Lagrangian Fibrations with designed singular fibres

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Abstract

We study Lagrangian Fibrations with designed singular fibers. The idea is to construct a $K3$ surface X as a minimal resolution of the singularities of a double cover Y of the plane branched along a reduced but possibly reducible singular sextic Σ. Moreover, we assume that Σ has at worst A-D-E singularities. This freeness of choosing Σ allows us to construct many examples of singular fibres with various singularities. We find an explicit description of the singular fibers of the Lagrangian Fibrations $f\colon M_X(0, 2H, \chi) \to |2H|$. The results shed also some light on the correlation between the degree of the discriminant divisor Δ and the topology of the corresponding moduli space.

Zusammenfassung

Wir studieren Lagrangefaserungen mit vorgegebenen singulären Fasern. Die Idee besteht darin, eine $K3$-Fläche X als minimale Auflösung von Singularitäten einer 2 : 1 Überlagerung zu konstruieren, die entlang einer reduzierten, aber möglicherweise reduziblen singulären Sextik Σ verzweigt ist. Außerdem nehmen wir an, dass Σ höchstens A-D-E Singularitäten besitzt. Diese Freiheit bei der Wahl von Σ erlaubt es uns, viele Beispiele von singulären Fasern mit verschiedenen Singularitäten zu konstruieren. Wir finden eine explizite Beschreibung der singulären Fasern der Lagrangefaserung $f\colon M_X(0, 2H, \chi) \to |2H|$. Die Ergebnisse beleuchten die Korrelation zwischen dem Grad des Diskriminantendivisors Δ und der Topologie des zugrundeliegenden Modulraumes.

Contents

1 Introduction

Irreducible symplectic varieties are defined as compact Kähler varieties having trivial fundamental group and being endowed with a unique, up to $\mathbb{C}^*$, global holomorphic 2-form which is nowhere degenerate. The simplest example are $K3$ surfaces. Fujiki constructed the first higher dimensional example of an irreducible symplectic manifold, namely the blow-up of the diagonal in $S^2(X)$ for a $K3$ surface X. His example was generalized by Beauville. If X is a $K3$ surface, then Beauville showed that for every n the Hilbert scheme $\mathrm{Hilb}^n(X)$ parametrizing 0-dimensional subschemes of length n on X is an irreducible holomorphic symplectic manifold. Beauville constructed also a second family of examples; the generalized Kummer varieties of 2-dimensional tori. Apart from Beauville's examples, O'Grady found two new examples of irreducible symplectic varieties up to deformation equivalence. They have been exhibited in [22] and [23] and their dimensions are 10 and 6, respectively. The O'Grady 10-dimensional moduli space is the main actor of our paper. Up to now, each known example of irreducible symplectic varieties can be deformed into one of the above examples. The O'Grady examples of irreducible symplectic varieties are deformation-equivalent to Lagrangian Fibrations (cf. [25]). So we study some special Lagrangian Fibrations on irreducible symplectic manifolds. They have become an important tool in understanding symplectic manifolds. Hwang–Oguiso ([11]) and Matsushita ([16]) classified the type of the general singular fibres of this fibrations.

O'Grady works with the singular moduli space $M_X(2,0,4)$ of rank 2 semistable sheaves on X with first Chern class 0 and second Chern class 4. Over the nonsingular part this moduli space can be equipped with a symplectic form. One can show that it has a symplectic resolution of singularities, i.e. that the symplectic form can be extended to the whole resolution and remains non-degenerate. The space $M_X(2,0,4)$ in fact turns out to be birational to the moduli space $M_X(0,2H,\chi)$ (if χ is an even number). The symplectic resolution of the moduli space $M_X(0,2H,\chi)$ is obtain by certain blow-ups (cf. [22, 15]). Since for χ odd the Mukai vector $(0,2H,\chi) \in H^{even}(X,\mathbb{Z})$ is primitive, the moduli space $M_X(0,2H,\chi)$ is an irreducible symplectic variety and by [28] it can be deformed into the Hilbert scheme $\mathrm{Hilb}^5(X)$ whose topology is better understood than that of O'Grady space. In this paper we assume that χ is an even number.

Let $\Sigma \subset \mathbb{P}^2$ be a plane sextic with at worst A-D-E singularities. Let $Y \to \mathbb{P}^2$ denote the double cover ramified along Σ. This is a normal surface with A-D-E singularities. Let $X \to Y$ be the minimal resolution, which is a $K3$ surface X. Let $\pi\colon X \to \mathbb{P}^2$ denote the composite map. Such a $K3$ surface X is polarized by a modification of the pull-back H of a line in the plane, involving small adjustment using the exceptional curves (see Lemma 4.4). This freeness of choosing Σ allows us to construct many examples of singular fibres with various singularities.

The space $M_X(0, 2H, \chi)$ is the moduli space of semistable sheaves with 1-dimensional support C on a $K3$ surface X and has the advantage that it admits a Lagrangian Fibration. This fibration is the map $f\colon M_X(0, 2H, \chi) \to |2H|$, which sends a sheaf to its support scheme. We have a Lagrangian Fibration, where the target space is the linear system $|2H|$, which is isomorphic to $\mathbb{P}^5$. An important open problem in the study of singular fibers of Lagrangian Fibration seems to be to understand the geometry of the discriminant locus. This is defined as the subset Δ of $|2H|$ given by:

$$\Delta = \{b \in |2H| : f^{-1}(b) \text{ is singular}\}.$$

The fact that Δ is a divisor if it is non-empty is a result of Hwang and Oguiso (cf. [11, Proposition 3.1]). In our situation this is a divisor in $\mathbb{P}^5$. We investigate the geometry of Δ. Moreover, we describe the structure of the singular fibres of the fibration f over the generic points of the connected components of Δ.

Each singular fibre of f can be interpreted as the moduli space of semistable sheaves on the curve $C = \pi^*(Q)$, where $Q \in |2H|$. Curves in this family are pull-backs of quadrics in the plane. If Q is the generic quadric then it intersects the ramification divisor Σ transversely in twelve different points. From Hurwitz's Theorem we obtain that the curve $C = \pi^*(Q)$ is a smooth curve of genus five and the generic fibre of f is the Jacobian of C. The family $\pi^*(Q)$ also contains reducible curves (pull-backs of pair of lines) and non-reduced curves. We will discuss in detail the structure of such fibres. We provide large repertoire of examples supporting the conjectures concerning the connection between the total degree of Δ and the weights of the singular fibres. Moreover, we hope that we shed some light on the correlation between the degree of Δ and the topology of the corresponding moduli space.

Let us describe the content and the structure of this paper. For more detailed explanations we refer to the introductions of the individual chapters.

In Chapter 2 we recall the basic definitions and results regarding irreducible symplectic manifolds and theory of sheaves. We end this chapter by collecting some facts about compactifications of Jacobians.
In Chapter 3 we recall some definitions and facts about moduli spaces of semistable sheaves. We introduce the Mukai vector. Then we summarize the results of Matsushita and Hwang concerning Lagrangian Fibrations. In the next section we recall from [19, Section 1.4] the construction of the map $f\colon M_X(0, 2H, \chi) \to |2H|$. At the end of this chapter we recall the result of Hwang and Oguiso regarding the discriminant locus Δ.
As we have mentioned before, we assume that a $K3$ surface X is a minimal resolution of the singularities of a double cover Y of the plane branched along a reduced but possibly reducible singular sextic Σ. We describe in Chapter 4 this construction. Then we discuss the choice of the polarization of a $K3$ surface X. The discriminant locus is a divisor, in our paper denoted by Δ. The singular fibres appear exactly over Δ. We want to consider only generic points $[Q] \in \Delta$. Chapter 5 provides a detailed description of the discriminant divisor. We find also the degree of its components. Moreover, our calculations support the conjectures concerning the connection between the total degree of Δ and the weights of the singular fibres.
Chapter 6 is devoted to the detailed study of the singular fibers of the fibration $f\colon M_X(0, 2H, \chi) \to |2H|$. This is the key chapter of our paper. In the first section of this chapter we collect some generalities on pure sheaves of dimension 1, which we need for our work. In Section 6.2 we describe the fibre of f over the components Δ_{Σ_i} of Δ. This is the case when a generic quadric Q intersect Σ (or, to be precise, one of its components Σ_i) with multiplicity 2 in a smooth point. In Section 6.3 we study the fibre over the component Δ_{deg}. This is the situation when a generic quadric Q degenerates into two lines. In the last Section 6.4 we give a description of the singular fibre over the components Δ_p, which means that a generic quadric Q passes through a singularity p of Σ_i. The non-reduced curves appearing in this case are the main technical difficulty of our work. The parts of the work build on an unpublished manuscript by Prof. Dr. Manfred Lehn.

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Notations and conventions

We try to stick to the following notations and conventions throughout this paper. Set is the category of sets, Sch is the category of algebraic schemes over k. An *algebraic scheme* is a separated scheme of finite type over $\mathbb{C}$. A *fibration* is a proper morphism with connected fibres from a variety to a normal variety. All sheaves will be assumed to be coherent.

2 Basic constructions and general set-up

In this chapter we recall some basic notions and facts. Let X be a projective scheme over $\mathbb{C}$, unless stated otherwise. All sheaves on X will be assumed to be coherent. The fundamental reference is [9].

2.1 Basic definitions

We recall basic definitions and results related to irreducible symplectic manifolds. The classification of surfaces gives that the only surfaces with a non-degenerate symplectic structure are $K3$ and abelian surfaces. Let X be a complex manifold and let $\omega \in H^0(X, \Omega_X^2)$ be a global holomorphic 2-form.

Definition 2.1 *An irreducible symplectic manifold is a compact, connected, simply connected Kähler manifold X with $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \omega$, where ω is symplectic, i.e. everywhere non-degenerate.*

As ω is symplectic, irreducible symplectic manifolds are even-dimensional. Let F be a coherent sheaf on X.

Definition 2.2 *[9] The support of F is the closed set supp$(F) = \{x \in X \mid F_x \neq 0\}$. Its dimension is called the dimension of the sheaf F and is denoted by* $\dim(F)$.

Definition 2.3 *[9] F is pure of dimension d if $\dim(E) = d$ for all non-trivial coherent subsheaves $E \subset F$.*

Let us recall that the Euler characteristic of a coherent sheaf F is given by

$$\chi(F) := \sum(-1)^i h^i(X, F) = \sum(-1)^i \dim_k H^i(X, F).$$

Then if we fix an ample line bundle $\mathcal{O}(1)$ on X, then the *Hilbert polynomial* $P(F)$ is given by (cf. [9])

$$n \mapsto \chi(F \otimes \mathcal{O}(n)).$$

The Hilbert polynomial can be uniquely written in the following form (see [9])

$$P(F, n) = \sum_{i=0}^{\dim(F)} \alpha_i(F) \frac{n^i}{i!},$$

where $\alpha_i(F)$ are rational coefficients for $i = 0, \ldots, \dim(F)$. The leading coefficient $\alpha_{\dim(F)}(F)$ is called the multiplicity.

Definition 2.4 *[9] The reduced Hilbert polynomial $p(F)$ of a coherent sheaf F of dimension d is defined by*

$$p(F,n) := \frac{P(F,n)}{\alpha_d(F)}.$$

We know that there is a natural ordering of polynomials which is given by the lexicographic order of their coefficients. We are now prepared to recall the definition of stability of sheaves.

Definition 2.5 *[9] A coherent sheaf F of dimension d is semistable if F is pure and for any proper non-trivial subsheaf $E \subset F$ one has $p(E) \leq p(F)$. F is called stable if F is semistable and the inequality is strict, i.e. $p(E) < p(F)$ for any proper nontrivial subsheaf $E \subset F$.*

Definition 2.6 *[9] A coherent sheaf F on an integral scheme X is torsion free if for each $x \in X$ and $s \in \mathcal{O}_{X,x} \backslash \{0\}$ multiplication by s is an injective homomorphism $F_x \longrightarrow F_x$.*

We recall the notion of a saturation.

Definition 2.7 *[9] The saturation of a subsheaf $E \subset F$ is the minimal subsheaf E' containing E such that F/E' is pure of dimension $d = \dim(F)$ or zero.*

The saturation of a subsheaf $E \subset F$ is the kernel of the surjection

$$F \longrightarrow F/E \longrightarrow (F/E)/T_{d-1}(F/E).$$

Definition 2.8 *[9] (Jordan-Hölder filtration) Let F be a semistable sheaf of dimension d. A Jordan-Hölder filtration of F is a filtration*

$$0 = F_0 \subset F_1 \subset \ldots \subset F_k = F,$$

such that the factors $gr_i(F) = F_i/F_{i-1}$ are stable with reduced Hilbert polynomial $p(F)$.

This filtration need not be unique. Jordan-Hölder filtrations always exist. Up to isomorphism, the sheaf $gr(F) := \bigoplus_i gr_i(F)$ does not depend on the choice of the Jordan-Hölder filtration (cf. [9]). Now we recall the notion of the S-equivalence of sheaves.

Definition 2.9 *[9] Two semistable sheaves F_1 and F_2 with the same reduced Hilbert polynomial are called S-equivalent if $gr(F_1) \cong gr(F_2)$ and then we write $F_1 \sim_S F_2$.*

The importance of this definition will become clear later. Below we recall the notion of a polystable sheaf.

Definition 2.10 *[9] A semistable sheaf F is called polystable if F is the direct sum of stable sheaves.*

Every S-equivalence class of semistable sheaves contains exactly one polystable sheaf up to isomorphism. The moduli space of semistable sheaves in fact parametrizes polystable sheaves, which means that closed points in the moduli space corresponds to the polystable sheaves.

2.2 Compactified Jacobians

Let C be a smooth and projective curve of genus g. Then the moduli space of isomorphism classes of invertible bundles of degree d is a smooth projective variety of dimension g (cf. [24, Sect. 8.5]) and we denote it by $\mathrm{Pic}^d(C)$. This space admits a Poincaré bundle over $\mathrm{Pic}^d(C) \times C$. The tensor product gives rise to a group structure on the Jacobian $\mathrm{Jac}(C)$ of C and $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$.

Lemma 2.11 *[24, Prop. 8.5.1] The variety $Jac(C)$ is irreducible.*

Moreover, all of the varieties $\mathrm{Pic}^d(C)$ are isomorphic to $\mathrm{Pic}^0(C)$.

Compactifications of Jacobians have been studied by many authors using different methods. We collect below some basic facts about compactified Jacobians. The main reference is [26]. Let now C be geometrically integral (reduced and irreducible) curve over an algebraically closed field k of arithmetic genus g, and let $\widetilde{C}$ denote the normalization of C. The Jacobian $\mathrm{Jac}(C)$ is the group scheme parametrizing rank one locally free sheaves on the curve C. Moreover, $\mathrm{Jac}(C)$ is an iterated extension of the Jacobian $\mathrm{Jac}(\widetilde{C})$ by copies of $\mathbb{C}^*$ and $\mathbb{C}$. There is a natural compactification of $\mathrm{Jac}(C)$ given by the moduli space M_C of rank one torsion free sheaves on C. For integral curves this moduli space was constructed by D'Souza in [5] (cf. also Altman and Kleiman [1]). If M_C is irreducible, its normalization will be an iterated $\mathbb{P}^1$-bundle over the abelian variety $\mathrm{Jac}(\widetilde{C})$, with the $\mathbb{P}^1$ fibres arising as compactifications of the $\mathbb{C}^*$ and $\mathbb{C}$ fibres (cf. [1, 21, 26]).

3 Moduli spaces and Lagrangian Fibrations

In this chapter we recall briefly the main facts about moduli spaces of semistable sheaves, which we need to our work. Historically the first construction of the moduli space is due to Gieseker and Maruyama. Then a different construction was found by Simpson. For more details see [9, Chapter 4]. In the first section we collect basic definitions concerning moduli spaces. Then we define the Mukai vector. In the Section 3.3 we collect some basic facts about complex Lagrangian Fibrations. In the Section 3.4 we recall the construction of the fibration $M_X(0, 2H, \chi) \to |2H|$. We end this chapter with the result of Hwang and Oguiso regarding the discriminant locus Δ.

3.1 Basic definitions

We denote by **Sch** the category of algebraic schemes over k (schemes of finite type over Spec k). Let $P \in \mathbb{Q}[x]$ be a fixed polynomial. We want to define a functor $\mathcal{M}_X(P)\colon \mathbf{Sch}^{op} \to \mathbf{Set}$. To begin with, we define first an auxiliary functor $\mathcal{M}'_X(P)\colon \mathbf{Sch}^{op} \to \mathbf{Set}$ as follows: for any $S \in \mathbf{Sch}$, $\mathcal{M}'_X(P)(S)$ is the set of all flat S-families of sheaves on X, i.e. coherent sheaves on $S \times X$ flat over S with all fibres being semistable with Hilbert polynomial P. For any morphism of schemes $f\colon T \to S$ define

$$\mathcal{M}'_X(P)(f)\colon \mathcal{M}'_X(P)(S) \to \mathcal{M}'_X(P)(T)$$

by pullbacks. Then define the functor $\mathcal{M}_X(P)$ as a quotient functor of $\mathcal{M}'_X(P)$ with respect to an equivalence relation: two families $F_1, F_2 \in \mathcal{M}'_X(P)(S)$ are equivalent if there exists an invertible sheaf L on S such that $F_2 \simeq F_1 \otimes p_S^* L$. Analogously we define a functor $\mathcal{M}^S_X(P)\colon \mathbf{Sch}^{op} \to \mathbf{Set}$ corresponding to families of stable sheaves on X with Hilbert polynomial P.

Theorem 3.1 *There exists a projective scheme $M_X(P)$ that universally corepresents the functor $\mathcal{M}_X(P)$. Closed points in $M_X(P)$ correspond to S-equivalence classes of semistable sheaves with Hilbert polynomial P. There exists an open subscheme $M^S_X(P)$ of $M_X(P)$ that universally corepresents $\mathcal{M}^S_X(P)$.*

The scheme $M_X(P)$ is called the moduli space.

