Degree Formula for the Discriminant Divisor of Lagrangian Fibrations of Irreducible Symplectic Manifolds

Dissertation
zur Erlangung des Grades
„Doktor der Naturwissenschaften“
am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität
in Mainz

Jonas Ehrhard

geboren in Biberach an der Riß
Mainz, den 25. März 2025

Datum der mündlichen Prüfung: 18.7.2025

Abstract

Let M be an irreducible holomorphic symplectic manifold with a Lagrangian fibration $f\colon M \to \mathbb{P}^n$, whose discriminant locus is $\Delta = \bigcup_i \Delta_i \subset \mathbb{P}^n$. This thesis defines weights $w_i \in \mathbb{Q}$ such that

$$24\left(\frac{n! \int_M \sqrt{\hat{A}}(M)}{d_1 \cdots d_n}\right)^{\frac{1}{n}} = \sum_i w_i \deg(\Delta_i),$$

where $(d_1, \dots, d_n)$ is the polarization type of f.

The definition of the w_i involves the cohomology sheaves of the ΩT complex

$$f^*\Omega_{\mathbb{P}^n} \to \Omega_M \cong T_M \to f^*T_{\mathbb{P}^n},$$

and this thesis gives an in-depth analysis of those sheaves.

Furthermore, the definition involves the choice of a Kähler form ω on M, which induces a polarization of type $(d_1, \dots, d_n)$ on the smooth fibers of f. To show that the w_i do not depend on the choice of ω is the main undertaking of this thesis.

If the characteristic cycle Θ_i over Δ_i is compact, then one can define the weights as

$$w_i = \frac{\chi(\Theta_i)}{\deg_{\Theta_i}(\omega)}.$$

In case of non-compact characteristic cycles one can choose an appropriate compact subcycle $\overline{\Theta}_i \subset \Theta_i$ to compute w_i in the same way.

Contents

Introduction

Irreducible holomorphic symplectic manifolds are one of the three building blocks of compact Kähler manifolds with vanishing first Chern class. One approach to study them is to study Lagrangian fibrations, i.e. holomorphic maps

$$f: M \to \mathbb{P}^n,$$

whose smooth fibers are Lagrangian submanifolds, which turn out to be complex n-dimensional tori. Suppose $n = 1$, then M is a K3-surface and f is an elliptic fibration, as studied by Kodaira. If $F_1, \dots, F_k \subset M$ are the singular fibers of f, then

$$24 = \chi(M) = \chi(F_1) + \dots + \chi(F_k), \tag{1}$$

because the smooth fibers have Euler characteristic zero. This thesis proposes a generalization of (1) to higher-dimensional irreducible holomorphic symplectic manifolds. The first step in this direction is a theorem by Hwang and Oguiso[17] that the set of critical values $\Delta \subset \mathbb{P}^n$ has codimension one.

Sawon[30] proved under the assumption that f has so called "good" singular fibers, that

$$24\left(\frac{n!}{d_1 \cdots d_n}\int_M \sqrt{\hat{A}}(M)\right)^{\frac{1}{n}} = \deg \Delta, \tag{2}$$

where $(d_1, \dots, d_n)$ is the polarization type of f, i.e. the minimal polarization on a smooth fiber that can be obtained by a Kähler class of M (cf. [36]).

If the singular fibers are not "good", it turns out that (2) fails. However, one can endow the irreducible components $\Delta_1, \dots, \Delta_k \subset \Delta$ with weights $w_1, \dots, w_k \in \mathbb{Q}$ to obtain

$$24\left(\frac{n!}{d_1 \cdots d_n}\int_M \sqrt{\hat{A}}(M)\right)^{\frac{1}{n}} = \sum_{i=1}^{k} w_i \deg \Delta_i. \tag{3}$$

The weights w_i were already defined by C. Lehn[20], but it remained an open question if they are independend of the choices involved in the definition. Proving the independence is the main endeavour of this thesis.

This thesis starts with a zeroeth chapter which recalls basic definitions and observations from algebraic geometry, which are used throughout the rest of the thesis. The reader might want to skip this at the first reading and come back to it when it is referenced.

The first chapter is a brief introduction to holomorphic symplectic manifolds, Lagrangian fibrations and irreducible holomorphic symplectic manifolds. It recalls important results by Hwang and Oguiso regarding characteristic cycles of Lagrangian fibrations, which are used intensively later on.

In chapter two, the weights w_i are derived from the cohomology sheaves of the ΩT-complex

$$f^*\Omega_{\mathbb{P}^n} \to \Omega_M \cong T_M \to f^*T_{\mathbb{P}^n}$$

in such a way, that (3) holds immediately. However, the definition also involves the choice of a Kähler class ω on M, and it is not clear whether the weights depend on the choice of ω. To analyze the weights, one has to study the cohomology sheaves of the ΩT-complex. Clearly $H^0(\Omega T) = 0$, as the first map is injective. The second cohomology sheaf, $H^2(\Omega T)$, is the cokernel of the differential map $T_M \to f^*T_{\mathbb{P}^n}$, and as such straightforward to compute, provided one has a local model for f. A classification of possible local models was done by Hwang and Oguiso. The first cohomology sheaf $H^1(\Omega T)$ can then be related to $H^2(\Omega T)$ by observing that the ΩT-complex is self-dual, which yields $H^1(\Omega T) \cong \mathcal{E}xt\, H^2(\Omega T), \mathcal{O}_M$. This analysis leads to a second description of the w_i in terms of combinatorial data of the singular fiber, and integrating ω over the singular tori Z, which are the components of the singular locus of the reduction of a general fiber over Δ_i.

Chapter three shows that for each singular torus Z, the inclusion map $Z \to M$ is homotopic to a C^∞-map $Z \to F_{\mathrm{sm}}$, where F_{sm} is a smooth fiber. This implies that the restriction $H^2(M, \mathbb{Q}) \to H^2(Z, \mathbb{Q})$ factors over $H^2(F_{\mathrm{sm}}, \mathbb{Q})$. By a theorem of Voisin, $H^2(M, \mathbb{Q}) \to H^2(F_{\mathrm{sm}}, \mathbb{Q})$ has rank one, and any two Kähler classes which induce the same polarization type on F_{sm} restrict to the same Kähler class on Z. This implies that w_i does not depend on the choice of ω.

The fourth chapter gives a third description of the weights which is more reminiscent of (1), namely

$$w_i = \frac{\chi(\Theta_i)}{\int_{\Theta_i} \omega},$$

if Θ_i is a compact characteristic cycle over Δ_i. If the characteristic cycles over Δ_i are not compact, one has to choose an appropriate compact subcycle $\overline{\Theta}_i \subset \Theta_i$.

The last chapter calculates the weights w_i explicitly for two examples, the Hilbert scheme of points on an elliptic K3 surface, and O'Grady's ten-dimensional irreducible symplectic holomorphic manifold, obtained as a moduli space of sheaves on a sextic K3 surface.

0 Generalities from algebraic geometry

This chapter collects standard results from algebraic and complex geometry. The experienced reader might skip it and come back when necessary.

Divisors and Chern classes

Definition 0.1. Let X be a complex analytic space, and let F be a coherent sheaf on X. A reduced closed analytic subspace $Y \subset X$ is called *associated to F* if for every $y \in Y$ the ideal sheaf $I_{Y,y} \subset \mathcal{O}_{X,y}$ is the intersection of associated primes of F_y.

Remark. Since $\mathcal{O}_{X,y}$ is Noetherian, the set of associated spaces is locally finite.

Lemma 0.2. *Let X be a complex analytic space and let*

$$0 \to F' \to F \to F'' \to 0$$

be a short exact sequence of coherent sheaves on X. If $D \subset X$ is an effective Cartier divisor that does not contain any associated subspace of F'', then the restricted sequence

$$0 \to F'|_D \to F|_D \to F''|_D \to 0$$

is still exact.

Proof. Pick a point $x \in D$, and a local equation $t \in \mathcal{O}_{X,x}$ for D. The following diagram has exact rows and columns.

$$\begin{array}{ccccccc}
0 & \longrightarrow & F'_x & \longrightarrow & F_x & \longrightarrow & F''_x & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot t} & & \downarrow{\scriptstyle \cdot t} & & \downarrow{\scriptstyle \cdot t} & & \\
0 & \longrightarrow & F'_x & \longrightarrow & F_x & \longrightarrow & F''_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (F'|_D)_x & \longrightarrow & (F|_D)_x & \longrightarrow & (F''|_D)_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

That D does not contain any associated subspace of F'' means that t is not a zero divisor for F''_x. Hence the kernel of $t\colon F'' \to F''$ is zero, and by the snake lemma $F'|_D \to F|_D$ is injective. □

Lemma 0.3. *If X is smooth and $i\colon D \hookrightarrow X$ is an effective Cartier divisor, then*

$$c_1(i_*\mathcal{O}_D) = c_1(\mathcal{O}(D)), \qquad c_2(i_*\mathcal{O}_D) = c_1(\mathcal{O}(D))^2$$
$$c_1(i_*\mathcal{O}(D)|_D) = c_1(\mathcal{O}(D)), \qquad c_2(i_*\mathcal{O}(D)|_D) = 0.$$

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to i_* \mathcal{O}_D \to 0.$$

Hence $\mathrm{ch}(i_*\mathcal{O}_D) = 1 - \mathrm{ch}(\mathcal{O}(-D))$, so that $c_1(i_*\mathcal{O}_D) = -c_1(\mathcal{O}(-D)) = c_1(\mathcal{O}(D))$. Furthermore

$$\frac{1}{2}c_1(i_*\mathcal{O}_D)^2 - c_2(i_*\mathcal{O}_D) = ch_2(i_*\mathcal{O}_D) = -\frac{1}{2}c_1(\mathcal{O}(-D))^2,$$

so that $c_2(i_*\mathcal{O}_D) = c_1(i_*\mathcal{O}_D)^2$. This proves the first two equalities. For the second line of equalities, consider the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(D) \to i_*\mathcal{O}(D)|_D \to 0,$$

which yields $\mathrm{ch}(i_*\mathcal{O}(D)|_D) = \mathrm{ch}(\mathcal{O}(D)) - 1$. So for $i \geq 1$ one gets $c_i(i_*\mathcal{O}(D)|_D) = c_i(\mathcal{O}(D))$. □

Normalization and fibers

Consider the following situation: Let $f\colon X \to Y$ be a proper dominant morphism of irreducible varieties over $\mathbb{C}$, and let $\nu\colon \tilde{X} \to X$ be the normalization map. Set $\tilde{f} = f \circ \nu$.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\nu} & X \\ & \underset{\tilde{f}}{\searrow} & \downarrow f \\ & & Y \end{array}$$

For $y \in Y$ let $X_y = f^{-1}(y)$ be the fiber in X, and let $\tilde{X}_y = \tilde{f}^{-1}(y)$ be the fiber in $\tilde{X}$. Let $\eta \in Y$ be the generic point.

Lemma 0.4. *The generic fiber $\tilde{X}_\eta$ is normal, and it is the normalization of X_η.*

Proof. The question is local, so one may assume that X and Y are affine, and f corresponds to an inclusion of integral domains, $A \subset B$. Let $\tilde{B}$ be the integral closure of B in $Q(B)$, and set $S = A \setminus \{0\}$. Then X_η corresponds to $Q(A) \subset S^{-1}(B)$, and $\tilde{X}_\eta$ corresponds to $S^{-1}\tilde{B}$. The latter ring is still integrally closed, hence $\tilde{X}_\eta$ is normal, and because $S^{-1}B$ and $S^{-1}\tilde{B}$ still have the same field of fractions, the map $\tilde{X}_\eta \to X_\eta$ is birational, hence it is the normalization map. □

Lemma 0.5. *There exists a non-empty open subset $U \subset Y$ such that for $y \in U$, the fiber $\tilde{X}_y$ is normal.*

Proof. By generic flatness one may assume that f is flat. Then by [11, 12.2.4 (iv)], the set $U \subset Y$ of points with geometrically connected fibers is open, and it suffices to show that U is non-empty. By Lemma 0.4, the generic fiber is normal, and since everything is in characteristic zero, normal implies geometrically normal. □

Proposition 0.6. *There exists a non-empty open subset $V \subset Y$ such that for $y \in V$, the fiber $\tilde{X}_y$ is the normalization of X_y.*

Proof. Let $W \subset X$ be the open set of normal points. By [11, 9.5.3], the set

$$E = \{\, y \in Y \mid X_y \cap W \text{ is dense in } X_y \,\}$$

is a constructible subset of Y. Note that X_η is irreducible (cf. proof of Lemma 0.4), and clearly $X_\eta \cap W$ contains the generic point of X. Hence $\eta \in E$, so that there exists an open subset $V' \subset E$. Let U be the open set from Lemma 0.5, and set

$$V = V' \cap U.$$

Then for $y \in V$ the map $\tilde{X}_y \to X_y$ is birational, and since $\tilde{X}_y$ is normal, it is the normalization map. □

Line bundles with constant transition functions

Proposition 0.7. *Let X be a compact complex manifold with a line bundle $L \in$ $\mathrm{Pic}(X)$. If there is an open cover $\{U_i\}$ of X, for which L admits trivializations $\varphi_i : L|_{U_i} \to \mathcal{O}_{U_i}$, such that the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \to U_i \cap U_j$ are locally constant, then L is numerically trivial.*

Proof. The diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp} & \mathcal{O}_X^* & \longrightarrow & 0 \end{array}$$

induces in cohomology a diagram with exact rows:

$$\begin{array}{ccccc} H^1(X, \mathbb{C}^*) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathbb{C}) \\ \downarrow & & \downarrow{=} & & \\ \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) & & \end{array}$$

The line bundle $L \in \mathrm{Pic}(X)$ admits a trivializing cover with locally constant transition functions if and only if it is in the image of $H^1(X, \mathbb{C}^*) \to \mathrm{Pic}(X)$. Since the upper row is exact, this implies $c_1(L) = 0$ in $H^2(X, \mathbb{C})$. □

$\mathbb{P}^1$-bundles with numerically trivial section

Proposition 0.8. *Let $\pi : Y \to X$ be a $\mathbb{P}^1$-bundle over a connected compact complex manifold X of dimension $\dim X = n - 1$. Suppose that $\sigma : X \to Y$ is a section of π, and that the normal bundle $N_{\sigma(X)/Y}$ is numerically trivial. Then for any cohomology class $\alpha \in H^2(Y, \mathbb{C})$,*

$$\int_Y \alpha^n = n \int_{\pi^{-1}(x)} \alpha \cdot \int_X (\sigma^* \alpha)^{n-1}.$$

Proof. By Cohomology and Base Change, the sheaf $E = \pi_*\mathcal{O}(\sigma X)$ is locally free of rank 2, and the canonical quotient

$$\pi^*E \to \mathcal{O}_Y(\sigma X) \to 0$$

identifies Y with $\mathbb{P}(E)$ such that $\mathcal{O}(\sigma X) = \mathcal{O}_\pi(1)$. Consider on Y the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(\sigma X) \to \mathcal{O}_Y(\sigma X)|_X \to 0.$$

Pushing-down to X yields the exact sequence

$$0 \to \mathcal{O}_X \to E \to N_{X/Y} \to 0,$$

since $R^1\pi_*\mathcal{O}_Y = 0$ by Cohomology and Base Change. As $N_{X/Y}$ is numerically trivial, also E is numerically trivial. Now consider $\xi = c_1(\mathcal{O}_\pi(1)) = c_1(\mathcal{O}(\sigma X))$, which satisfies the relation

$$\xi^2 - c_1(\pi^*E)\cdot\xi + c_2(\pi^*E) = 0.$$

This implies that ξ^2 is also numerically trivial.

Since X is connected, $H^2(Y,\mathbb{C}) = H^2(X,\mathbb{C}) \oplus \mathbb{C}\cdot\xi$, and so one can write $\alpha = \pi^*\beta + \varepsilon\xi$, with $\varepsilon = \int_{\pi^{-1}(x)}\alpha$. Hence

$$\int_Y \alpha^n = \int_Y (\pi^*\beta)^n + n\varepsilon \int_Y \xi\cdot(\pi^*\beta)^{n-1} + \int_Y \xi^2\cdot\gamma = n\int_{\pi^{-1}(x)}\alpha\cdot\int_X \beta^{n-1}. \qquad \square$$

Corollary 0.9. *Let $\pi\colon Y \to X$ be a $\mathbb{P}^1$-bundle with a multisection $Z \subset Y$, such that the restriction $\pi|_Z\colon Z \to X$ is étale of degree d. Suppose the normal bundle $N_{Z/Y}$ is numerically trivial. Then for any $\alpha \in H^2(Y,\mathbb{C})$,*

$$\int_Y \alpha^n = \frac{n}{d}\int_{\pi^{-1}(x)}\alpha\cdot\int_Z \alpha^{n-1}.$$

Proof. Make a base-change of π to Z to obtain a section $\sigma\colon Z \to Y_Z$.

$$\begin{array}{ccc} Y_Z & \xrightarrow{\varphi} & Y \\ \sigma \uparrow\downarrow \pi_Z & \lrcorner & \downarrow \pi \\ Z & \longrightarrow & X \end{array}$$

Applying Proposition 0.8 to π_Z yields

$$\int_Y \alpha^n = \frac{1}{d}\int_{Y_Z}(\varphi^*\alpha)^n = \frac{n}{d}\int_{\pi^{-1}(x)}\alpha\cdot\int_Z \sigma^*\alpha^{n-1} \qquad \square$$

Blowing up the diagonal $\Delta(X) \subset \mathrm{Sym}^2(X)$

Proposition 0.10. *Let X be a complex manifold with diagonal $\Delta\colon X \to \mathrm{Sym}^2(X)$, and consider the blow-up $\pi\colon \mathrm{Bl}_{\Delta(X)}(\mathrm{Sym}^2 X) \to \mathrm{Sym}^2(X)$. The exceptional divisor $E \subset \mathrm{Bl}_{\Delta(X)}(\mathrm{Sym}^2 X)$ is the projective bundle $\rho\colon \mathbb{P}(\Omega_X) \to X$ with normal sheaf $N_{E/\mathrm{Bl}(\mathrm{Sym}^2 X)} = \mathcal{O}_\pi(-1)|_E = \mathcal{O}_\rho(-2)$.*

Proof. Compare Δ to the ordinary diagonal $\delta\colon X \to X \times X$.

$$\begin{array}{ccc} X & \xrightarrow{\ \delta\ } & X \times X \\ & \searrow^{\Delta} & \downarrow p \\ & & \mathrm{Sym}^2(X) \end{array}$$

Both map Δ and δ are closed immersions. Denote the respective ideal sheaves by $I_\delta \subset \mathcal{O}_{X\times X}$ and $I_\Delta \subset \mathcal{O}_{\mathrm{Sym}^2(X)}$. By definition,

$$E = \mathrm{Proj}_X\Big(\bigoplus_{k\geq 0} I_\Delta^k/I_\Delta^{k+1}\Big) \quad \text{and} \quad \mathbb{P}(\Omega_X) = \mathrm{Proj}_X\Big(\bigoplus_{k\geq 0} I_\delta^k/I_\delta^{k+1}\Big).$$

The following lemma proves $I_\Delta^k/I_\Delta^{k+1} \cong I_\delta^{2k}/I_\delta^{2k+1}$, so the result follows from [13, Exercise II 5.13]. □

Lemma 0.11. *The canonical morphism $p^*I_\Delta \to I_\delta$ induces isomorphisms $I_\delta^k/I_\delta^{k+1} \cong I_\Delta^{2k}/I_\Delta^{2k+1}$.*

Proof. The question is local on X, so suppose $x_1, \dots, x_n$ are local coordinates on X. On $X \times X$ there are respective coordinates $x_1, \dots, x_n, y_1, \dots, y_n$, and $\Delta(X)$ is cut out by the equations $x_i = y_i$ for $i = 1, \dots, n$. Make the change of coordinates

$$\xi_i = x_i + y_i, \quad \eta_i = x_i - y_i,$$

so that $I_\Delta = (\eta_1, \dots, \eta_n)$. To $p\colon X \times X \to \mathrm{Sym}^2(X)$ corresponds the subring of invariants

$$\mathbb{C}\{\xi_1, \dots, \xi_n, \eta_i\eta_j \,|\, i, j = 1, \dots n\} \subset \mathbb{C}\{\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n\},$$

with $I_\delta = (\eta_i \cdot \eta_j \,|\, i, j = 1 \dots, n)$. So we see that the image of the natural map $p^*I_\delta \to I_\Delta$ is I_Δ^2. This leads to an exact sequence

$$I_\Delta \otimes p^*I_\delta^k \oplus p^*I_\delta^{k+1} \to p^*I_\delta^k \to I_\Delta^{2k}/I_\Delta^{2k+1} \to 0$$

whose first arrow becomes zero when restricting to $\Delta(X)$. □

Hilbert polynomials and stability

Here are the basic notions of stability for sheaves, as treated in [15]. Let X be a projective variety, and $\mathcal{O}_X(1)$ an ample line bundle.

Definition 0.12. Let E be a coherent sheaf on X of dimension d.

(a) The *Hilbert polynomial* of a coherent sheaf E on X is given by

$$P_E(n) = \chi(E \otimes \mathcal{O}_X(n)) = \frac{\alpha_d(E)}{d!} n^d + \ldots + \alpha_0(E), \quad n \in \mathbb{N}_0.$$

(b) The *reduced Hilbert polynomial* is $p_E(n) = \frac{1}{\alpha_d(E)} P_E(n)$.

(c) The sheaf E is called *semistable*, if E is pure and for any proper subsheaf $F \subset E$ one has $p(E) \leq p(F)$. E is called *stable* if the inequality is strict.

(d) If $d = \dim X$, then the *rank* of X is $\operatorname{rk} F = \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$. Otherwise, E has rank zero.

Lemma 0.13. *Let C be an irreducible reduced curve. Every pure sheaf F of rank one on C is stable.*

Proof. On an irreducible curve, being pure is equivalent to being torsion free. Recall that stability can be tested on pure quotients (cf. [15, Prop. 1.2.6]). Now suppose $\pi \colon F \twoheadrightarrow F''$ is a torsion-free non-zero quotient. Since F has rank one, π is generically an isomorphism. So the kernel $F' \subset F$ is a torsion sheaf, hence $F' = 0$. This shows that there are no destabilizing quotients of F. □

1 Foundations

This section contains the basic definitions, notations and properties of holomorphic symplectic manifolds and Lagrangian fibrations. It starts with a recap of the local theory of symplectic manifolds, before turning to Lagrangian fibrations over a local base. At this point important results by Hwang and Oguiso are recalled. The last part is dedicated to irreducible holomorphic symplectic manifolds and global Lagrangian fibrations.

1.1 Local theory of symplectic manifolds

Almost all manifolds in this thesis are complex manifolds and for this reason the word "holomorphic" is often omitted in the terminology.

Definition 1.1. Let M be a complex manifold. A *(holomorphic) symplectic form* is a nondegenerate, d-closed, holomorphic 2-form $\sigma \in H^0(M, \Omega^2_M)$. The pair (M, σ) is called a *(holomorphic) symplectic manifold.*

Remark. If M is a compact Kähler manifold, every holomorphic form is d-closed.

Definition 1.2. Let (M, σ) be a symplectic manifold. The form σ induces an isomorphism

$$\hat{\sigma} : T_M \to \Omega_M, \quad v \mapsto \sigma(v, -).$$

For a function $f \in \mathcal{O}_M(M)$, the *Hamiltonian vector field* is

$$X_f := \hat{\sigma}^{-1}(df).$$

Remark. By definition, the Hamiltonian vector field X_f satisfies

$$\sigma(X_f, Y) = Y(f)$$

for any other vector field Y, and in particular

$$X_f(f) = \sigma(X_f, X_f) = 0.$$

Thus X_f is tangent to the level sets $f^{-1}(c) = \{\, p \in M \mid f(p) = c \,\}, c \in \mathbb{C}$.

