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Urata's theorem in the logarithmic case and applications to integral points

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Abstract

Urata showed that a pointed compact hyperbolic variety admits only finitely many maps from a pointed curve. We extend Urata's theorem to the setting of (not necessarily compact) hyperbolically embeddable varieties. As an application, we show that a hyperbolically embeddable variety over a number field K with only finitely many $\mathcal{O}_{L,T}$-points for any number field L/K and any finite set of finite places T of L has, in fact, only finitely many points in any given $\mathbb{Z}$-finitely generated integral domain of characteristic zero. We use this latter result in combination with Green's criterion for hyperbolic embeddability to obtain novel finiteness results for integral points on symmetric self-products of smooth affine curves and on complements of large divisors in projective varieties. Finally, we use a partial converse to Green's criterion to further study hyperbolic embeddability (or its failure) in the case of symmetric self-products of curves. As a by-product of our results, we obtain the first example of a smooth affine Brody-hyperbolic threefold over $\mathbb{C}$ which is not hyperbolically embeddable.

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1 INTRODUCTION

A point of view emphasized by Lang, beginning with at least his seminal work [35], posits that Diophantine statements involving rational points over number fields (or integral points over rings of integers) should continue to hold over arbitrary finitely generated fields over $\mathbb{Q}$ (or their $\mathbb{Z}$-finitely generated subrings), and that this is the natural setting for such results. In this direction, after work of Siegel [52], Mahler [43], and Parry [49] in the classical setting of rings of (S-)integers, Lang showed that the unit equation

$$u + v = 1, \quad u, v \in A^*,$$

has only finitely many solutions when A is any $\mathbb{Z}$-finitely generated integral domain of characteristic zero. More generally, Lang [35] proved Siegel's theorem [53] on integral points on affine curves in this general setting (the unit equation corresponding to integral points on the affine curve $\mathbb{A}^1 \setminus \{0, 1\}$). Lang first proved these results for rings of S-integers in number fields, and then used a specialization argument to deduce the general case.

Our main theorem provides a systematic approach to such results for integral points on affine varieties, frequently allowing one to extend results known over number fields (where there are many arithmetic tools available) to the aforementioned general setting. More precisely, our method works for affine varieties that are *hyperbolically embeddable* (a purely complex-analytic notion due to Kobayashi). Thus, our work provides another manifestation of the deep conjectural connections between arithmetic geometry and complex-analytic notions of hyperbolicity (see Conjecture 1.2). We note that it had already been suggested by Lang that hyperbolic embeddability may be the correct notion of hyperbolicity to use in relation to integral points on affine varieties (Conjecture 1.3). Furthermore, a criterion of Green (Theorem 4.1) makes it practical to check hyperbolic embeddability in many cases.

Before stating our precise results, we introduce some terminology. Let k be an algebraically closed field of characteristic zero. By a *variety* over k, we mean a finite type separated scheme over k. Moreover, a variety X over k is said to be *arithmetically hyperbolic over* k if there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a finite type separated A-scheme $\mathcal{X}$ with $\mathcal{X}_k \cong X$ over k such that, for all $\mathbb{Z}$-finitely generated subrings $A' \subset k$ containing A, the set $\mathcal{X}(A')$ of A'-points on $\mathcal{X}$ is finite. Note that, if X is arithmetically hyperbolic over k, then, for every $\mathbb{Z}$-finitely generated subring $A \subset k$ and every finite type separated A-scheme $\mathcal{X}$ with $\mathcal{X}_k \cong X$ over k, we have that $\mathcal{X}(A)$ is finite; see [27, Lemma 4.8].

If X_L is arithmetically hyperbolic over L for all algebraically closed field extensions $L \supset k$, then we say that X is *absolutely arithmetically hyperbolic*. This extends to quasi-projective varieties Lang's notions [36] of the Mordell/Siegel property for projective/affine varieties; see also [3, 55, 59] and [6, 20, 21, 24–26, 30]. We refer to [3, 8, 14–16, 41, 45, 57, 58] for examples of arithmetically hyperbolic varieties.

Motivated by the previously mentioned principle, we are interested in the *persistence* of the arithmetic hyperbolicity of a variety along field extensions. The formal statement we are interested in is the 'Persistence Conjecture' for quasi-projective varieties (see also [6, Conjecture 1.20] or [20, Conjecture 17.5]):

Conjecture 1.1 (Persistence Conjecture). *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero, and let X be a variety over k. If X is arithmetically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*

If the Persistence Conjecture holds in full generality (that is, it holds for all extensions L/k of algebraically closed fields of characteristic zero and all varieties over k), then we may conclude that, for a variety X over k, we have that X is arithmetically hyperbolic over k if and only if X is absolutely arithmetically hyperbolic.

If X is a projective variety over k, then this conjecture is a consequence of a strong version of Lang–Vojta's conjectures as formulated in [20, Section 12]. Indeed, if X is a projective arithmetically hyperbolic variety over k, then every subvariety of X is of general type by this conjecture. Then, it follows that every subvariety of X_L is of general type (see, for instance, [29] for a proof of this well-known fact), so that (again by Lang–Vojta's conjecture), the variety X_L is arithmetically hyperbolic over L.

The Persistence Conjecture was first shown to hold for algebraically hyperbolic projective varieties in [21, Section 4]. It was subsequently shown to hold for varieties which admit a quasi-finite morphism to a semi-abelian variety [6, Theorem 7.4], projective surfaces with non-zero irregularity [6, Corollary 8.5], varieties which admit a quasi-finite period map [24], and certain moduli spaces of polarized varieties [28].

Our main result (Theorem 1.4) says that the Persistence Conjecture holds for hyperbolically embeddable smooth affine varieties. The proof relies on an extension to the affine case of a classical finiteness theorem of Urata for proper Kobayashi hyperbolic varieties. To explain this, we recall that, for proper varieties, Kobayashi's notion of hyperbolicity is the 'right' notion to consider. In this case, Lang [37] conjectured (see also [21, Conjecture 1.1]) that Kobayashi hyperbolicity and arithmetic hyperbolicity coincide:

Conjecture 1.2. *Let X be a projective variety over k. Then X is absolutely arithmetically hyperbolic if and only if for every subfield $k_0 \subset \mathbb{C}$, every embedding $k_0 \to k$, and every variety X_0 over k_0 with $X \cong X_0 \otimes_{k_0} k$, we have that $X_{0,\mathbb{C}}$ is Kobayashi hyperbolic.*

A well-known consequence of Conjecture 1.2 is that each conjugate of a Kobayashi hyperbolic variety is Kobayashi-hyperbolic. We stress that, given a Kobayashi hyperbolic (respectively, hyperbolically embeddable) variety over $\mathbb{C}$, it is not known whether the conjugates of X are Kobayashi hyperbolic (respectively, hyperbolically embeddable) in general.

Lang has extended this conjecture to affine varieties, but notes that in this case there are a priori three natural properties which may define 'hyperbolicity':

$$\text{Brody hyperbolic} \Leftarrow \text{Kobayashi hyperbolic} \Leftarrow \text{hyperbolically embeddable.} \tag{1.1}$$

Choosing the strongest property, Lang conjectures [38, p. 86] and [39, Conjecture 5.1, p. 225]:

Conjecture 1.3. *Let X be an affine variety over k. If there exists a subfield $k_0 \subset \mathbb{C}$, an embedding $k_0 \to k$, and a variety X_0 over k_0 with $X \cong X_0 \otimes_{k_0} k$ such that $X_{0,\mathbb{C}}$ is hyperbolically embeddable, then X is absolutely arithmetically hyperbolic.*

As a consequence of our main result, if X is a smooth affine variety over $\overline{\mathbb{Q}}$, then to prove Conjecture 1.3 it suffices to prove that X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ (that is, it suffices to prove finiteness of integral points in the classical case, over rings of S-integers of number fields).

