average markup within each market.

Considering the distribution within each market, this measure is influenced by only one channel, the number $n_i(t)$ of competitors in market i. As $\varepsilon < -1$ by (3.5), the markup $m_i(n_i)$ decreases in n_i and is bounded in the interval $m_i(n_i) \in [1, (1 + 1/\varepsilon)^{-1}]$, where the minimum value of 1 is reached for an infinite number of firms. The markup takes its maximum value, $(1 + 1/\varepsilon)^{-1}$, when there is only one firm produces in the market (the case of monopoly).

We note the one-to-one relationship between the markup m_i and the number n_i of firms in (3.20). As we know the probability distribution of n_i from (3.14), we are able to compute the probability $P_i^m(t, m)$ of observing a markup m in market i at t by

$$P_i^m(t, m) \equiv \text{Prob}\left(t, m = \frac{1}{1 + \frac{1}{n\varepsilon}}\right) = P_i^n(t, n). \tag{3.21}$$

In words, the probability of observing markup m in market i at some future point in time t is given by $P_i^n(t, n)$, where n is the number of firms that implies the markup m. The probability $P_i^n(t, n)$ comes from (3.14).

The conditional probability distribution of a productivity level φ that a firm z reveals in market i at t given that $s_{zi}(t) = 1$ is obviously represented by the distribution $G_i(\varphi)$ truncated at the level of the reservation productivity φ^*. The truncation appears because firms with productivity less than φ^* generate negative profit and therefore decide not to enter market i. This probability is crucially determined by the fixed costs f_{di} (see Appendix 3.C).

3.3.3 Cross-section analysis

In principle, our model predicts a distribution of markups for each individual market i. Our focus was exclusively on one firm or one market so far. However we are ultimately interested in predicting the distribution of firms or markups across markets. Therefore we formulate the following questions "How often the markup m is drawn in t?" or, equivalently, "How often the number of firms n is observed in a random market in t?". We thus seek the probability of observing n firms in a random market and of the corresponding markup, m.

In our model each market is fully determined by the entry and exit rates which, in turn, are functions of the number of firms. Therefore we *cannot* conclude that all markets are described by identical distributions. When we ask how often we draw a markup of m, we need to take into account how often this markup m may appear in each market and what is the probability to choose one particular market. In the probability theory and statistics, it is called a *mixture distribution*:

$$F(t,m) = \sum_{i=1}^{I} w_i P_i^m(t,m) = \frac{\sum_{i=1}^{I} P_i^m(t,m)}{I},$$

where $P_i^m(t,m)$ are distributions in each market i from (3.21), and the weights w_i, $\sum_{i=1}^{I} w_i = 1$, determine probabilities with which corresponding markets can be chosen. In our case weights are equal $w = w_i = \frac{1}{I}$ since the index of a chosen market, i, can be thought of as a uniformly distributed random variable.[9]

We obtain the expected markup in the economy referring to the definition of the expected value of a random variable and dealing with the mixture distribution only:

$$E[m(t)] = \sum_{m} mF(t,m) = \frac{\sum_m m \sum_{i=1}^{I} P_i^m(t,m)}{I}, \tag{3.22}$$

where m in the summation takes all values inside the interval $[1, (1+1/\varepsilon)^{-1}]$ according to (3.20) with $n = 1, 2, \ldots, Z$. As the variance of this value is zero for any $I = 1, 2, \ldots$, we will think of $E[m(t)]$ as the non-random prediction of our model to be compared with data.

3.4 Equilibrium in the closed economy

3.4.1 Definition of equilibrium

Equilibrium in our setup follows a simple sequential structure:

(i) The system of Fokker-Planck equations in (3.12) defines the probability that a firm is in a certain market. With an initial condition on probability distribution $[P_{z0}^s(0), P_{z1}^s(0), \ldots, P_{zI}^s(0)]$, this system can be solved and a unique solution $[P_{z1}^s(t), P_{z2}^s(t), \ldots, P_{zI}^s(t)]$ $\forall t > 0$ can be obtained.

(ii) The fundamental conditional probability distribution of productivity $P_z^\varphi(t,\varphi)$

[9] In a simplified case, all markup distributions are identical and since the weights are all $\frac{1}{I}$, the mixture distribution equals each distribution within a market. In this case it is trivial to perform the cross-section analysis because it coincides with an analysis carried out in each market.

is determined by $G_i(\varphi)$ and f_{di}.

(iii) Letting there be Z firms that initially all draw from the identical distribution $[p_0(0),\ p_1(0), \ldots, p_I(0)]$ to be in a market i, we then by (3.14) obtain the probability that there are n firms in market i at t.

(iv) This result immediately by (3.21) gives us the probability distribution for markups in a market.

(v) Going to a cross-section, equation (3.22) gives us the non-stochastic the expected markup value in the whole economy.

3.4.2 The long-run distributions

First, presenting our results, we start with the long-run probability of observing a single firm z in a market i. The long-run probability distribution obviously does not change over time. Thus we equate all time derivatives with 0 in (3.12) to obtain:

$$\begin{cases} \beta_1 P^s_{z0} - \xi_1 P^s_{z1} = 0, \\ \beta_2 P^s_{z0} - \xi_2 P^s_{z2} = 0, \\ \qquad \vdots \\ \beta_I P^s_{z0} - \xi_I P^s_{zI} = 0 \\ P^s_{z0} + \sum_{i=1}^{I} P^s_{zi} = 1. \end{cases}$$

If we express P^s_{z0} in the first I equations of the system above in terms of P^s_{zi} $i = 1, 2, \ldots, I$ and then plug the result into the last equation, we obtain the solution

$$P^s_{z0} = \frac{1}{1 + \sum_{j=1}^{I} \chi_j}, \tag{3.23}$$

$$P^s_{zi} = \frac{\chi_i}{1 + \sum_{j=1}^{I} \chi_j}, \tag{3.24}$$

where $\chi_i = \frac{\beta_i}{\xi_i}$ which is an important measure of a market i.

Definition In this paper we use the term *attractiveness* of market i to refer to the ratio between the entry and exit rates, $\chi_i \equiv \frac{\beta_i}{\xi_i}$.

According to (3.23), the probability to observe a firm z out of all markets, P^s_{z0}, decreases with any of $\{\chi_i\}_{i=1}^{I}$, $\frac{\partial P^s_{z0}}{\partial \chi_i} < 0$, $i = 1, 2, \ldots, I$. Economically, it means that when the entry rate β_i in the market i high, it is less probable that firm z remains out of all markets. The opposite is true for the exit rate ξ_i. Consequently, if the attractiveness of the market i is high, the probability that a potential producer enters this market i is also high. More precisely, a firm z produces in market i at t with probability P^s_{zi} that rises with χ_i, $\frac{\partial P^s_{zi}}{\partial \chi_i} > 0$. In contrast, this probability falls if another market becomes more attractive, i.e. if χ_j $\forall j \neq i$ increases, P^s_{zi} falls, $\frac{\partial P^s_{zi}}{\partial \chi_j} < 0$ $\forall j \neq i$. One important property of χ_i is that it does not change when the entry rate β_i and the exit rate ξ_i increase or decrease by the same amount.

These equations of course also show that the long-run probability of being in a market i is identical for all firms independent of their initial distributions. We denote these probabilities by

$$p_i \equiv \lim_{t\to\infty} p_i = P^s_{zi} = \frac{\chi_i}{1 + \sum_{j=1}^{I} \chi_j},$$

following (3.13). When we look at the long-run distribution for the number of firms, we obtain from (3.14)

$$P^n_i(n) = \begin{pmatrix} Z \\ n \end{pmatrix} p^n_i (1 - p_i)^{Z-n}. \tag{3.25}$$

Its mean value,

$$E[n_i] = Z \cdot p_i = \frac{Z\chi_i}{1 + \sum_{j=1}^{I} \chi_j},$$

rises in χ_i. It has an obvious economic interpretation: if the market is more attractive to potential producers (high χ_i), it will be on average occupied by more firms in the long run (high $E[n_i]$). Equation (3.25) determines the long-run markup in each market i according to (3.21).

We moreover obtain the long-run probability of observing a firm out of all markets. The equation (3.25) also hold with $p_0 \equiv P^s_{z0}$ from (3.23),

$$P^n_0(n) = \begin{pmatrix} Z \\ n \end{pmatrix} p^n_0 (1 - p_0)^{Z-n}, \tag{3.26}$$

with the mean value,

$$E[n_0] = Z \cdot p_0 = \frac{Z}{1 + \sum_{j=1}^{I} \chi_j},$$

which decreases in attractiveness of each market, χ_i $\forall i = 1, 2, \ldots, I$. The economic interpretation is also simple: if some market i is more attractive to potential producers (high χ_i), the number of outsiders on average is small in the long run (low $E[n_0]$).

3.4.3 The number of firms

The number of firms we can work with in our model for a given market i ranges from 0 (no firm – obviously), 1 firm (monopoly) up to infinity (perfect competition). Given any initial probability distribution of the number of firms in each market, $P_i^n(0, n)$, we note the convergence of the probability distribution to observe n firms in market i at t according to (3.12) and (3.14) to its steady state shape (3.25). Figure 3.2 presents this convergence.

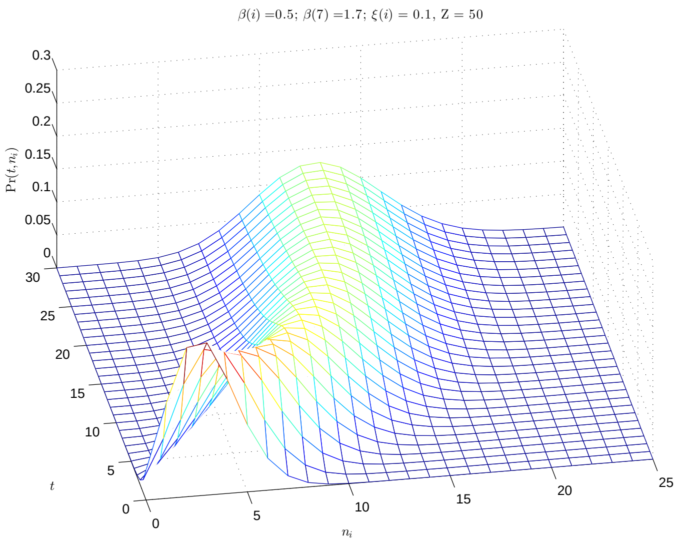

Figure 3.2: Probability to observe k firms in the market i at time t given constant arrival and exit rates, β_i and ξ_i

.

Suppose initially the attractiveness of the market i, χ_i, was low. It explains why the great mass of the probability distribution $P_i^n(0, n)$ initially locates on the small values of n. When χ_i increases, it causes the evolution of the probability distribution, which leads to the equilibrium distribution that locates on greater values of n. Figure 3.2 therefore shows not only the initial and terminal distributions, but also the path between them. It also allows us to estimate the time interval within which the transitional dynamics is completed.

While this is be calibrated below, we understand rates β_i, ξ_i (or jointly as χ_i) as monthly/quarterly/yearly rates at this point of market entry and exit. As is well-known, one divided by the rate gives the expected number of entries or exits for each individual firm (if arrival rates were not state dependent).

3.4.4 The dynamics of the markup distribution

We further analyze how we can quantitatively replicate the cross-section distribution of markups from Section 3.3.2 by our model. In doing so we provide a simple estimation of the model parameters $\{\{\beta_i\}, \{\xi_i\}, \sigma, Z\}$ to reach the average markup within the drugs and medicines market from U.S. data that is equal to 1.04 in the steady state. We obtain this markup by looking at market i with $\beta_i = 1.7$, $\beta_j = 0.5$ $\forall j \neq i$, and $\xi_j = 0.1$ $\forall j$, elasticity of market demand, $\varepsilon_i = -\sigma = -2$, and the total number of entrepreneurs $Z = 50$ (see Figure 3.3).

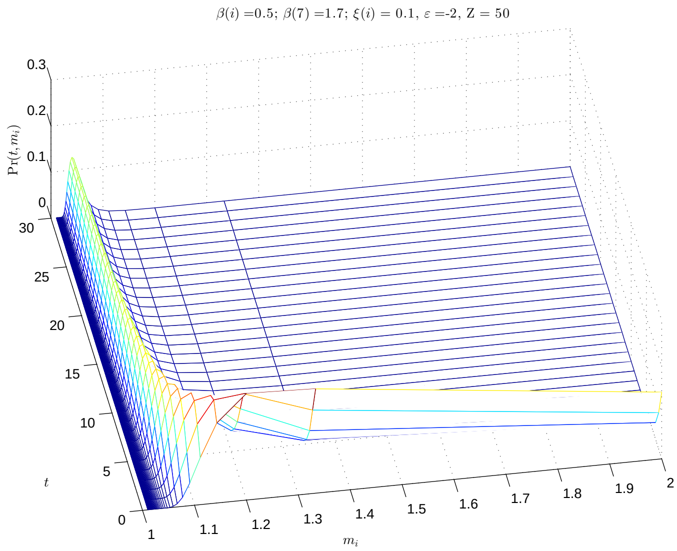

Figure 3.3: The endogenous evolution of markups from the initial (exogenous) distribution to the long-run (endogenous) distribution with mean 1.04 given constant arrival and exit rates, β_i and ξ_i

.

Figure 3.3 shows how the probability distribution of markups $P_i^m(t, m)$ evolves over time from an initial distribution $P_i^m(0, m)$ which corresponds to $P_i^n(0, n)$ discussed above. We thus observe that the desired average markup of 1.04 is reached after $15 - 20$ time periods.