Definition 1.3. For an open $U \subset M$, the *Poisson bracket* of two functions $f, g \in \mathcal{O}_M(U)$ is defined as

$$\{f, g\} = X_f(g) = -X_g(f).$$

Remark. Clearly the Poisson bracket is $\mathbb{C}$-bilinear, antisymmetric and satisfies the Leibniz rule

$$\{fg, h\} = \{f, h\}g + f\{g, h\}.$$

The following proposition is not hard, but too cumbersome to include a proof at this point. For it the assumption $d\sigma = 0$ is important.

Proposition 1.4 (e.g. [19, Proposition 22.19]). *The Poisson bracket is a Lie bracket on $\mathcal{O}_M(M)$, and the Hamiltonian vector fields satisfy*

$$X_{\{f,g\}} = -[X_f, X_g]. \qquad \square$$

Remark. If M is a symplectic manifold, then $c_1(M) = c_1(\Lambda^{2n} T_M) = 0$ in $H^2(M, \mathbb{Z})$, because the line bundle $\Lambda^{2n} T_M$ is isomorphic to its dual Ω_M^{2n}, hence it is trivial.

1.2 Lagrangian fibrations

Definition 1.5. A submanifold $X \subset M$ is called *Lagrangian*, if $\dim X = \frac{1}{2} \dim M$ and $\sigma|_X = 0$. Equivalently, for each $x \in X$ one has

$$T_{X,x} = T_{X,x}^{\perp},$$

where $T_{X,x}^{\perp}$ denotes the subspace of $T_{M,x}$ which is orthogonal to $T_{X,x}$ with respect to σ.

Definition 1.6. Let M be a symplectic manifold and let B be a complex manifold. A *Lagrangian fibration* is a proper surjective morphism $f : M \to B$ with connected fibers, such that the smooth fibers are Lagrangian submanifolds of M.

Remark. In particular, $\dim B = \frac{1}{2} \dim M$.

Theorem 1.7 (Liouville-Arnol'd [2]). *Let $f : M \to B$ be a Lagrangian fibration, and $b \in B$. If the fiber $F = f^{-1}(b)$ is smooth, then it is a complex torus.*

Proof. The fibration f is proper, so F is clearly compact. Choose local coordinates $z_1, \dots, z_n$ around b, set $f_i = f^* z_i$ and let $X_i = X_{f_i}$ be the corresponding Hamiltonian vector fields. At a point $x \in F$, the tangent space $T_{F,x} \subset T_{M,x}$ is cut out by the 1-forms $d f_i(x) : T_{M,x} \to \mathbb{C}$. As $\hat{\sigma}^{-1}(d f_i(x)) = X_i(x)$, one has

$$T_{F,x} = \langle X_1(x), \dots, X_n(x) \rangle^{\perp},$$

and since F is Lagrangian,

$$T_{F,x} = \langle X_1(x), \dots, X_n(x) \rangle.$$

This shows that the vector fields $X_1, \dots, X_n$ are tangent to F. In particular

$$\{f_i, f_j\} = X_i(f_j) = 0,$$

and so the vector fields $X_1, \dots, X_n$ commute by Proposition 1.4. Because F is compact this defines a global flow $\Phi : \mathbb{C}^n \times F \to F$, which we think of as a group action. Since $X_1, \dots, X_n$ are pointwise linearly independent, this group action has open orbits, and since F is connected, there is exactly one orbit. Choosing a point $x_0 \in F$ one obtains the universal covering

$$\nu : \mathbb{C}^n \to F, \quad t \mapsto \Phi(t, x_0).$$

Then $\Lambda = \nu^{-1}(x_0) \subset \mathbb{C}^n$ is a discrete subgroup, and $F = \mathbb{C}^n / \Lambda$. $\square$

The following are results by Hwang and Oguiso[17, 18], who studied the general singular fibers of f.

Definition 1.8. Given a Lagrangian fibration $f: M \to B$, the *discriminant locus* is the set of critical values, i.e.

$$\Delta = \{ b \in B \mid f^{-1}(b) \text{ is singular} \} \subset B.$$

Theorem 1.9 (Hwang, Oguiso [17, Prop. 3.1])**.** *If $f: M \to B$ is a Lagrangian fibration with non-empty discriminant locus $\Delta \subset B$, then Δ has codimension one.* □

Pick a general point $b \in \Delta$, let $z_n = 0$ be a local equation for Δ in b and let $z_1, \dots, z_{n-1}$ be complementary coordinates. Let $U \subset B$ an open neighbourhood of B where $z_1, \dots, z_n$ define an open embedding $U \to \mathbb{C}^n$, and set $M_U = f^{-1}(U)$, $f_U = f|_{M_U}$. Then the Hamiltonian vector fields

$$X_i = \hat{\sigma}^{-1}(f^* dz_i), \qquad i = 1, \dots, n-1$$

are still pointwise linearly independent and tangent to the fibers of f_U. Their flow gives a group action

$$\mathbb{C}^{n-1} \times M_U \to M_U,$$

which acts on the fiber $F = f^{-1}(b)$ and on its reduction F_{red}.

Lemma 1.10 ([17, Proposition 2.2])**.** *The connected components of* $\operatorname{Sing}(F_{\text{red}})$ *are complex tori.*

Proof. Every automorphism of F_{red} has to preserve the singular locus, so $\mathbb{C}^{n-1}$ acts on $\operatorname{Sing}(F_{\text{red}})$. Because the vector fields $X_1, \dots, X_{n-1}$ are pointwise linearly independent, the orbits are smooth of dimension $n-1$. But $\dim \operatorname{Sing}(F_{\text{red}}) \le n-1$, so that each connected component is an orbit. Conclude that the connected components are tori in the same way as in the proof of Theorem 1.7. □

Definition 1.11. A connected component $Z \subset \operatorname{Sing}(F_{\text{red}})$ is called a *singular torus.*

Remark. Note that singular tori are themselves smooth.

Exploiting the vector fields further, Hwang and Oguiso proved the following local analysis of the singularities a general singular fiber can have.

Proposition 1.12 (cf. [17])**.** *The reduction F_{red} of the general singular fiber $F = f^{-1}(b)$ is analytic-locally the product of a plane curve germ $\mathscr{C}$ with an $(n-1)$-dimensional manifold. If $\mathscr{C}$ is singular, it is of one of the four analytic isomorphism types given in Table 1.*

Proof. The first assertion is [17, Prop. 4.4 (2)]. The singularitiy types then follow from [17, Prop. 4.1] for the non-irreducible case, and [17, Prop. 4.7 (2)] for the cuspidal case. □

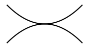
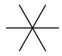
Singularity	Node	Cusp	Tangent	Triple intersection
Kodaira type	I	II	III	IV
Local equation	$xy = 0$	$x^3 = y^2$	$y(y - x^2) = 0$	$x(x - y)y = 0$
Illustration				
Milnor number	1	2	3	4

Table 1: Analytic isomorphism types of plane curve germs in local coordinates

Definition 1.13 ([17])**.** Set $Y = f^{-1}(\Delta)_{\mathrm{red}}$. The *characteristic vector field* of Y is the twisted vector field

$$\lambda_Y \in H^0(Y, T_M|_Y \otimes \mathcal{O}(Y)|_Y)$$

which is obtained by tensoring the composition

$$\mathcal{O}(-Y)|_Y \to \Omega_M|_Y \xrightarrow{\cong} T_M|_Y,$$

with $\mathcal{O}(Y)|_Y$.

Remark. If $h = 0$ is a local equation of Y at $p \in M$, the image of $\mathcal{O}(-Y)$ in $\Omega_M|_Y$ is generated by dh, and this trivialization identifies

$$\lambda_Y = \hat{\sigma}^{-1}(dh).$$

Note that this vector field extends to a neighbourhood of p in M that it is tangent to the fibers of f. In particular λ_Y is tangent to Y itself.

Proposition 1.14 ([17, Prop. 3.3, 3.5, 3.6, 3.7, 3.8])**.** *Let $F_0 \subset F_{\mathrm{red}}$ be an irreducible component of the reduced fiber, and let $\nu \colon \tilde{F}_0 \to F_0$ be the normalization. Then $\tilde{F}_0$ is smooth and the Albanese map $\alpha \colon \tilde{F}_0 \to \mathrm{Alb}(\tilde{F}_0)$ is surjective and has connected curves as fibers. If F_{red} is smooth, the fibers of α are elliptic curves. If F_{red} is singular, α is a $\mathbb{P}^1$-bundle.*

Moreover, the Hamiltonian vector fields $X_1, \dots, X_{n-1}$ lift to give a $\mathbb{C}^{n-1}$-action on $\tilde{F}_0$, which is compatible with the group structure on $\mathrm{Alb}(\tilde{F}_0)$. The characteristic vector field λ_Y also lifts to a twisted vector field $\tilde{\lambda}_Y \in H^0(\tilde{F}_0, T_{\tilde{F}_0} \otimes \nu^*\mathcal{O}(Y))$, which is tangent to the fibers of α and which vanishes exactly over the singularities $\mathrm{Sing}(F_{\mathrm{red}})$. □

Definition 1.15 ([17])**.** Let F be a general singular fiber of $f : M \to B$. A *characteristic curve* $\Theta \subset F$ is the image of an Albanese fiber of the normalization of one of the components of F. To each characteristic curve Θ one associates the *multiplicity* r_Θ, which is the multiplicity of the irreducible component $D_\Theta \subset f^{-1}(\Delta)$ which contains Θ. A *characteristic cycle* is a (possibly infinite) formal sum $\sum_s r_s \Theta_s$ of characteristic curves $\Theta_s \subset F$ with multiplicities $r_s = r_{\Theta_s}$, such that in $\bigcup_s \Theta_s \subset F$ every two points are connected by a finite number of characteristic curves, and $\bigcup_s \Theta_s$ is maximal with this property.

Theorem 1.16 (Classification of characteristic cycles [17, Thm 1.4][18, Thm 2.4])**.** *Let $\Theta = \sum_s r_s \Theta_s$ be the characteristic cycle, and let r be the greatest common divisor of the r_s. Then $\frac{1}{r}\Theta$ is either*

(a) a smooth elliptic curve,

(b) a non-multiple singular fiber of a minimal elliptic fibration as listed by Kodaira,

(c) or it consists of countably many components, each of which has multiplicity 1 and intersects exactly two other components transversely. □

See Table 2 for and overview of all possible characteristic cycles.

Corollary 1.17. *If a general singular fiber $F = f^{-1}(b)$ has a characteristic cycle of type III or IV, then all irreducible components of F have the same (scheme-theoretic) multiplicity.*

Proof. The characteristic curves all have the same multiplicity, which means that the components of $f^{-1}(\Delta)$ which intersect F all have the same multiplicity. Since F is general, the multiplicity of a component $F_0 \subset F$ equals the multiplicity of the component $D_0 \subset f^{-1}(\Delta)$ which contains F_0. □

1.3 Irreducible symplectic manifolds

Definition 1.18. A symplectic manifold M is called *irreducible* if it is a simply connected compact Kähler manifold, and σ spans $H^0(M, \Omega^2_M)$.

The importance of irreducible symplectic manifolds stems from the decomposition theorem due to Beauville and Bogomolov.

Theorem 1.19 ([4, Théorème 2])**.** *If M is a compact Kähler manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$, then there exists a finite étale covering*

$$T \times \prod_i X_i \times \prod_j Y_j \to M,$$

where

(a) T is a complex torus,

(b) the X_i are irreducible symplectic manifolds, and

(c) the Y_j are Calabi-Yau manifolds, i.e. $\dim Y_j \geq 3$, the canonical bundle ω_{Y_j} is trivial, and $H^0\big(Y_j, \Omega^p_{Y_j}\big) = 0$ for $0 < p < \dim Y_j$. □

Definition 1.20 ([4])**.** Let M be an irreducible symplectic manifold of complex dimension $2n$. The *Beauville-Bogomolov-Fujiki form* is the quadratic form q on $H^2(M, \mathbb{C})$, defined by

$$q(\alpha) = \frac{n}{2} \int_M (\sigma\bar{\sigma})^{n-1}\alpha^2 + (1-n) \int_M \sigma^{n-1}\bar{\sigma}^n \alpha \cdot \int_M \sigma^n \bar{\sigma}^{n-1}\alpha.$$

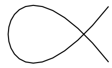
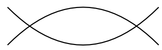
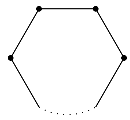
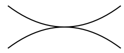
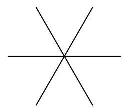
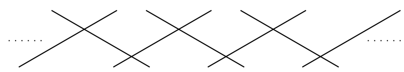
I_0	a nonsingular elliptic curve	
I_1	a rational curve with one node	
I_2	two smooth rational curves intersecting in two points transversely	
I_b ($\tilde{A}_{b-1}, b \geq 3$)	b smooth rational curves intersecting transversely with dual graph $\tilde{A}_{b-1}$	
II	a rational curve with one cusp	
III	two smooth rational curves, intersecting in one point tangentially with intersection multiplicity 2	
IV	three smooth rational curves intersecting in a single point, pairwise transversely	
I_b^* ($\tilde{D}_{b+4}, b \geq 0$)	$b+5$ smooth rational curves with dual graph $\tilde{D}_{b+4}$	1 \ 2 — ...... — 2 / 1 1 / 2 — ...... — 2 \ 1
II^* ($\tilde{E}_8$)	nine smooth rational curves intersecting transversely with dual graph $\tilde{E}_8$	1 — 2 — 3 — 4 — 5 — 6 — 4 — 2 | (from 6) 3
III^* ($\tilde{E}_7$)	eight smooth rational curves intersecting transversely with dual graph $\tilde{E}_7$	1 — 2 — 3 — 4 — 3 — 2 — 1 | (from 4) 2
IV^* ($\tilde{E}_6$)	seven smooth rational curves intersecting transversely with dual graph $\tilde{E}_8$	1 — 2 — 3 — 2 — 1 | (from 3) 2 | 1
A_∞	infinitely many smooth rational curves, each of which intersects two other rational curves transversely	

Table 2: Non-multiple characteristic cycles

Theorem 1.21 (Beauville[4]). *The form q is nondegenerate, and is up to a suitable muliple defined on $H^2(M, \mathbb{Z})$.* □

The following are called the *Fujiki relations.*

Theorem 1.22 (Fujiki[8]). *Assume $\beta \in H^{4k}(M, \mathbb{R})$ is of type $(2k, 2k)$ on all small deformations of M. Then there exists a constant $c_\beta \in \mathbb{R}$ such that for all $\alpha \in H^2(M, \mathbb{C})$*

$$\int_M \alpha^{2(n-k)}\beta = c_\beta \cdot q(\alpha)^{n-k}.$$ □

Theorem 1.23 (Matsushita[21, 22]). *Let M be an irreducible symplectic manifold, and let B be a complex manifold with $0 < \dim B < \dim M$. If $f : M \to B$ is a proper morphism with connected fibers, then f is a Lagrangian fibration.* □

Remark. One can more generally define Lagrangian fibrations for normal bases B, and Theorem 1.23 also extends to that case. It is an open problem if there exists a Lagrangian fibrations with a singular base space.

Theorem 1.24 (Hwang[16], Greb and Lehn[10]). *If M is an irreducible symplectic manifold, $f : M \to B$ is a Lagrangian fibration and B is smooth, then $B \cong \mathbb{P}^n$.*

Theorem 1.25 (Voisin via [6, Prop 2.1]). *Let $f : M \to \mathbb{P}^n$ be a Lagrangian fibration. There exists a Kähler class $[\omega] \in H^{1,1}(M) \cap H^2(M, \mathbb{R})$ whose restriction to a smooth fiber is rational. In particular smooth fibers of f are abelian varieties.*

Proof. Let F be a smooth fiber of f, and pick any Kähler class $[\omega_0]$ on M. Choose a rational class $\alpha \in H^2(M, \mathbb{Q})$ which approximates ω_0 so that its $(1, 1)$-part $\omega = \alpha^{1,1}$ is a Kähler class. Since F is a Lagrangian submanifold, the restriction map $r : H^2(M, \mathbb{C}) \to H^2(F, \mathbb{C})$ vanishes on $H^{2,0}(M) \oplus H^{0,2}(M)$. This implies that $r(\alpha) = r(\omega)$ is a rational Kähler class. □

The following was first observed by Oguiso[27] and is proved using results of Voisin[32] on the deformation theory of Lagrangian submanifolds, and results of Matsushita[23, 24] on deformations of Lagrangian fibrations.

Theorem 1.26. *If $f : M \to \mathbb{P}^n$ is a Lagrangian fibration of an irreducible holomorphic symplectic manifold M, then for any smooth fiber $F_{\text{sm}} \subset M$,*

$$\operatorname{rk}\left(H^2(M, \mathbb{C}) \to H^2(F_{\text{sm}}, \mathbb{C})\right) = 1.$$

Proof. Let $\operatorname{Def}(M)$ be the universal deformation space of M, and let $Z_1 \subset \operatorname{Def}(M)$ be the subset of deformations which preserve F_{sm}. Voisin showed[32, 1.2 and 1.5]

$$\operatorname{rk}\left(H^2(M, \mathbb{C}) \to H^2(F_{\text{sm}}, \mathbb{C})\right) = \operatorname{codim}(Z_1 / \operatorname{Def}(M)).$$

On the other hand, Matsushita proved[24] that the subset $Z_2 \subset \operatorname{Def}(M)$ of deformations which admit a deformation of f, has codimension one. But any deformation of f preserves the smooth fiber F_{sm}, so that $Z_2 \subset Z_1$ and

$$\operatorname{codim}(Z_1 / \operatorname{Def}(M)) = 1.$$ □

One key part of this thesis is to prove a similar theorem for singular tori $Z \subset F_{\mathrm{sm}}$, see Theorem 3.12.

Definition 1.27 ([36])**.** Let $f : M \to \mathbb{P}^n$ be a Lagrangian fibration. The *polarization type* $(d_1, \dots, d_n)$ of f is the polarization type on a smooth fiber F_{sm}, induced by a Kähler form ω on M which restricts to an integral, indivisible class on F_{sm}. By Theorem 1.25 such an ω exists, and by Theorem 1.26 the polarization type does not depend on the choice of ω.

Theorem 1.28 (Wieneck[36])**.** *The polarization type of f is a deformation invariant of f.* □

Remark. Note that the polarization type is not a deformation invariant of M. Wieneck[37] gave examples of generalized Kummer varieties which have different polarization types.

2 Weights of the discriminant divisor

Fix the following notation. Let M be an irreducible holomorphic symplectic manifold with symplectic form σ and let $f : M \to \mathbb{P}^n$ be a Lagrangian fibration. Let $\Delta = \bigcup_i \Delta_i \subset \mathbb{P}^n$ be the discriminant locus of f with irreducible components $\Delta_i \subset \Delta$. Set $H = f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let $(d_1, \dots, d_n)$ be the polarization type of f, and let ω be a Kähler form on M which induces this polarization type on the smooth fibers.

2.1 Derivation of a formula

The contents of this subsection are mostly due to Christian Lehn[20, VIII.2], following ideas by Sawon[30].

By Theorem 1.22, there exist constants $a_p \in \mathbb{C}$, depending only on the deformation type of M, such that for each $\alpha \in H^2(M, \mathbb{C})$,

$$q_M(\alpha)^{n-p} = a_p \cdot \int_M \alpha^{2(n-p)} c_{2p}(M).$$

For two classes $\alpha, \beta \in H^2(M, \mathbb{C})$ one obtains

$$(q_M(\alpha)^n)^{n-1}(q_M(\beta)^{n-1})^n = a_0^{n-1} a_1^n \left(\int_M \alpha^{2n}\right)^{n-1} \left(\int_M \beta^{2n-2} c_2(M)\right)^n.$$

The left hand side is symmetric in α and β, which yields on the right hand side

$$\left(\int_M \alpha^{2n}\right)^{n-1} \left(\int_M \beta^{2n-2} c_2(M)\right)^n = \left(\int_M \alpha^{2n-2} c_2(M)\right)^n \left(\int_M \beta^{2n}\right)^{n-1}. \tag{4}$$

Now specify $\alpha = \sigma + \bar{\sigma}$ and $\beta = t_1\omega + t_2 H$ for indeterminates t_1, t_2. To evaluate (4), first observe

$$(\sigma + \bar{\sigma})^{2n} = \binom{2n}{n} (\sigma\bar{\sigma})^n,$$

because $\sigma^{n+1} = 0$ and $\bar{\sigma}^{n+1} = 0$. Since $c_2(M)$ has Hodge type $(2, 2)$, similarly

$$(\sigma + \bar{\sigma})^{2n-2} c_2(M) = \binom{2n-2}{n-1} (\sigma\bar{\sigma})^{n-1} c_2(M).$$

Compare the coefficients of $(t_1 t_2)^{n(n-1)}$ in (4) and cancel the binomial coefficients to obtain

$$\left(\int_M (\sigma\bar{\sigma})^n\right)^{n-1} \left(\int_M c_2(M)\omega^{n-1} H^{n-1}\right)^n = \left(\int_M c_2(M)(\sigma\bar{\sigma})^{n-1}\right)^n \left(\int_M \omega^n H^n\right)^{n-1},$$

or

$$\frac{\left(\int_M c_2(M)\omega^{n-1} H^{n-1}\right)^n}{\left(\int_M \omega^n H^n\right)^{n-1}} = \frac{\left(\int_M c_2(M)(\sigma\bar{\sigma})^{n-1}\right)^n}{\left(\int_M (\sigma\bar{\sigma})^n\right)^{n-1}}. \tag{5}$$

Using Rozansky-Witten techniques, Sawon showed in [30, Lemma 2]

$$\frac{\left(\int_M c_2(M)(\sigma\overline{\sigma})^{n-1}\right)^n}{\left(\int_M (\sigma\overline{\sigma})^n\right)^{n-1}} = \frac{24^n (n!)^2}{n^n} \int_M \sqrt{\hat{A}}(M), \tag{6}$$

where $\sqrt{\hat{A}}(M) = 1 + \frac{c_2(M)}{24} + \frac{7c_2(M)^2 - 4c_4(M)}{5760} + \cdots$ is the $\sqrt{\hat{A}}$-genus of M. Combining (5) with (6), and taking n-th roots we obtain

$$\frac{24}{n}\left((n!)^2 \int_M \sqrt{\hat{A}}(M)\right)^{\frac{1}{n}} = \frac{\int_M c_2(M)\omega^{n-1}H^{n-1}}{\left(\int_M \omega^n H^n\right)^{\frac{n-1}{n}}}. \tag{7}$$

Since H^n is the class of a smooth fiber F, on which ω has polarization type $(d_1, \dots, d_n)$, this can be rephrased as

$$24\left(\frac{n! \int_M \sqrt{\hat{A}}(M)}{d_1 \cdots d_n}\right)^{\frac{1}{n}} = \frac{n \int_M c_2(M)\omega^{n-1}H^{n-1}}{\int_F \omega^n}. \tag{8}$$

Remark. It might seem odd to split up the contribution of ω on F to both sides of the equation. However, the left hand side is exactly what appears in Sawon's work[30], and calculating examples suggests that this might always be an integer.

To relate the Chern class $c_2(M)$ to the Lagrangian fibration f, let us introduce the ΩT*-complex*

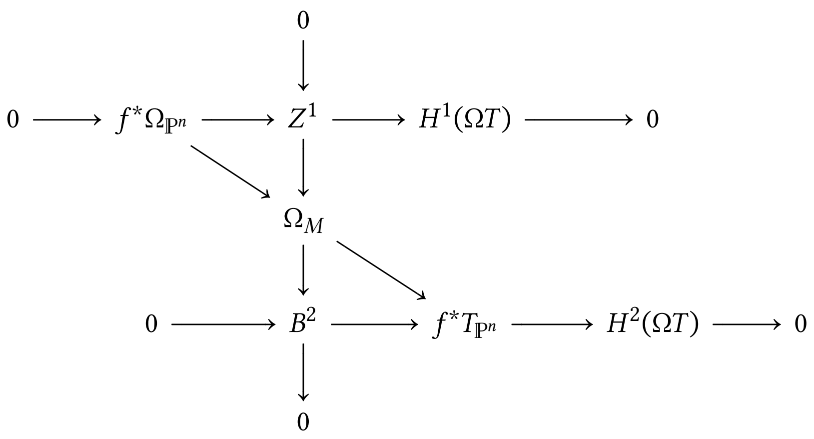
$$0 \to f^*\Omega_{\mathbb{P}^n} \xrightarrow{Df^\vee} \Omega_M \overset{\hat{\sigma}^{-1}}{\cong} T_M \xrightarrow{Df} f^*T_{\mathbb{P}^n} \to 0 \tag{ΩT}$$

as a cochain complex in degrees 0, 1 and 2.

