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Compressible Navier–Stokes Equations with Potential Temperature Transport: Stability of the Strong Solution and Numerical Error Estimates

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Abstract. We present a dissipative measure-valued (DMV)-strong uniqueness result for the compressible Navier–Stokes system with potential temperature transport. We show that strong solutions are stable in the class of DMV solutions. More precisely, we prove that a DMV solution coincides with a strong solution emanating from the same initial data as long as the strong solution exists. As an application of the DMV-strong uniqueness principle we derive a priori error estimates for a mixed finite element-finite volume method. The numerical solutions are computed on polyhedral domains that approximate a sufficiently a smooth bounded domain, where the exact solution exists.

Keywords. Compressible Navier–Stokes system, Dissipative measure-valued solution, DMV-strong uniqueness principle, Error estimates, Finite element-finite volume method.

1. Introduction

In meteorological applications the following system of compressible Navier–Stokes equations governing the motion of viscous Newtonian fluid is often used, see, e.g., [1,6,12,14],

$$\partial_t \varrho + \mathrm{div}_{\boldsymbol{x}}(\varrho \boldsymbol{u}) = 0 \tag{1.1}$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_{\boldsymbol{x}}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_{\boldsymbol{x}} p(\varrho\theta) = \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}} \boldsymbol{u})) \tag{1.2}$$

$$\partial_t(\varrho\theta) + \mathrm{div}_{\boldsymbol{x}}(\varrho\theta \boldsymbol{u}) = 0, \tag{1.3}$$

where $\varrho \geq 0$, $\boldsymbol{u}$, and $\theta \geq 0$, denote the *fluid density*, *velocity*, and *potential temperature*, respectively. The *viscous stress tensor* $\mathbb{S}(\nabla_{\boldsymbol{x}} \boldsymbol{u})$ is determined by the stipulation

$$\mathbb{S}(\nabla_{\boldsymbol{x}} \boldsymbol{u}) = \mu\left(\nabla_{\boldsymbol{x}} \boldsymbol{u} + (\nabla_{\boldsymbol{x}} \boldsymbol{u})^T - \frac{2}{d}\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\mathbb{I}\right) + \lambda\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathbb{I}\,, \tag{1.4}$$

where d is the space dimension, here $d = 2, 3$, and the *viscosity constants* μ and λ satisfy $\mu > 0$ and $\lambda \geq -\frac{2}{d}\,\mu$, respectively. The state equation for the *pressure* p reads

$$p(\varrho\theta) = a(\varrho\theta)^\gamma\,, \qquad a = \text{const.} > 0\,, \tag{1.5}$$

where $\gamma > 1$ is the so-called *adiabatic index*. System (1.1)–(1.3) is solved on $(0, T) \times \Omega$, where $T > 0$ is a given time and $\Omega \subset \mathbb{R}^d$ a bounded domain. It is accompanied with the initial data

$$\varrho(0, \cdot) = \varrho_0\,, \qquad \theta(0, \cdot) = \theta_0\,, \qquad \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0\,, \tag{1.6}$$

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and no-slip boundary conditions

$$\boldsymbol{u}|_{[0,T]\times\partial\Omega} = \mathbf{0}\,. \tag{1.7}$$

In the sequel, we shall call system (1.1)–(1.5) the *Navier–Stokes system with potential temperature transport*. For a brief overview of analytical results for this system we refer to our recent paper [15]. It is to be pointed out that the existence of global-in-time weak solutions to (1.1)–(1.5) is available in three space dimensions only for $\gamma \geq 9/5$, see Maltese et al. [17, Theorem 1 with $\mathcal{T}(s) = s^\gamma$]. However, physically relevant values of the adiabatic index γ lie in the interval $(1, 5/3]$ for $d = 3$. This drawback motivated our recent paper [15], where we have identified a larger class of generalized solutions, *dissipative measure-valued (DMV) solutions*, to the Navier–Stokes system with potential temperature transport. Analyzing the convergence of a suitable numerical scheme, the mixed finite element–finite volume method, we have proved global-in-time existence of DMV solutions for all adiabatic indices $\gamma > 1$ for $d = 2, 3$.

The first goal of the present paper is to show that the strong solutions to the Navier–Stokes system with potential temperature transport are stable in the class of DMV solutions. To this end we establish a *DMV-strong uniqueness principle*. This result states that the DMV and strong solutions emanating from the same initial data coincide. The key concept for the proof of this principle is the *relative energy*. This approach for proving weak-strong uniqueness is not new; see, e.g., [3], where DMV-strong uniqueness is proven for the Navier–Stokes system, and [7, Chapter 6], where DMV-strong uniqueness is proven for the barotropic Euler system, the complete Euler system, and the Navier–Stokes system. However, till now the weak-strong uniqueness principle was not available for the Navier–Stokes equations with potential temperature transport (1.1)–(1.5). The essential difficulty lies in the pressure law that only depends on the total potential temperature $\varrho\theta$, without any independent control of the density ϱ. To cure this problem, we will rewrite the pressure as a function of the density and total physical entropy. This allows us to separate the effects of the density and potential temperature in the derivation of the relative energy and finally to show the DMV-strong uniqueness principle.

The second goal is to derive a priori error estimates for the finite element–finite volume method proposed in [15]. To this end, we assume that the strong solution exists and apply a relative energy inequality and a consistency formulation for the numerical method. Such an approach has already been applied successfully to the compressible Navier–Stokes equations, see Kwon and Novotný [13], and to the compressible Euler system, see [16]. However, in those works, the approximation of a sufficiently smooth domain $\Omega \subset \mathbb{R}^d$ by a sequence of polygonal approximations $\Omega_h \subset \mathbb{R}^d$, $h \downarrow 0$, was not considered. In the present paper, novel consistency estimates are presented that allow to compare a strong solution on a smooth domain Ω with numerical solutions computed on polygonal domains Ω_h, $\Omega \subset \Omega_h$. Here, we only assume that $\mathrm{dist}(\boldsymbol{x}, \partial\Omega) = \mathcal{O}(h)$ for all $\boldsymbol{x} \in \partial\Omega_h$, see also Feireisl et al. [4,8] for related results for the compressible Navier–Stokes equations on general domains under slightly more restrictive assumptions.

The paper is organized as follows: In Sect. 2, we briefly repeat the relevant notation and our definition of DMV solutions to Navier–Stokes system with potential temperature transport proposed in [15]. Section 3 is devoted to the proof of the DMV-strong uniqueness principle. Further, the error estimates are derived in Sect. 4 where we also present some numerical results.

2. DMV Solutions

We start by introducing the *pressure potential* $P : [0, \infty) \to \mathbb{R}$ as

$$P(z) = \frac{a}{\gamma - 1}\, z^\gamma\,. \tag{2.1}$$

In what follows we write $\Omega_t = (0, t) \times \Omega$ whenever $t > 0$. If $\mathcal{V} = \{\mathcal{V}_{(t,\boldsymbol{x})}\}_{(t,\boldsymbol{x}) \in \Omega_T}$ is a space-time parametrized probability measure acting on $\mathbb{R}^{d+2}$, we write

$$\langle \mathcal{V}_{(t,\boldsymbol{x})}; g \rangle \equiv \int_{\mathbb{R}^{d+2}} g \, \mathrm{d}\mathcal{V}_{(t,\boldsymbol{x})} \equiv \int_{\mathbb{R}^{d+2}} g(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \, \mathrm{d}\mathcal{V}_{(t,\boldsymbol{x})}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}})$$

whenever $g \in C(\mathbb{R}^{d+2})$. In particular, we tend to write out the function g in terms of the integration variables $(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \cong \mathbb{R}^{d+2}$: if, for example, $g(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) = \tilde{\varrho}\,\tilde{\boldsymbol{u}}$, then we also write

$$\langle \mathcal{V}_{(t,\boldsymbol{x})}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \rangle \qquad \text{instead of} \qquad \langle \mathcal{V}_{(t,\boldsymbol{x})}; g \rangle .$$

We recall the definition of dissipative measure-valued solutions to the Navier–Stokes system with potential temperature transport (1.1)–(1.5) from [15].

Definition 2.1 (*DMV solutions,* [15, Definition 2.1]). A parametrized probability measure $\mathcal{V} = \{\mathcal{V}_{(t,\boldsymbol{x})}\}_{(t,\boldsymbol{x}) \in \Omega_T}$ that satisfies

$$\mathcal{V} \in L^\infty_{\text{weak}^\star}(\Omega_T; \mathcal{P}(\mathbb{R}^{d+2})), \qquad \mathbb{R}^{d+2} = \left\{ (\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \,\middle|\, \tilde{\varrho}, \tilde{\theta} \in \mathbb{R}, \tilde{\boldsymbol{u}} \in \mathbb{R}^d \right\},$$

[1]and for which there exists a constant $c_\star > 0$ such that

$$\mathcal{V}_{(t,\boldsymbol{x})}\big(\{\tilde{\varrho} \geq 0\} \cap \{\tilde{\theta} \geq c_\star\}\big) = 1 \quad \text{for a.a. } (t, \boldsymbol{x}) \in \Omega_T, \tag{2.2}$$

is called a *dissipative measure-valued (DMV) solution* to the Navier–Stokes system with potential temperature transport (1.1)–(1.5) with initial and boundary conditions (1.6) and (1.7) if it satisfies:

- **Energy inequality**

$$\boldsymbol{u}_{\mathcal{V}} \equiv \langle \mathcal{V}; \tilde{\boldsymbol{u}} \rangle \in L^2(0,T; W_0^{1,2}(\Omega)^d), \qquad \left\langle \mathcal{V}; \frac{1}{2}\,\tilde{\varrho}\,|\tilde{\boldsymbol{u}}|^2 + P(\tilde{\varrho}\,\tilde{\theta}) \right\rangle \in L^1(\Omega_T),$$

  and the integral inequality

$$\begin{aligned} \int_\Omega \left\langle \mathcal{V}_{(\tau,\cdot)}; \frac{1}{2}\,\tilde{\varrho}\,|\tilde{\boldsymbol{u}}|^2 + P(\tilde{\varrho}\,\tilde{\theta}) \right\rangle \mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}} \boldsymbol{u}_{\mathcal{V}}) : \nabla_{\boldsymbol{x}} \boldsymbol{u}_{\mathcal{V}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D} \leq \int_\Omega \left[\frac{1}{2}\,\varrho_0 |\boldsymbol{u}_0|^2 + P(\varrho_0 \theta_0) \right] \mathrm{d}\boldsymbol{x} \end{aligned} \tag{2.3}$$

  holds for a.a. $\tau \in (0,T)$ with the *energy concentration defect*

$$\mathfrak{E} \in L^\infty_{\text{weak}^\star}(0,T; \mathcal{M}^+(\overline{\Omega}))$$

  and the *dissipation defect*

$$\mathfrak{D} \in \mathcal{M}^+(\overline{\Omega_T});$$

- **Continuity equation**

$$\langle \mathcal{V}; \tilde{\varrho} \rangle \in C_{\text{weak}}([0,T]; L^\gamma(\Omega)), \quad \langle \mathcal{V}_{(0,\boldsymbol{x})}; \tilde{\varrho} \rangle = \varrho_0(\boldsymbol{x}) \ \text{ for a.a. } \boldsymbol{x} \in \Omega$$

  and the integral identity

$$\left[\int_\Omega \langle \mathcal{V}_{(t,\cdot)}; \tilde{\varrho} \rangle\, \varphi(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\langle \mathcal{V}; \tilde{\varrho} \rangle\, \partial_t \varphi + \langle \mathcal{V}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \rangle \cdot \nabla_{\boldsymbol{x}} \varphi \Big] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.4}$$

  holds for all $\tau \in [0,T]$ and all $\varphi \in W^{1,\infty}(\Omega_T)$[2];

- **Momentum equation**

$$\langle \mathcal{V}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \rangle \in C_{\text{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)^d), \quad \langle \mathcal{V}_{(0,\boldsymbol{x})}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \rangle = \varrho_0(\boldsymbol{x}) \boldsymbol{u}_0(\boldsymbol{x}) \ \text{ for a.a. } \boldsymbol{x} \in \Omega$$

  and the integral identity

$$\begin{aligned} \left[\int_\Omega \langle \mathcal{V}_{(t,\cdot)}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \rangle \cdot \boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \Big[\langle \mathcal{V}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \langle \mathcal{V}; \tilde{\varrho}\,\tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}} + p(\tilde{\varrho}\,\tilde{\theta})\mathbb{I} \rangle : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} \Big] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}} \boldsymbol{u}_{\mathcal{V}}) : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau \int_\Omega \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t)\,\mathrm{d}t \end{aligned} \tag{2.5}$$

[1] $\mathcal{P}(\mathbb{R}^{d+2})$ denotes the space of probability measures on $\mathbb{R}^{d+2}$.
[2] Here, the (Lipschitz) continuous representative of $\varphi \in W^{1,\infty}(\Omega_T)$ is meant.

holds for all $\tau \in [0,T]$ and all $\boldsymbol{\varphi} \in C^1(\overline{\Omega_T})^d$ satisfying $\boldsymbol{\varphi}|_{[0,T]\times\partial\Omega} = \mathbf{0}$, where the *Reynolds concentration defect* fulfills

$$\mathfrak{R} \in L^\infty_{\text{weak}^\star}(0,T;\mathcal{M}(\overline{\Omega})^{d\times d}_{\text{sym},+})^{3}$$
$$\text{and} \qquad \underline{d}\,\mathfrak{E} \le \operatorname{tr}(\mathfrak{R}) \le \overline{d}\,\mathfrak{E} \quad \text{for some constants } \overline{d} \ge \underline{d} > 0;$$

- **Potential temperature equation**

$$\langle \mathcal{V}; \tilde{\varrho}\,\tilde{\theta}\,\rangle \in C_{\text{weak}}([0,T];L^\gamma(\Omega))\,, \quad \langle \mathcal{V}_{(0,\boldsymbol{x})}; \tilde{\varrho}\,\tilde{\theta}\,\rangle = \varrho_0(\boldsymbol{x})\theta_0(\boldsymbol{x}) \text{ for a.a. } \boldsymbol{x} \in \Omega$$

and the integral identity

$$\left[\int_\Omega \langle \mathcal{V}_{(t,\,\cdot\,)}; \tilde{\varrho}\,\tilde{\theta}\,\rangle\, \varphi(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega \Big[\langle \mathcal{V}; \tilde{\varrho}\,\tilde{\theta}\,\rangle\,\partial_t\varphi + \langle \mathcal{V}; \tilde{\varrho}\,\tilde{\theta}\,\tilde{\boldsymbol{u}}\rangle \cdot \nabla_{\boldsymbol{x}}\varphi\Big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.6}$$

holds for all $\tau \in [0,T]$ and all $\varphi \in W^{1,\infty}(\Omega_T)$;

- **Entropy inequality**

$$\langle \mathcal{V}_{(0,\boldsymbol{x})}; \tilde{\varrho}\,\ln(\tilde{\theta})\rangle = \varrho_0(\boldsymbol{x})\ln(\theta_0(\boldsymbol{x})) \text{ for a.a. } \boldsymbol{x} \in \Omega$$

and for any $\psi \in W^{1,\infty}(\Omega_T)$, $\psi \ge 0$, the integral inequality

$$\left[\int_\Omega \langle \mathcal{V}_{(t,\,\cdot\,)}; \tilde{\varrho}\,\ln(\tilde{\theta})\rangle\, \psi(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} \ge \int_0^\tau\int_\Omega \Big[\langle \mathcal{V}; \tilde{\varrho}\,\ln(\tilde{\theta})\rangle\,\partial_t\psi + \langle \mathcal{V}; \tilde{\varrho}\,\ln(\tilde{\theta})\tilde{\boldsymbol{u}}\rangle \cdot \nabla_{\boldsymbol{x}}\psi\Big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{2.7}$$

is satisfied for a.a. $\tau \in (0,T)$;

- **Poincaré's inequality**

there exists a constant $C_P > 0$ such that

$$\int_0^\tau\int_\Omega \langle \mathcal{V}; |\tilde{\boldsymbol{u}} - \boldsymbol{U}|^2\rangle\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le C_P\left(\int_0^\tau\int_\Omega |\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{U})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(t)\,\mathrm{d}t + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D}\right) \tag{2.8}$$

for a.a. $\tau \in (0,T)$ and all $\boldsymbol{U} \in L^2(0,T;W^{1,2}_0(\Omega)^d)$.

Remark 2.2. As we shall see in the next section, the entropy inequality (2.7) and Poincaré's inequality (2.8) included in the definition of DMV solutions to the Navier–Stokes system with potential temperature transport are fundamental to guarantee DMV-strong uniqueness.

3. DMV-Strong Uniqueness

The aim of this section is to derive a DMV-strong uniqueness principle for our measure-valued solutions. For this purpose, we rely on the concept of relative energy. We introduce the *(physical) entropy* S as

$$S = S(\varrho,\theta) = \frac{\gamma}{\gamma-1}\,\varrho\ln(\theta) \tag{3.1}$$

and realize that the pressure $p = a(\varrho\theta)^\gamma$ can be rewritten with respect to ϱ, S as

$$p(\varrho,S) = \mathbb{1}_{\{\varrho>0\}}\,a\varrho^\gamma\exp\left((\gamma-1)\,\frac{S}{\varrho}\right). \tag{3.2}$$

We proceed by defining the relative energy between a triplet of arbitrary functions $(\varrho,\theta,\boldsymbol{u})$ belonging to a regularity class

$$\varrho,\theta \in C^1(\overline{\Omega_T})\,, \quad \varrho,\theta > 0\,, \quad \boldsymbol{u} \in C^1(\overline{\Omega_T}) \cap L^2(0,T;W^{2,\infty}(\Omega))\,, \quad \boldsymbol{u}|_{[0,T]\times\partial\Omega} = \mathbf{0}, \tag{3.3}$$

[3] $\mathcal{M}(\overline{\Omega})^{d\times d}_{\text{sym},+}$ denotes the set of bounded Radon measures defined on $\overline{\Omega}$ and ranging in the set of symmetric positive semi-definite matrices, i.e., $\mathcal{M}(\overline{\Omega})^{d\times d}_{\text{sym},+} = \{\mu \in \mathcal{M}(\overline{\Omega})^{d\times d}_{\text{sym}} \mid \int_{\overline{\Omega}} \phi(\xi\otimes\xi) : \mathrm{d}\mu \ge 0 \text{ for all } \xi \in \mathbb{R}^d,\ \phi \in C(\overline{\Omega}),\ \phi \ge 0\}$.

and a DMV solution $\mathcal{V}$ to the Navier–Stokes system with potential temperature transport (1.1)–(1.5) as

$$E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u}) = \Big\langle \mathcal{V}; \frac{1}{2}\,\tilde{\varrho}\,|\tilde{\boldsymbol{u}}-\boldsymbol{u}|^2 + P(\tilde{\varrho},\tilde{S}) - \frac{\partial P(\varrho,S)}{\partial \varrho}\,(\tilde{\varrho}-\varrho) - \frac{\partial P(\varrho,S)}{\partial S}\,(\tilde{S}-S) - P(\varrho,S)\Big\rangle, \tag{3.4}$$

where $P(\varrho,S) = \frac{1}{\gamma-1}\,p(\varrho,S)$ is the pressure potential expressed in terms of ϱ and S, $S = S(\varrho,\theta)$, and $\tilde{S} = S(\tilde{\varrho},\tilde{\theta})$.

Remark 3.1. We note that $P = P(\varrho,S)$ satisfies the following identity on $(0,\infty)\times\mathbb{R}$:

$$\frac{\partial P(\varrho,S)}{\partial \varrho}\,\varrho + \frac{\partial P(\varrho,S)}{\partial S}\,S - P(\varrho,S) = p(\varrho,S) \tag{3.5}$$

We further note that we only consider the case in which $\theta,\tilde{\theta}$ are bounded from below by some constant $c>0$ (for θ this is reflected by (3.3) and for $\tilde{\theta}$ by (2.2)). Consequently, (3.1) makes sense. In particular, $S(\varrho,\theta)$ and the composition $p(\varrho,S(\varrho,\theta))$ are continuous functions of (ϱ,θ) on $[0,\infty)\times[c,\infty)$ for every $c>0$. In addition, we shall always construe S and $\tilde{S}$ as functions of ϱ,θ and $\tilde{\varrho},\tilde{\theta}$, respectively. Accordingly, for example, $\langle\mathcal{V};\tilde{S}\rangle \equiv \langle\mathcal{V};\frac{\gamma}{\gamma-1}\,\tilde{\varrho}\,\ln(\tilde{\theta})\rangle$.

The relative energy inequality corresponding to (3.4) reads as follows.

Lemma 3.2 (Relative energy inequality). *Let $(\varrho,\theta,\boldsymbol{u})$ be a triplet of test functions, cf. (3.3), and $\mathcal{V}$ a DMV solution to (1.1)–(1.5) in the sense of Definition 2.1. Then the relative energy defined in (3.4) satisfies the inequality*

$$\begin{aligned}
&\Big[\int_\Omega E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\Big]_{t=0}^{t=\tau} + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D} + \int_0^\tau\!\!\int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}}-\boldsymbol{u})) : \nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad\le -\int_0^\tau\!\!\int_\Omega \big\langle \mathcal{V}; \tilde{\varrho}\,(\tilde{\boldsymbol{u}}-\boldsymbol{u})^T\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u})\big\rangle\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad - \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; p(\tilde{\varrho},\tilde{S}) - \frac{\partial p(\varrho,S)}{\partial\varrho}\,(\tilde{\varrho}-\varrho) - \frac{\partial p(\varrho,S)}{\partial S}\,(\tilde{S}-S) - p(\varrho,S)\Big\rangle\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad + \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}\,(\boldsymbol{u}-\tilde{\boldsymbol{u}})\Big\rangle\cdot\big[\varrho\partial_t\boldsymbol{u} + \varrho\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot\boldsymbol{u} + \nabla_{\boldsymbol{x}}p(\varrho,S) - \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad + \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; (\varrho-\tilde{\varrho})\,\frac{1}{\varrho}\,\frac{\partial p(\varrho,S)}{\partial\varrho}\Big\rangle\,\big[\partial_t\varrho + \mathrm{div}_{\boldsymbol{x}}(\varrho\boldsymbol{u})\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad + \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; (\varrho-\tilde{\varrho})\,\frac{1}{\varrho}\,\frac{\partial p(\varrho,S)}{\partial S}\Big\rangle\,\big[\partial_t S + \mathrm{div}_{\boldsymbol{x}}(S\boldsymbol{u})\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad + \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}\,S - \tilde{S}\Big\rangle\,\Big[\partial_t\vartheta + \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\vartheta + \frac{\partial p(\varrho,S)}{\partial S}\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\Big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad + \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; \Big(\frac{\tilde{\varrho}}{\varrho}\,S - \tilde{S}\Big)\,(\tilde{\boldsymbol{u}}-\boldsymbol{u})\Big\rangle\cdot\nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\!\!\int_\Omega \nabla_{\boldsymbol{x}}\boldsymbol{u} : \mathrm{d}\mathfrak{R}(t)\,\mathrm{d}t\\
&\qquad + \int_0^\tau\!\!\int_\Omega \Big\langle \mathcal{V}; (\tilde{\varrho}-\varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}})\Big\rangle\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned} \tag{3.6}$$

for a.a. $\tau\in(0,T)$. Here,

$$\vartheta = \frac{1}{\gamma-1}\,\frac{\partial p(\varrho,S)}{\partial S} = \frac{\partial P(\varrho,S)}{\partial S} \tag{3.7}$$

denotes the absolute temperature.