It is known that, in general, for quasi-projective varieties the three properties of (1.1) are inequivalent. However, Lang has raised the question of which properties are equivalent in the affine case (for example, [38, p. 80] and [39, pp. 225–226]). We give a partial answer to Lang's question, showing that there are smooth affine varieties which are Brody hyperbolic but not hyperbolically embeddable; see Theorem 1.15.

We follow [33] and say that a reduced complex analytic space $\mathfrak{X}$ is Kobayashi hyperbolic if the Kobayashi pseudo-metric $d_{\mathfrak{X}}$ is a metric on $\mathfrak{X}$. A variety X over $\mathbb{C}$ is Kobayashi hyperbolic if the analytification X^{an}_{red} of the reduced scheme X_{red} is Kobayashi hyperbolic.

Let $\overline{S}$ be a smooth projective variety over $\mathbb{C}$ and let $S \subset \overline{S}$ be a dense open subscheme. We say that S is *hyperbolically embedded in* $\overline{S}$ if the inclusion $S^{\mathrm{an}} \subset \overline{S}^{\mathrm{an}}$ of the complex analytic subspace S^{an} in $\overline{S}^{\mathrm{an}}$ is a hyperbolic embedding [33, Chapter 3.3]; note that this implies that S is Kobayashi hyperbolic.

A smooth affine variety S over $\mathbb{C}$ is *hyperbolically embeddable* if there is a projective variety $\overline{S}$ and an open immersion $S \to \overline{S}$ such that S is hyperbolically embedded in $\overline{S}$. Note that, if S is a hyperbolically embeddable variety, then S is Kobayashi (hence Brody) hyperbolic. A hyperbolically embeddable variety shares many common features with proper Kobayashi hyperbolic varieties, as is shown, for instance, in [23, 33, 34].

We extend the above definitions to varieties over k (an arbitrary algebraically closed field of characteristic zero). If $\overline{X}$ is a proper variety over k and D is a closed subset of $\overline{X}$, then we will say that $X := \overline{X} \setminus D$ is *(weakly-)hyperbolically embedded in* $\overline{X}$ if there is a subfield $k_0 \subset k$, an embedding $k_0 \to \mathbb{C}$, a model $\overline{\mathcal{X}}$ for $\overline{X}$ over k_0, and a model $\mathcal{D} \subset \overline{\mathcal{X}}$ for $D \subset \overline{X}$ over k_0 such that $\overline{\mathcal{X}}_{\mathbb{C}} \setminus \mathcal{D}_{\mathbb{C}}$ is hyperbolically embedded in $\overline{\mathcal{X}}_{\mathbb{C}}$. This means, in particular, that $\mathcal{X}_{\mathbb{C}}$ is Kobayashi hyperbolic (with respect to the fixed embedding $k_0 \to \mathbb{C}$). We stress that this notion depends a priori on the chosen embedding. As we are interested in algebraic properties of (weakly-)hyperbolically embedded varieties, this is harmless for our purposes.

Theorem 1.4 (Main Result, I). *Let $\overline{X}$ be a smooth projective variety over k, and let $D \subset \overline{X}$ be a divisor such that $X := \overline{X} \setminus D$ is hyperbolically embedded in $\overline{X}$. If X is arithmetically hyperbolic over k, then X is* absolutely *arithmetically hyperbolic.*

To prove Theorem 1.4, we work with the notion of geometric hyperbolicity as defined in [6, 20, 24, 30]. A variety X over k is *geometrically hyperbolic over* k if, for every smooth integral curve C over k, every c in $C(k)$, and every x in $X(k)$, the set $\mathrm{Hom}_k((C,c),(X,x))$ of morphisms $f : C \to X$ with $f(c) = x$ is finite. As is explained in [20, Remark 11.3], one may consider this property as a 'function field' analogue of arithmetic hyperbolicity. In this paper, we will exploit the properties of a variety which is simultaneously arithmetically hyperbolic and geometrically hyperbolic.

To explain our proof of Theorem 1.4, we note that the following result, obtained in [21, 24], implies that the Persistence Conjecture holds for all varieties X over $\overline{\mathbb{Q}}$ such that $X_{\mathbb{C}}$ is geometrically hyperbolic over $\mathbb{C}$. This result has, for example, already been successfully applied to proving the Persistence Conjecture for varieties which admit a quasi-finite period map in [24].

Theorem 1.5. *Let k be an algebraically closed subfield of $\mathbb{C}$, and let X be an arithmetically hyperbolic variety over k such that $X \otimes_k \mathbb{C}$ is geometrically hyperbolic over $\mathbb{C}$. Then, the variety X is* absolutely *arithmetically hyperbolic.*

Theorem 1.5 implies that, to prove Theorem 1.4, it suffices to show the geometric hyperbolicity of a hyperbolically embeddable variety. Toward this end, we first consider a classical finiteness theorem of Urata for *proper* varieties; see [33, Theorem 5.3.10] (or the original [56]).

Theorem 1.6 (Urata). *If X is a proper Brody hyperbolic variety over $\mathbb{C}$, Y is an integral variety over $\mathbb{C}$, $y \in Y(\mathbb{C})$, and $x \in X(\mathbb{C})$, then the set of morphisms $f : Y \to X$ with $f(y) = x$ is finite. In particular, the variety X is geometrically hyperbolic over $\mathbb{C}$.*

Urata's proof uses the *properness* of X in a crucial way, and it is currently not known whether a Kobayashi hyperbolic quasi-projective variety over $\mathbb{C}$ is geometrically hyperbolic over $\mathbb{C}$. Indeed, when transporting Urata's arguments to the non-proper setting, one is confronted with several technical difficulties. Our next result shows that one can prove the expected finiteness property for hyperbolically embedded varieties, under suitable assumptions.

Theorem 1.7 (Main Result, II). *Let $\overline{X}$ be a smooth projective variety over k and let $D \subset \overline{X}$ be a divisor such that $X := \overline{X} \setminus D$ is hyperbolically embedded in $\overline{X}$. If Y is an integral variety over k, $y \in Y(k)$, and $x \in X(k)$, then the set of morphisms $f : Y \to X$ with $f(y) = x$ is finite. In particular, the variety X is geometrically hyperbolic over k.*

The proof of Theorem 1.4 is thus achieved by combining our extension of Urata's theorem (Theorem 1.7) with Theorem 1.5. To prove Theorem 1.7, we use Kobayashi's result on the compactness of the moduli space of maps from a curve to X and a result of Pacienza-Rousseau on the logarithmic-algebraic hyperbolicity of X; see Section 2.

As a first application of Theorem 1.4, we consider a finiteness result of the second-named author for varieties over number fields [40, Theorems 6.1 A(b) and 6.2A(d)], [19, Theorem 1.4] (after work of Corvaja–Zannier [9–11]; see also work of Autissier [3, 4]), and extend the result to finitely generated fields.

Theorem 1.8 (Main Result, III). *Let m be a positive integer, let X be a smooth projective connected variety over $\overline{\mathbb{Q}}$ with $\dim X > 1$, and let $D = \sum_{i=1}^{r} D_i$ be a sum of r ample effective divisors on X such that at most m of the divisors D_i meet in a point. Suppose $r \geqslant 2m \dim(X)$ (or $r \geqslant 5$ if $m = \dim X = 2$). Then the affine variety $X \setminus D$ is absolutely arithmetically hyperbolic.*

More concretely, Theorem 1.8 says that, for every $\mathbb{Z}$-finitely generated integral domain A of characteristic zero and any model $\mathcal{X} \setminus \mathcal{D}$ for $X \setminus D$ over A, the set $(\mathcal{X} \setminus \mathcal{D})(A)$ of A-integral points on this model is finite. Assuming $\dim A = 1$, this finiteness is proven in [40] by the second-named author, building on seminal work of Corvaja–Zannier [9–11]. The proof in [40] relies crucially on Schmidt's Subspace Theorem (as does Corvaja–Zannier's work). Our proof of Theorem 1.8 combines the results of [40] (over number fields) with the above 'complex-analytic' results. It seems, however, interesting to note that one could also instead appeal to the function field version of the Subspace Theorem [5, 51, 61], and 're-do' some of the arguments in [40] to obtain Theorem 1.8. This line of reasoning is pursued in [42].