3.5 International trade

In this section we analyze one interesting trade scenario when impassable trade barriers in two closed-economy countries fall.[10] Importantly, both countries possess two kinds of markets: mutual and distinct. *Mutual* markets are characterized by products which are produced and consumed in both countries. In contrast, goods from *distinct* markets are produced within one country, but consumed in two countries. Moreover, both closed economies are initially observed in their long-run equilibrium that reveals the properties discussed in Section 3.4.

[10] The fall of the trade barriers is a simple case of trading without trade costs.

3.5.1 The model

We formulate the open-economy model now in detail where it is crucial that the number of producers influences the arrival and exit rates in both mutual and distinct markets.[11] Suppose countries are asymmetric: there are Z_1 potential entrepreneurs in country 1, and Z_2 in country 2. In these countries consumers reveal identical preferences, i.e. σ is the common elasticity of substitution between goods. It implies that only the number of firms Z_k and the number of markets I_k is different, where k is the country index, $k = 1, 2$.

There are $I_{1\&2}$ markets that have common characteristics among both countries, such that firms from both countries have an access to these markets (*mutual* markets). I_1 is the number of *distinct* goods where only firms from country 1 produce these goods. I_2 is the corresponding number of markets where only firms from country 2 manufacture goods (see Figure 3.4). Obviously, $I_{1\&2} + I_1$ is the total number of markets available in country 1, and $I_{1\&2} + I_2$ is the total number of markets available in country 2. We also assume no international mobility of labor. Besides, a company cannot run the business outside its country.

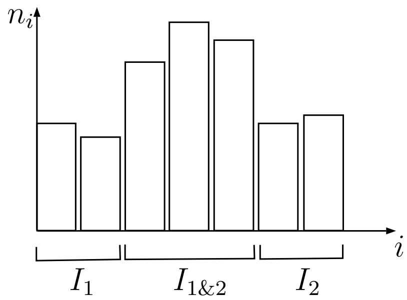

Figure 3.4: Mutual and distinct markets.

We therefore distinguish two situations when the trade barriers disappear. First, in mutual markets $i \in I_{1\&2}$, the number of firms n_i rises, prices p_i fall, and demand Y_i rises. Since the markup in this market solely depends on the value n_i, it will also fall. Second, in markers $i \in I_1$ or $i \in I_2$ the number of firms n_i remains constant, firms produce the same amount which is redistributed among all consumers from both countries. The markup in this market is therefore also constant. Although with the distinct markets the situation is clear, it is curious to see the change in the distribution of markups within a mutual market.

The numbers of firms in market i from the first country and the second country are generally different. We denote these values by n_{ik} where k is the country index, $k = 1, 2$. It means that n_{i1} and n_{i2} are random variables with their own probability distributions. Since the arrival rates in the market reveal the same functional form $\beta(n)$ and $\xi(n)$ in both countries as in the closed-economy case, the new arrival rates

[11] In this section we however use the constant rates when drawing figures.

after the integration of two economies are the functions of the sum $n_{i1} + n_{i2}$, i.e. $\beta(n_{i1} + n_{i2})$ and $\xi(n_{i1} + n_{i2})$.

Before we can perform dynamics of the number of firms with the new arrival rates, we have to define the initial distribution of the number of firms in the market i after the trade barriers fall. This initial distribution therefore is a combination of two long-run distributions in each country. More precisely, since n_i is the sum of two independent random variables n_{i1} and n_{i2}, its probability distribution is the convolution of individual distributions of n_{i1} and n_{i2}, i.e. of $P^n_{i1}(n)$ and $P^n_{i2}(n)$. We write the convolution as follows:

$$P^n_i(n) = P\left[\sum_{k=1}^{2} n_{ik} = n\right] = \sum_{m\in Z} P^n_{i1}(m) \cdot P^n_{i1}(n-m) \tag{3.27}$$

The dynamics of the probability distribution $P^n_{ik}(t, n)$ over time follows from (3.12) and (3.14). Besides, the dynamics of markups follows from (3.21).

3.5.2 Dynamics of international trade

This is the first example of a simple trade scenario, an integration of two economies. We consider two countries with a mutual market i.

The number of firms

We first consider two economies in their steady states before the trade barriers fall. Suppose the steady state distribution in country 1 reveals the low number of firms in some market i. In contrast, this market i is extremely competitive in country 2, i.e. we observe lots of firms producing the good i in the second country. We then plot the convolution of these two distributions according to (3.27) to find the initial distribution for the dynamics of the number of firms when economies open up (see Figure 3.5). We observe that the distribution of the sum $n_{i1} + n_{i2}$ (i.e. the convolution of these two distributions) shifts towards positive values of n_i relative to the distributions for country 1 and country 2.

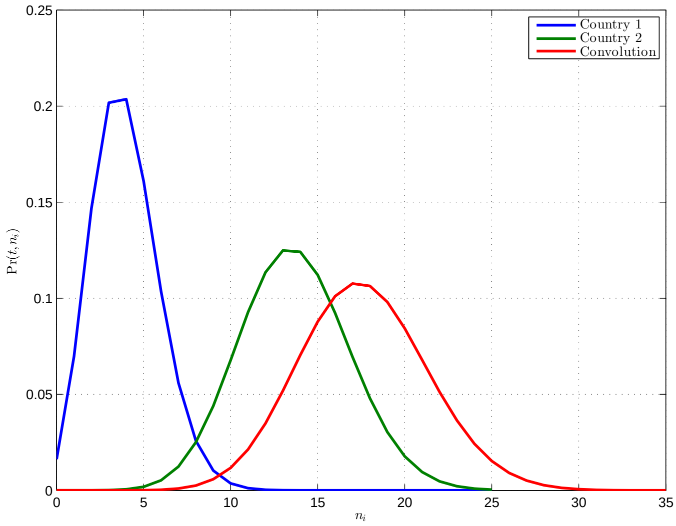

Figure 3.5: Convolution of the steady state distributions of the number of firms in a mutual market as an initial distribution for the international trade dynamics.

Looking at the dynamics of the probability distribution, we note that both partial derivatives play an important role in the redistribution of the number of firms since $\partial\beta_i(n_i)/\partial n_i < 0$ and $\partial\xi_i(n_i)/\partial n_i > 0$. Therefore, if the number of firms increases after the economy integration, the rate at which firms arrive in this market decreases, but the rate at which firms leave the market increases. It happens due to stronger competition in this market after the trade barriers disappear.

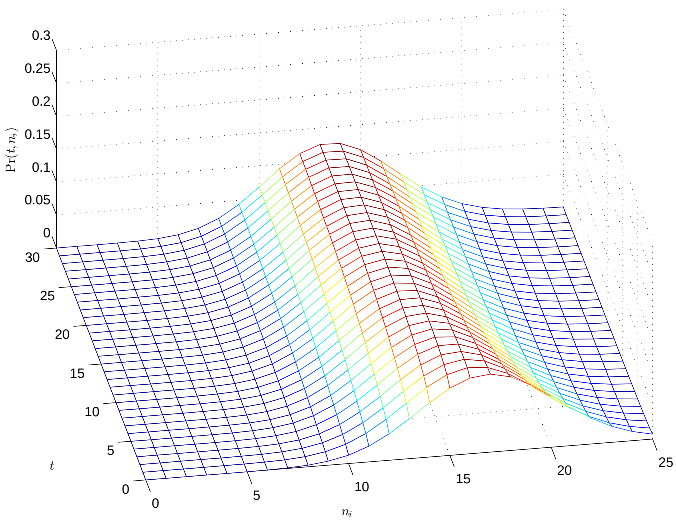

Figure 3.6: The evolution of the probability distribution $P_i^n(t,n)$ of the number of firms n over time t in the mutual market i after the trade barriers fall.

Therefore Figure 3.6 reveals the result of the severe competition such that some firms might have to leave this market. We observe that the distribution $P_i^n(t,n)$ shifts towards a smaller number of firms, i.e. the mean of the distribution $P_i^n(t,n)$, $E[n_i(t)]$, decreases.

The dynamics of the markup distribution

After obtaining the evolution of the probability distribution of the number of firms, it is now possible to draw a similar figure that shows how the markup changes in this mutual market i. To perform this analysis, we again exploit (3.21) which provides us with the one-to-one mapping between the number of firms and markups.

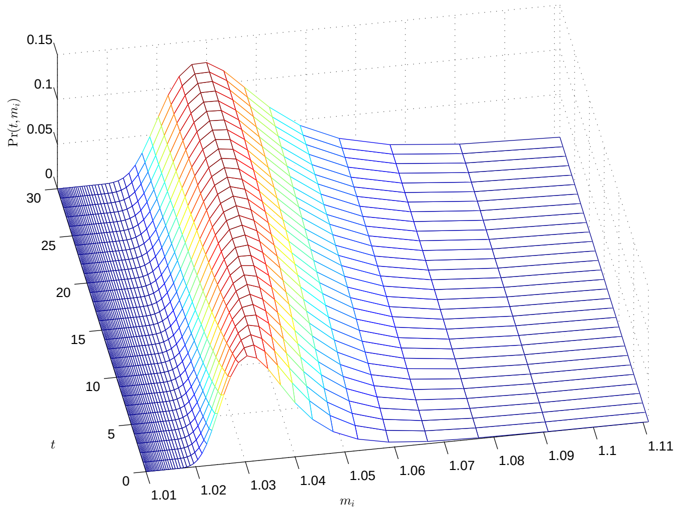

Figure 3.7: The evolution of the probability distribution $P_i^m(t, m)$ of the markup m over time t in the mutual market i after the trade barriers fall.

Figure 3.7 indicates that if the number of firms decreases, the markup will rise converging to its new steady state that is fully determined by parameters of the integrated economy with an updated number of potential entrepreneurs $Z_1 + Z_2$. This simple example, in addition, shows that the model is able to predict various movements in the economies – domestic and foreign. For instance, it can predict the drop in prices in the mutual markets, whereas the number of producers increases due to the integration, imposing stronger competition in these markets.

3.6 Conclusion

This paper studies the effects of international trade on the markup distribution in an economy. In order to perform this analysis, we first present a closed-economy general-equilibrium industry dynamics model, where firms enter and exit markets. The numbers of firms, goods and markets are discrete and finite. Within one market firms behave as Cournot competitors. As a result of such behavior, we observe an endogenous markup distribution in each market across markets.

When a firm is an outsider, it might start producing in each market with certain probability related to the market *attractiveness*. The latter depends on the number of firms already producing in this market. If this number is large, it is less rewarding for a potential producer to enter this market due to strong competition and low markup. Fully determined by the Kolmogorov equations, such dynamics give us a distinguishing feature: We can describe the markup and price fluctuations at any future point in time knowing only the initial conditions, the market attractiveness and the total number of entrepreneurs in the economy. Importantly, the markup and price distributions within a market result from the distribution of the number of firms only. The model also predicts that if a market is occupied by productive firms with low marginal costs, the price in this market will be also low. Moreover, a high number of firms producing in a market also sets lower prices. In the latter case, the price approaches average marginal costs as in the market with intense competition. We also provide a simple estimation of the closed-economy model parameters to reach the average steady-state markup within the drugs and medicines market equal to 1.04 (U.S. data).

Our relatively simple setup allows us to obtain firm dynamics out of steady states in both closed-economy and open-economy models. Having solved the closed-economy model, we notice that the movements at the firm level become especially important for the open-economy analysis due to difficulty in firm aggregation. We therefore present an open-economy model where two asymmetric countries start trading. In doing so, we analyze the evolution of markup distributions over time after the economies are opened up. As a result, we observe that opening up the economies leads to a decrease in markups in the mutual markets due to stronger competition.

Appendix

3.A Dixit-Stiglitz structure

A representative agent essentially chooses a quantity of goods $1, 2, \ldots, I$ that he/she desires to consume, $Y_1, Y_2, \ldots, Y_I$. We consider the Dixit-Stiglitz demand structure with the utility function:

$$u(Y_1, Y_2, \ldots, Y_I) = \left\{ \sum_i Y_i^{\rho} \right\}^{1/\rho},$$

where $\rho \in (0, 1)$ and with the budget constraint:

$$w = \sum_i p_i Y_i,$$

where w is a wage and p_i is the price of good i. We then solve the optimization problem of the representative agent who maximizes utility $u(Y_1, Y_2, \ldots, Y_I)$ subject to the budget constraint given above by writing the Lagrangian:

$$\mathcal{L} = \left\{ \sum_i Y_i^{\rho} \right\}^{1/\rho} - \Lambda \left[w - \sum_i p_i Y_i \right]$$

and maximizing it over the set of quantities $\{Y_i\}_{i=1}^{I}$. Λ is a Lagrange multiplier.

Taking a derivative with respect to Y_i, we obtain:

$$\frac{\partial \mathcal{L}}{\partial Y_i} = \left\{ \sum_i Y_i^{\rho} \right\}^{1/\rho - 1} Y_i^{\rho - 1} + \Lambda p_i = 0.$$

We introduce elasticity of substitution between any two goods by $\sigma = 1/(1-\rho) > 1$, or $\rho = (\sigma - 1)/\sigma$. Thus,

$$\Lambda p_i = -u^{1/\sigma} Y_i^{\rho - 1}$$

and

$$\frac{p_i}{p_j} = \left(\frac{Y_i}{Y_j} \right)^{\rho - 1} = \left(\frac{Y_i}{Y_j} \right)^{-1/\sigma},$$

which implies

$$Y_i = \left(\frac{p_i}{p_j} \right)^{-\sigma} Y_j.$$

If we plug it back into the budget constraint, we find

$$w = \sum_i p_i \left(\frac{p_i}{p_j}\right)^{-\sigma} Y_j = p_j^{\sigma} Y_j \sum_i p_i^{1-\sigma}.$$

We also define price index:

$$P = \left(\sum_i p_i^{1-\sigma}\right)^{-\frac{1}{\sigma}}$$

and write the demand function in market i:

$$Y_j = w\left[\frac{p_j}{P}\right]^{-\sigma} \Leftrightarrow Y_i = w\left[\frac{p_i}{P}\right]^{-\sigma}. \tag{3.28}$$

We are able now to find the link between elasticity of substitution between goods σ and elasticity of market demand ε_i. To do so, we first define the elasticity of market demand by $\varepsilon_i = \frac{dY_i/dp_i(Y_i)}{Y_i/p_i(Y_i)}$ (see Varian, 1992) and then write:

$$\varepsilon_i = \frac{dY_i/dp_i}{Y_i/p_i} = -\sigma w\left[\frac{p_i}{P}\right]^{-\sigma-1} \frac{1}{P}\frac{p_i}{Y_i} = -\sigma. \tag{3.29}$$

Since $\sigma > 1$, we conclude that $\varepsilon_i < -1$.