3.2 Mukai vector

Let $H^{even}(X,\mathbb{Z})$ be the Mukai lattice of a polarised K3 or abelian surface X. For every element $v \in H^{even}(X,\mathbb{Z})$ there is an associated moduli space $M(v)$ that parametrises polystable sheaves F with Mukai vector

$$v = v(F) := ch(F)\sqrt{\mathrm{td}(X)}.$$

Lemma 3.2 *(cf. [9, Chapter 6]) Let X be a $K3$ surface. If F is a coherent sheaf on X with $rk(F) = r$, $c_1(F) = c_1$ and $c_2(F) = c_2$, then*

$$v = (r, c_1, \frac{c_1^2 - 2c_2}{2}) \cup (1, 0, 1) = (r, c_1, \frac{c_1^2 - 2c_2}{2} + r) \in H^{even}(X,\mathbb{Z}).$$

We will use the notation $M(v)$ for the moduli space of semistable sheaves with $v = (r, c_1, \frac{c_1^2-2c_2}{2} + r)$. Let M^s denote the open subset parameterizing stable sheaves. From [9, Chapter 4.5] we get

Lemma 3.3 *The expected dimension of M^s is*

$$\dim(M^s) = 2rc_2 - (r-1)c_1^2 - 2(r^2 - 1) = (v,v) + 2,$$

where (v,v) is the intersection form on X.

Due to Mukai ([20]), if the polarization and the Mukai vector v are chosen in such a way that every semistable sheaf is automatically stable, then $M(v)$ is a projective holomorphically symplectic manifold. In the opposite case, $M(v)$ is singular and one may ask whether $M(v)$ at least admits a projective symplectic resolution. This question has been successfully answered in two cases by O'Grady ([22, 23]), leading to two new deformation classes of irreducible holomorphic symplectic manifolds. The complete answer to O'Grady's question for general ample divisors and moduli spaces whose expected dimension $2 + (v,v)$ is ≥ 4 was given by Kaledin, Lehn and Sorger in [12]. We assume that we have the following situation, which we fix for the rest of this paper. We consider the Mukai vector $v = (0, 2H, \chi)$ and χ is an even number. Then the intersection form $(v,v) = c_1^2 = (2H)^2 = 8$. Using Lemma 3.3 we have $\dim(M^s) = 10$. Let v_0 denote the primitive vector $(0, H, \frac{\chi}{2})$ (i.e. v_0 is not an integral multiple of another lattice element). Then we have $v = 2 \cdot v_0 = 2 \cdot (0, H, \frac{\chi}{2})$.

3.3 Complex Lagrangian Fibrations

To begin with we define a Lagrangian Fibration.

Definition 3.4 *A Lagrangian Fibration is a fibration* $f\colon M \to B$ *of a* $2n$*-dimensional holomorphic symplectic manifold* M *whose smooth fibers are Lagrangian with respect to the holomorphic symplectic structure.*

We recall the notion of a Lagrangian subvariety.

Definition 3.5 *[17] Let* M *be a manifold with a holomorphic symplectic form* ω*. A subvariety* U *is said to be a Lagrangian subvariety if* $\dim(U) = \frac{1}{2}\dim(M)$ *and there exists a resolution* $\nu\colon U' \to U$ *such that* $\nu^*\omega$ *is identically zero on* U'*.*

In [18] Matsushita proved that if $f\colon M \to B$ is a fibration on M then $\dim(B) = n$, every fibre must be Lagrangian with respect to the holomorphic symplectic form ω and the generic fibre must be an n-dimensional complex torus. With the assumption that B is projective Hwang ([10]) proved later that the base must be isomorphic to $\mathbb{P}^n$.

3.4 Fibration $f\colon M_X(0, 2H, \chi) \to |2H|$

In this section we recall briefly the construction of the map $M_X(0, 2H, \chi) \to |2H|$ (cf. [19, Section 1.4]). Let X be, as before, a $K3$ surface and $F \in M_X(0, 2H, \chi)$. We want to construct the map $f\colon M_X(0, 2H, \chi) \to |2H|$ that sends a sheaf to its support scheme.

We first recall the definition of the Fitting ideals (cf. [6, Section 20.2]). Given a finite module M over a noetherian ring R, consider a free presentation

$$P_1 \xrightarrow{\varphi} P_0 \longrightarrow M \longrightarrow 0$$

with $\operatorname{rk} P_0 = r$. If we choose bases for P_0 and P_1, then φ can be represented by a matrix. Then the Fitting ideal $\mathrm{Fitt}_i M$ is defined to be the image of the map

$$\wedge^{r-i} P_1 \otimes (\wedge^{r-i} P_0)^\vee \to R$$

induced by the map $\wedge^{r-i} P_1 \to \wedge^{r-i} P_0$. If bases of P_0 and P_1 are chosen, then the Fitting ideal $\mathrm{Fitt}_i M$ is generated by minors of φ of order $(r - i)$. The formation of Fitting ideals commutes with base change (cf. [6, Corollary 20.5]) and we have

$$\mathrm{Fitt}_0\, M \subset \mathrm{Fitt}_1\, M \subset \dots.$$

Moreover, we have $(\operatorname{ann} M)^r \subset \operatorname{Fitt}_0 M \subset \operatorname{ann} M$ (see [6, Proposition 20.7]).

We have $\operatorname{ann} F \neq 0$ so that already the zeroth Fitting ideal is nonzero and $I(F) = \operatorname{Fitt}_0 F$. We assume that F is pure which implies that $\operatorname{depth} F_x \geq 1$ for every $x \in X$. From the Auslander-Buchsbaum formula the projective dimension $\operatorname{pdim} F_x \leq 2 - 1$. So the minimal length of a projective resolution of F is 1. We want to show that $I(F)$ is an ideal of the curve in the linear system $|2H|$. Consider an exact sequence

$$0 \to E \to P \to F \to 0$$

of coherent sheaves over X with P locally free. But X is a surface and $\operatorname{pdim} F_x = 1$ hence E is locally free. So $I(F)$ is equal to the image of the map $\det E \otimes (\det P)^{-1} \to \mathcal{O}_X$ induced by $\det E \to \det P$. The corresponding curve is a divisor of the line bundle $(\det E)^{-1} \otimes \det P$. The first Chern class of this bundle equals $c_1(F) = 2H$ and so $I(F)$ defines a curve in the linear system $|2H|$.

3.5 The discriminant locus Δ

Let $f\colon M \to B$ be a holomorphic Lagrangian Fibration, where M and B are complex manifolds with $\dim(M) = 2n$ and $\dim(B) = n$. By discriminant locus Δ we mean the subset of B parameterizing singular fibres of f:

$$\Delta = \{b \in B,\ f^{-1}(b) \text{ is singular}\}.$$

The fact that Δ is a divisor if it is non-empty, is a result of Hwang and Oguiso:

Lemma 3.6 *[11, Proposition 3.1]*

(1) $M_s = f^{-1}(s)$ is an n-dimensional complex torus if $s \notin \Delta$.

(2) The critical set Δ is a hypersurface of B unless $\Delta = \emptyset$.

In our set-up Δ is a divisor in $\mathbb{P}^5$. In Chapter 5 we discuss the structure of Δ and degree of it's components.

4 Double covering of $\mathbb{P}^2$ and choice of the polarization of X

We assume that we have the following situation, which we fix for the rest of this paper. Let $\Sigma \subset \mathbb{P}^2$ be a plane sextic with at worst A-D-E singularities. Let $Y \to \mathbb{P}^2$ denote the double cover ramified along Σ. This is a normal surface with A-D-E singularities. Let $X \to Y$ be the minimal resolution, which is a $K3$ surface X. Moreover, let $\pi\colon X \to \mathbb{P}^2$ denote the composite map. This freeness of choosing Σ allow us to construct many examples of singular fibres with various singularities. In this chapter we recall briefly the construction of the above double cover Y. We end this chapter by description of the polarization of a $K3$ surface X.

4.1 Double covering of $\mathbb{P}^2$ ramified over a singular sextic Σ

Let $\Sigma \subset \mathbb{P}^2$ be a plane sextic given by a homogeneous polynomial f of degree 6. Let i denote the embedding of Σ in $\mathbb{P}^2$. We have the exact sequence

$$0 \longrightarrow I_{\Sigma\subset\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_\Sigma \longrightarrow 0$$

and

$$I_{\Sigma\subset\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-\Sigma) \cong \mathcal{O}_{\mathbb{P}^2}(-6).$$

The $K3$ surface X can be constructed in the following way:
let

$$\mathcal{A} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$$

be the sheaf of algebras where the multiplication $\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}$ is given by the equation of Σ

$$(s,i)\cdot(s',i') = (ss' + \varphi(i,i'), si' + s'i)$$

where $\varphi\colon \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}$ is given by

$$i \otimes i' \longrightarrow i \cdot i' \cdot f.$$

Now we define $Y := \mathrm{Spec}\mathcal{A}$ as a relative affine spectrum. Let $X \to Y$ be the minimal resolution of singularities. Since π is finite, X is projective and therefore a Kähler surface. So the canonical sheaf of X is given by

$$K_X = \pi^*(K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(3)) = \mathcal{O}_X.$$

Moreover, we have

$$b_1(X) = H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^2, \pi_*(\mathcal{O}_X)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)) = 0.$$

So X is a $K3$ surface. The singularities of Y appear exactly above the singularities of Σ. In analytic coordinates we obtain the following explicit equations for simple surface singularities (cf. [3, Section III.7]):

Type	Local equation
A_n $(n \geq 1)$:	$0 = w^2 + x^2 + y^{n+1}$
D_n $(n \geq 4)$:	$0 = w^2 + y(x^2 + y^{n-2})$
E_6 :	$0 = w^2 + x^3 + y^4$
E_7 :	$0 = w^2 + x(x^2 + y^3)$
E_8 :	$0 = w^2 + x^3 + y^5$

For each singular point $p \in \Sigma_{sing}$, let Γ_p denote the dual graph of the exceptional fibre $\pi^{-1}(p)$. It is well-known that to each vertex $i \in v(\Gamma_p)$ corresponds a (-2)-curve $E_i^{(p)}$.

Lemma 4.1 *(cf. [3, Chapter III]) Let $p \in Y$ be a normal surface singularity and let $\pi\colon X \to Y$ be a resolution. Then for every divisor C with support in $E = \pi^{-1}(p)$ we have $C^2 < 0$.*

From above Lemma 4.1 we obtain $E_i^2 < 0$. Using Adjunction formula and the fact that X is a $K3$ surface we deduce $2g - 2 = E_i^2$. So we have

$$E_i^2 = -2 \text{ and } g = 0. \tag{1}$$

Using (1) we conclude that all components are $\mathbb{P}^1$. Because of

$$(E_i + E_j)^2 = 2(E_iE_j - 2) < 0, \tag{2}$$

for all $i \neq j$ one has $E_iE_j \leq 1$, i.e. two such curves can intersect in at most one point and then transversely.

Lemma 4.2 *(cf. [3, Section III.7]) The dual graphs of the exceptional fibres $\pi^{-1}(p)$ are exactly the Dynkin diagrams.*

We come back to the above discussion in Section 6.4.1.

4.2 Polarization of a $K3$ surface X

Let H denote the preimage of the class of a line in $\mathbb{P}^2$, so that $H^2 = 2$. The divisor H is nef but not ample unless Σ is smooth. We want to modify the polarization H involving small adjustment using the exceptional curves. A natural choice of a polarisation of X which is very close to H can be obtained as follows: For each singular point $p \in \Sigma_{sing}$, let Γ_p denote the dual graph of the exceptional fibre $\pi^{-1}(p)$, so that to each vertex $i \in v(\Gamma_p)$ corresponds a (-2)-curve $E_i^{(p)}$. The intersection matrix $S^{(p)} = (E_i^{(p)} \cdot E_j^{(p)})_{ij}$ is negative definite. Moreover

Lemma 4.3 *There are unique positive integers $a_i^{(p)}$ such that the intersection between $E^{(p)} := \sum_i a_i^{(p)} E_i^{(p)}$ and each component $E_i^{(p)}$ is exactly -2.*

Proof. We want to consider the associated root system. Let E be an Euclidean space. For each p we have the root system (E, R) and let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the base for R. Then the fundamental weights for R (relative to Δ) are the vectors $\{\omega_1, \ldots, \omega_r\}$ determined by the following condition: For all i and j, we have

$$2\frac{(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{ij}.$$

As for A-D-E type all roots have length 2 we have $(\alpha_i, \omega_j) = \delta_{ij}$. Summing over all j we obtain $(\alpha_i, \sum_j \omega_j) = 1$. A weight of special importance is the one for which all Dynkin labels are equal to one

$$\rho = \sum_j \omega_j = (1, \ldots, 1).$$

This is the Weyl vector and is alternatively defined as

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

So we have

$$2 \sum_j \omega_j = 2\rho = \sum_{\alpha \in \Delta_+} \alpha.$$

The right side of the above equation is a linear combination $a_1\alpha_1 + \ldots + a_n\alpha_n$ and the coefficients a_i are integers.

□

We want to polarise the $K3$ surface X by

$$H' := H - \varepsilon \sum_p E^{(p)}, \tag{3}$$

where H denotes the preimage of the class of a line in $\mathbb{P}^2$ and $\varepsilon > 0$ is a small rational constant. Then

Lemma 4.4 *For any choice of a sufficiently small constant $\varepsilon > 0$, the $\mathbb{Q}$-divisor*

$$H' = H - \varepsilon \sum_p E^{(p)}$$

is ample.

Proof. As the surface X is projective, there exist an ample divisor $\hat{H}$ on X. We take this divisor. Then for every sufficiently small $\varepsilon > 0$ we can have the following properties:

$$H' \cdot \hat{H} > 0 \text{ and } H'^2 > 0.$$

Then for all $n >> 0$ there exist an effective divisor $D \in |nH'|$ (cf. [8, Corollary V.1.8]). We take this divisor D. Let C be any irreducible, not exceptional curve $C \subset X$. We consider the intersection $C \cdot D$. We have three possibilities:

i) $C \cdot D > 0$,

ii) $C \cdot D = 0$, which is equivalent to $C \cap D = \emptyset$.

iii) $C \cdot D < 0$

In the case *iii*) the curve C must be a component in D, so there are only finitely many such curves. For such curves we can modify H' by choosing ε a little bit smaller to achieve $C \cdot D > 0$ in the case *iii*). With this modification we get $C \cdot D > 0$ in all above cases.

□

Such a choice of H' somewhat simplifies later calculations but is of no real importance as long as H' is sufficiently close to H.

5 The discriminant divisor Δ and degree of it's components

One of the most important open problems in the study of singular fibers of Lagrangian Fibration seems to be to understand the geometry of the discriminant locus Δ (cf. Section 3.5). We define the set $\Delta \subset \mathbb{P}^5$. Due to Hwang and Oguiso, it is a hypersurface in $\mathbb{P}^5$ (see Lemma 3.6). In this chapter we describe the components of Δ and then we compute its degree. We want to consider only generic points of the divisor Δ.

Then we have the following strategy. There is a $2:1$ map $\pi\colon X \to \mathbb{P}^2$ branched along a sextic in $\mathbb{P}^2$ (see Chapter 4). The natural morphisms $\mathcal{O}_{\mathbb{P}^2}(m) \to \pi_*\pi^*\mathcal{O}_{\mathbb{P}^2}(m) = \pi_*\mathcal{O}_X(mH)$ induce the maps $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(X, \mathcal{O}_X(mH))$.

Lemma 5.1 *We can identify $|H|$ with $|\mathcal{O}_{\mathbb{P}^2}(1)|$ and $|2H|$ with $|\mathcal{O}_{\mathbb{P}^2}(2)|$.*

Proof. The maps $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(X, \mathcal{O}_X(mH))$ are always injective. Moreover, we have

$$\chi(\mathcal{O}_X(mH)) = \frac{1}{2}m^2H^2 + 2 = m^2 + 2$$

and

$$\chi(\mathcal{O}_{\mathbb{P}^2}(mH)) = \frac{1}{2}m^2H^2 + \frac{3}{2}mH^2 + 1 = \frac{1}{2}m^2 + \frac{3}{2}m + 1,$$

so for $m = 1$, 2 the spaces have the same dimension. So they are isomorphic and we can identify $|H|$ with $|\mathcal{O}_{\mathbb{P}^2}(1)|$ and $|2H|$ with $|\mathcal{O}_{\mathbb{P}^2}(2)|$. □

5.1 Description of the components of Δ

A generic quadric Q intersects the ramification divisor Σ transversely in twelve different points. Using Hurwitz's Theorem (cf. [8, Chapter IV.2]) we obtain in this situation that the curve C is a smooth curve of genus 5. We discuss below the possible degenerations. A generic quadric Q can degenerate in the following ways with respect to the ramification divisor Σ:

1. Q can intersect Σ with multiplicity 2 in some point. Such conics form the component

$$\Delta_{\Sigma_i} := \{Q \,|\, Q \text{ tangent to } \Sigma_i\},$$

one for each component Σ_i of Σ. For a generic element we may assume that Q is smooth and that it is tangent to Σ_i in a smooth point. We can also assume that Q does not pass through singularities of Σ and intersect Σ transversely in ten smooth points otherwise.

2. Q can degenerate into two lines l_1 and l_2. Such conics form the component

$$\Delta_{deg} := \{Q \,|\, Q = l_1 \cup l_2\}.$$

This includes the case that $Q \subset \mathbb{P}^2$ degenerate into a double line and hence is non-reduced and of the form $2L$, where $L \in |H|$. For a generic element we may assume that Q does not degenerate into a double line and that the intersection point does not lie on Σ, $l_1 \cap l_2 \notin \Sigma$. Moreover, we can also assume that both lines l_i are nowhere tangent to Σ and do not pass through singularities of Σ.

3. Q can pass through a singularity p of Σ. Such conics form the component

$$\Delta_p := \{Q \,|\, Q \text{ passes through a singularity } p \text{ of } \Sigma\},$$

one for each singular point of Σ. For a generic element we may assume that Q is smooth and should be nowhere tangent to Σ.

Our main goal is the description of the singular fibre $f^{-1}([Q])$ for generic points $[Q] \in \Delta$. In Chapter 6 we describe in details the structure of the singular fibres of the Lagrangian Fibration $f\colon M_X(0, 2H, \chi) \to |2H|$.

5.2 Degree of the components of Δ

We need to find the degree of the components of Δ. It is relative simple for the components Δ_{deg} and Δ_p.

