On Sandwiched Singularities

Dissertation zur Erlangung des Grades
"Doktor der Naturwissenschaften"

am Fachbereich Mathematik der
Johannes Gutenberg-Universität in Mainz

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am 26. November 2003

D77 Mainzer Dissertation

Tag der mündlichen Prüfung: 20. Februar 2004

Introduction

A sandwiched singularity is a surface singularity on the blowup of $\mathbb{C}^2$ in an ideal defined by infinitely near points. Sandwiched singularities have been studied by many authors including Zariski [Zar39], Lipman [Lip69], Hironaka [Hir83], and Spivakovsky [Spi90]. The deformation theory of sandwiched singularities has been studied by de Jong and van Straten in [dJvS98].

Sandwiched singularities are rational, in particular they are normal. In general, they are not complete intersections and there are no particularly simple or nice equations for them. For example, cyclic quotient singularities are sandwiched, and more generally, all rational singularities with reduced fundamental cycle (sometimes called minimal surface singularities) are sandwiched. Hence sandwiched singularities constitute a large class of rational singularities, and we cannot expect to get easy access to information about them by looking at their equations. On the other hand, that means that we can hope to study phenomena which might be typical for general rational singularities, but do not appear for hypersurfaces or complete intersection singularities which are by far the singularities best understood. For example, we will see that a general sandwiched singularity has many smoothing components, whereas the base space of the semiuniversal deformation of a hypersurface singularity is smooth. Instead of extracting information from the equations, we will use the geometry of the plane which we have to blow up to get the sandwiched singularity. More specifically, sandwiched singularities can be connected to plane curve singularities in the following way:

An ideal generated by infinitely near points is by definition an ideal generated by the equations of curves in $(\mathbb{C}^2, 0)$ with the property that their strict transforms under some blowups pass through certain points with prescribed multiplicities. By choosing a generic curve C with this property and attaching to each branch C_i a number $l(C_i)$ to specify the points on the exceptional divisors, we get an object (C, l) called a decorated curve. A decorated curve determines the singularity $X(C, l)$ on the blowup. Now the central idea is that there is a close connection between the geometry of the plane curve C and the geometry of $X(C, l)$. For example, we can easily read off the dual resolution graph of $X(C,l)$ from the equisingularity class of C and the numbers $l(C_i)$. Even more striking is the result of de Jong and van Straten which states that all deformations of a sandwiched singularity $X(C,l)$ are induced by deformations of the decorated curve (C,l). This enables us to answer many questions about sandwiched singularities by studying plane curve singularities, which are among the best understood geometric objects.

My intention while writing this thesis was two-fold. The first, of course, was to contribute to the solution of several open problems, most of which have been raised in [dJvS98]. My second intention was to write an introduction to sandwiched singularities from a classical geometrical point of view. The theory of infinitely near points in the study of plane curve singularities goes back to the nineteenth century and is a very beautiful subject. Inspired by the book [CA00] of Casas-Alvero which gives a modern account of Enriques' treatment of the subject, I have tried to give as many proofs as possible using only 'elementary' geometry of plane curves. I hope to convince the reader that one can expect to prove any correct statement about topological invariants of a sandwiched singularity, including statements which involve deformations, by studying plane curves and their behaviour at infinitely near points.

I will now give a general survey of the thesis and mention the main results. Each chapter also has a short introduction containing some more details.

Chapter 1 is an introduction to the theory of infinitely near points and complete ideals. Most of the material is contained in [CA00] except for some remarks, the examples and the following exceptions: The notion of a decorated curve and the associated ideal has been introduced in [dJvS98]. The description of the conductor of a plane curve as a complete ideal is probably well known, but I do not have any references for it. I also have not found the computations of the Hilbert-Samuel function of a complete ideal and of the multiplicity of an arbitrary $\mathfrak{m}_{\mathbb{C}^2,0}$-primary ideal via base points anywhere in the literature.

Chapter 2 starts with the definition of sandwiched singularities and the deduction of some of their most important properties. The notation for the representations $X(C,l)$ of a sandwiched singularity via decorated curves is introduced and it is shown how the dual resolution graph of $X(C,l)$ depends on the equisingularity class of C or more precisely of (C,l). The content of the first five sections is more or less known to the experts, but I had to rewrite most of the proofs which are scattered over various papers and often given in a very short form only. Also many of the proofs have not been given explicitly for the complex-analytic case. I hope that this summary of known results will be particularly helpful to someone who wishes to learn about sandwiched singularities. The theorem on the multiplicity of $X(C,l)$

in section 2.6 is new. Van Straten has informed me that he has an idea for a completely different proof. If we used his idea to prove the theorem, then my method of proof would give us a new proof (for sandwiched singularities) of the fact that every rational singularity of multiplicity n deforms into the cone over the rational normal curve of degree n.

Chapter 2 ends with the classification of taut and pseudotaut plane curve singularities. This classification has been obtained by reversing the usual direction of the arguments: Instead of deducing properties of sandwiched surface singularities from properties of plane curves, I use Laufer's classification of taut and pseudotaut surface singularities to obtain the corresponding lists of plane curve singularities. The lists have already been published by Gawlick in [Gaw92], but his proof is completely different. Equations and associated graphs of taut and pseudotaut curve singularities can be found in the appendices.

Chapter 3 contains one of the main results of the thesis. I start by reviewing the result from [dJvS98] which states that every deformation of the sandwiched singularity $X(C,l)$ is induced by a deformation of (C,l). Then I give some easy examples to demonstrate how this enables us to give easy proofs for some statements on adjacencies of sandwiched singularities. For example, it is almost trivial to see that cyclic quotient singularities only deform into cyclic quotients.

The biggest drawback of the result in [dJvS98] is that the precise statement is not very geometrical. Therefore, it was left as an open problem in [dJvS98] to find a direct geometrical construction of the induced deformation of $X(C,l)$ for a given 1-parameter deformation of (C,l). I solve this problem by showing that deformations of the decorated curve (C,l) correspond to equimultiple deformations of the fat point in which we have to blow up $(\mathbb{C}^2,0)$ to obtain $X(C,l)$. By a result of Teissier, equimultiplicity of this 1-parameter deformation implies that the blowup in the total space of the deformation is the deformation of $X(C,l)$ we are looking for. I also conjecture that the same construction works for deformations over an arbitrary reduced base space. This seems very probable, because for the deformation of the fat point corresponding to a deformation of (C,l), I have shown that the whole Hilbert-Samuel function is constant, not only the multiplicity. So some well known results on the connection between normal flatness and constant Hilbert-Samuel functions (Bennett's theorem) strongly support my conjecture.

Chapter 4 deals with multi-adjacencies of plane curve singularities. Since all deformations of sandwiched singularities are induced by deformations of plane curves, all the results of this chapter have a direct impact on the deformation theory of sandwiched singularities. The most important application is the study of smoothing components of a sandwiched singularity in section 4.6. The first sections of the chapter are devoted to certain combinatorial aspects associated to the problem of deciding whether a given plane curve singularity has a deformation into a curve with certain prescribed singularities. The results in the sections 'Semicontinuity of Multiplicity at Infinitely Near Points' and 'Cutting Enriques Graphs' may not be new but certainly very hard to find in the literature. The result on δ-constant deformations of a curve with four smooth branches is new.

Chapter 5 deals with the Kollár conjecture for sandwiched singularities. The first four sections give a survey of results which have motivated the Kollár conjecture. Then a result of de Jong is quoted which says that the Kollár conjecture for sandwiched singularities is true if and only if the symbolic power algebras of certain curves in three-space are finitely generated. In section 5.7 I collect and generalize some known criteria which are equivalent to the fact that the symbolic power algebra of certain curves is finitely generated. Finally, I show how to apply the general results to the case which is relevant for the Kollár conjecture and compute some examples. I think these examples help to understand the geometry of a generic smoothing of a sandwiched singularity. Unfortunately, I have not succeeded in proving the Kollár conjecture right or wrong.

I want to close the introduction by mentioning a possible subject of future work. Many of the results in the theory of complete ideals in the local ring of $(\mathbb{C}^2, 0)$ have been extended to complete ideals in the local ring of an arbitrary two-dimensional rational singularity by Lipman [Lip69]. The theory of infinitely near points on a rational singularity has been developed by Reguera [Reg97]. Therefore, it seems natural to generalize results on sandwiched singularities to singularities on the blowup of a rational singularity. For example, it might be possible to achieve the following for a rational surface singularity X: (1) Classify taut curves on X (compare chapter 2). (2) Relate deformations of a singularity on the blowup of X in a complete ideal to deformations of curves on X (compare chapter 3).

I thank everybody who has helped me in one way or another during the time I have been writing this thesis. This includes my advisor and just about everybody else working in pure mathematics at the university of Mainz, as well as several people who have been bothered by emails all over the world, and all my friends and family.

Contents

Notations and Conventions

A *singularity* is a complex space germ.

A *deformation* of a complex space germ $(X_0, 0)$ is a flat map germ $\pi : (X, 0) \to (S, 0)$ such that $(X_0, 0)$ is isomorphic to the fibre $(\pi^{-1}(0), 0)$ under a given isomorphism $i : (X_0, 0) \to (\pi^{-1}(0), 0)$.

If $\mathfrak{I} \subset \mathcal{O}_X$ is a coherent ideal sheaf, then

$$\Sigma(\mathfrak{I}) := (V(\mathfrak{I}), \mathcal{O}_X/\mathfrak{I}|_{V(\mathfrak{I})})$$

denotes the complex subspace of X defined by $\mathfrak{I}$. Analogously, if $I \subset \mathcal{O}_{X,x}$ is an ideal in the local ring of X at x, then $\Sigma(I)$ denotes the complex subgerm of (X, x) defined by I.

We often say *curve* for "plane curve singularity".

By $e(I)$ we denote the *multiplicity* of an ideal in the sense of *Hilbert-Samuel*.

By $e_p(I)$ we denote the *multiplicity* of an ideal $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ *in the infinitely near point* p, see 1.1.5 for the precise definition.

When we talk about the *components* of the base space of a semiuniversal deformation of a normal surface singularity, we always exclude the embedded components.

We often write *base space* of the singularity X instead of 'base space of a semiuniversal deformation of X'.

Chapter 1

Complete Ideals

Sandwiched singularities are singularities on the blowup of $(\mathbb{C}^2, 0)$ in complete ideals. In the first chapter we are going to study these ideals and give numerous examples.

Complete ideals in the local ring of $(\mathbb{C}^2, 0)$ are ideals defined by infinitely near points. They can be viewed as local analogues of complete linear systems whose base points are infinitely near points.

The concept of infinitely near points was successfully used by M. Noether (1844-1921) and systematically developed by Enriques, see [EC15, book IV], to study plane curve singularities. The general theory of complete ideals in two-dimensional regular rings comes from Zariski [ZS60, app. 4]. A modern treatment is given in the book [CA00], which we recommend as a general reference on the subject, including also more detailed historical information. A short introduction which covers most of the contents of this chapter is given in [LJ95]. We use the notations of [CA00].

Not included in the references on complete ideals mentioned above is the following material: The conductor of a plane curve singularity as an example of a complete ideal in section 1.2.2, and the notion of reduction of an ideal and the related results in section 1.5. Decorated curves have been introduced in [dJvS98].

In this thesis we only deal with complete ideals in the regular, two-dimensional ring $\mathcal{O}_{\mathbb{C}^2,0} \cong \mathbb{C}\{x, y\}$. Much of the theory, including for example the factorization into simple ideals, has been successfully generalized to local rings of rational, two-dimensional singularities by Lipman, see [Lip69]. This shows that the whole theory of sandwiched singularities may be generalized to singularities on the blowup of a rational surface singularity in a complete ideal. See [Reg97] for an exposition of complete ideals on a rational surface singularity which is close in spirit to this thesis. See [CPRL99] for some results on generalized sandwiched singularities over rational singularities.

1.1 Clusters of Infinitely Near Points

We need some definitions. Readers acquainted with infinitely near points and Enriques diagrams should skip this section.

Definition 1.1.1. (Infinitely near points)

1. Let p be a smooth point on a surface S. The points in the *first infinitely near neighbourhood* of p are the points on the exceptional divisor E_p of the blowup of S in p. A point in the $(i+1)$*-th infinitely near neighbourhood* is a point in the first infinitely near neighbourhood of a point in the i-th infinitely near neighbourhood of p.

   The set of all points infinitely near to p is denoted by $\mathcal{N}_p^*$. We define $\mathcal{N}_p := \{p\} \cup \mathcal{N}_p^*$.

2. There is a natural partial ordering on $\mathcal{N}_p$:

   $$q \prec q' \Leftrightarrow q' \in \mathcal{N}_q^*.$$

   We say that q *precedes* q'.

3. Assume $q, q' \in \mathcal{N}_p$. We say that the point q' is *proximate* to q and write $q' \to q$, if q' is a point on the exceptional divisor E_q of the blowup in q or a point on a strict transform of E_q.

4. If a point in $\mathcal{N}_p^*$ is proximate to more than one point, it is called a *satellite point*, else a *free point*.

5. Let $(C, p) \subset (S, p)$ be a reduced germ of a curve, $q \in \mathcal{N}_p$. We say that q *is on* C, if the strict transform of C in q is not empty.

   The *(effective) multiplicity* $e_q(C)$ of C in q is the multiplicity of the strict transform of C in q. So q is on C iff $e_q(C) > 0$.

From now on, 0 will be the zero in $\mathbb{C}^2$.

Definition 1.1.2. (Clusters)

1. A *cluster* is a set $K \subset \mathcal{N}_0$ such that the following two conditions hold:

   (a) K is finite.

   (b) If $p \in K$ and $q \prec p$, then $q \in K$.

2. A *weighted cluster* is a pair (K, μ), where K is a cluster and μ is a map $\mu : K \to \mathbb{Z}$.

   We sometimes view μ as a map on $\mathcal{N}_0$ by setting $\mu(p) = 0 \; \forall p \in \mathcal{N}_0 \setminus K$.

3. Let (K, μ) be a weighted cluster. The *excess* $\rho_p = \rho_p(K, \mu)$ of (K, μ) at $p \in K$ is
$$\rho_p = \mu(p) - \sum_{q \in K, q \to p} \mu(q).$$

4. A *consistent cluster* is a weighted cluster (K, μ), such that
$$\mu(p) \geq \sum_{q \in K, q \to p} \mu(q)$$
holds for all $p \in K$. So a weighted cluster (K, μ) is consistent if and only if $\rho_p(K, \mu) \geq 0$ for all p.

   This inequality is called the *proximity inequality* at p.

Proposition 1.1.3. *A weighted cluster (K, μ) is consistent if and only if there is a curve $(C, 0) \subset (\mathbb{C}^2, 0)$ such that $e_p(C) = \mu(p)$ for all $p \in K$.*

Proof. A proof is given in [CA00, Th. 4.2.2].
The idea of the proof is this: On one hand, the existence of a curve with $e_p(C) = \mu(p)$ implies that the cluster is consistent. On the other hand, assume that (K, μ) is consistent. For each $p \in K$ choose a curve which passes through p with multiplicity ρ_p and is transverse to the exceptional divisor in p. So these curves are on various different blowups of $(\mathbb{C}^2, 0)$. Blow down these curves to curves in $(\mathbb{C}^2, 0)$. The union C of these curves in $(\mathbb{C}^2, 0)$ has the property $e_p(C) = \mu(p)$ for all $p \in K$. □

Remark 1.1.4. In fact, [CA00, Th.4.2.2] is a statement which is slightly stronger: If a consistent cluster (K, μ) and a finite set T of infinitely near points with $K \cap T = \emptyset$ are given, then it is possible to choose a curve passing through the points of K with multiplicity $e_p(C) = \mu(p)$ and missing all points in T.

Lemma 1.1.5. *Let K be a cluster. Then there is a uniquely determined minimal set of positive, consistent weights on K, i.e. the set of all $\mu : K \to \mathbb{Z}$ such that (K, μ) is consistent and $\mu(p) > 0$ for all $p \in K$ has a unique minimal element.*

Proof. It is easy to see how to compute this minimal map min_K inductively, thus proving uniqueness: If no point in the first neighbourhood of p belongs to K, then $min_K(p) = 1$. Else $min_K(p) = \sum_{q \in K, q \to p} min_K(q)$. □

Remark 1.1.6. Following the proof of proposition 1.1.3, a curve having minimal positive multiplicity in each point of a cluster K can be constructed as follows: For each point p in K which is maximal with respect to the proximity relations, choose a smooth curve through p which is transversal to the exceptional divisor through p. If you blow down the union of these curves, you get a curve with the desired property.

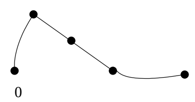
1.1.1 Enriques Diagrams

A cluster contains two sorts of information: the discrete information given by the number of points in the cluster and the proximity relations, and the analytic information of the exact positions of the free points. We encode the discrete information of a cluster K in an Enriques diagram.

An *Enriques diagram* of K is a tree-graph whose vertices correspond to the points of K. We draw an edge from the vertex p to the vertex q if q is in the first neighbourhood of p. To keep track of the proximity relations, there are two kinds of edges, which must be drawn in the following way, beginning at the root-vertex corresponding to 0:

- If q is a free point in the first neighbourhood of p, we join p and q by a smooth curve which is not straight. If $p \neq 0$ and p is in the first neighbourhood of p', then the tangent in p of the (old) edge from p' to p and the tangent in p of the (new) edge from p to q must be the same.
- Assume that q_0 is in the first neighbourhood of p, that q_i is in the first neighbourhood of q_{i-1} and that $q_1, \ldots, q_r$ are proximate to p. Then we draw the r edges from q_{i-1} to q_i onto a straight line through q_0 which is orthogonal to the tangent in q_0 of the edge from p to q_0.

Example 1.1.7. We consider a cluster consisting of five points, one each in the 0th, 1st, 2nd, 3rd and 4th neighbourhood. The points in the 0th, 1st and 4th neighbourhood shall be free, the points in the 2nd and 3rd neighbourhood proximate to $0 \in \mathbb{C}^2$. An Enriques diagram of such a cluster looks like this:



1.1.2 Example: Simple Clusters

Let $p \in \mathcal{N}_0$ be an infinitely near point in the k-th neighbourhood. We define the weighted cluster $\mathcal{K}(p)$ to be the smallest cluster which contains p with the minimal set of positive weights making it consistent.

So the set of points of $\mathcal{K}(p)$ is the union of $\{p\}$ with the set of all points in $\mathcal{N}_0$ preceding p, and the weights of p are inductively defined by $\mu(p) = 1$ and $\mu(q) = \sum_{q_i \to q} \mu(q_i)$.

We call a weighted cluster *simple* iff it is equal to $\mathcal{K}(p)$ for some $p \in \mathcal{N}_0$. So a consistent cluster which contains k points and has minimal positive weights is simple if and only if it has exactly one point each in the zero-th up to the $(k-1)$-th neighbourhood.

For example, if we assign the weights $3, 1, 1, 1, 1$ to the cluster of example 1.1.7, then we get a simple cluster.

1.1.3 Example: Enriques Cluster of a Curve

Definition 1.1.8. Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated curve singularity. A weighted cluster (K, μ) is called an *Enriques cluster* of C, iff

1. All points of K are on C.
2. K contains all points in which we have to blow up to get a minimal good embedded resolution of C.
3. K contains at least one point on each branch which is on no other branch.
4. $\mu(p) = e_p(C)$ for all $p \in K$.

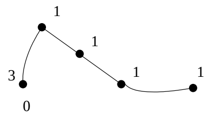
An *Enriques diagram* of an Enriques cluster of $(C, 0)$ is called an Enriques diagram of $(C, 0)$.

Remark 1.1.9.

1. The weights of an Enriques cluster (K, μ) are the minimal positive consistent weights on K.
2. An Enriques cluster of C must contain all satellite points on C as well as all points $p \in \mathcal{N}_0$ with $e_p(C) > 1$. Every isolated curve singularity does have Enriques clusters. This is equivalent to the fact that a finite number of point blowups gives us a good resolution of the singularity.

Example 1.1.10. An Enriques cluster of a smooth curve can have an arbitrary number of points $k \in \mathbb{N}_{>0}$. All points of an Enriques cluster are free and have multiplicity 1. If the cluster contains k points, then there is one point each in the zero-th up to the $(k-1)$-th neighbourhood.

Example 1.1.11. An isolated plane curve singularity C is irreducible if and only if there is only one point on C in each infinitely near neighbourhood. So Enriques clusters of C are simple if and only if C is irreducible. For example, we have already considered the simple cluster



This is an Enriques diagram of an Enriques cluster of the E_6-singularity $x^3 + y^4$.

Example 1.1.12. The infinitely near points on the A_2-singularity $V(y^2 - x^3)$ are shown in table 1.1. The minimal Enriques cluster consists of the three points $0, p_1, p_2$ with multiplicities $2, 1, 1$.

Various other examples of Enriques diagrams of curve singularities are shown in the appendices.

1.1.4 Example: Base Points of a Decorated Curve

Perhaps the most convenient way to describe a cluster of infinitely near points is to give a curve going through all points and to specify how many points on each branch belong to the cluster. The most natural way to count infinitely near points p on an irreducible curve C is to count them with their multiplicities $e_p(C)$. It is also a good idea to choose the curve as simple as possible. This leads to the following definition of a decorated curve. We use some notations from [dJvS98].

Notation 1.1.13. Let $(C, 0) \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an isolated plane curve singularity with r irreducible components C_i, $C = \bigcup_{i=1}^r C_i$.

1. $m(i) = m_C(i) = m_C(C_i)$, $i \in \{1, \ldots, r\}$, is the sum of the multiplicities of the i-th branch of C in all points we have to blow up to obtain the minimal resolution of C.

2. $M(i) = M_C(i) = M_C(C_i)$, $i \in \{1, \ldots, r\}$, is the sum of the multiplicities of the i-th branch of C in all points we have to blow up to obtain the minimal good resolution of C.

Let $n : \coprod_{i=1}^r (\mathbb{C}_i, 0_i) \to (C, 0)$ be a normalization of $(C, 0)$ with $(\mathbb{C}_i, 0_i)$ mapping onto $(C_i, 0)$. Since we are considering germs of curves, a divisor l on the normalization can only have support on $\{0_i |\ i = 1, \ldots, r\}$ and is determined by the r numbers

$$l(i) := l(C_i) := \text{degree of } l \text{ at } 0_i.$$

If l is a divisor on the normalization, we write $l = \sum l(i) \cdot 0_i$, i.e. we use the same letter l to denote the associated map from the set of branches to $\mathbb{Z}$. So the restriction of l to $(\mathbb{C}_i, 0_i)$ is the zero-dimensional space germ defined by $\mathfrak{m}_{\mathbb{C}_i,0_i}^{l(i)}$.

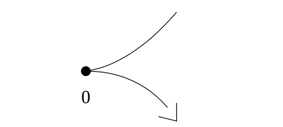
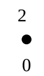
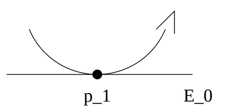
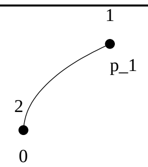
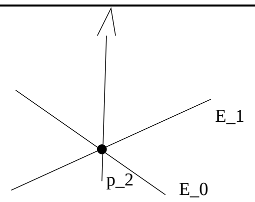
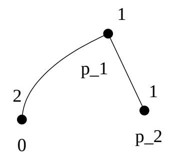
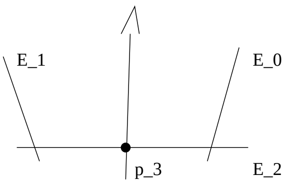
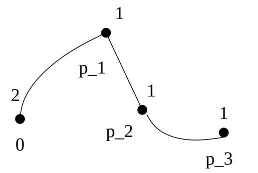
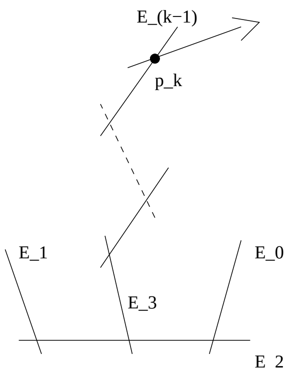
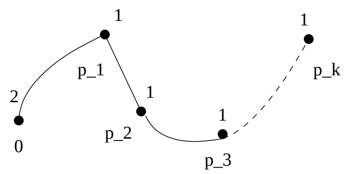
<table>
<tr><th></th><th>Infinitely near points on $V(y^2 - x^3)$</th><th>Enriques diagrams</th></tr>
<tr><td>0-th neighbourhood</td><td></td><td></td></tr>
<tr><td>1st neighbourhood</td><td></td><td></td></tr>
<tr><td>2nd neighbourhood</td><td></td><td></td></tr>
<tr><td>3rd neighbourhood</td><td></td><td></td></tr>
<tr><td>k-th neighbourhood</td><td></td><td></td></tr>
</table>

Table 1.1: The infinitely near points on an A_2-singularity and Enriques diagrams for the clusters of the points in the 0-th up to the k-th neighbourhood. For $k \geq 2$ we have an Enriques cluster of A_2.

Definition 1.1.14. 1. A *decorated curve* is a pair (C, l) consisting of an isolated plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$ and a divisor l on the normalization of C such that $l(C_i) \geq m(i)$.

2. Let (C, l) be a decorated curve.

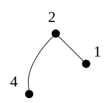
   $BP(C, l)$ is the weighted cluster with the following properties: All points of the cluster are on C, the weights of $BP(C, l)$ are the multiplicites of C, and for each branch C_i of C the sum of the multiplicities of C_i in the points of the cluster is equal to $l(C_i)$.

   $BP(C, l)$ is called the (weighted) *cluster of base points* of the decorated curve (C, l).

   We sometimes write $|BP(C, l)|$ for the underlying non-weighted cluster.

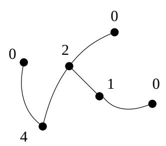
Remark 1.1.15. The condition $l(C_i) \geq m_C(i)$ ensures that there actually does exist a cluster such that for each branch C_i of C the sum of the multiplicities of C_i in the points of the cluster is equal to $l(C_i)$.

Example 1.1.16. We consider a Z_{12}-singularity $V\big(xy(x^2 + y^3)\big)$. An Enriques diagram of $BP\big(Z_{12}, (2, 1, 4)\big)$ looks like this:



Note that $BP\big(Z_{12}, (2, 1, 4)\big)$ is not an Enriques cluster of Z_{12}.

This example shows that it is sometimes convenient to add some points of multiplicity zero to a cluster. Weighted clusters obtained from one another by adding or deleting points of multiplicity zero should be considered as being equivalent. I expect that the reader will agree that the following Enriques diagram gives a better picture of the base points of the decorated curve $\big(Z_{12}, (2, 1, 4)\big)$:



1.1.5 Example: Base Points of an Ideal

This class of examples of weighted clusters is of central importance for this thesis. The definitions may seem rather technical at first sight, but there is a very geometrical way of understanding them, cf. remark 1.1.20.

Definition 1.1.17. Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated curve singularity and (K,μ) a weighted cluster. We give an inductive definition of the *virtual transforms* $\check{C}_p$ and the *virtual multiplicities* $\check{e}_p(C)$ of C in the points $p \in K$:

1. We set $\check{C}_0 := C_0$ and $\check{e}_0(C) := e_0(C)$.

2. If q is in the first neighbourhood of p, we define the virtual transform $\check{C}_q$ of C in q to be the strict transform of the virtual transform of C in p plus $\check{e}_p(C) - \mu(p)$ times the exceptional divisor E_p.

   So if $x^{\check{e}_p(C)} \cdot f(x,y)$ is an equation of the total transform of $\check{C}_p$ in q, where $V(x)$ is the exceptional divisor E_p and f an equation of the strict transform, then $x^{\check{e}_p(C)-\mu(p)} \cdot f(x,y)$ is an equation of the virtual transform of C in q.

   The virtual multiplicity $\check{e}_q(C)$ of C in q is the multiplicity of the virtual transform.

Remark 1.1.18. The definition can be extended to the case that the virtual multiplicities are negative, but we will not consider that case.

Definition 1.1.19. Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an ideal. We give an inductive definition of the set of *base points* of I and their *multiplicities*.

1. The multiplicity of I in zero is

   $$e_0(I) := \min\{e_0(f) |\ f \in I\}$$

   If $e_0(I) > 0$, then zero is a basepoint of I.

2. If p is a basepoint of I and q is in the first neighbourhood of p, then the multiplicity $e_q(I)$ is the minimum of the virtual multiplicities of functions $f \in I$ with respect to the multiplicities of I in the points preceding q. If $e_q(I) > 0$, then q is a basepoint of I.

We denote the weighted *cluster of base points* of I by $BP(I)$.

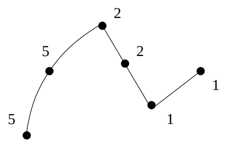
Remark 1.1.20. This definition is natural if you think of it in terms of linear systems as defined in [Zar71]. The (local) linear system of I is the set of curves $\{\big(\Sigma(f), \mathcal{O}_{\mathbb{C}^2,0}/(f)\big) \mid f \in I\}$. Assume that the base locus of the linear system has no fixed components, i.e. $V(I) = \{0\}$, otherwise remove the fixed part. Then $E_0^{e_0(I)}$ is the fixed part of the base locus of the linear system on the blowup of 0 defined by the total transforms of functions in I, so the virtual transforms are the variable part of the transformed linear system. Since the (isolated) base points of a linear system are defined to be the base points of the variable part of the system, the base points in the first neighbourhood are the base points of the transform of the linear system under a blowup. Inductively, we see that the infinitely near base points of I are all (isolated) base points of transforms of the linear system I under a finite number of point blow ups.

Example 1.1.21. The only base point of $(x, y)^n$ is $0 \in \mathbb{C}^2$, its multiplicity is $e_0\big((x, y)^n\big) = n$.

Example 1.1.22. The set of base points of a principal ideal (f) consists of all points on the curve $\Sigma(f)$. The multiplicity of (f) in a base point p is the multiplicity $e_p(\Sigma(f))$ of the curve in p, i.e. the multiplicity of the strict transform of $\Sigma(f)$ through p. Note that $\Sigma(f)$ must not be reduced.

(Recall that $\Sigma(I) \subset X$ denotes the subspace of X with structure sheaf $\mathcal{O}_X/I$, in contrast to $V(I)$ which always denotes a reduced space or the underlying analytic set.)

Example 1.1.23. An Enriques diagram of the cluster of base points of (x^{12}, y^5) is



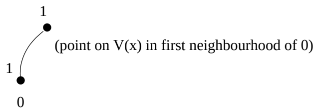
Example 1.1.24. If f defines a smooth curve, g has order k and $V(f)$ and $V(g)$ intersect transversally, then $BP(f, g)$ is the simple cluster consisting of k points on $V(f)$ with multiplicity 1.

More generally, if (C, l) is a decorated curve, C irreducible, f an equation for C and g a function of order l such that $V(g)$ and $V(f)$ intersect transversally, then $BP\big((f, g)\big) = BP(C, l)$.

1.2 Ideals Defined by Infinitely Near Points

If a weighted cluster (K, μ) is given, we would like to consider the set of all curves C going effectively through (K, μ), i.e. of all curves with the property that all points of K are on C and that $e_p(C) \geq \mu(p)$ for all $p \in K$. Unfortunately, if the number of base points is greater than or equal to two, the set of these curves does not form a linear system in general, i.e. the set of their equations is not an ideal.

Example 1.2.1. We consider the cluster consisting of zero and the point on $V(x)$ in the first infinitely near neighbourhood, both weighted with one. An Enriques diagram of the weighted cluster looks like this:



A curve goes effectively through this cluster if and only if its tangent cone at zero contains the y-axis or equivalently iff x divides the initial form of an equation. Obviously, the ideal generated by the equations of these curves is $(x) + (x, y)^2$, including every function of order two. Indeed, if $g(x, y)$ has order two, then $x + g(x, y)$ and x are members of the ideal, so the same is true for $g(x, y) = (x + g(x, y)) - x$.

So the best we can do is to look at the ideal which is generated by the equations of the curves going effectively through (K, μ). A very nice geometric interpretation of what it means for a curve to be in the linear system generated by those curves going effectively through a weighted cluster, and also a convenient way to compute whether f is in the ideal generated by the equations of all curves going through a weighted cluster is given by Enriques' theory of virtual multiplicities. We have already defined what virtual multiplicities are in definition 1.1.17.

Definition 1.2.2. Let (K, μ) be a weighted cluster. We write $\check{e}_p(C)$ for the virtual multiplicity of C in p with respect to (K, μ).
We say that C passes through (K, μ) iff $\check{e}_p(C) \geq \mu(p)$ for all $p \in K$.
We say that C passes strictly through (K, μ) iff $\check{e}_p(C) = \mu(p)$ for all $p \in K$.
We define

$$H_{(K,\mu)} := \{f \in \mathcal{O}_{\mathbb{C}^2,0} | \ \Sigma(f) \text{ passes through } (K, \mu).\}$$

Remark 1.2.3. A curve passing strictly through (K, μ) has virtual multiplicities equal to effective multiplicities, so it goes effectively through (K, μ).

Theorem 1.2.4. *Let (K, μ) be a weighted cluster.*

1. *$H_{(K,\mu)}$ is an ideal.*

2. *A non-negatively weighted cluster (K, μ) is consistent iff there is a curve passing strictly through (K, μ).*
   If (K, μ) is a non-negatively weighted, consistent cluster, then $H_{(K,\mu)}$ is generated by equations of curves passing strictly through (K, μ).

3. *There is exactly one consistent, positively weighted cluster (K', μ') s.t. $H_{(K,\mu)} = H_{(K',\mu')}$.*

For a proof see [CA00].

Remark 1.2.5. Enriques gave an easy-to-handle algorithm to compute the consistent cluster of the last statement of the theorem for a given (K, μ). It consists of performing so-called unloadings, see [CA00].

We now give an algebraic characterization of the ideals $H_{(K,\mu)}$.

Definition/Theorem 1.2.6. Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an ideal.

1. $f \in \mathcal{O}_{\mathbb{C}^2,0}$ is *integrally closed* over I, iff there is an equation
$$f^n + g_1 \cdot f^{n-1} + \cdots + g_n \cdot f^0 = 0$$
with $g_k \in I^k$.

2. The set of functions which are integrally closed over I form an ideal $\bar{I}$. This ideal is called the *integral closure* of I in $\mathcal{O}_{\mathbb{C}^2,0}$.

3. The ideal I is *complete* iff it is integrally closed, i.e. $I = \bar{I}$.

Remark 1.2.7. The theory of complete ideals (in a more general setting) goes back to Zariski. He defined complete ideals using valuations. Using his definition, our definition of complete ideals is a theorem. See [ZS60, App. 4], [Zar38] and [CA00, 8.3].

Remark 1.2.8. If X is a complex space and $\mathfrak{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals, then the ideal sheaf $\bar{\mathfrak{I}}$ whose stalk at $p \in X$ is the integral closure $\overline{\mathfrak{I}_p}$ of $\mathfrak{I}_p$ in $\mathcal{O}_{X,p}$ is again coherent. See [Tei77, p. 591] and [Hir74, L. 7] for different characterizations and geometric applications of integral closure in this more general context.

Theorem 1.2.9. *Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}_{\mathbb{C}^2,0}$.*

1. *An ideal $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ is $\mathfrak{m}$-primary iff $BP(I)$ is finite.*

2. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an $\mathfrak{m}$-primary ideal. Then the integral closure of I is $\bar{I} = H_{BP(I)}$.*

3. *There are bijections, inverse to each other, from the set of positively weighted, consistent clusters to the set of $\mathfrak{m}$-primary, complete ideals in $\mathcal{O}_{\mathbb{C}^2,0}$ and vice versa, given by*
$$(K, \mu) \mapsto H_{(K,\mu)} \qquad \textit{and} \qquad I \mapsto BP(I).$$

A proof of this theorem is in [CA00, chapter 8].

Corollary 1.2.10. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be a complete, $\mathfrak{m}$-primary ideal. Then I is the integral closure of an ideal which is generated by two elements, $I = \overline{(f, g)}$.*

Proof. Corollary 4.2.8 in [CA00] says that there are two functions f, g going strictly through $BP(I)$ and having no other common base points. $\square$

1.2.1 Example: Ideals Defined by Decorated Curves

Every weighted cluster defines a complete ideal, so in particular we can use the clusters $BP(C,l)$ of base points of decorated curves to define complete ideals.

Definition 1.2.11. $I(C,l)$ is the complete ideal $H_{BP(C,l)}$.

Remark 1.2.12. If f is a generic member of the complete ideal I, then $BP(I) = BP(V(f),l)$ for some l, so every complete ideal has a representation $I = I(C,l)$.

Example 1.2.13. If $f = \sum_{\alpha\in\mathbb{N}_0^2} c_\alpha x^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}_0^2} d_\alpha x^\alpha \in \mathbb{C}\{x\} = \mathbb{C}\{x_1,x_2\}$ are equations of two smooth curves in $(\mathbb{C}^2,0)$, then $V(f)$ and $V(g)$ pass through the same point in the k-th neighbourhood if and only if

$$\sum_{|\alpha|\le k} c_\alpha x^\alpha = \sum_{|\alpha|\le k} d_\alpha x^\alpha.$$

This implies that $I(V(f),k) = (f) + (x_1,x_2)^k$.

