Limit theorems for ancestral lineages in oriented percolation

Dissertation
zur Erlangung des Grades

"Doktor der Naturwissenschaften"

am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität
in Mainz

Timo Schlüter

geb. in Mainz
Mainz, den 23.11.2021

Datum der mündlichen Prüfung: 29. April 2022

Abstract

In this thesis we will consider two models of random walks in random environment. The first one is a directed random walk on the backbone of oriented percolation generated by the contact process. In Chapter 2 we prove a comparison result between the quenched and the annealed law on the level of constant boxes and use this to prove the existence of a measure Q on the environments that is invariant with respect to the point of view of the particle and absolutely continuous with respect to the a priori measure $\mathbb{P}$. We show that φ, the Radon-Nikodym derivative of Q with respect to $\mathbb{P}$, satisfies a certain concentration property and prove a quenched local limit theorem comparing the quenched law with the annealed law times φ. Additionally we prove an annealed local central limit theorem.

This model was introduced by Birkner et al. [BČDG13] and therein a quenched central limit theorem (CLT) proven. In [Ste17] Steiber improved on these results and proved estimates on the differences between quenched and annealed hitting probabilities of boxes of different sizes which are still growing in N, the number of steps.

The main difficulty stems from the fact that our environment is not i.i.d. and the random walk not uniformly elliptic, in fact not even elliptic. Uniform ellipticity would ensure that a quenched random walk can visit any site. Once it hits the backbone of oriented percolation our quenched random walk is unable to visit sites that lie outside. We overcome this difficulty by introducing so called "social" boxes which guarantee that two random walks starting in the same box can meet in some finite time depending on the size of the box. Furthermore, to deal with the correlation in the environment, we use the fact that the probability for the contact process started from a single site to die out after surviving for n steps falls exponentially in n. This allows us, with high probability, to approximate the original environment with one, where the correlations only have finite range. We can show that the density of "good" boxes is high with high probability. To show the existence of Q we consider the quenched law of the environment as seen from the particle after N steps. We then obtain Q as a weak limit of their Cesàro means along a subsequence. This has the advantage that Q is then invariant with respect to the point of view of the particle by construction. To prove the quenched local limit theorem we introduce hybrid measures and use space-time convolutions of these measures and the quenched law.

The second model is a class of random walks in an environment given by oriented percolation, where we make more general assumptions on the dynamics of the random walk and was introduced in [BČD16]. Here the random walk does not have to stay on the percolation cluster but rather its transition probabilities depend on the environment locally in some general way. Furthermore we assume the transitions to have finite range and, while on the cluster, the transition kernels are suitably close to a deterministic symmetric transition kernel. Lastly we have a symmetry assumption that leads to the annealed mean being 0.

In Chapter 3 we prove a quenched central limit theorem in the regime where the parameter p of the underlying Bernoulli-field, on which the percolation cluster is build, is close to 1. To prove the quenched CLT we control the the square of the quenched average under Lipschitz test functions. To that end we consider the dynamics of two random walks evolving in the same environment and therefore define a suitable regeneration construction for two random walks evolving in the same environment, as well as in independent environments. We use these regeneration times to compare the two different pairs of random walks. It turns out that, although the correlations in the environment have infinite range, they decay exponentially with distance and

we can prove that two random walks in a joint environment, that are far from each other, are behaving like they are in independent environments with high probability. To make use of this we show that two random walks in a joint environment separate fast and spend enough time apart such that we can couple them with two in independent environments for at least $N - N^{b_1}$ of the first N steps, where $0 < b_1 < 1/2$. Our approach to prove the quenched CLT requires us to split the proofs in dimensions $d \geq 2$ and $d = 1$. This is necessary because in $d = 1$ the random walks meet too often and we therefore need to control the time they spend while close to each other. Furthermore, for $d = 1$, we make use of a martingale decomposition for the dynamics of the two random walks evolving in a joint environment and show that cancellations in the predictable process of this decomposition, during the time at which the random walks are close to each other, lead to vanishing error terms.

Contents

Chapter 1

Introduction

The models considered in this thesis both fall under the broad umbrella of random walk in random environment (RWRE). In general RWRE describe the movement of particles in a disordered or inhomogeneous medium.

RWRE can be used to model a wide variety of processes occurring in physics and biology. We give some examples of those to give a more intuitive access to the topic. We begin with examples from physics such as disordered solids where atoms are placed at random points in the space $\mathbb{R}^d$ and thus the potential becomes random. In that case the positions of the atoms constitute a random environment and electrons moving in the solid are the random walks, see [Kir07]. Other examples include crystal growth, see [Tem69], electrical lines of random conductances, see [ABSO81] or transport processes, see [BG90], among a host of possible applications in physics.

Continuing with Biology, every population evolving in an environment where the individual's fitness depends on the spatial position or other factors such as age or size, see [BS09] or [MSDL$^+$17], can be modeled as a random environment. This is the case whenever individuals compete for resources, which results in a higher fitness for individuals in sparsely populated areas and a lower fitness in densely populated areas.

By studying the ancestral lineages of a population we can study, among other interesting things, the spatial distribution, the distribution of types and movement of ancestors of a sample of individuals [BR19]. This is the main motivation behind the models we will introduce later and we will therefore discuss it in a bit more detail.

As discussed in [BR19], we can extract a vast amount of information about a population from its pedigree. Ancestry can be used to identify relatives of a given individual, e.g. by finding the most recent common ancestor of a group of individuals. The ancestral lineages are random walks in random environment, where the environment is given by the population and its evolution over time. The dynamics of the ancestral lineages are then given by the dynamics of the population, i.e. migration and birth or death of individuals.

As a possible application consider ancestral lineages in a haploid stepping stone model. In the stepping stone model the population is arranged on a discrete array, e.g. $\mathbb{Z}$, of colonies of fixed size N. During each generation the old individuals are completely replaced by new individuals and we assign a new individual in colony x a parent in colony y with probability $p(x,y) = p(y-x)$, where $p(\cdot)$ is symmetric. Children assume the types of their parent and then independently of everything else mutate with probability u to a completely new type. Let $\psi(x,y)$ be the probability, in equilibrium, that two individuals randomly drawn from colonies x and y have the same type. Using the fact that ancestral lineages are random walks we can calculate

$$\psi(x,y) = \sum_{k=1}^{\infty}(1-u)^{2k}\mathbb{P}(M = k \mid X_0 = x, Y_0 = y) = \frac{\sum_{k=1}^{\infty}(1-u)^{2k}p_{2k}(x,y)}{N + \sum_{k=1}^{\infty}(1-u)^{2k}p_{2k}(0,0)},$$

where X and Y are random walks moving according to p with starting positions X_0 and Y_0, M is the meeting time of X and Y and p_k is the k-step transition kernel, see e.g. [Saw76] for more details. Now X and Y are the ancestral lineages of the two drawn individuals and M is the time that the first common ancestor is found. For the two individuals to have the same type there can not be any mutations along the ancestral lineages until the first common ancestor. Therefore we get the factor $(1-u)^{2k}$, that is no mutation for k generations for both ancestral lineages. Any mutation before the ancestral lineages meet would lead to different types. To summarize, we can use the behaviour of the ancestral lineages as random walks to determine the probability for two sampled individuals to have the same type.

In this thesis we will dedicate ourself to studying the long time behaviour of ancestral lineages in two models and derive results on the spatial distribution of the ancestors. Note that the evolution of the ancestral lineage depends on the modeling of the population since it depends on the dynamics of the evolution of the population. In the scope of this work we will focus on models with no selection nor mutation but add a spatial component for the individuals and thus introduce migration.

1.1 Discrete time contact process and its relation to oriented percolation

One vital ingredient for the models considered in this work is the discrete time contact process $\eta := (\eta_n)_{n\in\mathbb{Z}}$, since it will be the environment in which the random walks evolve.

It was first introduced in its continuous time version $(\eta_t)_{t\geq 0}$ by Harris in [Har74], where η_t is a random map taking values in $\{0,1\}^{\mathbb{Z}^d}$ that assigns every site $x \in \mathbb{Z}^d$ the value 1 or 0. Harris interpreted η as the spread of an infection. In that context we say that an individual at position x is infected at time t if $\eta_t(x) = 1$ and healthy if $\eta_t(x) = 0$. An infected individual can recover at rate 1 and become healthy, while a healthy individual can be infected by its nearest neighbours (according to the supremum norm) at rate $\lambda > 0$ for each infected neighbour, i.e. this yields the rate

$$r(\eta_t, x) = \begin{cases} 1, & \text{if } \eta_t(x) = 1, \\ \lambda \cdot |\{y \in \mathbb{Z}^d \colon \ \|x-y\|_\infty = 1, \eta_t(y) = 1\}|, & \text{if } \eta_t(x) = 0. \end{cases}$$

at which the state of η_t at site $x \in \mathbb{Z}^d$ is flipped.

Since we think of ancestral lineages we call a site (x, n) *inhabited* if $\eta_n(x) = 1$ and *uninhabited* if $\eta_n(x) = 0$. Similar to the continuous time contact process, in discrete time an uninhabited site will be inhabited in the next generation with probability p if there exits a site in its nearest neighbourhood that is inhabited. And an inhabited site will be uninhabited in the next generation with probability $1 - p$. In this section we will give a precise definition and introduce some known results for the discrete time contact process we will need later on.
The discrete time contact process will be build on an i.i.d. family of Bernoulli random variables for every space-time point $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$, where we refer to $x \in \mathbb{Z}^d$ as the spatial component and $n \in \mathbb{Z}$ as the time. We start with $\omega := \{\omega(x, n)\colon (x, n) \in \mathbb{Z}^{d+1}\}$ a family of i.i.d Bernoulli random variables with $\omega(x, n) \sim \mathrm{Ber}_p$ and $p \in [0, 1]$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus $\omega \in \{0, 1\}^{\mathbb{Z}^{d+1}}$ and $\mathbb{P} \circ \omega^{-1}$ is a probability measure on $\{0, 1\}^{\mathbb{Z}^{d+1}}$. Now we call a site $(x, n) \in \mathbb{Z}^{d+1}$ open if $\omega(x, n) = 1$ and closed otherwise.

Definition 1.1.1 (Directed open path)**.** Let $(x, m), (y, n) \in \mathbb{Z}^{d+1}$ be two sites and $m < n$. Given a realization of an environment ω we call a space-time sequence $(x_m, m), \ldots, (x_n, n) \in \mathbb{Z}^{d+1}$ a *directed open path* starting at (x, m) and ending in (y, n) if $x_m = x$, $x_n = y$, $\|x_k - x_{k-1}\| \le 1$ for all $k = m + 1, \ldots, n$ and $\omega(x_k, k) = 1$ for all $k = m, \ldots, n$. In that case we write $(x, m) \to^{\omega} (y, n)$.

A directed open path consists of nearest neighbour jumps, according to the supremum norm, on only open sites. For $x \in \mathbb{Z}^d$ we set $U(x) \coloneqq \{y \in \mathbb{Z}^d\colon \|x - y\| \le 1\}$, i.e. $U(x)$ is the neighbourhood of x according to the sup-norm. With that, given an environment ω, for a set $A \subset \mathbb{Z}^d$, we define $\eta^{A,m} := (\eta^{A,m}_n)_{n \ge m} := (\eta^{A,m}_n)_{n \ge m}(\omega)$ as the discrete time contact process starting at time $m \in \mathbb{Z}$ from the set A as

$$\eta^{A,m}_m(y) = \mathbb{1}_A(y), \qquad y \in \mathbb{Z}^d,$$

and then iteratively for $n \ge m$

$$\eta^{A,m}_{n+1}(x) = \begin{cases} 1 & \text{if } \omega(x, n+1) = 1 \text{ and } \eta^{A,m}_n(y) = 1 \text{ for some } y \in \mathbb{Z}^d \text{ with } \|x - y\| \le 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.1}$$

i.e. $\eta^{A,m}_n(y) = 1$ if and only if there exists a directed open path starting from some $(x, m) \in A \times \{m\}$ and ending in (y, n), or in short $(x, m) \to^{\omega} (y, n)$. The process $\eta^{A,m}$ starts from the configuration $\omega(x, m) = \mathbb{1}_A(x)$ for all $x \in \mathbb{Z}^d$, while for $n > m$ and $y \in \mathbb{Z}^d$ the $\omega(y, n)$ are again i.i.d. Bernoulli. Note that $\eta^{A,m}$ depends on ω but to shorten the notation we refrain from writing that dependency explicitly every time. Similar to the continuous version we call that a site (x, n) infected or inhabited if $\eta^{A,m}_n(x) = 1$ and healthy or uninhabited otherwise. In light of that we can interpret $\eta^{A,m}$ as a population process, where an individual lives at site (x, n), i.e. $\eta^{A,m}_n(x) = 1$, if there exists a possible parent in the previous generation, i.e. $\eta^{A,m}_{n-1}(y) = 1$ for some $y \in \mathbb{Z}^d$ with $\|x - y\| \le 1$. Sometimes we want to identify $\eta^{A,m}_n$ with the set of inhabited sites at time $n \ge m$.

We define $\tau^{A,m} := \inf\{n \ge m\colon \eta^{A,m}_n = \emptyset\}$, the time at which the contact process started from $A \times \{m\}$ dies out. Theorem 1 from [GH02] tells us that there exists a critical value $p_c \in (0, 1)$ for the success probability such that, the contact process survives with positive probability if and only if $p > p_c$.

Theorem 1.1.2. *For $d \ge 1$ and $p > p_c$ we have $\mathbb{P}(\tau^{A,m} = \infty) > 0$.*

We define $\tau^A := \inf\{n \geq 0 : \eta_n^{A,0} = \emptyset\}$ and in particular for $A = \{\mathbf{0}\}$ we write $\tau^0 = \tau^{\{\mathbf{0}\},0}$ and get $\mathbb{P}(\tau^0 = \infty) > 0$. For the rest of this thesis assume $p > p_c$ and if we require more, say p close to 1, we will specify that. For two sets $A, B \subset \mathbb{Z}^d$ with $A \subset B$ we obtain, by construction of the contact process

$$\eta_0^{A,0} \subset \eta_0^{B,0} \quad \text{and} \quad \eta_n^{A,0} \subset \eta_n^{B,0} \qquad \text{for all } n \in \mathbb{N}.$$

Let μ_n be the distribution of $\eta_n^{\mathbb{Z}^d,0}$, then, by the above monotonicity of the contact process and the Markov property, $\mu_n \leq \mu_0$, i.e. μ_0 stochastically dominates μ_n and by attractiveness $\mu_{n+m} \leq \mu_n$ for all $m \in \mathbb{N}$. Therefore, by compactness of the set of probability measures on $\{0,1\}^{\mathbb{Z}^d}$, there exists a unique weak limit $\nu = \lim_{n\to\infty} \mu_n$. This limit is non-trivial since $p > p_c$ and ν is called the upper invariant measure. This is a well known result; see Chapter IV and Theorem 2.3 in Chapter III from [Lig05] for the definition therein and existence. Thus for $\eta^{\mathbb{Z}^d,m}$, taking $m \to -\infty$, we obtain a stationary process

$$\eta := (\eta_n)_{n\in\mathbb{Z}} := (\eta_n^{\mathbb{Z}^d})_{n\in\mathbb{Z}} \tag{1.1.2}$$

where for a given realization of ω we have $\eta_n(x) = 1$ if and only if for every $m \leq n$ there exists some $y \in \mathbb{Z}^d$ such that $(y,m) \to^\omega (x,n)$. That means $\eta_n(x) = 1$ if there exists an infinitely long open path backwards in time starting from (x,n). In Chapter 2, for notational convenience, we consider the process $\xi = (\xi_n)_n$ on $\{0,1\}^{\mathbb{Z}^d}$, defined by $\xi_n(x) = 1$ iff $(x,n) \to \infty$, i.e. there exists an infinite directed open path connecting (x,n) to ∞, and $\xi_n(x) = 0$ otherwise. Note that $\mathcal{L}((\xi_n)_n) = \mathcal{L}((\eta_{-n})_n)$, since $\eta_{-n}(x) = 1$ iff there exists an open path $(x,-n) \to \infty$ which, in the time-reversed picture, translates to $(x,n) \to \infty$ and $\xi_n(x) = 1$. Therefore the process ξ can be interpreted as the time reversal of the stationary discrete time contact process η defined in (1.1.2). In particular, for any $n \in \mathbb{Z}$ the random field $\xi_n(\cdot)$ is distributed according to the upper invariant measure ν of the discrete time contact process, which is non-trivial in the case $p > p_c$.

We define by

$$\mathcal{C} := \big\{(x,n) \in \mathbb{Z}^d \times \mathbb{Z} : (x,n) \xrightarrow{\omega} \infty\big\} \tag{1.1.3}$$

the *backbone of the space-time cluster of oriented percolation*, i.e. the set of all space-time sites which are connected to "time $+\infty$" by an open directed path, see 1.1.1. Note that $\mathcal{C}$ depends on ω and that we have $\mathbb{P}(|\mathcal{C}| = \infty) = 1$ for $p > p_c$. The process $\xi := (\xi_n)_{n\in\mathbb{Z}}$ satisfies

$$\xi_n(x) = \mathbb{1}_{\mathcal{C}}\big((x,n)\big). \tag{1.1.4}$$

Furthermore for a measure μ on $\{0,1\}^{\mathbb{Z}^d}$ we write $\eta^{\mu,m}$ for the contact process started with initial configuration distributed according to μ at time m.

Lastly we want to mention a useful relation between the upper invariant measure ν and a Bernoulli product measure.

Lemma 1.1.3. *The upper invariant measure of the contact process stochastically dominates a Bernoulli product measure* $Ber_{p'}^{\otimes\mathbb{Z}^d}$ *for some* $p' > 0$.

Lemma 1.1.3 was proven for the continuous time contact process in [LS06] Theorem 1.1.

1.1.1 Connection to oriented percolation

We stick to a short definition that demonstrates a comparison between the discrete time contact process and oriented site percolation. A more detailed description of oriented site percolation and some results can be found in [Lig99]. We define oriented site percolation as a discrete time Markov chain A_n of subsets of $\mathbb{Z}^d$ with percolation parameter $p \in [0,1]$. The evolution of this process is defined by: conditioned on the past of the process $A_0, A_1, \dots, A_n$, for $x \in \mathbb{Z}^d$, the events $\{x \in A_{n+1}\}$ are independent and

$$\mathbb{P}(x \in A_{n+1} \,|\, A_0, A_1, \dots, A_n) = \begin{cases} p, & \text{if } A_n \cap U(x) \neq \emptyset, \\ 0, & \text{if } A_n \cap U(x) = \emptyset, \end{cases}$$

where $U(x)$ again is the neighbourhood of x. With the interpretation that a site x is occupied at time n if $x \in A_n$ and empty if $x \notin A_n$. Comparing oriented percolation with the contact process from equation (1.1.1) the similarities are obvious even in the definitions.

Since we consider two different models in chapters 2 and 3 respectively we present the required results about the contact process for each chapter in the two subsections in Section 1.5.

1.2 Directed random walk on the backbone of oriented percolation

The results presented in this section and the related proofs in Chapter 2 were obtained in a joint article [BBDS21] with Stein Andreas Bethuelsen, Matthias Birkner and Andrej Depperschmidt.

Our goal is to study the directed random walk on the cluster $\mathcal{C}$. This random walk was studied in [BČDG13] in the case that the initial point of the random walk belongs to the cluster. Here we want to compare the annealed and quenched laws for starting points without checking whether they are on the cluster or not. Thus, we define the random walk here differently: It behaves as a simple random walk (which jumps uniformly to one of the sites in the unit ball around the present site) as long as it is not on the cluster and once it hits the cluster it will behave as the random walk from [BČDG13]. For a site $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ we define its *neighbourhood* at time $(n + 1)$ by

$$U(x, n) \coloneqq \{(y, n + 1) : \|x - y\|_\infty \leq 1\}. \tag{1.2.1}$$

Since we mainly consider the sup-norm we set $\|\cdot\| \coloneqq \|\cdot\|_\infty$ for rest of this thesis. Given ω and therefore the random cluster $\mathcal{C}$ and $(y, m) \in \mathbb{Z}^d \times \mathbb{Z}$ we consider here random walks $(X_n)_{n=m,m+1,\dots}$ with initial position $X_m = y$ and transition probabilities for $n \geq m$ given by

$$\mathbb{P}(X_{n+1} = z \,|\, X_n = x, \omega) = \begin{cases} |U(x,n) \cap \mathcal{C}|^{-1} & \text{if } (x,n) \in \mathcal{C} \text{ and } (z, n+1) \in U(x,n) \cap \mathcal{C}, \\ 0 & \text{if } (x,n) \in \mathcal{C} \text{ and } (z, n+1) \notin U(x,n) \cap \mathcal{C}, \\ |U(x,n)|^{-1} & \text{if } (x,n) \notin \mathcal{C}. \end{cases} \tag{1.2.2}$$

We will write P_ω for the conditional law of $\mathbb{P}$ given ω and E_ω for the corresponding expectation. In particular, for the transition probabilities we have

$$P_\omega(X_{n+1} = z \,|\, X_n = x) = \mathbb{P}(X_{n+1} = z \,|\, X_n = x, \omega). \tag{1.2.3}$$

For the above random walk we denote by $P_\omega^{(y,m)}$ its *quenched law* and by $E_\omega^{(y,m)}$ the corresponding expectation. The *annealed (or averaged) law* of that random walk is denoted by $\mathbb{P}^{(y,m)}$ and its expectation by

$\mathbb{E}^{(y,m)}$. Note that for any $A \in \sigma(X_n : n = m, m+1, \dots)$ we have

$$\mathbb{P}^{(y,m)}(A) = \int P_\omega^{(y,m)}(A)\, d\mathbb{P}(\omega). \tag{1.2.4}$$

Annealed and quenched CLT for the random walk defined in (1.2.2) were obtained in [BČDG13] which in particular shows that the quenched and annealed laws after N steps are comparable on the level of boxes of side length $N^{1/2}$. These results were refined in [Ste17, Chapter 3]. In particular (and relevant for our purposes here), he obtained a comparison result between the quenched and annealed laws after N steps on the level of boxes of side length $N^{\theta/2}$ for $\theta \in (0,1)$. We recall this result here in Theorem 2.7.1 below. In fact in [Ste17, Section 3.4] he also studied such comparisons on boxes which grow on even slower scales.

In Theorem 1.1 in [BČDG13] it is shown that the random walk (X_n) starting in $0 \in \mathbb{Z}^d$ at time 0 satisfies an annealed central limit theorem and the limiting law is a non-trivial centred isotropic d-dimensional normal law. In particular its covariance matrix Σ is of the form $\sigma^2 I_d$ for a positive constant σ^2 and the d-dimensional identity matrix I_d. Recall that in [BČDG13] it is assumed that the space-time origin is contained in $\mathcal{C}$ so that the random walk starts and stays on $\mathcal{C}$. This is not a big constraint because the time a random walk needs to hit the cluster $\mathcal{C}$ has exponentially decaying tails; see Lemma 2.11.1.

The annealed CLT from [BČDG13] can be strengthened to an annealed *local* CLT. For a proof of the following theorem we refer to Section 2.2.

Theorem 1.2.1 (Annealed local CLT)**.** *For $d \geq 1$ and Σ as above we have*

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \Big| \mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Sigma}} \exp\Big(-\frac{1}{2n} x^T \Sigma^{-1} x\Big)\Big| = 0. \tag{1.2.5}$$

The main goal is to strengthen this further and prove a quenched local limit theorem which is an analogue of Theorem 1.2.1. In order to state the precise result, we need to introduce some notation. First, for $(y,m) \in \mathbb{Z}^d \times \mathbb{Z}$, we define the *space-time shift operator* σ on Ω by

$$\sigma_{(y,m)}\omega(x,n) \coloneqq \omega(x+y, n+m) \tag{1.2.6}$$

and we write $\xi_m(y;\omega)$ for $\xi_m(y)$ read off from a given realization ω as in (1.1.3) and (1.1.4). We define the transition kernel for the environment seen from the particle (compare this with (1.2.2)) by

$$\mathfrak{R} f(\omega) \coloneqq \sum_{\|y\|\leq 1} g(y;\omega) f(\sigma_{(y,1)}\omega) \tag{1.2.7}$$

acting on bounded measurable functions $f : \Omega \to \mathbb{R}$, where

$$g(y;\omega) \coloneqq \mathbb{1}_{\{\sum_{\|z\|\leq 1}\xi_1(z;\omega)>0,\, \omega(0,0)=1\}} \frac{\xi_1(y;\omega)}{\sum_{\|z\|\leq 1}\xi_1(z;\omega)} + \mathbb{1}_{\{\sum_{\|z\|\leq 1}\xi_1(z;\omega)=0 \text{ or } \omega(0,0)=0\}} \frac{1}{3^d}. \tag{1.2.8}$$

Definition 1.2.2. A measure Q on Ω is called *invariant with respect to the point of view of the particle* if for every bounded continuous function $f : \Omega \to \mathbb{R}$

$$\int_\Omega \mathfrak{R} f(\omega)\, dQ(\omega) = \int_\Omega f(\omega)\, dQ(\omega). \tag{1.2.9}$$

Theorem 1.2.3. *Let $d \geq 3$ and $p \in (p_c, 1]$. Then there exists a unique measure Q on Ω which is invariant with respect to the point of view of the particle satisfying $Q \ll \mathbb{P}$ and the concentration property (2.1.9) below.*

The concentration property tells us that on large boxes $[-M, M]^d$, with high probability (depending on M), the average of the Radon-Nikodym derivative $dQ/d\mathbb{P}$ evaluated for all possible shifts $x \in [-M, M]^d$ of the environment is close to 1.

