BCJ-relations in Quantum Field Theory

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Abstract

BCJ-relations have a series of important consequences in Quantum Field Theory and in Gravity. In QFT, one can use BCJ-relations to reduce the number of independent colour-ordered partial amplitudes and to relate non-planar and planar diagrams in loop calculations. In addition, one can use BCJ-numerators to construct gravity scattering amplitudes through a "squaring" procedure. For these reasons, it is important to find a prescription to obtain BCJ-numerators without requiring a diagram by diagram approach.

In this thesis, after introducing some basic concepts needed for the discussion, I will examine the existing diagrammatic prescriptions to obtain BCJ-numerators. Subsequently, I will present an algorithm to construct an effective Yang-Mills Lagrangian which automatically produces kinematic numerators satisfying BCJ-relations. A discussion on the kinematic algebra found through scattering equations will then be presented as a way to fix non-uniqueness problems in the algorithm.

Zusammenfassung

BCJ-Relationen haben eine Reihe wichtiger Konsequenzen in der Quantenfeldtheorie und der Gravitation. In der Quantenfeldtheorie können sie dazu verwendet werden, um die Anzahl unabhängiger farbgeordneter Partialamplituden zu reduzieren und um nicht-planare mit planaren Diagrammen in Schleifenrechnungen in Beziehung zu setzen. Darüber hinaus können BCJ-Relationen verwendet werden um Streuamplituden in der Gravitation durch "Quadrieren" zu konstruieren. Aus diesen Gründen ist es wichtig eine Vorschrift zu finden, mit der sogenannte BCJ-Nenner ermittelt werden können ohne auf einzelne Diagramme zurückgreifen zu müssen.

In dieser Arbeit stelle ich zunächst einige für die darauf folgende Diskussion benötigte, grundlegende Konzepte vor und untersuche danach bekannte Methoden um BCJ-Nenner aus Diagrammen zu ermitteln. Anschließend stelle ich einen Algorithmus vor, um eine effektive Yang-Mills-Lagrangedichte zu konstruieren, welche automatisch kinematische Nenner produziert, die BCJ-Relationen erfüllen. Schließlich wird die kinematische Algebra diskutiert, die man aus Streugleichungen ermitteln kann. Diese stellt eine Möglichkeit dar, Probleme bezüglich der Nicht-Eindeutigkeit des Algorithmus zu beheben.

To the unbroken chain that links the first cells to me.

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1

INTRODUCTION

1.1 A facade for the void?

"We define only out of despair, we must have a formula... to give a facade to the void."

– Emil Cioran, *A Short History of Decay*

As I am about to start my last thesis, I would like to take some time to write about human knowledge and how this concept stands in my view of life. The uninterested reader is free to skip this rather personal and philosophically oriented digression and immediately start with the physical content of the thesis which is presented starting from section 1.2.

The evolution of the brain is composed by a long succession of gradual steps, made by random genetic mutations in gene pools which survived - or not - resulting advantageous - or not - to the creatures they belonged to. This incredibly long and slow process shaped its functions and characteristics. The three-dimensional perception of space, the perception of bodies in a solid state as uniform and connected objects, can all be possibly accounted as favorable ways of experiencing the world through our bodily receptors.
How are these peculiar ways of perceiving the world related to our scientific description of it?
In other words, are our interpretation of the external signals, our limited perception of them and our limited size influencing our knowledge? How would we be describing the world if we could see it through the eyes of a being the size of a neutrino? How would our physical models be if we could see with our own eyes what we call the microscopic world?
These are of course rhetorical questions. It is not surprising that our early models for physical systems were entirely imbued with the prejudices given by the thinking structure that evolution imprinted in us. However, human brains showed a rather impressive capacity: the ability to think abstractly. Thanks to this quality, we were able - up to a certain degree - to detach ourselves from the peculiar perceptions of material reality and their interpre-

tations made by our brain. The idea that other kinds of abstraction might be possible is still an open question: what kind of thought processes might be developed by other forms of intelligence? Would they be just another re-casted version of mathematics and logic or are they something completely different, out of our reach?
A nice thought experiment to grasp this idea, is to consider the huge differences between us and apes[1]. Our DNA is different only for a really small percentage from theirs; what can we expect to be there in the depths of space, where completely different evolutionary processes might have happened, leading to all different kinds of minds and observers? If creatures out there are as far from us as we are from apes - in dna-evoution terms - would they be seeing us and our mathematical models as we see apes and their behaviours?
This is of course nothing we can speculate about; it is still interesting though to consider this idea and to get a good perspective on the relative value of our discoveries.

Leaving this fascinating idea aside for the moment, let us focus again on human knowledge.
The amazing work of science has something rather peculiar to it. Its transient state, its continual self improvement, its clear bounds of validity, predictivity and uncertainty. In this terms, science differs from any other kind of human knowledge. Being it a human product, however, it is still fallible and prone to mistakes, especially when research is done at the very limits of our understanding and technological power. One rather subtle mistake made in the interpretation of modern physical models, especially when quantum physics enters the game, is the idea that the objects we are describing would actually exist with the shapes and characteristics our brains give them. One example is the case of modern particle physics. The two ideas that help us visualizing the objects of our studies are particles and waves. However, none of the two is a real complete description of what is really being observed. At this level of abstraction, one must work with objects that can be described to an amazing degree of precision by our mathematical models, giving up, however, any kind of intuitive perception of the observed events. We introduced the use of fields, we introduced extra dimensions, we use regularly quantum mechanics. However, all these mathematical constructions, fail to satisfy our thirst for ultimate, complete, descriptions.

In a way, one could argue that there is a kind of analogy between early physics and modern quantum physics in the level of (conscious) conceptual-

[1] See for example the speech given by Neil DeGrasse Tyson at "Cosmic Quandaries", held at The Palladium in St. Petersburg College on March 2009.

approximation involved in the models created to describe systems. In classical physics, the models are faithful to what we can observe with our sight, making everything more intuitive and satisfying from a certain point of view. However, as mentioned before, this is purely a result of the prejudices that evolution instilled in our brains. Those objects we describe are made of smaller components we cannot distinguish and of the forces that keep them together. The fact that a cup on a table does not simply slip through it, is at a classical level just a consequence of the assumed solidity of the bodies involved, while at a deeper level of sophistication, can be understood as an interaction at the molecular level of the two bodies, with the bonding forces among the molecules in one body, preventing those of the other body to slip through. What kind of deception is at work in modern physics instead? We cannot observe the objects we are discussing directly with our senses. We can however observe them indirectly through the interactions with other physical systems that we equip with understandable interfaces. This is the case for example of modern particle scattering experiments. A series of physical systems called *sensors* interacts with the system we are studying, returning a signal that is then interpreted by interfaces which can finally be read by humans and given a proper interpretation in the view of some mathematical model. What is the picture of science and physics that comes out of this? Nothing more and nothing less than what every scientist should know from the beginning of his/her work.

In science, there is no presumption to access an *absolute truth*. The only expectation we can have is to increasingly better understand processes for the way they work, using mathematical models that have a finite level of precision and predictivity.

Does this mean that the Top quark or the Higgs boson do actually not exist and that what we use are just mathematical expedients to describe what we observe? No; or - better - not only. This rather means that in our human models, where particles and waves are implied (or quantum fields to be more precise), a Higgs particle or a Top quark are required for the model to work. Other models with other assumptions and structures would have a corresponding object performing somehow the same functions, perhaps in the same way or perhaps in a completely different fashion.

What I am arguing here is that a class of different theories that describe reality equally well might exist and that they could be so deeply different that our intuition would be completely unable to grasp them.

However, this whole thought experiment is just a matter of speculation. At the moment there is no evidence that other theories with the same precision and predictivity as ours might exist; and even less evidence that intelligent life on other planets might exist and have reached a technological and cul-

tural level suitable for scientific knowledge to arise.
The whole question is profoundly related to our incapacity to access an absolute level of understanding, to dig reality to its very *essence*; or - more likely - to our innate prejudice that such an absolute truth and essence exist. After all, the concept of *essence of reality* could most likely be a construct of our earthly brains, evolved and perfectioned to categorize items and creatures based on their absolute role in a restricted and short sighted perspective.
For example, a spoon is a spoon because it is used to pick up liquids. Being it a fully artificial concept, there is no deeper level of abstraction and we can completely understand the idea of a spoon to its very "essence". Reality however does not belong to these easily labellable categories and so, its understanding cannot be equally satisfying; especially when we enter the microscopic world. The rather useful idea of categorizing objects and to define precisely might be the root of our dissatisfaction.
Somehow, we ended up facing a very old and discussed problem. Backtracking all human definitions, we end up with empty hands, as the initial quote by Emil Cioran well describes.
To step out of this *impasse*, let me use the words of Richard Feynman, who recognized the same problem, but was able to amusingly find a solution:

> "We can't define anything precisely. If we attempt to, we get into the paralysis of thought that comes to philosophers... one saying to the other: «You don't know what you are talking about!» The second one says: «What do you mean by talking? What do you mean by you? What do you mean by know?»"

Which point of view shall one take?
One can focus on the unsettling truth of the well motivated but still arbitrary definitions mankind created to be able to understand the world in front of its eyes and on the scary void that stands behind them. This view can be frightful and demotivating, connecting us with our deepest ancestral fear of being alone in the universe, lost on this planet wihout a clue of what we are part of. On the other side, one can focus on the amazing enterprise human kind is part of. The search for a deeper and deeper understanding of what we live in; the hope for new astonishing discoveries that could radically change our future, perhaps enabling us to reach other regions of our universe that are nowadays too far and remote for our technology; the incredible challenge of uniting the world under the same hopes and efforts to reach these goals together, as the only romantic cure to the frightful perspective offered before. In some sense, we are charged with the responsibility of bringing on the rare and extraordinary selfawareness that characterizes us; to say it with the

inspiring and wise words of another person I really think highly of:

> "We are the local embodiment of a Cosmos grown to selfawareness. We have begun to contemplate our origins: starstuff pondering the stars; organized assemblages of ten billion billion billion atoms considering the evolution of atoms; tracing the long journey by which, here at least, consciousness arose. Our loyalties are to the species and the planet. We speak for Earth. Our obligation to survive is owed not just to ourselves but also to that Cosmos, ancient and vast, from which we spring."
>
> *Carl Sagan, Cosmos* [1].

In order to act accordingly, we have to accept the limitations of our definitions and discoveries and direct all our efforts to further research and refine what we have, contributing, even if only for a finite, tiny span of time to this great human adventure.
With this spirit, I welcome the reader to the following sections of the thesis.

1.2 Answers and mysteries in modern Physics

> "We make our world significant by the courage of our questions and the depth of our answers."
>
> *Carl Sagan, Cosmos* [1].

Despite what I said in the first section of this introduction, I believe that the universe we live in is wonderful in its rationality. It is an astonishing fact in itself that human brains, evolved on a tiny planet lost in space, are capable of such a degree of abstraction to formulate and investigate ideas well beyond our common earthly experience. The awe and thrilling excitement given by discovery and research has driven many scientists (whether or not the term can properly fit to some of them) on the pathway to the most precise and insightful description of reality. One of the two fundamental theories that describe reality is the *Standard Model* (SM), which describes electro-magnetic, strong and weak interactions through gauge theories and a symmetry breaking mechanism. The symmetry breaking mechanism adds in turn a boson particle to the model, the Higgs boson, which gives mass to fermions and to the W and Z bosons. The scenario of particle physics has been driven in the last years especially by the joint theoretical and experimental effort to find the Higgs boson, which was prized with a positive result in July 2012 (see for example [2]) and with the Nobel prize for Physics in 2013. This search has required an unprecedented effort from the experimental side, with the construction of the *Large Hadron Collider* (LHC) and other accelerators (previously constructed or still in construction) that are meant to further inspect the spectrum of reality. On the theoretical side, these efforts involved the precision calculation of scattering processes at the energy levels reached by the experiments, where loop corrections have a determinant role. The main efforts in these directions have been made using various analytical and numerical techniques like on shell methods [3–8], the subtraction method[2] [12–15] and many others, depending on the particular focus of the calculations.

The search is of course not over, since the SM still leaves a list of thrilling unsolved questions, mainly involving the hierarchy problem and the fine tuning of the model. A number of models have been proposed to solve these questions and are investigated at this very moment. The LHC, with its next run in 2015, will be one of the tools used to look for new physics, which might then give us new insights and directions towards the exclusion or further investigation of some of those theories.

[2] Used among others also by my colleagues and supervisor; see for example [9–11]

The second main theory that is nowadays considered the standard model for gravity is *General Relativity* (GR). GR is a classical theory. Its quantization encounters a series of problems that, at present time, are still being investigated. One of the main problems is given by the fact that if one tries to quantize the theory starting from a classical Lagrangian as one would do for a gauge theory, one obtains an infinite series of interaction terms which make in turn the theory non renormalizable and difficult to treat perturbatively. In addition to the proliferation of terms, we have to face also a more complicated expression for the Feynman rules, which further encumbers calculations. How to quantize the theory of relativity in order to look for a unified theory including gravity and the other fundamental interactions, is one of the most thrilling open questions and challenges of modern, and most likely future, physics and constituted one of the main motivations for my application and devotion to this field of research. In section 3.1 we will summarize the problem and present an intriguing idea offered by Zvi Bern and others [16].

How are these two theories related? Is there a Great Unified Theory that at some energy scale breaks down in the two different descriptions of reality? These are still open questions and are one of the driving aspects that motivated my curiousity and this PhD project.

In the past years, the issue brought people to inspect a puzzling aspect of these theories: *gauge-gravity duality*. The work of 't Hooft and Susskind on holography (see for example [17]) and successively of Maldacena [18] and others on AdS/CFT duality, gave us what is considered the most advanced attempt to obtain such a unified theory in the context of *String theories*. However, such theories are at the current state of the art impossible to test experimentally (except for some models with implications at the TeV scale; see for example [19]).

My thesis will involve instead another kind of gauge-gravity duality: that which arises from the *squaring relations* proposed in [20] by Bern, Carrasco and Johansson (BCJ), which can be seen as a recasted version of the *KLT relations*, first discovered in string theory by Kawai, Lewellen and Tye [21]. Underlying the structure of these relations, is a newly discovered duality: *color-kinematic duality*, which will be the real main focus of this work.
All the technical terms used in the following paragraphs, will be formally defined in the following chapters, so I invite the non-familiar reader to not expect a detailed and exhaustive discussion in this introduction. In particular, in chapter 2, a detailed description of kinematical and colour factors will be presented, as well as some other techniques employed in the calculation of amplitudes.

In [20], it was conjectured that tree amplitudes in massless gauge theories can always be put in a form of a pole expansion containing only three-valent vertices[3], such that the kinematical numerators, n_i, of this expansion satisfy anti-symmetry and Jacobi-like relations when the associated colour factors, c_i, do.

$$\frac{1}{g^{n-2}}\mathcal{A}_n^{tree}(1,2,\ldots,n) = \sum_i \frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}}$$

$$c_i + c_j + c_k = 0 \Longrightarrow n_i + n_j + n_k = 0$$

This pole expansion has some important consequences. First of all, it unveils additional relations between colour-ordered partial amplitudes, reducing the number of independent ones; these relations are called *BCJ relations.* As I will extensively explain in chapter 2, decomposing a n-point tree amplitude, brings in the game $(n-1)!$ independent colour-ordered partial amplitudes. Kleiss-Kuijf relations [22, 23] further reduce this number to $(n-2)!$ and finally BCJ relations lead us to $(n-3)!$ independent amplitudes. BCJ-relations have been proved first in string theory [24–29] and then within quantum field theories using on-shell recursion relations [30–32].
BCJ relations and colour-kinematic duality at loop level remain a conjecture, even though they have been verified in numerous examples [33–43].

The second important consequence of BCJ-relations, is that they can be used to simplify loop calculations. They can in fact be used on loop integrands in order to relate non-planar to planar graphs. An example will be shown in chapter 3.
The third consequence regards the forementioned gauge gravity duality. In fact, it is possible to exploit kinematical numerators that satisfy BCJ relations in the construction of gravity scattering amplitudes. In order to do so, one has to realise that gravity amplitudes can be written in the same forementioned pole expansion where, however, one has to replace the colour factors with another kinematical factor (from now on: *BCJ-numerator*).

$$\frac{-i}{(\kappa/2)^{n-2}}\mathcal{M}_n^{tree}(1,2,\ldots,n) = \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}$$

If one extracts the numerators from pure Yang-Mills theory amplitudes, one obtains as a result gravity amplitudes corresponding to Einstein gravity coupled to an anti-symmetric tensor and a dilaton. Other kinds of gravity can be obtained starting from different versions of Yang Mills theories, including Supersymmetric cases.

[3] The idea is to assign each contribution of a four-vertex to the right channel. More about this will be said in chapter 3 and 4.

Unfortunately, finding BCJ-numerators is not a trivial exercise. In fact, calculating partial amplitudes starting from the standard Yang Mills Lagrangian does in general not lead to numerators satisfying BCJ-relations. In addition, BCJ numerators are not unique, because they can be modified via generalized gauge transformations as one can see in [20, 44]. In literature, the main efforts to construct BCJ numerators have been made at the amplitude level [45–48] or using BRST covariant building blocks [28, 29]. The approach of this thesis will instead follow the steps of the work previously published with Stefan Weinzierl in [49].

1.3 This thesis

This thesis will follow another path compared to literature and aim for the construction of an effective *BCJ-manifest* Lagrangian, whose Feynman rules automatically lead to BCJ-numerators. One example of this approach can be found in literature in [50]. In order to do so, I will start in chapter 2 with the standard Yang Mills Lagrangian, containing only the kinetic term for the gauge fields and a gauge fixing term. The gauge fixing term will introduce ghosts in the theory. However, since I will be mainly concerned with Born amplitudes, ghosts will not enter the game and will be just mentioned in a separate section where loop order calculations will be examined.
When not differently stated, I will be working with $SU(N)$ as the symmetry group of the theory. However, for simplicity, I will be referring to the gauge particles as *gluons* and to the gauge degrees of freedom as *colour degrees of freedom*, as in the $SU(3)$ case of QCD.
In section 2.5 I will introduce some calculation procedures, notation and devices fot the efficient calculation of massless gauge theory amplitudes. These tools will be important in the prosecution of the thesis since they will simplify considerably the calculations and will allow us to draw a series of conclusions and comments on results otherwise concealed.
Section 3.1 will serve as a brief introduction to the quantization of General Relativity and to the related issues. Particular care will be taken in showing the underlying structures of gauge-gravity and colour-kinematic duality, as well as the difficulty involved in the calculation of massless scattering amplitudes starting from a general Einstein-Hilbert lagrangian for Gravity. Some results from [50] will be presented as a preliminary introduction to the construction of an effective BCJ-manifest Lagrangian.
BCJ-relations as they have originally been presented by Bern, Carrasco and Johansson will be introduced in chapter 3. In this section I will also present some examples for the validity of the pole expansion as well as for the diffi-

culty in obtaining the same result when the valency of the treated amplitudes (the number of external gluons) grows. Part of the chapter will be finally dedicated to the discussion of generalized gauge transformations and to the use of BCJ relations at loop level as presented in [20] and [51]. The original contribution of the author and of Stefan Weinzierl will be finally introduced in chapter 4, where an algorithm to construct an effective BCJ-manifest Lagrangian will be presented. The last chapters will be dedicated to a discussion and to some speculations on *scattering equations* used to construct a *kinematic algebra* [48, 52–56]. The chapter will be lead by the idea that the previously presented algorithm can be improved and refined by reproducing the canonical BCJ-numerators presented in [56].

2

YANG MILLS THEORIES

2.1 Non-abelian group theories

Most of the incredible theoretical achievements of the last century, have been reached only thanks to the power of that incredible labor saving device that is group theory. Mathematicians and theoretical physicists, used group theory with incredible effectiveness, even though, at the beginning at least, its proper application and interpretation looked quite puzzling. A quote by J.R.Newman taken from [57] in particular is quite suggestive of this feeling:

> *The theory of groups is a branch of mathematics in which one does something to something and then compares the results with the result of doing the same thing to something else, or something else to the same thing.*

as it is again this quote by Sir A.S.Eddington from the same book:

> *We need a super-mathematics in which the operations are as unknown as the quantities they operate on, and a super-mathematician who does not know what he is doing when he performs these operations. Such a super-mathematics is the Theory of Groups.*

In order to briefly introduce the vast topic and the essential tools and definitions that will be needed, I will mainly follow [58] and [59]. What is not clear from the two previous quotes is that, in physics, what actually leads to really insightful results is the study of group representations, and in particular, what makes it so effective is the fact that those representations live in linear spaces, allowing us to transform states conveniently using linear transformations. In many branches of physics, finite groups have been one of the most effective tools used to describe and characterize systems. In quantum field theories, the most important role is covered instead by Lie groups, i.e. groups that also satisfy the requirements for being differentiable manifolds. In particular, I will focus here on continuously generated groups and, in treating the specific case of Yang Mills theories, I will further restrict my attention to non-abelian groups. Continuously generated groups have the very useful property of containing elements that are infinitely close to the identity, allowing us in this way to introduce a convenient parametrization and express every general element g through the repeated action of those infinitesimal elements:

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2) \tag{2.1}$$

The coefficients T^a are called the *generators* of the symmetry group and are in fact Hermitian operators. The commutation relations of these operators can be easily found by noticing that they must span the full space of infinitesimal group transformations, which gives in turn:

$$[T^a, T^b] = i f^{abc} T^c \tag{2.2}$$

where f^{abc} are numbers called *structure constants* of the group, and they are totally antisymmetric in all indices. A *Lie Algebra* is composed by the vector space spanned by the generators, together with these commutation relations.

A crucial set of relations satisfied by the generators and in turn by the group structure constants is the set of Jacobi identities.

$$\big[T^a, [T^b, T^c]\big] + \big[T^b, [T^c, T^a]\big] + \big[T^c, [T^a, T^b]\big] = 0 \tag{2.3}$$

which can be written in terms of the structure constants as:

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \tag{2.4}$$

Their role will be essential in the prosecution of the thesis and in particular in the creation of the original algorithm published in [49].

As I stated before, what is really interesting in group theory is the study of particular representations of the group algebra.

For a general group G, a unitary finite dimensional representation is given through a set of $d \times d$ Hermitian matrices t^a satisfying the commutation relations 2.2. d here is the dimension of the representation.

Let us focus now on the case of SU(N).

The simplest irreducible representation is given by the N-dimensional complex vector and is called *Fundamental Representation.* Every simple Lie algebra also possesses another irreducible representation called the *Adjoint Representation*, which is the representation to which the generators of the algebra itself belong. As previously stated, the Lagrangian studied in the next chapters will be restricted to contain only fields living in this representation.

The adjoint representation has dimension $d_{adj} = N^2 - 1$ and its representation matrices are given by the structure constants of the group

$$(t^b)_{ac} = if^{abc} \tag{2.5}$$

which satisfy the commutation relations previously introduced:

$$\left([t^b, t^c]\right)_{ae} = if^{bcd}(t^d)_{ae} \tag{2.6}$$

It is interesting to notice that the last commutation relations are nothing but the Jacobi relations as expressed in 2.4. Since we will borrow abundantly the terminology of the $SU(3)$ case of QCD, let us pause for a moment and immediately introduce some terms (generalized to the $SU(N)$ case) that will accompany us throughout the whole thesis.
QCD describes strong interactions among quarks, anti-quarks, gluons and, through a gauge fixing procedure that I will introduce later, ghosts. Quarks and anti-quarks carry a N or $\overline{N}$ index $i, \overline{j} = 1, 2, \ldots, N$, while gluons carry an *adjoint color index* $a = 1, 2, \ldots, N^2 - 1$. These terms will recur often throughout the thesis and in particular they will form the language I will use when introducing the use of colour ordered amplitudes and Feynman rules.

2.2 Yang-Mills Lagrangian

This section will be mostly derived from standard textbooks and articles as [59] and [60]. In order to construct a gauge invariant Lagrangian containing only fields living in the adjoint representation, we need to introduce a set of gauge fields A^a_μ, labeled by a color index a, with one gauge field for each generator of the gauge group.
The Lagrangian I will start with can then be written as:

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{2.7}$$

where the field strength is defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2.8}$$

From now on, I will loosen a bit the notation and stop specifying the difference between the generators of the group T^a and their representation matrices t^a. I will instead generically refer to the matrices as to *generators* and I will indicate them with T^a.
The generators satisfy the commutation relations 2.2 and an additional arbitrary normalization equation:

$$Tr[T^a T^b] = \frac{1}{2}\delta^{ab} \tag{2.9}$$

It is useful to introduce the Lie-algebra valued matrix field

$$\boldsymbol{A}_\mu = \frac{g}{i} T^a A^a_\mu \tag{2.10}$$

and the corresponding field strength

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \tag{2.11}$$

In terms of these new quantities, we can rewrite the lagrangian as

$$\mathcal{L}_{YM} = \frac{1}{2g^2} Tr \boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu} \tag{2.12}$$

which is invariant under gauge transformations

$$\boldsymbol{A}_\mu \to U \boldsymbol{A}_\mu U^\dagger - (\partial_\mu U) U^\dagger \tag{2.13}$$

The explicit form of the gauge transformation is

$$U(x) = \exp(ig\alpha^a(x) T^a) \tag{2.14}$$

or in infinitesimal form

$$U(x) = 1 + ig\alpha^a(x) T^a + \mathcal{O}(\alpha^2) \tag{2.15}$$

where I restored for clarity the gauge group generators T^a and where α^a are the parameters of the transformation.

2.2.1 Gauge fixing and ghosts

In order to quantize the theory and to derive Feynman rules appropriately, one can define the functional integral:

$$\int \mathcal{D}A \exp \left[i \int d^4x \mathcal{L}_{YM} \right] \tag{2.16}$$

When doing so however, one must also consider that there is an infinite number of directions in the space of fields configurations along which the Lagrangian is unchanged (corresponding to the fact that performing a local gauge transformation leaves the Lagrangian invariate). The functional integral then, if performed as written above, will include a great amount of redundant information that will in particular lead to a wrong result for the gauge field propagator.
This problem was first solved in 1967 by Faddeev and Popov in the famous

[61].
Their method involves the application of a gauge fixing condition $G(A) = 0$ at each point x, in order to constrain the gauge directions in the functional integral. To do so, one can exploit the identity

$$1 = \int \mathcal{D}\alpha(x)\delta\left(G(A^{\alpha})\right)\det\left(\frac{\delta G(A^{\alpha})}{\delta\alpha}\right) \tag{2.17}$$

where A^{α} is the gauge field after a finite gauge transformation parametrized with $\alpha(x)$. The explicit calculation can be found in many textbooks like [59] and will not be presented here. However, what is important to notice is that the procedure adds a gauge fixing term to the Lagrangian as well as Faddeev-Popov ghosts in the theory.
When not otherwise specified, I will be working in the Feynman gauge.
The gauge fixing term in the Lagrangian reads

$$\mathcal{L}_{GF} = \frac{1}{g^2}Tr(\partial^{\mu}\boldsymbol{A}_{\mu})(\partial^{\nu}\boldsymbol{A}_{\nu}) \tag{2.18}$$

For now, ghosts will not be part of the discussion since the thesis will be focused on tree level gauge amplitudes, where ghosts do not contribute.
The concept of gauge fixing and how this might have an impact on algorithms for bcj-numerators will be treated again in the last chapter.

