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Emmy Noether and Her Theorems

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Today Noether's principal theorem occupies a prominent place in theoretical physics, though for a long time its significance was largely overlooked. Even now, relatively few physicists realize that Emmy Noether's original paper from 1918 contains two fundamental theorems. Moreover, both theorems are essential for understanding her original motivation, namely to distinguish between proper and improper conservation laws in physics.

1. Two Theorems and their Reception

For physicists, Emmy Noether's name is connected with her two theorems, famously linking symmetries and conservation laws. For mathematicians, she is the "mother of modern algebra," a field in which her work, but even more her dynamic personality, exerted a profound influence on many who followed in her footsteps. As the historian Mechthild Koreuber has emphasized, Noether's role in mathematics cannot be likened with someone who headed a conventional mathematical school. Her impact was far more pervasive; she saw herself as leading a broad movement that operated as a "thought collective" (*Denkkollektiv*)[1] and which promoted algebraic methods in nearly all branches of pure mathematics (**Figure 1**).

After her forced emigration from Nazi Germany in 1933, Noether taught at Bryn Mawr College for women, though only up until her untimely death in April 1935 at the age of 53. She also gave lectures at the Institute for Advanced Study (IAS) in Princeton. One of the young algebraists who attended Noether's final lectures at the IAS was Nathan Jacobson. Another was Richard Brauer, who in a letter to her brother, Fritz, reported on the circumstances surrounding her sudden death due to complications following surgery.[2] Jacobson, later an eminent figure in the American mathematical community as a Yale professor, edited Noether's Collected Papers[3] and wrote a brief introduction summarizing her most important works, except for two. He knew, of course, that Noether's Theorem was an important tool in particle physics, and so he asked his colleague, the Turkish physicist Feza Gürsey, to write about her two papers connected with physical theories. One of these, "Invariante Variationsprobleme,"[4] is today considered one of her most important works because of its relevance for particle physics (**Figure 2**).[5]

Today, in fact, Noether's name is probably better known among younger physicists than it is to mathematicians. That alone tells us something about new trends and massive shifts in perspective that have taken place during the last 40 years. In 1980, when Herbert Goldstein published the second edition of his textbook on classical mechanics,[6] he added a final section on "Noether's Theorem," which afterward gradually became a standard topic in physics courses. This marks part of the growing recognition that conserved quantities are closely related with the symmetries associated with Lie algebras or Lie groups. Gürsey summarized the general viewpoint of modern theoretical physicists when he wrote:

> Noether's Theorem associates each element of the Lie algebra (generator of a one parameter transformation of the group) with a corresponding conserved quantity. ...Before Noether's Theorem the principle of conservation of energy was shrouded in mystery, leading to the obscure physical systems of Mach and Ostwald. Noether's simple and profound mathematical formulation did much to demystify physics.[7]

This bird's-eye view from afar sheds no light on what actually happened. For one thing, the ideas of Ernst Mach and Wilhelm Ostwald were long since forgotten when Noether wrote her paper. But more to the point, if physicists and mathematicians who knew her work—like Albert Einstein, Hermann Weyl, and Wolfgang Pauli—had read it this way, then surely it would have made quite a splash. The fact that it did not cause much attention, however, has been firmly established by Yvette Kosmann-Schwarzbach, author of a detailed examination of the paper and the very weak reception it received for decades afterward.[8] This raises a serious question: why was Noether's paper presenting "Noether's Theorem" largely neglected and overlooked by the mathematical and physical communities until long after her death? Was this a simple case of sexist discrimination? That may have been a factor, but other conceptual issues were also involved here, and so this question is not at all easy to answer.

2. General Relativity during World War I

When I first wrote about how Noether's work was situated within the immediate reception of general relativity in Göttingen, I stressed that her seminal paper was not about one theorem but rather about two "Noether theorems" and that the less familiar

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Figure 1. Emmy Noether in 1930 aboard a ship crossing the Baltic to attend a conference of the German Mathematical Society held in Knigsberg. Photo by Helmut Hasse.

second theorem was her main result at that time.[9] She wrote the paper in order to clarify the distinction between conservation laws in special relativity and other types of derived relations in general relativity. In both cases, one utilizes an invariant variational principle to derive these results. In theories governed by the principle of general covariance this yields formal relations which have no analogue in special relativity. Thus, Noether's first theorem establishes a framework for deriving physically conserved quantities from an invariant variational system. Her second theorem, on the other hand, describes how additional relations arise when that invariance holds under general coordinate transformations.