The main result of Chapter 2 is a quenched local limit theorem which is an analogue of Theorem 1.11 in [BCR16] in the case of our model.

Theorem 1.2.4 (Quenched local limit theorem)**.** *Let $d \geq 3$ and $p \in (p_c, 1]$, let Q be the measure from Theorem 1.2.3 and denote by $\varphi = dQ/d\mathbb{P} \in L_1(\mathbb{P})$ the Radon–Nikodym derivative of Q with respect to $\mathbb{P}$. Then for $\mathbb{P}$ almost every ω we have*

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \left| P_\omega^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) \right| = 0. \tag{1.2.10}$$

Remark 1.2.5 (Uniqueness of Q). It will be proven in Lemma 2.9.1 that the function φ in (1.2.10) is $\mathbb{P}$ almost surely unique. Furthermore, it will follow from the arguments in the proofs (cf. also Remark 2.1.6) that if a measure Q' on Ω is invariant with respect to the point of view of the particle and satisfies $Q' \ll \mathbb{P}$ and (1.2.10) with $\varphi' = dQ'/d\mathbb{P}$ then this measure Q' satisfies the concentration property (2.1.9) as well and thus in particular agrees with Q.

Outlook and open questions While we do exhibit a measure Q which is invariant with respect to the point of view of the particle and absolutely continuous with respect to $\mathbb{P}$, we can currently establish uniqueness only in the class of such measures satisfying the additional property (2.1.9), see Remark 1.2.5. Furthermore, because of non-ellipticity, Q is not equivalent to $\mathbb{P}$, see the discussion in Remark 2.1.6 below. We leave open the questions whether property (2.1.9) is necessary for uniqueness and whether Q is equivalent to $\mathbb{P}$ when restricted to the set $\widetilde{\Omega}$ from (2.1.11) in Remark 2.1.6.

We restrict our analysis to the case $d \geq 3$. This is essentially owed to the fact that Theorem 2.7.1, which we quote from [Ste17, Thm. 3.24], is presently only available under this assumption. It was proved there using an environment exposure technique from [BS02], which was also used by [BCR16], and the proof exploited the fact that in dimension at least 3, two independent random walks will almost surely meet only finitely often, irrespective of the number N of steps they take. In spatial dimension $d = 2$, two independent walks will meet infinitely often, but the number of intersections up to time N grows quite slowly (of order $\log N$). It is conceivable that with substantial technical effort, the proof of [Ste17, Thm. 3.24] and also the results of the present investigation could be adapted to cover the case $d = 2$. We leave this question for future research. For our model, simulations suggest that Theorem 1.2.4 should hold even in spatial dimension $d = 1$. However, it seems that a rigorous analysis of the case $d = 1$ would require a completely new approach.

We prove in Theorem 1.2.4 a quenched local limit theorem for a very specific model of a non-elliptic random walk in a non-trivial dynamic random environment, and our proofs do exploit specific properties of this environment, namely the oriented percolation cluster. However, we think that this environment is prototypical for a large class of dynamic environments which can be "mapped" to it by suitable coarse-graining procedures, see [BČD16], Section 3 and the concrete example in Section 4 there. It seems quite possible that given substantial technical effort, our approach to Theorem 1.2.4 could be extended to the class of environments from [BČD16]. We leave this for future work.

Regeneration times

We recall the regeneration times for this model constructed in [BČDG13]. Recall that we think of the random walks as ancestral lineages. At this point we want to return to that interpretation. Since it is a priori not known which sites are on the cluster, the ancestral lineages are constructed locally with the aim to find times at which the local construction finds a "real" ancestor. For that we introduce new random variables that decide at which possible site the parent lived in the previous generation. For $(x,n) \in \mathbb{Z}^d$ let $\tilde{\omega}(x,n)$ be a uniformly chosen permutation of $U(x,n) \coloneqq \{(y,n+1) \in \mathbb{Z}^d \times \mathbb{Z} \colon \|x-y\| \leq 1\}$ the neighbourhood of x at time $n+1$, independent of everything else. Let $\tilde{\omega}$ be the family of all these permutations. Conditioned on the event that $(x,n) \in \mathcal{C}$ and given a starting site (x,n), k, ω and $\tilde{\omega}$, we define a path $\gamma_k^{(x,n)}$ of length k via

$$\gamma_k^{(x,n)}(j) = \begin{cases} (x,n), & \text{if } j = 0, \\ \tilde{\omega}(\gamma_k(j-1))[1] & \text{if } A_{n,k}(j) = \emptyset \text{ and } j = 1,\dots,k-1 \\ \tilde{\omega}(\gamma_k(j-1))[\min(A_{n,k}(j))] & \text{if } A_{n,k}(j) \neq \emptyset \text{ and } j = 1,\dots,k-1 \\ \tilde{\omega}(\gamma_k(j-1))[1], & \text{if } j = k \end{cases}$$

where here $\gamma_k(j) = \gamma_k^{(x,n)}(j)$ and $A_{n,k}(j) \coloneqq \{i \colon \tilde{\omega}(\gamma_k(j-1))[i] \to^{\omega} \mathbb{Z}^d \times \{n+k-1\}\}$ is the set of all sites in $U(\gamma_k(j-1))$ that are, in ω, connected to some site at time $n+k-1$. That means the path starts at position (x,n) and then chooses the first of the neighbouring sites in the previous generation, so a site in $U(x,n)$, that appears first in the permutation $\tilde{\omega}(x,n)$ if none of them are connected to $\mathbb{Z}^d \times \{n+k-1\}$. As soon as there is a site that is connected to $\mathbb{Z}^d \times \{n+k-1\}$ we restrict our choice in the previous generation to those and choose the first one of them according to $\tilde{\omega}$. Note that once the path finds a site that is connected to $\mathbb{Z}^d \times \{n+k-1\}$ it will stay on such sites. This is iterated for every step replacing (x,n) with its current position until $j = k$, where γ_k just chooses the first coordinate of the permutation. Note that $\gamma_k^{(x,n)}$ only depends on values of $\omega(y,m)$ and $\tilde{\omega}(y,m)$ for $(y,m) \in \mathbb{Z}^d \times \{n,\dots,n+k-1\}$ which means we can decide the position of the path without observing all of the environment ω. We interpret $\gamma_k^{(x,n)}(k)$ as a "potential" ancestor of (x,n) from k generations ago.

A few properties of $\gamma_k^{(x,n)}$ are discussed in Lemma 2.1 and Remark 2.2 in [BČDG13]. Due to those properties we can couple the random walk $(X_k,k)_k$ started in (x,n) with ω and $\tilde{\omega}$ by setting

$$(X_k,k) = \lim_{j\to\infty} \gamma_j^{(x,n)}(k).$$

There exist times at which the random walk and the path obtained from the local construction coincide. Let

$$\begin{aligned} &T_0 \coloneqq 0, \\ &T_j \coloneqq \inf\{k > T_{j-1} \colon \xi(\gamma_k^{(x,n)}(k)) = 1\}, \quad j \geq 1 \end{aligned}$$

with a slight abuse of notation in $\xi(y,m) = \xi_m(y)$ for $(y,m) \in \mathbb{Z}^d \times \mathbb{Z}$. The T_j's are exactly those times and thus the local construction finds a "real" ancestor of (x,n). We call those times regeneration times. In Figure 1.1 we illustrate how the construction finds the regeneration times using only local information about the environment. Let

$$\tau_i = T_i - T_{i-1} \quad \text{and} \quad Y_i \coloneqq X_{T_i} - X_{T_{i-1}},$$

then we have the following lemma from [BČDG13]

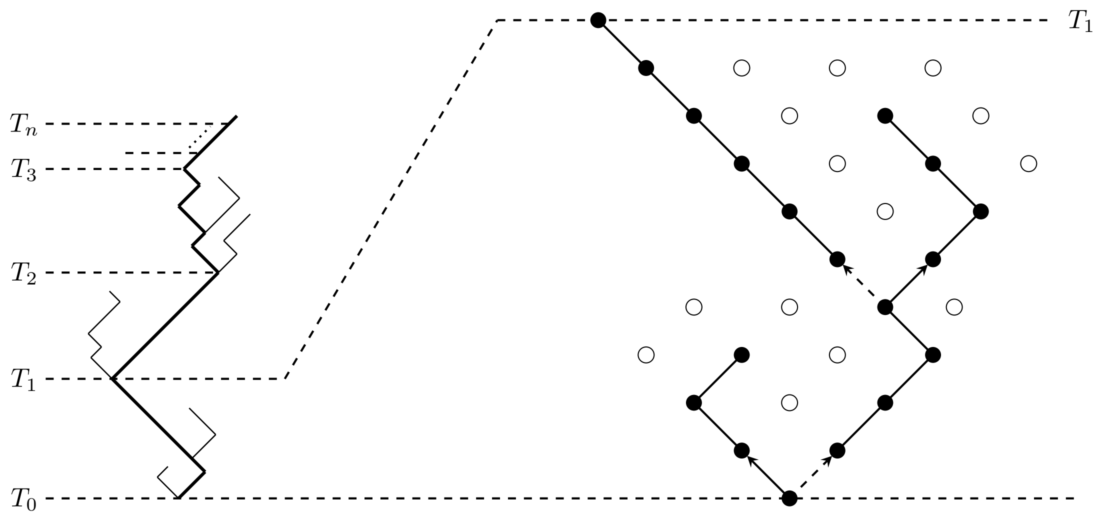

Figure 1.1: The left-hand picture shows how the construction discovers the trajectory of the random walk and the respective regeneration times $T_1, T_2, \dots, T_n$ locally. We restrict the paths and random walk to steps in $U = \{-1, 1\}$. On the right-hand side we zoom in on the environment between T_0 and T_1. There are two sites at which there exist two possibilities for the path to continue. At those sites we drew the first choice, according to $\tilde{\omega}$, with a solid arrow and the second choice with a dashed arrow.

Lemma 1.2.6. *Conditioned on the event $B_{(x,n)} := \{\omega \colon (x, n) \in \mathcal{C}\}$ the sequence $((Y_i, \tau_i))_{i\geq 1}$ is i.i.d. and Y_1 is symmetrically distributed. Furthermore, there exist constants $C, c \in (0, \infty)$, such that*

$$\mathbb{P}(\|Y_1\| > n \,|\, B_{(x,n)}) \leq C\mathrm{e}^{-cn} \quad \text{and} \quad \mathbb{P}(\tau_1 > n \,|\, B_{(x,n)}) \leq C\mathrm{e}^{-cn}.$$

Note that, as was mentioned above, the random walk introduced in [BČDG13] is a random walk on the the cluster $\mathcal{C}$ and thus its initial position has to be on the cluster. Birkner et. al already mention in Remark 2.3 that this can be expanded to a random walk not starting on $\mathcal{C}$. Using the fact that the random walk in our model hits the cluster fast, see Lemma 2.11.1, the same bounds on analogous regeneration times hold for our model.

1.3 A more general class of random walks in oriented percolation

The results presented in this section and the related proofs in Chapter 3 were obtained while working on an upcoming article with Matthias Birkner and Andrej Depperschmidt.

In this section we recall the auxiliary model from [BČD16] and present our main result. We aim to apply this result via coarse graining to the more general class of models that was introduced in Section 3 in [BČD16]. This will be included in the upcoming article with M. Birkner and A. Depperschmidt.

The model is a natural generalization of the model introduced above and in [BČDG13]. We will allow the random walks to step on 0's of the contact process and impose more general assumption on the transitions, e.g. finite range instead of only allowing nearest neighbour jumps. These assumptions also make sense for populations. We allow migration with a bounded range, where the movement is only depended locally on the environment, e.g. available resources and competition. We aim to prove a quenched CLT and the main tool will be to define appropriate regeneration times. In [BČD16] Birkner et al. introduced regeneration times for a single random walk and proved an annealed LLN and CLT. We expand on their idea to define simultaneous regeneration times for two random walks to apply the strategies used in [BČDG13] to prove a quenched CLT. For that we will consider two random walks evolving in the same environment and compare them to two random walks evolving in independent environments. It turns out that, although the environment has correlations with infinite range, these two pairs of random walks will behave almost equally as long as the starting distance (within each pair respectively) is large, see Lemma 3.2.1.

Recall the definition of the discrete time contact process η from (1.1.2) and let $p > p_c$. To define a random walk in the random environment generated by η, or more precisely by its time-reversal, let

$$\kappa \coloneqq \{\kappa_n(x,y) : n \in \mathbb{Z},\, x,y \in \mathbb{Z}^d\} \tag{1.3.1}$$

be a family of random transition kernels defined on the same probability space as η, in particular $\kappa_n(x,\,\cdot\,) \geq 0$ and $\sum_{y\in\mathbb{Z}^d} \kappa_n(x,y) = 1$ holds for all $n \in \mathbb{Z}$ and $x \in \mathbb{Z}^d$. Given κ, we consider a $\mathbb{Z}^d$-valued random walk $X \coloneqq (X_n)_{n=0,1,\ldots}$ with $X_0 = 0$ and transition probabilities given by

$$\mathbb{P}\big(X_{n+1} = y \,\big|\, X_n = x, \kappa\big) = \kappa_n(x,y), \tag{1.3.2}$$

that is, the random walk at time n takes a step according to the kernel $\kappa_n(x,\,\cdot\,)$ if x is its position at time n. We impose the following four assumptions on the distribution of κ. Recall that ξ, which we used above, is the time-reversal of η, i.e. $\eta_{-n}(\cdot) = \xi_n(\cdot)$. Since we aim to apply the results obtained in Chapter 3 to a more general class of models and in there might not exist a time-reversal equivalent to η in those, we will stick with η.

Assumption 1.3.1 (Locality)**.** The transition kernels in the family κ depend locally on the time-reversal of η, that is for some fixed $R_{\mathrm{loc}} \in \mathbb{N}$

$$\kappa_n(x,\cdot) \;\text{ depends only on }\; \big\{\omega(y,-n), \eta_{-n}(y) : \|x-y\| \leq R_{\mathrm{loc}}\big\}. \tag{1.3.3}$$

Assumption 1.3.2 (Closeness to a symmetric reference measure on $\eta_{-n}(x) = 1$)**.** There is a deterministic symmetric probability measure κ_{ref} on $\mathbb{Z}^d$ with finite range $R_{\mathrm{ref}} \in \mathbb{N}$, that is $\kappa_{\mathrm{ref}}(x) = 0$ if $\|x\| > R_{\mathrm{ref}}$, and a suitably small $\varepsilon_{\mathrm{ref}} > 0$ such that

$$\|\kappa_n(x, x+\,\cdot\,) - \kappa_{\mathrm{ref}}(\,\cdot\,)\|_{\mathrm{TV}} < \varepsilon_{\mathrm{ref}} \quad \text{whenever} \quad \eta_{-n}(x) = 1. \tag{1.3.4}$$

Here $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm.

Assumption 1.3.3 (Space-time shift invariance and spatial point reflection invariance)**.** The kernels in the family κ are shift-invariant on $\mathbb{Z}^d \times \mathbb{Z}$, that is, using notation

$$\theta^{z,m}\omega(\,\cdot\,,\,\cdot\,) = \omega(z+\,\cdot\,, m+\,\cdot\,),$$

we have

$$\kappa_n(x,y)(\omega) = \kappa_{n+m}(x+z, y+z)(\theta^{z,m}\omega). \tag{1.3.5}$$

Moreover, if ϱ is the spatial point reflection operator acting on ω, i.e., $\varrho\omega(x,n) = \omega(-x,n)$ for any $n \in \mathbb{Z}$ and $x \in \mathbb{Z}^d$, then

$$\kappa_n(0,y)(\omega) = \kappa_n(0,-y)(\varrho\omega). \tag{1.3.6}$$

Assumption 1.3.4 (Finite range)**.** There is $R_\kappa \in \mathbb{N}$ such that a.s.

$$\kappa_n(x,y) = 0 \quad \text{whenever} \quad \|y-x\| > R_\kappa. \tag{1.3.7}$$

As the main result we obtain the following quenched CLT.

Theorem 1.3.5 (Quenched CLT)**.** *For any $d \geq 1$ one can choose $0 < \varepsilon_{\text{ref}}$ sufficiently small and p sufficiently close to 1, so that if κ satisfies Assumptions 1.3.1–1.3.4 then X satisfies a quenched central limit theorem with non-trivial covariance matrix, i.e. for all bounded, continuous functions f*

$$\lim_{n\to\infty} \mathbb{E}_\omega\Big[f(X_n/\sqrt{n})\Big] = \Phi(f), \qquad \textit{for almost all } \omega, \tag{1.3.8}$$

where Φ is non-trivial normal law.

At first glance it seems unsatisfying that we obtain Theorem 1.3.5 only for ε_{ref} sufficiently small and p close to 1. A natural question arises: What do we gain with this theorem if we have such strict requirements on the parameters? The true value is shown by applying it through a coarse graining argument to a more general class of models.

Under certain assumptions on the environment and random walks, which allow us to compare the environment to oriented percolation on suitably large space-time scales via a coarse-graining construction, we can transfer our result for the random walks in an environment given by oriented percolation to a more abstract setting. This will be done in an upcoming article together with M. Birkner and A. Depperschmidt, but to show where this is roughly going we will present the assumptions. It is not necessary to read this upcoming part to understand the rest of this thesis, but we hope that the similarities to the percolation model can be seen.

These assumptions were stated in [BČD16], where they proved an annealed CLT and transferred this result to a more general class of models and we will state some of them to illustrate the abstract setting. In essence we want the environment to behave similar to oriented percolation when we zoom out, i.e. on a box-level.

Note that the following notation somewhat overlaps with the previous. The objects introduced here are just to illustrate where we aim to go and will not be used in the rest of this thesis. Starting with some notation to formulate the assumptions, let

$$U \coloneqq \{U(x,n) \colon x \in \mathbb{Z}^d, n \in \mathbb{Z}\}$$

be an i.i.d. random field with $U(0,0)$ taking values in some Polish space E_U, e.g. in our model here this would mean $E_U = \{0,1\}$. Furthermore for $R_\eta \in \mathbb{N}$ let $B_{R_\eta} \subset \mathbb{Z}^d$ be the ball of radius R_η around the origin with respect to the sup-norm. Let

$$\varphi \colon \mathbb{Z}_+^{B_{R_\eta}} \times E_U^{B_{R_\eta}} \to \mathbb{Z}_+$$

be a measurable function. Now we can state the first assumption.

Assumption 1.3.6 (Markovian, local dynamics, flow construction)**.** We assume that $\eta := (\eta_n)_{n\in\mathbb{Z}}$ is a Markov chain with values in $\mathbb{Z}_+^{\mathbb{Z}^d}$ whose evolution is local in the sense that $\eta_{n+1}(x)$ depends only on $\eta_n(y)$ for y in a finite ball around x. In particular we assume that η can be realised using the driving noise U as

$$\eta_{n+1}(x) = \varphi(\theta^x \eta_n|_{B_{R_\eta}}, \theta^x U(\cdot, n+1)|_{B_{R_\eta}}), \quad x \in \mathbb{Z}^d, n \in \mathbb{Z}. \tag{1.3.9}$$

Here θ^x denotes the spatial shift by x, i.e. $\theta^x\eta_n(\cdot) = \eta_n(\cdot + x)$ and $\theta^x U(\cdot, n+1) = U(\cdot + x, n+1)$. Furthermore $\theta^x \eta_n|_{B_{R_\eta}}$ and $\theta^x U(\cdot, n+1)|_{B_{R_\eta}}$ are the respective restrictions to the ball B_{R_η}.

The second assumption is what allows us the comparison between η and supercritical oriented percolation. By observing η on the level of space-time boxes we want good configurations to propagate. Furthermore if we have two good configurations at the bottom of a box then good noise inside the box produces a coupled region at the top.

In notation this reads, for $(\tilde{x}, \tilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}$ let

$$\mathrm{block}_m(\tilde{x}, \tilde{n}) := \{(y, k) \in \mathbb{Z}^d \times \mathbb{Z} \colon \; \|y - L_s\tilde{x}\| \le mL_s, \tilde{n}L_t < k \le (\tilde{n}+1)L_t\},$$

i.e. $\mathrm{block}_m(\tilde{x}, \tilde{n})$ is a space-time box placed on $(L_s\tilde{x}, \tilde{n}L_t)$, where we think of $L_t > L_s \gg R_\eta$. Furthermore for $A \in \mathbb{Z}^d \times \mathbb{Z}$ let $U|_A$ be the restriction of the random field U to A.

Assumption 1.3.7 ("Good" noise configurations and propagation of coupling)**.** There exist a finite set of "good" local configurations $G_\eta \subset \mathbb{Z}_+^{B_{2L_s}(0)}$ and a set of "good" local realisations of the driving noise $G_U \subset E_U^{B_{4L_s}(0)\times\{1,2,\dots,L_t\}}$ with the following properties:

- For a suitably small ε_U we have

$$\mathbb{P}\Big(U|_{\mathrm{block}_4(0,0)} \in G_U\Big) \ge 1 - \varepsilon_U.$$

- For any $(\tilde{x}, \tilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}$ and any configurations $\eta_{\tilde{n}L_t}, \eta'_{\tilde{n}L_t} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ at time $\tilde{n}L_t$,

$$\begin{aligned} &\eta_{\tilde{n}L_t}|_{B_{2L_s}(L_s\tilde{x})}, \eta'_{\tilde{n}L_t}|_{B_{2L_s}(L_s\tilde{x})} \in G_\eta \quad \text{and} \quad U|_{\mathrm{block}_4(\tilde{x},\tilde{n})} \in G_U \\ &\Rightarrow \eta_{(\tilde{n}+1)L_t}(y) = \eta'_{(\tilde{n}+1)L_t}(y) \quad \text{for all } y \text{ with } \|y - L_s\tilde{x}\| \le 3L_s \\ &\quad \text{and} \quad \eta_{(\tilde{n}+1)L_t}|_{B_{2L_s}(L_s(\tilde{x}+\tilde{e}))} \in G_\eta \quad \text{for all } \tilde{e} \text{ with } \|\tilde{e}\| \le 1, \end{aligned}$$

  and

$$\eta_{\tilde{n}L_t}|_{B_{2L_s}(L_s\tilde{x})} = \eta'_{\tilde{n}L_t}|_{B_{2L_s}(L_s\tilde{x})} \Rightarrow \eta_k(y) = \eta'_k(y) \text{ for all } (y,k) \in \mathrm{block}(\tilde{x},\tilde{n}),$$

  where $\eta = (\eta_n)$ and $\eta' = (\eta'_n)$ are given by (1.3.9) with the same noise U but possibly different initial conditions.

- There is a fixed reference configuration $\eta^{\mathrm{ref}} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ such that $\eta^{\mathrm{ref}}|_{B_{2L_s}(L_s\tilde{x})} \in G_\eta$ for all $\tilde{x} \in \mathbb{Z}^d$.

For the random walk we ask the following assumptions. Let $X = (X_k)_k$ be the random walk in the random environment generated by η. Moreover let $\widehat{U} := (\widehat{U}(x,k) \colon x \in \mathbb{Z}^d, k \in \mathbb{Z}_+)$ be an independent i.i.d. space-time field of random variables uniformly distributed on (0,1) and let

$$\varphi_X \colon \mathbb{Z}_+^{B_{R_X}} \times \mathbb{Z}_+^{B_{R_X}} \times (0,1) \to B_{R_X}$$

be a measurable function, where $R_X \in \mathbb{N}$ is the range of the jumps the random walk X as well as the range of the dependence. Given η, we define

$$X_{k+1} \coloneqq X_k + \varphi_X\big(\theta^{X_k}\eta_{-k}|_{B_{R_X}}, \theta^{X_k}\eta_{-k-1}|_{B_{R_X}}, \widehat{U}(X_k, k)\big), \quad k = 0, 1, \ldots,$$

with $X_0 = 0$. For the random walk we again assume it to be close to a simple random walk whenever it starts at a good box, i.e. a box with a good starting configuration and good noise.

