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Calabi–Yau operators of degree two

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Abstract
We show that the solutions to the equations, defining the so-called *Calabi–Yau condition* for fourth-order operators of degree two, define a variety that consists of ten irreducible components. These can be described completely in parametric form, but only two of the components seem to admit arithmetically interesting operators. We include a description of the 69 essentially distinct fourth-order Calabi–Yau operators of degree two that are presently known to us.

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1 Introduction

The hypergeometric series

$$\phi_0(x) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} x^n = 1 + 120x + 113400x^2 + \cdots \in \mathbf{Z}[[x]]$$

became famous after in the paper [20] its dual interpretation was discovered: on the one hand, the series encodes enumerative information on rational curves on the general quintic Calabi–Yau threefold in $\mathbf{P}^4$, and on the other hand, it can be identified as a normalised period of the mirror quintic Calabi–Yau threefold. The power series is the unique (up to scalar) series solution of the hypergeometric operator

$$L := \theta^4 - 5^5 x \left(\theta + \frac{1}{5}\right)\left(\theta + \frac{2}{5}\right)\left(\theta + \frac{3}{5}\right)\left(\theta + \frac{4}{5}\right), \quad \theta = x\frac{d}{dx}.$$

Gert Almkvist passed away in December 2018. Gert was a remarkable man and a great mathematician. It was a privilege to have known and worked with him.

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The operator L is the first of the family of 14 hypergeometric fourth-order operators related to mirror symmetry for complete intersections in weighted projective spaces. These operators are all of the form

$$\theta^4 - Nx(\theta+\alpha_1)(\theta+\alpha_2)(\theta+\alpha_3)(\theta+\alpha_4),$$

where the integer N is chosen as to make the coefficients of the normalised holomorphic solution integral in a minimal way. The properties of these operators have been well-studied from various points of view [8, 19, 26, 27, 34, 38, 39].

The operator L is also the first member of the ever growing list of *Calabi–Yau operators* [3, 41], a notion that was introduced by the first author and Zudilin [4] by abstracting the properties of the operator L. Calabi–Yau operators in this sense are essentially self-adjoint fourth-order Fuchsian operators with a point of maximal unipotent monodromy for which strong integrality properties are supposed to hold. In a sense, the term *Calabi–Yau operator* is somewhat of a misnomer, as there exist families of Calabi–Yau varieties, whose Picard–Fuchs operators have no MUM point and hence are not Calabi–Yau operators in the above sense of the word. For a recent account, see [23].

Let us consider a general Nth-order differential operator written *in θ-form*:

$$L = P_0(\theta) + xP_1(\theta) + \cdots + \cdots + x^r P_r(\theta), \quad \theta := x\frac{d}{dx},$$

where the P_i are polynomials of degree N. We call the largest r with $P_r \neq 0$ the *degree* of the operator. The differential equation

$$L\phi = 0, \quad \phi(x) = \sum_{n=0}^{\infty} a_n x^n$$

translates into the recursion relation

$$P_0(n)a_n + P_1(n-1)a_{n-1} + \cdots + P_r(n-r)a_{n-r} = 0$$

on the coefficients a_n of the series $\phi(x)$, so the degree of L is equal to the *length* of this recursion.

The roots of $P_0(\theta)$ are the *exponents* of the operator at 0. By translation, one can define exponents of P at each point of the Riemann sphere $\mathbf{P}^1$. For the point ∞ we have to apply inversion $x \mapsto 1/x$; the exponents then are the roots of $P_r(-\theta)$. The *Riemann symbol* is a table that contains for each singular point the corresponding exponents. These encode information about the local ramification of the solutions. Logarithmic terms may and usually do appear at points with exponents with integral difference. This leads to Jordan blocks in the local monodromy, but these are usually left implicit in the Riemann symbol.

We will usually assume that $N = 4$ and that all exponents at 0 vanish; $P_0(\theta) = \theta^4$. The most salient feature of this situation is that at 0 we have a canonical Frobenius basis of solution $\phi_0, \phi_1, \phi_2, \phi_3$, where ϕ_k contains terms with $\log^k(x)$. The local

monodromy has a Jordan block of maximal size, so 0 is a point of maximal unipotent monodromy: MUM.

The 14 hypergeometric operators have degree 1 and have a Riemann symbol of the form

$$\left\{\begin{array}{ccc} 0 & 1/N & \infty \\ \hline 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \\ 0 & 1 & \alpha_3 \\ 0 & 2 & \alpha_4 \end{array}\right\},$$

exhibiting a *conifold singularity* at the point $1/N$. At such a point the exponents are 0, 1, 1, 2 and the local monodromy is a *symplectic reflection*; there is a Jordan block of size two. The exponents at ∞ are $\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4$ and add up in pairs to one:

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 = 1.$$

Currently, we know over 500 Calabi–Yau operators with degrees running up to 40, but in no way do we expect the current list to be complete. An update of the AESZ list from [3] is in preparation [2].

The first Calabi–Yau operator of degree > 1 is #15 in the AESZ table and appeared in [10]:

$$\theta^4 - 3x(3\theta+1)(3\theta+2)(7\theta^2+7\theta+2) - 72x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5).$$

Its Riemann symbol is

$$\left\{\begin{array}{cccc} 0 & 1/216 & -1/27 & \infty \\ \hline 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 1 & \frac{2}{3} \\ 0 & 1 & 1 & \frac{4}{3} \\ 0 & 2 & 2 & \frac{5}{3} \end{array}\right\}.$$

Another operator of this type is #25 of the AESZ list. It arose from the quantum cohomology of the Grassmanian $G(2,5)$. The complete intersection $X(1,2,2)$ of hypersurfaces of degrees 1, 2, 2 in the Plücker embedding is a Calabi–Yau threefold with the following mirror Picard–Fuchs operator:

$$\theta^4 - 4x(2\theta+1)^2(11\theta^2+11\theta+3) - 16x^2(2\theta+1)^2(2\theta+3)^2.$$

Its holomorphic solution is

$$\phi_0(x) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 A_n x^n,$$

where

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

are the *small Apéry numbers* that were used in Apéry's proof of the irrationality of $\zeta(2)$.

At present we know 69 essentially distinct Calabi–Yau operators of order two, to which all further such degree two operators can be related by simple transformations. The operators for which the instanton numbers are 0 are obtained as Sym^3 of a second-order operator and do not count as a proper Calabi–Yau operator and do not appear in [3].

But contrary to the hypergeometric case, it appears that these degree two operators do not all fall in a *single* family. First of all, an operator of degree two can have four singularities, with a Riemann symbol of the form

$$\left\{\begin{array}{cccc} 0 & a & b & \infty \\ \hline 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 1 & \alpha_2 \\ 0 & 1 & 1 & \alpha_3 \\ 0 & 2 & 2 & \alpha_4 \end{array}\right\},$$

exhibiting *two distinct conifold points*, as in the previous two examples. But in many cases the points a and b coalesce, producing an operator with only three singular points.

As one can observe in the above two examples, the exponents at infinity are symmetrically centred around the value 1 and this seemed to be the case for all further examples we found. It was M. Bogner who first found an example of an Calabi–Yau operator for which this is *not* the case: it is the operator

$$\theta^4 - x(216\,\theta^4 + 396\,\theta^3 + 366\,\theta^2 - 168\,\theta + 30) + 36\,x^2\,(3\,\theta + 2)^2\,(6\,\theta + 7)^2\,.$$

with Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1/108 & \infty \\ \hline 0 & 0 & 2/3 \\ 0 & 1/6 & 2/3 \\ 0 & 1 & 7/6 \\ 0 & 7/6 & 7/6 \end{array}\right\},$$

and later two more such operators were found. The existence of such operators was initially a surprise to us, as it appears that, within the group of operators with three singular points, further distinctions can be made. It is this circumstance that led to this paper.

The structure of the paper is as follows. In the first section, we recall some basic facts about the Calabi–Yau condition and formulate the main result of this paper.

In the second section, we describe how the result can be obtained from a series of simple computations. In the third section we summarise the properties of the operators corresponding to each of the components that we find. In a final section, we review the list all degree two Calabi–Yau operators that are presently known to us. In two appendices we include some further monodromy and modular form information on these examples, together with some properties of the lower-order operators used in many constructions.

2 The Calabi–Yau condition

The *adjoint* L^* of a differential operator L of the form

$$L = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_1\frac{\mathrm{d}}{\mathrm{d}x} + a_0 \in \mathbf{Q}(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}\right]$$

is obtained by reading the operator *backwards with alternating signs*:

$$L^* = \frac{\mathrm{d}^n}{\mathrm{d}x^n} - \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}a_{n-1} + \cdots + \pm\frac{\mathrm{d}}{\mathrm{d}x}a_1 + (-1)^n a_0 \in \mathbf{Q}(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}\right],$$

so that

$$L^*(y) = y^{(n)} - (a_{n-1}y)^{(n-1)} + \cdots + (-1)^n a_0 y$$

The operator $\mathcal{P}$ is called *essentially self-adjoint*, if there exists a function $\alpha \neq 0$ (in an extension field of $\mathbf{Q}(x)$), such that

$$L\alpha = \alpha L^*.$$

It is easy to see that any such α has to satisfy the first-order differential equation

$$\alpha' = -\frac{2}{n}a_{n-1}\alpha.$$

This essential self-adjointness is equivalent to the existence of an invariant pairing on the solution space, symmetric if n is odd and alternating if n is even, [16, 17]. The condition of essential self-adjointness can be expressed by the vanishing of certain differential polynomials in the a_i, described in [1, 4, 16, 17], called the *Calabi–Yau condition*.

For an operator of order two, the essential self-adjointness does not impose any conditions, whereas for an differential equation of order three of the form

$$y''' + a_2 y'' + a_1 y' + a_0 y,$$

the Calabi–Yau condition comes down to the vanishing of the quantity

$$\mathcal{W} := \frac{1}{3}a_2'' + \frac{2}{3}a_2a_2' + \frac{4}{27}a_2^3 + 2a_0 - \frac{2}{3}a_1a_2 - a_1',$$

whereas for a fourth-order operator

$$y^{(iv)} + a_3y''' + a_2y'' + a_1y' + a_0y = 0,$$

the quantity

$$\mathcal{Q} := \frac{1}{2}a_2a_3 - a_1 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3(a_3)' - \frac{1}{2}a_3''$$

has to vanish. In that case the associated differential equation satisfied by the 2×2 Wronskians reduces its order from 6 to 5, [4].

For operators of higher order, one finds that more than one differential polynomial condition has to be satisfied; here we will be concerned with operators of three and mainly order four.

It is rather easy to fulfil the condition $\mathcal{Q} = 0$! For example, there exists one particularly nice family of operators of degree n that we call the *main family*.

Proposition 1 *If A, B, C are polynomials of degree n, then the operator*

$$\begin{aligned} L &= \theta^2A\theta^2 + \theta B\theta + C \\ &= A\theta^4 + 2A'\theta^3 + (B + A'')\theta^2 + B'\theta + C \end{aligned}$$

has degree n and satisfies the Calabi–Yau condition. If the constant term of A is 1 and of B and B is zero, then the exponents of L at 0 are all zero. If the roots of A are all distinct, L has n singular points with exponents $0, 1, 1, 2$ and a further singular point at infinity. (Here $A' := \theta(A) = x\partial A/\partial x$, etc.)

The operator of the proposition can be shown to satisfy the Calabi–Yau condition $Q = 0$ by a direct, but tedious computation. But the form of the operator makes the self-adjointness manifest. The statements about the exponents require a further direct calculation.

In this paper we take a closer look at differential operators of the form

$$\theta^4 + x(a\theta^4 + b\theta^3 + c\theta^2 + d\theta + e) + fx^2(\theta + \alpha)(\theta + \beta)(\theta + \gamma)(\theta + \delta).$$

If $f \neq 0$, then the numbers $\alpha, \beta, \gamma, \delta$ are then the characteristic exponents of the operator at ∞.

For this operator, the quantity $\mathcal{Q}$ is a rational function with

$$x^2(1 + ax + fx^2)^3$$

as denominator; the singular points of the operator are the roots of this polynomial, together with the point ∞.

The numerator is a polynomial Q of degree 5 in the variable x, so we can write

$$Q = Q_0 + Q_1 x + Q_2 x^2 + Q_3 x^3 + Q_4 x^4 + Q_5 x^5,$$

where the coefficients Q_i are rather complicated polynomials in the parameters a, b, c, d, e, f and the exponents $\alpha, \beta, \gamma, \delta$:

$$Q_0, \quad Q_1, \quad Q_2, \quad Q_3, \quad Q_4, \quad Q_5 \in \mathbf{Q}[a, b, c, d, e, f, \alpha, \beta, \gamma, \delta].$$

So, the Calabi–Yau condition is equivalent to six polynomial equations

$$Q_0 = Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = 0,$$

which define a certain *affine algebraic set*

$$X = V(Q_0, Q_1, \ldots, Q_5) \subset \mathbf{C}^{10}.$$

We will give a complete description of the set X, in particular we determine the irreducible components of X. As we have six equations in 10 variables, a first trivial remark is that all irreducible components of X have dimension *at least four*. We will give explicit parametrisations of all the irreducible components. Note also that the polynomials Q_i do not depend on the accessory parameter e, so the solution set X contains the e-line as trivial factor.

Theorem 1 *The algebraic set X defined by the condition* $Q(a, b, c, d, e, f, \alpha, \beta, \gamma, \delta) = 0$ *is the union of ten irreducible components, of which there are seven with* $f \neq 0$.

Note that if $f = 0$ we are dealing with an operator of lower degree (i.e. hypergeometric), so will from now on assume that $f \neq 0$ and we will ignore the three components with $f = 0$. The seven remaining components differ in the behaviour of the average exponents at infinity, i.e. the quantity:

$$\sigma := 2\frac{\alpha + \beta + \gamma + \delta}{4}.$$

There is a single component for which σ is not fixed, that we will call the *transverse component*. On each of the remaining six components, σ has a fixed value. The dimensions and values of σ that appear are summarised in the following diagram:

σ	0	1	2	3	4
dim	4	5	6 4	5	4

The two five-dimensional components are related by a simple transformation, as are the three four-dimensional components.

All Calabi–Yau operators known to us belong to the six-dimensional *main component* (M):

$$(M) : \theta^4 + x(a\theta^4 + 2a\theta^3 + (a+d)\theta^2 + d\theta + e) \\ + fx^2(\theta+\alpha)(\theta+\beta)(\theta+2-\alpha)(\theta+2-\beta)$$

or the *transverse component* (T) that we describe later.

We remark furthermore that the ideal

$$I = (Q_0, Q_1, Q_2, Q_3, Q_4, Q_5) \subset \mathbf{C}[a, b, c, d, e, f, \alpha, \beta, \gamma, \delta]$$

is, from an algebraic point of view, rather complicated. It is not radical and the components appear with non-trivial multiplicities in the primary decomposition.

For third-order operators of degree two

$$\theta^3 + x(a\theta^3 + b\theta^2 + c\theta + d) + fx^2(\theta+\alpha)(\theta+\beta)(\theta+\gamma),$$

the Calabi–Yau condition $\mathcal{W} = 0$ leads in a similar way to an algebraic set

$$Y \subset \mathbf{C}^8,$$

defined by a system of polynomial equations

$$W_0 = W_1 = W_2 = W_3 = W_4 = W_5 = 0,$$

where now the polynomials

$$W_0, \quad W_1, \quad W_2, \quad W_3, \quad W_4, \quad W_5 \in \mathbf{Q}[a, b, c, d, f, \alpha, \beta, \gamma].$$

are in eight variables. The analysis of this ideal is very similar to the ideal for the fourth-order operators, but hardly simpler. The result is, somewhat surprisingly, also very similar to the case of order four:

Theorem 2 *The algebraic set $Y \subset \mathbf{C}^8$ defined by the condition $W(a, b, c, d, f, \alpha, \beta, \gamma) = 0$ is the union of ten irreducible components, of which there are seven with $f \neq 0$.*

The seven remaining components differ in the behaviour of the (doubled) average exponent

$$\sigma := 2\frac{\alpha+\beta+\gamma}{3}.$$

Again there is a single *transverse component* on which the value of σ is not fixed. On each of the remaining six components, σ has a fixed value. The dimensions and values of σ that appear are summarised in the following diagram:

σ	0	1	2	3	4
dim	2	3	4 2	3	2

The main component (m) is the one with $\sigma = 2$, dim $= 4$ and is given by

$$(m) : \theta^3 + x(2\theta + 1)((c - 2d)\theta^2 + (c - 2d)\theta + d) + fx^2(\theta + \alpha)(\theta + 1)(\theta + 2 - \alpha),$$

whereas on the transverse component (t) we have the operator

$$(t) : \theta^3 + \frac{a}{4}x(4\theta^3 + 3(\alpha + \beta)\theta^2 + (\alpha + \beta + 2\alpha\beta)\theta + \alpha\beta) + \left(\frac{a}{4}\right)^2 x^2(\theta + \alpha)\left(\theta + \frac{\alpha + \beta}{2}\right)(\theta + \beta).$$

Again, the operators on the two three-dimensional components are related by a simple transformation, as are the three two-dimensional components.

One may speculate that the above patterns extends to higher-order operators, but we will not try to pursue it here.

3 Proof of Theorem 1

The proof of Theorem 1 is purely computational and consists of systematically solving of the equations, using a computer algebra system like MAPLE. The miracle here is that this procedure actually works. We will describe only the most important steps in the process.

3.1 Preliminary reductions $Q_0 = Q_5 = 0$

The value of b: The polynomial Q_0 is found to be

$$-4b - 8d + 8c,$$

and this allows us directly to eliminate the variable b and thus we will put from now on

$$b = 2(c - d).$$

Symmetry of the exponents: The polynomial Q_5 factors nicely as

$$-f^3(\alpha + \beta - \delta - \epsilon)(\alpha + \delta - \beta - \epsilon)(\alpha + \epsilon - \beta - \delta).$$

As we assume that $f \neq 0$, this only vanishes if one of the three other factors vanish, meaning that *the exponents come in two pairs that add up to the same value*, that we

shall call σ. So, without loss of generality we can assume to have the pairs $\alpha, \sigma - \alpha$ and $\beta, \sigma - \beta$ and we will take the operator to be of the form

$$\theta^4 + x(a\theta^4 + b\theta^3 + c\theta^2 + d\theta + e) + fx^2(\theta + \alpha)(\theta + \beta)(\theta + \sigma - \alpha)(\theta + \sigma - \beta),$$

with $b = 2(c - d)$ as our starting point.

