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Torusactions, Motives and Graphhypersurfaces

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Contents

Chapter 1

Introduction

The motivating background for the topic of this thesis comes from the complications that arise when one tries to evaluate integrals of rational functions arising from renormalization procedures in quantum field theory. Those integrals, the so-called Feynman-amplitudes, determine the coefficients of a series that is assumed to be an expansion of a solution of a given system in quantum field theory. In general the occuring rational functions depend on masses, positions and momenta. The cases we are concerned with are actually instances of a toy model, the so-called ϕ^4-theory, which is characterized by having equal masses for all particles and no external momenta.
If one applies standard techniques such as the Feynman (resp.Schwinger) trick one obtains, for the ϕ^4-theory toy model, a representation of the renormalization of the form $\int_{\sigma(\mathbb{R})} \frac{1}{P_\Gamma^2} d\Omega$, where Γ is a finite undirected graph -the Feynman graph- of the underlying physical system, P_Γ is a homogeneous polynomial of degree $h_1(\Gamma)$, the number of independent loops of Γ. (See [8] sections 5 and 6). One can evaluate these integrals explicitly only for a finite subset of all possible Feynman graphs, namely up to loop number of at most 3. However, Physicists evaluated a wide range of Feynman graphs beyond that restriction using numerical methods and they made the surprising observation that Feynman amplitudes seemed to always evaluate to multiple zeta values (see[10] and [11]). These numbers are -conjecturally- tightly associated to counting problems for algebraic varieties over finite fields on one side and to motives that decompose completely into tensor powers of motives of the projective space on the other side.
Kontsevich introduced the counting problem to a larger audience in [31] pointing out that the counting functions for hypersurfaces defined by the zero set of P_Γ should be polynomials in the order of the underlying field if the Feynman amplitude could be interpreted as a period of a suitable motive associated to the hypersurface defined by P_Γ which should be then so-called Tate-motives. He raised the question whether these counting functions are always polynomials. An exaustive computer aided search for counterexam-

ples for graphs with up to fourteen edges by Stembridge [45] did not provide a counterexample. In 2003 Belkale and Brosnan showed in [4] that this expectation is false in a very general sense. However the expectation that Feynman-amplitudes are geometric in nature has been shown to be correct in the seminal 2006 paper by Bloch, Esnault and Kreimer, which identified the Feynman-amplitude as a period in the sense of Hodge-Theory of the complement of (a certain blow-up of) the graph hypersurface relative to (the strict transform of) the standard simplex in the projective space. Since then research has been focused on making methods of algebraic geometry available to study the periods arising from renormalization. The complications in the study of the mixed Hodge-structure associated to a hypersurface defined by some P_Γ comes mainly from the complicated singularities of $Z(P_\Gamma) \subseteq \mathbb{P}^n$. Apart from a few special cases (notably are those where the dual hypersurface provides such a resolution, see [41] for details) there is no explicit resolution of singularities known. Three main approaches are known to obtain explicit results about the Hodge-Structures, counting functions or motives involved. One is mainly studying the graph hypersurfaces in the Grothendieck ring of motives, an approach employed by Aluffi and Marcolli [2]. The computations are usually very close to the computations needed to determine counting functions (strictly speaking only up to the counting functions of the Lefschetz motive). The second revolves around explicit reductions of the denominator of Feynman integrals and studying certain combinatorial invariants related to them. This has been done with extraordinal success by F. Brown, O. Schnetz and others ([13], [15]). The third is by directly studying the mixed Hodge-structure (resp. the underlying mixed Motive). This has been done by Bloch, Esnault, Kreimer and Doryn [8], [19]. Unfortunately the methods applied to obtain the results in [8] and [19] are not motivic (they involve e.g. Artin vanishing in etale cohomology) so the results do not directly generalize to mixed motives. For this more general methods are needed. One possible approach is to use group scheme actions to reduce the complexity to simpler, or at least smaller, invariant subschemes. For the case of smooth projective k-schemes X that are invariant under the action of an algebraic torus this is known as the motivic Bialynicki-Birula decomposition [7] which shows that the motive of such an X is a direct sum of motives of the connected components of the fixed point scheme of X. The motivation of this thesis is now to investigate the following three questions

- What graph hypersurfaces $Z(P_\Gamma)$ admit a torus action?
- What is the maximal dimension of such an action?
- How can one reduce the computation of the motive of such a $Z(P_\Gamma)$ to the motives of the fixed point loci?

In this thesis partial positive results are given for the first question. Further an example of a graph hypersurface that admits no torus action is provided.

We then classify the graphs that have a maximal torus action with respect to our construction of torus actions and finally we illustrate how torus actions can be applied provided one has a fairly explicit equivariant resolution of singularities to really compute the motive of a graph hypersurface in the triangulated category of mixed motives as defined by Voevodsky.

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Chapter 2

Determinants and schemes

The goal of this chapter is to introduce the basic notions we are concerned with throughout this work from a slightly more general angle than in most of the literature concerned with amplitudes and motives of Feynman graphs. Among the more general expositions of the main objects are the paper of Belkale and Brosnan [4] and the thesis of E. Patterson [41]. Our definition is essentially a straightforward generalization of the definitions in those works. Apart from discussing the central notions - determinantal schemes and hypersurfaces - we introduce some basic notions about actions of group schemes on schemes.

2.1 Graph polynomials

Below we introduce the notion of graph polynomials. They are defined purely in graph theoretical terms but they can be interpreted as determinantal hypersurfaces in several ways using the Matrix Tree Theorem [44] in some way. To my knowledge there are at least two ways of introducing the combinatorial class of graph polynomials. One is used for example by F. Brown (e.g. in [13]) and is obtained by building up a matrix of size $(|V(\Gamma)|-1)\times(|V(\Gamma)|-1)$, where $V(\Gamma)$ is the set of vertices, which is obtained from the so-called graph matrix that is basically a matrix with indeterminates on the diagonal extended by a representing matrix of an chosen orientation of the graph Γ. This approach is, w.r.t. the construction, quite different from the approach below, which essentially follows [8], but both approaches give the same polynomial.

Definition 2.1.1. *Recall the following notions:*

1. *A graph Γ is a tripel of a set $V(\Gamma)$ (whose elements are called vertices), a set $E(\Gamma)$ (the elements of $E(\Gamma)$ are called the edges of Γ) and a map $\varepsilon\colon E(\Gamma) \to V(\Gamma) \times V(\Gamma)$. A subgraph is simply a graph Δ such that $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. Let $\sim_\varepsilon := Im(\varepsilon) \subseteq V(\Gamma) \times V(\Gamma)$. We call Γ simple if ε is an injection and $\sim_\varepsilon$ is anti-reflexive.*

2. *For $x \in V(\Gamma)$ let $C(x)$ be the transitive hull of the set of all pairs (z, x), $(x, z') \in \sim_\varepsilon$. We say that two vertices v_1, v_2 are connected if $C(v_1) = C(v_2)$. A graph is connected if each pair of distinct vertices $v_1 \neq v_2$ is connected.*

3. *A graph Γ is a cycle if it is connected and if the simple graph*

$$\Gamma' := (V(\Gamma), \sim_\varepsilon \setminus \{(v_1, v_2)\})$$

is still connected for vertices $v_1 \neq v_2$.

4. *A graph is called acyclic if it contains no cycle.*

5. *A tree is an acyclic and connected graph. A spanning tree of a graph Γ is a subgraph τ that is a tree with $V(\tau) = V(\Gamma)$.*

Remark 2.1.2. 1. *For finite graphs, i.e. when $|V(\Gamma)| < \infty$, the connectedness property is equivalent to $x \sim_\varepsilon^* y$ or all vertices $x \neq y$. Here $\sim_\varepsilon^*$ denotes the transitive hull of $\sim_\varepsilon$.*

2. *An acyclic graph with anti-reflexive $\sim_\varepsilon$ is necessarily simple. Graphs such that $\sim_\varepsilon$ is not anti-reflexive contain so-called tadpole graphs, i.e. graphs with one vertex v and possibly multiple edges connecting v with itself.*

Definition 2.1.3. *(Graph polynomial) Let Γ be a finite, connected, not necessarily simple graph. The graph polynomial P_Γ of Γ is defined as*

$$P_\Gamma := \sum_\tau \prod_{e \notin \tau} X_e,$$

where τ runs through all spanning trees of Γ, and X_e is a polynomial variable for each edge $e \in E(\Gamma)$. The polynomial P_Γ is homogenous of degree $h = h_1(\Gamma)$ [8]. We define the graph hypersurface

$$X_\Gamma := \{P_\Gamma = 0\} \subset \mathbb{P}^{n-1}, \quad n = |E(\Gamma)|.$$

Following [8] we will now demonstrate that this polynomial is a configuration polynomial in the sense of section 2.2.

Construction : For Γ we choose an orientation of its edges. Define a map $\partial : \mathbb{Z}^{E(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$, by $e \mapsto \sum_{v \in V(\Gamma)} \operatorname{sgn}(v, e) v$, where $\operatorname{sgn}(v, e) = 1$ if v is the source of the edge e, further $\operatorname{sgn}(v, e) = -1$ if v is the target of E. This gives rise to a simplicial complex $\mathbb{Z}^{E(\Gamma)} \stackrel{\partial}{\to} \mathbb{Z}^{V(\Gamma)}$ and a corresponding exact sequence

$$0 \to H_1(\Gamma, \mathbb{Z}) \stackrel{\iota}{\to} \mathbb{Z}^{E(\Gamma)} \to \mathbb{Z}^{V(\Gamma)} \to H_0(\Gamma, \mathbb{Z}) \to 0.$$

The embedding $0 \to H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}^{E(\Gamma)}$ is a configuration space. Let $l_e(\cdot)$, $e \in E(\Gamma)$, denote the dual basis of the standard basis of all edges $e \in E(\Gamma) \subseteq \mathbb{Z}^{E(\Gamma)}$. Then we can consider the bilinear forms q_e of rank 1 given by

$$q_e := (l_e \circ \iota) \cdot (l_e \circ \iota) \; : \; H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}.$$

Choose a basis $B = (c_1, \ldots, c_{h_1(\Gamma)})$ of $H_1(\Gamma, \mathbb{Z})$, let $M_e = M_e(B)$ be the Gram matrix associated to q_e, and set

$$M_{\Gamma,B} := \sum_{e \in E(\Gamma)} X_e M_e \in \mathbb{Z}[X_e \colon e \in E(\Gamma)]_1 \otimes_{\mathbb{Z}} \operatorname{End}(\mathbb{Z}^{h_1(\Gamma)}).$$

Here, $\mathbb{Z}[X_e \colon e \in E(\Gamma)]_1$ denotes the degree 1 part of the algebra $\mathbb{Z}[X_e : e \in E(\Gamma)]$. We will usually abuse the notation and write M_Γ without the basis in the subscript. Alternatively with the language of section 2.2 we could just pass to the configuration polynomial of the configuration $0 \to H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}^{E(\Gamma)}$.

Lemma 2.1.4. *With the above notation one has* $P_\Gamma = \pm \det(M_\Gamma)$.

Proof. See [[8], Proposition (2.2)]. □

In this description of M_Γ, the diagonal entries contain sums of variables X_e of edges e contained in cycles in the basis B of $H_1(\Gamma, \mathbb{Z})$. The entries M_{ij} of M_Γ with $i < j$ contain variables X_e of edges which form the glueing data for the basis elements $c_i, c_j \in B$.

Lemma 2.1.5. *The diagonal entries of M_Γ generate a free $\mathbb{Z}$-submodule of rank $h_1(\Gamma)$ in $\mathbb{Z}[X_e \colon e \in E(\Gamma)]_1$ with free complement. In particular, if we tensor the same $\mathbb{Z}$-module with an arbitrary field k, then the diagonal entries remain k-linear independent.*

Proof. There is an isomorphism of free modules $\mathbb{Z}^{E(\Gamma)} \longrightarrow \mathbb{Z}[X_e \colon e \in E(\Gamma)]_1$ under the assignment $e \mapsto X_e$. There is a basis $B = (c_1, \ldots, c_{h_1(\Gamma)})$ of the free submodule $H_1(\Gamma, \mathbb{Z}) \subset \mathbb{Z}^{E(\Gamma)}$, where each c_i is a cycle in Γ such that the entries of $M_{\Gamma,B}$ on the diagonal are precisely the sums $\sum_{e \in c_i} X_e$:
First let $(\overline{c_1}, \ldots, \overline{c_l})$ be a basis of $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. Let

$$\mathcal{S}(c)(N) := \{e \in E(\Gamma) \colon pr_e \colon \langle c \rangle_{N \otimes_{\mathbb{Z}} H_1(\Gamma, \mathbb{Z})} \to N \otimes_{\mathbb{Z}} \mathbb{Z}^{E(\Gamma)} \to N \text{ is surjective}\},$$

where pr_e denotes the canonical projection to the e-th factor of $\mathbb{Z}^{E(\Gamma)}$. Since $\mathbb{Z}/2\mathbb{Z}$ is a flat $\mathbb{Z}$-module we have $\mathcal{S}(c_i)(\mathbb{Z}) = \mathcal{S}(\overline{c_i})(\mathbb{Z}/2\mathbb{Z})$ for a preimage of an element of a $\overline{c_i}$. Furthermore is the complement $E(\Gamma) \backslash \mathcal{S}(\overline{c_i})(\mathbb{Z}/2\mathbb{Z})$ the set of indices such that the image under canonical projection is trivial. Both, surjectivity and triviality are preserved under flat base change. This means that the non-zero coefficients of c_i in an $E(\Gamma)$-expansion are units in $\mathbb{Z}$, hence elements of $\{\pm 1\}$. The corresponding entry in M_Γ is then $\sum_{e\in c_i} X_e$. They generate a free submodule H of $\mathbb{Z}[X_e \colon e \in E(\Gamma)]_1$, and we obtain an isomorphism of free submodules

$$H_1(\Gamma, \mathbb{Z}) \longrightarrow H.$$

Any spanning tree τ in Γ induces a free submodule T in $\mathbb{Z}^{E(\Gamma)}$ generated by the edges of τ. Since the union of τ with any edge outside of τ contains a cycle, this free submodule is complementary to $H_1(\Gamma, \mathbb{Z})$. This shows that H has a free complement inside $\mathbb{Z}[X_e \colon e \in E(\Gamma)]_1$. □

Remark 2.1.6. *Subdivision of edges gives rise to affine fiber bundles over X_Γ: Let Γ' be the graph obtained from Γ by subdividing the edge e into e_1 and e_2. Then $P_{\Gamma'} = P_\Gamma|_{X_e = X_{e_1} + X_{e_2}}$.*

2.2 Determinantal Hypersurfaces

Since the later results in section 4.1 have not much to do with the fact that the polynomials studied there are coming from graph polynomials it is natural to not limit the discussion to these objects alone. A natural generalization in terms of multilinear algebra, that will be introduced below, has been successfully pursued in [8] and [41]. The bilinear families used below are a slight variation of this approach.

Construction: Let (W, B) be a based (finite dimensional) vector space over k, i.e. a pair consisting of a vector space W and a distinguished basis B of W. A *configuration* is a monomorphism $f \colon V \to (W, B)$. The choice of B provides a non-canonical isomorphism $W \to W^\vee$. The diagonal $\Delta \colon W^\vee \to W^\vee \otimes_k W^\vee$ yields a map

$$W \to W^\vee \to W^\vee \otimes W^\vee \to V^\vee \otimes V^\vee.$$

By selecting a basis C of V we can extend this sequence:

$$\beta_{f,C} \colon W \to W^\vee \to W^\vee \otimes W^\vee \to V^\vee \otimes V^\vee \to V^\vee \otimes V \simeq Hom(V, V).$$

If C' is another basis of V then there exists an isomorphism $\alpha \colon V \to V$ such that $(\alpha^\vee \otimes \alpha) \circ \beta_{f,C} = \beta_{f,C'}$.

Example 2.2.1. *Let us calculate what this definition means for the example of the trivial configuration $id_W \colon W \to (W, B)$. Pick an element $b_0 \in B$. Then*

$$b_0 \mapsto b_0^\vee \mapsto b_0^\vee \otimes b_0^\vee \in W^\vee \otimes W^\vee$$

Interpreting $b_0^\vee \otimes b_0^\vee$ as a dyadic product (and using $W \simeq W^\vee$) gives the desired element in $Hom_k(W, W)$.

By its construction $\beta_{f,C}$ defines a linear W-parametrized family of bilinear forms on V. By restricting V to one-dimensional subspaces respectively introducing linear coordinates one sees immediately that $\beta_{f,C}$ can be viewed as an element of $k[X_1, \ldots, X_{dim(W)}]_1 \otimes_k Hom_k(V, V)$.

Definition 2.2.2. *We call generators of the ideal $\langle det(\beta_{f,C})\rangle \trianglelefteq k[X_1, \ldots, X_n]$ configuration polynomials of the configuration $f\colon V \hookrightarrow (W, B)$.*

Let us give an elementary example.

Example 2.2.3. *Let $0 \to k^2 \xrightarrow{f} k^3$ be the injection defined by (and we identify the canonical basis (e_1, e_2) with the corresponding linear independent system in k^3)*

$$e_i \mapsto \begin{cases} e_2, & i = 1 \\ e_1, & i = 2 \end{cases} .$$

The Map $(k^3)^\vee \to (k^2)^\vee$ is given by

$$e_i^\vee \mapsto \begin{cases} e_2^\vee, & i = 1 \\ e_1^\vee, & i = 2 \\ 0, & else \end{cases} .$$

The representing matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which gives for the map $(k^3)^\vee \to (k^2)^\vee$

$$P \otimes P = \begin{pmatrix} 0_{2\times 3} & P & 0_{2\times 3} \\ P & 0_{2\times 3} & 0_{2\times 3} \end{pmatrix}.$$

The map $k^3 \to k^3 \otimes k^3$ is represented by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In total this gives the matrix

$$P \otimes P \cdot I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

If one reinterprets the columns as elements of $k^{2\times 2}$ one arrives at the parametric form

$$\beta_{f,C}(x_1,\, x_2,\, x_3) = \begin{pmatrix} x_2 & \\ & x_1 \end{pmatrix}$$

By the remarks preceeding the definition 2.2.2 we know that the determinant is unique up to multiplication with a non-zero element of k. For this reason we will use the following

Convention: Unless otherwise stated assume that for a configuration $f\colon V \hookrightarrow (W,\, B)$ a basis C of V has been chosen. In this situation we will drop the dependency on C in our notation and call

$$\Psi_f := \Psi_{f,C} := det(\beta_{f,C})$$

"the" configuration polynomial.

Remark 2.2.4. *The non-canonical choice of B is not the only part of the definition that is not canonical. We could have proceeded like this:*

$$W \to W^\vee \to V^\vee \to V^\vee \otimes V^\vee \simeq V^\vee \otimes V \simeq Hom(V,\, V).$$

However the diagram

$$\begin{array}{ccccc} W^\vee & \longrightarrow & W^\vee \otimes W^\vee & \longrightarrow & V^\vee \otimes V^\vee \\ \uparrow{\scriptstyle id} & & & & \uparrow{\scriptstyle id} \\ W^\vee & \longrightarrow & V^\vee & \longrightarrow & V^\vee \otimes V^\vee \end{array}$$

commutes (as one can immediately check).