Proof. Using Gauss's theorem we easily verify that

$$-\int_0^\tau\int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}) : \nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^\tau\int_\Omega \left\langle \mathcal{V}; (\tilde{\varrho} - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u} - \tilde{\boldsymbol{u}})\right\rangle \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}\,(\boldsymbol{u} - \tilde{\boldsymbol{u}})\right\rangle\cdot \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,. \tag{3.8}$$

Next, using the definition of the absolute temperature, cf. (3.7), and (3.5) we deduce that

$$\left[\int_\Omega E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} = \left[\int_\Omega \langle \mathcal{V}_{(t,\,\cdot\,)};\tilde{\varrho}\,\rangle\left(\frac{1}{2}\,|\boldsymbol{u}(t,\cdot)|^2 - \frac{\partial P(\varrho(t,\cdot),S(t,\cdot))}{\partial\varrho}\right)\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}$$
$$-\left[\int_\Omega \langle \mathcal{V}_{(t,\,\cdot\,)};\tilde{S}\rangle\,\vartheta(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} - \left[\int_\Omega \langle \mathcal{V}_{(t,\,\cdot\,)};\tilde{\varrho}\,\tilde{\boldsymbol{u}}\rangle\cdot\boldsymbol{u}(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}$$
$$+\left[\int_\Omega \left\langle \mathcal{V}_{(t,\,\cdot\,)};\frac{1}{2}\,\tilde{\varrho}\,|\tilde{\boldsymbol{u}}|^2 + P(\tilde{\varrho},\tilde{S})\right\rangle\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} + \left[\int_\Omega p(\varrho(t,\cdot),S(t,\cdot))\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}. \tag{3.9}$$

Combining (3.8) and (3.9) with (2.3)–(2.7), we obtain

$$\left[\int_\Omega E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} + \int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}}\mathrm{d}\mathfrak{D} + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})) : \nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\le \int_0^\tau\int_\Omega \left\langle \mathcal{V}; (\tilde{\varrho} - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u} - \tilde{\boldsymbol{u}})\right\rangle \mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \left[\langle\mathcal{V};\tilde{S}\rangle\,\partial_t\vartheta + \langle\mathcal{V};\tilde{S}\tilde{\boldsymbol{u}}\rangle\cdot\nabla_{\boldsymbol{x}}\vartheta\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \left[\langle\mathcal{V};\tilde{\varrho}\,\tilde{\boldsymbol{u}}\rangle\cdot\partial_t\boldsymbol{u} + \langle\mathcal{V};\tilde{\varrho}\,\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}} + p(\tilde{\varrho},\tilde{S})\mathbb{I}\rangle : \nabla_{\boldsymbol{x}}\boldsymbol{u}\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \nabla_{\boldsymbol{x}}\boldsymbol{u} : \mathrm{d}\mathfrak{R}(t)\,\mathrm{d}t$$
$$+ \int_0^\tau\int_\Omega \left[\langle\mathcal{V};\tilde{\varrho}\,\rangle\,\partial_t\left(\frac{1}{2}\,|\boldsymbol{u}|^2 - \frac{\partial P(\varrho,S)}{\partial\varrho}\right) + \langle\mathcal{V};\tilde{\varrho}\,\tilde{\boldsymbol{u}}\rangle\cdot\nabla_{\boldsymbol{x}}\left(\frac{1}{2}\,|\boldsymbol{u}|^2 - \frac{\partial P(\varrho,S)}{\partial\varrho}\right)\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int_0^\tau\int_\Omega \partial_t p(\varrho,S)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \left\langle\mathcal{V};\frac{\tilde{\varrho}}{\varrho}\,(\boldsymbol{u} - \tilde{\boldsymbol{u}})\right\rangle\cdot\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.$$

In the next step, we carry out the partial derivatives on the right-hand side of the above inequality. In doing so, we take into account that (3.5) implies that for any $w \in \{t, x_1, \ldots, x_d\}$,

$$\partial_w\,\frac{\partial P}{\partial\varrho} = \frac{\partial^2 P}{\partial\varrho^2}\,\partial_w\varrho + \frac{\partial^2 P}{\partial S\,\partial\varrho}\,\partial_w S = \left[\frac{1}{\varrho}\frac{\partial p}{\partial\varrho} - \frac{S}{\varrho}\frac{\partial^2 P}{\partial\varrho\,\partial S}\right]\partial_w\varrho + \left[\frac{1}{\varrho}\frac{\partial p}{\partial S} - \frac{S}{\varrho}\frac{\partial^2 P}{\partial S^2}\right]\partial_w S$$
$$= \frac{1}{\varrho}\left[\frac{\partial p}{\partial\varrho}\,\partial_w\varrho + \frac{\partial p}{\partial S}\,\partial_w S\right] + \frac{S}{\varrho}\,\partial_w\,\frac{\partial P}{\partial S} = \frac{1}{\varrho}\left[\frac{\partial p}{\partial\varrho}\,\partial_w\varrho + \frac{\partial p}{\partial S}\,\partial_w S\right] + \frac{S}{\varrho}\,\partial_w\vartheta\,,$$

where the last equality is due to (3.7). Consequently, we get

$$\left[\int_\Omega E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} + \int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}}\mathrm{d}\mathfrak{D} + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})) : \nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\le \int_0^\tau\int_\Omega \left\langle \mathcal{V}; (\tilde{\varrho} - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u} - \tilde{\boldsymbol{u}})\right\rangle \mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \left[\langle\mathcal{V};\tilde{S}\rangle\,\partial_t\vartheta + \langle\mathcal{V};\tilde{S}\tilde{\boldsymbol{u}}\rangle\cdot\nabla_{\boldsymbol{x}}\vartheta\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \left[\langle\mathcal{V};\tilde{\varrho}\,\tilde{\boldsymbol{u}}\rangle\cdot\partial_t\boldsymbol{u} + \langle\mathcal{V};\tilde{\varrho}\,\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}} + p(\tilde{\varrho},\tilde{S})\mathbb{I}\rangle : \nabla_{\boldsymbol{x}}\boldsymbol{u}\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \nabla_{\boldsymbol{x}}\boldsymbol{u} : \mathrm{d}\mathfrak{R}(t)\,\mathrm{d}t$$
$$+ \int_0^\tau\int_\Omega \langle\mathcal{V};\tilde{\varrho}\,\rangle\left[\boldsymbol{u}\cdot\partial_t\boldsymbol{u} - \frac{1}{\varrho}\left(\frac{\partial p(\varrho,S)}{\partial\varrho}\,\partial_t\varrho + \frac{\partial p(\varrho,S)}{\partial S}\,\partial_t S\right) + \frac{S}{\varrho}\,\partial_t\vartheta\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \langle \mathcal{V}; \tilde{\varrho}\,\tilde{\boldsymbol{u}}\rangle \cdot \left[\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot\boldsymbol{u} - \frac{1}{\varrho}\left(\frac{\partial p(\varrho,S)}{\partial\varrho}\nabla_{\boldsymbol{x}}\varrho + \frac{\partial p(\varrho,S)}{\partial S}\nabla_{\boldsymbol{x}}S\right) + \frac{S}{\varrho}\nabla_{\boldsymbol{x}}\vartheta\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \left[\frac{\partial p(\varrho,S)}{\partial\varrho}\partial_t\varrho + \frac{\partial p(\varrho,S)}{\partial S}\partial_t S\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\right\rangle\cdot \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.$$

To finish the proof of Lemma 3.2, we add and subtract the following terms on the right-hand side of the above inequality

$$\int_0^\tau\int_\Omega \langle \mathcal{V}; \tilde{\varrho}\,\boldsymbol{u}^T\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u})\rangle\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,, \qquad \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}S - \tilde{S}\right\rangle \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,$$
$$\int_0^\tau\int_\Omega p(\varrho,S)\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_0^\tau\int_\Omega \left(\frac{\partial p(\varrho,S)}{\partial\varrho}\nabla_{\boldsymbol{x}}\varrho + \frac{\partial p(\varrho,S)}{\partial S}\nabla_{\boldsymbol{x}}S\right)\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,$$
$$\int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}\boldsymbol{u}\right\rangle\cdot\nabla_{\boldsymbol{x}}p(\varrho,S)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\tilde{\varrho}}{\varrho}\right\rangle\left[\frac{p(\varrho,S)}{\partial\varrho}\nabla_{\boldsymbol{x}}\varrho + \frac{p(\varrho,S)}{\partial S}\nabla_{\boldsymbol{x}}S\right]\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,$$
$$\int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\partial p(\varrho,S)}{\partial\varrho}(\tilde{\varrho}-\varrho)\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\right\rangle\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,, \qquad \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\partial p(\varrho,S)}{\partial\varrho}\left(\tilde{S} - \frac{\tilde{\varrho}}{\varrho}S\right)\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\right\rangle\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,$$
$$\int_0^\tau\int_\Omega \left\langle \mathcal{V}; \frac{\partial p(\varrho,S)}{\partial\varrho}\left(\frac{\tilde{\varrho}}{\varrho}S - S\right)\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\right\rangle\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

and regroup the resulting expressions adequately. □

From the relative energy inequality we can deduce the DMV-strong uniqueness result.

Theorem 3.3 (DMV-strong uniqueness). *Let $\gamma > 1$, $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain of class C^3. Further, let $T^* > 0$ and $(\varrho,\theta,\boldsymbol{u})$ be a strong solution to system (1.1)–(1.5) on Ω_{T^*} belonging to the regularity class (3.3). Let $\mathcal{V}$ be a DMV solution in the sense of Definition 2.1 emanating from the same initial data. Then*

$$\mathfrak{E} = 0\,,\quad \mathfrak{D}([0,T^*)\times\overline{\Omega}) = 0\,,\quad \mathfrak{R} = \boldsymbol{0}\,,$$

and the DMV and strong solutions coincide on $[0,T^*)$, i.e.

$$\mathcal{V}_{(t,\boldsymbol{x})} = \delta_{(\varrho(t,\boldsymbol{x}),\theta(t,\boldsymbol{x}),\boldsymbol{u}(t,\boldsymbol{x}))} \quad \text{for a.a. } (t,\boldsymbol{x})\in\Omega_{T^*}.$$

Proof. Plugging the strong solution $(\varrho,\theta,\boldsymbol{u})$ into the relative energy inequality (3.6), we obtain

$$\begin{aligned}
&\left[\int_\Omega E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} + \int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}}\mathrm{d}\mathfrak{D} + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}}-\boldsymbol{u})):\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad\le \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \left(\frac{\tilde{\varrho}}{\varrho}S - \tilde{S}\right)(\tilde{\boldsymbol{u}}-\boldsymbol{u})\right\rangle\cdot\nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \langle \mathcal{V}; \tilde{\varrho}\,(\tilde{\boldsymbol{u}}-\boldsymbol{u})^T\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u})\rangle\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad\quad - \int_0^\tau\int_\Omega \left\langle \mathcal{V}; p(\tilde{\varrho},\tilde{S}) - \frac{\partial p(\varrho,S)}{\partial\varrho}(\tilde{\varrho}-\varrho) - \frac{\partial p(\varrho,S)}{\partial S}(\tilde{S}-S) - p(\varrho,S)\right\rangle\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad\quad + \int_0^\tau\int_\Omega \left\langle \mathcal{V}; (\tilde{\varrho}-\varrho)\frac{1}{\varrho}\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}})\right\rangle\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_\Omega \nabla_{\boldsymbol{x}}\boldsymbol{u}:\mathrm{d}\mathfrak{R}(t)\,\mathrm{d}t\\
&\quad\lesssim \int_0^\tau\int_\Omega E(\mathcal{V}\,|\,\varrho,\theta,\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}}\mathrm{d}\mathfrak{E}(t)\,\mathrm{d}t + \int_0^\tau\int_\Omega \left\langle \mathcal{V}; \left(\frac{\tilde{\varrho}}{\varrho}S - \tilde{S}\right)(\tilde{\boldsymbol{u}}-\boldsymbol{u})\right\rangle\cdot\nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad\quad + \int_0^\tau\int_\Omega \left\langle \mathcal{V}; (\tilde{\varrho}-\varrho)\frac{1}{\varrho}\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}})\right\rangle\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned} \tag{3.10}$$

for a.a. $\tau \in (0, T^*)$. To handle the last two integrals, we first observe that

$$\int_0^\tau \int_\Omega \mathbb{S}(\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})) : \nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^\tau \int_\Omega \left[\mu|\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})|^2 + \nu|\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})|^2\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\geq \mu \int_0^\tau \int_\Omega |\nabla_{\boldsymbol{x}}(\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,. \tag{3.11}$$

Next, we set

$$(\underline{\varrho}, \overline{\varrho}, \underline{\theta}, \overline{\theta}) = \left(\inf_{(t,\boldsymbol{x})\,\in\,\Omega_{T^*}} \{\varrho(t,\boldsymbol{x})\}, \sup_{(t,\boldsymbol{x})\,\in\,\Omega_{T^*}} \{\varrho(t,\boldsymbol{x})\}, \inf_{(t,\boldsymbol{x})\,\in\,\Omega_{T^*}} \{\theta(t,\boldsymbol{x})\}, \sup_{(t,\boldsymbol{x})\,\in\,\Omega_{T^*}} \{\theta(t,\boldsymbol{x})\} \right)$$

and apply Lemma A.1 to find constants $c_1, c_2, c_3 > 0$ that only depend on $\underline{\varrho}$, $\overline{\varrho}$, $\underline{\theta}$, $\overline{\theta}$, $c_\star$, and γ, and corresponding sets

$$\mathcal{R} = \left\{ (\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \in \mathbb{R}^{d+2} \,\middle|\, c_1\underline{\varrho} \leq \tilde{\varrho} \leq c_2\overline{\varrho}\,,\ c_\star \leq \tilde{\theta} \leq c_3\overline{\theta} \right\},$$
$$\mathcal{S} = \left\{ (\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \in \mathbb{R}^{d+2} \,\middle|\, \tilde{\varrho} \geq 0\,,\ \tilde{\theta} \geq c_\star \right\} \setminus \mathcal{R}$$

such that

$$\int_0^\tau \int_\Omega E(\mathcal{V}\,|\,\varrho, \theta, \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \gtrsim \int_0^\tau \int_\Omega \left\langle \mathcal{V}; \mathbb{1}_{\mathcal{R}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}})\big(|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + |\tilde{\varrho} - \varrho|^2 + |\tilde{S} - S|^2\big) \right\rangle \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}; \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}})\big(1 + \tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + (\tilde{\varrho}\tilde{\theta})^\gamma\big) \right\rangle \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,. \tag{3.12}$$

Seeing that

$$\left| \left(\frac{\tilde{\varrho}}{\varrho} S - \tilde{S} \right) (\tilde{\boldsymbol{u}} - \boldsymbol{u}) \right| \lesssim \big|(\tilde{\varrho}S - \tilde{S}\varrho)(\tilde{\boldsymbol{u}} - \boldsymbol{u})\big| \lesssim \big|S(\tilde{\varrho} - \varrho)(\tilde{\boldsymbol{u}} - \boldsymbol{u})\big| + \big|\varrho(S - \tilde{S})(\tilde{\boldsymbol{u}} - \boldsymbol{u})\big|$$
$$\lesssim |\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + |\tilde{\varrho} - \varrho|^2 + |\tilde{S} - S|^2$$

as well as

$$\left| \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(\frac{\tilde{\varrho}}{\varrho} S - \tilde{S} \right) (\tilde{\boldsymbol{u}} - \boldsymbol{u}) \right|$$
$$\lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(\tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}| + \tilde{S}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}| \right) \lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(\tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}| + \tilde{\varrho}\tilde{\theta}^{\,1/2}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}| \right)$$
$$\lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(\tilde{\varrho} + \tilde{\varrho}\tilde{\theta} + \tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 \right) \lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(1 + \tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + (\tilde{\varrho}\tilde{\theta})^\gamma \right),$$

we may use (3.12) to deduce

$$\left| \int_0^\tau \int_\Omega \left\langle \mathcal{V}; \left(\frac{\tilde{\varrho}}{\varrho} S - \tilde{S} \right) (\tilde{\boldsymbol{u}} - \boldsymbol{u}) \right\rangle \cdot \nabla_{\boldsymbol{x}}\vartheta \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| \lesssim \int_0^\tau \int_\Omega E(\mathcal{V}\,|\,\varrho, \theta, \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,. \tag{3.13}$$

We proceed by observing that

$$\big|(\tilde{\varrho} - \varrho)(\boldsymbol{u} - \tilde{\boldsymbol{u}})\big| \lesssim |\tilde{\varrho} - \varrho|^2 + |\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2$$

and

$$\big|\mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}})(\tilde{\varrho} - \varrho)(\boldsymbol{u} - \tilde{\boldsymbol{u}})\big| \lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}})(\tilde{\varrho} + 1)|\tilde{\boldsymbol{u}} - \boldsymbol{u}|$$
$$\lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(\tilde{\varrho} + \tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + \alpha|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + \alpha^{-1} \right)$$
$$\lesssim \mathbb{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{\boldsymbol{u}}) \left(1 + (\tilde{\varrho}\tilde{\theta})^\gamma + \tilde{\varrho}\,|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + \alpha|\tilde{\boldsymbol{u}} - \boldsymbol{u}|^2 + \alpha^{-1} \right)$$

for all $\alpha > 0$, where here and in the sequel the constant hidden in "$\lesssim$" does not depend on α. Together with (3.12) and Poincaré's inequality (2.8), these observations yield

$$\left| \int_0^\tau \int_\Omega \left\langle \mathcal{V}; (\tilde{\varrho} - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u})) \cdot (\boldsymbol{u} - \tilde{\boldsymbol{u}}) \right\rangle \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right|$$

$$\lesssim (1+\alpha^{-1}) \int_0^\tau \int_\Omega E(\mathcal{V} \,|\, \varrho, \theta, \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \mathrm{d}t + \alpha \left(\int_0^\tau \int_\Omega |\nabla_{\boldsymbol{x}} (\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})|^2 \, \mathrm{d}\boldsymbol{x} \mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(t) \mathrm{d}t + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D} \right). \tag{3.14}$$

Finally, combining (3.10), (3.11), (3.13), and (3.14), we arrive at

$$\begin{aligned} &\left[\int_\Omega E(\mathcal{V} \,|\, \varrho, \theta, \boldsymbol{u})(t, \cdot) \, \mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D} + \mu \int_0^\tau \int_\Omega |\nabla_{\boldsymbol{x}} (\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})|^2 \, \mathrm{d}\boldsymbol{x} \mathrm{d}t \\ &\quad \lesssim (1+\alpha^{-1}) \int_0^\tau \int_\Omega E(\mathcal{V} \,|\, \varrho, \theta, \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \mathrm{d}t + (1+\alpha) \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(t) \mathrm{d}t \\ &\qquad + \alpha \left(\int_0^\tau \int_\Omega |\nabla_{\boldsymbol{x}} (\boldsymbol{u}_{\mathcal{V}} - \boldsymbol{u})|^2 \, \mathrm{d}\boldsymbol{x} \mathrm{d}t + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D} \right) \end{aligned}$$

for a.a. $\tau \in (0, T^*)$ and all $\alpha > 0$. In particular, there exists a constant $C > 0$ such that

$$\begin{aligned} &\left[\int_\Omega E(\mathcal{V} \,|\, \varrho, \theta, \boldsymbol{u})(t, \cdot) \, \mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) + \int_{\overline{\Omega_\tau}} \mathrm{d}\mathfrak{D} \\ &\quad \le C \left(\int_0^\tau \int_\Omega E(\mathcal{V} \,|\, \varrho, \theta, \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(t) \mathrm{d}t + \int_0^\tau \int_{\overline{\Omega_t}} \mathrm{d}\mathfrak{D} \, \mathrm{d}t \right) \end{aligned}$$

for a.a. $\tau \in (0, T^*)$. Consequently, the desired result follows from Gronwall's lemma. □

Remark 3.4. The local existence of strong solutions to (1.1)–(1.5) for the Cauchy problem (i.e. $\Omega = \mathbb{R}^d$) is guaranteed by [11, Theorem 2.9] and the global existence for small initial data by [11, Theorem 3.6]. These results apply to a class of systems of hyperbolic-parabolic composite type. The local existence result just mentioned was generalized in [19]. We expect that these results can be transferred to the initial-boundary value problem considered here provided Ω is of class C^3 and the initial data satisfy suitable compatibility conditions. This can be an interesting topic for future studies.

4. Error Estimates for a Numerical Scheme

In our recent paper [15], we have introduced a mixed finite element-finite volume (FE-FV) numerical method and showed that in a suitable (weak) sense its solutions converge to a DMV solution to the Navier–Stokes equations with potential temperature transport (1.1)–(1.5). Moreover, we proved that if a strong solution exists, then the numerical solutions converge strongly to this strong solution, cf. [15, Theorem 6.1].

The ultimate goal of this section is to strengthen the just mentioned result and derive a priori error estimates for the finite element-finite volume method applying the relative energy method.

The section is organized as follows: In Sect. 4.1, we formulate *minimal regularity assumptions* required for the strong solution and the initial data. Sections 4.2 and 4.3 are devoted to the recapitulation of the numerical scheme presented in [15] and its properties. In Sect. 4.4, we present a novel consistency formulation taking the approximation of a smooth domain Ω by a sequence of polygonal computational domains Ω_h, $h \downarrow 0$, into account. The desired error estimates are presented in Sect. 4.5. We finish this section by presenting some numerical results illustrating the convergence of the scheme.

4.1. Regularity Class for the Strong Solution and the Initial Data

We will consider strong solutions $(\varrho, \theta, \boldsymbol{u})$ to (1.1)–(1.5) that belong to the regularity class

$$\varrho, \theta \in C^1(\overline{\Omega_T}), \quad \varrho, \theta > 0, \quad \boldsymbol{u} \in C^1(\overline{\Omega_T}) \cap L^2(0,T; W^{2,\infty}(\Omega)^d), \quad \boldsymbol{u}|_{[0,T]\times\partial\Omega} = \mathbf{0}. \tag{4.1}$$

Accordingly, the initial data satisfy

$$\varrho_0, \theta_0 \in C^1(\overline{\Omega}), \quad \varrho_0, \theta_0 > 0, \quad \boldsymbol{u}_0 \in C^1(\overline{\Omega}), \quad \boldsymbol{u}_0|_{\partial\Omega} = \mathbf{0}. \tag{4.2}$$

For functions such as ϱ in (4.1) and ϱ_0 in (4.2), we further introduce the following notation:

$$(\varrho_\star, \varrho^\star, (\varrho_0)_\star, (\varrho_0)^\star) = \left(\inf_{(t,\boldsymbol{x}) \in \Omega_T} \{\varrho(t,\boldsymbol{x})\}, \sup_{(t,\boldsymbol{x}) \in \Omega_T} \{\varrho(t,\boldsymbol{x})\}, \inf_{\boldsymbol{x} \in \Omega} \{\varrho_0(\boldsymbol{x})\}, \sup_{\boldsymbol{x} \in \Omega} \{\varrho_0(\boldsymbol{x})\} \right). \tag{4.3}$$

In addition, we consider the initial data $(\varrho_0, \theta_0, \boldsymbol{u}_0)$ to be extended by $((\varrho_0)_\star, (\theta_0)_\star, \mathbf{0})$ outside $\overline{\Omega}$.

4.2. Mixed Finite Element-Finite Volume Method

We recall the mixed FE-FV numerical method introduced in [15][4], see also [5, Chapter 7].

4.2.1. Spatial Discretization. Let $H \in (0,1)$. The spatial domain $\Omega \subset \mathbb{R}^d$ is approximated by a family $\{\Omega_h\}_{h \in (0,H]}$ that is connected to a family of (finite) meshes $(\mathcal{T}_h)_{h \in (0,H]}$ via the constraint

$$\overline{\Omega_h} = \bigcup_{K \in \mathcal{T}_h} K \qquad \text{for all } h \in (0,H].$$

We assume that the following conditions are satisfied:

- There is a constant $D > 0$ such that
$$\Omega \subset \Omega_h \subset \left\{ \boldsymbol{x} \in \mathbb{R}^d \,\middle|\, \operatorname{dist}(\boldsymbol{x}, \overline{\Omega}) < Dh \right\} \quad \text{for all } h \in (0,H]; \tag{4.4}$$
- Each element K of a mesh $\mathcal{T}_h$ is a d-simplex that can be written as
$$K = h\mathbb{A}_K(K_{\text{ref}}) + \mathbf{a}_K\,, \qquad \mathbb{A}_K \in \mathbb{R}^{d\times d}, \qquad \mathbf{a}_K \in \mathbb{R}^d\,,$$
where the reference element K_{ref} is the convex hull of the zero vector $\mathbf{0} \in \mathbb{R}^d$ and the standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$, i.e., $K_{\text{ref}} = \operatorname{conv}\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\}$;
- There exist constants $C > c > 0$ such that $\operatorname{spectrum}(\mathbb{A}_K^T \mathbb{A}_K) \subset [c, C]$ for all $K \in \bigcup_{h \in (0,H]} \mathcal{T}_h$;
- The intersection of two distinct elements K_1, K_2 of a mesh $\mathcal{T}_h$ is either empty, a common vertex, a common edge, or (in the case $d = 3$) a common face.

The symbol $\mathcal{E}_h$ denotes the set of all faces, $d = 3$, or all edges, $d = 2$, in the mesh $\mathcal{T}_h$. $\mathcal{E}_{h,\text{ext}}$ and $\mathcal{E}_{h,\text{int}}$ stand for the sets of exterior and interior faces, respectively, i.e.,

$$\mathcal{E}_{h,\text{ext}} = \left\{ \sigma \in \mathcal{E}_h \,\middle|\, \sigma \subset \partial\Omega_h \right\} \qquad \text{and} \qquad \mathcal{E}_{h,\text{int}} = \mathcal{E}_h \backslash \mathcal{E}_{h,\text{ext}}\,.$$

Moreover, for $K \in \mathcal{T}_h$, we put $\mathcal{E}_h(K) = \left\{ \sigma \in \mathcal{E}_h \,\middle|\, \sigma \subset K \right\}$ and $\mathcal{E}_{h,z}(K) = \left\{ \sigma \in \mathcal{E}_{h,z} \,\middle|\, \sigma \subset K \right\}$, where $z \in \{\text{int}, \text{ext}\}$. In connection with these sets, we shall use the abbreviations

$$\int_{\mathcal{E}_{h,\text{int}}} \equiv \sum_{\sigma \in \mathcal{E}_{h,\text{int}}} \int_\sigma \qquad \text{and} \qquad \int_{\mathcal{E}_h(K)} \equiv \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_h(K)} \int_\sigma .$$

Each face $\sigma \in \mathcal{E}_h$ is equipped with a unit vector $\boldsymbol{n}_\sigma$ that is determined as follows: We fix an arbitrary element $K_\sigma \in \mathcal{T}_h$ such that $\sigma \in \mathcal{E}_h(K_\sigma)$ and set $\boldsymbol{n}_\sigma = \boldsymbol{n}_{K_\sigma}(\boldsymbol{x}_\sigma)$. Here, $\boldsymbol{x}_\sigma$ denotes the center of mass of σ and $\boldsymbol{n}_{K_\sigma}(\boldsymbol{x}_\sigma)$ is the outward-pointing unit normal vector to the element K_σ at $\boldsymbol{x}_\sigma$. Finally, it will be convenient to write $A \lesssim B$ whenever there is an h-independent constant $c > 0$ such that $A \leq cB$ and $A \approx B$ whenever $A \lesssim B$ and $B \lesssim A$.

[4] Note that the numerical setting is slightly different from that in [15]. We now approximate the spatial domain Ω from the outside (cf. (4.4)). In addition, we introduce a new operator $\Pi^0_{V,h} : W^{1,2}(\Omega_h) \to V_{0,h}$ (cf. Sect. 4.2.3).