2.3 Feynman Rules

Starting from the Lagrangian density introduced before, we can derive a set of Feynman rules. Let us first re-write the Lagrangian in order to make the nonlinear terms manifest:

$$\mathcal{L}_{YM} = -\frac{1}{4}F^{a}_{\mu\nu}F^{a\mu\nu} = \mathcal{L}_0 - gf^{abc}(\partial_{\mu}A^{a}_{\nu})A^{b\mu}A^{c\nu} - g^2(f^{abc}A^{b}_{\mu}A^{c}_{\nu})(f^{ade}A^{d\mu}A^{e\nu}) \tag{2.19}$$

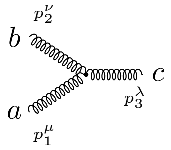
or, using the more compact formulation of 2.12 and including the gauge fixing term:

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2g^2}\left[\mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)}\right] \tag{2.20}$$

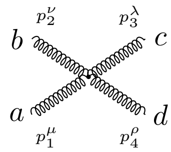
with

$$\begin{aligned}
\mathcal{L}^{(2)} &= -2Tr\boldsymbol{A}_{\mu}\Box\boldsymbol{A}^{\mu} \\
\mathcal{L}^{(3)} &= 4Tr(\partial_{\mu}\boldsymbol{A}_{\nu})[\boldsymbol{A}^{\mu},\boldsymbol{A}^{\nu}] \\
\mathcal{L}^{(4)} &= Tr[\boldsymbol{A}_{\mu},\boldsymbol{A}_{\nu}][\boldsymbol{A}^{\mu},\boldsymbol{A}^{\nu}]
\end{aligned} \tag{2.21}$$

It is important to mention once and for all that here and throughout the thesis, I will make the choice of always describing outgoing gluons.
The gluons interaction vertices are given by $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$, which respectively lead to the Feynman rules:



$$= gf^{abc}\left[g^{\mu\nu}(p_1 - p_2)^\lambda + g^{\nu\lambda}(p_2 - p_3)^\mu + g^{\lambda\mu}(p_3 - p_1)^\nu\right]$$



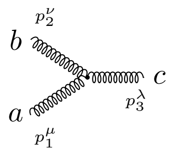
$$= -ig^2\left[f^{abe}f^{cde}(g^{\mu\lambda}g^{\mu\rho} - g^{\mu\rho}g^{\nu\lambda}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\rho}g^{\nu\lambda}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho})\right] \tag{2.22}$$

As we can see, the colour factor each three gluon vertex brings in a general YM diagram is given by a group theory structure constant f^{abc}. Each four gluon vertex instead brings in contracted pairs of structure constants $f^{abe}f^{cde}$. This formulation however keeps colour and kinematic information mixed together in the Feynman rules. For the purpose of this thesis, it will be convenient to introduce another way of formulating the vertices, such that only the kinematic part of the information will be present and it will correspond to a particular colour structure automatically associated with it. This formulation goes under the name of *colour decomposition of amplitudes.*

2.4 Colour decomposition

First of all we need to identify all the possible colour structures in a given amplitude. The next step is to find a prescription to construct the kinematic coefficients of these colour structures.
These kinematic coefficients are called *partial amplitudes* or *sub-amplitudes.*

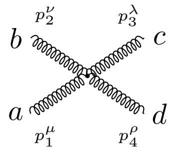
First of all, let us rewrite the Feynman rules in a more convenient way:



$$= gf^{abc}\left[g^{\mu\nu}(p_1-p_2)^\lambda + g^{\nu\lambda}(p_2-p_3)^\mu + g^{\lambda\mu}(p_3-p_1)^\nu\right]$$

$$= g(if^{abc})(i\left[g^{\mu\nu}(p_2-p_1)^\lambda + g^{\nu\lambda}(p_3-p_2)^\mu + g^{\lambda\mu}(p_1-p_3)^\nu\right]$$

$$= g(if^{abc})\left(iV_3^{\mu\nu\lambda}(p_1,p_2,p_3)\right)$$



$$= -ig^2\left[f^{abe}f^{cde}(g^{\mu\lambda}g^{\mu\rho} - g^{\mu\rho}g^{\nu\lambda}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\rho}g^{\nu\lambda})\right.$$

$$\left.+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho})\right]$$

$$= g^2\left[(if^{abe})(if^{cde})i(g^{\mu\lambda}g^{\mu\rho} - g^{\mu\rho}g^{\nu\lambda}) + (if^{ace})(if^{bde})i(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\rho}g^{\nu\lambda})\right.$$

$$\left.+(if^{ade})(if^{bce})i(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho})\right] \tag{2.23}$$

In order to apply colour decomposition we need to invert the commutation relations that define the structure constants:

$$[T^a, T^b] = if^{abd}T^d \tag{2.24}$$

Multiplying each side by another T^c and tracing, we obtain:

$$Tr\left(T^cT^aT^b - T^cT^bT^a\right) = if^{abd}Tr\left(T^dT^c\right) \tag{2.25}$$

Now, using the invariance of the trace under cyclic permutations and the normalization convention 2.9, we end up with:

$$if^{abc} = 2Tr\left(T^a[T^b, T^c]\right) \tag{2.26}$$

Another useful set of relations, used to manage contracted generators appearing from the use of the forementioned substitution, is given by Fierz identities.
Given two strings of $SU(N)$ generators S_1 and S_2, we can express the identities as:

$$Tr(T^aS_1)Tr(T^aS_2) = \frac{1}{2}\left(Tr(S_1S_2) - \frac{1}{N}Tr(S_1)Tr(S_2)\right)$$

or

$$Tr(T^aS_1T^aS_2) = \frac{1}{2}\left(Tr(S_1)Tr(S_2) - \frac{1}{N}Tr(S_1S_2)\right) \tag{2.27}$$

We can rewrite the vertices applying these rules, finally obtaining:

$$\begin{aligned}
&g(if^{abc})\left(iV_3^{\mu\nu\lambda}(p_1,p_2,p_3)\right)= \\
&=2g\left[Tr(T^aT^bT^c)iV_3^{\mu\nu\lambda}(p_1,p_2,p_3)-Tr(T^aT^cT^b)iV_3^{\mu\nu\lambda}(p_1,p_2,p_3)\right] \\
&=2g\left[Tr(T^aT^bT^c)iV_3^{\mu\nu\lambda}(p_1,p_2,p_3)+Tr(T^aT^bT^c)iV_3^{\mu\lambda\nu}(p_1,p_3,p_2)\right] \\
&=2g\sum_{P(2,3)}Tr(T^{a_1}T^{a_2}T^{a_3})iV_3^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)
\end{aligned} \tag{2.28}$$

where we used the antisymmetry of V_3 when any two legs are exchanged, and where I renamed indices in the last step to obtain a more compact expression. For the four-vertex we get:

$$\begin{aligned}
&2g^2\sum_{P(2,3,4)}Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4})i(2g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}-g^{\mu_1\mu_2}g^{\mu_3\mu_4}) \\
=&2g^2\sum_{P(2,3,4)}Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4})iV_4^{\mu_1\mu_2\mu_3\mu_4}
\end{aligned} \tag{2.29}$$

where we used the Fierz identity to calculate these expressions:

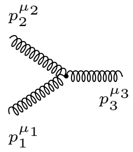
$$(if^{abe})(if^{cde})=2\left[Tr(T^aT^bT^cT^d)-Tr(T^aT^bT^dT^c)-Tr(T^bT^aT^cT^d)+Tr(T^bT^aT^dT^c)\right] \tag{2.30}$$

and where we defined the purely kinematic quantity $V_4^{\mu_1\mu_2\mu_3\mu_4}$.
In this way, we can always write a tree level n-gluons amplitude $\mathcal{A}_n$ as

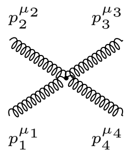
$$\mathcal{A}_n(p_i,\lambda_i,a_i)=g^{n-2}\sum_{\sigma\in S_n/Z_n}2Tr\left(T^{a_{\sigma(1)}},T^{a_{\sigma(2)}},\dots,T^{a_{\sigma(n)}}\right)A_n(\sigma(1)^{\lambda_1},\sigma(2)^{\lambda_2},\dots,\sigma(n)^{\lambda_n}) \tag{2.31}$$

Here p_i,λ_i are the momenta and helicities and $A_n(1^{\lambda_1},\dots,n^{\lambda_n})$ are the gauge invariant functions called *colour-ordered partial amplitudes*, which contain all the kinematic information. The partial amplitudes are colour-ordered, i.e. the only diagrams contributing to their calculations are those with a particular cyclic ordering of the gluons.
In order to calculate them we can exploit the decomposition of the vertices

derived before. Let us introduce then colour-ordered Feynman rules:



$$= i\left[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\right] \tag{2.32}$$



$$= i(2g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3} - g^{\mu_1\mu_2}g^{\mu_3\mu_4}) \tag{2.33}$$

Due to the cyclic property of partial amplitudes, we have:

$$A_n(1,2,3,\ldots,n) = A_n(2,3,\ldots,n,1) \tag{2.34}$$

The sum in eq. 2.31 is over $(n-1)!$ partial amplitudes. The cyclic property might be used to fix leg 1 in the first position, so that the sum in 2.31 corresponds to the $(n-1)!$ permutations of legs $2,\ldots,n$.

$$\mathcal{A}_n(p_i,\lambda_i,a_i) = g^{n-2}\sum_{\sigma\in S_{n-1}} 2Tr\left(T^{a_1},T^{a_{\sigma(2)}},\ldots,T^{a_{\sigma(n)}}\right)A_n(1^{\lambda_1},\sigma(2)^{\lambda_2},\ldots,\sigma(n)^{\lambda_n}) \tag{2.35}$$

In addition to cyclic invariance, partial amplitudes present other features, as the *reflection property*

$$A_n(1,2,3,\ldots,n) = (-1)^n A_n(n,\ldots,2,1) \tag{2.36}$$

the dual Ward identity (also called *photon decoupling identity*

$$A_n(1,2,3,\ldots,n)+A_n(2,1,3,\ldots,n)+A_n(2,3,1,\ldots,n)+\ldots+A_n(2,3,\ldots,1,n) = 0 \tag{2.37}$$

and the sub-cyclic sum

$$\sum_{Z_{n-1}(2,3,\ldots,n)} A_n(1,2,\ldots,n) = 0 \tag{2.38}$$

The $(n-1)!$ partial amplitudes are not independent. In fact, they satisfy another set of linear relations called *Kleiss-Kuijf relations* [22].

$$A_n^{tree}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta}\sum_{\{\sigma\}_i\in OP(\{\alpha\},\{\beta^T\})} A_n^{tree}(1,\{\sigma\}_i,n) \tag{2.39}$$

where we are summing over the *ordered permutations* $OP(\{\alpha\},\{\beta^T\})$, i.e. all the permutations of $\{\alpha\}\bigcup\{\beta^T\}$ which preserve the relative order of the elements of the two sets. $\{\beta^T\}$ here represents the set $\{\beta\}$ with the ordering reversed and n_β is the numeber of beta elements. Thanks to Kleiss-Kuijf (KK) relations we can fix the order of another leg, e.g. leg n in the last position, and express all the partial amplitudes where n does not appear in the last position as a linear combination of partial aplitudes where n instead does. For example, for $n = 4$ we have only two independent KK amplitudes: $A^{tree}(1, 2, 3, 4)$ and $A^{tree}(1, 3, 2, 4)$. Using KK-relations we can for example write:

$$A^{tree}(1, 2, 4, 3) = -A^{tree}(1, 2, 3, 4) - A^{tree}(1, 3, 2, 4) \tag{2.40}$$

All the other possible combinations of the external legs can be obtained using KK-relations, the reflection and the cyclic property. This means that the amplitudes $A^{tree}(1, \mathcal{P}(2, 3), 4)$ form a basis in which the remaining partial amplitudes can be expressed. In general, at valency n, we can write all the partial-amplitudes in terms of the $(n - 2)!$ KK-amplitudes where the order of two legs is fixed. This procedure is also part of the algorithm introduced by Bern Carrasco and Johansson in [20].

Now, before actually introducing BCJ relations and our alorithm, we will introduce a series of formalism devices that will simplify expressions and calculations in the following chapters.

2.5 Calculating amplitudes efficiently

2.5.1 Spinor helicity formalism

One of the tools utilized to obtain compact representations for tree and loop level amplitudes in QCD is the *Spinor Helicity formalism.* We'll start with massless fermions. Studying the solutions of the massless Dirac equation, one can see that the positive energy and negative energy solutions are identical up to normalization conventions.

$$i\gamma^\mu \partial_\mu \psi = 0 \tag{2.41}$$

Inserting the ansatz $\psi^+ = u(p)e^{-ipx}$ and $\psi^- = v(p)e^{ipx}$, we obtain:

$$\begin{aligned} \gamma^\mu p_\mu u(p) &= \not{p} u(p) = 0 \\ -\gamma^\mu p_\mu v(p) &= -\not{p} v(p) = 0 \end{aligned} \tag{2.42}$$

For the adjoint spinor we can procede in a totally similar way.

$$\bar{\psi}(-i\gamma^\mu \overleftarrow{\partial}_\mu) = 0 \tag{2.43}$$

which - in momentum space - translates to:

$$\begin{aligned} -\bar{u}(p)\gamma^{\mu}p_{\mu} &= -\bar{u}(p)\not{p} = 0 \\ \bar{v}(p)\gamma^{\mu}p_{\mu} &= \bar{v}(p)\not{p} = 0 \end{aligned} \tag{2.44}$$

What is interesting to notice is that, in the massless limit, the positive energy projector $\Lambda_+(p) \sim u(p) \otimes \bar{u}(p)$ and the negative energy projector $\Lambda_-(p) \sim v(p) \otimes \bar{v}(p)$ become both proportional to $\not{p}$. Because of this, we can choose the following solutions to be equal to each other:

$$u_\pm(p) = \frac{1}{2}(1 \pm \gamma_5)u(p) \quad v_\mp(p) = \frac{1}{2}(1 \pm \gamma_5)v(p) \tag{2.45}$$

The same is true for the conjugate spinors, where a similar relation holds:

$$\bar{u}_\pm(p) = \bar{u}(p)\frac{1}{2}(1 \mp \gamma_5) \quad \bar{v}_\mp(p) = \bar{v}(p)\frac{1}{2}(1 \mp \gamma_5) \tag{2.46}$$

In the next sections we will be working in general with a n-tuple of momenta. It is useful then to introduce a shorthand notation for these momenta and the definite helicity solutions associated to them. We will denote our set of momenta with p_i where $i = 1, \dots, n$ and we will - at least in the next few sections - use the following notation:

$$\left|i^\pm\right\rangle \equiv \left|p_i^\pm\right\rangle \equiv u_\pm(p_i) = v_\mp(p_i), \quad \left\langle i^\pm\right| \equiv \left\langle p_i^\pm\right| \equiv \bar{u}_\pm(p_i) = \bar{v}_\mp(p_i) \tag{2.47}$$

We are now finally ready to define the spinor products:

$$\langle ij\rangle \equiv \left\langle i^-\right| j^+\big\rangle = \bar{u}_-(p_i)u_+(p_j), \quad [ij] \equiv \left\langle i^+\right| j^-\big\rangle = \bar{u}_+(p_i)u_-(p_j) \tag{2.48}$$

Using an explicit representation for the spinors, we can also obtain explicit expressions for the spinor products. This, however, will not be object of our interest. For the interested reader, we suggest for example [62]. The spinor products satisfy a series of properties that will result being incredibly useful in the calculation of n-point amplitudes. First of all we can calculate the usual momentum dot products with:

$$\langle ij\rangle\,[ji] = \frac{1}{2}Tr[\not{p}_j\not{p}_i] = 2p_j \cdot p_i = s_{ji} \tag{2.49}$$

where the last identity is true for massless particles if we define

$$s_{ij} = (p_i + p_j)^2 \tag{2.50}$$

This is a good moment to introduce some other useful shorthand notations that we will be using:

$$\begin{aligned} p_{ij\ldots k} &= p_i + p_j + \ldots + p_k \\ s_{ij\ldots k} &= (p_{ij\ldots k})^2 = (p_i + p_j + \ldots + p_k)^2 \end{aligned} \tag{2.51}$$

We also have Gordon identity and a rewriting of the projection operator:

$$\left\langle i^{\pm}\right| \gamma^{\mu} \left|i^{\pm}\right\rangle = 2p_i^{\mu}, \qquad \left|i^{\pm}\right\rangle \left\langle i^{\pm}\right| = \frac{1}{2}(1 \pm \gamma_5)\not{p}_i \tag{2.52}$$

Spinor products are antisymmetric under exchange of the two arguments:

$$\langle ij\rangle = -\langle ji\rangle\,, \quad [ij] = -[ji]\,, \quad \langle ii\rangle = [ii] = 0 \tag{2.53}$$

In order to simplify expressions, we can use Fierz rearrangement:

$$\left\langle i^{+}\right| \gamma^{\mu} \left|j^{+}\right\rangle \left\langle k^{+}\right| \gamma_{\mu} \left|l^{+}\right\rangle = 2\,[ik]\,\langle lj\rangle \tag{2.54}$$

as well as charge conjugation of current:

$$\left\langle i^{+}\right| \gamma^{\mu} \left|j^{+}\right\rangle = \left\langle j^{-}\right| \gamma^{\mu} \left|i^{-}\right\rangle \tag{2.55}$$

and Schouten identity:

$$\langle ij\rangle\langle kl\rangle = \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle \tag{2.56}$$

In addition, in an n-point amplitude where momentum conservation is satisfied, we also have:

$$\sum_{\substack{i=1 \\ i\neq j,k}}^{n} [ji]\,\langle ik\rangle = 0 \tag{2.57}$$

If we want to describe massless gauge bosons however, we need to introduce a spinor representation also for the polarization vector that charachterizes them for a particular ± 1 helicity:

$$\epsilon_{\mu}^{\pm}(p,q) = \pm\frac{\left\langle q^{\mp}\right| \gamma_{\mu} \left|p^{\mp}\right\rangle}{\sqrt{2}\left\langle q^{\mp}\right| p^{\pm}\rangle} \tag{2.58}$$

where p is simply the momentum of the vector boson, while q is an auxiliary massless vector that we can call *reference momentum.* The freedom allowed in the choice of the reference momentum, simply reflects the freedom we already have in on-shell gauge transformations. To see that this is the case, let us calculate the difference between two polarization vectors where two different reference momenta have been used:

$$\epsilon^+_\mu(p,\tilde{q}) - \epsilon^+_\mu(p,q) = \frac{\langle \tilde{q}^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\,\langle \tilde{q}p\rangle} - \frac{\langle q^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\,\langle qp\rangle} = -\frac{\langle \tilde{q}^-|\,\gamma_\mu \not{p}\,|q^+\rangle + \langle \tilde{q}^-|\,\not{p}\gamma_\mu\,|q^+\rangle}{\sqrt{2}\,\langle \tilde{q}p\rangle\,\langle qp\rangle}$$
$$= -\frac{\sqrt{2}\,\langle \tilde{q}q\rangle}{\langle \tilde{q}p\rangle\,\langle qp\rangle} \times p_\mu \tag{2.59}$$

So, changing the reference momentum simply corresponds to shifting the polarization vector of an amount proportional to p_μ.

The expression 2.58 gives our polarization vector all the properties we expect. First of all, since $\not{p}\,|p^\pm\rangle = 0$, we have that $\epsilon^\pm(p,q)$ is transverse to p for any choice of the reference momentum:

$$\epsilon^\pm_\mu(p,q)\cdot p = 0 \tag{2.60}$$

Complex conjugation reverses the helicity:

$$(\epsilon^+_\mu(p,q))^* = \epsilon^-_\mu(p,q) \tag{2.61}$$

By using Fierz rearrangement, we can also verify that the standard normalization is implemented:

$$\begin{aligned} \epsilon^+\cdot(\epsilon^+)^* = \epsilon^+\cdot\epsilon^- &= -\frac{1}{2}\frac{\langle \tilde{q}^-|\,\gamma_\mu\,|p^-\rangle\,\langle \tilde{q}^+|\,\gamma_\mu\,|p^+\rangle}{\langle qp\rangle\,[qp]} = -1 \\ \epsilon^+\cdot(\epsilon^-)^* = \epsilon^+\cdot\epsilon^+ &= \frac{1}{2}\frac{\langle \tilde{q}^-|\,\gamma_\mu\,|p^-\rangle\,\langle \tilde{q}^-|\,\gamma_\mu\,|p^-\rangle}{\langle qp\rangle^2} = 0 \end{aligned} \tag{2.62}$$

Finally, using the set of rules described before and the anticommutation rules for gamma matrices, we can see that the completeness relations satisfied by the polarization vectors are the same ones of a light-like axial gauge:

$$\sum_{\lambda=\pm} \epsilon^\lambda_\mu(p,q)(\epsilon^\lambda_\nu(p,q))^* = -g_{\mu\nu} + \frac{p_\mu q_\nu + p_\nu q_\mu}{p\cdot q} \tag{2.63}$$

One is allowed to perform a different choice of reference momentum for each gluon (or in general momentum) in the quantity one wants to calculate. In doing so, one must be particularly cautius, since changing reference momentum corresponds to a gauge choice (i.e. changing reference momentum inside a gauge independent quantity is a wrong step).
This freedom will result particularly useful. Selecting carefully the reference momentum for the different gluons involved in a scattering amplitude can

often result in a simplification of calculations. In particular, smart choices of q can bring to the vanishing of terms or even full diagrams. The backbone of these simplifications lies in these identities (where we introduce the notation $\epsilon_i^{\pm}(q) \equiv \epsilon^{\pm}(k_i, q_i = q)$):

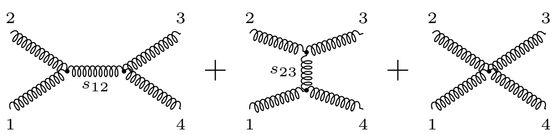
$$\begin{aligned} \epsilon_i^{\pm}(q) \cdot q &= 0 \\ \epsilon_i^{+}(q) \cdot \epsilon_j^{+}(q) &= \epsilon_i^{-}(q) \cdot \epsilon_j^{-}(q) = 0 \\ \epsilon_i^{+}(p_j) \cdot \epsilon_j^{-}(q) &= \epsilon_i^{+}(q) \cdot \epsilon_j^{-}(p_i) = 0 \\ \not{\epsilon}_i^{+}(p_j) \left| j^{+} \right\rangle &= \not{\epsilon}_i^{-}(p_j) \left| j^{-} \right\rangle = 0 \\ \left\langle j^{+} \right| \not{\epsilon}_i^{-}(p_j) &= \left\langle j^{-} \right| \not{\epsilon}_i^{+}(p_j) = 0 \end{aligned} \tag{2.64}$$

We are finally endowed with enough notation to calculate scattering amplitudes involving massless vector bosons (and massless fermions) in terms of spinor products. Let us start calculating some gluon amplitudes.
As previously stated, we will consider all the gluons as outgoing; the choice for helicities will be then defined accordingly. By requiring that all the momenta involved in the scattering event are real valued, we can immediately obtain:

$$A_3^{tree}(1, 2, 3) = 0 \tag{2.65}$$

The same quantity, however, does not vanish if we let momenta be complex valued quantities. This is for example the starting step of procedures involving complex kinematics like BCFW recursion relations [4], twistor techniques or other on-shell methods (see for example [63]).

Now, making use of all the machinery described before, one can try to calculate quantities like $A^{tree}(1^+, 2^+, 3^+, 4^+)$. First of all, let us consider the question from a diagrammatic point of view and write down all the colour ordered diagrams involved[1]:



(2.66)

Let us focus on the s_{12} channel to work out a calculation for the sake of

[1] The number of diagrams we need to consider is related to all the possible channels (i.e. to all the propagator structures) that can appear.

giving an example:

$$
\begin{aligned}
&A^{tree}_{s_{12}}(1^+,2^+,3^+,4^+) = \\
&= \epsilon^+_{\mu_1}(p_1,q_1)\epsilon^+_{\mu_2}(p_2,q_2) i \left[g^{\mu_1\mu_2}\left(p_1-p_2\right)^\rho + g^{\mu_2\rho}\left(2p_2+p_1\right)^{\mu_1} + g^{\rho\mu_1}\left(-2p_1-p_2\right)^{\mu_2}\right] \times \\
&\left(\frac{-ig_{\rho\sigma}}{s_{12}}\right) \times \\
&\epsilon^+_{\mu_3}(p_3,q_3)\epsilon^+_{\mu_1}(p_4,q_4) i \left[g^{\mu_3\mu_4}\left(p_3-p_4\right)^\sigma + g^{\mu_4\sigma}\left(2p_4+p_3\right)^{\mu_3} + g^{\sigma\mu_3}\left(-2p_3-p_4\right)^{\mu_4}\right] \\
& \\
&= i\left[\epsilon_1\cdot\epsilon_2\left(p_1-p_2\right)^\rho + \epsilon_2^\rho\left(2\epsilon_1\cdot p_2+\epsilon_1\cdot p_1\right) - \epsilon_1^\rho\left(2\epsilon_2\cdot p_1+\epsilon_2\cdot p_2\right)\right]\left(\frac{g_{\rho\sigma}}{s_{12}}\right) \times \\
&\left[\epsilon_3\cdot\epsilon_4\left(p_3-p_4\right)^\sigma + \epsilon_4^\sigma\left(2\epsilon_3\cdot p_4+\epsilon_3\cdot p_3\right) - \epsilon_3^\sigma\left(2\epsilon_4\cdot p_3+\epsilon_4\cdot p_4\right)\right] \\
& \\
&= \frac{i}{s_{12}}\left[\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\left(p_1-p_2\right)\cdot\left(p_3-p_4\right) + 2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot p_4\left(p_1-p_2\right)\cdot\epsilon_4\right. \\
&-2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\left(p_1-p_2\right)\epsilon_4\cdot p_3 + 2\epsilon_1\cdot p_2\epsilon_3\cdot\epsilon_4\epsilon_2\cdot\left(p_3-p_4\right) \\
&+4\epsilon_3\cdot p_4\epsilon_2\cdot\epsilon_4 - 4\epsilon_1\cdot p_2\epsilon_2\cdot\epsilon_3\epsilon_4\cdot p_3 - 2\epsilon_2\cdot p_1\epsilon_3\cdot\epsilon_4\epsilon_3\cdot\left(p_3-p_4\right) \\
&\left.-4\epsilon_1\cdot\epsilon_4\epsilon_3\cdot p_4\epsilon_2\cdot p_1 + 4\epsilon_1\cdot\epsilon_3\epsilon_2\cdot p_1\epsilon_4\cdot p_3\right]
\end{aligned}
\tag{2.67}
$$

where, in the second step, we used equation 2.60 (together with a substantial simplification of the notation). As we can see, each term in the final result contains a contraction $\epsilon_i \cdot \epsilon_j$. This immediately hints us to exploiting the freedom in the choice of the reference momenta $q_i's$.
In fact, choosing the same light-like vector for all the different polarization vectors and using the second identity in 2.64 will yeld an important result:

$$A^{tree}_{s_{12}}(1^+,2^+,3^+,4^+) = 0 \tag{2.68}$$

and the same can be shown to be true for the remaining diagrams, which yelds then to:

$$A^{tree}_{4}(1^+,2^+,3^+,4^+) = 0 \tag{2.69}$$

One important thing to notice is that we have to avoid setting all the $q_i's$ equal to one same external momentum, because this would yeld singularities in the expressions of the polarization vectors of those legs. This important result can be extended with some considerations to the case of n external legs.
In fact, each vertex can contribute the expression with at most one momentum vector p_i and since we have up to $n-2$ vertices in a diagram, we can

immediately see that the polarization vectors can be contracted with a maximum of $n-2$ momentum vectors ${p_i}$[2]. This means in turn that we will always have a $\epsilon_i^+ \cdot \epsilon_j^+$ contraction in each term and this allows us to set the whole expression to zero whenever we can set $\epsilon_i^+ \cdot \epsilon_j^+$ to zero choosing the reference momenta appropriately. So we can conclude that:

$$A_n^{tree}(1^+, 2^+, \ldots, n^+) = 0 \tag{2.70}$$

Using the same philosophy and the other identities in 2.64 we can also immediately see that also this evaluation holds:

$$A_n^{tree}(1^-, 2^+, \ldots, n^+) = 0 \tag{2.71}$$

in this case, the simple choice $q_2 = q_3 = \ldots = q_n = p_1$ and $q_1 = p_n$ is sufficient to obtain our result.
The first non-vanishing amplitude we can calculate is $A_4^{tree}(1^-, 2^-, 3^+, 4^+)$. Now we will cheat a bit by calculating once again just the s_{12} channel of the colour ordered amplitude. We will do so because we already know that - due to the reference momenta choice we will make - the contributions of the other diagrams will vanish. The expression is similar to the one we calculated before. This time however, we will reintroduce the full notation in order to make manifest that not all the terms will disappear after the choice of the reference momenta.

$$\begin{aligned}
&A_4^{tree}(1^-, 2^-, 3^+, 4^+) = \\
&= \frac{i}{s_{12}} \left[\epsilon_1^-(q_1) \cdot \epsilon_2^-(q_2)\epsilon_3^+(q_3) \cdot \epsilon_4^+(q_4)\, (p_1 - p_2) \cdot (p_3 - p_4)\right. \\
&+ 2\epsilon_1^-(q_1) \cdot \epsilon_2^-(q_2)\epsilon_3^+(q_3) \cdot p_4\, (p_1 - p_2) \cdot \epsilon_4^+(q_4) \\
&- 2\epsilon_1^-(q_1) \cdot \epsilon_2^-(q_2)\epsilon_3^+(q_3) \cdot (p_1 - p_2)\, \epsilon_4^+(q_4) \cdot p_3 \\
&+ 2\epsilon_1^-(q_1) \cdot p_2\epsilon_3^+(q_3) \cdot \epsilon_4^+(q_4)\epsilon_2^-(q_2) \cdot (p_3 - p_4) \\
&+ 4\epsilon_3^+(q_3) \cdot p_4\epsilon_2^-(q_2) \cdot \epsilon_4^+(q_4) \\
&- 4\epsilon_1^-(q_1) \cdot p_2\epsilon_2^-(q_2) \cdot \epsilon_3^+(q_3)\epsilon_4^+(q_4) \cdot p_3 \\
&- 2\epsilon_2^-(q_2) \cdot p_1\epsilon_3^+(q_3) \cdot \epsilon_4^+(q_4)\epsilon_3^+(q_3) \cdot (p_3 - p_4) \\
&- 4\epsilon_1^-(q_1) \cdot \epsilon_4^+(q_4)\epsilon_3^+(q_3) \cdot p_4\epsilon_2^-(q_2) \cdot p_1 \\
&\left. +4\epsilon_1^-(q_1) \cdot \epsilon_3^+(q_3)\epsilon_2^-(q_2) \cdot p_1\epsilon_4^+(q_4) \cdot p_3\right]
\end{aligned} \tag{2.72}$$