Lemma 5.2 *The degree of the component Δ_{deg} is 3.*

Proof. Let $x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$ be a basis of $H^0(\mathbb{P}^2, \mathcal{O}(2))$ and $z_0, z_1, z_2, z_3, z_4, z_5$ be the dual basis. So the universal quadric is given by

$$z_0x_0^2 + z_1x_1^2 + z_2x_2^2 + z_3x_0x_1 + z_4x_0x_2 + z_5x_1x_2 =$$

$$\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} z_0 & \frac{1}{2}z_5 & \frac{1}{2}z_4 \\ \frac{1}{2}z_5 & z_1 & \frac{1}{2}z_3 \\ \frac{1}{2}z_4 & \frac{1}{2}z_3 & z_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

The divisor Δ_{deg} is given by the following cubic equation

$$0 = \det \begin{pmatrix} z_0 & \frac{1}{2}z_5 & \frac{1}{2}z_4 \\ \frac{1}{2}z_5 & z_1 & \frac{1}{2}z_3 \\ \frac{1}{2}z_4 & \frac{1}{2}z_3 & z_2 \end{pmatrix}.$$

So Δ_{deg} is a component of degree 3. □

The simplest case is for the component Δ_p.

Lemma 5.3 *The degree of the component Δ_p is 1.*

Proof. Passing through a one given point translates into one linear equation. So the component Δ_p has degree 1. □

Now we want to discuss the components Δ_{Σ_i}. The case if Q become tangent to Σ (or, to be precise, tangent to one of its components Σ_i) in a smooth point requires some longer calculations. If Σ is a smooth sextic then $\deg \Delta_\Sigma = 42$ (cf. Fulton [7, Example 3.2.21] and Kleiman [13]). We do actually more. We find the degree of this component if Q become tangent to reduced, irreducible and possible singular curve $T \subset \mathbb{P}^2$ of degree d. Let us consider the following algebraic subset of $|2H|$:

$$\begin{aligned} \Delta_T := & \overline{\{Q \,|\, \text{Q smooth quadric, tangent to reduced, irreducible}} \\ & \overline{\text{and possible singular curve } T \text{ of degree } d \text{ in a smooth point } q\}}. \end{aligned} \tag{4}$$

The result can be formulated in the following lemma:

Lemma 5.4 *The degree of the component Δ_T is equal*

$$deg\,\Delta_T = d(d+1) - \sum_{p \in Sing\,T} (\mu_p + m_p - 1),$$

where m_p is the multiplicity of (the germ of) T at the singular point $p \in T$ and μ_p is its Milnor number.

Proof. We start with a linear system $|2H|$. We are looking for quadrics, which are smooth and tangent in a smooth point q to an reduced, irreducible and possibly singular curve T of degree d. We consider the following incidence variety

$$\mathbb{P}^2 \times (\mathbb{P}^2)^* \supset Z := \{(q, L) \,|\, q \in L\},$$

where Z is a divisor of type $\mathcal{O}(1,1)$. In coordinates, for $q = (q_0 : q_1 : q_2)$ and $L = (l_0 : l_1 : l_2)$ the condition is $\sum_i (q_i \cdot l_i) = 0$. Now we consider the following incidence variety over Z

$$\begin{array}{ccc} |2H| & \longleftarrow & \{(q,L,Q)\mid q\in L, Q \text{ quadric }, L \text{ tangent to } Q \text{ in } q\} \\ & & \big\downarrow{\scriptstyle \mathbb{P}^3} \\ & & Z=\{(q,L)\mid q\in L\} \end{array}$$

which is a $\mathbb{P}^3$-bundle. The general quadric is given by $Q = a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$. Let $I_L = (x_0)$ and $I_q = (x_0,\, x_1)$. The condition that $q \in Q$ translates into $a_{22} = 0$, L is tangent to Q gives $a_{12} = 0$. Let ν be the normalization of T. Consider the set $R := \{(\widetilde{s}, Q) \mid \widetilde{s} \in \widetilde{T}, \psi(\widetilde{s}) \text{ tangent to } Q \text{ in } s\}$ such that

$$\begin{array}{ccc} R & \xrightarrow{\;p_2\;} & |2H| \\ \big\downarrow{\scriptstyle \mathbb{P}^3} & & \\ \widetilde{T} & \searrow^{\psi} & \\ \big\downarrow{\scriptstyle \nu} & & \\ T & & Z \subset \mathbb{P}^2 \times (\mathbb{P}^2)^* \\ \big\uparrow & \nearrow_{\alpha} & \\ T_{reg} & & \end{array}$$

where $\alpha:\ s \mapsto (s, T_T)$, T_T is the tangent line to the curve T and ψ is the induced tangent direction. We have $\dim(R) = 4$. Moreover, Δ_T is the image of R. Our goal is the computation of $\deg \Delta_T = \int_R \mathrm{p}_2^*(h)^4$, where h is the class of an ample divisor on the linear system $|2H|$. Now we want to formalize the above construction.

Let now $\mathbb{P}^2 = \mathbb{P}(V) := \operatorname{Proj} S^*V$, $\dim(V) = 3$ and let $Z \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ be the incidence variety. Then we have

$$Z = \mathrm{Drap}(V) = \{0 \subset V_1 \subset V_2 \subset V \mid \dim(V_i) = i\}.$$

There are two natural projections

$$V_2 \overset{p_2}{\twoheadleftarrow} (V_1, V_2) \overset{p_1}{\twoheadrightarrow} V_1$$

$$\mathbb{P}(V) \overset{p_2}{\twoheadleftarrow} \mathrm{Drap}(V) \overset{p_1}{\twoheadrightarrow} \mathbb{P}(V^*)\ ,$$

where V_1 corresponds to choosing L, and V_2 corresponds to choosing q, as above. We have universal bundle sequence

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset V \otimes \mathcal{O}_Z.$$

The corresponding line bundles are

$$V \otimes \mathcal{O}_Z/\mathcal{V}_2 \cong \mathrm{p}_2^* \mathcal{O}(1) \cong \mathcal{O}(h_2),$$

$$\mathcal{V}_1 \cong \mathrm{p}_1^* \mathcal{O}(-1) \cong \mathcal{O}(-h_1),$$

$$\mathcal{V}_2/\mathcal{V}_1 \cong \mathcal{O}(h_1 - h_2).$$

Every quadric Q in $\mathbb{P}^2$ is an element of $S^2 V$. On Z there is a canonical subbundle $\mathcal{W}$ of rank 4

$$\mathcal{W} := V \cdot \mathcal{V}_1 + S^2 \mathcal{V}_2 \subset \mathcal{O}_Z \otimes S^2 V.$$

To see that $\mathcal{W}$ is the subbundle of $\mathcal{O}_Z \otimes S^2 V$ we can decompose V in 3 lines $V = V_1 \oplus N \oplus M$, where $V_2 = V_1 \oplus N$ and $V = V_2 \oplus M$. We can pointwise calculate

$$V \cdot V_1 + S^2 V_2 = (V_1 + N + M) \cdot V_1 + (V_1 + N)(V_1 + N) \subset S^2 V$$

is a 4-dimensional vector space. So we have the following filtration

$$0 \longrightarrow V \otimes \mathcal{V}_1 \longrightarrow \mathcal{W} \longrightarrow (\mathcal{V}_2/\mathcal{V}_1)^{\otimes 2} \longrightarrow 0. \tag{5}$$

Now we consider a projectivization $\widehat{Z} := \mathbb{P}(\mathcal{W}^*)$

$$\begin{array}{ccc} \widehat{Z} = \mathbb{P}(\mathcal{W}^*) & \longrightarrow & Z \\ \downarrow q & & \\ |2H| = \mathbb{P}(S^2 V^*) & & \end{array}$$

and $q^*\mathcal{O}(h) = \mathcal{O}_{W^*}(1)$, where h is an ample generator of $|2H|$. Now we want to find the cohomology ring

$$\mathrm{H}^*(\widehat{Z}, \mathbb{Z}) = \mathrm{H}^*(Z, \mathbb{Z})[h]/\sim$$

$$\frac{c(\mathcal{W}^*)}{1+h} = (1 + c_1\mathcal{W}^* + c_2\mathcal{W}^* + c_3\mathcal{W}^* + c_4\mathcal{W}^*) \cdot (1 - h + h^2 - h^3 + h^4).$$

So the fundamental relation $\sim$ is given by

$$0 = h^4 - c_1\mathcal{W}^* h^3 + c_2\mathcal{W}^* h^2 - c_3\mathcal{W}^* h + c_4\mathcal{W}^*.$$

Now we need to find the cohomology ring of Z and the Chern classes.
On $\mathbb{P}(V)$ we have

$$\mathrm{H}^*(\mathbb{P}(V), \mathbb{Z}) = \mathbb{Z}[h_2]/(h_2)^3$$

and the universal sequence

$$0 \subset \mathcal{V}_2 \subset V \otimes \mathcal{O}_{\mathbb{P}(V)},$$

where $\mathcal{V}_2$ is a rank 2 subbundle and let $Z = \mathbb{P}(\mathcal{V}_2) \to \mathbb{P}(V)$. On Z we have

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{O}(\xi).$$

So $\frac{c(\mathcal{V}_2)}{1+\xi} = (1 + c_1\mathcal{V}_2 + c_2\mathcal{V}_2) \cdot (1 - \xi + \xi^2)$ and

$$\mathrm{H}^*(Z, \mathbb{Z}) = \mathbb{Z}[\xi]/(\xi^2 - c_1\mathcal{V}_2 \cdot \xi + c_2\mathcal{V}_2).$$

We have an exact sequence

$$0 \to \mathcal{V}_2 \to V \otimes \mathcal{O} \to \mathcal{O}(h_2) \to 0.$$

So the total Chern class $c(\mathcal{V}_2) = \frac{1}{1+h_2} = 1 - h_2 + h_2^2$ and $c_1\mathcal{V}_2 = -h_2$, $c_2\mathcal{V}_2 = h_2^2$. We have

$$\mathrm{H}^*(Z, \mathbb{Z}) = \mathbb{Z}[h_2, \xi]/(h_2^3,\ \xi^2 + \xi \cdot h_2 + h_2^2).$$

Now using

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathcal{O}(\xi) \to 0$$

and $\mathcal{O}(\xi) = \mathcal{O}(h_1 - h_2)$

$$\mathrm{H}^*(Z, \mathbb{Z}) = \mathbb{Z}[h_1, h_2]/(h_2^3,\ h_1^2 - h_1h_2 + h_2^2, h_1^3).$$

Using (5) we obtain similar filtration for $\mathcal{W}^*$

$$(\mathcal{V}_2/\mathcal{V}_1)^{-2} \to \mathcal{W}^* \to V^* \otimes \mathcal{V}_1^* \to 0.$$

So the total Chern class

$$c(\mathcal{W}^*) = (1 - 2(h_1 - h_2)) \cdot (1 + h_1)^3 = (1 + (h_1 + 2h_2) + (-3h_1^2 + 6h_1h_2) + 6h_1^2h_2.$$

The cohomology ring is

$$\mathrm{H}^*(\widehat{Z}, \mathbb{Z}) = \mathbb{Z}[h_1, h_2, h]/I,$$

where I is generated by

$$(h_1^3,\ h_1^2 - h_1h_2 + h_2^2,\ h^4 - (h_1 + 2h_2) \cdot h^3 + (6h_1h_2 - 3h_1^2) \cdot h^2 - 6h_1^2h_2 \cdot h)$$

$$\begin{array}{ccccc} |2H| & \xleftarrow{p_2} & R & \longrightarrow & \widehat{Z} \\ & & \downarrow & & \downarrow \\ & & \widetilde{T} & \xrightarrow{\psi} & Z \end{array}$$

So we obtain

$$\psi^* h_1 = \deg(T^\vee) \cdot u = d^\vee \cdot u$$

and similarly

$$\psi^* h_2 = \deg(T) \cdot u = d \cdot u.$$

We deduce

$$\mathrm{H}^*(R, \mathbb{Z}) = \mathrm{H}^*(T, \mathbb{Z})[h]/(h^4 - (d^\vee + 2d) \cdot h^3 \cdot u).$$

The degree of $T^\vee$ is given by

$$d^\vee = d(d-1) - \sum_{p \in Sing\, T} (\mu_p + m_p - 1),$$

where m_p is the multiplicity of (the germ of) T at the point $p \in T$ and μ_p is its Milnor number (cf. [27, Thm. 7.2.2]). So we obtain the final result

$$\begin{aligned} \deg \Delta_T &= \int_R \mathrm{p}_2^*(h)^4 = (d^\vee + 2d) \cdot \int_R h^3 u \\ &= d(d+1) - \textstyle\sum_{p \in Sing\, T} (\mu_p + m_p - 1). \end{aligned}$$

□

6 Singular fibers of the fibration f

In this key chapter of our paper we study in details the singular fibres of the Lagrangian Fibration $f\colon M_X(0, 2H, \chi) \to |2H|$. We begin with some generalities on pure sheaves of dimension 1. In Section 6.2 we describe the fibre of f over the components Δ_{Σ_i} of Δ. This is the case when a generic quadric Q intersect Σ_i with multiplicity 2 in a smooth point. In Section 6.3 we study the fibre over the component Δ_{deg}. This is the situation when a generic quadric Q degenerate into two lines. In Section 6.4 we give a description of the singular fibre over the components Δ_p, which means that a generic quadric Q passes through a singularity p of Σ_i. To study these cases we use results from Section 6.1.

6.1 Generalities on pure sheaves of dimension 1

In this section we collect some generalities on pure sheaves of dimension 1 which we need for our work. If the sextic $\Sigma \subset \mathbb{P}^2$ has a singularity of type D_n or E_n then some components of $C = \pi^*(Q)$ have multiplicities > 1. Therefore we collect some general properties of pure sheaves on non-reduced curves.

Let F be a coherent sheaf of dimension 1 on a smooth surface X. The torsion subsheaf $\mathrm{Tors}(F)$ is the maximal subsheaf with zero-dimensional support, so that the quotient $F/\mathrm{Tors}(F)$ is a pure sheaf of dimension 1. If F is a pure sheaf then any nontrivial subsheaf F' is itself pure. The saturation of F' in F is defined as the kernel K of the epimorphism

$$F \longrightarrow (F/F')/\mathrm{Tors}(F/F').$$

By construction, K is a pure sheaf, and $K/F' \cong \mathrm{Tors}(F/F')$.

Assume that $C = \sum_{i=0}^{n} C_i$ is a simple normal crossing divisor with smooth components C_i. If F is a pure sheaf with $\mathrm{supp}(F)_{red} \subset C$, we denote by $\mu_i(F)$ the multiplicity of F along the component C_i. By $\mathrm{supp}(F)$ we mean $V(\mathrm{Ann}(F))$. We have $\mathrm{supp}(F) \subset \sum \mu_i(F) C_i$. The restriction of F to the non-reduced curve $\mu_j(F) C_j$ may contain torsion. We set

$$F_j := F|_{\mu_j(F)C_j}/\mathrm{Tors}(F|_{\mu_j(F)C_j}).$$

We have the natural surjections $F \to F_j$, which induce a morphism

$$F \to \bigoplus_j F_j$$

that is generically injective on each component of divisor C. As F is a pure sheaf this morphism is everywhere injective. The quotient is supported on the singular points x of C. We have an exact sequence

$$0 \longrightarrow F \longrightarrow \bigoplus_j F_j \longrightarrow \bigoplus_x T_x \longrightarrow 0.$$

So we define T_x as the local component of the cokernel scheme. Let x be the intersection point of the components C_i and C_j. Then we have the following.

Lemma 6.1 *Both sheaves F_i and F_j surject onto T_x.*

Proof. Let U be a small neighbourhood of the point x. We have the following exact sequence

$$0 \longrightarrow F|_U \xrightarrow{(a_i,a_j)} F_i|_U \oplus F_j|_U \xrightarrow{(\pi_i,\pi_j)} T_x \longrightarrow 0$$

and $\pi_i \circ a_i = -\pi_j \circ a_j$. So we have locally near x the following commutative diagram

$$\begin{array}{ccc} F_i & \xrightarrow{\pi_i} & T_x \\ {\scriptstyle a_i}\uparrow & \overset{\xi}{\nearrow} & \uparrow{\scriptstyle \pi_j} \\ F & \underset{-a_j}{\twoheadrightarrow} & F_j \end{array}$$

We deduce $\pi_i(F_i) = \xi(F) = \pi_j(F_j)$ and

$$T_x = \mathrm{Im}(\pi_i, \pi_j) = \mathrm{Im}(\pi_i) + \mathrm{Im}(\pi_j) = \mathrm{Im}(\pi_i) = \mathrm{Im}(\pi_j).$$

□

On each sheaf F_j, the ideal sheaf of the component C_j in X induces a natural decreasing filtration

$$F_j = F_j^0 \supset F_j^1 \supset \dots,$$

where F_j^s is the saturation of the subsheaf $F_j \cdot \mathcal{O}_X(-sC_j)$. The subquotients $\mathrm{gr}^s F_j := F_j^{s-1}/F_j^s$ are pure sheaves on the smooth reduced curve C_j and hence locally free. Let $r_{js} = \mathrm{rk}(\mathrm{gr}^s F_j)$ denote the rank of the s-th subquotient. Then

$$\sum_s r_{js} = \mu_j(F).$$

For each s there is a generically surjective map

$$\mathrm{gr}^s(F_j) \otimes \mathcal{O}_{C_j}(-C_j) \to \mathrm{gr}^{s+1}(F_j),$$

so the sequence of ranks is monotonously decreasing

$$r_{j1} \geq r_{j2} \geq \ldots . \tag{6}$$

Lemma 6.2 *If $x \in C_i \cap C_j$ then $\chi(T_x) \leq \sum_{s,s'} \min\{r_{js}, r_{is'}\}$.*

Proof. Assume first that the length of the filtrations on F_i and F_j equal 1, so that F_i and F_j are supported on the reduced curves C_i and C_j, respectively and are locally free. Locally near x we have the following cartesian square

$$\begin{array}{ccc} F_i & \twoheadrightarrow & T_x \\ \uparrow & & \uparrow \\ F & \twoheadrightarrow & F_j \end{array}$$

As T_x is a quotient both of F_i and F_j it must be of the form $\mathcal{O}_x^{\oplus r}$ with rank $r \leq \mathrm{rk}(F_i)$ and $r \leq \mathrm{rk}(F_j)$. So we have $\chi(T_x) \leq \min\{r_{i1}, r_{j1}\}$ and the assertion is clear in this case.