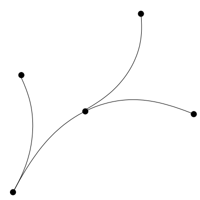
Remark 1.2.14. More generally, if C is irreducible, then $(x,y)^l$ is the smallest power of the maximal ideal which is contained in $I(C,l)$, i.e. $I(C,l) \supset (x,y)^l$ but $I(C,l) \not\supset (x,y)^{l-1}$.

Example 1.2.15. We consider the curve $C = V(x^2+y^3)$; the points on C can be seen in table 1.1 on page 7. We must choose $l \geq 3$ to get a decorated curve. $(C,3)$ has two base points and $I(C,3)$ is generated by curves passing through zero with multiplicity 2 and having the y-axis as a tangent. So $I(C,3) = (x,y)\cdot\big((x)+(x,y)^2\big)$.

For $l=4$ we get $I(C,4) = (x^2+y^3) + (x^2) + (x)\cdot(x,y)^2 + (x,y)^4$.

Remark 1.2.16. If the curve $C = \bigcup C_i$ is reducible, then the ideal $I(C,l)$ is the product $I(C,l) = \prod I(C_i,l_i)$, compare section 1.4.

Example 1.2.17. Consider the D_6-singularity $xy^2+x^5 = x(y+ix^2)(y-ix^2)$. An Enriques diagram of $BP(D_6,(2,3,3))$ looks like this:



The complete ideal $I(D_6,(2,3,3))$ is

$$\big((x)+(x,y)^2\big)\cdot\big((y+ix^2)+(x,y)^3\big)\cdot\big((y-ix^2)+(x,y)^3\big).$$

1.2.2 Example: The Conductor of a Curve

Notation 1.2.18. If M and N are two R-modules, we denote the transport ideal from N to M by

$$(M :_R N) := \{f \in R \mid f \cdot N \subset M\}.$$

Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated plane curve singularity and $n : \bar{C} \to C$ the normalization. The semilocal ring $\mathcal{O}_{\bar{C},n^{-1}(0)}$ is the integral closure of $\mathcal{O}_{C,0}$ in its full quotient ring and $(\mathcal{O}_{C,0} :_{\mathcal{O}_{\mathbb{C}^2,0}} \mathcal{O}_{\bar{C},n^{-1}(0)})$ is an $\mathfrak{m}$-primary ideal in $\mathcal{O}_{\mathbb{C}^2,0}$. One out of many references for this is [CA00].

Definition 1.2.19. The *conductor* of the plane isolated curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$ is the ideal $(\mathcal{O}_{C,0} :_{\mathcal{O}_{\mathbb{C}^2,0}} \mathcal{O}_{\bar{C},n^{-1}(0)}) \subset \mathcal{O}_{\mathbb{C}^2,0}$.

We define the δ-invariant of C as the degree of the conductor. So if J is the conductor,

$$\delta(C) := \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/J.$$

The following lemma is a well known formula of Max Noether for the δ-invariant of a plane isolated curve singularity, see e.g.[CA00].

Lemma 1.2.20. *Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated curve singularity. Then the δ-constant of C is*

$$\delta(C) = \sum_{p \in \mathcal{N}_0} \frac{e_p(C)(e_p(C) - 1)}{2}.$$

Theorem 1.2.21. *Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated plane curve singularity and $(K, e(C))$ an Enriques diagram of C weighted by the multiplicities of C. Then the conductor of C is the complete ideal $H_{(K,e(C)-1)}$.*

Proof. Using the notations of [CA00, 3.11], we let R^k be the ring of C in the k-th neighbourhood. For $p \in \mathcal{N}_0$, let R_p be the local ring of C in p. If $R = \mathcal{O}_{C,0}$ has multiplicity e_0, then $(R : R^1) = \mathfrak{m}_0^{e_0-1}$, so we see that the conductor J is included in

$$J = (R : \bar{R}) \subset \bigcap_{p \text{ on } C} (R_p : R_p^1) = H_{(K,e_p-1)}.$$

Note that the cluster $(K, e_p - 1)$ is consistent. By theorem 1.5.5 and Noethers formula for the δ-invariant

$$\begin{aligned} \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/H_{(K,e_p-1)} &= \sum \frac{(e_p - 1)e_p}{2} \\ &= \delta \\ &= \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/J. \end{aligned}$$

So $H_{(K,e_p-1)}$ must be equal to the conductor J. $\square$

1.3 Equisingularity

Definition 1.3.1. 1. Two clusters K, K' are *equisingular*, iff an Enriques diagram of K is also an Enriques diagram of K'. The equivalence class of K with respect to equisingularity is the *equisingularity class* of K and is denoted by $\mathcal{E}(K)$.

2. Two curves $C, C' \subset (\mathbb{C}^2, 0)$ are *equisingular* iff they have equisingular Enriques clusters. The equivalence class of C with respect to equisingularity is the *equisingularity class* of C and is denoted by $\mathcal{E}(C)$.

3. Two decorated curves (C, l) and (C', l') are *equisingular* iff an Enriques diagram of (C, l) is also an Enriques diagram of (C', l'). The equivalence class of (C, l) with respect to equisingularity is the *equisingularity class* of (C, l) and is denoted by $\mathcal{E}(C, l)$.

There are many different characterizations of equisingularity of plane curve singularities. Of some importance for this thesis are the following:

Theorem 1.3.2. *Let $(C_i, 0) \subset (\mathbb{C}^2, 0)$, $i = 1, 2$, be two isolated plane curve singularities. The following statements are equivalent.*

1. *C_1 is equisingular to C_2.*

2. *C_1 and C_2 are topologically equivalent in the following sense: There exist representatives (U_i, C_i) such that (U_1, C_1) is homeomorphic to (U_2, C_2).*

3. *There is a μ-constant path from C_1 to C_2 in the following sense: Let f_i be any equations of C_i. Then there exists an $F \in \mathbb{C}\{x, y, t\}$ such that the functions $F_t \in \mathbb{C}\{x, y\}$ with $F_t(x, y) = F(x, y, t)$ satisfy: $F_1 = f_1$, $F_2 = f_2$ and the function*

$$t \mapsto \mu_t := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(\partial_x F_t, \partial_y F_t)$$

is constant on $[1, 2]$.

The equivalence of the first two statements was proven by work of Brauner, Burau and Zariski. The equivalence with the third statement is essentially the so-called μ-constant theorem proven by Lê and Ramanujam in [LR73]. For more detailed references see [CA00, 3.8 and 7.3]. For twelve different characterizations of μ-constant paths see [Tei77, p. 623].

Remark 1.3.3. Statement 3 of the theorem says that the set of all equations of curves in a given equisingularity class is a μ-constant stratum in the sense of [AGLV98].

Normal forms for μ-constant strata with respect to right equivalence can be computed using techniques described in [AGZV85]. In particular, we can use the well-known normal forms with respect to right equivalence for the μ-constant strata of all singularities in Arnold's list, which are also given in [AGZV85].

Example 1.3.4. The set of all smooth curves is an equisingularity class.

Example 1.3.5. The sets of germs of curves having a fixed simple singularity A_k for $k \geq 1$, D_k for $k \geq 4$, E_6, E_7 or E_8 are one equisingularity class each.

Example 1.3.6. An ordinary singularity of multiplicity k is a singularity consisting of k smooth germs intersecting pairwise transversally. The set of all ordinary singularities of multiplicity k is an equisingularity class for all $k \in \mathbb{N}$.

Example 1.3.7. If $(C_i, 0) \subset (\mathbb{C}^2, 0)$, $i = 1, 2$, are two isolated plane curve singularities with smooth branches, then C_1 is equisingular to C_2 if and only if there is a bijection from the set of branches of C_1 to the set of branches of C_2 such that the intersection number of any two branches of C_1 is the same as the intersection number of the two corresponding branches of C_2.

1.4 Factorization into Simple Ideals

Zariski wrote in [ZS60, App. 5] that the culminating point of his theory of complete ideals was the theorem on unique factorization into simple complete ideals. We are interested in this factorization, because it gives us a description of the generic elements of a complete ideal. It will allow us to describe the blowup of $(\mathbb{C}^2, 0)$ in a complete ideal in the next chapter.

Definition 1.4.1. A non-trivial ideal I is *simple* if and only if it is not the product of two non-trivial ideals J_1, J_2.

We define the sum of two clusters (K, μ_K) and (L, μ_L) as $(K \cup L, \mu_K + \mu_L)$, where we put $\mu_K(p) = 0$ if $p \in L \setminus K$ and vice versa. The following proposition is immediately clear from the definition of base points of an ideal via virtual transforms:

Proposition 1.4.2. *Let $I, J \subset \mathcal{O}_{\mathbb{C}^2,0}$ be two $\mathfrak{m}$-primary ideals. Then*

$$BP(I \cdot J) = BP(I) + BP(J).$$

Corollary 1.4.3. *If an $\mathfrak{m}$-primary complete ideal $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ is the product $I = J_1 \cdot J_2$ of two not necessarily complete ideals J_1 and J_2, then I is also the product $I = \overline{J_1} \cdot \overline{J_2}$ of complete ideals.*

Proof. $\overline{J_1} \cdot \overline{J_2}$ contains $J_1 \cdot J_2$ and has the same base points. But since $I = J_1 \cdot J_2$ is complete, it is the biggest ideal with these base points. □

This result can also be found in [ZS60, App. 5, p. 385]. On the same page, we find the following theorem on complete ideals in a regular ring of dimension two:

Theorem 1.4.4. *The product of two complete ideals I, J is complete.*

Now we want to describe the set of simple complete ideals. Because of proposition 1.4.2, a complete ideal is simple if and only if its weighted cluster of base points can not be written as the sum of two non-trivial consistent clusters.

The key observation is the following: If (K, μ) is a non-empty, positively weighted, consistent cluster, then the total excess $\sum \rho_p(K, \mu) > 0$, because the excess at each point is non-negative and positive at points which are maximal with respect to the proximity relation. Proposition 1.4.2 implies $\sum \rho_p(BP(I \cdot J)) = \sum \rho_p(BP(I)) + \sum \rho_p(BP(J))$. We deduce that a complete ideal $H_{K,\mu}$ with $\sum \rho_p(K, \mu) = 1$ is simple. The only consistent clusters with positive weights and excess equal to one are the simple clusters $\mathcal{K}(p)$.

On the other hand, it is quite obvious that any consistent cluster (K, μ) with positive weights is the sum $(K, \mu) = \sum_{p \in K} \mathcal{K}(p)^{\rho_p}$, and that every decomposition of (K, μ) as a sum of positively weighted, consistent clusters can be refined to this decomposition.

This proves the following theorem, which is a special case of the theorem on unique factorization for complete ideals in regular rings of dimension 2 of Zariski, see [ZS60, App. 5, theorem 3]. The proof we have sketched here is in [CA00, 8.4].

Theorem 1.4.5. *A complete $\mathfrak{m}$-primary ideal in $\mathbb{C}\{x, y\}$ is simple if and only if its cluster of base points is simple.*

If $I \subset \mathbb{C}\{x, y\}$ is a complete, $\mathfrak{m}$-primary ideal and $\rho_p := \rho_p(BP(I))$, then

$$I = \prod_{\substack{p \in BP(I) \\ \rho_p > 0}} H_{\mathcal{K}(p)}^{\rho_p}$$

is the unique decomposition of I into a product of simple complete ideals.

The curves in $(\mathbb{C}^2, 0)$ going strictly through $\mathcal{K}(p)$ are those having $\mathcal{K}(p)$ as an Enriques diagram. They are irreducible. So taking strict transforms when blowing up the points of $\mathcal{K}(p)$ and blowing down again gives bijections, inverse to each other, from the set of curves in $(\mathbb{C}^2, 0)$ going strictly through $\mathcal{K}(p)$ and smooth curves on the blowup intersecting the exceptional divisor E_p in a free point.

The factorization into simple ideals generalizes this as follows:

Corollary 1.4.6. *Let $I \subset \mathbb{C}\{x, y\}$ be a complete, $\mathfrak{m}$-primary ideal. I is generated by the set of $f \in \mathbb{C}\{x, y\}$ such that the curve $V(f)$ has the following properties:*

1. *The number of branches is $\sum_p \rho_p(BP(I))$.*
2. *If we blow up all the base points of I, the strict transform consists of $\sum_p \rho_p(BP(I))$ smooth curves. Exactly $\rho_p(BP(I))$ of these intersect E_p and the intersections are transversal.*

Conversely, if we blow up $(\mathbb{C}^2, 0)$ in all the base points of I successively and pick any curve on the blowup that intersects each irreducible component E_p of the exceptional divisor with intersection multiplicity at least $\rho_p(BI(I))$, then it is the strict transform of a curve in I.

1.5 Some Numerical Invariants

The multiplicities of the base points of an $\mathfrak{m}$-primary ideal $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ determine several well known invariants of the ideal and of generic members of the ideal. In this section we define and compute some of these invariants. The results will be especially useful when studying deformations of the zero-dimensional space $\Sigma(I)$ defined by I. We also present results of Rees relating reductions, integral closure and the multiplicity of ideals.

Definition 1.5.1. Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an $\mathfrak{m}$-primary ideal. We define the *δ-invariant* of I to be

$$\delta(I) := \sum_{p \in BP(I)} \frac{e_p(I)(e_p(I) - 1)}{2}.$$

Corollary 1.5.2. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an $\mathfrak{m}$-primary ideal. Then*

$$\delta(I) = \min\{\delta(V(f)) \mid f \in I,\ f \text{ reduced}\}.$$

The set of those $f \in I$ whose δ-constant is $\delta(I)$ is Zariski-open in I.

Proof. A generic curve of I goes strictly through $BP(I)$ and is reduced, so its δ-invariant is equal to $\delta(I)$ by Noethers formula for the δ-invariant of a plane curve, lemma 1.2.20. This is the minimal value by the semicontinuity of the δ-invariant. □

A proof of the following lemma is also in [CA00].

Lemma 1.5.3. *For $f, g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $gcd(f,g) = 1$ the intersection multiplicity is*

$$\langle f, g\rangle = \sum_{p\in\mathcal{N}_0} e_p(f)\cdot e_p(g).$$

As with the δ-invariant we can deduce:

Corollary 1.5.4. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an $\mathfrak{m}$-primary ideal. For a generic pair $f, g \in I$ we have*

$$\begin{aligned}\langle f, g\rangle &= \min\{\langle f', g'\rangle \mid f', g' \in I\}\\ &= \sum_{p\in BP(I)} e_p(I)^2.\end{aligned}$$

For the next two theorems we restrict ourselves to complete ideals. The next theorem is proposition 4.7.1 in [CA00].

Theorem 1.5.5. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be a complete, $\mathfrak{m}$-primary ideal. Then the degree of I is*

$$\begin{aligned}\deg(I) :&= \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/I\\ &= \sum_{p\in BP(I)} \frac{e_p(I)^2 + e_p(I)}{2}.\end{aligned}$$

As a corollary we obtain

Theorem 1.5.6. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be a complete, $\mathfrak{m}$-primary ideal.*
The Hilbert-Samuel function of I is equal to the Hilbert-Samuel polynomial of I and is equal to

$$n \mapsto \sum_{p\in BP(I)} e_p(I)^2\cdot\frac{n^2}{2} + \frac{\sum_{p\in BP(I)} e_p(I)}{2}\cdot n.$$

In particular, the multiplicity of I is $e(I) = \sum_{p\in BP(I)} e_p(I)^2$.

Proof. If I is complete, then so is I^n and the multiplicity of I^n at an infinitely near point p is just $n \cdot e_p(I)$, see section 1.4. So the Hilbert-Samuel function of I is

$$\begin{aligned} n &\mapsto \dim_{\mathbb{C}} \mathbb{C}^2/I^n \\ &= \sum_{p \in BP(I)} \frac{(n \cdot e_p)^2 + n \cdot e_p}{2} \\ &= \left(\sum_{p \in BP(I)} e_p(I)^2 \right) \cdot \frac{n^2}{2} + \frac{\sum_{p \in BP(I)} e_p(I)}{2} \cdot n. \end{aligned}$$

□

Definition 1.5.7. Let $I \subset R$ be an ideal. An ideal $J \subset I$ is called a *reduction* of I iff $J \cdot I^n = I^{n+1}$ for $n \gg 0$.
It is called a *minimal reduction* if it is minimal among all reductions with respect to inclusion.

Recall that a local ring $(R, \mathfrak{m})$ is called *quasi-unmixed* or *formally equidimensional* if and only if all minimal primes of the $\mathfrak{m}$-adic completion of R have the same dimension. In particular, the local ring $\mathcal{O}_{X,p}$ of a complex space X at a point p is quasi-unmixed if and only if X is equidimensional at p.

Theorem 1.5.8 (Northcott, Rees). *Let $(R, \mathfrak{m})$ be a local ring of finite dimension and I and J two $\mathfrak{m}$-primary ideals.*

1. *J is a reduction of I iff $J \subset I$ and the integral closures are the same: $\bar{J} = \bar{I}$. If J is a reduction of I, then $e(J) = e(I)$.*

2. *If R is quasi-unmixed, then J is a reduction of I iff $J \subset I$ and $e(J) = e(I)$.*

Proof. See [NR54] and [Ree61]. □

Corollary 1.5.9. *Let $I \subset \mathbb{C}\{x, y\}$ be any $\mathfrak{m}$-primary ideal. Then the multiplicity of I is*

$$e(I) = \sum_{p \in BP(I)} e_p(I)^2.$$

The second coefficient of the Hilbert-Samuel polynomial is $\geq \frac{1}{2} \sum_{p \in BP(I)} e_p(I)$.

Corollary 1.5.10. *Let $I \subset \mathbb{C}\{x, y\}$ be a complete, $\mathfrak{m}$-primary ideal, $f, g \in I$. Then $\overline{(f, g)} = I$ if and only if their intersection multiplicity is*

$$\langle f, g\rangle = \sum_{p \in BP(I)} e_p(I)^2.$$

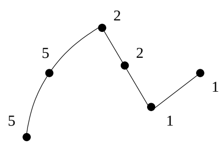
Proof. f and g have finite intersection multiplicity if and only if they generate an $\mathfrak{m}$-primary ideal and if and only if they form a regular sequence. In this case the multiplicity of the ideal (f, g) is

$$e\big((f, g)\big) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, g) = \langle f, g\rangle.$$

□

Remark 1.5.11. Of course we always have $\overline{(f, g)} = I$ if f and g both go strictly through $BP(I)$ without sharing any other infinitely near points. In this case $\langle f, g\rangle = \sum_{p \in BP(I)} e_p(I)^2$ is clear. Starting from this observation it is surely possible to give an elementary proof of the corollary without using the general result of Rees.

Example 1.5.12. It is also possible that $\overline{(f, g)} = I$ if neither f nor g go strictly through $BP(I)$. For example, the cluster of base points of (x^{12}, y^5) is



So $\overline{(x^{12}, y^5)}$ is the complete ideal with this cluster of base points. The multiplicity of this ideal is

$$\langle x^{12}, y^5\rangle = 12 \cdot 5 = 60 = 25 + 25 + 4 + 4 + 1 + 1 = \sum_p e_p^2.$$

Chapter 2

Sandwiched Singularities

In this chapter we introduce the main objects we wish to study: sandwiched singularities. Sandwiched singularities are singularities on the blowup of $(\mathbb{C}^2, 0)$ in a complete ideal. We use the results on complete ideals which we have summarized in chapter one to deduce some important properties of sandwiched singularities. In particular, we characterize sandwiched singularities by their dual resolution graphs in section 2.2. Then we introduce a convenient representation of sandwiched singularities via decorated curves and explore some of the relations between sandwiched singularities and the curves which we can use to represent them. We will pay special attention to the case of cyclic quotients and singularities with reduced fundamental cycle.

Finally, I will use Laufer's list of taut and pseudotaut surface singularities to give a new way of classifying taut and pseudotaut curves singularities. These have already been classified by Gawlick in [Gaw92], but his method is completely different. Equations and Enriques diagrams of taut and pseudotaut curve singularities as well as dual resolution graphs of the corresponding sandwiched singularities are in the appendices.

2.1 Definition and Construction

Definition 2.1.1. A *sandwiched singularity* is a two-dimensional singularity which is isomorphic to a singularity on the blowup of $(\mathbb{C}^2, 0)$ in a complete, $\mathfrak{m}$-primary ideal.

Recall that an $\mathfrak{m}$-primary ideal in $\mathcal{O}_{\mathbb{C}^2,0}$ is complete if and only if it is integrally closed. In two-dimensional, regular local rings, the powers of integrally closed ideals are also integrally closed. An ideal with the property that all powers are integrally closed is sometimes called normal, because the property is equivalent to the fact that the blowup in the ideal is normal. More precisely we can state:

Theorem 2.1.2. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an $\mathfrak{m}$-primary ideal. Then the normalization of the blowup of $\mathbb{C}^2$ in I is isomorphic to the blowup of $\mathbb{C}^2$ in the integral closure of I.*

A proof is in [Lê00]. I suppose that this was first noted by Zariski.

Remark 2.1.3. It is essential that $\mathcal{O}_{\mathbb{C}^2,0}$ is a two-dimensional ring, otherwise powers of an integrally closed ideal need not be integrally closed. Examples for this phenomenon occur in the last chapter of this thesis. The theorem does hold more generally though for ideals in the local ring of a two-dimensional rational singularity, see [Lip69] and the references given in [Vas94, 5.4, p.125].

Corollary 2.1.4. *Sandwiched singularities are normal. In particular they are isolated and Cohen-Macaulay.*

The last statement is true because all two-dimensional, normal singularities are isolated and Cohen-Macaulay.

We recall the definition of a rational surface singularity from [Art66]:

Definition 2.1.5. A surface singularity (X, x) is *rational* if and only if it is normal and the following holds: If $\pi : X' \to X$ is a resolution, then the first higher direct image sheaf $R^1 f_* \mathcal{O}_{X'}$ of the structure sheaf of X' vanishes at x.

We have just seen that sandwiched singularities are normal. Because they are on the blowup of a smooth point, it is then clear that sandwiched singularities are rational; more generally, if $X' \to X$ is a proper modification of normal surfaces and X has only rational singularities, then X' does only have rational singularities too, see e.g. [BR95, 2.4].

Corollary 2.1.6. *Sandwiched singularities are rational.*

Remark 2.1.7. Being rational is also equivalent to some numerical conditions on a dual resolution graph, see [Art66]. These conditions can be checked by Laufer's algorithm given in [Lau72, IV]. It follows easily from this algorithm that a normal surface singularity whose dual resolution graph is a subgraph of the dual resolution graph of a rational singularity is also rational. With this result, we can also derive the rationality of sandwiched singularities from the description of their dual resolution graphs as subgraphs of non-singular graphs we will give below.

Notation 2.1.8. Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be a complete, $\mathfrak{m}$-primary ideal. We set $(S_0, F_0) := (\mathbb{C}^2, 0)$. If (S_i, F_i) is defined, let $\pi_{i+1} : (S_{i+1}, F_{i+1}) \to (S_i, F_i)$ be the blowup of (S_i, F_i) in the base points of I in the i-th neighbourhood; $F_{i+1} = \pi_{i+1}^{-1}(F_i)$.

Denote the irreducible component of F_i which is the (strict transform of the) exceptional divisor of the point blowup in p by E_p.

Let n be minimal such that I has no base points in the n-th neighbourhood. We put $E := \bigcup\{E_p |\ \rho_p(BP(I)) = 0\} \subset F_n \subset S_n$.

We can blow down E to a point, because we can blow down such a divisor to a point iff the intersection matrix of the irreducible components is negative definite, see [Mum61]. Since the intersection matrix of the components of E is a submatrix of the intersection matrix of all components of F_n, it is negative definite. We denote this blowdown by $\hat{\pi} : (S_n, F_n) \to (S, F)$. We get the following diagram:

$$\begin{tikzcd}
(S_n,F_n) \arrow[d] \arrow[dr, "\hat{\pi}"] & \\
\cdots \arrow[d] & (S,F) \arrow[ddl, "\pi"] \\
(S_1,F_1) \arrow[d] & \\
(\mathbb{C}^2,0) &
\end{tikzcd}$$

Note that $E_p^2 = (-1)$ if and only if p is a base point of I such that no other base point is proximate to it. In this case $\rho_p(I) > 0$. So the set E which we contract contains no (-1)-curves.

Theorem 2.1.9. *The modification* $\pi : (S, F) \to (\mathbb{C}^2, 0)$ *is a blowup of* $(\mathbb{C}^2, 0)$ *in* I.

Remark 2.1.10. It is because of this construction that sandwiched singularities are called sandwiched singularities: They are being sandwiched between the two smooth surfaces S_n and $\mathbb{C}^2$.

Proof. The blowup in I is the minimal modification such that $\pi^*(I)$ has no base points. So to construct the blowup, we first blow up all the base points to get rid of them. We obtain (S_n, F_n). But generic curves in the linear system of I do only pass through those E_p with $\rho_p > 0$, so we can blow down E_p with $\rho_p = 0$. On the other hand every curve on S_n which intersects every E_p with multiplicity at least ρ_p is a strict transform of a curve in the linear system of I. So the modification is indeed minimal among those removing the base points of I. □

Theorem 2.1.11. *At each point of (S, F), the modification $\hat{\pi} : (S_n, F_n) \to (S, F)$ is a minimal resolution and also a minimal good resolution.*

Remark 2.1.12. Of course, the minimal and the minimal good resolution coincide for all rational surface singularities. This is clear from the characterization of rational singularities via their resolution graphs. But here we can see it directly from the construction.

Proof. S_n is smooth, so $\hat{\pi}$ is a resolution at each point of S. The exceptional divisor of $\hat{\pi}$ is the set E in F that we have contracted. We have noted above that E contains no (-1)-curves, so the resolution is minimal. We also see from the construction of (S_n, F) that no three components of the exceptional divisor meet in one point and that there are no cycles. □

Corollary 2.1.13. *Let the notations be as above.*

1. *The (isomorphism class of a) blowup of $\mathbb{C}^2$ in the complete ideal I only depends on the set of base points of I with $\rho_p(I) > 0$. In particular, if the I_k are simple complete ideals, then a blowup in $\prod_{k=1}^n I_k^{a_k}$ is also a blowup in $\prod_{k=1}^n I_k^{b_k}$ for all $a_k, b_k \in \mathbb{N}$.*

2. *The blowup of $\mathbb{C}^2$ in the complete ideal I is smooth iff $\rho_p(I) > 0$ for all base points p of I.*

3. *The dual resolution graphs of the minimal resolutions of the singularities on the blowup of $\mathbb{C}^2$ in the complete ideal I can be read off from a weighted Enriques diagram of the cluster of base points of I.*

4. *If (C, l) is a decorated curve with $l(C_i) > M(C_i)$ for all branches C_i of C, then either $I(C, l) = (x, y)$ and the blowup in $I(C, l)$ is smooth, or there is exactly one singular point on the blowup in $I(C, l)$.*

Proof. For the proof of the first statement note that the set of base points with $\rho_p(I) > 0$ determines the set of all base points, because those base points which are maximal with respect to the proximity relation have excess greater than zero.

The proof of the second and third statement is a direct consequence of the construction of the blowup in I as the blowup in all base points of I followed by the contraction of all exceptional divisors E_p with $\rho_p(I) > 0$.

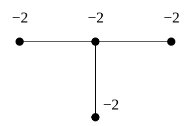
We come to the proof of the last statement. We see from the construction of the blowup that there is at most one singular point on the blowup in I if and only if the union of the $E_p \subset F_n$ which we contract, i.e. the union of the E_p with $\rho_p(I) = 0$, is connected.

The condition $l(C_i) > M(C_i)$ ensures that the set of base points with $\rho_p(I) > 0$ is the set of base points with the property that no other base points are proximate to them and furthermore that these base points are all free. This implies that the components E_p with $\rho_p(I) > 0$ only meet one other component of the exceptional divisor F_n each, so their complement is connected and we can have at most one singularity on the blowup. It is easy to see that under the above conditions the complement is empty if and only if $BP(I)$ consists of only one point with multiplicity one. $\square$

Remark 2.1.14. A special case of the fact that the blowup only depends on the set of simple ideals in the unique factorization is the well known theorem that blowups in a power of I are the same as blowups in I.

Example 2.1.15. Let C be a smooth curve. The blowup of $(\mathbb{C}^2, 0)$ in the complete ideal $I(C, 1) = (x, y)$ is smooth. For $k \geq 1$, the only singularity on the blowup of $(\mathbb{C}^2, 0)$ in the complete ideal $I(C, k+1)$ is an A_k-singularity.

Example 2.1.16. The D_4-singularity is not sandwiched. This can be seen from the dual resolution graph of D_4 which looks like this:



We will show that the dual graph of the exceptional divisor of a series of point blowups, starting with a smooth point, does not contain the D_4-graph as a subgraph.

The intersection number of the exceptional divisor of a point blowup is -1. Blowing up a point on an irreducible curve lowers the self intersection number by one. Now if the curve $E_{central}$ corresponding to the central vertex of the D_4-graph had been the blowup of a free point, we would have had to blow up at least two points on this curve and the self-intersection number would be ≤ -3. So assume that it was the blowup of the intersection point of two curves. One of these curves must have had intersection number at most -2, so after the blowup it would have had intersection number at most -3. If we wanted to get rid of this point by blowing up the intersection with $E_{central}$, then $E_{central}$ would have self intersection less than -2.

This argument shows that the dual resolution graph of a sandwiched singularity cannot contain the D_4-graph as a subgraph. So in particular, all D_k, $k \geq 4$, and E_6, E_7, E_8 are not sandwiched.

2.2 Dual Resolution Graphs

We have just seen how the dual resolution graph showed that the D_4-singularity is not sandwiched. On the other hand, theorem 2.2.3 states that every singularity whose dual resolution graph does not allow a similar argument to exclude the possibility that it is sandwiched, is in fact a sandwiched singularity! We introduce some notation from [Spi90]. Note that Spivakovsky uses positive weights whereas we use negative weights.

Definition 2.2.1. Let $\Gamma = (\Gamma, \mu_\Gamma)$ be a weighted graph.

1. Let x be a vertex of Γ. An *elementary modification of the first kind*, $\epsilon(y; x)$, is a weighted graph (Γ', μ'), which we obtain from Γ by adding the new vertex y, joining y with x by an edge and choosing the following weights:

$$\begin{aligned} \mu_{\Gamma'}(y) &:= -1, \\ \mu_{\Gamma'}(x) &:= \mu_\Gamma(x) - 1, \\ \mu_{\Gamma'}(z) &:= \mu_\Gamma(z) \qquad\qquad \forall z \notin \{x, y\}. \end{aligned}$$

2. Let x_1, x_2 be two vertices of Γ, joined by an edge. An *elementary modification of the second kind*, $\epsilon(y; x_1, x_2)$, is a weighted graph (Γ', μ'), which we obtain from Γ by adding the new vertex y, deleting the edge from x_1 to x_2, adding edges from x_1 and x_2 to y and choosing the following weights:

$$\begin{aligned} \mu_{\Gamma'}(y) &:= -1, \\ \mu_{\Gamma'}(x_i) &:= \mu_\Gamma(x_i) - 1, \\ \mu_{\Gamma'}(z) &:= \mu_\Gamma(z) \qquad\qquad \forall z \notin \{x_1, x_2, y\}. \end{aligned}$$

3. A weighted graph is called *non-singular*, if it can be obtained by performing a finite number of elementary modifications on the weighted graph $\overset{-1}{\bullet}$.

4. A weighted graph is called *sandwiched*, if it can be embedded into a non-singular graph as a connected component of the complement of the set of vertices with weight -1.

Lemma 2.2.2. *Let (Γ, μ) be a sandwiched graph.*

1. *Let (Γ, μ') be a graph with integer weights such that $\mu'(x) \leq \mu(x)$ for all vertices of Γ. Then (Γ, μ') is also a sandwiched graph.*

2. *Every connected subgraph of a sandwiched graph is a sandwiched graph.*

3. *There is an embedding of* Γ *into a non-singular graph* Δ *such that* Γ *is the only connected component of the complement of the set of vertices with weight* -1 *in* Δ*.*

   We call such an embedding a minimal embedding.

We leave the easy proof to the reader.

Theorem 2.2.3. *A normal surface singularity is sandwiched if and only if the dual resolution graph of its minimal resolution is a sandwiched graph.*

Proof. That the dual resolution graph of a sandwiched singularity is sandwiched follows from theorem 2.1.9 and the fact that every connected subgraph of a sandwiched graph is sandwiched.

Now let $(X, 0)$ be a normal surface singularity such that the dual resolution graph Γ of the minimal resolution $\pi : (S, F) \to (X, 0)$ is sandwiched. We choose a minimal embedding of Γ into a non-singular graph Δ. For each (-1)-vertex of Δ which is connected to $p \in \Gamma$ we choose a free point on the component F_p of F. In this point we transversally glue a (-1)-curve to F. It follows from the definition of non-singular graphs that we can blow down the union of F with the newly added (-1)-curves to $(\mathbb{C}^2, 0)$. We see that (S, F) is the blowup of $(\mathbb{C}^2, 0)$ in a complete ideal whose base points correspond to the vertices of Δ. □

Remark 2.2.4. Note that the choice of the free points corresponding to the (-1)-vertices was arbitrary!

2.3 Representations via Decorated Curves

The proof of theorem 2.2.3 in connection with lemma 2.2.2 also shows the following theorem.

Theorem 2.3.1. *Let* $(X, 0)$ *be a sandwiched singularity. Then there exists a complete ideal* $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ *with the following properties:*

1. $(X, 0)$ *is the only singularity on the blowup of* $(\mathbb{C}^2, 0)$ *in* I*.*

2. *For all base points of* I *with excess* $\rho_p > 0$ *the following holds:* $\rho_p = 1$*,* p *is free and* p *is maximal in* $BP(I)$ *with respect to the proximity relations.*

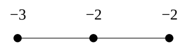
Remark 2.3.2. An ideal $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ has the two properties mentioned in the theorem if and only if $I = I(C, l)$ for a decorated curve with $l > M_C$, i.e. $l(C_i) > M_C(C_i)$ for all branches C_i of C.

Definition 2.3.3. If there is at most one singular point on the blowup of $(\mathbb{C}^2, 0)$ in $I(C, l)$, then we denote this sandwiched singularity by $X(C, l)$.

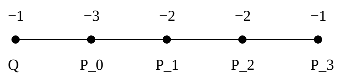
Example 2.3.4. If C is a smooth curve, then $X(C, k+1)$ is an A_k-singularity for $k \geq 1$.

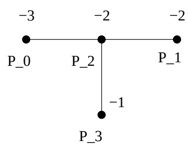
Example 2.3.5. If C is an A_2-singularity, then we must have $l \geq 2$ to get a decorated curve (C, l). The singularity $X(C, 2)$ is smooth. $X(C, 3)$ is an A_1-singularity. The blowup in $I(C, 4)$ has two A_1-singularities. The dual resolution graph of $X(C, 5)$ has three vertices with weights $-3, -2, -2$, so $X(C, 5)$ is the cyclic quotient singularity $A_{7,3}$ (see section 2.7 on cyclic quotients). Finally, for $l \geq 6$, $X(C, l)$ is a rational singularity with a star-formed dual resolution graph.

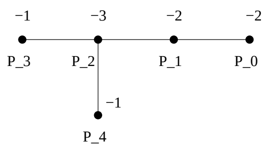
Example 2.3.6. We consider the following sandwiched graph:



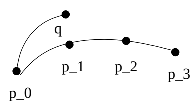
It is the dual resolution graph of the cyclic quotient singularity $A_{7,3}$. There are three different minimal embeddings into non-singular graphs:

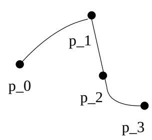
(a) 

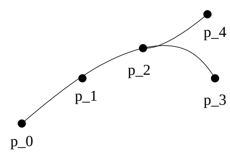
(b) 

(c) 

Enriques diagrams of the clusters of base points of corresponding complete ideals look like this:

(a) 

(b) 

(c) 

This gives us three essentially different representations $X(C, l)$ with $l > M_C$:

$$(a)\, X\big(xy, (4,2)\big) = X\big(A_1, (4,2)\big) \qquad (b)\, X(x^2 + y^3, 5) = X(A_2, 5) \qquad (c)\, X\big(x(x+y^3), (4,4)\big) = X\big(A_5, (4,4)\big)$$

The proof of theorem 2.2.3 also shows:

Theorem 2.3.7. *Let (C, l) be a decorated curve. Then $(C', l') \mapsto X(C', l')$ induces a surjective map*

$$\mathcal{E}(C, l)/\textit{analytic isomorphism} \to \left\{ \begin{array}{c} \textit{Isomorphism classes of normal} \\ \textit{surface singularities with the same} \\ \textit{dual resolution graph as } X(C, l) \end{array} \right\}$$

Remark 2.3.8. The use of the notation $X(C, l)$ implicitly assumes that there is only one singularity on the blowup in $I(C, l)$. But we see from the proof that the theorem is also true for arbitrary decorated curves if we let $X(C, l)$ denote the (finite) tuple of singularities on the blowup.

