INT. J. SYSTEMS SCI., 1974, VOL. 5, NO. 2, 179–186

The solution of a 'fixed-target'—model by an approach of system analysis

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[Received 7 May 1973]

A general approach for economic systems is combined with a concrete 'fixed-target' —model. The consideration of convergence leads—under conditions of a stable solution and two targets—to the result that five numerical restrictions must be recognized when treating the two instruments. Generalizations of the discussed illustrative model are possible.

1. Introduction

In economics characterized by free-market activity, decision–making is largely decentralized. This circumstance should be considered in models of quantitative economic policy.

It is possible to combine a general system approach (McFadden 1969) with a concrete 'fixed–target'—model so that concrete limits of the available instrument variables can be given to economic policy–makers. Moreover, the objectives are assumed to be known and the system 'economy' formulated in the model is to have a stable solution.

2. The 'fixed-target'—model

The following relationships describe a specific 'fixed–target'—model (Fox/Sengupta/Thorbecke 1966) :

$$Y = C + I + G + E - M \qquad (1)$$

$$X = Y - T_i \qquad (2)$$

$$C = bY = 0{\cdot}8Y \qquad (3)$$

$$M = dY = 0{\cdot}17Y \qquad (4)$$

$$I = kY_{t-1} \qquad (5)$$

$$T_i = f + hY = f + 0{\cdot}13Y \qquad (6)$$

$$B = E - M \qquad (7)$$

where

Y = National income at market prices,
X = National income at factor costs,
C = Private consumption,
I = Net private investment,
E = Exports,

M = Imports,
G = Government expenditures,
T_i = Indirect taxes,
B = Balance of payments surplus (deficit),
Y_{t-1} = National income for the period $t-1$,
f = Autonomous level of indirect taxes.

The numerical values in eqns. (3), (4) and (6) are used for illustrative purpose.

Let there be two given target variables, namely full employment X_v (for example a level of employment of 98%) and balance of payments equilibrium B_A. To reach these two targets two instruments are available, namely government expenditures G and the autonomous level of indirect taxes f. It is conceivable that the authorities for decisions concerning G and f fall in different controlling institutions.

The reduced form of the model may be obtained by substitution and transformation, so that the target variables are only functions of the instruments and the exogenous data, i.e. of the exogenous variables.

For X one obtains

$$X = \frac{1}{1-b+d}(1-h)(kY_{t-1}-E) - f + \frac{1}{1-b+d}(1-h)G \tag{8}$$

If we let a_1 (=2·7) represent $1/(1-b+d)$, a_2 (=0·87) represent $(1-h)$, and Z represent $kY_{t-1}-E$, then upon substitution in (8) follows

$$X = a_1a_2Z - f + a_1a_2G \tag{9}$$

For the second instrument variable B one obtains

$$B = E - d\frac{1}{1-b+d}(kY_{t-1}-E) - d\frac{1}{1-b+d}G \tag{10}$$

Since $d/(1-b+d) = a_3$ (=0·46), (10) may be expressed as

$$B = -a_3Z + E - a_3G \tag{11}$$

Equations (9) and (11) may then be represented in the following matrix form (12):

$$\begin{bmatrix} X \\ B \end{bmatrix} = \begin{bmatrix} a_1a_2 & -1 & a_1a_2 \\ -a_3 & 0 & -a_3 \end{bmatrix} \begin{bmatrix} Z \\ f \\ G \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} \tag{12}$$

In order to obtain a course of economic action, the inverse reduced form of (12) must be developed. Since the number of targets equals the number of instruments and the coefficient matrix of the equation system is non-singular, it is possible to solve this system for the instrument variables.