3.2 The remaining equations

After solving the equations $\mathbf{Q}_0 = 0, \mathbf{Q}_5 = 0$, one is left to solve the polynomial equations $Q_1 = Q_2 = Q_3 = Q_4 = 0$. For sake of concreteness, we give the explicit form of the polynomials:

$$\begin{aligned}
Q_1 :=\ & 8\,f\alpha^2\sigma + 16\,f\sigma\,\beta - 8\,f\sigma^2\beta - 16\,c^2 + 40\,cd - 24\,d^2 - 8\,f\alpha\,\sigma^2 - 16\,f\alpha^2 \\
& +16\,f\sigma^2 - 16\,f\beta^2 - 32\,f\sigma - 24\,da + 16\,ca + 16\,f\alpha\,\sigma + 8\,f\sigma\,\beta^2, \\
Q_2 :=\ & -8\,f\alpha^2 c - 8\,f\beta^2 c + 8\,f\beta^2 d + 8\,f\alpha^2 d - 8\,cad + 8\,f\sigma^2 c + 72\,f\sigma\,d \\
& -8\,f\sigma^2 d + 24\,c^2 d - 24\,cd^2 + 8\,c^2 a - 64\,cf\sigma + 24\,f\alpha\,\sigma\,a + 24\,f\sigma\,\beta\,a \\
& -16\,f\sigma^2\beta\,a + 16\,f\alpha^2\sigma\,a - 16\,f\alpha\,\sigma^2 a + 16\,f\sigma\,\beta^2 a \\
& -8\,f\alpha\,\sigma\,d + 8\,f\alpha\,\sigma\,c - 8\,c^3 + 8\,d^3 + 8\,f\sigma\,\beta\,c - 8\,f\sigma\,\beta\,d \\
& -64\,df - 8\,da^2 - 24\,f\sigma\,a - 24\,f\alpha^2 a + 48\,cf + 24\,f\sigma^2 a - 24\,f\beta^2 a, \\
Q_3 =\ & 8\,f\Big(2\,f\sigma\,\beta + 2\,f\alpha\,\sigma + f\sigma\,\beta^2 - f\sigma^2\beta + f\alpha^2\sigma - f\alpha\,\sigma^2 - 2\,f\beta^2 \\
& -3\,da + \sigma^2 a^2 + f\sigma^3 - \beta^2 a^2 - \sigma\,a^2 - \sigma\,\beta\,ad + \sigma\,\beta\,ac - \alpha\,\sigma\,ad + \alpha\,\sigma\,ac \\
& -\alpha^2 a^2 - 2\,c\sigma\,a - 2\,f\alpha^2 - 4\,f\sigma^2 + 4\,f\sigma + 4\,c^2 - 7\,cd + 3\,d^2 - \alpha\,\sigma^2 a^2 \\
& +\sigma\,\beta^2 a^2 + 6\,\sigma\,cd + 3\,\sigma\,ad - \beta^2 ac + \beta^2 ad - \alpha^2 ac + \alpha^2 ad + \sigma^2 ac \\
& -\sigma^2 ad - 3\,\sigma\,d^2 - 3\,\sigma\,c^2 + \alpha\,\sigma\,a^2 + \sigma\,\beta\,a^2 - \sigma^2\beta\,a^2 + \alpha^2\sigma\,a^2\Big), \\
Q_4 =\ & 8\,f^2\Big(-2\,c - \alpha^2 c - 2\,\sigma^2 c - 3\,\sigma\,d + 2\,\sigma^2 d + \beta^2 d - 2\,\sigma^2 a - \beta^2 a - \alpha\,\sigma\,d \\
& +\alpha\,\sigma\,c + \sigma\,\beta\,c - \sigma\,\beta\,d + \sigma\,\beta\,a - \alpha\,\sigma^2 a - \sigma^2\beta\,a + \alpha\,\sigma\,a + \alpha^2\sigma\,a + \beta^2\sigma\,a \\
& \sigma\,a - \alpha^2 a + \sigma^3 a + 4\,c\sigma - \beta^2 c + \alpha^2 d\Big).
\end{aligned}$$

Somewhat to our surprise, it turns out to be rather easy to give a complete solution to these equations. It will be useful to introduce the parameters

$$A := \alpha(\sigma - \alpha), \quad B := \beta(\sigma - \beta),$$

and use these instead of α and β. One has

$$\begin{aligned}
&(\theta + \alpha)(\theta + \sigma - \alpha)(\theta + \beta)(\theta + \sigma - \beta) = (\theta^2 + \sigma\theta + A)(\theta^2 + \sigma\theta + B) \\
&= \theta^4 + 2\sigma\theta^3 + (\sigma^2 + (A + B))\theta^2 + \sigma(A + B)\theta + AB.
\end{aligned}$$

We will see that most expressions only depend on the quantity

$$\Delta := -(A + B) = \alpha^2 + \beta^2 - \sigma(\alpha + \beta).$$

3.3 The equation $Q_4 = 0$

The polynomial Q_4 is an expression that is f^2 times an expression linear in the variable a. As we will only consider the cases with $f \neq 0$, we can solve for a and find

$$a = \frac{(2(\sigma-1)^2 + \Delta)c - (\sigma(2\sigma-3) + \Delta)d}{(\sigma-1)(\sigma^2 - \sigma + \Delta)},$$

Note that this value for a introduces the denominator

$$(\sigma - 1)(\sigma(\sigma - 1) + \Delta).$$

So, if we work further with this operator, we are implicitly assuming that these two factors do not vanish. If one of them does vanish, we should go back to the previous step, impose these conditions and compute further. For now, we will assume that these factors are nonzero and come back to these exceptional cases later.

3.4 The case $\sigma = 2$

It appears that with this value of a the quantity $\mathcal{Q}$ becomes divisible by $\sigma - 2$. So, in this remarkable $\sigma = 2$ case, we obtain an operator family with $\mathcal{Q} = 0$. For $\sigma = 2$ the value of a simplifies to

$$a = \frac{(2 + \Delta)c - (2 + \Delta)c}{1 \cdot (2 + \Delta)} = c - d,$$

and one obtains what we call the *main component* (M)*:*

$$\begin{aligned} P := {} & \theta^4 + x((c-d)\theta^4 + 2(c-d)\theta^3 + c\theta^2 + d\theta + e) \\ & + fx^2(\theta + \alpha)(\theta + \beta)(\theta + 2 - \alpha)(\theta + 2 - \beta), \end{aligned}$$

depending c, d, e, f and the exponents α and β as free parameters. Almost all operators of degree two from the AESZ list are of this type. In this way we obtain our first component of dimension 6.

3.5 The case $\sigma \neq 2$

If, however, $\sigma \neq 2$, we have to do a further analysis. Still using the above value for a, the coefficient of x of $\mathcal{Q}$ is a complicated expression in c, d, e, f and the exponents that, however, is factored by MAPLE instantly into four factors. First, there are the factors

$$(\sigma - 1),\ (\sigma - 2),\ (\sigma(\sigma - 1) + \Delta).$$

We assume first that these are nonzero, so one is forced to put to zero the fourth factor, which leads to a specific value for f:

$$f = (2c - 3d)\frac{((\sigma - 1)^2 + \Delta)c - (\sigma(\sigma - 2) + \Delta)c}{(\sigma - 1)\,(2\sigma + \Delta)\left(\sigma^2 - \sigma + \Delta\right)}.$$

Here a new factor $2\sigma + \Delta$ is introduced in the denominator, so we have to come back later to the case $2\sigma + \Delta = 0$.

3.5.1 The subcase $\sigma = 3$

Using the above values for a and f, the quantity $\mathcal{Q}$ only contains terms with x^2 and x^3. It factors out the factor $\sigma - 3$, which leads to another remarkable operator family with $\mathcal{Q} = 0$:

$$a = \frac{(8 + \Delta)c - (9 + \Delta)d}{2(6 + \Delta)}, \quad f = (2c - 3d)\frac{(4 + \Delta)c - (3 + \Delta)d}{2(6 + \Delta)^2}.$$

It has c, d, e and α, β as free parameters. This makes up a second component, of dimension 5.

3.5.2 The subcase $\sigma \neq 3$

Still with the given values for a and f, if $\sigma \neq 3$, one has to analyse the quantity $\mathcal{Q}$ further. MAPLE manages to factor the coefficient of x^2 into a product of eight factors, of only three were not supposed to be nonzero at this stage. Each of these factors leads to a linear dependence between c and d.

Case A:

$$c = (\Delta - \sigma)\frac{d}{\Delta}.$$

Upon substitution, we obtain an operator that satisfies the condition $\mathcal{Q} = 0$. For this operator one has

$$a = -2\frac{d}{\Delta}, \quad b = -2\sigma\frac{d}{\Delta}, \quad c = -(\sigma - \Delta)\frac{d}{\Delta}, \quad f = \left(\frac{d}{\Delta}\right)^2,$$

and has d, e and the exponents α, β, σ as free parameters. So, we obtain a further irreducible component of dimension 5. The remarkable thing is that now we have only Δ appearing in the denominator.

The operator of BOGNER mentioned in the introduction is an instance of case A for the parameter choice:

$$\alpha = \beta = 2/3, \quad \sigma = 11/6, \quad d = -168, \quad e = 30.$$

Case B:

$$c = d\frac{\sigma^2 - 2\sigma + \Delta}{(\sigma - 1)^2 + \Delta}.$$

In this case f reduces to 0, so we do not get an operator of degree two.
Case C:

$$c = d\frac{\sigma^2 - 6\sigma + 4 + \Delta}{(\sigma - 1)(\sigma - 4) + \Delta}$$

In this case the condition $\mathcal{Q} = 0$ reduces to special relations between the exponents. The exceptional cases are:

$$\sigma \in \{0, 1, 2, 3, 4\}$$

or

$$\left(\sigma^2 - 2\sigma + 4 + \Delta\right) = 0, \quad \left(\sigma^2 - 3\sigma + 4 + \Delta\right) = 0, \quad \left(\sigma^2 - 5\sigma + 4 + \Delta\right) = 0$$

This concludes the list of all possibilities. We still have to backtrack some of the cases.

3.6 Backtracking the remaining cases

Earlier, we use the value for a which involved the denominator

$$(\sigma - 1)(\sigma^2 - \sigma + \Delta).$$

The introduction of f led to a further denominator

$$2\sigma + \Delta.$$

The analysis of Case A gave Δ as denominator, and Case C gave further factor $(\sigma - 1)(\sigma - 4) + \Delta$ in the denominator. In each of the cases we have to backtrack and see if we find additional solutions.

3.6.1 The case $\sigma = 1$

Looking at the operator with $\sigma = 1$, we find from the coefficient of x of $\mathcal{Q}$ the value of f. Using this value, and factoring the coefficient of x^2 of $\mathcal{Q}$, we find there is a unique value of c:

$$c = d\frac{\Delta - 1}{\Delta}.$$

Here a denominator Δ appears, so we assume this to be $\neq 0$; but the case $\Delta = 0$ has to be analysed anyway.

3.6.2 The case $\Delta = 0$

To analyse the further cases, it is useful to write the operator in terms of A and B instead of α and β, so that $\Delta = C$ can be implemented by putting $B = C - A$. We start with $C = 0$. The following seven components make up the intersection with $\Delta = 0$:

Case 1 $\sigma = \Delta = 0,\ \ b,\ c,\ d = 0$:

$$\theta^4 + x(a\theta^4 + e) + fx^2(\theta^4 + u).$$

Case 2 $\sigma = 1,\ \ \Delta = 0,\ \ d = 0,\ \ f = (a - c)c$;

$$\theta^4 + x(a\theta^4 + 2c\theta^3 + c\theta^2 + e) + (a - c)c(\theta^4 + 2\theta^2 + 3 + \theta^2 + u).$$

Case 3 $\sigma = 2,\ \ \Delta = 0,\ \ a = c - d$. This is inside the main component.
Case 4 $\sigma = 2,\ \ \Delta = 0,\ \ a = d,\ \ c = 2d.\ \ f = d^2/4$.
Case 5 $\sigma = 3,\ \ \Delta = 0,\ \ a = 8c - 9d/12,\ \ f = (2c - 3d)(4c - 3d)/72$.
This is inside the $\sigma = 3$ component.
Case 6 $\sigma = 4, \Delta = 0,\ \ d = 0,\ \ c = 2a,\ \ f = a^2/4$.
Case 7 $d = \Delta = 0$, there is the family with parameters c, e, σ, α

$$a = \frac{2c}{\sigma},\quad b = 2c,\quad d = 0,\quad f = \frac{a^2}{4}.$$

3.6.3 Remaining cases

There are more cases to check and this could be done along the above sketched lines. Instead of this, we used the computer algebra system `Singular`. There is a library called `primdec.lib` containing the algorithms for doing primary decomposition. On our computer, `Singular` could not straight away compute the primary decomposition of the ideal $\mathcal{I}$ defined by the vanishing of the x-coefficients of $\mathcal{Q}$. But after taking the ideal quotient of $\mathcal{I}$ by the ideals already found, it turned out to be able to find the last component. In total we thus found seven irreducible components of the variety defined by $\mathcal{Q} = 0$.

3.7 Summary of the seven components

In total we have found seven components. Membership to a component is defined by certain polynomial relations between the coefficients. Below we give the defining relations for each of the components, the operator with its coefficients, together with its generic Riemann symbol.

The big $\sigma = 2$ component, also called the *main component* is defined by the linear equations

$$\sigma - 2 = 0, \quad a - (c - d) = 0.$$

For a general member of this component, the Riemann symbol is:

$$\left\{\begin{array}{cccc} 0 & * & * & \infty \\ \hline 0 & 0 & 0 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 1 & 1 & 2-\beta \\ 0 & 2 & 2 & 2-\alpha \end{array}\right\},$$

where $*$ are the two roots of $1 + ax + fx^2 = 0$. The operator depends on the six parameters $a, d, e, f, \alpha, \beta$. The parameters a, f determine the position of the singular fibres. The parameters α, β determine the four exponents at ∞ which are symmetric around the value 2. The parameters d, e are accessory parameters.

However, if the two roots coincide, $a^2 = 4f$, then we obtain an operator with three singular points.

The $\sigma = 1$ component is defined by the four conditions:

$$\begin{aligned} &\sigma - 1 = 0, \quad c\Delta + d(\Delta - 1) = 0, \\ &d(a - (c - d)) + f\Delta = 0, \quad c(a - (c - d)) + f(\Delta - 1) = 0. \end{aligned}$$

The Riemann symbol of the operator is:

$$\left\{\begin{array}{cccc} 0 & x & y & \infty \\ \hline 0 & 0 & 0 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 1 & 2 & 1-\beta \\ 0 & 2 & 3 & 1-\alpha \end{array}\right\},$$

where the special points x and y are located at:

$$x = \frac{\Delta}{d}, \quad y = \frac{\Delta}{d + a\Delta}.$$

The point y is not really a singular point of the differential operator, as the exponents $0,\ 1,\ 2,\ 3$ are those of a regular point of the differential equation. Indeed, it can be checked that no logarithms occur.

The $\sigma = 3$ component is defined by the conditions:

$$\begin{aligned} 0 &= \sigma - 3, \\ 0 &= 4a^2 - 4c^2 - 3ad + 11cd - 7d^2 - 3fA + 20f, \\ 0 &= 2aA - cA + dA - 12a + 8c - 9d, \\ 0 &= 2ac - 2c^2 - 3a^2aA - cA + dA - 12a + 8c - 9d, \\ 0 &= 2c^2A - 5cdA + 3d^2A + 2fA^2 - 8c^2 + 18cd - 9d^2 - 24fA + 72f. \end{aligned}$$

The Riemann symbol of the operator is:

$$\left\{\begin{array}{cccc} 0 & x & y & \infty \\ \hline 0 & 0 & -1 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 1 & 1 & 3-\beta \\ 0 & 2 & 2 & 3-\alpha \end{array}\right\},$$

where the singular points are located at:

$$x = \frac{6+\Delta}{2c-3d}, \quad y = \frac{-2(6+\Delta)}{4c-3d+(c-d)\Delta}.$$

The operator has five parameters c, d, e, α, β. α, β are parameters that determine the exponents at infinity, symmetric around 3. The parameters c, d determine together with α and β the position of the two singularities. e is a single accessory parameter.

The $\sigma = 0$ component is defined by the conditions:

$$\sigma = 0, \quad c = 0, \quad d = 0, \quad \Delta = 0.$$

The generic Riemann symbol for this operator is:

$$\left\{\begin{array}{cccc} 0 & x & y & \infty \\ \hline 0 & 0 & 0 & \alpha \\ 0 & 1 & 1 & i\alpha \\ 0 & 2 & 2 & -\alpha \\ 0 & 3 & 3 & -i\alpha \end{array}\right\},$$

where the special points x, y are located at the solutions of $1 + ax + fx^2 = 0$. The operator depends on four parameters a, e, f, α.

The small $\sigma = 2$ component is defined by the four conditions:

$$\sigma - 2 = 0, \quad 4ad - d^2 - 16f = 0, \quad 2c - 3d = 0, \quad \Delta + 2 = 0.$$

The Riemann symbol for a general member of this component is:

$$\left\{\begin{array}{cccc} 0 & x & y & \infty \\ \hline 0 & -1 & 0 & 1+\alpha \\ 0 & 0 & 1 & 1+i\alpha \\ 0 & 1 & 2 & 1-\alpha \\ 0 & 2 & 3 & 1-i\alpha \end{array}\right\},$$

where the special points are

$$x = -\frac{4}{d}, \quad y = -\frac{4}{4a-d}.$$

The $\sigma = 4$ component is defined by the four conditions:

$$\sigma - 4 = 0, \quad 6a - c = 0, \quad 2c - 3d, \quad \Delta + 8 = 0.$$

The Riemann symbol of this operator is:

$$\left\{\begin{array}{cccc} 0 & x & y & \infty \\ \hline 0 & -1 & -1 & 2+\alpha \\ 0 & 0 & 0 & 2+i\alpha \\ 0 & 1 & 1 & 2-\alpha \\ 0 & 2 & 2 & 2-i\alpha \end{array}\right\},$$

where the singular points x, y are solutions to $1 + ax + fx^2 = 0$.

The transverse component finally is defined by the equations

$$\begin{aligned} 0 &= a^2 - 4f, \\ 0 &= a\sigma - 2(c-d), \\ 0 &= a\Delta + 2d, \\ 0 &= -f\Delta^2 + d^2, \\ 0 &= ad + 2f\Delta, \\ 0 &= d\sigma - c\Delta + d\Delta, \\ 0 &= f\sigma\Delta - f\Delta^2 + cd, \\ 0 &= -f\sigma^2 + f\sigma\Delta + c^2 - cd, \\ 0 &= ac - 2f\sigma + 2f\Delta. \end{aligned}$$

Solving for a, b, c, f we find the family of operators

$$\begin{aligned} T(d,e,\alpha,\beta,\sigma) &:= \theta^4 \\ &-x\frac{1}{\Delta}\left(2d\theta^4 + 2d\sigma\theta^3 + (\sigma-\Delta)\theta)\right) + x(\theta+e) \\ &-\left(\frac{dx}{\Delta}\right)^2 (\theta+\alpha)(\theta+\beta)(\theta+\sigma-\beta)(\theta+\sigma-\beta) \end{aligned}$$

The Riemann symbol of the generic member in this family is:

$$\left\{\begin{array}{ccc} 0 & x & \infty \\ \hline 0 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 2-\sigma & \sigma-\beta \\ 0 & 3-\sigma & \sigma-\alpha \end{array}\right\},$$

where $x = \frac{\Delta}{d}$.