Using results of Noguchi–Winkelmann [48, Theorems 7.3.4 and 9.7.6] (see also [47]), Theorem 1.8 can be improved when the involved divisors generate a subgroup of small rank in the Néron–Severi group $\mathrm{NS}(X)$.

Theorem 1.9 (Main Result, IV). *Let X be a smooth projective connected variety over $\overline{\mathbb{Q}}$, and let $D = \sum_{i=1}^{r} D_i$ be a sum of r ample effective divisors on X in general position. Let $\mathrm{rank}\{D_i\}_{i=1}^{r}$ denote the (free) rank of the subgroup of $\mathrm{NS}(X)$ generated by the images of $D_1 \ldots, D_r$. Suppose that $r \geqslant 2 \dim X + \mathrm{rank}\{D_i\}_{i=1}^{r}$. Then the affine variety $X \setminus D$ is absolutely arithmetically hyperbolic.*

Lang conjectured that different notions of 'hyperbolicity' for projective algebraic varieties should be equivalent. As discussed earlier, for affine varieties it is not so clear whether one should expect all notions of hyperbolicity to be equivalent. In [41], the second-named author studied the case of symmetric powers of smooth affine curves and obtained the following version of Lang–Vojta's conjecture; see [2] for related results.

Theorem 1.10 [41]. *Let X be a smooth affine connected curve over $\overline{\mathbb{Q}}$, and let $d \geqslant 1$ be an integer. Then the following three statements are equivalent.*

1. *The smooth affine variety Sym^d_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$.*
2. *For every embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, the complex-algebraic variety $\mathrm{Sym}^d_{X_{\mathbb{C}}}$ is Brody hyperbolic.*
3. *For every dense open $U \subset X$, every finite morphism $U \to \mathbb{G}_{m,\overline{\mathbb{Q}}}$ is of degree at least $d+1$.*

We prove the following strengthening of Theorem 1.10, under a suitable assumption on the boundary of X in $\overline{X}$; see [54] for related results.

Theorem 1.11 (Main Result, V). *Let $d \geqslant 1$ be an integer, and let X be a smooth affine connected curve over $\overline{\mathbb{Q}}$, with smooth projective model $\overline{X}$, such that $\#\overline{X} \setminus X \geqslant 2d$. Then the following statements are equivalent.*

1. *The smooth affine variety Sym^d_X is absolutely arithmetically hyperbolic.*
2. *For every embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, the variety $\mathrm{Sym}^d_{X_{\mathbb{C}}}$ is hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}_{\mathbb{C}}}$.*
3. *For every dense open $U \subset X$, every finite morphism $U \to \mathbb{G}_{m,\overline{\mathbb{Q}}}$ is of degree at least $d+1$.*

If $\#\overline{X} \setminus X \geqslant 2d+1$, then it is easy to see that (3) in Theorem 1.11 always holds. In particular, we find (see Section 4 for the validity over $\mathbb{C}$):

Corollary 1.12. *Let X be a smooth affine connected curve over $\mathbb{C}$ and let $\overline{X}$ be its smooth projective compactification. If $\#\overline{X} \setminus X \geqslant 2d+1$, then Sym^d_X is hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}}$.*

The corollary may be viewed as a generalization of a well-known result, going back to Bloch, Cartan, and Dufresnoy (see also work of Green [18] and Fujimoto [17]) which asserts that the complement of $2n+1$ hyperplanes in general position in $\mathbb{P}^n$ is hyperbolically embedded in $\mathbb{P}^n$. Indeed, if $X \subset \overline{X} = \mathbb{P}^1$ and $r = \#\overline{X} \setminus X$, then Sym^d_X is isomorphic to the complement of r hyperplanes in general position in $\mathrm{Sym}^d_{\mathbb{P}^1} \cong \mathbb{P}^d$. Thus, Corollary 1.12 is a generalization of this result to symmetric powers of curves.

Quite interestingly, the condition on the boundary in Theorem 1.11 can be removed for the second symmetric self-product (see Corollary 4.4). Thus, we obtain the following strong version of Lang–Vojta's conjecture for the affine surface Sym^2_X.

Theorem 1.13 (Main Result, VI). *Let X be a smooth affine connected curve over $\overline{\mathbb{Q}}$ with smooth projective model $\overline{X}$. Then the following statements are equivalent.*

1. *The smooth affine surface Sym^2_X is absolutely arithmetically hyperbolic.*
2. *For every embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, the variety $\mathrm{Sym}^2_{X_{\mathbb{C}}}$ is hyperbolically embedded in $\mathrm{Sym}^2_{\overline{X}_{\mathbb{C}}}$.*
3. *For every dense open $U \subset X$, every finite morphism $U \to \mathbb{G}_{m,\overline{\mathbb{Q}}}$ is of degree at least 3.*

The next result shows that the second and third statements of Theorem 1.11 may be inequivalent when $d \geqslant 3$ and the numerical condition is not satisfied.

Theorem 1.14 (Main Result, VII). *For every $d \geqslant 3$, there exists a smooth affine connected curve X over $\overline{\mathbb{Q}}$, with smooth projective model $\overline{X}$, such that $\#\overline{X} \setminus X = 2d - 1$ and the following statements hold.*

(1) *The smooth affine variety Sym^d_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$.*
(2) *For every embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, the smooth affine variety $\mathrm{Sym}^d_{X_\mathbb{C}}$ is Brody hyperbolic, but* not *hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}_\mathbb{C}}$.*
(3) *For every dense open $U \subset X$, every finite morphism $U \to \mathbb{G}_{m,\overline{\mathbb{Q}}}$ is of degree at least $d + 1$.*

The fact that $\mathrm{Sym}^d_{X_\mathbb{C}}$ is not hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}_\mathbb{C}}$ for X as in Theorem 1.14 still leaves open the possibility that the smooth affine variety $\mathrm{Sym}^d_{X_\mathbb{C}}$ is hyperbolically embeddable. However, by carefully choosing the curve X, we can show that $\mathrm{Sym}^3_{X_\mathbb{C}}$ is not hyperbolically embeddable, thereby providing the first examples of Brody hyperbolic smooth affine threefolds which are *not* hyperbolically embeddable.

Theorem 1.15 (Main Result, VIII). *Let $\overline{X}$ be a smooth projective connected hyperelliptic curve of genus $g \geqslant 3$ over $\overline{\mathbb{Q}}$, and let $\iota : \overline{X} \to \overline{X}$ be the hyperelliptic involution. Let P_1, P_3, P_5 be pairwise distinct non-Weierstrass points, let $P_2 := \iota(P_1)$ and $P_4 := \iota(P_3)$, and write $X := \overline{X} \setminus \{P_1, P_2, P_3, P_4, P_5\}$. Then the following statements hold.*

(1) *The smooth affine threefold Sym^3_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$.*
(2) *For every embedding $\sigma : \overline{\mathbb{Q}} \to \mathbb{C}$, every projective variety Y over $\mathbb{C}$ and every open immersion $\mathrm{Sym}^3_{X_\sigma} \to Y$, the smooth affine threefold $\mathrm{Sym}^3_{X_\sigma}$ is Brody hyperbolic, but* not *hyperbolically embedded in Y.*

Conventions. Throughout this paper, we let k be an algebraically closed field of characteristic zero. A variety over k is a finite type separated scheme over k. If X is a variety over k and $A \subset k$ is a subring, then a *model for X* over A is a pair $(\mathcal{X}, \phi)$ with $\mathcal{X} \to \operatorname{Spec} A$ a finite type separated scheme and $\phi : \mathcal{X} \otimes_A k \to X$ an isomorphism of schemes over k. We omit ϕ from our notation. If X is a locally finite type scheme over $\mathbb{C}$, we let X^{an} denote the associated complex analytic space.