2.4 Non-Uniqueness of the Representations $X(C, l)$

Let's look at some examples to see how non-unique the representations $X = X(C, l)$ really are. For example, Corollary 2.1.13, (2) implies

Proposition 2.4.1. *$X(C, l)$ is smooth iff the excess of $BP(C, l)$ is greater than zero at all base points.*

In particular, $X(C, l)$ is smooth if C is an ordinary singularity and $l(C_i) = 1$ for each branch C_i of C.

So for smooth points, the non-uniqueness comes from the fact that the blow up in a complete ideal only depends on the set of base points and on the subset where the excess is equal to zero. But that is not the only possible cause for non-uniqueness. In general, as we have seen in example 2.3.6, there are different ways of embedding the dual resolution graph of a sandwiched graph into a non-singular graph, which give rise to different representations $X(C, l)$ where the ideals $I(C, l)$ have very different clusters of base points.

To eliminate the possible reasons for non-uniqueness we have noticed so far, we do now choose a fixed embedding of the dual resolution graph into a non-singular graph by adding only (-1)-curves, compute the Enriques diagram of the cluster of base points of a corresponding complete ideal and choose the minimal positive weights such that the cluster is consistent. The following theorem characterizes the set of representations $X(C, l)$ related to these choices.

Theorem 2.4.2. *Let (C, l) be a decorated curve with the property $l > M_C$. Let (C', l') be a decorated curve equisingular to (C, l).*

Then the singularities $X(C,l)$ and $X(C',l')$ are isomorphic iff there is an automorphism of $(\mathbb{C}^2,0)$ inducing a bijection between the two sets

$$\begin{aligned} &\{p \in BP(C,l) \mid \exists q \in BP(C,l) : p \prec q\} \\ \textit{and} \quad &\{p \in BP(C',l') \mid \exists q \in BP(C',l') : p \prec q\}. \end{aligned}$$

Proof. We start with $X(C,l)$ and try to reconstruct $I(C,l)$. Remember that we can do it like this: For each branch of C we glue a (-1)-curve to the divisor of the minimal resolution of $X(C,l)$. Successive blowdowns of (-1)-curves delete the added curves as well as all components of the minimal resolution, and we end up with a smooth point. The points to which we have blown down the irreducible curves are infinitely near points of this resulting smooth points; they are the base points of the complete ideal I in which we have to blow up to regain $X(C,l)$.

Non-uniqueness of the complete ideals stems from two facts: Choice of coordinates of the smooth point and the fact that the choice of the positions of the points where we glued the (-1)-curves to the minimal resolution was arbitrary. Changing coordinates is the same as letting an automorphism act on $(\mathbb{C}^2,0)$. The (-1)-curves correspond to those base points of the complete ideal I which are maximal with respect to the proximity relation, i.e. not in $\{p \in BP(C,l) \mid \exists q \in BP(C,l) : p \prec q\}$. □

Corollary 2.4.3. *Let C be an isolated plane curve singularity. If l is sufficiently big, then the map*

$$\mathcal{E}(C,l)/\textit{analytic isomorphism} \rightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of normal} \\ \textit{surface singularities with the same} \\ \textit{dual resolution graph as } X(C,l) \end{array} \right\}$$

is a bijection.

Proof. Curves equisingular to C all have the same Milnor number μ, as we see from the formula $\mu(C) = 2\delta(C) - \#(\text{branches of } C) + 1$, which holds for all plane curve singularities, see e.g. [CA00]. This implies that all curves equisingular to C are $(\mu(C)+1)$-determined. So if l is sufficiently big, then $\{p \in BP(C,l) \mid \exists q \in BP(C,l) : p \prec q\}$ determines the analytic type of C' for all (C',l') equisingular to (C,l). □

2.5 Reduced Fundamental Cycle

Theorem 2.5.1. 1. *Every rational surface singularity with reduced fundamental cycle is sandwiched.*

2. *A sandwiched singularity has reduced fundamental cycle if and only if the dual resolution graph of its minimal resolution can be minimally embedded into a non-singular graph which has been obtained from* $\overset{-1}{\bullet}$ *by performing only elementary modifications of the first kind, i.e. iff it is on the blowup in a complete ideal whose base points are all free.*

3. *A rational, normal surface singularity X has reduced fundamental cycle iff it is sandwiched and has a representation $X = X(C,l)$ where C is a curve whose branches are all smooth.*

Proof. We know that the dual resolution graph of the minimal resolution of a rational singularity is a tree and all self intersection numbers are ≤ -2. It follows from Laufer's algorithm for the computation of the fundamental cycle, given in [Lau72, IV], that a rational surface singularity has reduced fundamental cycle iff no component E_i of the exceptional divisor of the minimal resolution intersects more than $-E_i^2$ other components.

The reverse operation of an elementary modification of the first kind will be called blowing down.

Assume the dual resolution graph $\Gamma = (\Gamma, w)$ of a rational singularity with reduced fundamental cycle given. We use induction on the number of vertices of the dual resolution graph Γ:

We join each vertex with the number of new (-1)-vertices needed such that the weight is equal to the number of edges, except for one vertex v_0, which we join with one less. Call the new graph Δ. We claim that Δ is a non-singular graph which has been obtained from the graph whose only vertex is v_0, weighted by -1, by elementary modifications of the first kind.

If v_0 was the only vertex of Γ, the claim is obvious. Otherwise, remember that Γ is a tree. Choose an 'end'-vertex of Γ different from v_0 and blow down all (-1)-vertices you have just joined it with. It will end up to be a (-1)-vertex itself. This has reduced the number of vertices of Γ by one. The claim is thus proved by induction.

On the other hand, the graph $\overset{-1}{\bullet}$ has the property that no vertex v has more neighbours than its negative weight. This property is preserved if we perform elementary modifications of the first kind or delete vertices. That also proves the second part of the theorem.

The third statement is an immediate consequence of the second, because the points on a curve C are all free if and only if all branches of C are smooth. $\square$

2.6 Multiplicity of $X(C, l)$

Theorem 2.6.1. *Let $X = X(C, l)$ be a sandwiched singularity and $l > M_C$. Then the multiplicity of X is*

$$e\big(X(C, l)\big) = 1 + e(C).$$

I want to sketch a proof which makes use of the deformation theory of sandwiched singularities which we will study in the remaining chapters.

We exploit the fact that every rational surface singularity of multiplicity n deforms into the cone over the rational normal curve of degree n, which we denote by X_n. This has been conjectured by Wahl in [Wah79a] and proven by Karras in [Kar83]. X_n is the unique rational singularity whose dual resolution graph is one vertex with weight $-n$. The multiplicity of X_n is n, so by the semicontinuity of the multiplicity, the multiplicity of a rational surface singularity X is the maximal n such that there exists a deformation of X into X_n.

As we will see later on, every deformation of a sandwiched singularity $X(C, l)$ is induced by a deformation of the decorated curve (C, l). The existence of the Scott deformation of C described in section 4.4.1 implies that $X(C, l)$ with $l > M_C$ has a deformation to $X_{e(C)+1}$, so $e(X) \geq 1 + e(C)$. On the other hand, if p is a base point of $I(C, l)$ and E_p the corresponding component of the exceptional divisor of the minimal resolution of $X(C, l)$, then $-E_p^2$ is one plus the number of base points of $I(C, l)$ which are proximate to p. So $e(C) + 1 = e_0(C) + 1$ gives an upper bound for the $-E_i^2$ in the minimal resolution of $X(C, l)$. Since the multiplicity of C cannot increase under a deformation, it follows that $e(C) + 1$ is also an upper bound for the multiplicity of $X(C, l)$.

I also want to mention a proof for the special case of a rational singularity with reduced fundamental cycle:

Assume that we have a representation $X(C, l)$ where all branches of C are smooth. Then the multiplicity of C is the number of branches. We use [Art66, Cor. 6], which says that the multiplicity of a rational singularity is minus the self intersection number of the fundamental cycle. So if the fundamental cycle is reduced, the multiplicity is $2 + \sum(-E_i^2 - 2)$. If C has a single branch, the singularity is an A_k-singularity and this number is two. For every additional branch, the number goes up by one.

Remark 2.6.2. Of course, we can compute the multiplicity of any sandwiched singularity $X(C, l)$, because we know the dual resolution graph, so we can compute the fundamental cycle F. For a rational singularity, the multiplicity is $-F^2$. If the divisor l is small, the multiplicity will be smaller than $e(C) + 1$.

2.7 Example: Cyclic Quotient Singularities

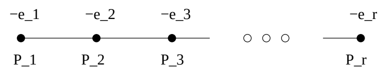
We introduce an interesting class of examples which has been very well studied.

A (two-dimensional) cyclic quotient singularity is a singularity that is isomorphic to the quotient $(\mathbb{C}^2, 0)/G$, where G is a finite cyclic group $G \subset Aut(\mathbb{C}^2, 0)$. If the order of G is n, then there is a change of coordinates such that G acts as

$$G = \langle \begin{pmatrix} \exp(2\pi i/n) & 0 \\ 0 & \exp(2\pi i q/n) \end{pmatrix} \rangle$$

with $0 < q < n$ and $gcd(q, n) = 1$. We call this singularity $A_{n,q}$. Cyclic quotient singularities are also called Hirzebruch-Jung singularities.

It is well known that the dual resolution graph is a chain

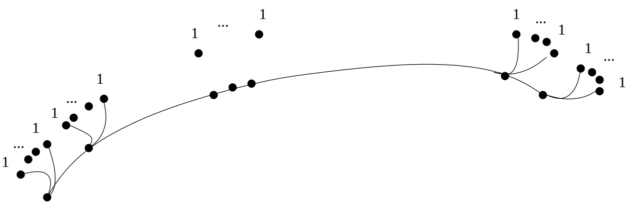


and the self-intersection numbers $-e_i$ can be computed from a continued fraction of n/q:

$$\frac{n}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\cdots}} =: [e_1, \ldots, e_r].$$

We have $e_i \geq 2$ for all i and we easily see that cyclic quotient singularities are sandwiched.

We want to find a representation via a decorated curve. Our first idea might be to add (-1)-vertices to the dual resolution graph to get a representation $X(C, l)$ where C has many branches which are all smooth. Let us do this in such a way that one of the end points of the graph corresponds to the exceptional divisor E_0 of the blowup of $0 \in \mathbb{C}^2$. An Enriques diagram of (C, l) then looks like this:



We have the following characterizations of these representations:

Lemma 2.7.1. *A two-dimensional singularity is a cyclic quotient if and only if it is a sandwiched singularity that can be represented by a decorated curve (C, l) with smooth branches C_i such that*

$$\min\{l(i), l(j)\} = \langle C_i, C_j \rangle + 1 \qquad \forall i \neq j.$$

Now let us look at another interesting representation:

Theorem 2.7.2. *Assume* $0 < q < n$ *and* $gcd(q, n) = 1$. *The singularity* $A_{n,q}$ *is isomorphic to the unique singularity lying over* $0 \in \mathbb{C}^3$ *in the normalization of the surface* $W_{n,q} = \{(x, y, z) \in \mathbb{C}^3 | \ x^{n-q}z - y^n = 0\}$.

For a proof see [BPdV84, p. 81].

Now $W_{n,q}$ is obviously a chart of the blowup of $\mathbb{C}^2$ in (x^{n-q}, y^n)! With theorem 2.1.2 this implies:

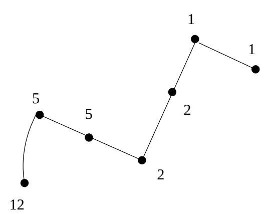
Corollary 2.7.3. *The singularity* $A_{n,q}$ *is isomorphic to a singularity on the blowup of* $(\mathbb{C}^2, 0)$ *in the integral closure of the ideal* (x^{n-q}, y^n).

We leave the verification of the following facts as an easy exercise to the reader: All base points of (x^{n-q}, y^n) are on the curve $x^{n-q} + y^n$. There are $\lceil \frac{n}{n-q} \rceil$ free base points, which are all on the smooth curve x, the other base points are all satellite points. The number of base points and the proximity relations can be read off from the numbers occurring in the Euclidean algorithm performed on n and $n - q$. In general there are two singularities on the blowup of $\overline{(x^{n-q}, y^n)}$.

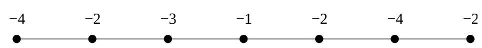
Example 2.7.4. Let's take a look at $A_{17,5}$. The Euclidean algorithm on n and $n - q$ yields:

$$\begin{aligned} 17 &= 1 \cdot 12 + 5 \\ 12 &= 2 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1. \end{aligned}$$

The cluster of base points of (x^{n-q}, y^n) has the following Enriques diagram:



Blowing up in all the base points leads to a divisor with the following dual graph:



Deleting the (-1)-vertex leaves two connected components. The left is the dual resolution graph of $A_{17,5}$ as we see from

$$\begin{aligned} 17 &= 4 \cdot 5 - 3 \\ 5 &= 2 \cdot 3 - 1 \\ 3 &= 3 \cdot 1 \\ \Longrightarrow \frac{17}{5} &= [4, 2, 3] \end{aligned}$$

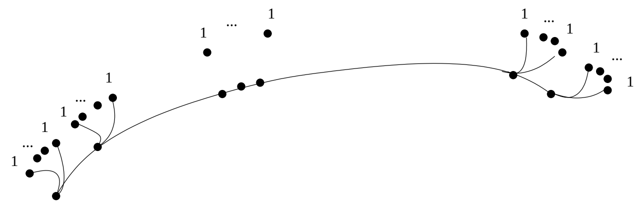
2.8 Taut and Pseudotaut Singularities

A surface singularity X is *taut* iff every normal surface singularity with the same dual resolution graph is isomorphic to X. It is called *pseudotaut* iff it is normal and there are only a finite number of isomorphism classes of normal surface singularities with the same dual resolution graph. Taut and pseudotaut surface singularities have been completely classified by Laufer in [Lau73].

In this section we will first give a few examples of how corollary 2.4.3 shows that certain $X(C, l)$ are taut or pseudotaut. In the next section we will apply our observations to obtain a complete list of those equisingularity classes of plane curves which contain only one respectively finitely many isomorphism classes.

Example 2.8.1. A_k-singularities have representations $X(\text{smooth curve}, k+1)$. Since all smooth curves are right-equivalent, we see that A_k is taut.

Example 2.8.2. A cyclic quotient singularity has a representation $X(C, l)$ such that (C, l) has an Enriques diagram:



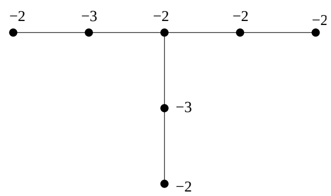
We see that the set of base points with the property that no base points are proximate to them consists of points lying all on one smooth curve. Assume that we are given two such decorated curves which are equisingular. We choose an automorphism of $(\mathbb{C}^2, 0)$ that maps the smooth curve which contains the free base points with no other points proximate to them of the first decorated curve onto that of the second decorated curve and conclude that any two cyclic quotient singularities $X(C, l)$ with the same dual resolution graph are isomorphic by theorem 2.4.2. (Of course, this result is well known.)

Remark 2.8.3. The argument used in the last example proves the following more general statement: If X is a taut sandwiched singularity with dual resolution graph (Γ, μ) and X' is a normal surface singularity with dual resolution graph (Γ, μ') such that $\mu' \leq \mu$, then X' is also taut.

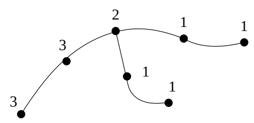
In fact, this statement is true for all taut singularities X, see [Lau73].

Example 2.8.4. We have seen that cyclic quotient singularities are also on the blowup of a complete ideal whose base points are all satellite points except for some points lying all on a single smooth curve. Given two equisingular such clusters (C, l) and (C', l'), any automorphism of $(\mathbb{C}^2, 0)$ mapping the smooth curve containing the free points of $BP(C, l)$ to the smooth curve of $BP(C', l')$ automatically maps all of $BP(C, l)$ bijectively onto $BP(C', l')$. So we have one more proof of the fact that cyclic quotient singularities are taut.

Example 2.8.5. We consider the following sandwiched graph:



There is only one minimal embedding of this graph into a non-singular graph (up to permutation of the left and the lower arm of the graph). The corresponding Enriques diagram with its minimal weights is



The curves going strictly through this cluster are the E_{13}-curve singularities. In [AGZV85], the following normal form for their equations with respect to right-equivalence has been given: $x^3 + xy^5 + ay^8$, $a \in \mathbb{C}$. So a normal surface singularity with the above dual resolution graph is of the form $X\big(x^3 + xy^5 + ay^8, (7, 5)\big)$, where 7 is the value on the singular branch and 5 is the value of the smooth branch.

The class E_{13} contains two analytic isomorphism classes, the one of the homogeneous equation $x^3 + xy^5$ and the class of the semiquasihomogeneous equations with $a \neq 0$. That the semiquasihomogeneous equations all define isomorphic curves can be seen via the change of coordinates $x \mapsto (\frac{b}{a})^5 x$, $y \mapsto (\frac{b}{a})^2 y$. So there can be at most two isomorphism classes of normal surface singularities with the given dual resolution graph.

In fact, the singularity $X\big(x^3 + xy^5 + ay^8, (7, 4)\big)$ is taut, because $x \mapsto x + y^3$, $y \mapsto y$ maps the reduced cluster of base points of $x^3 + xy^5$ to the reduced cluster of base points of $x^3 + xy^5 + y^8$. But if l is at least $(7, 5)$, the reduced cluster of base points is big enough to separate the two isomorphism classes of E_{13} and we get a double-indexed series of classes of pseudotaut singularities $X\big(E_{13}, (7 + a, 5 + b)\big)$, $a, b \in \mathbb{N}_0$, each class containing two isomorphism classes.

As in the above example, most other plane curve singularities which are unimodal with respect to right-equivalence are pseudotaut. The only exceptions are the simply elliptic curve singularities $\tilde{E}_7$ and $\tilde{E}_8$. For the other unimodal curve singularities a change of coordinates of the form $x \mapsto ax$, $y \mapsto by$ with $a, b \in \mathbb{C}$ depending on the weights of the homogeneous part of the equations in Arnold's normal forms shows that they are pseudotaut.

But these are not all the examples of pseudotaut curves, the complete list of taut and pseudotaut plane curve singularities is given in the next section. We will now demonstrate by a direct computation that the $J_{3,1}$-singularities, which are bimodal with respect to right-equivalence, are taut. In fact, it can be shown by similar computations that all $J_{k,i}$-singularities, $i > 0$, are taut, even though their modality with respect to right-equivalence is $k - 1$.

Example 2.8.6. Consider the equisingularity class $J_{3,1}$. The normal form of the $J_{3,1}$-singularities with respect to right-equivalence given by Arnold is $x^3 + x^2y^3 + a_0y^{10} + a_1y^{11}$, $a_0 \neq 0$.

Let us compute the normal form of $(1 + by)(x^3 + x^2y^3 + y^{10})$, $b \in \mathbb{C}$. We use the technique described in [AGZV85]. The terms which bother us are $bx^3y + bx^2y^3$, so we try to express them as a $\mathbb{C}[x, y]$-linear combination of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ plus higher order terms. The coefficients of the partial derivatives are $(b/3)xy$ and $(b/9)y^2$, so we make the change of coordinates $x \mapsto x - (b/3)xy$ and $y \mapsto y - (b/9)y^2$. The result is $x^3 + x^2y^3 + y^{10} - (b/9)y^{11} + f_1(y)x^3y^2 + f_2(y)x^2y^5 + f_3(y)y^{12}$ for some polynomials f_1, f_2, f_3. In the next step we want to eliminate the terms $f_1(0)x^3y^2 + f_2(0)x^2y^5 + f_3(0)y^{12}$. This is possible, because we can write this sum as a $\mathbb{C}$-linear combination of $xy^2 \cdot \partial f/\partial x$, $x \cdot \partial f/\partial y$ and $y^3 \cdot \partial f/\partial y$ plus higher order terms. The corresponding change of coordinates leads to $x^3 + x^2y^3 + y^{10} - (b/9)y^{11} + g_1(y)x^3y^3 + g_2(y)x^2y^6 + g_3(y)y^{13}$, so the powers of y in the terms we want to eliminate has been raised by one. Inductively, we can raise the power of y until the unwanted terms have order greater than the Milnor number, which implies that $(1+by)(x^3+x^2y^3+y^{10})$ is right-equivalent to $x^3 + x^2y^3 + y^{10} - (b/9)y^{11}$.

Now, similar as in the preceding example of E_{13}, the change of coordinates $x \mapsto c^3x, y \mapsto cy$ transforms $x^3 + x^2y^3 + y^{10} - (b/9)y^{11}$ into $c^9(x^3 + x^2y^3 + cy^{10} - (b/9)c^2y^{11})$. Solving $a_0 = c$ and $a_1 = -(b/9)c^2$, we get $c = a_0$ and

$b = -9(a_1/a_0^2)$ (remember that $a_0 \neq 0$). So we have shown:

$$\frac{a_0^2 - 9a_1 y}{a_0^{11}} \cdot \left(x^3 + x^2y^3 + y^{10}\right) \sim_R \left(x^3 + x^2y^3 + a_0y^{10} + a_1y^{11}\right),$$

which means that the $J_{3,1}$-singularity is taut.

2.9 Curves Determined by Their Topological Type

Definition 2.9.1.

1. We call an isolated plane curve singularity C *taut*, iff every singularity $C' \in \mathcal{E}(C)$ equisingular to C is analytically isomorphic to C.
2. We call an isolated plane curve singularity C *pseudotaut*, iff there is only a finite number of analytic isomorphism classes in the equisingularity class $\mathcal{E}(C)$ of C.

Remark 2.9.2. Two plane curve singularities are analytically isomorphic iff they are contact-equivalent or k-equivalent for short.

Remark 2.9.3. Remember from theorem 1.3.2 that two isolated plane curve singularities $(C_i, 0) \subset (\mathbb{C}^2, 0)$, $i \in \{1, 2\}$, are equisingular if and only if they are topologically equivalent in the following sense: There exist representatives (U_i, C_i) such that (U_1, C_1) is homeomorphic to (U_2, C_2). For references on this see [CA00, 3.8, p. 96].

So taut curve singularities are curves determined by their topological type. Pseudotaut curve singularities could be called curves which are almost determined by their topological type.

Remark 2.9.4. A singularity is called equisingularity-rigid if and only if every equisingular deformation is trivial. Obviously, taut curve singularities as well as generic members of a pseudotaut equisingularity class are equisingularity-rigid. It is shown in [Gaw92] that these are indeed all equisingularity-rigid plane curve singularities. In the same article, it is explicitly stated which pseudotaut singularities are equisingularity-rigid in terms of the normal forms of the equations from [AGZV85].

The following lemma is a consequence of Corollary 2.4.3. We have already exploited it in the last example of the previous section.

Lemma 2.9.5. *If C is a taut isolated plane curve singularity with r branches, then $X(C, m_C + \alpha)$, $\alpha \in \mathbb{N}_0^r$, is an r-indexed series of taut singularities.*

If C is a pseudotaut isolated plane curve singularity with r branches, then $X(\mathcal{E}(C), m_C + \alpha)$, $\alpha \in \mathbb{N}_0^r$, is an r-indexed series of pseudotaut singularities and there is an $l \geq m_C$ such that the number of isomorphism classes in $X(\mathcal{E}(C), l + \alpha)$ does not depend on $\alpha \in \mathbb{N}_0^r$.

As an application, we can go through the list of dual resolution graphs of taut and pseudotaut surface singularities given in [Lau73] to find all such series $X(C, l)$, thus getting complete lists of taut and pseudotaut plane curve singularities. The result is given by the following two theorems. They reproduce the result of [Gaw92].

Theorem 2.9.6. *An isolated plane curve singularity $(C, 0)$ is taut, if and only if it is in the following list:*

1. *A smooth curve: A_0.*
2. *A simple singularity: A_k, $k \geq 1$, D_k, $k \geq 4$ or E_6, E_7, E_8.*
3. *A hyperbolic triangle singularity: $T_{2,p,q}$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.*
4. *$J_{k,i}$ with $k > 1$ and $i > 0$.*
5. *$Z_{i,p}$ with $p > 0$.*
6. *$W_{1,p}$ with $p > 0$.*
7. *$W^{\#}_{1,p}$ with $p > 0$.*

A list with equations, Enriques diagrams and the dual resolution graphs of the corresponding taut sandwiched singularities is in appendix A.

Theorem 2.9.7. *An isolated plane curve singularity $(C, 0)$ is pseudotaut, if and only if it is taut or in the following list:*

1. *E_{6k}, E_{6k+1} or E_{6k+2} with $k \geq 2$.*
2. *Z_{6i+11}, Z_{6i+12} or Z_{6i+13} with $i \geq 0$.*
3. *W_{12}, W_{13}, W_{17} or W_{18}.*

A list with equations and Enriques diagrams of the curve singularities and the dual resolution graphs of the corresponding taut sandwiched singularities is in appendix B.

Remark 2.9.8.
1. We see from the lists that all taut and pseudotaut plane curve singularities have multiplicity ≤ 4. It would be nice to have a direct argument for this.

2. The singularities $J_{k,0}$, $Z_{i,0}$, $\tilde{E}_7 = T_{2,4,4}$ and $\tilde{E}_8 = T_{2,3,6}$ are missing in the above lists. This shows that taut and pseudotaut singularities can be adjacent to singularities which are not taut or pseudotaut.

   That these singularities are not pseudotaut can be seen directly as follows: Take the quasihomogeneous part of the normal form equation and add z^2. You get a class of quasihomogeneous, normal surface singularities. These singularities can be constructed from factors of automorphy, see e.g. [Dol75] and [Pin77]. For the above singularities, there is a modulus in the family of Fuchsian groups in the corresponding factors of automorphy, see [Möh00]. This is also a modulus for the analytic isomorphism types in the equisingularity class of plane curve singularities. So even if we restrict to the quasihomogeneous part of the above equisingularity classes, we can already see that the singularities are not taut or pseudotaut. The same argument does not work e.g. for the classes $W_{12k}, W_{12k+1}, W_{12k+5}$ and W_{12k+6}, $k \geq 2$; for these equisingularity classes all quasihomogeneous equations define isomorphic singularities.

Proof. To verify this proof, the reader will have to look at Laufer's list compiled in [Lau73]. We observe the following important facts:

1. The shape of the dual resolution graph of a taut surface singularity is a point, a line, a star or two stars, so it has no more than four arms; the dual resolution graph of a pseudotaut surface singularity which is not taut is a star. This restricts the possible number of branches of corresponding curves to four. A block of consecutive satellite points also gives an extra arm, which might coincide with the arm of another branch. This already restricts the set of Enriques diagrams we have to consider.

2. Consider the continued fractions

$$[b_1, \ldots, b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_r}}}$$

   If all $b_i \geq 2$, then an easy induction over n shows:

$$[b_1, \ldots, b_n - 1] < [b_1, \ldots, b_n, \ldots, b_r] < [b_1, \ldots, b_n] \quad \forall 1 \leq n < r.$$

   The limit of $[2, \ldots, 2]$ as the number of 2's goes to infinity is 1.

It is advisable to consider irreducible curves first. An irreducible taut curve can have no more than two blocks of consecutive satellite points; an irreducible pseudotaut curve no more than one. Using the above observations, it is an elementary exercise to show that all irreducible curves which are not in the above two lists are indeed not taut or pseudotaut.

Obviously, all branches of a reducible taut or pseudotaut curve must be taut or pseudotaut. To obtain a list of all taut and pseudotaut curves with two branches, we have to go through all pairs of irreducible curves in our lists and for each pair through all possible contact orders of the branches. In the same way, we obtain the lists of curves with three and four branches. □

Remark 2.9.9. The fact that an irreducible curve with three or more blocks of consecutive satellite points cannot be taut or pseudotaut has already been observed by Zariski, cf. [Gaw92].

Chapter 3

Deformations of Sandwiched Singularities

In this chapter we study deformations of a sandwiched singularity $X(C,l)$. We start by reviewing the main result of [dJvS98], which says that every deformation of a sandwiched singularity $X(C,l)$ is induced by a deformation of the decorated curve (C,l). After some technical preparations, we give a precise statement of their results in theorems 3.1.7 and 3.2.1.

In section 3.2 we give some examples of easy corollaries on (multi)-adjacencies of sandwiched singularities which follow immediately from the result. The main virtue of these examples is not to be seen in the results themselves, which are not new, but in demonstrating that we have gained complete control over multi-adjacencies of sandwiched singularities up to equisingularity, even though computations may be tough for complicated examples. Very similar is the application of theorem 3.2.1 in the study of smoothings and smoothing components of sandwiched singularities. This will be the main topic of the next chapter, in particular in sections 4.5 and 4.6.

Theorem 3.1.7 is a strong result, but it is not very geometrical. I cite from [dJvS98, p. 477]: 'As the above construction involves blowing up, it is not obvious how to obtain a flat family of surfaces $X(C_S,l_S)$ directly from any 1-parameter deformation of decorated curves (C_S,l_S).' I solve this problem by showing that a deformation of (C,l) describes an equimultiple deformation of the zero-dimensional space $\Sigma(I(C,l))$. For 1-parameter deformations, equimultiplicity is equivalent to the fact that the family of fibres of the blowup in the total space of the deformations is flat, thus giving us the direct construction we were looking for, see section 3.5, simultaneous blowup.

In fact, we get more. The deformation of $\Sigma(I(C,l))$ is not only equimultiple, but the whole Hilbert-Samuel function of the fibres is independent of the deformation parameter. I conjecture that this implies flatness of the fibres of the blowup for deformations over any reduced base space, see conjecture 3.5.4. At the end of the chapter there is a short review of some results on normal flatness which support the conjecture.

The study of deformations of the space $\Sigma(I(C,l))$ also yields another important result: For an equimultiple deformation $\Sigma(I(C_t,l_t))$ the number of base points of $I(C_t,l_t)$ is upper-semicontinuous in t. This gives an interesting and easy-to-handle restriction to the existence of multi-adjacencies of the plane curve singularity C, which we will study in chapter 4.

3.1 Deformations of Decorated Curves

Let (C,l) be a decorated curve, $n:\bar{C}\to C$ the normalization.

Definition 3.1.1. Let $(S,0)$ be normal. A *deformation of the decorated curve* (C,l) over S is a collection of the following data:

1. A δ-constant deformation $C_S\to S$ of C over S.
2. The simultaneous normalization $n:\bar{C}\times S\to C_S$.
3. A flat deformation l_S of the divisor l on $\bar{C}$, such that $(C_{s,p},l_{s,p})$ is a decorated curve for all $p\in C_s\subset C_S$ in a sufficiently small neighbourhood of $0\in C_S$.

The main result in [dJvS98] is that every deformation of the sandwiched singularity $X(C,l)$ is induced by a deformation of the decorated curve (C,l). The precise statement of their result will be given in theorem3.1.7. But first we have to discuss some technical problems.

The main problem is that we have only defined deformations over a normal base space. This is necessary for simultaneous normalization, compare section 4.2. But in classical deformation theory, the starting point for the construction of a deformation space for X are deformations over non-reduced, zero-dimensional base spaces, see e.g. [Pal90]. In fact, the first space to consider is the space $T^1(X)$ of first-order deformations, which is obtained by evaluating the deformation functor Def_X on *Specan* $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

The normalization $\tilde{X}$ of a complex space germ $(X,0)$ is constructed by taking the integral closure of the total quotient ring of $\mathcal{O}_{X,0}$. Since all nilpotent elements ε/x are integral, the straight forward approach to defining the functor of deformations which admit simultaneous normalization does not work. Instead, we define the functor of simultaneous normalization of X as the functor of deformations of the diagram of the normalization

$\operatorname{Def}(\tilde{X} \xrightarrow{n} X)$, see [dJvS90] and [Buc81]. Here a deformation of the diagram means that we are allowed to deform the map as well as both $\tilde{X}$ and X.

In order to define deformations of decorated curves over an arbitrary base, de Jong and van Straten used another way of generalizing deformations over a normal base which admit simultaneous normalizations. For the case of plane curves, the functor they define is equivalent to the functor of simultaneous normalization, see theorem 3.1.5. The basic idea is the following:

If the base space is normal, a deformation of a plane, isolated curve singularity admits a simultaneous normalization if and only if it is δ-constant, which is equivalent to the condition that the deformation of the curve induces a deformation of the conductor of the curve, see section 4.2.

The key observation of de Jong and van Straten is that these induced deformations of the conductor satisfy the so-called ring condition (R.C.), which makes sense over any base space, and that this condition gives us the correct generalization of deformations which admit simultaneous normalization over arbitrary base spaces.

3.1.1 R.C. Deformations

In this section we quote the definition and some basic properties of R.C. deformations from [dJvS90].

Let $n : \bar{C} \to C$ be the normalization of the curve C. By definition, $\mathcal{O}_{\bar{C}}$ is the integral closure of $\mathcal{O}$ in its total quotient ring. It is well known, that every $\mathcal{O}_C$-linear homomorphism from $\mathcal{O}_{\bar{C}}$ to $\mathcal{O}_C$ is just multiplication with an $h \in \mathcal{O}_C$. Indeed, if $\alpha \in Hom_{\mathcal{O}_C}(\mathcal{O}_{\bar{C}}, \mathcal{O}_C)$ and f is a non-zero divisor, then $h = \alpha(f)/f$ is independent of f, because $f\alpha(g) = \alpha(fg) = g\alpha(f)$. So we can identify the conductor of C with $Hom_{\mathcal{O}_C}(\mathcal{O}_{\bar{C}}, \mathcal{O}_C)$, compare definition 1.2.19.

The definition of the conductor only made use of the fact that $\mathcal{O}_{\bar{C}}$ is a $\mathcal{O}_C$-submodule of the total quotient ring. The fact that it is also a ring is reflected in the so-called ring condition which the conductor satisfies. We give the definition for a more general situation.

Definition 3.1.2. Let $(S, \mathfrak{m}_S)$ be a Noetherian local ring,
$(R, \mathfrak{m}_R)$ a Noetherian local S-algebra, M an R-module,
$\bar{R} = R \otimes_R (S/\mathfrak{m}_S)$ and $\bar{M} = M \otimes_R (S/\mathfrak{m}_S)$.

1. M is *Cohen-Macaulay over S* (*CM over S*), if and only if
   (a) $\bar{M}$ is Cohen-Macaulay as an $\bar{R}$-module and
   (b) M is S-flat.

M is a *maximal Cohen-Macaulay module* over S (*MCM over* S) if and only if M is CM over S and the codimension $\dim(\bar{R}) - \dim_{\bar{R}}(\bar{M})$ of M is zero.

2. A *fractional ideal* is a finitely generated R-module M which is contained in the total quotient ring of R and contains a non-zero divisor.

With the same proof as above, we see that if M, N are two fractional ideals, then $Hom_R(M, N)$ is again a fractional ideal. If a fractional ideal $\tilde{R} \supset R$ is also a ring, we call the ideal $Hom_R(\tilde{R}, R) \subset R$ the conductor of $\tilde{R}$. So with this notation, the conductor of a plane curve singularity is the conductor of the semilocal ring of its normalization.

Definition 3.1.3. The ideal $I \subset R$ satisfies the *ring condition (R.C.)* iff the natural injection $Hom_R(I, I) \hookrightarrow Hom_R(I, R)$ is an isomorphism.

Proposition 3.1.4. *Let $\tilde{R} \supset R$ be a fractional MCM over S and set $J := Hom_R(\tilde{R}, R) \subset R$.*
Then the following two statements are equivalent:

1. *$\tilde{R}$ is a ring (whose ring structure is induced by the ring structure of the total quotient ring of R).*

2. *J satisfies the ring condition.*

This equivalence is proved in [dJvS90, Prop. 1.8].
The next theorem is [dJvS90, Th. 1.1].

Theorem 3.1.5. *Let $\tilde{X} \to X$ be a finite surjective and generically injective mapping. Let $\Sigma = \Sigma(J)$ be the subspace of X defined by the conductor ideal $J = Hom_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. Assume that $\tilde{X}$ is Cohen-Macaulay and that X is Gorenstein.*

Then there is a natural equivalence of functors

$$\mathrm{Def}\,(\tilde{X} \to X) \to \mathrm{Def}\,(\Sigma \hookrightarrow X, R.C.).$$

Here the second functor describes deformations of the diagram $\Sigma \hookrightarrow X$ for which the ideal of Σ_S in X_S satisfies the ring condition (R.C.).

So the result says that the study of deformations which admit simultaneous normalization is the same as the study of R.C.-deformations of the conductor.

3.1.2 Normal Form Deformations

Let (C, l) be a decorated curve, Σ the fat point defined by the conductor and $g \in \mathcal{O}_{\mathbb{C}^2,0}$ a function whose intersection multiplicity with the i-th branch of C is $l(i)$. The last condition is equivalent to the fact that l is the divisor of zeroes of the pullback of g to the normalization of C. We also say that g cuts out the divisor l on C. So we can recover the decorated curve (C, l) from the triple (Σ, C, g); but for a given decorated curve, the triple is not unique, because we can choose different g.

Definition 3.1.6. Let S be a local analytic space. A triple (Σ_S, C_S, g_S) is called a *nice triple* or a *normal form deformation* if and only if

1. (Σ_S, C_S) is an R.C.-deformation of (Σ, C) over S and
2. $\big(\Sigma_S, \Sigma(g_S)\big)$ is an R.C. deformation of (Σ, g) over S.