It is time to make a choice for the reference momenta: $q_1 = q_2 = p_1$ and $q_3 = q_4 = p_4$. This choice will considerably simplify this expression. In fact,

[2] This particular consideration will be useful in chapter 4 (section 4.2), where it will be used in relation to the algorithm proposed in [49] and to define part of the notation.

due to the identities 2.64, only the contractions $\epsilon_2^- \cdot \epsilon_3^+$ will not vanish. This gives in the end:

$$A_4^{tree}(1^-,2^-,3^+,4^+) = \frac{i}{s_{12}} \left[-4\epsilon_1^-(p_4)\cdot p_2 \epsilon_2^-(p_4)\cdot \epsilon_3^+(p_1)\epsilon_4^+(p_1)\cdot p_3\right] \quad (2.73)$$

Now, applying the definition of the polarization vectors, Gordon identity and Fierz rearrangement, we finally obtain:

$$\begin{aligned}
&A_4^{tree}(1^-,2^-,3^+,4^+) = \frac{i}{s_{12}} \left[-4\epsilon_1^-(p_4)\cdot p_2 \epsilon_2^-(p_4)\cdot \epsilon_3^+(p_1)\epsilon_4^+(p_1)\cdot p_3\right] \\
&= \frac{-4i}{s_{12}} \left(-\frac{\langle p_4^+|\gamma_\mu|p_1^+\rangle}{\sqrt{2}\langle p_4^+|p_1^-\rangle}\cdot p_2^\mu\right)\left(-\frac{\langle p_4^+|\gamma_\nu|p_2^+\rangle}{\sqrt{2}\langle p_4^+|p_2^-\rangle}\cdot\frac{\langle p_1^-|\gamma^\nu|p_3^-\rangle}{\sqrt{2}\langle p_1^-|p_3^+\rangle}\right)\left(\frac{\langle p_1^-|\gamma_\rho|p_4^-\rangle}{\sqrt{2}\langle p_1^-|p_4^+\rangle}\cdot p_3^\rho\right) \\
&= -2i\frac{\langle 12\rangle [34]^2}{[12][14]\langle 14\rangle}
\end{aligned} \quad (2.74)$$

This result can be now put into a more revealing form. We will just need momentum conservation, antisymmetry of the spinor products and the good old trick of multiplying by 1 and in few steps we can re-express the amplitude as:

$$\begin{aligned}
A_4^{tree}(1^-,2^-,3^+,4^+) &= -2i\frac{\langle 12\rangle [34]^2}{[12][14]\langle 14\rangle} = -2i\frac{\langle 12\rangle [34]^2}{[12][14]\langle 14\rangle}\frac{\langle 23\rangle\langle 34\rangle}{\langle 23\rangle\langle 34\rangle} \\
&= 2i\frac{\langle 12\rangle(-\langle 21\rangle[14])([12]\langle 12\rangle)}{[12]\langle 23\rangle\langle 34\rangle\langle 41\rangle[14]} \\
&= 2i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned} \quad (2.75)$$

or, following a more standard notation:

$$A_4^{tree}(1^-,2^-,3^+,4^+) = 2i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad (2.76)$$

In order to obtain all the possible helicity configurations for the colour ordered 4-gluons amplitude, we have to calculate another amplitude: $A_4^{tree}(1^-,2^+,3^-,4^+)$. To obtain the remaining ones, in fact, we can simply flip all helicities simultaneously (i.e. $\langle\rangle \leftrightarrow [\ \ ]$). The missing amplitude can be calculated in a most simple way using the decoupling identity 2.37:

$$\begin{aligned}
A_4^{tree}(1^-,2^+,3^-,4^+) &= -A_4^{tree}(1^-,3^-,2^+,4^+) - A_4^{tree}(1^-,3^-,4^+,2^+) \\
&= -2i\left[\frac{\langle 13\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle} + \frac{\langle 13\rangle^4}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle}\right]
\end{aligned} \quad (2.77)$$

and, after using Schouten identity and some algebra:

$$A_4^{tree}(1^-,2^+,3^-,4^+) = 2i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{2.78}$$

Finally, one finds [64, 65] that this is a general rule (*Parke-Taylor formulae*) for expressing n-gluons amplitudes with 2 legs having helicity λ and all the others having opposite helicity:

$$A_4^{tree}(1^+,2^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i\left(\sqrt{2}^{n-2}\right)\frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle (n-1)n\rangle\langle n1\rangle} \tag{2.79}$$

or, flipping all the helicities:

$$A_4^{tree}(1^-,2^-,\ldots,j^+,\ldots,k^+,\ldots,n^-) = i\left(\sqrt{2}^{n-2}\right)\frac{[kj]^4}{[1n]\,[n(n-1)]\ldots[21]} \tag{2.80}$$

These particular amplitudes are called *MHV amplitudes* (Maximally Helicity Violating), because they accomplish for the maximum possible violation of helicity conservation. The reason to introduce this particular notation is that we will be using it in part of the calculations in the next chapter when we will finally present BCJ-relations and some particular examples. In addition, this particular way of describing gluon amplitudes, will represent a good analogy for the construction and relative discussion presented in chapter 5.

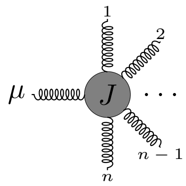
How can we treat amplitudes where three of the gluons have opposite helicity? We can call these amplitudes: *next to maximally helicity violating* (NMHV). In [66] the authors proposed a simple formulation making use just of propagators and some peculiar vertices which are in fact an off-shell continuation of the Parke Taylor amplitudes. That construction however does not show manifest Lorentz invariance which is instead auspicable, especially in view of using tree amplitudes as building blocks for loop calculations.[3]
In addition, in [67], in order to obtain a manifestly Lorentz invariant formulation, Kosower presented an improvement of the CSW method where the dependence on the reference momentum is shown to disappear.

[3] The importance of tree level amplitudes and BCJ-relations in the construction of both Yang Mills and Gravity scattering loop amplitudes will be underlined in the next chapter, with a special focus on how colour-kinematic duality and gauge-gravity duality coope to form a particularly intriguing picture.

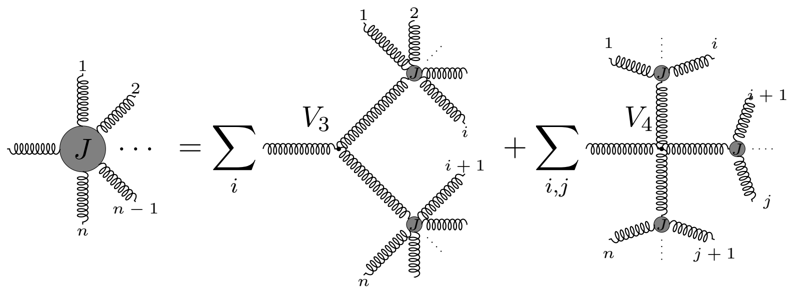
2.5.2 Berends Giele recursion

Another particular way of speeding up calculations[4], is given by the recursive techniques described by Berends and Giele in [65]. These recursive techniques have been largely used by the author to implement and test the algorithm of chapter 4.

The first step consists in defining the off-shell current $J^\mu(1, 2, \ldots, n)$ as the sum of the $(n + 1)$-point graphs where n gluons are on shell and the remaining "μ" leg is off shell.



The procedure to obtain such a current corresponds to taking an amplitude $A_n^{tree}(1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}, (n+1)^{\lambda_{n+1}})$, removing the polarization vector $\epsilon^\mu(p_{n+1})$ for the $n+1$ leg and multiplying by a propagator $\frac{-ig_{\mu\nu}}{p_{n+1}^2}$. On the other side, in order to obtain an on-shell amplitude from a Berends-Giele current, we have to perform the opposite steps. The recursive idea behind off-shell currents is that if we follow the off-shell line back into the diagram, we will encounter either a three vertex or a four vertex. Then, each line out of that vertex, will connect to a number of on-shell gluons and so on. This idea, can be seen in a pretty eloquent way as:



which is just a recast version of the formula:

$$
\begin{aligned}
J_\mu(1, \ldots, n) = & \frac{-ig_{\mu\alpha}}{p_{1,n}^2} \left[\sum_{i=1}^{n-1} V_3^{\alpha\nu\rho}(p_{1,i}, p_{i+1,n}) J_\nu(1, \ldots, i) J_\rho(i + 1 \ldots, n) \right. \\
& \left. + \sum_{j=i+1}^{n-1} \sum_{i=1}^{n-2} V_4^{\alpha\nu\rho\sigma} J_\nu(1, \ldots, i) J_\rho(i + 1, \ldots, j) J_\sigma(j + 1, \ldots, n) \right]
\end{aligned}
\tag{2.81}
$$

[4] In this approach, no Feynman diagrams have to be calculated.

where V_3 and V_4 are just:

$$\begin{aligned} V_3^{\mu\nu\rho}(P,Q) &= i\left(g^{\nu\rho}(P-Q)^\mu + 2g^{\rho\mu}Q^\nu - 2g^{\mu\nu}P^\rho\right) \\ V_4^{\mu\nu\rho\sigma} &= i\left(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \end{aligned} \tag{2.82}$$

and the base of the recursion is clearly:

$$J^\mu(p_i^{\lambda_i}) = \epsilon^\mu(p_i^{\lambda_i}, q) \tag{2.83}$$

The offshell currents satisfy a number of properties, partly inherited by partial amplitudes. We have the photon decoupling identity:

$$J^\mu(1,2,3,\ldots,n) + J^\mu(2,1,3,\ldots,n) + J^\mu(2,3,1,\ldots,n) + \ldots + J^\mu(2,3,\ldots,1,n) = 0 \tag{2.84}$$

the reflection identity:

$$J^\mu(1,2,3,\ldots,n) = (-1)^{n+1} J^\mu(n,\ldots,3,2,1) \tag{2.85}$$

and current conservation:

$$p_{1,n}^\mu J_\mu(1,2,3,\ldots,n) = 0 \tag{2.86}$$

This last property can be proved inductively and depends on the gauge fixing choice made. We invite the interested reader to read the details in appendix A. However, it is also important to notice that off-shell currents, contrary to partial amplitudes, are gauge dependent objects. This means for example that they depend on the choices of reference momenta made for the on-shell gluons. With respect to this, one should take particular care in not changing those reference momenta before extracting an on-shell quantity from the current.

3

BCJ RELATIONS

3.1 UV finiteness of gravity

We will see in the following sections how BCJ-relations can be used to simplify the exploration of the ultra violet properties of gravity. In view of this perspective, it is useful to compare briefly the structures of the two theories involved. This will in turn give us a first hint on the underlying idea of colour kinematic duality.

In order to give a compact and still good introduction to the topic and the relation to gravity, we will follow a review paper by Bern, Carrasco and Johansson [16].

The common knowledge of the previous decades was that constructing a point-like ultraviolet finite quantum field theory of gravity perturbatively was an impossible task in four dimensions (e.g. [68]). In fact, a series of paper clearly showed that gravity coupled to matter diverges at one loop in four dimensions [69–74]. In particular, the dimensionful character of the coupling implies that the divergences cannot be reabsorbed by a simple redefinition of a finite number of parameters, i.e. the theory is non-renormalizable. Moreover, pure Einstein theory also shows a divergence that cannot be cured with a viable counterterm at one loop (see for example [75]); this in turn delays the divergence to two loops [69, 76–80].
The whole work done through supersymmetry pointed to further delay divergences to higher loops. However, all the theories to date fail with this protection mechanism at some loop order.
The authors of [16] are challenging the belief that this finiteness search is destined to fail.
In particular, working with $\mathcal{N} = 8$ supergravity [81–83] at three loops order [84, 85], they found evidence that another mechanism to prevent divergencies indeed exists. In particular, the authors stress that every explicit complete calculation of $\mathcal{N} = 8$ supergravity scattering amplitudes [84–94] find an iden-

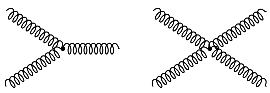
tical powercounting to that of $\mathcal{N} = 4$ super-Yang-Mills theory, which is finite in four dimensions.
The importance of this work - for our case - is given by the fact that exactly during those calculations BCJ-relations were discovered by noticing recurring structures among tree amplitudes used as building blocks.
The UV exploration of gravity is mainly performed making use of two categories of on-shell methods. However, before focusing on what these mechanisms are, we will briefly show why a standard Feynman diagrammatic approach is not optimal.

3.2 Yang Mills theories and Gravity compared

Let us start with the Einstein-Hilbert lagrangian:

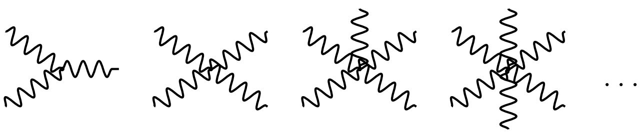
$$\mathcal{L}_{EH} = \frac{2}{\kappa^2}\sqrt{-g}R \tag{3.1}$$

where g is the determinant of the metric tensor and R is the curvature scalar.[1] κ is related to Newton's constant by $\kappa^2 = 32\pi^2 G_N$. Let us compare in a diagrammatic way the outcomes of Lagrangian 2.7 and Lagrangian 3.1. Starting from the Yang Mills lagrangian we have seen that, making standard gauge choices, one obtains three- and four-points interactions.



(3.2)

Instead, Einstein Hilbert's Lagrangian gives an infinite number of contact interactions:



[1] In this section, we will break for a while consistency with our previous and following notation. We will in fact indicate a general metric tensor with $g_{\mu\nu}$ while the Minkowski metric tensor will be given by $\eta_{\mu\nu}$.

In addition, by taking a look at the expression for the three graviton vertex, one is immediately stricken by the complexity of these interactions:

$$
\begin{aligned}
G_{3\mu\alpha,\nu\beta,\sigma\gamma} = \frac{i\kappa}{2} sym \Big[& -\frac{1}{2}P_3\left(p_1\cdot p_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\right) + 2P_3\left(p_{1\nu}p_{2\mu}\eta_{\gamma\alpha}\eta_{\beta\sigma}\right) \\
& + P_6\left(p_1\cdot p_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\right) - 2P_3\left(p_1\cdot p_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\right) - \frac{1}{2}P_6\left(p_{1\nu}p_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}\right) \\
& + 2P_3\left(p_{1\nu}p_{1\gamma}\eta_{\mu\alpha}\eta_{\sigma\beta}\right) - P_3\left(p_{1\beta}p_{2\mu}\eta_{\nu\alpha}\eta_{\sigma\gamma}\right) + P_3\left(p_{1\sigma}p_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\right) \\
& + P_6\left(p_{1\sigma}p_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\right) + 2P_6\left(p_{1\nu}p_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}\right) + \frac{1}{2}P_3\left(p_1\cdot p_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\right)\Big]
\end{aligned} \tag{3.3}
$$

"sym" stands for a symmetrization of the indices $\mu \leftrightarrow \alpha, \nu \leftrightarrow \beta, \sigma \leftrightarrow \gamma$, while P_3 and P_6 stand for symmetrization over the three external legs, generating respectively three or six terms. This means that the expression written above contains approximately one hundred terms. If one compared this to the expression for a gluon three vertex, the conclusion would be that gravity seems much more complicated than gauge theory.

3.3 Two important hints: On shell approach and KLT relations

One of the key aspects of on-shell methods, that partially explain their efficiency, is that the building blocks utilized are previously calculated on-shell amplitudes. In order to do so, one generally calculates tree level amplitudes making use of on-shell recursion [3, 4] and then moves on to loop level making use of the unitarity method [5–7].
Using an on-shell approach immediately allows for some simplifications in the expressions for the vertices. As an example, let us compare the three graviton vertex of equation 3.3 with its on-shell version. We will contract the expression for the vertex with polarization tensors $\epsilon^{\mu\nu}$ associated with the external legs and satisfying the on-shell conditions $p_i^2 = \epsilon_i^{\mu\nu}(p_i)_\mu = \epsilon_i^{\mu\nu}(p_i)_\nu = \epsilon^\mu_\mu = 0$:

$$
G_3(p_1,p_2,p_3) = -\frac{i\kappa}{2}\epsilon_1^{\mu\alpha}\epsilon_2^{\nu\beta}\epsilon_3^{\sigma\gamma}\left[(p_1-p_2)_\sigma\eta_{\mu\nu} + \text{ cyclic}\right]\left[(p_1-p_2)_\gamma\eta_{\alpha\beta} + \text{ cyclic}\right] \tag{3.4}
$$

as we can see, this is not really much more complicated than the on-shell three gluon vertex:

$$
V_3^{abc}(p_1,p_2,p_3) = \epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\sigma g f^{abc}\left[(p_1-p_2)_\sigma\eta_{\mu\nu} + \text{ cyclic}\right] \tag{3.5}
$$

At this point, it is important to make two remarks. First of all, as already mentioned in the previous chapter, the on-shell three gluon vertex - as well

as the three graviton vertex - actually vanish due to the kinematical impossibility of the process. This is the reason why on-shell methods involve the introduction of complex kinematics, which allows us to have momentum conservation, the on-shell condition and a non-vanishing quantity at the same time. The second remark is that the structure of the on-shell graviton vertex as a double copy of the Yang-Mills one is not a mere coincidence and reveals in fact an important property of gravity. This is the first hint one can consider for the existence of BCJ-relations and colour-kinematic duality as a basis for gauge-gravity duality.
Before we move on to a deeper analysis of the idea, we can make another technical remark about the polarization tensors for the gravitons. In the previous chapter, in eq. 2.58, we have seen how to construct polarization vectors for gluons. As it turns out, for gravitons one needs only to take a product of such vectors:

$$\epsilon^{\pm}_{\mu\alpha}(p,q) = \epsilon^{\pm}_{\mu}(p,q)\epsilon^{\pm}_{\alpha}(p,q) \tag{3.6}$$

The spinor product notation introduced before, as a consequence, applies to the treatment of gravity scattering amplitudes as we will see soon.

As we said in the introduction, one of the strinking properties of BCJ-relations is that they allow us to construct tree-level gravity amplitudes starting from gauge theory amplitudes. This fact was first discovered in string theory by Kawai, Lewellen and Tye (KLT) [95–102]. One can derive their validity in quantum field theory as a low-energy limit of string theory. As an example, we will show KLT relations in this limit for four-, five- and six-points amplitudes:

$$\begin{aligned}
M_4^{tree}(1,2,3,4) &= -is_{12}A_4^{tree}(1,2,3,4)\tilde{A}_4^{tree}(1,2,4,3) \\[2ex]
M_5^{tree}(1,2,3,4,5) &= is_{12}s_{34}A_5^{tree}(1,2,3,4,5)\tilde{A}_5^{tree}(2,1,4,3,5) \\
&\quad + is_{13}s_{24}A_5^{tree}(1,3,2,4,5)\tilde{A}_5^{tree}(3,1,4,2,5) \\[2ex]
M_6^{tree}(1,2,3,4,5,6) &= -is_{12}s_{45}A_6^{tree}(1,2,3,4,5,6)\Big[s_{35}\tilde{A}_6^{tree}(2,1,5,3,4,6) \\
&\quad +(s_{34}+s_{35})\tilde{A}_6^{tree}(2,1,5,4,3,6)\Big] + \mathcal{P}(2,3,4)
\end{aligned} \tag{3.7}$$

where the M_n are gravity amplitudes and A_n and $\tilde{A}_n$ are colour ordered amplitudes of two, possibly different, gauge theories. $\mathcal{P}(2,3,4)$ stands for a sum over all permutations of the labels 2,3 and 4.

It is also possible to generalize the relations to n-points [103]:

$$
\begin{aligned}
M_n^{tree}(1,2,\ldots,n) = i(-1)^{n+1}\Big[A_n^{tree}(1,2,\ldots,n) \sum_{perms} f(i_1,\ldots,i_j)\bar{f}(l_1,\ldots,l_{j'}) \\
\times \tilde{A}_n^{tree}(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_{j'},n)\Big] \\
+\mathcal{P}(2,\ldots,n-2)
\end{aligned}
\tag{3.8}
$$

where the sum is over all the permutations $\{i_1,\ldots,i_j\} \in \mathcal{P}\{2,\ldots,[n/2]\}$ and $\{l_1,\ldots,l_{j'}\} \in \mathcal{P}\{[n/2]+1,\ldots,n-2\}$ with $j = [n/2]-1$ and $j' = [n/2]-2$. This means that inside the square brackets there is a total of $([n/2]-1)!([n/2]-2)!$ terms.
The functions f and $\bar{f}$ are given by:

$$
\begin{aligned}
f(i_1,\ldots,i_j) &= s_{1,i_j}\prod_{m=1}^{j-1}\left(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m,i_k)\right) \\
\bar{f}(l_1,\ldots,l_{j'}) &= s_{l_1,n-1}\prod_{m=2}^{j'}\left(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k,l_m)\right)
\end{aligned}
\tag{3.9}
$$

with the function g being:

$$
g(i,j) = \begin{cases} s_{i,j} & \text{if } i<j \\ 0 & \text{else} \end{cases}
\tag{3.10}
$$

Let us see what KLT relations tell us for a practical example. Let us calculate $M_4^{tree}(1^-,2^-,3^+,4^+)$. We have already calculated the colour ordered amplitudes $A_4^{tree}(1^-,2^-,3^+,4^+)$ and $A_4^{tree}(1^-,2^-,4^+,3^+)$ in the previous chapter. According to eq. 3.7 we have then:

$$
\begin{aligned}
M_4^{tree}(1^-,2^-,3^+,4^+) &= -is_{12}A_4^{tree}(1^-,2^-,3^+,4^+)\tilde{A}_4^{tree}(1^-,2^-,4^+,3^+) \\
&= i4s_{12}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}
\end{aligned}
\tag{3.11}
$$

If one would calculate the same amplitude starting from the Einstein-Hilbert Lagrangian, one would obtain the same result. The amplitude shown above is stripped of the couplings. In order to obtain the full amplitude one needs to restore the $\left(\frac{\kappa}{2}\right)^{n-2}$ factor:

$$\mathcal{M}_4^{tree}(1^-,2^-,3^+,4^+) = i\kappa^2 s_{12} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle} \quad (3.12)$$

This means that to obtain full gravity amplitudes from eq. 3.8 one has to multiply the result in this way:

$$\mathcal{M}_n^{tree}(1,2,\ldots,n) = \left(\frac{\kappa}{2}\right)^{n-2} M_n^{tree}(1,2,\ldots,n) \quad (3.13)$$

As we will see in the next section, KLT relations can be reformulated in a - from a QFT point of view - clearer way, which makes at the same time manifest the existence of colour-kinematic duality. It is finally time to introduce BCJ-relations.

3.4 BCJ relations

In this section we will follow mainly reference [20], in order to give a pedagogical and yet illuminating description of BCJ-relations.
The idea is to reason on some peculiar properties of colour ordered aplitudes at four points. This will unveil deep and not manifest qualities of kinematical factors.
The next step will involve reasoning on a five-points example. This will have two effects for the purpose of this thesis. First of all, it will reveal the general idea of BCJ of as how to obtain BCJ-numerators and which freedom of choice is involved[2]. Secondly, it will make manifest how difficult and time consuming this process is at the level of amplitudes. The motivation for our effort to find a prescription at the Lagrangian level will then be a little bit clearer.

3.4.1 Four-points analysis

As we said, we will start at four-points. We introduced in section 2.4 the photon decoupling identity 2.37 satisfied by colour ordered gluon amplitudes. At four points, it reads:

$$A_4^{tree}(1,2,3,4) + A_4^{tree}(1,2,4,3) + A_4^{tree}(1,4,2,3) = 0 \quad (3.14)$$

[2] More on this topic however will be said in the following sections on general gauge transformations and in the next chapter.

Tree amplitudes are in general rational functions of polarization vectors, spinors, momenta and Mandelstam invariants $s \equiv s_{12} = (p_1 + p_2)^2$, $t \equiv s_{14} = (p_1 + p_4)^2$, $u \equiv s_{13} = (p_1 + p_3)^2$. The photon decoupling identity does not depend on the polarization choice or the number of dimensions. This in turn excludes that it could rely on four-dimensions spinor identities (as the ones introduced in the previous chapter). We are left with a dependence of the identity on Mandelstam variables. In particular, what ensures that the identity is satisfied in a non trivial way, is the formal equivalence of the photon decoupling identity to the vanishing of $s + t + u$:

$$A_4^{tree}(1,2,3,4) + A_4^{tree}(1,2,4,3) + A_4^{tree}(1,4,2,3) = (s+t+u)\chi = 0 \quad (3.15)$$

From this expression, we can see that the three amplitudes must be proportional to each other. In addition, we know that $A_4^{tree}(1,2,3,4)$ treats factors of s and t in the same way. This means in turn that it must be proportional to $u = -(s+t)$. Repeating the same steps for the other amplitudes involved in the identity finally gives:

$$A_4^{tree}(1,2,3,4) = \chi u \qquad A_4^{tree}(1,3,4,2) = \chi t \qquad A_4^{tree}(1,4,2,3) = \chi s \quad (3.16)$$

which is of course compatible with eq. 3.15.
Eliminating χ we obtain a new set of relations among these four-point amplitudes:

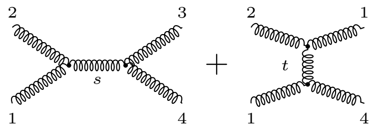
$$\begin{aligned} tA_4^{tree}(1,2,3,4) &= uA_4^{tree}(1,3,4,2) \\ sA_4^{tree}(1,2,3,4) &= uA_4^{tree}(1,4,2,3) \\ tA_4^{tree}(1,4,2,3) &= sA_4^{tree}(1,3,4,2) \end{aligned} \quad (3.17)$$

The next step is to consider the possibility we have to write colour-ordered partial amplitudes in a representation that makes manifest the poles that appear in each diagram, i.e. as a sum of the contributions of the different channels:

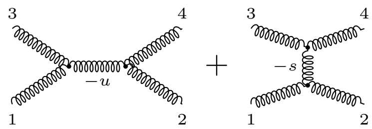
$$\begin{aligned} A_4^{tree}(1,2,3,4) &= \frac{n_s}{s} + \frac{n_t}{t} \\ A_4^{tree}(1,3,4,2) &= -\frac{n_s}{s} - \frac{n_u}{u} \\ A_4^{tree}(1,4,2,3) &= -\frac{n_t}{t} + \frac{n_u}{u} \end{aligned} \quad (3.18)$$

In order to use this representation, however, we have to take care of the four gluon vertices. In particular, we have to split them into contributions that sum to the channel contributions already present. This has been already done in literature with the use of a tensor particle [104].
A practical way to obtain such a representation is to consider the amplitudes from a diagramatic point of view (after the reabsorption of the quartic contact terms into the cubic ones):

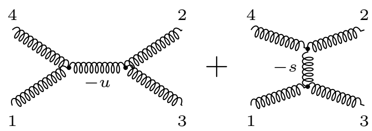
$A_4^{tree}(1,2,3,4) =$



$A_4^{tree}(1,3,4,2) =$ (3.19)



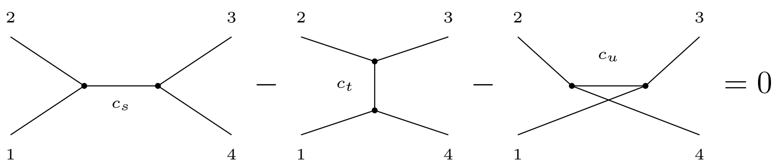
$A_4^{tree}(1,4,2,3) =$



Another way of seeing this is to consider eq. 3.18 as the expressions defining the n_i's; i.e. one has to calculate amplitudes with other techniques and afterwards solve for the n_i's.[3]
Up to now, we focused on the colour ordered sub-amplitudes. What about the colour factors related to this representation?
We are dealing with three different colour factors[4], composed of color structure constants, satisfying Jacobi identities as described in eq. 2.4.

$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4} \qquad c_t = f^{a_2 a_3 b} f^{b a_4 a_1} \qquad c_u = f^{a_4 a_2 b} f^{b a_3 a_1}$$
$$c_s - c_t - c_u = 0 \tag{3.20}$$

One can see the situation from a colour-diagrammatic point of view as:



(3.21)

[3] The sign of the n_i's is the result of a conventional starting choice plus the antisymmetry of colour ordered Feynman rules.
[4] Where the signs have been chosen in accordance with the convention used for the numerators above.