2 | URATA'S THEOREM IN THE LOGARITHMIC CASE

We follow Kovács–Lieblich's terminology and use the notion of 'weak boundedness' for quasi-projective schemes; see [31].

Definition 2.1 (Kovács–Lieblich). Let $\overline{X}$ be a projective scheme over k, let $\mathcal{L}$ be an ample line bundle on $\overline{X}$, and let $X \subset \overline{X}$ be a dense open subscheme. We say that X is *weakly bounded over k* in $\overline{X}$ with respect to $\mathcal{L}$ if, for every integer $g \geqslant 0$, and every $d \geqslant 0$, there is a real number $\alpha(X, \overline{X}, \mathcal{L}, g, d)$ such that, for every smooth projective connected curve $\overline{C}$ over k of genus g, every dense open subscheme $C \subset \overline{C}$ with $\#(\overline{C} \setminus C) = d$, and every morphism $f : C \to X$, the inequality

$$\deg_{\overline{C}} \overline{f}^* \mathcal{L} \leqslant \alpha(X, \overline{X}, \mathcal{L}, g, d)$$

holds, where $\overline{f} : \overline{C} \to \overline{X}$ is the (unique) extension of $f : C \to X$.

For Y and X projective varieties over k, we let $\underline{\mathrm{Hom}}_k(Y, X)$ be the locally finite type scheme parametrizing morphisms from Y to X; see [46]. Recall that a projective scheme X over k is 1-bounded over k if, for every smooth projective connected curve C over k, the scheme $\underline{\mathrm{Hom}}_k(C, X)$ is of finite type over k; see [22, § 4]. By [22, Theorem 1.14], a *projective* variety X is weakly bounded (in itself) over k if and only if it is 1-bounded over k (as defined in [22, Definition 4.1]). Thus, the notion of weakly boundedness extends the notion of 1-boundedness for projective varieties to quasi-projective varieties.

Our starting point is the following basic proposition. It says that the set of morphisms from a curve to a weakly bounded variety is (naturally) a quasi-compact (Zariski-)constructible subset of a certain Hom-scheme. More precisely, if $\overline{X}$ is a projective variety over k and $\overline{C}$ is a smooth projective connected curve over k, then we are interested in morphisms $C \to X$, where $C \subset \overline{C}$ is a dense open of $\overline{C}$ and $X \subset \overline{X}$ is a dense open of $\overline{X}$, respectively. The set $\mathrm{Hom}_k(C, X)$ of such morphisms is naturally a subset of the set $\mathrm{Hom}_k(\overline{C}, \overline{X})$ of morphisms $\overline{C} \to \overline{X}$ by the valuative criterion for properness. We identify the latter set with the set of k-points of the scheme $\underline{\mathrm{Hom}}_k(\overline{C}, \overline{X})$.

Proposition 2.2 [24, Proposition 3.2]. *Let $\overline{X}$ be a projective variety over k, and let $X \subset \overline{X}$ be a dense open subscheme. Let $\overline{C}$ be a smooth projective curve and let $C \subset \overline{C}$ be a dense open subscheme. If there is an ample line bundle $\mathcal{L}$ on $\overline{X}$ such that X is weakly bounded over k in $\overline{X}$ with respect to $\mathcal{L}$, then $\mathrm{Hom}_k(C, X)$ is a quasi-compact constructible subset of $\underline{\mathrm{Hom}}_k(\overline{C}, \overline{X})(k)$.*

Note that the property of being weakly bounded in $\overline{X}$ may depend a priori on the choice of the ample line bundle $\mathcal{L}$ on $\overline{X}$ (and we ignore whether this is really the case). However, the conclusion of Proposition 2.2 is *independent* of $\mathcal{L}$.

Our aim is to prove Urata's theorem in the logarithmic setting. Recall that Urata's theorem for a proper Brody hyperbolic variety X over $\mathbb{C}$ follows quite easily from the fact that the moduli space of maps $C \to X$ from any smooth proper curve C is a proper scheme over $\mathbb{C}$, and thus in particular of finite type.

In the logarithmic setting, when X is hyperbolically embeddable, Kobayashi proves that the connected components of the space $\mathrm{Hom}_{\mathbb{C}}((C, c), (X, x))$ of maps $f : C \to X$ with $f(c) = x$ are compact, but does not provide any information on the boundedness of this space. The additional ingredient we need in the 'logarithmic' setting is the following theorem of Pacienza–Rousseau [50]. This theorem actually follows from their extension of a theorem of Demailly [13] on the algebraic hyperbolicity of Brody hyperbolic varieties. Indeed, Demailly proved that, if X is a Brody hyperbolic projective variety over $\mathbb{C}$, then X is algebraically hyperbolic over $\mathbb{C}$ (in the sense of [13, 22]). Pacienza–Rousseau's theorem is the natural extension of Demailly's theorem to the logarithmic setting.

Theorem 2.3 (Pacienza-Rousseau). *Let $\overline{X}$ be a smooth projective variety over $\mathbb{C}$, let $\mathcal{L}$ be an ample line bundle on $\overline{X}$, and let D be a divisor on $\overline{X}$ with $X := \overline{X} \setminus D$. If X is hyperbolically embedded in $\overline{X}$, then X is weakly bounded over $\mathbb{C}$ in $\overline{X}$ with respect to $\mathcal{L}$.*

Proof. This follows from [50, Theorem 5] (although Pacienza–Rousseau require D to have simple normal crossings, this is irrelevant to the proof). As our notation differs a bit from [50, Theorem 5], we provide the details.

Let $\overline{C}$ be a smooth projective connected curve of genus g over k, let $C \subset \overline{C}$ be a dense open with $d := \#(\overline{C} \setminus C)$, and let $f : C \to X$ be a morphism. Let $\overline{f} : \overline{C} \to \overline{X}$ be the unique extension of this morphism. Let $\overline{C}'$ be the normalization of the irreducible curve $\overline{f}(\overline{C})$, and note that $\overline{f} : \overline{C} \to \overline{X}$ factors over a morphism $\overline{f}' : \overline{C}' \to \overline{X}$. Let $\nu : \overline{C}' \to \overline{f}(\overline{C})$ be the normalization map. We follow [50] and let $i(\overline{f}(\overline{C}), D)$ be the number of elements in $\nu^{-1}(D)$. Note that, as C lands in $X = \overline{X} \setminus D$, the inequality $i(\overline{f}(\overline{C}), D) \leqslant d'$ holds, where d' is the cardinality of the complement of the image of C in $\overline{C}'$. Therefore, as X is hyperbolically embedded in the smooth projective variety $\overline{X}$ with $D := \overline{X} \setminus X$ a divisor, it follows from [50, Theorem 5] that there is an ϵ depending only on $\overline{X}$, D, and $\mathcal{L}$ such that

$$\epsilon \deg_{\overline{C}'} \overline{f}'^{*} \mathcal{L} \leqslant 2g' - 2 + i(\overline{f}(\overline{C}), D) \leqslant 2g' - 2 + d',$$

where g' is the genus of $\overline{C}'$. It follows from this inequality that

$$\deg_{\overline{C}} \overline{f}^{*} \mathcal{L} \leqslant \epsilon^{-1} \deg(\overline{C}/\overline{C}')(2g' - 2 + d') \leqslant \epsilon^{-1}(2g - 2 + d).$$

We define $\alpha := \epsilon^{-1}(2g - 2 + d)$ and see that, as required, the smooth affine variety X is weakly bounded in $\overline{X}$ with respect to $\mathcal{L}$. $\square$