Two nice triples (Σ_S, C_S, g_S) and (Σ'_S, C'_S, g'_S) are isomorphic if there is a coordinate transformation in the x, y-plane over S that maps (Σ_S, C_S) to (Σ'_S, C'_S) and g_S to g'_S modulo some multiple of an equation for C_S.

We define the functor $\mathrm{Def}(\Sigma, C, g)$ of *normal form deformations* by putting

$$\mathrm{Def}(\Sigma, C, g)(S) := \{(\Sigma_S, C_S, g_S) \text{ nice triple over } S\}/\{\text{isomorphisms}\}$$

Over a normal base space, a normal form deformation induces a deformation of the decorated curve in the sense of section 3.1 and every deformation of a decorated curve is induced by a normal form deformation. So it would be natural to define a deformation of a decorated curve over an arbitrary base as an equivalence class of normal form deformations. However, since this approach is not very elegant and probably would not lead to a better understanding of the situation, we will not follow it any further. It would be interesting though to find a definition of deformations of decorated curves which does not make use of normal forms at all.

In [dJvS98, §3] it is shown that the functor of normal form deformations satisfies the condition of Schlessinger's theorem, so it has a hull, which means that there is a semiuniversal normal form deformation.

The main result of [dJvS98] is

Theorem 3.1.7. *There is a natural transformation of functors*

$$\mathrm{Def}(\Sigma, C, g) \to \mathrm{Def}\big(X(C, l)\big).$$

This transformation is formally smooth.

The proof is [dJvS98, §3].

Formal smoothness implies that every deformation of the sandwiched singularity $X(C, l)$ is induced by a normal form deformation and furthermore that the base spaces of the semiuniversal deformation spaces are the same up to a smooth factor.

3.2 Adjacencies of Sandwiched Singularities

If we restrict ourselves to deformations over a normal base space, theorem 3.1.7 implies the following:

Theorem 3.2.1 (de Jong, van Straten). *Let $(T, 0)$ be a normal complex space germ, for example $T = \mathbb{C}$.*

1. *For every deformation (C_T, l_T) of a decorated curve (C, l) there is a deformation X_T of $X(C, l)$, such that $X_t = X(C_t, l_t)$ for all t.*
2. *For every deformation X_T of a sandwiched singularity $X(C, l)$ over $(T, 0)$, there is a deformation (C_T, l_T) of (C, l) over $(T, 0)$, such that $X_t = X(C_t, l_t)$ for all t.*

Remark 3.2.2. Note that writing $X_t = X(C_t, l_t)$ is an abbreviation for the following: We can choose a representative of X_T such that the fibre X_t is isomorphic to the blowup of $\mathbb{C}^2$ in the ideals $I(C_{t,p}, l_{t,p}) \subset \mathcal{O}_{\mathbb{C}^2,p}$ where p ranges over all points of C_t where l_t is not zero.

Corollary 3.2.3. *Sandwiched singularities only deform into sandwiched singularities.*

Similarly, assume that we have a class of decorated curves of which we know that it is closed under deformations. By the above theorem it corresponds to a class of normal surface singularities which is closed under deformations. For example, rational singularities with reduced fundamental cycle are those sandwiched singularities which can be represented by a decorated curve with smooth branches, see theorem 2.5.1. So we get as a corollary that rational singularities with reduced fundamental cycle only deform into rational singularities with reduced fundamental cycle. Of course, this is rather trivial. But we also obtain a new proof of

Corollary 3.2.4. *Two-dimensional cyclic quotient singularities only deform into cyclic quotient singularities.*

That quotient singularities only deform into quotient singularities has been proved in [EV85]. That cyclic quotient singularities only deform into cyclic quotient singularities was conjectured by Riemenschneider and has been proved in [KSB88, §7].

Proof. This follows directly from the standard representation of a cyclic quotient via a decorated curve with smooth branches, see lemma 2.7.1. □

The above corollaries are very coarse statements on adjacencies of sandwiched singularities. But we can also choose to go more into details, because the theorem tells us that knowing all adjacencies of a curve C to multi-equisingularity classes gives us all multi-adjacencies of the sandwiched singularities $X(C, l)$ to classes of normal surface singularities with given resolution graphs.

Here is the easiest example:

Example 3.2.5. An A_k-singularity can be represented as $X(C, k+1)$, where C is a smooth curve. All deformations of a smooth curve are trivial, so all we can do is deform the divisor, i.e. split up the value $k+1$ at 0 into r parts $k_1, \ldots, k_r$ at different points with $\sum_{i=1}^{r} k_i = k+1$. We get a new proof of the following result:

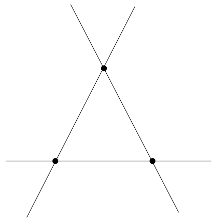
Proposition 3.2.6. *There is a multi-adjacency of the A_k-surface singularity to $(A_{k_1}, \ldots, A_{k_r})$ if and only if $\sum_{i=1}^{r} k_i \leq k+1-r$. This gives all multi-adjacencies of the A_k-singularities.*

This example is rather trivial, because an A_k-surface singularity is stably equivalent to the singularity $\Sigma(x^{k+1}) \subset (\mathbb{C}, 0)$, but it demonstrates the point.

In general, finding all adjacencies of a given isolated plane curve singularity C to multi-equisingularity classes is a very hard task. Chapter 4 contains several necessary and sufficient conditions for such adjacencies to exist.

The situation gets easier if we consider decorated curves (C, l) with a small divisor l, because this restricts the number of deformations of C we have to consider.

Example 3.2.7. Let X_n be the cone over the rational normal curve of degree n, i.e. the unique singularity whose dual resolution graph consists of a single vertex with weight $-n$. Let C be an ordinary singularity of multiplicity $n-1$. Then we have $X_n = X(C, (2, \ldots, 2))$. It is easy to see the following: If $n \neq 4$ and (C_t, l_t) is a 1-parameter deformation of $(C, (2, \ldots, 2))$, then C_T is a trivial deformation, i.e. we can transform C_T into $C \times T$ by a change of coordinates. If $n = 4$, then C_t can also look like this for $t \neq 0$:



For any $n \in \mathbb{N}$, we deduce that every 1-parameter deformation (C_t, l_t) of $(C, (2, \ldots, 2))$ with $(C_t, l_t) \notin \mathcal{E}(C, l)$ for $t \neq 0$ has the property that $BP(C_t, l_t)$ has excess greater than zero at every base point.

This shows that every non-trivial deformation of X_n over a reduced base space is a smoothing. The fact that there are two essentially different deformations of the decorated curve of X_4 corresponds to the fact that X_4 has two smoothing components, compare section 4.6.

Note that it is essential that we only consider reduced base spaces, because for $n \geq 5$, the base space of a semiuniversal deformation of X_n has an embedded zero-dimensional component, see [Pin74, Ch. II, 8]. This means that X_n has obstructed infinitesimal deformations i.e. non-trivial first-order deformations which cannot be lifted to 1-parameter deformations.

Example 3.2.8. Let $X_{a,b}$ be the cyclic quotient singularity whose dual resolution graph consists of two vertices with weights $-a$, $-b$.

As in the last example we can easily see that every fibre of a 1-parameter deformation of $X_{a,b}$ is either smooth or has a single singularity, which must be either $X_{a,b}$ or X_a or X_b or X_{a+b-2}. Here X_n denotes the cone over the rational normal curve of degree n, compare the above example.

3.3 Geometric Construction of Deformations

Theorem 3.2.1 tells us that a 1-parameter deformation (C_T, l_T) of a decorated curve (C, l) induces a deformation $X(C_T, l_T)$ of the sandwiched singularity $X(C, l)$, but it does not tell us how to construct $X(C_T, l_T)$ directly from (C_T, l_T). To find a direct construction of $X(C_T, l_T)$ using blowups was therefore left as an open problem in [dJvS98]. We devote the rest of this chapter to the solution of this problem. I give a short summary of the results we obtain:

To solve the problem, we will take a look at the family of zero-dimensional spaces $\Sigma(I(C_t, l_t))$. We prove that deformations of the decorated curve (C, l) correspond exactly to equimultiple deformations of the fat point $\Sigma(I(C, l))$. The notion of equimultiplicity is known to be important when considering blowups, see e.g. [HIO88] and [Lip82]. We will then see that for 1-parameter deformations equimultiplicity implies that the fibres of the blowup in $\Sigma(I(C_T, l_T))$ are a deformation of $X(C, l)$, thus giving us the direct construction via simultaneous blowup.

In fact the construction works for deformations over an arbitrary reduced base space: Our analysis of the deformation $\Sigma(I(C_T, l_T))$ shows that it is not only equimultiple but that the whole Hilbert-Samuel function is independent of the fibre. I conjecture that this implies that the blowup in $\Sigma(I(C_T, l_T))$ is a flat deformation of $X(C, l)$.

Along the way, we will also show that the number of base points of $I(C_t, l_t)$ is an upper semicontinuous function in t. This gives an interesting restriction to the existence of certain deformations of a decorated curve and also for the existence of δ-constant deformations of plane curve singularities without a decoration. We will make use of this application in the next chapter.

3.4 Deformations of Fat Points

3.4.1 Notations

Notation 3.4.1. A *fat point in the plane* is a zero-dimensional complex subspace of $\mathbb{C}^2$, whose support consists of a single point, say $0 \in \mathbb{C}^2$. This is the same as saying that a fat point is a complex space of the form $\Sigma(I)$, where $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ is an $\mathfrak{m}_{\mathbb{C}^2,0}$-primary ideal.

Let $(T, 0)$ be a complex space germ. A family of fat points over T whose special fibre is the fat point $\Sigma(I)$ is a complex space $\Sigma(\mathfrak{I}) \subset (\mathbb{C}^2 \times T, 0)$ with $\mathfrak{I} \otimes \mathcal{O}_{T,0}/\mathfrak{m}_{T,0} \cong I$. We will mostly consider base spaces $(T, 0)$ which are reduced or even normal. Note that the general fibre of such a family may consist of several fat points.

We always choose sufficiently small representatives in the usual way, so by abuse of notation we view $\mathfrak{I}$ as a coherent ideal sheaf on an open neighbourhood of $0 \in \mathbb{C}^2 \times T$. We let $\mathfrak{I}_t$ denote the restriction of $\mathfrak{I}$ to $\mathbb{C}^2 \times \{t\}$, i.e. the ideal sheaf of the fibre over t.

Definition 3.4.2. Let $\Sigma(\mathfrak{I})$ be a family of fat points in the plane over the reduced base space $(T, 0)$.

1. The family is *relative complete* if and only if $\mathfrak{I}_{t,p}$ is a complete ideal for all $t \in T$, $p \in V(\mathfrak{I}_t)$.

2. We define the *degree* of a fibre by

$$\begin{aligned}\deg \Sigma(\mathfrak{I}_t) &:= \sum_{p\in V(\mathfrak{I}_t)} \deg \mathfrak{I}_{t,p} \\ &= \sum_{p\in V(\mathfrak{I}_t)} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,p}/\mathfrak{I}_{t,p}.\end{aligned}$$

We recall the well known fact that a family of zero-dimensional complex spaces over a reduced base space $(T,0)$ is flat if and only if the degree is constant in t for t sufficiently small.

3. We define the *δ-constant* of a fibre by

$$\begin{aligned}\delta(\Sigma(\mathfrak{I}_t)) &:= \sum_{p\in V(\mathfrak{I}_t)} \delta(\mathfrak{I}_{t,p}) \\ &= \sum_{p\in V(\mathfrak{I}_t)} \frac{1}{2} \sum_{q\in\mathcal{N}} e_q^2(\mathfrak{I}_t) - e_q(\mathfrak{I}_t) \qquad \text{by 1.5.1.}\end{aligned}$$

The family is δ-constant if and only if the δ-invariant of the fibres is constant in t.

4. We define the *Hilbert-Samuel function* of a fibre to be

$$H_{\mathfrak{I}_t}(n) = \sum_{p\in V(\mathfrak{I}_t)} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,p}/(\mathfrak{I}_{t,p})^n.$$

5. We define the *multiplicity* of a fibre to be

$$\begin{aligned}e(\Sigma(\mathfrak{I}_t)) &:= \sum_{p\in V(\mathfrak{I}_t)} e(\mathfrak{I}_{t,p}) \\ &= \sum_{p\in V(\mathfrak{I}_t)} \sum_{q\in\mathcal{N}_p} e_q^2(\mathfrak{I}_t) \qquad \text{by corollary 1.5.9.}\end{aligned}$$

The family is equimultiple if and only if the multiplicity of the fibres is constant in t.

3.4.2 Decorated Curves and Equimultiplicity

Now we link the invariants just defined with the notion of a deformation of a decorated curve.

Proposition 3.4.3. *Let $(T, 0)$ be a normal complex space germ.*

For every deformation (C_T, l_T) of a decorated curve (C, l) over $(T, 0)$ there is a unique coherent ideal sheaf $\mathfrak{I}$ on an open neighbourhood of zero in $\mathbb{C}^2 \times T$ with $\mathfrak{I}_t = I(C_t, l_t)$. The space $\Sigma(\mathfrak{I})$ is a δ-constant, relative complete deformation of the fat point $\Sigma(I(C, l))$.

Conversely, if $\Sigma(\mathfrak{I})$ is a δ-constant, relative complete deformation over $(T, 0)$ such that $\mathfrak{I}_0 = \mathfrak{I}/(t)$ is the complete ideal $I(C, l)$, then there exists a deformation (C_T, l_T) of the decorated curve (C, l) such that $\mathfrak{I}_t = I(C_t, l_t)$.

Proof. Note that for a generic $g \in I(C, l)$ the intersection multiplicity of $\Sigma(g)$ with the i-th branch of C is $l(i)$. This is equivalent to the fact that l is the zero divisor of the pullback of g under the normalization of C, see e.g. [CA00, corollary 3.11.6]. So $g \in I(C, l)$ if and only if the pullback of g to the normalization of C is in $\mathcal{O}(-l)$.

Let (C_t, l_t) be a deformation of the decorated curve (C, l). We have to show the existence of a coherent ideal sheaf $\mathfrak{I}$ with $\mathfrak{I}_t = I(C_t, l_t)$. Choose $F \in \mathcal{O}_{\mathbb{C}^2 \times T}$ such that $C_t = V(F_t)$. Let J be the coherent ideal on the simultaneous normalization $\bar{C} \times T$ of C_t with the property $J_t = \mathcal{O}(-l_t)$. Then $n_*(J)$ is a coherent ideal on $V(F)$, so there is a coherent ideal $\mathfrak{I}$ with $V(\mathfrak{I}) = V(F)$ and $\mathfrak{I}/(F) = n_*(J)$. By construction $\mathfrak{I}_t = I(C_t, l_t)$, so $\Sigma(\mathfrak{I})$ is relative complete. The δ-invariant of the fibre $\Sigma(\mathfrak{I}_t)$ can be written as $\frac{1}{2} \sum (e_q^2 - e_q)$ where the sum is taken over all base points of $\mathfrak{I}_t$ and is equal to the δ-invariant of C_t, so it is constant by the definition of deformations of decorated curves. Also by definition, $\sum e_q =$ (the degree of l_t) is constant in t. Because all fibres are defined by complete ideals, this implies that the degree $\deg(\Sigma(\mathfrak{I})) = \frac{1}{2} \sum (e_q^2 + e_q)$ is constant in t, which means that the family is flat.

Conversely, let $\mathfrak{I}$ be a δ-constant, relative complete deformation of the complete ideal $I(C, l)$, $C = V(f)$ and $g \in \mathcal{O}_{\mathbb{C}^2, 0}$ a function such that l is the divisor of zeros of the pullback of g to the normalization of C. Then generic sections F, G in $\mathfrak{I}$ with $F_0 = f$ and $G_0 = g$ define (C_t, l_t). $\square$

The deformations of complete ideals that are associated to deformations of decorated curves have the property of being flat, relative complete and δ-constant. This may seem to be a rather strange condition. But for the case that the special fibre is defined by a complete ideal, we will now show that this condition is equivalent to equimultiplicity. Note that a priori equimultiplicity seems to be the weaker condition, because equimultiplicity means that $\sum e_q^2$ is constant in t, while the above conditions mean that $\sum e_q^2$ and $\sum e_q$ *both* are constant in t *plus* the fact that the family is relative complete.

Proposition 3.4.4. *Let $\Sigma(\mathfrak{I})$ be a family of fat points in the plane over the reduced base space $(T,0)$. Assume that the ideal $\mathfrak{I}_0 \subset \mathcal{O}_{\mathbb{C}^2,0}$ of the special fibre is a complete ideal.*

Then the family is equimultiple if and only if it is flat, relative complete and δ-constant.

The proof refers to the following lemma:

Lemma 3.4.5. *Let $\Sigma(\mathfrak{I})$ be a family of fat points in the plane over the reduced base space $(T,0)$.*

Then the degree, the δ-constant and the multiplicity of the fibres are upper-semicontinuous functions in t. In particular, for all sufficiently small $t \neq 0$:

$$\begin{aligned} \deg \mathfrak{I}_0 &\geq \deg \mathfrak{I}_t, \\ \delta(\mathfrak{I}_0) &\geq \delta(\mathfrak{I}_t), \\ e(\mathfrak{I}_0) &\geq e(\mathfrak{I}_t). \end{aligned}$$

Proof. Semicontinuity of the degree is well known.

Semicontinuity of the δ-invariant follows from the fact that the δ-invariant of an ideal describing a fat point in the plane is equal to the δ-invariant of a generic member of the ideal.

Semicontinuity of the multiplicity is also well known. For ideals describing fat points in the plane, the multiplicity is equal to the intersection multiplicity of a pair of generic elements of the ideal, which also shows the result. □

Proof of theorem 3.4.4. Assume that the family is equimultiple. Equimultiplicity (3.1) and semicontinuity of the δ-invariant (3.2) imply (3.3):

$$\sum_{q\in\mathcal{N}_0} e_q(\mathfrak{I}_0)^2 = \sum_{p\in V(\mathfrak{I}_t)} \sum_{q\in\mathcal{N}_p} e_q(\mathfrak{I}_{t,p})^2 \tag{3.1}$$

$$\frac{1}{2}\sum_{q\in\mathcal{N}_0} \big(e_q(\mathfrak{I}_0)^2 - e_q(\mathfrak{I}_0)\big) \geq \frac{1}{2} \sum_{p\in V(\mathfrak{I}_t)} \sum_{q\in\mathcal{N}_p} \big(e_q(\mathfrak{I}_{t,p})^2 - e_q(\mathfrak{I}_{t,p})\big) \tag{3.2}$$

$$\Longrightarrow \sum_{q\in\mathcal{N}_0} e_q(\mathfrak{I}_0) \leq \sum_{p\in V(\mathfrak{I}_t)} \sum_{q\in\mathcal{N}_p} e_q(\mathfrak{I}_{t,p}). \tag{3.3}$$

(3.3) implies

$$\begin{aligned}
\deg \mathfrak{I}_0 &= \frac{1}{2} \sum_{q \in \mathcal{N}_0} e_q^2(\mathfrak{I}_0) + e_q(\mathfrak{I}_0) \\
&\leq \frac{1}{2} \sum_{p \in V(\mathfrak{I}_t)} \sum_{q \in \mathcal{N}_p} e_q(\mathfrak{I}_{t,p})^2 + e_q(\mathfrak{I}_{t,p}) \qquad \text{by (3.1) and (3.3)} \\
&= \sum_{p \in V(\mathfrak{I}_t)} \deg(\overline{\mathfrak{I}_{t,p}}) \\
&\leq \sum_{p \in V(\mathfrak{I}_t)} \deg(\mathfrak{I}_{t,p}) \\
&= \deg(\mathfrak{I}_t).
\end{aligned}$$

But since the degree is upper-semicontinuous, we must have equality everywhere. The equality $\deg \mathfrak{I}_0 = \deg(\mathfrak{I}_t)$ is flatness, equality in (3.3) implies δ-constancy and the equality $\sum_{p \in V(\mathfrak{I}_t)} \deg(\overline{\mathfrak{I}_{t,p}}) = \sum_{p \in V(\mathfrak{I}_t)} \deg(\mathfrak{I}_{t,p})$ means that the family is relative complete.

Reversing the argument proves the other implication. □

We summarize our result as follows:

Theorem 3.4.6. *1. Every deformation (C_t, l_t) of a decorated curve (C, l) over a reduced base space $(T, 0)$ induces a deformation $\Sigma(I(C_t, l_t))$ of $\Sigma(I((C, l))$ over $(T, 0)$ with the property that the Hilbert-Samuel functions of the fibres are independent of t. In particular, the deformation $\Sigma(I(C_t, l_t))$ is equimultiple.*

2. Every equimultiple family *$\Sigma(\mathfrak{I})$ over $(T, 0)$ with special fibre $\Sigma(\mathfrak{I}_0) = I(C, l)$ is induced by a deformation (C_t, l_t) of the decorated curve (C, l), i.e. $\mathfrak{I}_t = I(C_t, l_t)$. In particular, the family is flat and the whole Hilbert-Samuel function of the fibres is constant in t.*

Proof. The only thing left to prove is the statement on the Hilbert-Samuel function. In the preceding proofs we have seen that $\sum e_q^2$ and $\sum e_q$ are constant in t and that all $\mathfrak{I}_{t,p}$ are complete. So the statement follows from the description of the Hilbert-Samuel function given in the first chapter, see theorem 1.5.6. □

The following proposition will be useful in the chapter on the Kollár conjecture.

Proposition 3.4.7. *Let $\Sigma(\mathfrak{I})$ be an equimultiple deformation of a zero-dimensional space in $\mathbb{C}^2$ over $(T,0)$. Assume that there are arbitrarily small t such that the stalks of $\mathfrak{I}_t$ are complete, but that $\mathfrak{I}_0$ is not complete.*

Then the deformation is not δ-constant.

Proof. The proof is essentially the same as that of theorem 3.4.4. We leave the details to the reader. □

3.4.3 Semicontinuity of the Number of Base Points

We prove one more theorem using the semicontinuity of invariants which can be expressed as sums of some multiplicities or squares of multiplicities of base points. It gives interesting restrictions on possible multi-adjacencies of decorated and also on non-decorated curves, compare section 4.3.

Theorem 3.4.8. *Let $I \subset \mathcal{O}_{\mathbb{C}^2,0}$ be an $\mathfrak{m}$-primary, complete ideal and $\Sigma(\mathfrak{I})$ an equimultiple deformation of $\Sigma(I) = \Sigma(\mathfrak{I}_0)$ over $(T,0)$. Then the number of base points*

$$\#BP(\mathfrak{I}_t) := \sum_{p \in V(\mathfrak{I}_t)} \#\{q \in \mathcal{N}_p |\; e_q(\mathfrak{I}_t) \geq 1\}$$

is upper-semicontinuous in t. In particular for all sufficiently small t:

$$\#BP(\mathfrak{I}_0) \geq \#BP(\mathfrak{I}_t).$$

The number of base points is constant if and only if the induced deformation of the conductor is δ-constant and if and only if the induced deformation of the conductor is equimultiple.

Remark 3.4.9. For non-equimultiple deformations of $\Sigma(I)$ the number of base points is not upper-semicontinuous, even if we assume that the deformation is relative complete. The simplest counter-example is given by the deformation of the fat point defined by $(x,y)^2$ into three reduced points.

Conjecture 3.4.10. The equisingularity class of a plane curve singularity C is coded in the so-called combinatorial representation of C, see section 4.3.1. I conjecture the following:

In the theorem 3.4.8, equality holds if and only if the combinatorial representation of the singularity $\Sigma(\mathfrak{I}_0)$ is equal to the sum of the combinatorial representations of the singularities of $\Sigma(\mathfrak{I}_t)$.

Remark 3.4.11. Assume $\mathfrak{I} = I(C_T, l_T)$. Then the conjecture is equivalent to the following statement: $\#BP(\mathfrak{I}_0) = \#BP(\mathfrak{I}_t)$ if and only if $X(C_t, l_t)$ is a deformation over the Artin component, see chapter 5. This shows that the conjecture is equivalent to conjecture 5.8.5.

If $\mathfrak{I} = I(C_T, l_T)$ and C_t for $t \neq 0$ has only ordinary singularities, then the conjecture is equivalent to the following statement: $\#BP(\mathfrak{I}_0) = \#BP(\mathfrak{I}_t)$ if and only if C_T is a Scott deformation.

Proof. An equimultiple deformation is δ-constant, so it implies a deformation of the (space defined by the) conductor. The conductor of a complete ideal I is by definition the conductor of a generic function in I and by theorem 1.2.21 equal to the complete ideal with base point cluster $(BP(I), e_p(I) - 1)$. So the δ-constant of a generic function in the conductor is

$$\delta = \frac{1}{2}\sum(e_q - 1)(e_q - 2) = \frac{1}{2}\sum e_q^2 - \frac{3}{2}\sum e_q + \#BP.$$

We have seen in the preceding proofs that equimultiplicity of the deformation $\Sigma(\mathfrak{I})$ implies that the first two terms on the right side are constant. The semicontinuity of the δ-invariant then gives the result.

Since the deformation of the conductor is relative complete, the deformation is δ-constant if and only if it is equimultiple. $\square$

Remark 3.4.12. In fact we have proven more by showing that the number of base points plus a constant is equal to the δ-constant of a generic function in the conductor, namely that the sets $\{t \in T \mid \#BP(\mathfrak{I}_t) \geq k\}$ are analytic subsets of T.

Remark 3.4.13. The number of base points is only defined for an ideal in $\mathcal{O}_{\mathbb{C}^2,0}$. It would be interesting to interpret the number of base points as a function of algebraic or homological invariants which generalizes to a more general situation.

3.5 Simultaneous Blowups

3.5.1 One-Parameter Deformations

The results in [dJvS98] include that every 1-parameter deformation of $X(C, l)$ is of the form $X(C_T, l_T)$, where (C_T, l_T) is a 1-parameter deformation of the decorated curve (C, l), theorem 3.2.1. It was left as an open problem though to find a direct construction of the deformation $X(C_T, l_T)$ for a given (C_T, l_T). We now prove that the deformation $X(C_T, l_T)$ can be constructed by a single blowup whose fibres are the $X(C_t, l_t)$.

Theorem 3.5.1. *Let (C_T, l_T) be a 1-parameter deformation of (C, l) over $(T, 0) = (\mathbb{C}, 0)$ and $\mathfrak{I}$ a coherent sheaf of ideals on a well chosen neighbourhood U of zero in $\mathbb{C}^2 \times T$ with fibres $I(C_t, l_t)$.*

Then the blowup of $\mathbb{C}^2 \times T$ respectively U in $\Sigma(\mathfrak{I})$ is flat over T with fibres $X(C_t, l_t)$.

Proof. Theorem 3.4.6 tells us that a deformation of $\Sigma(\mathfrak{I})$ over $(T, 0) = (\mathbb{C}, 0)$ is equimultiple if and only if it is of the form $\Sigma(I(C_T, l_T))$. The following result from [Tei82, ch.I, 5.1] completes the proof. □

Theorem 3.5.2 (Teissier). *Let $\Sigma(\mathfrak{I}) \subset \mathbb{C}^r \times \mathbb{C}$ be a family of zero-dimensional spaces in $\mathbb{C}^r$ over $(\mathbb{C}, 0)$. Let $\pi : X \to \mathbb{C}^r \times \mathbb{C}$ be the blowup in $\Sigma(\mathfrak{I})$ with exceptional divisor $E = \pi^{-1}(\Sigma(\mathfrak{I}))$. We define the divisor of vertical components E_{vert} to be the union of those irreducible components of E which map to zero. We define its degree to be $\deg E_{vert} := \deg(\mathcal{O}_{E_{vert}}(1))$.*

Then for all sufficiently small $t \neq 0$:

$$\deg E_{vert} = e(\Sigma(\mathfrak{I}_0)) - e(\Sigma(\mathfrak{I}_t)).$$

In particular, no component of the exceptional divisor is contained in the fibre over zero if and only if $\mathfrak{I}$ is equimultiple.

3.5.2 Conjecture on Multi-Parameter Deformations

We have proven that one-parameter deformations $X(C_t, l_t)$ can be constructed as the blowup of $\mathbb{C}^2 \times T$ in the total space of the deformation $\Sigma(I(C_T, l_T))$ of the fat point $\Sigma(I(C, l))$. I believe that the same can be done for deformations $X(C_t, l_t)$ over any reduced base space T.

Conjecture 3.5.3. Let (C_T, l_T) be a deformation of the decorated curve (C, l) over the reduced base space T.

Then the blowup of $\mathbb{C}^2 \times T$ in $\Sigma\big(I(C_T, l_T)\big)$ is a deformation of $X(C, l)$ over T.

More specifically, I believe that the following conjecture 3.5.4 holds, which implies conjecture 3.5.3. Recall that we have defined the Hilbert-Samuel function of a fibre Y_t of a deformation of a zero-dimensional space as the finite sum of the Hilbert-Samuel functions of Y_t at the points of Y_t.

Conjecture 3.5.4. Let $Y \subset \mathbb{C}^r \times T$ be a deformation of the zero-dimensional space $Y_0 \subset \mathbb{C}^r$ over the reduced base space T. Assume the Hilbert-Samuel function $HS(Y_t)$ is independent of t for sufficiently small t.

Then the blowup of $\mathbb{C}^r \times T$ in Y is flat over T and the fibres of the blowup are equal to the blowup of $\mathbb{C}^r$ in Y_t for sufficiently small t.

3.5.3 Normal Flatness

For the rest of this section, I want to report on normal flatness and its relation to Hilbert functions. This is related to conjecture 3.5.4, because normal flatness would imply flatness of the blowup as we need it.

Loosely speaking, normal flatness of X along Y means that the family of tangent cones is flat over Y. Normal flatness was introduced by Hironaka in his proof of the existence of a resolution of singularities in characteristic zero in [Hir64]. He used it as a condition on the center of blowups which guarantees that the singularities on the blowup are 'not worse' than the singularities you start from, see also [Hir74, lecture 3] and [HSV77].

Definition 3.5.5. Let X be a complex analytic space, $Y \subset X$ a closed complex subspace, $i : Y \hookrightarrow X$ the natural inclusion and $\mathfrak{I}_Y \subset \mathcal{O}_X$ the ideal sheaf of Y. We use the notation

$$gr_Y(X) = i^* \left(\bigoplus_{d=0}^{\infty} \mathfrak{I}_Y^d / \mathfrak{I}_Y^{d+1} \right).$$

X is *normally flat* along Y if and only if the graded ring $gr_Y(X)$ is flat over its degree zero part $gr_Y(X)_0 \cong \mathcal{O}_Y$.

There are various criteria for normal flatness in terms of Hilbert functions. The first results were obtained by Bennett, see [Ben70] and also [HSV77].

For the resolution of singularities it is sufficient to consider normal flatness along smooth subspaces. If $x \in Y$ is a point on a smooth subspace $Y \subset X$, the following conditions are equivalent, see [Hir74, lecture 3].

1. X is normally flat along Y at x.

2. $gr_Y(X)$ is locally free at x.

3. The Hilbert-Samuel function of X in $y \in Y$ is locally constant around x.

Now our situation is different in two respects: We are interested in subspaces $Y \subset X$ which are not smooth and we are in a relative situation.

Relative normal flatness has been studied in [LJT74]. One of their results is the following theorem:

Theorem 3.5.6 (Lejeune-Jalabert, Teissier). *Let X be a complex space over S and Y a subspace of X which is smooth over S. The following conditions are equivalent:*

1. *$(gr_Y X)_y$ is $\mathcal{O}_{Y,y}$-flat.*

2. *The application $y \mapsto H_{X/S,y}$ is constant in a neighbourhood of y in Y.*

Normal flatness along arbitrary subspaces and similar notions as well as their connections with Hilbert functions have been studied by various authors including M. Herrmann, B. Herzog, U. Orbanz and L. Robbiano, see e.g. [OR84] and the references given there. Unfortunately, I was unable to use their results to prove the above conjecture.

Chapter 4

Multi-Adjacencies of Equisingularity Classes of Plane Curves

Deformations of a sandwiched singularity $X(C,l)$ are induced by deformations of the decorated curve (C,l). So we can reduce the study of adjacencies of sandwiched singularities to the presumably simpler study of adjacencies of plane curves.

In fact, our main focus is not on adjacencies between singularities but on smoothings of an $X(C,l)$. A generic smoothing of $X(C,l)$ corresponds to a δ-constant deformation of C into a curve which has only ordinary singularities. We can define the combinatorial type of such a deformation of C. We call this type a combinatorial deformation, see section 4.3. As with all combinatorial matters, the notations may seem to be rather complicated at first sight, even though the idea behind them is simple.

It is clear that smoothings over the same component of the base space of a semiuniversal deformation of $X(C,l)$ are induced by deformations of the same combinatorial type. This has been observed in [dJvS98]. It is an open question whether combinatorial deformations always distinguish components for a general sandwiched singularity.

The largest part of this chapter deals with finding necessary and sufficient conditions for the existence of δ-constant deformations of a plane curve singularity C into curves with prescribed singularities, especially into curves with prescribed ordinary singularities respectively a given combinatorial type. We deal with necessary conditions until section 4.3.2, with sufficient conditions in sections 4.3.4 and 4.4. In sections 4.5 and 4.6 we explain the connection with smoothing components of sandwiched singularities mentioned above.

4.1 Multi-Adjacencies

A multi-equisingularity class is a finite set of equisingularity classes with assigned multiplicities in $\mathbb{N}$.

Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated curve singularity and consider a 1-parameter deformation (C_t). Then for a sufficiently small representative and $t \neq 0$ the number and the equisingularity classes of the singularities of C_t are fixed, i.e. C_t has a fixed multi-equisingularity class.

We call $(C,0)$ *adjacent to a multi-equisingularity class* $\mathcal{E}$ if and only if there exists a 1-parameter deformation (C_t) of C such that C_t has class $\mathcal{E}$ for small $t \neq 0$.

We call an *equisingularity class* $\mathcal{E}$ of plane curve singularities *adjacent to a multi-equisingularity class* $\mathcal{E}'$ if and only if every $C \in \mathcal{E}$ is adjacent to $\mathcal{E}'$.

Example 4.1.1. This example is taken from [dJvS98, Ex. 6.4]: An ordinary singularity of multiplicity 6 is adjacent to the multi-equisingularity class consisting of four ordinary triple points and three ordinary double points if and only if the tangent directions of the six branches are paired by an involution of $\mathbb{P}^1$. Note that this is an example of a δ-constant deformations.

This example proves:

Proposition 4.1.2. *There do exist isolated plane curve singularities C and multi-equisingularity classes $\mathcal{E}'$ such that C is adjacent to $\mathcal{E}'$, but $\mathcal{E}(C)$ is not adjacent to $\mathcal{E}'$.*

Remark 4.1.3. In [ACR01], necessary and sufficient conditions for linear adjacency between equisingularity classes are given in terms of Enriques diagrams. Unfortunately, the method fails for non-linear adjacencies. Also, the paper does not deal with multi-adjacencies, but Roé has told me that he knows how to extend the method to give sufficient criteria for multi-adjacencies of equisingularity classes as well.

4.2 Simultaneous Normalization

The most important fact about δ-constant deformations of plane curves is that they admit a simultaneous normalization. This implies some very strong necessary conditions for δ-constant deformations. These conditions are well known, but we give a short proof using the theory of infinitely near points, because it fits nicely into the overall exposition of this thesis.

Let $f : (X,0) \to (T,0)$ be a deformation over a reduced base $(T,0)$. A *simultaneous normalization* is a proper modification $\pi : \tilde{X} \to X$ such that for each sufficiently small representative the following holds:

1. The composed map $f \circ \pi : \tilde{X} \to T$ is flat, i.e. a deformation.
2. For each $t \in T$ the restriction $\pi_t : \tilde{X}_t \to X_t$ to the fibre over t is a normalization.

For a deformation of isolated curve singularities a simultaneous normalization is the same as a very weak simultaneous resolution and we have the following result from [Tei80]:

Theorem 4.2.1. *A deformation of isolated curve singularities over a normal base space admits a simultaneous normalization if and only if it is δ-constant.*

Furthermore, if it exists, a simultaneous normalization $\pi : \tilde{X} \to X$ is necessarily the normalization of the total space X of the deformation.

The normalization of an isolated curve singularity C with r branches C_i is a disjoint union of r smooth germs $(\mathbb{C}, 0)$, each mapping onto one irreducible component. So the fact that a δ-constant deformation C_t has a simultaneous normalization implies that C_t also has r branches $C_{t,i}$ where $C_{t,i}$ is a deformation of C_i. Assume that the number of branches is > 1 and denote by C' the union of the branches $C_2, \ldots, C_r$.