Otherwise we may assume that the length of the filtration on F_i is ≥ 2. So we are in the situation that $F_i^1 \neq 0$. Let $T' \subset T_x$ denote the image of F_i^1 in T_x, and let F_j' denote the preimage of T' under the projection $F_j \twoheadrightarrow T_x$. Let us consider the following diagrams.

$$\begin{array}{ccc} F_i^1 & \longrightarrow & T' \\ & & \uparrow \\ & & F_j' \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{gr}_i^1(F) & \longrightarrow & T_x/T' \\ & & \uparrow \\ & & F_j \end{array}$$

Note that all arrows are surjective by construction. As the inclusion $F_j' \to F_j$ is generically an isomorphism, the sheaves F_j' and F_j have the same rank sequences. Using the left diagram we obtain:

$$\chi(T') \leq \sum_{s,s',\, s' \geq 2} \min\{r_{js}, r_{is'}\}.$$

From the right diagram we get:

$$\chi(T_x/T') \leq \sum_{s} \min\{r_{js}, r_{i1}\}.$$

The assertion of the lemma follows now by induction on the length of the filtration.

□

We will be particularly interested in the case that F is a pure sheaf with the property that on each component C_j one has $r_{j1} = 1$ (see Lemma 6.11). As the ranks form a decreasing sequence this implies that $r_{js} = 1$ for all $1 \leq s \leq \mu_j(F)$ and $r_{js} = 0$ for $s > \mu_j(F)$. Equivalently, this is the case if and only if F is an invertible sheaf on a dense open subset of the scheme-theoretic support $\mathrm{supp}(F) = \sum_j \mu_j(F)C_j$. Assuming this property, there are monomorphism of line bundles on C_j:

$$\mathrm{gr}^s(F_j) \otimes \mathcal{O}_{C_j}(-C_j) \to \mathrm{gr}^{s+1}(F_j), \;\; 1 \leq s < \mu_j(F). \tag{7}$$

The sheaf F_j is locally free at $y \in \mu_j(F)C_j$ if and only if all these maps are isomorphisms at y.

Keeping the assumption that $r_{j1} = 1$ for all j, let x be an intersection point $x \in C_i \cap C_j$ and consider the scheme $\xi := (\mu_i(F)C_i \cap \mu_j(F)C_j)_x$. Then $\mathcal{O}_\xi \cong \mathbb{C}[z_1, z_2]/(z_1^{\mu_i(F)}, z_2^{\mu_j(F)})$, since C_i and C_j intersect transversely by assumption. Note that the socle $\kappa := (0 : m_x) \subset \mathcal{O}_\xi$ is a one-dimensional vector space spanned by the element corresponding to $z_1^{\mu_i(F)-1} z_2^{\mu_j(F)-1}$. Then T_x is an $\mathcal{O}_\xi$-module and the natural epimorphisms

$$\alpha_{xi} \colon F_i \to T_x \;\text{ and }\; \alpha_{xj} \colon F_j \to T_x$$

factor through the $\mathcal{O}_\xi$-modules $F_{xij} := (F_i|_{\mu_j(F)D_j})_x$ and $F_{xji} := (F_j|_{\mu_i(F)D_i})_x$. Note that using Lemma 6.2 we obtain

$$\chi(T_x) \leq \mu_i(F)\mu_j(F) = \chi(\mathcal{O}_\xi).$$

We have also

$$\chi(F_{xij}) = \chi(\mathcal{O}_\xi)$$

and

$$\chi(F_{xji}) = \chi(\mathcal{O}_\xi).$$

It is enough to prove the first equality. In the second equality the rules of i and j are exchanged. We take a restriction of the sequence

$$0 \longrightarrow F_i^1 \longrightarrow F_i^0 \longrightarrow F_i^0/F_i^1 \longrightarrow 0$$

to C_j and obtain the following exact sequence

$$0 \longrightarrow \mathrm{Tor}((F_i^0/F_i^1)|_{C_j}, \mathcal{O}_{C_j}) \longrightarrow F_i^1|_{C_j} \longrightarrow F_i^0|_{C_j} \longrightarrow (F_i^0/F_i^1)|_{C_j} \longrightarrow 0.$$

As $F_i^1|_{C_j}$ is pure and $\mathrm{Tor}((F_i^0/F_i^1)|_{C_j}, \mathcal{O}_{C_j})$ is 0-dimensional we must have $\mathrm{Tor}((F_i^0/F_i^1)|_{C_j}, \mathcal{O}_{C_j}) = 0$. Now the assertion follows by the additivity of the Euler characteristic on the exact sequences.

From this we conclude:

Lemma 6.3 *1. Assume that F_i is locally free at x. If $\chi(T_x) = \mu_i(F)\mu_j(F)$ then all sheaves F_{xij}, F_{xji} and T_x are isomorphic to $\mathcal{O}_\xi$. In particular, F_j is also locally free at x, and the structure of F at x is determined by a gluing isomorphism $F_{xij} \to F_{xji}$ determined by α_{xi} and α_{xj}.*

2. Assume that F_i and F_j are both locally free at x. If $\chi(T_x) = \mu_i(F)\mu_j(F) - 1$, then $T_x \cong \mathcal{O}_\xi/\kappa$, and the structure of F at x is determined by a gluing isomorphism $\overline{F}_{xij} \to \overline{F}_{xji}$, where $\overline{}$ denotes taking the quotient of the sheaf by its socle.

Proof. 1. If F_i is locally free at x then $F_{xij} \cong \mathcal{O}_\xi$. If $\chi(T_x) = \mu_i(F)\mu_j(F)$, then T_x and F_{xij} are sheaves of the same length, so that the epimorphism $F_{xij} \to T_x$ must be an isomorphism. Similarly, as $\chi(F_{xji}) = \mu_i(F)\mu_j(F)$, the epimorphism $F_{xji} \to T_x$ must be an isomorphism. But then F_x is generated by a single element and hence is locally free.
2. If $\chi(T_x) = \mu_i(F)\mu_j(F) - 1$, the kernels of both epimorphisms $F_{xij} \to T_x$ and $F_{xji} \to T_x$ are one-dimensional and hence are annihilated by m_x. As the socles of F_{xji} and F_{xij} are both isomorphic to the one-dimensional socle κ of $\mathcal{O}_\xi$, these kernels must coincide with the socles. □

After all preparations we describe the fibre $f^{-1}([Q])$ for a general point $[Q] \subset \Delta$ (see Chapter 5).

6.2 The components Δ_{Σ_i} of the discriminant divisor Δ

In this section we want to describe the fiber $f^{-1}([Q])$ for a general point $[Q] \in \Delta_{\Sigma_i} \subset \Delta$. So we are in the case when a generic quadric Q intersect Σ_i with multiplicity 2 in a smooth point (cf. Section 5.1). Under these genericity assumptions the curve $C = \pi^{-1}(Q)$ is a curve with a nodal singularity at the point q. Then we may take the normalization $\nu\colon \widetilde{C} \to C$. As $\widetilde{C} \to Q$ is ramified in 10 points, $\widetilde{C}$ has genus 4. Let q_1, $q_2 \in \widetilde{C}$ denote the two points that map to q.

Now we recall briefly the idea of compactification of generalised Jacobians, which we use for the description of the fiber of f over the components Δ_{Σ_i} and Δ_{deg}. The main reference is the paper of D'Souza ([5]).

Let C be a projective integral curve over $\mathbb{C}$ and let P be the space of isomorphism classes of line bundles of degree zero on C. There is a natural structure of a group variety on P. If C is smooth, P is projective, hence an abelian variety called the Jacobian of C. If C has singularities, P is a quasi-projective variety, called the generalised Jacobian of C. D'Souza showed that there exists a natural compactification $\bar{P}$ of P. Moreover, the points of P corresponds to invertible sheaves L on C such that $\chi(L) = \chi(\mathcal{O}_C)$. The points in the compactification $\bar{P}$ corresponds to rank 1, torsion free sheaves F on C such that $\chi(F) = \chi(\mathcal{O}_C)$.

Now we come back to our set-up and use the above idea of D'Souza. Sheaves in $f^{-1}([Q])$ correspond to torsion free sheaves on C of rank 1 and Euler characteristic χ. So our main goal is the description of the family of torsion free sheaves on C of rank 1 and Euler characteristic χ. Because of the fixed topological data for the sheaf F in the fibre over Δ_{Σ_i} we have

$$P(F, n) = 4n + \chi. \tag{8}$$

6.2.1 Local description of the fiber over Δ_{Σ_i}

Due to our set-up $C \subset X$ is a reduced curve. Moreover, for $q \in C$ singular, $\nu^{-1}(q)$ consists of two points: q_1 and q_2 corresponding to the different branches of C through q. On C there is a standard normalization sequence (cf. [3, Section II.1]):

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_{\widetilde{C}} \longrightarrow S \longrightarrow 0,$$

with S concentrated at the singularities of C.

Our idea is to consider a degree d line bundle $L := \nu^*F/\text{Tors}$. Using Riemann-Roch theorem for curves we obtain $d = \chi - r(1-g)$. Then the above sheaf F fits into an exact sequence

$$0 \longrightarrow F \longrightarrow \nu_*L \longrightarrow T \longrightarrow 0, \tag{9}$$

where L is a line bundle on $\widetilde{C}$ and T is $\mathcal{O}_q \cong \mathbb{C}$ or 0, depending on whether the sheaf F is locally free in the point q or not. If F is locally free at q then we say that we glue in the point q. The curve C has exactly two reduced branches which intersect transversely in q. This implies that we have only these two possibilities.

Below we consider the both cases separately. Then we want to handle both cases together as a family.

No gluing in q

We consider the first possibility. Let us assume that F is not locally free in q. Then the sheaf F arises from the choice of a line bundle $L \in \mathrm{Jac}_d(\widetilde{C})$ and $F = \nu_* L$. Moreover, we have $P(\nu_* L, n) = P(F, n)$, so $4n + d - 3 = 4n + \chi$, which gives $d = \chi + 3$.

Gluing in q

Let F be locally free in the point q. Then we have gluing in q. This means that we start with a map φ which is surjective on both components in point q. Then the sheaf $F = \ker(\nu_* L \xrightarrow{\varphi} \mathcal{O}_q)$ is locally free in all points. We have $P(\nu_* L, n) - 1 = P(F, n)$, so $4n + d - 3 - 1 = 4n + \chi$, which gives $d = \chi + 4$.

6.2.2 Description of the fiber over Δ_{Σ_i}

Now we want to handle both cases together. To simplify the situation we begin with the fixed line bundle L from $\mathrm{Jac}_{\chi+4}(\widetilde{C})$. Afterwards we let L vary. The crucial point is to understand the map

$$\nu_* L \xrightarrow{x} \mathcal{O}_q.$$

Let $L|_{q_i}$ be the restriction of the line bundle L to the point q_i for $i = 1, 2$.
The homomorphism $x\colon \nu_* L \to \mathcal{O}_q$ corresponds to two linear maps:

$$x_1\colon L|_{q_1} \to \mathcal{O}_q,\ x_2\colon L|_{q_2} \to \mathcal{O}_q.$$

Thus x may be consider as an element in

$$V = \mathrm{Hom}(L|_{q_1} \oplus L|_{q_2}, \mathcal{O}_q).$$

The automorphism group $\mathrm{Aut}(L) \times \mathrm{Aut}(\mathcal{O}_q) \cong (\mathbb{C}^*)^2$ act on V as follows: for

$$(t, s) \in \mathrm{Aut}(L) \times \mathrm{Aut}(\mathcal{O}_q)$$

and

$$(x_1, x_2) \in \mathrm{Hom}(L|_{q_1} \oplus L|_{q_2}, \mathcal{O}_q)$$

we have

$$(t, s) \cdot (x_1, x_2) = (t^{-1}s\, x_1, t^{-1}s\, x_2).$$

The invariant ring is equal $\mathbb{C}[x_1, x_2]^\chi$, where $\chi(t, s) := \frac{s}{t}$. The quotient $V/\!/_\chi G$ is isomorphic to $\operatorname{Proj} \mathbb{C}[x_1, x_2] \cong \mathbb{P}^1$.

After this preparation we want to describe the complete fibre as a family. Let $J := \operatorname{Jac}_{\chi+4}(\widetilde{C})$ and let P denote the Poincaré bundle on $J \times \widetilde{C}$. Let

$$E_{1q} = (\mathrm{id} \times \nu)_* P|_{J \times q_1} \quad \text{and} \quad E_{2q} = (\mathrm{id} \times \nu)_* P|_{J \times q_2}.$$

We need to study the quotient of

$$E_{1q}^* \oplus E_{2q}^* =: V$$

by G with respect to the action described above. This gives:

$$\begin{aligned} V^{ss}/\!/G &= \operatorname{Proj}_J((S^* V^\vee)^{G,\chi}) \\ &= \operatorname{Proj}_J(S^*(E_{1q} \oplus E_{2q})) \\ &= \mathbb{P}_J(E_{1q} \oplus E_{2q}) \end{aligned}$$

a $\mathbb{P}^1$-bundle over J.
Let $U := E_{1q} \oplus E_{2q}$. The two surjections

$$E_{1q} \overset{\psi_1}{\twoheadleftarrow} U \overset{\psi_2}{\twoheadrightarrow} E_{2q}$$

correspond to two disjoint sections

$$\sigma_1, \sigma_2 \colon J \to \mathbb{P}(U).$$

1. Points in $\mathbb{P}(U) \setminus (\sigma_1(J) \cup \sigma_2(J))$ correspond to stable locally free sheaves.

2. Points in $\sigma_1(J)$ correspond to maps x where only x_1 vanish. The corresponding sheaves are stable and isomorph to the sheaf $\nu_*(L(-q_2))$.

3. Points in $\sigma_2(J)$ correspond to maps x where only x_2 vanish. The corresponding sheaves are stable and isomorph to the sheaf $\nu_*(L(-q_1))$.

It follows, that $\sigma_1([L])$ and $\sigma_2([M])$ correspond to the same sheaf, if

$$L(-q_1) \cong M(-q_2)$$

or equivalently

$$[M] = [L] \otimes \mathcal{O}_{\widetilde{C}}(q_2 - q_1).$$

On $\mathbb{P}(U)$ we can define the equivalence relation $\sim$ in the following way:

$$\sigma_1(u \otimes \mathcal{O}_{\widetilde{C}}(q_2 - q_1)) \sim \sigma_2(u), u \in J. \tag{10}$$

After this preparation we can state the main result of this section:

Theorem 6.4 *Let $[Q]$ be a general point of Δ on the component Δ_{Σ_i}. Then the Lagrangian fibre $f^{-1}([Q])$ is isomorphic to the quotient $\mathbb{P}(U)/\sim$.*

Thus the general fibre over the component Δ_{Σ_i} is a $\mathbb{P}^1$-bundle over a Jacobian J of a genus 4 curve $\widetilde{C}$, where two sections are identified after a shift by the difference $q_2 - q_1$ in the group J.

Below we give a description of this fibre as a flat family of sheaves on C parametrised by $\mathbb{P}(U)$. Every such projectivization comes in the natural way with a surjective morphism to $\mathcal{O}_{\mathbb{P}(U)}(1)$. This gives a gluing data Λ. On $\mathbb{P}(U) \times C$ we define the sheaf F as a kernel $F = \ker(\phi)$ of the composition of the restriction map r (to the point q) and the map coming from the gluing data Λ:

$$\begin{array}{ccc} (pr_J \times id_C)^* \nu_* P & \xrightarrow{\;r\;} & pr_J^*(U) \otimes pr_C^* k(q) \\ & \searrow^{\phi} & \downarrow \Lambda \\ & & pr_{\mathbb{P}(U)}^* \mathcal{O}_{\mathbb{P}(U)}(1) \end{array}$$

6.3 The component Δ_{deg} of the discriminant divisor Δ

We study in this section the fiber $f^{-1}([Q])$ for a general point $[Q] \in \Delta_{deg} \subset \Delta$. So we are in the situation when a generic quadric Q degenerates into two lines (cf. Section 5.1). In this case, the preimage $C = \pi^*(Q)$ consists of two components C_1 and C_2 on X, so that $C = C_1 \cup C_2$. Under these genericity assumptions the curves C_1 and C_2 intersect transversely in two points p and q. Then we may take the normalization $\nu \colon \widetilde{C} \to C$, analogously to the previous case. The curve $\widetilde{C}$ has two connected smooth components $\widetilde{C_1}$, $\widetilde{C_2}$ that map isomorphically to C_1 and C_2, respectively. As $\widetilde{C}_i \to l_i$ is ramified in 6 points, $\widetilde{C}_i$ has genus 2. Let $p_i, q_i \in \widetilde{C}_i$ for $i = 1, 2$ be the points that map to p, q.

Our main goal, as previously, is the description of the family of torsion free sheaves on C of rank 1 and Euler characteristic χ. We assume that χ is an even number. Because of the above fixed topological data for the sheaf F in the fibre over Δ_{deg} we have

$$P(F, n) = 4n + \chi. \tag{11}$$

Any such sheaf F arises from the choice of line bundles $L_i \in \mathrm{Jac}_{d_i}(\widetilde{C}_i)$ and some gluing data. To simplify the situation we start with fixed line bundles L_i on the curves $\widetilde{C}_i$ of degrees d_i, respectively. The Riemann-Roch theorem for curves gives

$$\chi(L_1) = d_1 - 1 \text{ and } \chi(L_2) = d_2 - 1.$$

So the Hilbert polynomials are

$$P(\nu_* L_1, n) = 2n + d_1 - 1 \text{ and } P(\nu_* L_2, n) = 2n + d_2 - 1.$$

6.3.1 Local description of the fiber over Δ_{deg}

Locally there are three possibilities.

No gluing in p and no gluing in q

We assume that there is no gluing in p and no gluing in q. Then the sheaf F is given by

$$F = \nu_* L_1 \oplus \nu_* L_2. \tag{12}$$

This sheaf is neither locally free in the point p nor in the point q. We have

$$P(\nu_* L_1, n) + P(\nu_* L_2, n) = P(F, n), \text{ so}$$

$$2n + d_1 - 1 + 2n + d_2 - 1 = 4n + \chi \tag{13}$$

and from this we obtain the following condition

$$d_1 + d_2 = \chi + 2. \tag{14}$$

Semistability condition gives

$$d_1 - 1 + 2n \leq \frac{1}{2}(4n + \chi) \tag{15}$$

so

$$d_1 \leq 1 + \frac{\chi}{2} \text{ and analogously } d_2 \leq 1 + \frac{\chi}{2}. \tag{16}$$

Taking into account (14), we obtain $d_1 = d_2 = 1 + \frac{\chi}{2}$.