Theorem 2.4. *Let $\overline{X}$ be a smooth projective variety over $\mathbb{C}$ and let $D \subset \overline{X}$ be a divisor such that $X := \overline{X} \setminus D$ is hyperbolically embedded in $\overline{X}$. If C is a smooth quasi-projective connected curve over $\mathbb{C}$ with smooth projective model $\overline{C}$, $c \in \overline{C}(\mathbb{C})$, and x in $\overline{X}(\mathbb{C})$, then the set of morphisms $\overline{f} : \overline{C} \to \overline{X}$ with $\overline{f}(C) \subset X$ and $\overline{f}(c) = x$ is finite.*

Proof. If $D = \emptyset$, then $X = \overline{X}$ is a projective Kobayashi hyperbolic variety in which case the result follows from Urata's theorem [33, Theorem 5.3.10]. Thus, we may and do assume $D \neq \emptyset$. Let $\mathcal{L}$ be an ample line bundle on $\overline{X}$. By Pacienza–Rousseau's theorem (Theorem 2.3), the variety X is weakly bounded over $\mathbb{C}$ in $\overline{X}$ with respect to $\mathcal{L}$. Therefore, by Proposition 2.2, the subset $\mathrm{Hom}_{\mathbb{C}}(C, X)$ is a quasi-compact constructible subset of $\underline{\mathrm{Hom}}_{\mathbb{C}}(\overline{C}, \overline{X})(\mathbb{C})$.

Now, we consider the quasi-compact constructible subset $\mathcal{F} := \mathrm{Hom}_{\mathbb{C}}(C, X)$ as a subset of the complex-analytic space $\underline{\mathrm{Hom}}_{\mathbb{C}}(\overline{C}^{\mathrm{an}}, \overline{X}^{\mathrm{an}}) = \underline{\mathrm{Hom}}_{\mathbb{C}}(\overline{C}, \overline{X})^{\mathrm{an}}$. Let $\overline{\mathcal{F}}$ be the analytic-closure of $\mathcal{F}$ in the analytification $\underline{\mathrm{Hom}}_{\mathbb{C}}(\overline{C}, \overline{X})^{\mathrm{an}}$ of the scheme $\underline{\mathrm{Hom}}_{\mathbb{C}}(\overline{C}, \overline{X})$, and note that $\overline{\mathcal{F}}$ has only finitely many connected components (as $\mathcal{F}$ is quasi-compact). By a theorem of Kobayashi [33, Theorem 6.4.10.(4)], the universal evaluation map

$$\overline{C} \times \overline{\mathcal{F}} \to \overline{C} \times \overline{X}, \quad (c, f) \mapsto (c, f(c))$$

has finite fibers. This implies that $\overline{C} \times \mathcal{F} \to \overline{C} \times \overline{X}$ has finite fibers, thereby proving the required finiteness statement. $\square$

We can now deduce the desired finiteness statement (that is, Urata's finiteness theorem in the logarithmic case) over arbitrary algebraically closed fields of characteristic zero.

Theorem 2.5. *Let $\overline{X}$ be a smooth projective variety over k, and let $D \subset \overline{X}$ be a divisor such that $X := \overline{X} \setminus D$ is hyperbolically embedded in $\overline{X}$. Then, X is geometrically hyperbolic over k.*

Proof. By [24, Lemma 2.4], we may and do assume that $k = \mathbb{C}$. Then, the statement clearly follows from Theorem 2.4. □

Lemma 2.6. *Assume that k is uncountable. Let X be a geometrically hyperbolic variety over k. Then, for every integral variety Y over k, every y in $Y(k)$, and every x in $X(k)$, the set of morphisms $f : Y \to X$ with $f(y) = x$ is finite.*

Proof. We argue by contradiction. Let Y be a d-dimensional integral variety over k, let $y \in Y(k)$, let $x \in X(k)$, and let

$$f_1, f_2, \ldots \in \mathrm{Hom}_k((Y, y), (X, x))$$

be a sequence of pairwise distinct morphisms from Y to X which send y to x. Note that $d > 1$ by our assumption that X is geometrically hyperbolic over k. For $n, m \in \mathbb{Z}_{\geqslant 1}$, we define $Y^{n,m} = \{p \in Y(k) \mid f_n(p) = f_m(p)\}$. Note that, for every $n \neq m$, the subset $Y^{n,m} \subset Y(k)$ is a proper closed subset. In particular, by the uncountability of k, there is a point $P \in Y(k)$ in the complement of the union $\cup_{n\neq m} Y^{n,m}$. Let C be a smooth integral curve over k in Y containing y and P. Then, the morphisms $f_i|_C : C \to X$ are pairwise distinct morphisms sending y to x. This contradicts the geometric hyperbolicity of X, and concludes the proof. □

Proof of Theorem 1.7. We may and do assume that k is uncountable. Then, by Theorem 2.5, the variety X is geometrically hyperbolic over k, so that the result follows from Lemma 2.6. □

Proof of Theorem 1.4. Combine Theorems 1.5 and 1.7. □

3 | COMPLEMENTS OF LARGE DIVISORS AND A RESULT OF NOGUCHI–WINKELMANN

We start with a result of the second-named author [40, Section 6]; see also [3, 4, 8, 12, 19]. We follow the setup in [40, Section 4], and consider the following data fixed.

- A positive integer m and a positive integer r.
- A smooth projective variety X over k with $\dim X > 1$.
- An effective divisor $D = \sum_{i=1}^{r} D_i$ on X such that at most m of the divisors D_i meet in a point.

In this situation, the arithmetic hyperbolicity (when $k = \overline{\mathbb{Q}}$) and the analytic hyperbolicity of the complement of D in X were verified in [40], under suitable 'positivity' assumptions on D.

Theorem 3.1 (Levin). *Assume that $r \geqslant 2m \dim(X)$ (or $r \geqslant 5$ if $m = \dim X = 2$) and that, for every i, the divisor D_i is ample. Then the following statements hold.*

(1) *If $k = \overline{\mathbb{Q}}$, then $X \setminus D$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$.*
(2) *If $k = \mathbb{C}$, then the smooth affine variety $X \setminus D$ is hyperbolically embedded in X.*

Proof. This is [40, Theorem 6.1 A, Theorem 6.2A, Theorem 6.1 B and Theorem 6.2B], with a slight improvement coming from [3] (see [19, p. 2]). □

We establish the following extensions of Theorem 3.1 (with the notation as stated above Theorem 3.1).

Theorem 3.2 (Geometric hyperbolicity). *Assume that $r \geqslant 2m\dim(X)$ (or $r \geqslant 5$ if $m = \dim X = 2$) and that, for every i, the divisor D_i is ample. Then, for every integral variety Y over k, every y in $Y(k)$, and every x in $(X \setminus D)(k)$, the set of morphisms $f : Y \to X \setminus D$ with $f(y) = x$ is finite. In particular, the variety $X \setminus D$ is geometrically hyperbolic over k.*

Proof. To prove the geometric hyperbolicity of $X \setminus D$, by [24, Lemma 2.4], we may and do assume that k is the field of complex numbers. Then, by the second part of Theorem 3.1, the (complex algebraic) variety $X \setminus D$ is hyperbolically embedded in X. Thus, the required finiteness statement follows from the logarithmic version of Urata's theorem (Theorem 1.7). □

Now using this result and the first part of Theorem 3.1 combined with Theorem 1.5 (or alternatively, simply applying Theorem 1.4 with Theorem 3.1), we find Theorem 1.8:

Theorem 3.3 (Arithmetic hyperbolicity). *Assume that $r \geqslant 2m\dim(X)$ (or $r \geqslant 5$ if $m = \dim X = 2$) and that, for every i, the divisor D_i is ample. If $k = \overline{\mathbb{Q}}$ and $k \subset L$ is an extension of algebraically closed fields, then $X_L \setminus D_L$ is arithmetically hyperbolic over L.*

The proof of the extension (Theorem 1.9) of Noguchi–Winkelmann's results follows in the same manner from:

Theorem 3.4 (Noguchi-Winkelmann [48, Theorems 7.3.4 and 9.7.6] (see also [47])). *Let X be a smooth projective connected variety over k, and let $D = \sum_{i=1}^{r} D_i$ be a sum of r ample effective divisors on X in general position. Let $\operatorname{rank}\{D_i\}_{i=1}^{r}$ denote the (free) rank of the subgroup of $\mathrm{NS}(X)$ generated by the images of $D_1 \dots, D_r$. Suppose $r \geqslant 2\dim X + \operatorname{rank}\{D_i\}_{i=1}^{r}$. Then the following statements hold.*

(1) *If $k = \overline{\mathbb{Q}}$, then $X \setminus D$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$.*
(2) *If $k = \mathbb{C}$, the smooth affine variety $X \setminus D$ is hyperbolically embedded in X.*

4 | HYPERBOLIC EMBEDDINGS OF SYMMETRIC PRODUCTS

In this section, we prove that symmetric products of smooth affine curves are hyperbolically embedded in the symmetric product of their smooth projective model, under suitable assumptions; see Corollaries 1.12 and 4.6. To prove these results, we will use the following theorem of Green.