These types of relations were used by both Einstein and Hilbert in constructing two different formulations of energy conservation in general relativity. In a letter to Einstein, written in May

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918 1).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen 2). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17

Figure 2. Titlepage of Emmy Noether's paper,[4] in which she proved the Noether theorems.

1916, Hilbert conjectured that their two approaches were either identical or closely related, and he informed him that he had given this problem to Emmy Noether.[10] This marks the beginning of her involvement with problems relating to the physical status of energy laws in general relativity. Very little is known about her work from that time, though she was able to show formal similarities between Hilbert's approach to energy conservation and Einstein's. More than a year later, the discussions in Göttingen on general relativity became much more intense after Klein and Einstein began corresponding about these matters. Indeed, Noether's two papers were written in close consultation with Felix Klein.[11,12] So how did this all begin?

In the spring of 1915, Emmy Noether came to Göttingen from Erlangen, where her father was the well-known professor of mathematics, Max Noether. Emmy's brother, Fritz, had studied under Arnold Sommerfeld in Munich before gaining a position as a private lecturer in Karlsruhe; he later became an accomplished applied mathematician in Breslau. Both she and her brother had spent time in Göttingen, where their father and Klein first met in the late 1860s, the beginning of a lifelong friendship. Emmy's interests in algebraic invariant theory overlapped with Hilbert's, whereas both he and Klein had ample opportunity to appreciate her abilities as a researcher. Moreover, it was wartime. On 1 December 1914, Noether wrote a letter to Hilbert that ended with the news that her brother was stationed north of Reims and doing well.[13,19] When Hilbert and Klein decided to invite Emmy Noether to habilitate in Göttingen the following year, quite a few younger men were engaged in war service. This was probably the one and only time in Emmy Noether's life when she benefited from being a woman.

Hilbert and Klein also both had longstanding interests in physics, and they had funds available from the Wolfskehl Foundation to invite prominent foreign guest speakers. Henri Poincaré delivered the first series of Wolfskehl lectures in 1909, followed a year later by H.A. Lorentz's lectures, which the young Max Born wrote up for publication. During the early summer of 1915, the invited speaker was Albert Einstein, who spoke about his new theory of gravitation. There, Einstein gave an account of the state of the theory before his final breakthrough to full general covariance in the fall of 1915.[14] And although very little is known about what he said, Hilbert was genuinely excited by what he heard in those six lectures and probably even more by his discussions with Einstein during that week. But what about Emmy Noether? Did she interact with Einstein at all back then?

Unfortunately, like many other aspects of Noether's life, this relationship is not well documented, even though at the time of her death Einstein wrote a well-known letter for the *New York Times* in which he underscored her unique talents.[15,16] Sometimes people have even asked whether he knew her personally at all, and I recently discussed the possibility that they only first met in Princeton, after she was forced out of her position in Göttingen.[17] Of course, she would have met him already in the summer of 1915 had she attended his lectures, but until recently that seemed rather doubtful in view of the fact that she went back to Erlangen in May, at which time her mother passed away. Her father was an elderly invalid, who had been stricken with polio as a child, and Emmy was his only daughter. So we have no direct evidence that Emmy Noether was present when Einstein delivered his Wolfskehl lectures.

Thanks to recent research undertaken by Cordula Tollmien, however, there is no longer any doubt that Noether must have met Einstein during the week of his visit, because she had already returned to Göttingen the month before. This is important to know, because it helps to anchor some of the exchanges that took place afterward between Einstein and the mathematicians in Göttingen. During his visit there, Einstein stayed with the Hilberts, and so he was clearly made aware of an impending plan to petition the Prussian Ministry for an exemption from explicit regulations that prohibited women from being allowed to teach. He was surely also told about Noether's background and research interests, especially those aspects connected with Hilbert's own work. How she thereafter gradually took up various questions relating to the mathematics of general relativity is a complicated story. The literature listed below deals with it in considerable detail.

3. Emmy Noether steps back Onstage

In recent years, many new studies have appeared that have led to a better understanding of Emmy Noether's life, work, and times. Philosophers of modern physics have undertaken probing new investigations of the Noether theorems and their implications in ref. [18]. Moreover, the historian Cordula Tollmien has embarked on a multi-volume project, which will illuminate many new facets of Noether's life. Her first two books in this series,[19] and,[13] document in vivid detail the final obstacles Noether had to overcome in order to attain the lowly position of a private lecturer at a German university. The mathematicians who supported her case—above all David Hilbert and Felix Klein—were famously influential figures. Yet even they encountered such deeply entrenched resistance to the very notion of allowing a woman to teach that her appointment proved impossible. Only after the collapse of the German monarchy and the appointment of a Social Democrat as Prussian Minister of Education did this last barrier fall as well. Beginning in 1919, Noether taught her own courses, eventually attracting students from all over the world. Not once, however, was she ever offered a professorship in Germany.