The big and small $\sigma = 2$ components do intersect:

$$\sigma = 2, \quad \Delta = -2, \quad 2a = d, \; 2c = 3d, \; f = d^2/16.$$

The transverse component intersects all other components. No other pair of components can intersect, as they have a different σ-value.

3.8 The small components

It turns out that the components with $\sigma = 0, \quad 2, \quad 4$ are closely related to each other and the solutions of the corresponding differential equations can be related in a simple manner. The families of operators in question are:

$$\begin{aligned} P_0(a, e, f, A) &:= \theta^4 + x(a\theta^4 + e) + x^2 f(\theta^4 - A^2) \\ P_2(a, d, e, A) &:= \theta^4 + x(a\theta^4 + d\theta^3 + (3/2)d\theta^2 + d\theta + e) \\ &\quad + \tfrac{d(4a-d)}{16} x^2((\theta+1)^4 - (A-1)^2) \\ P_4(a, e, f, A) &:= \theta^4 + x(a\theta^4 + 4a\theta^3 + 6a\theta^2 + 4a\theta + e) \\ &\quad + x^2 f(\theta^4 + 8\theta^3 + 24\theta^2 + 32\theta + A(8 - A)) \end{aligned}$$

Their relation is most easily understood from the perspective of the operator $P_2(a, d, e, A)$ and its exponents. The discriminant $\Delta(x)$ of the operator $P_2(a, d, e, A)$ factors as $(1+dx/4)(1+(4a-d)x/4)$. The exponents at the first factor are $-1, 0, 1, 2$ and at the second factor $0, 1, 2, 3$. The exponents at infinity are of the form

$$1 + \alpha, \quad 1 - \alpha, \quad 1 + i\alpha, \quad 1 - i\alpha.$$

Multiplication of a solution to $P_2(a, d, e, A)$ by $(1 + dx/4)$ will shift the exponents at the first singular point by $+1$ and the exponents at infinity by -1, which then leads to an operator with two points with exponents $0, 1, 2, 3$ and

$$\alpha, -\alpha, i\alpha, -i\alpha$$

as exponents at infinity, i.e. with $\sigma = 0$ and $\Delta = 0$. Similarly, division of a solution to $P_2(a, d, e, A)$ by $(1 + (4a - d)x/4)$ leads to a shift at the second singularity by -1 and hence produces two singularities with exponents $-1, 0, 1, 2$. At infinity the exponents shift by $+1$ and are of the form

$$2 + \alpha, \quad 2 - \alpha, \quad 2 + i\alpha, \quad 2 - i\alpha,$$

i.e. with $\sigma = 4$ and $\Delta = -8$.

We remark that the exponents at infinity can only be all real in the very special case that these all coincide.

A precise statement relating these operators is the following:

Proposition 2

$$\begin{gathered} y(x) \text{ is a solution of } P_2(a,d,e,A) \\ \Longleftrightarrow \\ (1+dx/4).y_0(x) \text{ is a solution of} \\ P_0(a,e-d/4,(4a-d)d/16,A-1) \\ \Longleftrightarrow \\ (1+(4a-d)x/4)^{-1}y(x) \text{ is a solution of} \\ P_4(a,e+(4a-d)/4,(4a-d)d/16),A-3) \end{gathered}$$

It is a remarkable fact that the local monodromies around the special points $\neq 0$ and $\neq \infty$ are trivial. The monodromy around 0 is MUM, so has infinite order and thus the monodoromy group is the non-reductive group $\mathbf{Z}$, which clearly prevents these operators from being Picard–Fuchs operators of a family of algebraic varieties.

Something similar happens for the $\sigma = 1$ and $\gamma = 3$ components, given by the operator families

$$\begin{aligned} P_1(a,d,e,\alpha,\beta) := \theta^4 + x\left(a\theta^4 - \tfrac{2d}{\Delta}\theta^3 + \tfrac{\Delta-1}{\Delta}\theta^2 + d\theta + e\right) \\ -\tfrac{d(d+a\Delta)}{\Delta^2}(\theta+\alpha)(\theta+\beta)(\theta+1-\alpha)(\theta+1-\beta)\,, \end{aligned}$$

where $\Delta = \alpha^2 + \beta^2 - \alpha - \beta$, and

$$\begin{aligned} P_3(c,d,e,\alpha,\beta) := \theta^4 + x\left(\frac{(\Delta+8)c+(\Delta+9)d}{2(\Delta+6)} + 2(c-d)\theta^3 + c\theta^2 + d\theta + e\right) \\ +(2c-3d)\frac{(\Delta+4)c-(\Delta+3)d)}{2(\Delta+6)^2}x^2(\theta+\alpha) \\ (\theta+\beta)(\theta+3+\alpha)(\theta+3-\beta), \end{aligned}$$

with $\Delta = \alpha^2 + \beta^2 - 3\alpha - 3\beta$.

One verifies directly that

Proposition 3

$$\begin{gathered} y(x) \text{ is a solution of } P_1(a,d,e,\alpha,\beta) \\ \Longleftrightarrow \\ (\Delta-(d+a\Delta)x)y_0(x) \text{ is a solution of} \\ P_3(d(\Delta-1)/\Delta,d,e,\alpha+1,\beta+1) \end{gathered}$$

These operators are also remarkable. One of the roots of the polynomial $1+ax+fx^2$ is in fact a regular point; the other is a conifold point, so the three non-trivial monodromies are precisely as those for the hypergeometric operators. In fact, if we multiply the hypergeometric operator

$$\theta^4 - Nx(\theta+\alpha)(\theta)$$

with the linear factor $1 + Mx$, the result is the operator

$$P_1(M + N, -N\Delta, e, \alpha, \beta), \quad (\Delta = \alpha^2 + \beta^2 - \alpha - \beta)$$

for the specific value

$$e = N\alpha\beta(1 - \alpha)(1 - \beta)$$

of the accessory parameter e. If e takes on another value, the operator is no longer equal to linear factor $\times$ hypergeometric.

4 The known Calabi–Yau operators of degree 2

Below we will give an overview of the 69 Calabi–Yau operators of degree two that are currently known to us. We also include some further remarkable operators that are not strictly Calabi–Yau in the sense of [4]. Of course, this is just a small portion of the list of the still growing list Calabi–Yau operators that was started in [3] and now contains more than 500 members. An update of this list in preparation [2].

For operators of degree two, there are cases with three and with four singular points. Most of the known operators can be related to hypergeometric operators or operators that are *convolutions*, that is, can be obtained via Hadamard product of operators of lower order. Recall that the *Hadamard product* of two power series $\phi(x) := \sum_n a_n x^n$ and $\psi(x) := \sum_n b_n x^n$ is the series

$$\phi * \psi(x) = \sum_n a_n b_n x^n$$

and by a classical theorem, if ϕ and ψ satisfy a linear differential equations, then so does $\phi * \psi$. The Hadamard product corresponds to the multiplicative convolution of local systems, so are under very good control, see [24, 25].

Frobenius basis: For a fourth-order operator P with at MUM point at 0, there is a canonical basis of solutions y_0, y_1, y_2, y_3 to $Py = 0$ on a sufficiently small slit disc around the origin, called the *Frobenius basis*:

$$y_0(x) = f_0(x)$$

$$y_1(x) = \log(x) f_0(x) + f_1(x)$$

$$y_2(x) = \tfrac{1}{2}\log(x)^2 f_0(x) + \log(x) f_1(x) + f_2(x)$$

$$y_3(x) = \tfrac{1}{6}\log(t)^3 f_0(x) + \tfrac{1}{2}\log(x)^2 f_1(x) + \log(x) f_2(x) + f_3(x)$$

where $f_0(x) \in \mathbf{Q}[[x]]$, $f_i(x) \in x\mathbf{Q}[[x]]$, $i = 1, 2, 3$.

The Calabi–Yau operators from [4] are characterised by three integrality conditions:

(1) integrality of the *solution* y_0:

$$y_0(x) \in \mathbf{Z}[[x]].$$

(2) integrality of the *q-coordinate* $q(x)$:

$$q(x) := \exp(y_1(x)/y_0(x)) = x \exp(f_1(x)/f_0(x)) = x + \cdots \in \mathbf{Z}[[x]]$$

(3) integrality of the *instanton numbers* n_d.

These can be computed in several different ways. The second logarithmic derivative of y_2/y_0 expressed in the q-coordinate is called the *Yakawa-coupling* $K(q)$ of the operator P:

$$K(q) = \left(q\frac{d}{dq}\right)^2 \left(\frac{y_2}{y_0}\right)$$

Now expand $K(q)$ as a Lambert series

$$K(q) = 1 + \sum_{k=1}^{\infty} \frac{k^3 n_k q^k}{1-q^k}.$$

The numbers n_d are called the (normalised, $n_0 = 1$) *instanton numbers of* P and are required to be integral. It is also natural not to require strict integrality, but rather to allow small denominators in f_0, $q(x)$ and n_d to appear, that is, to require only N-integrality, for some denominator N.

4.1 Description of the operators

Many of the Calabi–Yau operators of degree 2 that we will describe involve special lower-order operators of Calabi–Yau type. These are introduced and discussed in "Appendix B".

4.1.1 Operators with three singular points

I. The 14 tilde operators There are 14 exponents $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with

$$\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4, \quad \alpha_1 + \alpha_4 = \alpha_2 + \alpha_4 = 1,$$

for which the hypergeometric operator, scaled by N,

$$\theta^4 - Nx(\theta+\alpha_1)(\theta+\alpha_2)(\theta+\alpha_3)(\theta+\alpha_4),$$

is a Calabi–Yau operator, [8]. Corresponding to these, there are also 14 hypergeometric *fifth-order* Calabi–Yau operator

$$\theta^5 - 4Nx(\theta+\alpha_1)(\theta+\alpha_2)(\theta+\frac{1}{2})(\theta+\alpha_3)(\theta+\alpha_4),$$

with Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1/4N & \infty \\ \hline 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \\ 0 & 3/2 & 1/2 \\ 0 & 2 & \alpha_3 \\ 0 & 3 & \alpha_4 \end{array}\right\}.$$

These operators have a *Yifan Yang pull-back* to 14 special fourth-order operators, called the *tilde*-operators $\widetilde{1}, \widetilde{2}, \ldots, \widetilde{14}$. These operators replace the more complicated *hat-operators* $\hat{i}, i = 1, 2, \ldots, 14$ that appeared in the list [3] into which they can be transformed. However, operator $\widetilde{3}$ is not 'new', as it can be reduced to operator 2.33.

II. The 16 operators of type $H * \mu(H)$

There are four special hypergeometric second-order operators called A, B, C, D. The Möbius transformation that interchanges the singularity with infinity leads to four *Möbius transformed hypergeometric operators*

$$\mu(A) := e, \quad \mu(B) = h, \quad \mu(C) = i, \quad \mu(D) = j,$$

where e, h, i, j refer to the names given in [4], see also "Appendix B".

Thus, we can form the 16 Hadamard product $A * \mu(B)$, etc., and obtain 16 fourth-order operators of degree two, with three singular points.

But there are two surprises: the operator $C * \mu(A)$ can be reduced to the hypergeometric operator AESZ#3 = 1.3 and the operator $C * \mu(B)$ has vanishing instanton numbers and in fact is the third symmetric power of the hypergeometric second-order operator

$$\theta^2 - 12\,x\,(12\,\theta+7)\,(12\,\theta+1)\,.$$

So, from this we obtain only 14 Calabi–Yau operators of degree two that are 'new'.

III. The four Hadamard products $I * \mu(H')$

There are also four special hypergeometric third-order operators A', B', C', D', analogous to second-order operators A, B, C, D. Interchanging the singularity with the point at infinity, we obtain third-order operators

$$\mu(A') = \beta, \quad \mu(B') = \iota, \quad \mu(C') = \theta, \quad \mu(D') = \kappa$$

of degree two, with three singular points. Taking Hadamard product with the central binomial coefficient produces four fourth-order operators of degree two. One of these,

$I * \beta$, can be transformed by a quadratic transformation to an operator $(I * \beta)^*$

$$\theta^4 - 16(4\theta+1)(4\theta+3)(8\theta^2+8\theta+3) + 4096x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7),$$

which in turn can be transformed to the hypergeometric operator AESZ #3. The operator $I*\theta$ can be transformed in 2.17, so only two of these give rise to 'new' Calabi–Yau operators, [9].

IV. Four sporadic operators

The operator $2.63 =$ #AESZ84 cannot be constructed in this way. Furthermore, there are three operators of Bogner-type that were obtained from transformations of the higher degree operators AESZ#245, AESZ#406 and AESZ #255. As these operators have unusual properties, they probably could better be understood in terms of their higher degree versions. We remark that the local monodromies around the two singularities $\neq 0$ are of finite order.

4.1.2 Operators with four singular points

I. The 24 *operators of type* $H * BZB$

There are six special second-order operators of degree two with four singular fibres and Riemann symbol of the form

$$\left\{\begin{array}{cccc} 0 & * & * & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right\}.$$

So, we are dealing here with special Heun operators. They appear in the work of Beukers [13] and Zagier [44] and are related to the rational elliptic surfaces described by Beauville [12], so we call them the BZB operators. They were denoted a, b, c, d, f, g in [4].

Taking the Hadamard product with the hypergeometric operators A, B, C, D produces 24 fourth-order operators of degree two, with four singular points.

II. The six Hadamard products $I * BZB'$

The BZB operators have also analogues as third-order operators that we will call the BZB'-operators, that were called $\alpha, \gamma, \delta, \epsilon, \zeta, \eta$ in [4]. Taking $I*$ with these operators produces six 'new' fourth-order operators of degree two with four singularities.

III. Six Sporadic operators with four singularities

III.a. Three operators $I*$ *sporadic third order*: There are three sporadic *third-order* operators of degree two, described in "Appendix B". These lead to fourth-order operators of degree two by taking $I*$ with them.

III.b: Three original operators: There are three operators that apparently cannot be constructed starting from simpler operators of lower order; we call them original operators.

The count

So, in total we have described

$$(14-1)+(16-2)+(4-2)+4=33$$

Calabi–Yau operators of degree two with three singular points and

$$24+6+3+3=36$$

with four singular points, making up a total of 69. Because these all have different instanton numbers, these are essentially different and there are no algebraic transformations mapping one to any of the other operators.

Miscellaneous operators

There are two further types of operators of degree two that deserve to be mentioned at this place, as they challenge the precise definition of Calabi–Yau operator, as formulated in [4].

Reducible operators: A large scale Zagier-type search for degree two operators with integral solutions has been performed by PAVEL METELITSYN. Apart from the operators equivalent to the ones mentioned above, there are some interesting reducible fourth-order operators. With regard to the q-coordinate and instanton numbers these examples behave very much in the same way as the irreducible ones. An example is the operator

$$\begin{aligned}&\theta^4-12x\,(6\theta+1)\,(6\theta+5)\left(2\theta^2+2\theta+1\right)\\&\quad+144x^2\,(6\theta+1)\,(6\theta+5)\,(6\theta+7)\,(6\theta+11)\,.\end{aligned}$$

One finds:

$$\begin{aligned}\phi(x)&=1+60x+13860x^2+4084080x^3+1338557220x^4\\&\quad+465817912560x^5+\cdots\\q(x)&=x+312x^2+107604x^3+39073568x^4+14645965026x^5+\cdots\\n_1&=-192,\quad n_2=4182,\quad n_3=-229568,\quad n_4=19136058,\\n_5&=-2006581440.\end{aligned}$$

But note that the above series $\phi(x)$ is just the solution of the second-order operator D, and the above operator is just the compositional square of D. (Doing this with the hypergeometric second-order operators A, B, C produces periodic instanton numbers.)

A strange operator: BOGNER [17] also found an example of an operator of degree two with an integer solution, integral q-coordinate, integral $K(q)$-series, but for which the instanton numbers are badly non-integral. We start from the third-order operator

$$L:=\theta^3-4x\,(2\theta+1)\left(5\theta^2+5\theta+2\right)+48x^2\,(3\theta+2)\,(3\theta+4)\,(\theta+1)$$

with integral solution

$$1 + 8x + 96x^2 + 1280x^3 + 17440x^4 + 231168x^5 + \cdots$$

The operator $I * L$ is the fourth-order operator

$$\theta^4 - 8x(2\theta+1)^2(5\theta^2+5\theta+2) + 192x^2(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3).$$

We find the following nice integral series:

$$\begin{aligned} y_0(x) &= 1 + 16x + 576x^2 + 25600x^3 + 1220800x^4 + 58254336x^5 + \cdots \\ q(x) &:= x + 40x^2 + 1984x^3 + 106496x^4 + 5863168x^5 + \cdots \\ K(q) &:= 1 + 8q^2 - 5632q^3 - 456064q^4 - 17708032q^5 + \cdots \end{aligned}$$

However, if we expand $K(q)$ as a Lambert series, we do not get integral instanton numbers:

$$\begin{aligned} &n_1 = 8, n_2 = -1, n_3 = -\frac{1880}{9}, n_4 = -7126, n_5 = -\frac{3541608}{25}, \ldots, \\ &n_{17} = \frac{2432475693294458880448632}{289}, \ldots \end{aligned}$$

Experimentally, the denominator p^2 appears in n_p for

$$p = 3,\ \ 5,\ \ 7,\ \ 11,\ \ 13,\ \ 17,\ \ 19, \ldots$$

This is rather puzzling. Conjecturally, already the fact that $\phi(x)$ is an integral series implies that the operator is a factor of a Picard–Fuchs equation of a family of varieties defined over $\mathbf{Q}$ and indeed our operator is of geometrical origin. It was conjectured in [4] that the integrality of the q-series implies the integrality of the instanton numbers, which in this example is not the case. The integrality of solution and mirror map clearly indicates that we have a rank four Calabi–Yau motive and one would expect the general arguments for the integrality of [43] to be applicable, but apparently they are not. There might exist a different scaling of the coordinate that repairs this defect, but up to now we have been unable to find it.

Acknowledgements This work for this paper was done in 2016 and we had hoped to find many new degree two operators. This did not happen and we never found the time to write the paper. We intended it to be a possible reference for this class of operators. The untimely death of Gert left the paper in a unfinished state and it is only now I found the time to complete it. We thank M. Bogner, J. Hofmann and K. Samol for help with computations, done years ago, and W. Zudilin and O. Gorodetsky for useful comments, corrections and showing interest in this paper.

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Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A

The following table lists the operators as they appear in the electronic database accessible at https://cydb.mathematik.uni-mainz.de/. (These are ordered by degree, so that the nth operator of degree d gets the code $d.n$ in that list.) The second column lists the AESZ numbers as appearing in [3]. The last three columns list the first three instanton numbers (normalised, $n_0 = 1$). As these do not change under transformations of the coordinate, the pair of numbers $|n_1|$, $|n_3|$ provide a simple 'fingerprint/superseeker' to identify an operator. Operators that are not 'new' are listed in their natural place, with xx added in their number. In "Appendix C" one finds some monodromy and modular form information for these operators.