Theorem 4.1 (Green). *Let Z be a smooth projective variety and let D be the union of Cartier divisors $D_1, \dots, D_m$. Then $Y = Z \setminus D$ is hyperbolically embedded in Z if the following two conditions are satisfied.*

(1) *Y is Brody hyperbolic.*
(2) *For any partition of indices $I \cup J = \{1, \dots, m\}$, the variety $\cap_{i\in I} D_i \setminus \cup_{j\in J} D_j$ is Brody hyperbolic.*

Proof. See [33, Theorem 3.6.13] (or the original [18]). □

We will also be interested in showing that certain symmetric products of curves are not hyperbolically embedded in their canonical model. To do so, we will use the following (partial) converse to Green's theorem due to Noguchi and Winkelmann [48, Theorem 7.2.13] (see also Zaidernberg's partial converse in [33, Theorem 3.6.18] and [62, 63]). (The reason we say 'partial converse' is because of the additional general position assumption.)

Theorem 4.2. *Let Z be a smooth projective variety and let D be the union of Cartier divisors $D_1, \dots, D_m$ in general position on Z. If $Y = Z \setminus D$ is hyperbolically embedded in Z, then:*

(1) *Y is Brody hyperbolic;*
(2) *for any partition of indices $I \cup J = \{1, \dots, m\}$, the variety $\cap_{i\in I} D_i \setminus \cup_{j\in J} D_j$ is Brody hyperbolic.*

We start with using Green's theorem and its partial converse to prove the following result.

Theorem 4.3. *Let X be a smooth affine connected curve over $\mathbb{C}$ and let $\overline{X}$ be its smooth projective compactification. Let d be a positive integer. Then Sym^d_X is hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}}$ if and only if for every subset $T \subset \overline{X} \setminus X$ with $0 \leqslant |T| < d$, the variety $\mathrm{Sym}^{d-|T|}_{X\cup T}$ is Brody hyperbolic.*

Proof. Let $\overline{X} \setminus X = \{P_1, \dots, P_r\}$. Let $\psi : \overline{X}^d \to \mathrm{Sym}^d_{\overline{X}}$ be the natural map and let $\pi : \overline{X}^d \to \overline{X}$ be one of the natural projections. Let $D_j = \psi_* \pi^* P_j$, $j = 1, \dots, r$. Then $D = \sum_{j=1}^r D_j$ is a normal crossings divisor on $\mathrm{Sym}^d_{\overline{X}}$ and $\mathrm{Sym}^d_X = \mathrm{Sym}^d_{\overline{X}} \setminus D$. In particular, the divisors $D_1, \dots, D_r$ are in general position.

Let $\emptyset \subset I \subset \{1, \dots, r\}$ and let $J = \{1, \dots, r\} \setminus I$. Then $\cap_{i\in I} D_i \cong \mathrm{Sym}^{d-|I|}_{\overline{X}}$ if $|I| \leqslant d$ and $\cap_{i\in I} D_i = \emptyset$ if $|I| > d$. In the first case, if $|I| \leqslant d$, then

$$\bigcap_{i\in I} D_i \setminus \bigcup_{j\in J} D_j \cong \mathrm{Sym}^{d-|I|}_{\overline{X}\setminus \cup_{j\in J}\{P_j\}} = \mathrm{Sym}^{d-|I|}_{\cup_{i\in I}\{P_i\}\cup X}.$$

Now the result follows from Green's theorem (Theorem 4.1) and its (partial) converse (Theorem 4.2), noting that when $|I| = d$, $\bigcap_{i\in I} D_i$ is a point. □

As we will show later, Brody hyperbolicity and being hyperbolically embeddable are not equivalent notions in general. However, it seems reasonable to suspect that in the case of smooth affine surfaces these two notions do in fact coincide. The following corollary is in accordance with this expectation.

Corollary 4.4. *Let X be a smooth affine connected curve over $\mathbb{C}$ and let $\overline{X}$ be its smooth projective compactification. Then Sym^2_X is hyperbolically embedded in $\mathrm{Sym}^2_{\overline{X}}$ if and only if Sym^2_X is Brody hyperbolic.*

Proof. By Theorem 4.3 with $d = 2$, the corollary is equivalent to showing that if Sym^2_X is Brody hyperbolic, then $X \cup \{P\}$ is Brody hyperbolic for any point $P \in \overline{X} \setminus X$. To do so, let $g(\overline{X})$ denote the genus of $\overline{X}$. If $g(\overline{X}) \geqslant 2$, then this is vacuous as $\overline{X}$ is Brody hyperbolic. If $g(\overline{X}) = 1$, then $X \cup \{P\}$ is Brody hyperbolic unless $X \cup \{P\} = \overline{X}$ and $\overline{X} \setminus X$ consists of a single point. But in this case Sym^2_X

is not Brody hyperbolic. If X is rational, then Sym^2_X is Brody hyperbolic if and only if $\#\overline{X}\setminus X \geqslant 5$, which implies that $X \cup \{P\}$ is Brody hyperbolic for any point $P \in \overline{X}\setminus X$. $\square$

Corollary 4.5. *Let $D = \overline{X}\setminus X$ and let d be a positive integer. If $\deg D \geqslant d$, then Sym^d_X is hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}}$ if and only if for every subset $T \subset D$, $t = |T|$, with $0 \leqslant t < d$, there is no finite morphism $f : \overline{X} \to \mathbb{P}^1_{\mathbb{C}}$ of degree at most $d - t$ such that $f(D\setminus T) \subset \{0, \infty\}$.*

Proof. Suppose that $\deg D \geqslant d$ and let $T \subset D$, $t = |T|$, with $0 \leqslant t < d$. Then, by the main (analytic) result of [41], $\mathrm{Sym}^{d-t}_{X\cup T}$ is Brody hyperbolic if and only if there is no finite morphism $f : \overline{X} \to \mathbb{P}^1_{\mathbb{C}}$ of degree at most $d - t$ such that $f(D\setminus T) \subset \{0, \infty\}$. Thus, the result follows from Theorem 4.3. $\square$

As a consequence, we obtain Corollary 1.12 from the introduction.

Proof of Corollary 1.12. Since $(2d + 1) - t > 2(d - t)$ for $t \geqslant 0$, this follows immediately from Corollary 4.5 as for any morphism $\phi : \overline{X} \to \mathbb{P}^1$, at most $2\deg\phi$ points of $\overline{X}$ can map to $\{0, \infty\}$. $\square$

Essentially the same proof gives:

Corollary 4.6. *Let X be a smooth affine connected curve over $\mathbb{C}$ and let $\overline{X}$ be its smooth projective compactification. If $\#\overline{X}\setminus X \geqslant 2d$ and Sym^d_X is Brody hyperbolic, then Sym^d_X is hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}}$.*

5 | INTEGRAL POINTS ON SYMMETRIC PRODUCTS

We follow [20, 22 29] and say that a variety V over k is *groupless over k* if, for every finite type connected group scheme G over k, every morphism $G \to V$ is constant. Note that Lang refers to such varieties as being 'algebraically hyperbolic'; see [37]. If V is affine, then V is groupless if and only if every morphism $\mathbb{G}_{m,k} \to V$ is constant [22, Lemma 2.5]. Moreover, by [22, Lemma 2.3], if L/k is an extension of algebraically closed fields, then V is groupless over k if and only if V_L is groupless over L.