Lemma 2.1. *The sequence* (ΩT) *is indeed a complex.*

Proof. We may check this on a stalk at $p \in M$. Let $z_1, \dots, z_n$ be local analytic coordinates on $\mathbb{P}^n$ around $f(p)$. The forms $f^*dz_1, \dots, f^*dz_n$ form a local basis of $(f^*\Omega_{\mathbb{P}^n})_p$, and $\hat{\sigma}^{-1}$ maps them to their corresponding Hamiltonian vector fields in $T_{M,p}$. Since f is a Lagrangian fibration, those vector fields are tangent to the fibers of f, hence they belong to the kernel of $T_M \to f^*T_{\mathbb{P}^n}$. □

Note that $f^*\Omega_{\mathbb{P}^n} \to \Omega_M$ is injective, because f is dominant. The complex (ΩT) is exact at a point $p \in M$ if and only if f is smooth at p, so the cohomology sheaves $H^1(\Omega T)$ and $H^2(\Omega T)$ are supported on the singular points of f. Splitting the ΩT-complex into short exact sequences we obtain the following commutative diagram of sheaves with exact rows and columns.



(9)

Lemma 2.2 ([20]). *One has*

$$c_2(M) = -(n+1)H^2 + c_2(H^1(\Omega T)) - c_2(H^2(\Omega T)).$$

Proof. The vertical short exact sequence in (9) implies

$$\begin{aligned} 0 = -c_1(M) &= c_1(Z^1) + c_1(B^2) \\ &= c_1(f^*\Omega_{\mathbb{P}^n}) + c_1(H^1(\Omega T)) + c_1(f^*T_{\mathbb{P}^n}) - c_1(H^2(\Omega T)) \\ &= c_1(H^1(\Omega T)) - c_1(H^2(\Omega T)). \end{aligned} \tag{10}$$

For the second Chern classes consider the degree two part of the Chern character, $\mathrm{ch}_2 = \frac{1}{2}(c_1^2 - 2c_2)$. As ch_2 is additive over short exact sequences, one obtains

$$\begin{aligned} -c_2(M) = \mathrm{ch}_2(M) &= \mathrm{ch}_2(Z^1) + \mathrm{ch}_2(B^2) \\ &= \mathrm{ch}_2(f^*\Omega_{\mathbb{P}^n}) + \mathrm{ch}_2(H^1(\Omega T)) + \mathrm{ch}_2(f^*T_{\mathbb{P}^n}) - \mathrm{ch}_2(H^2(\Omega T)) \\ &= (n+1)H^2 + \mathrm{ch}_2(H^1(\Omega T)) - \mathrm{ch}_2(H^2(\Omega T)). \end{aligned}$$

From (10) we get

$$\mathrm{ch}_2(H^1(\Omega T)) - \mathrm{ch}_2(H^2(\Omega T)) = c_2(H^2(\Omega T)) - c_2(H^1(\Omega T)),$$

which finishes the proof. □

As $H^{n+1} = 0$, the first summand from Lemma 2.2 disappears when multiplied by H^{n-1}, so that (8) becomes

$$24\left(\frac{n! \int_M \sqrt{\hat{A}(M)}}{d_1 \cdots d_n}\right)^{\frac{1}{n}} = \frac{n \int_M \big(c_2(H^1(\Omega T) - c_2(H^2(\Omega T)))\big)\, \omega^{n-1} H^{n-1}}{\int_F \omega^n} \tag{11}$$

The class H^{n-1} is the class of the preimage $N = f^{-1}(\ell)$ of a line $\ell \subset \mathbb{P}^n$. Choosing a general line one may assume that N is smooth, by Bertini's theorem[13, III Corollary 10.9]. Let $g\colon N \to \ell$ be the restriction of f to N.

$$\begin{array}{ccc} N & \hookrightarrow & M \\ \downarrow g & & \downarrow f \\ \ell & \hookrightarrow & \mathbb{P}^n \end{array}$$

Lemma 2.3. *For a general choice of ℓ, all short exact sequences in* (9) *stay exact when restricted to N. Furthermore one has*

$$c_2(H^k(\Omega T)) \cdot H^{n-1} = i_* c_2(H^k(\Omega T)|_N) = i_* c_2(H^k(\Omega T|_N)) \qquad (k = 1, 2),$$

where $\Omega T|_N$ denotes the ΩT-complex restricted to N, and $i\colon N \to M$ is the inclusion.

Proof. Let B^2 be the sheaf from (9), and let S be the set of associated subspaces of $H^1(\Omega T), H^2(\Omega T)$ and B^2 (cf. Definition 0.1). Since M is compact, S is a finite set. If ℓ is chosen such that it does not contain $f(Y)$ for any $Y \in S$, then N does not contain any of the associated subspaces. By an iterated application of Lemma 0.2, the short exact sequences in (9) stay exact when restricting to N.

The Chern classes of a coherent sheaf $\mathscr{F}$ on a complex manifold are define via finite resolutions $0 \to E^d \to \ldots \to E^0 \to \mathscr{F} \otimes C^\infty \to 0$, where the E^k are locally free C^∞-sheaves[34]. For each E^k one has the projection formula $c_j(E^k) \cap [N] = i_* c_j(E|_N)$. So if one chooses ℓ such that the resolutions remain exact when restricted to N, the second statement follows. □

A general line ℓ intersects the discriminant divisor Δ in finitely many points. For each irreducible component $\Delta_i \subset \Delta \subset \mathbb{P}^n$, choose a point $b_i \in \ell \cap \Delta_i$, and let $F_i = f^{-1}(b_i)$ be the scheme-theoretic fiber over b_i.

Lemma 2.4. *For a general choice of ℓ, and $k = 1, 2$ one has*

$$c_2(H^k(\Omega T|_N)) = \sum_i \deg(\Delta_i) \cdot c_2(H^k(\Omega T|_{F_i})).$$

Proof. The sheaf $H^k(\Omega T|_N)$ is supported on $N \cap f^{-1}(\Delta)$, which consists of a finite number of singular fibers. Write $c_2(H^k(\Omega T)) = \sum_i \alpha_i + \overline{\alpha}$, where α_i is a $(2n-2)$-cycle dominating Δ_i, and $f(|\overline{\alpha}|)$ has codimension ≥ 2. Here $|\overline{\alpha}|$ denotes the support of $\overline{\alpha}$ as a set. Then α_i is a family of cycles over Δ_i in the sense of [9, Chapter 10], so that for all $b_i, \tilde{b}_i \in \ell \cap \Delta_i$, $\alpha_i(b_i)$ and $\alpha_i(\tilde{b}_i)$ are algebraically equivalent. If one chooses ℓ disjoint from $f(|\overline{\alpha}|)$, the result follows. □

Plugging the last two propositions into (11) yields

$$24\left(\frac{n! \int_M \sqrt{\hat{A}}(M)}{d_1 \cdots d_n}\right)^{\frac{1}{n}} = \sum_i \deg(\Delta_i) \frac{n \int_N \left(c_2(H^1(\Omega T|_{F_i})) - c_2(H^2(\Omega T|_{F_i}))\right)\omega^{n-1}}{\int_F \omega^n} \tag{12}$$

which suggests the following definition, which was basically given by C. Lehn[20].

Definition 2.5. The *weight* associated to the component $\Delta_i \subset \Delta$ is

$$w_i := \frac{n \int \left(c_2(H^1(\Omega T|_{F_i})) - c_2(H^2(\Omega T|_{F_i}))\right)\omega^{n-1}}{\int_F \omega^n}. \tag{13}$$

This definition of w_i seemingly depends on the choice of ω, and proving that this is not the case is the main endeavour of this thesis, which will be completed in section 3.

Before this question can be seriously attacked, we need a better description for the weights. The following section 2.2 describes w_i in terms of combinatorial data of the singular fiber F_i and of the ratios

$$R_Z = \frac{\int_Z \omega^{n-1}}{\int_F \omega^n},$$

for the various singular tori $Z \subset F_i$. Since by assumption $[\omega|_{F_{\mathrm{sm}}}] \in H^2(F_{\mathrm{sm}}, \mathbb{Z})$ is indivisible for every smooth fiber $F_{\mathrm{sm}} \subset M$, every other such Kähler class $\tilde{\omega}$ differs from ω by an element in the kernel of $H^2(M, \mathbb{R}) \to H^2(F_{\mathrm{sm}}, \mathbb{R})$. So to show that R_Z does not depend on the choice of ω it will be sufficient to prove that the restriction map $H^2(M, \mathbb{C}) \to H^2(Z, \mathbb{C})$ has the same kernel as $H^2(M, \mathbb{C}) \to H^2(F_{\mathrm{sm}}, \mathbb{C})$.

2.2 Reducing the weights to singular tori

Fix a component $\Delta_i \subset \Delta$, a point $b = b_i \in \Delta_i \cap \ell$, and let $w = w_i$ be the associated weight as defined in Definition 2.5. Let $F = g^{-1}(b) \subset N$ be the singular fiber, with irreducible components $F_1, \dots, F_r \subset F$. Each F_i has a multiplicity $n_i \geq 1$, so that $F = \sum_i n_i F_i$ as a divisor in N. Let $F_{\mathrm{red}} = \sum_i F_i$ be the reduced fiber, and call $F^\flat = \sum_i (n_i - 1) F_i$ the *diminished fiber.* Set for convenience $\mathscr{H}^j = H^j(\Omega T|_F)$.

This section studies the sheaves $\mathscr{H}^1$ and $\mathscr{H}^2$ to reduce the computation of w to the intersection combinatorics of the components F_i and to the degree of the polarization induced by ω on the singular tori $Z \subset F$. The final result is Theorem 2.14.

Proposition 2.6. *There are isomorphisms* $H^1(\Omega T) \cong \mathscr{E}xt^1_M(H^2(\Omega T), \mathscr{O}_M)$ *and* $\mathscr{H}^1 \cong \mathscr{E}xt^1_N(\mathscr{H}^2, \mathscr{O}_N)$.

Remark. This is reminiscent of the Universal Coefficient Theorem[35, 3.6.5], which would assert the existence of a short exact sequence

$$0 \to \mathscr{E}xt^1(H^2(\Omega T), \mathscr{O}_M) \to H^{-1}(\mathscr{H}om(\Omega T, \mathscr{O}_M)) \to \mathscr{H}om(H^1(\Omega T), \mathscr{O}_M) \to 0.$$

Note that, up to a shift by 2, the ΩT-complex is isomorphic to its own dual, so the middle term is isomorphic to $H^1(\Omega T)$. Moreover, $H^1(\Omega T)$ is a torsion sheaf, so $\mathscr{H}om(H^1(\Omega T), \mathscr{O}_M) = 0$. However, the Universal Coefficient Theorem can not be applied directly, because the image of T_M in $f^* T_{\mathbb{P}^n}$ is not a locally free sheaf.

Proof. As noted, if one dualizes the ΩT-complex, the result is again the ΩT-complex, up to a shift by 2 and a sign in the isomorphism $\Omega_M \cong T_M$. Dualizing diagram (9) and writing $\mathscr{E}(-) = \mathscr{E}xt^1_M(-, \mathscr{O}_M)$ gives the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
0 & \longrightarrow & f^*\Omega_{\mathbb{P}^n} & \longrightarrow & (B^2)^\vee & \longrightarrow & \mathscr{E}(H^2(\Omega T)) & \longrightarrow & 0 & & \\
 & & & \searrow & \downarrow & & & & & & \\
 & & & & \Omega_M & & & & & & \\
 & & & & \downarrow & \searrow & & & & & \\
 & & 0 & \longrightarrow & (Z^1)^\vee & \longrightarrow & f^*T_{\mathbb{P}^n} & \longrightarrow & \mathscr{E}(H^1(\Omega T)) & \longrightarrow & \mathscr{E}(Z^1) \longrightarrow 0 \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathscr{E}(B^2) & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}
$$

Since the morphism $(Z^1)^\vee \to f^*T_{\mathbb{P}^n}$ is injective, one gets

$$Z^1 = \mathrm{Ker}(\Omega_M \to f^*T_{\mathbb{P}^n}) = \mathrm{Ker}(\Omega_M \to (Z^1)^\vee) = (B^2)^\vee,$$

and thus

$$H^1(\Omega T) = \mathscr{E}xt^1_M(H^2(\Omega T), \mathscr{O}_M).$$

Since diagram (9) stays exact when restricted to N, the same works for $\mathscr{H}^1$. □

Lemma 2.7. *The sheaf* $H^2(\Omega T|_N)$ *is canonically isomorphic to* $\mathrm{coker}(T_N \to g^*T_\ell)$, *and it is the structure sheaf of its own scheme-theoretic support.*

Proof. We have the following morphism of short exact sequences.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_N & \longrightarrow & T_M|_N & \longrightarrow & N_{N/M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & g^*T_\ell & \longrightarrow & g^*T_{\mathbb{P}^n}|_\ell & \longrightarrow & g^*N_{\ell/\mathbb{P}^n} & \longrightarrow & 0
\end{array}
$$

Since $N_{N/M}$ is isomorphic to $g^*N_{\ell/\mathbb{P}^n}$, the natural morphism $\mathrm{coker}(T_N \to g^*T_\ell) \to H^2(\Omega T|_N)$ is an isomorphism by the snake lemma.

The sheaf g^*T_ℓ is trivial in a neighbourhood of F, and hence trivial in a neighbourhood U of the support of $H^2(\Omega T)|_N$. There we get a surjection $\mathscr{O}_U \twoheadrightarrow H^2(\Omega T)|_U$, which means that $H^2(\Omega T)|_N$ is the structure sheaf of its own support. □

Let $\mathscr{T} \subset \mathscr{H}^2$ be the maximal subsheaf whose support has codimension ≥ 2 in N, and let X be the scheme theoretic support of $\mathscr{H}^2/\mathscr{T} = \mathscr{O}_X$. Then X is pure of codimension 1 in N, i.e. X is an effective Cartier divisor supported on the singular fiber F.

Lemma 2.8. *$c_2(\mathscr{H}^2) = c_2(\mathscr{T}) + c_2(\mathscr{O}_X)$ and $c_2(\mathscr{O}_X) = c_1(\mathscr{O}_N(X))^2$.*

Proof. Consider the short exact sequence $0 \to \mathscr{T} \to \mathscr{H}^2 \to \mathscr{O}_X \to 0$ of sheaves on N. As $\mathscr{T}$ has codimension ≥ 2, one has $c_1(\mathscr{T}) = 0$ and the first assertion follows from Whitney's formula. The second assertion is Lemma 0.3. □

Proposition 2.9. $c_2(\mathscr{H}^1) = 0$.

Proof. First note that by adjunction, the dualizing sheaf ω_N of N is

$$\omega_N = \det N_{N/M} = g^* \det N_{L/\mathbb{P}^n} = g^*\mathscr{O}(n-1).$$

In particular, ω_N is trivial in a neighbourhood of F, where the $\mathscr{H}^i$ are supported. Combining this with Proposition 2.6, one obtains

$$\mathscr{H}^1 = \mathscr{E}xt^1(\mathscr{H}^2, \mathscr{O}_N) \cong \mathscr{E}xt^1(\mathscr{O}_X, \omega_N) = \omega_X.$$

Again by adjunction we conclude $\mathscr{H}^1 \cong N_{X/N} = \mathscr{O}_N(X)|_X$, and by Lemma 0.3,

$$c_2(\mathscr{H}^1) = 0.$$

□

The local analysis by Hwang and Oguiso (cf. Table 1) allows us to locally describe the sheaves $\mathscr{T}$ and $\mathscr{O}_X$ in the following proposition.

Proposition 2.10. *The scheme-theoretic support of $\mathscr{T}$ is pure of codimension 2, the underlying set is $\mathrm{Sing}(F_{\mathrm{red}})$, and the multiplicity of each component is given by the Milnor number of the corresponding curve singularity. Furthermore, $X = F^\flat$.*

Proof. Choose a point $p \in F$. By Lemma 2.7 we can compute the stalk $\mathscr{H}^2_p$ as the cokernel of $Dg\colon T_{N,p} \to g^*T_{\ell,p}$. This can we done by choosing a local model for g in each of the cases $I - IV$.

If p is a smooth point of F_{red}, then in appropriate local coordinates, g is given by $(x_1, \dots, x_{n+1}) \mapsto x_1^k$, where k is the multiplicity of F in p. So $\mathscr{H}^2_p = \mathscr{O}_{N,p}/(x_1^{k-1})$ has no subsheaf of codimension 2, which shows $\mathrm{Supp}(\mathscr{T}) \subset \mathrm{Sing}(F_{\mathrm{red}})$ set-theoretically. We also immediately see $X = F^\flat$.

If F_{red} is singular of type I in p, the map g is given by $(x_1, \dots, x_{n+1}) \mapsto x_1^{k_1}x_2^{k_2}$. In that case

$$\mathscr{H}^2_p = \mathscr{O}_{N,p}/(x_1^{k_1-1}x_2^{k_2}, x_1^{k_1}x_2^{k_2-1}) = \mathscr{O}_{N,p}/(x_1, x_2) \cdot (x_1^{k_1-1}x_2^{k_2-1}).$$

Thus $\mathscr{T}_p = \mathscr{O}_{N,p}/(x_1, x_2)$, which is supported on $\mathrm{Sing}(F_{\mathrm{red}})$ with multiplicity 1.

If p is of type II, the map g is given by $(x_1, \dots, x_{n+1}) \mapsto (x_1^2 - x_2^3)^k$. So

$$\mathscr{H}^2_p = \mathscr{O}_{N,p}/(x_1(x_1^2 - x_2^3)^{k-1}, x_2^2(x_1^2 - x_2^3)^{k-1}) = \mathscr{O}_{N,p}/(x_1, x_2^2) \cdot ((x_1^2 - x_2^3)^{k-1}),$$

which shows that the multiplicity of $\mathscr{T}_p$ is 2.

If p is of type III, the irreducible components of F have the same multiplicity according to Corollary 1.17. The map g can be modeled by $(x_1, \dots, x_{n+1}) \mapsto (x_1^4 - x_2^2)^k$, so that

$$\mathscr{H}_p^2 = \mathscr{O}_{N,p}/(x^3, y) \cdot ((x^4 - y^2)^{k-1}),$$

and the multiplicity of $\mathscr{T}_p$ is 3.

If p is of type IV, the map g is given by $(x_1, \dots, x_{n+1}) \mapsto (x_1(x_1 - x_2)x_2)^k$. So

$$\mathscr{H}_p^2 = \mathscr{O}_{N,p}/\big((x_1(x_1 - x_2)x_2)^{k-1}\big) \cdot \big((2x_1 - x_2)x_2, x_1(x_1 - 2x_2)\big)$$

and the multiplicity of $\mathscr{T}_p$ is 4. □

Corollary 2.11. *If $F = n_0 F_0$ is an irreducible multiple fiber with smooth reduction F_0, then $w = 0$.*

Proof. As $\operatorname{Sing}(F_{\mathrm{red}}) = \operatorname{Sing}(F_0) = \emptyset$, the sheaf $\mathscr{T}$ is zero. Since we can always move F to another fiber, $(n_0 F_0)^2 = F^2 = 0$. So $X = F^\flat$ can at most self-intersect in a torsion class, so that $\int_N c_2(\mathscr{H}^2)\omega^{n-1} = \int_N c_1(\mathscr{O}(F^\flat))^2 \omega^{n-1} = 0$. □

Proposition 2.12. *If $Z_1, Z_2, \dots, Z_\ell$ are the components of $\operatorname{Sing}(F_{\mathrm{red}})$, with associated Milnor numbers $\mu_1, \dots, \mu_\ell$, then*

$$\int_N c_2(\mathscr{T})\omega^{n-1} = -\sum_k \mu_k \int_{Z_k} \omega^{n-1}.$$

Proof. Proposition 2.10 gives an equality of cycles

$$\operatorname{Supp} \mathscr{T} = \sum \mu_k Z_k.$$

Furthermore, $\mathscr{T}$ is a line bundle on its support. Indeed, the short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_N(-X) & \longrightarrow & \mathscr{O}_N & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \twoheaddownarrow & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{H}^2 & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0
\end{array}$$

provide a surjection $\mathscr{O}_N(-X) \to \mathscr{T}$. By Riemann-Roch without denominators[9, Thm 15.3],

$$c_2(\mathscr{T}) = -\sum_k \mu_k Z_k.$$ □

According to Lemma 2.8 and Proposition 2.10, $c_2(\mathscr{O}_X)$ is given by the self-intersection $F^\flat \cdot F^\flat$. The following lemma reduces this to the intersection of different components.

Lemma 2.13. *The self-intersection of the diminished fiber $F^\flat = \sum_i (n_i - 1)F_i$ is*

$$F^\flat \cdot F^\flat = -\sum_{i<j} \frac{(n_i - n_j)^2}{n_i n_j} F_i \cdot F_j.$$

Proof. Note that the full fiber F can always be moved away to another fiber, which implies $F \cdot F_i = 0$ for each i. Using this we obtain

$$F_i^2 = F_i \cdot \left(F - \sum_{j \neq i} n_j F_j - (n_i - 1) F_i \right) = -\sum_{j \neq i} n_j (F_i \cdot F_j) - (n_i - 1) F_i^2,$$

so that

$$F_i^2 = -\frac{1}{n_i} \sum_{j \neq i} n_j (F_i \cdot F_j). \tag{14}$$

For the diminished fiber we have

$$\begin{aligned} (F^\flat)^2 &= (F - F_{\mathrm{red}})^2 = (F_{\mathrm{red}})^2 \\ &= \sum_{i \neq j} F_i \cdot F_j + \sum_i F_i^2 \\ &\overset{(14)}{=} \sum_{i \neq j} \left(1 - \frac{n_j}{n_i}\right) F_i \cdot F_j = -\sum_{i<j} \frac{(n_i - n_j)^2}{n_i n_j} F_i \cdot F_j. \end{aligned}$$

□

Taking everything together we obtain the following theorem.

Theorem 2.14. *If $Z_1, \dots, Z_\ell \subset F$ are the singular tori with associated Milnor numbers $\mu_1, \dots, \mu_\ell$, then*

$$w = \left(\int_F \omega^n \right)^{-1} \cdot n \cdot \sum_{k=1}^{\ell} a_k \int_{Z_k} \omega^{n-1},$$

where

$$a_k = \begin{cases} 1 + \frac{(n_i - n_j)^2}{n_i n_j} & \text{if } Z_k \text{ is an intersection of type } I \text{ with local multiplicities } n_i, n_j \\ \mu_k & \text{if } Z_k \text{ is of type } II, III \text{ or } IV. \end{cases}$$

Proof. By Lemma 2.8, Proposition 2.9 and Proposition 2.10 one has

$$\int_N \left(c_2(\mathscr{H}^1) - c_2(\mathscr{H}^2) \right) \omega^{n-1} = -\int_N c_2(\mathscr{H}^2) \omega^{n-1} = -\int_N c_2(\mathscr{T}) \omega^{n-1} - \int_N (F^\flat)^2 \omega^{n-1}.$$

By Proposition 2.12,

$$-\int_N c_2(\mathscr{T}) \omega^{n-1} = \sum_{k=1}^{\ell} \mu_k \int_{Z_k} \omega^{n-1}. \tag{15}$$

Suppose first that all irreducible components have the same multiplicity. Then $(F^\flat)^2 = 0$ by Lemma 2.13. Furthermore $a_k = \mu_k$ (including the case $\mu_k = 1$ for type *I*), and so (15) proves the claim. This case occurs whenever F is irreducible (in particular type *II*), and by Corollary 1.17 it also applies to singularities of type *III* and *IV*.