Number	AESZ	Source	$\lvert\Sigma\rvert$	n_1	n_2	n_3
2.1	#45	$A * a$	4	12	163	3204
2.2	#15	$B * a$	4	21	480	15894
2.3	#68	$C * a$	4	52	2814	220220
2.4	#62	$D * a$	4	372	136182	71562236
2.5	#25	$A * b$	4	20	277	8220
2.6	#24	$B * b$	4	36	837	41421
2.7	#51	$C * b$	4	92	5052	585396
2.8	#63	$D * b$	4	684	253314	195638820
2.9	#58	$A * c$	4	16	142	11056/3
2.10	#70	$B * c$	4	27	432	18089
2.11	#69	$C * c$	4	64	2616	246848
2.12	#64	$D * c$	4	432	130842	78259376
2.13	#36	$A * d$	4	16	42	1232
2.14	#48	$B * d$	4	24	291/2	5832
2.15	#38	$C * d$	4	48	998	73328
2.16	#65	$D * d$	4	240	57102	19105840
2.17	#111, $\sim$ #40	$A * e$	3	32	-96	1440
2.18	#110	$B * e$	3	36	-144	8076
2.xx19	$\sim$ #3	$C * e$	3	32	608	26016
2.19	#112	$D * e$	3	-288	162504	-96055968
2.20	#133	$A * f$	4	12	-42	$-3284/3$
2.21	#134	$B * f$	4	18	$-207/2$	-5177
2.22	#135	$C * f$	4	36	-477	$-206716/3$
2.23	#136	$D * f$	4	180	-15615	-21847076
2.24	#137	$A * g$	4	20	2	1684/3
2.25	#138	$B * g$	4	27	189/4	2618
2.26	#139	$C * g$	4	44	607	22500
2.27	#140	$D * g$	4	108	54135	-4945756
2.28	#141	$A * h$	3	48	-438	2864
2.29	#142	$B * h$	3	45	$-3465/4$	27735
2.xx30	Sym^3	$C * h$	3	0	0	0
2.30	#143	$D * h$	3	-1008	499086	-607849200

Appendix B

In this appendix we collect some information on operators of lower order that are relevant to the construction of the order four Calabi–Yau operators of degree two.

Number	AESZ	Source		n_1	n_2	n_3
2.31	7^{**}	$A * i$	3	96	−3560	−12064
2.32		$B * i$	3	60	−7635	307860
2.33	6^{*}	$C * i$	3	−160	−6920	−539680
2.34		$D * i$	3	−3936	3550992	−10892932064
2.35	9^{**}, ∼ #67	$A * j$	3	480	−226968	−16034720
2.36		$B * j$	3	−36	−486279	128217204
2.37		$C * j$	3	−2592	−307800	81451104
2.38	#61	$D * j$	3	−41184	251271360	−5124430612320
2.39	$\sim \widehat{1}$	$\widetilde{1}$	3	−2450	−1825075/2	−623291900
2.40	$\sim \widehat{2}$	$\widetilde{2}$	3	−791200	−41486886600	−4288711075194400
2.*xx*41	$\sim \widehat{3}$	$\widetilde{3}$	3	−160	−6920	−539680
2.41	$\sim \widehat{4}$	$\widetilde{4}$	3	−522	−105291/2	−9879192
2.42	$\sim \widehat{5}$	$\widetilde{5}$	3	−288	−19260	−2339616
2.43	$\sim \widehat{6}$	$\widetilde{6}$	3	−736	−104512	−26911072
2.44	$\sim \widehat{7}$	$\widetilde{7}$	3	−57760	−354010600	−3869123234080
2.45	$\sim \widehat{8}$	$\widetilde{8}$	3	−10080	−11338740	−24400330080
2.46	$\sim \widehat{9}$	$\widetilde{9}$	3	−2710944	−717640301160	−302270555492914464
2.47	$\sim \widehat{10}$	$\widetilde{10}$	3	−3488	−1406056	−1142687008
2.48	$\sim \widehat{11}$	$\widetilde{11}$	3	−1344	−278040	−109320512
2.49	$\sim \widehat{12}$	$\widetilde{12}$	3	−264006	−52511160	−23039803408 0
2.50	$\sim \widehat{13}$	$\widetilde{13}$	3	−201888	−1567499400	−40177844666400
2.51	$\sim \widehat{14}$	$\widetilde{14}$	3	−5472	−4476528	−6444589536
2.52	#16	$I * \alpha$	4	4	20	644/3
2.53	#29	$I * \gamma$	4	14	303/2	10424/3
2.54	#41	$I * \delta$	4	2	−7	−104
2.55	#42	$I * \epsilon$	4	8	63	1000
2.56	#185	$I * \zeta$	4	6	93/2	608
2.57	#184	$I * \eta$	4	2	4	−8
2.*xx*58	∼ #3	$I * \beta$	3	0	4	0
2.*yy*58	∼ 2.*xx*58	$(I * \beta)^{*}$	3	32	608	26016
2.58	4∗	$I * \iota$	3	−6	−6	−104
2.*xx*59	∼ 2.17	$I * \theta$	3	−32	−88	−1440
2.59	13^{*}, ∼ 47	$I * \kappa$	3	−384	−1356	−164736
2.60	#18	$I *$ Spor1	4	4	39	364
2.61	#26	$I *$ Spor2	4	10	191/2	1724
2.62	#28	Original	4	5	28	312
2.63	#84	Original	3	−4	−11	−44
2.64	#182	Original	4	1	7/4	7
2.65	#183	$I *$ Spor3	4	4	7	556/9
2.66	Reducible!	$D \circ D$	3	−192	4182	−229568
2.67	∼ #245	Bogner1	3	6	−69/2	170
2.68	#406	Bogner2	3	−12	−186	−1668
2.69	#205	Original	4	1	7/4	5
2.70	∼ #255	Bogner3	3	20	290	28820/3

First-order operator

There is a single *first-order operator* that we call I:

$$\theta - 4x(\theta + 1/2).$$

The series

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

is the unique holomorphic solution of I and in fact coincides for $\|x\| \leq 1/4$ with the algebraic function

$$\frac{1}{\sqrt{1-4x}}.$$

Second-order operators

The four hypergeometric cases

There are four very remarkable hypergeometric second-order operators of the form

$$\theta^2 - Nx(\theta+\alpha)(\theta+\beta), \quad \alpha+\beta = 1, \quad \alpha = 1/2,\ 1/3,\ 1/4,\ 1/6$$

with Riemann symbol of the form

$$\left\{ \begin{array}{ccc} 0 & 1/N & \infty \\ \hline 0 & 0 & \alpha \\ 0 & 0 & \beta \end{array} \right\}.$$

These four operators appear in many different areas of mathematics, like the *alternative theories of Ramanujan* and are described at many places, see, for example, [22].

Name	Operator	$a_0, a_1, a_2, a_3, \ldots$	a_n
A	$\theta^2 - 16x(\theta+\frac{1}{2})^2$	1, 4, 36, 400, ...	$\frac{(2n!)^2}{n!^4}$
B	$\theta^2 - 27x(\theta+\frac{1}{3})(\theta+\frac{2}{3})$	1, 6, 90, 1680, ...	$\frac{(3n)!}{n!^3}$
C	$\theta^2 - 64x(\theta+\frac{1}{4})(\theta+\frac{3}{4})$	1, 12, 420, 18480, ...	$\frac{(4n)!}{(2n!)(n!)^2}$
D	$\theta^2 - 432x(\theta+\frac{1}{6})(\theta+\frac{5}{6})$	1, 60, 13860, 4084080, ...	$\frac{(6n)!}{(3n!)(2n!)(n!)}$

The above list of operators is in close correspondence with the six extremal elliptic surfaces with three singular fibres and of modular origin, [33, 37].

Here MW stands for the Mordell–Weil group of sections of the fibration. The matrices A, B and $(BA)^{-1}$ are monodromy matrices for loops around 0, $c \neq 0, \infty$ and ∞ respectively. In the monodromy column, the group generated by A and B is identified. The column H/MP denotes the notation used in [33, 37]. The last column denotes the type of the Picard–Fuchs operator. We see that there are three surfaces that

Fibres	MW	A	B	$(BA)^{-1}$	Monodromy	H/MP	
$I_2\ I_2\ I_2^*$	$(\mathbf{Z}/2)^2$	$\begin{pmatrix}1&2\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\-2&1\end{pmatrix}$	$\begin{pmatrix}1&-2\\2&-3\end{pmatrix}$	$\Gamma(2)$	$12/X_{222}$	A
$I_1\ I_1\ I_4^*$	$\mathbf{Z}/2$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&1\\-4&3\end{pmatrix}$	$\begin{pmatrix}-1&0\\4&-1\end{pmatrix}$	$\overline{\Gamma_0(4)}$	$11/X_{114}$	A
$I_1\ I_4\ I_1^*$	$\mathbf{Z}/4$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\-4&1\end{pmatrix}$	$\begin{pmatrix}-3&-1\\4&1\end{pmatrix}$	$\Gamma_0(4)$	$11/X_{141}$	A
$I_1\ I_3\ IV^*$	$\mathbf{Z}/3$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\-3&1\end{pmatrix}$	$\begin{pmatrix}1&-1\\3&-2\end{pmatrix}$	$\Gamma_0(3)$	$10/X_{134}$	B
$I_1\ I_2\ III^*$	$\mathbf{Z}/2$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\-2&1\end{pmatrix}$	$\begin{pmatrix}1&-1\\2&-1\end{pmatrix}$	$\Gamma_0(2)$	$9/X_{123}$	C
$I_1\ I_1\ II^*$	$\mathbf{Z}/1$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\-1&1\end{pmatrix}$	$\begin{pmatrix}1&-1\\1&0\end{pmatrix}$	$\Gamma_0(1)$	$8/X_{112}$	D

belong to operator A, but the surface $I_1\ \ I_4\ \ I_1^*$ shows it is a member of a very regular series of four surfaces, which we will say to belong to the operators A, B, C, D.

The surfaces belonging to the operators A, B, C, D have a clear relation to the geometry of the *simple elliptic singularities*. The fibres over ∞ correspond to configurations of rational curves intersecting in the pattern of the affine Dynkin diagrams. The corresponding simple elliptic singularities have a single modulus. These families of elliptic curves lead to elliptic surfaces that cover the corresponding extremal elliptic surfaces.

Case	Dynkin	Equation	Cover
A	$\widetilde{D_5}$	$x^2+y^2=suv,\ \ u^2+v^2=sxy$	$t=s^2$
B	$\widetilde{E_6}$	$x^3+y^3+z^3=sxyz$	$t=s^3$
C	$\widetilde{E_7}$	$x^2+y^4+z^4=sxyz$	$t=s^4$
D	$\widetilde{E_8}$	$x^2+y^3+z^6=sxyz$	$t=s^6$

The A case is slightly different, as the corresponding singularity is not a hypersurface, but a complete intersection of two quadrics. The last three cases correspond to the *Euclidean triples* (p,q,r) of integers ≥ 2 with the property that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1.$$

The transformed hypergeometric cases

The Möbius transformation

$$x \mapsto -\frac{x}{1-Nx}$$

preserves 0 and interchanges the points $1/N$ and ∞ of the Riemann sphere. If $y(x)$ solves one of the hypergeometric equations A, B, C, D, then

$$Y(x) := \frac{1}{1-Nx} y\left(-\frac{x}{1-Nx}\right)$$

satisfies a second-order equation of the form

$$\theta^2 - x(a\theta^2 + a\theta + b) + cx^2(\theta+1)^2$$

that we call *transformed hypergeometric equation* that we denote by $\mu(A)$, $\mu(B)$, $\mu(C)$, $\mu(D)$. These operators have Riemann symbol of the form

$$\left\{\begin{array}{cccc} 0 & 1/N & \infty \\ \hline 0 & -\alpha & 1 \\ 0 & -\beta & 1 \end{array}\right\}$$

Name	a	b	c	$a_0, a_1, a_2, \ldots$	a_n
$\mu(A) = e$	32	12	16^2	$1, 12, 164, \ldots$	$16^n \sum_{k=0}^n (-1)^k \binom{-1/2}{k}^2 \binom{-1/2}{n-k}$
$\mu(B) = h$	54	21	27^2	$1, 21, 495, \ldots$	$27^n \sum_{k=0}^n (-1)^k \binom{-1/3}{k}^2 \binom{-2/3}{n-k}$
$\mu(C) = i$	128	52	64^2	$1, 52, 2980, \ldots$	$64^n \sum_{k=0}^n (-1)^k \binom{-1/4}{k}^2 \binom{-3/4}{n-k}$
$\mu(D) = j$	864	372	432^2	$1, 372, 148644, \ldots$	$432^n \sum_{k=0}^n (-1)^k \binom{-1/6}{k}^2 \binom{-5/6}{n-k}$

The six Beukers–Zagier–Beauville operators

There are six special second-order operators of degree two with four singular points of the form

$$\theta^2 - t(a\theta^2 + a\theta + b) + c(\theta+1)^2$$

and hence have a Riemann symbol of the form

$$\left\{\begin{array}{cccc} 0 & * & * & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right\}$$

AZ-name	BZ-name	a	b	c	$a_0, a_1, a_2, a_3, \ldots$	a_n
a	A	7	2	-8	$1, 2, 10, 56, \ldots$	$\sum_{k=0}^{n} \binom{n}{k}^3$
c	C	10	3	9	$1, 3, 15, 93, \ldots$	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$
g	F	17	6	72	$1, 6, 42, 312, \ldots$	$\sum_{j,k=0}^{n} (-1)^j 8^{n-j} \binom{n}{j} \binom{j}{k}^3$
d	E	12	4	32	$1, 4, 20, 112, \ldots$	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k}$
f	B	9	3	27	$1, 3, 9, 21, \ldots$	$\sum_{k=0} n(-1)^k 3^{n-2k} \binom{n}{3k} \frac{(3k)!}{k!^3}$
b	D	11	3	-1	$1, 3, 19, 147, \ldots$	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$

We refer to [30] for a representation of these sequences as constant term of a Laurent polynomial.

Unfortunately, there is no natural naming for these operators. In the first column of the table we used the "AZ-names" used in [4]. In [14, 44] the same operators are named $A, B, \ldots, F$ (not to be confused with the hypergeometric operators A, B, C, D mentioned above!); these names appear in the second column as "BZ-names". These operators also appear as Picard–Fuchs operators for rational elliptic surfaces, namely the six semi-stable families of elliptic curves with four exceptional fibres that have been studied by Beauville [12] and described in detail in [42]. The modular level gives a more or less natural way order the operators.

Fibres	j	Pencil	Group	Operator
$I_3\ I_3\ I_3\ I_3$	0	$x^3 + y^3 + z^3 + txzy$	$\Gamma(3)$	f
$I_4\ I_4\ I_2\ I_2$	1728	$x(x^2 + z^2 + 2yz) + tz(x^2 - y^2)$	$\Gamma_1(4) \cap \Gamma(2)$	d
$I_5\ I_5\ I_1\ I_1$	$2^{14}31^3/5^3$	$x(x - z)(y - z) + tzy(x - y)$	$\Gamma_1(5)$	b
$I_6\ I_3\ I_2\ I_1$	$2^2.73^3/3^4$	$(x + y)(y + z)(z + x) + txyz$	$\Gamma_1(6)$	a, c, g
$I_8\ I_2\ I_1\ I_1$	1728	$(x + y)(xy - z) + txyz$	$\Gamma_0(8) \cap \Gamma_1(4)$	d
$I_9\ I_1\ I_1\ I_1$	0	$x^2y + y^2z + z^2x + txyz$	$\Gamma_0(9) \cap \Gamma_1(3)$	f

The coefficients of the holomorphic solution of the operator b are the Apéry numbers for the irrationality of π^2. It belongs to the elliptic surface with fibres I_5, I_5, I_1, I_1 over the modular curve $X_1(5)$, as was first noticed by Beukers [13].

The operators a, c, g differ by a Möbius transformations. The base of this family can be identified with the modular curve for the congruence subgroup $\Gamma_1(6)$. Over that curve we have the semi-stable elliptic surface with fibres of type I_6, I_3, I_2, I_1. The operator d belongs to the two isogenous elliptic surfaces with fibres I_8, I_2, I_1, I_1 and I_4, I_4, I_2, I_2. Operator f belongs to the two isogenous elliptic surfaces with fibres I_9, I_1, I_1, I_1 and I_3, I_3, I_3, I_3 which is nothing but the Hesse pencil.

Third-order operators

Hypergeometric operators

Closely related to the operators A, B, C, D, there are also four very remarkable hypergeometric third-order operators

$$A' = I * A, \quad B' = I * B, \quad C' = I * C, \quad D' = I * D$$

of the form

$$\theta^3 - Nx(\theta + \alpha)(\theta + 1/2)(\theta + \beta), \quad \alpha + \beta = 1, \quad \alpha = 1/2,\ 1/3,\ 1/4,\ 1/6$$

with Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1/N & \infty \\ \hline 0 & 0 & \alpha \\ 0 & 1/2 & 1/2 \\ 0 & 0 & \beta \end{array}\right\}.$$

Also important are the *symmetric squares* of the operators A, B, C, D. These are of the form

$$\theta^3 - x(2\theta + 1)(a\theta^2 + a\theta + b) + a^2x^2(\theta + \alpha)(\theta + 1)(\theta + \beta), \quad \alpha + \beta = 2,$$

and have Riemann symbol of the form

$$\left\{\begin{array}{ccc} 0 & 1/a & \infty \\ \hline 0 & 0 & 1 - \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 1 + \alpha \end{array}\right\}.$$

Name	a	b	α
$\mathrm{sym}^2(A)$	16	8	$1 - 2\frac{1}{2} = 0$
$\mathrm{sym}^2(B)$	27	12	$1 - 2\frac{1}{3} = 1/3$
$\mathrm{sym}^2(C)$	64	24	$1 - 2\frac{1}{4} = 1/2$
$\mathrm{sym}^2(D)$	432	120	$1 - 2\frac{1}{6} = 2/3$

The four transformed symmetric squares

More important for us are the *transformed symmetric squares* of the operators A, B, C, D. These are of the form

$$\theta^3 - x(2\theta+1)(a\theta^2+a\theta+b) + a^2x^2(\theta+1)^3$$

and have Riemann symbol of the form

$$\left\{\begin{array}{cccc} 0 & 1/a & \infty \\ \hline 0 & -\alpha & 1 \\ 0 & 0 & 1 \\ 0 & \alpha & 1 \end{array}\right\}.$$

Name	a	b	α	$a_0, a_1, a_2, \ldots$	a_n
$\mu(\mathrm{sym}^2(A)) = \beta$	16	8	1	1, 8, 88, 1088,	$16^n \sum_k \binom{-1/2}{k}^2 \binom{-1/2}{n-k}^2$
$\mu(\mathrm{sym}^2(B)) = \iota$	27	15	1/3	1, 15, 297, 6495	$27^n \sum_k \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2$
$\mu(\mathrm{sym}^2(C)) = \theta$	64	40	1/2	1, 40, 2008, 109120	$64^n \sum_k \binom{-1/4}{k}^2 \binom{-4/3}{n-k}^2$
$\mu(\mathrm{sym}^2(D)) = \kappa$	432	312	2/3	1, 312, 114264, 44196288	$432^m \sum_k \binom{-1/6}{k}^2 \binom{-5/6}{n-k}^2$

The names $\beta, \iota, \theta, \kappa$ stem from [4] and also appear in [9].