We are concerned with symmetric products of smooth affine connected curves. Let X be a smooth affine connected curve over k, and let $d \geqslant 1$. An obvious obstruction to the arithmetic hyperbolicity of Sym^d_X is the existence of a non-constant morphism $\mathbb{G}_{m,k} \to \mathrm{Sym}^d_X$, where $\mathbb{G}_m$ denotes the multiplicative group scheme over $\mathbb{Z}$. That is, if Sym^d_X is arithmetically hyperbolic over k, then Sym^d_X is groupless over k (see also [21, Proposition 3.9]). Lang–Vojta's conjectures predict that the affine variety Sym^d_X is arithmetically hyperbolic over k if (and only if) it is groupless over k. To make this more concrete, we now give an explicit criterion for the grouplessness of Sym^d_X.

Proposition 5.1. *There is a non-constant morphism $\mathbb{G}_{m,k} \to \mathrm{Sym}^d_X$ if and only if there is a dense open $U \subset X$ and a finite morphism $U \to \mathbb{G}_{m,k}$ of degree at most d.*

Proof. Let $\overline{X}$ be the smooth projective model for X over k, and let $\{P_1, \ldots, P_r\} := \overline{X}\setminus X$. Since abelian varieties do not contain rational curves, if there is a non-constant morphism $\varphi : \mathbb{G}_{m,k} \to \mathrm{Sym}^d_X$, then the image of φ must lie in a fiber of the Abel–Jacobi map on $\mathrm{Sym}^d_{\overline{X}}$. This fiber is isomorphic to some projective space $\mathbb{P}^n_k$ and corresponds to some complete linear system on the curve $\overline{X}$ (interpreting points of $\mathrm{Sym}^d_{\overline{X}}$ as degree d effective divisors). The intersection of this fiber with Sym^d_X is a complement of hyperplanes $H_1, \dots, H_r$ in $\mathbb{P}^n_k$ (one hyperplane H_i for each point P_i). If there exists a line L intersecting $\cup H_i$ in two or fewer points, then the line L corresponds to a linear system giving a finite morphism $f : \overline{X} \to \mathbb{P}^1_k$ of degree at most d which maps $\{P_1, \dots, P_r\}$ to a set of (at most) two points (which we may take to be $\{0, \infty\}$). It follows from [41, Theorem 4.1] that there is a dense open $U \subset X$ and a finite morphism $U \to \mathbb{G}_{m,k}$ of degree at most d. Thus, it suffices to show that if there exists a non-constant morphism $\mathbb{G}_{m,k} \to \mathbb{P}^n_k \setminus \cup_{i=1}^r H_i$ (that is, the complement of $H_1, \dots, H_r$ is non-groupless), then there exists a linear such $\mathbb{G}_{m,k}$ (equivalently, a line in $\mathbb{P}^n_k$ intersecting $\cup H_i$ in two or fewer points).

Suppose now that there exists a non-constant morphism $\varphi : \mathbb{G}_{m,k} \to \mathbb{P}^n_k \setminus \cup_{i=1}^r H_i$. By replacing $\mathbb{P}^n_k$ by an appropriate linear space, we may assume that the image of $\mathbb{G}_{m,k}$ is not contained in any hyperplane. Let $L_1, \dots, L_r$ be linear forms over k defining $H_1, \dots, H_r$. If $L_1, \dots, L_r$ are linearly independent, then $r \leqslant n+1$ and it is easy to see that the desired line exists (take any appropriate line containing a point in $\cap_{i=1}^{r-1} H_i$). Otherwise, let $L_{i_1}, \dots, L_{i_m}$ be linearly dependent over k with m minimal. Let $\phi_j = \frac{L_{i_j}}{L_{i_m}} \circ \varphi$, $j = 1, \dots, m$. Then ϕ_j and $1/\phi_j$ are regular functions on $\mathbb{G}_{m,k}$, and ϕ_j may be identified with $c_j t^{n_j} \in k(t)$ for some constant $c_j \in k^*$ and $n_j \in \mathbb{Z}$. Since the image of φ is not contained in any hyperplane, from the minimality of m, all of the powers n_j are distinct. However, this clearly contradicts that $\phi_1, \dots, \phi_m$ are linearly dependent over k (this last argument is an elementary case ($g = 0$, $|S| = 2$) of the function field S-unit equation height inequality of Mason [44], Brownawell–Masser [7], and Voloch [60]).

Conversely, the existence of an open $U \subset X$ and finite morphism $U \to \mathbb{G}_{m,k}$ of degree at most d obviously imply the existence of a non-constant morphism $\mathbb{G}_{m,k} \to \mathrm{Sym}^d_X$. $\square$

Proposition 5.1 says that Sym^d_X is not groupless over k if and only if there exists a dense open $U \subset X$ and a finite morphism $U \to \mathbb{G}_{m,k}$ of degree at most d. In [41] the second-named author showed that, if $k = \overline{\mathbb{Q}}$, then the obvious obstruction to the arithmetic hyperbolicity of Sym^d_X (that is, its non-grouplessness) is in fact the only one. That is, the variety Sym^d_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ if and only if Sym^d_X is groupless over $\overline{\mathbb{Q}}$. We now prove the following extension of Theorem 1.10, as stated in the introduction.

Proof of Theorem 1.11. Let X be a smooth affine connected curve over $\overline{\mathbb{Q}}$ with smooth projective model $\overline{X}$. Let $d \geqslant 1$ and assume that $2d \leqslant \#(\overline{X} \setminus X)$. Note that (3) holds if and only if Sym^d_X is groupless by Proposition 5.1. From this it is clear that (1) $\Rightarrow$ (3), as arithmetically hyperbolic varieties are groupless [21, Proposition 3.9]. Similarly, as Brody hyperbolic varieties are (obviously) groupless, we see that (2) $\Rightarrow$ (3).

Assume (3) holds, that is, the affine variety Sym^d_X is groupless over $\overline{\mathbb{Q}}$. Then, by [41], Sym^d_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ and the variety $\mathrm{Sym}^d_{X_\mathbb{C}}$ is Brody hyperbolic. Therefore, by Corollary 4.6, the variety $\mathrm{Sym}^d_{X_\mathbb{C}}$ is hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X}_\mathbb{C}}$. This shows that (2) holds. In particular, by our main result (Theorem 1.4), the Persistence Conjecture holds for Sym^d_X. Thus, as Sym^d_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$, it follows that for every algebraically closed field k of characteristic zero, the variety $\mathrm{Sym}^d_{X_k}$ is arithmetically hyperbolic over k. This shows that (1) also holds, and concludes the proof. $\square$

Proof of Theorem 1.13. Let X be a smooth affine connected curve over $\overline{\mathbb{Q}}$. To prove the theorem, we argue as in the proof of Theorem 1.11. Indeed, since (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3), we may and do assume that (3) holds. Then, by [41], the variety $\mathrm{Sym}^2_{X_{\mathbb{C}}}$ is Brody hyperbolic and Sym^2_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$. As $\mathrm{Sym}^2_{X_{\mathbb{C}}}$ is Brody hyperbolic, it follows from Corollary 4.4 that the surface $\mathrm{Sym}^2_{X_{\mathbb{C}}}$ is hyperbolically embedded in $\mathrm{Sym}^2_{\overline{X_{\mathbb{C}}}}$. This shows that (2) holds. Next, by Theorem 1.4, the Persistence Conjecture holds for the surface Sym^2_X, so that Sym^2_X is absolutely arithmetically hyperbolic. This shows that (1) holds, as required. □

6 | NON-HYPERBOLIC EMBEDDINGS OF BRODY HYPERBOLIC SYMMETRIC PRODUCTS

In this section, we prove Theorems 1.14 and 1.15. First, we show that for $d \geqslant 3$ the quantity $2d$ in Corollary 4.6 is sharp (see Corollary 4.4 for $d = 2$).