4.2.2. Function Spaces and Projection Operators. The space of piecewise constant functions is denoted by

$$Q_h = \big\{ v \in L^2(\Omega_h) \,\big|\, v|_K \in P_0(K) \text{ for all } K \in \mathcal{T}_h \big\}.$$

[5]For $v \in Q_h$ and $K \in \mathcal{T}_h$ we set $v_K = v(\boldsymbol{x}_K)$, where $\boldsymbol{x}_K$ denotes the center of mass of K. The projection $\Pi_{Q,h} \equiv \overline{\cdot} : L^2(\Omega_h) \to Q_h$ associated with Q_h is characterized by

$$(\Pi_{Q,h} v)\big|_K \equiv \overline{v}|_K \equiv \frac{1}{|K|} \int_K v \,\mathrm{d}\mathbf{y} \quad \text{for all } K \in \mathcal{T}_h\,.$$

The Crouzeix-Raviart finite element spaces are denoted by

$$V_h = \left\{ v \in L^2(\Omega_h) \,\middle|\, \begin{array}{c} v|_K \in P_1(K) \text{ for all } K \in \mathcal{T}_h \text{ and} \\ \displaystyle\int_\sigma \lim_{\delta \to 0+} \big(v(\boldsymbol{x} - \delta \boldsymbol{n}_\sigma) - v(\boldsymbol{x} + \delta \boldsymbol{n}_\sigma) \big) \,\mathrm{d}S_{\boldsymbol{x}} = 0 \quad \text{for all } \sigma \in \mathcal{E}_{h,\mathrm{int}} \end{array} \right\},$$

$$V_{0,h} = \left\{ v \in V_h \,\middle|\, \int_\sigma \lim_{\delta \to 0+} v(\boldsymbol{x} - \delta \boldsymbol{n}_\sigma) \,\mathrm{d}S_{\boldsymbol{x}} = 0 \quad \text{for all } \sigma \in \mathcal{E}_{h,\mathrm{ext}} \right\}.$$

With these spaces we associate the projections $\Pi_{V,h} : W^{1,2}(\Omega_h) \to V_h$, $\Pi^0_{V,h} : W^{1,2}(\Omega_h) \to V_{0,h}$ that are determined by

$$\int_\sigma \Pi_{V,h} v \,\mathrm{d}S_{\boldsymbol{x}} = \int_\sigma v \,\mathrm{d}S_{\boldsymbol{x}} \quad \text{for all } \sigma \in \mathcal{E}_h, \qquad \int_\sigma \Pi^0_{V,h} v \,\mathrm{d}S_{\boldsymbol{x}} = \int_\sigma v \,\mathrm{d}S_{\boldsymbol{x}} \quad \text{for all } \sigma \in \mathcal{E}_{h,\mathrm{int}},$$

respectively. Additionally, we agree on the notation

$$Q_h^+ = \big\{ v \in Q_h \,\big|\, v|_K > 0 \text{ for all } K \in \mathcal{T}_h \big\}, \qquad Q_h^{0,+} = \big\{ v \in Q_h \,\big|\, v|_K \geq 0 \text{ for all } K \in \mathcal{T}_h \big\},$$
$$\boldsymbol{Q}_h = (Q_h)^d, \qquad \boldsymbol{V}_h = (V_h)^d, \qquad \text{and} \qquad \boldsymbol{V}_{0,h} = (V_{0,h})^d.$$

4.2.3. Mesh-Related Operators. Next, we recall the necessary mesh-related operators. We start by repeating the definitions of the discrete counterparts of the differential operators $\nabla_{\boldsymbol{x}}$ and $\mathrm{div}_{\boldsymbol{x}}$. They are determined by the stipulations

$$(\nabla_h \boldsymbol{v})|_K = \nabla_{\boldsymbol{x}}(\boldsymbol{v}|_K) \quad \text{for all } \boldsymbol{v} \in (V_h \cup \boldsymbol{V}_h) \cup (W^{1,1}(\Omega_h) \cup W^{1,1}(\Omega_h)^d) \text{ and all } K \in \mathcal{T}_h$$
$$\text{and} \quad \mathrm{div}_h(\boldsymbol{v})|_K = \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{v}|_K) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_h \cup W^{1,1}(\Omega_h)^d \text{ and all } K \in \mathcal{T}_h,$$

respectively. We continue by recalling the trace operators. For arbitrary $\sigma \in \mathcal{E}_h$, $\boldsymbol{x} \in \sigma$, and

$$\boldsymbol{v} \in (Q_h \cup \boldsymbol{Q}_h) \cup (V_h \cup \boldsymbol{V}_h) \cup (C(\overline{\Omega_h}) \cup C(\overline{\Omega_h})^d)$$

we set

$$\boldsymbol{v}^{\,\mathrm{in},\,\sigma}(\boldsymbol{x}) = \lim_{\delta \to 0+} \boldsymbol{v}(\boldsymbol{x} - \delta \boldsymbol{n}_\sigma), \qquad \boldsymbol{v}^{\,\mathrm{out},\,\sigma}(\boldsymbol{x}) = \begin{cases} \lim\limits_{\delta \to 0+} \boldsymbol{v}(\boldsymbol{x} + \delta \boldsymbol{n}_\sigma) & \text{if } \sigma \in \mathcal{E}_{h,\mathrm{int}}, \\ \boldsymbol{0} & \text{else} \end{cases},$$

$$[\![\boldsymbol{v}]\!]_\sigma = \boldsymbol{v}^{\,\mathrm{out},\,\sigma} - \boldsymbol{v}^{\,\mathrm{in},\,\sigma}, \qquad \{\boldsymbol{v}\}_\sigma = \frac{\boldsymbol{v}^{\,\mathrm{out},\,\sigma} + \boldsymbol{v}^{\,\mathrm{in},\,\sigma}}{2} \qquad \text{and} \qquad \langle \boldsymbol{v} \rangle_\sigma = \frac{1}{|\sigma|} \int_\sigma \boldsymbol{v}^{\,\mathrm{in},\,\sigma} \,\mathrm{d}S_{\boldsymbol{x}}\,.$$

The convective terms shall be approximated by means of a dissipative upwind operator. For $\sigma \in \mathcal{E}_h$, $\boldsymbol{v} \in \boldsymbol{V}_{0,h}$, and $\boldsymbol{r} \in Q_h \cup \boldsymbol{Q}_h$ we put

$$\mathrm{Up}[\boldsymbol{r}, \boldsymbol{v}]_\sigma = \boldsymbol{r}^{\,\mathrm{out},\,\sigma} \left[\langle \boldsymbol{v} \cdot \boldsymbol{n}_\sigma \rangle_\sigma \right]^- + \boldsymbol{r}^{\,\mathrm{in},\,\sigma} \left[\langle \boldsymbol{v} \cdot \boldsymbol{n}_\sigma \rangle_\sigma \right]^+,$$
$$F_h^{\mathrm{up}}[\boldsymbol{r}, \boldsymbol{v}]_\sigma = \mathrm{Up}[\boldsymbol{r}, \boldsymbol{v}]_\sigma - \frac{h^\varepsilon}{2} [\![\boldsymbol{r}]\!]_\sigma = \{\boldsymbol{r}\}_\sigma \langle \boldsymbol{v} \cdot \boldsymbol{n}_\sigma \rangle_\sigma - \frac{1}{2} [\![\boldsymbol{r}]\!]_\sigma \big(h^\varepsilon + |\langle \boldsymbol{v} \cdot \boldsymbol{n}_\sigma \rangle_\sigma| \big),$$

where $\varepsilon > 0$ is a given constant, $[x]^+ = \max\{x, 0\}$ and $[x]^- = \min\{x, 0\}$.

Remark 4.1. As in [15], we tend to omit parts of the subscripts and superscripts of the operators defined in Sects. 4.2.2 and 4.2.3 if no confusion arises. This includes the letters h and σ as well as the word *in*.

[5] $P_n(K)$ denotes the set of all restrictions of polynomial functions $\mathbb{R}^d \to \mathbb{R}$ of degree at most n to the set K.

4.2.4. Time Discretization. To approximate the time derivatives, we employ the backward Euler method. Consequently, the discrete time derivative D_t is given by

$$D_t \boldsymbol{s}_h^k = \frac{\boldsymbol{s}_h^k - \boldsymbol{s}_h^{k-1}}{\Delta t}\,,$$

where $\Delta t > 0$ is a given time step and $\boldsymbol{s}_h^{k-1}$ and $\boldsymbol{s}_h^k$ are the numerical solutions at the time levels $t_{k-1} = (k-1)\Delta t$ and $t_k = k\Delta t$, respectively. For the sake of simplicity, we assume that Δt is constant and that there is a number $N_T \in \mathbb{N}$ such that $N_T \Delta t = T$.

4.2.5. Numerical Scheme. The mixed FE-FV method introduced in [15, Definition 3.2] reads as follows.

Given $(\varrho_h^0, \theta_h^0, \boldsymbol{u}_h^0) \in Q_h^+ \times Q_h^+ \times \boldsymbol{V}_h$, we search for a sequence $(\varrho_h^k, \theta_h^k, \boldsymbol{u}_h^k)_{k\in\mathbb{N}} \subset Q_h^+ \times Q_h^+ \times \boldsymbol{V}_{0,h}$ such that the following equations hold for all $k \in \mathbb{N}$, $\phi_h \in Q_h$ and $\boldsymbol{\phi}_h \in \boldsymbol{V}_{0,h}$:

$$\int_{\Omega_h} (D_t \varrho_h^k)\, \phi_h \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_{\mathrm{int}}} F_h^{\mathrm{up}}\left[\varrho_h^k, \boldsymbol{u}_h^k\right] [\![\phi_h]\!] \,\mathrm{d}S_{\boldsymbol{x}} = 0\,, \tag{4.5}$$

$$\int_{\Omega_h} D_t(\varrho_h^k \theta_h^k)\, \phi_h \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_{\mathrm{int}}} F_h^{\mathrm{up}}\left[\varrho_h^k \theta_h^k, \boldsymbol{u}_h^k\right] [\![\phi_h]\!] \,\mathrm{d}S_{\boldsymbol{x}} = 0\,, \tag{4.6}$$

$$\begin{aligned}\int_{\Omega_h} D_t\Big(\varrho_h^k \overline{\boldsymbol{u}_h^k}\Big) \cdot \boldsymbol{\phi}_h \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_{\mathrm{int}}} F_h^{\mathrm{up}}\Big[\varrho_h^k \overline{\boldsymbol{u}_h^k}, \boldsymbol{u}_h^k\Big] \cdot \Big[\!\!\Big[\overline{\boldsymbol{\phi}_h}\Big]\!\!\Big] \,\mathrm{d}S_{\boldsymbol{x}} + \mu \int_{\Omega_h} \nabla_h \boldsymbol{u}_h^k : \nabla_h \boldsymbol{\phi}_h \,\mathrm{d}\boldsymbol{x} \\ + \nu \int_{\Omega_h} \mathrm{div}_h(\boldsymbol{u}_h^k)\, \mathrm{div}_h(\boldsymbol{\phi}_h) \,\mathrm{d}\boldsymbol{x} - \int_{\Omega_h} \left(p(\varrho_h^k \theta_h^k) + h^\delta \left[(\varrho_h^k)^2 + (\varrho_h^k \theta_h^k)^2\right]\right) \mathrm{div}_h(\boldsymbol{\phi}_h) \,\mathrm{d}\boldsymbol{x} = 0\,,\end{aligned} \tag{4.7}$$

where $\delta > 0$ and $\nu = \frac{d-2}{d}\mu + \lambda \geq 0$.

4.2.6. Discrete Initial Data. The initial data for the mixed FE-FV method (4.5)–(4.7) are determined as follows:

$$\varrho_h^0 = \Pi_Q \varrho_0\,, \qquad \theta_h^0 = \Pi_Q \theta_0 \qquad \text{and} \qquad \boldsymbol{u}_h^0 = \Pi_V \boldsymbol{u}_0\,. \tag{4.8}$$

As a consequence of this stipulation, we observe that $(\varrho_h^0, \theta_h^0, \boldsymbol{u}_h^0) \in Q_h^+ \times Q_h^+ \times \boldsymbol{V}_{0,h}$ with

$$0 < (\varrho_0)_\star \leq \varrho_h^0 \leq (\varrho_0)^\star \qquad \text{and} \qquad 0 < (\theta_0)_\star \leq \theta_h^0 \leq (\theta_0)^\star\,. \tag{4.9}$$

4.3. Discrete Energy and Entropy Inequalities

The solvability of the FE-FV method (4.5)–(4.7) is guaranteed by [15, Lemma 3.4]. In particular, it follows from a combination of this lemma with (4.9) that for every $k \in \mathbb{N}_0$

$$\varrho_h^k > 0\,, \quad (\theta_0)_\star \leq \theta_h^k \leq (\theta_0)^\star\,, \quad ||\varrho_h^k||_{L^1(\Omega_h)} = ||\varrho_h^0||_{L^1(\Omega_h)}\,, \quad ||\varrho_h^k \theta_h^k||_{L^1(\Omega_h)} = ||\varrho_h^0 \theta_h^0||_{L^1(\Omega_h)}\,. \tag{4.10}$$

In addition, it turns out that the numerical solutions satisfy an energy balance and an entropy inequality that read as follows:

Discrete energy balance: (cf. [15, Lemma 4.2])
For every $k \in \mathbb{N}$ and suitably chosen $\xi_{\varrho\theta,P,k} \in Q_h$ and $(\xi^{(1)}_{\varrho\theta,P,k,\sigma})_{\sigma\in\mathcal{E}_{\mathrm{int}}}, (\xi^{(2)}_{\varrho\theta,P,k,\sigma})_{\sigma\in\mathcal{E}_{\mathrm{int}}} \subset \mathbb{R}$ it holds

$$
\begin{aligned}
&\int_{\Omega_h} D_t E_h^k \,\mathrm{d}\boldsymbol{x} + \int_{\Omega_h} \big[\mu |\nabla_h \boldsymbol{u}_h^k|^2 + \nu |\mathrm{div}_h(\boldsymbol{u}_h^k)|^2\big] \,\mathrm{d}\boldsymbol{x} \qquad (4.11)\\
&= -\frac{1}{2}\int_{\Omega_h} P''(\xi_{\varrho\theta,P,k}) \frac{(\varrho_h^k\theta_h^k - \varrho_h^{k-1}\theta_h^{k-1})^2}{\Delta t} \,\mathrm{d}\boldsymbol{x} - \frac{h^\varepsilon}{2}\int_{\mathcal{E}_{\mathrm{int}}} [\![\varrho_h^k\theta_h^k]\!]\,[\![P'(\varrho_h^k\theta_h^k)]\!] \,\mathrm{d}S_{\boldsymbol{x}}\\
&\quad - \frac{1}{2}\int_{\mathcal{E}_{\mathrm{int}}} \Big(P''(\xi^{(1)}_{\varrho\theta,P,k,\sigma})\big[\langle \boldsymbol{u}_h^k\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big]^+ - P''(\xi^{(2)}_{\varrho\theta,P,k,\sigma})\big[\langle \boldsymbol{u}_h^k\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big]^-\Big)[\![\varrho_h^k\theta_h^k]\!]^2 \,\mathrm{d}S_{\boldsymbol{x}}\\
&\quad - \int_{\Omega_h} \frac{\Delta t}{2}\,\varrho_h^{k-1}\left|\frac{\overline{\boldsymbol{u}_h^k} - \overline{\boldsymbol{u}_h^{k-1}}}{\Delta t}\right|^2 \mathrm{d}\boldsymbol{x} - \frac{h^\varepsilon}{2}\int_{\mathcal{E}_{\mathrm{int}}} \{\varrho_h^k\}\,\Big[\!\!\Big[\overline{\boldsymbol{u}_h^k}\Big]\!\!\Big]^2 \mathrm{d}S_{\boldsymbol{x}}\\
&\quad - \frac{1}{2}\int_{\mathcal{E}_{\mathrm{int}}} \Big((\varrho_h^k)^{\mathrm{in}}\big[\langle \boldsymbol{u}_h^k\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big]^+ - (\varrho_h^k)^{\mathrm{out}}\big[\langle \boldsymbol{u}_h^k\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big]^-\Big)\Big[\!\!\Big[\overline{\boldsymbol{u}_h^k}\Big]\!\!\Big]^2 \mathrm{d}S_{\boldsymbol{x}}\\
&\quad - h^\delta\int_{\Omega_h} \frac{(\varrho_h^k - \varrho_h^{k-1})^2}{\Delta t}\,\mathrm{d}\boldsymbol{x} - h^\delta\int_{\mathcal{E}_{\mathrm{int}}} \big(h^\varepsilon + |\langle \boldsymbol{u}_h^k\cdot\boldsymbol{n}_\sigma\rangle_\sigma|\big)[\![\varrho_h^k]\!]^2 \,\mathrm{d}S_{\boldsymbol{x}}\\
&\quad - h^\delta\int_{\Omega_h} \frac{(\varrho_h^k\theta_h^k - \varrho_h^{k-1}\theta_h^{k-1})^2}{\Delta t}\,\mathrm{d}\boldsymbol{x} - h^\delta\int_{\mathcal{E}_{\mathrm{int}}} \big(h^\varepsilon + |\langle \boldsymbol{u}_h^k\cdot\boldsymbol{n}_\sigma\rangle_\sigma|\big)[\![\varrho_h^k\theta_h^k]\!]^2 \,\mathrm{d}S_{\boldsymbol{x}}\,,
\end{aligned}
$$

where

$$
E_h^k \equiv E_h^k(\varrho_h^k,\theta_h^k,\boldsymbol{u}_h^k) = \frac{1}{2}\,\varrho_h^k|\overline{\boldsymbol{u}_h^k}|^2 + P(\varrho_h^k\theta_h^k) + h^\delta(\varrho_h^k)^2\big[1 + (\theta_h^k)^2\big]\,.
$$

Discrete entropy inequality: (cf. [15, Lemma 4.5])
For every $k \in \mathbb{N}$ and every pair $(\chi,\psi_h) \in C^2(0,\infty)\times Q_h^{0,+}$ it holds

$$
\begin{aligned}
0 &\le \int_{\Omega_h} D_t(\varrho_h^k\chi(\theta_h^k))\,\psi_h \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_{\mathrm{int}}} \mathrm{Up}\big[\varrho_h^k\chi(\theta_h^k), \boldsymbol{u}_h^k\big]\,[\![\psi_h]\!] \,\mathrm{d}S_{\boldsymbol{x}} \qquad (4.12)\\
&\quad + \frac{h^\varepsilon}{2}\int_{\mathcal{E}_{\mathrm{int}}} [\![\varrho_h^k\theta_h^k]\!]\,[\![\chi'(\theta_h^k)\psi_h]\!] \,\mathrm{d}S_{\boldsymbol{x}} + \frac{h^\varepsilon}{2}\int_{\mathcal{E}_{\mathrm{int}}} [\![\varrho_h^k]\!]\,\big[\!\!\big[\big(\chi(\theta_h^k) - \chi'(\theta_h^k)\theta_h^k\big)\psi_h\big]\!\!\big] \,\mathrm{d}S_{\boldsymbol{x}}\,.
\end{aligned}
$$

Given a solution $(\varrho_h^k,\theta_h^k,\boldsymbol{u}_h^k)_{k\in\mathbb{N}} \subset Q_h^+\times Q_h^+\times \boldsymbol{V}_{0,h}$ to the FE-FV method (4.5)–(4.7) starting from the initial data (4.8), we define the functions $\varrho_h^-, \varrho_h, \theta_h : \mathbb{R}\times\Omega_h \to (0,\infty)$, $\boldsymbol{u}_h : \mathbb{R}\times\Omega_h \to \mathbb{R}^d$ that are piecewise constant in time by setting

$$
(\varrho_h^-,\varrho_h,\theta_h,\boldsymbol{u}_h)(t,\cdot) = \begin{cases} (\varrho_h^{k-1},\varrho_h^k,\theta_h^k,\boldsymbol{u}_h^k) & \text{if } t\in(t_{k-1},t_k] \text{ for some } k\in\mathbb{N} \text{ and}\\ (\varrho_h^0,\varrho_h^0,\theta_h^0,\boldsymbol{u}_h^0) & \text{if } t\le 0. \end{cases}
$$

In addition, we introduce the functions $S_h, E_h : \mathbb{R}\times\Omega_h \to \mathbb{R}$ via the stipulations

$$
S_h = \frac{\gamma}{\gamma-1}\,\varrho_h\ln(\theta_h)\,, \qquad E_h = \frac{1}{2}\,\varrho_h|\overline{\boldsymbol{u}_h}|^2 + P(\varrho_h,S_h) + h^\delta\varrho_h^2\big[1+\theta_h^2\big]\,.
$$

Next, let us state two consequences of the discrete energy balance (4.11).

4.3.1. Stability Estimates. From (4.11) we obtain the subsequent energy estimates (cf. [15, Corollary 4.4]):

$$
\|\varrho_h|\overline{\boldsymbol{u}_h}|^2\|_{L^\infty(0,T;L^1(\Omega_h))} \lesssim 1\,, \quad \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega_h))} \lesssim 1\,, \quad \|\varrho_h\overline{\boldsymbol{u}_h}\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega_h)^d)} \lesssim 1\,, \qquad (4.13)
$$

$$
\|\nabla_h\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega_h)^{d\times d})} \lesssim 1\,, \quad \|\mathrm{div}_h(\boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega_h))} \lesssim 1\,, \quad \|\boldsymbol{u}_h\|_{L^2(0,T;L^q(\Omega_h)^d)} \lesssim 1\,, \qquad (4.14)
$$

$$||\varrho_h\theta_h||_{L^\infty(0,T;\,L^\gamma(\Omega_h))} \lesssim 1\,, \quad ||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))} \lesssim 1\,, \quad ||h^{\delta/2}\varrho_h\theta_h||_{L^\infty(0,T;\,L^2(\Omega_h))} \lesssim 1\,, \tag{4.15}$$

$$||\varrho_h\theta_h\overline{\boldsymbol{u}_h}||_{L^\infty(0,T;\,L^{2\gamma/(\gamma+1)}(\Omega_h)^d)} \lesssim 1\,, \quad ||\varrho_h\overline{\boldsymbol{u}_h}||_{L^2(0,T;\,L^2(\Omega_h)^d)} \lesssim h^{-\frac{d+3\delta}{6}}\,, \tag{4.16}$$

$$h^\delta \int_0^T\int_{\mathcal{E}_{\rm int}} \max\{h^\varepsilon, |\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma|\}\, [\![\varrho_h]\!]^2\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \lesssim 1\,, \tag{4.17}$$

$$h^\delta \int_0^T\int_{\mathcal{E}_{\rm int}} \max\{h^\varepsilon, |\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma|\}\, [\![\varrho_h\theta_h]\!]^2\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \lesssim 1\,, \tag{4.18}$$

$$\frac{h^\varepsilon}{2}\int_0^T\int_{\mathcal{E}_{\rm int}} \{\varrho_h\}[\![\overline{\boldsymbol{u}_h}]\!]^2\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \lesssim 1\,, \quad \Delta t\int_0^T\int_{\Omega_h} \varrho_h^-(D_t\overline{\boldsymbol{u}_h})^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim 1\,, \tag{4.19}$$

$$\frac{1}{2}\int_0^T\int_{\mathcal{E}_{\rm int}} \Big(\varrho_h^{\rm in}\big[\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big]^+ - \varrho_h^{\rm out}\big[\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big]^-\Big)\,[\![\overline{\boldsymbol{u}_h}]\!]^2\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \lesssim 1\,, \tag{4.20}$$

$$\int_0^T\int_{\mathcal{E}_{\rm int}} \big|[\![\varrho_h]\!]\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big|\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \lesssim h^{-\delta/2}(1+h^{-1/2})\,, \tag{4.21}$$

$$\int_0^T\int_{\mathcal{E}_{\rm int}} \big|[\![\varrho_h\theta_h]\!]\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big|\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \lesssim h^{-\delta/2}(1+h^{-1/2})\,, \tag{4.22}$$

$$h^\delta\Delta t\int_0^T\int_{\Omega_h} (D_t\varrho_h)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim 1\,, \quad h^\delta\Delta t\int_0^T\int_{\Omega_h} (D_t(\varrho_h\theta_h))^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim 1\,, \tag{4.23}$$

where $q \in [1,\infty)$ if $d=2$ and $q\in[1,6]$ if $d=3$.

Remark 4.2. Note that the proof of [15, Corollary 4.4] can be extended to include the estimates in (4.23). In addition, the different way of approximating the spatial domain Ω (we now have $\Omega\subset\Omega_h$ instead of $\Omega_h\subset\Omega$) requires some minor and straightforward modifications. We leave the details to the interested reader.

Moreover, for further application it is convenient to observe that the energy balance (4.11) provides us with the following

Energy inequality:

$$\left[\int_{\Omega_h} E_h(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} \le -\int_0^\tau\int_{\Omega_h}\left[\mu\,|\nabla_h\boldsymbol{u}_h|^2 + \nu\,|\mathrm{div}_h(\boldsymbol{u}_h)|^2\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \qquad \text{for every } \tau\ge 0. \tag{4.24}$$

4.4. Consistency

We proceed by stating a suitable consistency formulation of the numerical scheme (4.5)–(4.7).