The six BZB′ operators

Corresponding to the six BZB operators, there are also six special third-order operators of degree two with four singular points of the form

$$\theta^3 - x(2\theta+1)(a\theta^2+a\theta+b) + cx^2(\theta+1)^3,$$

and hence have a Riemann symbol of the form

$$\left\{\begin{array}{cccc} 0 & * & * & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right\}.$$

The closed form for the coefficients a_n appeared in [9], for a Laurent representation of these sequences we refer to [30].

The correspondence between the BZB operators and BZB' operators is determined by the following table:

a	c	g	d	f	b
δ	α	γ	ϵ	ζ	η

Name	a	b	c	$a_0, a_1, a_2, a_3, \ldots$	a_n
δ	7	3	81	$1, 3, 9, 3, -279$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{k} \frac{(3k)!}{k!^3}$
α	10	4	64	$1, 4, 28, 256, \ldots$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}$
γ	17	5	1	$1, 5, 73, 1445, \ldots$	$\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$
ϵ	12	4	16	$1, 4, 40, 544, \ldots$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$
η	11	5	125	$1, 5, 35, 275, \ldots$	$\sum_{k=0}^{\lfloor n/5 \rfloor} \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$
ζ	9	3	-27	$1, 3, 27, 309, \ldots$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$

There is a general formula expressing the relation between the BZB and BZB' operators:

$$Y_0(-x/Q(x)) = Q(x) y_0(x)^2$$

which shows that the third-order operators are twisted versions of the symmetric squares of the corresponding second-order operators. The factor $Q(x)$ is the discriminant of the second-order operator with y_0 as solution.

y_0	Y_0	Q
a	δ	$(1+x)(1-8x)$
c	α	$(1-x)(1-9x)$
g	γ	$(1-8x)(1-9x)$
d	ϵ	$(1-4x)(1-8x)$
f	ζ	$1-9x+27x^2$
b	η	$1-11x-x^2$

The three sporadic third-order operators

We know of three very special third-order operators that seem to be not directly related to the analogous of the second-order operators. These are the following ones:

Sporadic 1:

$$\theta^3 - 2x(2\theta+1)(3\theta^2+3\theta+1) - 4x^2(4\theta+3)(\theta+1)(4\theta+5),$$
$$\phi(x) = 1 + 2x + 18x^2 + 164x^3 + 1810x^4 + 21252x^5 + 263844x^6 + \cdots$$
$$\left\{ \begin{array}{cccc} 0 & 1/16 & -1/4 & \infty \\ \hline 0 & 0 & 0 & 3/4 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 1 & 5/4 \end{array} \right\},$$
$$a_n = \sum_{k=0}^{n} \binom{n}{k}^4.$$

This example has an A-incarnation as K3 surface that is the intersection of four hyperplane sections of type $(1, 1)$ in $\mathbf{P}^3 \times \mathbf{P}^3$. If we take $I*$, we obtain a fourth-order operator

2.60 with A-incarnation a CY threefold that appears as intersection of two (1, 1) and one (2, 2) hypersurface in $\mathbf{P}^3 \times \mathbf{P}^3$ [10].

Sporadic 2:

$$\theta^3 - x(2\theta+1)(13\theta^2+13\theta+4) - 3x^2(3\theta+2)(\theta+1)(3\theta+4),$$
$$\phi(x) = 1 + 4x + 48x^2 + 760x^3 + 13840x^4 + 273504x^5 + 5703096x^6 + \cdots$$

$$\left\{\begin{array}{cccc} 0 & 1/27 & -1 & \infty \\ \hline 0 & 0 & 0 & 2/3 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 1 & 4/3 \end{array}\right\},$$

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \binom{2k}{n}.$$

This example has an A-incarnation as K3 surface that is the intersection of six hyperplane sections of the Grassmanian $G(2, 6)$ in its Plücker embedding. If we take $I*$, we obtain a fourth-order operator 2.61 with A-incarnation a CY threefold that appears as intersection of four linear and a quadratic hypersurface in Grassmanian $G(2, 6)$ in its Plücker embedding [11].

Sporadic 3:

$$\theta^3 - 2x(2\theta+1)(7\theta^2+7\theta+3) + 12x^2(4\theta+3)(\theta+1)(4\theta+5),$$
$$\phi(x) = 1 + 6x + 54x^2 + 564x^3 + 6390x^4 + 76356x^5 + 948276x^6 + \cdots$$

$$\left\{\begin{array}{cccc} 0 & 1/16 & 1/12 & \infty \\ \hline 0 & 0 & 0 & 3/4 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 1 & 5/4 \end{array}\right\},$$

$$a_n = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\left(\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right).$$

These sporadic operators and sequences were also found by Cooper [21], where they are called s_{10}, s_7 and s_{18} and where the explicit description for a_n in the last case also can be found. We refer to [30] for representations as constant term sequence of a Laurent polynomial.

Miscellaneous third-order operators

There is a remarkable infinite series of third-order operators with an integral solution.

Theorem 3 *The differential operator*

$$P := \theta^3 - t(2\theta+1)(\theta^2+\theta-1/2d^2+1/2k^2+1/2) + x^2(\theta+1-d)(\theta+1)(\theta+1+d)$$

has

$$(1-x)^k \cdot_2 F_1\left(\frac{1-d+k}{2}, \frac{1+d+k}{2}, 1, x\right)^2$$

as solution.

Proof The Riemann symbol of the operator P is

$$\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \hline 0 & -k & (1-d) & \\ 0 & 0 & 1 & \\ .0 & k & 1+d & \end{array}\right\}$$

The operator is the symmetric square of a second-order operator with Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & -k/2 & (1-d)/2 \\ 0 & k/2 & (1+d)/2 \end{array}\right\}.$$

Multiplication by a factor $(1-x)^{k/2}$ gives a hypergeometric operator with Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & (1-d+k)/2 \\ 0 & k & (1+d+k)/2 \end{array}\right\}.$$

If we compare with the Riemann symbol of the standard hypergeometric operator for ${}_2F_1(a, b, c; x)$

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a \\ 1-c & c-a-b & b \end{array}\right\},$$

we read off $a = (1-d+k)/2, b = (1+d+k)/2, c = 1$; hence, the solution of our original operator is the square of

$$(1-x)^{k/2} \cdot_2 F_1((1-d+k)/2, (1+d+k)/2, 1; x).$$

□

Corollary *If k and d are rational numbers, then the only primes appearing in denominator of k and d appear in the denominator of the coefficients of the holomorphic solution of P. After an appropriate rescaling, the coefficients become integral.*

Appendix C

In this Appendix we list all essentially distinct Calabi–Yau operators of order four and degree two that are known to us. We also present some further information. To present the monodromy, it is sometimes convenient to use $u_i(x) = y_i(x)/(2\pi i)^3$ which we call the *scaled Frobenius basis*. The monodromy transformation around 0 in this basis is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix}.$$

In the cases with four singular points, the operator belongs to the main component and the Riemann symbol of the operator has the form

$$\left\{\begin{array}{cccc} 0 & c_1 & c_2 & \infty \\ \hline 0 & 0 & 0 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 1 & 1 & 2-\alpha \\ 0 & 2 & 2 & 2-\beta \end{array}\right\}$$

where c_1, c_2 are solutions to the equation

$$\Delta(x) := 1 + ax + fx^2 = 0$$

We will always assume that the exponents α, β are between 0 and 1. For each operator we give the monodromy transformations around the other singular points the with respect to this basis. This information was computed by Hofmann [31]. A very common transformation is the *symplectic reflection* $v \mapsto v - < v, w > w$ in a vector $w = (a, b, c, d)$ represented by the matrix

$$I - \frac{1}{a}\begin{pmatrix} -ad & ac & -ab & a^2 \\ -bd & bc & -b^2 & ab \\ -cd & c^2 & -bc & ac \\ -d^2 & cd & -bd & ad \end{pmatrix}.$$

Such a transformation is found at *conifold points*, where the exponents are $0\ \ 1\ \ 1\ \ 2$. For the conifold nearest to the origin, the vector is of the form

$$(a, 0, c, d),$$

and in that case the *characteristic numbers* $h^3, c_2h, c_3 = \chi$ of the operator are determined by the reflection vector via

$$a = h^3, \quad c = \frac{c_2 h}{24}, \quad d = c_3\lambda,$$

where

$$\lambda = \frac{\zeta(3)}{(2\pi i)^3}.$$

These numbers are formally attached to the differential operator, but have the interpretation as characteristic numbers of the mirror manifold, see [41].

Modular form information. For conifold points appearing at rational values of the parameters, one can determine the coefficients of a weight four modular form as described in the thesis of SAMOL [35, 36]. We include here the name of modular form from $S_4(\Gamma_0(N)$ as it appears in the list of MEYER [32]. For example, (6/1) denotes the first (and only) weight four new forms of level 6, which is the η-product

$$f = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + \cdots \in S_4(\Gamma_0(6))$$

For critical points at real quadratic irrationalities, one expects the appearance of Hilbert modular forms; at imaginary quadratic irrationalities, one would expect Bianchi modular forms. Another phenomenon that may occur is that of a so-called *K-point*, see [41]. Here two single logarithms are appearing (two size two Jordan blocks), and one expects the appearance of a modular form from $S_3(\Gamma_0(N))$.

Below we list the 90-degree rotated *extended Riemann symbols*; each row starts with the singular value, followed by the four exponents at that point, then monodromy information and finally modular form information. At the MUM point the monodromy is always standard, and we give the first few instanton numbers as a substitute for modular form information. If Hilbert modular, Bianchi or weight three modular forms are expected, we indicate this with h, b or k. An m is written for those cases where ordinary modular form from $S_4(\Gamma_0(N)$ is expected, but not yet determined. In case the local monodromy is of finite order we put $-$.

The operators with three singular points which are on the main components can all be written in the form

$$\begin{aligned}&\theta^4 + fx(-2\theta^4 - 4\theta^3 + (\gamma(1-\gamma) + \alpha^2 + \beta^2 - 4)\theta^2\\ &\quad +(\gamma(1-\gamma) + \alpha^2 + \beta^2 - 2)\theta + e +\\ &\quad +f^2(\theta + 1 - \alpha)(\theta + 1 - \beta)(\theta + 1 + \beta)(\theta + 1 + \alpha)\end{aligned}$$

which has Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1/f & \infty \\ \hline 0 & 0 & 1-\alpha \\ 0 & \gamma & 1-\beta \\ 0 & 1-\gamma & 1+\beta \\ 0 & 1 & 1+\alpha \end{array}\right\}$$

There is a single accessory parameter e.

2.1: $A * a = \#45 \qquad \chi = -120, \quad c_2H = 72, \quad H^3 = 24, \quad \dim|H| = 10.$

$$\theta^4 - 4x(2\theta+1)^2(7\theta^2+7\theta+2) - 128x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{\begin{array}{c|c||c||c} 0 & 0\ 0\ 0\ 0 & (24,\ 0,\ 3,\ -120\lambda) & 12,\ 163,\ 3204 \\ \frac{1}{128} & 0\ 1\ 1\ 2 & & (64/5) \\ -\frac{1}{16} & 0\ 1\ 1\ 2 & (48,\ -24,\ 10,\ -3-240\lambda) & (8/1) \\ \infty & \frac{1}{2}\ \frac{1}{2}\ \frac{3}{2}\ \frac{3}{2} & & k \end{array}\right\}$$

2.2: $B * a = \#15 \qquad \chi = -162, \quad c_2H = 72, \quad H^3 = 18, \quad \dim|H| = 9.$

$$\theta^4 - 3x(3\theta+1)(3\theta+2)(7\theta^2+7\theta+2) - 72x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{\begin{array}{c|c||c||c} 0 & 0\ 0\ 0\ 0 & (18,\ 0,\ 3,\ -162\lambda) & 21,\ 480,\ 15894 \\ \frac{1}{216} & 0\ 1\ 1\ 2 & & (54/2) \\ -\frac{1}{27} & 0\ 1\ 1\ 2 & (36,\ -18,\ 9,\ -3-324\lambda) & (27/2) \\ \infty & \frac{1}{3}\ \frac{2}{3}\ \frac{4}{3}\ \frac{5}{3} & & - \end{array}\right\}$$

2.3: $C * a = \#68 \qquad \chi = -228, \quad c_2H = 72, \quad H^3 = 12, \quad \dim|H| = 8.$

$$\theta^4 - 4x(4\theta+1)(4\theta+3)(7\theta^2+7\theta+2) - 128x(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$

$$\left\{\begin{array}{c|c||c||c} 0 & 0\ 0\ 0\ 0 & (12,\ 0,\ 3,\ -228\lambda) & 52,\ 2814,\ 220220 \\ \frac{1}{512} & 0\ 1\ 1\ 2 & & (256/3) \\ -\frac{1}{64} & 0\ 1\ 1\ 2 & (24,\ -12,\ 8,\ -3-456\lambda) & (32/3) \\ \infty & \frac{1}{4}\ \frac{3}{4}\ \frac{5}{4}\ \frac{7}{4} & & - \end{array}\right\}$$

2.4: $D * a = \#62 \qquad \chi = -336, \quad c_2H = 72, \quad H^3 = 6, \quad \dim|H| = 7.$

$$\theta^4 - 12x(6\theta+1)(6\theta+5)(7\theta^2+7\theta+2) - 1152x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$

$$\left\{\begin{array}{c|c||c||c} 0 & 0\ 0\ 0\ 0 & (6,\ 0,\ 3,\ -366\lambda) & 372,\ 136182,\ 71562236 \\ \frac{1}{3456} & 0\ 1\ 1\ 2 & & (1728/16) \\ -\frac{1}{432} & 0\ 1\ 1\ 2 & (12,\ -6,\ 7,\ -3-732\lambda) & (216/4) \\ \infty & \frac{1}{6}\ \frac{5}{6}\ \frac{7}{6}\ \frac{11}{6} & & - \end{array}\right\}$$

2.5: $A * b = \#25 \qquad \chi = -120, \quad c_2H = 68, \quad H^3 = 20, \quad \dim|H| = 9.$

$$\theta^4 - 4x(2\theta+1)^2(11\theta^2+11\theta+3) - 16x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{\begin{array}{c|c||c||c} 0 & 0\ 0\ 0\ 0 & (20,\ 0,\ 17/6,\ -120\lambda) & 20,\ 277,\ 8220 \\ 0.005636 & 0\ 1\ 1\ 2 & & h \\ -0.693136 & 0\ 1\ 1\ 2 & (80,\ -40,\ 46/3\ -13/3-480\lambda) & h \\ \infty & \frac{1}{2}\ \frac{1}{2}\ \frac{3}{2}\ \frac{3}{2} & & k \end{array}\right\}$$

2.6: $B * b = \#24 \qquad \chi = -150, \quad c_2H = 66, \quad H^3 = 15, \quad \dim|H| = 8.$

$$\theta^4 - 3x(3\theta+1)(3\theta+2)(11\theta^2+11\theta+3) - 9x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{ \begin{array}{c|c|c|c} 0 & \begin{matrix}0&0&0&0\\0&1&1&2\\0&1&1&2\\ \frac{1}{3}&\frac{2}{3}&\frac{4}{3}&\frac{5}{3}\end{matrix} & \begin{matrix}(15, & 0, & 11/4, & -150\lambda)\\(12, & -6, & 7, & -3-732\lambda)\end{matrix} & \begin{matrix}36, & 837, & 41421\\ & h & \\ & h & \\ & - & \end{matrix} \\ 0.003340 & & & \\ -0.410748 & & & \\ \infty & & & \end{array} \right\}$$

2.7: $C * b = \#51 \qquad \chi = -200, \quad c_2H = 64, \quad H^3 = 10, \quad \dim|H| = 7.$

$$\theta^4 - 4x(4\theta+1)(4\theta+3)(11\theta^2+11\theta+3) - 16x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$

$$\left\{ \begin{array}{c|c|c|c} 0 & \begin{matrix}0&0&0&0\\0&1&1&2\\0&1&1&2\\ \frac{1}{4}&\frac{1}{2}&\frac{3}{2}&\frac{3}{2}\end{matrix} & \begin{matrix}(10, & 0, & 8/3, & -200\lambda)\\(40, & -20, & 38/3, & -14/3-400\lambda)\end{matrix} & \begin{matrix}92, & 5052, & 585396\\ & h & \\ & h & \\ & - & \end{matrix} \\ 0.0014090 & & & \\ -0.173284 & & & \\ \infty & & & \end{array} \right\}$$

2.8 $D * b = \#63 \qquad \chi = -310, \quad c_2H = 62, \quad H^3 = 5, \quad \dim|H| = 6.$

$$\theta^4 - 12x(6\theta+1)(6\theta+5)(11\theta^2+11\theta+3) - 144x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$

$$\left\{ \begin{array}{c|c|c|c} 0 & \begin{matrix}0&0&0&0\\0&1&1&2\\0&1&1&2\\ \frac{1}{6}&\frac{5}{6}&\frac{7}{6}&\frac{11}{6}\end{matrix} & \begin{matrix}(5, & 0, & 31/12, & -310\lambda)\\(20 & -10 & 34/3 & -29/6-1240\lambda)\end{matrix} & \begin{matrix}684, & 253314, & 195638820\\ & h & \\ & h & \\ & - & \end{matrix} \\ 0.000208 & & & \\ -0.025671 & & & \\ \infty & & & \end{array} \right\}$$

2.9: $A * c = \#58 \qquad \chi = -112, \quad c_2H = 72, \quad H^3 = 24, \quad \dim|H| = 10.$

$$\theta^4 - 4x(2\theta+1)^2(10\theta^2+10\theta+3) + 144x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{ \begin{array}{c|c|c|c} 0 & \begin{matrix}0&0&0&0\\0&1&1&2\\0&1&1&2\\ \frac{1}{2}&\frac{1}{2}&\frac{3}{2}&\frac{3}{2}\end{matrix} & \begin{matrix}(24, & 0, & 3, & -112\lambda)\\(72, & -24, & 9, & -3-336\lambda)\end{matrix} & \begin{matrix}16, & 142, & 11056/3;\\ & (48/1) & \\ & (16/1) & \\ & k & \end{matrix} \\ \frac{1}{144} & & & \\ \frac{1}{16} & & & \\ \infty & & & \end{array} \right\}$$