It remains to consider the case where F is reducible, and the singularities are of type *I*. In that case, every Z_k is contained in the intersection of two different components of F. By Lemma 2.13,

$$-\int_N (F^\flat)^2 \omega^{n-1} = \sum_{k=1}^{\ell} (a_k - 1) \int_{Z_k} \omega^{n-1}.$$

Since $\mu_k = 1$, combining this with (15) concludes the proof. □

The numbers a_k may seem a bit strange. For extended ADE-graphs, the sum $\sum_k a_k$ has a natural graph-theoretic description, which connects the weights to the Euler characteristic of the characteristic cycle. For now, let's only verify this for elliptic fibrations of surfaces. This line of thought will be continued in section 4.

Proposition 2.15. *Let $\Gamma = (V, E)$ be an extended ADE-graph, i.e. of type $\tilde{A}_n$ ($n \in \mathbb{N}$), $\tilde{D}_n$ ($n \geq 4$) or $\tilde{E}_n$ ($6 \leq n \leq 8$). For each edge $e \in E$, define $a_e := 1 + \frac{(m-m')^2}{mm'}$, where $m, m' \in \mathbb{N}$ are the multiplicities of the two vertices incident to e. Then*

$$\sum_{e \in E} a_e = 2|V| - |E|.$$

Proof. Choose a numbering $V = \{v_1, \dots, v_{n+1}\}$, and let $Q = (q_{ij})$ be the symmetric negative semi-definite bilinear form given by

$$q_{ij} = \begin{cases} -2 & \text{if} \quad i = j, \\ 1 & \text{if} \quad (v_i, v_j) \in E, \\ 0 & \text{else.} \end{cases}$$

Let $m = (m_1, \dots, m_{n+1}) \in \mathbb{Q}^{n+1}$ be the vector whose entries are the multiplicities associated to the vertices of Γ. It is well-known that m generates the annihilator of Q[3, (2.12)], so for any vector $y \in \mathbb{Q}^{n+1}$,

$$0 = Q(m, y) = \sum_{i,j} q_{ij} m_i y_j.$$

Using this, calculate

$$\begin{aligned} \sum_{e \in E} a_e &= \sum_{i<j} q_{ij} \left(\frac{(m_i - m_j)^2}{m_i m_j} + 1 \right) = \sum_{i<j} q_{ij} \left(\frac{m_i}{m_j} + \frac{m_j}{m_i} - 1 \right) \\ &= \sum_{i,j} q_{ij} \frac{m_i}{m_j} - \sum_{i=1}^{n+1} q_{ii} - \sum_{i<j} q_{ij} = 2|V| - |E|. \end{aligned}$$ □

Corollary 2.16. *If $S \to \mathbb{P}^1$ is an elliptic K3 surface with singular fibers $F_1, \dots, F_n$, then the weights $w_1, \dots, w_n$ are exactly*

$$w_i = \chi(F_i) \quad (i = 1, \dots, n).$$

Proof. Since a general fiber F_{sm} is an elliptic curves, one has $\int_{F_{\mathrm{sm}}} \omega = 1$. Fix one singular fiber F. The "singular tori" are just the singular points of F_{red}, so $\int_{Z_k} \omega^0 = 1$. Hence Theorem 2.14 reduces to

$$w = \sum_k a_k,$$

where k runs through all singular points of F. If F has type I_b ($b \geq 1$), I_b^* ($b \geq 0$), II^*, III^* or IV^*, the dual graph $\Gamma = (V, E)$ of F is of extended ADE-type, so Proposition 2.15 applies. Since the components of F are smooth rational curves which intersect transversally, $\sum_k a_k = 2|V| - |E| = \chi(F)$.

If F has type II, III or IV, one checks by hand that $\chi(F) = \mu$, where μ is the Milnor number of the plane curve singularity (cf. Table 1). □

3 Independence of the weights from the choice of the Kähler form

Fix a component $\Delta_i \subset \Delta$, a general singular fiber $F \subset f^{-1}(\Delta_i)$, and let $w = w_i$ be the associated weight as defined in Definition 2.5. Then w seemingly depends on the choice of the Kähler form ω. If $\tilde{\omega}$ is any other Kähler class, which restricts to an indivisible integral class on a smooth fiber F_{sm}, then $\omega - \tilde{\omega}$ is contained in the kernel of $H^2(M, \mathbb{R}) \to H^2(F_{\mathrm{sm}}, \mathbb{R})$. Using the description of w from Theorem 2.14, we see that it suffices to show for each singular torus $Z \subset F$ the equality

$$\operatorname{Ker}\big(H^2(M,\mathbb{R}) \to H^2(Z,\mathbb{R})\big) = \operatorname{Ker}\big(H^2(M,\mathbb{R}) \to H^2(F_{\mathrm{sm}},\mathbb{R})\big).$$

This section presents two approaches to this problem. The first is a pretty short and conceptual argument involving the Beauville-Bogomolov form, closely following work by Matsushita[24]. However, this technique only applies if Z is the intersection of two irreducible components of F.

To treat the general case, section 3.2 show, that the inclusion map $Z \to M$ is, up to a covering of Z, homotopic to a C^∞-map $Z \to F_{\mathrm{sm}}$. This implies that the restriction $H^2(M, \mathbb{Q}) \to H^2(Z, \mathbb{Q})$ factor over $H^2(F_{\mathrm{sm}}, \mathbb{Q})$.

3.1 Independence à la Matsushita

As before, let $H = f^*\mathcal{O}(1)$ be the pull-back of the hyperplane bundle on $\mathbb{P}^n$, and let q be the Beauville-Bogomolov form on $H^2(M, \mathbb{C})$. Let $F_{\mathrm{sm}} \subset M$ be a smooth fiber. Matsushita proved the following lemma.

Lemma 3.1 ([24, Lemma 2.2])**.** *If $\alpha \in H^2(M, \mathbb{C})$ is such that $q(\alpha, H) = 0$, then $\alpha|_{F_{\mathrm{sm}}} = 0$.*

As not all classes of $H^2(M, \mathbb{C})$ restrict to zero on F_{sm}, the lemma immediately implies

$$H^\perp = \{\, \alpha \in H^2(M,\mathbb{C}) \,|\, q(\alpha, H) = 0 \,\} = \operatorname{Ker}\big(H^2(M,\mathbb{C}) \to H^2(F_{\mathrm{sm}},\mathbb{C})\big). \qquad (16)$$

Lemma 3.2. *Let $f: X \to D$ be a proper morphism of irreducible varieties (or irreducible complex spaces). If $F = f^{-1}(b)$ is a general fiber, then its irreducible components $F_1, \dots, F_k \subset F$ are algebraically equivalent in X.*

Proof. Consider the normalization

$$\nu: \tilde{X} \to X.$$

Since F is a general fiber, the preimage $\nu^{-1}(F)$ is the normalization of F by Proposition 0.6, so

$$\nu^{-1}(F) = \tilde{F} = \tilde{F}_1 \sqcup \dots \sqcup \tilde{F}_k,$$

where $\tilde{F}_i$ denotes the normalization of F_i. By Stein factorization, the composite map $f \circ \nu: \tilde{X} \to D$ factors over a morphism $g: \tilde{X} \to D'$ with connected fibers and a finite morphism $\pi\colon D' \to D$. Then $\pi^{-1}(b) = \{b_1, \dots, b_k\}$ with $g^{-1}(b_i) = \tilde{F}_i$. Because D' is connected, the $\tilde{F}_1, \dots, \tilde{F}_k$ are algebraically equivalent in $\tilde{X}$, and by proper push-forward their images $F_1, \dots, F_k$ are algebraically equivalent in X. □

Lemma 3.3. *For every irreducible component $F_0 \subset F$ there exists an irreducible, vertical effective divisor $X_0 \subset M$ which dominates the connected component of $\Delta \subset \mathbb{P}^n$ over which F lies, such that for some number $k \in \mathbb{N}$,*

$$kF_0 = X_0 \cdot H^{n-1} \in H^{2n}(M, \mathbb{Q}).$$

Proof. Let $D \subset \Delta$ be the irreducible component of the discriminant divisor which contains $f(F)$. The preimage $f^{-1}(D)$ is the union of vertical effective divisors $X_0, \dots, X_k$ which dominate D. One of the components contains F_0, say $F_0 \subset X_0$. Consider the preimage N of a general line $\ell \subset \mathbb{P}^n$ through $f(F)$. Then N intersects X_0 transversely, so that

$$X_0 \cdot H^{n-1} = X_0 \cdot N = X_0 \cap N.$$

The intersection $X_0 \cap N$ is supported over $D \cap \ell$, so it consists of $\deg D$ cycles, all algebraically equivalent to $X_0 \cap F$, so that $X_0 \cdot H^{n-1} = k_0[X_0 \cap F]$. The intersection $X_0 \cap F$ is the union of some irreducible components of F, and by Lemma 3.2, each component is algebraically equivalent to F_0 in X_0, and hence also in M. □

Proposition 3.4. *Let $Z \subset F$ be a singular torus. If set-theoretically $Z = F_1 \cap F_2$ for two irreducible components $F_1 \neq F_2$ of F, then*

$$\operatorname{Ker}\left(H^2(M, \mathbb{C}) \to H^2(Z, \mathbb{C})\right) = H^{\perp}.$$

Proof. The proof closely follows Matsushita's proof of Lemma 3.1.

As $H^{\perp} \subset H^2(M, \mathbb{C})$ has codimension 1, and $[\omega|_Z] \neq 0$, it is sufficient to show $H^{\perp} \subset \operatorname{Ker}(H^2(M, \mathbb{C}) \to H^2(Z, \mathbb{C}))$.

Suppose $\alpha \in H^{\perp}$. As q is up to a constant defined over $\mathbb{R}$, we may assume that α is a real class. Choose divisors $X_1, X_2 \subset M$ with $k_i[F_i] = [X_i][H]^{n-1}$. In $H^*(N, \mathbb{C})$ one has $k_0 Z = F_1 \cdot F_2$ for some $k_0 \in \mathbb{N}$, so that

$$k_0 k_1 k_2 Z = X_1 \cdot X_2 \cdot H^{n-1}. \tag{17}$$

Since f is Lagrangian, the symplectic form $\sigma \in H^2(M, \mathbb{C})$ restricts to zero on Z. Hence $\alpha|_Z \in H^{1,1}(Z)$. On the primitive cohomology

$$H^{1,1}(Z)_{\text{prim}} = \left\{ \alpha \in H^{1,1}(Z) \mid \int_Z \alpha \omega|_Z^{n-2} = 0 \right\},$$

there is a negative definite hermitian form[33, Theorem 6.32]

$$(\alpha, \beta) := \int_Z \alpha \bar{\beta} \omega|_Z^{n-3}.$$

So to prove that α restricts to zero on Z, it is sufficient to show that $\alpha|_Z$ is primitive, and that $(\alpha|_Z, \alpha|_Z) = 0$. Since α is real, the two conditions translate by (17) into

$$\int_M \alpha\omega^{n-2}H^{n-1}X_1X_2 = 0 \quad \text{and} \quad \int_M \alpha^2\omega^{n-3}H^{n-1}X_1X_2 = 0. \tag{18}$$

By Fujiki, there is a constant $c_M \in \mathbb{R}$, such that

$$\int_M (\alpha + s\omega + tH + uX_1 + vX_2)^n = c_M \cdot q(\alpha + s\omega + tH + uX_1 + vX_2)^n, \tag{19}$$

where s, t, u and v are indeterminates. Since X_1, X_2 and H are vertical divisors, their classes vanish when restricted to F_{sm}, so that by (16), $q(X_i, H) = 0$ and $q(H, H) = 0$. Expanding q on the right hand side of (19) gives

$$\begin{aligned} &q(\alpha + s\omega + tH + uX_1 + vX_2)^n \\ &\quad = \big(q(\alpha) + s^2q(\omega) + u^2q(X_1) + v^2q(X_2) \\ &\qquad + 2sq(\alpha,\omega) + 2uq(\alpha, X_1) + 2vq(\alpha, X_2) \\ &\qquad + 2stq(\omega, H) + 2suq(\omega, X_1) + 2svq(\omega, X_2) + 2uvq(X_1, X_2)\big)^n. \end{aligned}$$

As t only ever appears together with s, the coefficients in front of $s^{n-2}t^{n-1}uv$ and $s^{n-3}t^{n-1}uv$ vanish, which on the left hand side of (19) gives the equalities (18). □

3.2 Moving singular tori to smooth fibers

Let Z be a singular torus in a general singular fiber F. To prove

$$\operatorname{Ker}\big(H^2(M,\mathbb{C}) \to H^2(Z,\mathbb{C})\big) = \operatorname{Ker}\big(H^2(M,\mathbb{C}) \to H^2(F_{\mathrm{sm}},\mathbb{C})\big), \tag{20}$$

this section uses the Hamiltonian vector fields to constructs a homotopy

$$\mathscr{H} : [0,1] \times (S^1)^{2n-2} \to M,$$

which for $t = 0$ gives a topological covering of Z and for $t = 1$ maps $(S^1)^{2n-2}$ into a smooth fiber of f. Since topological coverings of tori induce isomorphisms in cohomology with rational coefficients, $\mathscr{H}$ induces a commutative diagram

$$\begin{array}{ccccc} & & H^2(Z,\mathbb{Q}) & & \\ & \nearrow & & \searrow^{\mathscr{H}_0^*}_{\cong} & \\ H^2(M,\mathbb{Q}) & & & & H^2((S^1)^{2n-2},\mathbb{Q}). \\ & \searrow & & \nearrow_{\mathscr{H}_1^*} & \\ & & H^2(F_{\mathrm{sm}},\mathbb{Q}) & & \end{array}$$

This proves the ”⊃” part of (20). Equality then follows because the right hand side is a hyperplane in $H^2(M,\mathbb{C})$, and restriction to Z is not the zero map.

3.2.1 Recurrence automorphisms

Let $z_1, \dots, z_n$ be local coordinates on $\mathbb{P}^n$ around $b \in \ell \cap \Delta_i$ with $F = f^{-1}(b)$, such that the line $\ell \subset \mathbb{P}^n$ is locally cut out by $z_1, \dots, z_{n-1}$. Choose a complex disc $D \subset \ell$ around b on which z_n is a coordinate, and set $N_D = f^{-1}(D)$. The Hamiltonian vector fields $X_i = \hat{\sigma}^{-1}(f^* dz_i)$, $(i = 1, \dots, n-1)$ induce a flow

$$\Phi \colon \mathbb{C}^{n-1} \times N_D \to N_D, \quad (t, q) \mapsto \Phi(t, q),$$

which preserves the fibers of $g \colon N_D \to D$ and which also preserves the singular torus $Z \subset F$. Choose a point $p \in Z$. By applying Φ to p one obtains the universal covering $\nu \colon \mathbb{C}^{n-1} \to Z$, with associated period lattice $\Lambda = \nu^{-1}(p) \subset \mathbb{C}^{n-1}$.

Let $D^2 = D^2_\varepsilon \subset N_D$ be a two-dimensional complex polydisc centered at p, which intersects Z transversely. The intersection of D^2 with F is of one of the types *I-IV* described in Table 1. So we may choose local functions $x, y \in \mathfrak{m}_p \setminus \mathfrak{m}_p^2 \subset \mathcal{O}_{N,p}$, such that, depending on the type of Z,

$$g = x^k y^\ell, \qquad g = (x^3 - y^2)^k, \qquad g = (y(y - x^2))^k \qquad \text{or} \qquad g = (xy(x - y))^k.$$

Then x and y form a local coordinate system of D^2 at p, and shrinking D^2 if necessary we may assume that x, y are coordinates on the full disc.

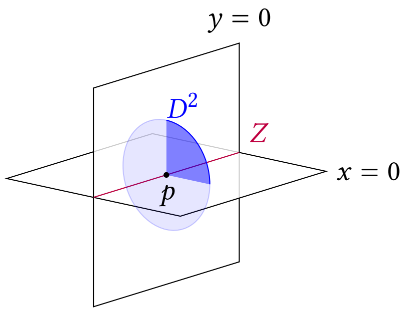


Figure 1: The local model for type *I* around a point p

Lemma 3.5. *For every period $\lambda \in \Lambda$, there exists a unique holomorphic map of germs $\tilde{\lambda} \colon (D^2, 0) \to (\mathbb{C}^{n-1}, \lambda)$ which satisfies*

$$\Phi\big(\tilde{\lambda}(q), q\big) \in D^2$$

for all $q \in D^2$ for which $\tilde{\lambda}(q)$ is defined.

Proof. Complete x, y to a local coordinate system $x, y, z_1, \dots, z_{n-1}$ of N at p in such a way, that $D^2 = \{ z_1 = \cdots = z_{n-1} = 0 \}$. Consider $z = (z_1, \dots, z_{n-1})$ as a map of germs $(N, p) \to (\mathbb{C}^{n-1}, 0)$. Since the X_i are transversal to D^2 at p, the differential matrix

$$D_\lambda(z \circ \Phi|_{\mathbb{C}^{n-1} \times \{p\}}) = \left(\frac{\partial (z_i \circ \Phi)}{\partial \lambda_j}(\lambda, p) \right)_{i,j=1,\dots,n-1} \in \mathbb{C}^{(n-1)\times(n-1)}$$

is invertible. By the theorem on implicit functions there exists a unique $\tilde{\lambda}$ with $\tilde{\lambda}(p) = \lambda$ and $z(\Phi(\tilde{\lambda}(q), q)) = 0$. □

Definition 3.6. For every period $\lambda \in \Lambda$, the automorphism $\hat{\lambda}\colon (D^2, 0) \to (D^2, 0)$ definied by $\hat{\lambda}(q) = \Phi\big(\tilde{\lambda}(q), q\big)$ is called the *recurrence automorphism* corresponding to λ.

Lemma 3.7.

(a) *For $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_3 = \lambda_1 + \lambda_2$ one has*

$$\tilde{\lambda}_3(q) = \tilde{\lambda}_1(\hat{\lambda}_2(q)) + \tilde{\lambda}_2(q).$$

(b) *The assignment $\Lambda \ni \lambda \mapsto \hat{\lambda} \in \operatorname{Aut}(D^2, 0)$ defines a group action of Λ on $(D^2, 0)$.*

(c) *The function g is invariant under this group action.*

Proof. Calculate

$$\begin{aligned} D^2 \ni \hat{\lambda}_1(\hat{\lambda}_2(q)) &= \Phi\big(\tilde{\lambda}_1(\hat{\lambda}_2(q)), \hat{\lambda}_2(q)\big) \\ &= \Phi\big(\tilde{\lambda}_1(\hat{\lambda}_2(q)), \Phi(\tilde{\lambda}_2(q), q)\big) \\ &= \Phi\big(\tilde{\lambda}_1(\hat{\lambda}_2(q)) + \tilde{\lambda}_2(q), q\big). \end{aligned}$$

Since $\tilde{\lambda}_1(\hat{\lambda}_2(0)) + \tilde{\lambda}_2(0) = \lambda_3$, the uniqueness from Lemma 3.5 implies

$$\tilde{\lambda}_3(q) = \tilde{\lambda}_1(\hat{\lambda}_2(q)) + \tilde{\lambda}_2(q)$$

for all q. Hence $\hat{\lambda}_1(\hat{\lambda}_2(q)) = \Phi(\tilde{\lambda}_3(q), q) = \hat{\lambda}_3(q)$. As the group action is defined by flowing along the Hamiltonian vector fields, it is clear that g is invariant under this group action. □

3.2.2 Recurrence homotopies

Let $\lambda \in \Lambda$ be a period. As $\hat{\lambda}$ is a map of germs, it defines a holomorphic map $\hat{\lambda}\colon D^2_\delta \to D^2_\varepsilon = D^2$ for some $\delta \leq \varepsilon$. This section aims to construct a homotopy

$$H\colon [0,1] \times D^2_\delta \to D^2_\varepsilon,$$

from $H_0 = \hat{\lambda}$ to the inclusion map $H_1 = (D^2_\delta \subset D^2_\varepsilon)$. This homotopy is supposed to be compatible with g, in the sense that $g \circ H_t = g$ for all $t \in [0,1]$. It turns out that constructing H is only possible if we replace λ by a multiple $d \cdot \lambda$, and this will be done freely.

Consider the action of $\hat{\lambda}$ on function germs

$$\hat{\lambda}^*\colon \mathbb{C}\{x,y\} \to \mathbb{C}\{x,y\}, \quad x \mapsto \xi, y \mapsto \eta.$$

Since $\hat{\lambda}$ commutes with g, one has

$$g(\xi, \eta) = g(x, y). \tag{21}$$

If g is a k-th power, say $g = h^k$, then replacing λ by $k\lambda$ one may always assume

$$h(\xi, \eta) = h(x, y). \tag{22}$$

In the following, the homotopy H will be constructed as a continuous family of automorphisms

$$H_t : x \mapsto \xi_t, y \mapsto \eta_t, \quad 0 \le t \le 1,$$

with $H_0 = \hat{\lambda}^*$ and $H_1 = \mathrm{id}$. This is done for each of the cases *I-IV* individually.

Case *I* We have $h(x, y) = x^k y^\ell$, with k and ℓ coprime. Then (22) reads

$$x^k y^\ell = \xi^k \eta^\ell. \tag{23}$$

As $\mathbb{C}\{x, y\}$ is a unique factorization domain, x divides either ξ or η. If $\ell \ne k$, then clearly x divides ξ and y divides η. If $k = \ell = 1$ it could happen that x divides η, in which case we replace λ by 2λ, which allows us to assume that x divides ξ.

For $0 \le t \le \frac{1}{2}$ and $s = 1 - 2t$ set

$$\xi_t = \frac{\xi(sx, sy)}{s} \quad \text{and} \quad \eta_t = \frac{\eta(sx, sy)}{s}. \tag{24}$$

Note that since x divides ξ and y divides η, the denominator cancels and this is well-defined even for $t = \frac{1}{2}$. Then

$$\xi_t^k \eta_t^\ell = \frac{1}{s^{k+\ell}} \big(\xi(sx, sy)^k \eta(sx, sy)^\ell\big) = \frac{1}{s^{k+\ell}} (sx)^k (sy)^\ell = x^k y^\ell,$$

so that the automorphism $H_t : x \mapsto \xi_t, y \mapsto \eta_t$ is compatible with g. For $t = \frac{1}{2}$, we get $\xi_{1/2} = ax$ and $\eta_{1/2} = by$ for some $a, b \in \mathbb{C}$ with

$$a^k b^\ell = 1.$$

Since k and ℓ are coprime, the set $\{\, \alpha^k \beta^\ell = 1 \,\} \subset \mathbb{C}^2$ is connected. Choose a path

$$[0, 1] \ni t \mapsto \big(a(t), b(t)\big) \in \{\, \alpha^k \beta^\ell = 1 \,\}, \quad 0 \le t \le 1$$

from (a, b) to $(1, 1)$, and set the second half of H as

$$H_t(x, y) = \big(a(2t - 1)x, b(2t - 1)y\big), \quad \frac{1}{2} \le t \le 1.$$

Case *II* In this case one has $h(x, y) = x^3 - y^2$, and so

$$x^3 - y^2 = \xi^3 - \eta^2.$$

Equivalently,

$$(\xi - x)(\xi - e^{\frac{2\pi i}{3}} x)(\xi - e^{\frac{4\pi i}{3}} x) = (\eta - y)(\eta + y). \tag{25}$$

Hence $(\eta - y)(\eta + y) \in \mathfrak{m}^3$, so that either

$$\eta = y + \text{terms of order} \geq 2 \quad \text{or} \quad \eta = -y + \text{terms of order} \geq 2.$$

Replacing λ by 2λ if necessary we may assume the former. Hence (25) vanishes of order ≥ 4 modulo y, which means that

$$\xi = e^{\frac{2\pi i d}{3}} x + ay + \text{terms of order} \geq 2$$

for some $d \in \{0, 1, 2\}$ and $a \in \mathbb{C}$. Replacing λ by 3λ if necessary we may assume $d = 0$. For $t \in [0, 1]$ and $s = 1 - t$ set

$$\xi_t = \frac{\xi(s^2 x, s^3 y)}{s^2} \quad \text{and} \quad \eta_t = \frac{\eta(s^2 x, s^3 y)}{s^3}, \tag{26}$$

so that $\xi_0 = \xi$ and $\eta_0 = \eta$. Note that

$$\xi_t = x + asy + s \cdot (\text{terms of order } \geq 2) \xrightarrow{t \to 1} x,$$

so that $\xi_1 = x$. Similarly, $\eta_1 = y$. Furthermore,

$$\xi_t^3 - \eta_t^2 = \frac{1}{s^6}\big(\xi(s^2 x, s^3 y)^3 - \eta(s^2 x, s^3 y)^2\big) = \frac{1}{s^6}\big(s^6 x^3 - s^6 y^2\big) = x^3 - y^2,$$

which means that the automorphism $x \mapsto \xi_t, y \mapsto \eta_t$ is compatible with g.