3.B The market share Θ_{zi}

To simplify the notation, we first define labor force $l_z \equiv l_z(t)$, output $y_{zi} \equiv y_{zi}(t)$ and productivity $\varphi_{zi} \equiv \varphi_{zi}(t)$ of a firm z. The number of firms in market i is $n_i \equiv n_i(t)$. We then consider the following profit maximization problem of firm z in market i:

$$\max_{y_{zi}} \pi_{zi}(y_{zi}) = \max_{y_{zi}} \{p(Y_i)y_{zi} - C_{zi}(y_{zi})\}, \tag{3.30}$$

where $C_{zi}(y_{zi}) = wl_z$ denotes costs of firm z to produce y_{zi} amount of good i in market i at t and

$$Y_i = \sum_{z=1}^{Z} y_{zi}$$

is the market clearing condition, which means that everything what is produced, is also consumed in the economy. Technically, the total output of good i, $\sum_{z=1}^{Z} y_{zi}$, is equal to its demand Y_i defined in (3.28).[12]

Given the production function, $y_{zi} = \varphi_{zi} l_z - f_{di}$ (see also (3.4)), costs are given

[12] Note that for a firm z which does not produce in the market i the output in this market is zero, $y_{zi} = 0$.

by:

$$C_{zi}(y_{zi}) = l_z w = \frac{y_{zi} + f_{di}}{\varphi_{zi}} w$$

The first order condition from (3.30) thus reads:

$$\frac{dp(Y_i)}{dY_i}\frac{dY_i}{dy_{zi}} y_{zi} + p(Y_i) - \frac{dC_{zi}(y_{zi})}{dy_{zi}} = 0,$$

where $p(Y_i)$ can be derived from the demand function (3.28). We thus rearrange terms further and obtain:

$$p(Y_i)\left[1 + \frac{dp(Y_i)}{dY_i}\frac{dY_i}{dy_{zi}}\frac{y_{zi}}{p(Y_i)}\right] = c_{zi},$$

where $c_{zi} \equiv c_{zi}(t) = \frac{dC_i(y_{zi})}{dy_{zi}} = \frac{w}{\varphi_{zi}(t)}$ are marginal costs of firm z producing good i at t linked to its productivity φ_{zi} and wage w. We further denote market share of production of firm z in market i by $\Theta_{zi} = y_{zi}/Y_i$. Then we write:

$$p(Y_i)\left[1 + \frac{dp(Y_i)}{dY_i}\frac{Y_i}{p(Y_i)}\Theta_{zi}\right] = c_{zi}$$

From the definition of the elasticity of market demand (3.29), we write the price equation:

$$p_i(Y_i) = \frac{c_{zi}}{1 + \frac{\Theta_{zi}}{\varepsilon_i}} \Leftrightarrow \Theta_{zi} = \varepsilon_i \left(\frac{c_{zi}}{p_i(Y_i)} - 1\right) \tag{3.31}$$

The economic interpretation is quite simple here: If firm z is highly productive in market i, its marginal costs c_{zi} are low. Thus, firm z gets larger market share Θ_{zi}.

To assess the market share Θ_{zi}, we first state its obvious property $\sum_{z=1}^{Z}\Theta_{zi} = 1$. We then sum market shares (3.31) of all producing firms and get:

$$1 = \sum_{z=1}^{Z}\Theta_{zi} = \varepsilon_i \sum_{z=1}^{Z}\left(\frac{c_{zi}(t)}{p_i} - 1\right) = \varepsilon_i n_i \left(\frac{\mu_{zi}^c(t)}{p_i} - 1\right),$$

where $\mu_i^c = \sum_{z=1}^{Z} c_{zi}/n_i$ is average marginal costs within market i. Combining further the latter result with (3.31), we obtain the price p_i expressed in two ways: through firm z's specific parameters, $c_{zi}(t)$ and Θ_{zi}, and through market i's characteristics, n_i and μ_i^c:

$$p_i = \frac{c_{zi}}{1 + \frac{\Theta_{zi}}{\varepsilon_i}} = \frac{\mu_i^c}{1 + \frac{1}{n_i \varepsilon_i}}.$$

Market specific parameters influence price p_i as follows: if market i is occupied by productive firms with low marginal costs, it will lower the price p_i. Given that $\varepsilon_i < -1$, a high number of producing in market i firms, n_i, set low prices as well.

3.C Market entry and survival condition

Outsiders need to pay R&D flow cost f_E to get the chance of drawing a productivity level from a distribution $G_i(\varphi)$. If they do not pay f_E, entry rates are zero for all markets, $\beta_i = 0$. If they pay f_E, entry rates are positive and given by $\beta_i > 0$. Successful R&D implies that the firm has found a technology for market i.

A firm would pay R&D flow cost f_E if the expected value of the firm at entry $E[V(0)]$ (present value of all future profits) multiplied by the probability of successful entry P^s exceed or equal to the value f_E. In equilibrium the equality holds:

$$P^s E[V(0)] = f_E.$$

The equilibrium cutoff conditions. Consider the profit of firm z in market i

$$\pi_{zi}(\varphi_{zi}) = p(Y_i)y_{zi} - w\frac{y_{zi}}{\varphi_{zi}} - f_{di},$$

using the optimal pricing equation (3.6) we obtain

$$\begin{aligned}\pi_{zi}(\varphi_{zi}) &= c_{zi}\left(\frac{1}{\frac{Y\sigma}{y_{zi}} - 1}\right) + f_{di} \\ &= \frac{w}{\varphi_{zi}}\left(\frac{1}{\frac{Y\sigma}{y_{zi}} - 1}\right) + f_{di}\end{aligned}$$

The equilibrium cutoff firms φ^*_{zi} will be pinned down by

$$\pi_{zi}(\varphi^*_{zi}) = 0 \text{ for } i = 1, 2, ...I. \tag{3.32}$$

3.D The probability to be in a certain market

So we define probabilities of observing a firm z in a market i at t via:

$$\begin{aligned}P^s_{z0}(t) &\equiv P(t, s_{z1} = 0, s_{z2} = 0, \dots, s_{zI} = 0), \\ P^s_{z1}(t) &\equiv P(t, s_{z1} = 1, s_{z2} = 0, \dots, s_{zI} = 0), \\ P^s_{z2}(t) &\equiv P(t, s_{z1} = 0, s_{z2} = 1, \dots, s_{zI} = 0), \\ &\vdots \\ P^s_{zI}(t) &\equiv P(t, s_{z1} = 0, s_{z2} = 0, \dots, s_{zI} = 1),\end{aligned}$$

or

$$P^s_{zi}(t) \equiv \begin{cases} P(t, s_{z1} = 0, s_{z2} = 0, \ldots, s_{zI} = 0), & \text{if } i = 0 \\ P(t, \ldots, s_{zi} = 1, \ldots), & \text{if } i = 1, 2, \ldots, I. \end{cases} \tag{3.33}$$

The property of complementary events must of course hold:

$$\sum_{i=0}^{I} P^s_{zi}(t) = 1.$$

We also note that $P^s_{zi}(t)$ is the probability mass function of the discrete random variable that denotes the market number in which firm produces at t.

We derive the Fokker-Planck equations to see the evolution of the probability mass function of the corresponding process (3.9) for one firm z. For any auxiliary function $f(\cdot)$ with a I-dimensional bounded support $\mathcal{S}$:

$$\begin{aligned} df(s_{z1}, s_{z2}, \ldots, s_{zI}) = & \sum_{i=1}^{I} \left[f(\ldots, s_{zi} + 1 - \sum_{j=1}^{I} s_{zj}, \ldots) - f(\ldots, s_{zi}, \ldots) \right] dq_{zi} \\ & + \sum_{i=1}^{I} [f(\ldots, s_{zi} - s_{zi}, \ldots) - f(\ldots, s_{zi}, \ldots)] dx_{zi} \end{aligned}$$

Since the arrival rates $\beta_i \equiv \beta_i(n_i(t^-))$ and $\xi_i \equiv \xi_i(n_i(t^-))$ are known at t,[13]

$$E[dq_{zi}] = \beta_i dt \text{ and } E[dx_{zi}] = \xi_i dt,$$

where $E[\cdot]$ is the expectation operator from the perspective of the time point t. Thus,

$$\begin{aligned} dE[f(s_{z1}, s_{z2}, \ldots, s_{zI})] = & \sum_{i=1}^{I} \left[f(\ldots, s_{zi} + 1 - \sum_{j=1}^{I} s_{zj}, \ldots) - f(\ldots, s_{zi}, \ldots) \right] \beta_i dt \\ & + \sum_{i=1}^{I} [f(\ldots, s_{zi} - s_{zi}, \ldots) - f(\ldots, s_{zi}, \ldots)] \xi_i dt \end{aligned}$$

$$\begin{aligned} \mathcal{A}(f) \equiv \frac{dE[f]}{dt} = & \sum_{i=1}^{I} \left[f(\ldots, s_{zi} + 1 - \sum_{j=1}^{I} s_{zj}, \ldots) - f(\ldots, s_{zi}, \ldots) \right] \beta_i \\ & + \sum_{i=1}^{I} [f(\ldots, s_{zi} - s_{zi}, \ldots) - f(\ldots, s_{zi}, \ldots)] \xi_i \end{aligned}$$

[13] The argument t^- denotes a time point which locates an instant before t on the time line.

Note that

$$E[\mathcal{A}(f)] = P^s_{z0}(t)\sum_{i=1}^{I}[f(\dots,1_i,\dots) - f(0,\dots,0)]\beta_i + \sum_{i=1}^{I} P^s_{zi}(t)[f(0,\dots,0) - f(\dots,1_i,\dots)]\xi_i,$$

where $f(\dots,1_i,\dots)$ has $s_{zj} = 0\ \forall\ j \neq i$ and only one $s_{zi} = 1$.

On the other hand,

$$\frac{dE[f(\cdot)]}{dt} = \frac{\partial P^s_{z0}(t)}{\partial t} f(0,\dots,0) + \sum_{i=1}^{I} \frac{\partial P^s_{zi}(t)}{\partial t} f(\dots,1_i,\dots).$$

The Dynkin formula states that:

$$E[\mathcal{A}(f)] = \frac{dE[f]}{dt}.$$

Then, combining terms for $f(0,\dots,0)$, we obtain:

$$\frac{\partial P^s_{z0}(t)}{\partial t} = -P^s_{z0}(t)\sum_{i=1}^{I}\beta_i + \sum_{i=1}^{I}\xi_i P^s_{zi}(t),$$

and for $f(\dots,1_i,\dots)$:

$$\frac{\partial P^s_{zi}(t)}{\partial t} = \beta_i P^s_{z0}(t) - \xi_i P^s_{zi}(t),$$

Thus, we finally write the system of FPEs:

$$\begin{cases} \frac{\partial P^s_{z0}(t)}{\partial t} & = -P^s_{z0}(t)\sum_{i=1}^{I}\beta_i + \sum_{i=1}^{I}\xi_i P^s_{zi}(t) \\ \frac{\partial P^s_{z1}(t)}{\partial t} & = \beta_1 P^s_{z0}(t) - \xi_1 P^s_{z1}(t) \\ & \vdots \\ \frac{\partial P^s_{zI}(t)}{\partial t} & = \beta_I P^s_{z0}(t) - \xi_I P^s_{zI}(t) \\ 1 & = P^s_{z0}(t) + \sum_{i=1}^{I} P^s_{zi}(t) \end{cases}$$

which states how probabilities of firm z to be out of the market or producing in one of I markets.

In the matrix notation the system reads:

$$\begin{cases} \dot{\boldsymbol{P}}^s_{\boldsymbol{z}} & = \boldsymbol{M}^s \boldsymbol{P}^s_{\boldsymbol{z}} \\ 1 & = \boldsymbol{1} \cdot \boldsymbol{P}^s_{\boldsymbol{z}} \end{cases} \tag{3.34}$$

where $\mathbf{1}$ is a row-vector of ones of size I,

$$\boldsymbol{P^s_z} = [P^s_{z0}, P^s_{z1}, P^s_{z2}, \dots, P^s_{zI}]',$$

and

$$\boldsymbol{M^s} = \begin{bmatrix} -\sum_{i=1}^{I} \beta_i & \xi_1 & \xi_2 & \dots & \xi_I \\ \beta_1 & -\xi_1 & 0 & \dots & 0 \\ \beta_2 & 0 & -\xi_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_I & 0 & 0 & \dots & -\xi_I \end{bmatrix}$$

We classify this system as a system of *first-order ordinary nonlinear differential equations*. Nonlinearity appears since rates β_i and ξ_i are not constant.