Gluing in the point p or in the point q

Let F be locally free in exactly one of the two points p and q. It suffices to treat the case that the sheaf F is locally free in the point p and not locally free in the point q. Then we have gluing in p. This means that we start with a map φ which is surjective on both components in point p. Then the sheaf

$$F = \ker(\nu_* L_1 \oplus \nu_* L_2 \xrightarrow{\varphi} \mathcal{O}_p)$$

is locally free in p and not locally free in q. We have

$$P(\nu_* L_1, n) + P(\nu_* L_2, n) - 1 = P(F, n),$$

so $d_1 - 1 + 2n + d_2 - 1 + 2n - 1 = 4n + \chi$, which gives

$$d_1 + d_2 = \chi + 3. \tag{17}$$

Semistability conditions translates into

$$d_1 \leq 2 + \frac{\chi}{2} \text{ and } d_2 \leq 2 + \frac{\chi}{2}.$$

So there are two possibilities

$$(d_1, d_2) = (2 + \frac{\chi}{2}, 1 + \frac{\chi}{2}) \text{ or } (d_1, d_2) = (1 + \frac{\chi}{2}, 2 + \frac{\chi}{2}).$$

If for example $(d_1, d_2) = (2 + \frac{\chi}{2}, 1 + \frac{\chi}{2})$ then for such strictly semistable sheaves we have an exact sequence

$$0 \longrightarrow \nu_*(L_1(-p)) \longrightarrow F \longrightarrow \nu_* L_2 \to 0.$$

The subsheaf

$$\nu_*(L_1(-p)) = \ker(\nu_* L_1 \to \mathcal{O}_p) \subset F$$

has reduced Hilbert polynomial $n + \frac{\chi}{4}$ and therefore is destabilizing.
F is S-equivalent to the $\nu_*(L_1(-p)) \oplus \nu_* L_2$, as discussed above

$$F \sim_S \nu_*(L_1(-p)) \oplus \nu_* L_2.$$

The case $(d_1, d_2) = (1 + \frac{\chi}{2}, 2 + \frac{\chi}{2})$ can be treated analogously.

Gluing in both points p and q

Let F be locally free in both points p and q. Then we have gluing in p and in q. This means that we start with a map φ which is surjective on both components in point p and φ is also surjective on both components in point q. Then the sheaf

$$F = \ker(\nu_* L_1 \oplus \nu_* L_2 \xrightarrow{\varphi} \mathcal{O}_p \oplus \mathcal{O}_q)$$

is locally free in all points. The Hilbert polynomials gives the following condition

$$P(\nu_* L_1, n) + P(\nu_* L_2, n) - 2 = P(F, n).$$

So that $4n + d_1 + d_2 - 4 = 4n + \chi$, or equivalently

$$d_1 + d_2 = \chi + 4. \tag{18}$$

The semistability condition translates into

$$d_1 \leq 3 + \frac{\chi}{2} \text{ and } d_2 \leq 3 + \frac{\chi}{2}.$$

So there are three possibilities for semistable sheaves:

1. $(d_1, d_2) = (3 + \frac{\chi}{2}, 1 + \frac{\chi}{2})$
2. $(d_1, d_2) = (1 + \frac{\chi}{2}, 3 + \frac{\chi}{2})$
3. $(d_1, d_2) = (2 + \frac{\chi}{2}, 2 + \frac{\chi}{2})$ and these are stable.

For strictly semistable sheaves in case (1) we have the following exact sequence

$$0 \longrightarrow \nu_*(L_1(-p-q)) \longrightarrow F \longrightarrow \nu_* L_2.$$

The subsheaf

$$\nu_*(L_1(-p-q)) = \ker(\nu_* L_1 \to \mathcal{O}_p \oplus \mathcal{O}_q) \subset F$$

is destabilizing and F is S-equivalent to

$$F \sim_S \nu_*(L_1(-p-q)) \oplus \nu_* L_2,$$

as in the first situation.
In case (2) the rules of L_1 and L_2 are exchanged.
In case (3) we have $(d_1, d_2) = (2 + \frac{\chi}{2}, 2 + \frac{\chi}{2})$ and F is stable. To see this we need to discuss two possibilities:

1. We could have a subsheaf $F' \subset F$ supported on both components C_1 and C_2. Then F' differs from F only by finite number of torsion points. But then we have $p(F') < p(F)$, which implies that F is a stable sheaf.

2. We could have a subsheaf F' supported only on one component, e.g. on C_1 and then $F' \subset \nu_* L_1$. But such a sheaf, due to our assumption, fits into an exact sequence

$$0 \longrightarrow F \longrightarrow \nu_* L_1 \oplus \nu_* L_2 \longrightarrow \mathcal{O}_p \oplus \mathcal{O}_q \longrightarrow 0.$$

So F' is also a subsheaf of $\nu_*(L_1(-p-q))$ and therefore is stable.

6.3.2 Description of the fiber over Δ_{deg}

Now we want to handle all these cases together. To simplify the situation we begin with fixed line bundles L_i from $\mathrm{Jac}_{2+\frac{\chi}{2}}(\widetilde{C}_i)$. Afterwards we let L_i, $i = 1, 2$ vary. The crucial point is to understand the map

$$\nu_* L_1 \oplus \nu_* L_2 \xrightarrow{x} \mathcal{O}_p \oplus \mathcal{O}_q.$$

Let $L_i|_{p_i}$ be the restriction of the line bundle L_i to the point p_i and similarly for $L_i|_{q_i}$. The homomorphism $x\colon \nu_* L_1 \oplus \nu_* L_2 \to \mathcal{O}_p \oplus \mathcal{O}_q$ corresponds to four linear maps:

$$x_1\colon L_1|_{p_1} \to \mathcal{O}_p \qquad x_3\colon L_2|_{p_2} \to \mathcal{O}_p$$
$$x_2\colon L_1|_{q_1} \to \mathcal{O}_q \qquad x_4\colon L_2|_{q_2} \to \mathcal{O}_q.$$

Thus x may be consider as an element in

$$V = \mathrm{Hom}(L_1|_{p_1} \oplus L_2|_{p_2}, \mathcal{O}_p) \oplus \mathrm{Hom}(L_1|_{q_1} \oplus L_2|_{q_2}, \mathcal{O}_q).$$

The automorphism groups $\mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2) \times \mathrm{Aut}(\mathcal{O}_p) \times \mathrm{Aut}(\mathcal{O}_q) \cong (\mathbb{C}^*)^4$ acts on V as follows:
for

$$(t_1, t_2, s_p, s_q) \in \mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2) \times \mathrm{Aut}(\mathcal{O}_p) \times \mathrm{Aut}(\mathcal{O}_q)$$

and

$$(x_1, x_2, x_3, x_4) \in \mathrm{Hom}(L_1|_{p_1} \oplus L_2|_{p_2}, \mathcal{O}_p) \oplus \mathrm{Hom}(L_1|_{q_1} \oplus L_2|_{q_2}, \mathcal{O}_q)$$

we have

$$(t_1, t_2, s_p, s_q) \cdot (x_1, x_2, x_3, x_4) = (t_1^{-1} s_p\ x_1,\ t_1^{-1} s_q\ x_2,\ t_2^{-1} s_p\ x_3,\ t_2^{-1} s_q\ x_4).$$

The diagonal group $\Delta\colon\ \mathbb{C}^* \to (\mathbb{C}^*)^4$ acts trivially. So only the factor group

$$G = (\mathbb{C}^*)^4/\Delta(\mathbb{C}^*) \tag{19}$$

acts effectively.

Lemma 6.5 *If x and x' have the same orbits, the sheaves $F = ker(x)$ and $F' = ker(x')$ are isomorphic.*

We consider the graded ring

$$\mathbb{C}[V]^{\chi,G} = \bigoplus_{n\geq 0} \{f \in \mathbb{C}[V] \mid f \circ g = \chi(g)^n \circ f\}$$

where $\chi(t_1, t_2, s_p, s_q) := \frac{s_p s_q}{t_1 t_2}$.
The invariant ring is spanned by $\mathbb{C}[x_1x_4, x_2x_3]$. The open subset $V_\chi^{ss} \subset V$ of semistable points is defined as follows:

$$x \in V_\chi^{ss} \ \Leftrightarrow\ x_1x_4 \neq 0 \text{ or } x_2x_3 \neq 0.$$

The quotient $V/\!/_\chi G$ is isomorphic to $\operatorname{Proj} \mathbb{C}[x_1x_4, x_2x_3] \cong \mathbb{P}^1$.

After this preparation we want to describe the complete fibre as a family. Let $T := \mathrm{Jac}_{2+\frac{\chi}{2}}(\widetilde{C_1}) \times \mathrm{Jac}_{2+\frac{\chi}{2}}(\widetilde{C_2})$. Moreover, let P_i denote the Poincaré line bundle on $\mathrm{Jac}_{2+\frac{\chi}{2}}(\widetilde{C}_i) \times \widetilde{C}_i$ and $\widetilde{P}_i \in \mathrm{Pic}(T \times \widetilde{C}_i)$ the pullback to $T \times \widetilde{C}_i$. Let

$$E_{ip} = (\mathrm{id} \times \nu)_*\, \widetilde{P}_i|_{T\times p_i} \quad\text{and}\quad E_{iq} = (\mathrm{id} \times \nu)_*\, \widetilde{P}_i|_{T\times q_i}$$

for $i = 1, 2$. We need to study the quotient of

$$E_{1p}^* \oplus E_{1q}^* \oplus E_{2p}^* \oplus E_{2q}^* =: V$$

by $G = (\mathbb{C}^*)^4/\Delta$ with respect to the action described above. This gives:

$$\begin{aligned} V^{ss}/\!/G &= \operatorname{Proj}_T((S^*V^\vee)^{G,\chi}) \\ &= \operatorname{Proj}_T(S^*(E_{1p} \otimes E_{2q} \oplus E_{1q} \otimes E_{2p})) \\ &= \mathbb{P}_T(E_{1p} \otimes E_{2q} \oplus E_{1q} \otimes E_{2p}) \end{aligned}$$

a $\mathbb{P}^1$-bundle over T.
Let $U := E_{1p} \otimes E_{2q} \oplus E_{1q} \otimes E_{2p}$. The two surjections

$$E_{1p} \otimes E_{2q} \xleftarrow{\psi_1} U \xrightarrow{\psi_2} E_{1q} \otimes E_{2p}$$

correspond to two disjoint sections

$$\sigma_1, \sigma_2 \colon T \to \mathbb{P}(U).$$

1. Points in $\mathbb{P}(U) \setminus (\sigma_1(T) \cup \sigma_2(T))$ correspond to stable locally free sheaves.
2. Points in $\sigma_1(T)$ correspond to maps x where x_2 or x_3 or both vanish. The corresponding sheaves are semistable only and in fact S-equivalent to the polystable sheaf

$$\nu_*(L_1(-q)) \oplus \nu_*(L_2(-p)).$$

3. Points in $\sigma_2(T)$ correspond to maps x where x_1 or x_4 or both vanish. The corresponding sheaves are semistable only and in fact S-equivalent to the polystable sheaf

$$\nu_*(L_1(-p)) \oplus \nu_*(L_2(-q)).$$

It follows, that $\sigma_1([L_1],[L_2])$ and $\sigma_2([M_1],[M_2])$ correspond to the same sheaf, if

$$L_1(-q) \cong M_1(-p),\ L_2(-p) \cong M_2(-q)$$

or equivalently

$$[M_1] = [L_1] \otimes \mathcal{O}_{\widetilde{C_1}}(p-q),\ [M_2] = [L_2] \otimes \mathcal{O}_{\widetilde{C_2}}(q-p).$$

The two sections in the projective bundle $\mathbb{P}(U)$ are identified after the composition of the two shifts: by the difference $p-q$ and $q-p$ in the groups $\mathrm{Jac}_{2+\frac{\chi}{2}}(\widetilde{C_1})$ and $\mathrm{Jac}_{2+\frac{\chi}{2}}(\widetilde{C_2})$, respectively. Similarly as for the components Δ_{Σ_i} (cf. relation (10)), we can define on $\mathbb{P}(U)$ the equivalence relation $\sim$ in the following way:

$$\sigma_1(u \otimes \mathcal{O}_{\widetilde{C_1}}(p-q) \otimes \mathcal{O}_{\widetilde{C_2}}(q-p)) \sim \sigma_2(u), u \in T. \tag{20}$$

The result of this section can be summarised as follows:

Theorem 6.6 *Let $[Q]$ be a general point of Δ on the component Δ_{deg}. Then the Lagrangian fibre $f^{-1}([Q])$ is isomorphic to the quotient $\mathbb{P}(U)/\sim$.*

Below we give a description of this fibre as a flat family of sheaves on C parametrised by $\mathbb{P}(U)$. Every such projectivization comes in the natural way with a surjective morphism to $\mathcal{O}_{\mathbb{P}(U)}(1)$. This gives a gluing data Λ. Similarly as in the previous section, we define on $\mathbb{P}(U) \times C$ the sheaf F as a kernel $F = \ker(\phi)$ of the composition of the restriction map r (to the points p and q) and the map coming from the gluing data Λ:

$$\begin{array}{ccc} (pr_T \times id_C)^*(\nu_* P_1 \oplus \nu_* P_2) & \xrightarrow{r} & pr_T^*(U) \otimes pr_C^* k(p) \otimes pr_C^* k(q) \\ & \searrow^{\phi} & \downarrow \Lambda \\ & & pr^*_{\mathbb{P}(U)} \mathcal{O}_{\mathbb{P}(U)}(1) \end{array}$$

6.4 The components Δ_p of the discriminant divisor Δ

We now turn to the more involved problem of describing the fiber $f^{-1}([Q])$ for a general point $[Q] \in \Delta_p \subset \Delta$, corresponding to a generic quadric Q passing through a singular point p of Σ. The point p is a point of the A-D-E-type. Let Γ be the corresponding Dynkin diagram of the resolution and let $\widetilde{\Gamma}$ be the corresponding extended Dynkin diagram. For singularities of type D_n and E_n in the resolution $X \to Y$ also appear non-reduced curves (see Section 4.1). To study these cases we use results from Section 6.1.

Roughly, the results can be summarised as follows:

Theorem 6.7 *Let $[Q]$ be a general point of Δ on the component Δ_p. Then the reduced fibre $f^{-1}([Q])$ is the union of several $\mathbb{P}^1$-bundles over the Jacobian of a genus 4 curve C_0, one for each vertex of the extended Dynkin diagram $\widetilde{\Gamma}$, which are glued along disjoint sections of the bundles according to the intersection pattern given by $\widetilde{\Gamma}$.*

More detailed information is given in the following sections.

6.4.1 Geometric situation for the components Δ_p

We describe below the geometric configurations for the components Δ_p. So we are in the situation that a smooth, generic quadric $Q \subset \mathbb{P}^2$ passes through a A-D-E singularity $p \in \Sigma$. Then we have the following situation.

Theorem 6.8 *The total transform of Q has the form $C = \pi^*(Q) = C_0 + \sum_i m_i C_i$ and the dual graph of C is the extended Dynkin diagram. Moreover, the strict transform C_0 has genus $g(C_0) = 4$ and corresponds to the new added vertex in the extended Dynkin diagram $\widetilde{\Gamma}$.*

Proof. The preimage $C = \pi^*(Q)$ consists of the components C_i of the exceptional fibre and the strict transform C_0. We assume that the quadric Q is reduced and smooth, so we deduce that the strict transform C_0 is reduced and smooth. Now we want to show that the strict transform C_0 corresponds to the *new added* vertex in the extended Dynkin diagram $\widetilde{\Gamma}$. Taking the pullback we have $\pi^*(Q) = C_0 + \sum_i m_i C_i$. In our set-up $E := \sum_i m_i C_i$ is the fundamental cycle, i.e. the smallest positive cycle such that $E \cdot C_j \leq 0$ for all j (cf. Artin [2, Prop. 2] and Laufer [14, Prop. 4.1.]). The quadric Q pass generically through the one A-D-E singularity $p \in \Sigma$ and C_0 is the strict transform of Q. Then, as Q moves in the linear system $|2H|$, for every j we must have $(C_0 + E) \cdot C_j = 0$. Using above facts we deduce that the dual graphs of C are the extended Dynkin diagrams as on the pictures below. Moreover, the strict transform C_0 corresponds to the *new added* vertex (depicted with the circle) and the weights on the vertices denote the multiplicities of the corresponding curves. Now we want to show that C_0 has genus $g(C_0) = 4$. As above, we have $\pi^*(Q) = C_0 + E$. Then we have

$$8 = [\pi^*(Q)]^2 = (C_0 + E)^2 = C_0^2 + 2C_0 \cdot E + E^2. \tag{21}$$

First we study the case for singularities of type A_n. We observe that in this situation C_0 intersect (transversely in one point) two reduced (-2)-curves: C_1 and C_n. So we have

$$8 = C_0^2 + 2C_0 \cdot (C_1 + C_n) + E^2$$

and the middle term on the right side of the above equation is equal 4. For the fundamental cycle E of an exceptional A-D-E curve we have $E^2 = -2$ (cf. [3, Prop. III.3.9]), which implies $C_0^2 = 6$. For the singularities of type D_n and E_n let C_1 be the (-2)-curve, which is not reduced (has multiplicity 2 for all singularities of type D_n and E_n) and intersect C_0 in exactly one point. So we have the following

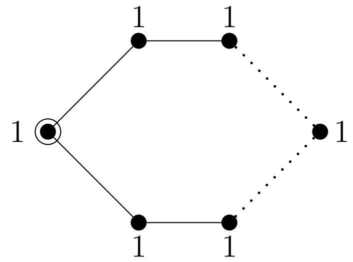
$$8 = C_0^2 + 2C_0 \cdot C_1 - 2,$$

which means that also for singularities of type D_n and E_n we have $C_0^2 = 6$. So for all A-D-E singularities we can deduce from above discussion that $C_0^2 = 6$.

Now using adjunction formula on a $K3$ surface X we obtain $2g(C_0) - 2 = C_0^2$, which implies $g(C_0) = 4$.