The notion of a configuration hypersurface would work just as well if we would allow general maps $V \to (W,\, B)$.

Definition 2.2.5. *(i) A generalized configuration is a map $V \to (W,\, B)$.*

(ii) Let β_f be defined as in the non-generalized configuration case. Then β_f can be considered as an element $k[X_1,\, \ldots,\, X_{dim(W)}]_1 \otimes_k Hom(V,\, V)$. The generalized configuration polynomial will be

$$\Psi_f = det(\beta_f) \in k[X_1,\, \ldots,\, X_{dim(W)}].$$

Example 2.2.1 shows that $Im(\beta_{f,C}) \subseteq Sym(W)$. This fact is clearly also valid for general and generalized configurations.

Lemma 2.2.6. *Let $f\colon V \to (W,\, B)$ be a generalized configuration with associated configuration hypersurface Ψ_f. Then there exists a configuration $g\colon U \hookrightarrow (W,\, B)$ such that $Z(\Psi_f) = Z(\Psi_g)$.*

Proof. Remark 2.2.4 reduces the problem immediately to trivial generalized configurations $f\colon W \to (W,\, B)$. Let $U \subseteq W$ such that $U^\vee = Im(W^\vee \to W^\vee)$. This leads to the commutative diagram

$$\begin{array}{ccccc} W^\vee & \longrightarrow & W^\vee \otimes W^\vee & \xrightarrow{f^\vee \otimes f^\vee} & W^\vee \otimes W^\vee \\ \downarrow & & \downarrow & & \downarrow \\ U^\vee & \longrightarrow & U^\vee \otimes U^\vee & \xrightarrow{f_U^\vee \otimes f_U^\vee} & U^\vee \otimes U^\vee \end{array}$$

where the vertical maps are surjective and the first horizontal maps are injective. Let now $w_1^\vee \otimes w_1^\vee \in Im(W^\vee \otimes W^\vee \overset{f \otimes f}{\to} W^\vee \otimes W^\vee)$. This means $w_1^\vee \otimes w_2^\vee \notin Ker(W \otimes W \overset{f \otimes f}{\to} W \otimes W)^\vee$. Clearly $Ker(W \otimes W \overset{f \otimes f}{\to} W \otimes W) = Ker(f) \otimes Ker(f)$ and hence

$$Ker(W \otimes W \overset{f \otimes f}{\to} W \otimes W)^\vee = (Ker(f) \otimes Ker(f))^\vee \simeq Ker(f)^\vee \otimes Ker(f)^\vee.$$

This means that $w_1^\vee \otimes w_2^\vee \in U^\vee \otimes U^\vee$ which immediately implies the assertion. □

Let us review some of the basic properties of configurations and configuration polynomials. These properties are either easy to see or contained in [41].

Proposition 2.2.7. *Let $0 \to V \to (W,\, B)$ be a configuration and $\beta_B\colon W \to Sym^2(V^\vee)$ be a composition as above.*

(i) $\Psi_{V \hookrightarrow (W,\, B)}$ *is homogeneous of degree $dim(V)$.*

(ii) Let $V_i \hookrightarrow (W_i,\, B_i)$ be non-zero configurations for $i = 1,\, 2$. Then $V_1 \oplus V_2 \hookrightarrow (W_1 \oplus W_2,\, B_1 \cup B_2)$ is also an non-zero configuration and we have the equation

$$\Psi_{V_1 \hookrightarrow (W_1,\, B_1)} \cdot \Psi_{V_2 \hookrightarrow (W_2,\, B_2)} = \Psi_{V_1 \oplus V_2 \hookrightarrow (W_1 \oplus W_2,\, B_1 \cup B_2)}$$

for the configuration polynomials.

(iii) There exist $c_F \in k$ for each $F \subseteq B$ such that

$$\Psi_{V \hookrightarrow (W,\, B)} = \sum_{F \subseteq B,\, |F| = dim(V)} c_F \prod_{f \in F} X_f.$$

(iv) Each variable X_e has degree at most 1 in $\Psi_{V \hookrightarrow (W,\, B)}$.

(v) Suppose $\Psi_{V \to (W,\, B)} = f \cdot g$ for non units $f,\, g \in k[X_1,\, \ldots,\, X_n]$. Then $deg_i(f),\, deg_i(g) \leq 1$ for all $1 \leq i \leq n$ and f and g have no variables in common.

Proof. (ii) and (iii) are [[41], 3.1.10] and [[41], 3.1.12]. (i) is a consequence of (iii). (iv) is [[8], 1.2]. (v) is an easy consequence of the degree formula. □

2.3 Determinantal schemes and torus actions

The defintion above also directly generalizes to vector bundles over a projective base X. We will implicitly give this generalization by showing how to generalize it to schemes over a (noetherian) base S. The approach via determinantal schemes to graph hypersurfaces has been used in [4] but in a slightly different form.

Let E and F be locally free $\mathcal{O}_S$-modules.

Definition 2.3.1. *Let $\varphi\colon E \to F$ be a morphism of locally free $\mathcal{O}_S$-modules of finite rank. The r-degeneracy locus (as a topological space) of the morphism φ is the closed subset*

$$D_r(\varphi) := \{s \in S\colon rank(\widehat{\varphi_s}\colon E_s \otimes k(s) \to F_s \otimes k(s)) \leq r\}.$$

$D_r(\varphi)$ carries a natural scheme structure as it is the support of the kernel of the induced map

$$\bigwedge^{r+1} \varphi\colon \bigwedge^{r+1} E \to \bigwedge^{r+1} F.$$

Consider now the pullback functor

$$Sch/S \to Ab$$

defined by

$$T \to S \mapsto Hom_{\mathcal{O}_T}(E_T^\vee, F_T),$$

on the objects and

$$T_1 \to T_2 \mapsto Hom_{\mathcal{O}_{T_2}}(E_{T_2}^\vee, F_{T_2}) \xrightarrow{\varphi^*} Hom_{\mathcal{O}_{T_1}}(E_{T_1}^\vee, F_{T_1})$$

on the morphisms. Here E_T, F_T are the pullbacks of the respective sheaves (as modules) to T.

Definition 2.3.2. *Let $\pi\colon Hom(E^\vee, F) \to S \in Sch/S$ be the representing object of the above functor. (This exists since this functor is a contravariant functor mapping colimits to limits.) Associated to this scheme via the Yoneda lemma is a morphism $\phi^*\colon \pi^*E^\vee \to \pi^*F$ (representing the universal element of the representation, in the sense of category theory). With this notation we call for any $r \in \mathbb{N}_0$ the schemes*

$$Hom_{\leq r}(E^\vee, F) := Hom(E^\vee, F) \times_S D_r(\phi^*)$$

r-generic determinantal schemes.

Note that in the definition above the dual is not necessary to specify the object $Hom(E^\vee, F)$ (at least not up to isomorphism) since locally free modules of finite rank are self-dual. We will still try to stick to the canonical representation to avoid unnecessary obfuscation in view of concrete computations and to display the relevant objects and constructions in their natural level of generality.

There is a natural automorphism, the transpose operation,

$$^t\colon Hom_{\mathcal{O}_T}(E_T^\vee, E_T) \to Hom_{\mathcal{O}_T}(E_T^\vee, E_T),$$

which is given by the usual transposition composed with dualization in the domain. By $Sym(E)$ we denote the representing object of the subfunctor induced by by the fixed points of ^{t}. We will call the schemes

$$Sym_{\leq r} := Sym(E) \times_S Hom_{\leq r}(E^\vee, E)$$

r-generic symmetric determinantal schemes.

Lemma 2.3.3. *Let E be a locally free $\mathcal{O}_S$-module (of finite rank) and let $\mathcal{V}(E^\vee)$ denote the vector bundle associated to E. Then there exist closed immersions $\mathcal{V}(E) \to Hom(E^\vee, E)$ and $\mathcal{V}(E^\vee) \times_S D_d(\phi^*_{\mathcal{V}(E^\vee)}) \to Hom_{\leq d}(E^\vee, E)$, where $\phi_{\mathcal{V}(E^\vee)} = \phi^* \otimes id_{\mathcal{V}(E^\vee)}$ and $\phi^*\colon \pi^* E^\vee \to \pi^* F$ is the universal morphism of 2.3.2.*

Proof. I will first give a description of $Hom(E^\vee, F)$ and $Hom_{\leq r}(E^\vee, F)$ in terms of coordinates and ideals (compare [[4], p.9]). Assume for simplicity $S = Spec(A)$. Let $T \to S$. Let $f\colon T \to S$, then the defining functor of $Hom(E^\vee, F)$ has stalkwise the form

$$\begin{aligned} Hom_{\mathcal{O}_{T,p}}(f^{-1}(E^\vee)_p \otimes_{f^{-1}(\mathcal{O}_{S,p})} \mathcal{O}_{T,p},\ f^{-1}(F)_p \otimes_{f^{-1}(\mathcal{O}_{S,p})} \mathcal{O}_{T,p}) &= \\ Hom_{\mathcal{O}_{T,p}}(E^\vee_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{T,p},\ F_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{T,p}) &= \\ Hom_{\mathcal{O}_{S,f(p)}}(E^\vee_{f(p)},\ \rho(\mathcal{O}_{S,f(p)})(F_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{T,p})) & \end{aligned}$$

with $\rho(\mathcal{O}_{S,f(p)})(-)$ being the restriction of to $\mathcal{O}_{S,f(p)}$. In our use case F is a locally free $\mathcal{O}_S$-module. Since $F_{f(p)}$ is a free module we have

$$\rho(\mathcal{O}_{S,f(p)})(F_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{T,p}) = F_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \rho(\mathcal{O}_{S,f(p)})(\mathcal{O}_{T,p})$$

Because free modules are projective (and hence reflexive) we can write

$$\begin{aligned} Hom_{\mathcal{O}_{S,f(p)}}(E^\vee_{f(p)},\ \rho(\mathcal{O}_{S,f(p)})(F_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \mathcal{O}_{T,p})) &= \\ Hom_{\mathcal{O}_{S,f(p)}}(E^\vee_{f(p)},\ F_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} \rho(\mathcal{O}_{S,f(p)})(\mathcal{O}_{T,p})) &= \\ Hom_{\mathcal{O}_{S,f(p)}}(E^\vee_{f(p)} \otimes_{\mathcal{O}_{S,f(p)}} F^\vee_{f(p)},\ \rho(\mathcal{O}_{S,f(p)})(\mathcal{O}_{T,p})) & \end{aligned}$$

From this one sees that (given $S = Spec(A)$) the scheme $Hom(E, F)$ is affine and given by $A[\underline{y}]$, where $\underline{y} = \{y_{ij} \colon 1 \leq i \leq rank(E), 1 \leq j \leq rank(F)\}$. Further if $\mathfrak{m}_d$ is the ideal generated by the $d \times d$-minors of the matrix $(y_{ij})_{1 \leq i \leq rank(E), 1 \leq j \leq rank(F)}$ then

$$Hom_{\leq d}(E^\vee, F) = Spec(A[\underline{y}]/\mathfrak{m}_d).$$

Assume now $E = F$ and (without loss of generality) $E = Spec_A(M)$ for some free A-module M. There is an A-algebra morphism $A[\underline{y}] \to \mathcal{S}(E^\vee)$ (with $\mathcal{S}(-)$ denoting the symmetric algebra functor) given by

$$y_{ij} \mapsto \delta_{ij} m_i$$

in degree 1. This homomorphism is obviously surjective, this gives a closed immersion $\mathcal{V}(E) \hookrightarrow Hom(E^\vee, E)$. The immersion $\mathcal{V}(E^\vee) \times_s D_r(\phi^*_{\mathcal{V}(E^\vee)}) \to Hom_{\leq r}(E^\vee, F)$ follows from the fact that (in suitable affine open subschemes)

$$\mathcal{S}(E^\vee)/\langle Im(A[\underline{y}] \to \mathcal{S}(E^\vee))(\mathfrak{m}_r)\rangle_{\mathcal{S}(E)} \simeq \mathcal{S}(E^\vee)/ann(Ker((\phi^*)^{\wedge^r}_{\mathcal{V}(E^\vee)}))$$

and that the map $A[\underline{y}] \to \mathcal{S}(E^\vee)$ obviously extends such that the diagram

$$\begin{array}{ccc} A[\underline{x}] & \longrightarrow & \mathcal{S}(E^\vee) \\ \downarrow & & \downarrow \\ A[\underline{x}]/\mathfrak{m}_r & \longrightarrow & \mathcal{S}(E^\vee)/\langle Im(A[\underline{y}] \to \mathcal{S}(E^\vee))(\mathfrak{m}_r)\rangle_{\mathcal{S}(E^\vee)} \end{array}$$

commutes (the vertical maps are the canonical surjections).

□

Definition 2.3.4. *Let E and F be locally free $\mathcal{O}_S$-modules of finite rank.*

(i) Let $\pi \colon Hom(E^\vee, F) \to S$ be the morphism making $Hom(E^\vee, F)$ into a S-scheme. Let further be $\phi \colon \pi^* E^\vee \to \pi^* F$ be the universal element corresponding to $Hom(E^\vee, F)$. A rank r determinantal scheme D is a scheme such that there exists a morphism $f \colon E^\vee \to F$ such that following diagram is a pullback diagram:

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{V}(E^\vee) \\ \Big\downarrow & & \downarrow \\ & & Hom(E^\vee, E) \\ & & \downarrow \\ & & Hom(E^\vee, F) \\ & & \downarrow \\ D_r(\phi) & \longrightarrow & S \end{array}$$

where the map $Hom(E^\vee, E) \to Hom(E^\vee, F)$ *is induced by* f. *With the above notation we will usually write* $D(f) := D$.

(ii) A rank r *symmetric determinantal scheme* D *is a scheme such that there exists a symmetric morphism* $f\colon E \to E$ *such that* D *fits as a pullback in the square*

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{V}(E^\vee) \\ \big\downarrow & & \big\downarrow \\ & & Sym(E) \\ & & \big\downarrow \\ & & Sym(E) \\ & & \big\downarrow \\ D_r(\pi^* f) & \longrightarrow & S \end{array}$$

where the map $Sym(E) \to Sym(E)$ *is induced by* f.

If $r \neq min(rk(E), rk(F)) - 1$ *we will mention the rank explicitly. In all other instances we will omit the reference to* r *and simply speak of determinantal schemes. The same goes for symmetric determinantal schemes.*

Remark 2.3.5. *Let* $f\colon E \to E$ *be a morphism of locally free* $\mathcal{O}_S$*-modules. Let* $f^2 := f^t f$, *where* t *is the transposition operation. Let further be* $D(f^2)$ *be the associated determinantal scheme. Then* $D(f^2)$ *is a symmetric determinantal scheme.*

Remark 2.3.6. $Hom_{\leq r}(E^\vee, E)$ *is, by definition, a pullback of* $Hom(E^\vee, E)$ *via a degeneracy locus over* S. *Suppose there is a map of locally free* $\mathcal{O}_S$*-modules* $F \to E$. *Since* $Qcoh(S)$, *the category of quasi-coherent sheaves on* S, *is a rigid tensor category we can write* $Hom(E^\vee, E) \simeq E^\vee \otimes_{\mathcal{O}_S} E^\vee$. *Since* $\otimes$ *is a covariant in each factor this yields a map*

$$Hom(E^\vee, E) \to Hom(F^\vee, F).$$

This means that symmetric determinantal schemes depend covariantly on the argument.

For the point of view of applications to graph hypersurfaces the notions of configuration hypersurfaces and symmetric determinantal schemes are fairly close:

Proposition 2.3.7. *(i) Let* $f\colon V \to (W, B)$ *be a configuration. The configuration hypersurface* $X\colon \Psi_f = 0 \subseteq \mathbb{P}(W)$ *is a symmetric determinantal scheme.*

(ii) *Let* $S = Spec(k)$ *and* X *be a symmetric determinantal scheme. Then* X *is a configuration hypersurface (induced by a generalized configuration).*

Proof. (i) The description of a configuration hypersurface as a variety together with the description of $Sym_{\leq r}(E)$ in terms of coordinates and ideals given in the proof of Lemma 2.3.3 immediately implies the assertion.

(ii) Obviously E is just a finite dimensional vector space over k and $\mathcal{V}(E) \simeq \mathbb{A}^{rk(E)}$. By defintion there exists a symmetric map $f\colon E \to E$ such that $D = D_r(\phi) \times_k \mathcal{V}(E)$ with $r = rk(E) - 1$. By using the description of the Hom-schemes in terms if their structure sheaves (compare the proof of Lemma 2.3.3) we can identify

$$Sym(E) = Spec(A[\underline{y}]/\langle y_{ij} - y_{ji}\colon 1 \leq i,\, j \leq rk(E)\rangle)$$

and

$$D_r(\phi) = Spec(B/\mathfrak{m}_{r+1}),$$

with $B = A[\underline{y}]/\langle y_{ij} - y_{ji}\colon 1 \leq i,\, j \leq rk(E)\rangle$ and $\mathfrak{m}_{r+1}$ being the ideal generated by the $(r+1) \times (r+1)$-minors of the generic $rk(E) \times rk(E)$-matrix. Hence follows the assertion.

□

It is well known (and true in a general categorical setting) that group actions correspond to natural injections into automorphism functors. In the case of representability of these functors this yields a morphism into the automorphism object. Concretely let $X \in Sch/S$ be a scheme, $\delta\colon G \times_S X \to X$ be a group scheme action of a group scheme $G \in Sch/S$. Assume further that the automorphism functor is representable. (Denote this representing scheme by $Aut(X)$.) Then there exists a morphism of group schemes

$$\rho\colon X \to Aut(X)$$

"realizing" the group action on (geometric) points.

Definition 2.3.8. *With the notation above we will call* δ *(respectively* ρ*) faithful iff* ρ *is an immersion.*

Remark 2.3.9. *The representability of the automorphism functor is a fairly well researched question. For our main objects of interest, (projective) determinantal varieties it is already contained in an old paper by Matsusaka [34]: The (abstract) group of automorphisms of a projective variety has the structure of a locally algebraic group with finite or countably many components (i.e. a group scheme with at most countable many components that is locally an algebraic group).*

To formulate concrete results we need to restrict the general case of group scheme actions to "linear" group scheme actions.

Definition 2.3.10. *Let X be a S-scheme and $G \in Sch/S$ be a group scheme. A group scheme action $\delta\colon G \times_S X \to X$ is called linear if there is a vector bundle $\mathcal{V} \to S$, an immersion $X \to \mathcal{V}$ and a group scheme action $\delta_0\colon G \times_S \mathcal{V} \to \mathcal{V}$ such that δ_0 yields a homomorphism $G_p \to GL(\mathcal{V}_p)$ on the stalks and*

$$\begin{array}{ccc} G \times_S \mathcal{V} & \longrightarrow & \mathcal{V} \\ \uparrow & & \uparrow \\ G \times_S X & \longrightarrow & X \end{array}$$

commutes. If x is linear with respect to a certain locally free $\mathcal{O}_S$-module E, i.e. $\mathcal{V} = \mathcal{V}(E)$ in the above diagram, then we will indicate this by saying X is linear w.r.t. to E.