This system holds for all firms $z = 1, 2, \dots, Z$. Therefore, all firms have identical distributions over states s_{zi} at t when the initial conditions are the same. The initial condition for a firm at $t = 0$ is given by the column vector

$$\boldsymbol{P^s_z}(0) = [P^s_{z0}(0), P^s_{z1}(0), \dots, P^s_{zI}(0)]'. \tag{3.35}$$

It can be intuitive or stochastic. The intuitive case is one where the firm is in, say, market i, i.e. $s_{zi}(0) = 1$ and in no other market, i.e. $s_{zj}(0) = 0$ for all j different from i. This would give

$$\boldsymbol{P^s_z}(0) = [0, 0, \dots, 0, 1, 0, \dots 0]'$$

where 1 stands at the i-th place. The stochastic initial condition allows for general values of $P^s_{zi}(0) = P^s_i(0)$ in (3.35), where $0 < P^s_i(0) < 1$ is the probability that any firm is in market i at $t = 0$. These probabilities are of course subject to summing up to one as in (3.34) and (3.35). We work with the stochastic version because it allows us to perform a cross-section analysis in the simplest way, where one distribution describes all firms without tracking where each firm exactly produces (see Sections 3.3.2 and 3.3.3).

Closed-form solutions for a special case.

We easily find the closed-form solution when we assume the constant exit rate identical in all markets $\xi \equiv \xi_i$. The first equation of the system (3.34) reads:

$$\frac{\partial P^s_{z0}(t)}{\partial t} = -\beta P^s_{z0}(t) + \xi(1 - P^s_{z0}(t)) = \xi - (\beta + \xi) P^s_{z0}(t),$$

where $\beta \equiv \sum_{i=1}^{I} \beta_i$, with solution:

$$P_{z0}^{s}(t) = \frac{\xi}{\beta + \xi} + \left(P_{z0}^{s}(0) - \frac{\xi}{\beta + \xi} \right) e^{-(\beta+\xi)t}. \tag{3.36}$$

The probability to observe the firm z in market i at t, $P_{zi}^{s}(t)$, can be easily obtained as a solution of the corresponding first-order differential equation of the system (3.34).[14]

As a next step, we compute the long-run properties at $t \to \infty$ as follows:

$$\begin{aligned} P_{z0}^{s*} &= \frac{\xi}{\sum_{i=1}^{I} \beta_i + \xi}, \\ P_{zi}^{s*} &= \frac{\beta_i}{\sum_{i=1}^{I} \beta_i + \xi}. \end{aligned} \tag{3.37}$$

3.E The productivity distribution in a market

As an extension of the closed-economy model, one can present the R&D activity of firms.[15] In doing so, we use an important attribute of the model, firm z's productivity, which is equal to zero when the firm is out of any market, but positive in any producing state. We therefore define the following process for $\varphi_{zi}(t)$, the productivity of firm z in the market i at time point t,

$$d\varphi_{zi}(t) = b(t) dq_z(t). \tag{3.38}$$

There is one new process, $q_z(t)$, that models the R&D activity of firms z once in a market. The arrival rate of $q_z(t)$ depend on the number of producing in market i firms, $\lambda = \lambda(n_i(t))$. Moreover, the jump sizes $b(t) \in \{b_1, b_2, \ldots, b_K\}$ form a series of i.i.d. finite discrete random variables with distribution $\pi^b = \{\pi_1, \pi_2, \ldots, \pi_K\}$, where K is the total number of realizations of $b(t)$.

The economic idea is straightforward: When a firm z is out of the market, i.e. when $\varphi_{zi}(t) = 0 \ \forall i = 1, 2, \ldots, I$, it cannot innovate. However, the firm can find a technology to enter one market i as the arrival rate for finding a technology is positive, β_i. The process q_{zi} captures this transition of an entrepreneur z into market i. As a firm simultaneously searches for technologies in all markets, transition into the production state is captured by the sum over all processes q_{zi}. As no two Poisson

[14]Note that we can combine all production states into one. Then we obtain only two states of a firm: it can be either a producer or an outsider. This situation coincides with the one discussed in Chapters 2 and 4 where an agent can also be found in one of two states, employment or unemployment. The equation (3.36) therefore closely corresponds to the solutions of the labor economics model in Chapter 4.

[15]We find this extension quite interesting. However we do not incorporate it into the main text because it only increases the model complexity.

processes ever jump at the same point in time,[16] a firm always ends up in one well-defined market i. In addition, for simplicity, we do not allow for a distribution of technologies at market entry. The initial level of productivity at market entry is thus assumed to be the same in all markets and equal to 1.

To illustrate the differential equation (3.38), consider a firm z that is out of the market (i.e. in market 0). For such a firm $s_{zi} = 0 \; \forall i$. According to (3.9) the firm waits for the process q_{zi} to jump which puts the firm in market i. As long as this jump does not take place, the firm does not produce, i.e. $s_{zi} = 0$. Once in market i, i.e. $s_{zi} = 1$, the firm innovates following q_z and might have to leave the market following x_{zi}. If the firm starts with $s_{zi} = 1$, the description is in analogy.

Once the firm has found a technology with a productivity drawn from $a_i(t)$ such that $\varphi_{zi} > 0$, the firm can improve her technology by doing R&D. Due to the random variable $b(t)$, the economic importance as measured by productivity increases differs from one innovation to the next. At the same time, there is the risk that the firm needs to leave the market as the death rate is now positive and given by ξ_i (see equations (3.9) and (3.38)).

Setup

This section considers a SDE with a Poisson process having a stochastic jump-size. In this first example the arrival rate itself is still fixed, but the change of the stochastic variable varies. This is an example of a Cox process, which is also known as a doubly stochastic Poisson process or mixed Poisson process in literature (cf. Cox, 1955 or for example Kingman, 1992).

Consider a random variable φ, e.g. productivity in a market, that evolves according to (3.38). The resulting FPE describing the probability $p(\varphi, t)$ of φ over time is a delay differential equation involving an expectations operator:

$$\frac{\partial}{\partial t} p(\varphi, t) = \lambda E p(\varphi - b, t) - \lambda p(\varphi, t).$$

Derivation of the FPE

Step 1 (An auxiliary function and its time-evolution)

Assume $f(\varphi) \in \mathcal{C}^2_{compact}(\mathbb{R})$ to be our arbitrary auxiliary function, which has as argument the state variable φ. This function is assumed to have a bounded support S, i.e. $f(\varphi) = 0$ outside this support. Heuristically, the differential of this function

[16] Poisson processes jump at random points in time. Assume that one jump takes place at some t. The probability that another jump takes place at this same point in time is zero, given that time is continuous.

using Itô's lemma gives

$$df(\varphi(t)) = \{f(\varphi(t)+b) - f(\varphi(t))\}\, dq_\lambda.$$

Applying the expectations operator E yields

$$Edf(\varphi(t)) = \lambda\, \{Ef(\varphi(t)+b) - f(\varphi(t))\}\, dt.$$

Note that expectations are formed with respect to two random events. One is the possibility of a jump and one is the uncertain jump size. The former yields λdt, the latter requires the expectations operator E in front of $f(\varphi(t)+b)$. As we have assumed that b is i.i.d., we can use the unconditional expectations operator E.

Dividing by dt yields

$$\mathcal{A}f(\varphi(t)) = \frac{Edf(\varphi(t))}{dt} = \lambda\left[Ef(\varphi(t)+b) - f(\varphi(t))\right]. \tag{3.39}$$

Step 2 (Derivation of Dynkin's formula)

Using Dynkin's formula and let $p(\varphi, t)$ be the probability density function for φ. The expectation operator E in (3.39) integrates over all possible states of $\varphi(t)$, hence

$$\frac{\partial}{\partial t} Ef(\varphi(t)) = E(\mathcal{A}f(\varphi(t))) = \int_{-\infty}^{\infty} \mathcal{A}f(\varphi)p(\varphi, t)d\varphi, \tag{3.40}$$

or using step 1

$$\frac{\partial}{\partial t} Ef(\varphi(t)) = \int_{-\infty}^{\infty} \lambda\left[Ef(\varphi(t)+b) - f(\varphi(t))\right] p(\varphi, t)d\varphi.$$

Step 3 (Differential equation of the density)

According to our assumption, b is independent from φ. This allows us to write the first term of the previous equation as

$$\begin{aligned}
\int_{-\infty}^{\infty} Ef(\varphi+b)p(\varphi,t)d\varphi &= \int_{-\infty}^{\infty}\int_{u}^{v} f(\varphi+b)g(b)db\,p(\varphi,t)d\varphi \\
&= \int_{-\infty}^{\infty}\int_{u}^{v} f(\varphi+b)p(\varphi,t)g(b)db\,d\varphi \\
&= \int_{u}^{v}\int_{-\infty}^{\infty} f(\varphi+b)p(\varphi,t)g(b)d\varphi\, db
\end{aligned}$$

or simply as

$$\int_{-\infty}^{\infty} Ef(\varphi+b)p(\varphi,t)d\varphi = E\int_{-\infty}^{\infty} f(\varphi+b)p(\varphi,t)d\varphi.$$

Now we can use our integration by substitution formula to get

$$\begin{aligned}\frac{\partial}{\partial t}Ef(\varphi(t)) &= E\int_{-\infty}^{\infty}\{\lambda f(\varphi)p(\varphi-b,t)-\lambda f(\varphi)p(\varphi,t)\}\,d\varphi \\ &= \int_{-\infty}^{\infty}\{\lambda f(\varphi)Ep(\varphi-b,t)-\lambda f(\varphi)p(\varphi,t)\}\,d\varphi, \qquad (3.41)\end{aligned}$$

where the last equation moved the expectations operator into the integral (using the same line of argument as above).

Step 4 (Differentiation of the expected value)

This step uses the differential version of Dynkin's formula, i.e.

$$\frac{\partial}{\partial t}Ef(\varphi(t)) = \int_{-\infty}^{\infty} f(\varphi)\frac{\partial}{\partial t}p(\varphi,t)d\varphi. \qquad (3.42)$$

This formula allows us to collect the terms of the left-hand side and the right-hand side.

Step 5 (Collecting results)

We now equate (3.41) with (3.42), which yields

$$\int_{-\infty}^{\infty}\{\lambda f(\varphi)Ep(\varphi-b,t)-\lambda f(\varphi)p(\varphi,t)\}\,d\varphi = \int_{-\infty}^{\infty} f(\varphi)\frac{\partial}{\partial t}p(\varphi,t)d\varphi,$$

which is equivalent to

$$\int_{-\infty}^{\infty} f(\varphi)\left[\lambda Ep(\varphi-b,t)-\lambda p(\varphi,t)-\frac{\partial}{\partial t}p(\varphi,t)\right]d\varphi = 0.$$

Using the same argument as in previous examples, this expression is satisfied for

$$\frac{\partial}{\partial t}p(\varphi,t) = \lambda Ep(\varphi-b,t)-\lambda p(\varphi,t) \qquad (3.43)$$

which is the Fokker-Planck equation we were looking for.

Understanding the Fokker-Planck equation

In order to understand the impact of the Cox process, assume that $b = 1$. Then φ is a standard Poisson process with arrival rate λ. The FPE then reads

$$\frac{\partial}{\partial t}p(\varphi,t) = \lambda p(\varphi-1,t)-\lambda p(\varphi,t),$$

which is the standard equation for the Poisson process.

If the jump size is fixed at $\bar{b}$, then the FPE reads

$$\frac{\partial}{\partial t}p(\varphi,t) = \lambda p(\varphi-\bar{b},t) - \lambda p(\varphi,t),$$

which also seems reasonable. Note that in those two cases $p(\cdot)$ denotes a probability and not a density as φ is discrete. Hence, we can start solving for the probability $p(\varphi,t)$ as we know that $p(\varphi-\bar{b},t)=0$. Once we know the solution for $p(\varphi,t)$, we can compute the solution for $p(\varphi+\bar{b},t)$.

In the general case, (3.43) is a delay differential equation with an integral or weighted sum. Trying to solve this in all generality would distract too much from economic questions. So let us assume b has a discrete distribution. This implies again that we can talk about $p(\cdot)$ as a probability. Then equation (3.43) can be written as

$$\frac{\partial}{\partial t}p(\varphi,t) = \lambda \sum_{i=1}^{n} p(\varphi-b_i,t)\pi_i - \lambda p(\varphi,t), \tag{3.44}$$

where we assume that we have n realizations of the random variable with values π_i. This is a delay differential equation with n delay terms. We solve it in the same way as we solved the case with a fixed $\bar{b}$:

If the initial technological level in t_0 is φ, then the probabilities $p(\varphi-b_i,t)$ are all zero. So we obtain an exponential distribution for φ, i.e. we can solve $\frac{\partial}{\partial t}p(\varphi,t) = -\lambda p(\varphi,t)$.

Analyzing (3.44) for $\varphi+b_1$ yields

$$\frac{\partial}{\partial t}p(\varphi+b_1,t) = \lambda \sum_{i=1}^{n} p(\varphi+b_1-b_i,t)\pi_i - \lambda p(\varphi+b_1,t).$$

Then all $p(\varphi+b_1-b_i,t)$ for which $\varphi+b_1-b_i<\varphi$ are equal to zero, i.e. for all $b_i>b_1$. So we end up with

$$\frac{\partial}{\partial t}p(\varphi+b_1,t) = \lambda p(\varphi,t)\pi_1 - \lambda p(\varphi+b_1,t).$$

As we have computed $p(\varphi,t)$ in the previous step, we can solve this *ODE* as well. Continuing with this recursive procedure allows us to compute the entire distribution of φ for all $t \geq t_0 = 0$.

Chapter 4

Unemployment in the OECD – Pure Chance or Institutions?

BY ALEXEY CHEREPNEV[1] AND KLAUS WÄLDE[2]

4.1 Introduction

It is commonly accepted that aggregate shocks play an important role in the labor market, e.g., in the overall behavior of the unemployment rate. These shocks normally explain positive and negative fluctuations in the unemployment rate during a short time period since these fluctuations are common to many individuals. Opposite to aggregate shocks, idiosyncratic shocks mainly influence long-run dynamics in the labor market.