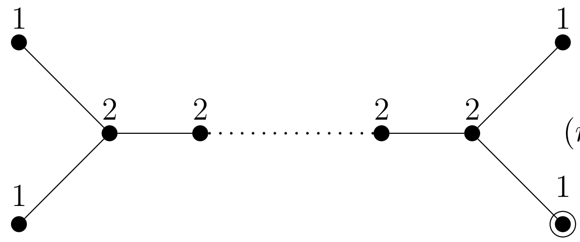
From above discussions we obtain following diagrams.

$\widetilde{A}_n$



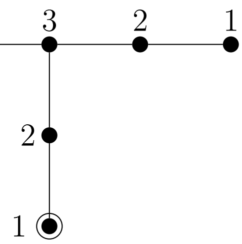
$(n+1$ vertices, $n \geq 2)$

$\widetilde{D}_n$

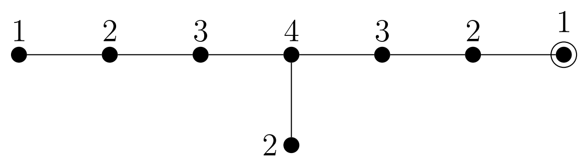


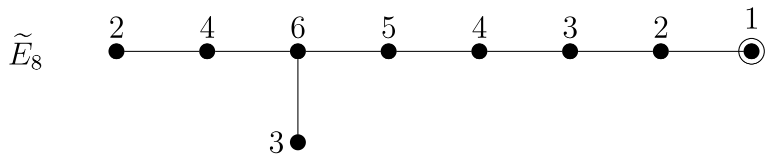
$(n+1$ vertices, $n \geq 4)$

$\widetilde{E}_6$



$\widetilde{E}_7$





Above pictures are the dual graphs of the exceptional fibres for the singularity $p \in \Sigma$ of type A-D-E. The strict transform C_0 corresponds to the vertex depicted with the circle. The weights of the vertices are the multiplicities m_i of the corresponding curves C_i.

□

Let $\widetilde{\Gamma}$, as above, denote the extended Dynkin diagram corresponding to the graph $\Gamma^{(p)}$, and let $S = (S_{ij})_{0 \le i,j \le n}$ denote the corresponding intersection matrix. Then

$$C_i \cdot C_j = S_{ij} + 8\delta_{i0}\delta_{j0}. \tag{22}$$

Let $\langle v, w \rangle = \sum_{i,j} S_{ij} v_i w_j$ denotes the negative semi-definite integral form on $\mathbb{Z}^{1+n}$. Then the radical of the corresponding quadratic form is one-dimensional and is generated by a positive minimal integral null vector of S. We call this vector m and its components m_i are equal the multiplicities m_i from the above Theorem 6.8 (cf. Artin [2, Prop. 2] and Laufer [14, Prop. 4.1.]). So we have

$$C = \sum_{i=0}^{n} m_i C_i.$$

One always has $m_0 = 1$. Note that we have in this situation

$$H'C_j = -\varepsilon E^{(p)} C_j = 2\varepsilon \text{ for } j \neq 0$$

and that

$$H'C = 2H^2 = 4,$$

so that

$$H'C_0 = 4 - 2\varepsilon \sum_{j=1}^{n} m_j.$$

6.4.2 Semistability condition for the components Δ_p

Assume now that F is a purely 1-dimensional sheaf supported on C with multiplicities $\mu_j(F)$ along C_j. Then the first Chern class of F is given by

$$c_1(F) = \sum_{j=0}^{n} \mu_j(F) C_j.$$

The notion of the stability depends on the choice of the polarization. The Hilbert polynomial for the polarization H' is given by

$$P(F,n) = \chi(F \otimes \mathcal{O}_X(nH')) = \int_X \mathrm{ch}(F \otimes \mathcal{O}_X(nH')) \cdot \mathrm{td}X,$$

which gives

$$P(F,n) = \chi(F) + n \cdot c_1(F) \cdot H'. \tag{23}$$

So the reduced Hilbert polynomial is given by

$$p(F,n) = n + \frac{\chi(F)}{c_1(F) \cdot H'}. \tag{24}$$

Using (23) we obtain

$$\begin{aligned} P(F,n) &= \chi(F) + n \cdot c_1(F) \cdot H' \\ &= \chi(F) + n[2\varepsilon \textstyle\sum_{j=1}^{n} \mu_j(F) + (4 - 2\epsilon \sum_{j=1}^{n} m_j)\mu_o(F)] \\ &= \chi(F) + n[4\mu_0(F) - 2\varepsilon \textstyle\sum_{j=1}^{n} (m_j \mu_0(F) - \mu_j(F))]. \end{aligned}$$

So the reduced Hilbert polynomial is

$$p(F,n) = n + \frac{\chi(F)}{4\mu_0(F) - 2\varepsilon \sum_{j=1}^{n} (m_j \mu_0(F) - \mu_j(F))}. \tag{25}$$

As F is a purely 1-dimensional sheaf we have

$$c_1(F) \cdot H' = 4\mu_0(F) - 2\varepsilon \sum_{j=1}^{n} (m_j \mu_0(F) - \mu_j(F)) \neq 0.$$

If $[F] \in f^{-1}([Q])$, one has

$$\mu_j(F) = m_j \text{ for all } j,$$

and the reduced Hilbert polynomial (25) takes the form

$$p(F,n) = n + \frac{\chi(F)}{4}. \tag{26}$$

Lemma 6.9 *The stability condition reads as follows: for a subsheaf F', $0 \subsetneq F' \subsetneq F$ with multiplicities m'_j, $0 \leq m'_j \leq m_j$, one must have*

$$\chi(F') < (m'_0 - \frac{1}{2}\varepsilon \sum_{j=1}^{n}(m_j m'_0 - m'_j))\chi(F).$$

Since $\chi(F')$ and $\chi(F) > 0$ are integers and ε is a very small positive rational number, we can rephrase this condition as follows.

Lemma 6.10 *Let F be a pure 1-dimensional sheaf with Euler characteristic $\chi(F)$ and with multiplicities m_j along the components C_j of C, where $j = 0, \ldots, n$. Then F is stable with respect to the polarisation H' if and only if every non-trivial proper subsheaf F' satisfies*

$$\chi(F') \leq \begin{cases} \chi(F) - 1, & \text{if } m'_0 = 1, \\ 0, & \text{if } m'_0 = 0, \end{cases}$$

where m'_0 is the multiplicity of F' along the component C_0.

6.4.3 Local description of the fiber over Δ_p

The preimage $C = \pi^*(Q) = C_0 + \sum_i m_i C_i$ consists of the components C_i and the strict transform C_0, which is a smooth curve of genus 4. As in the Section 6.1, let F_i denote the quotient of $F|_{m_i C_i}$ by its zero-dimensional torsion submodule, and consider the exact sequence

$$0 \longrightarrow F \longrightarrow \bigoplus_{j=0}^{n} F_j \longrightarrow \bigoplus_{x} T_x \longrightarrow 0, \tag{27}$$

with zero-dimensional sheaves T_x supported on the singular points x.

We want to use the following notational trick to simplify the calculations. Let $\psi(F_i) = \chi(F_i)$ if $i \neq 0$, and $\psi(F_0) = \chi(F_0) - \chi(F) + 1$. The additivity of the Euler characteristic for the sequence (27) gives

$$\sum_j \chi(F_j) = \chi(F) + \sum_x \chi(T_x).$$

Now using above notational trick we obtain

$$1 + \sum_x \chi(T_x) = \sum_j \psi(F_j). \tag{28}$$

We want to translate the stability condition from Lemma 6.10 by applying to the kernel of the surjection

$$\phi\colon F \longrightarrow F_j.$$

Let m_0' be the multiplicity of the *zero*-component of $\ker(\phi)$. By combining $\chi(\ker(\phi)) = \chi(F) - \chi(F_j)$ with Lemma 6.10:

1. If $m_0' = 1$: which is equivalent to $j \neq 0$ the following happens

$$\chi(F) - \chi(F_j) \leq \chi(F) - 1.$$

   In this case $\psi(F_j) = \chi(F_j)$, so for $j \neq 0$ we have $\psi(F_j) \geq 1$.

2. if $m_0' = 0$: which is equivalent to $j = 0$ the following happens

$$\chi(F) - \chi(F_j) \leq 0.$$

   In this case $\psi(F_0) = \chi(F_0) - \chi(F) + 1$: which is equivalent to

$$\chi(F_0) = \psi(F_0) + \chi(F) - 1.$$

   So the above inequality takes the form

$$\chi(F) - \chi(F_0) = \chi(F) - \psi(F_0) - \chi(F) + 1 \leq 0,$$

   or equivalently $\psi(F_0) \geq 1$.

So we obtain the following inequality

$$\psi(F_j) \geq 1 \ \text{ for all } j = 0, \dots, n. \tag{29}$$

This can be strengthened in the following way.

Let V be any non-empty proper subset of the set of vertices of the extended Dynkin diagram, and let E denote the set of all edges that connect vertices in V. Then the sheaf $F_V := (F|_{\sum_{i \in V} m_i C_i})/\text{torsion}$ fits into an exact sequence

$$0 \longrightarrow F_V \longrightarrow \bigoplus_{i \in V} F_i \longrightarrow \bigoplus_{x \in E} T_x \longrightarrow 0, \tag{30}$$

and the stability condition on F applied to $\ker(F \to F_V)$ yields the inequality

$$1 + \sum_{x \in E} \chi(T_x) \leq \sum_{i \in V} \psi(F_i). \tag{31}$$

As the first step towards a classification of stable sheaves F we can exclude the possibility that the scheme-theoretic support of F is a smaller scheme than the curve C itself. Of course, the lemma is clear if all multiplicities $m_j = 1$, i.e. if $\widetilde{\Gamma}$ is of type $\widetilde{A_n}$.

Lemma 6.11 *If F is a stable sheaf in our moduli space then F is locally free $\mathcal{O}_C$-module of rank 1 on an open dense subset of C.*

Proof. Let F_j, F_j^s and r_{js} have the same meaning as in the Section 6.1. Let $s_{jt} := \max\{s \,|\, r_{js} \geq t\}$. By definition

$$s_{j1} \geq s_{j2} \geq \ldots \geq 0.$$

For each t, we may form the vector

$$s_t = (s_{jt})_j \in \mathbb{Z}^{1+n}, \text{ so that } m = s_1 + s_2 + \ldots.$$

In order to prove the lemma we need to show that $m = s_1$. It suffices to show that the assumption that $s_2 \neq 0$ leads to a contradiction.

According to Lemma 6.2, one has for each point $x \in C_i \cap C_j$ the estimate

$$\chi(T_x) \leq \sum_{s,t} \min\{r_{is}, r_{jt}\} = \sum_r |\{s \,|r_{is} \geq r\}| \cdot |\{t \,|r_{jt} \geq r\}| = \sum_r s_{ir}s_{jr}. \quad (32)$$

Furthermore, if $j = 0$, one has $\psi(F_0) \geq 1$. If $j \neq 0$, the component C_j is a (-2)-curve. As C_j is smooth rational curve, all subquotients $\mathrm{gr}^s(F_j)$ decompose as sums of line bundles. By the stability property of F, each direct summand L of $\mathrm{gr}^1(F_j)$ satisfies $\chi(L) \geq 1$, and since $\mathcal{O}_{C_j}(-C_j) \cong \mathcal{O}_{C_j}(2)$, the generic surjectivity of the homomorphism

$$\mathrm{gr}^s(F_j) \otimes \mathcal{O}_{C_j}(-C_j) \to \mathrm{gr}^{s+1}(F_j)$$

shows that every direct summand L of $\mathrm{gr}^s(F_j)$ satisfies $\chi(L) \geq 2s - 1$. Adding up, one obtains

$$\psi(F_j) \geq \sum_s r_{js}(2s-1) = \sum_s \sum_{t \leq r_{js}} (2s-1) = \sum_t \sum_{s \leq s_{jt}} (2s-1) = \sum_t s_{jt}^2, \quad (33)$$

for all $j \neq 0$. As already stated, the same inequality holds for $j = 0$ for trivial reasons. Using (32) and (33)

$$2(\sum_x \chi(T_x) - \sum_j \psi(F_j)) \leq \sum_t (\sum_{i \neq j} S_{ij}s_{ti}s_{tj} + \sum_t S_{ii}s_{ti}^2) = \sum_t \langle s_t, s_t \rangle,$$

where $\langle v, w \rangle = \sum_{i,j} S_{ij}v_iw_j$ denotes the negative semi-definite integral form on $\mathbb{Z}^{1+n}$ defined by the matrix S associated to the extended Dynkin diagram $\widetilde{\Gamma}$. But the quadratic form $v \mapsto \langle v, v \rangle$ takes value 0 only on multiples of m and is

≤ -2 on any other integral vector. Thus if $m \neq s_1$ so that at least $s_1 \neq 0$ and $s_2 \neq 0$, we obtain

$$\sum_x \chi(T_x) - \sum_j \psi(F_j) \leq \frac{1}{2} \sum_t \langle s_t, s_t \rangle \leq -2. \tag{34}$$

This contradicts the relation

$$1 + \sum_x \chi(T_x) = \sum_j \psi(F_j).$$

Hence indeed $s_1 = m$. □

Assume that F is stable. Then the Lemma 6.11 has the following consequences:

1. Since $r_{js} = 1$ for all $s \leq m_j, j = 0, \ldots, n$, the inequality of the Lemma 6.2 specializes to $\chi(T_x) \leq m_i m_j$ for each $x \in C_i \cap C_j$. We will say that x is *ordinary* if $\chi(T_x) = m_i m_j$, and is *special* if $\chi(T_x) < m_i m_j$.

2. There are monomorphisms $\mathrm{gr}^s(F_j) \otimes \mathcal{O}_{C_j}(2) \to \mathrm{gr}^{s+1}(F_j)$ for all $1 \leq s < m_j$. Using the facts that: $\chi(\mathrm{gr}^1(F_j)) \geq 1$ for $j \neq 0$, $m = s_1$ and the relation (33) we deduce $\psi(F_j) \geq m_j^2$. We will say that F_j is *ordinary* if $\psi(F_j) = m_j^2$, and is *special* if $\psi(F_j) > m_j^2$.

Now using relations (28) and (34) yield the following piece of information

$$-1 = \sum_x \chi(T_x) - \sum_j \psi(F_j) \leq \frac{1}{2} \langle m, m \rangle. \tag{35}$$

So we have

$$1 = \sum_j \psi(F_j) - \sum_x \chi(T_x) \geq -\frac{1}{2} \langle m, m \rangle.$$

The term $-\frac{1}{2}\langle m, m \rangle$ on the right hand side is zero since m is the null vector of S. We deduce

$$\begin{aligned} 1 &= \textstyle\sum_j \psi(F_j) - \sum_x \chi(T_x) \\ &\geq |\{\text{special points } x\}| + |\{\text{special components } j\}|. \end{aligned}$$

Hence we have the following situation.

Lemma 6.12 *For each F stable exactly one of the following cases applies:*

(I) All F_j are ordinary, and all but one vertex x are ordinary. At the unique special vertex x_0 one has $\chi(T_{x_0}) = m_i m_j - 1$, if $x_0 \in C_i \cap C_j$.

(II) All x are ordinary, and all but one sheaf F_j are ordinary. For the unique special sheaf F_j one has $\psi(F_j) = m_j^2 + 1$.

We will see that sheaves of type (I) correspond to singular points in the fibre $f^{-1}([Q])$, whereas sheaves of type (II) correspond to regular points of the fibre. Moreover, for type (I) there will be no choice for the gluing of the components F_j up to isomorphism, and the isomorphism type of F will be completely determined by the discrete choice of the special point x_0 and a continuous choice of the line bundle F_0 in the Jacobian variety of C_0. In contrast to this, the construction of sheaves of type (II) also requires the choice of a gluing datum.

Below we discuss the *ordinary* and *special* sheaves F_j. We have the similar situation as described by Drézet (cf. [4, Section 5.2.]).

Lemma 6.13 *Let F_j be an ordinary sheaf. Then $F_j \cong \mathcal{O}_{m_j C_j}$ if $j \neq 0$, and F_j is a line bundle on C_0 of degree $\chi(F) + 3$ if $j = 0$.*

Proof. If $j = 0$, the sheaf F_0 is a line bundle, since the component C_0 has multiplicity 1, and the relation $\psi(F_0) = m_0^2 = 1$ is equivalent to

$$\deg(F_0) = \chi(F_0) + 3 = \psi(F_0) + \chi(F) - 1 + 3 = \chi(F) + 3.$$

If $j \neq 0$, F_j is ordinary if and only if $\chi(\mathrm{gr}^1(F_j)) = 1$ and all epimorphisms

$$\mathrm{gr}^s(F_j) \otimes \mathcal{O}_{C_j}(2) \to \mathrm{gr}^{s+1}(F_j) \tag{36}$$

are isomorphisms. Now we want to show that $F_j \cong \mathcal{O}_{m_j C_j}$. We tensor the standard sequence

$$0 \to \mathcal{O}(-C_j) \to \mathcal{O} \to \mathcal{O}_{C_j} \to 0$$

with $-(m_j - 1)C_j$ and obtain

$$0 \to \mathcal{O}(-m_j C_j) \to \mathcal{O}(-(m_j - 1)C_j) \to \mathcal{O}_{C_j}(-(m_j - 1)C_j) \to 0.$$

Then applying the snake lemma to

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{O}(-m_jC_j) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_{m_jC_j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-(m_j-1)C_j) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_{(m_j-1)C_j} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{O}_{C_j}(-(m_j-1)C_j) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

we get the following exact sequence

$$0 \longrightarrow \mathcal{O}_{C_j}(-(m_j-1)C_j) \longrightarrow \mathcal{O}_{m_jC_j} \longrightarrow \mathcal{O}_{(m_j-1)C_j} \longrightarrow 0\,. \qquad (37)$$

Using (36) we deduce that the graded pieces of F_j are $\mathcal{O}_{C_j}$, $\mathcal{O}_{C_j}(2)$, $\ldots$, $\mathcal{O}_{C_j}(2m_j)$. It suffices to show that $\mathrm{Ext}^1_X(\mathcal{O}_{(m_j-1)C_j}, \mathcal{O}_{C_j}(-(m_j-1)C_j) = 1$. So we want to compute $\mathrm{Ext}^1_X(\mathcal{O}_{(m_j-1)C_j}, \mathcal{O}_{C_j}(2(m_j-1))$. We use the sequence

$$0 \longrightarrow \mathcal{O}_X(-(m_j-1)C_j) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{(m_j-1)C_j} \longrightarrow 0.$$

Let now $L := \mathcal{O}_{C_j}(2(m_j-1))$, $l := m_j - 1$ and A be a (-2)-curve C_j. So we want to show that $\mathrm{Ext}^1(\mathcal{O}_{lA}, L) = 1$. We consider the following sequence

$$\begin{array}{ccccccc}
 & \mathrm{Hom}\,(\mathcal{O}_{lA}, L) & \to & \mathrm{Hom}\,(\mathcal{O}_X, L) & \to & \mathrm{Hom}\,(\mathcal{O}_X(-lA), L) & \to \\
\to & \mathrm{Ext}^1(\mathcal{O}_{lA}, L) & \to & \mathrm{Ext}^1(\mathcal{O}_X, L) & \to & \mathrm{Ext}^1(\mathcal{O}_X(-lA), L) & \to \\
\to & \mathrm{Ext}^2(\mathcal{O}_{lA}, L) & \to & \mathrm{Ext}^2(\mathcal{O}_X, L) & \to & \ldots\,. &
\end{array}$$

We have $\deg L \geq 0$, which implies $\mathrm{Ext}^i(\mathcal{O}_X, L) = \mathrm{H}^i(L) = 0$ for every $i \geq 1$. On the other hand

$$\mathrm{Hom}(\mathcal{O}_X(-lA), L) = \mathrm{H}^0(L \otimes \mathcal{O}(lA)) = \mathrm{H}^0(\mathcal{O}_A(2(m_j-1) - 2l)) = \mathrm{H}^0(\mathcal{O}_A) = \mathbb{C}.$$

We have $\mathrm{Hom}\,(\mathcal{O}_{lA}, L) \cong \mathrm{Hom}\,(\mathcal{O}_X, L)$. So we deduce that $\mathrm{Ext}^1(\mathcal{O}_{lA}, L) = 1$. □

Lemma 6.14 *Let F_j be a special sheaf. If $j = 0$ then F_j is a line bundle on C_0 of degree $\chi(F) + 4$. If $j \neq 0$ then there is a unique point $y \in C_j \setminus \bigcup_{i\neq j} C_i$ such that F_j is obtained from $\mathcal{O}_{m_jC_j}$ via a push-out diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{C_j}(-(m_j-1)C_j) & \longrightarrow & \mathcal{O}_{m_jC_j} & \longrightarrow & \mathcal{O}_{(m_j-1)C_j} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{C_j}(-(m_j-1)C_j)\otimes\mathcal{O}_{C_j}(y) & \longrightarrow & F_j & \longrightarrow & \mathcal{O}_{(m_j-1)C_j} & \longrightarrow & 0 \end{array}$$

If $m_j = 1$ this simply means that $F_j \cong \mathcal{O}_{C_j}(1)$.