2.10: $B * c = \#70 \qquad \chi = -156, \quad c_2H = 72, \quad H^3 = 18, \quad \dim|H| = 9.$

$$\theta^4 - 3x(3\theta+1)(3\theta+2)(10\theta^2+10\theta+3) - 81x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{ \begin{array}{c|c|c|c} 0 & \begin{matrix}0&0&0&0\\0&1&1&2\\0&1&1&2\\ \frac{1}{4}&\frac{3}{4}&\frac{5}{4}&\frac{7}{4}\end{matrix} & \begin{matrix}(18, & 0, & 3, & -156\lambda)\\(54, & -18, & 9, & -3-468\lambda)\end{matrix} & \begin{matrix}27, & 432, & 18089\\ & (243/1) & \\ & (27/1) & \\ & - & \end{matrix} \\ \frac{1}{243} & & & \\ \frac{1}{27} & & & \\ \infty & & & \end{array} \right\}$$

2.11: $C * c = \#69 \qquad \chi = -224, \quad c_2 H = 72, \quad H^3 = 12, \quad \dim |H| = 8.$

$$\theta^4 - 4x(4\theta+1)(4\theta+3)(10\theta^2+10\theta+3) + 144x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$

$$\left\{ \begin{array}{c|cccc||l||l} 0 & 0 & 0 & 0 & 0 & & 64, \quad 2616, \quad 246848; \\ \frac{1}{576} & 0 & 1 & 1 & 2 & (12, \quad 0, \quad 3, \quad -224\lambda) & (576/3) \\ \frac{1}{64} & 0 & 1 & 1 & 2 & (36, \quad -12, \quad 9, \quad -3-672\lambda) & (64/3) \\ \infty & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & & - \end{array} \right\}$$

2.12: $D * c = \#64 \qquad \chi = -364, \quad c_2 H = 72, \quad H^3 = 6, \quad \dim |H| = 7.$

$$\theta^4 - 12x(6\theta+1)(6\theta+5)(10\theta^2+10\theta+3) + 1296x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$

$$\left\{ \begin{array}{c|cccc||l||l} 0 & 0 & 0 & 0 & 0 & & 432, \quad 130842, \quad 78259376; \\ \frac{1}{3888} & 0 & 1 & 1 & 2 & (6, \quad 0, \quad 3, \quad -364\lambda) & (1944/5) \\ \frac{1}{432} & 0 & 1 & 1 & 2 & (18, \quad -6, \quad 9, \quad -3-1092\lambda) & (432/9) \\ \infty & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & & - \end{array} \right\}$$

2.13: $A * d = \#36 \qquad \chi = -88, \quad c_2 H = 80, \quad H^3 = 32, \quad \dim |H| = 12.$

$$\theta^4 - 16x(2\theta+1)^2(3\theta^2+3\theta+1) + 512x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{ \begin{array}{c|cccc||l||l} 0 & 0 & 0 & 0 & 0 & & 16, \quad 42, \quad 1232 \\ \frac{1}{128} & 0 & 1 & 1 & 2 & (32, \quad 0, \quad 10/3, \quad -88\lambda) & (64/4) \\ \frac{1}{64} & 0 & 1 & 1 & 2 & (64, \quad -16, \quad 20/3, \quad -7/3-176\lambda) & (32/2) \\ \infty & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & & k \end{array} \right\}$$

2.14: $B * d = \#48 \qquad \chi = -162, \quad c_2 H = 84, \quad H^3 = 24, \quad \dim |H| = 11.$

$$\theta^4 - 12x(3\theta+1)(3\theta+2)(3\theta^2+3\theta+1) + 288x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{ \begin{array}{c|cccc||l||l} 0 & 0 & 0 & 0 & 0 & & 4, \quad 291/2, \quad 5832 \\ \frac{1}{216} & 0 & 1 & 1 & 2 & (24, \quad 0, \quad 7/2, \quad -162\lambda) & (9/1) \\ \frac{1}{108} & 0 & 1 & 1 & 2 & (48, \quad -12, \quad 7, \quad -9/4-324\lambda) & (108/4) \\ \infty & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & & - \end{array} \right\}$$

2.15: $C * d = \#38 \qquad \chi = -268, \quad c_2 H = 88, \quad H^3 = 16, \quad \dim |H| = 10.$

$$\theta^4 - 16x(4\theta+1)(4\theta+3)(3\theta^2+3\theta+1) + 512x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$

$$\left\{ \begin{array}{c|cccc||l||l} 0 & 0 & 0 & 0 & 0 & & 48, \quad 998, \quad 73328 \\ \frac{1}{512} & 0 & 1 & 1 & 2 & (16, \quad 0, \quad 11/3, \quad -268\lambda) & (256/1) \\ \frac{1}{256} & 0 & 1 & 1 & 2 & (32, \quad -8, \quad 22/3, \quad -13/6-536\lambda) & (128/4) \\ \infty & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & & - \end{array} \right\}$$

2.16: $D * d = \#65 \qquad \chi = -470, \quad c_2H = 92, \quad H^3 = 8, \quad \dim|H| = 9.$

$$\theta^4 - 48x(6\theta+1)(6\theta+5)(3\theta^2+3\theta+1)$$
$$+4608x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$
$$\left\{\begin{array}{c|c||c|c} 0 & 0\ 0\ 0\ 0 & & 240,\ 57102,\ 19105840; \\ \frac{1}{3456} & 0\ 1\ 1\ 2 & (8,\ 0,\ 23/6,\ -470\lambda) & (576/8) \\ \frac{1}{1728} & 0\ 1\ 1\ 2 & (16,\ -4,\ 23/3,\ -25/12-940\lambda) & (864/3) \\ \infty & \frac{1}{6}\ \frac{5}{6}\ \frac{7}{6}\ \frac{11}{6} & & - \end{array}\right\}$$

2.17: $A * e = \#111$

$$\theta^4 - 16x(2\theta+1)^2(8\theta^2+8\theta+3) + 2^{12}x^2(2\theta+1)^2(2\theta+3)^2$$
$$\left\{\begin{array}{c|c||c} 0 & 0\ \ 0\ \ 0\ \ 0 & 32,\ -96,\ 1440 \\ 1/256 & 0\ \ 1/2\ \ 1/2\ \ 1 & m \\ \infty & 1/2\ \ 1/2\ \ 3/2\ \ 3/2 & k \end{array}\right\}$$

2.18: $B * e = \#110$

$$\theta^4 - 12x(3\theta+1)(3\theta+2)(8\theta^2+8\theta+3)$$
$$+2^8 3^2 x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$
$$\left\{\begin{array}{c|c||c} 0 & 0\ \ 0\ \ 0\ \ 0 & 36,\ -144,\ 8076 \\ 1/432 & 0\ \ 1/2\ \ 1/2\ \ 1 & m \\ \infty & 1/3\ \ 2/3\ \ 4/3\ \ 5/3 & - \end{array}\right\}$$

2.xx19: $C * e \sim 1.3 = \#3$

$$\theta^4 - 16x(4\theta+1)(4\theta+3)(8\theta^2+8\theta+3)$$
$$+2^{12}x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$
$$\left\{\begin{array}{c|c||c} 0 & 0\ \ 0\ \ 0\ \ 0 & 32,\ 608,\ 26016 \\ 1/1024 & 0\ \ 1/2\ \ 1/2\ \ 1 & m \\ \infty & 1/4\ \ 3/4\ \ 5/4\ \ 7/4 & - \end{array}\right\}$$

2.19: $D * e = \#112$

$$\theta^4 - 48x(6\theta+1)(6\theta+5)(8\theta^2+8\theta+3)$$
$$+2^{12}3^2x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$
$$\left\{\begin{array}{c|c||c} 0 & 0\ \ 0\ \ 0\ \ 0 & -288,\ 162504,\ -96055968 \\ 1/6912 & 0\ \ 1/2\ \ 1/2\ \ 1 & m \\ \infty & 1/6\ \ 5/6\ \ 7/6\ \ 11/6 & - \end{array}\right\}$$

2.20: $A * f = \#133$

$$\theta^4 - 12x(2\theta+1)^2(3\theta^2+3\theta+1) + 432x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{\begin{array}{c|cccc||l||ll} 0 & 0 & 0 & 0 & 0 & & 12, & -42, \quad -3284/3 \\ \alpha & 0 & 1 & 1 & 2 & (36, -6, 4, -5/6-120\lambda) & & b \\ \beta & 0 & 1 & 1 & 2 & (36, \quad 6, 4, -5/6-120\lambda) & & b \\ \infty & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & & & k \end{array}\right\}$$

2.21: $B * f = \#134$

$$\theta^4 - 9x(3\theta+1)(3\theta+2)(3\theta^2+3\theta+1)$$
$$+243x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{\begin{array}{c|cccc||l||ll} 0 & 0 & 0 & 0 & 0 & & 18, & -207/2, \quad -52177 \\ \alpha & 0 & 1 & 1 & 2 & (27, -9/2, 33/8, -13/16-198\lambda) & & b \\ \beta & 0 & 1 & 1 & 2 & (27, \quad 9/2, 33/8, -13/16-198\lambda) & & b \\ \infty & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & & & - \end{array}\right\}$$

2.22: $C * f = \#135$

$$\theta^4 - 12x(4\theta+1)(4\theta+3)(3\theta^2+3\theta+1)$$
$$+432x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$

$$\left\{\begin{array}{c|cccc||l||ll} 0 & 0 & 0 & 0 & 0 & & 36, & -477, \quad -206716/3 \\ \alpha & 0 & 1 & 1 & 2 & (18, -3, 17/4, -19/24-312\lambda) & & b \\ \beta & 0 & 1 & 1 & 2 & (18, \quad 3, 17/4, \quad 19/24-312\lambda) & & b \\ \infty & \frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & & & - \end{array}\right\}$$

2.23: $D * f = \#136$

$$\theta^4 - 36x(6\theta+1)(6\theta+5)(3\theta^2+3\theta+1) + 3888x^2(6\theta+1)(6\theta+5)(6\theta+7)$$
$$(6\theta+11)$$

$$\left\{\begin{array}{c|cccc||l||ll} 0 & 0 & 0 & 0 & 0 & & 180, & -15615, \quad -21847076 \\ \alpha & 0 & 1 & 1 & 2 & (9, -3/2, 35/8, -37/48-534\lambda) & & b \\ \beta & 0 & 1 & 1 & 2 & (9, \quad 3/2, 35/8, -37/48-534\lambda) & & b \\ \infty & \frac{1}{6} & \frac{5}{6} & \frac{7}{6} & \frac{11}{6} & & & - \end{array}\right\}$$

2.24: $A * g = \#137$ $\qquad \chi = -16, \quad c_2H = 96, \quad H^3 = 48, \quad \dim|H| = 16.$

$$\theta^4 - 4x(2\theta+1)^2(17\theta^2+17\theta+6) + 1152x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{\begin{array}{c|cccc||l||ll} 0 & 0 & 0 & 0 & 0 & & 20, & 2, \quad 1684/3 \\ \frac{1}{144} & 0 & 1 & 1 & 2 & (48, \quad 0, 4, \quad -16\lambda) & & (24/1) \\ \frac{1}{128} & 0 & 1 & 1 & 2 & (72, -12, 6, -2-24\lambda) & & (64/1) \\ \infty & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & & & k \end{array}\right\}$$

2.25: $B * g = \#138 \qquad \chi = -156, \quad c_2H = 108, \quad H^3 = 36, \quad \dim|H| = 15.$

$$\theta^4 - 3x(3\theta+1)(3\theta+2)(17\theta^2+17\theta+6) \\ +648x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (36,\ 0,\ 9/2,\ -156\lambda) & 27,\ 189/4,\ 2618 \\ \frac{1}{243} & 0\ 1\ 1\ 2 & & (243/2) \\ \frac{1}{216} & 0\ 1\ 1\ 2 & (54,\ -9,\ 27/4,\ -15/8-234\lambda) & (54/4) \\ \infty & \frac{1}{3}\ \frac{2}{3}\ \frac{4}{3}\ \frac{5}{3} & & - \end{array}\right\}$$

2.26: $C * g = \#139 \qquad \chi = -344, \quad c_2H = 120, \quad H^3 = 24, \quad \dim|H| = 14.$

$$\theta^4 - 4x(4\theta+1)(4\theta+3)(17\theta^2+17\theta+6) \\ +1152x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$$

$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (24,\ 0,\ 5,\ -344\lambda) & 44,\ 607,\ 22500 \\ \frac{1}{576} & 0\ 1\ 1\ 2 & & (288/10) \\ \frac{1}{512} & 0\ 1\ 1\ 2 & (36,\ -6,\ 15/2,\ -7/2-516\lambda) & (256/4) \\ \infty & \frac{1}{4}\ \frac{3}{4}\ \frac{5}{4}\ \frac{7}{4} & & - \end{array}\right\}$$

2.27: $D * g = \#140 \qquad \chi = -676, \quad c_2H = 132, \quad H^3 = 12, \quad \dim|H| = 13.$

$$\theta^4 - 12x(6\theta+1)(6\theta+5)(17\theta^2+17\theta+6) \\ +10368x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$

$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (12,\ 0,\ 11/2,\ -676\lambda) & 108,\ 54135,\ -494556 \\ \frac{1}{3888} & 0\ 1\ 1\ 2 & & (1944/6) \\ \frac{1}{3456} & 0\ 1\ 1\ 2 & (18,\ -3,\ 33/4,\ -13/8-1014\lambda) & (1728/15) \\ \infty & \frac{1}{6}\ \frac{5}{6}\ \frac{7}{6}\ \frac{11}{6} & & - \end{array}\right\}$$

2.28: $A * h = \#141$

$$\theta^4 - 12x(2\theta+1)^2(18\theta^2+18\theta+7) + 2^4 3^6 x^2(2\theta+1)^2(2\theta+3)^2$$

$$\left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & 48, -438, 2864 \\ 1/432 & 0\ \ 1/3\ 2/3\ \ 1 & - \\ \infty & 1/2\ 1/2\ 3/2\ 3/2 & k \end{array}\right\}$$

2.29: $B * h = \#142$

$$\theta^4 - 9x(3\theta+1)(3\theta+2)(18\theta^2+18\theta+7) \\ +3^8x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$$

$$\left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & 45, -3465/4, 27735 \\ 1/729 & 0\ \ 1/3\ 2/3\ \ 1 & - \\ \infty & 1/3\ 2/3\ 4/3\ 5/3 & - \end{array}\right\}$$

2.xx30: $C * h$

$$\theta^4 - 12x(4\theta+1)(4\theta+3)(18\theta^2+18\theta+7) \\ +2^4 3^6 x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7) \\ \left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & 0,0,0 \\ 1/1728 & 0\ \ 1/3\ 2/3\ \ 1 & - \\ \infty & 1/4\ 3/4\ 5/4\ 7/4 & - \end{array}\right\}$$

2.30: $D * h = \#143$

$$\theta^4 - 36x(6\theta+1)(6\theta+5)(18\theta^2+18\theta+7) \\ +2^4 3^8 x^2(6\theta+1)^(6\theta+5)(6\theta+7)(6\theta+11) \\ \left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & -1008, 499086, -607849200 \\ 1/11664 & 0\ \ 1/3\ 2/3\ \ 1 & - \\ \infty & 1/6\ 5/6\ 7/6\ 11/6 & - \end{array}\right\}$$

2.31: $A * i$

$$\theta^4 - 16x(2\theta+1)^2(32\theta^2+32\theta+13) + 2^{16}x^2(2\theta+1)^2(2\theta+3)^2 \\ \left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & 96, -3560, -12064 \\ 1/1024 & 0\ \ 1/4\ 3/4\ \ 1 & - \\ \infty & 1/2\ 1/2\ 3/2\ 3/2 & k \end{array}\right\}$$

2.32: $B * i$

$$\theta^4 - 12x(3\theta+1)(3\theta+2)(32\theta^2+32\theta+13) \\ +2^{12}3^2x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5) \\ \left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & 60, -7635, 307860 \\ 1/1728 & 0\ \ 1/4\ 3/4\ \ 1 & - \\ \infty & 1/3\ 2/3\ 4/3\ 5/3 & - \end{array}\right\}$$

2.33: $C * i$

$$\theta^4 - 16x(4\theta+1)(4\theta+3)(32\theta^2+32\theta+13) \\ +2^4 3^6 x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7) \\ \left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & -160, -6920, -539680 \\ 1/4096 & 0\ \ 1/4\ 3/4\ \ 1 & - \\ \infty & 1/4\ 3/4\ 5/4\ 7/4 & - \end{array}\right\}$$

2.34: $D * i$

$$\theta^4 - 48x(6\theta+1)(6\theta+5)(32\theta^2+32\theta+13) \\ +2^{16}3^2x^2(6\theta+1)^(6\theta+5)(6\theta+7)(6\theta+11) \\ \left\{ \begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -3936, 3550992, -10892932064 \\ 1/27648 & 0 & 1/4 & 3/4 & 1 & - \\ \infty & 1/6 & 5/6 & 7/6 & 11/6 & - \end{array} \right\}$$

2.35: $A * j$

$$\theta^4 - 48x(2\theta+1)^2(72\theta^2+72\theta+31) + 2^{12}3^6x^2(2\theta+1)^2(2\theta+3)^2 \\ \left\{ \begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & 480, -226968, -16034720 \\ 1/6912 & 0 & 1/6 & 5/6 & 1 & - \\ \infty & 1/2 & 1/2 & 3/2 & 3/2 & k \end{array} \right\}$$

2.36: $B * j$

$$\theta^4 - 36x(3\theta+1)(3\theta+2)(72\theta^2+72\theta+31) \\ +2^8 3^8 x^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5) \\ \left\{ \begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -36, -486279, 128217204 \\ 1/11664 & 0 & 1/6 & 5/6 & 1 & - \\ \infty & 1/3 & 2/3 & 4/3 & 5/3 & - \end{array} \right\}$$

2.37: $C * j$

$$\theta^4 - 48x(4\theta+1)(4\theta+3)(72\theta^2+72\theta+31) \\ +2^{12}3^6x^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7) \\ \left\{ \begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -2592, -307800, 81451104 \\ 1/27648 & 0 & 1/6 & 5/6 & 1 & - \\ \infty & 1/4 & 3/4 & 5/4 & 7/4 & - \end{array} \right\}$$

2.38: $D * j$

$$\theta^4 - 144x(6\theta+1)(6\theta+5)(72\theta^2+72\theta+31) \\ +2^{12}3^8x^2(6\theta+1)^(6\theta+5)(6\theta+7)(6\theta+11) \\ \left\{ \begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -41184, 251271360, -5124430612320 \\ 1/186624 & 0 & 1/6 & 5/6 & 1 & - \\ \infty & 1/6 & 5/6 & 7/6 & 11/6 & - \end{array} \right\}$$

2.52: $I * \alpha = \#16 \qquad \chi = -128, \quad c_2H = 96, \quad H^3 = 48, \quad \dim|H| = 16.$

$$\theta^4 - 4x(2\theta+1)^2(5\theta^2+5\theta+2) + 256x^2(2\theta)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (48,\ 0,\ 4,\ -128\lambda) & 4,\ 20,\ 644/3 \\ \frac{1}{64} & 0\ 1\ 1\ 2 & & (6/1) \\ \frac{1}{16} & 0\ 1\ 1\ 2 & (192,\ -48,\ 16,\ -4-512\lambda) & (12/1) \\ \infty & \frac{1}{2}\ 1\ 1\ \frac{3}{2} & & m \end{array}\right\}$$