Nowadays, when large companies start operating on an international scale, any financial shock can marginally affect unemployment rates in many countries. The Global Financial Crisis in 2008 that altered the unemployment rates in OECD member countries is a good example. However, during the crisis the unemployment rates in OECD member countries varied. There was a sharp increase in the unemployment rates in Spain and Greece, whereas the unemployment rate in Germany rose slightly.

In line with common globalization, governments impose similar labor market reforms. It is, however, interesting to study why despite this similarity there is a huge difference in the unemployment rates in countries. Is it just *pure chance*, i.e. due to randomness of the unemployment rate itself?

We see this model as a natural extension of the models with frictions in the

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labor market, in line with the standard textbook search and matching model (see Mortensen and Pissarides, 1994). We assume that all individual movements in the labor market happen due to both idiosyncratic and aggregate shocks.[3] We build the stochastic differential equation to model these movements. After doing so, we obtain the dynamics of the probability mass function that describes the probability of observing one individual either in the employment or unemployment state. Aggregation of all workers in the labor market allows us to predict the evolution of corresponding probability mass functions of the number of unemployed agents and of the unemployment rate. We can therefore discuss the effects of the idiosyncratic and aggregate shocks on the situation in the labor market.

Our findings shed some light on the variation in the unemployment rates. We claim that the difference in the unemployment rates among OECD member countries in the short-run happens by *chance* caused by the aggregate shocks. This partially explains the difference in the unemployment rates among OECD member countries. The policy however is important in the long-run. Therefore, according to our model, the long-run unemployment rate should be identical in countries that apply the same labor market reforms.

As a result, we find that, in contrast to the standard textbook model, our model can predict the long-run average unemployment rate at 5.86%, and that only 1% of the observed long-run variation in unemployment rates can be explained by aggregate shocks. The remaining 99% of variation is due to policy. The model moreover foretells that the effect of aggregate shocks on the distribution of unemployment rates is more significant in the short-run dynamics, i.e. just after aggregate shocks occur.

The remainder of the paper is organized as follows. In Section 4.2, we model the job-matching and job-separation process in the labor market with both idiosyncratic and aggregate shocks. In addition Section 4.2 describes the results of our qualitative analysis regarding the mean and variance of the unemployment rate. Section 4.3 demonstrates quantitative relevance of these results. Section 4.4 concludes.

4.2 The model

The reason why the variance of the unemployment rates in the standard textbook model approaches zero lies in the fact that the probabilities of becoming unemployed are considered to be independent of each other. This is pretty counterintuitive and against well-known facts. We therefore now consider a natural continuous-time generalization.

[3] Hopenhayn (1992), Kaas and Kircher (2011) and Burstein and Melitz (2011) study idiosyncratic and aggregate shocks jointly in various contexts.

4.2.1 Individual and aggregate shocks

Each individual can principally be found at each point in time in either of two states: employment or unemployment. Considering an individual i at time point t, we denote employment state by $x_i = 1$ and the unemployment state by $x_i = 0$, where $x_i \equiv x_i(t) \in \{0, 1\}$. We then write the dynamics between these two states formally via the following *stochastic differential equation* (SDE) using three Poisson processes $q^i_\lambda \equiv q^i_\lambda(t)$, $q^i_\mu \equiv q^i_\mu(t)$, $q_\nu \equiv q_\nu(t)$,[4]

$$dx_i = (1 - x_i)dq^i_\mu - x_i \left[dq^i_\lambda + \sigma dq_\nu\right].$$

The matching process q^i_μ with the constant arrival rate μ and the separation process q^i_λ with the constant arrival rate λ are individual processes, i.e. they stand for idiosyncratic shocks. The process q_ν is an aggregate separation process with the constant arrival rate ν. This process models a series of aggregate shocks. Each of these shocks hit many employed individuals at the same time such that they might lose their jobs.

According to the *SDE*, the matching process q^i_μ constantly jumps. It however influences the state x_i when the individual i is unemployed, i.e. $(1 - x_i) = 1$ or $x_i = 0$. The opposite is true for two separation processes, q^i_λ and q_ν. They exclusively affect individual i's state when the individual i is employed, $x_i = 1$. Therefore, within a certain period of time, our setup allows the matching process q^i_μ to change the state x_i and the separation processes, q^i_λ and q_ν, not to influence it and vice versa.

We make the *SDE* more precise by considering jointly two individuals i and j. The processes for their employment status x_i and x_j then read

$$\begin{aligned} dx_i &= (1 - x_i)dq^i_\mu - x_i \left[dq^i_\lambda + \sigma_i dq_\nu\right], \\ dx_j &= (1 - x_j)dq^j_\mu - x_j \left[dq^j_\lambda + \sigma_j dq_\nu\right]. \end{aligned} \tag{4.1}$$

In words, all individuals have their own processes q^i_μ and q^i_λ for matching and for separation respectively. They moreover share a joint separation process q_ν which might hit a certain fraction of individuals. As an aggregate shock will not make all workers unemployed, but just a fraction of them, we need to make sure that a jump in q_ν does not affect all i and j. This is achieved by making coefficient σ random. We, for example, set σ to be a Bernoulli random variable. This means that whenever q_ν jumps, Bernoulli distributed $\sigma \in \{0, 1\}$ takes value 1 with success probability p and value 0 with failure probability $1 - p$. In expectations an aggregate shock therefore influences a share p of the workers such that they become unemployed.

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[4]See the similar process in Chapter 2

How much two random variables change together is generally captured in probability theory by *covariance*. Random variables x_i and x_j exhibit nonzero covariance due to the common aggregate shock q_ν. As a first step, we obtain *long-run covariance* in a closed form (see rigorous derivations in Appendix 4.C):

$$cov^*(x_i, x_j) \equiv \lim_{t\to\infty} cov(x_i, x_j) = \frac{\mu^2 p^2 \nu}{(2\lambda + 2\mu + p(2-p)\nu)(\lambda + \mu + p\nu)^2}. \tag{4.2}$$

The long-run covariance[5] exhibits straight-forward properties. The probability that the aggregate shock hits a certain individual in a period of time depends on the arrival rate of this shock, identified by ν, and on the fraction of individuals that suffers from this shock, which expected value is given by p. In the case when $\nu = 0$ or $p = 0$, there is no aggregate shock and the processes for all individuals, x_i and x_j, become independent. Thus, the long-run covariance is obviously equals zero, i.e. $cov^*(x_i, x_j) = 0$. This result is also valid for short-run quantity $cov(x_i, x_j)$ (see the proof in Appendix 4.C). In the short run, the analysis moreover coincides with the simple case without the aggregate process q_ν discussed in Chapter 2.

4.2.2 Individual probabilities

In our model, the Fokker-Planck equations reflect all probability movements described by the individual processes (4.1). In this section, we focus on these probabilities of observing an agent in either employment or unemployment state, $P_0(t)$ and $P_1(t)$ respectively. We obtain them as a solution of the system of the Fokker-Planck equations (see Appendix 4.A):

$$P_0(t) = \frac{\lambda + p\nu}{\mu + \lambda + p\nu} + \left(P_0(0) - \frac{\lambda + p\nu}{\mu + \lambda + p\nu}\right) e^{-(\mu+\lambda+p\nu)t}, \tag{4.3}$$

$$P_1(t) = \frac{\mu}{\mu + \lambda + p\nu} + \left(P_1(0) - \frac{\mu}{\mu + \lambda + p\nu}\right) e^{-(\mu+\lambda+p\nu)t}. \tag{4.4}$$

The quantity $P_0(t) \in [0, 1]$ in (4.3) denotes the probability of finding one agent unemployed at any future point t, given the probability of being unemployed at $t = 0$, $P_0(0) \in [0, 1]$. The first term on the right-hand side of the equation expresses the long-run probability $P_0^* = \frac{\lambda+p\nu}{\mu+\lambda+p\nu}$. This summand remains in the long run, while the second term vanishes when the considered time point approaches infinity, $t \to \infty$. In the latter case, the initial probability $P_0(0)$ plays no role in the long-run distribution as expected. The long-run probability P_0^* increases with the job-separation rate, λ, and with the parameters of the aggregate shock, ν and

[5] We denote the *long-run* covariance, as well as other *long-run* quantities, by adding an asterisk to them, i.e. cov^*.

p, but falls with the job-matching rate μ. We interpret these obvious results as follows. Individuals are more likely observed unemployed in the long run if they constantly suffer from idiosyncratic and aggregate negative shocks. If, however, a match between an unemployed worker and an employer occurs often, the worker will be more probably found in the employment state in the long run.

Similar reasoning applies to (4.4) which describes the probability of observing an individual in the employment state, $P_1(t)$. This probability also evolves from its initial value $P_1(0)$ to the long-run quantity $P_1^* = \frac{\mu}{\mu+\lambda+p\nu}$. The change rate of $P_1(t)$ at time t is proportional to the level of $P_1(t)$ itself. The proportion coefficient is negative and given by $-(\mu+\lambda+\nu p)$. The long-run value P_1^* obviously increases in the job-matching rate μ, but falls in the job-separation rates λ and νp caused by the idiosyncratic and aggregate shocks respectively. The economic interpretation is similar to one given above for P_0^*.

Interestingly, probabilities $P_0(t)$ and $P_1(t)$ become identical to well-known probabilities of a birth-death process when there is no aggregate shock, i.e. when $p = 0$ or $\nu = 0$ (see Ross, 1993, Chapter 5).

4.2.3 Many individuals

We follow a straight-forward approach to obtain the distribution of the number of unemployed $U(t)$ and its properties. The idea behind this approach is based on two key ingredients of the model, initial probabilities $P_0(0)$ and the processes (4.1). Obviously, probabilities $P_0(t)$ of two agents with the identical initial probability $P_0(0)$ evolve over time in the same way due to identical dynamics defined by the processes (4.1). Since by the construction of the processes which are common to all individuals.

To illustrate our aggregation approach, we consider an economy with only two individuals. Suppose the first individual is initially unemployed and the second individual is initially employed, i.e. their initial probabilities $P_0^1(0) = 1$ and $P_0^2(0) = 0$ and $U(0) = 1$. Probabilities $P_0^1(t)$ and $P_0^2(t)$ follow (4.3) and are known at every point in time t. The discrete random variable $U(t)$ can take only three values, $U(t) \in \{0, 1, 2\}$, meaning that none of two agents is unemployed, only one of them is unemployed or both are unemployed.

$$\begin{cases} P(U(t) = 0) = [1 - P_0^1(t)] \cdot [1 - P_0^2(t)] \\ P(U(t) = 1) = P_0^1(t) \cdot [1 - P_0^2(t)] + [1 - P_0^1(t)] \cdot P_0^2(t) \\ P(U(t) = 2) = P_0^1(t) \cdot P_0^2(t) \end{cases} \tag{4.5}$$

In our example we assumed that only one agent is unemployed at $t = 0$. This result follows from the previous system and is reflected by $P(U(0) = 1) = 1$.

This system can obviously be extended to any number of agents N with any initial conditions $P_0^i(0) \in \{0,1\}\ \forall i = 1,2,\ldots,N$. This system consists of $N+1$ equations. We can also make some simplification by dividing all agents into two groups, initially unemployed and initially employed. Suppose we have n agents initially unemployed (group I) and $N-n$ agents initially employed (group II). Then the first and last equations of the system (4.5) can be simplified as follows[6]

$$\begin{cases} P(U(t)=0) & = \left[1-P_0^I(t)\right]^n \cdot \left[1-P_0^{II}(t)\right]^{N-n} \\ & \ldots \\ P(U(t)=N) & = \left[P_0^I(t)\right]^n \cdot \left[P_0^{II}(t)\right]^{N-n}. \end{cases} \tag{4.6}$$

For further derivations we denote the number of unemployed agents at time point t by $U \equiv U(t)$. We recall that value U is a discrete random variable, $U \in [0,N]$, but time t is continuous. Therefore, the function $P(U)$ is a probability mass function which is also discrete in U, but continuous in t.[7]

Number of unemployed

There are two ways to obtain the expected value of the distribution of U. First, by definition, it is given by[8]

$$E[U] = \sum_{U=0}^{N} U \cdot P(U).$$

Second, we might choose a simpler route that consists in writing down the number of unemployed at each point in time, U, via the state variables x_i and then apply the expectation operator to it. In doing so, U can be written as follows:

$$U = N - \sum_{i=1}^{N} x_i(t), \tag{4.7}$$

from which the mean value $E[U]$ is given by (see Appendix 4.D):

$$E[U] = NP_0(t), \tag{4.8}$$

[6]When writing this system, we simplify the aggregation process. The system is valid only when the employment states of agents at t are independent, i.e. when $\nu = 0$. Moreover, because of this simplification, we obtain the binomial probability mass functions presented in the figures given below, 4.3 and 4.5.

[7]The phrase "probability distribution" refers to the "probability mass function" hereafter.

[8]We use the expectation operator from perspective of time point 0.

where the aggregates shock obviously influences not only the short-run expected value of the number of unemployed agents, but also the long-run one,

$$E^*[U] = N\frac{\lambda + p\nu}{\mu + \lambda + p\nu}$$

in the same way as it affects the probability of observing an agent in the unemployment state, $P_0(t)$. It means that $E^*[U]$ increases in both p and ν.

The effect of the aggregate shock is certainly present in the variance of the number of unemployed agents, $var[U]$. Variance with the initial state of an agent being random is given by (see also Appendix 4.D):

$$var[U] = NP_0(t)[1 - P_0(t)] + N(N-1)cov(x_i, x_j). \tag{4.9}$$

Interestingly, covariance between states x_i and x_j plays an important role entering into variance of the number of unemployed individuals. Obviously, without the aggregate shock, variance $var[U]$ vanishes when the number of agents in the economy rises. The covariance term however remains in the result which makes $var[U]$ to become strictly positive.