Theorem 4.3 (Consistency of the FE-FV method). *Let $\beta = \min\left\{\varepsilon-1, \frac{1-2\delta}{4}\right\}$ and $\tau\in[0,T]$. Further, suppose $(\varrho_h,\theta_h,\boldsymbol{u}_h)_{h\in(0,H]}$ is a family of solutions to the FE-FV method (4.5)–(4.7) with*

$$\gamma>1\,, \qquad \Delta t\approx h\,, \qquad \varepsilon>1 \qquad \text{and} \qquad 0<\delta<\tfrac{1}{2} \tag{4.25}$$

starting from the initial data $(\varrho_h^0,\theta_h^0,\boldsymbol{u}_h^0)_{h\in(0,H]}$ defined in (4.8). Then

$$\left[\int_\Omega (\varrho_h\varphi)(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega\left[\varrho_h\partial_t\varphi + \varrho_h\overline{\boldsymbol{u}_h}\cdot\nabla_{\boldsymbol{x}}\varphi\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^\beta) \tag{4.26}$$

for all $\varphi\in C^1(\overline{\Omega_T})$ as $h\downarrow 0$,

$$\left[\int_\Omega (\varrho_h\overline{\boldsymbol{u}_h}\cdot\boldsymbol{\varphi})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega\left[\mu\nabla_h\boldsymbol{u}_h:\nabla_{\boldsymbol{x}}\boldsymbol{\varphi} + \nu\,\mathrm{div}_h(\boldsymbol{u}_h)\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\varphi})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^\beta)$$

$$= \int_0^\tau \int_\Omega \left[\varrho_h \overline{\boldsymbol{u}_h} \cdot \partial_t \boldsymbol{\varphi} + \varrho_h \overline{\boldsymbol{u}_h} \otimes \overline{\boldsymbol{u}_h} : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} + \big(p(\varrho_h \theta_h) + h^\delta \left[\varrho_h^2 + (\varrho_h\theta_h)^2\right]\big) \operatorname{div}_{\boldsymbol{x}}(\boldsymbol{\varphi}) \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{4.27}$$

for all $\boldsymbol{\varphi} \in C^1(\overline{\Omega_T})^d$, $\boldsymbol{\varphi}|_{[0,T]\times\partial\Omega} = \mathbf{0}$, *as* $h \downarrow 0$,

$$\left[\int_\Omega (\varrho_h \theta_h \varphi)(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho_h \theta_h \partial_t \varphi + \varrho_h \theta_h \overline{\boldsymbol{u}_h} \cdot \nabla_{\boldsymbol{x}} \varphi \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^\beta) \tag{4.28}$$

for all $\varphi \in C^1(\overline{\Omega_T})$ *as* $h \downarrow 0$ *and*

$$\left[\int_\Omega (\varrho_h \ln(\theta_h) \psi)(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_\Omega \left[\varrho_h \ln(\theta_h) \partial_t \psi + \varrho_h \ln(\theta_h)\, \overline{\boldsymbol{u}_h} \cdot \nabla_{\boldsymbol{x}} \psi \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^\beta) \tag{4.29}$$

for all $\psi \in C^1(\overline{\Omega_T})$, $\psi \geq 0$, *as* $h \downarrow 0$. *Here, the constants in the* $\mathcal{O}$ *notation do not depend on the particular time* $\tau \in [0,T]$.

Proof. The proof is given in Appendix A.3. □

4.5. Error Estimates

We continue with the derivation of a priori error estimates for the FE-FV method. For convenience, we agree that in this section the constants hidden in the $\lesssim$-symbols and the $\mathcal{O}$ notation neither depend on the times $\tau \in [0,T]$ nor on the number $\alpha > 0$ that will appear in the sequel.

4.5.1. Discrete Relative Energy. To begin with, we introduce a suitable extension of the relative energy $E(\,\cdot\,|\,\cdot\,)$ that we will refer to as the *discrete relative energy*. It will be used to measure a "distance" between a numerical solution $(\varrho_h, \theta_h, \boldsymbol{u}_h)$ and a triplet $(\varrho, \theta, \boldsymbol{u})$ of functions of the class (4.1) and reads

$$\begin{aligned} \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u}) &= \frac{1}{2}\, \varrho_h |\overline{\boldsymbol{u}_h} - \boldsymbol{u}|^2 + h^\delta (\varrho_h - \varrho)^2 + h^\delta (\varrho_h \theta_h - \varrho\theta)^2 \\ &\quad + P(\varrho_h, S_h) - \frac{\partial P(\varrho, S)}{\partial \varrho}\,(\varrho_h - \varrho) - \frac{\partial P(\varrho, S)}{\partial S}\,(S_h - S) - P(\varrho, S)\,. \end{aligned}$$

Our aim is to repeat the proof of Lemma 3.2 on the numerical level to obtain a version of the relative energy inequality for $\tilde{E}(\,\cdot|\cdot\,)$. First, our initial observation is that

$$\begin{aligned} &\left[\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_h \setminus \Omega} E_h(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \\ &= \left[\int_\Omega \varrho_h(t,\cdot) \left(\frac{1}{2}\,|\boldsymbol{u}(t,\cdot)|^2 - \frac{\partial P(\varrho(t,\cdot), S(t,\cdot))}{\partial \varrho} - 2h^\delta \varrho(t,\cdot) \right) \mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} + \left[\int_{\Omega_h} E_h(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \\ &\quad + \left[\int_\Omega \big(p(\varrho(t,\cdot), S(t,\cdot)) + h^\delta \varrho^2(t,\cdot)\left[1 + \theta^2(t,\cdot)\right] \big)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} - \left[\int_\Omega S_h(t,\cdot) \vartheta(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \\ &\quad - 2h^\delta \left[\int_\Omega \varrho_h(t,\cdot) \theta_h(t,\cdot) \varrho(t,\cdot) \theta(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} - \left[\int_\Omega \varrho_h(t,\cdot)\, \overline{\boldsymbol{u}_h}(t,\cdot) \cdot \boldsymbol{u}(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \end{aligned} \tag{4.30}$$

for every $\tau \in [0,T]$. To be able to transfer the next step in the proof of Lemma 3.2 to the discrete setting, we need to derive a suitable analogue of (3.8).

4.5.2. Partial Integration for Diffusion Terms. For the treatment of the diffusion terms, we extend the velocity $\boldsymbol{u} \in L^2(0,T;W^{2,\infty}(\Omega)^d)$ to a function $\widehat{\boldsymbol{u}} \in L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d)^d)$ using Stein's extension operator $\mathfrak{E}_{\text{Stein}}$, see [20, Chapter VI, Theorem 5], i.e., we put $\widehat{\boldsymbol{u}}(t,\cdot) = \mathfrak{E}_{\text{Stein}}[\boldsymbol{u}(t,\cdot)]$[6]. As a consequence,

$$||\widehat{\boldsymbol{u}}||_{L^2(0,T;W^{k,\infty}(\mathbb{R}^d)^d)} \leq C_{\text{Stein}}(\Omega,2)\,||\boldsymbol{u}||_{L^2(0,T;W^{k,\infty}(\Omega)^d)}, \qquad k=0,1,2, \tag{4.31}$$

where $C_{\text{Stein}}(\Omega,2) > 0$ is given by

$$C_{\text{Stein}}(\Omega,2) = \max_{k\in\{0,1,2\}} \left\{ ||\mathfrak{E}_{\text{Stein}}||_{W^{k,\infty}(\Omega)\to W^{k,\infty}(\mathbb{R}^d)} \right\}.$$

Having extended $\boldsymbol{u}$ as described above, we may use Gauss's theorem to observe that

$$\begin{aligned}
&-\int_0^\tau\int_{\Omega_h} \nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,\mathrm{div}_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \int_0^\tau\int_\Omega \nu\,|\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_{\Omega_h} \nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,\mathrm{div}_h(\boldsymbol{u}_h)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h) \\
&= -\int_0^\tau\int_\Omega \nu\,\nabla_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau \sum_{K\in\mathcal{T}_h}\int_K \nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u}_h)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h) \\
&= -\int_0^\tau\int_\Omega \nu\,\nabla_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau \sum_{K\in\mathcal{T}_h}\int_K \nu\,\nabla_{\boldsymbol{x}}\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\cdot\boldsymbol{u}_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad - \int_0^\tau\int_{\mathcal{E}(K)} \nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,\boldsymbol{u}_h\cdot\boldsymbol{n}_K\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t + \mathcal{O}(h) \\
&= \int_0^\tau\int_\Omega \nu\,\nabla_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\cdot(\boldsymbol{u}_h - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\int_{\mathcal{E}} \nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}_\sigma\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t + \mathcal{O}(h^{1/2}) \\
&= \int_0^\tau\int_\Omega \nu\,\nabla_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\cdot(\boldsymbol{u}_h - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{1/2})
\end{aligned}$$

for all $\tau \in [0,T]$. Here, the first, the fourth and the fifth equality are due to (4.4), (4.31) and the first and the last estimate in (4.14) which yield

$$\left|\int_0^\tau\int_{\Omega_h\setminus\Omega} \nu\,|\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \lesssim |\Omega_h\setminus\Omega|\,||\widehat{\boldsymbol{u}}||^2_{L^2(0,T;W^{1,\infty}(\mathbb{R}^d)^d)} \lesssim h\,||\boldsymbol{u}||^2_{L^2(0,T;W^{1,\infty}(\Omega)^d)} \lesssim h,$$

$$\begin{aligned}
\left|\int_0^\tau\int_{\Omega_h\setminus\Omega} \nu\,\nabla_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\cdot\boldsymbol{u}_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| &\lesssim |\Omega_h\setminus\Omega|^{1/2}\,||\widehat{\boldsymbol{u}}||_{L^2(0,T;W^{2,\infty}(\mathbb{R}^d)^d)}\,||\boldsymbol{u}_h||_{L^2(0,T;L^2(\Omega_h)^d)} \\
&\lesssim h^{1/2}\,||\boldsymbol{u}||_{L^2(0,T;W^{2,\infty}(\Omega)^d)}\,||\boldsymbol{u}_h||_{L^2(0,T;L^2(\Omega_h)^d)} \lesssim h^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_0^\tau\int_{\mathcal{E}} \nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}_\sigma\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\right| &= \left|\int_0^\tau\int_{\mathcal{E}} \nu\,\big(\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}}) - \mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})(\boldsymbol{x}_\sigma)\big)\,[\![\boldsymbol{u}_h]\!]\cdot\boldsymbol{n}_\sigma\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\right| \\
&\lesssim h\int_0^\tau ||\widehat{\boldsymbol{u}}(t,\cdot)||_{W^{2,\infty}(\mathbb{R}^d)^d}\int_{\mathcal{E}} \big|[\![\boldsymbol{u}_h]\!]\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \\
&\lesssim h\int_0^\tau ||\widehat{\boldsymbol{u}}(t,\cdot)||_{W^{2,\infty}(\mathbb{R}^d)^d}\int_{\mathcal{E}} \big|[\![\nabla_h\boldsymbol{u}_h]\!]\cdot(\boldsymbol{x}-\boldsymbol{x}_\sigma)\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \\
&\lesssim h^2\int_0^\tau ||\widehat{\boldsymbol{u}}(t,\cdot)||_{W^{2,\infty}(\mathbb{R}^d)^d}\int_{\mathcal{E}} \{|\nabla_h\boldsymbol{u}_h|\}\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t \\
&\lesssim h\int_0^\tau ||\widehat{\boldsymbol{u}}(t,\cdot)||_{W^{2,\infty}(\mathbb{R}^d)^d}\int_{\Omega_h} |\nabla_h\boldsymbol{u}_h|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned}$$

[6]This stipulation is to be understood componentwise.

$$
\begin{aligned}
&\lesssim h\,||\widehat{\boldsymbol{u}}||_{L^2(0,T;\,W^{2,\infty}(\mathbb{R}^d)^d)}\,||\nabla_h \boldsymbol{u}_h||_{L^2(0,T;\,L^1(\Omega_h)^{d\times d})}\\
&\lesssim h\,||\boldsymbol{u}||_{L^2(0,T;\,W^{2,\infty}(\Omega)^d)}\,||\nabla_h \boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})}\\
&\lesssim h.
\end{aligned}
$$

Similarly, we deduce that

$$
-\int_0^\tau\!\!\int_{\Omega_h} \mu \nabla_{\boldsymbol{x}} \widehat{\boldsymbol{u}} : \nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^\tau\!\!\int_\Omega \mu\,\Delta_{\boldsymbol{x}} \boldsymbol{u}\cdot(\boldsymbol{u}_h - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{1/2})
$$

for all $\tau \in [0,T]$. In addition, we may employ (4.4), (4.31) and the first and second estimate in (4.14) to observe that

$$
\begin{aligned}
&\left|\int_0^\tau\!\!\int_{\Omega_h\setminus\Omega} \left[\mu\nabla_h \boldsymbol{u}_h : \nabla_{\boldsymbol{x}} \widehat{\boldsymbol{u}} + \nu\,\mathrm{div}_h(\boldsymbol{u}_h)\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|\\
&\quad\lesssim |\Omega_h\setminus\Omega|^{1/2}\left(||\nabla_h \boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})} + ||\mathrm{div}_h(\boldsymbol{u}_h)||_{L^2(0,T;\,L^2(\Omega_h))}\right)||\widehat{\boldsymbol{u}}||_{L^2(0,T;\,W^{1,\infty}(\mathbb{R}^d)^d)}\\
&\quad\lesssim h^{1/2}\,||\boldsymbol{u}||_{L^2(0,T;\,W^{1,\infty}(\Omega)^d)} \lesssim h^{1/2}\,.
\end{aligned}
$$

Combining the previous estimates, we obtain the subsequent analogue of (3.8):

$$
\begin{aligned}
&-\int_0^\tau\!\!\int_{\Omega_h} \left[\nu\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\,\mathrm{div}_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}) + \mu\nabla_{\boldsymbol{x}}\widehat{\boldsymbol{u}} : \nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&-\int_0^\tau\!\!\int_{\Omega_h\setminus\Omega} \left[\mu\nabla_h \boldsymbol{u}_h : \nabla_{\boldsymbol{x}} \widehat{\boldsymbol{u}} + \nu\,\mathrm{div}_h(\boldsymbol{u}_h)\,\mathrm{div}_{\boldsymbol{x}}(\widehat{\boldsymbol{u}})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad= \int_0^\tau\!\!\int_\Omega \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u}_h - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{1/2})\\
&\quad= \int_0^\tau\!\!\int_\Omega \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\overline{\boldsymbol{u}_h} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{1/2})\\
&\quad= \int_0^\tau\!\!\int_\Omega (\varrho_h - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u} - \overline{\boldsymbol{u}_h}) - \frac{\varrho_h}{\varrho}\,(\boldsymbol{u} - \overline{\boldsymbol{u}_h})\cdot\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{1/2})
\end{aligned}
\tag{4.32}
$$

for all $\tau \in [0,T]$, where the second equality is due to the first estimate in (A.3) and the first estimate in (4.14) which imply

$$
\begin{aligned}
&\left|\int_0^\tau\!\!\int_\Omega \mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u}_h - \overline{\boldsymbol{u}_h})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right|\\
&\quad\lesssim ||\boldsymbol{u}||_{L^2(0,T;\,W^{2,\infty}(\Omega)^d)}\,||\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}||_{L^2(0,T;\,L^2(\Omega_h)^d)} \lesssim h\,||\nabla_h \boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})} \lesssim h\,.
\end{aligned}
$$

4.5.3. Relative Energy Inequality for $\tilde{E}$—General Form. It is now easy to transfer the remaining part of the proof of Lemma 3.2 to the discrete setting. Indeed, starting from (4.30) and ignoring the h^δ-terms, a repetition of the steps of the proof of Lemma 3.2 using (4.24), (4.26)–(4.29) and (4.32) instead of (2.3), (2.4)–(2.7) and (3.8) yields

$$
\begin{aligned}
&\left[\int_\Omega \tilde{E}(\varrho_h,\theta_h,\boldsymbol{u}_h\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_h\setminus\Omega} E_h(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}\\
&+\int_0^\tau\!\!\int_{\Omega_h} \left[\mu|\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 + \nu|\mathrm{div}_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad\le -\int_0^\tau\!\!\int_\Omega \varrho_h(\overline{\boldsymbol{u}_h} - \boldsymbol{u})^T\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot(\overline{\boldsymbol{u}_h} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\!\!\int_\Omega \left(\frac{\varrho_h}{\varrho}\,S - S_h\right)(\overline{\boldsymbol{u}_h} - \boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\qquad-\int_0^\tau\!\!\int_\Omega \left[p(\varrho_h,S_h) - \frac{\partial p(\varrho,S)}{\partial\varrho}\,(\varrho_h - \varrho) - \frac{\partial p(\varrho,S)}{\partial S}\,(S_h - S) - p(\varrho,S)\right]\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\tau\int_\Omega \frac{\varrho_h}{\varrho}\,(\boldsymbol{u}-\overline{\boldsymbol{u}_h})\cdot\left[\varrho\,\partial_t\boldsymbol{u}+\varrho\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot\boldsymbol{u}+\nabla_{\boldsymbol{x}}p(\varrho,S)-\operatorname{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega (\varrho-\varrho_h)\,\frac{1}{\varrho}\,\frac{\partial p(\varrho,S)}{\partial\varrho}\left[\partial_t\varrho+\operatorname{div}_{\boldsymbol{x}}(\varrho\boldsymbol{u})\right]\;+(\varrho-\varrho_h)\,\frac{1}{\varrho}\,\frac{\partial p(\varrho,S)}{\partial S}\left[\partial_t S+\operatorname{div}_{\boldsymbol{x}}(S\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\left(\frac{\varrho_h}{\varrho}\,S-S_h\right)\left[\partial_t\vartheta+\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\vartheta+\frac{\partial p(\varrho,S)}{\partial S}\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega(\varrho_h-\varrho)\,\frac{1}{\varrho}\operatorname{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\cdot(\boldsymbol{u}-\overline{\boldsymbol{u}_h})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\mathcal{O}(h^\beta)\\
&-h^\delta\int_0^\tau\int_\Omega\varrho_h^2\left[1+\theta_h^2\right]\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\left[h^\delta\int_\Omega\varrho^2(t,\cdot)\left[1+\theta^2(t,\cdot)\right]\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}\\
&-2h^\delta\left[\int_\Omega\varrho_h(t,\cdot)\varrho(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}-2h^\delta\left[\int_\Omega\varrho_h(t,\cdot)\theta_h(t,\cdot)\varrho(t,\cdot)\theta(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}
\end{aligned}
$$

for all $\tau\in[0,T]$. Then, using (4.26) and (4.28), we easily verify that

$$
\begin{aligned}
&-h^\delta\int_0^\tau\int_\Omega\varrho_h^2\left[1+\theta_h^2\right]\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\left[h^\delta\int_\Omega\varrho^2(t,\cdot)\left[1+\theta^2(t,\cdot)\right]\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}\\
&-2h^\delta\left[\int_\Omega\varrho_h(t,\cdot)\varrho(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}-2h^\delta\left[\int_\Omega\varrho_h(t,\cdot)\theta_h(t,\cdot)\varrho(t,\cdot)\theta(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}\\
&=2h^\delta\int_0^\tau\int_\Omega\varrho_h(\boldsymbol{u}-\overline{\boldsymbol{u}_h})\cdot\nabla_{\boldsymbol{x}}\varrho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+2h^\delta\int_0^\tau\int_\Omega\varrho_h\theta_h(\boldsymbol{u}-\overline{\boldsymbol{u}_h})\cdot\nabla_{\boldsymbol{x}}(\varrho\theta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+2h^\delta\int_0^\tau\int_\Omega(\varrho-\varrho_h)\left[\partial_t\varrho+\operatorname{div}_{\boldsymbol{x}}(\varrho\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+2h^\delta\int_0^\tau\int_\Omega(\varrho\theta-\varrho_h\theta_h)\left[\partial_t(\varrho\theta)+\operatorname{div}_{\boldsymbol{x}}(\varrho\theta\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\mathcal{O}(h^\beta)
\end{aligned}
$$

for all $\tau\in[0,T]$. Consequently,

$$
\begin{aligned}
&\left[\int_\Omega\tilde{E}(\varrho_h,\theta_h,\boldsymbol{u}_h\,|\,\varrho,\theta,\boldsymbol{u})(t,\cdot)\,\mathrm{d}\boldsymbol{x}+\int_{\Omega_h\setminus\Omega}E_h(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}\\
&+\int_0^\tau\int_{\Omega_h}\left[\mu|\nabla_h(\boldsymbol{u}_h-\widehat{\boldsymbol{u}})|^2+\nu|\operatorname{div}_h(\boldsymbol{u}_h-\widehat{\boldsymbol{u}})|^2\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\le-\int_0^\tau\int_\Omega\varrho_h(\overline{\boldsymbol{u}_h}-\boldsymbol{u})^T\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot(\overline{\boldsymbol{u}_h}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^\tau\int_\Omega\left(\frac{\varrho_h}{\varrho}\,S-S_h\right)(\overline{\boldsymbol{u}_h}-\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\left[p(\varrho_h,S_h)-\frac{\partial p(\varrho,S)}{\partial\varrho}(\varrho_h-\varrho)-\frac{\partial p(\varrho,S)}{\partial S}(S_h-S)-p(\varrho,S)\right]\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad-h^\delta\int_0^\tau\int_\Omega(\varrho_h-\varrho)^2\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-h^\delta\int_0^\tau\int_\Omega(\varrho_h\theta_h-\varrho\theta)^2\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega\frac{\varrho_h}{\varrho}\,(\boldsymbol{u}-\overline{\boldsymbol{u}_h})\cdot\left[\varrho\,\partial_t\boldsymbol{u}+\varrho\nabla_{\boldsymbol{x}}\boldsymbol{u}\cdot\boldsymbol{u}+\nabla_{\boldsymbol{x}}p(\varrho,S)-\operatorname{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u}))\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega(\varrho-\varrho_h)\,\frac{1}{\varrho}\left[\frac{\partial p(\varrho,S)}{\partial\varrho}+2h^\delta\varrho\right]\left[\partial_t\varrho+\operatorname{div}_{\boldsymbol{x}}(\varrho\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega(\varrho-\varrho_h)\,\frac{1}{\varrho}\,\frac{\partial p(\varrho,S)}{\partial S}\left[\partial_t S+\operatorname{div}_{\boldsymbol{x}}(S\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega\left(\frac{\varrho_h}{\varrho}\,S-S_h\right)\left[\partial_t\vartheta+\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\vartheta+\frac{\partial p(\varrho,S)}{\partial S}\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u})\right]\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned}
$$

$$+ 2h^\delta \int_0^\tau \int_\Omega (\varrho\theta - \varrho_h\theta_h) \left[\partial_t(\varrho\theta) + \mathrm{div}_{\boldsymbol{x}}(\varrho\theta\boldsymbol{u})\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ 2h^\delta \int_0^\tau \int_\Omega \varrho_h(\boldsymbol{u} - \overline{\boldsymbol{u}_h}) \cdot \nabla_{\boldsymbol{x}}\,\varrho \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 2h^\delta \int_0^\tau \int_\Omega \varrho_h\theta_h(\boldsymbol{u} - \overline{\boldsymbol{u}_h}) \cdot \nabla_{\boldsymbol{x}}(\varrho\theta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega (\varrho_h - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u})) \cdot (\boldsymbol{u} - \overline{\boldsymbol{u}_h})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^\beta) \tag{4.33}$$

for all $\tau \in [0, T]$.

4.5.4. Relative Energy Inequality for $\tilde{\boldsymbol{E}}$—Reduced form for Strong Solutions (Part I). In a particular situation when $(\varrho, \theta, \boldsymbol{u})$ is a strong solution to (1.1)–(1.5) of the class (4.1), the relative energy inequality (4.33) reduces to

$$\left[\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h\,|\,\varrho, \theta, \boldsymbol{u})(t, \cdot)\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_h\setminus\Omega} E_h(t, \cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau}$$
$$+ \int_0^\tau \int_{\Omega_h} \left[\mu|\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 + \nu|\mathrm{div}_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\le -\int_0^\tau \int_\Omega \varrho_h(\overline{\boldsymbol{u}_h} - \boldsymbol{u})^T \cdot \nabla_{\boldsymbol{x}}\boldsymbol{u} \cdot (\overline{\boldsymbol{u}_h} - \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau \int_\Omega \left(\frac{\varrho_h}{\varrho}\,S - S_h\right)(\overline{\boldsymbol{u}_h} - \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}}\vartheta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \left[p(\varrho_h, S_h) - \frac{\partial p(\varrho, S)}{\partial\varrho}(\varrho_h - \varrho) - \frac{\partial p(\varrho, S)}{\partial S}(S_h - S) - p(\varrho, S)\right] \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$- h^\delta \int_0^\tau \int_\Omega (\varrho_h - \varrho)^2\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - h^\delta \int_0^\tau \int_\Omega (\varrho_h\theta_h - \varrho\theta)^2\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ 2h^\delta \int_0^\tau \int_\Omega \varrho_h(\boldsymbol{u} - \overline{\boldsymbol{u}_h}) \cdot \nabla_{\boldsymbol{x}}\,\varrho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + 2h^\delta \int_0^\tau \int_\Omega \varrho_h\theta_h(\boldsymbol{u} - \overline{\boldsymbol{u}_h}) \cdot \nabla_{\boldsymbol{x}}(\varrho\theta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega (\varrho_h - \varrho)\,\frac{1}{\varrho}\,\mathrm{div}_{\boldsymbol{x}}(\mathbb{S}(\nabla_{\boldsymbol{x}}\boldsymbol{u})) \cdot (\boldsymbol{u} - \overline{\boldsymbol{u}_h})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^\beta)$$
$$=: T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + \mathcal{O}(h^\beta)\,. \tag{4.34}$$

Our goal is now to rewrite (4.34) in such a way that we can apply Gronwall's lemma. To this end, we first consider the terms T_j, $j \in \{1, \ldots, 8\}$. Clearly,

$$|T_1| + |T_3| + |T_4| + |T_5| \lesssim \int_0^\tau \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h\,|\,\varrho, \theta, \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.$$

Moreover, the second and third estimate in (4.13) yield

$$|T_6| + |T_7| \lesssim h^\delta\,\theta^\star ||\nabla_{\boldsymbol{x}}\,\varrho||_{L^\infty(\Omega_T)^d} \left(||\varrho_h||_{L^1(0,T;\,L^1(\Omega_h))}\,||\boldsymbol{u}||_{L^\infty(\Omega_T)^d} + ||\varrho_h\overline{\boldsymbol{u}_h}||_{L^1(0,T;\,L^1(\Omega_h)^d)}\right)$$
$$\lesssim h^\delta\,\theta^\star ||\nabla_{\boldsymbol{x}}\,\varrho||_{L^\infty(\Omega_T)^d} \left(||\varrho_h||_{L^\infty(0,T;\,L^\gamma(\Omega_h))}\,||\boldsymbol{u}||_{L^\infty(\Omega_T)^d} + ||\varrho_h\overline{\boldsymbol{u}_h}||_{L^\infty(0,T;\,L^{2\gamma/(\gamma+1)}(\Omega_h)^d)}\right) \lesssim h^\delta\,.$$

Then, exactly as in the proof of Theorem 3.3, we see that

$$|T_2| \lesssim \int_0^\tau \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h\,|\,\varrho, \theta, \boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.$$

To handle the term T_8, we need a suitable analogue of (2.8).