If we now combine eq. 3.18 and 3.17, we obtain for the corresponding kinematic factors:

$$n_s - n_t - n_u = 0 \tag{3.22}$$

We found the first sign of the correspondence between colour factors satisfying Jacobi identities and the related kinematic numerators doing so as well when the amplitudes are put in a particular representation.
Let us see this with an explicit example that will in turn illuminate us on another important characteristic of these relations. We can rewrite the amplitudes found above for a particular helicity configuration: let us choose $\{1^-, 2^-, 3^+, 4^+\}$.

$$A_4^{tree}(1,2,3,4) = 2i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -2i\frac{\langle 12\rangle^2[34]^2}{s_{12}s_{23}} = -2i\frac{\langle 12\rangle^2[34]^2}{st}$$

$$A_4^{tree}(1,3,4,2) = 2i\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle} = -2i\frac{\langle 12\rangle^2[34]^2}{s_{12}s_{13}} = -2i\frac{\langle 12\rangle^2[34]^2}{su}$$

$$A_4^{tree}(1,4,2,3) = 2i\frac{\langle 12\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle} = -2i\frac{\langle 12\rangle^2[34]^2}{s_{23}s_{13}} = -2i\frac{\langle 12\rangle^2[34]^2}{tu} \tag{3.23}$$

We can now comapare these expressions with eq. 3.18 and solve for the n_i's.

$$\begin{aligned} tn_s + sn_t &= -2i\langle 12\rangle^2[34]^2 \\ un_s + sn_u &= 2i\langle 12\rangle^2[34]^2 \\ -un_t + tn_s &= -2i\langle 12\rangle^2[34]^2 \end{aligned} \tag{3.24}$$

After some algebra, what we find is a rank 2 matrix for our system of three unknown quantities. The two relations we are left with are:

$$\begin{aligned} tn_s + sn_t &= -2i\langle 12\rangle^2[34]^2 \\ n_s - n_t - n_u &= 0 \end{aligned} \tag{3.25}$$

We need to make two remarks at this point. First of all, the second relation is exactly the BCJ-relation we were expecting and that corresponds to the colour factors Jacobi relation 3.20. Secondly, since we just have two relations

and three unknowns, we have an undefined system. This means that we are left with the freedom to set one of the n_i's to the value we prefer. We choose to set n_u to zero, which implies that $n_s = n_t = \frac{2i\langle 12\rangle^2[34]^2}{u}$.
The freedom we discovered to have in this example, is something deeply ingrained in the procedures to construct BCJ-numerators. We will see that in detail now, considering what that freedom actually corresponds to.[5]

Making use of expression 2.31 together with eq. 3.18 we also obtain an expression for the full colour dressed amplitude[6] as:

$$\mathcal{A}_4^{tree} = g^2\left(\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}\right) \tag{3.26}$$

Since terms like $\frac{n_s}{s}, \frac{n_t}{t}, \frac{n_u}{u}$ are in fact s-channel, t-channel and u-channel diagrams, we know that they cannot be gauge invariant quantities. In particular they depend on the field variable choices we made. The choice we made before when solving for the n_i's is one example of such freedom.
If we impose that the BCJ-numerators are local quantities, this freedom corresponds to all the terms the can be added that cancel the poles in the expansion 3.26. This operation can be called a *general gauge transformation*, cause it resembles the well known effect of gauge transformations of moving terms from one diagram to another.[7] One way to parametrize this transformation is:

$$n'_s = n_s + \alpha s \tag{3.27}$$

where $\alpha \equiv \alpha(p_i, \epsilon_i)$ is a parameter that depends on momenta and polarization vectors and is local.
If we want to maintain the amplitudes 3.18 unchanged, we must of course change the other numerators accordingly:

$$n'_t = n_t - \alpha t \qquad n'_u = n_u - \alpha u \tag{3.28}$$

This tern of transformations is exactly what we need to satisfy 3.22:

$$n'_s - n'_t - n'_u = n_s - n_t - n_u + \alpha(s + t + u) = 0 \tag{3.29}$$

One can see that 3.22 is true in any gauge at four points.
However this is not true at higher-points. In fact, only specific choices of the

[5] We will discuss in the last chapter a construction that makes use of scattering equations[52–54], where it seems possible to obtain a canonical set of BCJ-numerators free from ambiguities [56].

[6] In splitting the four gluon contact terms one must ensure that no cross terms appear.

[7] This however does not imply the existence of a gauge transformation that actually does so.

numerators will satisfy the higher-point Jacobi-like relations for kinematic numerators. In particular, if one starts from standard Feynman rules as introduced in the first chapter, the relations will in general not be satisfied. We decided before to keep α local. However, in 3.22 and 3.29 we never made use of locality. This means that in fact α can be non-local. This is for example the case of our example above, where setting $n_u = 0$ meant in fact performing a shift with $\alpha = \frac{n_u}{u}$. This gave in turn non local values for n_s and n_t. They both contained in fact a new u pole.

3.4.2 Higher points generalization

This section will cover three important roles. Here we will in fact generalize BCJ relations to higher points, show how time consuming and non-trivial is to obtain BCJ-numerators and at the same time introduce one of the first main consequences of the existence of BCJ relations.
The idea is that for every tern of dependent colour factors $c_\alpha, c_\beta, c_\gamma$ satisfying Jacobi identities, it is possible to write colour ordered sub-amplitudes in such a way that the corresponding kinematic factors $n_\alpha, n_\beta, n_\gamma$ satisfy the same relation.
We will follow once again [20] and use a five-point example to show how this is possible and what is the first visible consequence of this construction.
The first step is to write again the sub-amplitudes as sums over the possible diagrams that compose them[8].For example, we have

$$A_5^{tree}(1,2,3,4,5) = \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}} \tag{3.30}$$

In general, at five points, there are five diagrams that can appear for each colour-order considered. This corresponds to the general law that for an n-point colour-ordered amplitude, the number of diagrams appearing is given by $\frac{2^{n-2}(2n-5)!!}{(n-1)!}$.
We have a total of fifteen diagrams appearing in a general colour dressed amplitude at five points, corresponding to the n-law: $(2n-5)!!$. This means that we can write a five-point colour-dressed amplitude as:

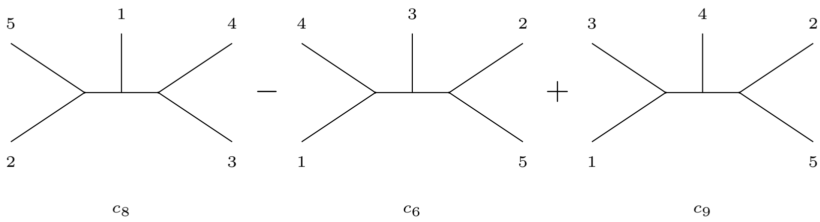
$$\begin{aligned}\mathcal{A}_5^{tree} = g^3\Big(&\frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{23}s_{51}} + \frac{c_3 n_3}{s_{34}s_{12}} + \frac{c_4 n_4}{s_{45}s_{23}} + \frac{c_5 n_5}{s_{51}s_{34}} + \frac{c_6 n_6}{s_{14}s_{25}} \\ &+ \frac{c_7 n_7}{s_{32}s_{14}} + \frac{c_8 n_8}{s_{25}s_{43}} + \frac{c_9 n_9}{s_{13}s_{25}} + \frac{c_{10} n_{10}}{s_{42}s_{13}} + \frac{c_{11} n_{11}}{s_{51}s_{42}} + \frac{c_{12} n_{12}}{s_{12}s_{35}} \\ &+ \frac{c_{13} n_{13}}{s_{35}s_{24}} + \frac{c_{14} n_{14}}{s_{14}s_{35}} + \frac{c_{15} n_{15}}{s_{13}s_{45}}\Big)\end{aligned} \tag{3.31}$$

[8] We are assuming again that only three-valent vertices are present

where the colour factors are given by:

$$
\begin{aligned}
&c_1 = f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5} && c_2 = f^{a_2a_3b}f^{ba_4c}f^{ca_5a_1} && c_3 = f^{a_3a_4b}f^{ba_5c}f^{ca_1a_2}\\
&c_4 = f^{a_4a_5b}f^{ba_1c}f^{ca_2a_3} && c_5 = f^{a_5a_1b}f^{ba_2c}f^{ca_3a_4} && c_6 = f^{a_1a_4b}f^{ba_3c}f^{ca_2a_5}\\
&c_7 = f^{a_3a_2b}f^{ba_5c}f^{ca_1a_4} && c_8 = f^{a_2a_5b}f^{ba_1c}f^{ca_4a_3} && c_9 = f^{a_1a_3b}f^{ba_4c}f^{ca_2a_5}\\
&c_{10} = f^{a_4a_2b}f^{ba_5c}f^{ca_1a_3} && c_{11} = f^{a_5a_1b}f^{ba_3c}f^{ca_4a_2} && c_{12} = f^{a_1a_2b}f^{ba_4c}f^{ca_3a_5}\\
&c_{13} = f^{a_3a_5b}f^{ba_1c}f^{ca_2a_4} && c_{14} = f^{a_1a_4b}f^{ba_2c}f^{ca_3a_5} && c_{15} = f^{a_1a_3b}f^{ba_2c}f^{ca_4a_5}
\end{aligned}
\tag{3.32}
$$

These colour factors will again satisfy Jacobi relations as in the example below:



$$= 0 \tag{3.33}$$

How can we move terms between diagrams in order to obtain the same set of identities for the kinematic part?
The first step is to recognize that - thanks to Kleiss-Kuijf relations 2.39 - in order to have enough information on the whole fifteen numerators, we will need just the set of $(5-2)! = 6$ independent amplitudes:

$$
\begin{aligned}
A_5^{tree}(1,2,3,4,5) &= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}\\
A_5^{tree}(1,4,3,2,5) &= \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}}\\
A_5^{tree}(1,3,4,2,5) &= \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}\\
A_5^{tree}(1,2,4,3,5) &= \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}\\
A_5^{tree}(1,4,2,3,5) &= \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}\\
A_5^{tree}(1,3,2,4,5) &= \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}}
\end{aligned}
\tag{3.34}
$$

The number of colour-factors that are independent corresponds to the number of amplitudes that are independent under Kleiss-Kuijf relations (see for example [23]). These means that of the fifteen c_i's only 6 are independent. If we are to generalize Jacobi relations among colour factors to kinematic numerators, we have then that the number of independent kinematic numerators is also $(n-2)!$ (6 in our case). We can then decide to choose 6 numerators among the fifteen ones and write all the KK-amplitudes in terms of those. This however is not the optimal way to do it, since every numerator corresponds in fact to a non-gauge-invariant quantity. The best way to minimize the gauge-dependent choices is to choose 6 numerators (e.g. $n_1, n_2, n_3, n_4, n_5, n_6$) and express 2 of them in terms of the gauge-invariant amplitudes they are part of:

$$\begin{aligned} n_5 &\equiv s_{51}s_{34}\Big(A_5^{tree}(1,2,3,4,5) - \frac{n_1}{s_{12}s_{45}} - \frac{n_2}{s_{23}s_{51}} - \frac{n_3}{s_{34}s_{12}} - \frac{n_4}{s_{45}s_{23}}\Big) \\ n_6 &\equiv s_{14}s_{25}\Big(A_5^{tree}(1,4,3,2,5) - \frac{n_5}{s_{43}s_{51}} - \frac{n_7}{s_{32}s_{14}} - \frac{n_8}{s_{25}s_{43}} - \frac{n_2}{s_{51}s_{32}}\Big) \end{aligned} \tag{3.35}$$

In this way, we ensure also the compatibility of $A_5^{tree}(1,2,3,4,5)$ and $A_5^{tree}(1,4,3,2,5)$. The fact that n_7 and n_8 appear in the description of n_6 does not constitute a problem because later we will see that they in fact only depend on n_1, n_2, n_3, n_4. In order to generalize BCJ-relations, we have to require that the numerators saitsfy Jacobi-like relations in accordance with what the corresponding colour factors do. So, for example, following the example of eq. 3.33:

$$c_8 - c_6 + c_9 = 0 \quad \Longrightarrow \quad n_8 - n_6 + n_9 = 0 \tag{3.36}$$

Following the same principle, we can evaluate all the relations among colour factors using Jacobi identities and subsequently impose the same relations on the kinematic factors:

$$\begin{aligned} &n_3 - n_5 + n_8 = 0, \\ &n_3 - n_1 + n_{12} = 0, \\ &n_4 - n_1 + n_{15} = 0, \\ &n_4 - n_2 + n_7 = 0, \\ &n_5 - n_2 + n_{11} = 0, \\ &n_7 - n_6 + n_{14} = 0, \\ &n_8 - n_6 + n_9 = 0, \\ &n_{10} - n_9 + n_{15} = 0, \\ &n_{10} - n_{11} + n_{13} = 0, \\ &n_{13} - n_{12} + n_{14} = 0 \end{aligned} \tag{3.37}$$

As we can see, we have ten numerator identities, corresponding to the n-law for the number of numerator identities: $\frac{(n-3)(2n-5)!!}{3}$.
We can now proceed to solve this system of equations for the nine dependent numerators:

$$\begin{aligned}
n_7 &= n_2 - n_4, \\
n_8 &= -n_3 + n_5, \\
n_9 &= n_3 - n_5 + n_6, \\
n_{10} &= -n_1 + n_3 + n_4 - n_5 + n_6, \\
n_{11} &= n_2 - n_5, \\
n_{12} &= n_1 - n_3, \\
n_{13} &= n_1 + n_2 - n_3 - n_4 - n_6, \\
n_{14} &= -n_2 + n_4 + n_6, \\
n_{15} &= n_1 - n_4
\end{aligned} \tag{3.38}$$

and, replacing the new-found terms in n_6:

$$\begin{aligned}
n_6 =& A_5^{tree}(1,4,3,2,5)s_{14}s_{25} - A_5^{tree}(1,2,3,4,5)(s_{15}+s_{25})s_{14} + n_1\frac{(s_{15}+s_{25})s_{14}}{s_{12}s_{45}} \\
&+ n_2\frac{s_{23}+s_{35}}{s_{23}} + n_3\frac{s_{14}}{s_{12}} + n_4\frac{s_{14}s_{15}+s_{14}s_{25}+s_{25}s_{45}}{s_{23}s_{45}}
\end{aligned} \tag{3.39}$$

Now it is also possible to re-write all the Kleiss-Kuijf amplitudes in terms of $n_1, n_2, n_3, n_4, A_5^{tree}(1,2,3,4,5), A_5^{tree}(1,4,3,2,5)$. Of the six parameters, the two colour-ordered amplitudes are gauge invariant quantities, while the other four are gauge-dependent numerators. This of course forces some tests in order to check if all the known properties of amplitudes are still satisfied.
In order to do that, we can inspect all the factorization channels of five-point amplitudes.[9] In these limits, relations 3.37 become simply the numerator identity 3.22 discussed in the previous section. Since - in these limits - numerators relations are satisfied and what we are left with are gauge-invariant quantities, we must consider possible violating terms that are proportional to quantities that vanish in those limits, i.e. the propagator structures themselves. We have to consider then transformations of this kind:

$$\begin{aligned}
n_1' &= n_1 + \alpha_1 s_{12}s_{45} \quad & n_2' &= n_2 + \alpha_2 s_{23}s_{51} \\
n_3' &= n_3 + \alpha_3 s_{34}s_{12} \quad & n_4' &= n_4 + \alpha_4 s_{45}s_{23}
\end{aligned} \tag{3.40}$$

[9] In the limit where one propagator structure goes to zero, we can see the five point amplitude as a product of a three- and a four-point amplitude.

Thanks to the construction performed before, the first two KK-amplitudes are invariant under these transformations, because the two numerators n_5, n_6 will change exactly in such a way to correctly reproduce them:

$$
\begin{aligned}
n'_5 &= n_5 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) s_{51} s_{34} \\
\\
n'_6 &= n_6 + \alpha_3 s_{12} s_{14} + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) s_{14} s_{15} + (\alpha_1 + \alpha_3 + \alpha_4) s_{14} s_{25} \\
&\quad - \alpha_2 s_{15} s_{25} + \alpha_4 s_{25} s_{45}
\end{aligned}
\tag{3.41}
$$

What is left to test is if also the other four KK-amplitudes are invariant under these transformations. To do so, we have to calculate the shifts for each numerator and then plug them into the expressions 3.34. Remarkably, a series of non-trivial cancellations happen, which make the other four amplitudes indeed invariant under the transormations described above:

$$
\begin{aligned}
\Delta A_5^{tree}(1,3,4,2,5) &= \frac{\Delta n_9}{s_{13}s_{25}} - \frac{\Delta n_5}{s_{34}s_{51}} + \frac{\Delta n_{10}}{s_{42}s_{13}} - \frac{\Delta n_8}{s_{25}s_{34}} + \frac{\Delta n_{11}}{s_{51}s_{42}} = 0, \\
\\
\Delta A_5^{tree}(1,2,4,3,5) &= \frac{\Delta n_{12}}{s_{12}s_{35}} + \frac{\Delta n_{11}}{s_{24}s_{51}} - \frac{\Delta n_3}{s_{43}s_{12}} + \frac{\Delta n_{13}}{s_{35}s_{24}} - \frac{\Delta n_5}{s_{51}s_{43}} = 0, \\
\\
\Delta A_5^{tree}(1,4,2,3,5) &= \frac{\Delta n_{14}}{s_{14}s_{35}} - \frac{\Delta n_{11}}{s_{42}s_{51}} - \frac{\Delta n_7}{s_{23}s_{14}} - \frac{\Delta n_{13}}{s_{35}s_{42}} - \frac{\Delta n_2}{s_{51}s_{23}} = 0, \\
\\
\Delta A_5^{tree}(1,3,2,4,5) &= \frac{\Delta n_{15}}{s_{13}s_{45}} - \frac{\Delta n_2}{s_{32}s_{51}} - \frac{\Delta n_{10}}{s_{24}s_{13}} - \frac{\Delta n_4}{s_{45}s_{32}} - \frac{\Delta n_{11}}{s_{51}s_{24}} = 0
\end{aligned}
\tag{3.42}
$$

In the four-point example we have seen that the parameters α_i's do not necessarily have to be local. The same applies at higher points. In fact, we can also set $n'_1 = n'_2 = n'_3 = n'_4 = 0$. By doing so, we can realize that the construction is in fact fully gauge invariant, since the only dependence now is on the gauge invariant amplitudes $A_5^{tree}(1,2,3,4,5), A_5^{tree}(1,4,3,2,5)$.
Inserting all the results in the expressions for the Kleiss-Kuijf basis ampli-

tudes, we finally obtain a new set of relations among them[10]:

$$
\begin{aligned}
A_5^{tree}(1,3,4,2,5) &= \frac{-s_{12}s_{45}A_5^{tree}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{tree}(1,4,3,2,5)}{s_{13}s_{24}} \\
A_5^{tree}(1,2,4,3,5) &= \frac{-s_{14}s_{25}A_5^{tree}(1,4,3,2,5) + s_{45}(s_{12}+s_{24})A_5^{tree}(1,2,3,4,5)}{s_{24}s_{35}} \\
A_5^{tree}(1,4,2,3,5) &= \frac{-s_{12}s_{45}A_5^{tree}(1,2,3,4,5) + s_{25}(s_{14}+s_{24})A_5^{tree}(1,4,3,2,5)}{s_{24}s_{35}} \\
A_5^{tree}(1,3,2,4,5) &= \frac{-s_{14}s_{25}A_5^{tree}(1,4,3,2,5) + s_{12}(s_{24}+s_{45})A_5^{tree}(1,2,3,4,5)}{s_{13}s_{24}}
\end{aligned}
\tag{3.43}
$$

This new set of relations reduces in fact the number of independent colour ordered amplitudes to two. The all-n pattern is that in general, n-point gauge theory amplitudes can be put in a form of a pole expansion such that whenever a tern of colour-factors satisfy Jacobi identities, the related kinematic numerator satisfy an identical relation:

$$
\frac{1}{g^{n-2}}\mathcal{A}_n^{tree}(1,2,\ldots,n) = \sum_i \frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}}
\tag{3.44}
$$

$$
c_i + c_j + c_k = 0 \quad \Longrightarrow \quad n_i + n_j + n_k = 0
$$

As we have just seen, bringing to new relations among tree-level gauge theory amplitudes, BCJ-relations reduce in fact the number of indipendent amplitudes to $(n-3)!$[11]. This also implies that the number of numerators that we have to rearrange simultaneously in order to obtain gauge-invariant amplitudes is at least $(n-3)!$. Using the set of BCJ-relations, one can fix a number of numerators, leaving however $(n-2)! - (n-3)!$ numerators unspecified. They can be local or non-local and in particular they can be set to zero in order to make the independence of gauge amplitudes on them manifest.
One can also obtain a all-n description of the relations among gauge theory amplitudes. We address the reader to [20] to find a complete description.
What one can also notice from the construction we just performed, is that finding BCJ-numerators is not a trivial exercise. In particular, when the number of external gluons grows, the difficulty arises quite quickly, making practical calculations unlikely. In order to summarize the number of diagrams, amplitudes and numerators one can encounter, we report a table taken from [20]:

[10] Notice that these relations should hold for any helicity configuration and for any number of dimensions.
[11] Considering that we started from the $(n-2)!$ Kleiss-Kuijf independent ones.

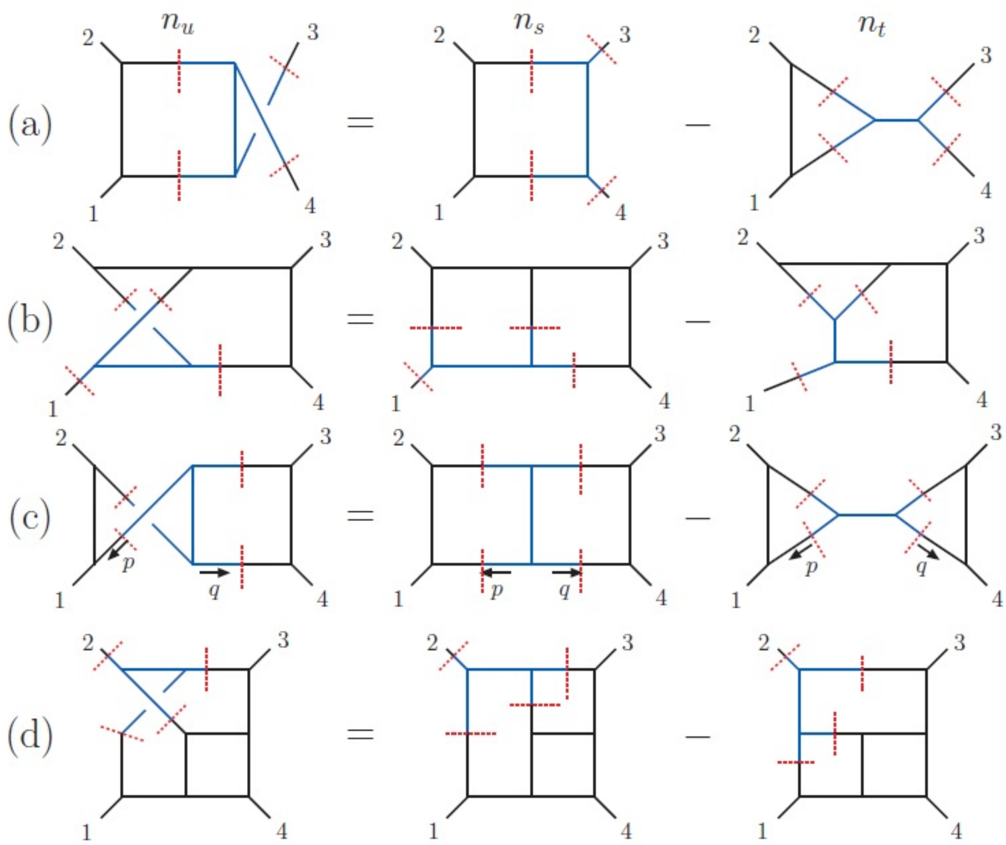
External legs	3	4	5	6	7	8	n
Ordered diagrams[12]	1	2	5	14	42	132	$\frac{2^{n-2}(2n-5)!!}{(n-1)!}$
Diagrams [13]	1	3	15	105	945	10395	$(2n-5)!!$
Kleiss-Kuijf amplitudes	1	2	6	24	120	720	$(n-2)!$
Basis numerators	1	2	6	24	120	720	$(n-2)!$
Numerator equations	0	1	10	105	1260	17325	$\frac{(n-3)(2n-5)!!}{3}$
Independent n-eqs	0	1	9	81	825	9675	$(2n-5)!! - (n-2)!$
Basis amplitudes[14]	1	1	2	6	24	120	$(n-3)!$

In [20], the last line was just a conjecture. BCJ-relations, however, have been proved first in string theory [24–29] and then within quantum field theories using on-shell recursion relations [30–32].
The second important consequence of BCJ-relations is their effect on loop level calculations.

3.4.3 Loop level consequences

The idea is to exploit BCJ-relations to relate non-planar to planar diagrams in a general unitarity method environment (making use in particular of maximal unitarity cuts [63]).The internal amplitudes that one observes at this point (the cut diagrams), can be considered as tree-level amplitudes that satisfy the new-found set of relations. One can then exploit the relations among BCJ-numerators to rewrite parts of loop amplitudes as sums over other - easier to treat - diagrams. As an example, we report a picture taken again from [20], where the simple four-point BCJ-relation 3.22 is being used to relate non-planar diagrams to easier planar diagrams:

[12] The propagator structures appearing in a single colour-ordered amplitude
[13] Propagator structures appearing in the full colour-dressed amplitude
[14] After imposing BCJ-relations



The identities in the picture can be seen both as Jacobi identities among colour diagrams (if we dress each vertex with a f^{abc}) or as the BCJ-identities of the corresponding kinematic numerators of the cut diagrams. In practice, when one calculates the numerator contributions on the right hand side, all the numerators on the left hand side are determined up to the cut conditions. The identities work also at higher points, i.e. n-points cuts satisfy the same relations as n-points tree level amplitudes. We address once again the reader to [20] to find a practical example of a two-loop calculation in QCD or to [84] to find these techniques applied in $\mathcal{N} = 8$ Supergravity theories.
It was also conjectured that BCJ-relations can be applied at loop level in a much deeper sense [51]. This will be briefly reviewed in section 3.4.5. First we will discuss how the newly found equations relate to KLT-relations.

3.4.4 Consequences on gravity: the squaring relations

As we have seen before, KLT relations allow us to construct gravity amplitudes from gauge theory amplitudes. Their validity is clear in string theory and in field theory as a limit of string theory. However, considering the question starting from the Yang-Mills Lagrangian and the Einstein-Hilbert Lagrangian, the link remains unclear.
BCJ-relations allow us to clarify the connection by seeing KLT-relations as a diagram by diagram *numerator squaring relations*. Let us see again a four-point example. We have seen how we can use four-point gluon amplitudes to construct a four-point gravity amplitude in 3.7. What happens if the amplitudes are written using the pole expansion introduced above? As it turns out, for pure gravity, we can write:

$$-iM_4^{tree}(1,2,3,4) = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} \tag{3.45}$$

where the n_i's are the gauge theory numerators described above. If one wants to consider other theories like supersymmetric versions of gravity or in general theories with other particle contents, one can have two different sets of BCJ-numerators:

$$-iM_4^{tree}(1,2,3,4) = \frac{n_s\tilde{n}_s}{s} + \frac{n_t\tilde{n}_t}{t} + \frac{n_u\tilde{n}_u}{u} \tag{3.46}$$

however, at least one of the sets of numerators will have to satisfy BCJ-relations:

$$n_s - n_t - n_u = 0 \qquad \tilde{n}_s - \tilde{n}_t - \tilde{n}_u = 0 \tag{3.47}$$

This way of expressing KLT relations as squaring relations can be extended at higher points. A five-point gravity amplitude can be written as a sum over the fifteen diagrams defined in 3.34, as in the colour dressed amplitude 3.31, where - however - the colour factors have been substituted by other "tilded" kinematic numerators[15].

$$\begin{aligned}-i\mathcal{M}_5^{tree} =& \frac{\tilde{n}_1 n_1}{s_{12}s_{45}} + \frac{\tilde{n}_2 n_2}{s_{23}s_{51}} + \frac{\tilde{n}_3 n_3}{s_{34}s_{12}} + \frac{\tilde{n}_4 n_4}{s_{45}s_{23}} + \frac{\tilde{n}_5 n_5}{s_{51}s_{34}} + \frac{\tilde{n}_6 n_6}{s_{14}s_{25}} \\ &+ \frac{\tilde{n}_7 n_7}{s_{32}s_{14}} + \frac{\tilde{n}_8 n_8}{s_{25}s_{43}} + \frac{\tilde{n}_9 n_9}{s_{13}s_{25}} + \frac{\tilde{n}_{10} n_{10}}{s_{42}s_{13}} + \frac{\tilde{n}_{11} n_{11}}{s_{51}s_{42}} + \frac{\tilde{n}_{12} n_{12}}{s_{12}s_{35}} \\ &+ \frac{\tilde{n}_{13} n_{13}}{s_{35}s_{24}} + \frac{\tilde{n}_{14} n_{14}}{s_{14}s_{35}} + \frac{\tilde{n}_{15} n_{15}}{s_{13}s_{45}}\end{aligned} \tag{3.48}$$

In the previous section we have seen how four of the six independent numerators could be set to zero, while two of them could be defined making

[15] And where of course the gauge theory couplings were removed.

use of gauge invariant KK-amplitudes. By varying the choice of the KK-amplitudes used to define the two numerators, we can obtain different sets of KLT-relations. For example if we use $A_5^{tree}(1,2,3,4,5)$ and $A_5^{tree}(1,3,2,4,5)$ when we solve for the n_i's and $\tilde{A}_5^{tree}(2,1,4,3,5)$ and $\tilde{A}_5^{tree}(3,1,4,2,5)$ when solving for the $\tilde{n}_i$'s, we obtain immediately the five-point KLT-relations described in 3.7.
The all-n pattern goes like this; starting from the expressions for two colour dressed amplitudes in two - possibly different - gauge theories

$$\begin{aligned} \frac{1}{g^{n-2}}\mathcal{A}_n^{tree}(1,2,\ldots,n) &= \sum_i \frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}} \\ \frac{1}{g^{n-2}}\tilde{\mathcal{A}}_n^{tree}(1,2,\ldots,n) &= \sum_i \frac{c_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}} \end{aligned} \tag{3.49}$$

we obtain a compact and illuminating expression for the gravity amplitudes:

$$-iM_n^{tree}(1,2,\ldots,n) = \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}} \tag{3.50}$$

where in both cases the sum runs over all the possible diagrams[16] that can appear. One is then left free to choose $2(n-3)!$ basis amplitudes to define the same number of numerators. Every gauge theory amplitude choice made to define the numerators will correspond to a different relation between gauge and gravity amplitudes[17].