1.4 Random walks in random environment

In this section we want to introduce the general concept of RWRE. Since the results and methods used in Chapter 2 are inspired by Berger et. al [BCR16] we want to highlight their model as an example for RWRE in this section. This serves two purposes, firstly we introduce a more general class of RWRE that has been well researched and can provide a good entry to RWRE, secondly we can emphasize the differences that required additional work for adjusting their approach to the model we consider later on in Chapter 2. For the proofs of the results in this subsection we refer mostly to Berger et. al [BCR16] and a review of the field by Drewitz and Ramírez [DR14].

Broadly speaking a random walk in random environment is a random experiment with two steps:

1. Choose an environment according to some probability measure.
2. Let a random walk evolve in the chosen environment.

The transition probabilities of a random walk depend on the environment.

We will only consider random walks on $\mathbb{Z}^d$. To illustrate the concept we recall the model introduced by Berger et. al in [BCR16]. Sticking to their notation, let $\mathcal{M}_d$ denote the space of all probability measures on $\mathcal{E}_d = \{\pm e_i\}_{i=1}^d$, the unit vectors, and define $\Omega = (\mathcal{M}_d)^{\mathbb{Z}^d}$. We call an element $\omega \in \Omega$ an environment. Then, for $x \in \mathbb{Z}^d$ and $e \in \mathcal{E}_d$, we identify with $\omega(x, e)$ the probability that $\omega(x)$ gives e. Often we are interested in probability measures P on Ω with the following two properties:

(IID) We call an environment i.i.d. if the coordinate maps on the product space Ω are i.i.d. under P.

(UE) We call an environment uniformly elliptic if there exits a constant $\delta > 0$ such that for all $x \in \mathbb{Z}^d$

$$P(\omega(x, e) > \delta \text{ for all } e \in \mathcal{E}_d) = 1.$$

Note that, if the environment is i.i.d., the property (UE) can be reduced to $P(\omega(0, e) > \delta \text{ for all } e \in \mathcal{E}_d) = 1$. These two properties are the key differences between this model and our models in Chapter 2 and Chapter 3. To deal with them in Chapter 2, we construct tools which enable us to be able to develop the ideas in the proofs from [BCR16]. For this section we assume P has the above properties.

Given an environment ω the quenched random walk is a Markov chain on $\mathbb{Z}^d$. Let $x \in \mathbb{Z}^d$, then the transition probabilities of the so called quenched random walk are

$$P_\omega^x\big(X_{n+1} = y + e \,|\, X_n = y\big) = \omega(y, e), \quad y \in \mathbb{Z}^d, e \in \mathcal{E}_d$$

and its starting position is x, i.e. $P_\omega^x(X_0 = x) = 1$. Whenever we fix an environment and consider a random walk in this environment, we call it quenched random walk and its distribution quenched law. Alongside the quenched law, there is also the so called annealed law. We define the annealed law of the random walk with starting position x as

$$\mathbb{P}^x(\cdot) = \int_\Omega P_\omega^x(\cdot)\, dP(\omega).$$

One process that is of particular interest is the environment viewed from the particle. For $x \in \mathbb{Z}^d$ let σ_x be the shift of ω in direction x, i.e. $\sigma_x \omega(y, \cdot) = \omega(x + y, \cdot)$ for every $y \in \mathbb{Z}^d$.

Definition 1.4.1. Let $(X_n)_n$ be a RWRE. We define the environment viewed from the particle as the discrete time process

$$\bar{\omega}_n = \sigma_{X_n} \omega,$$

for $n \geq 0$, with state space Ω.

Even under the annealed measure this process is Markovian, which was shown by Sznitman [BS02] in the following result.

Proposition 1.4.2. *Consider a RWRE in an environment with law P. Then, under $\mathbb{P}^0$, the process $(\bar{\omega}_n)_n$ is Markovian with state space Ω, initial law P, and transition kernel*

$$Rg(\omega) := \sum_{e \in \mathcal{E}_d} \omega(0, e) g(\sigma_e \omega),$$

defined for g bounded measurable on Ω.

Proof. For the proof we refer to Proposition 1 in [DR14] or [BS02]. □

Having defined $(\bar{\omega}_n)_n$ we can ask about properties of this process. One interesting object to study is the invariant measure for this process.

Definition 1.4.3. A probability measure Q on Ω, endowed with some topology (e.g. for $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ we will choose the product topology), is said to be invariant with respect to the point of view of the particle, if for every bounded continuous function $g \colon \Omega \to \mathbb{R}$

$$\int_\Omega Rg(\omega)\, dQ(\omega) = \int_\Omega g(\omega)\, dQ(\omega). \tag{1.4.1}$$

Furthermore we define the measure QR by the identity

$$\int_\Omega g(\omega)\, d(QR)(\omega) = \int_\Omega Rg(\omega)\, dQ(\omega). \tag{1.4.2}$$

One possible way to construct a measure that is invariant with respect to the point of view of the particle is given in the following lemma from [DR14]. This is also the method used in Chapter 2 to construct such a measure for the model studied there, as well as what Berger et. al used in [BCR16].

Lemma 1.4.4. *Consider a RWRE and the corresponding environmental process $(\bar{\omega}_n)_n$. Then, if Q is any probability measure on Ω, there exists at least one limit measure of the Césaro means*

$$\frac{1}{N+1} \sum_{k=0}^{N} QR^k.$$

Furthermore, every limit measure of these Césaro means is an invariant probability measure for the Markov chain $(\bar{\omega}_n)_n$.

Proof. We use the fact that the space of probability measures defined on Ω is compact under the topology of weak convergence. Therefore we can find a subsequence of the Césaro means that has a limit measure. Since by definition of QR in (1.4.2)

$$\int Rg\, d(QR^k) = \int g\, d(QR^{k+1})$$

we only get a shift inside the sum of the Césaro mean. This does not change the limit and we obtain (1.4.1). For more details see the proof of Lemma 1 in [DR14]. □

A measure Q that is invariant with respect to the point of view of the particle is particularly useful when it is equivalent to the original measure of the environment P. If such a measure exists, it can be used to prove a law of large numbers, see Corollary 1 in [DR14].

In the case that the measure P of the environments is elliptic and ergodic there is the following useful theorem by Kozlov [Koz85] that can be employed to show, among other things, equivalence of another measure ν on the environments and P.

Theorem 1.4.5 (Kozlov)**.** *Assume P is elliptic and ergodic with respect to the shift $\{\sigma_x\}_{x\in\mathbb{Z}^d}$. Assume there exists an invariant probability measure ν for the environment seen from the random walk which is absolutely continuous with respect to P. Then the following hold:*

(i) ν is equivalent to P.

(ii) The environment as seen from the random walk with initial law ν is ergodic.

(iii) ν is the unique invariant probability measure for the environment as seen from the particle which is absolutely continuous with respect to P.

(iv) The Césaro means $\{\frac{1}{N+1}\sum_{k=0}^{N} PR^k\}$ converge weakly to ν.

Proof. See the proof of Theorem 3 in [DR14]. □

Next we want to introduce the concept of ballisticity.

Definition 1.4.6. We call a RWRE ballistic in a given direction $l \in \mathbb{S}^d := \{x \in \mathbb{R}^d\colon\ \|x\|_2 = 1\}$ if $\mathbb{P}^0$-a.s.

$$\liminf_{n\to\infty} \frac{\langle X_n, l\rangle}{n} > 0.$$

Ballisticity ensures the existence of regeneration times for the random walk, see Theorem 2.8 in [BCR16], which are a powerful tool to analyze the long time behaviour of the random walk and are often used to prove laws of large numbers and central limit theorems. Constructing suitable regeneration times for two random walks walking in the same environment will play a central part in Chapter 3. More specifically we will consider directed random walks in random environment. We interpret the last coordinate as time and the random walk will evolve in the positive (in Chapter 2) or negative (in Chapter 3) time direction. Therefore ballisticity holds trivially. Despite this, since the environment is not i.i.d. the construction of suitable regeneration times requires a bit more work.

1.4.1 Random walks in dynamic random environment

In the special case where the environment itself changes over time, and thus the transition probabilities for the random walks also change over time, we speak of random walks in dynamic random environment. This is a subclass of RWRE but deserves its name because of the special role time plays in it. In this case the random walk is ballistic by definition, since it evolves directed in the time direction.

The contact process plays a vital part in the models we will consider in Chapter 2 and Chapter 3. In both chapters the environment in which the random walks evolve will be given by the contact process. As in (1.1.1) can be seen, the last coordinate, which we interpret as time, plays a special role. The contact process itself evolves over time and values at spatial sites can change, and therefore the environment changes over time. Thus, we consider random walks in dynamic random environment in both models.

Random walks in static and dynamic random environments is a very active research area. For a review of random walks in random environments and basic concepts and objects we refer the reader to [Szn04]; for a more recent review see [DR14].

The random walks that we consider here can be seen as a random walks in a dynamic random environment. Comparing the random walk considered in Chapter 2 to random walks in dynamic random environments in the literature we want to briefly mention some examples. In [JRA11] the authors consider environments that are "refreshed in each step", i.e. time slices are i.i.d., which does not hold for the contact process $(\xi_n)_n$. The contact process $(\xi_n)_n$ does not fulfil the uniform coupling conditions used in [RV13]. The main differences to the model considered in [BCR16] are that the random environment is not uniformly elliptic and is not i.i.d. In fact the environment that we consider here even has infinite range dependencies, due to the fact that the steps of the random walk are restricted to the backbone of the oriented percolation cluster once it hits the cluster. The environment also does not satisfy mixing conditions such as (conditional) cone-mixing in contrast to the model considered in [HSS13]. These differences extend to the setting in Chapter 3 since there we consider a generalization of the model from Chapter 2. In [BHT20] a much weaker mixing assumption than cone-mixing is introduced (literally for a continuous time model) and our models satisfy their assumption. However, they only prove a LLN for a nearest neighbour random walk in $d = 1$. Furthermore, in contrast to the models considered in [AHR11], [HdHS$^+$15], [BV16] and [SS18], our models are special in that the random walks and the random environment do not evolve in the same time direction, i.e. in our case forwards in time for the walk means backwards in time for the environment. A comprehensive overview of the recent literature on random walks in dynamic random environments can also be found in the introduction of [BHT20]. See also [BGS19, Remark 1.1].

Results on quenched local limit theorems for random walks in random environments are very recent. Our research in Chapter 2 is inspired by [BCR16] where a quenched local limit theorem was shown (in dimension $d \geq 4$) for the case of an i.i.d. random environment and where the random walk satisfies a ballisticity criterion and has uniformly elliptic transition probabilities. (Note that ballisticity is trivial in our models. Uniform ellipticity and the i.i.d. property are not satisfied.)

Other results on local limit theorems in random environments that we are aware of are concerned with specific models. In [DG19] the quenched local CLT is proven for random walks in a time-dependent balanced random environment. In [DG20b] and [DG20a] quenched local limit theorems are obtained for random walks in random environments on a strip. A different class of random walks in random media for which quenched local CLTs have been obtained are the so called random conductance models. In [ADS16] and [And14] the authors proved local limit theorems for the random conductance model. In [BS20] and [DNS18] a quenched functional CLT was obtained for the static random conductance model and in [ACDS18] for the dynamic random conductance model under ergodic degenerate conductances. For a recent work in this direction and a more detailed overview of the literature see [ACS21] and references therein.

1.5 Collection of results for the contact process

This section aims to accumulate results for the contact process needed in the later chapters.

1.5.1 Relevant results for the contact process for Chapter 2

The objective of this subsection is to collect all results about the contact process that are necessary to understand the proofs in Chapter 2. We want some form of control on the time at which the contact process $\eta^{\{0\},0}$ started from single site, here the origin $(0,0) \in \mathbb{Z}^d \times \mathbb{Z}$, evolved to look like the contact process started from every site. The following lemma from [DG82] gives us exactly that:

Lemma 1.5.1. *There exist positive constants C, c_1 and c_2 such that*

$$\mathbb{P}(\eta_n^{\{0\},0}(x) \neq \eta_n^{\mathbb{Z}^d,0}(x) \,|\, \tau^0 = \infty) \leq C\mathrm{e}^{-c_1 n} \tag{1.5.1}$$

for all $\|x\| \leq c_2 n$.

In [DG82] this was proved for the continuous time contact process but the methods can be adapted to the discrete time version.

Proof. The proof is an adaption of the arguments in [DG82] for the proof of equation (34) therein. ☐

Lemma 1.5.2. *For $p > p_c$ there exist $C, \gamma \in (0, \infty)$ such that*

$$\mathbb{P}(n \leq \tau^0 < \infty) \leq C\mathrm{e}^{-\gamma n} \tag{1.5.2}$$

and

$$\mathbb{P}(\tau^A < \infty) \leq C\mathrm{e}^{-\gamma |A|}. \tag{1.5.3}$$

Proof. Equation (1.5.2) is Lemma A.1 in [BČDG13] and a proof can be found there.
For a proof of (1.5.3) in the continuous case we refer to [Lig99] Theorem 2.30. ☐

1.5.2 Relevant results for the contact process for Chapter 3

We want to establish some results about the contact process that are needed for Chapter 3. The first ones being Theorem 1 and Theorem 2 from [CMS10] for a contact process restricted to a "wedge" in $d = 1$. There proven for the continuous time contact process the results can be adapted to the discrete time contact process. We start with some notation.
For $0 < \alpha_l < \alpha_r < 1$ and $M \geq 0$ we define so called "wedges" with slopes α_l and α_r as well as base M as $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$ by

$$\mathcal{W} = \{(x, n) \in \mathbb{Z} \times \mathbb{N}_0 \colon \alpha_l n \leq x \leq M + \alpha_r n\}.$$

Let $\eta^{\mathcal{W}} = (\eta_n^{\mathcal{W}})_{n\in\mathbb{N}}$ be the contact process restricted to the wedge $\mathcal{W}$, i.e. starting with $\eta_0^{\mathcal{W}} = \mathbb{1}_{[0,M]\cap\mathbb{Z}}$ and for $n \geq 1$ and $x \in \mathbb{Z}$

$$\eta_n^{\mathcal{W}}(x) = \begin{cases} 1, & \text{if } x \in [\alpha_l n, M + \alpha_r n], \omega(x.n) = 1 \text{ and } \eta_{n-1}^{\mathcal{W}}(y) = 1 \text{ for some } y \in U(x) \\ 0, & \text{otherwhise.} \end{cases}$$

Theorem 1.5.3. *Let $p > p_c$ and $0 < \alpha_l < \alpha_r < 1$, then, for $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$ and $\eta_0^{\mathcal{W}} = [0, M] \cap \mathbb{Z}$,*

$$\lim_{M\to\infty} \mathbb{P}(\eta_n^{\mathcal{W}} \neq \emptyset \text{ for all } n \geq 0) = 1 \tag{1.5.4}$$

Thus by increasing the base the probability for survival goes to 1. We can actually be a bit more precise. If $\eta^{\mathcal{W}}$ survives it looks like η^ν, the unrestricted contact process started from the upper invariant measure ν. Let

$$\begin{aligned} r_n^{\mathcal{W}} &:= \max\{x : \eta_n^{\mathcal{W}}(x) = 1\}, \\ l_n^{\mathcal{W}} &:= \min\{x : \eta_n^{\mathcal{W}}(x) = 1\}, \end{aligned} \tag{1.5.5}$$

be the rightmost and leftmost inhabited sites of $\eta^{\mathcal{W}}$ at time n.

Theorem 1.5.4. *Let $p > p_c$ and $0 < \alpha_l < \alpha_r < 1$, $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$ and $\eta_0^{\mathcal{W}} = [0, M] \cap \mathbb{Z}$. On the event $\{\eta_n^{\mathcal{W}} \neq \emptyset \text{ for all } n \geq 0\}$,*

$$\lim_{n\to\infty} \frac{r_n^{\mathcal{W}}}{n} = \alpha_r \text{ and } \lim_{n\to\infty} \frac{l_n^{\mathcal{W}}}{n} = \alpha_l \text{ a.s.} \tag{1.5.6}$$

Furthermore, $\eta_n^{\mathcal{W}}$ and η_n^ν can be coupled so that on the event $\{\eta_n^{\mathcal{W}} \neq \emptyset \text{ for all } n \geq 0\}$,

$$\eta_n^{\mathcal{W}}(x) = \eta_n^\nu(x) \text{ for all } x \in [l_n^{\mathcal{W}}, r_n^{\mathcal{W}}] \text{ for all large } n \text{ a.s.} \tag{1.5.7}$$

Equation (1.5.6) tells us that the rightmost inhabited site is close to the right border of the wedge (analogously for the leftmost), whereas equation (1.5.7) shows exactly that, after we let enough time pass, the contact process restricted to the wedge looks like the contact process started from ν in the area where it "lives".

Next we recall two Lemmas from [BČD16].

Lemma 1.5.5. *Let $\eta^{\{0\},0} = (\eta_n^{\{0\},0})_{n\geq 0}$ and let $\eta^\nu = (\eta_n^\nu)_{n\geq 0}$, where ν is the upper invariant measure. For p sufficiently close to 1 there exist constants $s_{\text{coupl}} > 0, C < \infty, c > 0$ such that*

$$\mathbb{P}(\eta_n^{\{0\},0}(x) = \eta_n^\nu(x) \text{ for all } \|x\| \leq s_{\text{coupl}} n \,|\, \eta_n^{\{0\},0} \neq \emptyset) \geq 1 - Ce^{-cn}, \quad n \in \mathbb{N}. \tag{1.5.8}$$

Lemma 1.5.5 tells us that the contact process started from the origin can be coupled to the contact process started from the upper invariant measure with high probability in a "small" box around the origin. Small is here in quotation marks since the box grows linearly in the time passed but with a small constant s_{coupl}.

Lemma 1.5.6. *For $p < 1$ large enough there exists $\varepsilon(p) \in (0, 1]$ satisfying $\lim_{p\nearrow 1} \varepsilon(p) = 0$ such that for any set $V = \{(x_i, t_i) : 1 \leq i \leq k\} \subset \mathbb{Z}^d \times \mathbb{Z}$ with $t_1 > t_2 > \cdots > t_k$, we have*

$$\mathbb{P}(\eta_t(x) = 0 \text{ for all } (x, t) \in V) \leq \varepsilon(p)^k. \tag{1.5.9}$$

1.6 FKG inequality and Coupling

In this section we want to briefly introduce two useful tools that we will use later on. The first one being the so called FKG inequality, a popular tool in random graphs and percolation theory. This inequality is a correlation inequality that, informally, says that increasing events are positively correlated. We will provide the FKG inequality adjusted to our setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $S = \{0,1\}^{\mathbb{Z}^{d+1}}$ be the state space of a random variable ω on Ω, where we think of x as the spatial coordinate and n the time coordinate, and, for $(x,n) \in \mathbb{Z}^{d+1}$, let $\omega(x,n)$ be i.i.d. Bernoulli random variables with parameter $p \in (0,1)$. We introduce the partial order

$$\omega \leq \omega' \qquad \text{if} \qquad \omega(x,n) \leq \omega'(x,n) \quad \text{for all } (x,n) \in \mathbb{Z}^{d+1}.$$

We call an event A increasing if for $\omega \leq \omega'$

$$\mathbb{1}_A(\omega) \leq \mathbb{1}_A(\omega').$$

The FKG inequality then states that for increasing events A and B we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Coupling is a very potent tool that we will make use of. Therefore we will give a short description on what coupling is to help understand the respective parts where it comes into play. Coupling allows to compare two random variables, more precisely their distributions, by creating a joint construction of them on a common probability space. To be more precise, let X_1 and X_2 be two random variables, each defined on probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ respectively. Then a coupling of X_1 and X_2 is a new probability space $(\Omega_3, \mathcal{F}_3, \mathbb{P}_3)$ on which there exist two random variables Y_1 and Y_2 such that $X_1 =^{\mathrm{d}} Y_1$ and $X_2 =^{\mathrm{d}} Y_2$.

At first glance this does not seem helpful, but this construction becomes particularly interesting if Y_1 and Y_2 are not independent. One way we will make use of this is, that we will couple two random walks by letting them evolve until they meet on a site and force them to stay together from that point on. This does not change the distribution for each of the random walks but they are now highly dependent.

1.7 Outline

The purpose for this section is to give a good overview of Chapters 2 and 3. We will describe the main goals for each section and reference the crucial points.

Outline Chapter 2 The proofs of the main results are long and quite technical. Let us describe the main ideas of the proofs and explain how this chapter is organised: In Section 2.1 we first give several auxiliary results which we then use for the proofs of Theorem 1.2.3 and of Theorem 1.2.4.

Annealed estimates: In Section 2.2 we prove several annealed derivative estimates which build on, and extend somewhat, previous work by [Ste17]. These estimates will be used for the proof of the annealed local CLT, Theorem 1.2.1, also presented in Section 2.2. Starting with Section 2.3 the chapter is devoted to the proofs of the auxiliary results from Section 2.1.

Comparison of the quenched and annealed laws: Lemma 2.1.1, proven in Section 2.3, provides a comparison between the *quenched* and *annealed* laws on the level of large (but finite) boxes. In particular it shows that the total variation distance between $\mathbb{P}(X_N \in \cdot)$ and $P_\omega(X_N \in \cdot)$ on the level of boxes of side length $M \gg 1$ is small with very high probability as $N \to \infty$ in a suitably quantified way; see equation (2.1.1). The starting point of the proof of Lemma 2.1.1 is [Ste17, Theorem 3.24], recalled in Theorem 2.7.1 below, which gives an analogous result for boxes whose size grows like $N^{\theta/2}$ with $0 < \theta < 1$ as $N \to \infty$, and therefore much slower than the diffusive scale $N^{1/2}$. We augment this with an iteration scheme that is guided by the proof of Theorem 5.1 in [BCR16]. The main argument towards the proof of Lemma 2.1.1 is formulated as Proposition 2.3.1. The proof of that proposition is long and relies to a large extent on ideas from [BCR16] and is postponed to Section 2.7. It requires a suitable control of the density of "good" boxes on which an estimate as in equation (2.1.1) from Lemma 2.1.1 holds locally uniformly, see Definition 2.7.2. This deviates from the set-up in [BCR16] because our environment is not i.i.d. and in fact here the boxes are in principle correlated over arbitrary lengths, albeit weakly.

Measure for the point of view of the particle: The function $\varphi = dQ/d\mathbb{P}$ from (1.2.10) is the density of a measure Q which is invariant with respect to the point of view of the particle and absolutely continuous with respect to $\mathbb{P}$. For the existence of such a measure Q we consider the quenched laws Q_N of the environment seen from the particle after N steps of the walk; see (2.1.4). The measure Q is constructed as a weak limit of the Cesàro average of the measures Q_N along a subsequence; see (2.1.6) and (2.1.8). In Proposition 2.1.2 and Corollary 2.1.4 we show that averages of $dQ_N/d\mathbb{P}$ and $dQ/d\mathbb{P}$ over large boxes are close to one with high probability depending on the size of the boxes. It will turn out that the measure Q which we obtain as described above is unique, i.e. it does not depend on the particular subsequence; see Remark 2.1.6.

Proposition 2.1.2 and Corollary 2.1.4 are proven in Section 2.4. To this end we construct a coupling of Q_N and P_N, the law of the environment viewed relative to the annealed walk (note that $P_N = \mathbb{P}$ for all N). Lemma 2.1.1 allows for a coupling which puts both walks in the same M-box with very high probability. We strengthen this to a coupling which puts both walks at exactly the same spatial position with uniformly non-vanishing probability; see the proof of Lemma 2.4.3.

Since we average over the environment in the definition of the annealed law of the random walk in equation (1.2.4) it is clear that the annealed random walk does not see any specific environment. In contrast to that the quenched random walk knows the exact environment it walks in. So, to compare the annealed and quenched laws of the random walk, the annealed walk needs to see the environment of the quenched random walk. This is done through reweighting by φ. In particular, a consequence of multiplying the annealed law with φ is that this product will be zero for all space-time point $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ in which the contact process ξ is 0 in the environment ω.

In Proposition 2.1.8 we show that the annealed law of the random walk at time n reweighted with the function φ converges for almost all ω to a probability law on $\mathbb{Z}^d$. It is proven in Section 2.5.

In Lemma 2.9.1 we will see that a prefactor φ satisfying (1.2.10) is unique. A quite general proof of that result is given in Section 2.9.

Hybrid measures: For the proof of Theorem 1.2.4, instead of comparing the quenched and annealed laws directly, we use the triangle inequality, some "hybrid" measures and space-time convolutions of quenched-annealed measures; see Definition 2.1.7. In Proposition 2.1.9, proven in Section 2.6, we show that the total variation distance of some of these measures converges to 0 as n, the number of steps of the random walk, goes to infinity. An essential tool of the proof of Proposition 2.1.9 is Lemma 2.6.1 in which we study the total variation distance of quenched laws of two random walks starting at different positions. The idea is to use couplings with the annealed measures on the level of large (growing) boxes combined with annealed derivative estimates in order to first ensure that the two walks are in the same box with probability bounded away from 0. Using connectivity properties of the oriented percolation clusters (see below) the above described procedure can be iterated to produce a literal coupling where the two walks coincide with high probability after sufficiently many steps. Lemma 2.6.1 is proven in Section 2.8.