4.5.5. A Discrete Analogue of Poincaré's Inequality (2.8). For the derivation of the discrete analogue of (2.8) it shall be convenient to introduce the following notation:

$$\mathcal{T}_{h,\,\mathrm{ext}} = \{K \in \mathcal{T}_h\,|\,\mathcal{E}_{\mathrm{ext}}(K) \neq \emptyset\} \qquad \text{and} \qquad \Omega_{h,\,\mathrm{ext}} = \mathrm{int}\left[\bigcup_{K\,\in\,\mathcal{T}_{h,\,\mathrm{ext}}} K\right].$$

With this notation at hand, we observe that

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_h} |\overline{\boldsymbol{u}_h} - \widehat{\boldsymbol{u}}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim \int_0^\tau \int_{\Omega_h} \left(|\overline{\boldsymbol{u}_h} - \boldsymbol{u}_h|^2 + |\boldsymbol{u}_h - \Pi^0_{V,h}\widehat{\boldsymbol{u}}|^2 + |\Pi^0_{V,h}\widehat{\boldsymbol{u}} - \Pi_{V,h}\widehat{\boldsymbol{u}}|^2 + |\Pi_{V,h}\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}|^2 \right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim h^2 + \int_0^\tau \int_{\Omega_h} |\nabla_h(\boldsymbol{u}_h - \Pi^0_{V,h}\widehat{\boldsymbol{u}})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim h^2 + \int_0^\tau \int_{\Omega_h} \left(|\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 + |\nabla_h(\widehat{\boldsymbol{u}} - \Pi_{V,h}\widehat{\boldsymbol{u}})|^2 + |\nabla_h(\Pi_{V,h}\widehat{\boldsymbol{u}} - \Pi^0_{V,h}\widehat{\boldsymbol{u}})|^2 \right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim h + \int_0^\tau \int_{\Omega_h} |\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,
\end{aligned} \tag{4.35}$$

where in the second step we have used (A.11) as well as the estimates

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_h} |\overline{\boldsymbol{u}_h} - \boldsymbol{u}_h|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim h^2 \,||\nabla_h \boldsymbol{u}_h||^2_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})} \lesssim h^2\,, \\
&\int_0^\tau \int_{\Omega_h} |\Pi^0_{V,h}\widehat{\boldsymbol{u}} - \Pi_{V,h}\widehat{\boldsymbol{u}}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau \int_{\Omega_h} |\Pi_{V,h}\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim h^2 \int_0^\tau ||\nabla_{\boldsymbol{x}} \widehat{\boldsymbol{u}}(t,\cdot)||^2_{L^\infty(\mathbb{R}^d)^{d\times d}} \,\mathrm{d}t \lesssim h^2\,||\widehat{\boldsymbol{u}}||^2_{L^2(0,T;\,W^{1,\infty}(\mathbb{R}^d)^d)} \lesssim h^2\,||\boldsymbol{u}||^2_{L^2(0,T;\,W^{1,\infty}(\Omega)^d)} \lesssim h^2\,,
\end{aligned}$$

which are based on the first estimate in (A.3) and the estimates (4.31), (A.13), (A.15). The last step in (4.35) is due to the estimates

$$\begin{aligned}
\int_0^\tau \int_{\Omega_h} |\nabla_h(\widehat{\boldsymbol{u}} - \Pi_{V,h}\widehat{\boldsymbol{u}})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &\lesssim h^2 \int_0^\tau ||\nabla^2_{\boldsymbol{x}} \widehat{\boldsymbol{u}}(t,\cdot)||^2_{L^\infty(\mathbb{R}^d)^{d\times d\times d}} \,\mathrm{d}t \lesssim h^2\,||\widehat{\boldsymbol{u}}||^2_{L^2(0,T;\,W^{2,\infty}(\mathbb{R}^d)^d)} \\
&\lesssim h^2\,||\boldsymbol{u}||^2_{L^2(0,T;\,W^{2,\infty}(\Omega)^d)} \lesssim h^2\,, \\
\int_0^\tau \int_{\Omega_h} |\nabla_h(\Pi_{V,h}\widehat{\boldsymbol{u}} - \Pi^0_{V,h}\widehat{\boldsymbol{u}})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \int_0^\tau \sum_{K\,\in\,\mathcal{T}_{h,\,\mathrm{ext}}} \int_K |\nabla_h(\Pi^0_{V,h}\widehat{\boldsymbol{u}} - \Pi_{V,h}\widehat{\boldsymbol{u}})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim \int_0^\tau ||\nabla_{\boldsymbol{x}} \widehat{\boldsymbol{u}}(t,\cdot)||^2_{L^\infty(\mathbb{R}^d)^{d\times d}} \sum_{K\,\in\,\mathcal{T}_{h,\,\mathrm{ext}}} \int_K \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\lesssim ||\nabla_{\boldsymbol{x}} \widehat{\boldsymbol{u}}||^2_{L^2(0,T;\,L^\infty(\mathbb{R}^d)^{d\times d})}\,|\Omega_{h,\,\mathrm{ext}}| \\
&\lesssim h\,||\widehat{\boldsymbol{u}}||^2_{L^2(0,T;\,W^{1,\infty}(\mathbb{R}^d)^d)} \lesssim h\,||\boldsymbol{u}||^2_{L^2(0,T;\,W^{1,\infty}(\Omega)^d)} \lesssim h
\end{aligned}$$

that are based on (A.14), (4.31), (A.15).

4.5.6. Relative Energy Inequality for $\tilde{E}$—Reduced form for Strong Solutions (Part II). With the help of (4.35) we can now estimate the term T_8 in (4.34) analogously to its continuous counterpart. We obtain

$$|T_8| \lesssim (1+\alpha^{-1}) \int_0^\tau \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \alpha \left(\int_0^\tau \int_{\Omega_h} |\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h) \right)$$

for all $\alpha > 0$. Together with the estimates for the terms T_j, $j \in \{1,\dots,7\}$, stated in Sect. 4.5.4, this observation allows us to rewrite (4.34) as

$$\begin{aligned}
&\left[\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(t,\cdot) \,\mathrm{d}\boldsymbol{x} + \int_{\Omega_h\setminus\Omega} E_h(t,\cdot) \,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \\
&+ \int_0^\tau \int_{\Omega_h} \left[\mu |\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 + \nu |\mathrm{div}_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t
\end{aligned}$$

$$\lesssim (1+\alpha^{-1}) \int_0^\tau \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \alpha \left(\int_0^\tau \int_{\Omega_h} |\nabla_h (\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \mathcal{O}(h) \right) + \mathcal{O}(h^{\min\{\beta,\delta\}}) \tag{4.36}$$

for all $\alpha > 0$. Next, let us turn to the first line in (4.36). Using Hölder's inequality, the first estimate in (A.3), (A.13) and (A.16), we deduce that

$$\begin{aligned}
&\int_\Omega \varrho_h^0 |\overline{\boldsymbol{u}_h^0} - \boldsymbol{u}_0|^2 \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_h \setminus \Omega} \varrho_h^0 |\overline{\boldsymbol{u}_h^0}|^2 \, \mathrm{d}\boldsymbol{x} \\
&\quad = \int_{\Omega_h} \varrho_h^0 |\overline{\boldsymbol{u}_h^0} - \boldsymbol{u}_0|^2 \, \mathrm{d}\boldsymbol{x} \lesssim ||\varrho_h^0||_{L^\infty(\Omega_h)} ||\overline{\boldsymbol{u}_h^0} - \boldsymbol{u}_0||^2_{L^2(\Omega_h)^d} \\
&\quad \lesssim ||\varrho_0||_{L^\infty(\Omega)} \left(||\overline{\boldsymbol{u}_h^0} - \boldsymbol{u}_h^0||^2_{L^2(\Omega_h)^d} + ||\boldsymbol{u}_h^0 - \boldsymbol{u}_0||^2_{L^2(\Omega_h)^d} \right) \lesssim h^2 \left(||\nabla_h \boldsymbol{u}_h^0||^2_{L^2(\Omega_h)^{d\times d}} + ||\boldsymbol{u}_0||^2_{W^{1,2}(\Omega_h)^d} \right) \\
&\quad \lesssim h^2 \left(||\nabla_h \boldsymbol{u}_h^0 - \nabla_{\boldsymbol{x}} \boldsymbol{u}_0||^2_{L^2(\Omega_h)^{d\times d}} + ||\nabla_{\boldsymbol{x}} \boldsymbol{u}_0||^2_{L^2(\Omega_h)^{d\times d}} + ||\boldsymbol{u}_0||^2_{W^{1,2}(\Omega_h)^d} \right) \\
&\quad \lesssim h^2 \, ||\boldsymbol{u}_0||^2_{W^{1,2}(\Omega_h)^d} \lesssim h^2 \, ||\boldsymbol{u}_0||^2_{W^{1,2}(\Omega)^d} \lesssim h^2 \,, \\
&\int_{\Omega_h \setminus \Omega} \left(P(\varrho_h^0, S_h^0) + h^\delta (\varrho_h^0)^2 \left[1 + (\theta_h^0)^2 \right] \right) \mathrm{d}\boldsymbol{x} \\
&\quad \lesssim |\Omega_h \setminus \Omega| \left(||\varrho_h^0||^\gamma_{L^\infty(\Omega_h)} \, ||\theta_h^0||^\gamma_{L^\infty(\Omega_h)} + h^\delta \, ||\varrho_h^0||^2_{L^\infty(\Omega_h)} \left[1 + ||\theta_h^0||^2_{L^\infty(\Omega_h)} \right] \right) \\
&\quad \lesssim h \left(||\varrho_0||^\gamma_{L^\infty(\Omega)} \, ||\theta_0||^\gamma_{L^\infty(\Omega)} + h^\delta \, ||\varrho_0||^2_{L^\infty(\Omega)} \left[1 + ||\theta_0||^2_{L^\infty(\Omega)} \right] \right) \lesssim h \,, \\
&h^\delta \int_\Omega (\varrho_h^0 - \varrho_0)^2 \, \mathrm{d}\boldsymbol{x} \lesssim h^\delta \left(||\varrho_h^0||^2_{L^\infty(\Omega_h)} + ||\varrho_0||^2_{L^\infty(\Omega)} \right) \lesssim h^\delta \, ||\varrho_0||^2_{L^\infty(\Omega)} \lesssim h^\delta \,, \\
&h^\delta \int_\Omega (\varrho_h^0 \theta_h^0 - \varrho_0 \theta_0)^2 \, \mathrm{d}\boldsymbol{x} \lesssim h^\delta \left(||\varrho_h^0||^2_{L^\infty(\Omega_h)} \, ||\theta_h^0||^2_{L^\infty(\Omega_h)} + ||\varrho_0||^2_{L^\infty(\Omega)} \, ||\theta_0||^2_{L^\infty(\Omega)} \right) \\
&\quad \lesssim h^\delta \, ||\varrho_0||^2_{L^\infty(\Omega)} \, ||\theta_0||^2_{L^\infty(\Omega)} \lesssim h^\delta \,.
\end{aligned}$$

Moreover, denoting

$$\mathcal{T}_{h,\Omega} = \{ K \in \mathcal{T} \,|\, K \subset \Omega \} \,, \qquad \Omega_{h,\Omega} = \operatorname{int} \left[\bigcup_{K \in \mathcal{T}_{h,\Omega}} K \right],$$

$$A(\varrho_0, S_0) = [(\varrho_0)_\star, (\varrho_0)^\star] \times \left[-(\varrho_0)^\star \max\{ |\ln((\theta_0)_\star)|, |\ln((\theta_0)^\star)| \}, (\varrho_0)^\star \max\{ |\ln((\theta_0)_\star)|, |\ln((\theta_0)^\star)| \} \right],$$

and employing Hölder's inequality, Taylor's theorem, (A.12) and (A.16), we observe that

$$\begin{aligned}
&\int_\Omega \left(P(\varrho_h^0, S_h^0) - \frac{\partial P(\varrho_0, S_0)}{\partial \varrho} (\varrho_h^0 - \varrho_0) - \frac{\partial P(\varrho_0, S_0)}{\partial S} (S_h^0 - S_0) - P(\varrho_0, S_0) \right) \mathrm{d}\boldsymbol{x} \\
&\lesssim \max_{(r,s) \in A(\varrho_0, S_0)} \left\{ ||\nabla^2_{(\varrho,S)} P(r,s)||_2 \right\} \left(||\varrho_h^0 - \varrho_0||^2_{L^2(\Omega)} + ||S_h^0 - S_0||^2_{L^2(\Omega)} \right) \\
&\lesssim ||\varrho_h^0 - \varrho_0||^2_{L^2(\Omega \setminus \Omega_{h,\Omega})} + ||S_h^0 - S_0||^2_{L^2(\Omega \setminus \Omega_{h,\Omega})} + ||\varrho_h^0 - \varrho_0||^2_{L^2(\Omega_{h,\Omega})} + ||S_h^0 - S_0||^2_{L^2(\Omega_{h,\Omega})} \\
&\lesssim |\Omega \setminus \Omega_{h,\Omega}| \left(||\varrho_h^0||^2_{L^\infty(\Omega_h)} \left(1 + ||\ln(\theta_h^0)||^2_{L^\infty(\Omega_h)} \right) + ||\varrho_0||^2_{L^\infty(\Omega)} \left(1 + ||\ln(\theta_0)||^2_{L^\infty(\Omega)} \right) \right) \\
&\quad + h^2 \, ||\varrho_0||^2_{W^{1,2}(\Omega_{h,\Omega})} + ||\varrho_h^0 - \varrho_0||^2_{L^2(\Omega_{h,\Omega})} \, ||\ln(\theta_h^0)||^2_{L^\infty(\Omega_{h,\Omega})} + ||\varrho_0||^2_{L^\infty(\Omega_{h,\Omega})} \, ||\ln(\theta_h^0) - \ln(\theta_0)||^2_{L^2(\Omega_{h,\Omega})} \\
&\lesssim \left(h \, ||\varrho_0||^2_{L^\infty(\Omega)} + h^2 \, ||\varrho_0||^2_{W^{1,2}(\Omega_{h,\Omega})} \right) \left(1 + \max\{ |\ln((\theta_0)_\star)|, |\ln((\theta_0)^\star)| \}^2 \right) \\
&\quad + ||\varrho_0||^2_{L^\infty(\Omega_{h,\Omega})} \, (\theta_0)_\star^{-2} \, ||\theta_h^0 - \theta_0||^2_{L^2(\Omega_{h,\Omega})} \\
&\lesssim h + h^2 \, (\theta_0)_\star^{-2} \, ||\varrho_0||^2_{L^\infty(\Omega)} \, ||\theta_0||^2_{W^{1,2}(\Omega_{h,\Omega})} \lesssim h + h^2 \, (\theta_0)_\star^{-2} \, ||\varrho_0||^2_{L^\infty(\Omega)} \, ||\theta_0||^2_{W^{1,2}(\Omega)} \lesssim h \,.
\end{aligned}$$

Consequently, we may rewrite (4.36) as

$$\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} + \int_0^\tau \int_{\Omega_h} \left[\mu |\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2 + \nu |\mathrm{div}_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\lesssim (1+\alpha^{-1}) \int_0^\tau \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \alpha \left(\int_0^\tau \int_{\Omega_h} |\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h) \right)$$
$$+ \mathcal{O}(h^{\min\{\beta,\delta\}}). \tag{4.37}$$

Fixing a sufficiently small $\alpha > 0$, we deduce from (4.37) the inequality

$$\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} + \int_0^\tau \int_{\Omega_h} |\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})|^2\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\lesssim \int_0^\tau \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{\min\{\beta,\delta\}}). \tag{4.38}$$

4.5.7. Error Estimates. We are now ready to apply Gronwall's lemma to (4.38) which yields

$$||\nabla_h(\boldsymbol{u}_h - \boldsymbol{u})||_{L^2(\Omega_T)^{d\times d}} \lesssim ||\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})||_{L^2((0,T)\,\times\,\Omega_h)^{d\times d}} \lesssim h^{\min\{\beta,\delta\}/2}, \tag{4.39}$$

$$\sup_{\tau\,\in\,[0,T]} \left\{ \int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} \right\} \lesssim h^{\min\{\beta,\delta\}}. \tag{4.40}$$

Combining (4.39) with (4.35), we get

$$||\boldsymbol{u}_h - \boldsymbol{u}||_{L^2(\Omega_T)^d} \lesssim ||\boldsymbol{u}_h - \widehat{\boldsymbol{u}}||_{L^2((0,T)\,\times\,\Omega_h)^d} \lesssim ||\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}||_{L^2((0,T)\,\times\,\Omega_h)^d} + ||\overline{\boldsymbol{u}_h} - \widehat{\boldsymbol{u}}||_{L^2((0,T)\,\times\,\Omega_h)^d}$$
$$\lesssim h\, ||\nabla_h \boldsymbol{u}_h||_{L^2((0,T)\,\times\,\Omega_h)^{d\times d}} + h^{1/2} + ||\nabla_h(\boldsymbol{u}_h - \widehat{\boldsymbol{u}})||_{L^2((0,T)\,\times\,\Omega_h)^{d\times d}} \lesssim h^{\min\{\beta,\delta\}/2}.$$

Furthermore, using Lemma A.1 and $\theta_\star \le \theta_h \le \theta^\star$, it is easy to see that for all $p \in [1,\gamma]$, all $q \in [1,\infty)$ and all $\tau \in [0,T]$

$$||(\varrho_h - \varrho)(\tau,\cdot)||_{L^p(\Omega)} \lesssim \left(\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} \right)^{1/2} + \left(\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} \right)^{1/p},$$
$$||(\theta_h - \theta)(\tau,\cdot)||_{L^q(\Omega)} \lesssim \left(\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} \right)^{1/2} + \left(\int_\Omega \tilde{E}(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho, \theta, \boldsymbol{u})(\tau, \cdot)\, \mathrm{d}\boldsymbol{x} \right)^{1/q}.$$

Consequently, (4.40) yields

$$||\varrho_h - \varrho||_{L^\infty(0,T;\,L^p(\Omega))} \lesssim h^{\min\{\beta,\delta\}/\max\{2,p\}} \qquad \text{for all } p \in [1,\gamma],$$
$$||\theta_h - \theta||_{L^\infty(0,T;\,L^q(\Omega))} \lesssim h^{\min\{\beta,\delta\}/\max\{2,q\}} \qquad \text{for all } p \in [1,\infty).$$

Remark 4.4. The optimal convergence rates are obtained for $\varepsilon \ge 7/6$ and $\delta = 1/6$. In this case, $\min\{\beta,\delta\} = 1/6$ and, in particular, the convergence rates for $\boldsymbol{u}_h$ in the L^2-norm, for ϱ_h in the L^∞-L^γ-norm (provided $\gamma \le 2$) and for θ_h in the L^∞-L^2-norm are $1/12$.

4.6. Numerical Results

We conclude this section by illustrating experimentally convergence behaviour of the FE-FV method (4.5)–(4.7). More specifically, motivated by the numerical experiments presented in [21, Section 5.1] and [7, Chapter 14.6.2], we simulate a vortex flow in $\Omega = [0,1]^2 \subset \mathbb{R}^2$ with the initial data

$$\varrho_0 \equiv 1, \qquad \theta_0(\boldsymbol{x}) = \frac{1}{2} + \frac{1}{4}\,\theta_r(\boldsymbol{x}), \qquad \boldsymbol{u}_0(\boldsymbol{x}) = u_r(\boldsymbol{x}) \begin{pmatrix} x_2 - \frac{1}{2} \\ \frac{1}{2} - x_1 \end{pmatrix},$$

where

$$r(\boldsymbol{x}) = \sqrt{\left(x_2 - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - x_1\right)^2}, \qquad u_r(\boldsymbol{x}) = \sqrt{\gamma} \begin{cases} 10 & \text{if } r(\boldsymbol{x}) < \frac{1}{10}, \\ 2\left(\frac{1}{r(\boldsymbol{x})} - 5\right) & \text{if } \frac{1}{10} \le r(\boldsymbol{x}) < \frac{1}{5}, \\ 0 & \text{if } r(\boldsymbol{x}) \ge \frac{1}{5}, \end{cases}$$

$$\theta_r(\boldsymbol{x}) = \begin{cases} 50 r(\boldsymbol{x})^2 & \text{if } r(\boldsymbol{x}) < \frac{1}{10}, \\ 4\ln(10 r(\boldsymbol{x})) + 4 - 40 r(\boldsymbol{x}) + 50 r(\boldsymbol{x})^2 & \text{if } \frac{1}{10} \le r(\boldsymbol{x}) < \frac{1}{5}, \\ 4\ln(2) - 2 & \text{if } r(\boldsymbol{x}) \ge \frac{1}{5}. \end{cases}$$

The parameters of the FE-FV method are chosen as $\mu = 0.1$, $\nu = 0$, $\varepsilon = 2.0$, $\delta = 0.1667$ and the final time for our convergence study is $T = 0.1$. The nonlinear algebraic system (4.5)–(4.7) is solved using a fixed point iteration. Thus, in each subiteration, the CFL stability condition

$$\Delta t^k \equiv t^k - t^{k-1} \le \frac{0.4h}{||\boldsymbol{u}_h^{k-1}||_{L^\infty(\Omega)^d} + ||c_h^{k-1}||_{L^\infty(\Omega)}}, \quad \text{where} \quad c_h^{k-1} = \sqrt{\gamma\, (\varrho_h^{k-1})^{\gamma-1} (\theta_h^{k-1})^{\gamma}},$$

is required. This is ensured by the choice $\Delta t = 16h/130$. We concentrate on the following errors:

$$\mathrm{Err}_{\varrho}^{\infty,\gamma}(h) = ||\varrho_h - \varrho_{h_{\mathrm{ref}}}||_{L^\infty(0,T;\,L^\gamma(\Omega))}, \quad \mathrm{Err}_{\varrho\theta}^{\infty,\gamma}(h) = ||\varrho_h\theta_h - \varrho_{h_{\mathrm{ref}}}\theta_{h_{\mathrm{ref}}}||_{L^\infty(0,T;\,L^\gamma(\Omega))},$$

$$\mathrm{Err}_{\boldsymbol{u}}^{2,2}(h) = ||\boldsymbol{u}_h - \boldsymbol{u}_{h_{\mathrm{ref}}}||_{L^2(0,T;\,L^2(\Omega)^d)}, \quad \mathrm{Err}_{\nabla\boldsymbol{u}}^{2,2}(h) = ||\nabla_h \boldsymbol{u}_h - \nabla_{h_{\mathrm{ref}}} \boldsymbol{u}_{h_{\mathrm{ref}}}||_{L^2(0,T;\,L^2(\Omega)^{d\times d})},$$

$$\mathrm{Err}_{\boldsymbol{u}}^{\infty,2}(h) = ||\boldsymbol{u}_h - \boldsymbol{u}_{h_{\mathrm{ref}}}||_{L^\infty(0,T;\,L^2(\Omega)^d)}, \quad \mathrm{Err}_{E}^{2,2}(h) = ||E(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho_{h_{\mathrm{ref}}}, \theta_{h_{\mathrm{ref}}}, \boldsymbol{u}_{h_{\mathrm{ref}}})||_{L^\infty(0,T;\,L^1(\Omega))},$$

where $h_{\mathrm{ref}} = 1/1024$ and

$$\begin{aligned} E(\varrho_h, \theta_h, \boldsymbol{u}_h \,|\, \varrho_{h_{\mathrm{ref}}}, \theta_{h_{\mathrm{ref}}}, \boldsymbol{u}_{h_{\mathrm{ref}}}) &= \frac{1}{2}\,\varrho_h |\overline{\boldsymbol{u}_h} - \overline{\boldsymbol{u}_{h_{\mathrm{ref}}}}|^2 \\ &\quad + P(\varrho_h, S_h) - \frac{\partial P(\varrho_{h_{\mathrm{ref}}}, S_{h_{\mathrm{ref}}})}{\partial \varrho}\,(\varrho_h - \varrho_{h_{\mathrm{ref}}}) \\ &\quad - \frac{\partial P(\varrho_{h_{\mathrm{ref}}}, S_{h_{\mathrm{ref}}})}{\partial S}\,(S_h - S_{h_{\mathrm{ref}}}) - P(\varrho_{h_{\mathrm{ref}}}, S_{h_{\mathrm{ref}}}). \end{aligned}$$

Tables 1 and 2 show the experimental order of convergence for two different values of the adiabatic exponent $\gamma = 1.4$ and $\gamma = 1.67$.

Here, the experimental orders of convergence were computed using the standard formula

$$\mathrm{EOC}(h) = \log_2\left(\frac{||\boldsymbol{s}_{2h} - \boldsymbol{s}_{h_{\mathrm{ref}}}||}{||\boldsymbol{s}_h - \boldsymbol{s}_{h_{\mathrm{ref}}}||}\right),$$

where $\boldsymbol{s}_h$ stands for a numerical solution on a mesh Ω_h, analogous notations are used for $\boldsymbol{s}_{2h}$ and $\boldsymbol{s}_{h_{\mathrm{ref}}}$. We observe that EOC for the density, velocity, gradient of velocity and potential temperature are around 1, while the second order EOC are obtained for the relative energy. Similarly as in theoretical analysis the convergence rates in the relative energy are twice as good as those of the density, velocity and potential temperature. Our numerical experiments indicate that theoretical results obtained in Sect. 4.5 might be suboptimal, such a behaviour was observed in the literature also for other numerical methods and models, see, e.g., [9,13,16]. Figure 1 illustrates time evolution of the solution computed at different times on a mesh with $h = 1/128$ and for $\gamma = 1.4$.