The attentive reader, will have noticed that in the introduction we stated that applying the squaring relations on two sets of pure YM numerators will lead to Einstein gravity plus an anti-symmetric tensor plus a dilaton. However, in these last sections we focused only on gravitons.
The idea is that the other two objects mentioned in the introduction are relevant and natural in string theories, where the graviton state is accompanied by an antisymmetric tensor $B_{\mu\nu}$ and by a scalar trace mode. The antisymmetric tensor possesses a dual 3-form field strength $H = dB$. This means

[16] where the four-point contact terms have been reabsorbed in the cubic ones

[17] This should not be confused with the freedom to use numerators belonging to different gauge theories. Different choices for the gauge theories from which the sets of numerators n_i or $\tilde{n}_i$ arise, will lead in fact, as stated above, to different gravity theories. For example, if one of the sets of numerators arises from $N = 4$ SYM amplitudes and the other set arises from $N = 0$ YM amplitudes, the gravity amplitudes constructed will be the ones of $N = 4$ Supergravity. If both the sets of numerators arise from $N = 4$ SYM theory, the gravity amplitudes constructed will be the ones of $N = 8$ Supergravity.

that it is dual to a scalar called axion in four dimensions. The scalar trace mode instead is the dilaton we mentioned in the introduction.

What we treated in the last sections is exclusively the Einstein-Hilbert Lagrangian, producing graviton interactions. The picture we analyzed is the following [105]:

$$graviton^{\pm 2}(p_i) = gluon^{\pm 1}(p_i) \otimes gluon^{\pm 1}(p_i). \tag{3.51}$$

This, in particular, fits the polarization scheme described in 3.6.

If one considers opposite helicities in the squaring relations (or equivalently in the KLT-relations), one obtains [105]:

$$\left.\begin{array}{r} dilaton \\ axion \end{array}\right\} = gluon^{\pm 1}(p_i) \otimes gluon^{\mp 1}(p_i). \tag{3.52}$$

This consideration should then clarify the generality of the statement made in the introduction.

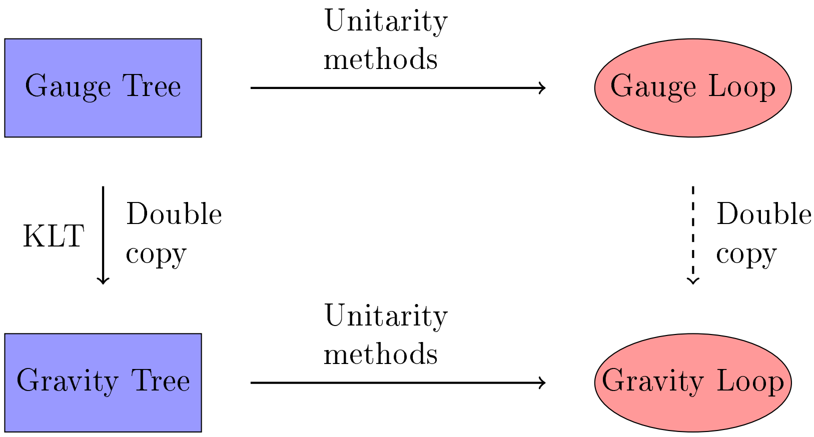
3.4.5 A loop-level conjecture

In [51] the authors proposed that what we just learnt about BCJ relations could be directly implemented at loop level with few modifications. In addition, in [50], was made a first attempt of constructing a BCJ-manifest Lagrangian up to five-points. The existence of such Lagrangians can be as well seen as another hint that the duality can extend to loop level. The idea in [51] is that we can write loop-level gauge theory amplitudes and -as a consequence - gravity loop-level amplitudes as:

$$\begin{gathered} \frac{(-1)^L}{g^{n-2+2L}} \mathcal{A}_n^{loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} s_{\alpha_j}} \\ \frac{(-1)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} s_{\alpha_j}} \end{gathered} \tag{3.53}$$

where the integrals are over the loop momenta, the sums are over the diagrams descibed in the previous sections and S_j are the internal symmetry factors of the diagrams. This formulation is at the moment a conjecture, even though it has been supported already by numerous calculations at different loop levels: [33–43]. In addition, as a consequence of the conjecture, one can see the expression for the gravity amplitudes as the application of the unitarity method on the tree-level expression 3.50. This idea is as well

supported by numerous calculations (see for example [86]).
This construction is not part of the main topic discussed in this thesis. It is however interesting to notice the picture of gauge-gavity duality that colour-kinematics duality is unfolding.
If one assumes the validity of the conjecture, what we are left with, is a rather easy way to construct gravity amplitudes both at tree- and loop-level; duality, in fact, creates the following schematic situation:



(3.54)

where the duality at loop level is described as a dashed line to stress once again the fact that it has not been proved yet.
In [50] the authors constructed also a BCJ manifest Lagrangian up to five-points. This Lagrangian, in addition to the hint to the validity of the loop conjecture, delivers also another information. It is possible to make the colour-kinematic duality manifest at the Lagrangian level, making the construction of BCJ-numerators an automatic process of derivation and application of a new set of Feynman rules. The idea of the authors has been thereafter improved and generalized to a n-point algorithm by the author of the thesis and Stefan Weinzierl. This will be the topic covered in the next chapter.

4

A BCJ-MANIFEST EFFECTIVE LAGRANGIAN

The structure of the chapter will be the following. First of all we will introduce some notation to re-formulate BCJ-relations and to make our approach as clear as possible.
Following the steps of the previous chapter, we will start from a four-point case and we will show - finally clearifying the problem completely - how to split efficiently four-point contact terms in the correct channels with the correct factors. We will then procede to the introduction of the general idea of the algorithm and to the treatment of a five-point example. This will be our way to finally introduce the complete n-point algorithm, together with some additional notation required for the generalization. In addition, the five point example will serve as a case study to treat the problem of non-uniqueness that these constructions present. Finally, some technical details on the construction of Feynman amplitudes will be covered.

4.1 Trees, rooted trees and Jakobi-like identities

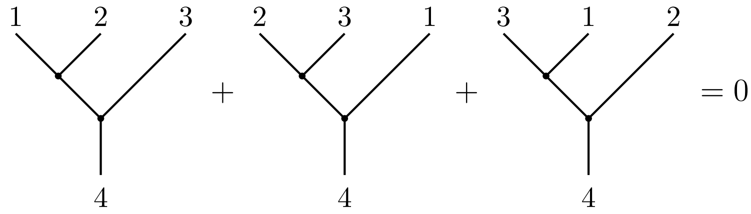
In this section we will treat trees characterized by a fixed cyclic order of the external legs and by the presence of three-valent vertices only.
They will serve - thanks to their properties - as the main tool to define and discuss our algorithm. First of all - as we have partially seen in the previous chapter - we can use trees to re-write Jacobi-relations as a Jacobi-like tree-relation. We have seen in chapter 2 that:

$$f^{a_1a_2b}f^{ba_3a_4} + f^{a_2a_3b}f^{ba_1a_4} + f^{a_3a_1b}f^{ba_2a_4} = 0 \tag{4.1}$$

For our purpose, it is interesting to recover their formulation in terms of the generators T^a's:

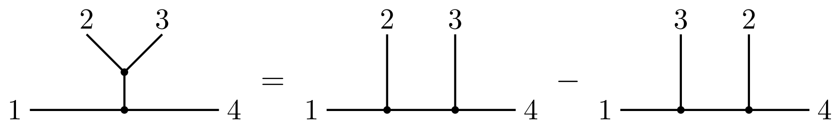
$$Tr\left([[T^{a_1}, T^{a_2}], T^{a_3}]T^{a_4} + [[T^{a_2}, T^{a_3}], T^{a_1}]T^{a_4} + [[T^{a_3}, T^{a_1}], T^{a_2}]T^{a_4}\right) = 0 \tag{4.2}$$

The graphical way of expressing this is:



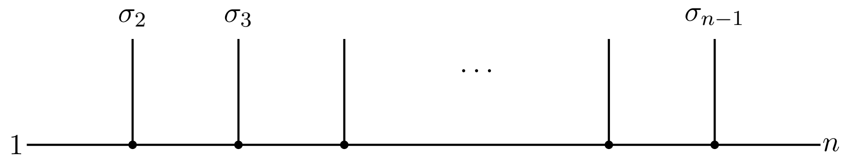
(4.3)

Since in our case the three-valent vertex is antisymmetric, we can also re-express them as the STU-relation we have seen in the previous chapter:



(4.4)

One of the interesting consequences of the validity of 4.4 is that its repeated use puts any tree-level n-legs graph containing only three-valent vertices in a multi-peripheral form with respect to 1 and n, i.e. in a form where all the other legs attach directly[1] to the line connecting 1 and n:

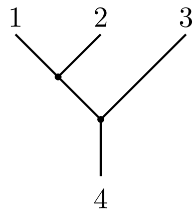


(4.5)

Let us consider a tree with n external legs labeled clockwise, where the last leg has been singled out. We call this tree a *rooted tree*, leg n being the root. The notation we choose to indicate rooted trees is made of square brackets that remind and symbolize also the antisymmetric properties of the vertices involved. For example, with

$$[[1,2],3] \tag{4.6}$$

we will indicate the rooted tree



(4.7)

[1] I.e. there are no non-trivial sub-trees attached to this line.

where the root is - as said - the leg labeled by 4.
Let us see some examples. For $n = 3$ we have one possible rooted tree

$$T_1^{(3)} = [1, 2], \tag{4.8}$$

for $n = 4$ we have two possible rooted trees

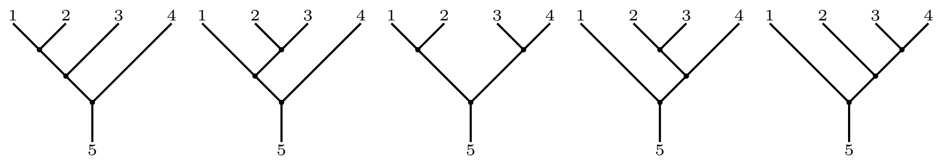
$$T_1^{(4)} = [[1, 2], 3], \qquad T_2^{(4)} = [1, [2, 3]] \tag{4.9}$$

for $n = 5$ we have five possible structures

$$T_1^{(5)} = [[[1, 2], 3], 4], \quad T_2^{(5)} = [[1, [2, 3]], 4] \quad T_3^{(5)} = [[1, 2], [3, 4]] \tag{4.10}$$

$$T_4^{(5)} = [1, [[2, 3], 4]], \quad T_5^{(5)} = [1, [2, [3, 4]]], \tag{4.11}$$

corresponding to the cyclic-ordered rooted trees



(4.12)

Of course, the possible cyclic-ordered rooted trees correspond to the possible propagator structures that can appear in a n-point amplitude containing only three-valent vertices.
In this notation, we write a multi-peripheral tree with respect to the line 1-n as

$$T_{m-p}^{(n)} = [[[\ldots[[[1, 2], 3], 4], \ldots], n-2], n-1] \tag{4.13}$$

The number of cyclic-ordered rooted trees with only three-valent vertices and n external legs can be obtained recursively through:

$$f(n) = \sum_{i=2}^{n-1} f(i)f(n-i+1), \qquad f(2) = 1 \tag{4.14}$$

or by the closed formula already presented in table 3.4.2:

$$f(n) = \frac{2^{n-2}(2n-5)!!}{(n-1)!} = \frac{(2n-4)!}{(n-1)!(n-2)!} \tag{4.15}$$

As we said before, this all-n behaviour corresponds to the number of propagator structures (channels) that can appear in a colour-ordered amplitude with n external legs and only three-valent vertices. This description makes once again clear why using trees to define our algorithm will be particularly useful.

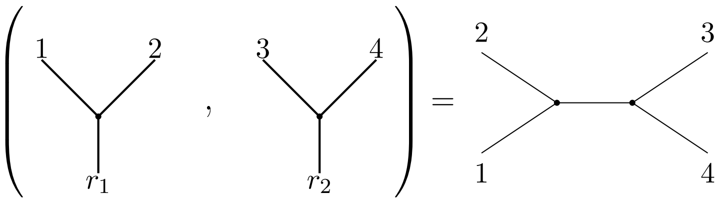
Rooted trees allow us to define some other operations. For example we can define two operators: L and R, which pick the left sub-tree and the right sub-tree respectively. To be precise, consider a rooted tree $T = [T_1, T_2]$ where T_1 and T_2 are sub-trees. Then:

$$L(T) = T_1, \qquad R(T) = T_2 \tag{4.16}$$

This definition has to be completed with the action of the operator on atomic trees $T = j$, for which we have:

$$L(j) = R(j) = 0 \tag{4.17}$$

The next operation we want to define is *concatenation*, an operation that allows us to construct non-rooted trees starting from rooted ones by connecting them with an edge. We denote this operation with round brackets. Let T_1 and T_2 be two rooted trees with roots being respectively r_1 and r_2. (T_1, T_2) is then the non-rooted tree obtained by joining r_1 and r_2 with an edge. This operation is symmetric in the two rooted trees: $(T_1, T_2) = (T_2, T_1)$. As an example we can see:



(4.18)

By applying the definitions above and the symmetric property of concatenation, we can see that the following relation is satisfied:

$$([T_1, T_2], T_3) = ([T_2, T_3], T_1) = ([T_3, T_1], T_2) \tag{4.19}$$

As a last step of this notation introduction, we want to denote by $\mathcal{T}_n$ the set of all cyclic-ordered rooted trees with n external legs and only three-valent vertices.

4.2 The BCJ-decomposition revisited

In chapter 2, in equations 2.51, we defined quantities like $p_{i\ldots j}$ and $s_{i\ldots j}$. It is time to re-write them using the newly introduced trees-formulation. This will allow us to rewrite BCJ-relations in a form that will make clearer how to implement them at the Lagrangian level.
So, let us get back to a n-gluons[2] born partial amplitude $A(1,\ldots,n)$ where we denote with $p_1, p_2, \ldots, p_n$ the momenta of the outgoing gluons.
Let us consider then a rooted tree $T \in \mathcal{T}_n$. Notice that, having n external legs and only three-valent vertices, it will have $(n-3)$ internal edges.
We define then a quantity $D(T)$ to be the product of the invariants corresponding to the $(n-3)$ internal edges. Let us see some simple examples:

$$D([[1,2],3]) = s_{12} \tag{4.20}$$
$$D([[[1,2],3],4]) = s_{12}s_{123} \tag{4.21}$$
$$D([[1,2],[3,4]]) = s_{12}s_{34} \tag{4.22}$$
$$D([[[1,2],[3,4]],5]) = s_{12}s_{34}s_{1234} \tag{4.23}$$

We denote the momentum that flows through the root by $p(T)$ and the related invariant quantity by $s(T) = p(T)^2$. Let us see again the same examples:

$$p([[1,2],3]) = p_{123} \qquad s([[1,2],3]) = s_{123} \tag{4.24}$$
$$p([[[1,2],3],4]) = p_{1234} \qquad s([[[1,2],3],4]) = s_{1234} \tag{4.25}$$
$$p([[1,2],[3,4]]) = p_{1234} \qquad s([[1,2],[3,4]]) = s_{1234} \tag{4.26}$$
$$p([[[1,2],[3,4]],5]) = p_{12345} \qquad s([[[1,2],[3,4]],5]) = s_{12345} \tag{4.27}$$

We are finally ready to re-write the BCJ-decomposition of partial amplitudes:

$$A_n^{tree}(1,\ldots,n) = \sum_{T\in\mathcal{T}_n} \frac{N(T)}{D(T)} \tag{4.28}$$

where the dominators $D(T)$ have been defined before and where the numerators $N(T)$ satisfy antisymmetry and Jacobi-like relations. In order to define the first property rigorously, we have to consider sub-trees T_1, T_2, T_3 such that

$$T_{12} = ([T_1,T_2],T_3), \qquad T_{21} = ([T_2,T_1],T_3), \tag{4.29}$$

are (non-rooted) trees with n external legs. The antisymmetry of numerators can be written then as

$$N(T_{12}) + N(T_{21}) = 0 \tag{4.30}$$

[2] Once again we are using the term *gluon* as a label for any gauge theory vector boson.

For Jacobi-like relations we need sub-trees T_1', T_2', T_3', T_4' such that

$$T_{123} = ([[T_1', T_2'], T_3'], T_4'), \qquad T_{231} = ([[T_2', T_3'], T_1'], T_4'), \qquad T_{312} = ([[T_3', T_1'], T_2'], T_4') \tag{4.31}$$

are again non-rooted trees with n external legs. The Jacobi-like relation can then be written as

$$N(T_{123}) + N(T_{231}) + N(T_{312}) = 0 \tag{4.32}$$

As we have already discussed, the statement that partial amplitudes A_n can be put in the form of 4.28 with numerators satisfying 4.30 and 4.32 is highly non-trivial one. As seen in the five point example of last chapter, starting from the standard Yang Mills Lagrangian will in general not produce this result. In particular, using a standard colour-ordered Feynman rules approach will fail because of the presence of four-gluon vertices. Each four gluon vertex, in fact, reduces the number of propagators by one, forcing us to perform the splitting procedure mentioned in the previous chapters by insertion of “smart ones” in the form of factors like $\frac{s_{ij}}{s_{ij}}$. How do we insert the right factors in the right places? That is where a systematic algorithm can help. In order to do so, let us explore the problem in a more rigorous way.
Consider a diagram contributing to the n-point colour-ordered amplitude A_n and containing only three-valent vertices. The number of vertices in this case will be given by $(n-2)$. We have then n polarization vectors ϵ_i^μ for the external legs to be contracted with $(n-2)$ momentum vectors q_j^μ[3] brought in by the $(n-2)$ three-gluons vertices. We have then at least one $\epsilon_{i_1} \cdot \epsilon_{i_2}$ product where two polarization vectors have been contracted. The maximum number of such products we can find in a n-point amplitude is given by $[n/2]$, indicating the largest integer smaller or equal to $n/2$.
These considerations allow us to decompose amplitudes in the following way:

$$A_n = \sum_{j=1}^{[n/2]} A_{n,j} \tag{4.33}$$

where $A_{n,j}$ contains j $\epsilon_{i_1} \cdot \epsilon_{i_2}$ products. To give an example, we can see how the lowest point amplitudes get decomposed:

$$A_4 = A_{4,1} + A_{4,2} \tag{4.34}$$
$$A_5 = A_{5,1} + A_{5,2} \tag{4.35}$$
$$A_6 = A_{6,1} + A_{6,2} + A_{6,3} \tag{4.36}$$
$$A_7 = A_{7,1} + A_{7,2} + A_{7,3} \tag{4.37}$$
$$A_8 = A_{8,1} + A_{8,2} + A_{8,3} + A_{8,4} \tag{4.38}$$

[3] Linear combination of the external momenta.

The terms $A_{n,1}$ are obtained from three-valent vertices only and satisfy BCJ-relations automatically. The presence of a four-gluon vertex will increase j by one. This means that the "problematic" terms we have to treat are the $A_{n,j}$ with $j \geq 2$.
As a first step, let us see how to cure the problem with $A_{4,2}$ at the level of the Lagrangian. This is formally equivalent to the splitting procedure described before at the level of the amplitudes.

4.3 The four-point case: an idea for a general algorithm

We have seen in chapter 2 that the Yang-Mills Lagrangian can be written as:

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2g^2}\left[\mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)}\right] \tag{4.39}$$

with

$$\begin{aligned} \mathcal{L}^{(2)} &= -2Tr\boldsymbol{A}_\mu \Box \boldsymbol{A}^\mu \\ \mathcal{L}^{(3)} &= 4Tr(\partial_\mu \boldsymbol{A}_\nu)[\boldsymbol{A}^\mu, \boldsymbol{A}^\nu] \\ \mathcal{L}^{(4)} &= Tr[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu][\boldsymbol{A}^\mu, \boldsymbol{A}^\nu] \end{aligned} \tag{4.40}$$

Our first step in the creation of a systematic algorithm is the splitting of four-gluon vertices into the right channels at four points. In order to do so, we re-write $\mathcal{L}^{(4)}$ as[4]:

$$\mathcal{L}^{(4)} = -g^{\mu_1\mu_3} g^{\mu_2\mu_4} g_{\nu_1\nu_2} \frac{\partial_{12}^{\nu_1}\partial_{34}^{\nu_2}}{\Box_{12}} Tr[\boldsymbol{A}_{\mu_1}, \boldsymbol{A}_{\mu_2}][\boldsymbol{A}_{\mu_3}, \boldsymbol{A}_{\mu_4}] \tag{4.41}$$

A brief pause to describe the new notation. The subscripts on the derivatives tell us on which fields they act. A derivative without subscripts, instead, acts on all the fields on its right. So, for example, $\partial_{12}^{\nu_1}$ acts on $\boldsymbol{A}_{\mu_1}$ and $\boldsymbol{A}_{\mu_2}$ but not on $\boldsymbol{A}_{\mu_3}$ and $\boldsymbol{A}_{\mu_4}$.
Let us look a bit more into the trick used to obtain the correct splitting. We introduced a factor $\frac{1}{\Box_{12}}$ as an intermediate propagator to the four-gluon vertex. However, this factor cancels out with $-g_{\nu_1\nu_2}\partial_{12}^{\nu_1}\partial_{34}^{\nu_2} = \Box_{12}$. In fact, the trick reduces once again to the insertion of a "smart one" at the level of the Lagrangian. This idea however, will allow us to generalize this procedure systematically. Let us see how.
The trace encodes the colour information and the propagator corresponds in fact to the tree structure of the colour. In this perspective, it is useful to

[4] From this chapter on, we re-adopt the usual notation concerning metrics, with $g_{\mu\nu}$ being the Minkowski metric.

define as D^{-1} the product of factors $\frac{(-1)}{\Box}$ correponding to the tree structure of the colour.
This definition is such that, in momentum space, the operator just defined agrees with the quantity D described in 4.2. Let us see some examples to get familiar with the operator:

$$D^{(-1)}Tr\big([[[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3}],\boldsymbol{A}_{\mu_4}]\boldsymbol{A}_{\mu_5}\big) = \frac{(-1)^2}{\Box_{12}\Box_{123}}Tr\big([[[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3}],\boldsymbol{A}_{\mu_4}]\boldsymbol{A}_{\mu_5}\big) \tag{4.42}$$

$$D^{(-1)}Tr\big([[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],[\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}]]\boldsymbol{A}_{\mu_5}\big) = \frac{(-1)^2}{\Box_{12}\Box_{34}}Tr\big([[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],[\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}]]\boldsymbol{A}_{\mu_5}\big) \tag{4.43}$$

$$D^{(-1)}Tr\big([[[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3}],[\boldsymbol{A}_{\mu_4},\boldsymbol{A}_{\mu_5}]]\boldsymbol{A}_{\mu_6}\big) = \tag{4.44}$$

$$= \frac{(-1)^3}{\Box_{12}\Box_{123}\Box_{45}}Tr\big([[[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3}],[\boldsymbol{A}_{\mu_4},\boldsymbol{A}_{\mu_5}]]\boldsymbol{A}_{\mu_6}\big) \tag{4.45}$$

Now we are finally ready to use the new notation to re-write $\mathcal{L}^{(4)}$:

$$\begin{aligned} \mathcal{L}^{(4)} &= \mathcal{O}^{\mu_1\mu_2\mu_3\mu_4}D^{-1}Tr[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}][\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}] \\ \mathcal{O}^{\mu_1\mu_2\mu_3\mu_4} &= g^{\mu_1\mu_3}g^{\mu_2\mu_4}g_{\nu_1\nu_2}\partial_{12}^{\nu_1}\partial_{34}^{\nu_2} \end{aligned} \tag{4.46}$$

The Feynman rule of this interaction term is given by:

$$V^{\mu_1\mu_2\mu_3\mu_4} = i\frac{p_{12}^2}{s_{12}}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) + i\frac{p_{23}^2}{s_{23}}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_2}g^{\mu_3\mu_4}) \tag{4.47}$$

One immediately notices that, simply by simplifying $p_{ij}^2 = s_{ij}$ with the corresponding s_{ij}, equation 4.47 reduces to the standard four-gluon vertex 2.33. However, if we do not perform those simplifications, eq. 4.47 performs the correct splitting of four-points interaction terms into s and t channels, generating the BCJ-decomposition correctly up to four-points. This simple trick will not be enough however for higher-points decompositions. We need then to generalize our trick.

4.4 Higher-points generalization and a five-points example

First of all, let us generalize eq. 4.39:

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2g^2}\sum_{n=2}^{\infty}\mathcal{L}^{(n)}. \tag{4.48}$$

where, with our notation, $\mathcal{L}^{(n)}$ contains n fields. Of course, we want eq. 4.48 to agree with the original Lagrangian 4.39; this means that we will have $\mathcal{L}^{(n)} = 0$ for $n \geq 5$. After using "smart ones" then, it is the moment to use "smart zeroes".
In the following sections, we will construct the terms $\mathcal{L}^{(n)}$ such that they generate the BCJ-decomposition up to n-gluons; their form is similar to the form of $\mathcal{L}^{(4)}$:

$$\mathcal{L}^{(n)} = \sum_{t} \sum_{j=2}^{[n/2]} \mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)} \hat{D}^{-1} Tr \boldsymbol{T}^{(n,t)}_{\mu_1 \cdots \mu_n} \tag{4.49}$$

where $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ is a differential operator of degree $(n-4)$, $\hat{D}^{-1}$ is a pseudo-differential operator of degree $(2n-8)$ that we will define later and $\boldsymbol{T}^{(n,t)}_{\mu_1 \cdots \mu_n}$ contains n fileds and all the colour information. As one can see from the presence of the two sums, more than one term is possible for a given n. The sum over t allows to consider inequivalent trees in $\boldsymbol{T}^{(n,t)}_{\mu_1 \cdots \mu_n}$, while the sum over j is related to the number of factors $g^{\mu_1 \mu_2}$ in $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ and it corresponds to the decomposition of amplitudes presented in 4.33. Notice how, in accordance with what said about the decomposition, we do not need to insert $j = 1$ terms.
As we said, the $\mathcal{L}^{(n)}$ terms have to vanish for $n \geq 5$. This is ensured by requiring that $\boldsymbol{T}^{(n,t)}_{\mu_1 \cdots \mu_n}$ vanishes due to the Jacobi identity. In order to clarify what we mean, let us see for example one term we can take at $n = 5$

$$\begin{aligned} \mathcal{L}^{(5)} =& 4 g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} \frac{\partial_1^{\mu_5}}{\Box_{123}} \big(Tr[[[\boldsymbol{A}_{\mu_1}, \boldsymbol{A}_{\mu_2}], \boldsymbol{A}_{\mu_3}], \boldsymbol{A}_{\mu_4}] \boldsymbol{A}_{\mu_5} \\ &+ Tr[[\boldsymbol{A}_{\mu_3}, \boldsymbol{A}_{\mu_4}], [\boldsymbol{A}_{\mu_1}, \boldsymbol{A}_{\mu_2}]] \boldsymbol{A}_{\mu_5} + Tr[[\boldsymbol{A}_{\mu_4}, [\boldsymbol{A}_{\mu_1}, \boldsymbol{A}_{\mu_2}]], \boldsymbol{A}_{\mu_3}] \boldsymbol{A}_{\mu_5} \big) \end{aligned} \tag{4.50}$$

which equals zero thanks to the Jacobi identity among the tern $[\boldsymbol{A}_{\mu_1}, \boldsymbol{A}_{\mu_2}], \boldsymbol{A}_{\mu_3}, \boldsymbol{A}_{\mu_4}$. However, when considered as a contact term, $\mathcal{L}^{(5)}$ will generate a five-gluon vertex that gives non-vanishing contributions to individual numerators. Of course, to preserve the values of partial amplitudes, the sum of the terms related to this five-gluons vertex will sum up to zero inside each partial amplitude.
Since Jacobi identity is the way through which we obtain the vanishing of this objects, it is convenient to introduce for arbitrary Lie algebra-valued expressions $\boldsymbol{T}_1$,$\boldsymbol{T}_2$ and $\boldsymbol{T}_3$ the notation:

$$J(\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3) = [[\boldsymbol{T}_1, \boldsymbol{T}_2], \boldsymbol{T}_3] + [[\boldsymbol{T}_2, \boldsymbol{T}_3], \boldsymbol{T}_1] + [[\boldsymbol{T}_3, \boldsymbol{T}_1], \boldsymbol{T}_2] \tag{4.51}$$

In particular, if $\boldsymbol{T}_1$,$\boldsymbol{T}_2$ and $\boldsymbol{T}_3$ are arbitrary trees, each term in the tern above contains one internal line that does not appear in the other two. We

will refer to this line as the "line[5] marked by the Jacobi identity". We are finally equipped to define the pseudo-differential operator $\hat{D}^{-1}$. This operator - when acting on a colour structure that contains a Jacobi identity - will give, for each term of the tern, the previously defined D^{-1} times a factor in the numerator given by $(-\Box_{mp})$, where "mp" stands for the marked propagator in that term.
As an example, let us consider the "Jacobi-sum" that appeared in the term 4.50:

$$\begin{aligned}\hat{D}^{-1}TrJ\left(\left[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}\right],\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}\right)\boldsymbol{A}_{\mu_5} =& -\frac{\Box_{123}}{\Box_{12}\Box_{123}}Tr\left[\left[\left[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}\right],\boldsymbol{A}_{\mu_3}\right],\boldsymbol{A}_{\mu_4}\right]\boldsymbol{A}_{\mu_5}\\ &-\frac{\Box_{34}}{\Box_{12}\Box_{34}}Tr\left[\left[\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}\right],\left[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}\right]\right]\boldsymbol{A}_{\mu_5}\\ &-\frac{\Box_{124}}{\Box_{12}\Box_{124}}Tr\left[\left[\boldsymbol{A}_{\mu_4},\left[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}\right]\right],\boldsymbol{A}_{\mu_3}\right]\boldsymbol{A}_{\mu_5}\end{aligned} \tag{4.52}$$

Exactly as for the four-point example, we do not simplify the common terms. The denominators define the correct tree structures, while the operators in the numerators contribute with the right quantitiy to BCJ-numerators. Example 4.50 can then be reformulated as:

$$\begin{aligned}\mathcal{L}^{(5)} &= \mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}\hat{D}^{-1}Tr\boldsymbol{T}^{(5,1)}_{\mu_1\mu_2\mu_3\mu_4\mu_5}\\ \mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{(5,1,2)} &= -4g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_1^{\mu_5}\\ Tr\boldsymbol{T}^{(5,1)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} &= TrJ([\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4})\boldsymbol{A}_{\mu_5}\end{aligned} \tag{4.53}$$

Let us exploit this example a bit more to discuss the non-uniqueness of the operators that we can use in our construction. $\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{(5,1,2)}$ is in fact not unique for three reasons. First of all, we can rewrite the same operator in a different way by making use of momentum conservation[6] and a suitable relabeling of indices. For example, we can obtain the same Feynman rule generated by $\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{(5,1,2)}$, by using instead:

$$\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{(5,1,2),alternative} = 4g^{\mu_1\mu_5}g^{\mu_2\mu_4}\partial_2^{\mu_3} \tag{4.54}$$

The second reason is that there are operators which generate Feynman rules in agreement with the previous one when restricted to the five-particles on-shell kinematics. This means in practice that the Feynman rules differ only by terms proportional to p_i^2 with $i = 1, \ldots, n$.[7]

[5] Or "propagator".
[6] Or equivalently the vanishing of a total derivative.
[7] n being 5 in our example.