Proof. If $j = 0$, the sheaf F_0 is a line bundle, since the component C_0 has multiplicity 1, and the relation $\psi(F_0) = m_0^2 + 1 = 2$ is equivalent to

$$\deg(F_0) = \chi(F_0) + 3 = \psi(F_0) + \chi(F) - 1 + 3 = \chi(F) + 4.$$

Now we want to study the case $j \neq 0$. For the unique special sheaf F_j we have $\psi(F_j) = m_j^2 + 1$. From this follows that exactly the last map

$$\mathrm{gr}^s(F_j) \otimes \mathcal{O}_{C_j}(2) \to \mathrm{gr}^{s+1}(F_j)$$

in the filtration of F_j is not an isomorphism – otherwise we would have $\psi(F_j) > m_j^2 + 1$, which contradict the relation (35). Moreover, for such special sheaf the degree of the last summand in its filtration is one bigger than in the ordinary case – otherwise we would have, as above, $\psi(F_j) > m_j^2 + 1$. We have the same extensions as in the *ordinary* case. So the first row of the above diagram is obtained as in the previous Lemma 6.13. In the last step we choose an homomorphism

$$\mathcal{O}_{C_j}(-(m_j-1)C_j) \to \mathcal{O}_{C_j}(-(m_j-1)C_j) \otimes \mathcal{O}_{C_j}(y),$$

where $y \in C_j \setminus \bigcup_{i\neq j} C_i$ is a unique point. Now the above diagram follows from easy diagram chasing.

□

From now on we assume that F is a sheaf which fits into the exact sequence

$$0 \longrightarrow F \longrightarrow \bigoplus_{j=0}^{n} F_j \longrightarrow \bigoplus_x T_x \longrightarrow 0.$$

For this sheaf we have the following.

Theorem 6.15 *Let F be the sheaf for which exactly one of the following cases applies:*

(I) All F_j are ordinary, which means that for all j: $F_j \cong \mathcal{O}_{m_jC_j}$. For all but one vertex x we have $T_x \cong \mathcal{O}_\xi$. At the unique special vertex x one has $T_x \cong \mathcal{O}_\xi/\kappa$ (cf. Section 6.1).

(II) All vertices x are ordinary, which means that for all x we have $T_x \cong \mathcal{O}_\xi$. Moreover, all but one sheaf F_j is ordinary, i.e isomorphic to $\mathcal{O}_{m_jC_j}$. For the unique special sheaf F_j:
If $j = 0$ then F_j is a line bundle on C_0 of degree $\chi(F) + 4$.
If $j \neq 0$ then there is a unique point $y \in C_j \setminus \bigcup_{i\neq j} C_i$ such that F_j is obtained from $\mathcal{O}_{m_jC_j}$ via a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{C_j}(-(m_j-1)C_j) & \longrightarrow & \mathcal{O}_{m_jC_j} & \longrightarrow & \mathcal{O}_{(m_j-1)C_j} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{C_j}(-(m_j-1)C_j)\otimes\mathcal{O}_{C_j}(y) & \longrightarrow & F_j & \longrightarrow & \mathcal{O}_{(m_j-1)C_j} & \longrightarrow & 0 \end{array}$$

If $m_j = 1$ this simply means that $F_j \cong \mathcal{O}_{C_j}(1)$.

Then F is stable.

Proof. We want to show that the sheaf F is stable, so that for every proper subsheaf $F' \subset F$ we have $p(F') < p(F)$. The sheaf F is locally free $\mathcal{O}_C$-module of rank 1 on an open dense subset of C. The case that F' differs from F only by finite number of torsion points is trivial. In this situation we have $p(F') < p(F)$, which implies that the sheaf F is stable. To study other cases we consider the quotient sheaf $\overline{F}$. We have the following exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow \overline{F} \longrightarrow 0. \quad (38)$$

We want to compute the reduced Hilbert polynomial of the quotient sheaf $\overline{F}$ and then compare this with the reduced Hilbert polynomial of the sheaf F. Moreover, using above sequence (38) we deduce

$$P(F', n) = P(F, n) - P(\overline{F}, n). \quad (39)$$

We want to show that in every case we have $p(F, n) < p(\overline{F}, n)$, which implies that $p(F', n) < p(F, n)$, i.e. F is stable. Without loss of generality we assume that $\overline{F}$ is torsion free. If F_j is locally free, then F_j is isomorphic to $\mathcal{O}_{m_jC_j}$. Every quotient sheaf of such F_j is isomorphic to the structure sheaf of a subscheme of

$\mathcal{O}_{m_jC_j}$. Such a saturated subscheme of a divisor is a subdivisor. We deduce that for $F_j \cong \mathcal{O}_{m_jC_j}$ the quotient sheaf $\overline{F_j}$ is of the form $\overline{F_j} \cong \mathcal{O}_{\overline{m_j}C_j}$, with $\overline{m_j} \leq m_j$. We have the following possibilities:

a) The sheaf F is of type (II). All vertices x are ordinary, and all but one sheaf F_j is ordinary. Then we need to study the following possibilities:

i) Let F_0 be special, i.e. F_0 is a line bundle on C_0 of degree $\chi + 4$. We have

$$\begin{aligned} P(F,n) &= H' \cdot c_1(F) \cdot n + \chi(F) \\ &= H'(\textstyle\sum_{j=0}^n m_jC_j)n + \sum_j S_{jj}m_j^2 - \sum_{i\neq j} S_{ij}m_im_j \\ &= H'(\textstyle\sum_{j=0}^n m_jC_j)n + (-\frac{1}{2}m^TSm) + \chi(F_0) - 1. \end{aligned}$$

As m is the null vector of S we have $\frac{1}{2}m^TSm = 0$. The Hilbert polynomial of the quotient sheaf $\overline{F}$ is equal

$$P(\overline{F},n) = H'(\sum_{j=0}^n \overline{m_j}C_j)n + (-\frac{1}{2}\overline{m}^TS\overline{m}) + \overline{m_0}(\chi(F_0) - 1).$$

Moreover, as S is negative definite, we have $(-\frac{1}{2}\overline{m}^TS\overline{m}) > 0$. We have two possibilities: $\overline{m_0} = 0$ or $\overline{m_0} = 1$. For $\overline{m_0} = 0$ we have

$$P(\overline{F},n) = H'(\sum_{j=0}^n \overline{m_j}C_j)n + (-\frac{1}{2}\overline{m}^TS\overline{m}),$$

which yields the following inequality for the reduced Hilbert polynomials of F and $\overline{F}$

$$p(F,n) = \frac{\chi(F_0) - 1}{2\varepsilon\sum_{j=1}^n m_j + 4 - 2\varepsilon} < \frac{-\frac{1}{2}\overline{m}^TS\overline{m}}{2\varepsilon\sum_{j=1}^n \overline{m_j}} = p(\overline{F},n),$$

since ε is a very small positive rational number, $-\frac{1}{2}\overline{m}^TS\overline{m}$ is a positive integer and $2\varepsilon\sum_{j=1}^n \overline{m_j}$ is a small positive number. This implies that $p(F',n) < p(F,n)$ and F is stable. For $\overline{m_0} = 1$ the sheaf $\overline{F_0}$ is also special and we have

$$P(\overline{F},n) = H'(\sum_{j=0}^n \overline{m_j}C_j)n + (-\frac{1}{2}\overline{m}^TS\overline{m}) + \chi(F_0) - 1$$

and

$$p(F,n) = \frac{\chi(F_0) - 1}{2\varepsilon\sum_{j=1}^n m_j + 4 - 2\varepsilon} < \frac{-\frac{1}{2}\overline{m}^TS\overline{m} + \chi(F_0) - 1}{2\varepsilon\sum_{j=1}^n \overline{m_j} + 4 - 2\varepsilon} = p(\overline{F},n),$$

since $-\frac{1}{2}\overline{m}^TS\overline{m}$ is a positive integer and for all j we have $\overline{m_j} < m_j$. This implies, as above, that $p(F',n) < p(F,n)$ and F is stable.

ii) Let F_j be special and $m_j \geq 2$. If $\overline{m_j} = m_j$ then the sheaf $\overline{F_j} \cong F_j$ and $\overline{F_j}$ is constructed via a push-out, in the same way as F_j. Similar calculations as above translates into the following inequalities: for $\overline{m_0} = 0$

$$p(F,n) = \frac{\chi(F_0)}{2\varepsilon \sum_{j=1}^n m_j + 4 - 2\varepsilon} < \frac{-\frac{1}{2}\overline{m}^T S \overline{m}}{2\varepsilon \sum_{j=1}^n \overline{m_j}} = p(\overline{F}, n),$$

since ε is a very small positive rational number, $-\frac{1}{2}\overline{m}^T S\overline{m}$ is a positive integer and $2\varepsilon \sum_{j=1}^n \overline{m_j}$ is a small positive number. This implies, as above, that $p(F', n) < p(F, n)$ and F is stable. We have for $\overline{m_0} = 1$

$$p(F,n) = \frac{\chi(F_0)}{2\varepsilon \sum_{j=1}^n m_j + 4 - 2\varepsilon} < \frac{-\frac{1}{2}\overline{m}^T S \overline{m} + \chi(F_0)}{2\varepsilon \sum_{j=1}^n \overline{m_j} + 4 - 2\varepsilon} = p(\overline{F}, n),$$

since $-\frac{1}{2}\overline{m}^T S\overline{m}$ is a positive integer and for all j we have $\overline{m_j} < m_j$. This implies, as above, that $p(F', n) < p(F, n)$ and F is stable.

iii) Let F_j be special with $m_j \geq 2$ and let $\overline{m_j} < m_j$. Then the scheme-theoretic support $\overline{m_j}C_j$ of $\overline{F_j}$ is smaller than m_jC_j. Moreover, $\overline{F_j}$ as a quotient sheaf is locally free and isomorph to $\mathcal{O}_{\overline{m_j}C_j}$. Analogous calculations as above gives $p(F', n) < p(F, n)$ and F is stable.

iv) Let F_j be special and $m_j = 1$. Similarly as above we have $p(F', n) < p(F, n)$ and F is stable.

b) The sheaf F is of type (I). All F_j are ordinary, and all but one vertex x is ordinary. Let $x_{ij} \in m_iC_i \cap m_jC_j$ be a special vertex. We need to discuss two possibilities for the corresponding vertex $\overline{x_{ij}} \in \overline{m_i}C_i \cap \overline{m_j}C_j$ of the quotient sheaf $\overline{F_j}$:

i) the multiplicities $\overline{m_i} = m_i$ and $\overline{m_j} = m_j$. In this case $T_{\overline{x_{ij}}}$ is isomorphic to $T_{x_{ij}}$ and $T_{\overline{x_{ij}}} \cong T_{x_{ij}} \cong \mathcal{O}_\xi/\kappa$, where $\kappa = (0 : m_{x_{ij}}) \subset \mathcal{O}_\xi$ is the socle. Therefore in this situation the vertex $\overline{x_{ij}}$ is also special.

ii) the multiplicities $\overline{m_i} < m_i$ or $\overline{m_j} < m_j$. In this case $T_{\overline{x_{ij}}} \subset \mathcal{O}_\xi/\kappa$ and the vertex $\overline{x_{ij}}$ is ordinary.

In both cases: i) and ii) after similar calculations as in the case a) we obtain $p(F, n) < p(\overline{F}, n)$, which implies that $p(F', n) < p(F, n)$ and F is stable. $\square$

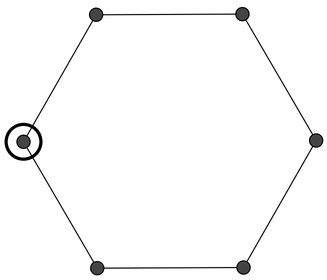
We are prepared to give a detailed description of the fiber $f^{-1}([Q])$ for a general point $[Q] \in \Delta_p$, corresponding to a general quadric Q passing through a singular point p of Σ.

6.4.4 Description of the fiber over Δ_p

We consider first the case that p is a singular point of the sextic Σ that produces a singularity of type A_n on the double cover $Y \to \mathbb{P}^2$.

Extended Dynkin diagrams of type $\widetilde{A}_n$

The extended Dynkin diagram $\widetilde{\Gamma}$ has the following form (cf. Section 6.4.1):



drawn here for $n = 5$. The marked vertex corresponds to the curve C_0, and $g(C_0) = 4$. We choose a cyclic enumeration $C_0, C_1, \ldots, C_n, C_{n+1} = C_0$, so that C_j intersects C_{j+1} in the point x_{j+1}. If $n = 1$, one has $C_0 \cap C_1 = \{x_0, x_1\}$. Moreover, all multiplicities m_j are equal one. Using Lemma 6.12 we have to discuss two possibilities:

(I) All sheaves F_j are line bundles. There exists exactly one point $x' \in C_i \cap C_j$, in which we do not glue the line bundles: F_i and F_j. Moreover, the sheaf F is not locally free in the point x'.

(II) We have $T_{x_j} = \mathcal{O}_{x_j}$ for every intersection point x_j. All components are reduced, so all sheaves F_j are line bundles, regardless whether they are ordinary or special in the sense of the Lemma 6.12.

Using above discussion we can reconstruct our fibre. We start with the sheaves of type (I). The gluing of two sheaves F_j and F_{j+1} in the intersection point x_{j+1} means that we choose an isomorphism

$$\alpha_j \colon\ F_j \otimes \mathcal{O}_{x_{j+1}} \longrightarrow F_{j+1} \otimes \mathcal{O}_{x_{j+1}}.$$

The gluing cycle is not closed. There is up to isomorphism for every choice of sheaves $F_0, F_1, \ldots, F_n$ exactly one choice of such isomorphisms. The sheaves are uniquely determined by their Euler characteristic. So the isomorphism type of F depends only on the discrete choice of the special point x_0 and a continuous choice the line bundle F_0 in the Jacobian variety of C_0. On this way we obtain $n+1$ families, the base of which are isomorph to the Jacobian variety of C_0.

For the sheaves of type (II) we choose at each point x_{j+1} an isomorphisms

$$\alpha_j \colon\ F_j \otimes \mathcal{O}_{x_{j+1}} \longrightarrow F_{j+1} \otimes \mathcal{O}_{x_{j+1}}$$

and define $F(\alpha)$ as the line bundle on C obtained by gluing the sheaves F_j via the isomorphisms $\alpha = (\alpha_j)$. So the gluing data is a tuple of isomorphisms $\alpha_j \in (\mathbb{C}^*)^{n+1}$. On this gluing data acts the automorphism group G which consist of a tuple of automorphisms $t_j \in \mathrm{Aut}(F_j) = \mathbb{C}^*$. The isomorphism type of $F(\alpha)$ will not change if α is replaced by α' with $\alpha'_j = t_{j+1}\alpha_j t_j^{-1}$. Due to this freeness of choosing gluing isomorphisms we obtain a Torsor over $\mathbb{C}^*$. The base is $\mathrm{Jac}_{\chi+4}(C_0)$ or $\mathrm{Jac}_{\chi+3}(C_0)$, depending on whether the line bundle F_0 is special or ordinary.