5. Conclusions

In the present paper, we have proved the DMV-strong uniqueness principle for the Navier–Stokes system with potential temperature transport (1.1)–(1.5). This result shows that strong solutions are stable in the class of DMV solutions introduced in [15]. We have derived the relative energy by taking the total physical entropy into account. More precisely, the pressure was rewritten as a function of the density and entropy, instead of the total potential temperature only. Moreover, we also require the entropy inequality

Table 1. Convergence study for the FE-FV method with $\gamma = 1.4$

h	$\mathrm{Err}_{\varrho}^{\infty,\gamma}$	EOC	$\mathrm{Err}_{\boldsymbol{u}}^{2,2}$	EOC	$\mathrm{Err}_{\nabla\boldsymbol{u}}^{2,2}$	EOC	$\mathrm{Err}_{\varrho\theta}^{\infty,\gamma}$	EOC	$\mathrm{Err}_{\boldsymbol{u}}^{\infty,2}$	EOC	$\mathrm{Err}_{E}^{\infty,1}$	EOC
1/16	3.71e-03	–	4.28e-03	–	1.55e-01	–	3.95e-03	–	6.42e-02	–	1.58e-09	–
1/32	2.66e-03	0.48	2.20e-03	0.96	8.36e-02	0.89	2.42e-03	0.71	3.63e-02	0.82	4.07e-10	1.96
1/64	1.90e-03	0.49	1.21e-03	0.86	4.48e-02	0.90	1.54e-03	0.65	2.00e-02	0.86	1.02e-10	2.00
1/128	1.31e-03	0.54	6.92e-04	0.81	2.38e-02	0.91	9.83e-04	0.65	1.05e-02	0.93	3.76e-11	1.44
1/256	8.11e-04	0.69	3.85e-04	0.85	1.23e-02	0.96	5.84e-04	0.75	5.02e-03	1.07	9.53e-12	1.98
1/512	3.82e-04	1.09	1.73e-04	1.15	5.53e-03	1.15	2.69e-04	1.12	1.84e-03	1.44	2.97e-12	1.68

TABLE 2. convergence study for the FE-FV method with $\gamma = 1.67$

h	$\mathrm{Err}_{\varrho}^{\infty,\gamma}$	EOC	$\mathrm{Err}_{\boldsymbol{u}}^{2,2}$	EOC	$\mathrm{Err}_{\nabla \boldsymbol{u}}^{2,2}$	EOC	$\mathrm{Err}_{\varrho\theta}^{\infty,\gamma}$	EOC	$\mathrm{Err}_{\boldsymbol{u}}^{\infty,2}$	EOC	$\mathrm{Err}_{E}^{\infty,1}$	EOC
1/16	5.00e-03	–	4.71e-03	–	1.70e-01	–	5.50e-03	–	7.03e-02	–	1.89e-09	–
1/32	3.57e-03	0.49	2.42e-03	0.96	9.14e-02	0.89	3.30e-03	0.74	3.98e-02	0.82	4.80e-10	1.98
1/64	2.54e-03	0.49	1.31e-03	0.88	4.89e-02	0.90	2.08e-03	0.67	2.19e-02	0.86	1.23e-10	1.96
1/128	1.74e-03	0.54	7.42e-04	0.82	2.59e-02	0.91	1.32e-03	0.66	1.15e-02	0.93	3.71e-11	1.73
1/256	1.08e-03	0.69	4.09e-04	0.86	1.33e-02	0.96	7.76e-04	0.76	5.49e-03	1.07	9.82e-12	1.92
1/512	5.08e-04	1.09	1.82e-04	1.16	5.99e-03	1.15	3.56e-04	1.12	2.02e-03	1.44	2.92e-12	1.75

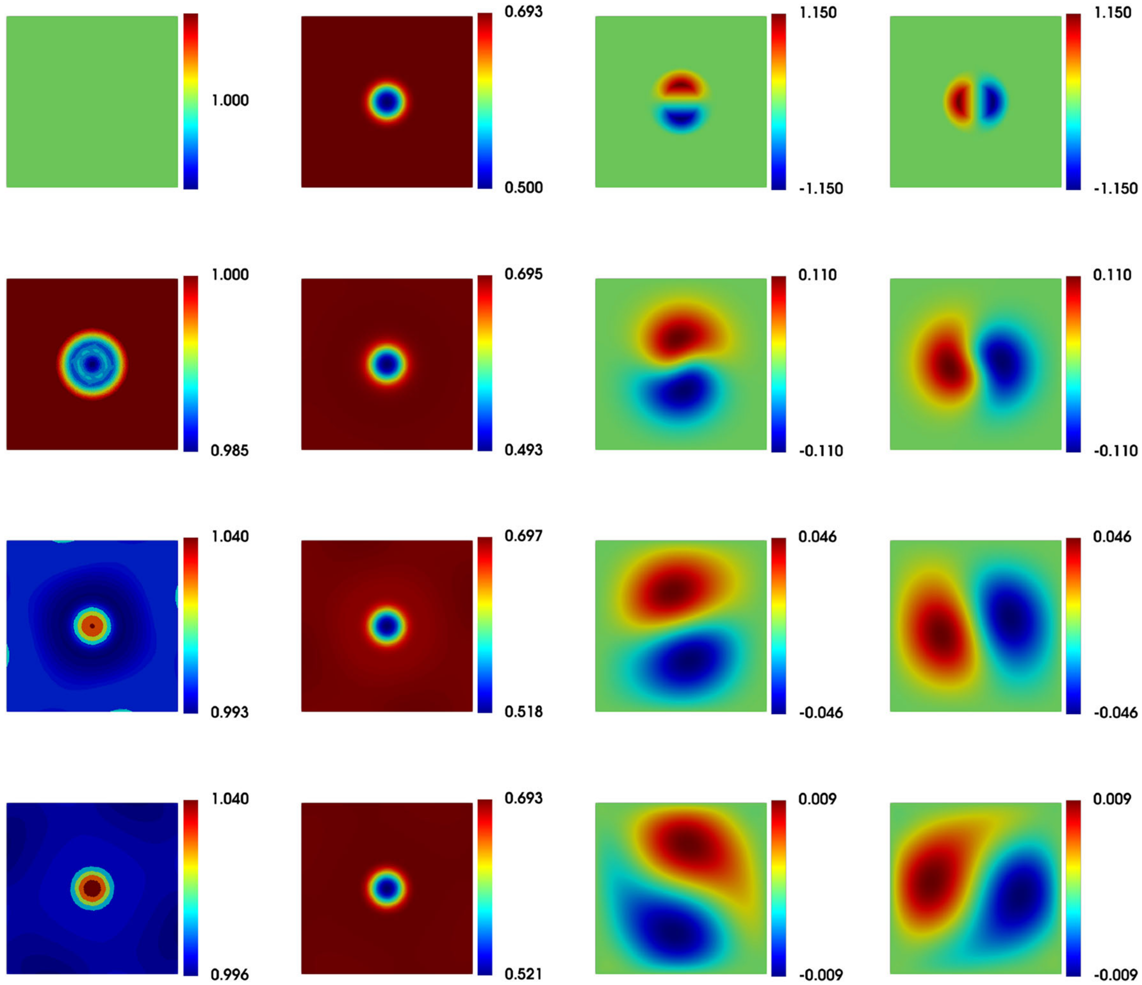

FIG. 1. Numerical solutions for ϱ, $\varrho\theta$, u_1, u_2 at times $t = 0, 0.1, 0.2, 0.5$

(2.7) that is included in our definition of DMV solutions. The importance of Poincaré's inequality (2.8) became clear from the proof of DMV-strong uniqueness: It allowed us to rewrite viscosity terms in such a way that Gronwall's lemma was applicable and yield the DMV-strong uniqueness principle.

As an application of the DMV-strong uniqueness principle we derive a priori error estimates by applying the relative energy to numerical solutions. Our theoretical error estimates include not only the errors between the numerical and the strong solutions but also the so-called variational crime errors due to the approximation of a smooth domain Ω by polygonal approximations Ω_h, $\Omega \subset \Omega_h$ such that $\mathrm{dist}(\boldsymbol{x}, \partial\Omega) = \mathcal{O}(h)$ for all $\boldsymbol{x} \in \partial\Omega_h$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A. Appendix

A.1. An Auxiliary Result Concerning the Relative Energy

Here, we prove the auxiliary result used in the proof of DMV-strong uniqueness.

Lemma A.1. *Let $\tilde\varrho \geq 0$, $\tilde\theta \geq c_\star > 0$, $0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho}$, $0 < \underline{\theta} \leq \theta \leq \overline{\theta}$, and $\gamma > 1$. Then there exist constants $c_1, c_2, c_3, c_4 > 0$ that only depend on $\underline{\varrho}, \overline{\varrho}, \underline{\theta}, \overline{\theta}, c_\star$ and γ, and corresponding sets*

$$\mathcal{R} = \left\{ (\tilde\varrho, \tilde\theta) \in \mathbb{R}^2 \,\middle|\, c_1 \underline{\varrho} \leq \tilde\varrho \leq c_2 \overline{\varrho}\,,\ c_\star \leq \tilde\theta \leq c_3 \overline{\theta} \right\}, \qquad \mathcal{S} = \left\{ (\tilde\varrho, \tilde\theta) \in \mathbb{R}^2 \,\middle|\, \tilde\varrho \geq 0\,,\ \tilde\theta \geq c_\star \right\} \setminus \mathcal{R}$$

such that

$$\begin{aligned} F(\tilde\varrho, \tilde S \,|\, \varrho, S) &\equiv P(\tilde\varrho, \tilde S) - \frac{\partial P(\varrho, S)}{\partial \varrho}(\tilde\varrho - \varrho) - \frac{\partial P(\varrho, S)}{\partial S}(\tilde S - S) - P(\varrho, S) \\ &\geq c_4 \left[\mathbb{1}_{\mathcal{R}}(\tilde\varrho, \tilde\theta)\big(|\tilde\varrho - \varrho|^2 + |\tilde S - S|^2\big) + \mathbb{1}_{\mathcal{S}}(\tilde\varrho, \tilde\theta)\big(1 + (\tilde\varrho\tilde\theta)^\gamma\big) \right], \end{aligned} \tag{A.1}$$

where $P(\varrho, S) = \frac{1}{\gamma - 1} p(\varrho, S)$ with p from (3.2), $S = S(\varrho, \theta)$ is defined in (3.1), and $\tilde S = S(\tilde\varrho, \tilde\theta)$.

Proof. To begin with, let $0 < c_1 \leq c_2$, and $c_3 \geq c_\star / \overline{\theta}$ be arbitrary numbers. Further, let $\mathcal{R}$, $\mathcal{S}$ be defined as described in the lemma. We decompose $\mathcal{S}$ into the sets

$$\mathcal{S}^- = \left\{ (\tilde\varrho, \tilde\theta) \in \mathcal{S} \,|\, \tilde\varrho < c_1 \underline{\varrho} \right\}, \quad \mathcal{S}^+ = \left\{ (\tilde\varrho, \tilde\theta) \in \mathcal{S} \,|\, \tilde\varrho > c_2 \overline{\varrho} \right\}, \quad \mathcal{S}^0 = \mathcal{S} \setminus (\mathcal{S}^+ \cup \mathcal{S}^-)$$

and observe that

$$\begin{aligned} F(\tilde\varrho, \tilde S \,|\, \varrho, S) &= a \left((\varrho\theta)^\gamma - \frac{\gamma}{\gamma - 1} \varrho^{\gamma - 1} \theta^\gamma \tilde\varrho \big(1 - \ln(\theta) + \ln(\tilde\theta)\big) + \frac{1}{\gamma - 1} (\tilde\varrho\tilde\theta)^\gamma \right) \\ &\geq a \left((\varrho\theta)^\gamma - \frac{\gamma}{\gamma - 1} \varrho^{\gamma - 1} \theta^\gamma \tilde\varrho \big(1 + |\ln(\theta)| + \tilde\theta^{1/2}\big) + \frac{1}{\gamma - 1} (\tilde\varrho\tilde\theta)^\gamma \right) \end{aligned}$$

wherefore

$$\begin{aligned} \mathbb{1}_{\mathcal{S}^-}(\tilde\varrho, \tilde\theta) F(\tilde\varrho, \tilde S \,|\, \varrho, S) \geq{}& a(\underline{\varrho}\underline{\theta})^\gamma - \frac{a c_1 \gamma}{\gamma - 1} \big(1 + \max\{|\ln(\underline{\theta})|, |\ln(\overline{\theta})|\}\big) (\overline{\varrho}\overline{\theta})^\gamma - c_1^{1/2} \frac{a(2\gamma - 1)}{2(\gamma - 1)} \overline{\varrho}^{\,\gamma - 1 + \gamma/(2\gamma - 1)} \overline{\theta}^{\,\gamma} \\ &+ \frac{a}{\gamma - 1} \left(1 - \frac{c_1^{1/2}}{2} \overline{\varrho}^{\,\gamma - 1} \overline{\theta}^{\,\gamma} \right) (\tilde\varrho\tilde\theta)^\gamma, \end{aligned}$$

$$\mathbb{1}_{\mathcal{S}^+}(\tilde{\varrho},\tilde{\theta})F(\tilde{\varrho},\tilde{S}\,|\,\varrho,S) \geq a(\underline{\varrho}\underline{\theta})^\gamma + \frac{a}{\gamma-1}\left(1-\gamma c_2^{1-\gamma}\left(\frac{\overline{\theta}}{c_\star}\right)^\gamma\left(1+\max\left\{|\ln(\underline{\theta})|,|\ln(\overline{\theta})|\right\}+c_\star^{1/2}\right)\right)(\tilde{\varrho}\tilde{\theta})^\gamma\,,$$

$$\mathbb{1}_{\mathcal{S}^0}(\tilde{\varrho},\tilde{\theta})F(\tilde{\varrho},\tilde{S}\,|\,\varrho,S) \geq a(\underline{\varrho}\underline{\theta})^\gamma + \frac{a}{\gamma-1}\left(1-\gamma c_3^{-\gamma}\left(\frac{\overline{\varrho}}{c_1\underline{\varrho}}\right)^{\gamma-1}\left(1+\max\left\{|\ln(\underline{\theta})|,|\ln(\overline{\theta})|\right\}+(c_3\overline{\theta})^{1/2}\right)\right)(\tilde{\varrho}\tilde{\theta})^\gamma\,.$$

Here, the first inequality is obtained using Young's inequality. Together, the above observations show that we can specify c_1, c_2, c_3 in dependence of $\underline{\varrho}, \overline{\varrho}, \underline{\theta}, \overline{\theta}, c_\star, \gamma$ such that

$$\mathbb{1}_{\mathcal{S}}(\tilde{\varrho},\tilde{\theta})F(\tilde{\varrho},\tilde{S}\,|\,\varrho,S) \geq c_{4,1}\mathbb{1}_{\mathcal{S}}(\tilde{\varrho},\tilde{\theta})\big(1+(\tilde{\varrho}\tilde{\theta})^\gamma\big)\,,$$

where $c_{4,1} > 0$ solely depends on $\underline{\varrho}, \overline{\varrho}, \underline{\theta}, \overline{\theta}, c_\star, \gamma$. Having fixed c_1, c_2, c_3 as described above, it remains to show that

$$\mathbb{1}_{\mathcal{R}}(\tilde{\varrho},\tilde{\theta})F(\tilde{\varrho},\tilde{S}\,|\,\varrho,S) \geq c_{4,2}\mathbb{1}_{\mathcal{R}}(\tilde{\varrho},\tilde{\theta})\big(|\tilde{\varrho}-\varrho|^2+|\tilde{S}-S|^2\big)\,,$$

where $c_{4,2} > 0$ only depends on $\underline{\varrho}, \overline{\varrho}, \underline{\theta}, \overline{\theta}, c_\star, \gamma$. This inequality is a direct consequence of the fact that $P = P(\varrho, S)$ is strongly convex on every compact convex subset of $(0,\infty)\times\mathbb{R}$ which, in turn, follows from the positive definiteness of the Hessian of P on $(0,\infty)\times\mathbb{R}$. □

A.2. Mesh-Related Estimates

We recall several important mesh-related estimates; see, e.g., [7] and the references therein. We begin with the discrete trace and inverse inequalities. We have

$$\|r_K\|_{L^p(\sigma)} \lesssim h^{-1/p}\|r\|_{L^p(K)} \qquad \text{and} \qquad \|r\|_{L^p(\Omega_h)} \lesssim h^{d\left(\frac{1}{p}-\frac{1}{q}\right)}\|r\|_{L^q(\Omega_h)} \tag{A.2}$$

for all $r \in Q_h$, all $K \in \mathcal{T}_h$, all $\sigma \in \mathcal{E}_h(K)$, and all $1 \leq q \leq p \leq \infty$. In addition,

$$\|v-\overline{v}\|_{L^p(K)} \lesssim h\|\nabla_h v\|_{L^p(K)^d}\,, \qquad \|v-\langle v\rangle_\sigma\|_{L^p(\sigma)} \lesssim h\|\nabla_h v\|_{L^p(\sigma)^d}\,, \tag{A.3}$$

$$\text{and} \quad \|\langle v\rangle_\sigma\|_{L^p(\sigma)} \lesssim h^{-1/p}\left(\|v\|_{L^p(K)} + h\|\nabla_h v\|_{L^p(K)^d}\right) \tag{A.4}$$

are valid for all $p \in [1,\infty]$, all $v \in V_{0,h}$, all $K \in \mathcal{T}_h$, and all $\sigma \in \mathcal{E}_h(K)$. Moreover, given $\phi \in C(\overline{\Omega_h}) \cap W^{1,\infty}(\Omega_h)$, it is easy to see that

$$\|[\![\overline{\phi}]\!]\|_{L^\infty(\sigma)} \lesssim h\|\phi\|_{W^{1,\infty}(\Omega_h)} \qquad \text{for all } \sigma \in \mathcal{E}_{h,\text{int}}, \tag{A.5}$$

$$\|\phi-\overline{\phi}_K\|_{L^\infty(\sigma)} \lesssim h\|\phi\|_{W^{1,\infty}(\Omega_h)} \qquad \text{for all } K \in \mathcal{T}_h \text{ and all } \sigma \in \mathcal{E}_h(K), \tag{A.6}$$

$$\|\phi-\langle\phi\rangle_\sigma\|_{L^\infty(\sigma)} \lesssim h\|\phi\|_{W^{1,\infty}(\Omega_h)} \qquad \text{for all } K \in \mathcal{T}_h \text{ and all } \sigma \in \mathcal{E}_h(K), \tag{A.7}$$

$$\|[\![\overline{\Pi_{V,h}\phi}]\!]\|_{L^\infty(\sigma)} \lesssim h\|\phi\|_{W^{1,\infty}(\Omega_h)} \qquad \text{for all } \sigma \in \mathcal{E}_{h,\text{int}}, \tag{A.8}$$

$$\|\phi-\overline{\Pi_{V,h}\phi}_K\|_{L^\infty(\sigma)} \lesssim h\|\phi\|_{W^{1,\infty}(\Omega_h)} \qquad \text{for all } K \in \mathcal{T}_h \text{ and all } \sigma \in \mathcal{E}_h(K), \tag{A.9}$$

$$\text{and} \qquad \|\phi-\overline{\Pi_{V,h}\phi}\|_{L^\infty(\Omega_h)} \lesssim h\|\phi\|_{W^{1,\infty}(\Omega_h)}\,. \tag{A.10}$$

Next, combining [18, Theorem 6.1] with [10, Lemma 2.2] we obtain a discrete version of Poincaré's inequality, namely

$$\|v\|_{L^q(\Omega_h)} \lesssim \|\nabla_h v\|_{L^2(\Omega_h)^d} \tag{A.11}$$

for all $v \in V_{0,h}$, where $q \in [1,\infty)$ if $d = 2$ and $q \in [1,6]$ if $d = 3$. Due to [2, Theorem 5], we have the following estimates for the projection operators $\Pi_{Q,h}$ and $\Pi_{V,h}$:

$$\|\phi-\overline{\phi}\|_{L^q(\Omega_h)} \equiv \|\phi-\Pi_{Q,h}\phi\|_{L^q(\Omega_h)} \lesssim h\|\phi\|_{W^{1,q}(\Omega_h)}\,, \tag{A.12}$$

$$\|\phi-\Pi_{V,h}\phi\|_{L^q(\Omega_h)} + h\|\nabla_{\boldsymbol{x}}\phi-\nabla_h\Pi_{V,h}\phi\|_{L^q(\Omega_h)^d} \lesssim h\|\phi\|_{W^{1,q}(\Omega_h)}\,, \tag{A.13}$$

$$||\psi - \Pi_{V,h}\psi||_{L^q(\Omega_h)} + h\,||\nabla_{\boldsymbol{x}}\psi - \nabla_h \Pi_{V,h}\psi||_{L^q(\Omega_h)^d} \lesssim h^2\,||\phi||_{W^{2,q}(\Omega_h)} \tag{A.14}$$

for all $q \in [1,\infty]$, all $\phi \in W^{1,q}(\Omega_h)$, and all $\psi \in W^{2,q}(\Omega_h)$.

Furthermore, we record the following estimate concerning the comparison of the operators $\Pi_{V,h}$ and $\Pi^0_{V,h}$, see [4, Corollary 2.12]:

$$||\Pi_{V,h}\phi - \Pi^0_{V,h}\phi||_{L^\infty(K)} + h\,||\nabla_{\boldsymbol{x}}(\Pi_{V,h}\phi - \Pi^0_{V,h}\phi)||_{L^\infty(K)} \lesssim h\,||\nabla_{\boldsymbol{x}}\phi||_{L^\infty(\mathbb{R}^d)^d} \tag{A.15}$$

[7]for all $\phi \in C^1(\mathbb{R}^d)$ and all $K \in \mathcal{T}_h$.

Finally, we report the boundedness of the projection operators $\Pi_{Q,h}$ and $\Pi_{V,h}$. It follows from Jensen's inequality, cf. [5, p.90], that

$$||\Pi_{Q,h}v||_{L^q(\Omega_h)} \le ||v||_{L^q(\Omega_h)} \qquad \text{for all } v \in L^q(\Omega_h), q \in [1,\infty]. \tag{A.16}$$

Furthermore, we may use (A.13) and the triangle inequality to deduce that there exists an h-independent constant $C > 0$ such that

$$||\Pi_{V,h}v||_{L^q(\Omega_h)} \le (1 + Ch)\,||v||_{W^{1,q}(\Omega_h)} \qquad \text{for all } v \in W^{1,q}(\Omega_h), q \in [1,\infty]. \tag{A.17}$$

A.3. Proof of Theorem 4.3

This appendix is devoted to the proof of Theorem 4.3. Its proof is an adaption of the proof of [15, Theorem 5.1]. Apart from the estimates listed in Appendix A.2, we need the subsequent results.

Lemma A.2. *Let $\phi \in C^1(\overline{\Omega_T})$, $\tau \in \{\Delta t, \dots, N_T\Delta t\}$, $(r_h^k)_{k\in\mathbb{N}_0} \subset Q_h$, and define the functions r_h, $\tilde{r}_h : \mathbb{R}\times\Omega_h \to \mathbb{R}$ via*

$$(r_h, \tilde{r}_h)(t,\cdot) = \begin{cases} (r_h^k, r_h^{k-1} + (t-t_k)D_t r_h^k) & \text{if } t \in (t_{k-1}, t_k] \text{ for some } k \in \mathbb{N} \text{ and} \\ (r_h^0, r_h^0) & \text{if } t \le 0. \end{cases}$$

Then

$$\left| \int_0^\tau \int_\Omega \left[(D_t r_h)\,\phi + r_h\,\partial_t\phi \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \left[\int_\Omega (r_h\phi)(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \right|$$
$$\lesssim ||\partial_t\phi||_{L^\infty(\Omega_T)} \int_0^T \int_\Omega \Delta t\,|D_t r_h|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim ||\partial_t\phi||_{L^\infty(\Omega_T)} (\Delta t)^{1/2} h^{-\delta/2} \left(h^\delta \Delta t \int_0^T \int_{\Omega_h} (D_t r_h)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right)^{1/2}. \tag{A.18}$$

Proof. Let $m \in \mathbb{N}$ be such that $\tau = m\Delta t$. Then

$$\begin{aligned}\int_0^\tau \int_\Omega \tilde{r}_h\,\partial_t\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \sum_{k=1}^m \int_\Omega \left(r_h^{k-1} \int_{t_{k-1}}^{t_k} \partial_t\phi\,\mathrm{d}t + \frac{r_h^k - r_h^{k-1}}{\Delta t} \int_{t_{k-1}}^{t_k} (t - t_{k-1})\,\partial_t\phi\,\mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\ &= \sum_{k=1}^m \int_\Omega \left(r_h^{k-1}\big(\phi(t_k,\cdot) - \phi(t_{k-1},\cdot)\big) + \phi(t_k,\cdot)\big(r_h^k - r_h^{k-1}\big)\right) \mathrm{d}\boldsymbol{x} \\ &\quad - \sum_{k=1}^m \int_\Omega \frac{r_h^k - r_h^{k-1}}{\Delta t} \int_{t_{k-1}}^{t_k} \phi\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} \\ &= \left[\int_\Omega (r_h\phi)(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega (D_t r_h)\,\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.\end{aligned}$$

[7]Compared to [4, Corollary 2.12] we only have the factor h instead of h^2 on the right-hand side which is due to the fact that (4.4) only ensures that there is a constant $d_\Omega > 0$ such that $\mathrm{dist}[\boldsymbol{x}, \partial\Omega] \le d_\Omega h$ for all $\boldsymbol{x} \in \partial\Omega_h$.