The third possibility is given by operators which generate non-vanishing Feynman rules but whose contribution to BCJ-relations actually vanishes. One example of such operators is given by:

$$4\lambda(g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_1^{\mu_5} - g^{\mu_1\mu_3}g^{\mu_4\mu_5}\partial_4^{\mu_2}) \tag{4.55}$$

where λ is a free parameter. This particualr case was already noted in [50]. It is time to explain all the steps that are required to construct an effective Lagrangian which shows manifest BCJ-symmetry[8] up to n-points.

4.5 The n-points algorithm: a BCJ-manifest effective Lagrangian

In this section we will present a step by step algorithm to construct the n^{th} term of the Lagrangian 4.48

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2g^2}\sum_{n=2}^{\infty}\mathcal{L}^{(n)}. \tag{4.56}$$

in order to do so, we will assume that all the previous terms $\mathcal{L}^{(2)}, \mathcal{L}^{(3)}, \ldots, \mathcal{L}^{(n-1)}$ have already been constructed.
We already made it clear that this construction is not unique. In this perspective, the reader is informed that our algorithm will give a specific choice as a result. In addition, it is important to notice that the choices made for the terms $\mathcal{L}^{(k)}$ with $k < n$ will affect the term $\mathcal{L}^{(n)}$ and the following ones. The form of $\mathcal{L}^{(n)}$ is the one presented in 4.49

$$\mathcal{L}^{(n)} = \sum_t \sum_{j=2}^{[n/2]} \mathcal{O}^{\mu_1\cdots\mu_n}_{(n,t,j)}\hat{D}^{-1}Tr\boldsymbol{T}^{(n,t)}_{\mu_1\cdots\mu_n} \tag{4.57}$$

The three steps that compose our algorithm are the following.

- First of all we have to construct all the inequivalent tree topologies[9] $\boldsymbol{T}^{(n,t)}_{\mu_1\cdots\mu_n}$ for the Jacobi relations. For a given n we obtain then a set:

$$\left\{\boldsymbol{T}^{(n,1)}_{\mu_1\cdots\mu_n}, \ldots, \boldsymbol{T}^{(n,t_{max})}_{\mu_1\cdots\mu_n}\right\} \tag{4.58}$$

[8] Which automatically produces amplitudes that can be BCJ-decomposed as defined in this chapter.
[9] In appendix B we will present an algorithm to do that.

- Then, for a given n, a given j and a given t, we consider[10] all the possible forms that the operator $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ can take. In general, $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ can be written as sum of terms where each term, as already said, contains j metric tensor factors. Since the operator is of order $(n-4)$, we know that it must contain $(n-2j)$ derivatives with open indices and $(2j-4)$ derivatives contracted into each other. It is useful then to denote each term in $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ through one permutation σ of the set $(1, 2, \ldots, n)$ and a multi-index $\boldsymbol{i} = (i_1, \ldots, i_{n-4})$ where each component i_j takes values $1 \leq i_j \leq n$. The form of the terms composing $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ is then

$$\mathcal{O}^{\mu_1 \cdots \mu_n}(\sigma, \boldsymbol{i}) = \left(\prod_{k=1}^{j} g^{\mu_{\sigma(2k-1)} \mu_{\sigma(2k)}} \right) \left(\prod_{k=1}^{n-2j} \partial_{i_k}^{\mu_{\sigma(2j+k)}} \right) \prod_{k=1}^{j-2} \left(\partial_{i_{n-2j+2k-1}} \cdot \partial_{i_{n-2j+2k}} \right) \tag{4.59}$$

  For each term, we must generate the corresponding Feynman rule through a procedure that will be described in section 4.5.1. If two terms lead to the same Feynman rule up to a sign in the on-shell kinematics, we consider them to be equivalent and we dismiss one. In this way, for each equivalence class we only have one representative.
  At this point, we define a function $\theta(\sigma, \boldsymbol{i})$ which takes values one and zero. We will use this function to decide whether a term is kept or not. We finally have a complete ansatz for $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$:

$$\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)} = \sum_{\sigma} \sum_{\boldsymbol{i}} c_{n,t,j,\sigma,\boldsymbol{i}} \theta(\sigma, \boldsymbol{i}) \mathcal{O}^{\mu_1 \cdots \mu_n}(\sigma, \boldsymbol{i}) \tag{4.60}$$

  where the coefficients $c_{n,t,j,\sigma,\boldsymbol{i}}$ are unknown.

- We can now insert our ansatz 4.60 into the t_{max} BCJ-relations defined by the set 4.58. The next step is to extract the coefficients of the independent scalar products $\epsilon_i \cdot \epsilon_j$, $\epsilon_i \cdot p_j$ and $p_i \cdot p_j$. By requiring that these coefficients vanish (ensuring in this way the validity of the BCJ-relations) we obtain a system of linear equations for the unknown coefficients $c_{n,t,j,\sigma,\boldsymbol{i}}$. We solve then for the $c_{n,t,j,\sigma,\boldsymbol{i}}$'s.
  As we have already remarked, the solutions will not be unique. The non-uniqueness in these solutions reflects the freedom to add terms like 4.55 at five-points. Since we are only interested in finding one solution, we make a choice and this defines in turn $\mathcal{O}^{\mu_1 \cdots \mu_n}_{(n,t,j)}$ and $\mathcal{L}^{(n)}$.

Let us discuss some technical details. First of all, in the second step, the number of terms to consider can be reduced by some simple considerations.

[10] With $2 < j < [n/2]$ and $1 \leq t \leq t_{max}$

Permutation symmetries and momentum conservation will allow us to impose some useful restrictions:

Restriction	for	Motivation
$\sigma(2k-1) < \sigma(2k)$	$k \in \{1, \ldots, j\}$	$g^{\mu_{\sigma(2k-1)}\mu_{\sigma(2k)}}$ $= g^{\mu_{\sigma(2k)}\mu_{\sigma(2k-1)}}$
$\sigma(2k-1) < \sigma(2k)$	$k \in \{1, \ldots, j-1\}$	$g^{\mu_{\sigma(2k-1)}\mu_{\sigma(2k)}} g^{\mu_{\sigma(2k+1)}\mu_{\sigma(2k+2)}}$ $= g^{\mu_{\sigma(2k+1)}\mu_{\sigma(2k+2)}} g^{\mu_{\sigma(2k-1)}\mu_{\sigma(2k)}}$
$\sigma(2k-1) < \sigma(2k)$	$k \in \{1, \ldots, n-2j\}$	$\partial_{i_k}^{\mu_{\sigma(2j+k)}} \partial_{i_k+1}^{\mu_{\sigma(2j+k+1)}}$ $= \partial_{i_k+1}^{\mu_{\sigma(2j+k+1)}} \partial_{i_k}^{\mu_{\sigma(2j+k)}}$
$i_{n-2j-+2k-1} < i_{n-2j-+2k}$	$k \in \{1, \ldots, j-2\}$	$\partial_{i_{l-1}} \cdot \partial_{i_l} = \partial_{i_l} \cdot \partial_{i_{l-1}}$
$i_{n-2j-+2k-1} < i_{n-2j-+2k+1}$	$k \in \{1, \ldots, j-3\}$	$\left(\partial_{i_{l-1}} \cdot \partial_{i_l}\right) \left(\partial_{i_{l+1}} \cdot \partial_{i_{l+2}}\right)$ $= \left(\partial_{i_{l+1}} \cdot \partial_{i_{l+2}}\right) \left(\partial_{i_{l-1}} \cdot \partial_{i_l}\right)$
$i_k < n$	$k \in \{1, \ldots, n-4\}$	Momentum conservation

The second technical detail is that, in the third step, one finds a block-triangular system of equations. This means that the coefficients of a monomial involving j products of the type $\epsilon_{i_1} \cdot \epsilon_{i_2}$ will bring to equations containing variables $c_{n,t,j',\sigma,\boldsymbol{i}}$, with $j' < j$. In particular then, if we consider only monomials involving exactly two scalar products $\epsilon_{i_1} \cdot \epsilon_{i_2}$, we will obtain equations containing only variables $c_{n,t,2,\sigma,\boldsymbol{i}}$.

4.5.1 *Feynman rules from the new Lagrangian terms*

We described in detail how to obtain the terms $\mathcal{L}^{(n)}$ in the new formulation for the Yang-Mills Lagrangian. How do we obtain the corresponding Feynman rules? This will be the topic covered in this section.
Let us start with a toy example in order to gradually construct our master formula. Consider an interaction term of the form:

$$\mathcal{L}_{int} = \mathcal{O}^{a_1 \ldots a_n, \mu_1 \ldots \mu_n}(\partial_1, \ldots, \partial_n) A_{\mu_1}^{a_1}(x) \ldots A_{\mu_n}^{a_n}(x) \tag{4.61}$$

where $\mathcal{O}^{a_1 \ldots a_n, \mu_1 \ldots \mu_n}$ is a pseudo-differential operator of degree (4-n) depending on derivatives ∂_j that act only on the corresponding $A_{\mu_j}^{a_j}(x)$ fields. The full[11] Feynman rule for the vertex is given by:

$$V_{full} = \sum_{\sigma \in S_n} \mathcal{O}^{a_{\sigma(1)} \ldots a_{\sigma(n)}, \mu_{\sigma(1)} \ldots \mu_{\sigma(n)}}(ip_{\sigma(1)}, \ldots, ip_{\sigma(n)}) \tag{4.62}$$

where, once again, all the momenta p_j are outgoing. Let us suppose now that the operator $\mathcal{O}^{a_1 \ldots a_n, \mu_1 \ldots \mu_n}(\partial_1, \ldots, \partial_n)$ can be colour decomposed in the

[11] Including colour.

following way:

$$\mathcal{O}^{a_1\dots a_n,\mu_1\dots\mu_n}(\partial_1,\dots,\partial_n) = 2g^{n-2}Tr(T^{a_1}\dots T^{a_n})\mathcal{O}^{\mu_1\dots\mu_n}(\partial_1,\dots,\partial_n) \quad (4.63)$$

The Feynman rule for the vertex can then be written, splitting the sum over the permutations S_n into $S_n/\mathbb{Z}_n$ and $\mathbb{Z}_n$, as

$$V_{full} = ig^{n-2}\sum_{\pi\in S_n/\mathbb{Z}_n} Tr(T^{a_{\pi(1)}}\dots T^{a_{\pi(n)}})\sum_{\sigma\in\mathbb{Z}_n}\mathcal{O}^{\mu_{\sigma\pi(1)}\dots\mu_{\sigma\pi(n)}}(ip_{\sigma\pi(1)},\dots,ip_{\sigma\pi(n)}) \quad (4.64)$$

It is now possible to extract the colour-ordered Feynman rule:

$$V = i\sum_{\sigma\in\mathbb{Z}_n}\mathcal{O}^{\mu_{\sigma(1)}\dots\mu_{\sigma(n)}}(ip_{\sigma(1)},\dots,ip_{\sigma(n)}) \quad (4.65)$$

Since we will be working with trees, we need to reintroduce and expand part of the notation. Let us consider a tree T with n external legs cyclically ordered as $(1,2,\dots,n)$ and let us denote by $\boldsymbol{T}_{\mu_1\dots\mu_n}$ a representation of this tree made of commutators and fields A_{μ_i}. Consider as an example the tree

$$T = ([[1,2],[3,4]],5). \quad (4.66)$$

Its representation will be given by

$$\boldsymbol{T}_{\mu_1\dots\mu_n} = ([[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],[\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}]],\boldsymbol{A}_{\mu_5}). \quad (4.67)$$

As we discussed already before, a general tree T with n external legs, has (n-2) vertices and, when drawn as a rooted tree, there are $2^{(n-2)}$ ways of swapping at each vertex the two branches not connected to the root. By writing all the different swapping possibilities, we obtain a set of permutations of the external legs which we denote by $B_n(T)$ and which correspond to different cyclic orders for the external legs.
Let us consider now a term of the form

$$\mathcal{L}_{int} = \frac{1}{2g^2}\mathcal{O}^{\mu_1\dots\mu_n}(\partial_1,\dots,\partial_n)Tr\boldsymbol{T}_{\mu_1\dots\mu_n} \quad (4.68)$$

By exploiting the antisymmetry of commutators[12], we can rewrite it as

$$\mathcal{L}_{int} = \frac{1}{2g^2}\mathcal{O}^{\mu_1\dots\mu_n}(\partial_1,\dots,\partial_n)\sum_{\pi\in B_n(T)}(-1)^{n_{swap}(\pi)}Tr\boldsymbol{A}_{\mu_{\pi(1)}}\dots\boldsymbol{A}_{\mu_{\pi(n)}} \quad (4.69)$$

where it is important not to confuse $n_{swap}(\pi)$ with the sign of normal permutations. To give an example, let us take the tree $([[1,2],3],4)$. If we swap

[12] Or, the properties of trees.

the branches $[1,2]$ and 3, we obtain $([3,[1,2]],4)$, with $n_{swap}(\pi)=1$ and therefore $(-1)^{n_{swap}(\pi)}=-1$. This should not be confused with the sign of the permutation for the order of the external legs $\pi=(3,1,2,4)$, whose sign is $(+1)$.
By relabelling the indices we obtain:

$$\mathcal{L}_{int}=\frac{1}{2g^2}\sum_{\pi\in B_n(T)}(-1)^{n_{swap}(\pi)}\mathcal{O}^{\mu_{\pi^{-1}(1)}\cdots\mu_{\pi^{-1}(n)}}(\partial_{\pi^{-1}(1)},\ldots,\partial_{\pi^{-1}(n)})Tr\boldsymbol{A}_{\mu_1}\ldots\boldsymbol{A}_{\mu_n} \tag{4.70}$$

The colour-ordered Feynman rule becomes then

$$V=\frac{(-1)^n}{4}i^{n+1}\sum_{\pi\in B_n(T)}(-1)^{n_{swap}(\pi)}\sum_{\sigma\in\mathbb{Z}_n}\mathcal{O}^{\mu_{\sigma\pi^{-1}(1)}\cdots\mu_{\sigma\pi^{-1}(n)}}(ip_{\sigma\pi^{-1}(1)},\ldots,ip_{\sigma\pi^{-1}(n)}) \tag{4.71}$$

Up to now we have defined a way to construct Feynman rules starting from particular tree structures, taking care of the colour-information separately and managing all the possible swappings allowed keeping the root fixed. We only miss some notation now to treat another kind of permutation needed in our case: "Jacobi permutations", corresponding to the presence inside of $\boldsymbol{T}_{\mu_1\ldots\mu_n}$ of Jacobi-sums[13]. So, let us consider again interaction terms of the form 4.68, where this time $\boldsymbol{T}_{\mu_1\ldots\mu_n}$ is given by

$$\boldsymbol{T}_{\mu_1\ldots\mu_n}=J(\boldsymbol{T}_{\mu_1\ldots\mu_{j_1}},\boldsymbol{T}_{\mu_{j_1+1}\ldots\mu_{j_2}},\boldsymbol{T}_{\mu_{j_2+1}\ldots\mu_{j_3}})\boldsymbol{T}_{\mu_{j_3+1}\ldots\mu_{j_n}} \tag{4.72}$$

The symbol J denotes once again a sum over the three permutations of the Jacobi identity. It is convenient then to introduce three trees which correspond to the three terms in the Jacobi sum. We will denote these trees with T_{1234}, T_{2314} and T_{3124}

$$T_{1234}=[[\boldsymbol{T}_{\mu_1\ldots\mu_{j_1}},\boldsymbol{T}_{\mu_{j_1+1}\ldots\mu_{j_2}}],\boldsymbol{T}_{\mu_{j_2+1}\ldots\mu_{j_3}}]\boldsymbol{T}_{\mu_{j_3+1}\ldots\mu_{j_n}} \tag{4.73}$$
$$T_{2314}=[[\boldsymbol{T}_{\mu_{j_1+1}\ldots\mu_{j_2}},\boldsymbol{T}_{\mu_{j_2+1}\ldots\mu_{j_3}}],\boldsymbol{T}_{\mu_1\ldots\mu_{j_1}}]\boldsymbol{T}_{\mu_{j_3+1}\ldots\mu_{j_n}} \tag{4.74}$$
$$T_{1234}=[[\boldsymbol{T}_{\mu_{j_2+1}\ldots\mu_{j_3}},\boldsymbol{T}_{\mu_1\ldots\mu_{j_1}}],\boldsymbol{T}_{\mu_{j_1+1}\ldots\mu_{j_2}}]\boldsymbol{T}_{\mu_{j_3+1}\ldots\mu_{j_n}} \tag{4.75}$$

With these definitions, we can rewrite

$$\boldsymbol{T}_{\mu_1\ldots\mu_n}=J(\boldsymbol{T}_{\mu_1\ldots\mu_{j_1}},\boldsymbol{T}_{\mu_{j_1+1}\ldots\mu_{j_2}},\boldsymbol{T}_{\mu_{j_2+1}\ldots\mu_{j_3}})\boldsymbol{T}_{\mu_{j_3+1}\ldots\mu_{j_n}}=T_{1234}+T_{2314}+T_{3124} \tag{4.76}$$

Moreover, it is also convenient to define three permutations

$$\tau_{1234}=(1,\ldots,j_1,j_1+1,\ldots,j_2,j_2+1,\ldots,j_3,j_3+1,\ldots,n) \tag{4.77}$$
$$\tau_{2314}=(j_1+1,\ldots,j_2,j_2+1,\ldots,j_3,1,\ldots,j_1,j_3+1,\ldots,n) \tag{4.78}$$
$$\tau_{1234}=(j_2+1,\ldots,j_3,1,\ldots,j_1,j_1+1,\ldots,j_2,j_3+1,\ldots,n) \tag{4.79}$$

[13] As defined through the quantity "J" in the previous section.

We also define the set

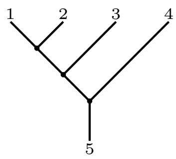
$$I = \{1234, 2314, 3124\}. \tag{4.80}$$

In order to not confuse the order of the indices, since we will have three different kinds of permutations at work, we need some more notation. We denote by $T(1, \dots, n)$ a tree with cyclic order $(1, \dots, n)$. We will denote by (πT) a tree obtained from the original tree by swapping the branches at the vertices to reach the permutation π. (πT) will have the cyclic order $(\pi(1), \dots, \pi(n))$. We then denote by

$$(\pi T)(1, \dots, n) \tag{4.81}$$

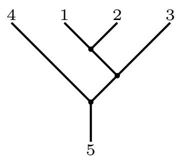
the above constructed tree, where the indices have been in the end re-labeled into $(1, \dots, n)$. Let us see some examples to avoid confusion. At five-points, we start with

$$T(1, 2, 3, 4, 5) = \tag{4.82}$$



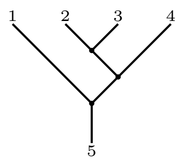
We decide to swap the branches at the lowest vertex in order to obtain the cyclic order $(\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)) = (4, 1, 2, 3, 5)$

$$(\pi T) = \tag{4.83}$$



We finally relabel indices to obtain:

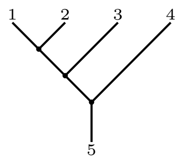
$$(\pi T)(1, 2, 3, 4, 5) = \tag{4.84}$$



The next notation we need to define is the one for Jacobi operations τ_j with $j \in I$.

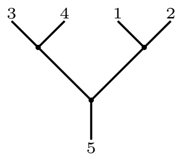
We denote by $(\tau_j T)$ the tree obtained from T performing the Jacobi operation τ_j. The cyclic order of this tree will then be $(\tau(1), \dots \tau(n))$. Thanks to the definitions given above, we can see for example that $T_{2314} = \tau_{2314} T_{1234}$. Once again, we denote by $(\tau T)(1, \dots, n)$ this tree where, however, we re-labeled external legs as $(1, \dots, n)$. Some graphical examples can help to clarify once again. Let us start from a tree $T(1,2,3,4,5) = ([[[1,2],3],4],5)$ corresponding in our notation to T_{1234}.

$$T_{1234} = T(1,2,3,4,5) = \tag{4.85}$$



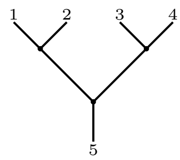
We are considering a case where the Jacobi permutations are performed on the three sub-trees $[1,2], 3$ and 4. We decide to apply τ_{2314} on T, in order to obtain $T_{2314} = (\tau_{2314}T)$

$$(\tau_{2314}T) = \tag{4.86}$$



We finally relabel indices to obtain $(\tau_{2314}T)(1,2,3,4,5)$

$$(\tau_{2314}T)(1,2,3,4,5) = \tag{4.87}$$



The Feynman rule for an interaction term of the type 4.68 with $\boldsymbol{T}_{\mu_1 \dots \mu_n}$ as in 4.72 can then be written as

$$\begin{aligned} V =& \frac{(-1)^n}{4} i^{n+1} \sum_{j \in I} \sum_{\pi \in B_n((\tau_j T)(1,\dots,n))} (-1)^{n_{swap}(\pi)} \\ & \sum_{\sigma \in \mathbb{Z}_n} \mathcal{O}^{\mu_{\sigma\pi^{-1}\tau_j^{-1}(1)} \cdots \mu_{\sigma\pi^{-1}\tau_j^{-1}(n)}} (ip_{\sigma\pi^{-1}\tau_j^{-1}(1)}, \dots, ip_{\sigma\pi^{-1}\tau_j^{-1}(n)}) \end{aligned} \tag{4.88}$$

where $T = T_{1234}$.