Unemployment rate

Then the unemployment rate defined as $u \equiv u(t)$

$$u = \frac{U(t)}{N} \tag{4.10}$$

reveals similar properties. The probability mass function of the unemployment rate, $P(u)$, is related one-to-one to the probability distribution of the number of unemployed agents, $P(U)$. It means that in order to find $P(u)$ we first find $U = uN$ and corresponding probability $P(U)$. We then equate $P(u) = P(U)$.

The expected value of u given by (see Appendix 4.E):

$$E[u] = P_0(t), \tag{4.11}$$

which long-run value

$$E^*[u] = \frac{\lambda + p\nu}{\mu + \lambda + p\nu} \tag{4.12}$$

also rises in p and ν. Variance:

$$var[u] = \frac{1}{N^2}var[U] = \frac{1}{N}P_0(t)[1 - P_0(t)] + \left(1 - \frac{1}{N}\right)cov(x_i, x_j) \tag{4.13}$$

depends on covariance between states x_i and x_j (see its long-run value in (4.2)).

This result nicely shows that variance of the unemployment rates without aggregate shocks ($p = 0$ or $\nu = 0$) would approach zero as the number of workers in a country, N, rises. The long-run variance of the unemployment rates obviously reveals the same property

$$var^*[u] = \frac{1}{N}\frac{\mu(\lambda + p\nu)}{(\mu + \lambda + p\nu)^2} + \left(1 - \frac{1}{N}\right)\frac{\mu^2 p^2 \nu}{\chi(\mu + \lambda + p\nu)^2}, \tag{4.14}$$

where $\chi = 2(\mu + \lambda) + p(2 - p)\nu$.

4.2.4 Many countries

We try to explain why in reality the unemployment rates in European countries are not identical. The difference in the unemployment rates across countries might be due to dissimilarity in regulation of labor markets. A chance might also partially explain this difference. It means that the unemployment rates are taken from the same distribution which is characterized by variance, $var[u]$, from (4.13), and what we observe is just realizations from this distribution. If we moreover assume that all countries are large in population, i.e. $N \to \infty$, variance of the unemployment rates across countries might be a result of correlation between agents' employment states. Correlation then appears due to the presence of the aggregate shocks, e.g. due to an economic crisis. In summary, the results in this paper indicate that the aggregate shocks might cause dissimilarity in the unemployment rates across European countries.

4.3 Quantitative relevance

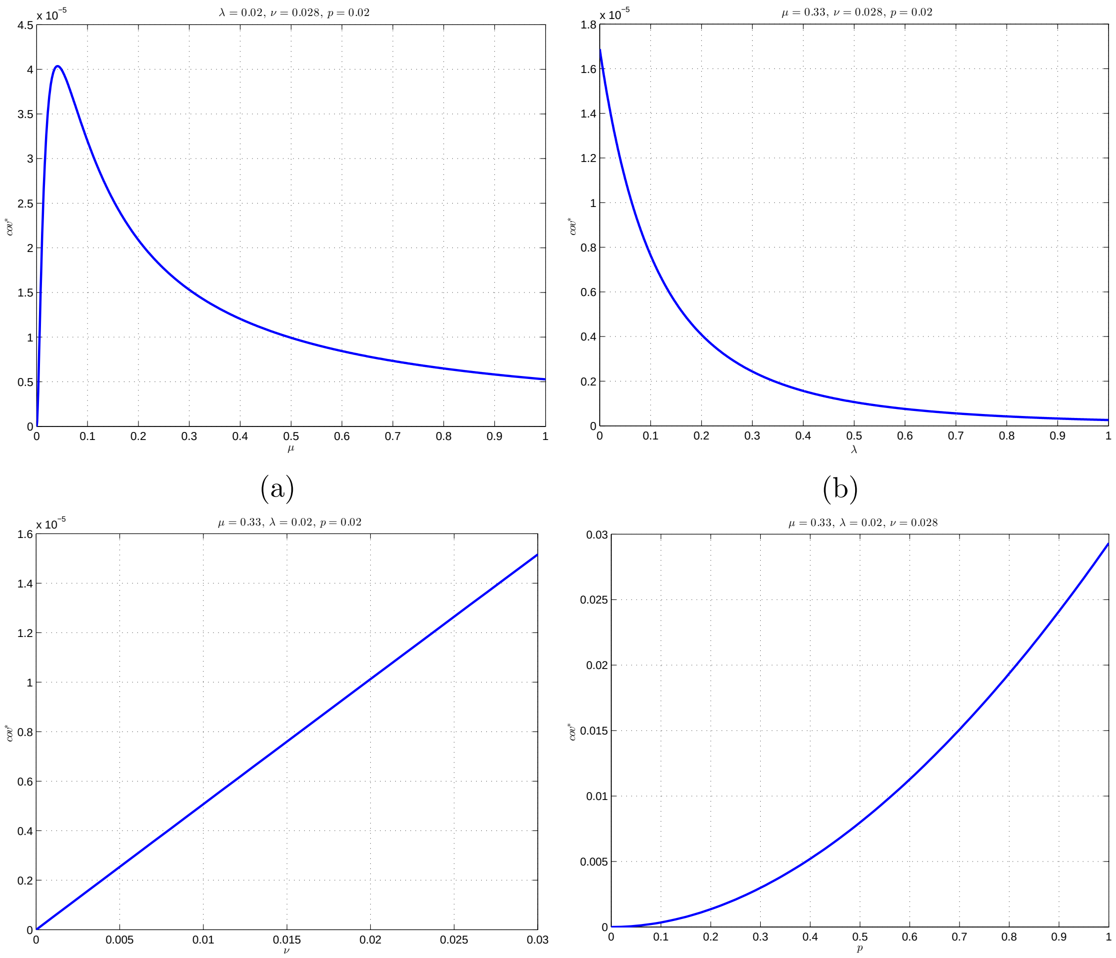
In this section we perform the quantitative analysis of the long-run covariance (4.1), of the dynamics of the individual probabilities (4.3) and (4.4), of the number of unemployed agents (4.6) and of the unemployment rate over time. We also study the effect of an increase in the number of agents in the economy, N, on aggregate quantities U and u. Out ultimate goal is to show that the distribution of the unemployment rates does not vanish when aggregate shocks are present.

4.3.1 Long-run covariance

Figures 4.1 complete the analysis of long-run covariance given by (4.2). First, Figure 4.1a shows that the long-run covariance is a non-monotonic function with respect to the job-matching rate μ.[9] Second, the long-run covariance is a decreasing function

[9]We keep all other parameters of the model constant while changing the job-matching rate μ (see Figure 4.1a). We proceed in the same way when analyzing how $cov^*(x_i, x_j)$ changes with

in the job-separation rate λ (see Figure 4.1b). Interestingly, $cov^*(x_i, x_j)$ approaches zero when either of μ and λ take high values. In these situations aggregate shocks do not influence long-run covariance much since almost all movements in the labor market are solely characterized by the idiosyncratic shocks. This leads to zero long-run covariance, i.e. $cov^*(x_i, x_j) = 0$.



(a) (b) (c) (d)

Figure 4.1: Change in the long-run covariance with respect to parameters of the model, (a) μ, (b) λ, (c) ν and (d) p.

Third, Figures 4.1c and 4.1d present how the properties of the aggregate shocks affect the long-run covariance. In particular, they show an increase in the long-run covariance $cov^*(x_i, x_j)$ when both ν and p rise. This is certainly quite intuitive. Higher values of ν and p lead to a higher effect of the aggregate shocks on the individual movements. In these cases reallocations of individuals from employment to unemployment states are solely caused by the aggregate shocks. The long-run covariance $cov^*(x_i, x_j)$ therefore rises.

respect to λ, ν and p (see Figures 4.1b, 4.1c and 4.1d respectively).

4.3.2 Individual probabilities

Figure 4.2 show dynamics of both probabilities $P_0(t)$ and $P_1(t)$ given by (4.3) and (4.4) where the time unit represents one month. The dynamics show that the agent unemployed at $t = 0$ almost surely becomes employed in 20 months.

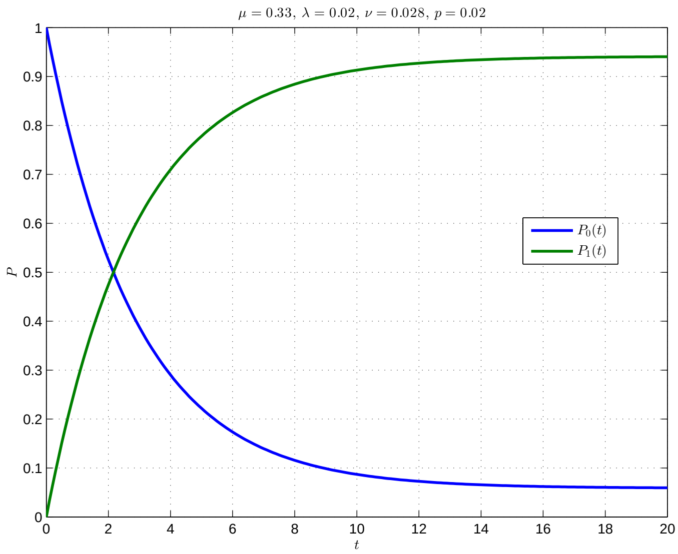


Figure 4.2: Evolution of the probabilities of observing an individual in unemployment and employment states over time, $P_0(t)$ and $P_1(t)$ respectively. The described agent is initially unemployed, i.e. $P_0(0) = 1$ and $P_1(0) = 0$.

The probabilities $P_0(t)$ and $P_1(t)$ represent complementary events: an individual is either unemployed or employed. It means that $P_0(t) + P_1(t) = 1$, $\forall t \geq 0$, in particular, $P_0(0)+P_1(0) = 1$ and $P_0^* + P_1^* = 1$. We exploit this elementary property quite often in derivations.

4.3.3 Number of unemployed agents

As we have shown above, the distribution of the number of agents is described by (4.6), has its mean (4.8) and variance (4.9). We now present how this distribution and its properties change over time t as predicted by our model.

We consider an economy with the labor market being initially in a bad state. By the bad state, we understand the high initial individual probability of being unemployed, $P_0(0) = 0.1$. The total number of agents in the economy is set to $N = 1000$. We also set the job-matching rate equal to $\mu = .33$ which means that an unemployed worker gets a new job on average after 3 months $\left(\frac{1}{\mu} = 3\right)$. The job-separation rate is then $\lambda = .02$ that refers on average to 50 months of continuous employment $\left(\frac{1}{\lambda} = 50\right)$.

Given all these parameters, we are able to plot the dynamics of the probability mass function from (4.6) for the economy with $N = 1000$ agents within the two-year time period, i.e. $t \in [0, 24]$, where time units are measured in months (see Figure 4.3).

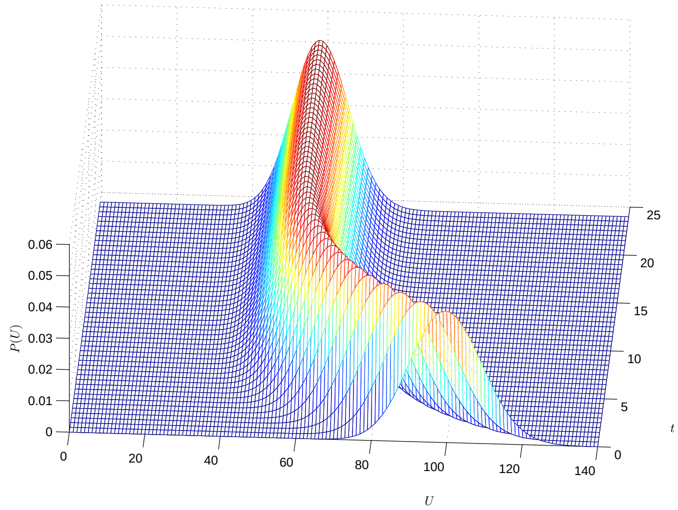


Figure 4.3: The theoretically predicted dynamics of the distribution of the number of unemployed workers, U, over time t.

Figure 4.3 shows dynamics of the distribution of U. We observe that the great mass of this distribution shifts towards zero which means that the average number of unemployed agents decreases over time. This fact can be better seen in Figure 4.4 where the mean value $E[U]$ drops from 100 unemployed agents to approximately 59 within the two-year time period and remains at its steady state thereafter.

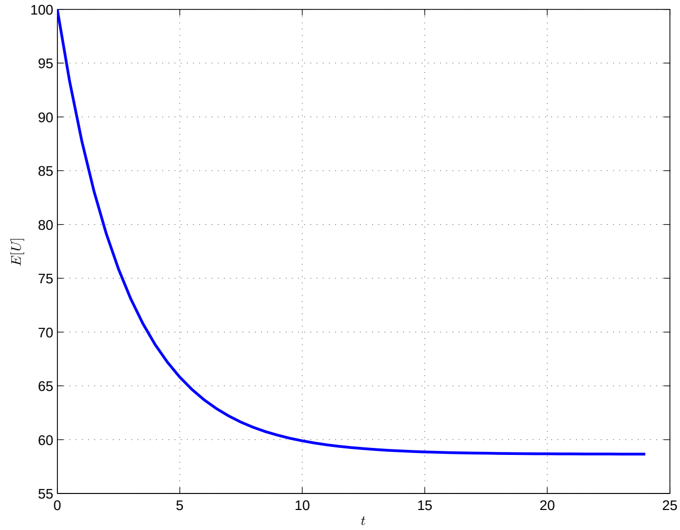


Figure 4.4: Dynamics of the expected value of the number of agents, $E[U]$ over time t.

4.3.4 Unemployment rates

The dynamics of the distribution of the unemployment rates, $P(u)$, depends on $P(U)$ according to (4.10). To plot the dynamics of $P(u)$ as discussed in Section

4.2, we map $P(U)$ from one support, $U \in [0, N]$, onto another one, $u \in [0, 1]$ (see Figure 4.5).