Oriented percolation results: In Section 2.10, we show that two infinite percolation clusters intersect with high probability within a finite time. This result was pointed out in [GH02], who proved that two infinite clusters do intersect almost surely, but without the quantification of the time of intersection. Finally, in Section 2.11, we show that the probability that a random walk started off the cluster does not hit the cluster within time t decays exponentially with t.

Outline of Chapter 3 To give a good overview of Chapter 3 we describe here the steps to proof Theorem 1.3.5.

Regeneration construction: In Section 3.1 we provide a framework and the necessary auxiliary tools to prove the quenched CLT. We follow the ideas used in [BČDG13] and expand the regeneration construction introduced in [BČD16] to two random walks. An essential tool in the construction of regeneration times of a random walk in [BČD16] was a cone based at the current position of the random walk. This cone consists of an inner and outer cone and the region between the inner and the outer cone is referred to as the cone shell. This is needed in the construction because the cone shell separates the information collected by the random walk on the random environment inside the inner cone and outside the outer cone.

The construction is expanded to a double cone (see Figure 3.1) and the cone shell is extended to a double cone shell: we consider two cones based at the current respective positions of the two random walks, see (3.1.4). This object is used for the definition of joint regeneration times of two random walks and to combine ideas from [BČDG13] and [BČD16]. Similarly to the case of the single walk, the double cone shell separates the information collected by the two random walks together on the random environment inside the inner cones and outside the outer cones of the double cone.

Next we define a sequence of stopping times $(\sigma_n^{\mathrm{sim}})_n$ at which the reasons for 0's of the contact process η in the vicinity of the two random walks are explored and thus no "negative correlation" carries over to the future of the random walks. We prove that the increments of this sequence have exponential tails.

Intuitively the cone shell together with the sequence σ^{sim} isolate the part of the environment that the random walks have explored from everything outside the shell and this is a central building block for the construction of regeneration times.

Finally we define regeneration times and in Lemma 3.1.15 we show that their increments have polynomial tails with exponent $-\beta < 0$. Furthermore, β is large when p is close to 1 and $\varepsilon_{\mathrm{ref}}$ is small. The details can be found at the end of the proof of Lemma 3.1.15.

Auxiliary results: In Section 3.2 we prove some auxiliary results that are useful for all dimensions and give the main proposition, Proposition 3.2.2, that we use to prove Theorem 1.3.5.

The QCLT 1.3.5 in dimensions $d \geq 2$: In Section 3.3 we prove the quenched CLT, Theorem 1.3.5, for $d \geq 2$. The reason we need to split the proof for dimension $d = 1$ essentially boils down to the fact that the random walks meet too often for $d = 1$ and thus we require more detailed calculations. The arguments are however robust enough to treat the cases $d = 2$ and $d \geq 3$ together.

We prove a comparison result between two random walks in the same environment and in two independent environments in Lemma 3.2.1. This lemma enables us to couple two random walks in the same environment with two independent random walks as long as their starting positions have a large enough distance. The main tool for the proof of the quenched CLT, Theorem 1.3.5, is Proposition 3.2.2, which proves a quenched CLT along regeneration times and most of Section 3.3 is used to prove this proposition. Equation (3.2.6) hints to the reason why we need to be able to control the behaviour of two random walks in the same environment: expand the square in the expectation in order to see that once we can replace two walks in the same environment by two walks in independent environments up to a small error term, we obtain bounds on the variance of the quenched transition operator.

To ensure that we can effectively use the coupling argument provided by Lemma 3.2.1 the random walks need to spend most of the time at enough distance to each other. Lemma 3.3.1 tells us, that the random walks will separate "fast enough" to a suitable distance at which we can start the coupling argument. These arguments are made more specific in Lemma 3.3.2 and its proof.

Lastly we need to make sure that the random walk behaves well between regeneration times and that the convergence to a normal distribution carries over.

The special case for $d = 1$*:* Section 3.4 aims to fill in the gaps to prove Proposition 3.2.2 for $d = 1$. Note that the proof of Theorem 1.3.5 then does not require any additional work for $d = 1$. The main difficulty arises from the fact that two random walks will meet often in $d = 1$. Thus, we need to calculate the duration of suitable "excursions of separation" during which the random walks have a certain minimal distance to each other so that we can use the coupling argument via Lemma 3.2.1. We split the time axis into 2 alternating "phases", a "black box phase", where the random walks are close to each other and we cannot use the coupling argument and a "white box phase" where we know that the random walks have at least a certain distance, see definitions (3.4.10) and (3.4.11). We show that the pair of random walks will spend most the time in the "white box phase", more precisely during n steps the number R_n, see (2.7.8), of steps spent in a "black box" is of order $o(n)$ and a certain moment condition holds, see Lemma 3.4.1. To prove convergence to a normal law along the joint regeneration times we consider Doob decompositions of X and X'. Using a quantitative version of a martingale CLT from [Rac95] and Lemma 3.4.1 we obtain upper bounds on the decompositions which are good enough to carry over the convergence to a normal distribution to X and X' along joint regeneration times.

Contributions

Since both Chapter 2 and Chapter 3 are based on a joint work with other Authors, I want to point out my own contributions. For Chapter 2 I adapted the results from [BCR16] to our model working around the fact that we do not have uniform ellipticity and i.i.d. environments. For Chapter 3 I expanded the construction for the regeneration times to two random walks which allowed me to adapt the results from [BČDG13] to the model from [BČD16] and to then go on and prove the QCLT.

Chapter 2

Quenched local limit theorem for a directed random walk on the backbone of oriented percolation

This chapter is mainly concerned with proving equation (1.2.10), that is

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \left| P_\omega^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)\right| = 0,$$

from Theorem 1.2.4, a quenched local limit theorem for a directed random walk on directed percolation.

We show the existence of a measure Q on Ω that is invariant with respect to the point of view of the particle. Furthermore it is absolutely continuous with respect to $\mathbb{P}$ and the Radon–Nikodym derivative $dQ/d\mathbb{P}$ satisfies a certain concentration property (2.1.9). Moreover we show that $\varphi = dQ/d\mathbb{P}$ is the unique prefactor satisfying (1.2.10).

We start by refining a comparison between quenched and annealed law on slowly growing boxes, see Theorem 3.24 in [Ste17], to boxes of constant size.

2.1 Proofs of the main results

In this section we collect several important auxiliary results and present towards the end of this section how to utilise them to prove Theorem 1.2.3 and Theorem 1.2.4. The proofs of the auxiliary results are postponed to the subsequent sections.

Our starting point is a lemma which can be seen as an adaptation of Theorem 5.1 in [BCR16] to our setting. Recall between (1.2.3) and (1.2.4) the definitions of the quenched measure $P_\omega^{(x,m)}$ and the annealed measure $\mathbb{P}^{(x,m)}$ for the random walk $(X_n)_{n=m,m+1,\dots}$ with $X_m = x$. For any positive real number L we denote by Π_L a partition of $\mathbb{Z}^d$ into boxes of side length $\lfloor L \rfloor$.

Lemma 2.1.1. *Let $d \geq 3$. For $N, M \in \mathbb{N}$, $c, C > 0$ denote by $K(N) := K(N, M, c, C)$ the set of environments*

$\omega \in \Omega$ such that for every $x \in \mathbb{Z}^d$ satisfying $\|x\| \le N$

$$\sum_{\Delta \in \Pi_M} \big| P_\omega^{(x,0)}(X_N \in \Delta) - \mathbb{P}^{(x,0)}(X_N \in \Delta) \big| \le \frac{C}{M^c} + \frac{C}{N^c}. \tag{2.1.1}$$

If $c > 0$ is small enough and $C < \infty$ large enough, there are universal positive constants $\tilde{c}$, $\widetilde{C}$, for which we have

$$\mathbb{P}\big(K(N)\big) \ge 1 - \widetilde{C} N^{-\tilde{c} \log N} \quad \textit{for all } N. \tag{2.1.2}$$

In words, Lemma 2.1.1 shows that the total variation distance between the annealed measure $\mathbb{P}^{(x,0)}(X_N \in \cdot)$ and the quenched measure $P_\omega^{(x,0)}(X_N \in \cdot)$ on the level of boxes of side length $M \gg 1$ is small with very high probability as $N \to \infty$. The proof of Lemma 2.1.1 is given in Section 2.3. It builds on a preliminary result by Steiber [Ste17, Theorem 3.24] which we recall in Theorem 2.7.1 below. The latter gives an analogous result to Lemma 2.1.1 for boxes of side length $N^{\theta/2}$ with $0 < \theta < 1$ for large N. In particular, for $N \to \infty$ the side length of these boxes grows much more slowly than the diffusive scale $N^{1/2}$.

Lemma 2.1.1 allows to construct a coupling of the quenched walk under $P_\omega^{(x,0)}$ and the annealed walk under $\mathbb{P}^{(x,0)}$ which puts both walks in the same M-box with very high probability. We strengthen this coupling to a coupling which puts both walks at exactly the same spatial position with uniformly non-vanishing probability; see Lemma 2.4.3 below. This, in turn, is essential for the next statement which concerns the difference between the annealed and quenched law of the environment viewed relative to the walk after N steps, which we denote by P_N and Q_N respectively. More precisely, for $N \in \mathbb{N}$, we define Q_N and P_N by

$$P_N(A) := \mathbb{E}\Big[\sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) \mathbb{1}_{\{\sigma_{(x,N)}\omega \in A\}} \Big] \tag{2.1.3}$$

and

$$Q_N(A) := \mathbb{E}\Big[\sum_{x \in \mathbb{Z}^d} P_\omega^{(0,0)}(X_N = x) \mathbb{1}_{\{\sigma_{(x,N)}\omega \in A\}} \Big]. \tag{2.1.4}$$

Note that, in fact we have $P_N = \mathbb{P}$ for all N; see (2.4.9).

The following proposition is proven in Section 2.4.

Proposition 2.1.2. *For $M \in \mathbb{N}$ let $\Delta_0(M)$ denote a d-dimensional cube of side length M in $\mathbb{Z}^d$ centred at the origin. There exists a universal constant $c > 0$ so that for every $\varepsilon > 0$ there is $M_0 = M_0(\varepsilon) \in \mathbb{N}$ so that for $M \ge M_0$ and all $N \in \mathbb{N}$*

$$\mathbb{P}\Big(\Big| \frac{1}{|\Delta_0(M)|} \sum_{x \in \Delta_0(M)} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega) - 1 \Big| > \varepsilon \Big) \le M^{-c \log M}. \tag{2.1.5}$$

Corollary 2.1.3. *Let $d \ge 3$ and $p > p_c$. Then, for every $k \in \mathbb{N}$, $\sup_N \mathbb{E}[(\frac{dQ_N}{d\mathbb{P}})^k] < \infty$.*

Proof. For $M \in \mathbb{N}$ large enough, Proposition 2.1.2 implies

$$\mathbb{P}\Big(\frac{dQ_N}{d\mathbb{P}}(\omega) > 2(2M+1)^d \Big) \le \mathbb{P}\Big(\frac{1}{(2M+1)^d} \sum_{x \in \{-M,\dots,M\}^d} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,n)}\omega) > 2 \Big) \le M^{-c \log M},$$

which implies the assertion. ∎

We equip Ω with the product topology and consider the Cesàro sequence

$$\widetilde{Q}_n := \frac{1}{n}\sum_{N=0}^{n-1} Q_N, \quad n = 1, 2, \dots. \tag{2.1.6}$$

Using Corollary 2.1.3 and the Cauchy-Schwarz inequality for some finite positive constant $\tilde{c}$ we have

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{N=0}^{n-1}\frac{dQ_N}{d\mathbb{P}}\Big)^2\Big] = \frac{1}{n^2}\sum_{N,N'=0}^{n-1}\mathbb{E}\Big[\frac{dQ_N}{d\mathbb{P}}\frac{dQ_{N'}}{d\mathbb{P}}\Big] \le \tilde{c}. \tag{2.1.7}$$

For $\varepsilon > 0$ let $K \subset \Omega$ be a compact subset such that $\mathbb{P}(K^{\mathsf{c}}) < \varepsilon$. Then by the Cauchy-Schwarz inequality we obtain

$$\widetilde{Q}_n(K^{\mathsf{c}}) = \int_\Omega \mathbb{1}_{K^{\mathsf{c}}}\frac{d\widetilde{Q}_n}{d\mathbb{P}}\, d\mathbb{P} \le \sqrt{\tilde{c}}\mathbb{P}(K^{\mathsf{c}})^{1/2} = \sqrt{\tilde{c}\varepsilon}.$$

Thus, the sequence $(\widetilde{Q}_n)_n$ is tight. In particular, there is a weakly converging subsequence, say $(\widetilde{Q}_{n_k})_k$, and we set

$$Q := \lim_{k\to\infty}\widetilde{Q}_{n_k}. \tag{2.1.8}$$

A standard argument shows that Q is invariant with respect to the point of view of the particle; see Proposition 1.8 in [Lig85] for an abstract argument or the proof of Lemma 1 in [DR14] for the argument in the case of random walks in random environments.

The proof of the following analogue of Proposition 2.1.2 for Q instead of Q_n is given in Section 2.4.

Corollary 2.1.4. *Recall the notation of Proposition 2.1.2 and let Q be the measure obtained as a limit in (2.1.8). There exists a universal constant $c > 0$ so that for every $\varepsilon > 0$ there is $M_0 = M_0(\varepsilon) \in \mathbb{N}$ and for every $M \ge M_0$ we have*

$$\mathbb{P}\Big(\Big|\frac{1}{|\Delta_0(M)|}\sum_{x\in\Delta_0(M)}\frac{dQ}{d\mathbb{P}}(\sigma_{(x,0)}\omega) - 1\Big| > \varepsilon\Big) \le M^{-c\log M}. \tag{2.1.9}$$

Proof of Theorem 1.2.3. By construction and shift invariance of $\mathbb{P}$ we have $Q_N \ll \mathbb{P}$ for every N and therefore $\tilde{Q}_n \ll \mathbb{P}$ for every n. Furthermore, by (2.1.7) the family of Radon-Nikodym derivatives $(d\tilde{Q}_n/d\mathbb{P})_{n=1,2,\dots}$ is uniformly integrable. These facts together imply that we also have $Q \ll \mathbb{P}$ for any Q obtained as in (2.1.8). The concentration property is the assertion of Corollary 2.1.4. For the question of uniqueness of Q see Remark 2.1.6 below. □

Remark 2.1.5. Using shift-invariance of $\mathbb{P}$, it is easy to see that for Q_N from (2.1.4) a version of $dQ_N/d\mathbb{P}$ is given by

$$\varphi_N(\omega) = \sum_{x\in\mathbb{Z}^d} P_\omega^{(-N,x)}(X_0 = 0) \tag{2.1.10}$$

(we have $P^{(0,0)}_{\sigma_{(-x,-N)}\omega}(X_N = x) = P_\omega^{(-N,-x)}(X_0 = 0)$, recall the notation introduced below (1.2.3)). This formula is the analogue of [BS02, Proposition 1.2] in our context. In particular, φ_N is a local function of the space-time values of ξ which themselves can be obtained as limits of local functions of ω. Thus, $dQ/d\mathbb{P}$ can be considered as an almost sure limit of local functions of ω.

Remark 2.1.6 (Uniqueness of invariant $Q \ll \mathbb{P}$ with concentration properties of the density).
A measure Q obtained as in (2.1.8) may in principle depend on a particular subsequence. In the proof of Theorem 1.2.4 we will show that the density $\varphi = dQ/d\mathbb{P}$ of any measure Q satisfying the concentration property (2.1.9) also satisfies (1.2.10). By Lemma 2.9.1 below, such a measure is unique. In particular, in (2.1.8) we have weak convergence towards the unique Q along any subsequence and therefore we have weak convergence of the Cesàro sequence $(\widetilde{Q}_n)_{n\in\mathbb{N}}$ from (2.1.6) towards Q. However, we currently do not know whether the sequence $(Q_N)_{N\in\mathbb{N}}$ from (2.1.4) converges itself.

Using Lemma 2.11.1 and (2.1.10) from Remark 2.1.5 one can show that Q is concentrated on

$$\widetilde{\Omega} = \{\omega \in \Omega : \omega \text{ contains a doubly infinite directed open path through } (0,0)\} \tag{2.1.11}$$

and thus Q is not equivalent to $\mathbb{P}$ because $0 < \mathbb{P}(\widetilde{\Omega}) < 1$. Note that Kozlov's classical argument concerning equivalence, see e.g. [DR14, Thm. 2.12], does not apply because our walks are not elliptic. We do not know whether Q is equivalent to $\mathbb{P}(\,\cdot\,|\widetilde{\Omega})$.

To prove Theorem 1.2.4 we want to make use of the good control of the difference between the quenched and annealed law on the level of boxes and various properties of the prefactor φ that we have formulated above in Lemma 2.1.1 and Corollary 2.1.4. Furthermore, instead of comparing $\mathbb{P}^{(0,0)}(X_N \in \cdot)$ and $P_\omega^{(0,0)}(X_N \in \cdot)$ directly, we compare both of these two measures with auxiliary "hybrid" measures which are introduced in the following definition.

Definition 2.1.7. Let Q be the measure on Ω defined in (2.1.8), which by Theorem 1.2.3 and its proof is invariant with respect to the point of view of the particle with $Q \ll \mathbb{P}$. Let $\varphi = dQ/d\mathbb{P}$ be the corresponding Radon-Nikodym derivative. For $\omega \in \Omega$ and a given partition Π of $\mathbb{Z}^d$ into boxes of a fixed side length we define the following measures on $\mathbb{Z}^{d+1}$:

$$\nu_\omega^{\mathrm{ann}\times\mathrm{pre}}(x,n) := \nu_\omega^{\mathrm{ann}\times\mathrm{pre}}(\{(x,n)\}) := \frac{1}{Z_{\omega,n}}\mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega), \tag{2.1.12}$$

$$\nu_\omega^{\mathrm{que}}(x,n) := \nu_\omega^{\mathrm{que}}(\{(x,n)\}) := P_\omega^{(0,0)}(X_n = x), \tag{2.1.13}$$

$$\nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}(x,n) := \nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}(\{(x,n)\}) := P_\omega^{(0,0)}(X_n \in \Delta_x)\frac{\varphi(\sigma_{(x,n)}\omega)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n)}\omega)}. \tag{2.1.14}$$

Here, $Z_{\omega,n} = \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)$ is the normalizing constant in (2.1.12) and Δ_x in (2.1.14) is the unique d-dimensional box that contains x in the partition Π.

All of the measures introduced in the above definition are different measures of the random walk after n steps: $\nu_\omega^{\mathrm{ann}\times\mathrm{pre}}(\cdot,n)$ is the annealed measure with a prefactor, $\nu_\omega^{\mathrm{que}}(\cdot,n)$ is the quenched measure and $\nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}(\cdot,n)$ is a "hybrid" measure, where the box is chosen according to the quenched measure but then the point inside the box is chosen according to the (annealed) normalised prefactor. Of course the measure $\nu_\omega^{\mathrm{box-que}\times\mathrm{pre}}(\cdot,n)$ does depend on the particular partition Π but it will be clear from the context which partition is used.

First we study the behaviour of the normalizing constant in (2.1.12); see Section 2.5 for a proof of the following result.

Proposition 2.1.8. *For $\mathbb{P}$-almost all $\omega \in \Omega$ the normalizing constant $Z_{\omega,n}$ satisfies*

$$\lim_{n\to\infty} Z_{\omega,n} = 1. \tag{2.1.15}$$

The following proposition is the key result for the proof of Theorem 1.2.4. It states that for large n the above introduced measures are close to each other in a suitable norm. To state this precisely, for $\omega \in \Omega$ and any two probability measures ν^1_ω and ν^2_ω on $\mathbb{Z}^d \times \mathbb{Z}$ (more precisely these are transition kernels from Ω to $\mathbb{Z}^d \times \mathbb{Z}$) let the *$L^1$ distance of ν^1_ω and ν^2_ω* at time $n \in \mathbb{Z}$ be defined by

$$\left\|\nu^1_\omega - \nu^2_\omega\right\|_{1,n} := \sum_{x\in\mathbb{Z}^d} |\nu^1_\omega(x,n) - \nu^2_\omega(x,n)|. \tag{2.1.16}$$

Furthermore, for $k \le n$ the *space-time convolution of ν^1_ω and ν^2_ω* is defined by

$$(\nu^1 * \nu^2)_{\omega,k}(x,n) := \sum_{y\in\mathbb{Z}^d} \nu^1_\omega(y, n-k)\nu^2_{\sigma_{(y,n-k)}\omega}(x-y,k). \tag{2.1.17}$$

We can interpret (2.1.17) as follows: A random walk takes $n-k$ steps in the random medium ω according to ν^1_ω, then re-centers the medium at its current position in space-time and takes the remaining k steps according to ν^2_ω.

Proposition 2.1.9. *Fix $0 < 2\delta < \varepsilon < \frac{1}{4}$, and for $n \in \mathbb{N}$ set $k = \lceil n^\varepsilon \rceil$ and $\ell = \lceil n^\delta \rceil$. Let $\Pi = \Pi(\ell)$ be a partition of $\mathbb{Z}^d$ into boxes of side length ℓ. For $\mathbb{P}$-almost every $\omega \in \Omega$ the measures from Definition 2.1.7 satisfy*

$$\lim_{n\to\infty} \left\|\nu^{\text{ann}\times\text{pre}}_\omega - (\nu^{\text{ann}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}\right\|_{1,n} = 0, \tag{L1}$$

$$\lim_{n\to\infty} \left\|(\nu^{\text{ann}\times\text{pre}} * \nu^{\text{que}})_{\omega,k} - (\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}\right\|_{1,n} = 0, \tag{L2}$$

$$\lim_{n\to\infty} \left\|(\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k} - (\nu^{\text{que}} * \nu^{\text{que}})_{\omega,k}\right\|_{1,n} = 0. \tag{L3}$$

The proof of the above proposition is given in Section 2.6. With the results stated in the present section we can give a proof of the quenched local limit theorem.

Proof of Theorem 1.2.4. Using the triangle inequality we have

$$\sum_{x\in\mathbb{Z}^d} |P^{(0,0)}_\omega(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)|$$

$$\le \sum_{x\in\mathbb{Z}^d} |P^{(0,0)}_\omega(X_n = x) - (\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n)| \tag{2.1.18}$$

$$+ \sum_{x\in\mathbb{Z}^d} |(\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n) - (\nu^{\text{ann}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n)| \tag{2.1.19}$$

$$+ \sum_{x\in\mathbb{Z}^d} |(\nu^{\text{ann}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n) - \nu^{\text{ann}\times\text{pre}}_\omega(x,n)| \tag{2.1.20}$$

$$+ \sum_{x\in\mathbb{Z}^d} |\nu^{\text{ann}\times\text{pre}}_\omega(x,n) - \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega)|. \tag{2.1.21}$$

By Proposition 2.1.9 the terms in (2.1.18), (2.1.19) and (2.1.20) tend to 0 as n goes to infinity. In order to compare (2.1.18) with (L3) literally note that we have $P^{(0,0)}_\omega(X_n = x) = \nu^{\text{que}} * \nu^{\text{que}})_{\omega,k}(x,n)$ by construction. Finally, by definition of $\nu^{\text{ann}\times\text{pre}}_\omega(x,n)$ the term in (2.1.21) can be written as

$$\Big|\frac{1}{Z_{\omega,n}} - 1\Big| \sum_{x\in\mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)\varphi(\sigma_{(x,n)}\omega) = \Big|\frac{1}{Z_{\omega,n}} - 1\Big| Z_{\omega,n}. \tag{2.1.22}$$

By Proposition 2.1.8 it follows that the expression in (2.1.22) converges to 0 as n tends to infinity. ☐

2.2 Annealed estimates and the proof of Theorem 1.2.1

In this section we collect estimates for the annealed walk that will be needed later in the proofs, and present a proof of Theorem 1.2.1.

Lemma 2.2.1 (Annealed derivative estimates)**.** *For $d \geq 3$, $j = 1, \dots, d$, $x, y \in \mathbb{Z}^d$, $m, n \in \mathbb{Z}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ denoting by e_j the j-th (canonical) unit vector we have*

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y+e_j,m)}(X_{n+m} = x)| \leq Cn^{-(d+1)/2}, \tag{2.2.1}$$

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y,m+1)}(X_{n+m} = x)| \leq Cn^{-(d+1)/2}, \tag{2.2.2}$$

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y,m)}(X_{n+m} = x + e_j)| \leq Cn^{-(d+1)/2}, \tag{2.2.3}$$

$$|\mathbb{P}^{(y,m)}(X_{n+m} = x) - \mathbb{P}^{(y,m)}(X_{n-1+m} = x)| \leq Cn^{-(d+1)/2}. \tag{2.2.4}$$

Proof. The estimates (2.2.1) and (2.2.2) are from [Ste17]; see Lemma 3.9 and its proof in Appendix A.2 there. By translation invariance we have

$$\mathbb{P}^{(y+e_j,m)}(X_{n+m} = x) = \mathbb{P}^{(y,m)}(X_{n+m} = x - e_j)$$

and

$$\mathbb{P}^{(y,m+1)}(X_{n+m} = x) = \mathbb{P}^{(y,m)}(X_{n-1+m} = x).$$

Thus, the estimates (2.2.3) and (2.2.4) follow from (2.2.1) and (2.2.2). □

We will also need the following generalization of the annealed derivate estimates in the previous lemma.