Now we just need to take care of one last step. In fact, we never explicitely treated any term containing the operator $\hat{D}^{-1}$. In principle we can think of it as being part of $\mathcal{O}^{\mu_1 \dots \mu_n}(\partial_1, \dots, \partial_n)$ in 4.68. However, for our purposes it is better to treat it explicitely. Let us consider then a Lagrangian term of the form

$$\mathcal{L}_{int} = \frac{1}{2g^2} \mathcal{O}^{\mu_1 \dots \mu_n}(\partial_1, \dots, \partial_n) \hat{D}^{-1} Tr \boldsymbol{T}_{\mu_1 \dots \mu_n} \tag{4.89}$$

Notice that $\hat{D}^{-1}$ depends only on the tree-structures it acts upon.
The Feynman rule is then given by:

$$V = \frac{(-1)^n}{4} i^{n+1} \sum_{j\in I} \sum_{\pi\in B_n((\tau_j T)(1,\ldots,n))} (-1)^{n_{swap}(\pi)} \sum_{\sigma\in\mathbb{Z}_n} \mathcal{O}^{\mu_{\sigma\pi^{-1}\tau_j^{-1}(1)}\cdots\mu_{\sigma\pi^{-1}\tau_j^{-1}(n)}}(ip_{\sigma\pi^{-1}\tau_j^{-1}(1)},\ldots,ip_{\sigma\pi^{-1}\tau_j^{-1}(n)})\hat{D}^{-1}\left((\pi\tau_j T)(\sigma(1),\ldots,\sigma(n))\right) \tag{4.90}$$

4.6 Some results

In this section we will present results up to $n = 6$, following the choices made in [49].
The effective Lagrangian takes the form

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2g^2}\sum_{n=2}^{\infty}\mathcal{L}^{(n)}. \tag{4.91}$$

The terms $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)}$ are the ones presented with the original Yang-Mills Lagrangian

$$\begin{aligned}\mathcal{L}^{(2)} &= -2Tr\boldsymbol{A}_\mu\Box\boldsymbol{A}^\mu\\ \mathcal{L}^{(3)} &= 4Tr(\partial_\mu\boldsymbol{A}_\nu)[\boldsymbol{A}^\mu,\boldsymbol{A}^\nu]\end{aligned} \tag{4.92}$$

The term for $n = 4$ has been slightly modified from the original form in order to assign four-point contact terms to the right channels

$$\begin{aligned}\mathcal{L}^{(4)} &= \mathcal{O}^{\mu_1\mu_2\mu_3\mu_4}D^{-1}Tr[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}][\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}]\\ \mathcal{O}^{\mu_1\mu_2\mu_3\mu_4} &= g^{\mu_1\mu_3}g^{\mu_2\mu_4}g_{\nu_1\nu_2}\partial_{12}^{\nu_1}\partial_{34}^{\nu_2}\end{aligned} \tag{4.93}$$

$\mathcal{L}^{(5)}$ and $\mathcal{L}^{(6)}$ are instead of the form

$$\mathcal{L}^{(n)} = \sum_t\sum_{j=2}^{[n/2]}\mathcal{O}^{\mu_1\cdots\mu_n}_{(n,t,j)}\hat{D}^{-1}Tr\boldsymbol{T}^{(n,t)}_{\mu_1\cdots\mu_n} \tag{4.94}$$

and they both are formally zero due to the Jacobi identity contained in $\boldsymbol{T}^{(n,t)}_{\mu_1\cdots\mu_n}$.
For $\mathcal{L}^{(5)}$ we have only one tree structure ($t = 1$) (see appendix B) and j can only take the value two ($j = 2$). The tree structure is

$$Tr\boldsymbol{T}^{(5,1)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = TrJ([\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4})\boldsymbol{A}_{\mu_5} \tag{4.95}$$

Since $t = 1$ and $j = 2$, we just have one operator $\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{(5,1,2)}$. As we said in the previous sections, however, the operator is nonetheless not unique. We make once again the choice presented in 4.53

$$\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{(5,1,2)} = -4g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_1^{\mu_5} \tag{4.96}$$

For $n = 6$, the situation is a bit more complicated because t and j can take more then one value. We have in fact two different tree structures

$$Tr\boldsymbol{T}^{(6,1)}_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} = TrJ([[\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],\boldsymbol{A}_{\mu_3}],\boldsymbol{A}_{\mu_4},\boldsymbol{A}_{\mu_5})\boldsymbol{A}_{\mu_6} \tag{4.97}$$
$$Tr\boldsymbol{T}^{(6,2)}_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} = TrJ([\boldsymbol{A}_{\mu_1},\boldsymbol{A}_{\mu_2}],[\boldsymbol{A}_{\mu_3},\boldsymbol{A}_{\mu_4}],\boldsymbol{A}_{\mu_5})\boldsymbol{A}_{\mu_6} \tag{4.98}$$
$$\tag{4.99}$$

In addition, we can have $j = 2$ or $j = 3$. This means that overall we will have four possibilities for $\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(6,t,j)}$. Once again, we have to make a choice among the possible equivalent terms. We confirm the choice made in [49]:

$$\begin{aligned}
\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(6,1,2)} &= \\
&-8g^{\mu_1\mu_2}g^{\mu_3\mu_4}\partial_1^{\mu_5}\partial_2^{\mu_6} - 4g^{\mu_1\mu_2}g^{\mu_3\mu_4}\partial_1^{\mu_5}\partial_4^{\mu_6} - 4g^{\mu_1\mu_2}g^{\mu_3\mu_4}\partial_1^{\mu_5}\partial_5^{\mu_6} + 4g^{\mu_1\mu_2}g^{\mu_4\mu_5}\partial_1^{\mu_3}\partial_4^{\mu_6} \\
&+8g^{\mu_1\mu_2}g^{\mu_4\mu_5}\partial_4^{\mu_3}\partial_1^{\mu_6} + 8g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_2^{\mu_5}\partial_3^{\mu_6} - 24g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_2^{\mu_5}\partial_4^{\mu_6} - 8g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_2^{\mu_5}\partial_5^{\mu_6} \\
&+8g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_4^{\mu_5}\partial_5^{\mu_6} + 8g^{\mu_1\mu_3}g^{\mu_4\mu_5}\partial_1^{\mu_2}\partial_4^{\mu_6} - 8g^{\mu_1\mu_3}g^{\mu_4\mu_5}\partial_4^{\mu_2}\partial_2^{\mu_6} - 8g^{\mu_1\mu_3}g^{\mu_4\mu_5}\partial_4^{\mu_2}\partial_3^{\mu_6} \\
&+8g^{\mu_1\mu_4}g^{\mu_2\mu_5}\partial_1^{\mu_3}\partial_4^{\mu_6} + 24g^{\mu_1\mu_4}g^{\mu_2\mu_5}\partial_4^{\mu_3}\partial_2^{\mu_6} + 16g^{\mu_1\mu_4}g^{\mu_2\mu_5}\partial_4^{\mu_3}\partial_3^{\mu_6} + 32g^{\mu_1\mu_4}g^{\mu_2\mu_5}\partial_4^{\mu_3}\partial_4^{\mu_6} \\
&+32g^{\mu_1\mu_4}g^{\mu_2\mu_5}\partial_4^{\mu_3}\partial_5^{\mu_6} + 4g^{\mu_1\mu_4}g^{\mu_2\mu_6}\partial_5^{\mu_3}\partial_3^{\mu_5} - 16g^{\mu_1\mu_4}g^{\mu_2\mu_6}\partial_5^{\mu_3}\partial_4^{\mu_5} - 8g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_1^{\mu_2}\partial_5^{\mu_6} \\
&+8g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_3^{\mu_2}\partial_1^{\mu_6} + 8g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_3^{\mu_2}\partial_2^{\mu_6} + 8g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_3^{\mu_2}\partial_3^{\mu_6} + 16g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_5^{\mu_2}\partial_1^{\mu_6} \\
&+8g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_5^{\mu_2}\partial_2^{\mu_6} - 8g^{\mu_1\mu_4}g^{\mu_3\mu_5}\partial_5^{\mu_2}\partial_3^{\mu_6} - 8g^{\mu_1\mu_4}g^{\mu_3\mu_6}\partial_5^{\mu_2}\partial_1^{\mu_5} + 16g^{\mu_1\mu_4}g^{\mu_5\mu_6}\partial_5^{\mu_2}\partial_1^{\mu_3} \\
&-16g^{\mu_3\mu_4}g^{\mu_5\mu_6}\partial_2^{\mu_1}\partial_5^{\mu_2} - 16g^{\mu_3\mu_4}g^{\mu_5\mu_6}\partial_4^{\mu_1}\partial_5^{\mu_2} + 16g^{\mu_3\mu_6}g^{\mu_4\mu_5}\partial_4^{\mu_1}\partial_5^{\mu_2}
\end{aligned} \tag{4.100}$$

$$\begin{aligned}
\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(6,2,2)} &= \\
&-2g^{\mu_1\mu_2}g^{\mu_3\mu_4}\partial_1^{\mu_5}\partial_3^{\mu_6} - 12g^{\mu_1\mu_2}g^{\mu_3\mu_5}\partial_1^{\mu_4}\partial_4^{\mu_6} - 4g^{\mu_1\mu_2}g^{\mu_3\mu_5}\partial_1^{\mu_4}\partial_4^{\mu_6} - 8g^{\mu_1\mu_2}g^{\mu_5\mu_6}\partial_1^{\mu_3}\partial_5^{\mu_4} \\
&-12g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_1^{\mu_5}\partial_3^{\mu_6} - 4g^{\mu_1\mu_3}g^{\mu_2\mu_4}\partial_1^{\mu_5}\partial_5^{\mu_6} - 8g^{\mu_1\mu_3}g^{\mu_2\mu_5}\partial_1^{\mu_4}\partial_4^{\mu_6} + 16g^{\mu_1\mu_3}g^{\mu_2\mu_5}\partial_3^{\mu_4}\partial_1^{\mu_6} \\
&-8g^{\mu_1\mu_3}g^{\mu_2\mu_5}\partial_5^{\mu_4}\partial_4^{\mu_6} + 8g^{\mu_1\mu_3}g^{\mu_5\mu_6}\partial_1^{\mu_2}\partial_5^{\mu_4} + 8g^{\mu_1\mu_5}g^{\mu_2\mu_6}\partial_1^{\mu_3}\partial_3^{\mu_4} + 8g^{\mu_1\mu_5}g^{\mu_2\mu_6}\partial_1^{\mu_3}\partial_5^{\mu_4} \\
&-8g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_1^{\mu_2}\partial_2^{\mu_4} - 4g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_1^{\mu_2}\partial_3^{\mu_4} - 16g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_1^{\mu_2}\partial_5^{\mu_4} + 12g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_3^{\mu_2}\partial_1^{\mu_4} \\
&+8g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_3^{\mu_2}\partial_2^{\mu_4} - 4g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_4^{\mu_2}\partial_2^{\mu_4} - 8g^{\mu_1\mu_5}g^{\mu_3\mu_6}\partial_5^{\mu_2}\partial_3^{\mu_4}
\end{aligned} \tag{4.101}$$

$$\begin{aligned}
\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(6,1,3)} &= \\
&-2g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6}(\partial_1\cdot\partial_4) - 4g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6}(\partial_1\cdot\partial_5) - 6g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6}(\partial_4\cdot\partial_5)
\end{aligned} \tag{4.102}$$

$$
\begin{aligned}
&\mathcal{O}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(6,2,3)} = \\
&\quad -2g^{\mu_1\mu_3}g^{\mu_2\mu_4}g^{\mu_5\mu_6}(\partial_1\cdot\partial_5) - 2g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6}(\partial_1\cdot\partial_3) - 2g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6}(\partial_2\cdot\partial_4) \\
&\quad + 2g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6}(\partial_2\cdot\partial_5) + 4g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6}(\partial_3\cdot\partial_5)
\end{aligned}
\tag{4.103}
$$

In addition, we obtain also the set of terms that can be added to the Lagrangian without violating BCJ-relations[14] for $n = 6$. These results do not seem to offer any particular hint on the existence of some smarter way to formulate them or of a particular structure that can be investigated.
Some proposals for ways to improve the algorithm and to, perhaps, find something more revealing about the Lagrangian structures that generate BCJ-numerators will be the topic of the next chapter. At the present state, the algorithm is still a useful tool to further investigate BCJ-relations and its implications for gravity.

[14] Corresponding to the term 4.55 for $n = 5$.

5

SCATTERING EQUATIONS AND KINEMATIC ALGEBRAS

In the previous chapters, we have seen how the construction of BCJ-numerators presents a considerable amount of freedom, which takes the form of generalized gauge transformations at the level of amplitudes and takes the form of additional terms that do not contribute to BCJ-relations or the choice of free parameters in our Lagrangian formulation. This freedom might seem a positive feature. However, on the contrary, it should be seen as the fact that we have not found yet additional relations to further constrain those choices. In particular, in our Lagrangian-construction algorithm, the freedom we are left with, complicates calculations and every choice made at n valency further influences the successive results.

The last part of my doctoral studies involved the search of a way to further constrain our algorithm in order to improve its efficiency and to obtain a canonical formulation for BCJ-numerators. A first attempt was made with corolla polynomials [106, 107]. The tree formulation of this idea is in a way prone to be exploited in terms of our construction. However, the attempts made did not go further than the correct splitting for the four-point contact terms.

In the rest of the chapter instead, we will discuss another viable way to improve our algorithm: the kinematic algebra based on scattering equations analysed by Monteiro and O'Connell first in the context of the (anti-)self-dual sector of Yang Mills theory [48] and then in the general Yang-Mills gauge theory[56].

In the next section we will briefly review scattering equations. Afterwars, we will show how the construction of BCJ-numerators work in the (anti)-self-dual sector of Yang Mills theories. Finally, we will present the - so called in [56] - canonical construction of BCJ-numerators based on solutions of the scattering equations.

The last section will be dedicated to a discussion of possible ways to exploit these results from the perspective of our algorithm and to difficulties and caveats that can be foreseen on this path.

5.1 The scattering equations

The scattering equations have been shown to be particularly relevant for tree-level scattering of massless particles [52–54]. In this section we will see a short review - following mainly [56] - of why this is the case.
The scattering equations at n-points read

$$\sum_{b\neq a} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} = 0 \tag{5.1}$$

where the p_a's are the momenta of the extenal legs. This set of equations has to be solved for the n complex variables σ_a. Of this n equations, however, only $(n-3)$ are independent. A key feature is their invariance under $SL(2,\mathbb{C})$. This means in particular that givena solution σ_a, another valid solution is given by

$$\sigma_a' = \frac{A\sigma_a + B}{C\sigma_a + D} \tag{5.2}$$

where $AD - BC = 1$. This property allows us in turn to interpret the solutions σ_a as points on a sphere S^2. Up to this redundancy, there are always $(n-3)!$ different solutions of the scattering equations.
One way to see this is through the recursive algorithm proposed in [52]. In fact, each solution for the $(n-1)$-points system yelds $(n-3)$ solutions for the n-points system for a total of $(n-3)!$ solutions. Exploiting scattering equations, one can easily write n-points colour-ordered gluon amplitudes and graviton amplitudes

$$A_n = \int \frac{d^n\sigma}{\text{vol } SL(2,\mathbb{C})} {\prod_a}' \delta\left(\sum_{b\neq a} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b}\right) \frac{E_n(\{p,\epsilon,\sigma\})}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} \tag{5.3}$$

$$M_n = \int \frac{d^n\sigma}{\text{vol } SL(2,\mathbb{C})} {\prod_a}' \delta\left(\sum_{b\neq a} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b}\right) E_n^2(\{p,\epsilon,\sigma\}) \tag{5.4}$$

where $\sigma_{ab} = \sigma_a - \sigma_b$ and where

$${\prod_a}' \delta\left(\sum_{b\neq a} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b}\right) = \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b}\right) \tag{5.5}$$

In particular, this last definition can be shown to be independent of the choice of i, j, k and is therefore permutation-symmetric. Let us define now the object $E_n(\{p,\epsilon,\sigma\})$. It is a gauge invariant object depending on the momenta and the polarizations ϵ_a of the particles and it is as well symmetric

under permutations of the particles.
It can be described in terms of a $2n \times 2n$ matrix Ψ which can be given in a block form

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \tag{5.6}$$

where the different $n \times n$ blocks are defined as

$$A_{ab} = \begin{cases} \dfrac{p_a \cdot p_b}{\sigma_{ab}^2} & a \neq b \\ 0 & a = b \end{cases} \tag{5.7}$$

$$B_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \tag{5.8}$$

$$C_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot p_b}{\sigma_{ab}} & a \neq b \\ -\displaystyle\sum_{c \neq a} \dfrac{\epsilon_a \cdot p_c}{\sigma_{ac}} & a = b \end{cases} \tag{5.9}$$

We denote by Ψ_{ij}^{ij} the matrix where rows i and j and the corresponding i and j columns have been left out. With this notation, we can finally define $E_n(\{p, \epsilon, \sigma\})$

$$E_n(\{p, \epsilon, \sigma\}) \equiv Pf'(\Psi_{ij}^{ij}) \equiv 2\frac{(-1)^{i+j}}{\sigma_{ij}} Pf(\Psi_{ij}^{ij}) \tag{5.10}$$

where "Pf" is the pfaffian of the matrix.
The integrations over the delta functions have two functions. First of all they completely localise the integrals in 5.3 and 5.4, reducing them in fact to a sum over the $(n-3)!$ solutions of the scattering equations. Secondly, the integration introduces a Jacobian. In order to describe it, we introduce the matrix Φ, with components

$$\Phi_{ab} = \begin{cases} \dfrac{p_a \cdot p_b}{\sigma_{ab}^2} & a \neq b \\ -\displaystyle\sum_{c \neq a} \dfrac{p_a \cdot p_c}{\sigma_{ac}^2} & a = b \end{cases} \tag{5.11}$$

The delta functions in 5.5 tell us to leave out rows i, j and k. For this reason, we omit them from the Jacobian determinant. In addition, we have to gauge fix $SL(2,\mathbb{C})$. To do so, we can fix the positions of three points σ_r, σ_s and σ_t and this procedure will introduce a factor $\sigma_{rs}\sigma_{st}\sigma_{tr}$. The final result is that the Jacobian determinant is given by the minor determinant of Φ where rows i, j and k and columns r, s and t are left out. We introduce then the notation

$$\det{}'\Phi = \frac{|\Phi|^{ijk}_{rst}}{\sigma_{rs}\sigma_{st}\sigma_{tr}\sigma_{ij}\sigma_{jk}\sigma_{ki}} \tag{5.12}$$

The amplitudes can then be written as sums over the solutions of the scattering equations

$$A_n = \sum_{solutions} \frac{1}{\sigma_{12}\sigma_{23}\dots\sigma_{n1}} \frac{Pf'\Psi}{\det' \Phi} \tag{5.13}$$

$$M_n = \sum_{solutions} \frac{(Pf'\Psi)^2}{\det' \Phi} \tag{5.14}$$

5.1.1 The light-cone gauge

First of all, we need to introduce some new notation, since we will be working in the light cone gauge. This choice can be motivated in different ways. First of all it is one of the most promising concerning the study and the construction of a gravity Lagrangian since the explicit independent fields that appear are two in both theories in four space-time diensions[1]. Secondly, it is the most natural gauge in which to perform the following study of the kinematic algebra.

We will use light cone coordinates $(u, v, w, \bar{w})$. The metric is such that, given any two four-vectors A, B

$$2A \cdot B = A_u B_v + A_v B_u - A_w B_{\bar{w}} - A_{\bar{w}} B_w \tag{5.15}$$

The polarization vectors are given by

$$\epsilon_a^+ = (0, p_{aw}, 0, p_{au}), \quad \epsilon_a^- = (0, p_{a\bar{w}}, p_{au}, 0) \tag{5.16}$$

and they satisfy

$$\epsilon_a^\pm \epsilon_b^\pm = 0, \quad 2\epsilon_a^+ \epsilon_b^- = -p_{au}p_{bu} \tag{5.17}$$

We can now define two important quantities that will constitute the main objects we will work with

$$2\epsilon_a^+ \cdot p_b = p_{aw}p_{bu} - p_{au}p_{bw} = X_{a,b} \tag{5.18}$$

$$2\epsilon_a^- \cdot p_b = p_{a\bar{w}}p_{bu} - p_{au}p_{b\bar{w}} = \bar{X}_{a,b} \tag{5.19}$$

[1] This is true in general for light-like gauge of which the light-cone gauge is a particular case.

If p_a and p_b are on-shell, we have also

$$s_{ab} = \frac{X_{a,b}\bar{X}_{a,b}}{p_{au}p_{bu}} \tag{5.20}$$

Equations 5.18 and 5.19 can be instead extended to off-shell quantities, using for example the simple rule $X_{a+b,c} = X_{a,c} + X_{b,c}$. Off-shell momenta will be later denominated by capital letters $A, B, \ldots$.
How do these quantities relate to the scattering equations?
First of all let us specify how we write a four-vector with spinorial indices

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p_u & p_w \\ p_{\bar{w}} & p_v \end{pmatrix} \tag{5.21}$$

When the four-vector is on shell, we can write it in terms of two spinors as $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. One can choose

$$\lambda_\alpha = \begin{pmatrix} 1 \\ \sigma \end{pmatrix} \quad \tilde{\lambda}_{\dot{\alpha}} = \begin{pmatrix} p_u \\ p_w \end{pmatrix} \quad \text{where } \sigma = \frac{p_{\bar{w}}}{p_u} = \frac{p_v}{p_w} \tag{5.22}$$

We are now ready to write a new set of spinor products anologous to the ones introduced in chapter 2 in 2.48

$$\langle ab \rangle = \epsilon^{\alpha\beta}\lambda^{(a)}_\alpha \lambda^{(b)}_\beta = \sigma_a - \sigma_b = \frac{\bar{X}_{a,b}}{p_{au}p_{bu}} \tag{5.23}$$

$$[ab] = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{(a)}_{\dot{\alpha}} \tilde{\lambda}^{(b)}_{\dot{\beta}} = -X_{a,b} \tag{5.24}$$

From eq. 5.20 we also obtain

$$X_{a,b} = \frac{s_{ab}}{\sigma_a - \sigma_b} \tag{5.25}$$

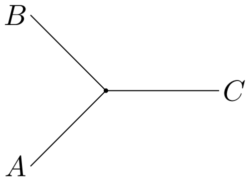
Now, using eq. 5.18 and 5.19 and momentum conservation we have

$$\sum_{b \neq a} X_{a,b} = 0 \qquad \sum_{b \neq a} \bar{X}_{a,b} = 0 \tag{5.26}$$

which are a re-casted version of the scattering equations in terms of the quantities of interest $X_{a,b}, \bar{X}_{a,b}$.
It is finally time then to see why they are particularly interesting quantities.

5.2 The (anti)-self-dual sector

The quantity $X_{a,b}$ can be shown to be the kinematic part of the vertex of self-dual gauge theory [108, 109]



$$= (2\pi)^4 \delta^{(4)}(P_A, P_B, P_C) X_{A,B} f^{a_A, a_B, a_C} \tag{5.27}$$

We can see the vertex as part of a three-valent graph, where P_A, P_B and P_C are sums of momenta of the external particles (e.g. $P_A = p_1 + p_2$ and $P_B = p_3$ and as a consequence $X_{A,B} = X_{1+2,3} = X_{1,3} + X_{2,3}$). In order for $X_{A,B}$ to be a vertex, it has to be independent of the two legs chosen to represent it, i.e.

$$X_{A,B} = X_{B,C} = X_{C,A} \tag{5.28}$$

This is guaranteed by momentum conservation and by the fact that $X_{A,B}$ is an anti-symmetric bilinear form. Where does the relation with colour-kinematics duality arise? As shown in [48], $X_{A,B}$ is the structure constant of a Lie Algebra. In particular the one that arises with area-preserving diffeomorphisms in the $w - u$ plane and generated by the vectors

$$V_A^+ = -2e^{-iP_A \cdot x} \epsilon_A^+ \cdot \partial, \qquad [V_A^+, V_B^+] = iX_{A,B} V_{A+B}^+ \tag{5.29}$$

These generators satisfy as well a Jacobi relation, which in turn yelds a Jacobi relation for the structure constants $X_{A,B}$

$$X_{A,B} X_{A+B,C} + X_{B,C} X_{B+C,A} + X_{C,A} X_{C+A,B} = 0 \tag{5.30}$$

What we just observed is the simplest manifestation of colour-kinematics duality. In fact, the self-dual gauge theory has one three-vertex whose kinematic part satisfies Jacobi relations exactly as the colour part does.

How can we obtain BCJ-numerators then? The construction is pretty straightforward. Let us see some examples. At four-points, we have already met the colour factors[2]

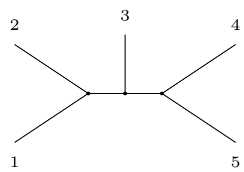
$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4} \qquad c_t = f^{a_2 a_3 b} f^{b a_1 a_4} \qquad c_u = f^{a_3 a_1 b} f^{b a_2 a_4}. \tag{5.31}$$

The corresponding numerators are then given by

$$n_s = \alpha X_{1,2} X_{3,4} \qquad n_t = \alpha X_{2,3} X_{1,4} \qquad n_u = \alpha X_{3,1} X_{2,4} \tag{5.32}$$

[2] Up to a sign convention.

where α stands for the normalization coming from the polarizations.
To generalize this idea to higher points, we can see the issue from a tree-graphs point of view. Let us take for example the five-points colour factor $f^{a_1 a_2 b} f^{b a_3 d} f^{d a_4 a_5}$, corresponding to the graph representation



. (5.33)

One way to construct the BCJ-numerators is to associate a $X_{A,B}$ factor to each vertex encountered. What we obtain for graph 5.33 is then

$$n = \alpha X_{1,2} X_{1+2,3} X_{4,5} \tag{5.34}$$

The tree-level amplitudes obtained from the self dual gauge theory however vanish. They correspond in fact to helicity configurations where one single particle has negative helicity. As we have seen in chapter 2 these amplitudes vanish in four dimensions. This property has been used in [56] as a way to test the validity of the generalization to the general gauge theory we will present in the following section.

5.3 BCJ-numerators from the scattering equations

5.3.1 A first generalization

Given a solution of the scattering equation σ_a, we define

$$X_{a,b} \equiv \frac{s_{a,b}}{\sigma_a - \sigma_b}, \qquad \bar{X}_{a,b} \equiv (\sigma_a - \sigma_b) h_a h_b \tag{5.35}$$

with the additional condition $X_{a,a} = 0$. The quantities h_a have been introduced to generalize the p_a's present in the previous relations. They need to satisfy the additional constraints

$$\sum_{a=1}^{n} h_a = \sum_{a=1}^{n} \sigma_a h_a = 0 \tag{5.36}$$

in order to allow us to still have

$$\sum_{b \neq a} X_{a,b} = 0 \qquad \sum_{b \neq a} \bar{X}_{a,b} = 0. \tag{5.37}$$

If we want to generalize the role of X (or $\bar{X}$) as a vertex we need to implement once again the consistency condition described in 5.28. First of all we have to define an off-shell version of X

$$X_{A,B} = \sum_{a\in\{A\}} X_{a,B} = \sum_{b\in\{B\}} X_{A,b} = \sum_{a\in\{A\}} \sum_{b\in\{B\}} X_{a,b} \tag{5.38}$$

where $\{A\}$ and $\{B\}$ are sets of external particles. Consider now three sets of external particles $\{A\}$, $\{B\}$ and $\{C\}$, connected to the lines A, B and C of the vertex 5.27. The consistency condition

$$X_{A,B} = X_{B,C} = X_{C,A} \tag{5.39}$$

is then satisfied thanks to the scattering equations

$$\begin{aligned} X_{A,B} &= \sum_{a\in\{A\}} \sum_{b\in\{B\}} X_{a,b} = -\sum_{a\in\{A\}} \sum_{c\notin\{B\}} X_{a,c} \\ &= -\sum_{a\in\{A\}} \sum_{c\in\{A\}} X_{a,c} - \sum_{a\in\{A\}} \sum_{c\in\{C\}} X_{a,c} = -X_{A,C} = X_{C,A} \end{aligned} \tag{5.40}$$

In order to treat Jacobi relation we need a fourth set of external particles $\{D\}$.
We have then

$$\begin{aligned} &X_{A,B}X_{C,D} + X_{B,C}X_{A,D} + X_{C,A}X_{B,D} = \\ &= -X_{A,B}(X_{A,D} + X_{B,D}) - X_{B,C}(X_{B,D} + X_{C,D}) - X_{C,A}(X_{C,D} + X_{A,D}) \\ &= 2X_{A,B}X_{D,D} = 0 \end{aligned} \tag{5.41}$$

where in the second last step we used eq. 5.39 and in the last step the fact that $X_{D,D} = 0$.
Since we have Jacobi identities, once again, we have a Lie Algebra involved

$$[\hat{V}_A^+, \hat{V}_B^+] = iX_{A,B}\hat{V}_{A+B}^+ \qquad [\hat{V}_A^-, \hat{V}_B^-] = i\bar{X}_{A,B}\hat{V}_{A+B}^-. \tag{5.42}$$

In this case, however, we cannot simply extend the explicit representation 5.29.

With this generalization of the quantity X, it is possible to show that "X-amplitudes"[3] still vanish. We address the reader to [56] to find a proof of this statement. We will instead procede to show how to construct BCJ-numerators in the general case, making use of the newly defined quantities X and $\bar{X}$.

[3] Amplitudes containing only X vertices, corresponding to the vanishing amplitudes constructible in the self-dual gauge theory.

5.3.2 The general idea

Let us start, following [56], from a toy model where an explicit representation for the generators exists and let us give the general idea underlying the construction. For this reason we will first treat vectors of the type

$$V_a = -2e^{-ip_a\cdot x}\epsilon_a \cdot \partial, \quad a = 1, 2, 3 \tag{5.43}$$

The most natural Lorentz-invariant quantity one can construct at three-points is

$$\begin{aligned} &V_1 \cdot [V_2, V_3] + V_2 \cdot [V_3, V_1] + V_3 \cdot [V_1, V_2] = \\ &- ie^{-i(p_1+p_2+p_3)\cdot x}\big((\epsilon_1 \cdot \epsilon_2)(p_1 - p_2) \cdot \epsilon_3 + (\epsilon_2 \cdot \epsilon_3)(p_2 - p_3) \cdot \epsilon_1 + (\epsilon_3 \cdot \epsilon_1)(p_3 - p_1) \cdot \epsilon_2\big) \end{aligned} \tag{5.44}$$

One immediately notices that this is in fact the three-points gluon amplitude. Moving to four-points, we can generalise the quantity in the following way[4]

$$n_{12,34} = V_1 \cdot [V_2, [V_3, V_4]] + V_2 \cdot [V_3, [V_4, V_1]] + V_3 \cdot [V_4, [V_1, V_2]] + V_4 \cdot [V_1, [V_2, V_3]] \tag{5.45}$$

Here we chose to follow [56]'s notation for the numerators. If we want to connect this quantity to the notation of the previous chapter, we have to consider a tree $T(1, 2, 3, 4)$ or, considering our necessity to treat Jacobi identities, the related general notation T'_{123} introduced in 4.31. The numerator $n_{12,34}$ corresponds then in our notation to the term $N(T'_{123})$ appearing in the Jacobi relation[5] [6]. One should also notice that the commutator structures, as in the notation of chapter 4, reflect the orientation of the vertices.

The BCJ Jacobi-like relations, in this construction, follow directly from the standard Jacobi identities of the algebra of spacetime vectors

$$\begin{aligned} &n_{12,34} + n_{23,14} + n_{31,24} = \\ &= V_1 \cdot ([V_2, [V_3, V_4]] + [V_3, [V_4, V_2]] + [V_1, [V_2, V_3]]) \\ &+ V_2 \cdot ([V_3, [V_1, V_4]] + [V_1, [V_4, V_3]] + [V_4, [V_3, V_1]]) \\ &+ V_3 \cdot ([V_1, [V_2, V_4]] + [V_2, [V_4, V_1]] + [V_4, [V_1, V_2]]) \\ &+ V_4 \cdot ([V_1, [V_2, V_3]] + [V_2, [V_3, V_1]] + [V_3, [V_1, V_2]]) = 0 \end{aligned} \tag{5.46}$$

The all-n rule to construct the numerator for the trivalent graph α will then be

$$n_\alpha = \sum_{a=1}^{n} V_a \cdot \mathfrak{G}_a^{(\alpha)} \tag{5.47}$$

[4] In the case of the s_{12} channel.

[5] This correspondence is valid up to minus signs given by the anti-symmetry of the three-valent vertex.

[6] However, the real correspondence will be underlined in the following, where the construction will be valid for n-points gluon amplitudes.

where $\mathfrak{G}_a^{(\alpha)}$ stands for the commutator structure as read from leg a. In this sense, one can apply - also in this approach - exactly the same diagrammatical way used for representing colour factors. In fact, substituting the terms V_i with T^{a_i}, the $\cdot$ product with standard matrix multiplication and finally taking the trace, gives us the colour factor corresponding to the numerator just found. The BCJ-numerators for gluon amplitudes will be constructed following the same idea. However, an explicit representation as 5.43 will not in general be available.