Lemma 2.3.11. *Let $X \in Sch/S$ and G a S-group scheme. There is an equivalence of categories*

$$\{Linear\ G - actions\ on\ X\} \longleftrightarrow \left\{ \begin{array}{c} Group\ scheme\ morphisms \\ G \to Aut(X) \times_S GL(\mathcal{V}), \\ for\ some\ vector\ bundle\ \mathcal{V} \to S \\ such\ that\ X \hookrightarrow \mathcal{V} \end{array} \right\}$$

Proof. (Sketch) Without the "linearity" attributes on both sides the lemma is well known folklore. To see the linearity preservation note that passing from actions to representations, i.e. group scheme morphisms $G \to Aut(X)$ is covariant. This is the direction "$\to$". For a functor in the other direction note that the immersion condition $X \hookrightarrow \mathcal{V}$ turns the fibre product $Aut(X) \times_S GL(\mathcal{V})$ locally into an intersection. From this it is easy to see that the action corresponding to the morphism $G \to GL(\mathcal{V})$ induces an action $G \times_S X \to X$ yielding a commutative diagram as in Defintion 2.3.10. □

Let us consider the case of torus actions on hypersurfaces $Z(f) \subseteq \mathbb{A}^n(k)$ over a field k. We assign to each such f a $\mathbb{Z}$-module as follows:

Definition 2.3.12 (Weight lattice)**.** *Let $s \in \mathbb{N}_0$ and $k[X_1, \ldots, X_n]_s$ be the vector space of homogeneous polynomials in n variables of degree s. For any*

$$f = \sum_{|\alpha|=s} c_\alpha X^\alpha \in k[X_1, \ldots, X_n]_s$$

we call

$$\Lambda_f := \{\eta \in \mathbb{Z}^n \ :\ \eta \cdot \alpha = c \text{ independent of } \alpha \text{ for any } \alpha \text{ with } c_\alpha \neq 0\}$$

the weight lattice of the polynomial f. We will call the elements of Λ_f the weight vectors of f. Each weight vector $\eta \in \Lambda_f$ gives rise to a $\mathbb{G}_m$-operation on $X = \{f = 0\} \subset \mathbb{P}^{n-1}$ by $X_i \mapsto t^{\eta_i} X_i$. An operation corresponding to a weight vector ω if trivial on $\mathbb{P}^{n-1}$ if and only if $\omega \in \mathbb{Z} \cdot (1, \dots, 1)$. Since we consider homogeneous polynomials f, we have $\mathbb{Z} \cdot (1, \dots, 1) \subseteq \Lambda_f$. Thus, the weight lattice Λ_f has rank $r+1$ if and only if X admits a linear rank r torus operation in the coordinates X_i.

Remark 2.3.13. *In the notation of Definition 2.3.12: A $\mathbb{Z}$-basis $(\omega_1, \dots, \omega_s)$ of Λ_f gives rise to a multigrading of $k[X_1, \dots, X_n]$ in the sense of [28] by forming dot products as in the definition 2.3.12. On the other hand it is well known that $\mathbb{G}_m^s$-actions correspond to homogeneous ideals, where homogeneity is determined by a multigrading.*

For our purposes following two questions are relevant:

- Under which assumptions on symmetric morphisms of locally free $\mathcal{O}_S$-modules $f\colon E \to E$ is a given action $\mathbb{G}_m^r \times_S D(f) \to D(f)$ faithful?
- Let $f\colon E \to E$ as above. What is the maximal r such that $\mathbb{G}_m^r \hookrightarrow Aut(X)$; respectivly what is the maximal r such that there is an immersion of group schemes $\mathbb{G}_m^r \hookrightarrow Aut(X) \times_S GL(E)$?

Unfortunately we can only give partial answers to these questions. But note the following remark.

Remark 2.3.14. *Let $S = Spec(k)$ and let $X \subseteq \mathbb{A}^n$ be a hypersurface together with a (w.r.t. the ambient space) linear $\mathbb{G}_m^s$-action that is faithful on $\mathbb{A}^n$. Let Ω be a $\mathbb{G}_m^s$-orbit in $\mathbb{A}^n$ of maximal dimension. Then $\overline{\Omega}$ is a variety of dimension s. By the affine dimension theorem this implies $dim(X \cap \overline{\Omega}) \geq s-1$, which implies that the kernel $\mathbb{G}_m^s \to Aut(D(f)) \cap GL(E)$ is at most 1-dimensional.*

Definition 2.3.15. *Let $\delta_X\colon G \times_S X \to X$ and $\delta_Y\colon G \times_S Y \to Y$ be group scheme actions. Let $f\colon X \to Y$ be a G-equivariant morphism. Then we will call δ_Y faithful relative f if the kernel of the action on Y is the image of the kernel on X.*

Lemma 2.3.16. *Let $X = Z(f) \subseteq \mathbb{A}^n$ by a hypersurface and $\mathbb{G}_m^s$ acting (linearly) on X. Assume there exists a ($\mathbb{G}_m^s$-equivariant) projection $\pi\colon \mathbb{A}^n \to \mathbb{A}^{n-r}$ (for some $1 \leq r \leq n$) such that $\mathbb{G}_m^s$ acts faithful on $\mathbb{A}^{n-r}$ relative π. Then $X' := \pi(X)$ is faithful relative $\pi|_X$.*

Proof. This is a direct consequence of the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{A}^n & \longrightarrow & \mathbb{A}^{n-r} \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' = \pi(X) \end{array}$$

where the vertical arrows are given by inclusions. □

There is a systematic way of checking the faithfulness using the above Lemma 2.3.16:

Lemma 2.3.17. *Let* $Z(f) \subseteq \mathbb{A}^n$ *be an irreducible hypersurface that is invariant under* $\mathbb{G}_m^s$ *acting linear w.r.t. the underlying linear space of* $\mathbb{A}^n$*. Then either there exists a projection* $\pi\colon \mathbb{A}^n \to \mathbb{A}^{n-r}$ *such that the action on* $\mathbb{A}^{n-r}$ *is relatively faithful, or* f *is a monomial.*

Proof. There exists a relative faithful projection if the multigrading associated to $\mathbb{G}_m^s$ is of rank $< n$. If not then the ideal $\langle f \rangle \trianglelefteq k[X_1, \ldots, X_n]$ is monomial by [[28], Prop. 2.3]. □

Chapter 3

Motives, Graphs and the Theorem of Bialynicki-Birula

3.1 Motives

As explained earlier, there are several ways of narrowing down the complexity of evaluating Feynman amplitudes by exploiting possible motivic interpretations. Let us first describe briefly the underlying idea and the most important properties that are relevant to our context.

Contrary to algebraic topology, where the Eilenberg-Steenrod axioms largely (up to some normalization) determine cohomology functors, in algebraic geometry there is a plethora of cohomology functors such as Hodge cohomology, crystalline cohomology, l-adic cohomology and de Rahm cohomology. All these theories have in common that they are Weil cohomology theories, i.e.:

Definition 3.1.1. *(Weil Cohomology Theories, [18]) Let k and K be fields. Let further $char(K) = 0$. A Weil cohomology functor H is a contravariant functor*

$$H\colon \{Smth.\ proj.\ varieties\ over\, k\} \to \{Graded\ comm.\, K - algebras\}$$

together with the following data

(D1) A 1-dimensional space $K(1)$. Using this space we introduce $V(n) := V \otimes_K K(1)^{\otimes n}$, for any K-vector space V and $n \in \mathbb{N}_0$. Further we define $V(-n) := V \otimes_K Hom(K(1)^{\otimes -n}, K)$.

(D2) For every smooth projective variety X over k there is a trace map $Tr\colon H^{2dim(X)}(X)(dim(X)) \to K$.

(D3) For every smooth projective variety X and every codimension c subvariety $Z \subseteq X$ there is a cohomology class $cl(Z) \in H^{2c}(X)(c)$

that has to satisfy the following properties

(W1) $H^i(X)$ *is a finite dimensional* K*-vector space.*

(W2) $H^\bullet(X)$ *is concentrated in the degrees* $[0,\, 2dim(X)]$, *i.e.* $H^i(X) = 0$ *for all* $i \notin [0,\, 2dim(X)]$.

(W3) There is a canonical Künneth isomorphism

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times_k Y)$$

given via contravariance and the existence of the cup product (as H takes values in the category of graded K*-algebras) by the formula*

$$\alpha \otimes \beta \mapsto pr^*X(\alpha) \cup pr_Y^*(\beta).$$

(W4) The composition of the cup product and the trace map

$$H^i(X) \times H^{2dim(X)-i}(X) \overset{\cup}{\to} H^{2dim(X)}(X) \overset{Tr}{\to} K$$

induces a non-degenerate pairing.

(W5) For non-singular projective varieties X, Y *the map*

$$Tr_{X \times_k Y} \colon H^{2dim(X)+2dim(Y)}(X \times_k Y)(dim(x) + dim(y)) \to K$$

satisfies

$$Tr_{X \times_k Y}(pr_X^*(\alpha) \cup pr_Y^*(\beta)) = Tr_X(\alpha) Tr_Y(\beta).$$

(W6) Given X, Y *with closed subvarieties* $Z \subseteq X$, $W \subseteq Y$ *we have*

$$cl(Z \times_k W) = pr_X^*(cl(Z)) \cup pr_Y^*(cl(W)).$$

(W7) Let $f \colon X \to Y$ *and* $Z \subseteq Y$ *closed such that* $dim(f^{-1}(Z)) = dim(Z) + dim(X) - dim(Y)$. *Let* $[f^{-1}(Z)]_k$ *denote the algebraic cycle corresponding to* $f^{-1}(Z)$. *Then* $[f^{-1}(Z)]_k = \sum n_i Z_i$, *where* $k = dim(Z) + dim(X) - dim(Y)$. *Then (W7) requires*

$$f^*(cl(Z)) = \sum n_i cl(Z_i).$$

(W8) $Tr_{Spec(k)}(cl(Spec(k))) = 1.$

Grothendieck observed that many important applications of cohomology theory can be derived using the above described list of properties of a cohomology theory. This led him to conjecture the exitence of a universal such cohomology theory, that was supposed to arise in the same way as the other cohomology theories, namely by specifying a suitable "motivic" coefficient sheaf functor on the category of (smooth) schemes over a field k and defining the associated motivic cohomology as the right derived functor of the global sections functor of that sheaf. The category of these motivic coefficient sheaves is then expected to be an abelian tensor category plus a list of nice properties. Before defining universal Weil cohomology theories it is worth pointing out that one of the desired properties of a universal Weil cohomology theory is that it factors over the category of smooth correspondences. There are several reasons why one would want this. I'll name two. First the Hom-sets in SmProj_k (the smooth maps between smooth projective varieties) tend to be rather "small". Another reason is that one can infer the correctness of formulas in H^*, where H denotes a Weil cohomology theory, from the so-called Manin-identity-principle (which is (almost) identical to the Yoneda-lemma if the underlying category is the category of smooth correspondences modulo an adequate equivalence) for algebraic cycles (see [[20], p.312]).

Definition 3.1.2. *A universal Weil cohomology is a contravariant functor*

$$\mathcal{H}\colon SmCorr_k \to \mathcal{A}$$

with $\mathcal{A}$ being a pseudoabelian tensor category such that for each Weil cohomology theory $H\colon SmProj_k \to \{Graded\ comm.\, K-algebras\}$ there exists a (unique) tensor functor $\mathcal{A} \to \{Graded\ comm.\, K-algebras\}$ such that the following diagram commutes:

$$\begin{array}{ccc} SmCorr_k & \longrightarrow & \mathcal{A} \\ \uparrow & & \downarrow \\ SmProj_k & \longrightarrow & \{Graded\ comm.\, K-algebras\} \end{array}$$

Based on this definition one constructs the category of pure motives as follows.

1. One constructs an additive category with certain properties by dividing out an equivalence relation $\sim$ from SmCorr_k.
2. One passes to the pseudoabelian envelop of $\mathrm{SmCorr}_k/\sim$. Let us denote that envelop by $(\mathrm{SmCorr}_k/\sim)^\sharp$
3. One inverts formally the "motive" $\mathbb{L} := [\mathbb{P}^1] \in (\mathrm{SmCorr}_k/\sim)^\sharp$. (The $\mathbb{L}$ is for "Lefschetz".)

Then the category $(\mathrm{SmCorr}_k/\sim)^\sharp\,[\mathbb{L}]$ is called the category of pure motives over the field k with respect to $\sim$. One suitable choice of $\sim$ such that the resulting category of pure motives provides a universal Weil cohomology theory is to set $\sim := \sim_{rat}$, the rational equivalence on algebraic cycles. The corresponding theory of pure motives in this case is called the category of Chow motives. Note that it is not a Weil cohomology theory itself (since $(\mathrm{SmCorr}_k/\sim)^\sharp[\mathbb{L}]$ is not abelian).

For many applications the restriction to smooth projective is too restrictive and wouldn't be sufficient as a unifying cohomology theory, since it would leave out e.g. the theory of mixed Hodge structures. As a way to extend the category of pure motives over k Alexander Beilinson conjectured the existence of an abelian tensor category $\mathrm{MM}(k)$ such the *Ext*-groups give rise to so-called motivic cohomology.

Definition 3.1.3. *(Motivic Cohomology, [21]) We will use $H^\bullet$ to denote ordinary cohomology and $\mathbb{H}^\bullet$ for hypercohomology. There exists a complex of Zariski sheaves $\mathbb{Z}(n)$ for all $n \geq 0$ such that the following properties hold:*

(MC1) *$\mathbb{Z}(0)$ is the constant sheaf.*

(MC2) *$\mathbb{Z}(1)$ is the sheaf $\mathcal{O}^*$ placed in cohomological degree 1.*

(MC3) *For any field F/k one has*

$$H^n(Spec(F), \mathbb{Z}(n)) = K_n^M(F),$$

where K_n^M is the Milnor K-group.

(MC4) *$\mathbb{H}^{2n}_{Zar}(X, \mathbb{Z}(n)) = CH^n(X)$, where CH^n denotes the Chow group of cycles of codimension n.*

(MC5) *For any smooth scheme X over k there is a canonical spectral sequence with E_2-term*

$$E_2^{p,q} = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q))$$

and differentials $d_r\colon E_r^{p,q} \to E^{p+1,q+2r-1}$ converging to Quillen K-groups $K_{2q-p}(X)$. After tensoring with $\mathbb{Q}$ this spectral sequence degenerates yielding

$$\mathbb{H}^i_{Zar}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq gr^n_\gamma K_{2n-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where gr^n_γ is the γ-filtration on K-theory.

The cohomology functor

$$\mathcal{H}^\bullet_{\mathcal{M}}(-, \mathbb{Z}(n))\colon Sch/k \to \{Graded\ k - Vector\ spaces\},$$

$$X \mapsto \mathbb{H}^\bullet_{Zar}(X, \mathbb{Z}(n))$$

is called motivic cohomology.

Remark 3.1.4. *Originally Beilinson required two further axioms. These axioms are also known as the Beilinson-Lichtenbaum Conjecture, which roughly requires the motivic cohomology groups to coincide with l-adic cohomology when changing to* $\mathbb{Z}/l\mathbb{Z}$ *coefficients, and the Beilinson-Soule vanishing conjecture that requires motivic cohomology to be concentrated in non-negative degrees for smooth schemes.*

Instead of directly trying to construct $\mathrm{MM}(k)$ Deligne proposed to first construct a triangulated category $\mathrm{DM}(k)$ that containes an object $\mathbb{Z}(n)$ such that the cohomology functor induced by

$$X \mapsto Hom_{\mathrm{DM}(k)}(X, \mathbb{Z}(i)[j])$$

yields a cohomology satisfying the cohomological constraints of Definition 3.1.3. Note that j denotes the cohomological shift. In a second step one should try to find a t-structure such that the associated heart gives an abelian tensor category that would satisfy the requirements of $\mathrm{MM}(k)$. I am now going to report briefly how Voevodsky constructs $\mathrm{DM}(k)$. For the background from homological algebra see [47] and for (localization of) triangulated categories [38].

Construction of the triangulated category of motives over a field.

1. Take the pseudoabelian envelop of the category of smooth correspondences $\mathrm{SmCorr}(k)$.
2. E.g. by using abelianization of the pseudoabelian category $\mathrm{SmCorr}(k)$ or by directly constructing a model structure on $Ch^b(\mathrm{SmCorr}(k))$ (where the weak equivalences are homotopy equivalences of complexes and cofibrations are Kan-complexes) we can pass to the homotopy category of the bounded complexes $\mathcal{H}^b(\mathrm{SmCorr}(k))$ over $\mathrm{SmCorr}(k)$.
3. Take T to be the minimal thick category containing for each smooth x the complexes

$$[X \times_k \mathbb{A}^1] \to [X]$$

and open coverings $X = U \cup V$

$$[U \cap V] \stackrel{[j_U] \oplus [j_V]}{\to} [U] \oplus [V] \stackrel{[i_U] \oplus (-[i_V])}{\to} [X]$$

for the obvious open embeddings j_U, j_V, i_U, i_V. Then the category of effective geometric motives over k is the localization $\mathrm{DM}_{gm}^{eff}(k) := \mathcal{H}^b(\mathrm{SmCorr}(k))[T]$.

This category enjoys many properties that give rise to the well-known features that algebraic cohomoloy theories have, which includes homotopy invariance, Mayer-Vietoris, the Künneth-formula and Blow-up distinguished triangles. For more details see subsection 3.1.1.

One very important property that is expected to hold for MM(k) is semi-simplicity.

Definition 3.1.5. *A general additive category $\mathcal{C}$ is called semi-simple if there exists a class of objects $\mathcal{S}$ in $\mathcal{C}$ that are simple, i.e. for $S_1,\, S_2 \in \mathcal{S}$ we have*

$$Hom_{\mathcal{C}}(S_1,\, S_2) = \begin{cases} 0, & S_1 \neq S_2 \\ Aut(S_1) \cup \{0\}, & S_1 = S_2 \end{cases}$$

such that each object $X \in \mathcal{C}$ decomposes into $\mathcal{S}$-factors, i.e. there exists a (set-theoretical) map $f\colon \alpha \to \mathcal{S}, \beta \mapsto S_\beta$ for a suitable ordinal number α such that

$$X \simeq \bigoplus_{\beta\in\alpha} S_\beta.$$

In the following lemma we want to assume that the additive category is closed under biproducts sufficiently general to represent all cases in the Definition 3.1.5, e.g. if arbitrary small biproducts occur in decompositions then the category should be closed under small biproducts. Recall that a category is called pseudoabelian iff all idempotents split.