Lemma 2.2.2. *Let $\varepsilon > 0$. For $n \in \mathbb{N}$ large enough and every partition $\Pi_n^{(\varepsilon)}$ of $\mathbb{Z}^d$ into boxes of side length $\lfloor n^\varepsilon \rfloor$, we have*

$$\sum_{\Delta \in \Pi_n^{(\varepsilon)}} \sum_{x \in \Delta} \max_{y \in \Delta} \big[\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)\big] \leq Cn^{-\frac{1}{2}+3d\varepsilon}. \tag{2.2.5}$$

Proof. We consider the following set of boxes around the origin of $\mathbb{Z}^d$

$$\widetilde{\Pi}_n^{(\varepsilon)} := \{\Delta \in \Pi_n^{(\varepsilon)} : \Delta \cap [-\sqrt{n}\log^3 n, \sqrt{n}\log^3 n]^d \neq \emptyset\}. \tag{2.2.6}$$

With this notation we can write the sum on the left hand side of (2.2.5) as

$$\sum_{\Delta \in \widetilde{\Pi}_n^{(\varepsilon)}} \sum_{x \in \Delta} \max_{y \in \Delta} \big[\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)\big] \tag{2.2.7}$$

$$+ \sum_{\Delta \in \Pi_n^{(\varepsilon)} \setminus \widetilde{\Pi}_n^{(\varepsilon)}} \sum_{x \in \Delta} \max_{y \in \Delta} \big[\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)\big]. \tag{2.2.8}$$

So, it is enough to prove suitable upper bounds for these two sums. By Lemma 3.6 from [Ste17] we have

$$\sum_{\Delta \in \Pi_n^{(\varepsilon)} \setminus \widetilde{\Pi}_n^{(\varepsilon)}} \mathbb{P}^{(0,0)}(X_n \in \Delta) \leq Cn^{-c \log n} \tag{2.2.9}$$

for some positive constants C and c. Thus, the double sum (2.2.8) is bounded from above by

$$\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\widetilde{\Pi}_n^{(\varepsilon)}}\sum_{x\in\Delta}\left[\mathbb{P}^{(0,0)}(X_n\in\Delta)-\mathbb{P}^{(0,0)}(X_n=x)\right]$$
$$=\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\widetilde{\Pi}_n^{(\varepsilon)}}(|\Delta|-1)\mathbb{P}^{(0,0)}(X_n\in\Delta)\le Cn^{d\varepsilon}n^{-c\log n}\le\widetilde{C}n^{-\tilde{c}\log n}$$

for suitably chosen constants $\tilde{c}$ and $\widetilde{C}$. Using annealed derivative estimates from Lemma 2.2.1 the double sum (2.2.7) is bounded above by

$$\sum_{\Delta\in\widetilde{\Pi}_n^{(\varepsilon)}}\sum_{x\in\Delta}Cn^{\varepsilon}n^{-\frac{d+1}{2}}\le C(n^{\varepsilon}+\sqrt{n}\log^3 n)^d n^{\varepsilon}n^{-\frac{d+1}{2}}\le Cn^{3d\varepsilon}n^{-1/2}.$$

Combination of the last two displays completes the proof. □

Proof of Theorem 1.2.1. Let $\varepsilon,\delta>0$ be small (they will later be tuned appropriately). Let $\Pi_n^{(\varepsilon)}$ be a partition of $\mathbb{Z}^d$ in boxes of side length $\lceil\varepsilon\sqrt{n}\,\rceil$. Let $C_\delta>0$ be a constant such that $\mathbb{P}^{(0,0)}(\|X_n\|>C_\delta\sqrt{n})<\delta$; such a constant exists by Lemma 3.6 from [Ste17]. Furthermore denote by $\Pi_n^{(\varepsilon,\delta)}$ the subset of boxes in $\Pi_n^{(\varepsilon)}$ intersecting $\{x\in\mathbb{Z}^d:\|x\|\le C_\delta\sqrt{n}\}$. Then

$$\sum_{x\in\mathbb{Z}^d}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|$$
$$=\sum_{\Delta\in\Pi_n^{(\varepsilon)}\setminus\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big| \tag{2.2.10}$$
$$+\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}\Big|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|. \tag{2.2.11}$$

We will show that ε can be chosen so small that the above sum is bounded by 4δ for large enough n. We first find an upper bound for (2.2.10). By definition of $\Pi_n^{(\varepsilon,\delta)}$ if $\Delta\in\Pi_n^{(\varepsilon)}\setminus\Pi_n^{(\varepsilon,\delta)}$ then we have $\|x\|>C_\delta\sqrt{n}$ for all $x\in\Delta$. Thus, (2.2.10) is bounded from above by

$$\sum_{\substack{x\in\mathbb{Z}^d\\ \|x\|>C_\delta\sqrt{n}}}\Big(\mathbb{P}^{(0,0)}(X_n=x)+\frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big)\le\delta+C\exp\Big(-\frac{c}{2}C_\delta^2\Big).$$

By choosing C_δ large enough we can ensure that (2.2.10) is bounded by 2δ.

Turning to (2.2.11) we first compare the two terms in $|\cdot|$ with the averages over appropriate boxes. First, let $x\in\mathbb{Z}^d$ be fixed and let $\Delta\in\Pi_n^{(\varepsilon)}$ be the box containing x. Using annealed derivative estimates from Lemma 3.9 in [Ste17] we obtain

$$|\mathbb{P}^{(0,0)}(X_n=x)-\frac{1}{\lceil\varepsilon\sqrt{n}\,\rceil^d}\mathbb{P}^{(0,0)}(X_n\in\Delta)|$$
$$=\frac{1}{\lceil\varepsilon\sqrt{n}\,\rceil^d}\Big|\sum_{y\in\Delta}\mathbb{P}^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_n=y)\Big|$$
$$\le\frac{1}{\lceil\varepsilon\sqrt{n}\,\rceil^d}\sum_{y\in\Delta}\|x-y\|\,n^{-(d+1)/2}\le\lceil\varepsilon\sqrt{n}\,\rceil\cdot n^{-(d+1)/2}\le\frac{\varepsilon}{n^{d/2}}.$$

Now consider $\Delta \in \Pi_n^{(\varepsilon,\delta)}$. For every $x \in \Delta$ we have

$$\begin{aligned}
&\Big|\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big| \\
&\quad = \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big|1 - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \exp\Big(-\frac{1}{2n}(y^T\Sigma^{-1}y - x^T\Sigma^{-1}x)\Big)\,dy\Big| \\
&\quad \le \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d} \\
&\qquad\qquad \times \int_\Delta \Big|1 - \exp\Big(-\frac{1}{2n}((y-x)^T\Sigma^{-1}(y-x) + 2x^T\Sigma^{-1}(y-x))\Big)\Big|\,dy \\
&\quad \le \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \Big|1 - \exp\Big(-\frac{1}{2n}(C\varepsilon^2 n + CC_\delta\varepsilon n)\Big)\Big|\,dy \\
&\quad \le \exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\cdot C\varepsilon \le C\varepsilon,
\end{aligned}$$

where we have used $\|x-y\| \le \varepsilon\sqrt{n}$ and $\|x\| \le C_\delta\sqrt{n}$. Using first the triangle inequality and then combining the last two estimates we see that each summand in (2.2.11) is bounded from above by

$$\begin{aligned}
&\big|\mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\mathbb{P}^{(0,0)}(X_n \in \Delta)\big| \\
&\quad + \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\Big|\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big) - \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big| \\
&\quad + \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\Big|\mathbb{P}^{(0,0)}(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big| \\
&\le \frac{C\varepsilon}{n^{d/2}} + \frac{C\varepsilon}{(2\pi n)^{d/2}\sqrt{\det\Sigma}} \\
&\quad + \frac{1}{\lceil \varepsilon\sqrt{n}\,\rceil^d}\Big|\mathbb{P}^{(0,0)}(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big|.
\end{aligned} \tag{2.2.12}$$

The number of vertices summed over all $\Delta \in \Pi_n^{(\varepsilon,\delta)}$ is bounded by $((C_\delta+\varepsilon)\sqrt{n})^d \le C(C_\delta\sqrt{n})^d$. Thus,

$$\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}\Big(\frac{C\varepsilon}{n^{d/2}} + \frac{C\varepsilon}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\Big) \le C\cdot C_\delta^d\varepsilon. \tag{2.2.13}$$

Summing the last line in (2.2.12) with the double sum $\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\sum_{x\in\Delta}$ gives

$$\sum_{\Delta\in\Pi_n^{(\varepsilon,\delta)}}\Big|\mathbb{P}^{(0,0)}(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\int_\Delta \exp\Big(-\frac{1}{2n}y^T\Sigma^{-1}y\Big)\,dy\Big|. \tag{2.2.14}$$

By applying the annealed CLT from [BČDG13] (and approximating the indicator $\mathbb{1}_\Delta$ appropriately by continuous and bounded functions) and noting that for fixed ε and δ the set $\Pi_n^{(\varepsilon,\delta)}$ is finite implies that (2.2.14) goes to zero as n tends to infinity. In particular it is smaller than δ for large enough n.

Combining the estimates above we obtain

$$\sum_{x\in\mathbb{Z}^d}\Big|\mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det\Sigma}}\exp\Big(-\frac{1}{2n}x^T\Sigma^{-1}x\Big)\Big| \le 2\delta + C\cdot C_\delta^d\varepsilon + \delta < 4\delta$$

for large enough n and choosing $\varepsilon > 0$ so that $C\cdot C_\delta^d\varepsilon < \delta$. This concludes the proof. □

2.3 Proof of Lemma 2.1.1

For the proof of of Lemma 2.1.1 we follow closely the proof of Theorem 5.1 in [BCR16] and adapt their arguments to our model. The general idea is to implement an iteration scheme that carries the annealed-quenched comparison from Theorem 2.7.1 below along a sequence of more and more slowly growing box scales.

Let us introduce some notation first. Let $\theta > 0$ be a (small) constant to be determined in the proof. For $j \in \mathbb{N}$, we set $n_j \coloneqq \lfloor N^{\frac{1}{2^j}} \rfloor$ and $r(N) \coloneqq \lceil \log_2(\frac{\log N}{\theta \log M}) \rceil$. Note that $r(N)$ is the smallest integer satisfying $n_{r(N)}^\theta \leq M$. Furthermore we set

$$N_0 \coloneqq N - \sum_{j=1}^{r(N)} n_j \quad \text{and} \quad N_k \coloneqq \sum_{j=1}^{k} n_j + N_0 = N_{k-1} + n_k, \text{ for all } 1 \leq k \leq r(N). \tag{2.3.1}$$

Finally, for $0 \leq k \leq r(N)$, abusing the notation and suppressing the dependence on θ and n we write for the rest of this section $\Pi_k \coloneqq \Pi_{n_k^\theta}$ and define

$$\lambda_k(\omega) \coloneqq \sum_{\Delta \in \Pi_k} \big| P_\omega^{(0,0)}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta) \big|. \tag{2.3.2}$$

Note in particular that $\lambda_{r(N)}$ is twice the total variation distance between the quenched and the annealed measures on boxes of side length $\leq M$, which is the term we wish to bound from above to show (2.1.1). If one wishes to be slightly more precise, then one should replace $N_{r(N)}$ by M, thus obtaining the total variation for boxes of side length M exactly. This, however, does not influence the estimates to follow.

The proof of the following proposition is long and technical and will be given in Section 2.7.

Proposition 2.3.1. *There exists constants $C, c, \alpha > 0$ and events $G(N), N \in \mathbb{N}$, with $\mathbb{P}(G(N)) \geq 1 - CN^{-c\log N}$ such that for all $\omega \in G(N)$ we have*

$$\lambda_k \leq \lambda_{k-1} + Cn_k^{-\alpha}, \qquad \forall\, 1 \leq k \leq r(N). \tag{2.3.3}$$

In particular, $\lambda_{r(N)} \leq \lambda_1 + C\sum_{k=1}^{r(N)} n_k^{-\alpha}$ for $\omega \in G(N)$.

Proof of Lemma 2.1.1. The assertion is a consequence of Proposition 2.3.1 and can be proven analogously to the argument in the last part of the proof of Theorem 5.1 in [BCR16], page 35. ☐

2.4 Concentration from coupling: Proofs of Proposition 2.1.2 and Corollary 2.1.4

In this section we prove some analogues of the results of Section 6 in [BCR16] and present proofs of Proposition 2.1.2 and Corollary 2.1.4.

Lemma 2.4.1. *There exists a constant $c > 0$ and set of environments $K(N, c)$ satisfying*

$$\mathbb{P}(K(N, c)) \geq 1 - N^{-c\log N} \tag{2.4.1}$$

such that for every ω there exists a coupling $\Theta_{\omega,N}$ of $\mathbb{P}^{(0,0)}(X_N = \cdot)$ and $P_\omega^{(0,0)}(X_N = \cdot)$ with the property

$$\Theta_{\omega,N}(\Lambda) > c \text{ for every } \omega \in K(N,c), \tag{2.4.2}$$

where $\Lambda := \{(x,x) : x \in \mathbb{Z}^d\}$.

Proof. For $\varepsilon > 0$ and $M \in \mathbb{N}$ denote by $K(N) = K(N, M, \varepsilon)$ the set of environments $\omega \in \Omega$ satisfying

$$\sum_{\Delta \in \Pi_M} |P_\omega^{(0,0)}(X_N \in \Delta) - \mathbb{P}^{(0,0)}(X_N \in \Delta)| < \varepsilon, \tag{2.4.3}$$

where Π_M is a partition of $\mathbb{Z}^d$ into d-dimensional boxes of side length M. By Lemma 2.1.1, for every $\varepsilon \in (0,1)$ there exists a $M \in \mathbb{N}$ such that $\mathbb{P}(K(N)) \geq 1 - N^{-c \log N}$. On the event $K(N)$, the inequality (2.4.3) tells us that twice the total variation distance between $\mathbb{P}^{(0,0)}(X_N \in \cdot)$ and $P_\omega^{(0,0)}(X_N \in \cdot)$ on Π_M is less than ε and therefore there exists a coupling $\widetilde{\Theta}_{\omega,N,M}$ of both measures on $\Pi_M \times \Pi_M$ such that $\widetilde{\Theta}_{\omega,N,M}(\Lambda_{\Pi_M}) > 1 - \varepsilon$, where $\Lambda_{\Pi_M} = \{(\Delta, \Delta) : \Delta \in \Pi_M\}$.

Using the coupling $\widetilde{\Theta}$ we construct a new coupling of $\mathbb{P}^{(0,0)}(X_N = \cdot)$ and $P_\omega^{(0,0)}(X_N = \cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ which puts positive probability on the diagonal $\Lambda = \{(x,x) : x \in \mathbb{Z}^d\}$. We define $\Theta_{\omega,N}$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ by

$$\Theta_{\omega,N}(x,y) := \sum_{\Delta, \Delta' \in \Pi_M} \widetilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta') \cdot \mathbb{P}^{(0,0)}(X_N = x | X_{N-M} \in \Delta) \cdot P_\omega^{(0,0)}(X_N = y | X_{N-M} \in \Delta'). \tag{2.4.4}$$

Since $\widetilde{\Theta}_{\omega,N-M,M}$ is a coupling of $\mathbb{P}^{(0,0)}$ and $P_\omega^{(0,0)}$ on $\Pi_M \times \Pi_M$ one can easily see that by the formula of total probability $\Theta_{\omega,N}$ is indeed a coupling of $\mathbb{P}^{(0,0)}(X_N = \cdot)$ and $P_\omega^{(0,0)}(X_N = \cdot)$.

For $x \in \mathbb{Z}^d$, let Δ_x be the unique cube which contains x in the partition Π_M. Since the side length of each box in the partition Π_M is M it follows that the annealed random walk can reach x from each point in the box Δ_x in less than M steps.

Next we want to show that the coupling gives us a positive chance for the two walks to end up at the same position. In [BCR16] this is done by showing that $\Theta_{\omega,N}(x,x)$ is bounded away from zero for all $x \in \mathbb{Z}^d$. This is not true in our model because we do not have uniform ellipticity for the quenched measure. The idea here is to show that for "typical" ω the measure $\Theta_{\omega,N}(x,x)$ is bounded away from zero for "many" $x \in \mathbb{Z}^d$. To this end for given ω we define the set $\Pi_\omega^x \subset \Pi_M$ as the set of boxes $\Delta \in \Pi_M$ satisfying

$$P_\omega^{(0,0)}(X_N = x | X_{N-M} \in \Delta) > 0. \tag{2.4.5}$$

Note that if $\Pi_\omega^x = \emptyset$ for x and ω then we have $\Theta_{\omega,N}(x,x) = 0$. Furthermore, by definition of $P_\omega^{(0,0)}(X_N = x | X_{N-1} = y)$ we have

$$P_\omega^{(0,0)}(X_N = x | X_{N-M} \in \Delta) \geq \left(\frac{1}{3^d}\right)^M \tag{2.4.6}$$

for all $\Delta \in \Pi_\omega^x$. Now using (2.4.4), (2.4.6) and uniform ellipticity of the annealed measure we obtain

$$\begin{aligned}\Theta_{\omega,N}(x,x) &= \sum_{\Delta \in \Pi_\omega^x} \widetilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \cdot \mathbb{P}^{(0,0)}(X_N = x | X_{N-M} \in \Delta) \cdot P_\omega^{(0,0)}(X_N = x | X_{N-M} \in \Delta) \\ &\geq \sum_{\Delta \in \Pi_\omega^x} \widetilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \eta^M \left(\frac{1}{3^d}\right)^M,\end{aligned}$$

where $\eta \in (0,1)$ is the "uniform ellipticity bound" of the annealed random walk. Now it suffices to show

$$\sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi_\omega^x}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta) \geq \sum_{\Delta\in\Pi_M}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta). \tag{2.4.7}$$

This follows immediately if we can show that for all $\Delta \in \Pi_M \setminus \cup_{x\in\mathbb{Z}^d}\Pi_\omega^x$ we have

$$\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta) = 0.$$

For that consider a box $\Delta \in \Pi_M \setminus \cup_{x\in\mathbb{Z}^d}\Pi_\omega^x$, i.e. there is no $x \in \mathbb{Z}^d$ with $\Delta \in \Pi_\omega^x$ for the fixed ω. Thus, we have $P_\omega^{(0,0)}(X_N = x|X_{N-M} \in \Delta) = 0$ for all $x \in \mathbb{Z}^d$. It follows that $P_\omega^{(0,0)}(X_{N-M} \in \Delta) = 0$, because there can be no infinitely long open path starting from Δ. We obtain

$$\begin{aligned}\Theta_{\omega,N}(\Lambda) &= \sum_{x\in\mathbb{Z}^d}\Theta_{\omega,N}(x,x) \geq \sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi_\omega^x}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta)\eta^M\left(\frac{1}{3^d}\right)^M \\ &\geq \sum_{\Delta\in\Pi_M}\widetilde{\Theta}_{\omega,N-M,M}(\Delta,\Delta)\eta^M\left(\frac{1}{3^d}\right)^M \geq (1-\varepsilon)\eta^M\left(\frac{1}{3^d}\right)^M\end{aligned} \tag{2.4.8}$$

for every $\omega \in K(N)$. □

Recall the definitions of P_N and Q_N from (2.1.3) respectively (2.1.4). Note that for every $N \in \mathbb{N}$ the measure P_N is in fact the measure $\mathbb{P}$ since for every measurable event $A \in \Omega$ we have by translation invariance

$$\begin{aligned}P_N(A) &= \mathbb{E}\Big[\sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{1}_{\{\sigma_{(x,N)}\omega\in A\}}\Big] = \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{E}[\mathbb{1}_{\{\sigma_{(x,N)}\omega\in A\}}] \\ &= \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{P}(\sigma_{(-x,-N)}A) = \sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_N = x)\mathbb{P}(A) = \mathbb{P}(A).\end{aligned} \tag{2.4.9}$$

Definition 2.4.2. Given two environments $\omega, \omega' \in \Omega$ we define their distance by

$$\operatorname{dist}(\omega,\omega') = \inf\big\{\|(x,n)\| : \omega' = \sigma_{(x,n)}\omega\big\},$$

where the infimum over an empty set is defined to be infinity.

We denote by Ψ_N the coupling of P_N and Q_N from Lemma 2.4.1 extended to $\Omega\times\Omega$, that is,

$$\Psi_N(A) = \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\omega,N}(x,y)\mathbb{1}_{\{(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)\in A\}}\Big]. \tag{2.4.10}$$

The following result is an analogue to Lemma 6.6 in [BCR16].

Lemma 2.4.3. *For $M, N \in \mathbb{N}$ let $D^{(1)}_{M,N} : \Omega \to [0,\infty]$ and $D^{(2)}_{M,N} : \Omega \to [0,\infty]$ be defined by*

$$D^{(i)}_{M,N}(\omega_i) := \mathbb{E}_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega_1,\omega_2)>M\}}|\mathfrak{F}_{\omega_i}](\omega_i), \qquad i = 1,2,$$

where $\mathfrak{F}_{\omega_1}$, $\mathfrak{F}_{\omega_2}$ are the σ-algebras generated by the first, respectively, second coordinate in $\Omega\times\Omega$ and Ψ_N is defined in (2.4.10). For $M \in \mathbb{N}$, there exists an event F_M with the following properties:

(1) $\mathbb{P}(F_M) \geq 1 - M^{-c\log M}$.

(2) For every $\varepsilon > 0$ one can choose $M = M(\varepsilon)$ large enough

$$\max\Big\{D^{(1)}_{M,N}(\omega),\ \frac{dQ_N}{d\mathbb{P}}(\omega)D^{(2)}_{M,N}(\omega)\Big\} \le \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^{\complement}}(\omega). \tag{2.4.11}$$

Proof. Let

$$F_M = \bigcap_{k>M/2} \Big\{\omega \in \Omega : \forall x \in [-k,k]^d \cap \mathbb{Z}^d,\\ \sum_{\Delta\in\Pi_M} |\mathbb{P}^{(x,0)}(X_k \in \Delta) - P^{(x,0)}_{\omega}(X_k \in \Delta)| \le \frac{C_2}{M^{c_1}} + \frac{C_2}{k^{c_1}}\Big\}$$

where Π_M is a partition of $\mathbb{Z}^d$ into boxes of side length M and C_2, c_1 are the (renamed) constants from Lemma 2.1.1. Thus, $\mathbb{P}(F_M) \ge 1 - M^{-c\log M}$. Fix $\varepsilon > 0$. Then, by the definition of F_M and the coupling $\widetilde{\Theta}_{\omega,k,M}$ constructed in the proof of Lemma 2.4.1, for every $\omega \in F_M$, every $k > M/2$ and every $x \in [-k,k]^d \cap \mathbb{Z}^d$ we have

$$\widetilde{\Theta}_{\sigma_{(x,k)}\omega,k,M}(\Lambda_{\Pi_M}) > 1 - \frac{2C_2}{M^{c_1}} > 1 - \varepsilon \tag{2.4.12}$$

for large enough M, where $\Lambda_{\Pi_M} = \{(\Delta,\Delta) : \Delta \in \Pi_M\}$. Note that for $k \le M/2$ the left hand side of (2.4.12) is 1 and therefore (2.4.11) is trivially true for $N \le M/2$.