Using the formula of Noether for the δ-invariant, lemma 1.2.20, we get

$$\begin{aligned}
2\delta(C_t) &= 2 \sum_{p \in \mathrm{Sing}(C_t)} \delta(C_{t,p}) \\
&= \sum_{p \in \mathrm{Sing}(C_t)} \sum_{q \in \mathcal{N}_p} e_q(C_{t,p})^2 - e_q(C_{t,p}) \\
&= \sum_{p \in \mathrm{Sing}(C_t)} \sum_{q \in \mathcal{N}_p} \big(e_q(C_{t,1,p}) + e_q(C'_{t,p})\big)^2 - \big(e_q(C_{t,1,p}) + e_q(C'_{t,p})\big) \\
&= \sum_{p \in \mathrm{Sing}(C_t)} \sum_{q \in \mathcal{N}_p} \big(e_q(C_{t,1,p})^2 - e_q(C_{t,1,p})\big) + \big(e_q(C'_{t,p})^2 - e_q(C'_{t,p})\big) \\
&\qquad\qquad + 2e_q(C_{t,1,p}) \cdot e_q(C'_{t,p}) \\
&= \sum_{p \in \mathrm{Sing}(C_t)} 2\delta(C_{t,1,p}) + 2\delta(C'_{t,p}) + 2\langle C_{t,1,p}, C'_{t,p}\rangle \\
&= 2\delta(C_{t,1}) + 2\delta(C'_t) + 2\langle C_{t,1}, C'_t\rangle.
\end{aligned}$$

By semicontinuity of the δ-invariant and the intersection multiplicity, this proves the following well known corollary by induction:

Corollary 4.2.2. *Let $(C, 0)$ be an isolated plane curve singularity. A deformation C_t of $C = C_0$ over a reduced base is δ-constant if and only if the following holds: The number of branches and the pairwise intersection multiplicities of the branches are constant in t, and each branch is deformed δ-constant.*

4.3 Combinatorial Restrictions

4.3.1 Combinatorial Representations and Deformations

Corollary 4.2.2 gives several restrictions on the possible adjacencies of equisingularity classes. We introduce some notation to handle these.

Definition 4.3.1. 1. Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated curve singularity with r numbered branches $C_1,\dots,C_r$. The *combinatorial representation* of the equisingularity class of C is the map $\phi_C : \mathbb{N}_0^r \mapsto \mathbb{N}_0$ with

$$\phi_C(x) := \#\{p \in \mathcal{N}_0|\ e_p(C) > 1 \text{ and } x = (e_p(C_1),\dots,e_p(C_r))\}.$$

2. Let (C,l) be a decorated curve. The *combinatorial representation* of the equisingularity class of (C,l) is the map $\phi_{C,l} : \mathbb{N}_0^r \to \mathbb{N}_0$ with

$$\phi_{C,l}(x) := \#\{p \in |BP(C,l)|\ |\ x = (e_p(C_1),\dots,e_p(C_r))\}.$$

Example 4.3.2. Let C be a smooth curve. Then

$$\phi_C \equiv 0, \qquad \phi_{C,k}(1) = k, \qquad \phi_{C,k}(x) = 0 \quad \forall x \neq 1.$$

Example 4.3.3. Let C be an A_2-singularity. Then

$$\begin{aligned} &\phi_C(2) = 1, && && \phi_C(x) = 0 \quad \forall x \neq 2,\\ &\phi_{C,k}(2) = 1, && \phi_{C,k}(1) = k-2, && \phi_C(x) = 0 \quad \forall x \notin \{1,2\}. \end{aligned}$$

Example 4.3.4. Let C be an A_1-singularity. Then

$$\begin{aligned} &\phi_C\big((1,1)\big) = 1, && \phi_C(x) = 0 \quad \forall x \neq (1,1),\\ &\phi_{C,(a,b)}\big((1,1)\big) = 1, && \phi_{C,(a,b)}\big((1,0)\big) = a-1,\\ &\phi_{C,(a,b)}\big((0,1)\big) = b-1, && \phi_{C,(a,b)}(x) = 0 \quad \forall x \notin \{(1,1),(1,0),(0,1)\}. \end{aligned}$$

Calling ϕ_C and $\phi_{C,l}$ combinatorial representations of equisingularity classes is justified by the following obvious lemma. As usual, we let the symmetric group S_r act on $\mathbb{N}_0^r$ by permutations of the standard basis.

Lemma 4.3.5. *Let $(C_i,0) \subset (\mathbb{C}^2,0)$ be two isolated curve singularities. Then there exists a permutation $\sigma \in S_r$ such that $\phi_{C_1} = \phi_{C_2} \circ \sigma$ if and only if $(C_1,0)$ is equisingular to $(C_2,0)$.*

Let (C_i,l_i) be two decorated curves. Then there exists a permutation $\sigma \in S_r$ such $\phi_{C_1,l_1} = \phi_{C_2,l_2} \circ \sigma$ if and only if (C_1,l_1) is equisingular to (C_2,l_2).

We also note:

$$\begin{aligned} \phi_C(x) &= \phi_{C,l}(x) && \forall x \text{ with } |x| \neq 1. \\ \phi_C(x) &= 0 && \forall x \text{ with } |x| = 1. \\ \phi_{C,l}(e_k) &= l_C(C_k) - m_C(C_k), \end{aligned}$$

where e_k is the k-th unit vector in $\mathbb{N}_0^r$, i.e. $e_{k,i} = \delta_{k,i}$.

Now consider δ-constant deformations C_t and deformations of decorated curves (C_t, l_t). Since each branch of C is deformed separately, we have well-defined maps

$$\begin{aligned} \phi_{C_t} : \mathbb{N}_0^r &\to \mathbb{N}_0 \\ x &\mapsto \sum_{q \in Sing(C_t)} \#\{p \in \mathcal{N}_q |\ e_p(C_t) > 1 \text{ and } x = (e_p(C_{t,1}), \ldots, e_p(C_{t,r}))\} \\ \phi_{C_t,l_t} : \mathbb{N}_0^r &\to \mathbb{N}_0 \\ x &\mapsto \sum_{q \in n(Supp(l_t))} \#\{p \in |BP(C_{t,q}, l_{t,q})|\ |\ x = (e_p(C_{t,1}), \ldots, e_p(C_{t,r}))\} \end{aligned}$$

for all sufficiently small $t \neq 0$. If the deformation is a 1-parameter-deformation, the map is independent of t.

Examples of combinatorial deformations are given in the next section.

Remark 4.3.6. Unfortunately, the combinatorial deformation of a 1-parameter deformation C_t does not determine the (multi)-equisingularity class of a generic fibre C_t, $t \neq 0$. One of the reasons for this is, of course, that we do not treat each singularity of a fibre C_t separately. But that is not the main problem: Even if we did not sum over all singular points on C_t, but defined maps $\mathbb{N}_0^r \to \mathbb{N}_0$ for each singular point q of C_t separately, these could not determine the equisingularity type of C_t at q in general, because $C_{t,i}$ might be locally reducible at q. For example, $C_t = V(x^2 + y^3 - ty^2)$ is a deformation of the irreducible A_2-singularity into an ordinary double point A_1 which locally has two branches. But since globally the two branches are on the same curve, the maps ϕ_{C_t} do not detect the difference between the A_2- and the A_1-singularity.

We deal with this problem by considering only deformations into a curve which has only ordinary singularities. This is the case we are most interested in anyway, because our main application is the study of smoothing components in section 4.6, and generic smoothings are induced by deformations into ordinary singularities. In this case, the maps ϕ_{C_t} and ϕ_{C_t,l_t} carry even more information than just the multi-equisingularity type: They also tell us which branches pass through which of the singular points.

The main task now is to find a good characterization of the set of maps $\mathbb{N}_0^r \to \mathbb{N}_0$ which occur as ϕ_{C_t} or ϕ_{C_t,l_t} for a given singularity $(C,0)$ or decorated curve (C,l). The existence of the Scott deformation proved in section 4.4.1 shows that if $\psi = \phi_{C_t}$ or ϕ_{C_t,l_t} for some deformation, then we can choose C_t or (C_t,l_t) such that C_t has only ordinary singularities.

Proposition 4.2.2 implies the following restrictions on ϕ_{C_t} and ϕ_{C_t,l_t}:

Corollary 4.3.7. *1. Let C_t be a δ-constant 1-parameter-deformation of an isolated plane curve singularity C with r branches C_i. For sufficiently small t, the map $\phi_{C_t} : \mathbb{N}_0^r \to \mathbb{N}_0$ satisfies:*

(a) $\sum_x \phi_{C_t}(x) \cdot \big(x_i^2 - x_i\big) = 2\delta(C_i) \qquad \forall i \in \{1,\dots,r\}.$

(b) $\sum_x \phi_{C_t}(x) \cdot \big(x_i \cdot x_k\big) = \langle C_i, C_k\rangle \qquad \forall i \neq k.$

2. Let (C_t,l_t) be a 1-parameter-deformation of a decorated curve with r branches. For sufficiently small t, the map $\phi_{C_t,l_t} : \mathbb{N}_0^r \to \mathbb{N}_0$ satisfies conditions (a) and (b) above and

(c) $\sum_x \phi_{C_t,l_t}(x) \cdot x_i = l(i) \qquad \forall i \in \{1,\dots,r\}.$

Definition 4.3.8. 1. A *combinatorial deformation* of a decorated curve (C,l) with r branches is a map $\phi : \mathbb{N}_0^r \to \mathbb{N}_0$ satisfying properties (a), (b) and (c) above as well as

(d) $\sum_x \phi(x) \leq \#|BP(C,l)|$.

2. A *combinatorial deformation* of a plane, isolated curve singularity C with r branches is a map $\phi : \mathbb{N}_0^r \to \mathbb{N}_0$ satisfying properties (a) and (b) above such that altering the values $\phi(x)$ of the vectors $x \in \mathbb{N}_0^r$ with $|x| = 1$ gives a combinatorial deformation of a decorated curve (C,l) for some divisor l.

3. We say that a combinatorial deformation $\phi : \mathbb{N}_0^r \to \mathbb{N}_0$ *can be realized* if and only if there exists a 1-parameter-deformation C_t respectively (C_t,l_t), where C_t has only ordinary singularities, with $\phi = \phi_{C_t}$ respectively $\phi = \phi_{C_t,l_t}$.

4. We say that the combinatorial deformation ϕ contains n points through which the branches C_i pass with multiplicity x_i if and only if $\phi(x) = n$.

So a combinatorial deformation is nothing but an a priori possible δ-constant adjacency to a multi-equisingularity class of ordinary singularities. We have seen that the first three conditions are necessary. The necessity of condition (d) is given by theorem 3.4.8.

We give examples of combinatorial deformations in section 4.3.2 below.

Remark 4.3.9. In [dJvS98] combinatorial deformations are defined in a similar way. The only difference is that they did not impose condition (d) and that they used a different notation.

Note that conditions (a)–(c) imply condition (d) if all branches of C are smooth.

Remark 4.3.10. The idea behind our definition of a combinatorial deformation is the following: ψ should be a combinatorial deformation of C if and only if ψ can be realized as a combinatorial deformation of some curve C' that is equisingular to C. Our examples below show that this is the best we can hope for. But they also show that there are examples of combinatorial deformations of a curve C which cannot be realized for any C' which is equisingular to C. So we regard our definition as being preliminary only. A more detailed discussion of this problem is in the next section, 4.3.2.

By definition, the set of combinatorial deformations of a decorated curve (C, l) is a subset of the set of combinatorial deformations of C. One should note:

Lemma 4.3.11. *A combinatorial deformation ϕ of (C, l) can be realized if and only if it can be realized as a combinatorial deformation of C.*

We know that certain classes of combinatorial deformations can always be realized, see theorem 4.3.27 and section 4.4.

Note that a combinatorial deformation is a map with only finitely many values unequal zero.

Notation 4.3.12.

1. We write maps $\phi : \mathbb{N}_0^r \to \mathbb{N}_0$ as formal products
$$\prod_{x \in \mathbb{N}_0^r} x^{\phi(x)}.$$

2. If all components of $x \in \mathbb{N}_0^r$ are zero or one, then we write $(i_1, \ldots, i_s)$ for x, where $i_1, \ldots, i_s$ are the indices of the non-zero components.

   More generally, we denote a vector $x \in \mathbb{N}_0^r$ by $(1^{(x_1)}, \ldots, r^{(x_r)})$, skipping the numbers with $x_i = 0$ and omitting exponents equal to (1).

3. If $\phi, \psi : \mathbb{N}_0^r \to \mathbb{N}_0$ are two maps, then we write
$$\phi \to \psi$$
if ψ is a combinatorial deformation of a curve with combinatorial representation ϕ.

4.3.2 Examples and Missing Restrictions

Example 4.3.13. Every plane isolated curve singularity C hat at least one combinatorial deformation, namely its combinatorial representation ϕ_C. It can always be realized by a Scott deformation, see proposition 4.4.1.

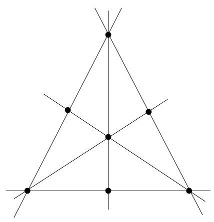
Example 4.3.14. An ordinary triple point has combinatorial representation (123). It has two combinatorial deformations, (123) and (12)(13)(23). Both can be realized.

Example 4.3.15. An A_{2k}-singularity has combinatorial representation $(1^{(2)})^k$, which is its only combinatorial deformation. Without condition (d) there would be more combinatorial deformations for $k \geq 3$, for example $(1^{(2)})^3 \to (1^{(3)})$, which would mean an adjacency $A_6 \to D_4$. Of course, it is obvious that A_6 does not deform into D_4, because A_6 has multiplicity 2 and D_4 has multiplicity 3. Note that the non-existence of the adjacency $A_6 \to D_4$ cannot be seen from the semicontinuity of the singularity spectrum.

Example 4.3.16. Example 4.1.2 can now be written as follows: An ordinary 6-fold point has the combinatorial deformation

$$(123456) \to (126)(135)(234)(456)\ (14)(25)(36).$$

This combinatorial deformation can only be realized, if the six tangent directions of the ordinary 6-fold point are paired by an involution of $\mathbb{P}^1$ as follows: $1 \leftrightarrow 4, 2 \leftrightarrow 5, 3 \leftrightarrow 6$. A picture of a deformation with this combinatorial deformations looks like this:



Example 4.3.17. This example is taken from [dJvS98, Ex. 4.20]. Set $C = V(x^4 + y^5)$. The decorated curve $(C, 9)$ has combinatorial representation $(1^{(4)})(1)^5$. There are three combinatorial deformations: $(1^{(4)})(1)^5$, $(1^{(3)})(1^{(2)})^3$ and $(1^{(3)})^2(1)^3$. The last combinatorial deformation, $(1^{(3)})^2(1)^3$, represents a deformation into two ordinary triple points and cannot be realized even if we are allowed to replace C by an arbitrary singularity $C' \in \mathcal{E}(C)$.

One way of proving this is the following: Assume that we have a deformation into two D_4-singularities. Without loss of generality we can assume that the two D_4-singularities are on the line $V(y)$, say at zero and at $(t, 0)$. Let J be the induced deformation of the conductor. Then $J_0 = (x, y)^3$ and $J_{t,0} = (x, y)^2$, $J_{t,(t,0)} = (x - t, y)^2$. So $y^2 \in J_t$ for all $t \neq 0$, but $y^2 \notin J_0$.

Remark 4.3.18. I believe that example 4.3.17 is typical in that it is a deformation into few singular points with high multiplicities. Supporting evidence for this is a result of Alexander and Hirschowitz proven in [AH00]: Suppose you have a fat point in $(\mathbb{C}^2, 0)$ defined by $(x, y)^N$ and r numbers n_i such that $N^2 + N = \sum_i n_i^2 + n_i$. Then a priori there could be a deformation of the fat point $\Sigma((x, y)^N)$ into r fat points of the form $\Sigma((x, y)^{n_i})$, but there are cases where such a deformation does not exist; e.g. we used the fact that the fat point defined by $(x, y)^3$ does not deform into two points of the form $\Sigma((x, y)^2)$ in example 4.3.17. The result in [AH00] says that such a deformation does exist if N is big and the number r of points is much bigger than the maximum of the n_i.

The last two examples show that there are combinatorial deformations which cannot be realized. But in the example with smooth branches, at least there did exist some curves in the equisingularity class for which the combinatorial deformation could be realized. I do not know any example of a combinatorial deformation of a curve C with smooth branches which cannot be realized for any curve in the equisingularity class $\mathcal{E}(C)$.

Question 4.3.19. Does there exist a combinatorial deformation of a plane curve singularity C with smooth branches which cannot be realized for any $C' \in \mathcal{E}(C)$?

If the answer is yes, then we say that we are 'missing some restrictions to combinatorial deformations' of curves with smooth branches. We already know, see example 4.3.17, that we are missing restrictions to combinatorial deformations in general.

Question 4.3.20. What are 'the' missing restrictions to combinatorial deformations of curves, especially curves with non-smooth branches?

Remark 4.3.21. A well known restriction to the existence of multi-adjacencies is provided by the semicontinuity of the singularity spectrum. The spectrum of an isolated hypersurface or complete intersection singularity does not change under μ-constant deformations; so in particular, the spectrum of an isolated plane curve singularity only depends on its equisingularity class, see [Ste85]. An algorithm to compute the spectrum from an Enriques diagram of an irreducible plane curve singularity has been given by M. Saito, see [Sai00].

Another well known restriction to the existence of multi-adjacencies is given by the semicontinuity of the multiplicity. We can generalize this to a statement which includes multiplicities at infinitely near points.

4.3.3 Semicontinuity of Multiplicity at Infinitely Near Points

One of the restrictions that is missing is given by the fact that the multiplicity is upper-semicontinuous. We can easily prove a generalization which includes multiplicities at infinitely near points. We begin by stating some trivial facts:

- The multiplicity of a curve C does not increase when blowing up points, so $e_p(C) \geq e_q(C)$ for all $p \prec q$. In particular, if we define

  $$x \geq y \Leftrightarrow x_i \geq y_i \quad \forall i \in \{1, \ldots, r\}$$

  for vectors $x, y \in \mathbb{N}_0^r$, then $(e_0(C_1), \ldots, e_0(C_r))$ is the maximum of $\{x \in \mathbb{N}_0^r | \ \phi_C(x) \neq 0\}$.

- Assume that C_t is a deformation of C into an ordinary singularity of the same multiplicity as C. Semicontinuity of multiplicity then implies that the multiplicity is constant on each branch.

  This implies the following: Assume that C_t is a δ-constant deformation of C into ordinary singularities such that one of the ordinary singularities has the same multiplicity as C. Then $\phi_{C_t}\big(e_0(C_1), \ldots, e_0(C_r)\big) > 0$.

- Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated plane curve singularity. Let $\phi_1, \ldots, \phi_k$ be the combinatorial representations of the strict transforms of C under the point-blowup of $\mathbb{C}^2$ in zero. Then the combinatorial representation of C is the sum $\phi_C = (e_0(C_1), \ldots, e_0(C_r)) + \sum_{j=1}^{k} \phi_j$.

  $\sum_{j=1}^{k} \phi_j$ is the finest partition of $\phi_C - (e_0(C_1), \ldots, e_0(C_r))$ with the following property: If $i \neq j$ and $x, y \in \mathbb{N}_0^r$ with $\phi_i(x) \neq 0$ and $\phi_j(y) \neq 0$, then x_k or y_k is zero for all $k \in \{1, \ldots, r\}$.

We leave the proof of the following easy lemma to the reader. The main arguments are similar to those in the proof of the existence of Scott deformations, theorem 4.4.1.

Lemma 4.3.22. *Let C_t be a δ-constant deformation of the isolated plane curve singularity C into ordinary singularities. Assume that one of the ordinary singularities has the same multiplicity as C. Without loss of generality we assume that this ordinary singularity is at zero, i.e. that the multiplicities $e_0(C_{t,i})$ of the branches in $0 \in \mathbb{C}^2$ is constant under the deformation.*

Then C_t induces induces δ-constant deformations of the strict transforms of C under the point-blowup of $\mathbb{C}^2$ in zero. These induced deformations are deformations into ordinary singularities such that the deformed transforms intersect the exceptional divisor transversally.

Let $\psi_1, \ldots, \psi_k$ be the combinatorial deformations of the induced deformations of the strict transforms. Then the combinatorial deformation ϕ_{C_t} is the sum $(e_0(C_1), \ldots, e_0(C_r)) + \sum_{j=1}^{k} \psi_j$.

We have now proven the following:

Theorem 4.3.23 (Semicontinuity of Multiplicities). *Let C_t be a δ-constant deformation of the isolated plane curve singularity C into ordinary singularities. Let the ϕ_j be the combinatorial representations of the strict transforms of C under a point-blowup. Note that the ϕ_j only depend on the combinatorial representation of ϕ_C of C and that $\phi_C = (e_0(C_1), \ldots, e_0(C_r)) + \sum_{j=1}^{k} \phi_j$.*

Then the following holds for the combinatorial deformation ϕ_{C_t}:

1. $\phi_{C_t}(x) = 0 \quad \forall x \in \mathbb{N}_0^r$ *with* $|x| > \sum e_0(C)$.
2. $\phi_{C_t}(x) = 0 \quad \forall x \in \mathbb{N}_0^r$ *with* $|x| = \sum e_0(C)$ *and* $x \neq (e_0(C_1), \ldots, e_0(C_r))$.
3. *If* $\phi_{C_t}\big((e_0(C_1), \ldots, e_0(C_r))\big) \neq 0$, *then* $\phi_{C_t} = (e_0(C_1), \ldots, e_0(C_r)) + \sum_{j=1}^{k} \psi_j$, *where ψ_j is a combinatorial deformation of ϕ_j which can be realized. In particular, the statements of this theorem hold for the ψ_j.*

Remark 4.3.24. This does not give additional restrictions for combinatorial deformations of a curve with smooth branches.

Remark 4.3.25. For curves with non-smooth branches this does give additional restrictions, but example 4.3.17 shows that these are still not all.

4.3.4 Results Proved Via Sandwiched Singularities

The basic idea from [dJvS98], which we have pursued further in this thesis, is to use easy-to-prove facts about plane curve singularities to obtain results on sandwiched surface singularities. But sometimes we can also go the other way, which is what we did for example in the classification of taut and pseudotaut plane curve singularities. In this section we mention two more results obtained in this way.

We cite the following theorem from [dJvS98, th. 4.16]:

Theorem 4.3.26. *If $X(C, l)$ is isomorphic to $X(C', l')$ and all branches of C and C' are smooth, then there is a natural bijection from the set of combinatorial deformations of (C, l) to the set of combinatorial deformations of (C', l').*

The proof uses elementary combinatorics to associate to each combinatorial deformation a so-called Γ-representation of the dual resolution graph of $X(C,l)$. For details we refer to [dJvS98]. The article also gives an example that shows that the condition that all branches of C and C' are smooth cannot be dropped. But in the example, the combinatorial deformation that is 'excluded from the bijection' is the one from example 4.3.17 that cannot be realized. So we might hope that we will always get a bijection if we are able to find all necessary restrictions to combinatorial deformations.

In general, a sandwiched singularity with reduced fundamental cycle will have different representations $X(C,l)$ and $X(C',l')$ with both C and C' having smooth branches and such that C and C' are not equisingular. If a combinatorial deformation of C cannot be realized, it seems to be quite mysterious why there should be a corresponding combinatorial deformation of C'. So the theorem makes it seem rather likely that all combinatorial deformations of curves C with smooth branches can be realized, at least for some D equisingular to C. This has only been proven for some special cases. One case is proven in this thesis: Every combinatorial deformation of a curve with up to four smooth branches can be realized, see theorem 4.4.16. The other class of examples I know is given by [dJvS98, theorem 6.18]:

Theorem 4.3.27. *If $X(C,l)$ is a cyclic quotient singularity and all branches of C are smooth, then every combinatorial deformation of (C,l) can be realized.*

4.4 Existence of Special Deformations

In this section we show how to construct δ-constant 1-parameter-deformations of plane curves such that the general fibre has certain prescribed singularities. Essentially, the technique is to blow up, translate strict transforms and to blow down again. The method is quite old, it dates back to the work of Ch. Scott in the late nineteenth century. Many mathematicians have used this or similar techniques, including e.g. S. Gusein-Zade in [GZ74] and N. A'Campo in [A'C75].

An important principle when searching for multi-adjacencies of isolated singularities is the so-called openness of versality, proved in great generality in [Bin80]. It implies that if we have an adjacency from S to a multi-class of singularities $(S_i)_i$ and adjacencies from S_i to $(S_{ij})_j$, then we have an adjacency from S to $(S_{ij})_{i,j}$.

4.4.1 Scott Deformations

Theorem 4.4.1. *Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated plane curve singularity and $p_1,\dots,p_r \in \mathcal{N}_0$ the infinitely near points where C has multiplicity greater than one.*

Then there is a 1-parameter-deformation C_t of C such that C_t, for $t \neq 0$, has r ordinary singularities with multiplicities $e_{p_1}(C),\dots,e_{p_r}(C)$ and no other singularities.

Remark 4.4.2. We call such deformations *Scott deformations* after the work of Charlotte A. Scott, who constructed these deformations in the 1892-article [Sco92]; see also [Sco93]. Her motivation was to give a geometrical meaning to M. Noethers purely algebraically deduced formula $\delta(C) = \sum_{i=1}^{r} \frac{1}{2} e_{p_i}(C)(e_{p_i}(C)-1)$.
It is also quite common to call them *deformations which make the infinitely near points visible.* Some mathematicians, especially Russians, call these deformations *sabirfications* after the work of Sabir Gusein-Zade.

Remark 4.4.3. We can rephrase the proposition as follows: Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated plane curve singularity with combinatorial representation ϕ_C. Then there is a deformation C_t into ordinary singularities with $\phi_{C_t} = \phi_C$.

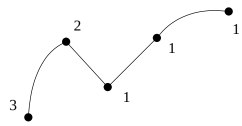
Proof. The proof is elementary. We successively blow up singular points on the curve until we get to an infinitely near point p such that the strict transform in p has smooth branches and such that the curve has multiplicity not greater than 1 in all points proximate to p. Then we move the singularity away from the exceptional divisor, which is to say that we deform the strict transform by translating it, such that the ordinary singularity is not on the exceptional divisor any more and such that the intersections of the strict transform with the exceptional divisor are all transversal. Since we only translate the strict transform, the intersection multiplicity with each component of the exceptional divisor stays constant by the Bézout theorem. So the deformation blows down to a deformation of the curve. We have thus succeeded in splitting off an ordinary singularity with multiplicity $e_p(C)$.

Furthermore, if p is in the k-th infinitely near neighbourhood and q is the point in the $(k-1)$-st infinitely near neighbourhood to which p is proximate, then the strict transforms at q of the deformations of the branches which pass through p form an ordinary singularity, because we arranged their strict transforms after the blowup in q to intersect the exceptional divisor E_q transversally. Induction on the number of infinitely near points in an Enriques diagram of C finishes the proof. ☐

Remark 4.4.4. Note that the proof is constructive. The construction involves nothing more than translations and blowing up, which makes the computations so easy that we don't even need to use computer algebra to construct interesting examples.

Lemma 4.3.22 shows that all Scott deformations can be constructed by successive blowups and translations as in the proof.

Example 4.4.5. The E_8-singularity has the following Enriques diagram:



So a Scott deformation is a deformation into an ordinary triple and an ordinary double point. The construction is shown in table 4.1. As can be seen directly from the construction, the ordinary triple point of the deformed curve is at $(0,0) \in \mathbb{C}^2$ and the ordinary double point is at $(-t,0) \in \mathbb{C}^2$.

Corollary 4.4.6. *Let (C,l) be a decorated curve. Then there exists a deformation (C_t,l_t) such that l_t is a reduced divisor on the normalization of C_t for all $t \neq 0$. It follows that C_t has only ordinary singularities.*

Proof. Obviously, curves with ordinary singularities are the only curves with the property that a reduced divisor l on the normalization may define a decorated curve. We choose C_t to be a Scott deformation. On each branch, the sum of the multiplicities in the singular points stays constant under a Scott deformation, so we can deform the divisor l such that (C_t,l_t) is a deformation of the decorated curve. □

Corollary 4.4.7. *Let $(C,0) \subset (\mathbb{C}^2,0)$ be an isolated plane curve singularity.*

Then there is a 1-parameter-deformation C_t of C such that C_t, for $t \neq 0$, has $\delta(C)$ ordinary double points as its only singularities.

Proof. The statement is trivial for ordinary singularities, so the corollary follow from the existence of Scott deformations by openness of versality. □

Remark 4.4.8. Note that the statement is not true for decorated curves, because the divisor l might be too small. For example, let C be an ordinary singularity with four branches and $l(C_i) = 2$, for $i = 1,2,3,4$. If we deform C into $\delta = 6$ ordinary double points, then there are three such points on each branch, but the degree of a deformation l_t of l on each branch would have to be 2. Contradiction!

Corollary 4.4.7 implies the following well known fact:

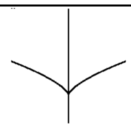
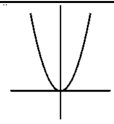
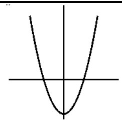
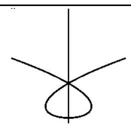
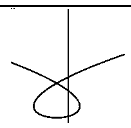
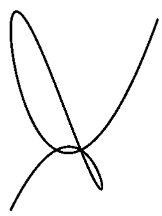
E_8-singularity	$x^5 - y^3$	
1st blowup	$x^3(x^2 - y^3)$	
2nd blowup	$y^2x^3(x^2 - y)$	
translation	$y^2x^3(x^2 - y - 2t)$	
blowing down	$x^3(x^2 - y^3 - 2ty^2)$	
translation	$x^3\big((x+t)^2 - y^3 - 2ty^2\big)$	
blowing down	$x^3(x+t)^2 - y^3 - 2txy^2$	

Table 4.1: Scott deformation of E_8-singularity

Corollary 4.4.9. *A generic δ-constant deformation of a plane curve singularity is a deformation into δ ordinary double points. This means that there is a Zariski-open subset of the δ-constant stratum, such that deformations over this subset are deformations into δ ordinary double points.*

4.4.2 'Cutting Enriques Graphs'

Of course, the principle that blowing up and translating strict transforms gives rise to δ-constant deformations produces not only the Scott deformation but also a lot of others. A Scott deformation is a generic such deformation, because moving singularities away from the exceptional divisors and having only transversal intersections is generic. Looking at an Enriques diagram of the curve, we see that a Scott deformation 'cuts the Enriques diagram into pieces', each piece consisting of just one vertex. In general, we can get the following result for singularities with smooth branches, which we will need for the proof of theorem 4.4.16.

By slight abuse of notation, we might also call a map $\phi : \mathbb{N}_0^r \to \mathbb{N}_0$ a combinatorial representation of a curve singularity with fewer than r branches if for some indices $i \in \{1, \ldots, r\}$ we always have $\phi(x) = 0$ if $x_i \neq 0$.

Theorem 4.4.10. *Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated curve singularity with smooth branches and combinatorial representation $\phi_C : \mathbb{N}_0^r \to \mathbb{N}_0$. If $\phi_C = \sum \phi_i$ and $\phi_i : \mathbb{N}_0^r \to \mathbb{N}_0$, for $i = 1, \ldots, k$, are combinatorial representations of singularities $(C_i, 0)$ with up to r (necessarily smooth) branches, then $\mathcal{E}(C)$ is multi-adjacent to $(\mathcal{E}(C_1), \ldots, \mathcal{E}(C_k))$.*

Remark 4.4.11. Of course, we can also use blowing up and translations of the strict transforms to construct deformations for singularities with non-smooth branches. But in that case, the use of combinatorial representations is not appropriate for a statement of the result; just think of the deformation $A_2 \to A_1$, cf. remark 4.3.6.

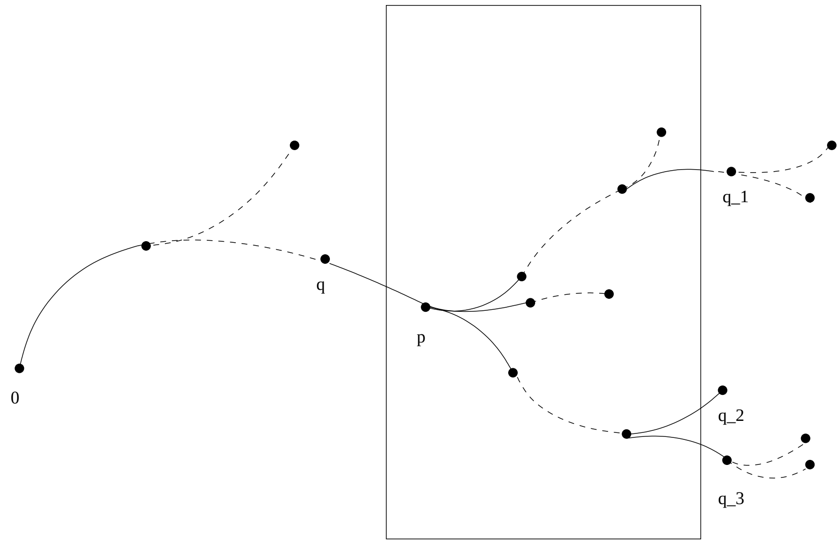
Proof. The proof is essentially the same as the proof of the existence of Scott deformations. I start with some preliminary remarks:

1. Each summand ϕ_i of $\phi_C = \sum \phi_i$ corresponds to an essentially unique subset of the set of vertices of an Enriques diagram of C. For this reason we speak of splitting or cutting the Enriques diagram into pieces.

2. There exists a singularity $(C_i, 0)$ such that ϕ_i is a combinatorial representation of $(C_i, 0)$ if and only if the subset of vertices corresponding to ϕ_i contains a vertex p such that all other vertices of the subset are in the unique maximal subdiagram with root p.

3. It is sufficient to show that we can split the Enriques diagram into two pieces. The general case follows by induction.

First assume that we want to split off a singularity corresponding to a connected subdiagram Γ_{C_2} of the Enriques diagram Γ_C. Let $p \neq 0$ be the root of Γ_{C_2}, $p \to q$ and $\{0, q_1, \dots q_r\}$ the roots of the connected components of $\Gamma_C \setminus \Gamma_{C_2}$. In the picture, Γ_{C_2} is the subdiagram in the box:



As in the proof of the existence of a Scott deformation, we can certainly do the following. We can blow up until we get to the point p on E_q and translate the strict transform through p to split off a singularity whose Enriques diagram is the maximal subdiagram of Γ_C with root p. Then we can blow up further to translate the strict transforms through the q_i. In this way we have split off a singularity with Enriques diagram Γ_{C_2}, but without wanting it we have also dissected the rest into $r+1$ singularities, whose Enriques diagrams are the connected components of $\Gamma_C \setminus \Gamma_{C_2}$.

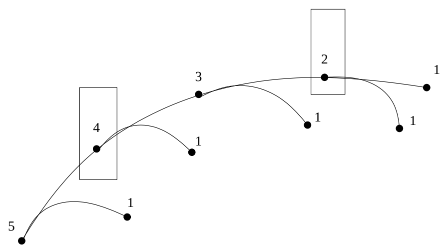
So what we need is the following: When pushing the singularity through p off the exceptional divisor E_q while at the same time splitting off the singularities in the points q_i, we have to arrange for the singularities through the q_i to be translated in such a way that they are always on E_q for each value of the deformation parameter t.

To see that this is possible, consider the total transform of E_q under the blowups leading from p to q_i. This total transform contains all the exceptional divisors of the point blowups. When pushing the singularity through p off the exceptional divisor (or equivalently when translating E_q such that p is no longer on E_q), the total transform deforms in a flat way.

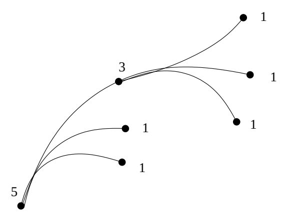
Since the singularities through the q_i are on the total transform of E_q for $t = 0$, we can translate them in such a way that they are on E_q for all t.

The situation is demonstrated in table 4.2.

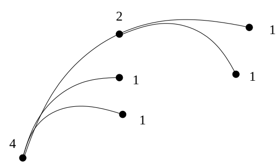
The proof for the case that ϕ_2 corresponds to a subgraph of the Enriques diagram which is not connected follows by induction. I believe that the best way to demonstrate this is to give an example. Consider a singularity with combinatorial representation (12345)(1234)(123)(12). It has the following Enriques diagram:



We choose $\phi_2 = (1234)(12)$, i.e. we want to split off a singularity corresponding to the two boxed vertices. In other words, we are trying to construct a deformation into two singularities with Enriques diagrams



and



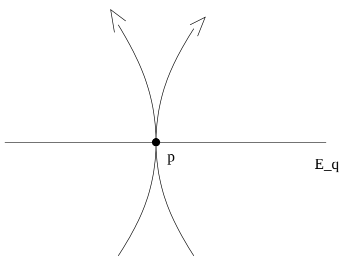
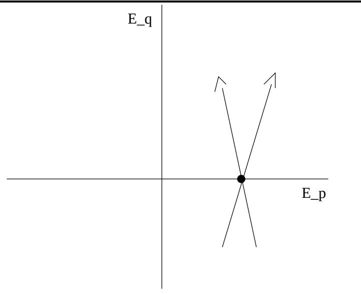
The first step is to deform the singularity on the first blowup, which has combinatorial deformation (1234)(123)(12) into two singularities with combinatorial representations (1234)(12) and (123). This can be done, because (123) corresponds to a connected subdiagram. Now for the second step the situation is as above: When splitting off the ordinary triple point with representation (123), we have to move the singularity of type (1234)(12) away from the exceptional divisor E_0 while letting the ordinary triple point stay on E_0. Blowing down gives the deformation we are looking for.