Case *III* In this case one has $h = y(y - x^2)$, so that

$$y(y - x^2) = \eta(\eta - \xi^2). \tag{27}$$

Thus y divides either η or $\eta - \xi^2$. Replacing λ by 2λ we may assume that y divides η and so

$$\eta = bv(x, y)y$$

for some $b \in \mathbb{C}$ and a power series v with $v(0, 0) = 1$. Plugging this into (27) one sees $b^2 = 1$. Replacing λ by 2λ we may assume $b = 1$. Note that the linear term of ξ is of the form $ax + cy$ with $a \neq 0$. Then (27) implies $a^2 = 1$, and replacing λ by 2λ we may assume $a = 1$. For $t \in [0, 1]$ and $s = 1 - t$ set

$$\xi_t = \frac{\xi(sx, s^2 y)}{s} \quad \text{and} \quad \eta_t = \frac{\eta(sx, s^2 y)}{s^2}, \tag{28}$$

so that $\xi_0 = \xi$ and $\eta_0 = \eta$. Note that

$$\eta_t = v(sx, s^2y) \cdot y \xrightarrow{t \to 1} y$$

and

$$\xi_t = x + scy + s \cdot (\text{terms of order } \geq 2) \xrightarrow{t \to 1} x,$$

which means $\xi_1 = x$ and $\eta_1 = y$. Furthermore,

$$\eta_t(\eta_t - \xi_t^2) = \frac{1}{s^4}\eta(sx, s^2y)(\eta(sx, s^2y) - \xi(sx, s^2y)^2) = y(y - x^2),$$

so that the homotopy $H_t(x, y) = (\xi_t, \eta_t)$ preserves g.

Case *IV* In this case one has $h = xy(x - y)$, so that

$$xy(x - y) = \xi\eta(\xi - \eta).$$

Replacing λ by a suitable multiple one may assume that x divides ξ and y divides η. For $t \leq \frac{1}{2}$ and $s = 1 - 2t$ set

$$\xi_t(x, y) = \frac{\xi(sx, sy)}{s} \quad \text{and} \quad \eta_t(x, y) = \frac{\eta(sx, sy)}{s}, \tag{29}$$

so similarly to the cases $I - III$ one has $\xi_t\eta_t(\xi_t - \eta_t) = xy(x - y)$. For $t = \frac{1}{2}$ we get

$$\xi_{1/2} = ax \quad \text{and} \quad \eta_{1/2} = by,$$

for $a, b \in \mathbb{C}$ satisfying $ab(a - b) = 1$. Since the algebraic curve

$$C = \{\, (\alpha, \beta) \,|\, \alpha\beta(\alpha - \beta) = 1 \,\} \subset \mathbb{C}^2$$

is connected, one may choose a path $[0, 1] \ni t \mapsto (a(t), b(t)) \in C$ connecting (a, b) with $(1, 1)$. Set for $t \geq \frac{1}{2}$

$$\xi_t(x, y) = a(2t - 1) \cdot x \quad \text{and} \quad \eta_t(x, y) = b(2t - 1) \cdot y. \tag{30}$$

Definition 3.8. The homotopy H_t is called *recurrence homotopy* for $d\lambda$.

With the help of a recurrence homotopy, one can easily move a 1-cycle represented by a period $\lambda \in \Lambda$ from Z to a smooth fiber. Suppose λ admits a recurrence homotopy H. Choose a path $\gamma(s) \in D^2$ which goes from the origin to a smooth fiber, and write down a singular square

$$\sigma\colon [0, 1]^2 \to M, \quad \sigma(s, t) = \begin{cases} \Phi(2t\tilde{\lambda}(\gamma(s)), \gamma(s)) & \text{for } t \leq \frac{1}{2} \\ H_{2t-1}(\hat{\lambda}(\gamma(s))) & \text{for } t \geq \frac{1}{2}, \end{cases}$$

To move higher-dimensional cycles, the following lemma is needed.

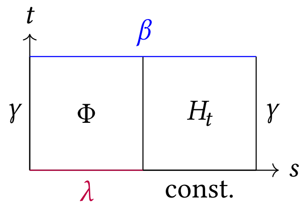


Figure 2: Singular square to move a 1-cycle

Lemma 3.9. *If $\lambda^1, \lambda^2 \in \Lambda$ are two periods for which recurrence homotopies H^1, H^2 exist, then the maps H^1_t and H^2_t commute for each t.*

Proof. Write $\hat{\lambda}^i(x,y) = (\xi^i(x,y), \eta^i(x,y))$, and $H^i_t = (\xi^i_t, \eta^i_t)$. For $t = 0$, the recurrence automorphisms $\hat{\lambda}^1$ and $\hat{\lambda}^2$ commute by Lemma 3.7, so that

$$\begin{cases} \xi^1(\xi^2(x,y), \eta^2(x,y)) = \xi^2(\xi^1(x,y), \eta^1(x,y)) \\ \eta^1(\xi^2(x,y), \eta^2(x,y)) = \eta^2(\xi^1(x,y), \eta^1(x,y)) \end{cases} \tag{31}$$

holds. Suppose for now that Z is of type *I*. Then by (24) one has

$$\begin{aligned} \xi^1_t(\xi^2_t(x,y), \eta^2_t(x,y)) &= \frac{1}{s}\xi^1(\xi^2(sx,sy), \eta^2(sx,sy)) \\ &\overset{(31)}{=} \frac{1}{s}\xi^2(\xi^1(sx,sy), \eta^1(sx,sy)) \\ &= \xi^2_t(\xi^1_t(x,y), \eta^1_t(x,y)). \end{aligned}$$

Similarly

$$\eta^1_t(\xi^2_t(x,y), \eta^2_t(x,y)) = \eta^2_t(\xi^1_t(x,y), \eta^2_t(x,y)),$$

which means that H^1_t and H^2_t commute for $t \leq \frac{1}{2}$. For $t \geq \frac{1}{2}$, the maps H^1_t and H^2_t multiply the coordinates by the $a^i(t), b^i(t)$, so they commute.

The cases *II-IV* work the same, replacing the use of (24) by (26), (28) or (29). □

3.2.3 Construction of the homotopy $[0,1] \times (S^1)^{2n-2} \to M$

Let $\frac{1}{d^1}\lambda^1, \frac{1}{d^2}\lambda^2, \ldots, \frac{1}{d^{2n-2}}\lambda^{2n-2} \subset \Lambda$ be a basis of Λ, such that $\lambda^1, \ldots, \lambda^{2n-2}$ admit recurrence homotopies $H^1, \ldots, H^{2n-2}$. Choose a path

$$[0,1] \ni s \mapsto \gamma(s) \in D^2$$

with $\gamma(0) = (0,0)$, such that $g \circ \gamma : [0,1] \to \mathbb{C}$ is injective. For example $s \mapsto (s, -s)$ works for small s in all cases *I-IV*. We want to define a map

$$\mathscr{C} : [0,1] \times [0,1]^{2n-2} \to N, \quad (s, t_1, \ldots, t_{2n-2}) \mapsto \mathscr{C}(s, t_1, \ldots, t_{2n-2}).$$

For $0 \le k \le 2n-2$, consider the region $0 \le t_1 \le t_2 \le \cdots \le t_k \le 1 \le t_{k+1} \le \cdots \le t_{2n-2} \le 2$ and define there

$$\mathscr{C}(s, t_1, \dots, t_{2n-2}) = \Phi\Big(\sum_{i=1}^{k} t_i \tilde{\lambda}^i\big(\hat{\lambda}^{i+1} \circ \cdots \circ \hat{\lambda}^k \circ H^{k+1}_{t_{k+1}-1} \circ \cdots \circ H^{2n-2}_{t_{2n-2}-1} \circ \gamma(s)\big), \\ H^{k+1}_{t_{k+1}-1} \circ \cdots \circ H^{2n-2}_{t_{2n-2}-1} \circ \gamma(s)\Big). \quad (32)$$

Doing this for all k gives a well-defined map on the region $0 \le t_1 \le t_2 \le \cdots \le \cdots \le t_{2n-2} \le 2$, because if $t_k = 1$, then $\hat{\lambda}^k = H^k_{t_k-1}$, and the relation

$$\Phi(\lambda + \tilde{\lambda}^k(q), q) = \Phi(\lambda, \hat{\lambda}^k(q))$$

allows us to move the last summand in (32) from the first to the second argument of Φ. Note that $\Phi(0, q) = q$, so that on the region $1 \le t_1 \le \cdots \le t_{2n-2}$, one has

$$\mathscr{C}(s, t_1, \dots, t_{2n-2}) = H^1_{t_1-1} \circ \cdots \circ H^{2n-2}_{t_{2n-2}-1} \circ \gamma(s).$$

To extend the definition of $\mathscr{C}$ to all of $[0,1] \times [0,2]^{2n-2}$, apply a permutation to the indices in (32). To verify that this is well-defined, one has to show that if $t_i = t_j$, then the definitions for $t_i \le t_j$ and $t_j \le t_i$ agree.

Suppose $t_i = t_j \le 1$. By Lemma 3.7, $\hat{\lambda}^i$ and $\hat{\lambda}^j$ commute and satisfy

$$\tilde{\lambda}^i(\hat{\lambda}^j(q)) + \tilde{\lambda}^j(q) = \tilde{\lambda}^j(\hat{\lambda}^i(q)) + \tilde{\lambda}^i(q).$$

Together those facts imply that the two definitions agree.

If $1 \le t_i = t_j$, one has to replace $H^i_{t_i-1} \circ H^j_{t_j-1}$, by $H^j_{t_j-1} \circ H^i_{t_i-1}$, which is possible by Lemma 3.9.

Example 3.10. If $n = 2$, the $t_1 t_2$-plane is divided into the six regions depicted in Figure 3. In that case,

$$\mathscr{C}(s, t_1, t_2) = \begin{cases} \Phi\big(t_1 \tilde{\lambda}_1(\hat{\lambda}_2(\gamma(s))) + t_2 \tilde{\lambda}_2(\gamma(s)), \gamma(s)\big), & 0 \le t_1 \le t_2 \le 1 \\ \Phi\big(t_1 \tilde{\lambda}_1(H^2_{t_2-1}(\gamma(s))), H^2_{t_2-1}(\gamma(s))\big), & 0 \le t_1 \le 1 \le t_2 \le 2 \\ H^1_{t_1-1}(H^2_{t_2-1}\big(\gamma(s))\big), & 1 \le t_1 \le t_2 \le 2 \\ \Phi\big(t_1 \tilde{\lambda}_1(\gamma(s)) + t_2 \tilde{\lambda}_2(\hat{\lambda}_1(\gamma(s))), \gamma(s)\big), & 0 \le t_2 \le t_1 \le 1 \\ \Phi\big(t_2 \tilde{\lambda}_2(H^1_{t_1-1}(\gamma(s))), H^1_{t_1-1}(\gamma(s))\big), & 0 \le t_2 \le 1 \le t_1 \le 2 \\ H^2_{t_2-1}\big(H^1_{t_1-1}(\gamma(s))\big), & 1 \le t_2 \le t_1 \le 2. \end{cases}$$

If one identifies $S^1 = [0,1]/\partial[0,1]$, it is clear from the construction that $\mathscr{C}$ descends to a map

$$[0,1] \times (S^1)^{2n-2} \to N,$$

which for $s = 0$ is homotopic to a finite topological covering map of Z. This proves the following proposition.

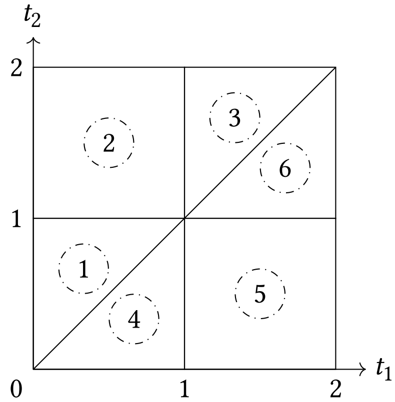


Figure 3: Division of the t_1t_2-plane for $n = 2$

Proposition 3.11. *There exists a homotopy*

$$\mathscr{H} : [0,1] \times (S^1)^{2n-2} \to M$$

such that $\mathscr{H}_0$ is a finite topological covering of Z, and $\mathscr{H}_1$ maps $(S^1)^{2n-2}$ into a smooth fiber. ☐

Theorem 3.12. *Let Z be a singular torus in a general singular fiber of $f : M \to \mathbb{P}^n$, and let F_{sm} be a smooth fiber. Then*

$$\operatorname{Ker}\big(H^k(M,\mathbb{Q}) \to H^2(Z,\mathbb{Q})\big) = \operatorname{Ker}\big(H^k(M,\mathbb{Q}) \to H^2(F_{\mathrm{sm}},\mathbb{Q})\big).$$

Proof. The homotopy $\mathscr{H}$ induces a commutative diagram

$$\begin{array}{ccc} H^k(M,\mathbb{Q}) & \longrightarrow & H^k(Z,\mathbb{Q}) \\ \big\downarrow & & \big\downarrow \cong \\ H^k(F_{\mathrm{sm}},\mathbb{Q}) & \longrightarrow & H^k((S^1)^{2n-2},\mathbb{Q}), \end{array}$$

which shows

$$\operatorname{Ker}\big(H^k(M,\mathbb{Q}) \to H^2(F_{\mathrm{sm}},\mathbb{Q})\big) \subset \operatorname{Ker}\big(H^k(M,\mathbb{Q}) \to H^2(Z,\mathbb{Q})\big).$$

As $\operatorname{Ker}\big(H^k(M,\mathbb{Q}) \to H^2(F_{\mathrm{sm}},\mathbb{Q})\big)$ has codimension one, and $H^k(M,\mathbb{Q}) \to H^k(Z,\mathbb{Q})$ is not the zero morphism, this proves the theorem. ☐

Corollary 3.13. *The definition of the weight w does not depend on the choice of the Kähler class ω.* ☐

4 Weights via Euler characteristic of the characteristic cycle

Fix an irreducible component $\Delta_i \subset \Delta$ and a general singular fiber $F \subset f^{-1}(\Delta_i)$. Let $w = w_i$ be the weight associated to Δ_i in Definition 2.5. This section relates w to the Euler characteristic of a characteristic cycle $\Theta \subset F$. The following proposition is the key technical ingredient, which allows one to apply Corollary 0.9 to the Albanese morphisms $\tilde{F}_0 \to \mathrm{Alb}(\tilde{F}_0)$ of the irreducible components $F_0 \subset F$.

Proposition 4.1. *Let $F \subset M$ be a general singular fiber of f, let $F_0 \subset F_{\mathrm{red}}$ be an irreducible component, and let $\nu \colon \tilde{F}_0 \to F_0$ be the normalization map. Every connected component Z of $\nu^{-1}(\mathrm{Sing}(F_{\mathrm{red}}))$ is a complex torus and its normal bundle $N_{Z/\tilde{F}_0}$ is numerically trivial .*

Proof. By Hwang and Oguiso, $\tilde{F}_0$ is smooth and the Albanese map $\alpha \colon \tilde{F}_0 \to \mathrm{Alb}(\tilde{F}_0)$ is a $\mathbb{P}^1$-bundle (cf. Proposition 1.14). Let $\tilde{X}_1, \dots, \tilde{X}_{n-1}$ be the lifts of the Hamiltonian vector fields $X_1, \dots, X_{n-1}$ to $\tilde{F}_0$. Similarly to Lemma 1.10, the $\tilde{X}_i$ are pointwise linearly independent and tangent to Z, and so Z is a complex torus. Choose a point $p \in Z$, and local coordinates $z_1, \dots, z_n$ of $\tilde{F}_0$ in p such that

$$\tilde{X}_i = \frac{\partial}{\partial z_i} \quad \text{for} \quad i = 1, \dots, n-1$$

and

$$\mathrm{Ker}(D\alpha) = \langle \frac{\partial}{\partial z_n} \rangle.$$

For $Y = f^{-1}(\Delta)_{\mathrm{red}}$, let $\tilde{\lambda}_Y \in H^0(\tilde{F}_0, T_{\tilde{F}_0} \otimes \nu^* \mathcal{O}(Y))$ be the lift of the characteristic vector field (cf. Proposition 1.14). Choosing a local equation of Y one trivializes $\nu^* \mathcal{O}(Y)$ and since $\tilde{\lambda}_Y$ is tangent to the fibers of α, this trivialization identifies

$$\tilde{\lambda}_Y = g \frac{\partial}{\partial z_n}$$

for some function $g \in \mathcal{O}_{\tilde{F}_0, p}$. By Proposition 1.14, g vanishes exactly along Z, so that $g = h^k$ for a local equation $h = 0$ of Z. By the construction of λ_Y as a Hamiltonian vector field, one has $0 = [\tilde{X}_i, g\frac{\partial}{\partial z_n}] = \tilde{X}_i(g)\frac{\partial}{\partial z_n}$ for $i = 1, \dots, n-1$, so that also

$$\tilde{X}_i(h) = 0. \tag{33}$$

The transition functions for the line bundle $\mathcal{O}(Z)$ are given by quotients $\varphi = \frac{h_1}{h_2}$, where h_1 and h_2 are local equations for Z. By (33) we may assume $\tilde{X}_i(h_j) = 0$, so that also $\tilde{X}_i(\varphi) = 0$. Hence the restriction $\varphi|_Z$ is locally constant, and by Proposition 0.7 the normal bundle $N_{Z/\tilde{F}_0} = \mathcal{O}(Z)|_Z$ is numerically trivial. □

Theorem 4.2. *Let F be a general singular fiber of $f \colon M \to \mathbb{P}^n$ with characteristic cycle Θ. If Θ is compact, then*

$$w = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

Proof. Let r be the multiplicity of F and write $F = r\sum_i n_i F_i$. Set $\overline{F} = \sum_i n_i F_i$, and $\Theta = r\overline{\Theta}$. Let $Z_1, \dots, Z_\ell \subset F$ be the singular tori. By Theorem 2.14 one has

$$w = \frac{n \sum_k a_k \int_{Z_k} \omega^{n-1}}{\int_F \omega^n}. \tag{34}$$

Note that the coefficients a_k are insensitive to the multiplicity r. So the numerators in both formulations do not depend on r. The denominator on the other hand satisfies $\int_F \omega^n = r \int_{\overline{F}} \omega^n$. But since also $\int_\Theta \omega = r \int_{\overline{\Theta}} \omega$ one may reduce to the case $r = 1$.

If the characteristic cycle is a smooth elliptic curve (i.e. type I_0), then the reduction of F is smooth. In that case, Corollary 2.11 tells us that $w = 0 = \chi(\Theta)$.

Suppose now that F contains an irreducible component $F_0 \subset F$ which is singular, and suppose the singularity in F is of type I.

Claim. In that case, F is irreducible and Θ has type I_b ($b \geq 1$).

Proof of Claim. Let $Z_0 \subset F_0$ be the corresponding singular torus, and let Θ_0 be a characteristic curve on F_0. Then one of the two following cases occurs.

(a) Θ_0 is a singular nodal curve, in which case $\Theta = \Theta_0$ is of type I_1, and $F = F_0$ is irreducible.

(b) Θ_0 is smooth. Then Θ_0 intersects Z_0 in two distinct points $p, q \in Z_0$. There is an element $\lambda \in \mathbb{C}^{n-1}$ such that $q = \lambda.p$. Then Θ contains all the curves $\Theta_k = \lambda^k.\Theta_0$ for $k \in \mathbb{Z}$. Since Θ is assumed to be compact, this is a finite set of characteristic curves, intersecting cyclically. By the classification of characteristic cycles, Θ has to be of type I_b ($b \geq 2$), and is completely contained in F_0. Hence $F = F_0$ is irreducible.

Now consider the normalization $\nu\colon \tilde{F} \to F$, and set $\tilde{Z} = \nu^{-1}(Z_0)$. The induced map $\tilde{Z} \to Z_0$ is étale of degree two, just as is the map $\tilde{Z} \to \operatorname{Alb}(\tilde{F})$. By Corollary 0.9,

$$\int_F \omega^n = \int_{\tilde{F}} \nu^*\omega^n = \frac{n}{2} \int_{\nu^{-1}\Theta_0} \omega \cdot \int_{\tilde{Z}} \nu^*\omega^{n-1},$$

so that

$$w = \frac{n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{\frac{n}{2} \int_{\tilde{Z}} \nu^*\omega^{n-1}}{\frac{n}{2} \int_{\nu^{-1}\Theta_0} \omega \cdot \int_{\tilde{Z}} \nu^*\omega^{n-1}} = \frac{1}{\int_{\Theta_0} \omega} = \frac{\chi(\Theta)}{\int_\Theta \omega}$$

Suppose now that the characteristic cycle Θ is of one of the types I_b ($b \geq 2$), I_b^*, II^*, III^* or IV^*, and suppose that each irreducible component $F_0 \subset F$ is smooth.

Furthermore, assume for now that each component $F_i \subset F$ contains exactly one component Θ_i of Θ. Then each singular torus $Z_k \subset F_i$ is a section of the Albanese map $F_i \to \operatorname{Alb}(F_i)$, and by Proposition 0.8

$$\int_{F_i} \omega^n = n \int_{\Theta_i} \omega \cdot \int_{Z_k} \omega^{n-1}.$$

If there is a second singular torus $Z_{k'} \subset F_i$, this implies

$$\int_{Z_k} \omega^{n-1} = \int_{Z_{k'}} \omega^{n-1}, \tag{35}$$

and since F is connected, (35) holds for arbitrary pairs (k, k'). This yields

$$w = \frac{n \sum_k a_k \int_{Z_k} \omega^{n-1}}{\sum_i n_i \int_{F_i} \omega^n} = \frac{\left(\sum_k a_k\right) \int_{Z_1} \omega^{n-1}}{\int_\Theta \omega \cdot \int_{Z_1} \omega^{n-1}} = \frac{\sum_k a_k}{\int_\Theta \omega}.$$

Since every component F_i contains exactly one component of Θ, the edges in the dual graph of Θ correspond bijectively to the singular tori. Thus Proposition 2.15 yields

$$\sum_k a_k = \chi(\Theta).$$

If some components of F contain more than one component of Θ, consider the subgroup

$$\Lambda = \{\, \lambda \mid \lambda.\Theta = \Theta \,\} \subset \mathbb{C}^{n-1}.$$

Note that Λ is a lattice. Indeed, Λ is discrete since Θ is compact, and for each component $\Theta_i \subset \Theta$ it contains the lattice $\Lambda_i = \{\, \lambda \mid \lambda.\Theta_i = \Theta_i \,\}$. If $F_i \subset F$ is the component which contains Θ_i, then $\mathrm{Alb}(F_i) \cong \mathbb{C}^{n-1}/\Lambda_i$. So if one sets

$$T = \mathbb{C}^{n-1}/\Lambda,$$

one has composite maps $F_i \to \mathrm{Alb}(F_i) \to T$, which fit together to a map

$$\vartheta \colon F_{\mathrm{red}} \to T.$$

The fibers of ϑ are exactly the reduced characteristic cycles, and for each singular torus $Z_k \subset F$ the induced map $Z_k \to T$ is finite étale. Making a base-change to Z_k produces a section $\sigma_k \colon Z_k \to F_{\mathrm{red}} \times_T Z_k$. Doing this for all singular tori iteratively, one obtains a cartesian diagram

$$\begin{array}{ccc} \tilde{F}_{\mathrm{red}} & \xrightarrow{\tilde{\beta}} & F_{\mathrm{red}} \\ \tilde{\vartheta}\downarrow & & \downarrow\vartheta \\ \tilde{T} & \xrightarrow[\beta]{} & T \end{array} \tag{36}$$

such that the preimage $\tilde{\beta}^{-1}(Z_k)$ of each singular torus Z_k is a disjoint union of tori $\tilde{Z}_{i_k}, \dots, \tilde{Z}_{i_{k+1}-1}$, each of which is a section of $\tilde{\vartheta}$. Set $\tilde{a}_\ell = a_k$ for $i_k \le \ell < i_{k+1}$. Similarly, $\tilde{\beta}^{-1}(F_i)$ is a disjoint union of components $\tilde{F}_{j_i}, \dots, \tilde{F}_{j_{i+1}-1}$. Set $\tilde{n}_j = n_i$ for $j_i \le j < j_{i+1}$. Since $\deg(\tilde{\beta}) \int_{Z_k} \omega^{n-1} = \int_{\tilde{\beta}^{-1} Z_k} \tilde{\beta}^* \omega^{n-1}$ and $\deg(\tilde{\beta}) \int_{F_i} \omega^n = \int_{\tilde{\beta}^{-1} F_i} \tilde{\beta}^* \omega^n$ one gets

$$\begin{aligned} w &= \frac{n \sum_k a_k \int_{Z_k} \omega^{n-1}}{\int_F \omega^n} = \frac{n \sum_k a_k \int_{Z_k} \omega^{n-1}}{\sum_i n_i \int_{F_i} \omega^n} \\ &= \frac{n \sum_\ell \tilde{a}_\ell \int_{\tilde{Z}_\ell} \tilde{\beta}^* \omega^{n-1}}{\sum_j \tilde{n}_j \int_{\tilde{F}_j} \tilde{\beta}^* \omega^n}. \end{aligned}$$

This reduces the problem to the case where each component of F contains exactly one component of Θ, which was discussed above.