Let us now verify the estimates (2.4.11) for $D^{(1)}_{M,N}$ and $\frac{dQ_N}{d\mathbb{P}} D^{(2)}_{M,N}$ and $N > M/2$. Note that for $\mathbb{P}$-almost every environment $\omega \in \Omega$ we have

$$D^{(1)}_{M,N}(\omega) = \sum_{x,y\in\mathbb{Z}^d} \Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}} \tag{2.4.13}$$

and for Q_N-almost every ω we have

$$D^{(2)}_{M,N}(\omega) = \Big(\frac{dQ_N}{d\mathbb{P}}(\omega)\Big)^{-1} \sum_{x,y\in\mathbb{Z}^d} \Theta_{\sigma_{-(y,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}}. \tag{2.4.14}$$

Using (2.4.10) we have for every measurable event $A \subset \Omega$

$$\begin{aligned}
&E_{\Psi_N}[\mathbb{1}_{\{(\omega_1,\omega_2)\in A\times\Omega\}}\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}}]\\
&\quad= \Psi_N(A\times\Omega \cap \{(\omega_1,\omega_2) : \mathrm{dist}(\omega_1,\omega_2) > M\})\\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d} \Theta_{\omega,N}(x,y)\mathbb{1}_{\{(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)\in A\times\Omega\}}\mathbb{1}_{\{\mathrm{dist}(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)>M\}}\Big]\\
&\quad= \sum_{x,y\in\mathbb{Z}^d} \mathbb{E}\big[\Theta_{\omega,N}(x,y)\mathbb{1}_{\{\sigma_{(x,N)}\omega\in A\}\}}\mathbb{1}_{\{\|x-y\|>M\}}\big]\\
&\quad= \sum_{x,y\in\mathbb{Z}^d} \mathbb{E}\big[\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\omega\in A\}\}}\mathbb{1}_{\{\|x-y\|>M\}}\big],
\end{aligned}$$

where the last equality follows by translation invariance of $\mathbb{P}$. Since Ψ_N is a coupling of $P_N = \mathbb{P}$ and Q_N the last term equals

$$E_{\Psi_N}\Big[\mathbb{1}_{\{(\omega,\omega')\in A\times\Omega\}} \sum_{x,y\in\mathbb{Z}^d} \Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}}\Big],$$

which implies (2.4.13).

For $B_N := \{\omega\colon \frac{dQ_N}{d\mathbb{P}}(\omega) \neq 0\}$ we have $Q_N(B_N^\complement) = \Psi_N(\Omega \times B_N^\complement) = 0$, and we get similarly

$$
\begin{aligned}
&E_{\Psi_N}[\mathbb{1}_{\{\Omega\times A\}}\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}}] \\
&\quad= E_{\Psi_N}[\mathbb{1}_{\{\Omega\times A\cap B_N\}}\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>M\}}] \\
&\quad= \Psi_N(\Omega\times(A\cap B_N)\cap\{(\omega_1,\omega_2):\mathrm{dist}(\omega_1,\omega_2)>M\}) \\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\omega,N}(x,y)\mathbb{1}_{\{(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)\in\Omega\times A\cap B_N\}}\mathbb{1}_{\{\mathrm{dist}(\sigma_{(x,N)}\omega,\sigma_{(y,N)}\omega)>M\}}\Big] \\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\omega,N}(x,y)\mathbb{1}_{\{\sigma_{(y,N)}\omega\in A\cap B_N\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big] \\
&\quad= \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega,N}(x,y)\mathbb{1}_{\{\omega\in A\cap B_N\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big] \\
&\quad= E_{Q_N}\Big[\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}(\omega)\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega,N}(x,y)\mathbb{1}_{\{\omega\in A\cap B_N\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big] \\
&\quad= E_{\Psi_N}\Big[\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}(\omega_2)\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega_2,N}(x,y)\mathbb{1}_{\{(\omega_1,\omega_2)\in\Omega\times(A\cap B_N)\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big] \\
&\quad= E_{\Psi_N}\Big[\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}(\omega_2)\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega_2,N}(x,y)\mathbb{1}_{\{(\omega_1,\omega_2)\in\Omega\times A\}}\mathbb{1}_{\{\|x-y\|>M\}}\Big],
\end{aligned}
$$

which shows (2.4.14)

If $\Theta_{\sigma_{-(x,N)}\omega,N}(x,y) > 0$ then necessarily $x \in [-N,N]^d \cap \mathbb{Z}^d$ because in N steps the annealed walk can only reach points in this box. It follows that for large enough M, every $\omega \in F_M$ and every $N \geq M$ we have

$$
\begin{aligned}
&\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}} \\
&\quad= 1-\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|\leq M\}} \\
&\quad\leq 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(z,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|\leq M\}} \\
&\quad\leq 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\sum_{\Delta\in\Pi_M}\sum_{x,y\in\Delta}\Theta_{\sigma_{-(z,N)}\omega,N}(x,y) \\
&\quad= 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\sum_{\Delta\in\Pi_M}\widetilde{\Theta}_{\sigma_{-(z,N)}\omega,N,M}(\Delta,\Delta) \\
&\quad= 1-\min_{z\in[-N,N]^d\cap\mathbb{Z}^d}\widetilde{\Theta}_{\sigma_{-(z,N)}\omega,N,M}(\Lambda_{\Pi_M}) < \varepsilon.
\end{aligned}
$$

Thus,

$$
D^{(1)}_{M,N}(\omega) = \sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(x,N)}\omega,N}(x,y)\mathbb{1}_{\{\|x-y\|>M\}} \leq \varepsilon\mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega).
$$

For $\omega \in F_M \cap B_N$ we have shown

$$
\frac{dQ_N}{d\mathbb{P}}(\omega)D^{(2)}_{M,N}(\omega) = \sum_{x,y\in\mathbb{Z}^d}\Theta_{\sigma_{-(y,N)}\omega,N}\mathbb{1}_{\{\|x-y\|>M\}} \leq \varepsilon
$$

whereas for $\omega \in F_M \cap B_N^{\complement}$

$$\frac{dQ_N}{d\mathbb{P}}(\omega) D^{(2)}_{M,N}(\omega) = 0$$

and thus

$$\frac{dQ_N}{d\mathbb{P}}(\omega) D^{(2)}_{M,N}(\omega) \le \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^{\complement}}(\omega).$$

□

Proof of Proposition 2.1.2. We follow the ideas of the proof of Lemma 6.5 in [BCR16]. To this end, we consider the events

$$\begin{aligned} B_\varepsilon^- &= \{\omega \in \Omega : \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega) < 1 - \varepsilon\} \\ B_\varepsilon^+ &= \{\omega \in \Omega : \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega) > 1 + \varepsilon\}. \end{aligned}$$

First we consider B_ε^-. We decompose this event into two events, first of which has probability $M^{-c\log M}$ and the second is a $\mathbb{P}$ null set. We assume without loss of generality that Δ_0 is centred at the (spatial) origin, set $M_\varepsilon = \frac{\varepsilon}{6d^2} M$, define $\Delta_0^- = \{x \in \mathbb{Z}^d : \|x\| < M - M_\varepsilon\}$ and

$$S_\varepsilon^- = \{\omega \in B_\varepsilon^- : \sigma_{(x,0)}\omega \in F_{M_\varepsilon}, \forall x \in \Delta_0\},$$

where F_{M_ε} is the event from Lemma 2.4.3. Due to property (1) of F_{M_ε} from Lemma 2.4.3

$$\begin{aligned} \mathbb{P}(S_\varepsilon^-) &\ge \mathbb{P}(B_\varepsilon^-) - |\Delta_0| \mathbb{P}(F_{M_\varepsilon}^{\complement}) \\ &\ge \mathbb{P}(B_\varepsilon^-) - M^d (M_\varepsilon)^{-c\log M_\varepsilon} \ge \mathbb{P}(B_\varepsilon^-) - M^{-\tilde{c}\log M}, \end{aligned}$$

where $\tilde{c}$ is a positive constant. Therefore it is enough to show that $\mathbb{P}(S_\varepsilon^-) = 0$.

We claim that there exists an event $K^- \subset S_\varepsilon^-$ such that

$$\mathbb{P}(K^-) \ge \mathbb{P}(S_\varepsilon^-) \cdot ((4d)^d |\Delta_0|)^{-1} \tag{2.4.15}$$

and

$$\text{if } \omega, \omega' \in K^-,\ \omega \ne \omega', \text{ then } \operatorname{dist}(\omega, \omega') > 4M. \tag{2.4.16}$$

For every $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ let $U_{(x,n)}$ be an independent (of everything else defined so far) random variable uniformly distributed on $[0, 1]$, and define

$$K^- := \big\{\omega \in S_\varepsilon^- : \forall (x,n) \in 4\Delta_0 \ \text{ if } \ \sigma_{(x,n)}\omega \in B_\varepsilon^- \ \text{ then } \ U_{(x,n)} < U_{(0,0)}\big\}.$$

This means informally, that from each family of environments whose distance is smaller than $4dM$ we choose one uniformly. This implies that property (2.4.16) for K holds. Property (2.4.15) holds because due to translation invariance of $\mathbb{P}$ we have

$$\mathbb{P}(S_\varepsilon^-) \le \mathbb{P}\Big(\bigcup_{x \in 4d\Delta_0} \sigma_{(x,0)} K^-\Big) \le \sum_{x \in 4d\Delta_0} \mathbb{P}\big(\sigma_{(x,0)} K^-\big) = (4d)^d |\Delta_0| \mathbb{P}(K^-).$$

Now, let

$$G = \bigcup_{x\in\Delta_0} \sigma_{(x,0)} K^- \quad \text{and} \quad G^- = \bigcup_{x\in\Delta_0^-} \sigma_{(x,0)} K^-.$$

By property (2.4.16) of K^- these are in both cases disjoint unions and therefore we have

$$\begin{aligned} \mathbb{P}(G) &= \sum_{x\in\Delta_0} \mathbb{P}(\sigma_{(x,0)}K^-) = |\Delta_0|\mathbb{P}(K^-) \quad \text{and} \\ \mathbb{P}(G^-) &= |\Delta_0^-|\mathbb{P}(K^-) = |\Delta_0|\big(1-\frac{\varepsilon}{6d^2}\big)^d\mathbb{P}(K^-) > \big(1-\frac{\varepsilon}{6}\big)\mathbb{P}(G). \end{aligned} \tag{2.4.17}$$

Going back to the definition of the event B_ε^- and recalling that $K^- \subset S_\varepsilon^- \subset B_\varepsilon^-$ we obtain

$$\begin{aligned} Q_N(G) &= \int_G \frac{dQ_N}{d\mathbb{P}}(\omega)\,d\mathbb{P}(\omega) = \sum_{x\in\Delta_0}\int_{\sigma_{(x,0)}K^-} \frac{dQ_N}{d\mathbb{P}}(\omega)\,d\mathbb{P}(\omega) = \int_{K^-}\sum_{x\in\Delta_0}\frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega)\,d\mathbb{P}(\omega) \\ &\le \int_{K^-}(1-\varepsilon)|\Delta_0|\,d\mathbb{P}(\omega) = (1-\varepsilon)|\Delta_0|\mathbb{P}(K^-) = (1-\varepsilon)\mathbb{P}(G) \end{aligned}$$

Combining this with (2.4.17), for small enough $\varepsilon > 0$ we obtain

$$Q_N(G) \le (1-\varepsilon)\mathbb{P}(G) = \frac{1-\varepsilon}{1-\varepsilon/6}\Big(1-\frac{\varepsilon}{6}\Big)\mathbb{P}(G) < \frac{1-\varepsilon}{1-\varepsilon/6}\mathbb{P}(G^-) < \Big(1-\frac{\varepsilon}{3}\Big)\mathbb{P}(G^-). \tag{2.4.18}$$

Let $A^- = \{(\omega,\omega') : \omega\in G^-, \omega'\notin G\}$. Then by (2.4.17) and (2.4.18)

$$\begin{aligned} \Psi_N(A^-) &\ge \mathbb{P}(G^-) - Q_N(G) \ge \mathbb{P}(G^-) - \Big(1-\frac{\varepsilon}{3}\Big)\mathbb{P}(G^-) \\ &\ge \frac{\varepsilon}{3}\mathbb{P}(G^-) > \frac{\varepsilon}{3}\Big(1-\frac{\varepsilon}{6}\Big)\mathbb{P}(G) > \frac{\varepsilon}{4}\mathbb{P}(G). \end{aligned} \tag{2.4.19}$$

By construction of K^-, for every $(\omega,\omega')\in A^-$ we have $\operatorname{dist}(\omega,\omega') > M_\varepsilon$ and, therefore,

$$\begin{aligned} \int_G D^{(1)}_{M_\varepsilon,N}\,d\mathbb{P}(\omega) &= \int_{G\times\Omega} D^{(1)}_{M_\varepsilon,N}\,d\Psi_N(\omega,\omega') \ge \int_{G^-\times\Omega} D^{(1)}_{M_\varepsilon,N}\,d\Psi_N(\omega,\omega') \\ &= \int_{\Omega\times\Omega} E_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\,|\,\mathfrak{F}_\omega](\omega)\mathbb{1}_{\{G^-\times\Omega\}}(\omega,\omega')\,d\Psi_N(\omega,\omega') \\ &= \int_{\Omega\times\Omega} E_{\Psi_N}[\mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{G^-\times\Omega\}}(\omega,\omega')\,|\,\mathfrak{F}_\omega](\omega)\,d\Psi_N(\omega,\omega') \\ &= \int_{\Omega\times\Omega} \mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{G^-\times\Omega\}}(\omega,\omega')\,d\Psi_N(\omega,\omega') \\ &\ge \int_{\Omega\times\Omega} \mathbb{1}_{\{\operatorname{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{A^-\}}(\omega,\omega')\,d\Psi_N(\omega,\omega') \\ &= \int_{\Omega\times\Omega} \mathbb{1}_{A^-}(\omega,\omega')\,d\Psi_N(\omega,\omega') \\ &= \Psi_N(A^-) > \frac{\varepsilon}{4}\mathbb{P}(G). \end{aligned} \tag{2.4.20}$$

Since $G \subset F_{M_\varepsilon}$ by definition, using Lemma 2.4.3 with M_ε and $\varepsilon/5$ instead of M and ε we obtain

$$\int_G D^{(1)}_{M_\varepsilon,N}(\omega)\,d\mathbb{P}(\omega) \le \int_G \frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}}(\omega) + \mathbb{1}_{F^{\mathsf{c}}_{M_\varepsilon}}(\omega)\,d\mathbb{P}(\omega) = \int_G \frac{\varepsilon}{5}\,d\mathbb{P}(\omega) = \frac{\varepsilon}{5}\mathbb{P}(G). \tag{2.4.21}$$

Combining (2.4.20) and (2.4.21) we conclude that $\mathbb{P}(G) = 0$ and, therefore $\mathbb{P}(K^-) = 0$. By property (2.4.15) of K^- this implies that $\mathbb{P}(S_\varepsilon^-) = 0$ and finally $\mathbb{P}(B_\varepsilon^-) \le M^{-c\log M}$.

Next we turn to the event B_ε^+. As before we set $M_\varepsilon = \frac{\varepsilon}{6d^2} M$ and assume that Δ_0 is centred at the origin. Define $\Delta_0^+ \coloneqq \{x \in \mathbb{Z}^d : \|x\| < M + M_\varepsilon\}$ and let

$$S_\varepsilon^+ = \left\{\omega \in B_\varepsilon^+ : \sigma_{(x,0)}\omega \in F_{M_\varepsilon}, \forall x \in \Delta_0^+\right\},$$

where F_{M_ε} is, as before, the event from Lemma 2.4.3. Due to property (1) of F_{M_ε}

$$\begin{aligned}\mathbb{P}(S_\varepsilon^+) &\geq \mathbb{P}(B_\varepsilon^+) - |\Delta_0^+|\mathbb{P}(F_{M_\varepsilon}^\complement) \geq \mathbb{P}(B_\varepsilon^+) - (1+\frac{\varepsilon}{6d^2})^d M^d (M_\varepsilon)^{-c\log M_\varepsilon} \\ &\geq \mathbb{P}(B_\varepsilon^+) - M^{-\tilde{c}\log M}\end{aligned}$$

and again it is enough to show that $\mathbb{P}(S_\varepsilon^+) = 0$. As for S_ε^- we claim that there exists an event $K^+ \subset S_\varepsilon^+$ such that

$$\mathbb{P}(K^+) \geq \mathbb{P}(S_\varepsilon^+) \cdot ((4d)^d |\Delta_0^+|)^{-1} \tag{2.4.22}$$

and

$$\text{if } \omega, \omega' \in K^+ \text{ with } \omega \neq \omega', \text{ then } \operatorname{dist}(\omega, \omega') > 4(M + M_\varepsilon). \tag{2.4.23}$$

Let

$$H = \bigcup_{x \in \Delta_0} \sigma_{(x,0)} K^+ \quad \text{and} \quad H^+ = \bigcup_{x \in \Delta_0^+} \sigma_{(x,0)} K^+.$$

Both are, by property (2.4.23) of K^+ disjoint unions. Therefore we have for $\varepsilon > 0$ small enough

$$\begin{aligned}\mathbb{P}(H) &= |\Delta_0|\mathbb{P}(K^+) \quad \text{and} \\ \mathbb{P}(H^+) &= |\Delta_0^+|\mathbb{P}(K^+) = \left(1 + \frac{\varepsilon}{6d^2}\right)^d |\Delta_0|\mathbb{P}(K^+) < \left(1 + \frac{\varepsilon}{5}\right)\mathbb{P}(H).\end{aligned} \tag{2.4.24}$$

From $K^+ \subset S_\varepsilon^+ \subset B_\varepsilon^+$ we obtain

$$\begin{aligned}Q_N(H) &= \int_H \frac{dQ_N}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) = \sum_{x \in \Delta_0} \int_{\sigma_{(x,0)}K^+} \frac{dQ_N}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) \\ &= \int_{K^+} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma_{(x,0)}\omega)\, d\mathbb{P}(\omega) \\ &> \int_{K^+} |\Delta_0|(1+\varepsilon)\, d\mathbb{P}(\omega) = (1+\varepsilon)|\Delta_0|\mathbb{P}(K^+) = (1+\varepsilon)\mathbb{P}(H).\end{aligned} \tag{2.4.25}$$

Combination of this with (2.4.24), for small enough $\varepsilon > 0$ then yields

$$Q_N(H) > (1+\varepsilon)\mathbb{P}(H) = \frac{1+\varepsilon}{1+\varepsilon/5}\left(1 + \frac{\varepsilon}{5}\right)\mathbb{P}(H) > \frac{1+\varepsilon}{1+\varepsilon/5}\mathbb{P}(H^+) > \left(1 + \frac{\varepsilon}{3}\right)\mathbb{P}(H^+). \tag{2.4.26}$$

Let $A^+ \coloneqq \{(\omega, \omega') : \omega \notin H^+, \omega' \in H\}$. Then by (2.4.26)

$$\begin{aligned}\Psi_N(A^+) &\geq Q_N(H) - \mathbb{P}(H^+) > Q_N(H) - \frac{1}{1+\varepsilon/3} Q_N(H) = \frac{\varepsilon/3}{1+\varepsilon/3} Q_N(H) \\ &\geq \frac{\varepsilon}{4} Q_N(H).\end{aligned} \tag{2.4.27}$$

By the construction of K^+, for every $(\omega,\omega')\in A^+$ we have $\mathrm{dist}(\omega,\omega')>M_\varepsilon$ and, therefore,

$$\begin{aligned}\int_H D^{(2)}_{M_\varepsilon,N}(\omega)\,dQ_N(\omega) &= \int_{\Omega\times H} D^{(2)}_{M_\varepsilon,N}(\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} D^{(2)}_{M_\varepsilon,N}(\omega')\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} E_{\Psi_N}[\mathbb{1}_{\{\mathrm{dist}(\omega,\omega')>M_\varepsilon\}}\,|\,\mathfrak{F}_{\omega'}](\omega')\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} E_{\Psi_N}[\mathbb{1}_{\{\mathrm{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,|\,\mathfrak{F}_{\omega'}](\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} \mathbb{1}_{\{\mathrm{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{\{\Omega\times H\}}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &\geq \int_{\Omega\times\Omega} \mathbb{1}_{\{\mathrm{dist}(\omega,\omega')>M_\varepsilon\}}\mathbb{1}_{A^+}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \int_{\Omega\times\Omega} \mathbb{1}_{A^+}(\omega,\omega')\,d\Psi_N(\omega,\omega')\\ &= \Psi_N(A^+)\geq \frac{\varepsilon}{4}Q_N(H).\end{aligned} \tag{2.4.28}$$

Since $H\subset F_{M_\varepsilon}$ by definition, $\mathbb{P}(H)\leq Q_N(H)$ by (2.4.25), and using Lemma 2.4.3 with M_ε and $\frac{\varepsilon}{5}$ instead of M and ε we obtain

$$\begin{aligned}\int_H D^{(2)}_{M_\varepsilon,N}\,dQ_N(\omega) &\leq \int_{H\cap B_N}\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}\Big[\frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}\cap B_N}+\mathbb{1}_{(F_{M_\varepsilon}\cap B_N)^{\complement}}\Big]\,dQ_N(\omega)\\ &= \int_{H\cap B_N}\Big(\frac{dQ_N}{d\mathbb{P}}\Big)^{-1}\Big[\frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}\cap B_N}+\mathbb{1}_{(F_{M_\varepsilon}\cap B_N)^{\complement}}\Big]\,dQ_N(\omega)\\ &= \int_{H\cap B_N}\Big[\frac{\varepsilon}{5}\mathbb{1}_{F_{M_\varepsilon}\cap B_N}+\mathbb{1}_{(F_{M_\varepsilon}\cap B_N)^{\complement}}\Big]\,d\mathbb{P}(\omega)\\ &= \int_{H\cap B_N}\frac{\varepsilon}{5}\,d\mathbb{P}(\omega)\\ &= \frac{\varepsilon}{5}\mathbb{P}(H\cap B_N)\leq\frac{\varepsilon}{5}\mathbb{P}(H)\leq\frac{\varepsilon}{5}Q_N(H),\end{aligned} \tag{2.4.29}$$

where we recall from Lemma 2.4.3 that $B_N=\{\omega:\frac{dQ_N}{d\mathbb{P}}(\omega)\neq 0\}$ and note that $B_N^{\complement}$ is a Q_N null set. Combining (2.4.28) and (2.4.29), we conclude that $Q_N(H)=0$ and, therefore, by (2.4.25) we have $\mathbb{P}(H)=0$. It follows that $\mathbb{P}(K^+)=0$, which by property (2.4.22) of K^+ implies that $\mathbb{P}(S^+_\varepsilon)=0$ and finally that (2.1.5) holds. ☐

Proof of Corollary 2.1.4. To show that Proposition 2.1.2 holds for Q as well we define Ψ as the weak limit of $\{\frac{1}{n}\sum_{N=0}^{n-1}\Psi_N\}_{n=1}^{\infty}$ along any converging sub-sequence $\{n_k\}_{k\geq 1}$ (tightness of Ψ_N follows similarly to the discussion below Corollary 2.1.3). Note that Ψ is a coupling of $\mathbb{P}$ and Q on $\Omega\times\Omega$. Furthermore let

$$D^{(i)}_M(\omega_i):=E_\Psi[\mathbb{1}_{\mathrm{dist}(\omega_1,\omega_2)>dM}\,|\,\mathcal{F}_{\omega_i}](\omega_i),\qquad i=1,2.$$

Now we want to prove inequality (2.4.11) from Lemma 2.4.3 for $D^{(1)}_M$ and $D^{(2)}_M$. It is enough to show that along some sub-sequence $\{n_\ell\}_{l\geq 1}$ of $\{n_k\}_{k\geq 1}$

$$D^{(1)}_M(\omega)=\lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}D^{(1)}_{M,N}(\omega)\quad \mathbb{P}\text{-a.s.} \tag{2.4.30}$$

and

$$D_M^{(2)}(\omega) = \Big(\frac{dQ}{d\mathbb{P}}(\omega)\Big)^{-1} \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell -1} \frac{dQ_N}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \quad Q\text{-a.s.} \tag{2.4.31}$$

In fact, if the above equalities hold, then for $\mathbb{P}$-almost every ω we have

$$\begin{aligned}
D_M^{(1)}(\omega) &= \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \\
&= \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[\sum_{N=0}^{M-1} D_{M,N}^{(1)}(\omega) + \sum_{N=M}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \Big] \\
&\le \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[M + \sum_{N=M}^{n_\ell-1} D_{M,N}^{(1)}(\omega) \Big] \\
&\le \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[M + \sum_{N=M}^{n_\ell-1} (\varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega)) \Big] \\
&= \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega).
\end{aligned}$$

In addition for $D_M^{(2)}$ we have for Q almost all ω

$$\begin{aligned}
\frac{dQ}{d\mathbb{P}}(\omega) D_M^{(2)}(\omega) &= \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \frac{dQ_N}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \\
&\le \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[\sum_{N=0}^{M-1} \frac{dQ_N}{d\mathbb{P}}(\omega) + \sum_{N=M}^{n_\ell-1} \frac{dQ_N}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \Big] \\
&\le \lim_{\ell\to\infty} \frac{1}{n_\ell} \Big[\sum_{N=0}^{M-1} \frac{dQ_N}{d\mathbb{P}}(\omega) + \sum_{N=M}^{n_\ell-1} (\varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega)) \Big] \\
&\le \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^\complement}(\omega).
\end{aligned}$$

Let us now prove (2.4.30) and (2.4.31). Starting with (2.4.30) let $A \subset \Omega$ be a measurable event. We have

$$\begin{aligned}
&\mathbb{E}[D_M^{(1)}(\omega_1)\mathbb{1}_A(\omega_1)] \\
&\quad = E_\Psi[\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>dM\}} \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)] \\
&\quad = \Psi(\{(\omega_1,\omega_2) \in \Omega\times\Omega : \mathrm{dist}(\omega_1,\omega_2) > dM\} \cap A\times\Omega) \\
&\quad = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \Psi_N(\{(\omega_1,\omega_2) \in \Omega\times\Omega : \mathrm{dist}(\omega_1,\omega_2) > dM\} \cap A\times\Omega) \\
&\quad = \lim_{\ell\to\infty} \frac{1}{n_\ell} E_{\Psi_N}[\mathbb{1}_{\{\mathrm{dist}(\omega_1,\omega_2)>dM\}} \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)] \\
&\quad = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \mathbb{E}[D_{M,N}^{(1)}(\omega_1)\mathbb{1}_A(\omega_1)] \\
&\quad = \lim_{\ell\to\infty} \mathbb{E}\Big[\frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_{M,N}^{(1)}(\omega_1)\mathbb{1}_A(\omega_1)\Big]
\end{aligned}$$

where we used the definitions of Ψ and of $D^{(1)}_{M,N}$ as the conditional expectation. This implies convergence of $\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1} D^{(1)}_{M,N}$ to $D^{(1)}_M$ in $L^1(\mathbb{P})$. Thus, by standard arguments we can choose a subsequence that converges $\mathbb{P}$-almost surely. For $D^{(2)}_M$ we get in a similar way

$$\begin{aligned}
&E_Q[D^{(2)}_M(\omega)\mathbb{1}_A(\omega_2)]\\
&\quad= E_\Psi[\mathbb{1}_{\mathrm{dist}(\omega_1,\omega_2)>dM}\mathbb{1}_{\Omega\times A}(\omega_1,\omega_2)]\\
&\quad= \Psi(\{\mathrm{dist}(\omega_1,\omega_2)>dM\}\cap\Omega\times A)\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}\Psi_N(\{\mathrm{dist}(\omega_1,\omega_2)>dM\}\cap\Omega\times A)\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_{\Psi_N}[\mathbb{1}_{\mathrm{dist}(\omega_1,\omega_2)>dM}\mathbb{1}_{\Omega\times A}(\omega_1,\omega_2)]\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_{\Psi_N}[D^{(2)}_{M,N}(\omega_2)\mathbb{1}_{\Omega\times A}(\omega_1,\omega_2)]\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_{Q_N}[D^{(2)}_{M,N}(\omega_2)\mathbb{1}_A(\omega_2)]\\
&\quad= \lim_{\ell\to\infty}\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}E_Q\Big[\big(\frac{dQ}{d\mathbb{P}}(\omega_2)\big)^{-1}\cdot\frac{dQ_N}{d\mathbb{P}}(\omega_2)\cdot D^{(2)}_{M,N}(\omega_2)\mathbb{1}_A(\omega_2)\Big]\\
&\quad= \lim_{\ell\to\infty}E_Q\Big[\big(\frac{dQ}{d\mathbb{P}}(\omega_2)\big)^{-1}\cdot\frac{1}{n_\ell}\sum_{N=0}^{n_\ell-1}\frac{dQ_N}{d\mathbb{P}}(\omega_2)\cdot D^{(2)}_{M,N}(\omega_2)\mathbb{1}_A(\omega_2)\Big]
\end{aligned}$$

Q-almost surely. Thus, Lemma 2.4.3 holds for $D^{(1)}_M$ and $D^{(2)}_M$ instead of $D^{(1)}_{M,N}$ and $D^{(2)}_{M,N}$ respectively.