Lemma 3.1.6. *Let $\mathcal{C}$ be a semi-simple additive category. Then $\mathcal{C}$ is pseudoabelian.*

Proof. Let $p = p^2\colon \bigoplus_{\beta\in\alpha} S_\beta \to \bigoplus_{\beta\in\alpha} S_\beta$ be an idempotent. After possible reordering there exists a minimal ordinal number $\beta_0 \in \alpha$ such that $\pi\colon \bigoplus_{\beta\in\alpha} S_\beta \to \bigoplus_{\beta_0\leq\beta} S_\beta$ has the property $\pi \circ p = 0$. This implies for the projections $\pi_{\beta_1}\colon \bigoplus_{\beta\in\alpha} S_\beta \to S_{\beta_1}$, where $\beta_1 < \beta_0$, that $S_{\beta_1} \overset{\iota_{\beta_1}}{\to} \bigoplus_{\beta\in\alpha} S_\beta \overset{p}{\to} \bigoplus_{\beta\in\alpha} S_\beta \overset{\pi_{\beta_1}}{\to} S_{\beta_1}$ is not 0. Hence it must be an idempotent isomorphism and thus the identity. □

The current state in the theory of motives is the following famous theorem of Jannsen:

Theorem 3.1.7. *(Jannsen, [25]) Let $M(k)$ be the category of pure motives over the field k, w.r.t. the adequate equivalence relation $\sim$. Then $M(k)$ is semi-simple if and only if $\sim = \sim_{num}$, the so-called numerical equivalence relation.*

We will now give a brief overview over the Grothendieck and Denef-Loeser rings of motives. The relevance of these objects lies, mainly, in their relation to point counting functions.

Definition 3.1.8. 1. *Let k be a field. Let $\mathcal{V}_k$ be the category of varieties over k. First consider the free abelian group generated by $\mathcal{V}_k$. Then the Grothendieck ring of varieties is defined by identifying*

$$X \sim Y + X \backslash Y,$$

for every closed $Y \subseteq X$.

2. *Now consider $M(k)$, the category of pure numerical motives. The Grothendieck ring of (pure) motives is then obtained by dividing out the relation*

$$M(X) \sim M(Y) + M(X \backslash Y),$$

for every closed $Y \subseteq X$, from the free abelian group generated by the objects of $M(k)$.

Functions on $\mathcal{V}_k$ resp. on $\mathrm{M}(k)$ that respect the structure of the Grothendieck ring play a special role as we will see in some examples.

Definition 3.1.9. *Let R be a commutative ring. A function $\chi\colon M(k) \to R$ (resp. $\mathcal{V}_k\colon R$) is called an additive invariant if for each $X \in \mathcal{V}_k$ and each closed subvariety $Y \subseteq X$ we have*

$$\chi(M(X)) = \chi(M(Y)) + \chi(M(X \backslash Y)).$$

Let us give some familiar examples.

Example 3.1.10.

- *Euler characteristic: $\chi\colon M(k) \to \mathbb{Z}$, $X \mapsto \sum(-1)^i h_i(X)$, where $h_i(X)$ is the i-th Betti-number of X. Then χ is an additive invariant.*
- *Counting functions: Let p be a prime power. Let $\chi\colon M(k) \to \mathbb{F}_p$, $X \mapsto \#[X(\mathbb{F}_p)]$. Then χ is also an additive invariant. It is also called the point counting function over $\mathbb{F}_p$.*

Computations on the level of graph polynomials and counting functions tend to correspond closely to computations in the Grothendieck ring of motives. Non-trivial applications of this approach can be found in the recent paper by Brown and Schnetz [15] where the use point counting techniques and computations in the Grothendieck ring of motives to obtain explicit formulas for the so-called c_2 invariant of graph hypersurfaces, which contains much information about the possible "Tate"-motive character of graph hypersurfaces. The Denef-Loeser ring of motive is now obtained by formally inverting the motive of $\mathbb{P}^1$, the Lefschetz motive. There is the famous theorem of Belkale and Brosnan disproving the hope/conjecture of Kontsevich that point counting functions of graph hypersurfaces are polynomial.

Theorem 3.1.11. *(Belkale-Brosnan, [4]) The classes of the graph hypersurfaces $[X_G]$ generate the Denef-Loeser ring of motives.*

3.1.1 Nisnevich sheaves with transfers and $DM_-^{eff}(k)$

In this section we will introduce $DM_-^{eff}(k)$ and explain how the notion of motives of (smooth) $k$schemes is extended to the singular case. We follow closely [46]. The category $DM_-^{eff}(k)$ is larger than $DM_{gm}^{eff}(k)$, however one has the following property:

Let $A \in DM_-^{eff}$. Then there exist direct sums B, C of objects of $DM_{gm}^{eff}(k)$ and a distinguished triangle

$$A \to B \to C \to A[1].$$

In order to define the category of motivic complexes $DM_-^{eff}(k)$ we need to first introduce the notion of Nisnevich sheaves with transfers.

Definition 3.1.12. *A presheaf with transfers over Sm_k is a presheaf that extends to an additive contravariant functor on $SmCorr(k)$, the category of smooth correspondences over k. The category of presheaves with transfers will be denoted by $PreSh(SmCorr(k))$.*

Definition 3.1.13. *A family of etale morphisms $\{U_\alpha \overset{u_\alpha}{\to} X\}$ is called a Nisnevich cover if for each $x \in X$ there is a $x_\alpha \in U_\alpha$, for a suitable α, such that $u_\alpha(x_\alpha) = x$ and the induced map $k(x) \to k(x_\alpha)$ is an isomorphism.*

This notion of a coveriing leads to a Grothendieck topology, the Nisnevich topology.

Definition 3.1.14. *A presheaf with transfers is a Nisnevich sheaf with transfers if the underlying presheaf on Sm_k is a Nisnevich sheaf (i.e. a sheaf w.r.t. the Nisnevich topology). The category of Nisnevich sheaves with transfers will be denoted by $Sh_{Nis}(SmCorr(k))$.*

$Sh_{Nis}(SmCorr(k))$ has the properties one would readily expect if we would consider plain Nisnevich sheaves.

- $Sh_{Nis}(SmCorr(k))$ admits sheafication of presheaves with transfers, i.e. the inclusion $Sh_{Nis}(SmCorr(k)) \hookrightarrow PreSh(SmCorr(k))$ has an exact left adjoint.
- $Sh_{Nis}(SmCorr(k))$ has sufficiently many injectives.

Convention : The representable presheaf with transfers $Hom_{SmCorr(k)}(-, X)$ will be denoted by $L(X)$ (following Voevodsky's notation).

A key property derived from the Nisnevich topology is the following.

Proposition 3.1.15. *([46]) Let X be a scheme of finite type over k and $\{U_i \to X\}$ be a Nisnevich covering of X. Denote the coproduct $\coprod_i U_i$ by U and consider the complex of presheaves:*

$$\ldots \to L(U \times_X U) \to L(U) \to L(X) \to 0$$

with differential given by alternated sums of morphisms induced by projections. Then it is exact as a complex of Nisnevich sheaves.

Next we need to define homotopy invariant Nisnevich sheaves.

Definition 3.1.16. *A presheaf with transfers $F \in PreSh(SmCorr(k))$ is called homotopy invariant if the morphism $F(X) \to F(X \times \mathbb{A}^1)$ induced by $X \times \mathbb{A}^1 \to X$ is an isomorphism for all $X \in Sm_k$. A Nisnevich sheaf with transfers is called homotopy invariant if the underlying presheaf is homotopy invariant.*

Then one has the following results.

Theorem 3.1.17. *([46]) Let F be a homotopy invariant presheaf with transfers on Sm_k. Then the Nisnevich sheaf with transfers F^{Nis} associated with F is homotopy invariant. Moreover as a presheaf on Sm_k it coincides with the Zariski sheaf F^{Zar} associated with F. If in addition the field k is perfect one has:*

1. *The presheaves with transfers $H^i_{Nis}(, F^{Nis})$ have canonical structures of homotopy invariant presheaves with transfers.*
2. *For any smooth scheme over k one has*

$$H^i_{Zar}(X, F^{Zar}) = H^i_{Nis}(X, F^{Nis}).$$

Proposition 3.1.18. *For any perfect field k the full subcategory $HI(k)$ of the category $Sh_{Nis}(SmCorr(k))$ which consists of homotopy invariant sheaves is abelian and the inclusion functor $HI(k) \hookrightarrow Sh_{Nis}(SmCorr(k))$ is exact.*

This leads then to the following definition of $DM^{eff}_{-}(k)$.

Definition 3.1.19. *The category of motivic complexes $DM^{eff}_{-}(k)$ is the full subcategory of $D^{-}(Sh_{Nis}(SmCorr(k)))$ consisting of complexes with homotopy invariant cohomology sheaves. It follows from 3.1.18 that this is a triangulated category.*

Remark 3.1.20. *Voevodsky shows in [46] that $DM^{eff}_{gm}(k)$ is embedded in $DM^{eff}_{-}(k)$ and constructs further a tensor product stucture on $DM^{eff}_{-}(k)$. Then it is shown that this category contains the expected distinguished triangles.*

- *Mayer-Vietoris*
- *Gysin*
- *Blow-up*
- *Künneth*
- *Projective bundle theorem.*

We will conclude this subsection by explaining how the notion of motives in $DM_-^{eff}(k)$ is extended to schemes of finite type over k.

Definition 3.1.21. *Let $X \in Sch_k$ be a scheme of finite type over k. For $U \in Sm_k$ consider the abelian groups $L(X)(U)$ of closed integral subschemes $Z \subseteq X \times U$ such that Z is finite over U and dominant over an irreducible component of U. This defines a presheaf with transfers and further a functor*

$$L(-): Sch_k \to PreSh(SmCorr(k)).$$

To actually associate a motivic complex (in a functorial way) to an object X of Sch_k we need a functorial representation of X as a complex of Nisnevich sheaves with transfers that preserves the (motivic) cohomology. This is achieved by the so-called Suslin-complex.

Definition 3.1.22. *([46]) For each $n \in \mathbb{N}_0$ let Δ^n be the standard n-simplex in Sm_k. For any presheaf with transfers F on Sm_k let $C_*(F)$ be the complex of presheaves on Sm_k of the form $C_n(F)(X) := F(X \times \Delta^n)$ with differentials given by alternated sums of morphisms which correspond to the duals of the coface morphisms $\Delta^l \to \Delta^{l-1}$. This complex is called the singular simplicial complex of F. If $F = L(X)$ then $C_*(L(X))$ is called the Suslin complex of X.*

The Suslin complex gives a functor

$$C_*(-): Sch_k \to DM_-^{eff}(k).$$

Voevodsky shows then (among other things) homotopy invariance and that the Mayer-Vietoris and the Blow-up distinguished triangles carry over to (motivic) Suslin complexes of schemes of finite type. Furthermore in the case k admitting resolution of singularities one has a extension of the notion of motives of schemes to schemes of finite type over k.

Theorem 3.1.23. *[46, Cor. 4.1.6] Let k be a field which admits resolution of singularities. Then for any scheme X of finite type over k the object $C_*^c(X)$ belongs to $DM_{gm}^{eff}(k)$.*

Remark 3.1.24. *$C_*^c(X)$ is the analog of $C_*(X)$ for motives with compact support. As usual in the case X projective it coincides in $DM_-^{eff}(k)$ with $C_*(X)$.*

3.1.2 Levine's motives and Hyperresolutions

As hyperresolutions are used in section 4.3 we will give a short description of hyperresolutions following the exposition of M. Levine in [40] and the overview part of [24].

Levine uses hyperresolutions to extend the definition of mixed motives to the case of (not necessarily smooth) schemes of finite type over k. The fact that it yields a definition of mixed motives over k means that it can be used as a tool to descents properties of morphisms of smooth projective schemes on the singular scheme from which they are constructed.

Recall the following definition.

Definition 3.1.25. *(i) A triangulated category $\mathfrak{T}$ is called a tensor triangulated category if there exists a functor $\otimes_{\mathfrak{T}}\colon \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ that is a functor of triangulated categories on each factor. This is often strengthened in the literature, e.g. in Mazza, Voevodsky, Weibel [35] by requiring that the functor $\otimes_{\mathfrak{T}}$ is also compatible with the shift autoequivalence $\Sigma\colon \mathfrak{T} \to \mathfrak{T}$.*

(ii) Consider an additive functor $F\colon \mathfrak{D} \to \mathfrak{T}$, where $\mathfrak{D}$ is a tensor DG category and $\mathfrak{T}$ is a tensor triangulated category. Then F is called a pseudo-tensor functor if there is a natural transformation $F \otimes_{\mathfrak{T}} F \to F \circ \otimes_{\mathfrak{D}}$ making the usual diagrams (associativity and commutativity constraint, preservation of the unital object) commute. Other authors call this sometimes a lax tensor functor. (See [40].)

Convention: To distinguish between Levine's and Voevodsky's triangulated categories of motives we will write $\mathcal{DM}(k)$ to denote Levine's category and $DM_{gm}(k)$ to denote Voevodsky's category of geometric motives. In this notation k denotes the ground field.

We are now sketching Levine's construction of his triangulated category of mixed motives over a field k and then describe its relation to Voevodsky's $DM(k)$.
Levine's approach is, contrary to Voevodsky's approach, to construct the triangulated category of mixed motives via a series of localizations of commutative DG categories that themselves are build from cycle cohomology theories. He obtains a "motivic" commutative DG category $\mathcal{A}_{mot}(Sm_k)$ as a subcategory of the constructed DG category. Then he passes to the derived category over this motivic DG category and enforces the expected distinguished triangles by dividing out the necessary relations.

The construction of $\mathcal{A}_{mot}(Sm_k)$ consists of five steps.

Step 1.
The first goal is to obtain a differential graded category $\mathcal{A}_1^{eff}(Sm_k)$ together with a functor $M_1\colon Sm_k \to \mathcal{A}_1^{eff}(Sm_k)$ such that cohomological pullback holds on the induced homotopy category. To do this one first replaces the category Sm_k by a category $\mathcal{L}(Sm_k)$ that is a symmetric monoidal category fibered over Sm_k and consists of pairs $(X,\, f)$ of smooth k-schemes and morphisms $f\colon X' \to X$ having a smooth section $s\colon X \to X'$.

Step 2.
Take $\mathcal{A}_2(Sm_k) := \mathcal{A}_1(Sm_k)^{\otimes,\mathfrak{c}}$, the universal commutative DG tensor category without unit. Associated with it come external products $\boxtimes\colon \otimes \to \times_{Spec(k)}$.

Step 3.
There is a category $\mathfrak{E}$, called the homotopy one point DG tensor category, that essentially models the category of mixed Tate motives. Let $\mathcal{A}_2[\mathfrak{E}]$ be the coproduct of $\mathcal{A}_2(Sm_k)$ and $\mathfrak{E}$ in the category of DG categories. Then $\mathcal{A}_3(Sm_k)$ is obtained by adjoining maps to each $(X,\, f) \in \mathcal{L}(Sm_k)$ and $Z \in \mathcal{Z}(X)_f$ a map $[Z]\colon \mathfrak{e} \to \mathbb{Z}_X(d)_f$. Here $\mathbb{Z}_X(d)$ is just a notation for an object $((X,\, f),\, d) \in \mathcal{L}(Sm_k)^*$.

Step 4.
There is a functor $\mathcal{Z}_1\colon \mathcal{L}(Sm_k) \to Mod(\mathbb{Z})$ defined by $(X,\, f) \mapsto \mathcal{Z}^d(X)_f$ on objects and $(X, f) \xrightarrow{p} (X',\, f') \mapsto \mathcal{Z}^d(X)_f \xrightarrow{p^*} \mathcal{Z}^d(X')_{f'}$ on morphisms, the cycle functor. This functor extends to a functor $\mathcal{Z}_1\colon \mathcal{A}_1(Sm_k) \to Mod(\mathbb{Z})$. Then $\mathcal{A}_4(Sm_k)$ is obtained from $\mathcal{A}_3(Sm_k)$ by adjoining morphisms as listed below:

1. Let $(X,\, f)$ and $(Y,\, g) \in \mathcal{L}(Sm_k)$ and $p\colon \mathbb{Z}_X(q) \to \mathbb{Z}_Y(q) \in \mathcal{A}_1(Sm_k)$. Let Z be a non-zero cycle in $\mathcal{Z}^q(X)$. This induces via the cycle functor (see [40]) a non-zero cycle in $\mathcal{Z}^q(Y)_g$. Then adjoin the map of degree $2q-1$
$$h_{X,\,Y,\,[Z],\,p}\colon \mathfrak{e} \to \mathbb{Z}_Y(q)_g$$
such that
$$dh_{X,\,Y,\,[Z],\,p} = p \circ [Z] - [\mathcal{Z}_1(p)(Z)].$$

2. For algebraic cycles $Z_1 \in \mathcal{Z}^p(X)$, $Z_2 \in \mathcal{Z}^q(Y)$ one adjoins morphisms of degree $2(p+q)+1$
$$h^l_{X,\,Y,\,[Z_1],\,[Z_2]}\colon \mathfrak{e} \otimes \mathfrak{e} \to \mathbb{Z}_{(X,\,f)\times(Y,\,g)}(p)_r,$$
$$h^r_{X,\,Y,\,[Z_1],\,[Z_2]}\colon \mathfrak{e} \otimes \mathfrak{e} \to \mathbb{Z}_{(X,\,f)\times(Y,\,g)}(p)_r,$$

where r in the subscript of $\mathbb{Z}_{(X,\,f)\times(Y,\,g)}(p)_r$ is the morphism of $(X,\ f)\times(Y,\ g)$ and satisfying

$$dh^l_{X,Y,[Z_1],[Z_2]} = \boxtimes_{\mathbb{Z}_X(p)_f,\mathbb{Z}_Y(q)_g}\circ([Z_1]\otimes[Z_2]) - \boxtimes_{\mathbb{Z}_X(p)_f\times\mathbb{Z}_Y(q)_g,1}\circ([(Z_1\times_S Z_2)]\otimes[S])$$

and

$$dh^r_{X,,Y,[Z_1],[Z_2]} = \boxtimes_{\mathbb{Z}_X(p)_f,\mathbb{Z}_Y(q)_g}\circ([Z_1]\otimes[Z_2]) - \boxtimes_{1,\mathbb{Z}_X(p)_f\times\mathbb{Z}_Y(q)_g}\circ([S]\otimes[Z_1\times_S Z_2]).$$

The $\boxtimes$ symbols represent the relevant external products.

3. Let $(X,\ f)\in\mathcal{L}(Sm_k)$, $Z_1,\ Z_2\in\mathcal{Z}^q(X)_f$ and $n_1,\ n_2\in\mathbb{Z}$. Adjoin the map of degree $2q1$

$$h_{n_1,\,n_2,\,[Z_1],\,[Z_2]} : \mathfrak{e}\mathbb{Z}_X(q)_f$$

with

$$dh_{n_1,\,n_2,\,[Z_1],\,[Z_2]} = [n_1Z_1+n_2Z_2]-n_1[Z_1]n_2[Z_2].$$

Step 5.
Construct $\mathcal{A}_5(Sm_k)$ from $\mathcal{A}_4(Sm_k)$ by adding successively morphisms $h\colon\mathfrak{e}^{\otimes k}\to\mathbb{Z}_X(n)_f$:

- $\mathcal{A}_5(Sm_k) := \mathcal{A}_4(Sm_k)$.
- Suppose we have constructed the DG tensor category without unit $\mathcal{A}_5(Sm_k)^{(r-1)}$, $r\geq 1$. Let $\mathcal{A}_5(Sm_k)^{(r,0)} := \mathcal{A}_5(Sm_k)^{(r-1)}$ and suppose we have formed $\mathcal{A}_5(Sm_k)^{(r,k-1)}$ for some $k\geq 1$. Then define the DG tensor category $\mathcal{A}_5^{(r,\,k)}$ by adjoining morphisms of degree $2n-r-1$

$$h\colon\mathfrak{e}^{\otimes k}\to\mathbb{Z}_X(q)_f$$

to $\mathcal{A}_5^{(r,\,k-1)}$, with $dh=s$ for each non-zero morphism $s\colon\mathfrak{e}^{\otimes k}\to\mathbb{Z}_X(n)_f$ in $\mathcal{A}_4(Sm_k)^{(r,\,k-1)}$ such that s has degree $2nr$ and $ds=0$. Finally let

$$\mathcal{A}_5^r := lim_{k\to\infty}\mathcal{A}_5^{(r,\,k)}(Sm_k)$$

and

$$\mathcal{A}_5(Sm_k) := lim_{r\to\infty}\mathcal{A}_5^r(Sm_k).$$

Definition 3.1.26. *The motivic DG tensor category $\mathcal{A}_{mot}(Sm_k)$ of Sm_k is the full additive subcategory of $\mathcal{A}_5(Sm_k)$ generated by tensor products of objects of the form $\mathbb{Z}_X(n)_f$, or $\mathfrak{e}^{\otimes a}\otimes\mathbb{Z}_X(n)_f$.*

The actual triangulated category of motives is then obtained by suitable localizations of the category of bounded chain complexes $Ch^b(\mathcal{A})$ over the just constructed category $\mathcal{A}_{mot}(Sm_k)$:

Let $\mathcal{T}$ be the thick subcategory of $D^b(\mathcal{A}_{mot}(Sm_k))$ containing the maps for

1. *Homotopy*
2. *Excision*
3. *Künneth morphisms*
4. *Gysin morphisms*
5. *Moving lemma*
6. *Tensor product unit.*

We omit the exact representation of these relations as it is not needed later.