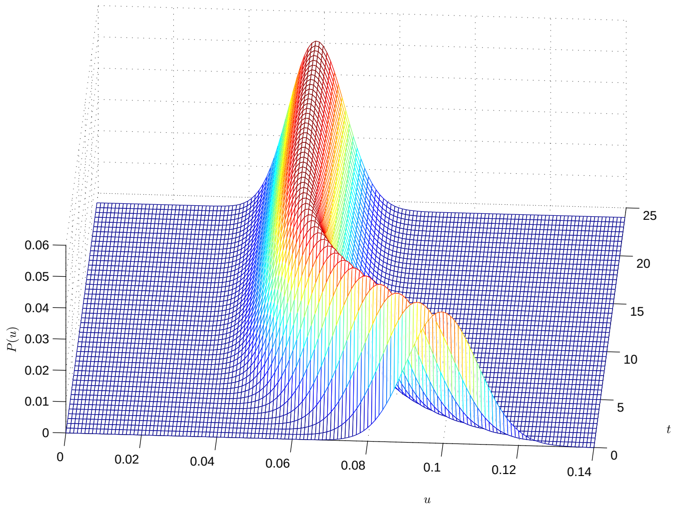


Figure 4.5: The theoretically predicted dynamics of the distribution of the unemployment rates, $u(t)$.

The long-run average unemployment rate, $E^*[u]$, is a quantity that we can compare to the real data. This quantity depends on the long-run distribution of u which, in turn, heavily relies on presence of the aggregate shocks and on the total number of agents in the economy.

The shape of the long-run distribution of unemployment rates plays a crucial role in our analysis. We now obtain this shape for different numbers of workers. In doing so, we continuously increase the total number of workers, N, to study the effect of this rise on the long-run distribution of unemployment rates (see Figure 4.6).

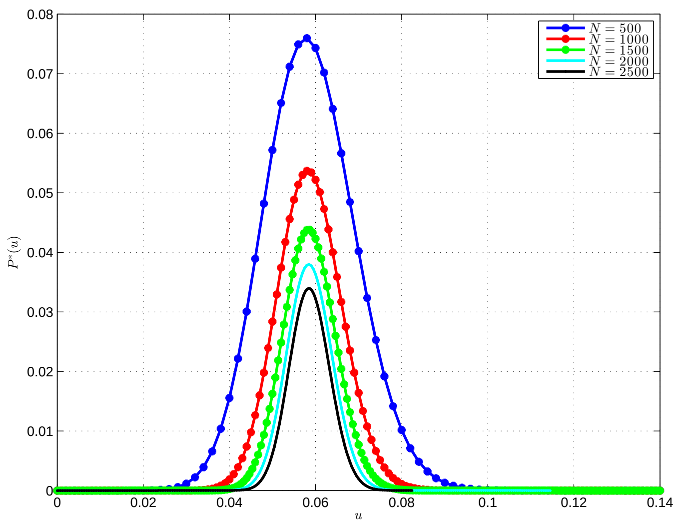


Figure 4.6: The theoretically predicted long-run distribution of the unemployment rates, $P^*(u)$, for the different total number of agents N.

Figure 4.6 shows the terminal distribution of the unemployment rates after the two-year time period that also can be seen in Figure 4.5 at $t = 24$. This distribution is plotted for $N = 500, 1000, 1500, 2000, 2500$ with "more and more ticks" between 0 and 1 as the number of agents increases. This distribution converges to the steady state distribution described via $E^*[u]$ and $var^*[u]$ in (4.12) and (4.15) respectively.

Interestingly, the distribution of the unemployment rates does not vanish in the long run for the high number of firms as if it would do in the case of no aggregate shocks. According to the discussion that follows equation (4.13), the increase in N together with absence of the aggregate shocks leads to the zero variance of the unemployment rates. This obviously cannot explain the real unemployment rate variation across the OECD member countries. Out model however shows that the presence of the aggregate shocks makes it possible.

4.3.5 Aggregate shocks – Comparing to Europe

Since the population of the OECD member counties is large, we assume $N \to \infty$. In this case variance of the unemployment rates coincides with covariance according to (4.13), i.e. (see Appendix 4.C)

$$\lim_{N \to \infty} var[u] = cov(x_i, x_j), \tag{4.15}$$

where covariance is fully explained by presence of the aggregate shocks. It means that without shocks of this type, covariance $cov(x_i, x_j)$ and variance $var[u(t)]$ vanish:

$$\lim_{p \to 0} cov(x_i, x_j) = 0,$$
$$\lim_{\nu \to 0} cov(x_i, x_j) = 0.$$

Thus, the long-run variance for each OECD member country is finite and given by:

$$var^*[u(t)] = \frac{\mu^2 p^2 \nu}{\chi(\mu + \lambda + p\nu)^2}.$$

This result is quite impressive. The long-run variance $var^*[u(t)]$ helps us to interpret why the unemployment rates in various OECD member countries change in a different way despite similar regulation changes.

4.3.6 Distributions of unemployment rates in OECD countries

Let us now ask how relevant our results are from a quantitative perspective in order to understand cross-country variation in the unemployment rates. Consider the evolution of the distribution of unemployment rates in OECD member countries in the following figure.

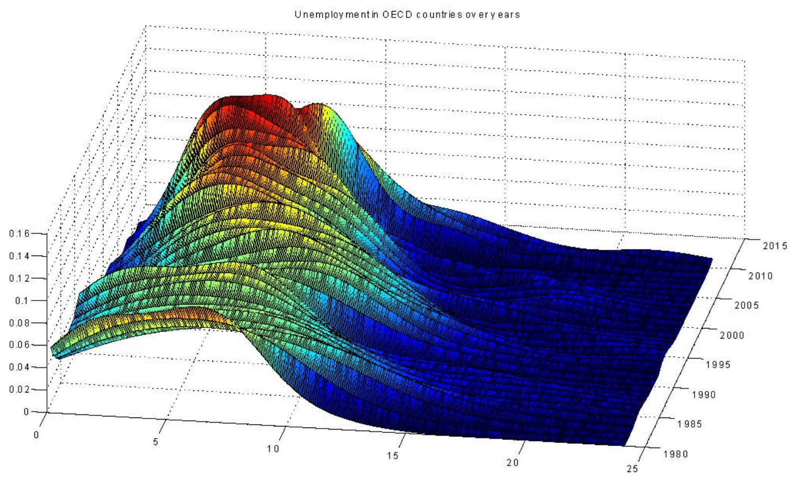


Figure 4.7: Densities of standardized unemployment rates for OECD member countries between 1980 and 2011. Rates are given in percentage points.

Some summary statistics are as follows: The average unemployment rate is .05, the variance is .0016 (see London Institutional Database, Layard and Nickell, 2006, see more details in Appendix 4.F).

To make this figure comparable to Figure 4.5, we need to go beyond the one-country analysis and consider many OECD member countries. The number of unemployed agents in each of these countries is assumed to be distributed as described by (4.6).

4.3.7 Quantitative check of long-run empirical and theoretical values

Absence of aggregate shocks

We can now ask how large the number of workers, N, needs to be in one country such that, in the absence of the aggregate shocks, the empirical mean and the theoretical mean given by (4.11), empirical variance and theoretical variance given by (4.13) coincide. So we require

$$E[u] = P_0(t) = .05,$$
$$var[u] = \frac{P_0(t)(1 - P_0(t))}{N} = .0016.$$

This system gives $N \approx 30$. In other words, the theoretical model can replicate the empirical mean and variance of the distribution of unemployment rates across countries if the number of workers per country equals 30.

As the real number of workers per country is by a factor of about one million larger, the variance predicted by the theoretical model is by around one million smaller. In other words, we observe that 0% of the observed heterogeneity in unemployment rates across countries can be replicated by pure chance and only institutions make them different across countries.

Presence of aggregate shocks

To check the quantitative relevance of our model, we need to choose plausible values for the arrival rates. They are shown in the following table

μ	λ	ν	p
.33	.02	1/36	.02

Table 4.1: Values of parameters of the model. The time unit is one month.

These values imply that the average duration of unemployment is $1/\mu = 3$ months, individuals lose their jobs on average every $1/\lambda = 50$ months, an aggregate shock occurs every 3 years and this shock makes 2% of the currently employed workers unemployed. The long-run average unemployment rate is given by 5.71% in the absence of aggregate shocks ($\nu = 0$ or $p = 0$) and by 5.86% in the presence of aggregate shocks. The aggregate shocks explain only 1% of the long-run variance (0.0016) because the long-run covariance from (4.2) equals $1.4 \cdot 10^{-5}$. The effect of the aggregate shocks becomes stronger if this type of shocks occurs more often or it hits more agents (see Figures 4.1c and 4.1d).

The effect of aggregate shocks on the distribution of the unemployment rates is of course more relevant in the short run. In order to observe this effect, a deep analysis of frequently collected data has to be performed. These data must include a time period just after an aggregate shock occurs. This analysis is obviously outside of the scope of the current theoretical work.

4.4 Conclusion

In this paper we present the model that extends the standard textbook models with frictions in the labor market. In doing so, we assume that all individual movements in the labor market happen due to both idiosyncratic and aggregate shocks. We build the stochastic differential equation to model these movements. As the result, we obtain the dynamics of the probability mass function that describes the probability of observing one individual either in the employment or unemployment state at each point in time. Aggregation of all workers in the labor market allows us to predict the evolution of corresponding probability mass functions of the number of unemployed agents and of the unemployment rate. After obtaining these results, we discuss the effects of the idiosyncratic and aggregate shocks on the situation in the labor market.

Our findings shed some light on the variation in the unemployment rates across OECD member countries. We claim that the difference in the unemployment rates among these countries in the short-run happens by *chance* caused by the aggregate shocks. This partially explains the difference in the unemployment rates among OECD member countries. The policy however is important in the long-run. Therefore, according to our model, the long-run unemployment rate should be identical in countries that apply the same labor market reforms.

As a result, we find that, in contrast to the standard textbook model, our model can predict the long-run average unemployment rate at 5.86%, and that only 1% of the observed long-run variation in unemployment rates can be explained by aggregate shocks. The remaining 99% of variation is due to policy. The model moreover foretells that the effect of aggregate shocks on the distribution of unemployment rates is more significant in the short-run dynamics, i.e. just after aggregate shocks occur.

Appendix

4.A Probabilities

Consider the process

$$dx_i(t) = (1 - x_i)dq_\mu^i - x_i \left[dq_\lambda^i + \sigma_i dq_\nu\right]. \tag{4.16}$$

We prefer to express it by using the indicator function and by suppressing the index i,

$$dx(t) = \mathbf{1}_{x=0}dq_\mu - \mathbf{1}_{x=1}\left[dq_\lambda + \sigma dq_\nu\right].$$

We would like to understand the distribution of $x(t)$.

The probabilities for $x(t)$

To this end, compute the differential for some given function $f(.)$,

$$\begin{aligned} df(x(t)) =& [f(x + \mathbf{1}_{x=0}) - f(x)]dq_\mu + [f(x - \mathbf{1}_{x=1}) - f(x)]dq_\lambda \\ &+ [f(x - \mathbf{1}_{x=1}\sigma) - f(x)]dq_\nu. \end{aligned}$$

Taking into account that $f(x+\mathbf{1}_{x=0}) = f(1)$, $f(x-\mathbf{1}_{x=1}) = f(0)$, and $f(x-\mathbf{1}_{x=1}\sigma) = pf(0) + (1-p)f(x)$, $\forall x \in \{0, 1\}$, we write

$$dEf = [f(1) - f(x)]\mu dt + [f(0) - f(x)]\lambda dt + [pf(0) + (1 - p)f(x) - f(x)]\nu dt,$$

$$\begin{aligned} \mathcal{A}(f) \equiv \frac{dE[f]}{dt} =& [f(1) - f(x)]\mu + [f(0) - f(x)]\lambda \\ &+ [pf(0) + (1 - p)f(x) - f(x)]\nu, \end{aligned}$$

$$\begin{aligned} E[\mathcal{A}(f)] =& P(t, x = 0)\left\{[f(1) - f(0)]\mu + [pf(0) + (1 - p)f(0) - f(0)]\nu\right\} \\ &+ P(t, x = 1)\left\{[f(0) - f(1)]\lambda + p[f(0) - f(1)]\nu\right\} \\ =& P(t, x = 0)[f(1) - f(0)]\mu + P(t, x = 1)[f(0) - f(1)]\left\{\lambda + p\nu\right\}. \end{aligned}$$

On the other hand,

$$\frac{dE[f]}{dt} = \frac{\partial P(t, x = 0)}{\partial t}f(0) + \frac{\partial P(t, x = 1)}{\partial t}f(1).$$

To simplify notation we define

$$P_0(t) \equiv P(t, x = 0),\ P_1(t) \equiv P(t, x = 1), \tag{4.17}$$

$\dot{P}_0 \equiv \frac{\partial P(t,x=0)}{\partial t}$ and $\dot{P}_1 \equiv \frac{\partial P(t,x=1)}{\partial t}$.