Since the only tools we need for the proof of Proposition 2.1.2 are Lemma 2.1.1 and Lemma 2.4.3, we can walk through the proof of Proposition 2.1.2 and repeat the same steps for $\frac{dQ}{d\mathbb{P}}$ to show Corollary 2.1.4. ☐

The following proposition is an analogue to Proposition 7.1 from [BCR16]. Note that the assertion expresses a general property of the density of a measure which is invariant for the point of view of the particle in the setting of a random walk in random environment. It is not model-specific.

Proposition 2.4.4. *For $\mathbb{P}$-almost every ω, every $n\in\mathbb{N}_0$, every $x\in\mathbb{Z}^d$ and all $k\le n$*

$$\varphi(\sigma_{(x,n)}\omega)=\sum_{y\in\mathbb{Z}^d}P^{(x+y,n-k)}_\omega(X_n=x)\varphi(\sigma_{(x+y,n-k)}\omega).$$

Proof. Let $n\in\mathbb{N}$. First we consider the case $k=1$. For every bounded measurable function $h:\Omega\to\mathbb{R}$ we have (recall the notation in (1.2.7) and (1.2.8))

$$\begin{aligned}
\int_\Omega h(\omega)\varphi(\sigma_{(x,n)}\omega)\,d\mathbb{P}(\omega) &= \int_\Omega h(\sigma_{(-x,-n)}\omega)\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega h(\sigma_{(-x,-n)}\omega)\,dQ(\omega)\\
&= \int_\Omega \Re h(\sigma_{(-x,-n)}\omega)\,dQ(\omega)\\
&= \int_\Omega (\Re h(\sigma_{(-x,-n)}\omega))\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega \sum_{\|y\|\le 1} g(\omega,y)h(\sigma_{(-x+y,1-n)}\omega)\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega \sum_{\|y\|\le 1} g(\sigma_{(x-y,n-1)}\omega,y)h(\omega)\varphi(\sigma_{(x-y,n-1)}\omega)\,d\mathbb{P}(\omega).
\end{aligned}$$

Thus

$$\begin{aligned}
\varphi(\sigma_{(x,n)}\omega) &= \sum_{\|y\|\le 1} g(\sigma_{(x-y,n-1)}\omega)\varphi(\sigma_{(x-y,n-1)}\omega)\\
&= \sum_{\|y\|\le 1} P^{(0,0)}_{\sigma_{(x-y,n-1)}\omega}(X_1=y)\varphi(\sigma_{(x-y,n-1)}\omega)\\
&= \sum_{\|y\|\le 1} P^{(x-y,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x-y,n-1)}\omega)\\
&= \sum_{y\in\mathbb{Z}^d} P^{(x+y,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x+y,n-1)}\omega).
\end{aligned}$$

By applying the operator $\Re$ a second time we see that

$$\begin{aligned}
\int_\Omega h(\omega)\varphi(\sigma_{(x,n)}\omega)\,d\mathbb{P} &= \int_\Omega h(\omega)\sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x+y_1,n-1)}\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega h(\sigma_{(-x-y_1,-n+1)}\omega)\sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\sigma_{(-x-y_1,-n+1)}\omega}(X_1=x)\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega \Big[\Big(\Re(h(\sigma_{(-x-y_1,-n+1)}\omega)\sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\sigma_{(-x-y_1,-n+1)}\omega}(X_1=x))\Big)\Big]\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega \sum_{\|y_2\|\le 1} g(\omega,y_2)h(\sigma_{(-x-y_1+y_2,-n+2)}\omega)\\
&\qquad \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\sigma_{(-x-y_1+y_2,-n+2)}\omega}(X_1=x)\varphi(\omega)\,d\mathbb{P}(\omega)\\
&= \int_\Omega \sum_{\|y_2\|\le 1} g(\sigma_{(x+y_1-y_2,n-2)}\omega,y_2)h(\omega)\\
&\qquad \sum_{\|y_1\|\le 1} P^{(x+y_1,n-1)}_{\omega}(X_1=x)\varphi(\sigma_{(x+y_1-y_2,n-2)}\omega)\,d\mathbb{P}(\omega)
\end{aligned}$$

$$= \int_\Omega \sum_{\|y_2\|\le 1} P_\omega^{(x+y_1+y_2,n-2)}(X_1 = x+y_1) \sum_{\|y_1\|\le 1} P_\omega^{(x+y_1,n-1)}(X_1 = x) h(\omega)\varphi(\sigma_{(x+y_1+y_2,n-2)}\omega)\, d\mathbb{P}(\omega).$$

Thus,

$$\begin{aligned}\varphi(\sigma_{(x,n)}\omega) &= \sum_{\|y_1\|\le 1}\sum_{\|y_2\|\le 1} P_\omega^{(x+y_1+y_2,n-2)}(X_1 = x+y_1) P_\omega^{(x+y_1,n-1)}(X_1 = x)\varphi(\sigma_{(x+y_1+y_2,n-2)}\omega)\\ &= \sum_{y\in\mathbb{Z}^d} P_\omega^{x+y,n-2}(X_2 = x)\varphi(\sigma_{(x+y,n-2)}\omega).\end{aligned}$$

Inductively we obtain

$$\varphi(\sigma_{(x,n)}\omega) = \sum_{y\in\mathbb{Z}^d} P_\omega^{(x+y,n-k)}(X_k = x)\varphi(\sigma_{(x+y,n-k)}\omega)$$

for all $k \le n$. □

2.5 Proof of Proposition 2.1.8

Let Π be a partition of $\mathbb{Z}^d$ into boxes of side length $\lfloor n^\delta \rfloor$ with $0 < \delta < \frac{1}{6d}$. Since $\mathbb{P}^{(0,0)}(X_n = x) = 0$ for $\|x\| > n$ only boxes in $\Pi_n := \{\Delta \in \Pi : \Delta \cap [-n,n]^d \ne \emptyset\}$ have to be considered. We have

$$\begin{aligned}|Z_{\omega,n} - 1| &= \Big|\sum_{x\in\mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big|\\ &= \Big|\sum_{\Delta\in\Pi_n}\sum_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big|.\end{aligned} \tag{2.5.1}$$

By the annealed CLT from [BČDG13] for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\mathbb{P}^{(0,0)}(\|X_n\| \ge C_\varepsilon\sqrt{n}) < \varepsilon$$

We want to use this fact below and separate the sum in the last line of (2.5.1) into boxes in $\widehat{\Pi}_n = \{\Delta \in \Pi_n : \Delta \cap \{x \in \mathbb{Z}^d : \|x\| \le C_\varepsilon\sqrt{n}\} \ne \emptyset\}$ and in $\Pi_n \setminus \widehat{\Pi}_n$. Using triangle inequality we obtain

$$|Z_{\omega,n} - 1| \le \Big|\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big| \tag{2.5.2}$$

$$+ \Big|\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\Big(\frac{1}{|\Delta|}\sum_{y\in\Delta}[\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)]\Big)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big| \tag{2.5.3}$$

$$+ \Big|\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\frac{1}{|\Delta|}\sum_{y\in\Delta} \mathbb{P}^{(0,0)}(X_n = y)[\varphi(\sigma_{(x,n)}\omega) - 1]\Big|. \tag{2.5.4}$$

We start with an upper bound of (2.5.2). By Corollary 2.1.4 there exists a constant C, such that, due to translation invariance of $\mathbb{P}$, with $\mathbb{P}$ probability of a least $1 - Cn^{-c\log n}$ for every $\Delta \in \Pi_n$ we have $\sum_{y\in\Delta}[\varphi(\sigma_{(y,n)}\omega) + 1] \le C|\Delta|$. Under this event we can bound (2.5.2) from above by

$$\sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n}\sum_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) + 1] \le C \sum_{\Delta\in\Pi_n\setminus\widehat{\Pi}_n} \max_{x\in\Delta} \mathbb{P}^{(0,0)}(X_n = x)|\Delta|.$$

Using Lemma 2.2.2 with $\delta > 0$ replacing ε there we see that (2.5.2) is bounded from above by

$$\begin{aligned} C \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \sum_{y \in \Delta} \Big[\max_{x \in \Delta} \mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = y)\Big] + C \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \sum_{y \in \Delta} \mathbb{P}^{(0,0)}(X_n = y) \\ \leq C\varepsilon + C \sum_{\Delta \in \Pi_n} \sum_{y \in \Delta} \Big[\max_{x \in \Delta} \mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = y)\Big] \\ \leq C\varepsilon + C n^{-\frac{1}{2} + 3d\delta}. \end{aligned}$$

Since $\delta < \frac{1}{6d}$ it follows by the Borel–Cantelli lemma that

$$\limsup_{n \to \infty} \Big| \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \sum_{x \in \Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma_{(x,n)}\omega) - 1] \Big| \leq C\varepsilon, \qquad \mathbb{P}\text{-a.s.} \tag{2.5.5}$$

Next we turn to (2.5.3). First note that by the annealed derivative estimates from Lemma 2.2.1 we have for $x, y \in \Delta$, $\Delta \in \widehat{\Pi}_n$

$$|\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = y)| \leq C \|x - y\| n^{-\frac{d+1}{2}} \leq C n^{-\frac{d+1}{2} + \delta}. \tag{2.5.6}$$

By triangle inequality, (2.5.6) and again, as above, using Corollary 2.1.4 for the bound $\sum_{y \in \Delta} [\varphi(\sigma_{(y,n)}\omega) + 1] \leq C|\Delta|$ the expression (2.5.3) is bounded from above by

$$\begin{aligned} \sum_{\Delta \in \widehat{\Pi}_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} |\mathbb{P}^{(0,0)}(X_n = y) - \mathbb{P}^{(0,0)}(X_n = x)| |\varphi(\sigma_{(x,n)}\omega) - 1| \\ \leq C n^{-\frac{d+1}{2} + \delta} \sum_{\Delta \in \widehat{\Pi}_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} [\varphi(\sigma_{(x,n)}\omega) + 1] \\ \leq C n^{-\frac{d+1}{2} + \delta} \sum_{\Delta \in \widehat{\Pi}_n} \sum_{y \in \Delta} C \\ \leq \widetilde{C} (C_\varepsilon \sqrt{n})^d n^{-\frac{d+1}{2} + \delta} \leq \widehat{C}_\varepsilon n^{-\frac{1}{2} + \delta}. \end{aligned}$$

with probability at least $1 - Cn^{-c \log n}$. Thus, as $n \to \infty$, by the Borel–Cantelli lemma the expression (2.5.3) tends to 0 $\mathbb{P}$-almost surely.

Finally we consider (2.5.4). By triangle inequality and $\mathbb{P}^{(0,0)}(X_n = y) \leq Cn^{-d/2}$ for all y we have

$$\begin{aligned} \Big| \sum_{\Delta \in \widehat{\Pi}_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} \mathbb{P}^{(0,0)}(X_n = y)[\varphi(\sigma_{(x,n)}\omega) - 1] \Big| \\ \leq \sum_{\Delta \in \widehat{\Pi}_n} \frac{1}{|\Delta|} \sum_{y \in \Delta} \mathbb{P}^{(0,0)}(X_n = y) \Big| \sum_{x \in \Delta} [\varphi(\sigma_{(x,n)}\omega) - 1] \Big| \\ \leq C n^{-d/2} \sum_{\Delta \in \widehat{\Pi}_n} \Big| \sum_{x \in \Delta} [\varphi(\sigma_{(x,n)}\omega) - 1] \Big| \\ = C n^{-d(1/2 - \delta)} \sum_{\Delta \in \widehat{\Pi}_n} \frac{1}{|\Delta|} \Big| \sum_{x \in \Delta} [\varphi(\sigma_{(x,n)}\omega) - 1] \Big|. \end{aligned}$$

Using Corollary 2.1.4 we obtain

$$\begin{aligned}
&\mathbb{P}\Big(Cn^{d(1/2-\delta)} \sum_{\Delta\in\widehat{\Pi}_n} \frac{1}{|\Delta|}\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big| > \varepsilon\Big)\\
&\qquad\qquad \le \mathbb{P}\Big(\exists \Delta\in\widehat{\Pi}_n : \sum_{\Delta\in\widehat{\Pi}_n} \frac{1}{|\Delta|}\Big|\sum_{x\in\Delta}[\varphi(\sigma_{(x,n)}\omega)-1]\Big| > \frac{\varepsilon}{CC_\varepsilon^d}\Big)\\
&\qquad\qquad \le n^{-d(1/2-\delta)}\mathbb{P}\Big(\frac{1}{|\Delta_0|}|\sum_{x\in\Delta_0}[\varphi(\sigma_{(x,n)}\omega)-1]| > \frac{\varepsilon}{CC_\varepsilon^d}\Big)\\
&\qquad\qquad \le n^{-d(1/2-\delta)} n^{-c\delta^2\log n} \le \tilde{C} n^{-\tilde{c}\log n},
\end{aligned}$$

where $\Delta_0 \in \widehat{\Pi}_n$ is an arbitrarily fixed box. Thus, for $\varepsilon > 0$ as $n\to\infty$ the lim sup of (2.5.4) is bounded from above by ε $\mathbb{P}$-almost surely. Combining all three bounds of (2.5.2)–(2.5.4), we see that there is a constant $\widehat{C}$ so that for all $\varepsilon > 0$

$$\limsup_{n\to\infty} |Z_{\omega,n} - 1| \le \widehat{C}\varepsilon, \quad \mathbb{P}\text{-almost surely,}$$

which concludes the proof. □

2.6 Proof of Proposition 2.1.9

The following result is an essential tool to prove Proposition 2.1.9 and will be proven in Section 2.8.

Lemma 2.6.1. *Let $0 < \theta < 1/2$ and $b > 0$. Define the set*

$$D(n) := \bigcap_{\substack{x,y\in\mathbb{Z}^d:\\ \|x\|,\|y\|\le n^b,\\ \|x-y\|\le n^\theta}} \Big\{ \Big\| P_\omega^{(x,0)}(X_n\in\cdot) - P_\omega^{(y,0)}(X_n\in\cdot)\Big\|_{\mathrm{TV}} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}}\Big\}. \tag{2.6.1}$$

Then there are constants $C, c > 0$ so that $\mathbb{P}(D(n)) \ge 1 - Cn^{-c\log n}$.

Note that the restriction $\|x\|, \|y\| \le n^b$ in the definition of $D(n)$ in (2.6.1) is necessary because with probability 1 we have an environment where there exist (somewhere far out in space) two neighbouring points $x, y \in \mathbb{Z}^d$ so that the sites $(x,0)$ and $(y,0)$ are both connected to infinity but the respective clusters do not intersect for the first n time steps.

Remark 2.6.2. The above lemma is the analogue of Lemma 7.7 from [BCR16] in our setting. Note that the bound stated in Lemma 7.7 from [BCR16] is too optimistic to hold in general. However, its assertion can be weakened and one obtains a bound which is still strong enough to prove Lemma 7.5 in [BCR16] by going a similar route as in the proof of Lemma 2.6.1 here.

Proof of Proposition 2.1.9, (L1). For this part we make use of the fact that, due to the annealed derivative estimates from Lemma 2.2.1 for $|x-y|\le k$, $|\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y)| \le Ck/(n-k)^{(d+1)/2} \approx n^{-(d+1)/2+\varepsilon}$, since $k = \lceil n^\varepsilon\rceil \ll n$. Furthermore we use the fact that by definition as a density of the invariant measure of the environment with respect to the point of view of the particle, the prefactor can be "transported" along the quenched transition probabilities; see Proposition 2.4.4. Finally we use the concentration property of Corollary 2.1.4; see equation (2.1.9).

We have to show

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d}\Big|\frac{1}{Z_{\omega,n}}\mathbb{P}^{(0,0)}(X_n=x)\varphi(\sigma_{(x,n)}\omega) - \frac{1}{Z_{\omega,n-k}}\sum_{y\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_{n-k}=y)\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big| = 0. \quad (2.6.2)$$

Note that the by the triangle inequality the sum on the left hand side is bounded from above by

$$\sum_{x\in\mathbb{Z}^d}\Big|\frac{1}{Z_{\omega,n}}-\frac{1}{Z_{\omega,n-k}}\Big|\mathbb{P}^{(0,0)}(X_n=x)\varphi(\sigma_{(x,n)}\omega) + \frac{1}{Z_{\omega,n-k}}\sum_{x\in\mathbb{Z}^d}\Big|\mathbb{P}^{(0,0)}(X_n=x)\varphi(\sigma_{(x,n)}\omega) - \sum_{y\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_{n-k}=y)\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big|.$$

By definition of $Z_{\omega,n}$, recall from Definition 2.1.7, the first sum in the above display equals to

$$\Big|\frac{1}{Z_{\omega,n}}-\frac{1}{Z_{\omega,n-k}}\Big|Z_{\omega,n},$$

which by Proposition 2.1.8 almost surely goes to 0 as n and $n-k$ both tend to ∞. Thus, taking also into account the trivial deterministic bound on the speed of the random walk, for (2.6.2) it suffices to show

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d}\Big|\mathbb{P}^{(0,0)}(X_n=x)\varphi(\sigma_{(x,n)}\omega) - \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\mathbb{P}^{(0,0)}(X_{n-k}=y)\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big| = 0. \quad (2.6.3)$$

Denoting by $B_n=\{x\in\mathbb{Z}^d : \|x\|\le\sqrt{n}\log^3 n\}$ and using the triangle inequality an upper bound of the sum in (2.6.3) is given by

$$\sum_{x\in B_n}\Big|\sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\big[\mathbb{P}^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_{n-k}=y)\big] \times\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big| \quad (2.6.4)$$

$$+\sum_{x\in B_n}\mathbb{P}^{(0,0)}(X_n=x) \times\Big|\varphi(\sigma_{(x,n)}\omega)-\sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big| \quad (2.6.5)$$

$$+\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\Big|\mathbb{P}^{(0,0)}(X_n=x)\varphi(\sigma_{(x,n)}\omega) - \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\mathbb{P}^{(0,0)}(X_{n-k}=y)\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big|. \quad (2.6.6)$$

By the annealed derivative estimates (2.6.4) is bounded from above by

$$\begin{aligned}
&\sum_{x\in B_n}\Big|\sum_{\substack{y\in\mathbb{Z}^d\\ \|x-y\|\le k}}\left[\mathbb{P}^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_{n-k}=y)\right]\\
&\qquad\qquad\times\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\Big|\\
&\le\frac{2Ck}{(n-k)^{(d+1)/2}}\sum_{x\in B_n}\sum_{\substack{y\in\mathbb{Z}^d\\ \|x-y\|\le k}}\varphi(\sigma_{(y,n-k)}\omega)P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k=x-y)\\
&\le\frac{2Ck(\sqrt{n}\log^3 n+k)^d}{(n-k)^{(d+1)/2}}\frac{1}{(\sqrt{n}\log^3 n+k)^d}\sum_{\substack{y\in\mathbb{Z}^d\\ \mathrm{dist}(y,B_n)\le k}}\varphi(\sigma_{(y,n-k)}\omega).
\end{aligned}$$

Now using Corollary 2.1.4 and the fact that $k=\lceil n^{\varepsilon}\rceil<n^{1/4}$ for $\mathbb{P}$-almost every ω the last term tends to zero as n tend to infinity.

Next we deal with (2.6.5). Recall that by Proposition 2.4.4 we have

$$\varphi(\sigma_{(x,n)}\omega)=\sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(x,n-k)}\omega)P^{(y,n-k)}_{\omega}(X_k=x)$$

for every $x\in\mathbb{Z}^d$ such that $x+[-k,k]^d\cap\mathbb{Z}^d\subset[-n,n]^d\cap\mathbb{Z}^d$. This holds for every $x\in B_n$ and therefore the expression (2.6.5) equals 0.

Finally, for (2.6.6), using Lemma 3.6 from [Ste17], we have $\mathbb{P}^{(0,0)}(X_n\notin B_n)\le Cn^{-c\log n}$. Recall that $k=\lceil n^{\varepsilon}\rceil$ and note that if $P^{(y,n-k)}_{\omega}(X_k=x)>0$ then $\|x-y\|\le k$. Thus, for $x\in[-n,n]^d\cap\mathbb{Z}^d\setminus B_n$ and large enough n

$$\|y\|\ge\|x\|-\|x-y\|\ge\sqrt{n}\log^3 n-k\ge\frac{1}{2}\sqrt{n}\log^3 n.$$

This implies, again due to Lemma 3.6 from [Ste17] that $\mathbb{P}^{(0,0)}(X_{n-k}=y)\le Cn^{-c\log n}$. Therefore, the expression (2.6.6) is bounded from above by

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\mathbb{P}^{(0,0)}(X_n=x)\varphi(\sigma_{(x,n)}\omega)\\
&\qquad+\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\ \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\mathbb{P}^{(0,0)}(X_{n-k}=y)\varphi(\sigma_{(y,n-k)}\omega)P^{(y,n-k)}_{\omega}(X_k=x)\\
&\le Cn^{-c\log n}\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\varphi(\sigma_{(x,n)}\omega)\\
&\quad+Cn^{-c\log n}\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus B_n}\ \sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(y,n-k)}\omega)P^{(y,n-k)}_{\omega}(X_k=x)\\
&\le Cn^{-c\log n}\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(x,n)}\omega)+Cn^{-c\log n}\sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(y,n-k)}\omega).
\end{aligned}$$

By Corollary 2.1.4 we have

$$\mathbb{P}\Big(\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(x,n)}\omega)\le 2n^d\Big)>1-n^{-c\log n},$$

as well as

$$\mathbb{P}\Big(\sum_{y\in\mathbb{Z}^d\cap[-n,n]^d}\varphi(\sigma_{(y,n-k)}\omega)\leq 2n^d\Big)>1-Cn^{-c\log n}.$$

Thus, the probability of the event that (2.6.6) is bounded above by $4Cn^{-c\log n}n^d$ converges to 1 super-algebraically fast. Hence the expression (2.6.6) converges to 0 $\mathbb{P}$-almost surely. □

Proof of Proposition 2.1.9, (L2). First note that, it is enough to show that

$$\left\|\nu_\omega^{\text{ann}\times\text{pre}}-\nu_\omega^{\text{box}-\text{que}\times\text{pre}}\right\|_{1,n-k}\xrightarrow{n\to\infty}0,$$

since the last k steps are according to the quenched law for both hybrid measures. Then, as the measure $\nu^{\text{box}-\text{que}\times\text{pre}}$ suggests, we make use of the comparison between the quenched and the annealed laws on the level of boxes we derived from Lemma 2.1.1. We also use the concentration properties of φ from Corollary 2.1.4.