It remains to treat the cases where Θ is of type *II*, *III* or *IV*.

II. In this case Θ is a cuspidal rational curve, F is irreducible and there is one singular torus $Z \subset F$. Consider the normalization $\nu\colon \tilde{F} \to F$, with $\tilde{Z} = \nu^{-1}(Z)$. Then $\tilde{Z}$ is a section of $\tilde{F} \to \mathrm{Alb}(\tilde{F})$, and so

$$\int_F \omega^n = \int_{\tilde{F}} \nu^* \omega^n = n \int_{\nu^{-1}\Theta} \nu^* \omega \cdot \int_{\tilde{Z}} \nu^* \omega = n \int_\Theta \omega \cdot \int_Z \omega^{n-1}.$$

The torus Z has the coefficient $a = 2$, and so

$$w = \frac{2n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{2}{\int_\Theta \omega} = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

III. The characteristic cycle Θ consists of two smooth curves which touch. There is one singular torus $Z \subset F$ with coefficient $a = 3$. Distinguish two cases:

(a) F is irreducible. Then F is singular, so consider the normalization $\nu\colon \tilde{F} \to F$, and set $\tilde{Z} = \nu^{-1}(Z)$. Then $\tilde{Z}$ is a section of $\tilde{F} \to \mathrm{Alb}(\tilde{F})$, and the map $\tilde{Z} \to Z$ has degree 2. Hence

$$\int_F \omega^n = \int_{\tilde{F}} \nu^* \omega^n = n \int_{\nu^{-1}\Theta} \nu^* \omega \cdot \int_{\tilde{Z}} \nu^* \omega^{n-1} = n \int_\Theta \omega \cdot \int_Z \omega^{n-1},$$

and so

$$w = \frac{3n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{3}{\int_\Theta \omega} = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

(b) F is not irreducible, say $F = F_1 + F_2$. Then Z is a section of $F_i \to \mathrm{Alb}(F_i)$ for both $i = 1, 2$, and so

$$\int_{F_i} \omega^n = n \int_{\Theta_i} \omega \cdot \int_Z \omega^{n-1}.$$

Thus

$$w = \frac{3n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{3}{\int_\Theta \omega} = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

IV. The characteristic cycle Θ consists of three smooth rational curves $\Theta = \Theta_1 + \Theta_2 + \Theta_3$, and there is one singular torus $Z \subset F$ with coefficient $a = 4$. Distinguish three cases:

(a) F is irreducible. Consider the normalization $\nu\colon \tilde{F} \to F$ with $\tilde{Z} = \nu^{-1}(Z)$. Then $\tilde{Z}$ is a section of $\tilde{F} \to \mathrm{Alb}(\tilde{F})$, and the degree of $\tilde{Z} \to Z$ is three. Thus

$$\int_F \omega^n = \int_{\tilde{F}} \nu^* \omega^n = n \int_{\nu^{-1}\Theta_1} \nu^* \omega \cdot \int_{\tilde{Z}} \nu^* \omega^{n-1} = n \int_\Theta \omega \cdot \int_{\tilde{Z}} \omega^{n-1},$$

and so

$$w = \frac{4n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{4}{\int_\Theta \omega} = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

(b) F consists of two components, $F = F_1 + F_2$. Supppose F_1 contains Θ_1 and F_2 contains Θ_2 and Θ_3. Then F_1 is smooth and Z is a section of $F_1 \to \mathrm{Alb}(F_1)$. Hence

$$\int_{F_1} \omega^n = n \int_{\Theta_1} \omega \cdot \int_Z \omega^{n-1}.$$

The component F_2 is singular, so consider its normalization $\nu\colon \tilde{F}_2 \to F_2$, and set $\tilde{Z} = \nu^{-1}(Z)$. Then $\tilde{Z}$ is a section of $\tilde{F}_2 \to F_2$, and $\tilde{Z} \to Z$ has degree 2. So

$$\int_{F_2} \omega^n = n \int_{\nu^{-1}\Theta_2} \nu^*\omega \cdot \int_{\tilde{Z}} \nu^*\omega^{n-1} = n \int_{\Theta_2+\Theta_3} \omega \cdot \int_Z \omega^{n-1}$$

Taking everything together yields

$$w = \frac{4n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{4 \int_Z \omega^{n-1}}{\int_\Theta \omega \cdot \int_Z \omega^{n-1}} = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

(c) F consists of three components $F = F_1 + F_2 + F_3$. Then each component F_i is smooth, and Z is a section of $F_i \to \mathrm{Alb}(F_i)$, so that

$$\int_{F_i} \omega^n = n \int_{\Theta_i} \omega \cdot \int_Z \omega^{n-1}.$$

Hence

$$w = \frac{4n \int_Z \omega^{n-1}}{\int_F \omega^n} = \frac{4 \int_Z \omega^{n-1}}{\int_\Theta \omega \cdot \int_Z \omega^{n-1}} = \frac{\chi(\Theta)}{\int_\Theta \omega}.$$

□

Corollary 4.3. *If the characteristic cycle Θ is compact, then $\int_\Theta \omega$ is independent from the choice of ω. Equivalentley, the map*

$$H^2(M, \mathbb{C}) \to \mathbb{C}, \quad \alpha \mapsto \int_\Theta \alpha$$

has the same kernel as $H^2(M, \mathbb{C}) \to H^2(F_{\mathrm{sm}}, \mathbb{C})$. □

It remains to discuss the case where Θ is not compact, i.e. $\Theta = \sum_{k \in \mathbb{Z}} \Theta_k$, and Θ_k intersects Θ_{k-1} and Θ_{k+1} in one point each.

Definition 4.4. The dual graph of Θ is denoted by Γ_Θ, and the dual graph of F by Γ_F. Those graphs have one vertex for each component, and one edge whenever two components intersect. There is a natural surjective map

$$\Gamma_\Theta \twoheadrightarrow \Gamma_F.$$

Observation 4.5. If two characteristic curves Θ_i, Θ_j are contained in a common irreducible component $F_0 \subset F$, then there exists $\lambda \in \mathbb{C}^{n-1}$ with $\lambda.\Theta_i = \Theta_j$. So Θ and $\lambda.\Theta$ are both characteristic cycles which contain Θ_j, which implies $\Theta = \lambda.\Theta$. So λ acts on the dual graph Γ_Θ, but clearly fixes Γ_F. This implies that the map $\Gamma_\Theta \twoheadrightarrow \Gamma_F$ is a quotient by some subgroup $G < \mathrm{Aut}(\Gamma_\Theta)$.

Lemma 4.6. *If Θ is not compact, then F falls into one of the following categories:*

(a) F is irreducible, with one singular torus $Z \subset F$.

(b) F has ≥ 2 components, the dual graph Γ_F is cyclic.

(c) F has ≥ 3 components, the dual graph Γ_F is a string.

Proof. The automorphism group of Γ_Θ is $\mathrm{Aut}(\Gamma_\Theta) = \mathbb{Z} \rtimes \mathbb{Z}/2$, generated by the shift $\Theta_k \mapsto \Theta_{k+1}$ and by the reflection $\Theta_k \mapsto \Theta_{-k}$ which fixes Θ_0.

Suppose G does not act transitively, so F is not irreducible. If G is generated by a shift $\Theta_k \mapsto \Theta_{k+b}$, then Γ_F is cyclic of length b.

If G contains a reflection $r \in G$, note that r cannot fix an edge of Γ_Θ, since then two characteristic curves on the same component would intersect, which is only possible if F is irreducible. Also note that G has to contain a shift, since Γ_F is a finite graph. Let $s \in G$ be a minimal shift, so that $G = \langle s, r\rangle$. Then $\Gamma_\Theta/\langle s\rangle$ is cyclic, and modding out r yields a string whose end points correspond to the fixed points of r and srs^{-1}. □

Definition 4.7. Suppose Θ is not compact. Define a cycle $\overline{\Theta} \subset \Theta$ for each of the cases from Lemma 4.6.

(a) If F is irreducible, $\overline{\Theta} \coloneqq \Theta_0$ is a single characteristic curve.

(b) If F is a cycle of b components, $\overline{\Theta} \coloneqq \Theta_0 + \ldots + \Theta_{b-1}$.

(c) If F is a string of b components, $\overline{\Theta} \coloneqq \Theta_0 + \ldots + \Theta_{2b-3}$.

Proposition 4.8. *If Θ is not compact, let $\chi \in \mathbb{N}$ be the number of components of $\overline{\Theta}$. Then*

$$w = \frac{\chi}{\int_{\overline{\Theta}} \omega.}$$

Proof. Distinguish the three cases from Lemma 4.6.

(a) F is irreducible with one singular torus $Z \subset F$. Consider the normalization $\nu\colon \tilde{F} \to F$ with $\tilde{Z} = \nu^{-1}(Z)$. Then $\tilde{Z}$ is a two-section of the Albanese map $\tilde{F} \to \mathrm{Alb}(\tilde{F})$, and covers Z two-to-one. By Corollary 0.9,

$$\int_F \omega^n = \int_{\tilde{F}} \nu^*\omega^n = \frac{n}{2}\int_{\nu^{-1}\overline{\Theta}} \nu^*\omega \cdot \int_{\tilde{Z}} \nu^*\omega^{n-1} = n \int_{\overline{\Theta}} \omega \cdot \int_Z \omega^{n-1},$$

and so

$$w = \frac{n\int_Z \omega^{n-1}}{\int_F \omega} = \frac{1}{\int_{\overline{\Theta}} \omega} = \frac{\chi}{\int_{\overline{\Theta}} \omega}.$$

(b) Γ_F is a cycle of length $b = \chi$, and $\Theta_i \subset F_i$ for $i = 0, \dots, b-1$. There are b singular tori $Z_i = F_i \cap F_{i+1}$ (with $F_b = F_0$). Each Z_i is a section of both $F_i \to \mathrm{Alb}(F_i)$ and $F_{i+1} \to \mathrm{Alb}(F_{i+1})$, so that

$$\int_{F_i} \omega^n = n \int_{\Theta_i} \omega \cdot \int_{Z_i} \omega^{n-1}$$

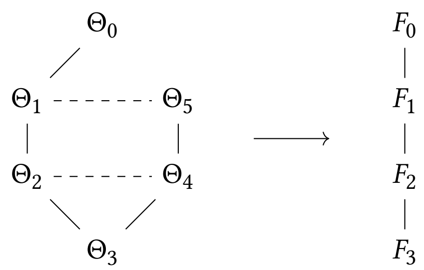
and

$$\int_{Z_i} \omega^{n-1} = \int_{Z_{i+1}} \omega^{n-1}.$$

Together,

$$w = \frac{n \sum_i \int_{Z_i} \omega^{n-1}}{\int_F \omega^n} = \frac{b}{\sum_{i=0}^{b-1} \int_{\Theta_i} \omega} = \frac{\chi}{\int_{\overline{\Theta}} \omega}.$$

(c) Γ_F is a string with end points F_0 and F_{b-1}. We may assume without loss of generality that $\Theta_0 \subset F_0$ and $\Theta_{b-1} \subset F_{b-1}$. For $0 < i < b-1$ the component F_i contains two characteristic curves of $\overline{\Theta}$, namely Θ_i and Θ_{2b-2-i}.



The singular tori are $Z_i = F_{i-1} \cap F_i$ for $i = 1, \dots, b-1$. Since the characteristic curve Θ_0 on F_0 intersects the two characteristic curves $\Theta_{-1}, \Theta_1 \subset F_1$, the torus Z_1 is a two-section of $F_0 \to \mathrm{Alb}(F_0)$, but a section of $F_1 \to \mathrm{Alb}(F_1)$. Similarly Z_{b-1} is a two-section of $F_{b-1} \to \mathrm{Alb}(F_{b-1})$, but a section of $F_{b-2} \to \mathrm{Alb}(F_{b-2})$. The other tori are sections of the involved Albanese map. This leads to

$$\int_{F_0} \omega^n = \frac{n}{2} \int_{\Theta_0} \omega \cdot \int_{Z_1} \omega^{n-1} \quad \text{and} \quad \int_{F_{b-1}} \omega^n = \frac{n}{2} \int_{\Theta_{b-1}} \omega \cdot \int_{Z_{b-1}} \omega^{n-1}$$

as well as

$$\int_{Z_i} \omega^{n-1} = \int_{Z_{i+1}} \omega^{n-1} \qquad (i = 1, \dots, b-2).$$

Thus

$$\begin{aligned} w &= \frac{n \sum_k \int_{Z_k} \omega^{n-1}}{\int_F \omega^n} = \frac{(b-1) \int_{Z_1} \omega^{n-1}}{\left(\frac{1}{2} \int_{\Theta_0} \omega + \sum_{k=1}^{b-2} \int_{\Theta_k} \omega + \frac{1}{2} \int_{\Theta_{b-1}} \omega \right) \int_{Z_1} \omega^{n-1}} \\ &= \frac{2b-2}{\int_{\overline{\Theta}} \omega} = \frac{\chi}{\int_\Theta \omega}. \end{aligned}$$

□

Remark. Note that this approach is also valid if Θ is of the compact type I_b and F has less than b irreducible components. In that case one can calculate w by considering the same cycle $\overline{\Theta}$, which contains one characteristic curve from each component of F.

5 Examples

We have already seen in Corollary 2.16 that if $\varphi\colon X \to \mathbb{P}^1$ is an elliptic K3 surface with singular fibers $F_1, \dots, F_n$, the weights w_i are exactly the Euler characteristics $w_i = \chi(F_i)$. So (12) reduces to the classical formula

$$\sum_i \chi(F_i) = 24 = \chi(X).$$

5.1 Hilbert schemes of elliptic K3 surfaces

Let

$$\varphi\colon X \to \mathbb{P}^1$$

be an elliptic K3 surface, with singular fibers $X_i = X_{p_i} = \varphi^{-1}(p_i)$. For every $n \in \mathbb{N}$, the Hilbert scheme $\mathrm{Hilb}^n(X)$ is an irreducible symplectic manifold[4], and the composition

$$f\colon \mathrm{Hilb}^n(X) \xrightarrow{\pi} \mathrm{Sym}^n(X) \xrightarrow{\mathrm{Sym}^n(\varphi)} \mathrm{Sym}^n(\mathbb{P}^1) = \mathbb{P}^n$$

is a Lagrangian fibration, where $\pi\colon \mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$ is the Hilbert-Chow morphism. Indeed, if $x = a_1 + \dots + a_n \in \mathrm{Sym}^n(\mathbb{P}^1)$ is a point where each curve $E_i = \varphi^{-1}(a_i)$ is smooth, then its fiber is the torus $f^{-1}(x) = E_1 \times \cdots \times E_n$. By Haiman[12], the Hilbert-Chow morphism π is the blowing-up of $\mathrm{Sym}^n(X)$ along the big diagonal $D \subset \mathrm{Sym}^n(X)$.

$$\begin{array}{ccccc} & & \mathrm{Hilb}^2(X) & & \\ & & \big\downarrow \pi & \searrow f & \\ X \times X & \xrightarrow{p} & \mathrm{Sym}^2(X) & \longrightarrow & \mathrm{Sym}^2(\mathbb{P}^1) = \mathbb{P}^2 \end{array}$$

Assume for now that $n = 2$ and that X is projective K3 surface, so that $\mathrm{Hilb}^2(X)$ is also projective. Choose an ample line bundle L on X. Then $L \boxtimes L$ is ample on $X \times X$ and descends to an ample bundle $\overline{A}$ on $\mathrm{Sym}^2(X)$. For some $N \gg 0$,

$$A = \pi^* \overline{A}^{\otimes N} \otimes \mathcal{O}_\pi(1)$$

will be an ample line bundle on $\mathrm{Hilb}^2(X)$. The discriminant divisor $\Delta \subset \mathrm{Sym}^2(\mathbb{P}^1)$ consists of a line $L_i = \{p_i + q \,|\, q \in \mathbb{P}^1\}$ for each p_i, and the diagonal $D \subset \mathrm{Sym}^2(\mathbb{P}^1)$, which is a conic. Each of the L_i is tangent to D.

Let $E, F \subset X$ be two distinct smooth fibers. Then $E \times F$ is a smooth fiber of $\mathrm{Hilb}^2(X) \to \mathbb{P}^2$, and since $E \times F$ is disjoint from the diagonal, $\int_{E\times F} c_1(\mathcal{O}_\pi(1)) = 0$. If L induces polarizations of type d on E and F, then $\pi^*\overline{A}|_{E\times F} = L \boxtimes L|_{E\times F}$ induces a polarization of type (d, d). So the Kähler class

$$[\omega] = \frac{1}{dN} c_1(A)$$

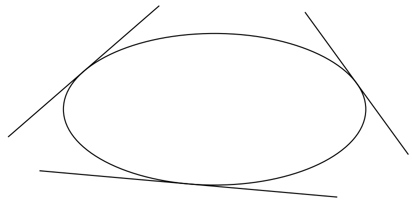


Figure 4: Discriminant divisor $\Delta \subset \mathbb{P}^2$

induces principal polarizations on the smooth fibers of $f\colon \mathrm{Hilb}^2(X) \to \mathbb{P}^2$.

Over the line L_i, the general fiber is a product

$$F_{\mathrm{sing}} = E \times X_i,$$

where E is a smooth fiber of φ. Then $\Theta = \{p\} \times X_i$ is a characteristic cycle, and as for the smooth fibers, $\int_\Theta \omega = \frac{1}{d} \int_{X_i} c_1(L) = 1$. Thus

$$w_i = \chi(X_i).$$

A general point of $D \subset \mathrm{Sym}^2(\mathbb{P}^1)$ is of the form $[2P]$, such that the fiber of φ over $P \in \mathbb{P}^1$ is a smooth curve $E \subset X$. By Proposition 0.10,

$$f^{-1}([2P]) = 2\,\mathrm{Sym}^2(E) + \mathbb{P}(\Omega_X|_E).$$

To define a characteristic cycle, let $\Theta_2 \subset \mathrm{Sym}^2(E)$ be the image of the antidiagonal

$$\Delta^- = \{\, (z, -z) \mid z \in E \,\} \subset E \times E.$$

Then Θ_2 intersects $\mathbb{P}(\Omega_X|_E)$ in the four branch points of $\Delta^- \to \Theta_2$. Let $\Theta_0, \Theta_1, \Theta_3$ and Θ_4 be the characteristic curves in $\mathbb{P}(\Omega_X|_E)$ at those intersection points. Then $\Theta = \Theta_0 + \Theta_1 + 2\Theta_2 + \Theta_3 + \Theta_4$ is a characteristic cycle. Clearly,

$$\int_{\Theta_2} c_1(\mathcal{O}_\pi(1)) = -4.$$

By Proposition 0.10, $\int_{\Theta_i} c_1(\mathcal{O}_\pi(1)) = 2$ for $i = 0, 1, 3, 4$ so that

$$\int_{\Theta} c_1(\mathcal{O}_\pi(1)) = 0.$$

For $i = 0, 1, 3, 4$ the curve Θ_i is contracted by π, hence $\int_{\Theta_i} c_1(\pi^* \overline{A}) = 0$. Also $\int_{\Delta^-} c_1(L \boxtimes L) = 2d$ implies

$$\int_{\Theta_2} c_1(\pi^* \overline{A}) = d.$$

Taking everything together,

$$w_D = \frac{\chi(\Theta)}{\int_\Theta \omega} = \frac{6}{\frac{2}{d} \int_{\Theta_2} c_1(\pi^* \overline{A})} = 3.$$

Finally one computes

$$\sum_i w_i \deg(L_i) + w_D \deg(D) = \chi(X) + 3 \cdot 2 = 30.$$

Sawon computed[29, Prop. 19] $\int_{\mathrm{Hilb}^2(X)} \sqrt{\hat{A}} = \frac{25}{32}$, so that

$$24\left(2\int_{\mathrm{Hilb}^2(X)} \sqrt{\hat{A}}\right)^{\frac{1}{2}} = 30,$$

which verifies the formula.

5.2 O'Grady's 10-dimensional example

Let $\varphi\colon X \to \mathbb{P}^2$ be a sextic K3 surface, i.e. there is a smooth sextic $\Sigma \subset \mathbb{P}^2$, such that X is the double cover branched over Σ. Let $H \subset X$ be the preimage of a general line in $\mathbb{P}^2$, and let $\mathcal{O}_X(1) = \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$ be the corresponding ample line bundle. Consider the moduli space of semistable sheaves on X which have rank zero, first Chern class $2H$ and some fixed, even Euler characteristic $\chi \in 2\mathbb{N}$,

$$M = M_X(r = 0, c_1 = 2H, \chi).$$

Equivalently, M is the moduli space of semistable sheaves with Hilbert polynomial $P(t) = 4t + \chi$. Then M is a singular 10-dimensional variety, and O'Grady has shown[26] that its resolution

$$\pi : \tilde{M} \to M$$

is a smooth irreducible holomorphic symplecic manifold.

5.2.1 Lagrangian fibration by Fitting supports

A Lagrangian fibration $f : \tilde{M} \to \mathbb{P}^5$ can be obtained by sending a semistable sheaf E to the subscheme defined by its zeroth Fitting ideal $\mathrm{Fitt}_0(E) \subset \mathcal{O}_X$ (cf. [7, Section 20.2]).

Definition 5.1. The subscheme $Z \subset X$ defined by $\mathrm{Fitt}_0(E)$ is called the *Fitting support* of E.