For example, one equation of a singularity with combinatorial representation (12345)(1234)(123)(12) and the above Enriques diagram is

$$y(y + x^4)(y + x^3)(y + x^2)(y + x).$$

The reader is invited to check that the construction leads to the deformation given by the following equation (I have used the translations $x \mapsto x \pm t$):

$$y \cdot \big(y + x^2(x + t)^2\big) \cdot \big(y + x^2(x + t)\big) \cdot \big(y + x(x + t)\big) \cdot \big(y + x\big).$$

<table>
<tr><th></th><th>blowup in q</th><th>blowup in p</th></tr>
<tr><td></td><td>
</td><td>
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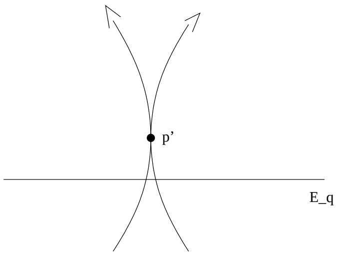
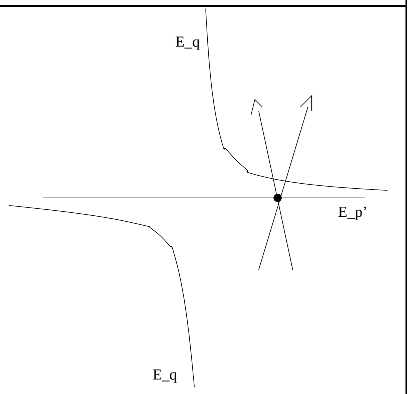
<tr><td>translating the singularity in p</td><td>
</td><td>
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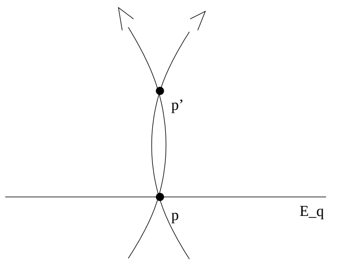
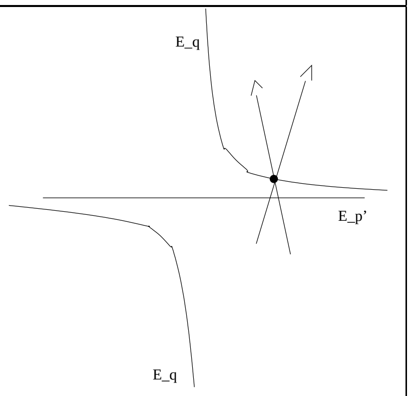
<tr><td>translating the singularity in q_i</td><td>
</td><td>
</td></tr>
</table>

Table 4.2: A schematic demonstration of the construction principle

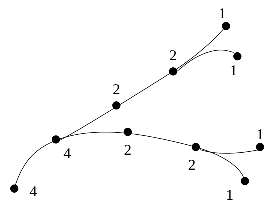
As can be easily seen, for $t \neq 0$ the singularity of type (12345)(123) is at zero and the singularity of type (1234)(12) is at the point given by $x + t = 0$ and $y = 0$. □

Example 4.4.12. The combinatorial representation of A_{2k-1} is $(1,1)^k$. So A_{2k-1} is multi-adjacent to $(A_{2k_1-1}, \ldots, A_{2k_s-1})$ with $\sum_{i=1}^{s} k_i = k$. These are all δ-constant multi-adjacencies of A_{2k-1}.

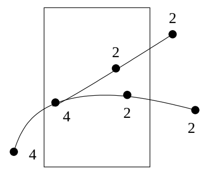
Remark 4.4.13. The A_k-singularities are the only singularities with the property that all δ-constant deformations can be obtained by cutting Enriques graphs. For all other curve singularities, a deformation into δ double points cannot be constructed with this technique.

Example 4.4.14. The combinatorial representation of D_{4+2k} is $(1,1,1)\cdot(1,1,0)^k$ So D_{4+2k} is multi-adjacent to $(D_{4+k_0}, A_{2k_1-1}, \ldots, A_{2k_s-1})$ with $\sum_{i=0}^{s} k_i = k$.

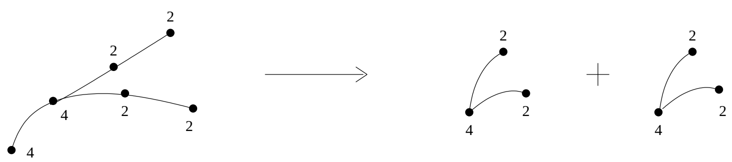
Example 4.4.15. Consider $C = V\big(x(x+y^4)(x-y^2)(x-y^2+y^4)\big)$. The combinatorial type is $(1234)^2(12)^2(34)^2$. An Enriques diagram looks like this:



From now on, I will always omit the points of multiplicity one. Let us cut out the following subdiagram:



So we want to find a deformation into two singularities of type (1234)(12)(34), which we picture like this:



The construction of the deformation is shown in table 4.3. As can be seen directly from the construction, the singularities of the deformed curve are at $(0,0)$ and $(t,0)$.

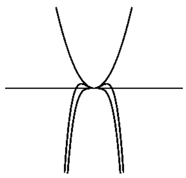
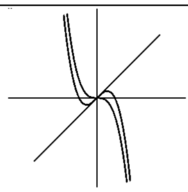
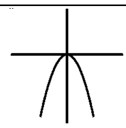
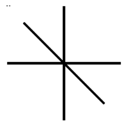
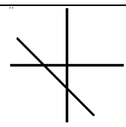
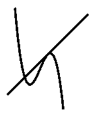
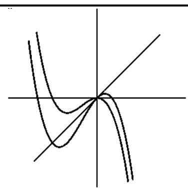
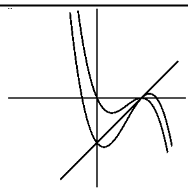
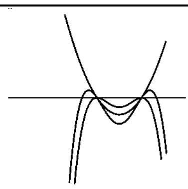
starting point	$y(y+x^4)(y-x^2)(y-x^2+x^4)$	
1st blowup	$x^4y(y+x^3)(y-x)(y-x+x^3)$	
2nd blowup of 3rd and 4th branch	$x^2(y-1)(y-1+x^2)$	
3rd blowup of 3rd and 4th branch. (Strict transform of E_1 is in the other chart)	$x^4y(y+x)$	
translation	$x^4y(y+x+t)$	
blowing down twice	$(y-x)(y-x+x^3+tx^2)$	
same with 1st & 2nd branch $\Rightarrow$ double points at $(-t,0)$ and $(-t,-t)$	$x^4y(y+x^3+tx^2)\,(y-x)(y-x+x^3+tx^2)$	
translation	$x^4y(y+(x-t)^3+t(x-t)^2)\,(y-(x-t))\,(y-(x-t)+(x-t)^3+t(x-t)^2)$	
final blowdown	$y(y+x(x-t)^3+tx(x-t)^2)\,(y-x(x-t))\,(y-x(x-t)+x(x-t)^3+tx(x-t)^2)$	

Table 4.3: An example for cutting Enriques diagrams. The exceptional divisor is always the y-axis.

4.4.3 Curves with Four Smooth Branches

In this section I prove the following theorem:

Theorem 4.4.16. *Let $(C,0)$ be an isolated plane curve singularity with no more than four branches, all branches being smooth. Then every combinatorial deformation of C can be realized.*

Remark 4.4.17. Example 4.1.2 shows that the statement is not true for curves with six smooth branches. I do not know whether it holds for curves with five smooth branches.

Remark 4.4.18. The corresponding sandwiched singularities $X(C,l)$ are the singularities with reduced fundamental cycle and multiplicity less or equal to five, see theorem 2.6.1.

Proof. The case of one smooth branch is trivial.

A curve with two smooth branches has only one combinatorial deformation, the Scott deformation, which can be realized. This is a deformation of an A_{2k-1}-singularity into k ordinary double points.

A curve with three smooth branches has a combinatorial representation $(123)^a(12)^b$. There are $a+1$ combinatorial deformations; these are $(123)^{a-k}(12)^k(13)^k(23)^k(12)^b$, $k \in \{0,\dots a\}$. By openness of versality, they can all be realized by making a Scott deformation first and then splitting up k of the triple points into three double points each.

Now we come to the case of four smooth branches. The idea is the same as for the case of three branches: I claim that there is a finite set of multi-adjacencies of equisingularity classes $\mathcal{E}(C_i) \to \mathcal{E}(C_{ij})$ such that the following holds: Let ϕ_i be a combinatorial representation of C_i and $\phi_i \to \psi_i$ the combinatorial deformation of $\mathcal{E}(C_i) \to \mathcal{E}(C_{ij})$. Assume that C is a curve with four smooth branches, ϕ the combinatorial representation of C and $\phi \to \psi$ a combinatorial deformation. Then there exist $n_i \in \mathbb{N}_0$ such that $\phi = \sum \phi_i$ and $\psi = \sum \psi_i$.

This implies that $\phi \to \psi$ can be realized. Indeed, by cutting Enriques graphs we can construct a δ-constant deformation of C such that the generic fibre has n_i singularities of class $\mathcal{E}(C_i)$, and by openness of versality we can deform each C_i separately into the multi-class $(C_{ij})_j$.

For the case of three branches, the finite set of multi-adjacencies consisted of the adjacency with combinatorial deformation $(123) \to (12)(13)(23)$ only. For the case of four branches, we need the adjacencies with combinatorial deformations

$$\begin{aligned}(123) &\to (12)(13)(23)\\ (1234) &\to (123)(14)(24)(34)\\ (1234)^2 &\to (123)(124)(134)(234)\\ (1234)(12) &\to (123)(124)(34)\end{aligned}$$

as well as the last three adjacencies followed by a deformation of one or two triple points into three ordinary double points. All these combinatorial deformations can be realized, the non-trivial cases being dealt with in lemma 4.4.20.

According to lemma 4.4.19, we have to distinguish two cases. We are left with some rather boring combinatorial details, which we leave to the reader. □

Lemma 4.4.19. *An isolated, plane curve singularity with four smooth branches has one of the following combinatorial representations:*

$$\begin{array}{rlr} (i) & (1234)^a\ (123)^b\ (12)^c, & a,b,c \in \mathbb{N}_0,\\ (ii) & (1234)^a\ (12)^b\ (34)^c, & a,b,c \in \mathbb{N}_0.\end{array}$$

Proof. This lemma is trivial. □

Lemma 4.4.20. *The combinatorial deformations*

$$\begin{aligned}(1234)^2 &\to (123)(124)(134)(234)\\ \textit{and } (1234)(12) &\to (123)(124)(34)\end{aligned}$$

of plane curve singularities can always be realized.

Proof. Let C_1 and C_2 be two curves whose combinatorial representations are $(1234)^2$ and $(1234)(12)$ and put $l_1 = (3,3,3,3)$, $l_2 = (3,3,2,2)$. Then $X(C_1, l_1)$ and $X(C_2, l_2)$ are cyclic quotients, so all their combinatorial deformations can be realized by theorem 4.3.27. □

Remark 4.4.21. Of course, we could have proven the preceding lemma directly. For example, consider the singularity

$$x(x+ay^2)(x+by^2)(x+cy^2),$$

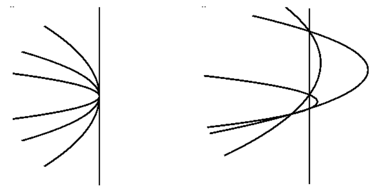
with $0, a, b, c \in \mathbb{C}$ pairwise different. It has combinatorial representation $(1234)^2$. The combinatorial deformation $(1234)^2 \to (123)(124)(134)(234)$ can be realized by

$$(x-t^2)(x+ay^2 - t\cdot a\frac{b+c}{\sqrt{abc}}y)(x+by^2 - t\cdot b\frac{a+c}{\sqrt{abc}}y)(x+cy^2 - t\cdot c\frac{a+b}{\sqrt{abc}}y).$$

For $t \neq 0$ the curve has four ordinary triple points at

$$(0,0), (t^2, \frac{a}{\sqrt{abc}}t), (t^2, \frac{b}{\sqrt{abc}}t), (t^2, \frac{c}{\sqrt{abc}}t),$$

with the i-th branch passing through all but the i-th of these points. The equation for this deformation was found using a blow up. Here is a picture of such a deformation:



4.5 Smoothings of Sandwiched Singularities

The basic idea in the study of sandwiched singularities is that we can reduce the study of surface singularities to the presumably easier study of plane curve singularities. Application of this principle to deformations has already led to very simple new proofs of some well known facts in section 3.2. Here is one more example: The existence of Scott deformations implies that every decorated curve (C, l) has a deformation (C_t, l_t) such that C_t has only ordinary singularities and such that l_t is reduced for $t \neq 0$, see corollary 4.4.6. By openness of versality this implies

Proposition 4.5.1. *A generic 1-parameter-deformation of a sandwiched singularity is a smoothing of the form $X(C_t, l_t)$, where C_t has only ordinary singularities and l_t is reduced.*

Remark 4.5.2. In [dJvS98] smoothings of this form are called picture deformations.

Here generic means that there is a Zariski-open subset of the base space of a semiuniversal deformation of $X(C, l)$ whose complement is nowhere dense, such that 1-parameter-deformations over this set can be represented in the above form.

In particular we see that we have smoothings over every component of the base space of a semiuniversal deformation. Once again, this is a result which is well known for all rational singularities, but here we have a very simple, 'sandwiched' proof.

4.6 Smoothing Components of Sandwiched Singularities

The main result from [dJvS98], which we have cited as theorem 3.1.7, says that there is a smooth transformation of functors from the functor of normal form deformations of (C, l) to the deformation functor of $X(C, l)$. This implies that the base spaces of their semiuniversal deformations are the same up to a smooth factor.

In particular, we can study the set of (non-embedded) irreducible components of the base space of $X(C, l)$ by studying deformations of the decorated curve (C, l). Since we have excluded the embedded components, we do not have to bother with normal form deformations but can use deformations over a reduced base space, in particular one-parameter-deformations.

Recall that every component of a rational singularity is a smoothing component. A generic smoothing is of the form $X(C_t, l_t)$ where C_t has only ordinary singularities and l_t is reduced. Such deformations are classified by their associated combinatorial deformations $\phi_{C_t,l_t} : \mathbb{N}_0^r \to \mathbb{N}_0$.

Because the transformation of functors is smooth, combinatorial deformations associated to smoothings over the same component are equal. So we have a well-defined map from the set of smoothing components of $X(C, l)$ to the set of combinatorial deformations. In [dJvS98] this map was denoted by $\varphi : \mathcal{S}(X(C, l)) \to \mathcal{I}(C, l)$. The image of this map is the set of combinatorial deformations which can be realized. It is an open question whether this map is injective. The question is equivalent to the following:

Question 4.6.1. Let $(C, 0)$ be an isolated plane curve singularity and $(S, 0)$ the base space of a semiuniversal deformation $(\mathcal{C}, 0)$ of $(C, 0)$. Let ϕ be a combinatorial deformation of $(C, 0)$.

Is it always possible to choose arbitrarily small representatives, such that the (possibly empty) set of $s \in S$, for which the fibres $\mathcal{C}_s$ have combinatorial type ϕ, is connected?

I do not know the answer to this question. I believe that the answer is yes for curves with smooth branches, but might be no in general.

The following partial results are known:

1. The δ-constant-stratum of a plane curve singularity C is the subspace of the base space of a semiuniversal deformation of C whose fibres have the same δ-constant as C. If C is irreducible, then the δ-constant-stratum is irreducible.

   From this it follows by corollary 4.4.9 that the subspace of the base space of a semiuniversal deformation of C whose fibres have δ ordinary double points is connected.

2. Lemma 4.3.22 shows that the subspace corresponding to the Scott deformations is irreducible. This is also clear from the fact that the Scott deformations correspond to generic smoothings over the Artin component, which is known to be smooth, see section 5.1.

3. If the multiplicity of C is less than four, the corresponding sandwiched singularities are rational double, triple or quadruple points. Since these surface singularities are well understood, we know that in this case the answer to question 4.6.1 is yes, see [dJvS98, 4.13, (3)].

4. If $X(C,l)$ is a cyclic quotient and C has smooth branches, the above map is a bijection from the set of combinatorial deformations to the set of components of the base space of $X(C,l)$, see [dJvS98, Th. 6.18].

Remark 4.6.2. The subspace corresponding to equisingular deformations, the so-called μ-constant-stratum, is smooth, see [Wah74]. The equisingular deformations of the curve C do not correspond to a smoothing component but to the equisingular deformations of $X(C,l)$, i.e. the deformations where all fibres have the same dual resolution graph.

Chapter 5

The Kollár Conjecture for Sandwiched Singularities

For a cyclic quotient singularity X, Kollár and Shepherd-Barron have found a natural bijection from the set of certain partial resolutions of X to the set of irreducible components of the base space of a semiuniversal deformation of X. This has led to an easy combinatorial description of the set of components of a cyclic quotient. The Kollár conjecture 5.5.1 grew out of an attempt to extend this result from cyclic quotients to arbitrary rational singularities. So far, the conjecture has only been verified for a few cases, see section 5.5. I will review the result on cyclic quotients and other results which are necessary in order to understand the Kollár conjecture in sections 5.1 to 5.5.

Now consider a sandwiched singularity $X(C, l)$. In [dJ02], de Jong has shown that the statement of the Kollár conjecture for $X(C, l)$ is equivalent to a condition on the conductor of a δ-constant 1-parameter deformation of C, namely that the symbolic power algebra of the induced deformation of the conductor is finitely generated.

In section 5.7 I collect conditions which are equivalent to the fact that the symbolic power algebra of a curve is finitely generated. The main result is theorem 5.7.24 which extends work of Huneke and Morales, who have considered symbolic power algebras of reduced, irreducible curves. Theorem 5.7.24 generalizes their results to certain non-reduced curves with an arbitrary number of branches. Finally I show how to apply this to the Kollár conjecture for sandwiched singularities and compute some examples.

5.1 Simultaneous Resolutions

In this section, all deformations will be over a reduced base space.

Let X be a rational surface singularity and $\tilde{X} \to X$ a minimal resolution. Then every deformation of $\tilde{X}$ "blows down" to a deformation of X. We get an induced map *Res* between the base spaces of semiuniversal deformations. We say that a deformation of X allows a simultaneous resolution if and only if it can be obtained by blowing down a deformation of $\tilde{X}$. For the rational double points, Brieskorn has shown that the map *Res* is a surjective Galois covering and that its group of automorphisms is the Weyl group of the Lie group having the same name as the rational double point. This implies that every deformation of a rational double point allows a simultaneous resolution *after a finite base change* (the base change is *Res*).

This has been generalized by joint work of Artin and Schlessinger, see [Art70]: If X is a rational surface singularity, then the image of *Res* is an irreducible, non-embedded component of the base space of the semiuniversal deformation of X; this component is now commonly called the Artin component. The map *Res* is a finite Galois map and every deformation of X allows a simultaneous resolution after a finite base change. The Galois group is a direct product of the Weyl groups associated to the (-2)-configurations in the dual resolution graph.

Finally, in [Lip79] the following is proved:

Theorem 5.1.1 (Lipman). *Let X be a rational surface singularity and $\tilde{X} \to X$ a minimal resolution. We call the contraction of all (-2)-curves in $\tilde{X}$ the* rational double point resolution *of X (RDP-resolution for short).*

Every deformation of the RDP-resolution of X blows down to a deformation over the Artin component. The induced map from the base space of a semiuniversal deformation of the RDP-resolution to the Artin component is bijective. Conversely, every deformation over the Artin component allows a unique simultaneous double point resolution (without base change!).

Remark 5.1.2.

1. The Artin component is smooth.
2. Assume that the equations for the rational surface singularity X are given by the 2×2 minors of a $2 \times d$ matrix. Then the deformations over the Artin component can be characterized as those deformations which are given by a perturbation of the entries of the matrix, see [Wah79b].

5.2 Small Modifications and Symbolic Blowups

The total space of a 1-parameter deformation of a surface singularity is a threefold. So results from three-dimensional geometry can be applied to the study of such deformations. This has been done by Kawamata, Kollár and

Shepherd-Barron in [Kaw88], [KSB88] and [Kol91]. In the following three sections, we present some of their results.

Definition 5.2.1. A *small modification* of a three-dimensional, normal singularity $(X, 0)$ is the contraction $f : (Y, C) \to (X, 0)$ of a curve C to $0 \in X$.

Examples for small modifications are simultaneous resolutions of 1-parameter smoothings of normal surface singularities: Because the general fibre is smooth, a simultaneous resolution is biholomorphic outside the special fibre, and in the special fibre we have the contraction of a curve to a point.

Proposition 5.2.2 (Kawamata). *Let $X = (X, 0)$ be a three-dimensional, normal singularity.*

1. *Let $f : Y \to X$ be a small modification and $D \subset Y$ an f-ample divisor. Then the following holds:*
   (a) *$mf(D) \subset X$ is not Cartier for all $m > 0$,*
   (b) *$f_*\mathcal{O}_Y(mD) = \mathcal{O}_X(mf(D))$ for all $m \geq 0$, and*
   (c) *$\sum_{m=0}^{\infty} \mathcal{O}_X(mf(D))$ is a finitely generated $\mathcal{O}_X$-algebra.*
2. *Assume that $D' \subset X$ is a divisor such that*
   (a) *no multiple of D' is Cartier, and*
   (b) *$\sum_{m=0}^{\infty} \mathcal{O}_X(mD')$ is a finitely generated $\mathcal{O}_X$-algebra.*

   Then the projection from $Y = Proj \sum_{m=0}^{\infty} \mathcal{O}_X(mD')$ to X is a small modification of X.

A proof is in [Kaw88, section 3]. See also [Kol91, §6] for further comments. The ideal sheaf $\mathcal{O}_X(mD')$ is the m-th symbolic power of the ideal sheaf $\mathcal{O}_X(D')$. See section 5.6 for definitions and properties of symbolic powers. An important special case is the symbolic power algebra of the canonical divisor.

Notation 5.2.3.
1. The algebra $\sum_{m=0}^{\infty} \mathcal{O}_X(mD')$ is called the symbolic power algebra of D'. The modification $Proj \sum_{m=0}^{\infty} \mathcal{O}_X(mD')$ is called symbolic blowup.
2. Let X_0 be a rational surface singularity, X a 1-parameter smoothing of X_0 and K_X the canonical divisor of X. The canonical algebra of the smoothing X is the $\mathcal{O}_X$-algebra $\bigoplus_{n=0}^{\infty} \mathcal{O}(nK_X)$.

5.3 Smoothing Components of Cyclic Quotient Singularities

Let $f : Y \to X$ be a small modification of a 1-parameter smoothing of a normal surface singularity X_0. The restriction to the special fibre is a modification of X_0. Set $Y_0 := f^{-1}(X_0)$. Since f is biholomorphic outside the special fibre, Y is a smoothing of Y_0.

A normal variety Y is called $\mathbb{Q}$-Gorenstein iff some non-zero integral multiple mK_Y of the canonical divisor K_Y is Cartier and Y is Cohen-Macaulay. In the situation of proposition 5.2.2, we know that Y is $\mathbb{Q}$-Gorenstein, see [Kaw88, Lemma 3.1].

Now we consider the special case that X_0 is a cyclic quotient singularity. The study of small modifications of one-parameter smoothings of cyclic quotients in [KSB88] gave rise to the following definition:

Definition 5.3.1. Let X_0 be a two-dimensional cyclic quotient singularity. A *P-resolution* of X_0 is a partial resolution $f : Z_0 \to X_0$ such that

1. All singularities of Z_0 are quotient singularities and admit a one-parameter smoothing which is $\mathbb{Q}$-Gorenstein.

2. The canonical divisor K_{Z_0} is ample relative to f.

The following theorem is a generalization of the characterization of the Artin component given in theorem 5.1.1.

Theorem 5.3.2 (Kollár, Shepherd-Barron). *Let X_0 be a two-dimensional cyclic quotient singularity.*
The canonical algebra of any 1-parameter smoothing of X_0 is finitely generated. The special fibre of the symbolic blowup in the canonical algebra is a P-resolution. If two smoothings give rise to isomorphic P-resolution, then the two smoothings are over the same component of the base space of a semiuniversal deformation of X_0.

This induces a bijection between isomorphism classes of P-resolutions and components of the base space of a semiuniversal deformation of X_0.

The proof of this theorem is in [KSB88]. See also [Kol91].

So for each component of the base space, there is a unique 'partial resolution' with the property that certain smoothings of this partial resolution blow down to smoothings over the given component. The P-resolutions corresponding to the Artin-component are the RDP-resolutions.

Remark 5.3.3. In general, the base space of a semiuniversal deformation will have embedded components. These are always excluded when we talk of components of the base space.

Remark 5.3.4. Recall that all componentsof the base space of a semiuniversal deformation are smoothing components for any rational surface singularity.

This theorem has led to a very nice combinatorial description of the components of the base space of a cyclic quotient singularity in [Chr91] and [Ste91a].

5.4 P-modifications

In [Kol91], Kollár asks whether the above results on cyclic quotient singularities can be generalized to the class of all rational surface singularities. This leads to the following open questions.

Question 5.4.1. Let X_0 be any rational surface singularity. Is the canonical algebra of an arbitrary 1-parameter smoothing of X_0 finitely generated?

Kollár conjectures that the answer is yes. We call this conjecture the Kollár conjecture. We will discuss the conjecture in the next section and the rest of the chapter. But before we do that, we want to mention some more open questions.

If the canonical algebra is finitely generated, then the symbolic blowup in the canonical algebra is a small modification and the special fibre is a modification of X_0.

Definition 5.4.2. These modifications of X_0 are the *P-modifications* of X_0.

If the smoothing is over the Artin component, then the P-modification is an RDP-resolution.

Question 5.4.3. How can we characterize P-modifications?

This is the central question in [Kol91, 6.2]. The normal P-modifications $f : Y_0 \to X_0$ are characterized by the property that the canonical divisor of Y_0 is f-ample and that every singularity of Y_0 has a 1-parameter smoothing Y' such that the canonical divisor of Y' is $\mathbb{Q}$-Cartier. For non-normal P-modifications, the situation is not so clear. Kollár works with the assumption that P-modifications are reduced and Cohen-Macaulay with at worst double normal crossing points in codimension one, but it is not clear at all if these conditions are always satisfied.

If the canonical algebra of a smoothing is always finitely generated, then we can ask

Question 5.4.4. Let X_0 be any rational surface singularity. Is there a natural bijection from the set of isomorphism classes of P-modifications to the set of smoothing components of X_0?

Actually, this might be asking for too much because we cannot exclude the possibility that a P-modification has several smoothing components which correspond to different smoothing components of X_0. So we might have to refine the question to ask for a bijection not from the set of isomorphism classes of P-modifications but of smoothing components of P-modifications over which there are $\mathbb{Q}$-Gorenstein deformations. Again, Kollár has conjectured that the answer is yes, see [Kol91, conjecture 6.2.15].

The idea behind all these questions is to get control over the components of the base space of a rational surface singularity. This has worked very well for cyclic quotient singularities. Unfortunately, it has turned out that P-modifications of arbitrary rational singularities can behave quite unpleasantly; for example, they might be non-normal. So even if there is a bijection from isomorphism classes of P-modifications to components of the base space of a semiuniversal deformation, it might be quite hard to get information about the components by studying P-modifications.

We will now return to the first of these questions and see what we can say for the case of sandwiched singularities.

5.5 The Kollár Conjecture

The *Kollár Conjecture* is Conjecture 6.2.1 in [Kol91] and reads:

Conjecture 5.5.1. Let X be a 1-parameter smoothing of a rational surface singularity and K_X the canonical divisor on X. Then the canonical algebra $\bigoplus_{n=0}^{\infty} \mathcal{O}(n \cdot K_X)$ is a finitely generated $\mathcal{O}_X$-algebra.

We have discussed the motivation to study this conjecture in the preceding sections.

The following partial results are known:

1. The statement of the Kollár conjecture is true for smoothings over the Artin component. So the Kollár conjecture is true if the base space of the semiuniversal deformation is irreducible. This is e.g. the case for all rational double and triple points and for hypersurface singularities. Kollár has conjectured that the base space of a rational surface singularity is irreducible if every vertex of the dual resolution graph has weight ≤ -5. In [dJvS98] the theory of sandwiched singularities has been used to show that this is true if the fundamental cycle is reduced.

2. The Kollár conjecture is true for cyclic quotient singularities and for quotients of simple elliptic and cusp singularities, see [KSB88].

3. The Kollár conjecture is true for rational quadruple points, see [Ste91b].

5.5.1 The Case of Sandwiched Singularities

Let $X(C_t, l_t)$ be a 1-parameter smoothing of the sandwiched singularity $X(C, l)$. Since the deformation C_t is δ-constant, it implies a flat deformation of the conductor of C. We denote the ideal of the deformation of the conductor by J, so $J \subset \mathcal{O}_{\mathbb{C}^2 \times T,0} \cong \mathbb{C}\{x, y, t\}$ and $J_0 = J/(t)$ is the conductor of C. $J^{(n)}$ denotes the n-th symbolic power of J.

Theorem 5.5.2 (de Jong). *Let the notations be as above. The canonical algebra of the smoothing $X(C_t, l_t)$ is finitely generated if and only if the symbolic algebra $\bigoplus_{n=0}^{\infty} J^{(n)}$ is a finitely generated $\mathbb{C}\{x, y, t\}$-algebra.*

A proof is given in [dJ02].

Remark 5.5.3. There are irreducible curves in three-dimensional regular rings whose symbolic algebra is not finitely generated, see [Nag60], [Rob85] and [GNW94]. For example it is shown in [GNW94] that the symbolic algebra of the monomial prime ideal $(t^{25}, t^{72}, t^{29}) = (x^{11} - yz^7, y^3 - x^4z^4, z^{11} - x^7y^2)$ in $\mathbb{C}[[x, y, z]]$ is not finitely generated. If we replace $\mathbb{C}$ by a field of characteristic > 0, then the symbolic algebra is finitely generated, but the blow up ring is not Cohen-Macaulay.

These examples cannot occur as the deformation of the conductor of a plane curve singularity under a δ-constant deformation though, because they are reduced curves. This means that the corresponding curve deformation would have to be a deformation into δ ordinary double points. But for such a deformation, the symbolic algebra of the conductor is always finitely generated, see proposition 5.9.1.

Still, it might be possible to construct a counterexample to the Kollár conjecture from one of these examples. To do so, it would suffice to find a deformation into ordinary singularities, all having the same multiplicity $n \geq 3$, such that the conductor is the $(n-1)$-st symbolic power of one the reduced curves of which we know that the symbolic algebra is not finitely generated.

Corollary 5.5.4. *If the Kollár conjecture is true for $X(C, l)$ and $l \geq l'$, then the Kollár conjecture is true for $X(C, l')$.*
If the divisor l is so big that every δ-constant deformation of C implies a deformation of $X(C, l)$, then the Kollár conjecture is true for $X(C, l)$ if and only if it is true for all $X(C, l')$.

Proof. The algebra $\bigoplus_{n=0}^{\infty} J^{(n)}$ does not depend on the divisor l or l_t but only on the deformation of C. The only thing that does depend on l is whether a certain deformation of C implies deformations of the sandwiched singularity or not. □

5.5.2 Construction of P-modifications

Theorem 5.5.2 only gives a criterion to decide whether a P-modification does exist or not. Now we would like to have more information on the P-modification and we would like to read off this information directly from the deformation (C_t, l_t) of the decorated curve respectively of the associated combinatorial deformation.

Let us take a look at the proof of theorem 5.5.2 given in [dJ02]: Under the assumption that the symbolic algebra of the conductor is finitely generated, de Jong constructs a small modification of the smoothing as follows: First you have to take the symbolic blowup in the conductor J; call this blowup W. De Jong shows that the deformation l_T of l can be seen as a subspace of W and that it makes sense to define the set D of irreducible components of l_T which are not contained in the zero set of the conductor. The blowup W_D of W in D is a small modification of the smoothing $X(C_t, l_t)$.

Example 5.5.5. Let $X(C_t, l_t)$ be a smoothing over the Artin component, J the induced deformation of the conductor. Then $J^{(n)} = J^n$ for all n, see corollary 5.8.4, so the symbolic blowup is equal to the ordinary blowup. The conductor $J_0 = J/(t)$ of C is the complete ideal whose multiplicity at the infinitely near point p is $e_p(C)-1$, see theorem 1.2.21. So the base points p of J_0 with excess $\rho_p(J_0) > 0$ are the points on C with the property that there are ≥ 2 points on C proximate to p. These points correspond to the irreducible components E_p of the minimal resolution of $X(C,l)$ with $-E_p^2 > 2$.

Using the description of the blow up in a complete ideal from chapter 2, we see that the special fibre of the small modification W_D can be obtained as follows: Blow up all base points of (C,l), then contract the (-2)-curves. This gives the RDP-resolution of $X(C,l)$.

Smoothings over the Artin component are of course the easiest to handle. In general, the computation of the P-modification is more complicated, but can be done in a similar way:

If the symbolic algebra of the conductor is finitely generated, then there is a k such that the symbolic blowup in J is equal to the ordinary blowup in $J^{(k)}$, see theorem 5.7.24. The hardest task is to find this k and to compute $J^{(k)}/(t)$. The special fibre of the symbolic blowup W is the blowup of $\mathbb{C}^2$ in $J^{(k)}/(t)$. If $J^{(k)}/(t)$ is complete, we have a good description of the special fibre of the symbolic blowup W and can compute the special fibre of W_D as in the case of the Artin component, i.e. as the blowup of W_0 in those base points of (C, l) which are not base points of $J^{(k)}/(t)$.

If $J^{(k)}/(t)$ is not complete, then the special fibre of the symbolic blowup W is not normal, and the P-modification probably will not be normal either. But at least we will still get some information on the P-modification if we know the cluster of base points of $J^{(k)}/(t)$. In particular, I expect that replacing $J^{(k)}/(t)$ by its integral closure will lead to the normalization of the P-modification.

We compute several examples in section 5.9. The most interesting phenomena occur in example 5.9.4.

5.6 Computation of Symbolic Powers

We remind the reader of the definition and some basic properties of symbolic powers, see e.g. [Eis94, chapter 3].

Definition 5.6.1. Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring. The n-th *symbolic power* $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$.

Let I be an arbitrary ideal in a Noetherian ring and $I^n = \bigcap \mathfrak{q}_i$ a primary decomposition of the n-th power of I. Let $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary. Then the primary components of the primes $\mathfrak{p}_i$ minimal in $Ass(I)$ are uniquely determined by I and we define

$$I^{(n)} := \bigcap_{\substack{\mathfrak{p}_i \text{ minimal} \\ \text{in } Ass(I)}} \mathfrak{q}_i$$

to be the n-th symbolic power of I.

So the n-th symbolic power is the ordinary n-th power without the embedded components. The geometric idea is the following: If I is a reduced ideal, then $I^{(n)}$ consists of the functions having vanishing order $\geq n$ in a generic point of the zero set $V(I)$.

Proposition 5.6.2. *Let I be an ideal in a Noetherian ring R, $\mathfrak{p}$ a minimal prime of I. Then the $\mathfrak{p}$-primary component of I is the contraction $R \cap (IR_{\mathfrak{p}})$ of $IR_{\mathfrak{p}}$. So*

$$I^{(n)} = \bigcap_{\mathfrak{p} \in Ass(\sqrt{I})} (I^n \cdot R_p) \cap R.$$

A curve $V(I)$ in $(\mathbb{C}^3, 0)$ can have no more but one embedded component, a zero-dimensional component in zero. We can eliminate this component by passing to the saturation of I in the maximal ideal at zero:

Proposition 5.6.3. *Let $I \subset \mathbb{C}\{x, y, t\}$ be a one-dimensional ideal. Then the symbolic power $I^{(n)}$ is the saturation of I^n with respect to $\mathfrak{m} = (x, y, t)$:*

$$I^{(n)} = \bigcap_{i=1}^{\infty} (I^n : \mathfrak{m}^i).$$

Passing to the saturation with respect to a maximal ideal is an operation which is well suited for the use of computer algebra, see e.g. [GP02].

Proposition 5.6.4. *Let R be a Noetherian ring, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \subset R$ prime ideals such that $R_{\mathfrak{p}_i}$ is regular and $n_i \in \mathbb{N}$.*

Then $I = \bigcap_{i=1}^{r} \mathfrak{p}_i^{(n_i)}$ is integrally closed.

Proof. Since a finite intersection of integrally closed ideals is integrally closed, we only have to consider a single prime $\mathfrak{p} \subset R$. In a regular local ring, the powers of the maximal ideal are integrally closed. For every ideal $I \subset R_{\mathfrak{p}}$ we have $\bar{I} \cap R \supset \overline{I \cap R}$, because an integral equation in R over the contraction $I \cap R$ is also an integral equation in $R_{\mathfrak{p}}$ over I itself.

We obtain

$$\mathfrak{p}^{(n)} = (\mathfrak{p}^n R_{\mathfrak{p}}) \cap R = (\overline{\mathfrak{p}^n R_{\mathfrak{p}}}) \cap R \supset \overline{\mathfrak{p}^{(n)}},$$

so $\mathfrak{p}^{(n)}$ is integrally closed. □

5.7 When is the Symbolic Algebra Finitely Generated?

We now gather criteria to decide when the symbolic algebra of a curve is finitely generated. The result is summarized in theorem 5.7.24, which the reader may prefer to read before going through the technical sections on multiplicities and analytic spread.