5.3.3 BCJ-numerators for gluon-amplitudes

In the following, we will construct BCJ-numerators for each contribution to the gauge theory amplitudes corresponding to solutions of the scattering equations. From eq. 5.3, we can schematically rewrite each of these contributions as

$$\text{Parke-Taylor factor} \quad \times \quad \text{permutation invariant factor} \tag{5.48}$$

The permutation invariant part can be ignored for the moment, because it will automatically satisfy BCJ-relations.What we really need instead are BCJ-numerators reproducing the Parke-Taylor amplitudes

$$A_{PT}^{(I)} = \sum_{\beta \in S_n/\mathbb{Z}_n} \frac{Tr(T^{a_{\beta(1)}}T^{a_{\beta(2)}} \dots T^{a_{\beta(n)}})}{\sigma_{\beta(1)\beta(2)}^{(I)} \dots \sigma_{\beta(n)\beta(1)}^{(I)}} \tag{5.49}$$

where I stands for the particular solution to the scattering equations that is being considered. The Lie algebras we will consider, are the ones introduced in 5.42. In addition, we define a set of rules for the action of elements on one another. This will be particularly useful for our calculations in the absence of an explicit representation. We have - in the case of the X-algebra -

$$\hat{V}_A^+ \hat{V}_B^+ = iX_{A,B} \hat{V}_{A|B}^+ \tag{5.50}$$

where $\hat{V}_{A|B}^+$ is not an element of the X-algebra but satisfies instead

$$\hat{V}_{A|B}^+ + \hat{V}_{B|A}^+ = \hat{V}_{A+B}^+ \tag{5.51}$$

which is consistent with the commutation relations. By further defining $\hat{V}_{\emptyset|A}^{\pm} = \hat{V}_A^{\pm}$, we obtain another set of rules

$$\hat{V}_{A|B}^+ \hat{V}_{C|D}^{\pm} = iX_{B,C+D} \hat{V}_{A+B+C|D}^{\pm} \tag{5.52}$$

$$\hat{V}_{A|B}^- \hat{V}_{C|D}^{\pm} = i\bar{X}_{B,C+D} \hat{V}_{A+B+C|D}^{\pm} \tag{5.53}$$

Moreover, we need to define a symmetric product "$\star$" such that

$$\hat{V}^{\pm}_{A|B} \star \hat{V}^{\pm}_{C|D} = 0 \qquad \hat{V}^{+}_{A|B} \star \hat{V}^{-}_{C|D} = \begin{cases} -2h_B h_D & \text{if } P_A + P_B + P_C + P_D = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.54}$$

We are finally ready to calculate complete numerators such as - at four-points -

$$\begin{aligned} n_{1^-,2^+,3^+,4^+} =& \hat{V}^-_1 \star [\hat{V}^+_2, [\hat{V}^+_3, \hat{V}^+_4]] + \hat{V}^+_2 \star [\hat{V}^+_3, [\hat{V}^+_4, \hat{V}^-_1]] \\ &+ \hat{V}^+_3 \star [\hat{V}^+_4, [\hat{V}^-_1, \hat{V}^+_2]] + \hat{V}^+_4 \star [\hat{V}^-_1, [\hat{V}^+_2, \hat{V}^+_3]] \end{aligned} \tag{5.55}$$

which satisfy Jacobi identities for the same reasons as in 5.46. In order to actually calculate the numerator, we can apply the rules introduced above to see that

$$\hat{V}^-_1 \star \hat{V}^+_2 \hat{V}^+_3 \hat{V}^+_4 = 2X_{2,3+4} X_{3,4} h_1 h_4 = 2X_{1,2} X_{3,4} h_1 h_4 \tag{5.56}$$

and as a consequence

$$\hat{V}^-_1 \star [\hat{V}^+_2, [\hat{V}^+_3, \hat{V}^+_4]] = 2X_{1,2} X_{3,4} h_1 (h_2 + h_3 + h_4) = -2X_{1,2} X_{3,4} h_1^2 \tag{5.57}$$

Now, by considering the other terms in 5.55, one can verify that

$$n_{1^-,2^+,3^+,4^+} = -4X_{1,2} X_{3,4} h_1^2 = 2\hat{V}^-_1 \star [\hat{V}^+_2, [\hat{V}^+_3, \hat{V}^+_4]]. \tag{5.58}$$

This turns out to be a general rule for numerators containing only one single "-" particle[7], which can then be written as

$$n_{\alpha,r} = \sum_{a=1}^{n} \hat{V}_a \star \mathfrak{S}^{(\alpha)}_a = 2\hat{V}^-_r \star \hat{\mathfrak{S}}^{(\alpha)}_r \tag{5.59}$$

where r labels the "-" particle. An important observation to make is that numerators with only one "-" particle contain only X-vertices. This means that they actually correspond to numerators of the X-amplitudes mentioned in the previous sections that actually vanish[8].

[7] Mind that "$\pm$" here do not refer to the helicity of the particles, but rather correspond to the notation introduced in the previous sections and in a deeper sense, to the notation of [110] and [48]. The helicity information for the external particles is instead contained in the permutation invariant part of the amplitude.

[8] Notice also that the numerators containing only "+" particles vanish due to the symmetry of $\star$.

We can now finally introduce the general formula to obtain BCJ-numerators for the Parke-Taylor amplitude. The idea is to consider this time two particles of the "-" type. We will denote these particles with r and s. We start by rewriting 5.49 in a BCJ-decomposed fashion

$$A_{PT} = \beta_{rs} \sum_{\alpha \in cubic} \frac{n_{\alpha,rs} c_\alpha}{D_\alpha} \tag{5.60}$$

where the factor β_{rs} is independent of the particle ordering and is given by

$$\beta_{rs} = -\left[4i^n (\sigma_r - \sigma_s)^2 h_r h_s (h_r + h_s) \sum_{a=1}^{n} \sigma_a^2 h_a\right]^{-1} \tag{5.61}$$

The numerators are characterized once again by the property found above so that they can be written by calculating only the parts related to the "-" particles

$$n_{\alpha,rs} = \sum_{a=1}^{n} \hat{V}_a \star \hat{\mathfrak{S}}_a^{(\alpha)} = 2\left(\hat{V}_r^- \star \hat{\mathfrak{S}}_r^{(\alpha)} + \hat{V}_s^- \star \hat{\mathfrak{S}}_s^{(\alpha)}\right) \tag{5.62}$$

The dependence on r and s of the numerators cancels out with the same dependence in 5.61. In addition, the results are independent on the choice of the quantities h_a as long as they satisfy the constraints described above.

The conclusion of [56] is then that a natural canonical choice of BCJ-numerators for a graph α in a gauge theory amplitude is

$$n_\alpha = \sum_{I=1}^{(n-3)!} \beta_{rs}^{(I)} n_{\alpha,rs}^{(I)} \gamma^{(I)}, with \qquad \gamma^{(I)} = \frac{Pf'\Psi^{(I)}}{\det'\Phi^{(I)}}. \tag{5.63}$$

where one should notice that, since $\beta_{rs}^{(I)}$ and $\gamma^{(I)}$ are independent of the particle ordering, the numerators n_α satisfy the same Jacobi-like relations as the $n_{\alpha,rs}^{(I)}$.

5.4 A possible path

The construction proposed by Monteiro and O'Connell (MO), is the first able to identify a set of canonical BCJ-numerators. This stands in contrast with the existing algorithms (both at the amplitudes- and Lagrangian-level) where the constructions are characterized by a peculiar freedom in the choices one can make.

In which way can MO's construction help us to improve the algorithm introduced in chapter 4[9]?
The idea of the author is that a new - light-cone gauge - formulation of the Lagrangian algorithm is possible and that the comparison of the new set of $\mathcal{L}^{(n)}$ terms with the prescription offered by MO will offer further insights in the issue. In particular, it is possible that, by obtaining the new - light-cone gauge - $\mathcal{L}^{(n)}$'s , an explicit representation of the vectors $\hat{V}_a^{\pm}$ could be found.

As no rigorous calculations have been performed to date, the section will rather serve as a discussion aimed at outlining a strategy of improvement for the next cycle of research on the topic.

The first step of this improvement strategy involves of course a re-writing of the Lagrangian in a light-cone gauge. A recent paper by Diana Vaman and York-Peng Yao [111] might be useful in this process. In fact, the authors there pursued a similar strategy to the one just presented. Their opinion is that working in a - more general - light-like gauge is the correct choice in order to obtain results at the Lagrangian-level that can be translated into a systematic construction of a Gravity Lagrangian. Despite obtaining a BCJ-manifest Lagrangian formulation up to five-points, the procedure applied in [111] rather follows the steps of BCJ in [20], with the calculation of the necessary shifts to be applied at the level of the numerators[10] and corresponding to the freedom allowed by the general gauge transformations mentioned in previous chapters.

Since our aim is rather to obtain a systematic algorithm at the Lagrangian-level, we decide to follow a different path. The idea is to exploit the light-cone formulation of the Lagrangian of [110][11] and to - step by step - obtain once again our systematic algorithm.

Using light-cone coordinates $(u, v, w, \bar{w})$, with a metric such that, given two arbitrary vectors A, B, defines an inner product[12]

$$A \cdot B = A_u B_v + A_v B_u - A_w B_{\bar{w}} - A_{\bar{w}} B_w \tag{5.64}$$

we can rewrite a gauge field A_μ as

$$A_\mu = (A_u, A_v, A_w, A_{\bar{w}}). \tag{5.65}$$

[9] The content of this section is highly speculative and is - at the moment - not suppoted by higher-points calculations.
[10] The authors report particular difficulties in translating the algorithm presented in chapter 4 and [49] into their own formulation.
[11] Or, equivalently, the one used in [48] and [111].
[12] Notice the factor "2" missing in the definition. In this section, we want to be as consistent as possible to our previous notation of chapter 2 and 4. We will also try to be consistent with the notations and normalizations of [111, 112].

In particular, we are interested in the gauge-fixing condition $A_u = 0$. We also decide to redefine some components of A_μ to avoid the proliferation of indices

$$A_w = \bar{A}, \qquad A_{\bar{w}} = A. \tag{5.66}$$

With this new notation, the gauge-fixed Yang-Mills Lagrangian reads[13][14]

$$\mathcal{L} = Tr\Big\{ -\bar{A}\partial^2 A + 2g\left(\frac{\bar{\partial}}{\partial_u}A\right)[A, \partial_u \bar{A}] - ig\left(\frac{\partial}{\partial_u}\bar{A}\right)[\bar{A}, \partial_u A] - g^2[A, \partial_u \bar{A}]\frac{1}{\partial_u^2}[\bar{A}, \partial_u A]\Big\} \tag{5.67}$$

The idea now is to recognize the terms composing the Lagrangian and to subsequently perform an analogue procedure to the one applied in 4.3. In order to do so, we switch to the more convenient notation introduced in chapter 2. The Lagrangian becomes then

$$\mathcal{L} = \frac{1}{2g^2}Tr\Big\{ -2\bar{\boldsymbol{A}}\partial^2 \boldsymbol{A} + 8\left(\frac{\bar{\partial}}{\partial_u}\boldsymbol{A}\right)[\boldsymbol{A}, \partial_u \bar{\boldsymbol{A}}] + 8\left(\frac{\partial}{\partial_u}\bar{\boldsymbol{A}}\right)[\bar{\boldsymbol{A}}, \partial_u \boldsymbol{A}] - 8[\boldsymbol{A}, \partial_u \bar{\boldsymbol{A}}]\frac{1}{\partial_u^2}[\bar{\boldsymbol{A}}, \partial_u \boldsymbol{A}]\Big\} \tag{5.68}$$

We address the reader to [111] to find the set of Feynman rules that can be derived from the interaction terms presented here. We can instead focus on three ideas that can outline a possible research strategy.

First of all, the Lagrangian now contains only the physical degrees of freedom of a gauge field in four space-time dimensions. We have in fact positive and negative helicities corresponding respectively to A and $\bar{A}$ components [48, 111].

Secondly, we notice that the interaction terms are given by a self-dual and an anti-self-dual three-valent vertices and by the four-points vertex. This consideration hints to a possibility related to the decomposition of gluon amplitudes 4.33. In particular, we interpret the presence of the self-dual and anti-self-dual cubic vertices as a hint to the fact that the $j = 1$ terms in the decomposition 4.33 can be in fact seen as a sum on self- and anti-self-dual contributions.

Finally, the four-gluon vertex can be once again rewritten so that it reproduces correctly the BCJ-splitting up to four-points [49, 111]. Using the notation of chapter 4

$$\mathcal{L}^{(4)} = -8Tr[\boldsymbol{A}, \partial_u \bar{\boldsymbol{A}}]\frac{1}{\partial_u^2}[\bar{\boldsymbol{A}}, \partial_u \boldsymbol{A}] \tag{5.69}$$

[13] We also used the equation of motion for A_v to eliminate it from the expression. After fixing the gauge, A_v becomes in fact non-dynamical and can be treated as an auxiliary field.

[14] The notation for derivative operators is chosen to be consistent with the one used for four-vectors.

can be rewritten as

$$\mathcal{L}^{(4)} = 8\Lambda_1^{\mu_1\mu_2}\Lambda_2^{\mu_3\mu_4} g_{\nu_1\nu_2}\frac{\partial_{12}^{\nu_1}\partial_{34}^{\nu_2}}{\Box_{12}} Tr[\boldsymbol{A}_{\mu_1}, \partial_u \boldsymbol{A}_{\mu_2}]\frac{1}{\partial_u^2}[\boldsymbol{A}_{\mu_3}, \partial_u \boldsymbol{A}_{\mu_4}] \tag{5.70}$$

where we introduced once again the splitting thanks to a "smart one" and where we introduced the two new 4×4 matrices Λ_1 and Λ_2. They can be seen as simple ways to obtain the correct component of the field $\boldsymbol{A}_\mu$. In the example above we have

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.71}$$

The term 5.70 correctly reproduces the splitting of the four-gluon vertex [49, 50, 111]. As a pure speculation[15] then, we imagine an algorithm that considers all the inequivalent tree topologies as in chapter 4, but where the correspondence between j and the number of Minkowski metric factors is replaced by a similar correspondence with the matrices Λ[16]. In absence of a full tested algorithm, any other comment would be purely speculative and its value would therefore drastically reduce. In particular, we still lack a complete picture of how to treat the different helicity configurations in a systematic way analogue to the one presented in chapter 4.

The ideas presented above are the fetal stage of one of the possible ways we can take to implement our algorithm in a light-cone gauge. Further research steps and in particular in depths calculations are needed to corroborate these speculations.

Moreover, once obtained an analogue of our prescription in a light-cone gauge, one still needs to investigate and figure out how the results of Monteiro and O'Connell of [56] can be reproduced and how scattering equations enter the game.

These intriguing steps and the confirmation or rejection of the just-presented speculations will be just some of the future exciting proceedings in this fruitful field of research.

[15] Actual calculations at higher points have not been performed yet. This should rather be taken as a personal conjecture of the author.

[16] We stress once again that this has not been tested yet for higher points.

6

CONCLUSIONS

In this thesis we analyzed BCJ-relations in Quantum Field Theories. In particular, we examined some construction algorithm for the BCJ-numerators that can be used through squaring relations to construct Gravity amplitudes. We first reviewed the method described by Bern, Carrasco and Johansson in [20] where a amplitude by amplitude approach is required and where a systematic construction was impaired by the necessity to perform choices at the level of the amplitudes[1] as well as when deciding the final values for the numerators.

We then analyzed the systematic algorithm presented by the author and Stefan Weinzierl in [49], where a systematic approach was taken to generate all the possible terms who produce an effective Lagrangian with manifest BCJ-duality. We realized, however, that also this method is made ambiguous by the amount of freedom left to the user to decide which terms have to be kept in the Lagrangian and which instead are dismissed, as well as which terms giving non-vanishing Feynman rules but with vanishing contributions to the BCJ-relations should be included.

We moved then on to the first algorithm to construct BCJ-numerators that does not present the same ambiguities. The kinematic algebra approach taken by Monteiro and O'Connell in [56] making use of the scattering equations, allowed us - in fact - to define a canonical set of BCJ-numerators.

Finally, we discussed possible ways to improve the systematic algorithm of [49] exploiting the new results of [56] and [111] in a light-cone gauge setting. Considered the highly speculative character of the last sections, one has to mind some important caveats. As the story of BCJ-relations taught us so far, the approaches that sound more easily constructible contain unfortnuately unpredictable complications that are not so easily bypassable. In this regard, the conjectures presented in section 5.4 are a mere collection of naïve considerations motivated by the construction made in [49]. Further

[1] Deciding in particular which independent KK-amplitudes to use to define $(n-3)!$ numerators.

research and - especially - detailed calculations, will allow us to sort out if this path to improvement is actually a safe and fruitful one.

The potential of BCJ-numerators to make further investigations in gauge theories and gravity[2] easier, faster and clearer[3], motivated this research project. There is still a lot to uncover and a lot to see. This will surely motivate any successive attempt to improve what was done by the author of this thesis in the spirit of science. We can only wish a good, fruitful research process to the next generation of graduate and undergraduate students who will accept the challenge.

[2] And on the links between the two.

[3] See for example the beautiful picture presented in section 3.4.5.

Appendix A

OFF-SHELL CURRENTS

A.1 Off-shell current conservation proof

The base of the induction is simply given by:

$$J_\alpha(1) = \epsilon_\alpha(1) = \epsilon^\lambda_\alpha(p_1, q) \tag{A.1}$$

which of course satisfies:

$$p_1^\alpha J_\alpha(1) = p_1^\alpha \epsilon_\alpha(1) = 0 \tag{A.2}$$

Let us now assume that $p_{1,m}^\alpha J_\alpha(1,\dots,m) = 0$ with $m < n$. We have then:

$$\begin{aligned}
&p_{1,n}^\alpha J_\alpha(1,\dots,n) \propto \\
&\propto p_{1,n}^\alpha g_{\mu\alpha} \sum_{i=1}^{n-1} \left[g^{\nu\rho}(p_{1,i} - p_{i+1,n})^\mu + 2g^{\rho\mu} p_{i+1,n}^\nu - 2g^{\mu\nu} p_{1,i}^\rho\right] J_\nu(1,\dots,i) J_\rho(i+1\dots,n) \\
&+ p_{1,n}^\alpha g_{\mu\alpha} \sum_{j=i+1}^{n-1} \sum_{i=1}^{n-2} [2g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}] J_\nu(1,\dots,i) J_\rho(i+1,\dots,j) J_\sigma(j+1,\dots,n) \\
&= \sum_{i=1}^{n-1} \left[g^{\nu\rho}(p_{1,i}^2 - p_{i+1,n}^2) + 2p_{1,n}^\rho p_{i+1,n}^\nu - 2p_{1,i}^\nu p_{1,i}^\rho\right] J_\nu(1,\dots,i) J_\rho(i+1\dots,n) \\
&+ \sum_{j=i+1}^{n-1} \sum_{i=1}^{n-2} \left[2p_{1,n}^\rho g^{\nu\sigma} - p_{1,n}^\nu g^{\rho\sigma} - p_{1,n}^\sigma g^{\nu\rho}\right] J_\nu(1,\dots,i) J_\rho(i+1,\dots,j) J_\sigma(j+1,\dots,n)
\end{aligned} \tag{A.3}$$

Now, using the assumption we made at the beginning, we obtain:

$$
\begin{aligned}
=&\sum_{i=1}^{n-1}\left[(p_{1,i}^2-p_{i+1,n}^2)J(1,\dots,i)\cdot J(i+1\dots,n)\right]\\
&+\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2}[2(p_{1,i}+p_{j+1,n})\cdot J(i+1,\dots,j)J(1,\dots,i)\cdot J(j+1,\dots,n)\\
&-p_{i+1,n}\cdot J(1,\dots,i)J(i+1,\dots,j)\cdot J(j+1,\dots,n)\\
&-p_{1,j}\cdot J(j+1,\dots,n)J(1,\dots,i)\cdot J(i+1,\dots,j)]\\
\\
=&\chi_1+\chi_2
\end{aligned}
$$

where we defined:

$$
\begin{aligned}
\chi_1=&\sum_{i=1}^{n-1}\left[(p_{1,i}^2-p_{i+1,n}^2)J(1,\dots,i)J(i+1\dots,n)\right]\\
\\
\chi_2=&\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2}[2(p_{1,i}+p_{j+1,n})\cdot J(i+1,\dots,j)J(1,\dots,i)\cdot J(j+1,\dots,n)\\
&-p_{i+1,n}\cdot J(1,\dots,i)J(i+1,\dots,j)\cdot J(j+1,\dots,n)\\
&-p_{1,j}\cdot J(j+1,\dots,n)J(1,\dots,i)\cdot J(i+1,\dots,j)]
\end{aligned}
\tag{A.4}
$$

Now, one can easily verify that χ_2 can be rewritten as:

$$
\begin{aligned}
\chi_2=-i&\left(\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}J_\mu(1,\dots,i)V_3^{\mu\nu\rho}(p_{i+1,j},p_{j+1,n})J_\nu(i+1,\dots,j)J_\rho(j+1,\dots,n)\right.\\
&\left.-\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}J_\mu(j+1,\dots,n)V_3^{\mu\nu\rho}(p_{1,i},p_{i+1,j})J_\nu(1,\dots,i)J_\rho(i+1,\dots,j)\right)
\end{aligned}
\tag{A.5}
$$

Part of the terms present in A.5 can be re-written once again, by making use of the formal definition of the off-shell current 2.81:

$$
\begin{aligned}
&V_3^{\mu\nu\rho}(p_{i+1,j}, p_{j+1,n})J_\nu(i+1,\ldots,j)J_\rho(j+1,\ldots,n) = \\
=&ip^2_{i+1,n}J^\mu(i+1,\ldots,n) - \sum_{k=i+1}^{n-2}\sum_{l=k+1}^{n-1} V_4^{\mu\nu\rho\sigma}J_\nu(i+1,\ldots,k)J_\rho(k+1,\ldots,l)J_\sigma(l+1,\ldots,n) \\
\\
&V_3^{\mu\nu\rho}(p_{1,i}, p_{i+1,j})J_\nu(1,\ldots,i)J_\rho(i+1,\ldots,j) = \\
=&ip^2_{i,j}J^\mu(i,\ldots,j) - \sum_{k=1}^{j-2}\sum_{l=k+1}^{j-1} V_4^{\mu\nu\rho\sigma}J_\nu(1,\ldots,k)J_\rho(k+1,\ldots,l)J_\sigma(l+1,\ldots,j)
\end{aligned}
\tag{A.6}
$$

Now, after explicitly writing the sums, rearranging terms and renaming indices, we can see that:

$$
\chi_2 = \sum_{i=1}^{n-2}(p^2_{i+1,n} - p^2_{1,i})J(1,\ldots,i)\cdot J(i+1,\ldots,n) \tag{A.7}
$$

which means in turn that:

$$
p^\mu_{1,n}J_\mu(1,\ldots,n) \propto (\chi_1 + \chi_2) = 0 \tag{A.8}
$$

A.1.1 Off-shell currents in different gauges

In chapter 2 I stated that the current conservation depends on the gauge fixing choice we made. Let us see why this is the case.
Up to now we worked in Feynman gauge, where the expression for the off-shell current was:

$$
\begin{aligned}
J^F_\mu(1,\ldots,n) =&\frac{-ig_{\mu\gamma}}{p^2_{1,n}}\left[\sum_{i=1}^{n-1} V_3^{\gamma\nu\rho}(p_{1,i}, p_{i+1,n})J_\nu(1,\ldots,i)J_\rho(i+1\ldots,n)\right. \\
&\left.+\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2} V_4^{\gamma\nu\rho\sigma}J_\nu(1,\ldots,i)J_\rho(i+1,\ldots,j)J_\sigma(j+1,\ldots,n)\right] \\
\\
=&\frac{-ig_{\mu\gamma}}{p^2_{1,n}}J^\gamma(V_3,V_4)
\end{aligned}
\tag{A.9}
$$

If we make one step backward and just treat the case of a general covariant gauge, we obtain:

$$\begin{aligned} J_\mu^{CG}(1,\ldots,n) =& \frac{-i}{p_{1,n}^2}\left(g_{\mu\gamma} - (1-\xi)\frac{p_{1,n_\mu}p_{1,n_\gamma}}{p_{1,n}^2}\right) J^\gamma(V_3,V_4) \\ =& \frac{-ig_{\mu\gamma}}{p_{1,n}^2} J^\gamma(V_3,V_4) = J_\mu^F(1,\ldots,n) \end{aligned} \tag{A.10}$$

where the second term vanishes due to the conservation of current shown in the previous section. So, in the case of a general covariant gauge, the conservation of off-shell currents is preserved:

$$p_{1,n}^\mu J_\mu^{CG}(1,\ldots,n) = p_{1,n}^\mu J_\mu^F(1,\ldots,n) = 0 \tag{A.11}$$

As a counter-example, we will now briefly discuss the case of an axial gauge where we introduce a reference vector $\boldsymbol{n}$ [1]:

$$\begin{aligned} J_\mu^A(1,\ldots,n) =& \frac{-i}{p_{1,n}^2}\left(g_{\mu\gamma} - \frac{p_{1,n_\mu}\boldsymbol{n}_\gamma + p_{1,n_\gamma}\boldsymbol{n}_\mu}{p_{1,n}\cdot\boldsymbol{n}} + \frac{\boldsymbol{n}^2 + \xi p_{1,n}^2}{(p_{1,n}\cdot\boldsymbol{n})^2}p_{1,n_\mu}p_{1,n_\gamma}\right) J^\gamma(V_3,V_4) = \\ =& \frac{-i}{p_{1,n}^2}\left(g_{\mu\alpha} - \frac{p_{1,n_\mu}\boldsymbol{n}_\gamma}{p_{1,n}\cdot\boldsymbol{n}}\right) J^\gamma(V_3,V_4) = \\ =& J_\mu^F(1,\ldots,n) + \frac{i}{p_{1,n}^2}\frac{p_{1,n_\mu}}{p_{1,n}\cdot\boldsymbol{n}}(\boldsymbol{n}\cdot J(V_3,V_4)) \end{aligned} \tag{A.12}$$

which means in turn that

$$p_{1,n}^\mu J_\mu^A(1,\ldots,n) = \frac{i}{p_{1,n}\cdot\boldsymbol{n}}(\boldsymbol{n}\cdot J(V_3,V_4)) \tag{A.13}$$

This shows how the conservation of the current is also a gauge-dependent property of the gauge dependent off-shell currents.

[1] The special case $\boldsymbol{n}^\mu = (1,0,-1,0)$ - the light cone gauge - will be particularly important in the last chapter.

Appendix B

INEQUIVALENT TREE TOPOLOGIES

When one considers the possible Jacobi relations that can arise among trees, one has to find first of all the inequivalent tree topologies for the specific number of external legs treated.
Since this is our aim, we will consider trees with at least four external legs[1].
As already described in chapter 3, each of the three terms in a Jacobi relation has a specific marked propagator. We decide then to represent such terms in the following way

$$T = (T_{12}, T_{34}) = ([T_1, T_2], [T_3, T_4]) \tag{B.1}$$

In this notation, the marked propagator corresponds to the root of the subtree T_{12}[2]. Given two trees T, T' of the form B.1, they are called inequivalent, if they cannot be obtained from one another through the following three operations:

1 The cyclic property of $(\ldots, \ldots)$

$$(T_{12}, T_{34}) = (T'_{34}, T'_{12}) \tag{B.2}$$

2 The anti-symmetry of $[\ldots, \ldots]$

$$[T_a, T_b] \longrightarrow [T_b, T_a] \tag{B.3}$$

where T_a, T_b are sub-trees at one vertex of T_{12} (or T_{34}). [3]

3 Jacobi operation

$$([T_1, T_2], [T_3, T_4]) = ([T'_2, T'_3], [T'_1, T'_4]) \tag{B.4}$$

---

[1] The minimum required to have Jacobi relations at all.

[2] Or equivalently of sub-tree T_{34}.

[3] If we can obtain T'_{12} (or T'_{34}) from T_{12} (or T_{34}) with a sequence of swaps like B.3, the trees are equivalent.

Using these rules, we can construct all the classes of inequivalent trees for a given n. We have the following inequivalent classes up to $n = 8$

$$
\begin{array}{llll}
n=4 & ([1,2],[3,4]), & & \\
n=5 & ([[1,2],3],[4,5]), & & \\
n=6 & ([[[1,2],3],4],[5,6]), & ([[1,2],[3,4]],[5,6]), & \\
n=7 & ([[[[1,2],3],4],5],[6,7]), & ([[[1,2],[3,4]],5],[6,7]), & ([[[1,2],3],[4,5]],[6,7]), \\
 & ([[1,2],[3,4]],[[5,6],7]), & & \\
n=8 & ([[[[[1,2],3],4],5],6],[7,8]), & ([[[[1,2],[3,4]],5],6],[7,8]), & ([[[[1,2],3],[4,5]],6],[7,8]), \\
 & ([[[[1,2],3],4],[5,6]],[7,8]), & ([[[1,2],[3,4]],[5,6]],[7,8]), & ([[[1,2],3],[[4,5],6]],[7,8]), \\
 & ([[[1,2],3],[4,5]],[[6,7],8]), & ([[1,2],[3,4]],[[5,6],[7,8]]). &
\end{array}
\tag{B.5}
$$

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