Beginning in 2019, many people have gained a new appreciation for Emmy Noether's unique career through a play produced by Portraittheater Vienna, an ensemble that specializes in bringing famous or lesser-known women back to life onstage. The original German version premiered on the 100th anniversary of Noether's final qualifying lecture, after which she gained the right to teach courses as a member of the Göttingen faculty. Since then, "Diving into Math with Emmy Noether" (the English version) has been performed at fourteen colleges and universities throughout the United States (see ref. [20]) and at numerous localities in Europe. For information about past and future performances, see https://www.portraittheater.net/.

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Conflict of Interest

The author declares no conflict of interest.

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[1] M. Koreuber, *Emmy Noether, die Noether-Schule und die moderne Algebra. Zur Geschichte einer kulturellen Bewegung*, Springer, Berlin, Heidelberg **2015**.
[2] D. E. Rowe, *Emmy Noether: Mathematician Extraordinaire*, Springer Nature, Cham **2021**, pp. 258–260.
[3] E. Noether, *Gesammelte Abhandlungen – Collected Papers*, Springer, Berlin, Heidelberg **1983**.
[4] E. Noether, *Nachr. Akad. Wiss. Goettingen, Math.-Phys. Kl.* **1918**, *2*, 235.
[5] A. Borrelli, in *Wie kommt das Neue in die Welt? Emmy Noether, die Noether-Theoreme und die moderne Algebra* (Ed: M. Koreuber), Springer, Cham to appear **2024**.
[6] H. Goldstein, *Classical Mechanics*, 2nd ed., Addison-Wesley, Reading, MA **2017**.
[7] E. Noether, in *Gesammelte Abhandlungen – Collected Papers*, Springer, Berlin, Heidelberg **1983**, p. 23.
[8] Y, Kosmann-Schwarzbach, *Les Théorèmes de Noether. Invariance et lois de conservation au XXe siècle, avec une traduction de l'article original Invariante Variationsprobleme*, Éditions de l'École Polytechnique, deuxième édition, révisée et augmentée, Springer, Berlin, Heidelberg **2004**.
[9] D. E. Rowe, in *The Symbolic Universe. Geometry and Physics, 1890-1930* (Ed: J. Gray), Oxford University Press, Oxford **1999**, pp. 189–233.
[10] David Hilbert, in *The Collected Papers of Albert Einstein. Vol. 8A, The Berlin Years, Correspondence 1914–1917* (Eds: R. Schulmann, A. J. Kox, M. Janssen, J. Illy), Princeton University Press, Princeton, NJ **1998**, pp. 290–292.
[11] D. E. Rowe, in *The Philosophy and Physics of Noether's Theorems* (ref. 18), pp. 25–50.
[12] T. Sauer, in *Wie kommt das Neue in die Welt? Emmy Noether, die Noether-Theoreme und die moderne Algebra* (Ed: M. Koreuber), Springer, Cham to appear **2024**.
[13] C. Tollmien, *Die Lebens- und Familiengeschichte der Mathematikerin Emmy Noether in Einzelaspekten: "Wir bitten nur um Dispens für den vorliegenden einzigartig liegenden Fall" – die Habilitation Emmy Noethers*, tredition, Hamburg **2022**.
[14] M. Janssen, J. Renn, *How Einstein found his Field Equations*, Birkhuser, Basel **2022**.
[15] A. Einstein, "The Late Emmy Noether" *The New York Times*, 4 May **1935**, https://www.nytimes.com/1935/05/04/archives/the-late-emmy-noether-professor-einstein-writes-in-appreciation-of.html (accessed: November 2023).
[16] R. Siegmund-Schultze, *Mitt. Dtsch. Math.-Ver. 15*, 221.
[17] D. E. Rowe, *Emmy Noether: Mathematician Extraordinaire*, Springer Nature, Cham **2021**, pp. 262–266.
[18] *The Philosophy and Physics of Noether's Theorems* (Eds: James Read, Nicholas Teh), Cambridge University Press, Cambridge **2022**.
[19] C. Tollmien, *Die Lebens- und Familiengeschichte der Mathematikerin Emmy Noether in Einzelaspekten: "Kann eine Frau Privatdozentin werden?" – die Umfrage des Preußischen Kultusministeriums zur Habilitation von Frauen 1907*, tredition, Hamburg **2021**.
[20] D. E. Rowe, *Math. Intell.* **2023**, *45*, 177.