Doing so, one obtains (13)

$$\begin{bmatrix} f \\ G \end{bmatrix} = \begin{bmatrix} -1 & a_1a_2 \\ 0 & -a_3 \end{bmatrix}^{-1} \begin{bmatrix} X - a_1a_2Z \\ B + a_3Z - E \end{bmatrix} \tag{13}$$

The equation system (12) may then be written as (14)

$$\begin{bmatrix} X \\ B \end{bmatrix} = \begin{bmatrix} -1 & a_1a_2 \\ 0 & -a_3 \end{bmatrix} \begin{bmatrix} f \\ G \end{bmatrix} \tag{14}$$

3. Integration of the 'fixed-target'—model with the system approach

In dynamic form of the system approach system (14) may be represented in short form as (15)

$$\Delta \mathbf{x} = \mathbf{A}\mathbf{u} \tag{15}$$

where

$$\mathbf{x} = \begin{bmatrix} X \\ B \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} f \\ G \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -1 & a_1a_2 \\ 0 & -a_3 \end{bmatrix} = \begin{bmatrix} -1 & 2{\cdot}35 \\ 0 & -0{\cdot}46 \end{bmatrix}$$

Moreover is

$$\Delta \mathbf{x} = \mathbf{x}_{t+1} - \mathbf{x}_t$$
$$\Delta f = f_{t+1} - f_t$$
$$\Delta G = G_{t+1} - G_t$$

From a certain initial state x_0, the system shall be led to the target constellation (16)

$$\mathbf{x}^+ = \begin{bmatrix} X_v \\ B_A \end{bmatrix} \tag{16}$$

This means that the balance of payments is balanced ($B = B_A = 0$) and full employment is reached ($X = X_v$). These are the control variables of the model. The system continuously compares the actual values of X and B with the desired values of X_v and B_A. The instruments G and f are applied until the difference between actual and desired values are zero. When this occurs, the system has reached its objective.

4. Determination of the numerical values for the instruments

In the model let $\mathbf{u}$ have the linear, time–independent form represented in (17)

$$\mathbf{u} = \mathbf{S}(\mathbf{x} - \mathbf{x}^+) \tag{17}$$

$\mathbf{S}$ is a 2×2 matrix in which all off–diagonal elements are zero :

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

Let M be a set of matrices $\mathbf{S}$, which contains possible numerical values of the controlling institutions s_1 and s_2. The question however is whether

$$\mathbf{S} \in M$$

such that the dynamic system (18)

$$\Delta \mathbf{x} = \mathbf{AS}(\mathbf{x} - \mathbf{x}^{+})$$

is stable.

AS is here

$$\mathbf{AS} = \begin{bmatrix} a_{11}s_1 & a_{12}s_2 \\ a_{21}s_1 & a_{22}s_2 \end{bmatrix}.$$

The initial state is x_0 (x vectors not now written in bold type), i.e.

$$\Delta x_0 = \mathbf{AS}(x_0 - x^{+}) \tag{19}$$

From that follows

$$x_1 = x_0 + \Delta x_0$$

and

$$\Delta x_1 = \mathbf{AS}(x_1 - x^{+}).$$

This results in

$$x_2 = x_1 + \Delta x_1 = x_0 + \Delta x_0 + \Delta x_1.$$

Generally one obtains

$$\Delta x_n = \mathbf{AS}(x_n - x^{+}) \tag{20}$$

or for

$$x_{n+1} = x_n + \Delta x_n = x_0 + \sum_{i=0}^{n} x_i \tag{21}$$

It is now desirable to know whether the expression with the summation sign in (21) converges, i.e. whether an approximate state of equilibrium exists and under what conditions. It must therefore be determined which values the instrument variables of the model may take on, i.e. which restrictions they must have that the model remains stable.

First, it is maintained that eqn. (22) is fulfilled

$$\Delta x_i = (\mathbf{I} + \mathbf{AS})^i \Delta x_0 \tag{22}$$

where **I** is the identity matrix.