Moreover, using Hölder's inequality, we deduce that

$$\left| \int_0^\tau \int_\Omega (\tilde{r}_h - r_h)\, \partial_t \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| \lesssim ||\partial_t \phi||_{L^\infty(\Omega_T)} \int_0^T \int_\Omega |\tilde{r}_h - r_h| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\lesssim ||\partial_t \phi||_{L^\infty(\Omega_T)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \int_\Omega \left| \frac{r_h^k - r_h^{k-1}}{\Delta t} (\Delta t + t_{k-1} - t) \right| \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\lesssim ||\partial_t \phi||_{L^\infty(\Omega_T)} \int_0^T \int_\Omega \Delta t\, |D_t r_h| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\lesssim ||\partial_t \phi||_{L^\infty(\Omega_T)} (\Delta t)^{1/2} h^{-\delta/2} \left(h^\delta \Delta t \int_0^T \int_{\Omega_h} (D_t r_h)^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right)^{1/2} .$$

Together, the previous computations yield the desired result. □

Lemma A.3. *Let $r, f \in Q_h$, $\boldsymbol{v} \in \boldsymbol{V}_{0,h}$ and $\phi \in C(\overline{\Omega_h}) \cap W^{1,\infty}(\Omega_h)$. Then*

$$\int_{\Omega_h} r\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_{\text{int}}} F_h^{\text{up}}[r, \boldsymbol{v}] \, [\![f]\!] \,\mathrm{d}S_{\boldsymbol{x}}$$
$$= \int_{\Omega_h} r\,(f - \phi)\,\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{E}(K)} (f - \phi)\, [\![r]\!] \left[\langle \boldsymbol{v} \cdot \boldsymbol{n}_K \rangle_\sigma\right]^- \mathrm{d}S_{\boldsymbol{x}}$$
$$+ \int_{\mathcal{E}(K)} (\phi - \langle \phi \rangle_\sigma)\, r \left(\boldsymbol{v} \cdot \boldsymbol{n}_K - \langle \boldsymbol{v} \cdot \boldsymbol{n}_K \rangle_\sigma\right) \mathrm{d}S_{\boldsymbol{x}} + \frac{h^\varepsilon}{2} \int_{\mathcal{E}_{\text{int}}} [\![r]\!]\, [\![f]\!] \,\mathrm{d}S_{\boldsymbol{x}} \,. \tag{A.19}$$

[8]

Corollary A.4. *Let $\boldsymbol{s}, \boldsymbol{g} \in \boldsymbol{Q}_h$, $\boldsymbol{w} \in \boldsymbol{V}_{0,h}$, and $\boldsymbol{\psi} \in C(\overline{\Omega_h})^d \cap W^{1,\infty}(\Omega_h)^d$. Then*

$$\int_{\Omega_h} \boldsymbol{s} \otimes \boldsymbol{w} : \nabla_{\boldsymbol{x}} \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}_{\text{int}}} F_h^{\text{up}}[\boldsymbol{s}, \boldsymbol{w}] \cdot [\![\boldsymbol{g}]\!] \,\mathrm{d}S_{\boldsymbol{x}}$$
$$= \int_{\Omega_h} \boldsymbol{s} \cdot (\boldsymbol{g} - \boldsymbol{\psi})\,\mathrm{div}_h(\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{E}(K)} (\boldsymbol{g} - \boldsymbol{\psi}) \cdot [\![\boldsymbol{s}]\!] \left[\langle \boldsymbol{w} \cdot \boldsymbol{n}_K \rangle_\sigma\right]^- \mathrm{d}S_{\boldsymbol{x}}$$
$$+ \int_{\mathcal{E}(K)} (\boldsymbol{\psi} - \langle \boldsymbol{\psi} \rangle_\sigma) \cdot \boldsymbol{s} \left(\boldsymbol{w} \cdot \boldsymbol{n}_K - \langle \boldsymbol{w} \cdot \boldsymbol{n}_K \rangle_\sigma\right) \mathrm{d}S_{\boldsymbol{x}} + \frac{h^\varepsilon}{2} \int_{\mathcal{E}_{\text{int}}} [\![\boldsymbol{s}]\!] \cdot [\![\boldsymbol{g}]\!] \,\mathrm{d}S_{\boldsymbol{x}} \,. \tag{A.20}$$

Lemma A.5. *Let $r \in Q_h$, $v \in V_{0,h}$, $\phi \in W_0^{1,2}(\Omega_h)$, and $\boldsymbol{\phi} \in W_0^{1,2}(\Omega_h)^d$. Then*

$$\int_{\Omega_h} \nabla_h v \cdot \nabla_h \Pi_V \phi \,\mathrm{d}\boldsymbol{x} = \int_{\Omega_h} \nabla_h v \cdot \nabla_{\boldsymbol{x}} \phi \,\mathrm{d}\boldsymbol{x} \quad \textit{and} \quad \int_{\Omega_h} r\,\mathrm{div}_h(\Pi_V \boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x} = \int_{\Omega_h} r\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\phi}) \,\mathrm{d}\boldsymbol{x} \,.$$

For the proof of the Lemmata A.3 and A.5, we refer to [5, Chapter 9.2, Lemma 7 with $\chi = 1$] and [5, Chapter 9.3, Lemma 8], respectively. For the proof of Lemma A.3, we additionally need to observe that

$$\int_\sigma \langle \phi \rangle_\sigma\, r \left(\boldsymbol{v} \cdot \boldsymbol{n}_K - \langle \boldsymbol{v} \cdot \boldsymbol{n}_K \rangle_\sigma\right) \mathrm{d}S_{\boldsymbol{x}} = 0 \,,$$

which follows from the fact that $r \in Q_h$. Corollary A.4 can be proven by applying Lemma A.3 with $(r, f, \boldsymbol{v}, \phi) = (s_i, g_i, \boldsymbol{w}, \psi_i)$, $i \in \{1, \ldots, d\}$.

Having all necessary tools at our disposal, we can approach the proof of Theorem 4.3.

[8]In integrals of the form $\int_{\mathcal{E}(K)}$ we consider the vector $\boldsymbol{n}_\sigma$ in the definition of the trace operators $(\cdot)^{\text{in},\sigma}$ and $(\cdot)^{\text{out},\sigma}$ to be replaced by $\boldsymbol{n}_K$.

Proof of Theorem 4.3 (Part I). In this part, we only consider the case $\tau \in \{\Delta t, \dots, N_T \Delta t\}$. We choose arbitrary test functions $\varphi, \psi \in C^1(\overline{\Omega_T})$, $\psi \geq 0$, and $\boldsymbol{\varphi} \in C^1(\overline{\Omega_T})^d$, $\boldsymbol{\varphi}|_{[0,T]\times\partial\Omega} = \mathbf{0}$. By extension with $\mathbf{0}$, we consider $\boldsymbol{\varphi} \in C([0,T]\times\mathbb{R}^d)^d \cap L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d)^d)$. Moreover, since Ω is a smooth domain, Ω_T is also a smooth domain. Thus, we may use Stein's extension operator $\mathfrak{E}_{\text{Stein}}$, see [20, Chapter VI, Theorem 5], to extend φ, ψ in such a way that $\varphi, \psi \in W^{1,\infty}(\mathbb{R}^{d+1}) \cap C(\mathbb{R}^{d+1})$ and

$$||\varphi||_{W^{k,\infty}(\mathbb{R}^{d+1})} \leq C_{\text{Stein}}(\Omega_T,1)\, ||\varphi||_{C^k(\overline{\Omega_T})}\,, \qquad ||\psi||_{W^{k,\infty}(\mathbb{R}^{d+1})} \leq C_{\text{Stein}}(\Omega_T,1)\, ||\psi||_{C^k(\overline{\Omega_T})} \tag{A.21}$$

for all $k \in \{0,1\}$, where $C_{\text{Stein}}(\Omega_T,1) > 0$ is given by

$$C_{\text{Stein}}(\Omega_T,1) = \max_{k\,\in\,\{0,1\}} \left\{ ||\mathfrak{E}_{\text{Stein}}||_{W^{k,\infty}(\Omega_T)\,\to\,W^{k,\infty}(\mathbb{R}^{d+1})} \right\}.$$

Putting $\varphi_h = \Pi_Q \varphi$, $\psi_h = \Pi_Q \psi$ and $\boldsymbol{\varphi}_h = \Pi_V \boldsymbol{\varphi}$, we make the following observations.

The continuity equation.

From (4.5) we deduce that

$$\int_0^\tau \int_{\Omega_h} (D_t \varrho_h)\, \varphi_h \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau \int_{\mathcal{E}_{\text{int}}} F_h^{\text{up}}[\varrho_h, \boldsymbol{u}_h]\, [\![\varphi_h]\!]\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t = 0\,. \tag{A.22}$$

Using the fact that $\varrho_h(t,\cdot) \in Q_h$ for every $t \in [0,\tau]$, we see that

$$\int_0^\tau \int_{\Omega_h} (D_t \varrho_h)\, \varphi_h \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^\tau \int_{\Omega_h} (D_t \varrho_h)\, \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,. \tag{A.23}$$

Next, we observe that

$$\begin{aligned}
\int_0^\tau \int_{\Omega_h\setminus\Omega} (D_t\varrho_h)\,\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \int_0^\tau \int_{\Omega_h\setminus\Omega} \frac{\varrho_h(t,\cdot)-\varrho_h(t-\Delta t,\cdot)}{\Delta t}\,\varphi(t,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \frac{1}{\Delta t}\int_0^\tau \int_{\Omega_h\setminus\Omega} (\varrho_h\varphi)(t,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \frac{1}{\Delta t}\int_{-\Delta t}^{\tau-\Delta t} \int_{\Omega_h\setminus\Omega} \varrho_h(t,\cdot)\,\varphi(t+\Delta t,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= -\int_0^{\tau-\Delta t} \int_{\Omega_h\setminus\Omega} \varrho_h(t,\cdot)\,\frac{\varphi(t+\Delta t,\cdot)-\varphi(t,\cdot)}{\Delta t}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad + \frac{1}{\Delta t}\int_{\tau-\Delta t}^{\tau} \int_{\Omega_h\setminus\Omega} \varrho_h(\tau,\cdot)\,\varphi(t,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad - \frac{1}{\Delta t}\int_0^{\Delta t} \int_{\Omega_h\setminus\Omega} \varrho_h(0,\cdot)\,\varphi(t,\cdot)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,.
\end{aligned}$$

Consequently, (4.4), the second estimate in (4.15), (A.16) and (A.21) yield

$$\begin{aligned}
\left| \int_0^\tau \int_{\Omega_h\setminus\Omega} (D_t\varrho_h)\,\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| &\lesssim T\,|\Omega_h\setminus\Omega|^{1/2}\,h^{-\delta/2}\,||\varphi||_{W^{1,\infty}(\mathbb{R}^{d+1})}\,||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))} \\
&\quad + |\Omega_h\setminus\Omega|^{1/2}\,h^{-\delta/2}\,||\varphi||_{L^\infty(\mathbb{R}^{d+1})}\,||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))} \\
&\quad + |\Omega_h\setminus\Omega|\,||\varphi||_{L^\infty(\mathbb{R}^{d+1})}\,||\varrho_h^0||_{L^\infty(\Omega_h)} \\
&\lesssim h^{(1-\delta)/2}\,||\varphi||_{C^1(\overline{\Omega_T})} \lesssim h^{(1-\delta)/2}\,.
\end{aligned}$$

Combining (A.23), (A.3) and the first estimate in (4.23) with $\Delta t \approx h$ and Lemma A.2 applied to $(r_h,\phi) = (\varrho_h,\varphi)$, we obtain

$$\int_0^\tau \int_{\Omega_h} (D_t\varrho_h)\,\varphi_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \left[\int_\Omega (\varrho_h\varphi)(t,\cdot)\,\mathrm{d}\boldsymbol{x} \right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \varrho_h \partial_t\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{(1-\delta)/2})\,.$$

Next, let us consider the second term on the left-hand side of (A.22). Employing Hölder's inequality, the first estimate in (A.3), the second estimate in (4.15), the first estimate in (4.14) and (A.21), we see that

$$\begin{aligned}\left|\int_0^\tau\int_{\Omega_h}\varrho_h\big(\boldsymbol{u}_h-\overline{\boldsymbol{u}_h}\big)\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| &\lesssim ||\varphi||_{W^{1,\infty}(\mathbb{R}^{d+1})}\,||\varrho_h||_{L^2(0,T;\,L^2(\Omega_h))}\,||\boldsymbol{u}_h-\overline{\boldsymbol{u}_h}||_{L^2(0,T;\,L^2(\Omega_h)^d)}\\ &\lesssim h^{1-\delta/2}\,||\varphi||_{C^1(\overline{\Omega_T})}\,||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))}\,||\nabla_h\boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})}\\ &\lesssim h^{1-\delta/2}\,.\end{aligned}$$

Moreover, applying (A.21), (4.4), the second estimate in (4.15) and the first estimate in (4.13), we obtain

$$\begin{aligned}\left|\int_0^\tau\int_{\Omega_h\setminus\Omega}\varrho_h\overline{\boldsymbol{u}_h}\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| &\lesssim ||\varphi||_{W^{1,\infty}(\mathbb{R}^{d+1})}\,|\Omega_h\setminus\Omega|^{1/4}\int_0^T||\sqrt{\varrho_h}(t,\cdot)||_{L^4(\Omega_h)}\,||(\sqrt{\varrho_h}|\overline{\boldsymbol{u}_h}|)(t,\cdot)||_{L^2(\Omega_h)}\,\mathrm{d}t\\ &\lesssim h^{(1-\delta)/4}\,||\varphi||_{C^1(\overline{\Omega_T})}\int_0^T||h^{\delta/2}\varrho_h(t,\cdot)||^{1/2}_{L^2(\Omega_h)}\,||(\varrho_h|\overline{\boldsymbol{u}_h}|^2)(t,\cdot)||^{1/2}_{L^1(\Omega_h)}\,\mathrm{d}t\\ &\lesssim h^{(1-\delta)/4}\,T\,||h^{\delta/2}\varrho_h||^{1/2}_{L^\infty(0,T;\,L^2(\Omega_h))}\,||\varrho_h|\overline{\boldsymbol{u}_h}|^2||^{1/2}_{L^\infty(0,T;\,L^1(\Omega_h))}\lesssim h^{(1-\delta)/4}\,.\end{aligned}$$

Consequently,

$$\int_0^\tau\int_{\Omega_h}\varrho_h\boldsymbol{u}_h\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\int_0^\tau\int_{\Omega}\varrho_h\overline{\boldsymbol{u}_h}\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\mathcal{O}(h^{(1-\delta)/4})\qquad\text{as } h\downarrow 0.$$

Then, using Lemma A.3 with $(r,\boldsymbol{v},f,\phi)=(\varrho_h,\boldsymbol{u}_h,\varphi_h,\varphi)(t,\cdot)$, $t\in[0,\tau]$, as well as the estimates (A.5)–(A.7), (A.12) and (A.21), we deduce that

$$\int_0^\tau\int_{\mathcal{E}_{\mathrm{int}}}F_h^{\mathrm{up}}[\varrho_h,\boldsymbol{u}_h]\,[\![\varphi_h]\!]\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t=\int_0^\tau\int_{\Omega_h}\varrho_h\boldsymbol{u}_h\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\sum_{j=2}^5 I_{j,h}\,,$$

where

$$|I_{2,h}|\lesssim h\,||\varphi||_{C^1(\overline{\Omega_T})}\int_0^T\int_{\mathcal{E}(K)}\big|[\![\varrho_h]\!]\big[\langle\boldsymbol{u}_h\cdot\boldsymbol{n}_K\rangle_\sigma\big]^-\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\,,$$

$$|I_{3,h}|\lesssim h\,||\varphi||_{C^1(\overline{\Omega_T})}\int_0^T\int_{\mathcal{E}(K)}\big|\,\varrho_h\big(\boldsymbol{u}_h-\langle\boldsymbol{u}_h\rangle_\sigma\big)\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\,,$$

$$|I_{4,h}|\lesssim h\,||\varphi||_{C^1(\overline{\Omega_T})}\int_0^T\int_\Omega\big|\varrho_h\,\mathrm{div}_h(\boldsymbol{u}_h)\big|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,\qquad |I_{5,h}|\lesssim h^{1+\varepsilon}\,||\varphi||_{C^1(\overline{\Omega_T})}\int_0^T\int_{\mathcal{E}_{\mathrm{int}}}\big|[\![\varrho_h]\!]\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\,.$$

These terms can be further estimated as follows.

- **Term** $|I_{2,h}|$. Due to (4.21), we obtain

$$|I_{2,h}|\lesssim h\int_0^T\int_{\mathcal{E}_{\mathrm{int}}}\big|[\![\varrho_h]\!]\,\langle\boldsymbol{u}_h\cdot\boldsymbol{n}_\sigma\rangle_\sigma\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\lesssim h^{1-\delta/2}(1+h^{-1/2})\,.$$

- **Term** $|I_{3,h}|$. By means of Hölder's inequality, the second estimate in (A.3), the first estimate in (A.2), the second estimate in (4.15), and the first estimate in (4.14), we derive

$$\begin{aligned}|I_{3,h}|&\lesssim h\,||\varrho_h||_{L^2(0,T;\,L^2(\Omega_h))}\,||\nabla_h\boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})}\\ &\lesssim h^{1-\delta/2}\,||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))}\,||\nabla_h\boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})}\lesssim h^{1-\delta/2}.\end{aligned}$$

- **Term** $|I_{4,h}|$. Employing Hölder's inequality, the second estimate in (4.14), and the second estimate in (4.15), we conclude that

$$\begin{aligned}|I_{4,h}|&\lesssim h\,||\varrho_h||_{L^2(0,T;\,L^2(\Omega_h))}\,||\mathrm{div}_h(\boldsymbol{u}_h)||_{L^2(0,T;\,L^2(\Omega_h))}\\ &\lesssim h^{1-\delta/2}\,||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))}\,||\mathrm{div}_h(\boldsymbol{u}_h)||_{L^2(0,T;\,L^2(\Omega_h))}\lesssim h^{1-\delta/2}\,.\end{aligned}$$

- **Term** $|I_{5,h}|$. Applying the first estimate in (A.2) and the second estimate in (4.13), we get

$$|I_{5,h}| \lesssim h^{\varepsilon}\, ||\varrho_h||_{L^1(0,T;\,L^1(\Omega_h))} \lesssim h^{\varepsilon}\, ||\varrho_h||_{L^\infty(0,T;\,L^\gamma(\Omega_h))} \lesssim h^{\varepsilon}\,.$$

Consequently,

$$\left[\int_\Omega (\varrho_h \varphi)(t,\cdot)\, \mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega \left[\varrho_h \partial_t \varphi + \varrho_h \overline{\boldsymbol{u}_h}\cdot\nabla_{\boldsymbol{x}}\varphi\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{\alpha_1}) \qquad \text{as } h\downarrow 0, \tag{A.24}$$

where $\alpha_1 = \min\left\{\varepsilon, \frac{1-\delta}{4}\right\}$.

The potential temperature equation.

The proof of (4.28) can be done by repeating the proof of (4.26) with ϱ_h and ϱ_h^0 replaced by $\varrho_h\theta_h$ and $\varrho_h^0\theta_h^0$, respectively.

The momentum equation.

Realizing that $\boldsymbol{\varphi}_h(t,\cdot) \in \boldsymbol{V}_{0,h}$ for all $t\in[0,T]$, we deduce from (4.7) that

$$\begin{aligned}\int_0^\tau\int_{\Omega_h} D_t(\varrho_h\overline{\boldsymbol{u}_h})\cdot\boldsymbol{\varphi}_h\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_{\mathcal{E}_{\mathrm{int}}} F_h^{\mathrm{up}}[\varrho_h\overline{\boldsymbol{u}_h}, \boldsymbol{u}_h]\cdot[\![\overline{\boldsymbol{\varphi}_h}]\!]\, \mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t + \mu\int_0^\tau\int_{\Omega_h} \nabla_h\boldsymbol{u}_h : \nabla_h\boldsymbol{\varphi}_h\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \nu\int_0^\tau\int_{\Omega_h} \mathrm{div}_h(\boldsymbol{u}_h)\,\mathrm{div}_h(\boldsymbol{\varphi}_h)\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_{\Omega_h} \left(p(\varrho_h\theta_h) + h^\delta\left[\varrho_h^2 + (\varrho_h\theta_h)^2\right]\right)\mathrm{div}_h(\boldsymbol{\varphi}_h)\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0\,.\end{aligned} \tag{A.25}$$

Let us consider the first term on the left-hand side of (A.25). Since $\boldsymbol{\varphi}$ vanishes on $[0,T]\times(\Omega_h\backslash\Omega)$, we have

$$\begin{aligned}\int_0^\tau\int_{\Omega_h} D_t(\varrho_h\overline{\boldsymbol{u}_h})\cdot\boldsymbol{\varphi}_h\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^\tau\int_{\Omega} D_t(\varrho_h\overline{\boldsymbol{u}_h})\cdot\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^\tau\int_{\Omega_h} \varrho_h^-\, D_t\overline{\boldsymbol{u}_h}\cdot(\Pi_V\boldsymbol{\varphi}-\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_0^\tau\int_{\Omega_h} (D_t\varrho_h)\,\overline{\boldsymbol{u}_h}\cdot(\Pi_V\boldsymbol{\varphi}-\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,,\end{aligned}$$

where by Hölder's inequality, the second information in (4.10), (A.16), the second estimate in (4.19), (A.13) and $\Delta t\approx h$

$$\begin{aligned}&\left|\int_0^\tau\int_{\Omega_h} \varrho_h^-\, D_t\overline{\boldsymbol{u}_h}\cdot(\Pi_V\boldsymbol{\varphi}-\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \\ &\quad\lesssim h(\Delta t)^{-1/2}\, ||\boldsymbol{\varphi}||_{C^1(\overline{\Omega_T})^d} \left(\int_0^T\int_{\Omega_h} \varrho_h^-\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right)^{1/2} \left(\Delta t\int_0^T\int_{\Omega_h} \varrho_h^-(D_t\overline{\boldsymbol{u}_h})^2\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right)^{1/2} \lesssim h^{1/2}\end{aligned}$$

and by Hölder's inequality, the first estimate in (4.23), the third estimate in (4.14), (A.16), (A.5) and $\Delta t\approx h$

$$\begin{aligned}&\left|\int_0^\tau\int_{\Omega_h} (D_t\varrho_h)\,\overline{\boldsymbol{u}_h}\cdot(\Pi_V\boldsymbol{\varphi}-\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \\ &\quad\lesssim h^{1-\delta/2}(\Delta t)^{-1/2}\, ||\boldsymbol{\varphi}||_{C^1(\overline{\Omega_T})^d} \left(h^\delta\Delta t\int_0^T\int_{\Omega_h} (D_t\varrho_h)^2\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right)^{1/2} ||\overline{\boldsymbol{u}_h}||_{L^2(0,T;\,L^2(\Omega_h)^d)} \lesssim h^{(1-\delta)/2}\,.\end{aligned}$$

In view of the previous two computations, it is easy to verify that

$$\int_0^T\int_\Omega \Delta t\,|D_t(\varrho_h\overline{\boldsymbol{u}_h})|\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim \int_0^T\int_\Omega \Delta t\,|\varrho_h^- D_t\overline{\boldsymbol{u}_h}|\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T\int_\Omega \Delta t\,|(D_t\varrho_h)\overline{\boldsymbol{u}_h}|\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \lesssim h^{(1-\delta)/2}\,,$$

whence Lemma A.2 applied to $(r_h,\phi) = (\varrho_h\overline{u_{h,i}}, \varphi_i)$, $i\in\{1,\dots,d\}$, yields

$$\int_0^\tau\int_\Omega D_t(\varrho_h\overline{\boldsymbol{u}_h})\cdot\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \left[\int_\Omega (\varrho_h\overline{\boldsymbol{u}_h}\cdot\boldsymbol{\varphi})(t,\cdot)\, \mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} - \int_0^\tau\int_\Omega \varrho_h\overline{\boldsymbol{u}_h}\cdot\partial_t\boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{(1-\delta)/2})$$

as $h \downarrow 0$. Next, we turn to the last three terms on the left-hand side of (A.25). It follows from Lemma A.5 that

$$\mu \int_0^\tau \int_{\Omega_h} \nabla_h \boldsymbol{u}_h : \nabla_h \boldsymbol{\varphi}_h \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \int_0^\tau \int_{\Omega_h} \big(\nu \operatorname{div}_h(\boldsymbol{u}_h) - p(\varrho_h\theta_h) - h^\delta \big[\varrho_h^2 + (\varrho_h\theta_h)^2\big]\big) \operatorname{div}_h(\boldsymbol{\varphi}_h) \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t$$
$$= \mu \int_0^\tau \int_{\Omega_h} \nabla_h \boldsymbol{u}_h : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \int_0^\tau \int_{\Omega_h} \big(\nu \operatorname{div}_h(\boldsymbol{u}_h) - p(\varrho_h\theta_h) - h^\delta \big[\varrho_h^2 + (\varrho_h\theta_h)^2\big]\big) \operatorname{div}_{\boldsymbol{x}}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t\,.$$

Finally, let us examine the second term on the left-hand side of (A.25). Using Hölder's inequality, the first estimate in (A.3), the second estimate in (4.16) and the first estimate in (4.14), we deduce that

$$\left| \int_0^\tau \int_{\Omega_h} \varrho_h \overline{\boldsymbol{u}_h} \otimes \big(\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\big) : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \right| \lesssim \|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega_T})^d} \|\varrho_h \overline{\boldsymbol{u}_h}\|_{L^2(0,T;\,L^2(\Omega_h)^d)} \|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_{L^2(0,T;\,L^2(\Omega_h)^d)}$$
$$\lesssim h \|\varrho_h \overline{\boldsymbol{u}_h}\|_{L^2(0,T;\,L^2(\Omega_h)^d)} \|\nabla_h \boldsymbol{u}_h\|_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})} \lesssim h^{1-(d+3\delta)/6}\,.$$

Applying Corollary A.4 with $(\boldsymbol{s}, \boldsymbol{w}, \boldsymbol{g}, \boldsymbol{\psi}) = (\varrho_h \overline{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\varphi})(t,\cdot)$, $t \in [0,\tau]$, as well as the estimates (A.8)–(A.10) and (A.13), we deduce that

$$\int_0^\tau \int_{\mathcal{E}_{\mathrm{int}}} \boldsymbol{F}_h^{\mathrm{up}}[\varrho_h \overline{\boldsymbol{u}_h}, \boldsymbol{u}_h] \cdot [\![\overline{\boldsymbol{\varphi}_h}]\!] \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t = \int_0^\tau \int_{\Omega_h} \varrho_h \overline{\boldsymbol{u}_h} \otimes \boldsymbol{u}_h : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t + \sum_{j=2}^5 J_{j,h}\,,$$

where

$$|J_{2,h}| \lesssim h \|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega_T})^d} \int_0^T \int_{\mathcal{E}(K)} \big| [\![\varrho_h \overline{\boldsymbol{u}_h}]\!] \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_K \rangle_\sigma\big]^- \big| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t\,,$$
$$|J_{3,h}| \lesssim h \|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega_T})^d} \int_0^T \int_{\mathcal{E}(K)} |\varrho_h \overline{\boldsymbol{u}_h}| \, \big|\boldsymbol{u}_h - \langle \boldsymbol{u}_h \rangle_\sigma\big| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t\,,$$
$$|J_{4,h}| \lesssim h \|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega_T})^d} \int_0^T \int_{\Omega_h} \big|\varrho_h \overline{\boldsymbol{u}_h} \operatorname{div}_h(\boldsymbol{u}_h)\big| \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t\,,$$
$$|J_{5,h}| \lesssim h^{1+\varepsilon} \|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega_T})^d} \int_0^T \int_{\mathcal{E}_{\mathrm{int}}} \big| [\![\varrho_h \overline{\boldsymbol{u}_h}]\!] \big| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t\,.$$

We continue by estimating the above terms.