Definition 3.1.27. *With the above notation the triangulated tensor category of motives over Sm_k is*

$$\mathcal{T}^{-1}D^b(\mathcal{A}_{mot}(Sm_k)).$$

The relation between Voevodsky's category DM and Levine's is addressed by Levine himself: In [40, Ch VI, Thm 2.5.5.] Levine shows the following:

Theorem 3.1.28. *There is an equivalence of triangulated categories*

$$\mathcal{DM}(k) \to DM_{gm}(k)$$

that comes from an isomorphism of the "motives" functors $X \mapsto \mathbb{Z}_X(n)$ (for $\mathcal{DM}(k)$) resp. $X \mapsto M(X)^\vee \in DM_{gm}(k)$, where $M(X)^\vee$ denotes the dual of the motive of X.

This leads to a notion of motives for singular k-schemes coming from Hanamura's [22] extension of mixed motives for singular schemes.
The main technical notion to achieve this are the so-called cubical hyperresolutions. First we need to introduce the notion of 2-resolutions and reduction operations.

Definition 3.1.29. *Let I be the diagram category of a finite partially ordered set. Let X be an I-diagram of (reduced) finite type k-schemes, i.e. a functor $X\colon I \to Sch_k$. Let $\square_n$ be the opposite category of the partially ordered set $\{0 < 1\}^{n+1}$ and $\square_n^+$ be the full subcategory obtained from $\square_n$ by deleting the object $(0, \ldots, 0)$. $\square_n^+$ is isomorphic to the $n+1$ cube.*

(i) Let Y be another I diagram and $f\colon X \to Y$ be a map of I-diagrams. The discriminant locus of f is the I-diagram

$$i \mapsto disc(f)(i),$$

where $disc(f)(i)$ is the complement of the largest open subset $U(i)$ of $Y(i)$ over which $f(i)$ is an isomorphism.

(ii) 2*-resolution of* X *is a cartesian* $\square_1^+ \times I$*-diagram of the form*

$$\begin{array}{ccc} Z_{11} & \longrightarrow & Z_{01} \\ \downarrow & & \downarrow f \\ Z_{10} & \longrightarrow & Z_{00} \end{array},$$

where

(a) $Z_{00} = X$

(b) $Z_{01} \in Sm_k$

(c) the horizontal maps are closed embeddings

(d) f *is proper*

(e) Z_{10} *contains the discriminant locus of* X*.*

The 2*-resolution is called strict if, for each* iI*,* $dim_k Z_{01}(i) = dim_k Z_{00}(i)$*, and the restriction of* f *to the components of* $Z_{01}(i)$ *and* $Z_{00}(i)$ *of maximal dimension is birational.*

Definition 3.1.30. *Let* $\square_n^+ \times I$*-diagrams* X_*^n *for* $1 \leq n < r$ *such that the* $\square_{n-1}^+ \times I$*-diagrams* X_{00*}^{n+1} *and* X_{11*}^n *are the same for all* $1 \leq n < r$*. Define the* $\square_r^+ \times I$*-diagram*

$$Z_* := rd(X_*^1, \ldots, X_*^r)$$

by letting set $Z_* := X_*^1$ *for* $r = 1$*, for* $r = 2$ *define*

$$Z_{ab} := \begin{cases} X_{0b}^1, & a = (0, 0) \\ X_{ab}^2, & a \in \square_1 \end{cases}$$

and for $r > 2$ *define* $rd(X_*^1, \ldots, X_*^r) := rd(rd(X_*^1, \ldots, X_*^{r-1}), X_*^r)$ *where one identifies* $\square_r^+ \times I$ *and* $\square_2^+ \times (\square_{r-2}^+ \times I)$ *and* $\square_{r-1}^+ \times I$ *with* $\square_1^+ \times (\square_{r-2}^+ \times I)$ *via the obvious morphisms.*

Definition 3.1.31. *Let* X *be an* I*-diagram of reduced finite type* k*-schemes. A cubical hyperresolution of* X *is a* $\square_r \times I$*-diagram of reduced finite type* k*-schemes* Z_**, such that* Z_* *is the restriction to* $\square_r \times I$ *of* $rd(X_*^1, \ldots, X_*^r)$*, where*

1. X_*^1 *is a* 2*-resolution of* X
2. *For* $1 \leq n < r$*,* X_*^{n+1} *is a* 2*-resolution of* X_*^n
3. Z *a is in* Sm_k *for all* $a \square_r \times I$*.*

The hyperresolution is called strict if the 2*-resolutions in (1) and (2) are strict.*

The definition of motives can be extended to hyperresolutions:

Definition 3.1.32. *We define an object of $C^b_{mot}(Sm_k)$ corresponding to a cubical hyperrresolution of a scheme X by defining the motive of cubical schemes. Let $(X_*, f_*)\colon \square_r \to Sm_k$ be a cubical scheme. Define the motive of this cubical scheme, $\mathbb{Z}_{X_*}(0)_{f_*} = \bigoplus_{s=0}^{\infty} \mathcal{Z}^s$, by setting*

$$\mathcal{Z}^s := \bigoplus_{|I|=s} \mathbb{Z}_{X_I}(0)_{f_I}$$

and differentials $\partial^s\colon \mathcal{Z}^s \to \mathcal{Z}^{s+1}$, $\partial^s := \sum_{|I|=s} \sum_{i=1}^{n} (-1)^i \partial^s_{I,\,i}$, where

$$\partial^s_{I,\,i}\colon \mathbb{Z}_{X_I}(0)_{f_I} \to \mathbb{Z}_{X_{I\cup\{i\}}}(0)_{f_{I\cup\{i\}}}$$

is defined by

$$\partial^s_{I,\,i} = \begin{cases} \text{the map corresponding to } X_{I\cup\{i\}} \to X_I, & i \notin I \\ 0, & i \in I \end{cases}.$$

Now for the case of hyperresolutions we need to adapt this definition to reflect the removed origin $(0, \ldots, 0) \notin \square_r^+$. This is done by setting $\partial_{I,\,(0,\ldots,0)} = 0$ in the above definition.

This notion is independent of the choice of the hyperresolution:

Theorem 3.1.33. *([40]) Let X be a (reduced) scheme of finite type over k.Let Z_* and Z'_* be hyperresolutions of X. Then the associated motives coincide:*

$$\mathbb{Z}_{Z_*} = \mathbb{Z}_{Z'_*}$$

Further hyperresolutions provide a way to define motives for all finite type k-schemes.

Theorem 3.1.34. *Let k be a perfect field admitting resolution of singularities. The functor $\mathbb{Z}(-)\colon Sm_k D^b_{mot}(Sm_k)$ extends to the functor*

$$\mathbb{Z}(-)\colon Sch^{fin}_k \to D^b_{mot}(Sm_k).$$

3.2 Decompositions arising from the Theorem of Bialynicki-Birula

In this section we will describe the decompositions on the level of schemes, cohomology and motives arising from the famous decomposition theorem of A. Bialynicki-Birula.

Let X be a k-scheme and $G \times_k X \to X$ a group scheme action on X. For each point $p \in X$ consider the Zariski-tangent space $T_p(X) := \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^{\vee}$.

If $p \in Fix_G(X)$ is a closed point than the module $\mathfrak{m}_p \trianglelefteq \mathcal{O}_{X,p}$ is $G(k(p))$-invariant and the same is obviously true for $\mathfrak{m}_p^2$. This shows that there is a linear action $G(k(p)) \times_{k(p)} T_p(X) \to T_p(X)$. It is a well known fact in the theory of algebraic groups/group schemes that split algebraic tori can be simultaneously diagonalized. This means that there are vectors $\omega_1, \omega_r \in \mathbb{Z}^{dim(X)}$ and a basis $(v_1, \ldots, v_{dim(X)})$ of $T_p(X)$ such that

$$v_i \mapsto \lambda_1^{\omega_{i1}} \cdots \lambda_r^{\omega_{ir}} v_i.$$

Let us introduce some standard notions. We will follow [5].

Definition 3.2.1. *Let M be a $G := \mathbb{G}_m^r$-module. Let $\omega_1, \ldots, \omega_r$ as above corresponding to decomposition of M. Then M is called positive if the following two conditions are satisfied:*

(i) $\omega_{ij} \geq 0$ for all i, j.

(ii) For every $1 \leq i \leq n = dim(M)$ there exists a $1 \leq j \leq r$ such that $\omega_{ij} \neq 0$.

M is called non-negative if only condition (i) holds. Further M is called fully definite (resp. definite) if there exists a G-equivariant isomorphism $M \to M'$, where M' is positive (resp. non-negative).

Remark 3.2.2. *For any abelian group scheme A the inversion morphism ${}^{-1}\colon A \to A$ is an isomorphism of group schemes. In the case $A = \mathbb{G}_m^r$ is acting on a free module M with weights $\omega_1, \ldots, \omega_r$ via the representation $\delta\colon A \times_k M \to M$, then $\delta' := \delta \circ ({}^{-1}, id_M)\colon A \times_k M \to M$ is also a linear representation with weigths $-\omega_1, \ldots, -\omega_r$. Because of this it is clearly sufficient to only consider the notion of a fully definite (resp. definite) module for positive (resp. non-negative) actions.*

We want to explicit how an action on a quasi-projective k-scheme X induces a k-linear $\mathbb{G}_m$-action on $T_p(X)$.
From the definition there is a k-linear $\mathbb{G}_m$-action on $\mathcal{O}_X$, hence especially a k-linear action on $k[X] = \mathcal{O}_X(X)$. Suppose p is a closed $\mathbb{G}_m$-invariant point then also $\mathfrak{m}_p^n \trianglelefteq \mathcal{O}_{X,p}$ admits a k-linear $\mathbb{G}_m$-action that is obtained by restriction for each $n \in \mathbb{N}_0$. Hence $\mathfrak{m}_p/\mathfrak{m}_p^n$ is a $k[\mathbb{G}_m]$-module.
One has the following lemma:

Lemma 3.2.3. *Let $G := \mathbb{G}_m^r$ be an algebraic torus acting on a smooth irreducible and quasi-affine k-scheme X. Then if there is a closed G-invariant point $p \in X$ such that $T_p(X)$ is definite, then so is the induced $G(k)$-module $k[X]$.*

Proof. As discussed above the action of $\mathbb{G}_m$ on $k[X]$ is directly induced by the definition. Any $\mathbb{G}_m$-module can be decomposed into one dimensional invariant subspaces. By smoothness one has $dim_k(T_p(X)) = dim_{\text{Krull}}(k[X])$. Because of this we only need to check that definiteness on one dimensional invariant subspaces of $k[X]$ (that are not killed by passing to $T_p(X)$) is preserved (or inverted over all weights). It is well known that the map on the tangent space of quasi-affine varieties is given by differentiating (w.r.t. the local coordinates) the pullback map on the structure sheaves. In our case this leads to expressions of the form $\omega_i \lambda^{\omega_i - 1} d_p X_i$. In case $char(k) \not| \ \omega_i$ for all i this implies the definiteness. This condition is not a real restriction as (powers of) the Frobenius automorphism can be applied in each coordinate. □

Remark 3.2.4. *That the action on the tangent space ie given by the differential makes it obvious that this condition is sufficient but not necessary.*

Let $G := \mathbb{G}_m$ and let $G(k)$ act on a finite dimensional vector space V. As already mentioned the $G(k)$-module decomposes in invariant one dimensional subspaces. Let

$$V^0 := C_V(G) = \{v \in V : G(k).v = v\},$$

$$V^+ := \{v \in V : \exists m \in \mathbb{N} \,\text{s.th.}\, \lambda.v = \lambda^m \cdot v, \text{ for all}\, \lambda \in G(k)\}$$

and

$$V^- := \{v \in V : \exists m \in -\mathbb{N} \,\text{s.th.}\, \lambda.v = \lambda^m \cdot v, \text{ for all}\, \lambda \in G(k)\}$$

Then we have the decomposition

$$V = V^- \oplus V^0 \oplus V^+.$$

We will now explain how in a common special case the Bialynicki-Birula decomposition arises naturally. For this let $Z \subseteq X$ be a closed subset and let $G := \mathbb{G}_m$ act on X. Let $X \subseteq \mathbb{P}^n$ be a projective variety and the action of G be the restriction of an appropriate G-action on $\mathbb{P}^n$. Let $V := k^{n+1}$. Then V is a G-module by extending the G to the affine cone of $\mathbb{P}^n$. Further we can as above decompose $V = V^- \oplus V^0 \oplus V^+$. We will for simplicity assume that each of these spaces is non-trivial.
Since the action on X is induced by a linear action we have $\overline{G(k).p} = \overline{k^*} = \mathbb{P}^1$. Let $\mathbb{A}^n \simeq U \subseteq \mathbb{P}^n$ be the open affine neighborhood of p where the coordinate with the largest weight $\omega_i > 0$ such that $p_i \neq 0$. As one can see by direct computation $U \cap \overline{G(k).p}$ contains exactly one element. We will denote this element by

$$lim_{\lambda \to \infty} \lambda.p \in U.$$

Similarly if we choose the smallest weight we get a notion of $lim_{\lambda \to 0} \lambda.p$. Now we can consider more generally

$$Z^+ := \{p \in \mathbb{P}^n : \exists z \in Z \,\text{s.th.}\, lim_{\lambda \to 0} \lambda . p = z \,\text{or}\, p \in G(k).z\}.$$

If one restricts the choice of Z to irreducible components F of the fixed point scheme $Fix_X(G)$ then these sets Z^+ become constructible schemes. (See [30].) In this case the schemes F^+ are called plus-cells. For projective varieties X with a linear G-action it is not hard to see that the plus-cells form a decomposition of the ambient projective space $\mathbb{P}^n$ and therefore also a decomposition of X.

In the general (smooth) case Bialynicki-Birula characterizes these cells by the following theorem.

Theorem 3.2.5. *(Bialynicki-Birula) Let X be a smooth projective k-scheme, with k algebraically closed. Let $G := \mathbb{G}_m$ act on X and $Fix_X(\mathbb{G}_m) = \bigcup F_i$ be a decomposition into irreducible components. Then there exist unique smooth G-invariant subschemes $F_i^+ \to X$ together with a unique morphism $\gamma_i : F_i^+ \to F_i$ such that*

(i) F_i is a closed subscheme of F_i^+ and $\gamma_i|_{F_i}$ is the identity.

(ii) $F_i^+ \to F_i$ is with respect to the given G-action a G-fibration.

(iii) For any closed point $p \in F_i$ one has $T_p(F_i) = T_p(X)^0 \oplus T_p(X)^+$.

The dimension of the fibres defined in (ii) equals $dim T_p(X)^+$, for any closed $p \in F_i$.

Then he concludes $\overline{G(k).p} \cap F_i \neq \emptyset$ if and only if $p \in F_i^+$ or $p \in F_i^-$, which is one expects from the projective variety examples. The actual Bialynicki-Birula decomposition is now contained in the following theorem

Theorem 3.2.6. *(Bialynicki-Birula) Let X be a smooth $\mathbb{G}_m$-invariant and complete k-scheme. Let $Fix_X(\mathbb{G}_m) = \bigcup F_i$ be a decomposition into connected components. Then there exists a decomposition $X = \bigcup F_i^+$ with the F_i^+ satisfying the conditions (i), (ii), (iii) of theorem 3.2.5.*

These theorems are not quite sufficient to obtain decompositions on the level of cohomology or even on the level of motives. One way to express $H^\bullet(X)$ in terms of cohomology groups of the fixed point components F_i would be to apply a version of the Gysin long exact sequence w.r.t. $X_1 \subseteq X$:

$$\ldots \to H^n(X_1)(X \backslash X_1) \to H^n(X) \to H^n(X_1)(dim(X/X_1)) \to H^{n+1}(X \backslash X_1) \to \ldots$$

In a second paper ([6]) Bialynicki-Birula provided such an approach by proving that the Bialynicki-Birula decomposition gives as in 3.2.6 gives rise to an equivariant filtration:

Theorem 3.2.7. *Let X be a smooth projective k-scheme an $G := \mathbb{G}_m$ act on X. Then there exists a filtration*

$$\emptyset \subseteq X_r \subseteq \ldots \subseteq X_1 \subseteq X$$

such that $X_i \backslash X_{i+1}$ is a G-equivariant fibration over some fixed point component F_i.

Using Gysin long exact sequences as above shows that cohomology rings of such schemes are successive extensions of cohomology groups of the fixed point components.

Karpenko generalized this as follows.

Definition 3.2.8. *Let X be a projective k-scheme. X is called a cellular scheme if there exists a filtration*

$$\emptyset \subseteq X_r \subseteq \ldots \subseteq X_1 \subseteq X$$

of closed subschemes X_i such that $X_i \backslash X_{i+1}$ is an affine fibration over some projective scheme F_i.

Then one has the following theorem.

Theorem 3.2.9. *Let X be a smooth projective k-scheme. Assume further that X is cellular such that all occuring morphisms and schemes are in the category of smooth schemes. Then one has*

$$M(X) = \bigoplus_i M(F_i)(dim(X_i/X_{i+1}))$$

in the category of smooth correspondences.

To apply this theorem to the decompositions arising from the theorems of Bialynicki-Birula one needs a further result of Iversen [26] that says that the connected components of the fixed point scheme $Fix_X(G)$ are smooth (provided X is smooth).

3.3 Motives associated to Graph hypersurfaces

We will now explain some background information on how mixed motives are contructed for a given graph hypersurface. The essential or motivating part of the problem here is to construct a mixed motive (together with a realization as a mixed Hodge structure) such that Feynman amplitudes can be seen as periods of the underlying Hodge structure. To be useful for applications in Physics it would be moreover necessary to directly relate the Feynman period with the (algebraic) structure of the mixed Hodge structure.