Proof of Theorem 1.14. Let $d \geqslant 3$. Let $\overline{X}$ be a smooth projective curve over $\overline{\mathbb{Q}}$ of genus $g(\overline{X}) > (d-2)(d-1)$ and gonality $d-1$ (such curves are easily constructed). Let $\phi : \overline{X} \to \mathbb{P}^1$ be a morphism of degree $d-1$, which after an automorphism of $\mathbb{P}^1$ we can assume is unramified above 0 and ∞. Let $P \in \overline{X} \setminus \phi^{-1}(\{0,\infty\})$ and let $D = \phi^{-1}(\{0,\infty\}) \cup \{P\}$. Let $X = \overline{X} \setminus D$ and note that $\#\overline{X} \setminus X = 2d-1$. By Corollary 4.5 with $T = \{P\}$, for every embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, we see that $\mathrm{Sym}^d_{X_{\mathbb{C}}}$ is not hyperbolically embedded in $\mathrm{Sym}^d_{\overline{X_{\mathbb{C}}}}$. On the other hand, it follows from Castelnuovo's inequality [1, p. 366] and our assumptions on the genus and gonality of $\overline{X}$ that there does not exist a morphism $\overline{X} \to \mathbb{P}^1$ of degree d. Since $\#\overline{X} \setminus X = 2d-1 > 2(d-1)$, this immediately implies that (3) holds. Therefore, by Theorem 1.10, Sym^d_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ (so that (1) holds) and, for every embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, the variety $\mathrm{Sym}^d_{X_{\mathbb{C}}}$ is Brody hyperbolic (so that (2) holds). This concludes the proof. □

Theorem 6.1. *Let $\overline{X}$ be a smooth projective connected hyperelliptic curve of genus $g \geqslant 3$ over $\mathbb{C}$, and let $\iota : \overline{X} \to \overline{X}$ be the hyperelliptic involution. Let P_1, P_3, P_5 be pairwise distinct non-Weierstrass points, let $P_2 := \iota(P_1)$ and $P_4 := \iota(P_3)$, and write $X := \overline{X} \setminus \{P_1, P_2, P_3, P_4, P_5\}$. Then Sym^3_X is not hyperbolically embeddable.*

Proof. Let Y be a projective variety over $\mathbb{C}$ and let $\mathrm{Sym}^3_X \subset Y$ be a hyperbolic embedding (so that, in particular, the threefold Sym^3_X is Kobayashi hyperbolic). Consider the natural embedding $\mathrm{Sym}^3_X \subset \mathrm{Sym}^3_{\overline{X}}$, and note that its complement is a normal crossings divisor. Therefore, as Sym^3_X is hyperbolically embedded in Y, the extension theorem of Kiernan–Kobayashi–Kwack [33, Theorem 6.3.9] implies that the identity map $\mathrm{Sym}^3_X \to \mathrm{Sym}^3_X$ extends to a morphism $\mathrm{Sym}^3_{\overline{X}} \to Y$.

As in the proof of Theorem 4.3, if $D_1, \ldots, D_5$ are the effective divisors on $\mathrm{Sym}^3_{\overline{X}}$ naturally corresponding to the points $P_1, \ldots, P_5$, then

$$D_5 \setminus \cup_{j=1}^4 D_j \cong \mathrm{Sym}^2_{\overline{X}\setminus\{P_1,P_2,P_3,P_4\}}.$$

The unique g^1_2 on $\overline{X}$ corresponds to a $\mathbb{P}^1$ in $\mathrm{Sym}^2_{\overline{X}}$, and the intersection with $\mathrm{Sym}^2_{\overline{X}\setminus\{P_1,P_2,P_3,P_4\}}$ yields a curve isomorphic to $\mathbb{G}_m$ (corresponding to $g^1_2 \setminus \{P_1+P_2, P_3+P_4\}$). Under the isomorphism above, we let $C \cong \mathbb{G}_m \subset D_5 \setminus \cup_{j=1}^4 D_j$ be the corresponding curve in the boundary of Sym^3_X (in $\mathrm{Sym}^3_{\overline{X}}$).

From the proof of Theorem 4.2 (see [48, Theorem 7.2.13]), for any two points $Q, R \in C$ there are sequences $\{Q_i\}$ and $\{R_i\}$ in Sym^3_X converging to Q and R, respectively, such that

$$d_{\mathrm{Sym}^3_X}(Q_i, R_i) \to 0.$$

It follows immediately that if Sym^3_X is hyperbolically embedded in Y, then the morphism $\mathrm{Sym}^3_{\overline{X}} \to Y$ must contract C to a point. (In particular, the presence of this $\mathbb{G}_m$ is enough to conclude that $\mathrm{Sym}^3_X \subset \mathrm{Sym}^3_{\overline{X}}$ is not a hyperbolic embedding and, as we will show now, its presence is also enough to conclude that $\mathrm{Sym}^3_X \subset Y$ is not a hyperbolic embedding.)

Fix a Weierstrass point P_0 in $\overline{X}$ and consider the morphism $f : \mathrm{Sym}^3_{\overline{X}} \to \mathrm{Jac}(\overline{X})$ given by $(P, Q, R) \mapsto [P + Q + R - 3P_0]$. Furthermore, consider the embedding of $\overline{X}$ in $\mathrm{Jac}(\overline{X})$ given by $P \mapsto [P - P_0]$ and identify $\overline{X}$ with its image. Let $S = f^{-1}(\overline{X})$, and note that S is a surface in $\mathrm{Sym}^3_{\overline{X}}$. Explicitly, the surface S consists of points of the form $P + \iota(P) + Q$ in $\mathrm{Sym}^3_{\overline{X}}$ (identifying points with degree 3 effective divisors). The closure of the curve C is precisely the fiber F of S above $[P_5 - P_0]$. Then, by restriction, we obtain a morphism $S \to Y$ which contracts the fiber F (of the morphism $S \to \overline{X}$). In a diagram

$$\begin{tikzcd} S \arrow[rr, "\text{contracts the fiber } F"] \arrow[d] & & Y \\ \overline{X} & & \end{tikzcd}$$

Since the morphism $S \to Y$ contracts the fiber F, it follows from the Rigidity Lemma [32, Lemma 1.6] that it contracts every fiber. However, since $S \to Y$ is birational onto its image, we obtain a contradiction, and conclude that Sym^3_X is not hyperbolically embeddable. $\square$

Proof of Theorem 1.15. Let $\overline{X}$ be a smooth projective connected hyperelliptic curve of genus $g \geqslant 3$ over $\overline{\mathbb{Q}}$, and let $\iota : \overline{X} \to \overline{X}$ be the hyperelliptic involution. Let P_1, P_3, P_5 be pairwise distinct non-Weierstrass points, and let $P_2 := \iota(P_1)$ and $P_4 := \iota(P_3)$. Define $X := \overline{X} \setminus \{P_1, P_2, P_3, P_4, P_5\}$. From the proof of Theorem 1.14, the variety Sym^3_X is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ and $\mathrm{Sym}^3_{X_\sigma}$ is Brody hyperbolic for every $\sigma : \overline{\mathbb{Q}} \to \mathbb{C}$.

Let Y be a projective variety over $\mathbb{C}$ and let $\mathrm{Sym}^3_{X_\sigma} \subset Y$ be an embedding. By Theorem 6.1, this embedding $\mathrm{Sym}^3_{X_\sigma} \subset Y$ is not a hyperbolic embedding. This concludes the proof. $\square$

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