- **Term** $|J_{2,h}|$. We observe that $[\![\varrho_h \overline{\boldsymbol{u}_h}]\!] = \varrho_h^{\mathrm{out}} [\![\overline{\boldsymbol{u}_h}]\!] + [\![\varrho_h]\!] \overline{\boldsymbol{u}_h}^{\,\mathrm{in}}$, which implies

$$|J_{2,h}| \lesssim h \int_0^T \int_{\mathcal{E}(K)} \big| \varrho_h^{\mathrm{out}} [\![\overline{\boldsymbol{u}_h}]\!] \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_K \rangle_\sigma\big]^- \big| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t + h \int_0^T \int_{\mathcal{E}(K)} \big| [\![\varrho_h]\!] \overline{\boldsymbol{u}_h} \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_K \rangle_\sigma\big]^- \big| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t\,. \quad \text{(A.26)}$$

Employing Hölder's inequality, (4.20), (A.4), the first estimate in (A.2), the first and third estimate in (4.14), and the second estimate in (4.15), we see that

$$h \int_0^T \int_{\mathcal{E}(K)} \big| \varrho_h^{\mathrm{out}} [\![\overline{\boldsymbol{u}_h}]\!] \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_K \rangle_\sigma\big]^- \big| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t$$
$$\lesssim h \left(\int_0^T \int_{\mathcal{E}(K)} -\varrho_h^{\mathrm{out}} [\![\overline{\boldsymbol{u}_h}]\!]^2 \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_K \rangle_\sigma\big]^- \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t \right)^{1/2} \left(\int_0^T \int_{\mathcal{E}(K)} \varrho_h^{\mathrm{out}} |\langle \boldsymbol{u}_h \rangle_\sigma| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t \right)^{1/2}$$
$$= h \left(\int_0^T \int_{\mathcal{E}_{\mathrm{int}}} \Big(\varrho_h^{\mathrm{in}} \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_\sigma \rangle_\sigma\big]^+ - \varrho_h^{\mathrm{out}} \big[\langle \boldsymbol{u}_h \cdot \boldsymbol{n}_\sigma \rangle_\sigma\big]^- \Big) [\![\overline{\boldsymbol{u}_h}]\!]^2 \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t \right)^{1/2} \times$$
$$\times \left(\int_0^T \int_{\mathcal{E}(K)} \varrho_h^{\mathrm{out}} |\langle \boldsymbol{u}_h \rangle_\sigma| \, \mathrm{d}S_{\boldsymbol{x}}\, \mathrm{d}t \right)^{1/2}$$
$$\lesssim h \big(h^{-1} \|\varrho_h\|_{L^2(0,T;\,L^2(\Omega_h))} \big(\|\boldsymbol{u}_h\|_{L^2(0,T;\,L^2(\Omega_h)^d)} + h \|\nabla_h \boldsymbol{u}_h\|_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})} \big) \big)^{1/2}$$

$$\lesssim h\big(h^{-1-\delta/2}\,||h^{\delta/2}\varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))}\big(||\boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^d)} + h\,||\nabla_h \boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})}\big)\big)^{1/2}$$
$$\lesssim h^{1/2-\delta/4} + h^{1-\delta/4}\,. \tag{A.27}$$

Next, using Hölder's inequality, the estimates (A.2), (A.4), (4.17), the first and third estimate in (4.14), the second estimate in (4.15), and the fact that $\Delta t \approx h$, we deduce that

$$h\int_0^T\int_{\mathcal{E}(K)} \big|[\![\varrho_h]\!]\overline{\boldsymbol{u}_h}\big[\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_K\rangle_\sigma\big]^-\big|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t$$
$$\lesssim h^{1-\delta/2}\left(h^\delta\int_0^T\int_{\mathcal{E}(K)} [\![\varrho_h]\!]^2|\langle \boldsymbol{u}_h\cdot\boldsymbol{n}_K\rangle_\sigma|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\mathcal{E}(K)} \overline{\boldsymbol{u}_h}^2|\langle \boldsymbol{u}_h\rangle_\sigma|\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\right)^{1/2}$$
$$\lesssim h^{1-\delta/2}\big(h^{-1}\,||\boldsymbol{u}_h||^2_{L^2(0,T;\,L^6(\Omega_h)^d)}\big(||\boldsymbol{u}_h||_{L^\infty(0,T;\,L^{3/2}(\Omega_h)^d)} + h\,||\nabla_h\boldsymbol{u}_h||_{L^\infty(0,T;\,L^{3/2}(\Omega_h)^{d\times d})}\big)\big)^{1/2}$$
$$\lesssim h^{1-\delta/2}\big(h^{-1}(\Delta t)^{-1/2}\,\big(||\boldsymbol{u}_h||_{L^2(0,T;\,L^{3/2}(\Omega_h)^d)} + h\,||\nabla_h\boldsymbol{u}_h||_{L^2(0,T;\,L^{3/2}(\Omega_h)^{d\times d})}\big)\big)^{1/2}$$
$$\lesssim h^{1/4-\delta/2} + h^{3/4-\delta/2}\,. \tag{A.28}$$

Consequently, plugging (A.27) and (A.28) into (A.26), we obtain

$$|J_{2,h}| \lesssim h^{1/2-\delta/4} + h^{1-\delta/4} + h^{1/4-\delta/2} + h^{3/4-\delta/2}\,.$$

- **Term** $|J_{3,h}|$. Applying Hölder's inequality, the first estimate in (A.2), the second estimate in (A.3), the first estimate in (4.14), and the second estimate in (4.16), we conclude that
$$|J_{3,h}| \lesssim h\,||\varrho_h\overline{\boldsymbol{u}_h}||_{L^2(0,T;\,L^2(\Omega_h)^d)}\,||\nabla_h\boldsymbol{u}_h||_{L^2(0,T;\,L^2(\Omega_h)^{d\times d})} \lesssim h^{1-(d+3\delta)/6}\,.$$
- **Term** $|J_{4,h}|$. Employing Hölder's inequality, the first estimate in (4.14), and the second estimate in (4.16), we obtain
$$|J_{4,h}| \lesssim h\,||\varrho_h\overline{\boldsymbol{u}_h}||_{L^2(0,T;\,L^2(\Omega_h)^d)}\,||\mathrm{div}_h(\boldsymbol{u}_h)||_{L^2(0,T;\,L^2(\Omega_h))} \lesssim h^{1-(d+3\delta)/6}\,.$$
- **Term** $|J_{5,h}|$.
Using the first estimate in (A.2) and the third estimate in (4.13), we deduce that
$$|J_{5,h}| \lesssim h^\varepsilon\,||\varrho_h\overline{\boldsymbol{u}_h}||_{L^1(0,T;\,L^1(\Omega_h)^d)} \lesssim h^\varepsilon\,||\varrho_h\overline{\boldsymbol{u}_h}||_{L^\infty(0,T;\,L^{2\gamma/(\gamma+1)}(\Omega_h)^d)} \lesssim h^\varepsilon\,.$$

Keeping in mind that $\boldsymbol{\varphi}$ vanishes on $[0,T]\times(\Omega_h\backslash\Omega)$, we may summarize the previous observations as follows:

$$\left[\int_\Omega(\varrho_h\overline{\boldsymbol{u}_h}\cdot\boldsymbol{\varphi})(t,\cdot)\,\mathrm{d}\boldsymbol{x}\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega\big[\mu\nabla_h\boldsymbol{u}_h:\nabla_{\boldsymbol{x}}\boldsymbol{\varphi} + \nu\,\mathrm{div}_h(\boldsymbol{u}_h)\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\varphi})\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \mathcal{O}(h^{\alpha_2})$$
$$= \int_0^\tau\int_\Omega\big[\varrho_h\overline{\boldsymbol{u}_h}\cdot\partial_t\boldsymbol{\varphi} + \varrho_h\overline{\boldsymbol{u}_h}\otimes\overline{\boldsymbol{u}_h}:\nabla_{\boldsymbol{x}}\boldsymbol{\varphi} + \big(p(\varrho_h\theta_h) + h^\delta\big[\varrho_h^2 + (\varrho_h\theta_h)^2\big]\big)\,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\varphi})\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{A.29}$$

as $h\downarrow 0$ with $\alpha_2 = \min\left\{\varepsilon, \frac{1-2\delta}{4}\right\} > 0$.

The entropy inequality.
Taking $\psi_h^\star(t,\cdot) = \psi_h(t,\cdot) + C_{\mathrm{Stein}}(\Omega_T,1)Dh\,||\psi||_{C^1(\overline{\Omega_T})}$, $t\in[0,\tau]$, as a test function in (4.12) with $\chi = \ln$, we deduce that

$$0 \le \int_0^\tau\int_{\Omega_h} D_t(\varrho_h\ln(\theta_h))\,\psi_h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^\tau\int_{\mathcal{E}_{\mathrm{int}}}\mathrm{Up}[\varrho_h\ln(\theta_h),\boldsymbol{u}_h]\,[\![\psi_h]\!]\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t - \sum_{j=1}^4 H_{j,h}\,, \tag{A.30}$$

where

$$H_{1,h} = \frac{h^\varepsilon}{2}\int_0^\tau\int_{\mathcal{E}_{\mathrm{int}}}[\![\varrho_h]\!]\,[\![\psi_h]\!]\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\,, \qquad H_{2,h} = -\frac{h^\varepsilon}{2}\int_0^\tau\int_{\mathcal{E}_{\mathrm{int}}}[\![\varrho_h]\!]\,[\![\ln(\theta_h)\psi_h^\star]\!]\,\mathrm{d}S_{\boldsymbol{x}}\,\mathrm{d}t\,,$$

$$H_{3,h} = -\frac{h^\varepsilon}{2} \int_0^\tau \int_{\mathcal{E}_{\rm int}} [\![\varrho_h \theta_h]\!] \left[\!\!\left[\frac{\psi_h^\star}{\theta_h} \right]\!\!\right] \, {\rm d}S_{\boldsymbol{x}} \, {\rm d}t \,,$$

$$H_{4,h} = -C_{\rm Stein}(\Omega_T, 1) D h \, ||\psi||_{C^1(\overline{\Omega_T})} \left[\int_{\Omega_h} (\varrho_h \ln(\theta_h))(t, \cdot) \, {\rm d}\boldsymbol{x} \right]_{t=0}^{t=\tau} .$$

Now we may rewrite the first two integrals in (A.30) following the procedure used to handle the continuity equation. The error terms appearing during this process are exactly the same as in the case of the continuity equation with ϱ_h replaced by $\varrho_h \ln(\theta_h)$ and φ_h replaced by ψ_h. However, the analogue of the error term $I_{5,h}$ will not be there since (A.30) contains the usual upwind operator ${\rm Up}[\,\cdot\,,\cdot\,]$ instead of the dissipative upwind operator $F_h^{\rm up}[\,\cdot\,,\cdot\,]$. Since $(\theta_0)_\star \leq \theta_h \leq (\theta_0)^\star$,

$$\begin{aligned} \left| [\![\varrho_h^k \ln(\theta_h^k)]\!]_\sigma \right| &= \left| [\![\varrho_h^k]\!]_\sigma \{ \ln(\theta_h^k) \}_\sigma + \{ \varrho_h^k \}_\sigma [\![\ln(\theta_h^k)]\!]_\sigma \right| = \left| [\![\varrho_h^k]\!]_\sigma \{ \ln(\theta_h^k) \}_\sigma + \frac{1}{\eta_{\theta,k,\sigma}} \{ \varrho_h^k \}_\sigma [\![\theta_h^k]\!]_\sigma \right| \\ &= \left| [\![\varrho_h^k]\!]_\sigma \{ \ln(\theta_h^k) \}_\sigma + \frac{1}{\eta_{\theta,k,\sigma}} \left([\![\varrho_h^k \theta_h^k]\!]_\sigma - [\![\varrho_h^k]\!]_\sigma \{ \theta_h^k \}_\sigma \right) \right| \lesssim \left| [\![\varrho_h^k]\!]_\sigma \right| + \left| [\![\varrho_h^k \theta_h^k]\!]_\sigma \right| \end{aligned}$$

for every $(k, \sigma) \in \mathbb{N} \times \mathcal{E}_{\rm int}$ and suitably chosen values $(\eta_{\theta,k,\sigma})_{\sigma \in \mathcal{E}_{\rm int}} \subset [(\theta_0)_\star, (\theta_0)^\star]$ and, analogously,

$$|D_t(\varrho_h \ln(\theta_h))| \lesssim |D_t \varrho_h| + |D_t(\varrho_h \theta_h)| \,,$$

it is easy handle these error terms. Thus,

$$\left[\int_\Omega (\varrho_h \ln(\theta_h) \psi)(t, \cdot) \, {\rm d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_\Omega \left[\varrho_h \ln(\theta_h) \partial_t \psi + \varrho_h \ln(\theta_h) \overline{\boldsymbol{u}_h} \cdot \nabla_{\boldsymbol{x}} \psi \right] \, {\rm d}\boldsymbol{x} \, {\rm d}t + \mathcal{O}(h^{\alpha_1}) + \sum_{j=1}^4 H_{j,h} \tag{A.31}$$

as $h \downarrow 0$. For the error terms $H_{j,h}$, $j \in \{1, \ldots, 4\}$ we proceed as follows. Since $H_{1,h} = -I_{5,h}$, we have $|H_{1,h}| \lesssim h^{\alpha_1}$. Combining $(\theta_0)_\star \leq \theta_h \leq (\theta_0)^\star$ with Hölder's inequality, the first estimate in (A.2), the second estimate in (4.13) and the first estimate in (4.15), we deduce that

$$|H_{2,h}| \,, |H_{3,h}| \lesssim h^{\varepsilon - 1} \,.$$

Moreover, using $(\theta_0)_\star \leq \theta_h \leq (\theta_0)^\star$, the second information in (4.10) and (A.16), we easily verify that

$$|H_{4,h}| \lesssim h \, ||\psi||_{C^1(\overline{\Omega_T})} \, ||\varrho_h^0||_{L^1(\Omega_h)} \lesssim h \, ||\psi||_{C^1(\overline{\Omega_T})} \, ||\varrho_0||_{L^1(\Omega_h)} \lesssim h \,.$$

Consequently, we may rewrite (A.31) as

$$\left[\int_\Omega (\varrho_h \ln(\theta_h) \psi)(t, \cdot) \, {\rm d}\boldsymbol{x} \right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_\Omega \left[\varrho_h \ln(\theta_h) \partial_t \psi + \varrho_h \ln(\theta_h) \, \overline{\boldsymbol{u}_h} \cdot \nabla_{\boldsymbol{x}} \psi \right] \, {\rm d}\boldsymbol{x} \, {\rm d}t + \mathcal{O}(h^{\alpha_3}) \,,$$

as $h \downarrow 0$ with $\alpha_3 = \min\{\alpha_1, \varepsilon - 1\}$. □

Proof of Theorem 4.3 (Part II). In this part, we turn to the situation in which $\tau \in [0, T]$ is arbitrary. Since the case $\tau = 0$ is trivial, we may assume without loss of generality that $\tau \in (0, T]$. Let $m \in \{1, \ldots, N_T\}$ be the smallest number such that $t_m = m \Delta t \geq \tau$.

The continuity equation.
Using Hölder's inequality, (A.16) and $\Delta t \approx h$, we deduce that

$$\left| \left[\int_\Omega (\varrho_h \varphi)(t, \cdot) \, {\rm d}\boldsymbol{x} \right]_{t=\tau}^{t=t_m} \right| \lesssim \Delta t \, ||\varphi||_{C^1(\overline{\Omega_T})} \, ||\varrho_0||_{L^1(\Omega)} \lesssim h \,.$$

Moreover, employing Hölder's inequality the second and third estimate in (4.13), we see that

$$\begin{aligned} \left| \int_\tau^{t_m} \int_\Omega \varrho_h \partial_t \varphi + \varrho_h \overline{\boldsymbol{u}_h} \cdot \nabla_{\boldsymbol{x}} \varphi \, {\rm d}\boldsymbol{x} \, {\rm d}t \right| &\lesssim \Delta t \, ||\varphi||_{C^1(\overline{\Omega_T})} \left(||\varrho_h||_{L^\infty(0,T;L^1(\Omega_h))} + ||\varrho_h \overline{\boldsymbol{u}_h}||_{L^\infty(0,T;L^1(\Omega_h)^d)} \right) \\ &\lesssim h \,. \end{aligned}$$

Consequently, (4.26) holds for any $\tau \in [0, T]$.

The potential temperature equation and the entropy inequality.
Keeping in mind that $(\theta_0)_\star \leq \theta_h \leq (\theta_0)^\star$, one easily reduces the setting of the potential temperature equation and the entropy inequality to that of the continuity equation.

The momentum equation.
From Hölder's inequality, the third estimate in (4.13) and $\Delta t \approx h$ we deduce that

$$\left| \left[\int_\Omega (\varrho_h \overline{\boldsymbol{u}_h} \cdot \boldsymbol{\varphi})(t, \cdot) \,\mathrm{d}\boldsymbol{x} \right]_{t=\tau}^{t=t_m} \right| \lesssim \Delta t \,||\varphi||_{C^1(\overline{\Omega_T})} \,||\varrho_h \overline{\boldsymbol{u}_h}||_{L^\infty(0,T;\,L^1(\Omega_h)^d)} \lesssim h\,.$$

Furthermore, Hölder's inequality, $\Delta t \approx h$, the first two estimates in (4.14) as well as estimates in (4.13) and (4.15) yield

$$\begin{aligned}
&\left| \int_\tau^{t_m} \int_\Omega \left[\mu \nabla_h \boldsymbol{u}_h : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} + \nu \,\mathrm{div}_h(\boldsymbol{u}_h) \,\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\varphi}) \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| \\
&+ \left| \int_\tau^{t_m} \int_\Omega \left[\varrho_h \overline{\boldsymbol{u}_h} \cdot \partial_t \boldsymbol{\varphi} + \varrho_h \overline{\boldsymbol{u}_h} \otimes \overline{\boldsymbol{u}_h} : \nabla_{\boldsymbol{x}} \boldsymbol{\varphi} + \left(p(\varrho_h \theta_h) + h^\delta \left[\varrho_h^2 + (\varrho_h \theta_h)^2 \right] \right) \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\varphi}) \right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right| \\
&\lesssim (\Delta t)^{1/2} \,||\boldsymbol{\varphi}||_{C^1(\overline{\Omega_T})^d} \left(||\nabla_h \boldsymbol{u}_h||_{L^2(0,T;\,L^1(\Omega_h)^{d\times d})} + ||\mathrm{div}_h(\boldsymbol{u}_h)||_{L^2(0,T;\,L^1(\Omega_h))} \right) \\
&\quad + \Delta t \,||\boldsymbol{\varphi}||_{C^1(\overline{\Omega_T})^d} \left(||\varrho_h \overline{\boldsymbol{u}_h}||_{L^\infty(0,T;\,L^1(\Omega_h)^d)} + ||\varrho_h |\overline{\boldsymbol{u}_h}|^2||_{L^\infty(0,T;\,L^1(\Omega_h))} + ||\varrho_h \theta_h||_{L^\infty(0,T;\,L^\gamma(\Omega_h))}^\gamma \right. \\
&\quad \left. + ||h^{\delta/2} \varrho_h||_{L^\infty(0,T;\,L^2(\Omega_h))}^2 + ||h^{\delta/2} \varrho_h \theta_h||_{L^\infty(0,T;\,L^2(\Omega_h))}^2 \right) \lesssim h^{1/2}\,,
\end{aligned}$$

which implies that (4.27) holds for any $\tau \in [0, T]$. □

References

[1] Chertock, A., Kurganov, A., Lukáčová-Medvid'ová, M., Spichtinger, P., Wiebe, B.: Stochastic Galerkin method for cloud simulation. Math. Clim. Weather Forecast. **5**(1), 65–106 (2019)

[2] Ciarlet, P.G., Raviart, P.A.: General Lagrange and Hermite interpolation in $\mathbb{R}^n$ with applications to finite element methods. Arch. Ration. Mech. Anal. **46**, 177–199 (1972)

[3] Feireisl, E., Gwiazda, P., Świerczewska Gwiazda, A., Wiedemann, E.: Dissipative measure-valued solutions to the compressible Navier-Stokes system. Calc. Var. Partial Differ. Equ. **55**(141) (2016)

[4] Feireisl, E., Hošek, R., Maltese, D., Novotný, A.: Error estimates for a numerical method for the compressible Navier–Stokes system on sufficiently smooth domains. ESAIM Math. Model. Numer. Anal. **51**(1), 279–319 (2017)

[5] Feireisl, E., Karper, T.G., Pokorný, M.: Mathematical Theory of Compressible Viscous Fluids: Analysis and Numerics. Advances in Mathematical Fluid Mechanics. Springer International Publishing AG, Cham (2016)

[6] Feireisl, E., Klein, R., Novotný, A., Zatorska, E.: On singular limits arising in the scale analysis of stratified fluid flows. Math. Models Methods Appl. Sci. **26**(3), 419–443 (2016)

[7] Feireisl, E., Lukáčová-Medvid'ová, M., Mizerová, H., She, B.: Numerical Analysis of Compressible Fluid Flows, volume 20 of MS & A. Springer International Publishing, Berlin (2021)

[8] Feireisl, E., Michálek, M., Karper, T.K.: Convergence of a numerical method for the compressible Navier–Stokes system on general domains. Numer. Math. **134**, 667–704 (2016)

[9] Gallouët, T., Maltese, D., Novotný, A.: Error estimates for the implicit MAC scheme for the compressible Navier–Stokes equations. Numer. Math. **141**, 495–567 (2019)

[10] Gallouët, T., Herbin, R., Latché, J.-C.: A convergent finite element-finite volume scheme for the compressible Stokes problem. Part I: the isothermal case. Math. Comp. **78**(267), 1333–1352 (2009)

[11] Kawashima, S.: Systems of a hyperbolic parabolic type, with applications to the equations of magnetohydrodynamics. Ph.D. thesis, Kyoto University (1983)

[12] Klein, R.: An applied mathematical view of meteorological modelling. In: Applied mathematics entering the 21st century, pp. 227–269. SIAM, Philadelphia (2004)

[13] Kwon, Y.-S., Novotný, A.: Consistency, convergence and error estimates for a mixed finite element-finite volume scheme to compressible Navier-Stokes equations with general inflow/outflow boundary data. IMA J. Numer. Anal. **42**(1), 107–164 (2021)

[14] Lukáčová-Medvid'ová, M., Rosemeier, J., Spichtinger, P., Wiebe, B.: IMEX finite volume methods for cloud simulation. In: Finite Volumes for Complex Applications VIII—Hyperbolic, Elliptic and Parabolic Problems, pp. 179–187. Springer International Publishing, Cham (2017)

[15] Lukáčová-Medvid'ová, M., Schömer, A.: Existence of dissipative solutions to the compressible Navier–Stokes system with potential temperature transport. J. Math. Fluid Mech. **24**(82) (2022)
[16] Lukáčová-Medvid'ová, M., She, B., Yuan, Y.: Error estimates of the Godunov method for the multidimensional compressible Euler system. J. Sci. Comput. **91**(71) (2022)
[17] Maltese, D., Michálek, M., Mucha, P.B., Novotný, A., Pokorný, M., Zatorska, E.: Existence of weak solutions for compressible Navier–Stokes equations with entropy transport. J. Differ. Equ. **261**(8), 4448–4485 (2016)
[18] Pietro, D.A., Ern, A.: Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations. Math. Comp. **79**, 1303–1330 (2010)
[19] Serre, D.: Local existence for viscous system of conservation laws: H^s-data with $s>1 + d/2$. In: Nonlinear PDEs and Hyperbolic Wave Phenomena, volume 526 of Contemp. Math., pp. 339–358. American Mathematical Society, Providence (2010)
[20] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions (PMS-30). Princeton University Press, Princeton (2016)
[21] Zeifang, J., Schütz, J., Kaiser, K., Beck, A., Lukáčová-Medvid'ová, M., Noelle, S.: A novel full-Euler low Mach number IMEX splitting. Commun. Comput. Phys. **27**(1), 292–320 (2019)

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