For instance a useful goal is to determine the precise graded piece of the underlying MHS (mixed Hodge structure).

To avoid confusion let us describe how Feynman diagrams differ from the graphs we consider as Feynman graphs. Let us recall first:

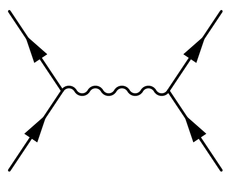
Definition 3.3.1. *A Feynman graph is a connected, finite, undirected graph Γ consisting of vertices $V(\Gamma)$ and edges $E(\Gamma)$. A Feynman graph is called*

- *convergent if* $2h_1(\Gamma) < |E(\Gamma)|$
- *logarithmically convergent if* $2h_1(\Gamma) = |E(\Gamma)|$
- *divergent if* $2h_1(\Gamma) > |E(\Gamma)|$

On the other hand a Feynman diagram is in general a richer structure: It is a tupel $(\Gamma, \tau, \mathfrak{p}, \mathfrak{m})$, where

- Γ is a Feynman graph as in 3.3.1,
- τ is a map $E(\Gamma) \to \{\text{Types of particles}\}$,
- $\mathfrak{p}$ is a map $E(\Gamma) \to \mathbb{R}^d$, which is representing the momenta of the particles and
- $\mathfrak{m}$ finally is a map $E(\Gamma) \to \mathbb{R}_0^+$, representing the masses of the particles.

To represent the various particle types one draws the connection of two vertices using special types of lines. A typical example is:



In the cases we are interested in we will always omit all "colorings" (particle types) and weights (masses/momenta) and thus making it sufficient to study Feynman graphs as in 3.3.1.
As we mentioned in the Introduction Feynman amplitudes are essentially (after applying several standard reduction steps and the so-called Feynman trick) integrals of the form

$$I(D, \Gamma, \nu) = \int_{\sigma^{|E(\Gamma)|-1}(\mathbb{R})} \left(\prod_{i=1}^{|E(\Gamma)|} x_i^{\nu_i - 1} \right) \frac{Q_\Gamma^{\overline{\nu} - \frac{D(h_1(\Gamma)+1)}{2}}}{P_\Gamma^{\overline{\nu} - \frac{h_1(\Gamma)D}{2}}} \Omega_{|E(\Gamma)|-1},$$

here Γ is a Feynman graph, D is the dimension parameter, which is used in regularization procedures, ν is some exponent vector ($\overline{\nu} = \sum_{i=1}^{|E(\Gamma)|} \nu_i$) and

$$\Omega_{2n-1} = \sum_{i=1}^{2n} (-1)^i X_i dX_1 \wedge \ldots \wedge \widehat{dX_i} \wedge \ldots \wedge dX_{2n}.$$

Finally P_Γ is the graph polynomial defined in 2.1.3 and Q_Γ is the so-called second graph polynomial and it is defined by

$$Q_\Gamma = \sum_{\substack{\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2: \\ \text{sp.2-Forrest}}} \prod_{e\notin E(\mathcal{F})} X_e \left(\sum_{\substack{p\in\mathfrak{P}_{\mathcal{F}_1}, \\ q\in\mathfrak{P}_{\mathcal{F}_2}}} p \cdot q \right) + \left(\sum_{e\in E(\Gamma)} m_e^2 X_e \right) P_\Gamma.$$

The set $\sigma^{|E(\Gamma)|-1}(\mathbb{R})$ is defined by

$$\sigma^{2n-1}(\mathbb{R}) = \{p = [p_1 : \ \ldots : p_{2n}] \in \mathbb{P}^{2n-1} : p_i > 0 \, \forall \, i\}.$$

The set $\mathfrak{P}_{\mathcal{F}_i}$ denotes the indexed set of momenta of the underlying Feynman graph.
Spanning Forests of Γ are subgraphs $\mathcal{F} = \bigcup_{i=1}^{l} \mathcal{F}_i$ such that

- $\mathcal{F}_i$ is connected and acyclic, i.e. a tree.
- $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ for all $i \neq j$.
- $\mathcal{F}$ contains all vertices of Γ, i.e. $V(\mathcal{F}) = V(\Gamma)$.

In our discussion we limit ourselves to a QFT toy model of the so-called ϕ^4-Quantum field theory without external momenta and with equal masses. After applying some standard simplifications one finally arrives at integrals of the form

$$I(\Gamma) = \frac{c}{\pi^2} \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}}{P_\Gamma^2}, \tag{3.1}$$

where c is an expression of values of the Γ function on integer numbers.

In [8] it is shown that the Feynman amplitudes 3.1 in this simplified situation are convergent if and only if the underlying Feynman graph is logarithmically divergent and each proper subgraph of it is convergent (see [8] Proposition 5.2 and section 6).

Let us first recall the definiton of Kontsevich-Zagier periods:

Definition 3.3.2. *(see [32]) Let X be a smooth quasi-projective variety. Let $Y \subseteq X$ be a subvariety and ω be a closed algebraic n-form defined over $\overline{O}$ and vanishing along Y. Let σ be a singular chain on $X(\mathbb{C})$ such that the boundary $\partial(\sigma)$ is contained in $Y(\mathbb{C})$. Then the integral*

$$\int_\sigma \omega$$

is called a period (of X).

The importance of this notion comes -in a simple form- from the following

Proposition 3.3.3. *(See [43], Proposition 4.4.1) Let X be a projective manifold over $\mathbb{C}$. Then the primitive integral cohomology groups $H^\bullet_{prim}(X, \mathbb{Z})$ carries a natural Hodge structure. Let $\{\gamma_i\}$ be an integral basis of $H^w_{prim}(X, \mathbb{Z})$ and let (for $w \in \{2m, wm+1\}$) the set $\{\omega_j\}$ be a basis of F^m adapted to the filtration*

$$F^w \subseteq F^{w-1} \subseteq \ldots \subseteq F^m$$

in the sense that $\{\omega_j\} \cap F^{w-l}$ is a basis of F^{w-l}. Then the Hodge structure is determined by the matrix of periods.

In other words: The period matrix provides enough information to *recognize* (pure) Hodge-structures.

It is known that there is a realization functor from the category of mixed Tate motives mapping to the category of mixed Hodge structures, which form an abelian subcategory of Voevodsky's triangulated category of motives $DM(k)$, mapping $\mathcal{MTM}$ to the category of mixed Tate-Hodge structures. This functor is explicitly constructed e.g. in [9].
In that sense one can associate to each mixed Tate motive a corresponding period matrix. Periods of mixed Tate motives enjoy then the remarkable property that they are $\mathbb{Q}[\frac{1}{2\pi}]$ linear combinations of multiple zeta values:

Theorem 3.3.4. *(F. Brown [14]) The periods of every mixed Tate motive over $\mathbb{Z}$ are $\mathbb{Q}[\frac{1}{2\pi}]$-linear combinations of multiple zeta values*

$$\zeta(n_1, \ldots, n_r) := \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$$

with $n_1, \ldots, n_r \in \{2, 3\}$.

We will now introduce the functorial construction of a mixed Hodge-structure such that the Feynman-amplitude 3.1 is a period of that Hodge-structure. If one is able to relate the known cases of Feynman-amplitudes that evaluate to multiple zeta values to periods of Tate-Hodge structures then this would, in the light of Theorem 3.3.4, give a geometric explanation for them.

Before we describe the construction of the graph motive let us recall how the notion of periods extends to the smooth but non-projective case.

Remark 3.3.5. *For each smooth variety defined over $\mathbb{C}$ there exists a smooth compact variety $X \hookrightarrow \widehat{X}$ such that $D := \widehat{X} \backslash X$ is a simple normal crossings divisor. Compactifications of that type are called good compactifications. As shown e.g. in [42] Thm 4.2, one can compute the ordinary Cohomology of X using the hypercohomology groups of the associated log-deRahm complex:*

$$H^\bullet(X) = \mathbb{H}^\bullet(\Omega_{\widehat{X}}(log(D)))$$

Moreover the weight filtration on $\Omega_{\widehat{X}}(log(D))$ induces a weight filtration on the cohomology groups. Finally the Hodge-filtration is induced by

$$Im(\mathbb{H}^k(F^\bullet \Omega_{\widehat{X}}(log(D))) \to H^k(X))$$

in each cohomological degree k. Then a the statement of 3.3.3 has a corresponding statement for the mixed Hodge-structure case:
Let $F^\bullet$ be the Hodge-filtration and $W_\bullet$ the weight filtration of the mixed Hodge-structure of $H^\bullet(U)$. For $k \in \mathbb{Z}$ let $\{\omega_i \colon 1 \leq i \leq rk(H^k(U))\}$ be a basis that is adapted to the increasing Hodge-filtration $F^\bullet$ and at the same time gives a basis of the graded pieces of the mixed Hodge-structure

$$Gr_r^W(H^k(U)) = W_r / W_{r-1}.$$

That such a basis exists can be easily seen inductively. Let finally $\{\sigma_j\}$ be an integral basis of $H_k(U)$. Then the pure Hodge-structures $Gr_r^W(H^k(U))$ are uniquely determined by the period matrix defined by the pairings

$$\int_{\sigma_j} \omega_i.$$

Let Γ be a Feynman graph let $N := |E(\Gamma)|$. The integrand of the Feynman-amplitude 3.1 is a differential form having poles exactly along X_Γ, the graph hypersurface. Thus it can naturally be viewed as an element of the differentially graded deRahm algebra $\Omega^*_{\mathbb{P}^{N-1} \backslash X_\Gamma}$. From its defining formula it is clear that it actually represents a closed form and therefore an element in the cohomology ring $H^*(\mathbb{P}^{N-1} \backslash X_\Gamma, \mathbb{C})$. The integral in 3.1 is almost a cohomology-homology pairing.

Lemma 3.3.6. *([8]) $X_\Gamma \cap \sigma^{N-1}(\mathbb{R}) = \bigcup_{L \subseteq X_\Gamma} L(\mathbb{R}_{\geq 0})$, where the union is taken over all linear coordinate subspaces of $\mathbb{P}^{N-1}$.*

Note that the intersection in Lemma 3.3.6 contains in general a lot of affine simplexes of linear subspaces of $\mathbb{P}^{N-1}$:

Proposition 3.3.7. *([8]) A coordinate linear space L is contained in X_Γ if and only if $h_1(G(L)) > 0$. Here $G(L)$ is the graph defined by the edges $e \in E(\Gamma)$ such that $X_e \in Z(L)$.*

To interpret the Feynman-amplitude as a proper period (as in definition 3.3.2) Bloch, Esnault and Kreimer apply an inductive procedure to move the poles of the (pullback of the) Feynman differential form away from the domain of integration:

Construction:
Let

$$S := \{L \subseteq \mathbb{P}^{N-1} \colon \text{maximal linear coordinate subspaces of}\, X_\Gamma\}$$

and

$$\mathcal{F}^{(0)} := \{\bigcap_{L \in M} L \colon M \subseteq S\}.$$

Assume $\mathcal{F}^{(j)}$, $0 \leq j \leq l$ have already been constructed. Let $\mathcal{F}^{(l+1)}_{min}$ be the set of minimal elements of $\mathcal{F}^{(l+1)}$. Now blow-up the successively the elements of $\mathcal{F}^{(l+1)}_{min}$ to obtain a blow-up morphism $P_{l+1} \to \mathbb{P}^{N-1}$. This blow-up is independent of the chosen order of blow-ups of elements of $\mathcal{F}^{(l)}_{min}$ since these elements are pairwise disjoint. Define then

$$\mathcal{F}^{(l+1)} := \{\text{strict transforms of the elements of}\, \mathcal{F}^{(l)} \backslash \mathcal{F}^{(l)}_{min}\}.$$

Note that this process terminates because $\mathcal{F}^{(0)}$ is a finite set and each step removes elements. The resulting blow-up has the properties needed to define a (relative) mixed Hodge-structure such that the pullback of the Feynman differential form and $\sigma^{N-1}(\mathbb{R})$ yield indeed a representation of the Feynman-amplitude as a period in the sense of mixed Hodge-structures.

Theorem 3.3.8. *([8], Prop. 7.3) Notation as above. Let* $\eta = \frac{\Omega_{N-1}}{P_\Gamma^2}$ *be the differential form of 3.1. There exists a tower of blow-ups (obtained by applying the successive blow-ups of the above construction.)*

$$P = P_r \to P_{r-1} \to \ldots \to P_1 \to P_0 = \mathbb{P}^{N-1}$$

such that

(i) The pullback of η *to the i-th blow-up* $P_i \to \mathbb{P}^{N-1}$ *has no poles along the exceptional divisors of this blow-up.*

(ii) The total transform B *of the projective standard simplex*

$$\Delta^{N-1} \colon \prod_{e \in E(\Gamma)} X_e = 0 \subseteq \mathbb{P}^{N-1}$$

in P *is a simple normal crossings divisor and no face of* B *is contained in the strict transform* Y *of* X_Γ *in* P*.*

(iii) The strict transform of $\sigma^{N-1}(\mathbb{R})$ *in* P *does not meet* Y*.*

The sought after mixed Hodge-structure realizing the Feynman-amplitude as a period is then defined as follows.

Definition 3.3.9. *The graph mixed Hodge-structure is the mixed Hodge-structure is the induced MHS on*

$$H^{\bullet}(P\backslash Y,\, B\backslash(Y \cap B)).$$

Integrating $W_0 H^*_{Betti}(P\backslash Y,\, B\backslash(Y \cap B))$ *of Thm 3.3.8 over the strict transform of* $\sigma^{N-1}(\mathbb{R})$ *yields an isomorphism of* $\mathbb{Q}$ *(see [8], Prop. 7.5).*

Remark 3.3.10. *Note that this whole discussion only applies once one has applied the Feynman-trick to a amplitude integral. In fact in [8] section 9 it is claimed, but not proven, that both values are equal up to some factor in* $\mathbb{Q}^{\times}[\frac{1}{2\pi}]$.

Chapter 4

Torus actions on graph hypersurfaces

This chapter contains the main results of this work. We first present what we can say about the existence problem of torus actions on graph hypersurfaces by discussing the question for determinantal hypersurfaces and then applying this to graph hypersurfaces. One existence result is shown to determine a particular simple class of graphs, which means that -in the context of our results- large torus actions only exist for a fairly restricted class of graphs. We conclude the chapter by illustrating a possible computational approach to apply these results to compute motives of graph hypersurfaces. We follow here closely our paper with S. Müller-Stach [36].

4.1 Existence of Torus actions on determinantal hypersurfaces

Let us start with some observations on the existence of torus actions on determinantal hypersurfaces.

Remark 4.1.1. *(a) In our situation, all tori are split over* k*, i.e., isomorphic to* $\mathbb{G}_m^r$*.*
(b) Any linear operation of a torus is diagonizable, i.e., the image is conjugate (w.r.t. linear transformations) to a subgroup of the maximal standard torus in $\mathrm{PGL}_n(k)$*.*
(c) Consider an operation $\mathbb{G}_m =: T \to \mathrm{Aut}(X)$*, where* X *is projective. Assume that it is the restriction of a linear, diagonal operation on some embedding* $X \hookrightarrow \mathbb{P}^{n-1}$*. Then the operation* T *is determined by a vector* $\eta = (\eta_1, ..., \eta_n)$ *of integers - usually called the weights of the operation - such that the operation is given by* $X_i \mapsto t^{\eta_i} X_i$ *for* $1 \leq i \leq n$*.*

Let

$$D = \{((t_1,\, \ldots,\, t_n),\, (s_1,\, \ldots,\, s_n)) \colon \prod_{1 \le i \le n} s_i t_i = 1\} \subseteq \mathbb{G}_m^n \times \mathbb{G}_m^n$$

be a subgroup (of dimension $2n-1$). Suppose

$$M = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix} \in k[y_{ij} \colon 1 \le i,\, j \le n]_1^{n \times n}$$

for $n \in \mathbb{N}$. The associated determinantal hypersurface $X \colon det(M) = 0 \subseteq \mathbb{P}^{n^2-1}$ is usually called *generic* determinantal hypersurface. By defining

$$D \times \mathbb{P}^{n^2-1} \xrightarrow{\delta} \mathbb{P}^{n^2-1},$$

$$((s,\, t),\, [p_{11} \colon y_{12} \colon \ldots \colon p_{nn}]) \mapsto [s_1 t_1 p_{11} \colon s_1 t_2 p_{12} \colon \ldots \colon s_n t_n p_{nn}]$$

we obtain a torus action on $X \subseteq \mathbb{P}^{n^2-1}$, since $X = Z(det(M))$ and (if we write $M(p)$ for the matrix where each variable y_{ij} is substituted by p_{ij})

$$\begin{aligned} det(M(\delta((s,\, t),\, p))) &= \begin{pmatrix} s_1 t_1 p_{11} & \ldots & s_1 t_n p_{1n} \\ \vdots & & \vdots \\ s_n t_1 p_{n1} & \ldots & s_n t_n p_{nn} \end{pmatrix} \\ &= det\left(\begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} M(p) \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \right) \\ &= det(M(p)) \prod_{1 \le i \le n} s_i t_i = det(M(p)). \end{aligned}$$

Having shown that the action of D restricts to X is not sufficient. We need to determine the "effectiveness" of the action. As usually we can (without losing information) pass from the action δ to a representation of D as a subgroup of $Aut(X)$. The relevant question is then: What is the dimension of $Im(D \to Aut(X))$? We will answer that question together with giving the general existence results, but before doing so let us introduce more possible constructions of torus actions on hypersurfaces.

Let us now start with the general discussion.

Proposition 4.1.2. *Let M be the generic symmetric $(h \times h)$-matrix over the polynomial ring $k[Y_{ij} \mid 1 \le i \le j \le h]$, i.e. the matrix $M = (Y_{ij})_{1 \le i,\, j \le h}$ Its entries are $M_{ij} = Y_{ij}$ for $i \le j$ and $M_{ij} = Y_{ji}$ for $i > j$. Then the number of variables is $n = \binom{h+1}{2}$, and $f = \det(M)$ has the weight lattice*

$$\Lambda_h = \{\omega = (\omega_{ij})_{i \le j} \in \mathbb{Z}^n \;:\; 2\omega_{ij} = \omega_{ii} + \omega_{jj} \text{ for all } i < j\}.$$

The lattice Λ_h has rank h, since an element is already determined by the diagonal entries. The corresponding torus operation on $X = \{\det(M) = 0\}$ is given by

$$Y_{ij} \mapsto t_i t_j Y_{ij}$$

for an element $(t_1, \ldots, t_h)$ in the diagonal torus $\mathbb{G}_m^{h-1} \subset \mathrm{PGL}_h$ of rank $h-1$.

Proof. The explicitly defined torus operation above shows immediately that $\Lambda_h \subseteq \Lambda_{\det(M)}$. To prove the converse, one looks at

$$\det(M) = \sum_{\sigma \in \Sigma_h} \mathrm{sgn}(\sigma) Y_{1\,\sigma(1)} \cdots Y_{h\,\sigma(h)}.$$

The permutation $\sigma = \mathrm{id}$ corresponds the monomial $Y_{1\,1} \cdots Y_{h\,h}$ in $\det(M)$, the transposition (ij) corresponds to $Y_{1\,1} \cdots Y_{i\,j} \cdots Y_{j\,i} \cdots Y_{h\,h}$. Therefore, the definition of the weight lattice $\Lambda_{\det(M)}$ implies that $\Lambda_{\det(M)} \subseteq \Lambda_h$. □

We now construct torus operations on general determinantal hypersurfaces (with the restriction to the case of linear entries of the defining matrix). We first look at the special case where all entries are linearly independent linear homogenous polynomials and the matrix $M = (m_{ij})$ is full in the sense that $m_{ij} \neq 0$, $\forall i, j$.