By complete induction one obtains for

$$i = 0: \quad \Delta x_0 = (\mathbf{I} + \mathbf{AS})^0 \Delta x_0$$

and for

$$i = n + 1: \Delta x_{n+1} = \mathbf{AS}(x_{n+1} - x^{+})$$

or, after some transformations

$$\Delta x_{n+1} = (\mathbf{I} + \mathbf{AS})^n \Delta x_0$$

Therefore

$$\sum_{i=0}^{n} x_n = \sum_{i=0}^{n} (\mathbf{I} + \mathbf{AS})^i \Delta x_0 \tag{23}$$

From the theory of linear operators follows that

$$\sum_{i=0}^{n} (\mathbf{I} + \mathbf{AS})^i$$

converges, if

$$(\mathbf{I} + \mathbf{AS}) \tag{24}$$

has only eigenvalues whose absolute values are less than one and that the following expression is valid

$$\lim_{n\to\infty} \sum_{i=0}^{n} (\mathbf{I}+\mathbf{AS})^i = -\mathbf{AS}^{-1}$$

From this follows

$$\lim_{n\to\infty} x_{n+1} = x_0 - \mathbf{AS}^{-1}\Delta x_0 = x^+ \tag{25}$$

x^+ in (25) corresponds to the value which the target variables should have (compare expression (16)).

From $(\mathbf{I}+\mathbf{AS})$ one can construct the characteristic determinant for $\mathbf{AS}$, i.e.

$$|I-\lambda I+\mathbf{AS}| = \begin{vmatrix} (1-\lambda)+a_{11}s_1 & a_{12}s_2 \\ a_{21}s_1 & (1-\lambda)+a_{22}s_2 \end{vmatrix} \tag{26}$$

The characteristic equation for this is

$$(1-\lambda+a_{11}s_1)(1-\lambda+a_{22}s_2)-a_{12}s_2a_{21}s_1=0$$

For the solution of the resulting quadratic equation follows that

$$\lambda_{1,\,2} = 1+\left[\frac{Sp(AS)}{2} \pm \left(\frac{Sp^2(AS)}{4} - \det\,(AS)\right)^{1/2}\right] \tag{27}$$

where $Sp(AS)$ is the trace of $\mathbf{AS}$.
To fulfil the condition $\lambda \leqslant +1$, the expression in the square brackets must be smaller than zero, i.e. $(Sp(AS))/2$ and $\det\,(AS)$ have to be smaller than zero.

Since

$$Sp(AS) = a_{11}s_1 + a_{22}s_2 < 0$$

and

$$a_{22} < 0$$

follows that

$$s_2 > -\frac{a_{11}s_1}{a_{22}} = -2{\cdot}17 \tag{28}$$

For the decentralized controlling institutions to reach a stable solution, (28) is the first restriction (I) which must be met.

In order that $\det\,(AS) = s_1s_2a_{11}a_{22}$ is smaller than zero, i.e. in order that the minus sign before the expression remains within the root, $\det\,(AS)$ itself must be greater than zero. Since a_{11} and a_{22} are negative in this model, s_1 and s_2 must have the same sign (restriction II). (29)

On the other hand, λ may not be less than -1. Therefore the expression in the square brackets of (27) must be greater than -2 (i.e. the numerical value of the bracket lies between 0 and -2).

With (27) follows (30)

$$2+\frac{Sp(AS)}{2} > \left(\frac{Sp^2(AS)}{4} - \det\,(AS)\right)^{1/2} \tag{30}$$

From (30) follows restriction III, namely

$$s_2 < -\frac{4}{a_{22}} - \frac{a_{11}}{a_{22}} s_1 = 8{\cdot}7 - 2{\cdot}17 s_1 \tag{31}$$

Expression (30) may be developed into (32)

$$s_2(2a_{22} + s_1 a_{11} a_{22}) > -4 - 2 s_1 a_{11} \tag{32}$$

For the determination of s_2 two cases must now be considered :