The Dynkin formula states that

$$E[\mathcal{A}(f)] = \frac{dE[f]}{dt}.$$

Then, combining terms for $f(0)$, we obtain

$$\dot{P}_0 = -\mu P_0 + \lambda + p\nu P_1,$$

and for $f(1)$

$$\dot{P}_1 = \mu P_0 - (\lambda + p\nu) P_1,$$

or as a system:

$$\begin{cases} \dot{P}_0 & = -\mu P_0 + \xi P_1 \\ \dot{P}_1 & = \mu P_0 - \xi P_1 \\ 1 & = P_0 + P_1 \end{cases}$$

where ξ is defined as

$$\xi \equiv \lambda + p\nu.$$

We next solve the system by replacing P_1 in the first equation using the third one,

$$\dot{P}_0 = \xi - (\mu + \xi) P_0,$$

$$P_0(t) = C_0(t) e^{-(\mu+\xi)t},$$

$$C_0(t) = \frac{\xi}{\mu + \xi} e^{(\mu+\xi)t} + C,$$

where $C = P_0(0) - \frac{\xi}{\mu+\xi}$. Thus,

$$P_0(t) = \frac{\xi}{\mu + \xi} + \left(P_0(0) - \frac{\xi}{\mu + \xi} \right) e^{-(\mu+\xi)t}, \quad (4.18)$$

$$P_1(t) = \frac{\mu}{\mu + \xi} + \left(P_1(0) - \frac{\mu}{\mu + \xi} \right) e^{-(\mu+\xi)t}, \quad (4.19)$$

Note that this allows for stochastic initial conditions $P_0(0)$.

Long-run properties

Obviously as $\xi > 0$, we find for the long run, i.e. for $t \to \infty$,

$$P_0^* = \frac{\xi}{\mu + \xi}, \quad (4.20)$$

$$P_1^* = \frac{\mu}{\mu + \xi}. \quad (4.21)$$

4.B Mean and variance of $x(t)$

By definition and using (4.17)

$$E[x(t)] = P_0(t) \cdot 0 + P_1(t) \cdot 1 = P_1(t). \tag{4.22}$$

In the long run,

$$E^*[x] = \frac{\mu}{\mu + \xi}.$$

Similarly, the variance is defined by

$$var[x(t)] = E[x^2(t)] - (E[x(t)])^2.$$

We can compute

$$E[x^2(t)] = P_0(t) \cdot 0 + P_1(t) \cdot 1 = P_1(t).$$

Hence,

$$var[x(t)] = P_1(t) - P_1^2(t).$$

In the long run,

$$var^*[x(t)] = \frac{\mu\xi}{(\mu + \xi)^2}.$$

4.C Covariance

General derivations

Computing the covariance is not as straightforward as computing the mean or variance. We start from the definition

$$cov(x_i, x_j) = E[x_i x_j] - E[x_i]E[x_j] = E[x_i x_j] - (P_1(t))^2.$$

To find $E[x_i x_j]$ we define $f(x_i, x_j) = x_i x_j$, where

$$dx_i(t) = (1 - x_i)dq_\mu^i - x_i[dq_\lambda^i + \sigma_i dq_\nu]$$
$$dx_j(t) = (1 - x_j)dq_\mu^j - x_j[dq_\lambda^j + \sigma_j dq_\nu]$$

Ito's lemma gives

$$\begin{aligned} df(x_i, x_j) =& [(x_i + 1 - x_i)x_j - x_i x_j]dq_\mu^i \\ &+ [(x_i - x_i)x_j - x_i x_j]dq_\lambda^i \\ &+ [x_i(x_j + 1 - x_j) - x_i x_j]dq_\mu^j \\ &+ [x_i(x_j - x_j) - x_i x_j]dq_\lambda^j \\ &+ [(x_i - \sigma_i x_i)(x_j - \sigma_j x_j) - x_i x_j]dq_\lambda^j \end{aligned}$$

With $f(x_i, x_j) = x_i x_j$, we get

$$\begin{aligned} d[x_i(t)x_j(t)] &= x_j(t)dq_\mu^i(t) + x_i(t)dq_\mu^j(t) \\ &\quad - x_i(t)x_j(t)\{dq_\mu^i(t) + dq_\mu^j(t) + dq_\lambda^i(t) + dq_\lambda^j(t) + (\sigma_i + \sigma_j - \sigma_i\sigma_j)dq_\nu(t)\}. \end{aligned}$$

Note that if $\sigma_i = \sigma_j = 0$, dq_ν has no influence on x_i, x_j, and $x_i x_j$.

Writing this as an integral equation yields

$$\begin{aligned} x_i(t)x_j(t) - x_i(0)x_j(0) =& \int_0^t x_j(\tau)dq_\mu^i(\tau) + \int_0^t x_i(\tau)dq_\mu^j(\tau) \\ &- \int_0^t x_i(\tau)x_j(\tau)\{dq_\mu^i(\tau) + dq_\mu^j(\tau) + dq_\lambda^i(\tau) + dq_\lambda^j(\tau) \\ &+ (\sigma_i + \sigma_j - \sigma_i\sigma_j)dq_\nu(\tau)\}. \end{aligned}$$

Forming expectations and using the martingale result for Poisson processes(see García and Griego, 1994 or Wälde, 2012)

$$\begin{aligned} E[x_i(t)x_j(t)] - x_i(0)x_j(0) =& \int_0^t Ex_j(\tau)\mu ds + \int_0^t Ex_i(\tau)\mu d\tau \\ &- \int_0^t E[x_i(\tau)x_j(\tau)]\{\mu + \mu + \lambda + \lambda + E[\sigma_i + \sigma_j - \sigma_i\sigma_j]\nu\}ds. \end{aligned}$$

Note that the latter also used that $\sigma_k(t)$ are independent of $x_l(t)$ for all k and l. As $E\sigma_i = p$ and $E\sigma_i\sigma_j = E\sigma_i E\sigma_j = p^2$ as σ_i and σ_j are independent (while the aggregate jump is of course the same for both i and j), we get

$$\begin{aligned} E[x_i(t)x_j(t)] - x_i(0)x_j(0) =& \int_0^t Ex_j(\tau)\mu d\tau + \int_0^t Ex_i(\tau)\mu d\tau \\ &- \int_0^t E[x_i(\tau)x_j(\tau)]\left\{2(\mu + \lambda) + (2p - p^2)\nu\right\} d\tau. \end{aligned}$$

Computing the time t derivatives yields

$$\frac{d}{dt}E[x_i(t)x_j(t)] = (Ex_j(t) + Ex_i(t))\mu + E[x_i(t)x_j(t)]\chi,$$

where $\chi \equiv 2(\mu+\lambda)+(2p-p^2)\nu$. Defining $\Omega(t) \equiv E[x_i(t)x_j(t)]$ and as $Ex(t) = P_1(t)$ for any i from (4.22), we get

$$\frac{d}{dt}\Omega(t) = -\chi\Omega(t) + 2\mu P_1(t).$$

This is a standard inhomogeneous linear ODE for $\Omega(t)$ whose solution is (see e.g. Wälde, 2012, and Braun, 1975)

$$\Omega(t) = \Omega(0)e^{-\chi t} + 2\mu \int_0^t e^{-\chi[t-\tau]} P_1(\tau) d\tau.$$

When we plug in (4.18) for $P_1(\tau)$, we obtain

$$\Omega(t) = \Omega(0)e^{-\chi t} + 2\mu \int_0^t e^{-\chi[t-\tau]} \left(\frac{\mu}{\mu+\xi} + \left(P_1(0) - \frac{\mu}{\mu+\xi} \right) e^{-(\mu+\xi)\tau} \right) d\tau.$$

Computing the integral, we obtain:

$$\begin{aligned}\Omega(t) = E[x_i(t)x_j(t)] =& E[x_i(0)x_j(0)]e^{-\chi t} + \frac{2\mu^2}{(\mu+\xi)\chi}\left(1 - e^{-\chi t}\right) \\ &+ \left[P_1(0) - \frac{\mu}{\mu+\xi}\right] \frac{2\mu}{\chi - (\mu+\xi)} \left(e^{-(\mu+\xi)t} - e^{-\chi t}\right)\end{aligned}$$

Hence, we can write the covariance as

$$cov(x_i, x_j) = \Omega(t) - (P_1(t))^2.$$

The long-run properties

$$E^*[x_i x_j] = \frac{2\mu^2}{(\mu+\xi)\chi}$$

Long-run covariance:

$$\begin{aligned}cov^*(x_i, x_j) =& \frac{2\mu^2}{(\mu+\xi)\chi} - \left[\frac{\mu}{\mu+\xi}\right]^2 \\ =& \frac{\mu}{\mu+\xi}\left[\frac{2\mu}{\chi} - \frac{\mu}{\mu+\xi}\right] \\ =& \frac{\mu}{\mu+\xi} \frac{2\mu^2 + 2\mu s + 2\mu p\nu - 2\mu^2 - 2\mu s + p(2-p)\nu\mu}{\chi(\mu+\xi)} \\ =& \frac{\mu^2 p^2 \nu}{\chi(\mu+\xi)^2}\end{aligned}$$

Check: $p = 0$

When $p = 0$, we obtain $\xi = \mu + \lambda$, $\chi = 2\mu + 2\lambda$.

$$E[x_i x_j] = \left(\frac{\mu}{\mu+\lambda}\right)^2 \left(1 - e^{-2(\mu+\lambda)t}\right) + \left[E[x(0)] - \frac{\mu}{\mu+\lambda}\right] \frac{2\mu}{\mu+\lambda} \left(e^{-(\mu+\lambda)t} - e^{-2(\mu+\lambda)t}\right) + E[x_i(0)x_j(0)]e^{-2(\mu+\lambda)t}$$

Note that:

$$E[x_i]E[x_j] = \left(\frac{\mu}{\mu+\lambda}\right)^2 + \left[E[x(0)] - \frac{\mu}{\mu+\lambda}\right] \frac{2\mu}{\mu+\lambda} e^{-(\mu+\lambda)t} + \left(\frac{\mu}{\mu+\lambda}\right)^2 e^{-2(\mu+\lambda)t} - 2E[x(0)]\frac{\mu}{\mu+\lambda} e^{-2(\mu+\lambda)t} + E[x_i(0)x_j(0)]e^{-2(\mu+\lambda)t}$$

Showing that:

$$-\left(\frac{\mu}{\mu+\lambda}\right)^2 e^{-2(\mu+\lambda)t} - \left[E[x(0)] - \frac{\mu}{\mu+\lambda}\right] \frac{2\mu}{\mu+\lambda} e^{-2(\mu+\lambda)t} = \left(\frac{\mu}{\mu+\lambda}\right)^2 e^{-2(\mu+\lambda)t} - 2E[x(0)]\frac{\mu}{\mu+\lambda} e^{-2(\mu+\lambda)t},$$

we obtain that $E[x_i x_j] = E[x_i]E[x_j]$ and $cov(x_i, x_j) = 0$ when $p = 0$. Similar results are obtained when $\nu = 0$. Thus,

$$\lim_{p \to 0} cov(x_i, x_j) = 0$$

$$\lim_{\nu \to 0} cov(x_i, x_j) = 0$$

4.D Unemployment

Given in (4.7):

$$U(t) = N - \sum_{i=1}^{N} x_i(t)$$

Mean

$$E[U(t)] = N - NE[x_i(t)] = N[1 - P_1(t)] = NP_0(t),$$

where $E[x_i(t)] = P_1(t)$ (see Appendix 4.B) and $P_0(t)$ is defined in (4.18).

Variance

$$var[U(t)] = var\left[-\sum_{i=1}^{N} x_i(t)\right] = var\left[\sum_{i=1}^{N} x_i(t)\right]$$
$$= \sum_{i=1}^{N} var[x_i(t)] + \sum_{i}\sum_{j\neq i} cov[x_i, x_j]$$
$$= NP_0(t)[1 - P_0(t)] + N(N-1)cov(x_i, x_j)$$

In the forth equality we exploit that all x_i have the same initial distributions described by $P_0(0)$ and $P_1(0)$. That means that $var[x_i(t)]$ is identical for all $i = 1, 2, ..., N$. Thus, the summation of $var[x_i(t)]$ may be replaced by $Nvar[x_i(t)]$ (see Appendix 4.B). Similarly we get the expression for the second term with covariance (see Appendix 4.C).

The long-run properties

$$E^*[U(t)] = N\frac{\xi}{\mu + \xi}$$
$$var^*[U(t)] = N\frac{\mu\xi}{(\mu + \xi)^2} + N(N-1)\frac{\mu^2 p^2 \nu}{\chi(\mu + \xi)^2}$$

4.E Unemployment rate

$$E[u(t)] = P_0(t)$$
$$var[u(t)] = \frac{1}{N^2}var[U(t)] = \frac{1}{N}P_0(t)[1 - P_0(t)] + (1 - \frac{1}{N})cov(x_i, x_j)$$
$$\lim_{N\to\infty} var[u(t)] = cov(x_i, x_j)$$

The long-run properties

$$E^*[u(t)] = \frac{\xi}{\mu + \xi}$$
$$var^*[u(t)] = \frac{1}{N}\frac{\mu\xi}{(\mu + \xi)^2} + (1 - \frac{1}{N})\frac{\mu^2 p^2 \nu}{\chi(\mu + \xi)^2}$$

4.F Summary statistics for OECD unemployment rates

	Percentiles	Smallest		
1%	0	0		
5%	.007	0		
10%	.013	0	Obs	805
25%	.021	0	Sum of Wgt.	805
50%	.045		Mean	.0532497
		Largest	Std. Dev.	.040936
75%	.074	.22		
90%	.104	.227	Variance	.0016758
95%	.123	.229	Skewness	1.355119
99%	.2	.241	Kurtosis	5.418556

Table 4.2: Summary statistics of data from the LSE institutional database for OECD countries from 1960 to 2000 on an annual basis.

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Solemn Declaration

Bei der vorliegenden Arbeit handelt es sich um eine kumulative Dissertation. Kapitel 3 und 4 wurden in Zusammenarbeit mit Herrn Prof. Dr. K. Wälde verfasst. Kapitel 2 wurde von mir alleine verfasst. Ich habe nur die von mir angegebenen Quellen und Hilfsmittel benutzt. Alle Textstellen, die wörtlich oder sinngem aus veröffentlichten oder nicht veröffentlichten Schriften entnohmen sind, sowie alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht.

Alexey Cherepnev