Proposition 4.1.3. *Suppose $M = (M_{ij}) \in k[X_1, \ldots, X_n]^{h \times h}$ is a full symmetric matrix, such that the entries M_{ij} in the upper triangle are all non-zero and linearly independent linear homogenous polynomials. Assume also that $n = \ell(M) = \binom{h+1}{2}$ is the number of all entries. Consider the maximal torus $\mathbb{G}_m^{h-1}(k) \subset \mathrm{PGL}_h(k)$ given by diagonal matrices. Then the variety*

$$X := \{\det(M) = 0\} \subseteq \mathbb{P}^{n-1}$$

carries a $\mathbb{G}_m^{h-1}$-operation of rank $h-1$.

Proof. By assumption, there is an isomorphism

$$k[Y_{ij} \mid i \leq j] \xrightarrow{\cong} k[X_1, \ldots, X_n]$$

by substitution, and hence a linear k-isomorphism

$$\mathrm{Proj}\, k[X_1, \ldots, X_n] \xrightarrow{\cong} \mathrm{Proj}\, k[Y_{ij} \mid i \leq j]$$

between projective spaces of dimension $n-1$. Thus we may work with the Y_{ij}-coordinates and may assume that we are in the situation of Example 4.1.2. Define a torus operation on $\mathrm{Proj}\, k[Y_{ij} \mid i \leq j]$ by $Y_{ij} \mapsto t_i t_j Y_{ij}$. We have to show that this operation is of rank $h-1$. But the point $P = (1 : \ldots : 1)$ lies on X and its orbit is given by the image of the morphism

$$\varphi : (t_1, \ldots, t_h) \mapsto (t_1^2 : \ldots : t_h^2 : t_1 t_2 : \ldots) \in X.$$

Taking the differential of φ at P, we see that it is an immersion. Hence $\dim \mathrm{Im}(\varphi) = h-1$. □

In general, one has $n \geq \ell(M)$ and we get a slightly more general result for matrices which have some vanishing entries off the diagonal but all non-zero entries are linearly independent linear homogenous as above.

Theorem 4.1.4. *Let $M = (M_{ij}) \in k[X_1, \ldots, X_n]^{h \times h}$ be a symmetric matrix such that all non-zero entries M_{ij} for $i \leq j$ are linearly independent linear homogenous polynomials, and all diagonal entries M_{ii} are non-zero. Then the hypersurface*

$$X := \{\det(M) = 0\} \subset \mathbb{P}^{n-1}$$

admits a linear $\mathbb{G}_m^r$-operation with $r = h - 1 + n - \ell(M)$.
This operation is of rank r, if there is a point $P \in X$ such that P is not contained in the union of the linear hypersurfaces defined by the diagonal entries of M.

Remark 4.1.5. *For the torus operation defined in Example 4.1.3, the number $r = h - 1 + n - \ell(M)$ is maximal with this property. However, there may be examples with extra operations, see example 4.1.10.*

Proof. Let us first assume that $n = \ell(M)$. As in the proof of Prop. 4.1.3, we may work with the variables Y_{ij}, and assume that $X = \det(Y_{ij})$, where some variables Y_{ij} for $i \neq j$ are set to be zero. The $\mathbb{G}_m^{h-1}$-operation from Prop. 4.1.3 can be restricted to X, since X is the zero locus of the $\mathbb{G}_m^{h-1}$-invariant hyperplanes $Y_{ij} = 0$. Therefore, the determinantal hypersurface X admits an operation of $\mathbb{G}_m^{h-1}$ defined by the weight lattice Λ_h from Prop. 4.1.3. To show that the operation is still of rank $h-1$ in this case, where some entries vanish, look at the given point $P = (P_{i,j}) \in X$. Let Σ be the set of all indices $i \leq j$ such that the entry M_{ij} in M is non-zero. Consider the morphism

$$\varphi_\Sigma : \mathbb{G}_m^{h-1} \longrightarrow X \subset \mathbb{P}^{n-1}, \quad (t_1, ..., t_h) \mapsto (P_{i,j} t_i t_j)_{(i,j) \in \Sigma} \in X.$$

Differentiating at P as in Prop. 4.1.3, we see that the Jacobi matrix of φ_Σ contains a diagonal submatrix of rank h, since all P_{ii} are non-zero by assumption.
Suppose now that $n > \ell(M)$. The matrix M defines a k-linear surjection

$$q : \tilde{X} \longrightarrow \tilde{X}_\Sigma$$

of the affine cone $\tilde{X}$ over X to $\tilde{X}_\Sigma$, the affine cone of $X_\Sigma \subset \mathbb{P}^{\ell(M)-1}$ which is the determinantal hypersurface defined by the symmetric matrix with non-zero entries $Y_{i,j}$ for $(i,j) \notin \Sigma$. Since q is induced by the projection

$$\mathbb{A}^n \longrightarrow \mathbb{A}^{\ell(M)},$$

the morphism q is a trivial vector bundle of rank $n - \ell(M)$ whose fibers are linearly embedded in $\mathbb{A}^n$. We have already shown that $\tilde{X}_\Sigma$ admits a rank $h-1$ torus operation. The $\mathbb{G}_m^{h-1}$-operation on X_Σ induces a $\mathbb{G}_m^{h-1} \times \mathbb{G}_m^{n-\ell(M)}$ operation on $\tilde{X} = \tilde{X}_\Sigma \times \mathbb{A}^{n-\ell(M)}$ which is of rank r when restricted to X. $\square$

Example 4.1.6. *Wheels WS_h with h spokes and $2h$ edges satisfy Theorem 4.1.4, since*

$$M(WS_h) = \begin{pmatrix} Y_{11} & -X_2 & 0 & \cdots & -X_1 \\ -X_2 & Y_{22} & -X_3 & \cdots & 0 \\ 0 & -X_3 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & Y_{h-1,h-1} & -X_h \\ -X_1 & 0 & \cdots & -X_h & Y_{hh} \end{pmatrix},$$

with $Y_{ii} = X_i + X_{i+1} + X_{h+i}$. Here, $i+1$ is to be considered $\bmod h$.
As a consequence, the associated hypersurfaces X_h admits a torus operation of rank $h-1$. This bound is sharp, e.g. in the case $h = 3$, the hypersurface $X_3 \subseteq \mathbb{P}^5 \simeq \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ is the complement of the 5-dimensional homogenous space $PSL_3(\mathbb{C})/SO_3(\mathbb{C})$, which admits a rank 2 torus operation. There is no larger linear torus operation, since the group $PSL_3(\mathbb{C})$ is the stabilizer of $\mathbb{P}^5 \backslash X_3$ in $\mathrm{PSL}_6(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^5)$ and it has rank 2 (see [7]).

In general, the condition of linear independence in Theorem 4.1.4 is too restrictive. We need to define a new invariant for symmetric matrices M to formulate a more general result. The proof of Theorem 4.1.4 then implies much more as we will see now.
Let us first fix a certain normal form that we need to formulate the setting and the result in an economical manner. One can always pass from M to a certain normal form by a unique linear transformation as follows. Let $M \in k[X_1, \dots, X_n]_1^{h\times h}$ be a symmetric matrix of linear forms such that all diagonal entries are non-zero and linearly independent. As above, we denote by $\ell(M)$ the dimension of the span of all upper-triangular entries. The h diagonal entries of M are linearly independent by assumption, so we label them (in this order) by $X_1, ..., X_h$. Then we pass to the next parallel diagonal with $i = j-1$. If the entry M_{12} is linearly independent of $X_1, ..., X_h$, then we replace it by X_{h+1}, otherwise it is a linear form $L_{12}(X_1, ..., X_h)$. We continue in the obvious way by going from top to bottom in all diagonals in the upper triangle. For the entries below the diagonal we take the mirror image. Each non-zero entry M_{ij} of M is either a variable $X_1, ..., X_{\ell(M)}$, if it occurs for the first time, or a linear form $L_{ij}(X_1, ..., X_{\ell(M)})$ in those variables. If L_{ij} equals a repeated variable (which may happen), we nevertheless call it a linear form. Hence, the entries which are called ”variables” are the first occurences in the chosen ordering. We say that the resulting symmetric matrix is in *quasi-lexicographic normal form.* Note that this procedure really gives a normal form for these matrices, i.e. it associates to each such M exactly one lexicographic normal form.

Definition 4.1.7. *Let M be in quasi-lexicographic normal form. We define an equivalence relation on indices (ij) $(i \leq j)$ of the non-zero entries M_{ij} as* *the transitive hull of the symmetric relation given by*

$(ij) \sim (kl) \Leftrightarrow$ *a common variable* $X \in \{X_1, ..., X_{\ell(M)}\}$ *occurs in* M_{ij} *and* M_{kl}.

The equivalence classes are called clusters.
An element (ij) *with* $i < j$ *in a cluster* C *is called excessive, if* X_i *or* X_j *do not occur in* $L_{ij}(X_1, ..., X_{\ell(M)})$. *Let*

$$\delta(M) := \sum_{\text{clusters } C} (|C| - 1) + \sharp \text{ excessive entries in } M$$

be the excess of M.

Theorem 4.1.8. *Let* M *be in quasi-lexicographic normal form. Then the hypersurface*

$$X := \{\det(M) = 0\} \subset \mathbb{P}^{n-1},$$

admits a rank r *torus operation which is diagonal in the variables* $X_1, ..., X_n$*, and where*

$$r \geq \max\left(0, h - 1 + n - \ell(M) - \delta(M)\right),$$

if there is a point $P \in X$ *such that* P *is not contained in the union of the linear hypersurfaces defined by the diagonal entries of* M.

Proof. By our convention, all variables $X_1, \ldots, X_{\ell(M)}$ occur for the first time at a unique position M_{ij} in M, and $X_1, \ldots, X_h$ are the diagonal entries. Substituting new variables Y_{ij} for each remaining linear form $L_{ij}(X_1, ..., X_{\ell(M)})$, we arrive at an inclusion

$$i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N+n-\ell(M)-1},$$

where $N - \ell(M)$ is the number of additional variables Y_{ij} with $i < j$. This inclusion maps X to a codimension $N - \ell(M) + 1$ subvariety

$$X' = i(X) = \{\det(M') = 0\} \bigcap \{H_{ij} = 0\},$$

where M' is the matrix obtained by the same substitutions, and H_{ij} are the linear hyperplanes

$$H_{ij} = Y_{ij} - L_{ij}(X_1, ..., X_{\ell(M)}).$$

Theorem 4.1.4 implies the existence of a torus T of rank $\geq h - 1 + N + n - \ell(M) - \ell(M') = h - 1 + n - \ell(M)$ acting on $\{\det(M') = 0\}$. Now we count conditions to estimate the minimal dimension dimension of a torus stabilizing $X' = i(X)$. For the variables X_i in each cluster C to have equal weight amounts to at most $|C| - 1$ conditions. The weights ω_{ij} of the new variables Y_{ij} with $i > j$ are related to the weights of the diagonal entries by the formula $2\omega_{ij} = \omega_{ii} + \omega_{jj}$. Hence, if (ij) is not excessive, one has

$\omega_{ij} = \omega_{ii} = \omega_{jj}$ which satisfies the formula. If (ij) is excessive, then the equation $2\omega_{ij} = \omega_{ii} + \omega_{jj}$ imposes one new extra condition on the weights ω_{ii} and ω_{jj}.
In total, this gives $\delta(M)$ conditions, and hence we obtain a torus operation of rank $\geq n - \ell(M) + h - 1 - \delta(M)$. □

Remark 4.1.9. *We cannot prove that the coordinate system suggested in our proof does always yield a torus operation of the highest possible rank. For example, there could be an operation which is not diagonal in our chosen coordinates, or the cluster conditions are not independent. The latter would be detected in the computations of the weights following the algorithm implicit in the proof though. Therefore, the bounds in this theorem are not sharp. We provide a corresponding example below.*

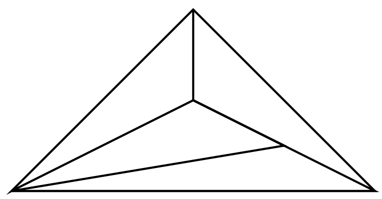
Example 4.1.10. *Consider the graph which is the wheel with* 3 *spokes with one additional triangle subdivided (see figure below). This gives rise to the matrix*

$$M = \begin{pmatrix} X_2 + X_6 + X_8 & X_2 + X_6 & -X_2 & X_2 \\ X_2 + X_6 & X_1 + X_2 + X_4 + X_6 + X_7 & -X_1 - X_2 - X_4 & X_1 + X_2 \\ -X_2 & -X_1 - X_2 - X_4 & X_1 + X_2 + X_4 + X_5 & -X_1 - X_2 \\ X_2 & X_1 + X_2 & -X_1 - X_2 & X_1 + X_2 + X_3 \end{pmatrix}.$$

Substituting as in Theorem 4.1.8 we arrive at

$$M = \begin{pmatrix} Y_1 & Y_5 & Y_8 & -Y_8 \\ Y_5 & Y_2 & Y_6 & -Y_7 \\ Y_8 & Y_6 & Y_3 & Y_7 \\ -Y_8 & -Y_7 & Y_7 & Y_4 \end{pmatrix}.$$

Obviously we have two clusters of length 2 *and* 6 *clusters of length* 1*. By the theorem this means we can expect $X_\Gamma = \{\det(M) = 0\} \subseteq \mathbb{P}^7$ to have no torus operation. However, there is a* 1*-dimensional operation given by the weight vector $\omega := (3, -1, -1, -1, 1, -1, -1, 1)$. The algorithm would give the same result, as Y_7 and Y_8 are in excessive positions but impose no extra relation.*



Let us now describe how to construct examples of graph hypersurfaces with no torus actions, by giving a necessary condition for the existence.

Computation : Let $P_\Gamma = det(M_\Gamma)$ be irreducible. An element $g \in GL(\mathbb{C}^{E(\Gamma)})$ leaves X_Γ invariant if

$$\lambda(g)P_\Gamma = P_\Gamma(g\underline{X}) \qquad (**)$$

for a suitable $\lambda(g) \in \mathbb{C}^*$. For $x \in \mathbb{C}^{E(G)}$ consider the (holomorphic) map $P_{G,x} : GL(\mathbb{C}^{E(G)}) \to \mathbb{C}$. Then we have

$$P_\Gamma((id + a)x) = P_\Gamma(x) + grad(P_\Gamma)(x)a \cdot x + O(\|a\|^2).$$

Now assume that $\mathrm{Aut}(X_\Gamma) \cap \mathrm{GL}(\mathbb{C}^{E(G)})$ contains a copy of $\mathbb{C}^*$, i.e. let $\iota : \mathbb{C}^* \to \mathrm{Aut}(X_\Gamma) \cap \mathrm{GL}(\mathbb{C}^{E(G)})$ an injection (which is automatically holomorphic). Then

$$P_\Gamma(\iota(1 + h)x) = P_\Gamma(x) + grad(P_\Gamma)(x)\iota'(1)h + O(|h|^2)$$

and

$$\lambda(\iota(1 + h)) = \lambda(id) + \mathrm{d}\lambda(id)\iota'(1)h + O(|h|^2).$$

Comparing first degree terms (both sides are power series expansions of the same holomorphic function $\mathbb{C}^* \to \mathbb{C}$) yields

$$grad(P_\Gamma)(x)\iota'(1)h = \mathrm{d}\lambda(id)\iota'(1)hP_\Gamma(x).$$

This gives a necessary condition for the existence of a $\mathbb{C}^*$ action: Namely assume there is an injective $\iota : \mathbb{C}^* \to \mathrm{Aut}(X_\Gamma) \cap \mathrm{GL}(\mathbb{C}^{E(G)})$ such that $\mathrm{Im}(\iota) \not\leq Z(\mathrm{GL}(\mathbb{C}^{E(G)}))$, then there is a $\zeta \in \mathbb{C}$ and a

$$id + a \in \mathrm{Aut}(X_\Gamma) \cap \mathrm{GL}(\mathbb{C}^{E(G)}) \backslash Z(\mathrm{GL}(\mathbb{C}^{E(G)}))$$

such that

$$grad(P_\Gamma)(\underline{X})a\underline{X} = \zeta P_\Gamma(\underline{X}).$$

(The latter equation necessarily holds for the Zariski-open subset $\mathbb{P}(E(G)\backslash X_\Gamma$ resp. for the associated affine cone.)
Without loss of generality let now $\lambda = 1$ for all $g \in \mathrm{Im}(\iota)$ (else consider $g' := \lambda^{-\frac{1}{h_1(G)}}g$). Then the equation

$$grad(P_\Gamma)(\underline{X})a\underline{X} = 0$$

must be satisfied.

Example 4.1.11. *Banana graphs $\Gamma = B_N$ are those graphs with 2 vertices and N edges between those edges. The graph polynomial (of degree $h_1 = N - 1$) is given by*

$$P_{B_N} = \sum_{i=1}^{N} \prod_{j \neq i} X_j.$$

This is an irreducible polynomial $P_\Gamma = \sum_{i=1}^{n} \prod_{j \neq i} X_j$ *in* $A = \mathbb{C}[X_e : e \in E(G)]$. *The partial derivatives of that polynomial are the polynomials* ${P_{B_{N-1}}}^{(i)}$ *in variables* X_j *with* $j \neq i$. *For* $N = 4$ *we get a* 16×16 *system of linear equations* $grad(P_\Gamma)(\underline{X})a\underline{X} = 0$. *By using a solver (e.g. using the Maple CAS) we see that this system does not have a solution* a.

Lemma 4.1.12. *Let* Γ *be a graph such that the non-zero entries in the upper triangle of* M_Γ *are linearly independent. Then, for any faithful operation of* $T := \mathbb{G}_m^r$ *with* $r = h - 1 + n - \ell(M)$ *on* X_Γ, *as described in Theorem 4.1.4, the variety* $\mathrm{Fix}_{\mathbb{P}^{n-1}}(T)$ *is zero-dimensional, and consists of points contained in* X_Γ.

Proof. We may assume that $n = \ell(M)$, since the operation on the $n - \ell(M)$ extra variables is effective. By Example 4.1.3, the operation on the generic symmetric matrix with independent linear entries is given by $(t, x) \mapsto (t_i t_j x_{ij})$. Choosing special values for t_i and t_j with $\prod_i t_i = 1$, one sees that the fixed points in this case are just the points corresponding to the usual standard basis of the underlying space $\mathbb{P}^{N-1}$ with $N = \binom{h+1}{2}$. In our more general situation, the graph hypersurfaces are intersections of the generic zero set of the determinant of the generic symmetric matrix with (T-invariant) linear coordinate subspaces. Hence the fixed point set $\mathrm{Fix}_{\mathbb{P}^{n-1}}(T)$ is given by points in $\mathbb{P}^{n-1}$ with exactly one non-zero entry, i.e., a vertex of the coordinate simplex. Obviously these points are contained in X_Γ. □

4.2 Examples: *-graphs

At the beginning of this section we need to introduce a few conventions. We will call a basis $B \subseteq H_1(\Gamma)$ a cycle basis if it consists only of simple cycles. That such a basis exists is a standard fact in graph theory. Since the matrix M_Γ associated to a graph Γ depends on the chosen basis of $H_1(\Gamma)$ we will make this dependence explicit in this section by writing $M_{\Gamma,B}$.