The part of the main theorem which cannot be found in the literature is a generalization of work of Huneke and Morales, see [Hun87] and [Mor91]. Huneke gives a criterion for curves in three-space, Morales generalized this to curves in an arbitrary analytically unramified and formally equidimensional local domain of dimension $d \geq 3$ with regular localization. The proof given here follows Morales proof closely and generalizes further from irreducible curves to curves with several components and a certain non-reduced structure. At the same time we restrict ourselves to curves in a regular local ring. I believe that it would also work under Morales' assumptions, but reading the proof probably would become a burden for a reader who is not into the subject.

5.7.1 Multiplicities

The basics on multiplicities can be found in [Nor68, chapter 7]. Our notations are the same as in [Mor91].

We denote the *length* of an R-module M by $\lambda_R(M)$.

Definition/Theorem 5.7.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d and E a finitely generated R-module of dimension r.

Then $x = x_1, \ldots, x_r$ is a *parameter system* of E if and only if $x_1, \ldots, x_r \in E$ and the length $\lambda_R(E/(x)E)$ is finite.

Now assume that $x = x_1, \ldots, x_r$ is a parameter system of E. Then for $n \gg 0$ the function $n \mapsto \lambda_R(E/(x)^n E)$ coincides with a polynomial whose leading term is of the form $e(x|E)\frac{n^r}{r!}$ with $e(x|E)$ an integer.

We define the *multiplicity of E with respect to x* to be the integer $e(x|E)$.

Definition/Theorem 5.7.2. Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d, $I \subset R$ an $\mathfrak{m}$-primary ideal.

Then for $n \gg 0$ the function $n \mapsto \lambda_R(R/I^n)$ coincides with a polynomial whose leading term is of the form $e(I)\frac{n^d}{d!}$ with $e(I)$ an integer.

We define the *multiplicity of I* to be the integer $e(I)$.

Remark 5.7.3. If I is generated by a regular sequence x, then obviously $e(I) = e(x|R)$.

Remark 5.7.4. We have $e(x|E) \leq \lambda_R\big(E/(x)E\big)$ and equality if and only if E is Cohen-Macaulay. So in particular, if $E = R/I$ is a curve, i.e. one-dimensional, we have equality if and only if the curve has no embedded components.

Remark 5.7.5. We have $e(x_1^{n_1}, \ldots, x_r^{n_r}|E) = (\prod_{i=1}^r n_i) \cdot e(x|E)$, see [Nor68, ch. 7, Cor. 1, p. 311]. Some people call this the associativity of the multiplicity symbol.

Definition/Theorem 5.7.6. Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d, $I \subset R$ an ideal, $\dim R/I = r$ and $x = x_1, \ldots, x_r$ a parameter system of R/I.

Then for $n \gg 0$ the function $n \mapsto e(x|R/I^n)$ coincides with a polynomial whose leading term is of the form $e(x; I)\frac{n^r}{r!}$ with $e(x; I)$ an integer.

We define the *relative multiplicity of I with respect to x* to be the integer $e(x; I)$.

Theorem 5.7.7. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, E a Noetherian R-module and $x = x_1, \ldots, x_r$ a parameter system of E. Then*

$$e(x|E) = \sum \lambda_{R_\mathfrak{p}}(E_\mathfrak{p}) e(x|R/\mathfrak{p}),$$

where the sum is being taken over all primes $\mathfrak{p} \supset Ann(E)$ *containing the annihilator of* E *with* $\dim(R/\mathfrak{p}) = r$.

For a proof see [Mor91, Theorem 1.2] or [Nor68, Ch. 7, Prop. 11, p. 341].

Corollary 5.7.8. *With the above notations:*

$$e(x;I) = \sum e_{R_{\mathfrak{p}_i}}(I_{\mathfrak{p}_i}) \cdot e(x|R/\mathfrak{p}_i).$$

Proof. The relative multiplicity is defined as the $(r!)$-multiple of the leading coefficient of the polynomial in n whose values for $n \gg 0$ are

$$e(x|R/I^n) = \sum \lambda_{R_\mathfrak{p}}(R_{\mathfrak{p}_i}/I^n_{\mathfrak{p}_i}) \cdot e(x|R/\mathfrak{p}_i).$$

Now $e(x|R/\mathfrak{p}_i)$ doesn't depend on n, while $\lambda_{R_\mathfrak{p}}(R_{\mathfrak{p}_i}/I^n_{\mathfrak{p}_i})$ is a polynomial with leading coefficient $e_{R_{\mathfrak{p}_i}}(I_{\mathfrak{p}_i})/r!$ for $n \gg 0$. □

Corollary 5.7.9. *Let* $x, f_1, \ldots, f_{d-1}$ *be a regular sequence in the regular, local, d-dimensional ring* $(R, \mathfrak{m})$. *Then*

$$e\big(x;(f_1,\ldots,f_{d-1})\big) = e(x, f_1, \ldots, f_{d-1}).$$

Proof. Let $\mathfrak{p}$ be an associated prime of the ideal $(f_1, \ldots, f_{d-1})$. Then $f_1, \ldots, f_{d-1}$ is a regular sequence in $R_\mathfrak{p}$. We deduce

$$\begin{aligned}
e\big(x;(f_1,\ldots,f_{d-1})\big) &= \sum_{\mathfrak{p}} e_{R_\mathfrak{p}}\big((f_1,\ldots,f_{d-1})_\mathfrak{p}\big) \cdot e(x|R/\mathfrak{p}) && \text{by 5.7.8}\\
&= \sum_{\mathfrak{p}} e_{R_\mathfrak{p}}\big(f_1,\ldots,f_{d-1}|R_\mathfrak{p}\big) \cdot e(x|R/\mathfrak{p}) && \text{by 5.7.3}\\
&= \sum_{\mathfrak{p}} \lambda\big(R_\mathfrak{p}/(f_1,\ldots,f_{d-1})\big) \cdot e(x|R/\mathfrak{p}) && R_\mathfrak{p} \text{ is CM}\\
&= e\big(x|R/(f_1,\ldots,f_{d-1})\big) && \text{by 5.7.7}\\
&= e(x, f_1, \ldots, f_{d-1}|R)\\
&= e(x, f_1, \ldots, f_{d-1}).
\end{aligned}$$

□

Corollary 5.7.10. *Let* J *and* I *be of pure dimension* r, $J \subset I$ *be a reduction. Then*

$$e(x;I) = e(x;J).$$

Proof. Since J is a reduction of I and both are pure dimensional, they have the same associated primes. Localizing we see that $J_{\mathfrak{p}_i}$ is a reduction of $I_{\mathfrak{p}_i}$, so according to remark 1.5.8

$$e_{R_{\mathfrak{p}_i}}(J_{\mathfrak{p}_i}) = e_{R_{\mathfrak{p}_i}}(I_{\mathfrak{p}_i}).$$

Corollary 5.7.8 implies

$$\begin{aligned} e(x;I) &= \sum e_{R_{\mathfrak{p}_i}}(I_{\mathfrak{p}_i}) \cdot e(x|R/\mathfrak{p}_i) \\ &= \sum e_{R_{\mathfrak{p}_i}}(J_{\mathfrak{p}_i}) \cdot e(x|R/\mathfrak{p}_i) \\ &= e(x;J). \end{aligned}$$

□

Corollary 5.7.11. *Let $(R,\mathfrak{m})$ be a regular, Noetherian local ring of dimension d and $I = \bigcap \mathfrak{p}_i^{(n_i)} \subset R$ the ideal of a curve without embedded components and $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$. Then*

$$e(x;I^{(k)}) = k^{d-1} \cdot \sum n_i^{d-1} \lambda\big(R/(x) + \mathfrak{p}_i\big).$$

Proof. The condition $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$ means that x is a regular sequence of length one on $R/\mathfrak{p}_i$, for all $\mathfrak{p}_i$. So we get

$$\begin{aligned} e(x;I^{(k)}) &= \sum e_{R_{\mathfrak{p}_i}}(I^{(k)}_{\mathfrak{p}_i}) \cdot e(x|R/\mathfrak{p}_i) && \text{by Corollary 5.7.8} \\ &= \sum e_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i\,\mathfrak{p}_i}^{kn_i}) \cdot e(x|R/\mathfrak{p}_i) \\ &= \sum (kn_i)^{d-1} \cdot e(x|R/\mathfrak{p}_i) \\ &= \sum (kn_i)^{d-1} \cdot \lambda\big(R/(x) + \mathfrak{p}_i\big), && \text{because } R/\mathfrak{p}_i \text{ is CM.} \end{aligned}$$

□

Lemma 5.7.12. *Let $(R,\mathfrak{m})$ be a regular, Noetherian local ring, $I = \bigcap \mathfrak{p}_i^{(n_i)} \subset R$ the ideal of a curve without embedded components, $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$ and $S = R/(x)$.*
Then for all $k \in \mathbb{N}$ the function $n \mapsto \lambda_R(S/I^{(kn)}S)$ coincides for $n \gg 0$ with a polynomial with leading term $e\big(x;I^{(k)}\big)\frac{n^{d-1}}{(d-1)!}$.

Proof. We localize to separate the primary components from each other. After localizing we only have to deal with the maximal ideals in the local rings $R_{\mathfrak{p}_i}$, so life gets fairly easy. Since I was of the form $I = \bigcap \mathfrak{p}_i^{(n_i)}$, the same hold for $I^{(k)}$; we set $I^{(k)} = \bigcap_i \mathfrak{p}_i^{(k_i)}$.

Theorem 5.7.7 implies

$$\begin{aligned}
e(x|R/I^{(k)n}) &= \sum_i \lambda_{R_{\mathfrak{p}_i}}\big((R/\bigcap_j \mathfrak{p}_j^{(k_j)n}) \otimes R_{\mathfrak{p}_i}\big) \cdot e(x|R/\mathfrak{p}_i) \\
&= \sum_i \lambda_{R_{\mathfrak{p}_i}}\big(R_{\mathfrak{p}_i}/\bigcap_j (\mathfrak{p}_j^{(k_j)n} \cdot R_{\mathfrak{p}_i})\big) \cdot e(x|R/\mathfrak{p}_i) \\
&= \sum_i \lambda_{R_{\mathfrak{p}_i}}\big(R_{\mathfrak{p}_i}/(\mathfrak{p}_i^{(k_i)n} \cdot R_{\mathfrak{p}_i})\big) \cdot e(x|R/\mathfrak{p}_i) \\
&= \sum_i \lambda_{R_{\mathfrak{p}_i}}\big(R_{\mathfrak{p}_i}/(\mathfrak{p}_i^{k_i n} \cdot R_{\mathfrak{p}_i})\big) \cdot e(x|R/\mathfrak{p}_i).
\end{aligned}$$

In the line before the last line of the preceding computation we have made use of the fact that $\mathfrak{p}_j \cdot R_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ for all $i \neq j$. The last equality is a consequence of the characterization of symbolic powers via contractions, see proposition 5.6.2.

By definition, $e(x; I^{(k)})$ is $(d-1)!$ times the leading coefficient of the polynomial in n which coincides with $e(x|R/I^{(k)n})$ for $n \gg 0$. In the expression for $e(x|R/I^{(k)n})$ that we have just computed, the factor $e(x|R/\mathfrak{p}_i)$ does not depend on n, while the other factor is, for $n \gg 0$, an polynomial with leading term

$$e_{R_{\mathfrak{p}_i}}\big(\mathfrak{p}_i^{k_i}\big)\frac{n^{d-1}}{(d-1)!} = k_i^{d-1} e_{R_{\mathfrak{p}_i}}\big(\mathfrak{p}_i\big)\frac{n^{d-1}}{(d-1)!}.$$

We conclude:

$$e\big(x; I^{(k)}\big) = \sum_i k_i^{d-1} e_{R_{\mathfrak{p}_i}}(\mathfrak{p}_i) \cdot e(x|R/\mathfrak{p}_i). \tag{5.1}$$

Now let's take look at the function the theorem is about:

$$\begin{aligned}
\lambda_R(S/I^{(kn)}S) &= \lambda_R\big((R/(x))/I^{(kn)}\big) \\
&= \lambda_R\big(R/(x) + I^{(kn)}\big) \\
&= \lambda_R\big((R/I^{(kn)})/(x)\big) \\
&= e(x|R/I^{(kn)}) && R/I^{(kn)} \text{ is CM} \\
&= \sum_i \lambda_{R_{\mathfrak{p}_i}}\big((R/\mathfrak{p}_i^{(k_i n)}) \otimes R_{\mathfrak{p}_i}\big) \cdot e(x|R/\mathfrak{p}_i) && \text{by theorem 5.7.7.} \\
&= \sum_i \lambda_{R_{\mathfrak{p}_i}}\big(R_{\mathfrak{p}_i}/(\mathfrak{p}_i^{k_i n} \cdot R_{\mathfrak{p}_i})\big) \cdot e(x|R/\mathfrak{p}_i).
\end{aligned}$$

For $n \gg 0$, the expression $\lambda_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/(\mathfrak{p}_i^{k_i n} \cdot R_{\mathfrak{p}_i}))$ in the last sum is a polynomial with leading term $e_{R_{\mathfrak{p}_i}}\big(\mathfrak{p}_i\big)\frac{(k_i n)^{d-1}}{(d-1)!}$. So for $n \gg 0$, $n \mapsto \lambda_R(S/I^{(kn)}S)$ coincides with a polynomial with leading term

$$\sum_i e_{R_{\mathfrak{p}_i}}(\mathfrak{p}_i)\frac{(k_i n)^{d-1}}{(d-1)!}\cdot e(x|R/\mathfrak{p}_i).$$

Substituting $e\big(x; I^{(k)}\big)$ for the expression on the right of equation 5.1 on page 102 completes the proof. □

5.7.2 Analytic Spread

Definition/Theorem 5.7.13. Let I be an ideal in a local Noetherian ring $(R,\mathfrak{m})$. The function $n \mapsto$ (minimal number of generators needed for I^n) coincides for $n \gg 0$ with a polynomial $P(n)$.

We define the *analytic spread* of I to be the integer

$$\mathfrak{l}(I) = (\text{degree of } P) + 1.$$

Remark 5.7.14. Obviously $\mathfrak{l}(I) = \mathfrak{l}(I^n)$ for all $n \in \mathbb{N}$.

Remark 5.7.15. The analytic spread is the dimension of the special fibre of the Rees-algebra in I, i.e. the Krull dimension of $R[tI] \otimes_R R/\mathfrak{m}$, see [Vas94, 5.1]. So the geometric meaning of the analytic spread is that $\mathfrak{l}(I) - 1$ is the dimension of the special fibre of the blowup in I.

Recall from section 1.5 that an ideal $J \subset I$ is called a *reduction* of I if and only if $J \cdot I^n = I^{n+1}$ for $n \gg 0$, and a *minimal reduction* if it is minimal among all reductions with respect to inclusion.

Theorem 5.7.16. *If $R/\mathfrak{m}$ is infinite, then $\mathfrak{l}(I)$ is the minimal number of generators of a minimal reduction of I.*

Proof. See [NR54] or [McA83]. □

The following theorem is 1.4.2 in [Mor91].

Theorem 5.7.17. *Let I be an ideal in a formally equidimensional ring, such that the height of I is equal to the analytic spread, $h(I) = \mathfrak{l}(I)$. Then the integral closure of I has no embedded components.*

The next theorem is 1.4.3 in [Mor91], where it is called Dade's theorem.

Theorem 5.7.18. *Let $(R,\mathfrak{m})$ be a formally equidimensional ring, whose residue field is infinite. For an ideal I the following conditions are equivalent:*

1. $\mathfrak{l}(I) = h(I)$.

2. *There is a sequence* $x = x_1, \ldots, x_r$ *in* R *such that*

   (a) $(x) + I$ *is* $\mathfrak{m}$*-primary.*

   (b) $\dim\big(R/(x)\big) = \dim R - r = \dim R - \dim R/I$.

   (c) If $r < \dim R$, *then* $e(x; I) = e_{R/(x)}\big(I \cdot \big(R/(x)\big)\big)$.

   (d) If $r = \dim R$, *then* $e(x; I) = e\big((x)\big)$.

Theorem 5.7.19. *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local Noetherian ring,* $\mathfrak{p}_i \subset R$ *primes such that* $R_{\mathfrak{p}_i}$ *is regular and* $I = \bigcap_{i=1}^r \mathfrak{p}_i^{(m_i)}$ *an ideal containing non-zero-divisors. Assume that* R *is excellent and reduced.*

If $\mathfrak{l}(I^{(k)}) = h(I^{(k)})$, *then the symbolic algebra of* $I^{(k)}$ *is finitely generated.*

Remark 5.7.20. Let X be a reduced, equidimensional complex space, $p \in X$. Then the local ring $\mathcal{O}_{X,p}$ satisfies the conditions of the theorem.

For the proof we need the following lemma, which is included in [LJT73, prop. 1.14].

Lemma 5.7.21. *Assume that* A *is an excellent, reduced ring and* $I \subset A$ *an ideal containing a non-zero divisor. Then there exists an integer* N *such that for all* $n \geq N$: $I \cdot \bar{I}^n = \overline{I^{n+1}}$.

Proof of theorem 5.7.19. It follows trivially from $\mathfrak{l}(I^{(k)}) = h(I^{(k)})$ that $\mathfrak{l}(I^{(k)n}) = \mathfrak{l}(I^{(k)}) = h(I^{(k)}) = h(I^{(k)n})$. The symbolic powers of I are integrally closed by theorem 5.6.4, so we have an inclusion $I^{(kn)} \supset \overline{I^{(k)n}}$. By theorem 5.7.17, we have an inclusion $I^{(kn)} \subset \overline{I^{(k)n}}$, so the symbolic powers of $I^{(k)}$ are the integral closures of the ordinary powers. By lemma 5.7.21, this implies that the symbolic algebra is finitely generated.

□

Remark 5.7.22. Under the additional assumptions that $V(I)$ is a curve and R is a three-dimensional, regular ring, $\mathfrak{l}(I^{(k)}) = h(I^{(k)})$ implies $I^{(k)n} = I^{(kn)}$ for all n, which is stronger than the statement in the theorem.

If I is the ideal of a curve in a ring with infinite residue field, the converse of the theorem also holds:

Theorem 5.7.23. *Let* $(R, \mathfrak{m})$ *be a formally equidimensional, local Noetherian ring of dimension* d *with infinite residue field,* $I \subset R$ *an ideal of height* 1 *and* $k \in \mathbb{N}$.

If $(I^{(k)n}) = (I^{(kn)})$ *for all* n, *then* $\mathfrak{l}(I^{(k)}) = h(I^{(k)}) = d - 1$.

Proof. We want to use theorem 5.7.18, so all we have to do is to show the following: Let $x \in R$ such that $(x)+I$ is an $\mathfrak{m}$-primary ideal. Then $e(x; I^{(k)}) = e\big(I^{(k)} \cdot (R/(x))\big)$.

We have

$$\begin{aligned} \lambda\big(R/(x) + I^{(k)n}\big) &= \lambda\big(R/(x) + I^{(kn)}\big) && \text{by assumption} \\ &= e\big((x)|R/I^{(kn)}\big) && \text{because } R/I^{(kn)} \text{ is Cohen-Macaulay} \\ &= e\big((x)|R/I^{(k)n}\big) && \text{by assumption} \\ &= e\big(x|R/I^{(k)n}\big). \end{aligned}$$

So the polynomials whose leading terms define $e_{R/(x)}\big(I^{(k)} \cdot (R/(x))\big)$ and $e(x; I^{(k)})$ are the same. □

5.7.3 Main Theorem

Theorem 5.7.24. *Let $(R, \mathfrak{m})$ be a d-dimensional, regular local ring with infinite residue field and $I = \bigcap \mathfrak{p}_i^{(n_i)} \subset R$ an ideal of height 1 without embedded components.*

Then the following eleven statements are equivalent:

1. *The symbolic algebra $\bigoplus I^{(n)}$ is finitely generated.*
2. *The ring $\bigoplus I^{(n)}$ is Noetherian.*
3. *$\exists k \in \mathbb{N}$ such that $I^{(k)n} = I^{(kn)}$ for all $n \in \mathbb{N}$.*
4. *$\exists k \in \mathbb{N}$ such that $\mathfrak{l}(I^{(k)}) = d - 1 = h(I^{(k)})$.*
5. *$\exists k \in \mathbb{N}$ such that the dimension of the special fibre of the symbolic blowup of R in $I^{(k)}$ is equal to $d - 2$.*
6. (a) *$\exists k \in \mathbb{N}$ and $f_1, \dots, f_{d-1} \in I^{(k)}$, such that for all $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$:*
$$e(x, f_1, \dots, f_{d-1}) = e(x; I^{(k)}).$$
   (b) *$\exists k \in \mathbb{N}$ and $f_1, \dots, f_{d-1} \in I^{(k)}$, such that for some $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$:*
$$e(x, f_1, \dots, f_{d-1}) = e(x; I^{(k)}).$$
7. (a) *$\exists k \in \mathbb{N}$ and $f_1, \dots, f_{d-1} \in I^{(k)}$, such that for all $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$:*
$$\lambda\big(R/(x, f_1, \dots, f_{d-1})\big) = k^{d-1} \sum n_i^{d-1} \lambda\big(R/(x) + \mathfrak{p}_i\big).$$

(b) *$\exists k \in \mathbb{N}$ and $f_1, \dots, f_{d-1} \in I^{(k)}$, such that for some $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$:*

$$\lambda\big(R/(x, f_1, \dots, f_{d-1})\big) = k^{d-1} \sum n_i^{d-1} \lambda\big(R/(x) + \mathfrak{p}_i\big).$$

8. (a) *$\exists k \in \mathbb{N}$ and $f_j \in I^{(k_j)}$, $j = 1, \dots, d-1$, such that for all $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$:*

$$\lambda\big(R/(x, f_1, \dots, f_{d-1})\big) = (\prod_{j=1}^{d-1} k_j) \cdot \sum n_i^{d-1} \lambda\big(R/(x) + \mathfrak{p}_i\big).$$

(b) *$\exists k \in \mathbb{N}$ and $f_j \in I^{(k_j)}$, $j = 1, \dots, d-1$, such that for some $x \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i$:*

$$\lambda\big(R/(x, f_1, \dots, f_{d-1})\big) = (\prod_{i=j}^{d-1} k_j) \cdot \sum n_i^{d-1} \lambda\big(R/(x) + \mathfrak{p}_i\big).$$

Proof. The equivalence of the first three statements is well known, see [Sch88].

The equivalence of 3 and 4 is given by theorem 5.7.19 using the equivalence of 1 and 3 and theorem 5.7.23.

The equivalence of 4 and 5 is given by remark 5.7.15.

The implications from a to b in the last three statements are trivial.

4 $\implies$ 6a: By theorem 5.7.16, the condition $\mathfrak{l}(I^{(k)}) = d-1$ implies that there is a reduction $(f_1, \dots, f_{d-1})$ of $I^{(k)}$ generated by $d-1$ elements. Corollaries 5.7.9 and 5.7.10 imply

$$e(x; I^{(k)}) = e\big(x; (f_1, \dots, f_{d-1})\big) = e(x, f_1, \dots, f_{d-1}).$$

6b $\implies$ 4: Dade's Theorem 5.7.18 says that we have to show that

$$e(x; I^{(k)}) = e_{R/(x)}(I^{(k)}).$$

The residue classes $\bar{f}_1, \dots, \bar{f}_{d-1}$ von $f_1, \dots, f_{d-1}$ in $R/(x)$ are a system of parameters of $I^{(k)} \cdot R/(x)$. According to [Nor68, S.300] $e_R(x, f_1, \dots, f_{d-1}) = e_{R/(x)}(\bar{f}_1, \dots, \bar{f}_{d-1})$. We conclude:

$$e_{R/(x)}(I^{(k)}) \leq e_{R/(x)}(\bar{f}_1, \dots, \bar{f}_{d-1}) = e_R(x, f_1, \dots, f_{d-1}) = e(x; I^{(k)}).$$

On the other hand, we notice $I^{(k)n} \cdot R/(x) \subset I^{(kn)} \cdot R/(x)$, so

$$\lambda\big((R/(x))/I^{(k)n}\big) \geq \lambda\big((R/(x))/I^{(kn)}\big).$$

By lemma 5.7.12 a comparison of leading terms implies

$$e_{R/(x)}(I^{(k)}) \geq e(x; I^{(k)}).$$

6 $\Leftrightarrow$ 7: Since R is regular and Cohen-Macaulay, we have $e(x, f_1, \ldots, f_{d-1}) = \lambda\big(R/(x, f_1, \ldots, f_{d-1})\big)$. Corollary 5.7.11 implies

$$e(x; I^{(k)}) = k^{d-1} \cdot \sum n_i^{d-1} \lambda\big(R/(x) + \mathfrak{p}_i\big).$$

7 $\Leftrightarrow$ 8: The equivalence follows from

$$e(x_1^{n_1}, \ldots, x_r^{n_r} | E) = (\prod_{i=1}^{r} n_i) \cdot e(x|E),$$

cf. remark 5.7.5. □

5.7.4 Application to the Kollár Conjecture

Let us now apply theorem 5.7.24 to the symbolic power algebra of the conductor considered in theorem 5.5.2.

Theorem 5.7.25. *Let $X(C_t, l_t)$ be a 1-parameter smoothing of $X(C, l)$ and J the induced deformation of the conductor of C.*

The following statements are equivalent:

1. *The statement of the Kollár conjecture holds for the smoothing $X(C_t, l_t)$.*
2. *There is an $k \in \mathbb{N}$ such that $J^{(k)n} = J^{(kn)}$ for all $n \in \mathbb{N}$.*
3. *There is an $k \in \mathbb{N}$ such that $J^{(k)}$ is an equimultiple deformation.*

If, in addition to the above assumptions, C_t has r singularities which are ordinary of multiplicity $n_1, \ldots, n_r$, then the above three statements are also equivalent to the following:

4. *There exist $k_1, k_2 \in \mathbb{N}$ and $f_1 \in J_0^{(k_1)}, f_2 \in J_0^{(k_2)}$ with intersection multiplicity*

$$\langle f_1, f_2 \rangle = k_1 k_2 \cdot \sum_{i=1}^{r} (n_i - 1)^2.$$

Proof. Theorem 5.5.2 says that the Kollár conjecture holds if and only if the symbolic algebra of the conductor J is always finitely generated.

By theorem 5.7.24, the symbolic algebra is finitely generated if and only if $\exists k \in \mathbb{N}$ such that $J^{(k)n} = J^{(kn)}$ for all $n \in \mathbb{N}$.

Also by theorem 5.7.24, the symbolic algebra of the conductor J is finitely generated if and only if $\exists k \in \mathbb{N}$ such that the exceptional divisor of the blowup in $J^{(k)}$ has no components which are contained in the special fibre. By theorem 3.5.2 this is equivalent to the fact that $J^{(k)}$ is an equimultiple deformation.

Finally, we show that the third and the fourth statement are equivalent. We may assume that $k_1 = k_2 = k$ in the fourth statement; otherwise set $k = lcm(k_1, k_2)$ and replace f_i by its (k/k_i)-th power. The term $k^2 \sum(n_i - 1)^2$ is the multiplicity of a fibre $J_t^{(k)}$ with $t \neq 0$. From corollary 1.5.4 and the semicontinuity of the multiplicity we get

$$\langle f_1, f_2 \rangle \geq e(J_0^{(k)}) \geq e(J_t^{(k)}) = k^2 \sum (n_i - 1)^2.$$

Equality holds if and only if f_1 and f_2 are generic and $J^{(k)}$ is an equimultiple deformation, thus proving the equivalence. □

Remark 5.7.26. We see from the proof that the relative multiplicity $e(t; J^{(k)})$ is equal to the multiplicity of a fibre $J_t^{(k)}$ for small $t \neq 0$.

In the proof we have not used the last criteria of theorem 5.7.24. To see how they fit into the picture, I will now show how we could have deduced the fourth statement of theorem 5.7.25 directly from the last statement of theorem 5.7.24:

We identify

$$\begin{aligned}
R &= \mathbb{C}\{x, y, t\} \\
I &= \text{deformation of the conductor} \\
&= \bigcap_i \mathfrak{p}_i^{n_i - 1} \\
x &= t \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_i \\
\lambda\big(R/(x, f_1, \dots, f_{d-1})\big) &= \dim_{\mathbb{C}} \mathbb{C}\{x, y, t\}/(t, f_1, f_2) \\
&= \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f_1, f_2) \\
&= \langle f_1, f_2 \rangle \\
\lambda\big(R/(x) + \mathfrak{p}_i\big) &= \dim_{\mathbb{C}} \mathbb{C}\{x, y, t\}/\big((t) + \mathfrak{p}_i\big) \\
&= \text{number of points in } V(\mathfrak{p}_i) \cap (\mathbb{C}^2 \times \{t\}) \text{ for } t \neq 0
\end{aligned}$$

The last equality holds because $\mathfrak{p}_i$ has no embedded components, so the curve $V(\mathfrak{p}_i)$ is a flat deformation over t, which means that the degree of $(\mathfrak{p}_i)_t$ is constant in t. For $t \neq 0$, the points are all reduced, so the degree is equal to the number of points in $V(\mathfrak{p}_i) \cap (\mathbb{C}^2 \times \{t\})$.

5.8 Deformations of the Conductor

Recall that the conductor of the isolated plane curve singularity $(C,0) \subset (\mathbb{C}^2,0)$ is the $\mathfrak{m}_{\mathbb{C}^2,0}$-primary, complete ideal whose multiplicity at an infinitely near point p is $e_p(C)-1$, see 1.2.2. Since the degree of the conductor is $\delta(C)$, a deformation C_t of C induces a deformation of the conductor if and only if C_t is a δ-constant deformation.

Lemma 5.8.1. *Let C_t be a δ-constant 1-parameter deformation of C into s ordinary singularities with multiplicities $m_1,\dots,m_s$ and $J \subset \mathcal{O}_{\mathbb{C}^2\times\mathbb{C}}$ the induced deformation of the conductor. Then*

$$\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\big(J^{(n)}/(t)\big) = \sum_{i=1}^{s} \frac{n(m_i-1)\cdot\big(n(m_i-1)+1\big)}{2}.$$

Proof. The conductor of an ordinary plane curve singularity of multiplicity m is the $(m-1)$-th power of the maximal ideal. So the term on the right is the degree of a fibre $J_t^{(n)}$ with $t \neq 0$. Since the symbolic power has no embedded components, this is equal to the degree of $J_0^{(n)} = J^{(n)}/(t)$. $\square$

Theorem 5.8.2. *Let $X(C_t,l_t)$ be a (generic) smoothing of $X(C,l)$, J the induced deformation of the conductor of C. Assume that $J^{(k)}/(t) \subset \mathcal{O}_{\mathbb{C}^2,0}$ is a complete ideal.*

Then $J^{(k)n} = J^{(kn)}$ for all $n \in \mathbb{N}$ if and only if $\Sigma(J^{(k)})$ is a δ-constant deformation.

Proof. This follows directly from theorem 3.4.4 on page 55. $\square$

There are cases where $J^{(k)}$ is not relative complete, but $J^{(k)n} = J^{(kn)}$ for all $n \in \mathbb{N}$. This corresponds to non-normal P-modifications of the sandwiched singularity $X(C,l)$, see section 5.5.2 and example 5.9.4.

For these cases we know:

Theorem 5.8.3. *Let $X(C_t,l_t)$ be a generic smoothing of $X(C,l)$ and J the induced deformation of the conductor of C. Assume that $J^{(k)}/(t)$ is not complete and that $J^{(k)n} = J^{(kn)}$ for all $n \in \mathbb{N}$.*

Then the deformation $\Sigma(J^{(k)})$ is not δ-constant.

Proof. This follows from proposition 3.4.7 on page 58. $\square$

Smoothings over the Artin component of $X(C,l)$ correspond to Scott deformations of C. This implies that the induced deformation of the conductor is relative complete and δ-constant, which gives a 'sandwiched proof' of the well known fact that the statement of the Kollár conjecture is true for smoothings over the Artin component:

Corollary 5.8.4. *Let $X(C_t, l_t)$ be a smoothing of $X(C, l)$ over the Artin component, J the induced deformation of the conductor of C. Then $J^n = J^{(n)}$ for all n, so the symbolic algebra is generated by its degree-one part.*

Proof. This follows from the last statement of theorem 3.4.8. □

Conjecture 5.8.5. $J^n = J^{(n)}$ for all n if and only if the deformation is over the Artin component.

Remark 5.8.6. This conjecture is true if and only if conjecture 3.4.10 is true.

5.9 Examples

Proposition 5.9.1 (Deformation into ordinary double points). *Let C_t be a 1-parameter deformation of an isolated plane curve singularity C into $\delta(C)$ ordinary double points, J the induced deformation of the conductor.*

Then the symbolic algebra of J is finitely generated (by $J \oplus J^{(2)}$).

Proof. Let $F \in \mathcal{O}_{\mathbb{C}^2 \times T, 0}$ be an equation of the deformation C_t. Since the general fibre of the deformation has only ordinary double points, the curve defined by J is the reduced set of singular points $\bigcup_{t \in T} Sing(C_t)$ and $F \in J^{(2)}$.

By theorem 1.2.21, the conductor $J_0 \subset \mathcal{O}_{\mathbb{C}^2}$ of the curve C is the complete ideal whose base points are the infinitely near points with $e_p(C) > 1$ with multiplicities $e_p(J_0) = e_p(C) - 1$.

So the intersection multiplicity of F_0, which is an equation of C, with a generic element of J_0 is $\sum_{p \in \mathcal{N}_0} e_p(C)(e_p(C) - 1) = 2\delta(C)$, satisfying the last criterion of theorem 5.7.25. □

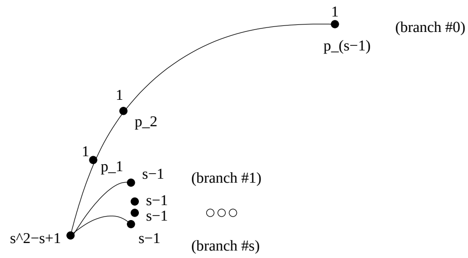
Proposition 5.9.2. *Let C be an ordinary singularity of multiplicity $s + 1$ and C_t a deformation with combinatorial representation*

$$(0, 1, \ldots, s) \to (0, 1)(0, 2) \ldots (0, s)(1, 2, \ldots s).$$

In other words, C_t is a δ-constant deformation into one ordinary singularity of multiplicity s and s ordinary double points, obtained by 'moving away one branch from the singularity'. Let J denote the induced deformation of the conductor.

Then the symbolic algebra of J is finitely generated (by $\bigoplus_{n=1}^{s} J^{(n)}$).

Proof. Let $C_0 = V(f_0)$ be the branch which is moved by the deformation, $C_i = V(f_i)$, $i = 1, \ldots, s$ the branches that are fixed and F_i equations of the deformations. Then the restriction of $F_0 \cdot \prod_{i=1}^{s} F_i^{s-1}$ to a fibre with $t \neq 0$ passes through the s-fold point of C_t with multiplicity $s(s-1)$ and through each of the double points with multiplicity s, so $F_0 \cdot \prod_{i=1}^{s} F_i^{s-1} \in J^{(s)}$. For this to work, we only require F_0 to pass through the s double points and F_i to pass through two singular points. That means that if we alter f_0 by adding a function of order s and f_i by adding a function of order 2, then we end up with functions f_0' and f_i' for which we can find deformations F_i' passing through the same points, i.e. $F_0' \cdot \prod_{i=1}^{s} F_i'^{s-1} \in J^{(s)}$. So $J_0^{(s)}$ contains the complete ideal $I = \big((f_0) + (x,y)^s\big) \cdot \prod_{i=1}^{s} \big((f_i) + (x,y)^2\big)^{s-1}$, whose weighted cluster of base points looks like this:



The multiplicity of the complete ideal I is equal to $\sum e_p^2(I)$, which is:

$$\begin{aligned}(s^2 - s + 1)^2 + s \cdot (s-1)^2 + (s-1) \cdot 1^2 &= s^4 - s^3 + s^2 \\ &= s^2 \cdot \big((s-1)^2 + \underbrace{1^2 + \cdots + 1^2}_{s \text{ times}}\big).\end{aligned}$$

So the criterion of theorem 5.7.25 is satisfied. We can also deduce that $J^{(s)}$ is equal to this complete ideal, because otherwise $e(J_0^{(s)}) < e(J_t^{(s)})$ for $t \neq 0$, but the multiplicity is upper-semicontinuous. (We could also check that $J^{(s)}$ is the given complete ideal by computing the degree.) ☐

Example 5.9.3. Now let us look at δ-constant deformations of ordinary singularities with low multiplicity.

A δ-constant deformation of an ordinary double point is trivial.

An ordinary triple point has two kinds of δ-constant deformations: the trivial deformation, which in this case is a Scott deformation, and the deformation into three double points. We know for both that the symbolic power is finitely generated.

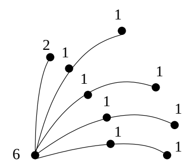
An ordinary quadruple point has three kinds of δ-constant deformations: the trivial or Scott deformation, the deformation into six double points and deformations into one triple and three double points. By the above results, the symbolic algebra is finitely generated in all these cases.

An ordinary quintuple point has five kinds of δ-constant deformations: the first three are the Scott deformation, the deformation into δ ordinary double points and the deformations into an ordinary quadruple point and four double points. Of these we know that the symbolic power is finitely generated by the above results. We are left with deformations into one triple and seven double points and with deformations into two triple and four double points.