Now we want to formalize the above discussion. Let $D_j := \bigcup_{k \neq j} C_k$. We begin with the projective bundle $\mathbb{P}_0$. The sheaf F_0 on the component C_0 is special. Let $J' := \mathrm{Jac}_{\chi+4}(C_0)$ and let $\mathcal{L}' \in \mathrm{Pic}(J' \times C_0)$ be the Poincaré bundle. We want to glue the bundle $\mathcal{L}'|_{J' \times \{x_i\}}$ with the trivial sheaf $\mathcal{O}_{J' \times \{x_i\}}$ for $i = 0, 1$. Let

$$l_i := \mathcal{H}om(\mathcal{L}'|_{J' \times \{x_i\}}, \mathcal{O}_{J' \times \{x_i\}}) = \mathcal{L}'^*|_{J' \times \{x_i\}} \text{ for } i = 0, 1.$$

and we define $U_0 := l_1^* \oplus l_0^* = \mathcal{L}'|_{J' \times \{x_1\}} \oplus \mathcal{L}'|_{J' \times \{x_0\}}$. Now we consider the projectivization

$$\mathbb{P}_0 := \mathbb{P}(U_0) \xrightarrow{\ p\ } J'.$$

We use the following notation

$$\begin{array}{ccccc} \mathbb{P}_0 \times C & \xrightarrow{p \times id_C} & J' \times C & \xrightarrow{pr_C} & C \\ \downarrow{\scriptstyle pr_{\mathbb{P}_0}} & & \downarrow{\scriptstyle pr_{J'}} & & \\ \mathbb{P}_0 & \xrightarrow{\quad p \quad} & J' & & \end{array}$$

Every such projectivization comes in the natural way with a surjective morphism

$$p^* U_0 \longrightarrow \mathcal{O}_p(1)\ .$$

Dualizing we get the map $\mathcal{O}_p(-1) \hookrightarrow p^*U_0^* = \mathcal{L}'^*|_{J'\times\{x_1\}} \oplus \mathcal{L}'^*|_{J'\times\{x_0\}}$. We want to describe the family of sheaves on the curve C, which is parameterized by $\mathbb{P}_0$. We have two maps: the gluing $\varphi := (\varphi', \varphi'')$

$$\mathrm{pr}_{\mathbb{P}_0}^*\mathcal{O}_p(-1) \otimes (\mathrm{p} \times \mathrm{id}_C)^*\mathcal{L}' \xrightarrow{\varphi=(\varphi',\varphi'')} \mathcal{O}_{\mathbb{P}_0\times\{x_1\}} \oplus \mathcal{O}_{\mathbb{P}_0\times\{x_0\}}$$

and the restriction r

$$\mathcal{O}_{\mathbb{P}_0\times D_0} \xrightarrow{r} \mathcal{O}_{\mathbb{P}_0\times\{x_1\}} \oplus \mathcal{O}_{\mathbb{P}_0\times\{x_0\}}.$$

So we have the map $\phi_0 := ((\varphi', \varphi'')|r)$:

$$\begin{matrix} \mathrm{pr}_{\mathbb{P}_0}^*\mathcal{O}_p(-1) \otimes (\mathrm{p} \times \mathrm{id}_C)^*\mathcal{L}' & & \mathcal{O}_{\mathbb{P}_0\times\{x_1\}} \\ \oplus & \xrightarrow{\phi_0=((\varphi',\varphi'')|r)} & \oplus \\ \mathrm{pr}_C^*\mathcal{O}_{D_0} & & \mathcal{O}_{\mathbb{P}_0\times\{x_0\}} \end{matrix}$$

and we define the sheaf ε_0 on $\mathbb{P}_0 \times C$ as the kernel $\varepsilon_0 := \ker(\phi_0)$ of the composition of the restriction map r and the map coming from the gluing data. The bundle $\mathbb{P}_0$ has two disjoint sections

$$\sigma_0 \colon \mathbb{P}(\mathcal{L}'|_{J'\times\{x_0\}}) \hookrightarrow \mathbb{P}_0 \quad \text{and} \quad \sigma_0' \colon \mathbb{P}(\mathcal{L}'|_{J'\times\{x_1\}}) \hookrightarrow \mathbb{P}_0 .$$

1. Points in $\mathbb{P}_0 \setminus (\sigma_0 \cup \sigma_0')$ correspond to sheaves, which are locally free in both points: x_0 and x_1. The gluing map is surjective on both components.

2. Points in σ_0 correspond to maps φ where φ'' vanish. The corresponding sheaf is isomorph to the sheaf $\mathcal{L}'(-x_1)$.

3. Points in σ_0' correspond to maps φ where φ' vanish. The corresponding sheaf is isomorph to the sheaf $\mathcal{L}'(-x_0)$.

Now we study in details the projective bundles $\mathbb{P}_j$, for $j \neq 0$. The trivial sheaf on the component C_j is special, i.e. isomorphic to $\mathcal{O}_{C_j}(1)$. Let $J := \mathrm{Jac}_{\chi+3}(C_0)$ and let $\mathcal{L} \in \mathrm{Pic}(J \times C_0)$ be the Poincaré bundle. Moreover, let D_j', D_j'' denote the connected components of $\overline{D_j \setminus C_0}$ such that $D_j' \cap C_0 = \{x_0\}$ and $D_j'' \cap C_0 = \{x_1\}$. Let $\mathcal{M}_j \in \mathrm{Pic}(J \times D_j)$ be the line bundle with $\mathcal{M}_j|_{J\times C_0} = \mathcal{L}$ and

$$\mathcal{M}_j|_{J\times D_j'} = pr_J^*(\mathcal{L}|_{J\times\{x_0\}}),\ \mathcal{M}_j|_{J\times D_j''} = pr_J^*(\mathcal{L}|_{J\times\{x_1\}}).$$

Let

$$m_{j+1} := Hom(\mathcal{M}_j|_{J\times\{x_{j+1}\}}, \mathcal{O}_{J\times\{x_{j+1}\}}) \text{ and } m_j := Hom(\mathcal{M}_j|_{J\times\{x_j\}}, \mathcal{O}_{J\times\{x_j\}}).$$

Let $U_j := m_{j+1}^* \oplus m_j^*$. Now we consider the projectivization

$$\mathbb{P}_j := \mathbb{P}(U_j) \xrightarrow{p} J.$$

We use, as above, the following notation

$$\begin{array}{ccccc} \mathbb{P}_j \times C & \xrightarrow{p \times id_C} & J \times C & \xrightarrow{pr_C} & C \\ \downarrow{\scriptstyle pr_{\mathbb{P}_j}} & & \downarrow{\scriptstyle pr_J} & & \\ \mathbb{P}_j & \xrightarrow{p} & J & & \end{array}$$

Every such projectivization comes in the natural way with a surjective morphism

$$p^* U_j \twoheadrightarrow \mathcal{O}_p(1) .$$

Dualizing we get the map $\mathcal{O}_p(-1) \hookrightarrow p^* U_j^*$. We want to describe the family of sheaves on the curve C, which is parameterized by $\mathbb{P}_j$. We have, as above, two maps: the restriction r and the gluing $\varphi = (\varphi', \varphi'')$:

$$\begin{array}{ccc} \mathrm{pr}_{\mathbb{P}_j}^* \mathcal{O}_p(-1) \otimes (\mathrm{p} \times \mathrm{id}_C)^* \mathcal{M}_j & & \mathrm{pr}_C^* \mathcal{O}_{x_{j+1}} \\ \oplus & \xrightarrow{\phi_j := ((\varphi', \varphi'')|r)} & \oplus \\ \mathrm{pr}_C^*(\mathcal{O}_{C_j}(1)) & & \mathrm{pr}_C^* \mathcal{O}_{x_j} \end{array}$$

We define the sheaf ε_j on $\mathbb{P}_j \times C$ as the kernel $\varepsilon_j := \ker(\phi_j)$ of the composition of the restriction map r and the map coming from the gluing data. The bundle $\mathbb{P}_j$ has two disjoint sections

$$\sigma_j \colon \mathbb{P}(m_{j+1}^*) \hookrightarrow \mathbb{P}_j \ \text{ and } \ \sigma_j' \colon \mathbb{P}(m_j^*) \hookrightarrow \mathbb{P}_j .$$

1. Points in $\mathbb{P}_j \setminus (\sigma_j \cup \sigma_j')$ correspond to sheaves, which are locally free in both points: x_{j+1} and x_j. The gluing map is surjective on both components.

2. Points in σ_j correspond to maps φ where φ'' vanish.

3. Points in σ_j' correspond to maps φ where φ' vanish.

Similarly as for the singular fibers over Δ_{Σ_i} and Δ_{deg} we can introduce the equivalence relation $\sim$ in the following way: We obtain, as described above, $n+1$ $\mathbb{P}^1$-bundles $\mathbb{P}_j$, $j = 0, \ldots, n$ over a Jacobian of a genus 4 curve C_0. Every projective bundle $\mathbb{P}_j$ has two sections: σ_j and σ_j'. The fibre $f^{-1}([Q])$ will then be obtained by cyclicly gluing these bundles along their sections. In all ordinary points x_{j+1} we identify the section σ_j' with σ_{j+1}. For the sheaves of the first type the gluing circle is not closed. Without loss of generality, let $x_{j+1} \in C_j \cap C_{j+1}$ be the special point. Then we need to glue

$$\varepsilon_j|_{\sigma'_j \times C} \cong \varepsilon_0|_{\sigma'_0 \times C} \cong \mathcal{L}'(-x_0)$$

with

$$\varepsilon_{j+1}|_{\sigma_{j+1} \times C} \cong \varepsilon_0|_{\sigma_0 \times C} \cong \mathcal{L}'(-x_1).$$

Using above notation we can introduce the following equivalence relation $\sim$ in the following way:

$$\sigma'_0(x) = \sigma_1(x), \sigma'_1(x) = \sigma_2(x), \ldots, \sigma'_n(x) = \sigma_0(t(x)),$$

where $x \in J$ and $t\colon \mathrm{Jac}_{\chi+3}(C_0) \to \mathrm{Jac}_{\chi+3}(C_0)$ means the shift by the difference $\mathcal{O}_{C_0}(x_0 - x_1)$ in the group J. Now we can formulate detailed description of the singular fibre of f for a general quadric Q passing through the singularity of type A_n:

Theorem 6.16 *Let $[Q]$ be a general quadric Q passing through the A_n-singularity $p \in \Sigma$, and $C_0 \subset X$ be a strict transform of Q. Then there are $\mathbb{P}^1$-bundles $\mathbb{P}_j \to Jac(C_0)$, $j = 0, \ldots, n$ and disjoint sections $\sigma_j, \sigma'_j\colon Jac(C_0) \to \mathbb{P}_j$ such that the reduced fiber $f^{-1}([Q])$ over $[Q]$ has the following structure:*

$$f^{-1}([Q]) = \coprod \mathbb{P}_j / \sim .$$

We discuss below the case $\widetilde{D}_4$, which is more complicated than the $\widetilde{A}_n$ cases because of the non reduced curves occurring in its configuration. In the future work we want to solve this problem completely by describing explicitly as a family the singular fibres of f for all A-D-E singularities.

Extended Dynkin diagram of type $\widetilde{D}_4$

Let us now consider the case that p is a singular point of the sextic Σ that produces a singularity of type D_4 on the double cover $Y \to \mathbb{P}^2$.

We use the following notation: Let C_4 denote the component with multiplicity 2. So F_4 is a pure sheaf on the non-reduced curve $2C_4$. Let C_0, as before, denote the strict transform and C_1, C_2, C_3 denotes the remaining (-2)-curves in the configuration. Moreover, let x_i denotes the intersection point of C_4 with C_i for $i = 1, 2, 3$ and let $x_0 \in C_4 \cap C_0$. We have locally the following possibilities:

(I) All F_j are ordinary, and all but one vertex x is ordinary. At the unique special vertex x' one has $\chi(T_{x'}) = m_i m_j - 1$, if $x' \in C_i \cap C_j$. Without loss of generality, let x_1 be special. The gluing data in the three points: x_0, x_2 and x_3 consists of $(\mathbb{C}[\epsilon]/\epsilon^2)^*$. Moreover, in the point x_1 we have $\mathcal{O}_{x_1} \cong \mathbb{C}[\epsilon, \eta]/(\epsilon^2, \eta) \cong \mathbb{C}[\epsilon]/\epsilon^2$ and we glue only modulo the socle $\kappa = (0 : m_{x_1}) = \epsilon$, so in this point we glue only $\mathbb{C}^*$ with $\mathbb{C}^*$ (cf. Lemma 6.3). Moreover, $F_0 \in \mathrm{Jac}_{\chi+3}(C_0), F_i \cong \mathcal{O}_{C_i}$ for $i = 1, 2, 3$ and $F_4 \cong \mathcal{O}_{2C_4}$. The automorphism groups are $\mathrm{Aut}(F_i) = \mathbb{C}^*$ for $i = 0, 1, 2, 3$. The sheaf $F_4 \cong \mathcal{O}_{2C_4}$ fit into the following exact sequence

$$0 \to \mathcal{O}_{C_4}(2) \to \mathcal{O}_{2C_4} \to \mathcal{O}_{C_4} \to 0.$$

Now taking the global invertible sections we obtain the following exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_{C_4}(2) \to \mathrm{H}^0\, (\mathcal{O}_{2C_4})^* \to \mathbb{C}^* \to 0. \tag{40}$$

So the automorphism group $\mathrm{Aut}(F_4)$ is a semidirect product $\mathbb{C}^3 \rtimes \mathbb{C}^*$ and

$$\mathrm{Aut}(F_4)/\mathbb{C}^* \cong 1 + \varepsilon\, \mathrm{H}^0(\mathcal{O}_{C_4}(2)).$$

The choice of the gluing parameter is represented by the cokernel of the map

$$\mathrm{Aut}(F_4)/\mathbb{C}^* \longrightarrow \prod_{x_i} \mathrm{Aut}(\mathcal{O}_{m_i C_i \cap m_j C_j, x_i})/\mathbb{C}^* \cong 1 + \varepsilon\, \mathbb{C}^3.$$

In this situation we do not have any free parameter. In the point x_1 we glue less ($\mathbb{C}^*$ with $\mathbb{C}^*$) and from (I) we become as locus inside the fibre a copy of $\{x_1\} \times \mathrm{Jac}_{\chi+3}(C_0)$. Analogously in other three cases $\{x_i\} \times \mathrm{Jac}_{\chi+3}(C_0)$ for $i = 0, 2, 3$.

(II) All intersection points are ordinary, and all but one sheaf F_j is ordinary. According to Lemma 6.12 for the unique special sheaf F_{j_0} one has $\psi(F_{j_0}) = m_{j_0}^2 + 1$.

Basically we need to discuss the following three cases:

(1) F_0 special, then $F_0 \in \mathrm{Jac}_{\chi+4}(C_0)$. Moreover, $F_i \cong \mathcal{O}_{C_i}$ for $i = 1, 2, 3$ and $F_4 \cong \mathcal{O}_{2C_4}$. After dividing out globally $\mathbb{C}^*$ action, as gluing data we have $\bigoplus_{i=0}^{3} \mathrm{H}^0\mathcal{O}_{C_4}(2)|_{x_i}$ and as automorphism group $\mathrm{H}^0\, \mathcal{O}_{C_4}(2)$. From

$$\bigoplus_{i=0}^{3} \mathrm{H}^0\mathcal{O}_{C_4}(2)|_{x_i}/\mathrm{H}^0\, \mathcal{O}_{C_4}(2) \text{ for } i = 0, 1, 2, 3$$

follows that there is only one parameter free and this parameter comes from the gluing. The choice of the gluing parameter is represented by the cokernel of the map

$$\operatorname{Aut}(F_4)/\mathbb{C}^* \longrightarrow \prod_{x_i} \operatorname{Aut}(\mathcal{O}_{m_iC_i\cap m_jC_j,x_i})/\mathbb{C}^* \cong 1+\varepsilon\,\mathbb{C}^4.$$

This gives a $\mathbb{A}^1$-bundle over $\operatorname{Jac}_{\chi+4}(C_0)$.

(2) F_k is special for $k=1,2,3$. Without loss of generality, let F_1 be special. Then from Lemma 6.14 we conclude that $F_1 \cong \mathcal{O}_{C_1}(1)$. Moreover, $F_0 \in \operatorname{Jac}_{\chi+3}(C_0)$, $F_i \cong \mathcal{O}_{C_i}$ for $i=2,3$ and $F_4 \cong \mathcal{O}_{2C_4}$. Similar, as above

$$\bigoplus_{i=0}^{3} \mathrm{H}^0\,\mathcal{O}_{C_4}(2)|_{x_i}/\mathrm{H}^0\,\mathcal{O}_{C_4}(2) \text{ for } i=0,1,2,3.$$

The choice of the gluing parameter is represented by the cokernel of the map

$$\operatorname{Aut}(F_4)/\mathbb{C}^* \longrightarrow \prod_{x_i} \operatorname{Aut}(\mathcal{O}_{m_iC_i\cap m_jC_j,x_i})/\mathbb{C}^* \cong 1+\varepsilon\,\mathbb{C}^4.$$

We obtain the same parameter as in the previous case. This gives a $\mathbb{A}^1$-bundle over $\operatorname{Jac}_{\chi+3}(C_0)$.

(3) F_4 is special, $F_i \cong \mathcal{O}_{C_i}$ for $i=1,2,3$ and $F_0 \in \operatorname{Jac}_{\chi+3}(C_0)$. Moreover, the sheaf F_4 is obtained via the following push-out diagram by multiplication with y and $y \in C_4 \backslash \bigcup C_i$, $i=0,1,2,3$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{C_4}(2) & \longrightarrow & \mathcal{O}_{2C_4} & \longrightarrow & \mathcal{O}_{C_4} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \otimes y} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathcal{O}_{C_4}(3) & \longrightarrow & F_4 & \longrightarrow & \mathcal{O}_{C_4} & \longrightarrow & 0
\end{array}$$

Now we consider the following sequence

$$\mathcal{O}_{C_4}(-1) \to \mathcal{O}_{C_4}(3) \to \bigoplus_{i=0}^{3} \mathcal{O}_{C_4}(3)|_{x_i}.$$

Taking the associated long exact cohomology sequence we deduce that the last map is surjective and hence an isomorphism. So the group action is transitive and we do not have any gluing parameter. The choice of the gluing parameter is represented by the cokernel of the map

$$\mathrm{Aut}(F_4)/\mathbb{C}^* \cong 1+\varepsilon \mathrm{H}^0(\mathcal{O}_{C_4}(3)) \longrightarrow \prod_{x_i} \mathrm{Aut}(\mathcal{O}_{m_iC_i\cap m_jC_j,x_i})/\mathbb{C}^* \cong 1+\varepsilon\,\mathbb{C}^4.$$

The free parameter comes from the multiplication with the linear equation of the point y. So we obtain a $\mathbb{P}^1$-bundle over $\mathrm{Jac}_{\chi+3}(C_0)$ without four intersection points (vertices) x_i, $i = 0, 1, 2, 3$.

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