A class of examples which have linearly independent entries in $M_{\Gamma,B}$ and which contains the wheels with n spokes are the $*$-graphs:

Definition 4.2.1. *A polygonal graph* Γ *is a connected, not necessarily simple, graph which has a decomposition* $\Gamma = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_h$ *as a successive glueing (in the sense of topological spaces) along non-empty, connected sets of edges inside given cycles* Δ_i, *and such that no edge is used twice for glueing. Let* $E_0 \subset \Gamma$ *be the union of all edges used for the glueing. A* $*$*-graph* Γ *is a polygonal graph such that every such decomposition has the property* $h_1(E_0) = 0$.

Note that there are also other, but different, notions of polygonal graphs in the literature.

Example 4.2.2. *In the literature dealing with the motives of graph hypersurfaces one calls a connected graph* Γ *a banana graph (denoted by* B_n*) if and only if it consists of exactly two vertices and* n *edges connecting both vertices. This implies that* $h_1(\Gamma) = n - 1$*. The example of a banana graph with* $n = 4$ *edges and* 3 *loops shows that the condition* $h_1(E_0) = 0$ *depends on the glueing order. To see this, label the edges* 1, 2, 3, 4*. This gives as candidates for cycles the graphs consisting of exactly two edges, e.g.* (1, 2)*. Then* $B_4 = ((1,\,2) \amalg_{\{2\}} (2,\,3)) \amalg_{\{3\}} (3,\,4)$*. But also* $B_4 = ((1,\,2) \amalg_{\{2\}} (2,\,3)) \amalg_{\{2\}} (2,\,4)$*. Hence,* $E_0 = \{2, 3\}$ *(and* $h_1(E_0) = 1$*) in the first case and* $E_0 = \{2\}$ *(and* $h_1(E_0) = 0$*) in the second. This shows that we have to require that* $h_1(E_0) = 0$ *for all decompositions. The matrix* $M_{\Gamma,B}$ *(corresponding to the basis* B *obtained from the* 3 *obvious loops) has linearly dependent entries for this graph. One can verify that the hypersurface corresponding to* B_4 *does not admit any non-trivial linear* $\mathbb{G}_m$*-operation.*

Lemma 4.2.3. *Assume that* Γ *is a polygonal graph.*
(i) If there is a decomposition with $h_1(E_0) = 0$*, then*

$$h_1(\Gamma) = \sharp \text{ cycles } \Delta_i = h.$$

(ii) For all edges e *in* Γ*, one has*

$$h_1(\Gamma \setminus e) < h_1(\Gamma).$$

We will call a graph satisfying (ii) a *homology model.* In the literature this is sometimes called 1-particle irreducible without external edges [8]. We prefer to call it a homology model, since this captures in a better way the topological nature of the definition.

Proof. (i) We use the Mayer-Vietoris Theorem and induction on the number of cycles. Assume $\Gamma = \Gamma' \cup \Delta$, where Δ is a cycle. Then the intersection $\Gamma' \cap \Delta$ is a connected and contractible union of edges, in particular $h_1(\Gamma' \cap \Delta) = 0$ and $h_0(\Gamma' \cap \Delta) = 1$. Hence there is an isomorphism $H_1(\Gamma') \oplus H_1(\Delta) \cong H_1(\Gamma)$.
(ii) A trivial induction on the decomposition of a polygonal graph shows that $\Gamma \setminus e$ is still connected. Let U be an open subset of Γ which contains $\Gamma \setminus e$ and is homotopy equivalent to it. Also, let V be a contractible open subset containing e. Then $U \cup V = \Gamma$, and the assertion follows from the Mayer-Vietoris sequence for open coverings. □

While it is natural to define $*$-graphs as polygonal graphs with an additional property, we remark that they form a subclass of planar graphs:

Lemma 4.2.4. *A graph* Γ *is polygonal if and only if it is planar and a homology model.*

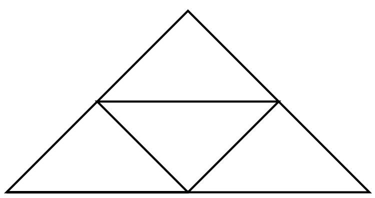
Proof. Γ is polygonal if and only if $\Gamma = \coprod_{E_0} \Delta_i$, where all Δ_i are simple cycles, every edge belongs to at most two Δ_i's, and no edge in E_0 is used twice for glueing. This condition means that the set $\{\Delta_i\}$ contains a simple basis of the cycle space $H_1(\Gamma)$ of Γ. For the definition of a simple basis, see [17, sect. 4.5]. Hence Γ is planar by MacLane's planarity criterion [17, Thm. 4.5.1]: a graph is planar if and only if $H_1(\Gamma)$ contains a simple basis.
Conversely, consider a plane embedding $\Gamma \to \mathbb{R}^2$. Choose a compact disc $\Gamma \subseteq D \subseteq \mathbb{R}^2$ such that $\partial D \cap \Gamma = \emptyset$ (here ∂ means "boundary of"). Define the equivalence relation $\sim$ on $D \times D$ by requiring $x \sim y$ if and only if x and y are connected by a path inside $D \backslash \Gamma$ or inside Γ. This gives a partition $D = \Gamma \cup A \cup \bigcup_{i=1}^{h_1(\Gamma)} \Delta_i^\circ$, where A is the unique component with $\partial D \subseteq A$ and $\partial \Delta_i$ are cycles [17, Prop. 4.2.6]. Then, $(\partial \Delta_1^\circ, \ldots, \partial \Delta_{h_1(\Gamma)}^\circ)$ is a cycle basis of $H_1(\Gamma)$, and no edge of Γ lies in more than two $\partial \Delta_i$ [17, Lem. 4.2.2]. Since Γ is a homology model, every edge of Γ is contained in some $\partial \Delta_i$. Hence, glueing in the given order shows that Γ is polygonal. □

Definition 4.2.5. *We will call a simple cycle $\Delta \subseteq \Gamma$ an inner cycle of Γ if there exist simple cycles $\Delta_2, \ldots, \Delta_{h_1(\Gamma)}$ such that $B := (\Delta, \Delta_2, \ldots, \Delta_{h_1(\Gamma)})$ is a cycle basis of $H_1(\Gamma)$ and $\Delta = \sum_{i=2}^{h_1(\Gamma)} \Delta \cap \Delta_i \in H_1(\Gamma, \mathbb{F}_2)$.*

Lemma 4.2.6. *A graph Γ with no inner cycles is planar.*

Proof. Note that the class of graphs without inner cycles is closed under taking subgraphs and that (all subdivisions of) the complete bipartite graph $K_{3,3}$ and the complete graph K_5 have inner cycles. Thus the assertion follows from Kuratowski's planarity criterion that states that a graph is planar if and only if it does not contain neither $K_{3,3}$ (complete bipartite graph) nor K_5 (complete graph) [17, Thm. 4.4.6]. □

The converse does not hold, since a typical graph with an inner cycle is



This graph is not a $*$-graph, as E_0 is the inner triangle. For $*$-graphs, the following characterization holds.

Theorem 4.2.7. *Let Γ be a graph. Then the following conditions are equivalent:*
(i) Γ is a $*$-graph.
(ii) Γ is a homology model, and there exists a cycle basis $B \subseteq H_1(\Gamma)$ such that the non-zero upper-triangular matrix entries M_{ij} of $M_{\Gamma,B}$ are linearly independent polynomials in $k[X_1, \ldots, X_n]_1$.

Proof. (ii) ⇒ (i): We will first show that Γ is planar. To this end, we show that Γ has no inner cycles, hence Γ is planar by Lemma 4.2.6. Suppose Γ has inner cycles. This means that, in addition to the cycle basis $B = (\Delta_1, \ldots, \Delta_{h_1(\Gamma)})$, there is another cycle basis $B' = (\Delta'_1, \ldots, \Delta'_{h_1(\Gamma)})$ of $H_1(\Gamma, \mathbb{F}_2)$ such that

$$\Delta'_1 = \sum_{i=2}^{h_1(\Gamma)} \Delta'_1 \cap \Delta'_i.$$

In the special case where $B = B'$, this relation immediately leads to a linear dependence between the matrix entry M_{11} and other entries in the first row or column, and hence contradicts the assumption.

In general, since $GL(H_1(\Gamma, \mathbb{F}_2))$ is generated by transvections, we can always find $t \in GL(H_1(\Gamma, \mathbb{F}_2))$ such that $t(B) = B'$, and t is product $t = t_1 \cdots t_l$ of transvections. In addition, we will now show that we can reduce to the case where $t_i \cdots t_l(B)$ is a cycle basis for all i. In the following, we shall do only one iteration of the reduction, since one obtains the full reduction by simply repeating this step. Hence, assume that $\Delta_i = \Delta'_i$ for all $i > 1$ and $\Delta'_1 = \sum_{i=1}^{h_1(\Gamma)} \alpha_i \Delta_i$, with $\alpha_1 = 1$. For $i = 2, \ldots, h_1(\Gamma)$ define $t_i = 1 + \alpha_i E_{1i}$, where E_{1i} is the matrix with 1 at entry $(1, i)$ and 0 else. Then $t(B) = B'$, where $t = \prod_{i=2}^{h_1(\Gamma)} t_i$. Note that the t_i commute pairwise. Suppose (after reordering if necessary) for some $i > 2$ (if $i = 2$ we are done) that $t_j \cdots t_{h_1(\Gamma)}(B)$ is a cycle basis for all $i \leq j$. Then there exists $2 \leq k < i$ such that $t_i \cdots t_{h_1(\Gamma)}(\Delta_1)$ shares edges with the cycle Δ_k, since otherwise Δ'_1 would not be a simple cycle. Now, swap the indices of t_{i-1} and t_k and proceed inductively.

Having shown this reduction for $t(B) = B'$, and assuming $B \neq B'$, this reduces us without loss to the situation $\Delta'_1 = \Delta_1 + \Delta_2$, $\Delta'_j = \Delta_j$ for $j \geq 2$, and $\Delta_1 \cap \Delta_2 \neq \emptyset$. Hence, $\Delta'_1 \cap \Delta_j = (\Delta_1 \cap \Delta_j) + (\Delta_2 \cap \Delta_j)$ for all j. In particular, $\Delta'_1 \cap \Delta_2 = (\Delta_1 \cap \Delta_2) + \Delta_2$.

This implies that the relation

$$\Delta'_1 = \sum_{i=2}^{h_1(\Gamma)} \Delta'_1 \cap \Delta'_i = \Delta'_1 \cap \Delta_2 + \sum_{i=3}^{h_1(\Gamma)} \Delta'_1 \cap \Delta_j$$

from the beginning yields the equation

$$\Delta_1 = \Delta_1 \cap \Delta_2 + \sum_{j \geq 3} (\Delta_1 + \Delta_2) \cap \Delta_j.$$

This is a non-trivial relation among the elements of

$$\{\Delta_1, \ldots, \Delta_{h_1(\Gamma)}, \Delta_i \cap \Delta_j : \forall i, j\},$$

i.e., matrix entries of M, a contradiction.

Hence, Γ is planar, and therefore polygonal by Lemma 4.2.4. Assume $\Gamma = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_h$, but $h_1(E_0) > 0$. Let $\Delta_1, ..., \Delta_h$ be the natural basis of

$H_1(\Gamma)$ given by the cycles Δ_i. Given a simple non-zero loop $\gamma \subset E_0$, there is a linear relation between the diagonal entries for all Δ_i meeting γ and all off-diagonal entries carrying glueing data for these Δ_i.
(i) $\Rightarrow$ (ii): Conversely, suppose that Γ is a $*$-graph and we have given a linear relation among the entries of $M_{\Gamma,B}$. By definition of $*$-graphs, this relation involves a diagonal element, since every edge is only used once for glueing. Hence, we get an equation

$$\sum_{i=1}^{h} a_i M_{ii} = \sum_{i<j} b_{ij} M_{ij},$$

with at least one a_i and one b_{ij} non-zero by Lemma 2.1.5. This is a contradiction, since each Δ_i occurring on the left with $a_i \neq 0$ has an edge which is not contained in E_0. □

Corollary 4.2.8. *The $*$-graphs admit a torus operation of dimension $r \geq h-1+n-\ell(M_{\Gamma,B})$. It is faithful under the condition given in Theorem 4.1.4, i.e., if the graph hypersurface is not a union of h linear hyperplanes.*

Proof. By Theorem 4.2.7, the entries of $M_{\Gamma,B}$ satisfy the assumptions of Thm. 4.1.4. □

4.3 Motivic Bialynicki-Birula decompositions

In this section we discuss how to apply high dimensional torus operations on X_Γ to compute the motive of a graph hypersurface $X_\Gamma = \{\det(M_\Gamma) = 0\}$ using a motivic version of the decomposition theorem of Bialynicki-Birula [5]. For simplicity assume that k is algebraically closed and of characteristic zero.
In the following we use (cohomological) motives $M(X)$ in the sense of Voevodsky's triangulated category $DM(k) = DM_{gm}(k)$ attached to any k-scheme X. The motive $M(X)$ for a possibly singular variety X is defined in [46, chap. 5]. We want to give a criterion when the motive of a graph hypersurface $M(X_\Gamma) \in DM(k)$ is mixed Tate. An object $M \in DM(k)$ is called mixed Tate, if it is in the image of

$$DMT(k) \to DM(k) \otimes \mathbb{Q},$$

where $DMT(k)$ is the full $\mathbb{Q}$-linear triangulated subcategory of $DM(k) \otimes \mathbb{Q}$ generated by the Tate objects $\mathbb{Q}(n)$ as defined by Levine [39].

Example 4.3.1. *The simplest example which is not entirely trivial is $\Gamma = WS_3$, the wheel with 3 spokes. The graph hypersurface X_Γ for $\Gamma = WS_3$ is isomorphic to $\mathrm{Sym}^2\mathbb{P}^2$, and admits a 2-dimensional torus operation. The motive of X_Γ is mixed Tate by [7, Sect. 9].*

In view of the classical Bialynicki-Birula theorem [5] and its motivic versions [12, 27], one might expect that the motive of X_Γ should be determined by the components F of the fixed point set, if X_Γ carries a non-trivial torus operation. In the smooth case, the theorem of Bialynicki-Birula takes the form

$$M(X) \cong \bigoplus_F M(F)(n_F)$$

in $DM(k)$ with appropriate Tate twists n_F depending on each F. In the presence of singularities, we have to use equivariant cubical hyperresolutions to obtain a useful version ob Bialynicki-Birula's theorem. The idea is to replace a singular variety X by a simplicial variety $X_\bullet \to X$ with smooth components X_α and an equivariant torus operation on each X_α.

Proposition 4.3.2. *For every integral closed subvariety $X \subset \mathbb{P}^{n-1}$ with an algebraic operation of a torus T, there is an equivariant cubical hyperresolution*

$$X_\bullet \longrightarrow X$$

in the sense of [23]. Every component X_α in the hyperresolution $X_\bullet$ can be chosen smooth and projective. The motive $M(X) \in DM(k)$ can be obtained from $X_\bullet$ by descent, i.e., the morphism $M(X) \to M(X_\bullet)$ is an isomorphism.

Proof. See [23] or [42, Thm. 5.2.6] for the explicit construction of a cubical hyperresolution via a resolution of singularities. The construction is inductive, and in each step some varieties are replaced by several smooth components. Levine has used this in the context of motives, see for example [40, Thm. 3.2.5, pg. 246]. For X one has now two motives: $M(X)$ as defined in [46, chap. 5], and $M(X) := M(X_\bullet)$ as defined in Levine. However, there is a descent statement for the cdh-topology in $DM(k)$ [46, chap.5, sect.4], and this implies, by inductive application in the abtract blow-up squares of a cubical hyperresolution, that $M(X)$ and $M(X_\bullet)$ are isomorphic in the triangulated category $DM(k)$. A resolution of singularities, hence a cubical hyperresolution, can be made equivariant using equivariant resolution of singularities, see [48]. □

Example 4.3.3. *A nice example with a torus operation is the nodal rational curve C with desingularization $\mathbb{P}^1$, where the points 0 and ∞ on $\mathbb{P}^1$ are identified to the singular point P in C. The associated cube is the square*

$$\begin{array}{ccc} \{0,\infty\} & \longrightarrow & \{P\} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & C. \end{array}$$

Over a perfect field of positive characteristic, alterations in the sense of de Jong give another way of constructing such a hyperresolution.

We assume that we are in this situation now.

Proposition 4.3.4. *Assume that all fixed point loci in all smooth, proper components X_α of $X_\bullet$ induce mixed Tate motives $M(X_\alpha)$. Then $M(X)$ is mixed Tate.*

Proof. All components X_α in the cubical hyperresolution give a mixed Tate motive $M(X_\alpha)$ by assumption. The arrows in the simplicial variety $X_\bullet$ are contained in the full subcategory $DMT(k)$. Hence $M(X_\bullet)$ descends to a mixed Tate motive $M(X)$. □

Proposition 4.3.4 reduces the complexity of the motive of X_Γ with this method to that of the fixed point loci in some resolution of singularities. This method should be successful provided there is some sufficiently high dimensional torus operation.

Example 4.3.5. *Let us revisit $\Gamma = WS_3$, the wheel with 3 spokes. The graph hypersurface X_Γ for $\Gamma = WS_3$ is isomorphic to $\mathrm{Sym}^2\mathbb{P}^2$, which has a resolution by a single blow-up of the diagonal. By Lemma 4.1.12, the fixed point locus $\mathrm{Fix}_{\mathbb{P}^5}(T)$ consists of points, hence $M(X_\Gamma)$ is a mixed Tate motive.*

However, besides the wheel with 3 spokes explicit examples are very hard to obtain computationally since this requires to compute an explicit equivariant resolution of singularities respectively the evolution of the fixed point loci in this resolution. Doing this algorithmically involves solving many instances of NP-hard problems such as computing the irreducible components of subvarieties of graph hypersurfaces. Note that the equivariant resolution of a singular hypersurface X can have a fixed point set which is a not mixed Tate motive, even if the fixed point set in X consists of isolated points. The cone over an elliptic curve gives such an example. However in special situations it seems to be possible to use the approach of computing the motive via a resolution of singularities in special situations. One such situation, which is not related to graph hypersurfaces, would be the case of a Springer resolution of the variety of nilpotent points of a semi-simple Lie-algebra.

Remark 4.3.6. *The torus actions constructed in section 4.1 are equivariant w.r.t. the blow-ups used in the construction of $P \to \mathbb{P}^{N-1}$ resp. $Y \to X_\Gamma$. (See 3.3.8.) The centers of these blow-ups are linear spaces hence their motives are in particular mixed Tate. Together with 4.3.4 this means that whether the Feynman motive 3.3.9 is mixed Tate is already determined by the motive of the fixed point loci of a $\mathbb{G}_m^r$-action on X_Γ.*

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