Let us deal with the deformations into two triple points and four double points first. Up to a permutation of branches the corresponding combinatorial deformation is $(12345) \to (123)(145)\,(24)(25)(34)(35)$. As in the proof of the preceding proposition we see that $F_1^2 \cdot \prod_{i=2}^5 F_i \in J^{(2)}$ and

$$J_0^{(2)} = \big(f_1 + (x,y)^2\big)^2 \cdot \prod_{i=2}^{5} \big(f_i + (x,y)^3\big).$$

The cluster of base points of $J_0^{(2)}$ has Enriques diagram

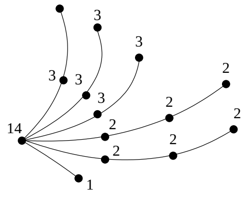


and the multiplicity of $J_0^{(2)}$ is 48, which satisfies our criterion for the finite generation of the symbolic power.

With the deformation into one triple and seven double points, a new phenomenon occurs. Up to a permutation of branches the corresponding combinatorial deformation is $(12345) \to (123)\,(14)(15)(24)(25)(34)(35)(45)$. Let f_i be equations of the five branches as before and g the equation of a smooth branch having a tangent direction such that it can be deformed to pass through the triple point and the intersection of the fourth and the fifth branch for $t \neq 0$. Then $F_1^3 F_2^3 F_3^3 F_4^2 F_5^2 G \in J^{(5)}$ and by the same kind of computations as before we see

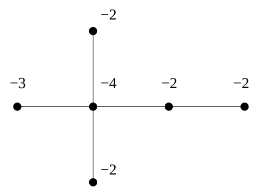
$$\begin{aligned} J_0^{(5)} = {} & \big(f_1 + (x,y)^3\big)^3 \cdot \big(f_2 + (x,y)^3\big)^3 \cdot \big(f_3 + (x,y)^3\big)^3 \\ & \cdot \big(f_4 + (x,y)^4\big)^2 \cdot \big(f_5 + (x,y)^4\big)^2 \cdot \big(g + (x,y)^2\big). \end{aligned}$$

(The reader may find it helpful to draw himself a picture of a fibre with $t \neq 0$.) So unlike before, the fibre over $t = 0$ of the symbolic powers of the deformed conductor has base points which are not on the curve we have started with! An Enriques diagram of $J_0^{(5)}$ is

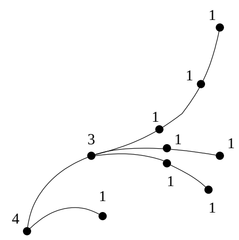


The multiplicity is $275 = 5^2 \cdot (2^2 + 7 \cdot 1^2)$, so our criterion is satisfied.

Example 5.9.4. Now I want to study example 6.3.3 of [Kol91]. Consider a normal surface singularity with the following dual resolution graph:



This singularity is sandwiched. It is the only singularity on the blowup of $\mathbb{C}^2$ in a complete ideal with the following Enriques diagram:



The complete ideal can be represented as $I = I(C, l)$ where (C, l) is a decorated with four smooth branches such that the first three branches have pairwise intersection multiplicity two, the fourth is transversal to the others and $l(1) = 5$, $l(2) = l(3) = 4$ and $l(4) = 2$. The combinatorial deformations of this decorated curve are the following:

1. (1234) (123) $(1)^3(2)^2(3)^2(4)$ (Scott deformation).

2. (1234) (12)(13)(23) $(1)^3(2)^2(3)^2(4)$.

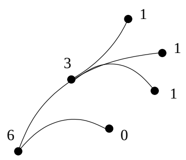
3. (123) (124) (13)(23)(34) $(1)^2(2)$,
   (123) (134) (12)(23)(24) $(1)^2$ (3) and
   (123) (234) (12)(13)(14) (1) (2)(3).

4. (12)(13)(23) (234) (12)(13)(14) = (234) $(12)^2(13)^2(14)(23)$.

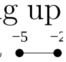
Since C has four smooth branches, all combinatorial deformations can be realized, which means that the singularity has six smoothing components. In [Kol91] six P-modifications have been constructed, so there is one for each smoothing component.

The first combinatorial deformation is the Scott deformation.

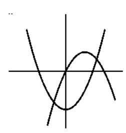
The second combinatorial deformation corresponds to the second P-modification in [Kol91], which is constructed from the minimal resolution by contracting the (-4)-curve and the (-2)-curve on the right. As in the previous examples we note that $f_1 f_2 f_3 f_4^3 \in J_0^{(2)}$ and deduce that $J_0^{(2)}$ is a complete ideal with Enriques diagram



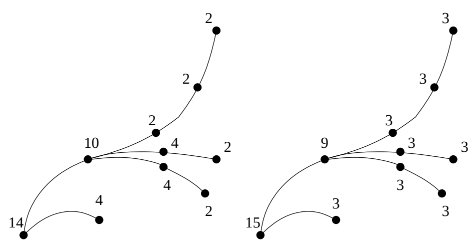
The multiplicity is $e(J_0^{(2)}) = 48 = 2^2(9+1+1+1) = e(J_t^{(2)})$, which means our criterion for finite generation of the symbolic power algebra is satisfied. Compare this with the general description of the construction of a P-modification corresponding to a smoothing $X(C_t, l_t)$ in section 5.5.2, and note that indeed all base points of $J^{(2)}/(t) = J_0^{(2)}$ are also base points of (C, l), and that the curves of the minimal resolution which we had to contract to obtain the P-modification are those which correspond to base points where the excess of $J^{(2)}/(t)$ is zero.

The next three P-modifications in [Kol91, example 6.3.3] are obtained from the minimal resolution by blowing up the intersection point of the (-3)- and the (-4)-curve and contracting a  configuration and the (-2)-curve on the right, if it does not intersect the contracted curves. They are normal P-modifications which are not dominated by the minimal resolution. Obviously, these three P-modifications correspond to the next three combinatorial deformations in our list. We see that $J^{(k)}/(t)$ seems to have an additional satellite base point in these three cases.

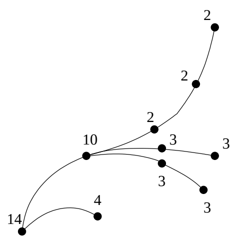
The last combinatorial deformation must correspond to the non-normal P-modification. The general fibre of a realization of the last combinatorial deformation looks like this:



We see that $J_0^{(6)}$ contains the two complete ideals with Enriques diagrams



So the integral closure of $J_0^{(6)}$ must be contained in the complete ideal with Enriques diagram



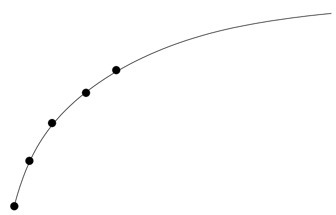
The multiplicity of this ideal is 360, satisfying our criterion, so it must be the integral closure of $J_0^{(6)}$. But the degree of this complete ideal is 203, while the degree of $J_0^{(6)}$ is equal to the degree of the fibres $J_t^{(6)}$ with $t \neq 0$, which is 204.

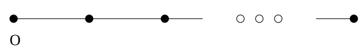
We find that the sixth symbolic power is a deformation with constant multiplicity which is not relative complete. So as conjectured in section 5.5.2, the fact that the P-modification is not normal corresponds to the fact that $J_0^{(6)}$ is not complete.

Appendix A

List of Taut Curves

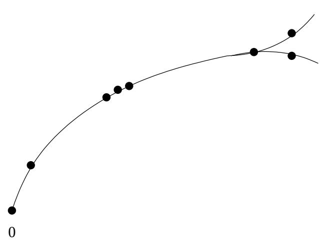
1. A_0 (smooth curve)
   Equation: x.
   Enriques diagram and graph of corresponding taut sandwiched singularities, class (II) in [Lau73]:

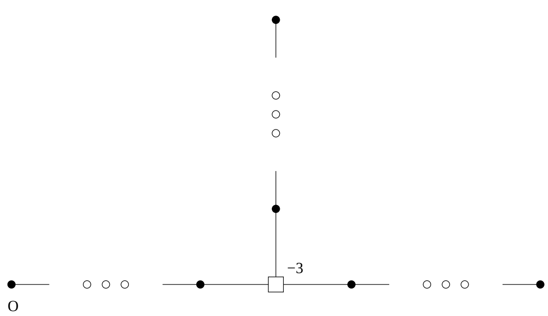




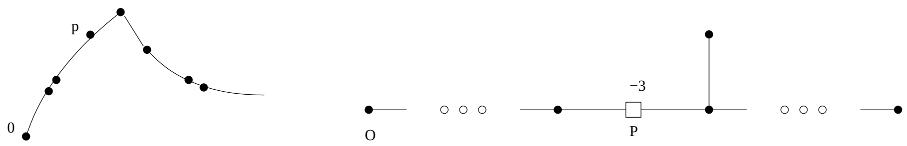
2. A_k, $k \geq 1$ (curve with two smooth branches)
   Equation: $x^2 + y^{k+1}$.

   (a) k odd.
   Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.1) in [Lau73]:



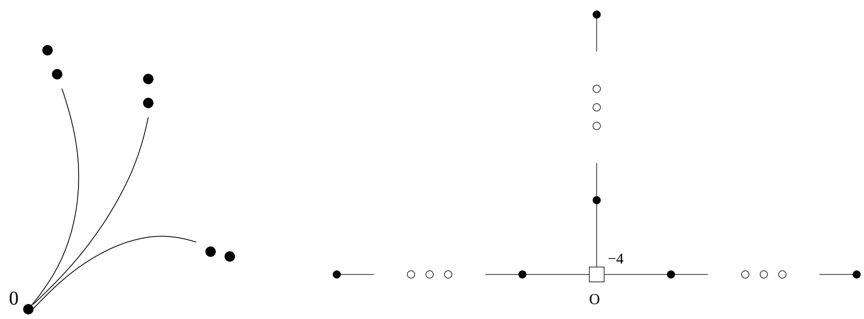


(b) k even.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.3) in [Lau73]:

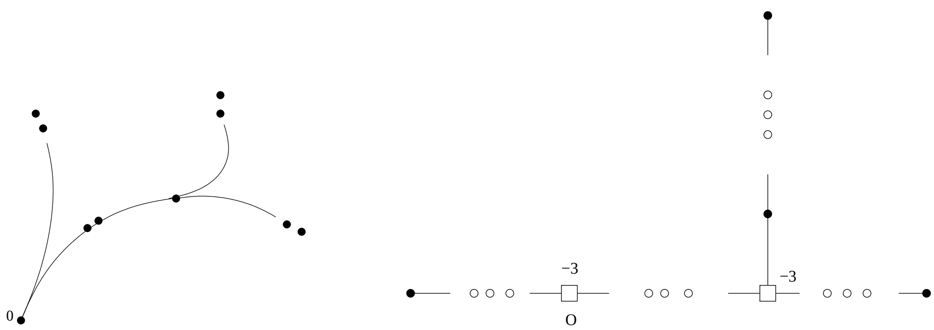


3. D_k, $k \geq 4$.
Equation: $x^2y + y^{k-1})$.

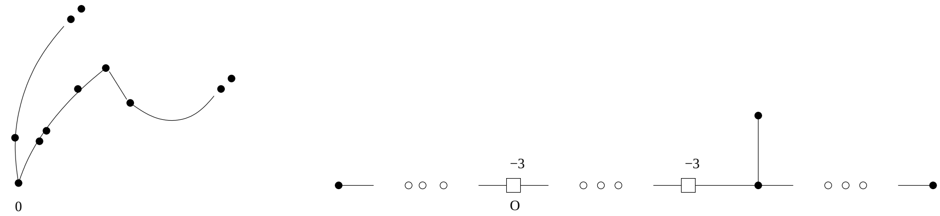
(a) $k = 4$.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.1) in [Lau73]:



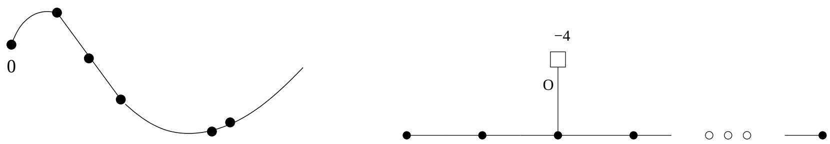
(b) $k > 4$ even.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.1) in [Lau73]:



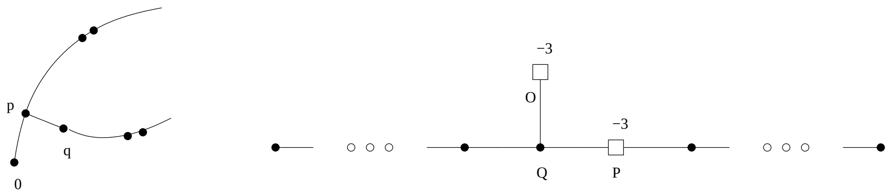
(c) k odd.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.3) in [Lau73]:



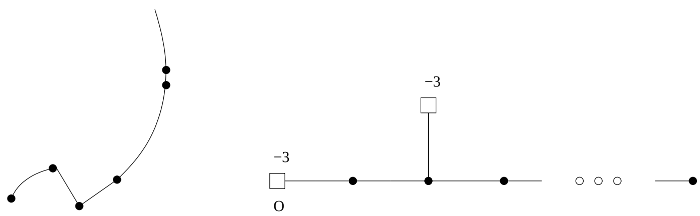
4. The E_6-singularity
   Equation: $x^3 + y^4$.
   Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.4) in [Lau73]:



5. The E_7-singularity
   Equation: $x(x^2 + y^3)$.
   Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.3) in [Lau73]:

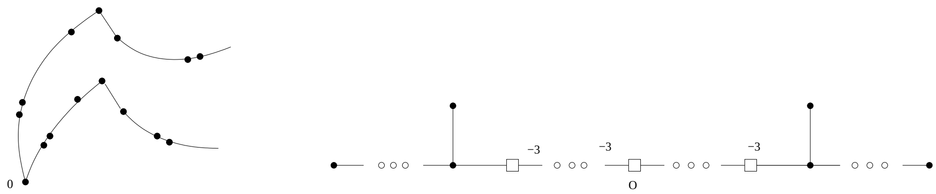


6. The E_8-singularity
   Equation: $x^3 + y^5$.
   Enriques diagram and graph of corresponding taut sandwiched singularities, class (III.4) in [Lau73]:

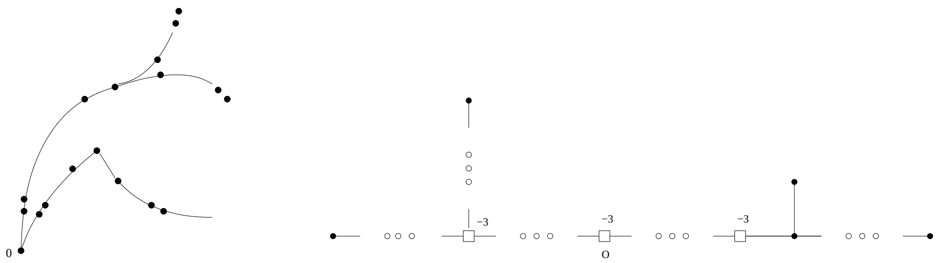


7. $T_{2,p,q}$, $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, $3 < p \leq q$ (two A_k-singularities A_{p-3} and A_{q-3}, transversal to each other)
   Remark: The $T_{2,3,q}$, $q \geq 7$, are the $J_{2,i}$-singularities, $i > 0$.
   Normal form: $(ax^2 + y^{p-2})(ay^2 + x^{q-2})$, $a \neq 0$.

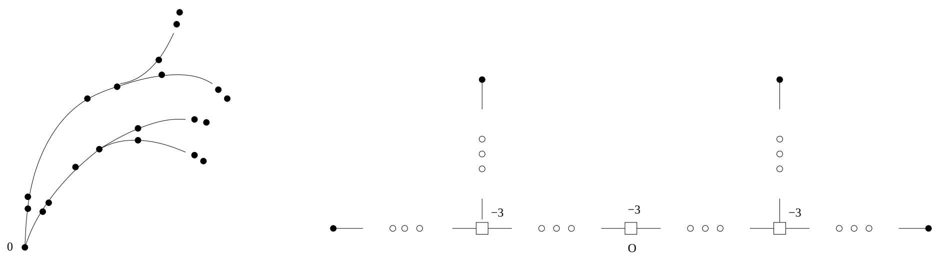
(a) $p-3$ and $q-3$ even.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_8, R_8) in [Lau73]:



(b) $p-3k$ even, $q-3$ odd.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_1, R_8) in [Lau73]:

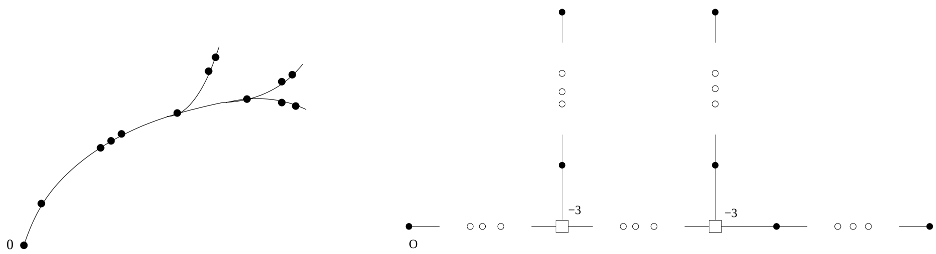


(c) $p-3$ and $q-3$ odd.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_1, R_1) in [Lau73]:

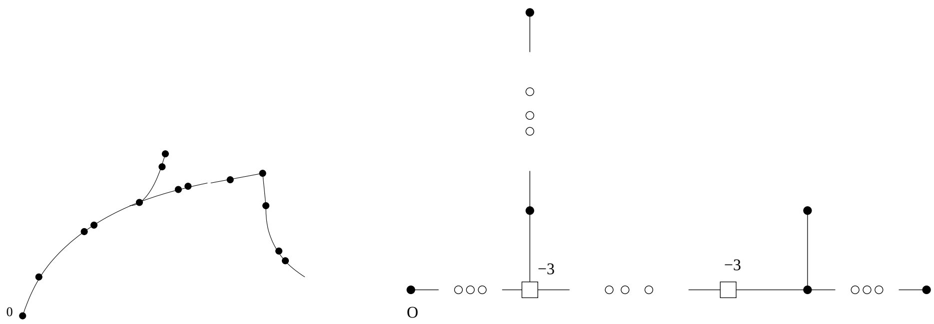


8. $J_{k,i}$, $k \geq 2$, $i > 0$ (An A_{2k+i-1}-singularity plus a smooth branch having contact order k)
Remark: The $J_{2,i}$, $i > 0$ are the $T_{2,3,6+i}$-singularities.
Normal form from [AGZV85]: $x^3 + x^2y^k + (a_0 + \cdots + a_{k-2}y^{k-2})y^{3k+i}$, $a_0 \neq 0$. (Note that there is a printing error in [AGZV85].)

(a) i even.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_1, R_1) in [Lau73]:

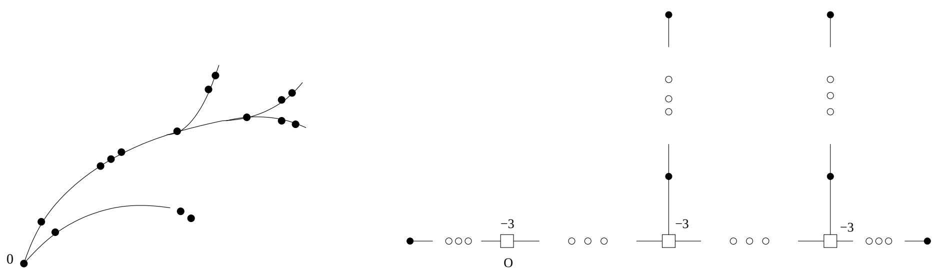


(b) i odd.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_1, R_8) in [Lau73]:

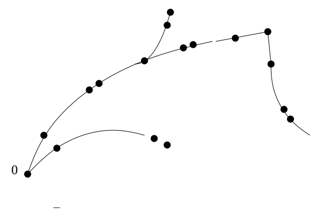


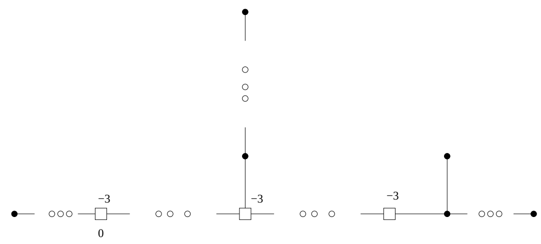
9. $Z_{i,p}$, $p > 0$ ($J_{i+1,p}$ plus a smooth branch transversal to it)
Normal form from [AGZV85]: $y\big(x^3 + x^2y^{i+1} + (b_0 + \cdots + b_iy^i)y^{3i+p+3}\big)$, $b_0 \neq 0$.

(a) p even.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_1, R_1) in [Lau73]:



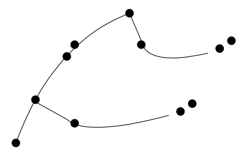
(b) p odd.
Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_1, R_8) in [Lau73]:

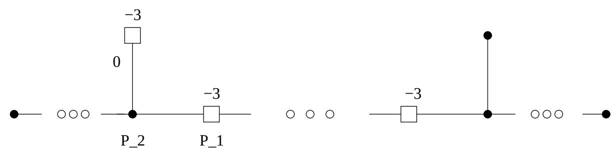




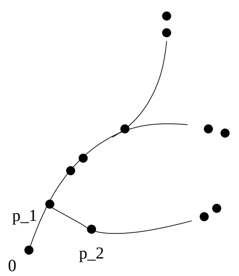
10. $W_{1,p}$, $p > 0$. (An A_2-singularity tangential to an A_{p+2}-singularity)
Normal form from [AGZV85]: $x^4 + x^2y^3 + (b_0 + b_1y)y^{6+p}$, $b_0 \neq 0$.

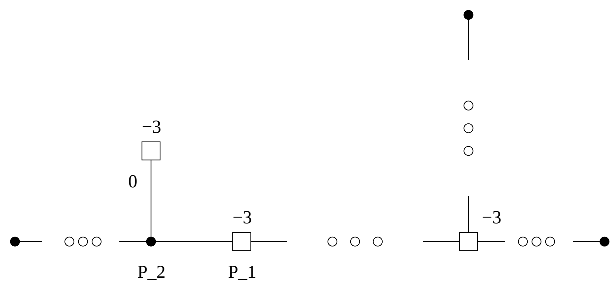
    (a) p even.
    Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_8, R_8) in [Lau73]:





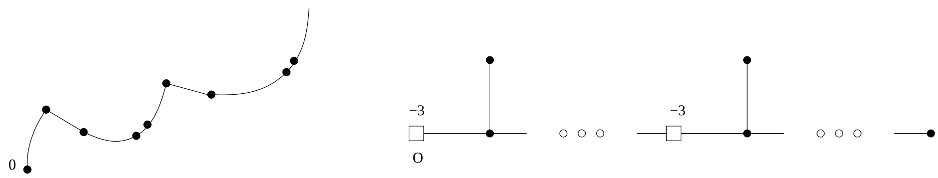
    (b) p odd.
    Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_8, R_1) in [Lau73]:



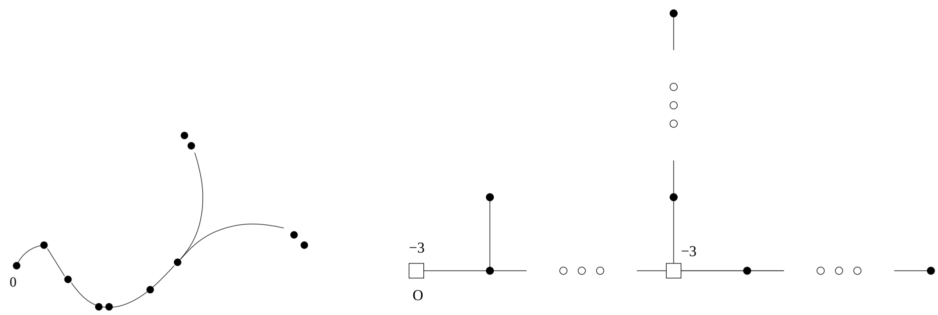


11. $W^{\#}_{1,p}$, $p > 0$.

    (a) $p = 2q - 1$ odd. (irreducible curve with two satellite points, the first one proximate to zero, the second one not proximate to zero or the first)
    Normal form from [AGZV85]: $(x^2+y^3)^2+(a_0+a_1y)xy^{4+q}$, $a_0 \neq 0$.
    Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_2, R_8) in [Lau73]:



(b) $p = 2q$ even. (Two A_2-singularities having contact order $q + 3$) Normal form from [AGZV85]: $(x^2+y^3)^2+(a_0+a_1y)x^2y^{3+q}$, $a_0 \neq 0$. Enriques diagram and graph of corresponding taut sandwiched singularities, class (IV, L_2, R_1) in [Lau73]:

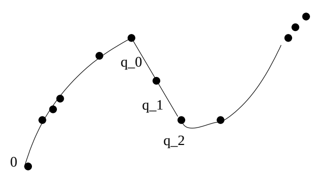


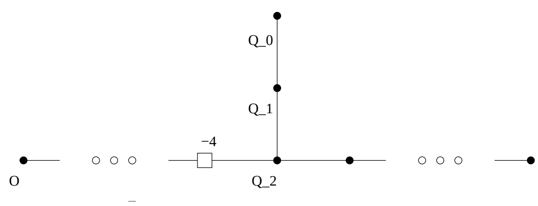
Appendix B

List of Pseudotaut Curves

This appendix contains all pseudotaut isolated plane curve singularities which are not taut.

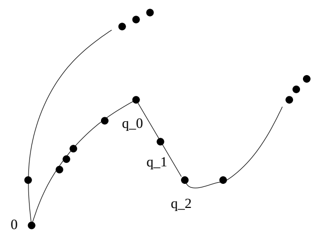
1. E_{6k}, $k > 1$ (irreducible curve with exactly two satellite points, both proximate to the same point in the $(k-1)$-th neighbourhood)
   Normal form from [AGZV85]: $x^3 + y^{3k+1} + (a_0 + \cdots + a_{k-2}y^{k-2})xy^{2k+1}$
   Enriques diagram and graph of corresponding taut sandwiched singularities:

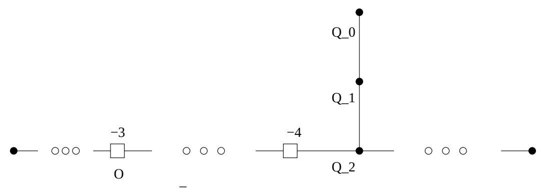
   

   The continued fraction of the left arm is > 3.
   The continued fraction of the upper arm is $\frac{3}{2}$.
   The continued fraction of the right arm tends to 1 from above.
   Graph is included in line 18 in [Lau73, table 3.2].

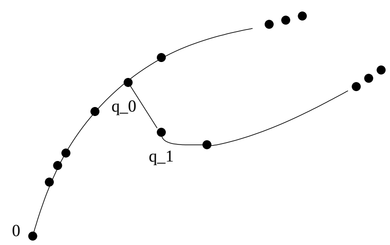
2. Z_{6i+11}, $i > 0$ ($E_{6(i+1)}$ singularity plus a transversal, smooth branch)
   Normal form from [AGZV85]: $x^3y + y^{3i+5} + (b_0 + \cdots + b_iy^i)xy^{3i+4}$
   Enriques diagram and graph of corresponding taut sandwiched singularities:

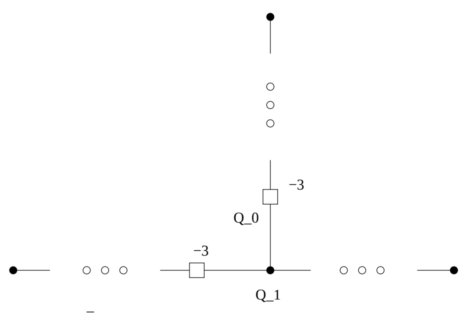




If $r = 1$, then the weight of $P_1 = P_r$ is -5.
The continued fraction of the left arm is > 3.
The continued fraction of the upper arm is $\frac{3}{2}$.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 18 in [Lau73, table 3.2].

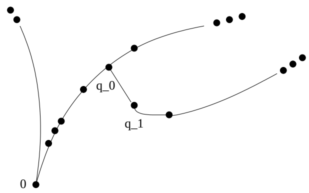
3. E_{6k+1}, $k > 1$ (irreducible A_{2k}-singularity plus a smooth branch having contact order k)
   Normal form from [AGZV85]: $x^3 + y^{2k+1} + (a_0 + \cdots + a_{k-2}y^{k-2})xy^{3k+2}$
   Enriques diagram and graph of corresponding taut sandwiched singularities:

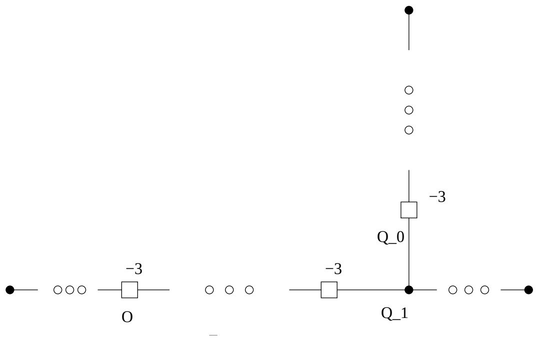




The continued fraction of the left arm is > 2.
The continued fraction of the upper arm is > 2.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 21 in [Lau73, table 3.2].

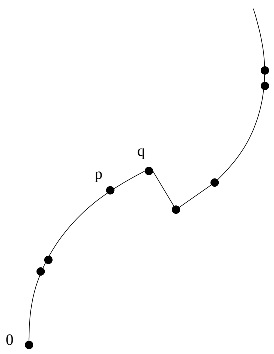
4. Z_{6i+12}, $i > 0$ ($E_{6(i+1)+1}$ singularity plus a transversal, smooth branch)
   Normal form from [AGZV85]: $y(x^3 + xy^{2i+3} + (b_0 + \cdots + b_iy^i)y^{5+3i})$
   Enriques diagram and graph of corresponding taut sandwiched singularities:

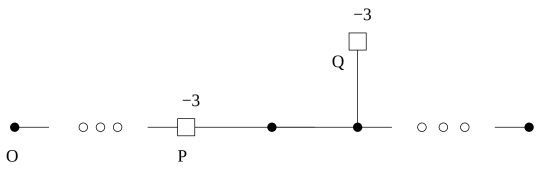




If $k = 1$, the weight of $P_1 = P_k$ is -4.
The continued fraction of the left arm is > 2.
The continued fraction of the upper arm is > 2.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 21 in [Lau73, table 3.2].

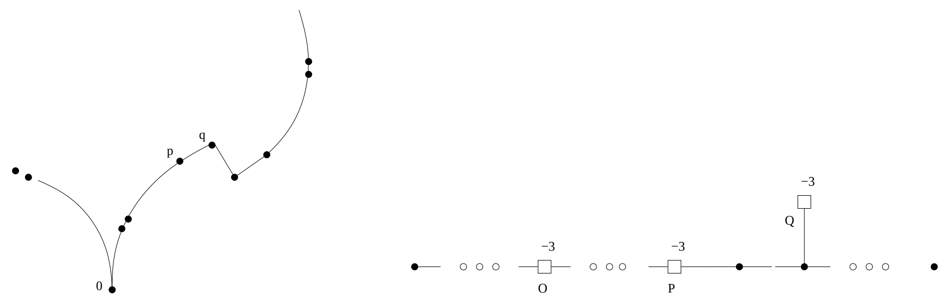
5. E_{6k+2}, $k > 1$ (irreducible curve with exactly two satellite points p and q, p proximate to q, q in the k-th neighbourhood)
Normal form from [AGZV85]: $x^3 + y^{3k+2} + (a_0 + \cdots + a_{k-2}y^{k-2})xy^{2k+2}$
Enriques diagram and graph of corresponding taut sandwiched singularities:





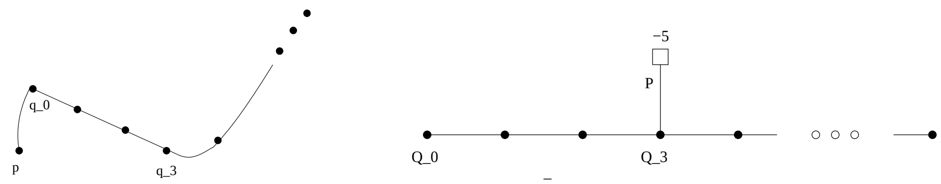
The continued fraction of the left arm is $\geq \frac{5}{3}$.
The continued fraction of the upper arm is 3.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 13 in [Lau73, table 3.2].

6. Z_{6i+13} oder Z_{6i+11}, $i > 0$ ($E_{6(i+1)+2}$ singularity plus a transversal, smooth branch)
Normal form from [AGZV85]: $x^3y + y^{3i+6} + (b_0 + \cdots + b_i y^i)xy^{2i+5}$
Enriques diagram and graph of corresponding taut sandwiched singularities:



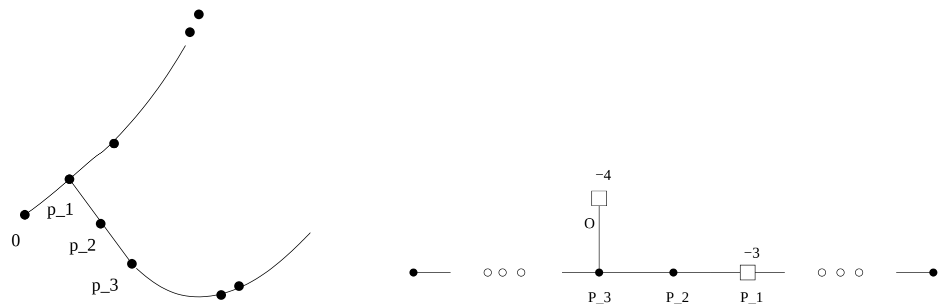
(If $k = 1$, then the weight of $O = P$ is -4.)
The continued fraction of the left arm is $\geq \frac{5}{3}$.
The continued fraction of the upper arm is 3.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 13 in [Lau73, table 3.2].

7. W_{12} (irreducible curve with exactly three satellite points, all proximate to zero)
   Normal form from [AGZV85]: $x^4 + y^5 + ax^2y^3$
   Enriques diagram and graph of corresponding taut sandwiched singularities:



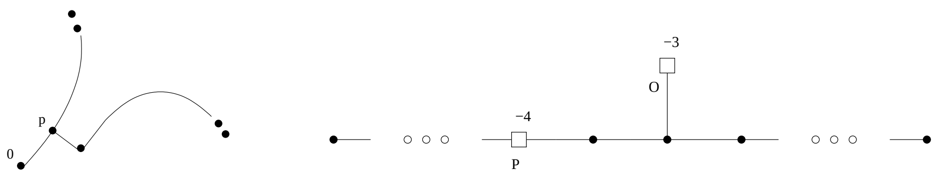
The continued fraction of the left arm is $\frac{4}{3}$.
The continued fraction of the upper arm is 5.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 7 in [Lau73, table 3.2].

8. W_{13}. (An E_6-singularity with a smooth branch tangent to it)
   Normal form from [AGZV85]: $x^4 + xy^4 + ay^6$.
   Enriques diagram and graph of corresponding taut sandwiched singularities:



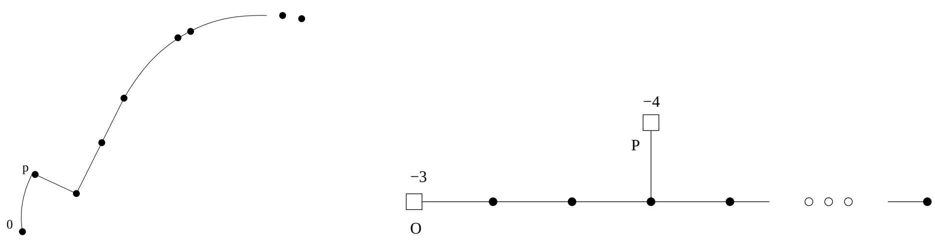
The continued fraction of the left arm tends to 1 from above.
The continued fraction of the upper arm is 4.
The continued fraction of the right arm tends to $\frac{3}{2}$ from above.
Graph is included in line 13 in [Lau73, table 3.2].

9. W_{17} (An E_8-singularity with a smooth branch tangent to it.)
   Normal form from [AGZV85]: $x^4 + xy^5 + (a_0 + a_1 y)y^7$
   Enriques diagram and graph of corresponding taut sandwiched singularities:



The continued fraction of the left arm is $\geq \frac{7}{4}$.
The continued fraction of the upper arm is 4.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 13 in [Lau73, table 3.2].

10. W_{18} (irreducible curve with exactly three satellite points, one proximate to zero, the other two proximate to the point in the first neighbourhood)
    Normal form from [AGZV85]: $x^4 + y^7 + (a_0 + a_1 y)x^2 y^4$
    Enriques diagram and graph of corresponding taut sandwiched singularities:



The continued fraction of the left arm is $\frac{7}{5}$.
The continued fraction of the upper arm is 4.
The continued fraction of the right arm tends to 1 from above.
Graph is included in line 7 in [Lau73, table 3.2].

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