2.53: $I * \gamma = \#29 \qquad \chi = -116, \quad c_2H = 72, \quad H^3 = 24, \quad \dim|H| = 10.$

$$\theta^4 - 2x(2\theta+1)^2(17\theta^2+17\theta+5) + 4x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (24,\ 0,\ 3,\ -116\lambda) & 14,\ 303/2,\ 10424/3 \\ 0.00737 & 0\ 1\ 1\ 2 & & - \\ 8.49263 & 0\ 1\ 1\ 2 & (600,\ -240,\ 75\ -20-2900\lambda) & - \\ \infty & \frac{1}{2}\ 1\ 1\ \frac{3}{2} & & m \end{array}\right\}$$

2.54: $I * \delta = \#41 \qquad \chi = -116, \quad c_2H = 72, \quad H^3 = 24, \quad \dim|H| = 10.$

$$\theta^4 - 2x(2\theta+1)^2(7\theta^2+7\theta+3) + 324x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (72,\ -12,\ 6,\ -1-180\lambda) & 14,\ 303/2,\ 10424/3 \\ * & 0\ 1\ 1\ 2 & & b \\ * & 0\ 1\ 1\ 2 & (72,\ 12,\ 6,\ 1-180\lambda) & b \\ \infty & \frac{1}{2}\ 1\ 1\ \frac{3}{2} & & m \end{array}\right\}$$

2.55: $I * \epsilon = \#42 \qquad \chi = -116, \quad c_2H = 80, \quad H^3 = 32, \quad \dim|H| = 12.$

$$\theta^4 - 8x(2\theta+1)^2(3\theta^2+3\theta+1) + 64x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (32,\ 0,\ 10/3,\ -116\lambda) & 8,\ 63,\ 1000 \\ \alpha & 0\ 1\ 1\ 2 & & - \\ \beta & 0\ 1\ 1\ 2 & (288,\ -96,\ 30,\ -8-1044\lambda) & - \\ \infty & \frac{1}{2}\ 1\ 1\ \frac{3}{2} & & \end{array}\right\}$$

2.56: $I * \zeta = \#185 \qquad \chi = -120, \quad c_2H = 84, \quad H^3 = 36, \quad \dim|H| = 13.$

$$\theta^4 - 6x(2\theta+1)^2(3\theta^2+3\theta+1) - 108x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (36,\ 0,\ 7/2,\ -120\lambda) & 6,\ 93/2,\ 608 \\ \alpha & 0\ 1\ 1\ 2 & & h \\ \beta & 0\ 1\ 1\ 2 & (144,\ -72,\ 26,\ -7-480\lambda) & h \\ \infty & \frac{1}{2}\ 1\ 1\ \frac{3}{2} & & m \end{array}\right\}$$

2.57: $I * \eta = \#184$

$$\theta^4 - 2x(2\theta+1)^2(11\theta^2+11\theta+5) + 500x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c|c} 0 & 0\ 0\ 0\ 0 & (100,\ -10,\ 20/3,\ -5/6-200\lambda) & 2,\ 4,\ -8 \\ \alpha & 0\ 1\ 1\ 2 & & b \\ \beta & 0\ 1\ 1\ 2 & (100,\ \ 10,\ 20/3,\ \ 5/6-200\lambda) & b \\ \infty & \frac{1}{2}\ 1\ 1\ \frac{3}{2} & & m \end{array}\right\}$$

2.58: $I * \iota = 4*$

$$\theta^4 - 6x(2\theta+1)^2(9\theta^2+9\theta+5) + 2916x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c} 0 & 0\ \ 0\ \ 0\ \ 0 & -6, -6, -104 \\ 1/108 & 0\ 1/6\ 5/6\ 1 & - \\ \infty & 1/2\ \ 1\ \ 1\ \ 3/2 & m \end{array}\right\}$$

2.xx59: $I * \theta \sim 2.17$.

$$\theta^4 - 16\,x\,(2\,\theta+1)^2\left(8\,\theta^2+8\,\theta+5\right) + 16384\,x^2\,(2\,\theta+1)\,(\theta+1)^2\,(2\,\theta+3)$$
$$\left\{\begin{array}{c|c|c} 0 & 0\ \ 0\,0\ \ 0 & -32, -88, -1440 \\ 1/256 & 0\ \ 0\,1\ \ 1 & k \\ \infty & 1/2\ 1\ 1\ 3/2 & m \end{array}\right\}$$

2.59: $I * \kappa$

$$\theta^4 - 48(2\theta+1)^2(18\theta^2+18\theta+13) + 746496x^2(2\theta+1)(\theta+1)^2(2\theta+3)$$
$$\left\{\begin{array}{c|c|c} 0 & 0\ \ 0\,0\ \ 0 & -384, -1356, -164736 \\ 1/1728 & -1/6\ 0\ 1\ 7/6 & - \\ \infty & 1/2\ \ 1\ 1\ 3/2 & m \end{array}\right\}$$

2.60: $= I * Sporadic1 = \#18$ $\chi = -128$, $c_2H = 88$, $H^3 = 40$, $\dim|H| = 14$.

$$\theta^4 - 4x(2\theta+1)^2(3\theta^2+3\theta+1) - 16x^2(2\theta+1)(4\theta+3)(4\theta+5)(2\theta+3)$$
$$\left\{\begin{array}{c|c|c} 0 & 0\ 0\ 0\ 0 & (40,\ \ 0,\ 11/3,\ \ -128\lambda) & 4,\ 39,\ 364 \\ \frac{1}{64} & 0\ 1\ 1\ 2 & & (80/3) \\ -\frac{1}{16} & 0\ 1\ 1\ 2 & (80,\ -40,\ 46/3,\ -13/3-256\lambda) & (40/2) \\ \infty & \frac{1}{2}\ \frac{3}{4}\ \frac{5}{4}\ \frac{3}{2} & & - \end{array}\right\}$$

2.61: $I * Sporadic2 = \#26$ $\chi = -116$, $c_2H = 76$, $H^3 = 28$, $\dim|H| = 11$

$$\theta^4 - 2x(2\theta+1)^2(13\theta^2+13\theta+4) - 12x^2(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3)$$
$$\left\{\begin{array}{c|c|c} 0 & 0\ 0\ 0\ 0 & (28,\ \ 0,\ 19/6,\ \ -116\lambda) & 10,\ 191/2,\ 1724 \\ \frac{1}{108} & 0\ 1\ 1\ 2 & & (252/3) \\ -\frac{1}{4} & 0\ 1\ 1\ 2 & (112,\ -56,\ 62/3,\ -17/3-464\lambda) & (28/1) \\ \infty & \frac{1}{2}\ \frac{2}{3}\ \frac{4}{3}\ \frac{3}{2} & & - \end{array}\right\}$$

2.62: = #28 $\chi = -96,\ \ c_2H = 84,\ \ H^3 = 42,\ \ \dim|H| = 14.$

$$\theta^4 - x(65\theta^4 + 130\theta^3 + 105\theta^2 + 40\theta + 6) + 4x^2(4\theta + 3)(\theta + 1)^2(4\theta + 5)$$

$$\left\{ \begin{array}{c|c||c|c} 0 & 0\ 0\ 0\ 0 & & 5,\ \ 28,\ \ 312 \\ \frac{1}{64} & 0\ 1\ 1\ 2 & (42,\ \ 0,\ 7/2,\ \ -96\lambda) & (14/2) \\ 1 & 0\ 1\ 1\ 2 & (756,\ -252,\ \ 63,\ -15 - 1728\lambda) & (7/1) \\ \infty & \frac{3}{4}\ 1\ 1\ \frac{5}{4} & & m \end{array} \right\}$$

$$a_n = \sum_{j,k} \binom{n}{k}^2 \binom{n}{j}^2 \binom{k+j}{n}^2$$

The operator comes from the mirror symmetry of the intersection of six general hyperplanes of the Grassmanian $G(3, 6)$, Plücker-embedded in $\mathbf{P}^{19}$ [11].

2.63: #84

$$\theta^4 - 4x(32\theta^4 + 64\theta^3 + 63\theta^2 + 31\theta + 6) + 256x^2(4\theta + 3)(\theta + 1)^2(4\theta + 5)$$

$$\left\{ \begin{array}{c|c||c} 0 & 0\ \ 0\ 0 & -4,\ \ -11,\ \ -44, \ldots \\ 1/64 & 0\ \ 0\ 1 & k \\ \infty & 3/4\ 1\ 1\ 5/4 & m \end{array} \right\}$$

$$A_n = \sum_k \binom{n}{k}^2 \binom{2n}{2k}^{-1} \frac{(4k!)}{(2k!)k!^2} \frac{(4n-4k)!}{(2n-2k)!(n-k)!^2}$$

2.64: #182 $\chi = -96,\ \ c_2H = 132,\ \ H^3 = 132,\ \ \dim|H| =$

$$\theta^4 - x(43\theta^4 + 86\theta^3 + 77\theta^2 + 34\theta + 6) + 12x^2(6\theta + 5)(\theta + 1)^2(6\theta + 7)$$

$$\left\{ \begin{array}{c|c||c|c} 0 & 0\ 0\ 0\ 0 & & 1,\ \ 7/4,\ \ 7 \\ \frac{1}{16} & 0\ 1\ 1\ 2 & (132,\ \ 0,\ 11/2,\ \ -96\lambda) & (22/3) \\ \frac{1}{27} & 0\ 1\ 1\ 2 & (396,\ -66,\ 33/2,\ -13/4 - 288\lambda) & (33/2) \\ \infty & \frac{5}{6}\ 1\ 1\ \frac{7}{6} & & m \end{array} \right\}$$

This operator was found by a brute-force search. We do not know an explicit formula for the coefficient A_n.

2.65 $= I * Sporadic3 =$ #183 $\chi = -128,\ \ c_2H = 120,\ \ H^3 = 72,\ \ \dim|H| =$ 22.

$$\theta^4 - 4x(2\theta + 1)^2(7\theta^2 + 7\theta + 3) + 48x^2(2\theta + 1)(4\theta + 3)(4\theta + 5)(2\theta + 3)$$

$$\left\{ \begin{array}{c|c||c|c} 0 & 0\ 0\ 0\ 0 & & 4,\ \ 7,\ \ 556/9 \\ \frac{1}{64} & 0\ 1\ 1\ 2 & (72,\ \ 0,\ \ 5,\ \ -128\lambda) & (16/1) \\ \frac{1}{48} & 0\ 1\ 1\ 2 & (144,\ -24,\ 10,\ -7/3 - 256\lambda) & (72/1) \\ \infty & \frac{1}{2}\ \frac{3}{4}\ \frac{5}{4}\ \frac{3}{2} & & - \end{array} \right\}$$

2.66 Reducible operator (does not really count).

$$\theta^4 - 12x(6\theta+1)(6\theta+5)(2\theta^2+2\theta+1)$$
$$+144x^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$
$$\left\{\begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -192, \ 4182, \ -229568 \\ 1/432 & 0 & 0 & 1 & 1 & k \\ \infty & 1/6 & 5/6 & 7/6 & 11/6 & - \end{array}\right\}$$

This operator is the square of the second-order hypergeometric operator D.

2.67: $\sim$ #245 (Bogner 1)

$$\theta^4 - 2x(108\theta^4+198\theta^3+183\theta^2+84\theta+15) + 36x^2(3\theta+2)^2(6\theta+7)^2$$
$$\left\{\begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -6, \ -33, \ -170 \\ 1/108 & 0 & 1/6 & 1 & 7/6 & - \\ \infty & 2/3 & 2/3 & 7/6 & 7/6 & - \end{array}\right\}$$

We do not know a formula for A_n.

2.68: $\sim$ #406 (Bogner 2)

$$\theta^4 - 4x(128\theta^4+224\theta^3+197\theta^2+85\theta+14)$$
$$+128x^2(2\theta+1)(4\theta+5)(8\theta+5)(8\theta+9)$$
$$\left\{\begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & -12\ , -186\ , -1668, \\ 1/256 & 0 & 1/4 & 1 & 5/4 & - \\ \infty & 1/2 & 5/8 & 9/8 & 5/4 & - \end{array}\right\}$$

We do not know a formula for A_n. **2.69:** $= \#205 \quad \chi = -128, \quad c_2H = 160, \quad H^3 = 160, \quad \dim|H| = 40.$

$$\theta^4 - x(59\theta+118\theta^3+105\theta^2+46\theta+8) + 96x^2(3\theta+2)(\theta+1)^2(3\theta+4)$$
$$\left\{\begin{array}{c|cccc||cccc||c} 0 & 0 & 0 & 0 & 0 & & & & & 1, \ 7/4, \ 5 \\ \frac{1}{32} & 0 & 1 & 1 & 2 & (160, & 0, & 20/3, & -128\lambda) & (5/1) \\ \frac{1}{27} & 0 & 1 & 1 & 2 & (320, & -40, & 40/3, & -7/3-256\lambda) & (15/2) \\ \infty & \frac{2}{3} & 1 & 1 & \frac{4}{3} & & & & & m \end{array}\right\}$$
$$A_n = 4\sum_k^{\lfloor n/4\rfloor} \frac{n-2k}{3n-4k}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{3n-4k}{2n}$$

2.70: $\sim$ #255 (Bogner 3)

$$\theta^4 - 4x(128\theta^4+160\theta^3+125\theta^2+45\theta+6)$$
$$+128x^2(8\theta+7)(2\theta+1)(4\theta+3)(8\theta+3)$$
$$\left\{\begin{array}{c|cccc||c} 0 & 0 & 0 & 0 & 0 & 20, \ 290, \ 28820/3 \\ 1/256 & 0 & 3/4 & 1 & 7/4 & - \\ \infty & 3/8 & 1/2 & 3/4 & 7/8 & - \end{array}\right\}$$

We do not know a formula for A_n.

The 14 *tilde operators* There are 14 exponents $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with

$$\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4, \quad \alpha_1 + \alpha_4 = \alpha_2 + \alpha_4 = 1,$$

for which the hypergeometric operator, scaled by N,

$$\theta^4 - Nx(\theta + \alpha_1)(\theta + \alpha_2)(\theta + \alpha_3)(\theta + \alpha_4)$$

is a Calabi–Yau operator [8]. Corresponding to these, there are also 14 hypergeometric *fifth-order* Calabi–Yau operator

$$\theta^5 - 4Nx(\theta + \alpha_1)(\theta + \alpha_2)\left(\theta + \frac{1}{2}\right)(\theta + \alpha_3)(\theta + \alpha_4)$$

with Riemann symbol

$$\left\{\begin{array}{ccc} 0 & 1/4N & \infty \\ \hline 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \\ 0 & 3/2 & 1/2 \\ 0 & 2 & \alpha_3 \\ 0 & 3 & \alpha_4 \end{array}\right\}$$

These operators have a *Yifan Yang pull-back* to 14 special fourth-order operators, called the *tilde*-operators $\widetilde{1}, \widetilde{2}, \ldots, \widetilde{14}$. These operators replace the more complicated *hat-operators* $\hat{i}, i = 1, 2, \ldots, 14$ that appeared in the list [3] into which they can be transformed.

An explicit formula for the tilde operators can be found in [5]:

$$\begin{aligned} &\theta^4 - 4Nx\left(2\left(\theta + \frac{1}{2}\right)^4 + \frac{1}{2}\left(\frac{7}{2} - \mu^2 - \nu^2\right)\left(\theta + \frac{1}{2}\right)^2\right. \\ &\left. + \frac{1}{16} - \frac{1}{4}\left(\mu^2 + \frac{1}{4}\right)\left(\nu^2 + \frac{1}{4}\right)\right) \\ &+(4N)^2x^2\left(\theta + 1 + \frac{\mu + \nu}{2}\right)\left(\theta + 1 - \frac{\mu + \nu}{2}\right) \\ &\left(\theta + 1 + \frac{\mu - \nu}{2}\right)\left(\theta + 1 - \frac{\mu - \nu}{2}\right) \end{aligned}$$

where $\mu := \alpha_3 - \frac{1}{2} = \frac{1}{2} - \alpha_2$, $\nu := \alpha_4 - \frac{1}{2} = \frac{1}{2} - \alpha_1$. Its Riemann symbol is of the form

$$\left\{\begin{array}{ccc} 0 & 1/4N & \infty \\ \hline 0 & -1/2 & 1-(\mu+\nu)/2 \\ 0 & 0 & 1+(\mu-\nu)/2 \\ 0 & 1 & 1-(\mu-\nu)/2 \\ 0 & 3/2 & 1+(\mu+\nu)/2 \end{array}\right\}$$

A general formula for the holomorphic solution is also given in [5]. The 2×2 Wronskians for this operator, multiplied by a factor $(1-4Nx)^{3/2}$, are solutions to the hypergeometric fifth-order equation.