Lemma 5.2. *The Fitting support Z of E contains the scheme-theoretic support* $\operatorname{supp} E$*, and both have the same reduction. In particular, if Z is reduced, then $Z = \operatorname{supp} E$.*

Proof. The scheme-theoretic suppport of E is defined by the annihilator ideal $\mathrm{Ann}(E) \subset \mathcal{O}_X$. By [7, Prop. 20.7], there exists an $n \in \mathbb{N}$ such that

$$(\mathrm{Ann}\, E)^n \subset \mathrm{Fitt}_0(E) \subset \mathrm{Ann}\, E. \qquad \square$$

Lemma 5.3. *If E is purely one-dimensional and $c_1(E) = 2H$, then the Fitting support is a curve in the linear system $|2H|$.*

Proof. Let P be a locally free sheaf on X, which surjects onto E, and so gives a short exact sequence of sheaves

$$0 \to K \to P \to E \to 0.$$

For each $x \in X$, one has the Auslander-Buchsbaum formula

$$\operatorname{pd} E_x + \operatorname{depth} E_x = \dim \mathcal{O}_{X,x} = 2. \tag{37}$$

Since E is purely one-dimensional, the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is not an associated prime ideal of E_x. By prime avoidance, there exists an element $f \in \mathfrak{m}_x \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}(E_x)} \mathfrak{p}$. Then f is not a zero divisor for E_x, so it forms a regular sequence of length one. This proves $\operatorname{depth} E_x \geq 1$, so that $\operatorname{pd} E_x = 1$ by (37), which implies that K is locally free. But then $\operatorname{Fitt}_0(E) \cong \det K \otimes (\det P)^\vee$ is a line bundle whose first Chern class is $-c_1(E) = -2H$. □

Lemma 5.4. *The pull-back map* $|\mathcal{O}_{\mathbb{P}^2}(2)| \to |2H|, Q \mapsto \varphi^{-1}(Q)$ *is an isomorphism, and smooth curves in* $|2H|$ *have genus* 5.

Proof. Clearly, the map $|\mathcal{O}_{\mathbb{P}^1}(2)| \to |2H|$ is injective and embeds $|\mathcal{O}_{\mathbb{P}^1}(2)|$ as a linear subspace of $|2H|$. So it suffices to show $\dim |\mathcal{O}_{\mathbb{P}^1}(2)| = \dim |2H|$.

Note that the curve $C = \varphi^{-1}(Q)$ is smooth if Q intersects Σ transversely. In that case, the double cover $C \to Q$ is ramified in the twelve points of $C \cap \varphi^{-1}(\Sigma)$. By Riemann-Hurwitz,

$$2 - 2g(C) = 2 \cdot 2 - 12 = -8,$$

so that $g(C) = 5$. By [14, Lemma 2.1],

$$\dim |2H| = \dim H^0(X, \mathcal{O}(C)) - 1 = g(C) = 5. \qquad \square$$

Given any scheme T and a flat family of semistable sheaves $\mathcal{E} \in \operatorname{Coh}(T \times X)$ with first Chern class $2H$ and Euler characteristic χ, one can form the Fitting ideal $\operatorname{Fitt}_0(\mathcal{E}) \subset \mathcal{O}_{T\times X}$. Since the formation of Fitting ideals commutes with base change ([7, Cor. 20.5]), one obtains a morphism $T \to |2H|$ in a functorial way. As M corepresents the functor of families of semistable sheaves up to $\operatorname{Pic} T$, this induces a map $\bar{f}\colon M \to |2H|$. Composing with the resolution map $\pi\colon \tilde{M} \to M$ defines a Lagrangian fibration

$$f\colon \tilde{M} \to |2H|.$$

The fiber $\bar{f}^{-1}(Q)$ over a conic $Q \subset \mathbb{P}^2$ consists of all S-equivalence classes of semistable sheaves whose Fitting support is $C = \varphi^{-1}(Q)$. If Q is reduced, so is C, and by Lemma 5.2, $\bar{f}^{-1}(Q)$ consists of all semistable sheaves whose scheme-theoretic support is C. In that case one can identify $\bar{f}^{-1}(Q)$ with the moduli space of semistable sheaves of rank 1 and Euler characteristic χ on C. The only non-reduced conics are double lines, which form a subset of codimension two in $|2H|$, so they will not play a role in the calculation of the weights associated to f.

If Q is a smooth conic which intersects Σ transversely, then C is a smooth curve of genus 5 (cf. the proof of Lemma 5.4). A pure sheaf E supported on C is necessarily a vector bundle on C, and if $c_1(E) = 2H = [C]$, then E is a line bundle on C by Riemann-Roch without denominators[9, Example 15.3.1]. Thus $f^{-1}(Q) = \mathrm{Pic}^{\chi}(C)$ is an abelian variety, and the inclusion $\mathrm{Pic}^{\chi}(C) \subset M$ is given by a Poincaré line bundle.

This can go wrong in two ways, each of which gives an irreducible component of the discriminant locus:

(a) The conic Q is not transverse to Σ. The divisor

$$\Delta_1 = \{ Q \subset \mathbb{P}^2 \mid Q \text{ is tangent to } \Sigma \} \subset |2H|$$

has degree $\deg \Delta_1 = 42$ (cf. [9, Chapter 10.4]).

A general element $Q \in \Delta_1$ is a smooth conic tangent to Σ in a single point. In that case, C is has a single node over the tangent intersection point.

(b) The conic Q itself is singular. The divisor

$$\Delta_2 = \{ Q \subset \mathbb{P}^2 \mid Q \text{ is singular} \} \subset |2H|$$

has degree 3: A conic Q is the vanishing locus of a quadratic form, whose polar form defines a symmetric (3×3)-matrix A. Then Q is singular if and only if $\det A = 0$, which is a condition of degree 3 on the entries of A.

A general element $Q \in \Delta_2$ is the union of two lines, $Q = \ell_1 \cup \ell_2$, which intersects Σ transversely. In that case, $C = C_1 \cup C_2$ where each $C_i = \varphi^{-1}(\ell_i)$ is a smooth curve of genus two by Riemann-Hurwitz.

Recall, that the space M is smooth at all points which correspond to stable sheaves[15, Chapter 6.1], and since every pure sheaf of rank one on an irreducible curve is stable by Lemma 0.13, M is smooth over $|2H| \setminus \Delta_2$.

5.2.2 Ample class on smooth fibers

Similarly to the example of $\mathrm{Hilb}^2(X)$, an ample line bundle A on $\tilde{M}$ can be constructed as

$$A = \pi^* \overline{A}^{\otimes N} \otimes \mathcal{O}_\pi(1) \qquad (N \gg 0),$$

where $\overline{A}$ is an ample bundle on M. The ample bundle $\overline{A}$ can be obtained using Le Potier's method[15, Chapter 8.1], with the input provided by Álvarez-Cónsul and King.

Proposition 5.5 ([1, Prop. 7.7]). *For $m \gg n \gg 0$ there exists an ample line bundle $\overline{A} = \overline{A}(m,n)$ on M, such that for every scheme T and every family of semistable sheaves $E \in \mathrm{Coh}(T \times X)$ with Hilbert polynomial P, which induces a classifying map $\psi_E \colon T \to M$, one has*

$$\psi_E^* \overline{A}(m,n) = \det\left(\mathbb{C}^{P(m)} \otimes \mathrm{pr}_{1*}(E \otimes \mathrm{pr}_2^* \mathcal{O}_X(n))\right)^{-1} \otimes \det\left(\mathbb{C}^{P(n)} \otimes \mathrm{pr}_{1*}(E \otimes \mathrm{pr}_2^* \mathcal{O}_X(m))\right). \qquad \square$$

Proposition 5.6 (Schwarzenberger[31]). *Let C be a smooth projective curve with a point $p \in C$, and suppose $\mathscr{P}$ is a Poincaré line bundle on* $\mathrm{Pic}^0(C) \times C$. *Then for $n > 2g(C) - 2$, the class*

$$-c_1\big(\, \mathrm{pr}_{1*}(\mathscr{P} \otimes \mathrm{pr}_2^* \mathscr{O}_C(np))\big)$$

is a principal polarization on $\mathrm{Pic}^0(C)$.

Proof. Set $E_n = \mathrm{pr}_{1*}(\mathscr{P} \otimes \mathrm{pr}_2^* \mathscr{O}_C(np))$. By Schwarzenberger[31, Theorem 4],

$$\mathbb{P}(E_n^\vee) \cong \mathrm{Sym}^n(C),$$

where the map $\mathrm{Sym}^n(C) \to \mathrm{Pic}^0(C)$ is given by

$$p_1 + \ldots + p_n \mapsto \mathscr{O}(p_1 + \ldots + p_n - np). \tag{38}$$

By Mattuk[25] (which is also reproduced in [31]), this implies

$$c_1(E_n) = -[W_{g-1}],$$

where W_{g-1} is the image of $\mathrm{Sym}^{g-1}(C) \to \mathrm{Pic}^0(C)$, defined analogously to (38). Finally, by Poincaré's formula[5, 11.2.1], W_{g-1} is a theta divisor on $\mathrm{Pic}^0(C)$. □

Corollary 5.7. *For a smooth fiber $F = f^{-1}(Q) = \mathrm{Pic}^\chi(C)$ and $m \gg n \gg 0$,*

$$\int_F \overline{A}(n,m) = P(m) - P(n) = 4(m-n),$$

so that the Kähler form

$$\omega = \frac{1}{4N(m-n)} c_1\big(\pi^* \overline{A}(n,m)^{\otimes N} \otimes \mathscr{O}_\pi(1)\big) \qquad (N \gg 0)$$

induces a principal polarisation on the fibers of $f : \tilde{M} \to |2H|$.

Proof. Combine Proposition 5.5 and Proposition 5.6. □

5.2.3 General fibers over Δ_1

Let C be a projective curve with a nodal singularity $p \in C$, such that $C \setminus \{p\}$ is smooth. For $\chi \in \mathbb{Z}$, consider the moduli space of semistable sheaves of rank one and Euler characteristic χ,

$$M_C = M_C(r = 1, \chi).$$

Let $\mathscr{O}_p$ denote the structure sheaf of the point p, which is a skyscraper sheaf with fiber $\mathbb{C}$ supported on p. Let

$$\nu : \tilde{C} \to C$$

be the normalization, and let $p', p'' \in \tilde{C}$ be the two preimages of p.

Construction 5.8. Let L be a line bundle on $\tilde{C}$ with $\chi(L) = \chi + 1$. By Lemma 0.13, every quotient $q\colon \nu_* L \twoheadrightarrow \mathcal{O}_p$ defines a stable sheaf E of rank one and Euler characteristic χ by the exact sequence

$$0 \to E \to \nu_* L \to \mathcal{O}_p \to 0.$$

Note that $\nu_* L \otimes \mathcal{O}_p = \nu_*(L \otimes \mathcal{O}_{p'}) \oplus \nu_*(L \otimes \mathcal{O}_{p''}) \cong \mathbb{C}^2 \otimes \mathcal{O}_p$. In particular, for each L there is a $\mathbb{P}^1$-choice of quotients q.

Lemma 5.9. *The sheaf E from this construction is a line bundle if and only if q is nonzero on both summands $\nu_*(L \otimes \mathcal{O}_{p'})$ and $\nu_*(L \otimes \mathcal{O}_{p''})$. If E is not a line bundle, then the natural map $E \to \nu_*(\nu^* E / \operatorname{Tors}(\nu^* E))$ is an isomorphism.*

Proof. Localizing at p one may assume $L = \mathcal{O}(ap' + bp'')$. Taking completions, one can furthermore identify $\mathcal{O}_{C,p} = \mathbb{C}[\![x,y]\!]/(xy)$, with normalization $\widetilde{\mathcal{O}}_{C,p} = \mathbb{C}[\![x]\!] \times \mathbb{C}[\![y]\!]$ and

$$L = \left\{ (f,g) \in \mathbb{C}(\!(x)\!) \times \mathbb{C}(\!(y)\!) \mid x^a f \in \mathbb{C}[\![x]\!], y^b g \in \mathbb{C}[\![y]\!] \right\}.$$

In this notation, the map $\nu_* L \to \nu_* L \otimes \mathcal{O}_p \cong \mathbb{C}^2$ is given by

$$\Big(\sum_{n \geq -a} f_n x^n, \sum_{n \geq -b} g_n y^n \Big) \mapsto (f_{-a}, g_{-b}).$$

Assume q is zero on one of the two summands, say it is given by $(f,g) \mapsto f_{-a}$. Then the kernel E is just $\nu_* \mathcal{O}((a-1)p' + bp'')$. But if q is given by $(f,g) \mapsto (\lambda f_{-a} + \mu g_{-b})$ for $\lambda, \mu \neq 0$, then E is generated by $\left(x^{-a}, -\frac{\lambda}{\mu} y^{-b}\right)$ as a $\mathcal{O}_{C,p}$-module. □

Lemma 5.10. *Every stable sheaf E of rank one can by obtained by Construction 5.8.*

Proof. For a stable sheaf E of rank one, consider the natural homomorphism

$$E \to \nu_*(\nu^* E / \operatorname{Tors}(\nu^* E)).$$

Since $\nu^* E / \operatorname{Tors}(\nu^* E)$ is a line bundle on $\tilde{C}$, the cokernel of this homomorphism is either $\mathcal{O}_p$ or zero. If the cokernel is $\mathcal{O}_p$, then $L = \nu^* E / \operatorname{Tors}(\nu^* E)$ does it. Otherwise, set $L = (\nu^* E / \operatorname{Tors}(\nu^* E)) \otimes \mathcal{O}(p')$, so on $\tilde{C}$ there is a short exact sequence

$$0 \to \nu^* E / \operatorname{Tors} \nu^* E \to L \to \mathcal{O}_{p'} \to 0.$$

Apply ν_* to get the desired sequence on C. □

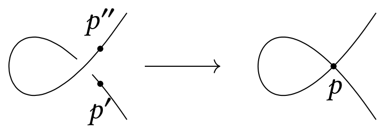


Figure 5: The curve C and its normalization $\tilde{C}$

Lemma 5.11. *If E is a line bundle on C, then L and $q\colon \nu_*L \to \mathcal{O}_p$ from Construction 5.8 are uniquely determined up to a scalar.*

Proof. The map $E \to \nu_*L$ is adjoint to a map $\nu^*E \to L$. Pushing this map down gives the following commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \nu_*\nu^*E & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & \nu_*L & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \end{array}$$

By the Snake lemma, $\nu_*\nu^*E \to \nu_*L$ is an isomorphism. Since ν is affine, this implies that $\nu^*E \to L$ is also an ismorphism, so that L is determined up to isomorphism. Now assume $q\colon \nu_*L \to \mathcal{O}_p$ and $\tilde{q}\colon \nu_*L \to \mathcal{O}_p$ are two quotients, and both kernels $E, \tilde{E}$ are isomorphic. Using the isomorphisms above, one obtains an automorphism $L \to L$, and a diagram of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \nu_*L & \xrightarrow{q} & \mathcal{O}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\cong & & \| & & \\ 0 & \longrightarrow & \tilde{E} & \longrightarrow & \nu_*L & \xrightarrow{\tilde{q}} & \mathcal{O}_p & \longrightarrow & 0 \end{array}$$

Since C is irreducible, the automorphism $L \to L$ is multiplication with a nonzero number, and hence $E = \tilde{E}$ as subobjects in ν_*L. □

Characteristic curves over Δ_1 Fix a line bundle $L \in \mathrm{Pic}(\tilde{C})$ and isomorphisms $L \otimes \mathcal{O}_{p'} \cong \mathbb{C}, L \otimes \mathcal{O}_{p''} \cong \mathbb{C}$, which give an identification $\nu_*L \otimes \mathcal{O}_p \cong \mathbb{C}^2$. Composing with the tautological quotient $\mathcal{O}_{\mathbb{P}^1}^2 \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$ one obtains a family $\mathcal{E}$ of semistable sheaves on C parametrized by $\mathbb{P}^1$, which fits into the short exact sequence of sheaves on $\mathbb{P}^1 \times C$,

$$0 \to \mathcal{E} \to \mathrm{pr}_2^* \nu_*L \to \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_p \to 0. \tag{39}$$

The family $\mathcal{E}$ induces a nonconstant morphism

$$\psi\colon \mathbb{P}^1 \to M,$$

whose image is a characteristic curve on M. Conversely, Lemma 5.10 implies that if one lets L vary, the resulting curves cover all of M, so those are exactly the characteristic curves.

Proposition 5.12. *The characteristic cycle is either of type A_∞ or of type I_b for some $b \geq 2$, and one can take $\overline{\Theta} = \psi(\mathbb{P}^1)$ (cf. Definition 4.7).*

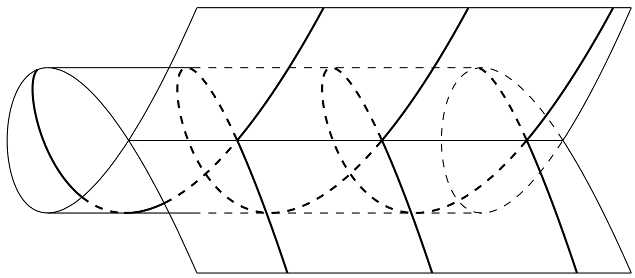


Figure 6: Singular fiber and characteristic cycle over Δ_1

Proof. First note that ψ is injective. On $\mathbb{P}^1 \setminus \{0, \infty\}$ this is asserted by Lemma 5.11. Moreover, the sheaves E_0 and E_∞ over 0 and ∞, which are both not locally free at p, are not isomorphic, since $\nu^* E_0$ has torsion at p', whereas $\nu^* E_\infty$ is locally free at p' and has torsion at p''. In particular, the characteristic cycle cannot be of type I_1.

On the other hand, note that E_0 can also be constructed from the line bundle $\tilde{L} = L(p'' - p')$ (cf. proof of Lemma 5.10). If $\tilde{\mathscr{E}}$ denotes the family of sheaves on $\mathbb{P}^1$ corresponding to $\tilde{L}$, then $E_0 \cong \tilde{E}_\infty$, and hence $\psi(\mathbb{P}^1)$ intersects $\tilde{\psi}(\mathbb{P}^1)$. Continuing this one obtains a (possibly infinite) cycle of characteristic curves. □

To calculate the weight associated to Δ_1, apply Proposition 5.5 to the map ψ. First twist the sequence (39) by $\mathscr{O}_C(n)$ for $n \gg 0$,

$$0 \to \mathscr{E} \otimes \mathrm{pr}_2^* \mathscr{O}_C(n) \to \mathrm{pr}_2^*(\nu_* L \otimes \mathscr{O}_C(n)) \to \mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_p \to 0.$$

Note that the cokernel is not affected by the twisting, as it is supported over $p \in C$. If n is big enough, $R^1 \mathrm{pr}_{1*}(\mathscr{E} \otimes \mathrm{pr}_2^* \mathscr{O}(n)) = 0$, so pushing-down yields a short exact sequence

$$0 \to \mathrm{pr}_{1*}(\mathscr{E} \otimes \mathrm{pr}_2^* \mathscr{O}_C(n)) \to H^0(C, \nu_* L \otimes \mathscr{O}_C(n)) \otimes \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1) \to 0,$$

and hence

$$\det \mathrm{pr}_{1*}(\mathscr{E} \otimes \mathrm{pr}_2^* \mathscr{O}_C(n)) \cong \mathscr{O}_{\mathbb{P}^1}(-1).$$

Thus

$$\psi^*(\overline{A}(m,n)) \cong \mathscr{O}_{\mathbb{P}^1}(P(m) - P(n)) = \mathscr{O}_{\mathbb{P}^1}(4(m-n)).$$

It follows that $\int_{\overline{\Theta}} \omega = 1$, and the weight associated to Δ_1 is

$$w_1 = 1.$$

5.2.4 General fiber over Δ_2

Let C be a reducible curve which has two irreducible components $C = C_1 \cup C_2$, each of which is smooth, and which intersect transversely in two points $C_1 \cap C_2 = \{ p_1, p_2 \}$. Suppose $\mathscr{O}_C(1)$ is an ample line bundle, which has the same degree on both C_1 and C_2. This implies that a sheaf of rank one restricts to a sheaf of rank one on both of the C_i. Again one wants to understand the moduli space

$$M_C = M_C(r = 1, \chi).$$

The approach is similar to the case of a nodal irreducible curve, but the presence of semistable sheaves which are not stable makes the situation a bit more complicated. Note that the proof of Lemma 0.13 fails in this sitation, because even a line bundle on C does admit pure 1-dimensional quotients, supported on one of the components.

Set $\tilde{C} = C_1 \sqcup C_2$, so that $\nu\colon \tilde{C} \to C$ is the normalization map. Let $p_{ij} \in C_j \subset \tilde{C}$ be the preimage of $p_i \in C$.

Construction 5.13. Fix a line bundle $L \in \operatorname{Pic}(\tilde{C})$ such that each $L_i = L|_{C_i}$ has Euler characteristic $\frac{\chi}{2} + 1$. Consider sheaves E on C obtained by a short exact sequence

$$0 \to E \to \nu_* L \xrightarrow{A} \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \to 0. \tag{40}$$

Note that $\nu_* L \otimes \mathcal{O}_{p_i} = L_1 \otimes \mathcal{O}_{p_{i1}} \oplus L_2 \otimes \mathcal{O}_{p_{i2}}$, and the map A restricted to a summand gives a map $a_{ij}\colon L_j \otimes \mathcal{O}_{p_{ij}} \to \mathcal{O}_{p_i}$ of 1-dimensional $\mathbb{C}$-vector spaces. Choosing bases for the L_j at p_{ij} one can identify A with a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} \in \mathbb{C}^{2\times 4}$$

The condition that A has full rank is equivalent to

$$a_{11}a_{21} \neq 0 \quad \text{or} \quad a_{11}a_{22} \neq 0 \quad \text{or} \quad a_{12}a_{21} \neq 0 \quad \text{or} \quad a_{12}a_{22} \neq 0. \tag{41}$$

Lemma 5.14. *The sheaf E from Construction 5.13 is locally free at the point p_i if and only if $a_{i1}a_{i2} \neq 0$.*

Proof. The question is local in p_i, and the proof is identical to that of Lemma 5.9. □

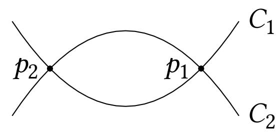


Figure 7: The curve C

Lemma 5.15. *Let E be a sheaf obtained from Construction 5.13. Then*

(a) E semistable if and only if $a_{11}a_{22} \neq 0$ or $a_{12}a_{21} \neq 0$;

(b) E is stable if and only if $a_{ij} \neq 0$ for all $i, j = 1, 2$.

Proof. (a) Suppose $a_{11}a_{22} = 0 = a_{12}a_{21}$, and assume without loss of generality $a_{11} = 0$. Then by (41), $a_{12}a_{22} \neq 0$, so that $a_{21} = 0$. Thus A maps L_1 to zero, so there is a short exact sequence

$$0 \to L_1 \to E \to L_2(-p_1 - p_2) \to 0.$$

Then L_1 is a destabilizing subsheaf of E, since $\chi(L_1) = \frac{\chi}{2} + 1$.

Suppose $a_{11}a_{22} \neq 0$ or $a_{12}a_{21} \neq 0$. In the first case, the induced maps $L_1 \to \mathcal{O}_{p_1}$ and $L_2 \to \mathcal{O}_{p_2}$ are nonzero, in the second case the maps $L_1 \to \mathcal{O}_{p_2}$ and $L_2 \to \mathcal{O}_{p_1}$ are nonzero. In any case, the sheaves $L_1, L_2 \subset L_1 \oplus L_2$ are not contained in E. Now suppose $0 \neq F \subsetneq E$ is a saturated subsheaf, so E/F is pure of dimension one. Then E/F is supported on either C_1 or C_2, and assume without loss of generality $\operatorname{supp}(E/F) = C_2$. Then F is supported on C_1, hence it is mapped to zero under the projection $L_1 \oplus L_2 \to L_2$, so that $F \subset L_1$.

$$\begin{array}{ccccccccc} & & F & \hookrightarrow & E & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & L_1 \oplus L_2 & \longrightarrow & L_2 & \longrightarrow & 0 \end{array}$$

Note that F is a proper subsheaf of L_1, because L_1 is not a subsheaf of E. Then

$$\chi(F) < \chi(L_1) = \frac{\chi}{2} + 1,$$

so that $\chi(F) \leq \frac{\chi}{2}$, which implies the desired inequality of the reduced Hilbert polynomials of F and E.