(1) The expression in the brackets of (32) is smaller than zero, so that

$$s_1 < -\frac{2}{a_{11}} = 2{\cdot}0 \text{ (restriction IV)} \tag{33}$$

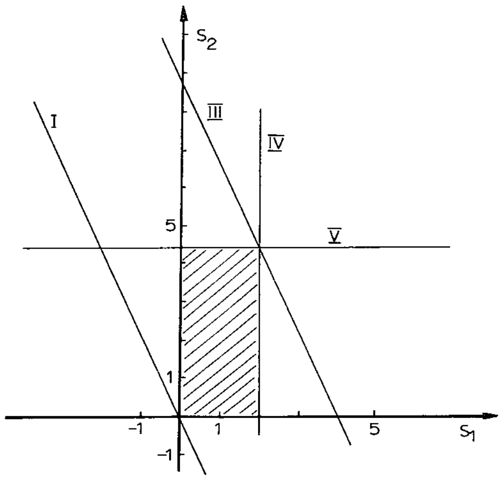
For s_2 then restriction V is valid

$$s_2 = -\frac{2}{a_{22}} = 4{\cdot}35 \tag{34}$$

(2) The expression in the brackets of (32) is greater than zero. Here it may be seen that this area for s_2 is eliminated by restriction III (31).

Imaginary roots are not possible because of the specific structure of matrix A.

Fig. 1



All possible restrictions are therefore accounted for. The stable area of solution for the numerical values of the controlling institutions s_1 and s_1, i.e. the values for G and f, under restrictions I–V is shown graphically in fig. 1. If the system is to be stable in time, the decentralized decision-makers may choose only values in the hatched area of the first quadrant.

5. Conclusion

According to the basic equations of the model it is seen from the form of the hatched area in fig. 1 that the application of government expenditures G

and autonomous level of indirect taxes f in the stable area may be independant. The decision of the controlling institutions are not influenced by each other. Generally, it is, however, possible that a stable solution exists even if the instrument variables are dependent because of a connection of the controlling institutions by information channels. The computation of the discussed approach served to illustrate and analyse the questions formulated.

The analysis may be expanded for complex problems with n instruments and m controlling institutions. In such cases, the application of computers is necessary.

The significant result of this approach is that policy–makers can be given concrete, quantitative limits for the instruments to be applied, under consideration of the desired, given objectives.

ACKNOWLEDGMENTS

I would like to thank E. Oswald, B. Turner and R. Zimmer for their helpful comments.

Derivations and proofs

Page 182 Transformation from row 29 to row 31 :

$i = n+1$:

$$\begin{aligned}\Delta x_{n+1} &= \mathbf{AS}(x_{n+1} - x^+)\\ &= \mathbf{AS}(x_n - x^+ + \Delta x_n)\\ &= \mathbf{AS}(x_n - x^+) + \mathbf{AS}\Delta x_n\\ &= (\mathbf{I} + \mathbf{AS})\Delta x_i\\ \Delta x_{n+1} &= (\mathbf{I} + \mathbf{AS})^n \Delta x_0\end{aligned}$$

Page 183/184 Transformations from (30) to (31) :

(30) may be written as

$$2 + \frac{Sp(AS)}{2} > 0$$

or

$$a_{11}s_1 + a_{22}s_2 > -4$$

or

$$a_{22} > -4 - a_{11}s_1$$

Since $a_{22} < 0$, one obtains (31).

Page 183/184 Transformation from (30) to (32) :

(30) may be written as

$$\left(2 + \frac{Sp(AS)}{2}\right)^2 > \frac{Sp^2(AS)}{4} - \det(AS)$$

or

$$4 + 2Sp(AS) > -\det(AS)$$

or

$$4 + 2s_1a_{11} + 2s_2a_{22} + a_{11}a_{22}s_1s_2 > 0$$

S.S.

Page 184 Question of imaginary roots :

For imaginary roots it should be

$$\sqrt{\left[\frac{Sp^2(AS)}{4} - \det(AS) < 0\right]}$$

or

$$Sp^2(AS) < 4 \det(AS)$$

It is

$$(s_1 a_{11} + s_2 a_{22})^2 < 4 s_1 s_2 a_{11} a_{22}$$

or

$$s_1{}^2 a_{11}{}^2 + 2 s_1 s_2 a_{11} a_{22} + s_2{}^2 a_{22}{}^2 < 4 s_1 s_2 a_{11} a_{22}$$

or

$$(s_1 a_{11} - s_2 a_{22})^2 < 0$$

This is impossible.

References

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