The monodromy around the central point $1/4N$ is of order two, and the matrix in the scaled Frobenius basis is of the form

$$\begin{pmatrix} x & 0 & y & 0 \\ z & -x & 0 & y \\ t & 0 & -x & 0 \\ 0 & t & -z & x \end{pmatrix},$$

where x, y, t satisfy the relation $x^2+yt=1$. The invariants x, y, z, t can be expressed in terms of the reflection vector:

$$(a,0,c,d) = (h^3, 0, c_2 h/24, \chi\lambda), \quad \lambda := \frac{\zeta(3)}{(2\pi i)^3}$$

of the corresponding hypergeometric operator by the formulas

$$x = \frac{\sqrt{a}}{24} + \frac{c}{\sqrt{a}}, \quad y = \sqrt{a} \quad z = \frac{2d-4a\lambda}{\sqrt{a}}, \quad t = (1-x^2)/y.$$

The fourteen hypergeometric and the corresponding tilde operators

Case		Exponents			N	Monodromy data	Number
1	1/5	2/5	3/4	4/5	5^5	$5, 0, 25/12, -200\lambda$	1.1
$\widetilde{1}$	4/5	9/10	11/10	6/5		$11\sqrt{5}/24, \sqrt{5}, -84\sqrt{5}\lambda - 29\sqrt{5}/2880$	2.39
2	1/10	3/10	7/10	9/10	$2^8 5^5$	$1, 0, 17/12, -288\lambda$	1.2
$\widetilde{2}$	7/10	9/10	11/10	13/10		$35/24, 1, -580\lambda, -649/576$	2.40
3	1/2	1/2	1/2	1/2	2^8	$16, 0, 8/3, -128\lambda$	1.3
$\widetilde{3}$	1	1	1	1		$5/6, 4, -80\lambda,$	$\tilde{2}$.33
4	1/3	1/3	2/3	2/3	3^6	$9, 0, 9/4, -144\lambda$	1.4
$\widetilde{4}$	5/6	1	1	7/6		$7/8, 3, -108\lambda, 5/64$	2.41
5	1/3	1/2	1/2	2/3	$2^4 3^3$	$12, 0, 5/2, -144\lambda$	1.5
$\widetilde{5}$	11/12	11/12	13/12	13/12		$\sqrt{3}/2, 2\sqrt{3}, -56\sqrt{3}\lambda, \sqrt{3}/24$	2.42
6	1/4	1/2	1/2	3/4	2^{10}	$8, 0, 7/3, -296\lambda$	1.6
$\widetilde{6}$	7/8	7/8	9/8	9/8		$2\sqrt{2}/3, 2\sqrt{2}, -96\sqrt{2}\lambda, \sqrt{2}/36$	2.43
7	1/8	3/8	5/8	7/8	2^{16}	$2, 0, 11/6, -296\lambda$	1.7
$\widetilde{7}$	3/4	7/8	9/8	5/4		$23\sqrt{2}/24, \sqrt{2}, -300\sqrt{2}\lambda, -241\sqrt{2}/576$	2.44

Case		Exponents			N	Monodromy data	Number
8	1/6	1/3	2/3	5/6	$2^4 3^6$	$3, 0, 7/4, -204\lambda$	2.45
$\widetilde{8}$	3/4	11/12	13/12	5/4		$5\sqrt{3}/8, \sqrt{3}, -140\sqrt{3}\lambda, -11\sqrt{3}/192$	2.45
9	1/12	5/12	7/12	11/12	$2^{12}3^6$	$1, 0, 23/12, -484\lambda$	1.9
$\widetilde{9}$	3/4	5/6	7/6	5/4		$47/24, 1, -972\lambda, -1633/576$	2.46
10	1/4	1/4	3/4	3/4	2^{12}	$4, 0, 5/3, -144\lambda$	1.10
$\widetilde{10}$	3/4	1	1	5/4		$11/12, 2, -152\lambda, 23/288$	2.47
11	1/4	1/3	2/3	3/4	$2^6 3^3$	$6, 0, 2, -156\lambda$	1.11
$\widetilde{11}$	19/24	23/24	25/24	29/24		$3\sqrt{6}/8, \sqrt{6}, -56\sqrt{6}\lambda, 5\sqrt{6}/192$	2.48
12	1/5	1/4	3/4	5/6	$2^{10}3^3$	$2, 0, 4/3, -156\lambda$	1.12
$\widetilde{12}$	17/24	23/24	25/24	31/24		$17\sqrt{2}/24, \sqrt{2}, -160\sqrt{2}\lambda, -\sqrt{2}/576$	2.49
13	1/6	1/6	5/6	5/6	$2^8 3^6$	$1, 0, 11/12, -120\lambda$	1.13
$\widetilde{13}$	2/3	1	1	4/3		$23/24, 1, -244\lambda, 47/576$	2.50
14	1/6	1/2	1/2	5/6	$2^8 3^3$	$4, 0, 13/6, -256\lambda$	1.14
$\widetilde{14}$	5/6	5/6	7/6	7/6		$7/6, 2, -264\lambda, -13/72$	2.51

List of Calabi–Yau operators of degree 2

We give the parameters $a, b, c, d, e, f, \alpha, \beta, \gamma, \delta$ for the Calabi–Yau operators written in the form (note the sign change)

$$\theta^4 - x(a\theta^4 + b\theta^3 + c\theta^2 + d\theta + e) + fx^2(\theta+\alpha)(\theta+\beta)(\theta+\gamma)(\theta+\delta).$$

Number	a	b	c	d	e	f	α	β	γ	δ
2.1	112	224	172	60	8	$-(2)^{11}$	1/2	1/2	3/2	3/2
2.2	189	378	285	96	12	$-(2)^3(3)^6$	1/3	2/3	4/3	5/3
2.3	448	896	660	212	24	$-(2)^{15}$	1/4	3/4	5/4	7/4
2.4	3024	6048	4308	1284	120	$-(2)^{11}(3)^6$	1/6	5/6	7/6	11/6
2.5	176	352	268	92	12	$-(2)^8$	1/2	1/2	3/2	3/2
2.6	4752	9504	6708	1956	180	$-(3)^6$	1/3	2/3	4/3	5/3
2.7	704	1408	1028	324	36	$-(2)^{12}$	1/4	3/4	5/4	7/4
2.8	4752	9504	6708	1956	180	$-(2)^8(3)^6$	1/6	5/6	7/6	11/6
2.9	160	320	248	88	12	$(2)^8(3)^2$	1/2	1/2	3/2	3/2
2.10	270	540	411	141	18	$(3)^8$	1/3	2/3	4/3	5/3
2.11	640	1280	952	312	36	$(2)^{12}(3)^2$	1/4	3/4	5/4	7/4
2.12	4320	8640	6216	1896	180	$(2)^8(3)^8$	1/6	5/6	7/6	11/6
2.13	192	384	304	112	16	$(2)^{13}$	1/2	1/2	3/2	3/2
2.14	324	648	504	180	24	$(2)^5(3)^6$	1/3	2/3	4/3	5/3
2.15	768	1536	1168	400	48	$(2)^{17}$	1/4	3/4	5/4	7/4
2.16	5184	10368	7632	2448	240	$(2)^{13}(3)^6$	1/6	5/6	7/6	11/6
2.17	512	1024	832	320	48	$(2)^{16}$	1/2	1/2	3/2	3/2
2.18	864	1728	1380	516	72	$(2)^8(3)^6$	1/3	2/3	4/3	5/3
2.*xx*19	−144	−1152	−3200	−4096	−2048	$(2)^{20}$	1/4	3/4	5/4	7/4
2.19	13824	27648	20928	7104	720	$(2)^{16}(3)^3$	1/6	5/6	7/6	11/6
2.20	144	288	228	84	12	$(2)^8(3)^3$	1/2	1/2	3/2	3/2

Number	*a*	*b*	*c*	*d*	*e*	*f*	α	β	γ	δ
2.21	243	486	378	135	18	$(3)^9$	1/3	2/3	4/3	5/3
2.22	576	1152	876	300	36	$(2)^{12}(3)^3$	1/4	3/4	5/4	7/4
2.23	3888	7776	5724	1836	180	$(2)^8(3)^9$	1/6	5/6	7/6	11/6
2.24	272	544	436	164	24	$(2)^{11}(3)^2$	1/2	1/2	3/2	3/2
2.25	459	918	723	264	36	$(2)^3(3)^8$	1/3	2/3	4/3	5/3
2.26	1088	2176	1676	588	72	$(2)^{15}(3)^2$	1/4	3/4	5/4	7/4
2.27	7344	14688	10956	3612	360	$(2)^{11}(3)^8$	1/6	5/6	7/6	11/6
2.28	864	1728	1416	552	84	$(2)^8(3)^6$	1/2	1/2	3/2	3/2
2.29	1458	2916	2349	891	126	$(3)^{12}$	1/3	2/3	4/3	5/3
2.30	23328	46656	35640	12312	1260	$(2)^8(3)^{12}$	1/6	5/6	7/6	11/6
2.31	2048	4096	3392	1344	208	$(2)^{20}$	1/2	1/2	3/2	3/2
2.32	3456	6912	5628	2172	312	$(2)^{12}(3)^6$	1/3	2/3	4/3	5/3
2.33	8192	16384	13056	4864	624	$(2)^{24}$	1/4	3/4	5/4	7/4
2.34	55296	110592	85440	30144	3120	$(2)^{20}(3)^6$	1/6	5/6	7/6	11/6
2.35	13824	27648	23232	9408	1488	$(2)^{16}(3)^6$	1/2	1/2	3/2	3/2
2.36	23328	46656	38556	15228	2232	$(2)^8(3)^{12}$	1/3	2/3	4/3	5/3
2.37	55296	110592	89472	34176	4464	$(2)^{20}(3)^6$	1/4	3/4	5/4	7/4
2.38	373248	746496	585792	212544	22320	$(2)^{16}(3)^{12}$	1/6	5/6	7/6	11/6
2.39	25000	50000	58750	33750	7380	$(2)^4(5)^{10}$	4/5	6/5	9/10	11/10
2.40	6400000	12800000	14880000	8480000	1824880	$(2)^{20}(5)^{10}$	7/10	9/10	11/10	13/10
2.41	5832	11664	13770	7938	1746	$(2)^4(3)^{12}$	1	1	5/6	7/6
2.42	3456	6912	8184	4728	1044	$(2)^{12}(3)^6$	11/12	11/12	13/12	13/12
2.43	8192	16384	19328	11136	2448	$(2)^{24}$	7/8	7/8	9/8	9/8
2.44	524288	1048576	1224704	700416	151920	$(2)^{36}$	3/4	5/4	7/8	9/8
2.45	93312	186624	218376	125064	27180	$(2)^{12}(3)^{12}$	3/4	5/4	11/12	13/12
2.46	23887872	47775744	55655424	31767552	6870384	$(2)^{28}(3)^{12}$	3/4	5/4	5/6	7/6
2.47	32768	65536	76800	44032	9584	$(2)^{28}$	3/4	1	1	5/4
2.48	13824	27648	32520	18696	4092	$(2)^{16}(3)^6$	19/24	23/14	25/24	29/24
2.49	221184	442368	515712	294528	63600	$(2)^{24}(3)^6$	17/24	23/24	25/24	31/24
2.50	1492992	2985984	3462912	1969920	421488	$(2)^{20}(3)^{12}$	1	1	2/3	4/3
2.51	55296	110592	129792	74496	16272	$(2)^{20}(3)^6$	5/6	5/6	7/6	7/6
2.52	80	160	132	52	8	$(2)^{10}$	1/2	1	1	3/2
2.53	136	272	210	74	10	$(2)^4$	1/2	1	1	3/2
2.54	56	112	94	38	6	$(2)^4(3)^4$	1/2	1	1	3/2
2.55	96	192	152	56	8	$(2)^8$	1/2	1	1	3/2
2.56	72	144	114	42	6	$-(2)^4(3)^3$	1/2	1	1	3/2
2.57	88	176	150	62	10	$(2)^4(5)^3$	1/2	1	1	3/2
2.xx58	128	256	224	96	16	$(2)^{12}$	1/2	1	1	3/2
2.yy58	2048	4096	3200	1152	144	$(2)^{20}$	1/4	3/4	5/4	7/4
2.58	216	432	390	174	30	$(2)^4(3)^6$	1/2	1	1	3/2
2.59	3456	6912	6816	3360	624	$(2)^{12}(3)^6$	1/2	1	1	3/2
2.60	48	96	76	28	4	$-(2)^{10}$	1/2	3/4	5/4	3/2
2.61	104	208	162	58	8	$-(2)^4(3)^3$	1/2	2/3	4/3	3/2
2.62	65	130	105	40	6	$(2)^6$	3/4	1	1	5/4
2.63	128	256	252	124	24	$(2)^{12}$	3/4	1	1	5/4
2.64	43	86	77	34	6	$(2)^4(3)$	1	1	5/6	7/6

Number	a	b	c	d	e	f	α	β	γ	δ
2.65	112	224	188	76	12	$(2)^{10}(3)^6$	1/2	3/4	5/4	3/2
2.66	864	1728	1416	552	60	$(2)^4(3)^4$	1/6	5/6	7/6	11/6
2.67	216	396	366	168	30	$(2)^4(3^6)$	2/3	2/3	7/6	7/6
2.68	512	896	788	340	56	$(2)^{16}$	1/2	5/4	5/8	9/8
2.69	59	118	105	46	8	$(2)^5(3)^3$	1	1	2/3	4/3
2.70	512	1408	1652	948	216	$(2)^{16}$	5/4	3/2	9/8	13/8

References

1. Almkvist, G., Bogner, M., Guillera, J.: About a class of Calabi–Yau differential equations. arXiv:1310.6658 [math.NT]
2. Almkvist, G., Cynk, S., van Straten, D.: Update on Calabi–Yau operators (in preparation)
3. Almkvist, G., van Enckevort, C., van Straten, D., Zudilin, W.: Tables of Calabi–Yau operators. arXiv:math/0507430
4. Almkvist, G., Zudilin, W.: Differential equations, mirror maps and zeta values. In: Mirror symmetry. V, 481515, AMS/IP Studies in Advanced Mathematics, vol. **38**. American Mathematical Society, Providence, RI (2006)
5. Almkvist, G.: Calabi–Yau differential equations of degree 2 and 3 and Yifan Yang's pullback. arXiv:math.AG/0612215
6. Almkvist, G.: Fifth order differential equations related to Calabi–Yau differential equations. arXiv:math.AG/0703261v1
7. Almkvist, G.: The art of finding Calabi–Yau differential equations, Dedicated to the 90-th birthday of Lars Gårding. In: Gems in Experimental Mathematics, Contemporary Mathematics **517**. American Mathematical Society, Providence, RI (2010)
8. Almkvist, G.: Strängar i månsken I. Normat **51**(2), 63–79 (2003)
9. Almkvist, G., van Straten, D., Zudilin, W.: Generalizations of Clausen's formula and algebraic transformations of Calabi–Yau differential equations. Proc. Edinb. Math. Soc. (2) **54**(2), 273–295 (2011)
10. Batyrev, V., van Straten, D.: Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in Toric varieties. Comm. Math. Phys. **168**(3), 493–533 (1995)
11. Batyrev, V., Ciocan-Fontanine, I., Kim, B., van Straten, D.: Conifold transitions and mirror symmetry for Calabi–Yau complete intersections in Grassmannians. Nuclear Phys. B **514**(3), 640–666 (1998)
12. Beauville, A.: Les familles stables de courbes elliptiques sur P^1 admettant quarte fibres singulières. C. R. Acad. Sci. Paris **294**, 657–660 (1982)
13. Beukers, F.: Irrationality proofs using modular forms. In: Journées Arithmétique de Besançon, Besançon (Besançon, 1985). Astérisque No. **147–148**, pp. 271–283 (1997)
14. Beukers, F.: On Dwork's accessory parameter problem. Math. Z. **241**(2), 425–444 (2002)
15. Beukers, F., Peters, C.: A family of K3 surfaces and $\zeta(3)$. J. Reine Angew. Math. **351**, 42–54 (1984)
16. Bogner, M.: Algebraic characterization of differential operators of Calabi–Yau type. arXiv:1304.5434 [math.AG]
17. Bogner, M.: On differential operators of Calabi–Yau type. Thesis, Mainz (2012)
18. Bogner, M., Reiter, S.: On symplectically rigid local systems and Calabi–Yau operators. J. Symbolic Comput. **48**, 64–100 (2013)
19. Brav, C., Thomas, H.: Thin Monodromy in Sp(4). Compos. Math. **150**(3), 333–343 (2014)
20. Candelas, P., de la Ossa, X., Green, P., Parkes, L.: An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds. Phys. Lett. B **258**(1–2), 118–126 (1991)
21. Cooper, S.: Sporadic sequences, modular forms and new series for $1/\pi$. Ramanujan J. **29**, 163–183 (2012)
22. Cooper, S.: Ramanujan's Theta Functions. Springer, Cham (2017)
23. Cynk, S., van Straten, D.: Picard–Fuchs operators for octic arrangements (the case of Orphans). arXiv:1709.09752v1 [math.AG]
24. Dettweiler, M., Reiter, S.: Middle convolution of Fuchsian systems and the construction of rigid differential systems. J. Algebra **318**(1), 1–24 (2007)

25. Dettweiler, M., Sabbah, C.: Hodge theory of the middle convolution. Publ. Res. Inst. Math. Sci. **49**(4), 761–800 (2013)
26. Doran, C., Morgan, J.: Mirror symmetry and integral variations of hodge structure underlying one parameter families of Calabi–Yau threefolds. In: Mirror Symmetry V, AMS/IP Studies in Advanced Mathematics, vol. **38**, pp. 517–537. American Mathematical Society, Providence, RI (2006)
27. Eskin, A., Kontsevich, M., Möller, M., Zorich, A.: Lower bounds for Lyapunov exponents of flat bundles on curves. Geom. Topol. **22**(4), 2299–2338 (2018)
28. Golyshev, V.: Classification problems and mirror duality. In: Surveys in Geometry and Number Theory: Reports on Contemporary Russian Mathematics, London Mathematical Society Lecture Note series vol. **338**, pp. 88–121. Cambridge University Press, Cambridge (2007)
29. Golyshev, V.: Modularity of the D3 equations and the Iskovskikh classification. Dokl. Akad. Nauk **396**(6), 733–739 (2004). (**Russian**)
30. Gorodetsky, O.: New representations for all sporadic Apéry-like sequences, with applications to congruences. arXiv:2102.11839v1
31. Hofmann, J.: Monodromy Calculations for Some Differential Equations. Thesis, Mainz (2013)
32. Meyer, C.: Modular Calabi–Yau threefolds. In: Fields Institute Monographs, vol. **22**. American Mathematical Society, Providence, RI (2005)
33. Miranda, R., Persson, U.: On extremal rational elliptic surfaces. Math. Z. **193**, 537–558 (1986)
34. Rodriques-Villegas, F.: Hypergeometric families of Calabi–Yau manifolds, In: Yui, N., Lewis, J.D. (eds.) Calabi–Yau Varieties and Mirror Symmetry, Fields Institute Communications, vol. **38**, pp. 223–231. American Mathematical Society, Providence, RI, (2003)
35. Samol, K.: Frobenius polynomial for Calabi–Yau equations. Thesis, Mainz (2010)
36. Samol, K., van Straten, D.: Frobenius polynomials for Calabi–Yau equations. Commun. Number Theory Phys. **2**(3), 537–561 (2008)
37. Schmickler-Hirzebruch, U.: Elliptische Flächen über P1 mit drei Ausnahmefasern und die hypergeometrische Differentialgleichung, Schriftenreihe des Mathematischen Instituts der Universität Münster (1985)
38. Singh, S.: Arithmeticity of the 4 monodromy groups associated to the Calabi–Yau threefolds. Int. Math. Res. Not. **2015**(18), 8874–8889 (2015)
39. Singh, S., Venkataramana, T.N.: Arithmeticity of certain symplectic hypergeometric groups. Duke Math. J. **163**(3), 591–617 (2014)
40. Stienstra, J., Beukers, F.: On the Picard–Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces. Math. Ann. **271**(2), 269–304 (1985)
41. van Straten, D.: Calabi–Yau operators. In: Ji, Lizhen, Yau, Shing-Tung. (eds.) Uniformization, Riemann–Hilbert Correspondence, Calabi–Yau Manifolds and Picard–Fuchs Equations, Advanced Lectures in Mathematics, vol. 42. International Press, Boston (2018)
42. Verrill, H.: Picard–Fuchs equations of some families of elliptic curves. In: Proceedings on Moonshine and Related Topics (Montréal, QC, 1999), CRM Proceedings and Lecture Notes, vol. 30, pp. 253–268. American Mathematical Society, Providence, RI (2001)
43. Vologodsky, V.: On the N-integrality of instanton numbers (2008) arXiv:0707.4617 [math.AG]
44. Zagier, D.: Integral solutions of Apéry-like recurrence equations. In: Groups and Symmetries, CRM Proceedings and Lecture Notes, vol. 47, pp. 349–366. American Mathematical Society, Providence, RI (2009)