(b) Suppose without loss of generality, that $a_{11} = 0$. Then E contains $L_1(-p_2)$ as a subsheaf, and $\chi(L_1(-p_2)) = \frac{\chi}{2}$. Thus E cannot be stable.

Assume conversely, that $a_{ij} \neq 0$ for all $i, j = 0, 1$. Let $F \subset E$ be a saturated subsheaf, supported without loss of generality on C_1. As in case (a), this implies $F \subset L_1$, and F is contained in the kernel of the surjection $L_1 \twoheadrightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2}$. Hence $\chi(F) + 2 \leq \chi(L_1) = \frac{\chi}{2} + 1$, so that $\chi(F) < \frac{\chi}{2}$. □

Proposition 5.16. *Up to S-equivalence, every semistable sheaf of rank one and Euler characteristic χ on C can be obtained by Construction 5.13.*

Proof. Let E be a semistable sheaf of rank one and Euler characteristic χ on C. Consider the line bundles $F_i = E|_{C_i} / \operatorname{Tors} E|_{C_i}$, which form a short exact sequence

$$0 \to E \to F_1 \oplus F_2 \to T \to 0. \tag{42}$$

Then T is possibly supported on p_1 and p_2. Distinguish the following cases.

(a) $T \cong \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2}$, i.e. E is locally free. Then (42) is a natural candidate for (40). It remains to determine $\chi(F_i)$. Clearly,

$$\chi + 2 = \chi(F_1) + \chi(F_2).$$

On the other hand, note that $F_i(-p_1 - p_2) \subset E$, and so the semistability of E implies

$$\chi(F_i) - 2 \leq \frac{\chi}{2}.$$

This leaves the following possibilities:

(i) $\chi(F_1) = \chi(F_2) = \frac{\chi}{2} + 1$. Then (42) is indeed an instance of (40).

(ii) One of the F_i has Euler characteristic $\frac{\chi}{2} + 2$, the other one has $\frac{\chi}{2}$, say $\chi(F_1) = \frac{\chi}{2} + 2, \chi(F_2) = \frac{\chi}{2}$. In this case, E is not stable, but S-equivalent to $F_1(-p_1 - p_2) \oplus F_2$. This sheaf can be realized by Construction 5.13 with $L_1 = F_1(-p_2), L_2 = F_2(p_2)$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) $T \cong \mathcal{O}_{p_i}$ for $i = 1$ or 2. Then

$$\chi + 1 = \chi(F_1) + \chi(F_2), \tag{43}$$

and $F_j(-p_i) \subset E$ for both $j = 1, 2$. Since E is semistable, this leads to the estimates

$$\chi(F_j) - 1 \leq \frac{\chi}{2} \qquad (i = 1, 2).$$

By (43), one of the two estimates is an equality, the other one is strict, say $\chi(F_1) = \frac{\chi}{2} + 1$. Then F_1 surjects onto $\mathcal{O}_{p_i}$, because otherwise F_1 would be a subsheaf of E, contradicting semistability. Thus the short exact sequence

$$0 \to F_1(-p_i) \to E \to F_2 \to 0$$

is a Jordan-Hölder filtration of E, so E is S-equivalent to $F_1(-p_i) \oplus F_2$. This sheaf can be realized by Construction 5.13 with $L_1 = F_1, L_2 = F_2(p_2)$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) $T = 0$, in which case $E = F_1 \oplus F_2$. Then

$$\chi = \chi(F_1) + \chi(F_2),$$

and since E is semistable,

$$\chi(F_i) \leq \frac{\chi}{2} \qquad (i = 1, 2).$$

This is only possible if $\chi(F_i) = \frac{\chi}{2}$. The sheaf E can be realized by Construction 5.13 with $L_1 = F_1(p_1), L_2 = F_2(p_2)$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. □

Characteristic curves over Δ_2 Fix line bundles L_1, L_2 on C_1 and C_2 respectively, with $\chi(L_i) = \frac{\chi}{2} + 1$, and choose bases for $L_i \otimes \mathcal{O}_{p_{ji}}$. Consider the submanifold

$$\mathcal{U} = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} : a_{11}a_{22} \neq 0 \text{ or } a_{12}a_{21} \neq 0 \right\} \subset \mathbb{C}^{2\times 4}.$$

Following Construction 5.13, there exists a family of semistable sheaves on C parametrized by $\mathcal{U}$,

$$0 \to \mathcal{E}_{\mathcal{U}} \to \mathrm{pr}_2^*(L_1 \oplus L_2) \xrightarrow{\mathrm{pr}_1^* \mathscr{A}} \mathrm{pr}_2^*(\mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2}) \to 0, \tag{44}$$

where pr_i denotes the projections from $\mathcal{U} \times C$, and $\mathscr{A} : \mathcal{O}_{\mathcal{U}}^4 \to \mathcal{O}_{\mathcal{U}}^2$ is the tautological matrix on $\mathcal{U}$. To get rid of the choice of bases, consider the action by the group $G = \mathrm{Aut}(\mathcal{O}_{p_1}) \times \mathrm{Aut}(\mathcal{O}_{p_2}) \times \mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2) = (\mathbb{C}^*)^4$, which acts on $\mathcal{U}$ by

$$\begin{aligned}(\lambda_1, \lambda_2, \mu_1, \mu_2).A &= \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mu_1^{-1} & & & \\ & \mu_2^{-1} & & \\ & & \mu_1^{-1} & \\ & & & \mu_2^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1\mu_1^{-1}a_{11} & \lambda_1\mu_2^{-1}a_{12} & 0 & 0 \\ 0 & 0 & \lambda_2\mu_1^{-1}a_{21} & \lambda_2\mu_2^{-1}a_{22} \end{pmatrix}.\end{aligned}$$

Lemma 5.17. *Suppose $A, \tilde{A}$ are two matrices, which define stable line bundles $E, \tilde{E}$. Then E is isomorphic to $\tilde{E}$ if and only if A and $\tilde{A}$ are in the same G-orbit.*

Proof. Clearly, if A and $\tilde{A}$ are in the same G-orbit, they define isomorphic line bundles.

Conversely, suppose there is an isomorphism $\varphi \colon E \to \tilde{E}$. Since the natural maps $\nu^*E \to L$ and $\nu^*\tilde{E} \to L$ are isomorphisms, φ induces an isomorphism $L \to L$, which which gives a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \nu_* L \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \cong} \\ \tilde{E} & \longrightarrow & \nu_* L. \end{array}$$

This in turn induces an isomorphism on cokernels, $\mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \to \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_1}$, and introducing bases gives the action by an element $g \in G$. □

The map

$$\vartheta \colon \mathcal{U} \to \mathbb{P}^1, \quad A \mapsto [a_{11}a_{22} : a_{12}a_{21}].$$

is clearly invariant with respect to the G-action. It is easy to verify that ϑ is a geometric quotient on the open set $\mathcal{U}_0 = \{\forall i, j \colon a_{ij} \neq 0\}$, which parametrizes stable line bundles. Over $0 \in \mathbb{P}^1$ the map ϑ identifies the three orbits $\{a_{11} = 0, a_{22} \neq$

$0\}, \{a_{11} \neq 0, a_{22} = 0\}$ and $\{a_{11} = a_{22} = 0\}$; a similar thing happens over $\infty \in \mathbb{P}^1$. Note that ϑ is a good quotient in the sense of Geometric Invariant Theory.

To define a family of semistable sheaves on $\mathbb{P}^1$, consider the local sections

$$s_0 : \mathbb{P}^1 \setminus \{\infty\} \to \mathscr{U}, \quad t \mapsto \begin{pmatrix} t & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and

$$s_1 : \mathbb{P}^1 \setminus \{0\} \to \mathscr{U}, \quad t^{-1} \mapsto \begin{pmatrix} 1 & t^{-1} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then a family on $\mathbb{P}^1$ can be defined by pulling-back (44) via s_0 and s_1, and gluing along the intersection. For $t \neq 0, \infty$, the two families get glued by the group element $g(t) = (t^{-1}, 1, 1, 1) \in G$, corresponding to the following diagram.

$$\begin{array}{ccc} \mathbb{C}^4 & \xrightarrow{\begin{pmatrix} t & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & \mathbb{C}^2 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix}} \\ \mathbb{C}^4 & \xrightarrow[\begin{pmatrix} 1 & t^{-1} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}]{} & \mathbb{C}^2 \end{array}$$

This leads to a family $\mathscr{E}$ of semistable sheaves on C parametrized by $\mathbb{P}^1$, which fits into a short exact sequence

$$0 \to \mathscr{E} \to \mathrm{pr}_2^*(L_1 \oplus L_2) \to \mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{p_1} \oplus \mathscr{O}_{\mathbb{P}^1 \times p_2} \to 0. \tag{45}$$

This family induces a map $\psi : \mathbb{P}^1 \to M_C$.

Lemma 5.18. *The map ψ is injective.*

Proof. Over $\mathbb{P}^1 \setminus \{0, \infty\}$, the family $\mathscr{E}$ paramatrizes stable line bundles by Lemma 5.15. By Lemma 5.17, ψ is injective on $\mathbb{P}^1 \setminus \{0, \infty\}$.

Over $0 \in \mathbb{P}^1$, the matrix defining E_0 is $A_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Hence the square

$$\begin{array}{ccc} L_1 \oplus L_2 & \xrightarrow{A_0} & \mathscr{O}_{p_1} \oplus \mathscr{O}_{p_2} \\ \uparrow & & \uparrow \\ L_1 & \longrightarrow & \mathscr{O}_{p_2} \end{array}$$

is commutative, and leads to a short exact sequence

$$0 \to L_1(-p_2) \to E_0 \to L_2(-p_1) \to 0,$$

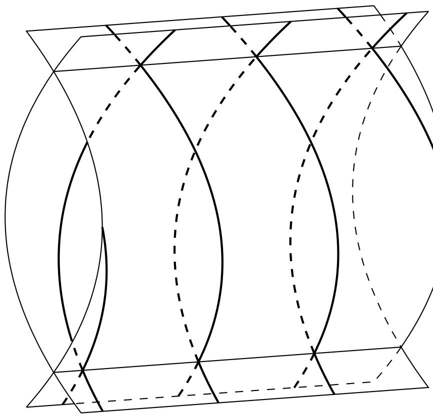


Figure 8: Singular fiber and characteristic cycle over Δ_2

which is a Jordan-Hölder filtration for E_0. Similarly, for $\infty \in \mathbb{P}^1$, one obtains a Jordan-Hölder filtration

$$0 \to L_2(-p_2) \to E_\infty \to L_1(-p_1) \to 0.$$

Because $g(C_i) = 2$, the line bundles $L_i(-p_1)$ and $L_i(-p_2)$ are not isomorphic, so that E_0 and E_∞ are not S-equivalent. □

Proposition 5.19. *The characteristic cycle Θ on a general singular fiber over Δ_2 is of type A_∞ or I_{2b} for some $b \geq 1$. The weight associated to Δ_2 is*

$$w_2 = 2.$$

Proof. It was shown by O'Grady[26], that the singularity of $M = M_X(0, 2H, \chi)$ in a semistable point of M_C is locally a product of an A_1-singularity with a manifold. At those points, the fibers of the resolution $\pi \colon \tilde{M} \to M$ are rational (-2)-curves, on which $\mathcal{O}_\pi(1)$ has degree 2.

Now let Θ_1 be the strict transform of $\psi(\mathbb{P}^1)$ in $\tilde{M}$. Since $\psi(\mathbb{P}^1)$ contains two distinct strictly semistable points, Θ_1 intersects two (-2)-curves. Each of those intersects another curve similar to $\psi(\mathbb{P}^1)$, constructed from different line bundles L_i' etc.

To calculate the weight, let Θ_2 be one of the (-2)-curves which intersects Θ_1, and set $\overline{\Theta} = \Theta_1 + \Theta_2$. Since Θ_1 intersects the exceptional divisor of π in two points, $\int_{\Theta_1} c_1 \mathcal{O}_\pi(1) = -2$. Furthermore, (45) implies

$$\int_{\Theta_1} \pi^* \overline{A}(m, n) = \int_{\mathbb{P}^1} \psi^* \overline{A}(m, n) = 4(m - n)$$

by exactly the same argument as for Δ_1. Thus

$$\int_{\overline{\Theta}} \omega = \frac{1}{4N(m-n)} \left(N \int_{\Theta_1} \pi^* \overline{A}(m, n) + \int_{\Theta_1} c_1 \mathcal{O}_\pi(1) + \int_{\Theta_2} c_1 \mathcal{O}_\pi(1) \right) = 1.$$

By Proposition 4.8,
$$w_2 = \frac{2}{\int_\Theta \omega} = 2$$ □

Finally, (12) yields

$$24\left(5! \int_{\tilde{M}} \sqrt{\hat{A}}(\tilde{M})\right)^{\frac{1}{5}} = w_1 \deg \Delta_1 + w_2 \deg \Delta_2 = 42 + 2 \cdot 3 = 48.$$

On the other hand, the Chern numbers of $\tilde{M}$ were computed by Cao and Jiang[28, Appendix A] and can be used to calculate

$$\sqrt{\hat{A}}(\tilde{M}) = \frac{4}{15}.$$

Hence
$$24\left(5! \int_{\tilde{M}} \sqrt{\hat{A}}(\tilde{M})\right)^{\frac{1}{5}} = 24 \cdot \sqrt[5]{32} = 48,$$

which finishes the verification of the formula for this Lagrangian fibration of $\tilde{M}$.

Conventions

The set of natural numbers is $\mathbb{N} = \{ 1, 2, 3, \dots \}$, and $\mathbb{N}_0 = \{ 0 \} \cup \mathbb{N}$.

The end of proofs is marked with □, theorems marked with □ are given without proof.

The rings $\mathbb{C}[\![x]\!]$ and $\mathbb{C}(\!(x)\!)$ denote the ring of formal power series and its field of fractions, the field of formal Laurent series. The ring $\mathbb{C}\{x\}$ is the ring of convergent power series.

If E is a locally free sheaf of rank $r + 1$ on X, then $\mathbb{P}(E) \to X$ is the $\mathbb{P}^r$-bundle whose points corresond to quotients $E_x \to k(x) \to 0$ for $x \in X$.

If F is a sheaf on X and G is a sheaf on Y, then $F \boxtimes G$ denotes the sheaf $pr_1^* F \otimes pr_2^* G$ on $X \times Y$.

If not mentioned otherwise, Chern classes are considered as elements in singular cohomology $H^*(M, \mathbb{Q})$, if necessary over $\mathbb{R}$ or $\mathbb{C}$. Similarly, all intersection theoretic calculations take place in singular cohomology $H^*(M, \mathbb{Q})$. If $f : X \to Y$ is a smooth map of compact oriented manifolds, the push-forward in cohomology, $f_* : H^c(X, \mathbb{Q}) \to H^{c+(\dim X - \dim Y)}(Y, \mathbb{Q})$, is obtained by composing the push-forward in homology with Poincaré duality $H_k(X, \mathbb{Q}) \cong H^{\dim X - k}(X, \mathbb{Q})$. The natural pairing $H_k(X, \mathbb{Q}) \times H^k(X, \mathbb{Q}) \to \mathbb{Q}$ is denoted by $(\alpha, \beta) \mapsto \int_\alpha \beta$.

References

[1] Luis Álvarez-Cónsul and Alastair King. "A functorial construction of moduli of sheaves". In: *Inventiones Mathematicae* 168.3 (Mar. 2007), pp. 613–666. ISSN: 0020-9910. DOI: 10.1007/s00222-007-0042-5.

[2] V. I. Arnol′d. *Mathematical methods of classical mechanics*. 2nd ed. Graduate texts in mathematics 60. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition. New York: Springer, 2009. ISBN: 0387968903.

[3] Wolf P. Barth et al. *Compact Complex Surfaces*. 2nd enlarged ed. Vol. 4. Ergeb. Math. Grenzgeb., 3. Folge. Berlin, Heidelberg: Springer, 2004. ISBN: 3-540-00832-2.

[4] Arnaud Beauville. "Variétés Kähleriennes dont la première classe de Chern est nulle". French. In: *Journal of Differential Geometry* 18.4 (Jan. 1983), pp. 755–782. ISSN: 0022-040X. DOI: 10.4310/jdg/1214438181.

[5] Christina Birkenhake and Herbert Lange. *Complex Abelian varieties*. Ed. by Herbert Lange. 2nd ed. Vol. 302. Grundlehren Math. Wiss. 302. Berlin: Springer, 2004. ISBN: 3-540-20488-1.

[6] Frédéric Campana. "Isotrivialité de certaines familles kählériennes de variétés non projectives". In: *Mathematische Zeitschrift* 252.1 (2006), pp. 147–156. ISSN: 1432-1823. DOI: 10.1007/s00209-005-0851-4.

[7] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Vol. 150. Graduate texts in mathematics 150. New York, NY: Springer, 1995. ISBN: 0387942688.

[8] Akira Fujiki. "On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold". In: *Algebraic Geometry, Sendai, 1985*. Vol. 10. Adv. Stud. Pure Math. Mathematical Society of Japan, 1987, pp. 105–165. DOI: 10.2969/aspm/01010105.

[9] William Fulton. *Intersection Theory*. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. New York: Springer, 1998. ISBN: 9781461217008. DOI: 10.1007/978-1-4612-1700-8.

[10] Daniel Greb and Christian Lehn. "Base Manifolds for Lagrangian Fibrations on Hyperkähler Manifolds". In: *International Mathematics Research Notices. IMRN* 2014.19 (July 2013), pp. 5483–5487. ISSN: 1073-7928. DOI: 10.1093/imrn/rnt133.

[11] A. Grothendieck and J. Dieudonné. "Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas. (Troisième partie). Rédigé avec la colloboration de J. Dieudonné". French. In: *Publications Mathématiques* 28.1 (1966). ISSN: 0073-8301. DOI: 10.1007/bf02684343.

[12] Mark Haiman. "t, q-Catalan numbers and the Hilbert scheme." In: *Discrete Mathematics* 193.1-3 (1998), pp. 201–224. ISSN: 0012-365X. DOI: 10.1016/S0012-365X(98)00141-1.

[13] Robin Hartshorne. *Algebraic geometry.* 3rd ed. Graduate texts in mathematics 52. New York: Springer, 1983. ISBN: 0387902449.

[14] Daniel Huybrechts. *Lectures on K3 Surfaces.* Camb. Stud. Adv. Math. 158. Cambridge: Cambridge University Press, 2016. ISBN: 9781316594193. DOI: 10.1017/CBO9781316594193.

[15] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves.* 2nd ed. Cambridge Mathematical Library. Cambridge: Cambridge University Press, 2010. ISBN: 9780511711985. DOI: 10.1017/cbo978051171 1985.

[16] Jun-Muk Hwang. "Base manifolds for fibrations of projective irreducible symplectic manifolds". In: *Inventiones Mathematicae* 174.3 (2008), pp. 625–644. ISSN: 0020-9910. DOI: 10.1007/s00222-008-0143-9.

[17] Jun-Muk Hwang and Keiji Oguiso. "Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration". In: *Amer. J. Math.* 131.4 (2009), pp. 981–1007. ISSN: 0002-9327. DOI: 10.1353/ajm.0.0062.

[18] Jun-Muk Hwang and Keiji Oguiso. "Multiple fibers of holomorphic Lagrangian fibrations". In: *Communications in Contemporary Mathematics* 13.2 (2011), pp. 309–329. ISSN: 0219-1997. DOI: 10.1142/S0219199711004269.

[19] John M. Lee. *Introduction to smooth manifolds.* 2nd ed. Graduate texts in mathematics 218. New York: Springer, 2013. ISBN: 978-1-4419-9981-8.

[20] Christian Lehn. "Symplectic lagrangian fibrations". PhD thesis. Johannes Gutenberg-Universität Mainz, 2011. DOI: 10.25358/OPENSCIENCE-3153.

[21] Daisuke Matsushita. "On fibre space structures of a projective irreducible symplectic manifold". In: *Topology. An International Journal of Mathematics* 38.1 (1999), pp. 79–83. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(98)00003-2.

[22] Daisuke Matsushita. "Addendum: "On fibre space structures of a projective irreducible symplectic manifold" [Topology **38** (1999), no. 1, 79–83; MR1644091 (99f:14054)]". In: *Topology. An International Journal of Mathematics* 40.2 (2001), pp. 431–432. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(99)00048-8.

[23] Daisuke Matsushita. "Higher direct images of dualizing sheaves of Lagrangian fibrations". In: *Amer. J. Math.* 127.2 (2005), pp. 243–259. ISSN: 0002-9327,1080-6377. DOI: 10.1353/ajm.2005.0009.

[24] Daisuke Matsushita. "On Deformations of Lagrangian Fibrations". In: *K3 Surfaces and Their Moduli.* Springer, 2016, pp. 237–243. ISBN: 9783319299594. DOI: 10.1007/978-3-319-29959-4_9.

[25] Arthur Mattuck. "Symmetric Products and Jacobians". In: *American Journal of Mathematics* 83.1 (1961), pp. 189–206. ISSN: 0002-9327. DOI: 10.2307/2372727.

[26] Kieran G. O'Grady. "Desingularized moduli spaces of sheaves on a $K3$". In: *Journal für die Reine und Angewandte Mathematik* 512 (1999), pp. 49–117. ISSN: 0075-4102. DOI: 10.1515/crll.1999.056.

[27] Keiji Oguiso. "Picard number of the generic fiber of an abelian fibered hyperkähler manifold". In: *Mathematische Annalen* 344.4 (2009), pp. 929–937. ISSN: 0025-5831. DOI: 10.1007/s00208-009-0335-7.

[28] Ángel David Ríos Ortiz. "Riemann-Roch polynomials of the known hyperkähler manifolds". In: *Bulletin de la Société Mathématique de France* 152.2 (2024), pp. 169–184. ISSN: 0037-9484. DOI: 10.24033/bsmf.2887.

[29] Justin Sawon. "Rozansky-Witten invariants of hyperkähler manifolds". PhD thesis. Trinity College, University of Cambridge, Oct. 1999. URL: https://arxiv.org/abs/math/0404360.

[30] Justin Sawon. "On the discriminant locus of a Lagrangian fibration". In: *Mathematische Annalen* 341.1 (2008), pp. 201–221. ISSN: 0025-5831. DOI: 10.1007/s00208-007-0189-9.

[31] R. L. E. Schwarzenberger. "Jacobians and symmetric products". In: *Illinois Journal of Mathematics* 7 (1963), pp. 257–268. ISSN: 0019-2082. URL: http://projecteuclid.org/euclid.ijm/1255644637.

[32] Claire Voisin. "Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes". In: *Complex Projective Geometry*. Vol. 179. London Math. Soc. Lecture Note Ser. Cambridge University Press, 1992, pp. 294–303. DOI: 10.1017/CBO9780511662652.022.

[33] Claire Voisin. *Hodge theory and complex algebraic geometry, I.* Camb. Stud. Adv. Math. 76. Translated from the French original by Leila Schneps. Cambridge: Cambridge University Press, 2002. ISBN: 0-521-80260-1. DOI: 10.1017/CBO9780511615344.

[34] Claire Voisin. "Recent progresses in Kähler and complex algebraic geometry". In: *Proceedings of the 4th European congress of mathematics (ECM), Stockholm, Sweden, June 27–July 2, 2004.* Zürich: European Mathematical Society (EMS), 2005, pp. 787–807. ISBN: 3-03719-009-4. DOI: 10.4171/009-1/50.

[35] Charles A. Weibel. *An introduction to homological algebra.* Camb. Stud. Adv. Math. 38. Cambridge: Cambridge University Press, 1994. ISBN: 0-521-43500-5.

[36] Benjamin Wieneck. "On polarization types of Lagrangian fibrations". In: *Manuscripta Math.* 151.3-4 (2016), pp. 305–327. DOI: 10.1007/s00229-016-0845-z.

[37] Benjamin Wieneck. “Monodromy invariants and polarization types of generalized Kummer fibrations”. In: *Mathematische Zeitschrift* 290.1–2 (2018), pp. 347–378. ISSN: 1432-1823. DOI: 10.1007/s00209-017-2020-y.