Let $k\in\{0,\dots,n\}$ be fixed. Note that we have

$$\begin{aligned}\left\|(\nu^{\text{ann}\times\text{pre}}*\nu^{\text{que}})_{\omega,k}-(\nu^{\text{box}-\text{que}\times\text{pre}}*\nu^{\text{que}})_{\omega,k}\right\|_{1,n}&\leq\left\|\nu_\omega^{\text{ann}\times\text{pre}}-\nu_\omega^{\text{box}-\text{que}\times\text{pre}}\right\|_{1,n-k}\\&=\sum_{x\in\mathbb{Z}^d}\varphi(\sigma_{(x,n-k)}\omega)\left|\frac{\mathbb{P}^{(0,0)}(X_{n-k}=x)}{Z_{\omega,n-k}}-\frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\right|.\end{aligned}$$

By Proposition 2.1.8 it is enough to show that $\mathbb{P}$-almost surely

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d}\varphi(\sigma_{(x,n-k)}\omega)\left|\mathbb{P}^{(0,0)}(X_{n-k}=x)-\frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\right|=0. \tag{2.6.7}$$

Let $A_n=\{x\in\mathbb{Z}^d:\|x\|\leq C_\varepsilon\sqrt{n}\}$, with C_ε chosen so that $\mathbb{P}^{(0,0)}(\|X_{n-k}\|>\frac{C_\varepsilon}{2}\sqrt{n-k})<\varepsilon$ for n large enough. Using the triangle inequality the sum in (2.6.7) is bounded by

$$\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus A_n}\varphi(\sigma_{(x,n-k)}\omega)\left|\mathbb{P}^{(0,0)}(X_{n-k}=x)-\frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\right| \tag{2.6.8}$$

$$+\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\left|\mathbb{P}^{(0,0)}(X_{n-k}=x)-\frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{|\Delta_x|}\right| \tag{2.6.9}$$

$$+\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\left|\frac{1}{|\Delta_x|}\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)-\frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\right| \tag{2.6.10}$$

$$+\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\left|\frac{\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}-\frac{P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\right|. \tag{2.6.11}$$

Now we deal with the four terms separately. Expression (2.6.8) is bounded from above by

$$\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus A_n}\mathbb{P}^{(0,0)}(X_{n-k}=x)\varphi(\sigma_{(x,n-k)}\omega)+P_\omega^{(0,0)}(\|X_{n-k}\|>C_\varepsilon\sqrt{n}).$$

The term $\sum_{x\in\mathbb{Z}^d\cap[-n,n]^d\setminus A_n}\mathbb{P}^{(0,0)}(X_{n-k}=x)\varphi(\sigma_{(x,n-k)}\omega)$ goes to zero as n goes to infinity by the same arguments used to bound (2.5.2) in the proof of Proposition 2.1.8. For the second term we can argue as in the proof of Claim 2.15 from [BCR16], to obtain that for a set of environments, with $\mathbb{P}$ probability $> 1-\sqrt{\varepsilon}$, for large enough n

$$P_\omega^{(0,0)}(\|X_{n-k}\| > C_\varepsilon\sqrt{n}) \leq P_\omega^{(0,0)}\Big(\|X_n\| > \frac{C_\varepsilon}{2}\sqrt{n}\Big) \leq \sqrt{\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, this proves that (2.6.8) goes to zero as n goes to infinity.

Next we turn to (2.6.9). The annealed derivative estimates yield that it is bounded from above by

$$\begin{aligned}
\sum_{x\in A_n} \varphi(\sigma_{(x,n-k)}\omega)\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x} &|\mathbb{P}^{(0,0)}(X_{n-k}=x) - \mathbb{P}^{(0,0)}(X_{n-k}=y)| \\
&\leq C\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}\frac{1}{(n-k)^{(d+1)/2}}\|x-y\| \\
&\leq Cdn^\delta\frac{1}{(n-k)^{(d+1)/2}}\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega) \\
&= \frac{Cn^{\delta+d/2}}{(n-k)^{(d+1)/2}}\Big(\frac{1}{n^{d/2}}\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\Big) \xrightarrow{n\to\infty} 0, \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where for the limit we use Proposition 2.1.2, the fact that $k = \lceil n^\varepsilon\rceil$ and $\delta < \varepsilon < \frac{1}{4}$.

Next we deal with (2.6.10). Writing $\widehat{\Pi}_n = \{\Delta \in \Pi : \Delta \cap A_n \neq \emptyset\}$, using annealed derivative estimates and Corollary 2.1.4 we see that (2.6.10) is bound by

$$\begin{aligned}
\sum_{x\in A_n}&\varphi(\sigma_{(x,n-k)}\omega)\frac{1}{|\Delta_x|}\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x)\Big|1-\frac{1}{\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big| \\
&\leq \frac{C}{(n-k)^{d/2}}\sum_{x\in A_n}\varphi(\sigma_{(x,n-k)}\omega)\Big|1-\frac{1}{\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big| \\
&\leq C\Big(\frac{n-k}{n}\Big)^{-d/2}\frac{1}{n^{d/2}}\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\varphi(\sigma_{(x,n-k)}\omega)\Big|1-\frac{1}{\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big| \\
&= C\Big(1-\frac{k}{n}\Big)^{-d/2}\frac{1}{n^{d/2}}\sum_{\Delta\in\widehat{\Pi}_n}\sum_{x\in\Delta}\frac{\varphi(\sigma_{(x,n-k)}\omega)|\Delta_x|}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}\Big|\frac{1}{|\Delta_x|}\sum_{y\in\Delta_x}\varphi(\sigma_{(x,n-k)}\omega)-1\Big| \\
&= C\Big(1-\frac{k}{n}\Big)^{-d/2}\frac{1}{n^{(d/2)(1-2\delta)}}\sum_{\Delta\in\widehat{\Pi}_n}\Big|\frac{1}{|\Delta|}\sum_{x\in\Delta}\varphi(\sigma_{(x,n-k)}\omega)-1\Big|.
\end{aligned}$$

Using the same argument that was used for (2.5.4), we get that by the Borel–Cantelli lemma the last term goes to zero $\mathbb{P}$-a.s.

Finally, we estimate (2.6.11). It is bounded from above by

$$\begin{aligned}
\sum_{x\in A_n}\frac{\varphi(\sigma_{(x,n-k)}\omega)}{\sum_{y\in\Delta_x}\varphi(\sigma_{(y,n-k)}\omega)}&|\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta_x) - P_\omega^{(0,0)}(X_{n-k}\in\Delta_x)| \\
&= \sum_{\Delta\in\widehat{\Pi}_n}|\mathbb{P}^{(0,0)}(X_{n-k}\in\Delta) - P_\omega^{(0,0)}(X_{n-k}\in\Delta)|.
\end{aligned}$$

For the last term we can use Theorem 2.7.1 which implies that it is bounded by $Cn^{-\frac{1}{3}\delta}$ for $\mathbb{P}$-almost every ω and large enough n. Therefore $\mathbb{P}$ almost surely it converges to zero as n tends to infinity. $\square$

Proof of Proposition 2.1.9, (L3). Note that the first measure chooses, at time $n-k$, a box according to the quenched law and a point in that box weighted by the prefactor, whereas the second measure chooses a box and a point in that box according to the quenched law at time $n-k$. These points are then the starting points for the quenched random walks for the remaining k steps. We use the fact that, given enough time (much more than the square of the starting distance), the total variation distance for two quenched random walks starting from any pair of sites in a box with side length $\lceil n^\ell \rceil$ is, given enough time, i.e. much more than the square of the side length of the box, is small with high probability, see Lemma 2.6.1.

The proof follows along the same lines as in [BCR16]. We will highlight the point in the proof where we deviate. We have

$$\begin{aligned}
&\big\|(\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k} - (\nu^{\text{que}} * \nu^{\text{que}})_{\omega,k}\big\|_{1,n} \\
&= \sum_{x\in\mathbb{Z}^d} \big|(\nu^{\text{box}-\text{que}\times\text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n) - (\nu^{\text{que}} * \nu^{\text{que}})_{\omega,k}(x,n)\big| \\
&= \sum_{x\in\mathbb{Z}^d} \Big| \sum_{y\in\mathbb{Z}^d} P_\omega^{(0,0)}(X_{n-k}\in\Delta_y) \frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta_y}\varphi(\sigma_{(z,n-k)}\omega)} P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x-y) \\
&\qquad\qquad - \sum_{y\in\mathbb{Z}^d} P_\omega^{(0,0)}(X_{n-k}=y) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_k = x-y)\Big| \\
&= \sum_{x\in\mathbb{Z}^d} \Big| \sum_{\Delta\in\Pi}\sum_{y\in\Delta} P_\omega^{(y,n-k)}(X_k = x) P_\omega^{(0,0)}(X_{n-k}\in\Delta) \\
&\qquad\qquad \cdot \Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y \,|\, X_{n-k}\in\Delta)\Big)\Big| \\
&\le \sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi} \Big| \sum_{y\in\Delta} P_\omega^{(y,n-k)}(X_k = x) P_\omega^{(0,0)}(X_{n-k}\in\Delta) \\
&\qquad\qquad \cdot \Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y \,|\, X_{n-k}\in\Delta)\Big)\Big|.
\end{aligned} \tag{2.6.12}$$

Since for every $\Delta\in\Pi$ and $x\in\mathbb{Z}^d$ we have

$$\sum_{y\in\Delta}\frac{1}{|\Delta|}\sum_{v\in\Delta} P_\omega^{(v,n-k)}(X_k=x)\Big[\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y \,|\, X_{n-k}\in\Delta)\Big] = 0$$

it follows that (2.6.12) equals

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\Big|\sum_{y\in\Delta}\Big[P_\omega^{(y,n-k)}(X_k=x) - \Big(\frac{1}{|\Delta|}\sum_{w\in\Delta}P_\omega^{(w,n-k)}(X_k=x)\Big)\Big] \\
&\qquad\qquad \Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y \,|\, X_{n-k}\in\Delta)\Big)\Big| \\
&= \sum_{x\in\mathbb{Z}^d}\sum_{\Delta\in\Pi} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\Big|\frac{1}{|\Delta|}\sum_{y\in\Delta}\sum_{w\in\Delta}\Big[P_\omega^{(y,n-k)}(X_k=x) - P_\omega^{(w,n-k)}(X_k=x)\Big] \\
&\qquad\qquad \Big(\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y \,|\, X_{n-k}\in\Delta)\Big)\Big| \\
&\le \sum_{\Delta\in\Pi}\sum_{x\in\mathbb{Z}^d} P_\omega^{(0,0)}(X_{n-k}\in\Delta)\sum_{y\in\Delta}\frac{1}{|\Delta|}\sum_{w\in\Delta}\Big|P_\omega^{(y,n-k)}(X_k=x) - P_\omega^{(w,n-k)}(X_k=x)\Big| \\
&\qquad\qquad \Big|\frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z\in\Delta}\varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k}=y \,|\, X_{n-k}\in\Delta)\Big|
\end{aligned} \tag{2.6.13}$$

Until this point the steps are basically the same as in [BCR16]. Here we deviate from their proof. Note that $P_\omega^{(0,0)}(X_{n-k} \in \Delta) = 0$ if $\Delta \cap [-n+k, n-k]^d = \emptyset$. For $\Delta \cap [-n+k, n-k]^d \neq \emptyset$ we have $y, w \in \Delta$ implies that $\|y\|, \|w\| \leq n = k^{1/\varepsilon}$ and $\|y - w\| \leq n^\delta = k^{\delta/\varepsilon}$.

Using Lemma 2.6.1 we see that (2.6.13) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta \in \Pi} P_\omega^{(0,0)}(X_{n-k} \in \Delta) \sum_{y \in \Delta} \Big| \frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z \in \Delta} \varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k} = y \,|\, X_{n-k} \in \Delta) \Big| \\
&\qquad \frac{1}{|\Delta|} \sum_{w \in \Delta} \sum_{x \in \mathbb{Z}^d} \Big| P_\omega^{(y,n-k)}(X_k = x) - P_\omega^{(w,n-k)}(X_k = x) \Big| \\
&\leq \mathrm{e}^{-c \frac{\log k}{\log \log k}} \sum_{\Delta \in \Pi} P_\omega^{(0,0)}(X_{n-k} \in \Delta) \sum_{y \in \Delta} \Big| \frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z \in \Delta} \varphi(\sigma_{(z,n-k)}\omega)} - P_\omega^{(0,0)}(X_{n-k} = y \,|\, X_{n-k} \in \Delta) \Big| \\
&\leq 2\mathrm{e}^{-c \frac{\log k}{\log \log k}} \sum_{\Delta \in \Pi} P_\omega^{(0,0)}(X_{n-k} \in \Delta) = 2\mathrm{e}^{-c \frac{\log k}{\log \log k}} \leq C\mathrm{e}^{-\tilde{c} \frac{\log n}{\log \log n}}
\end{aligned}$$

since $k = \lceil n^\varepsilon \rceil$. The right hand side goes to 0 for $n \to \infty$. $\square$

2.7 Proof of Proposition 2.3.1

The starting point is a result from [Ste17]. Define

$$\mathcal{P}(N) := \Big(\Big[-\frac{1}{24}\sqrt{N} \log^3 N, \frac{1}{24}\sqrt{N} \log^3 N \Big]^d \times \Big[0, \frac{1}{3}N \Big] \Big) \cap (\mathbb{Z}^d \times \mathbb{Z}). \tag{2.7.1}$$

For $\theta \in (0,1)$ and $(x,m) \in \mathcal{P}(N)$ let $G'((x,m),N)$ denote the event that for every box $\Delta \subset \mathbb{Z}^d$ of side length $N^{\theta/2}$ we have

$$\big| P_\omega^{(x,m)}(X_{m+N} \in \Delta) - \mathbb{P}^{(x,m)}(X_{m+N} \in \Delta) \big| \leq CN^{-d(1-\theta)/2 - \frac{1}{6}\theta}. \tag{2.7.2}$$

Furthermore set

$$G'(N) := \bigcap_{(x,m) \in \mathcal{P}(N)} \big(G'((x,m),N) \cup \{\xi_m(x) = 0\} \big). \tag{2.7.3}$$

Theorem 2.7.1 (Theorem 3.24 in [Ste17])**.** *Let $d \geq 3$. There exist positive constants c and C, such that for all $(x,m) \in \mathcal{P}(N)$ we have*

$$\mathbb{P}^{(x,m)}\big(G'((x,m),N)\big) \geq 1 - CN^{-c \log N} \tag{2.7.4}$$

and

$$\mathbb{P}\big(G'(N)\big) \geq 1 - CN^{-c \log N}. \tag{2.7.5}$$

The following notion of *good sites* and *good boxes* will be needed in the proof of Proposition 2.3.1. On such boxes the annealed and quenched laws are "close" to each other. Recall the process $\xi = (\xi_n)_{n \in \mathbb{Z}}$ from (1.1.4) and the definition of n_k from the beginning of Section 2.3. Recall also that Π_k is a partition of $\mathbb{Z}^d$ into the boxes of side length $\lfloor n_k^\theta \rfloor$.

Definition 2.7.2. For a given realisation $\omega \in \Omega$, we say that $(x,m) \in \mathbb{Z}^d \times \mathbb{Z}$ is $(k-1,\theta,\varepsilon)$-*good* if either $\xi_m(x;\omega) = 0$ or $\xi_m(x;\omega) = 1$ and the following two conditions are satisfied

$$\sup_{\Delta' \in \Pi_k} \left| P_\omega^{(x,m)}(X_{m+n_k} \in \Delta') - \mathbb{P}^{(x,m)}(X_{m+n_k} \in \Delta') \right| \leq n_k^{\theta d - \frac{d}{2} - \varepsilon}, \tag{2.7.6}$$

$$P_\omega^{(x,m)}\left(\max_{s \leq n_k} \|X_{m+s} - x\| > \sqrt{n_k} \log^3 n_k \right) \leq C n_k^{-c \log n_k}. \tag{2.7.7}$$

Otherwise the site is said to be $(k-1,\theta,\varepsilon)$-*bad*. We say that for $\Delta \in \Pi_{k-1}$ and $m \in \mathbb{Z}$ the box $\Delta \times \{m\}$ is $(k-1,\theta,\varepsilon)$-*good* if each $(x,m) \in \Delta \times \{m\}$ is $(k-1,\theta,\varepsilon)$-*good*. Otherwise we say that $\Delta \times \{m\}$ is $(k-1,\theta,\varepsilon)$-*bad*.

The following lemma is a direct consequence of Theorem 2.7.1.

Lemma 2.7.3. *For all $\Delta \in \Pi_{k-1}$ there are positive constants C and c so that*

$$\mathbb{P}\left(\Delta \textit{ is } (k-1,\theta,\varepsilon)\textit{-good}\right) \geq 1 - C n_k^{-c \log n_k}. \tag{2.7.8}$$

The assertion of Proposition 2.3.1 is the analogue of the inequality (5.1) in [BCR16]. The strategy of the proof there is as follows. First, using the triangle inequality and the Markov property an upper bound of λ_k is obtained which is given by a sum of four terms (5.2) – (5.5) in [BCR16]. Second, for each of these four terms an upper bound is shown. Three of these upper bounds, the ones for (5.2), (5.4) and (5.5), are not difficult and can be proven in the same way as in [BCR16]. For (5.3) Berger et. al use a notion of "good" boxes and the fact that they are independent at a large but finite distance. The definition of those good boxes translates to our Definition 2.7.2, where it is clear that the dependence on ξ prevents us from directly using any argument hinging on independence at a finite distance. We circumvent this problem by defining a new type of boxes for which we are able to work with independence, see the ideas below Proposition 2.7.4. Using those boxes as an approximation for the good boxes we prove a lower bound on the probability of hitting a good box in Proposition 2.7.4.

Proof of the analogue of an upper bound of (5.2) in [BCR16]. Consider

$$\begin{aligned} \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \Bigg| \sum_{u \in \Delta'} & P_\omega^{(u,N_{k-1})}(X_{N_k} \in \Delta) \\ & \times \left[P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \right] \Bigg|. \end{aligned} \tag{2.7.9}$$

To get an upper bound for (2.7.9) the arguments in [BCR16] do not require any specific properties of the model and apply to our model as well. The steps are as follows: by the triangle inequality followed by elementary computations (2.7.9) is bounded from above by

$$\begin{aligned} & \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} P_\omega^{(u,N_{k-1})}(X_{N_k} \in \Delta) \\ & \qquad \times \left| P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \right| \\ & = \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} |P_\omega^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta')| \\ & = \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} P_\omega^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') |P_\omega^{(0,0)}(X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')| \\ & = \sum_{\Delta' \in \Pi_{k-1}} |P_\omega^{(0,0)}(X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta')| = \lambda_{k-1}. \end{aligned} \tag{2.7.10}$$

□

Proof of the analogue of an upper bound of (5.3) in [BCR16]. Consider

$$\sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta') \times[P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)]\Big|. \tag{2.7.11}$$

First, by the triangle inequality (2.7.11) is bounded from above by

$$\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta') \sum_{\Delta\in\Pi_k}\big|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\big|. \tag{2.7.12}$$

Next we define Π^1_{k-1} as the set of boxes $\Delta'\in\Pi_{k-1}$ with the property

$$\Delta'\cap\{x\in\mathbb{Z}^d:\|x\|\le\sqrt{N_{k-1}}\log^3 N_{k-1}\}\neq\emptyset.$$

By Lemma 3.6 in [Ste17] it follows

$$\sum_{\Delta'\notin\Pi^1_{k-1}}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\le CN_{k-1}^{-c\log N_{k-1}} \tag{2.7.13}$$

and consequently (2.7.11) is bounded from above by

$$CN_{k-1}^{-c\log N_{k-1}}+\sum_{\Delta'\in\Pi^1_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta') \sum_{\Delta\in\Pi_k}\big|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\big|. \tag{2.7.14}$$

Recall Definition 2.7.2. We will write "good" for $(k-1,\theta,\varepsilon)$-good to simplify the notation. By Lemma 2.7.3 we have $\mathbb{P}\big(\Delta \text{ is } (k-1,\theta,\varepsilon)\text{-good}\big)\ge 1-Cn_k^{-c\log n_k}$. For $u\in\mathbb{Z}^d$ define by $\Pi_k^{(1,u)}$ the set of boxes $\Delta\in\Pi_k$ satisfying (note that $\mathbb{E}^{(u,0)}[X_{n_k}]=u$)

$$\Delta\cap\big\{x\in\mathbb{Z}^d:\big\|x-u\big\|\le\sqrt{n_k}\log^3 n_k\big\}\neq\emptyset. \tag{2.7.15}$$

If a box $\Delta'\in\Pi^1_{k-1}$ is $(k-1,\theta,\varepsilon)$-good, then for $u\in\Delta'$

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&=\sum_{\Delta\in\Pi_k^{(1,u)}}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&\quad+\sum_{\Delta\in\Pi_k\setminus\Pi_k^{(1,u)}}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&\le\sum_{\Delta\in\Pi_k^{(1,u)}}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|+Cn_k^{-c\log n_k}\\
&\le|\Pi_k^{(1,u)}|Cn_k^{\theta d-\frac{d}{2}-\varepsilon}+Cn_k^{-c\log n_k}\\
&\le Cn_k^{\frac{d}{2}-\theta d+\theta d-\frac{d}{2}-\varepsilon}(\log n_k)^{3d}+Cn_k^{-c\log n_k}\\
&\le C(n_k^{-\varepsilon}(\log n_k)^{3d}+n_k^{-c\log n_k})\le Cn_k^{-\varepsilon/2},
\end{aligned} \tag{2.7.16}$$

where we used in the first inequality that by Lemma 3.6 from [Ste17]

$$\mathbb{P}^{(0,0)}\big(\|X_n\| > \sqrt{n}\log^3 n\big) \le Cn^{-c\log n}$$

and that $|\Pi_k^{(1,u)}| \le Cn_k^{d/2-\theta d}(\log n_k)^{3d}$.

It follows that (2.7.11) is bounded from above by

$$\begin{aligned}
&CN_{k-1}^{-c\log N_{k-1}} + \sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is good}}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')Cn_k^{-\varepsilon/2}\\
&\qquad + \sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is bad}}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')\\
&\qquad\qquad\times\sum_{\Delta\in\Pi_k}|P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)|\\
&\le CN_{k-1}^{-c\log N_{k-1}} + Cn_k^{-\varepsilon/2} + C\sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is bad}}}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta').
\end{aligned} \tag{2.7.17}$$

Now we want to find an estimate for the probability of hitting a bad box. For some $\beta>0$, to be chosen later, we consider the following event

$$G_{N,n_{k-1}} \coloneqq \Big\{\sum_{\Delta\in\Pi_{k-1}}\mathbb{1}_{\{\Delta\text{ is}(k-1,\theta,\varepsilon)\text{-good}\}}\mathbb{P}^{(0,0)}\big(X_{N_{k-1}}\in\Delta\big)\ge 1-C'n_k^{-\beta}\Big\} \tag{2.7.18}$$

and define

$$G_N \coloneqq \bigcap_{k=1}^{r(N)} G_{N,n_k}. \tag{2.7.19}$$

We want to mimic the proof in [BCR16] and for that we need to define a new type of boxes to approximate the density of bad boxes. The problem with following the proof in [BCR16] arises from the fact that our environment is, due to the dependence on infinitely long open paths, not i.i.d. To overcome that problem the idea is to exchange the environment ξ with a process that only has finite range dependencies. We will use this idea to show in Proposition 2.7.4 below that

$$\mathbb{P}(G_N)\ge 1-CN^{-c\log(N)}. \tag{2.7.20}$$

Note that $n_{k-1}=n_k^2$. Thus, on G_N the expression (2.7.11) is bounded from above by

$$\begin{aligned}
&CN_{k-1}^{-c\log N_{k-1}} + Cn_k^{-\varepsilon/2} + C\sum_{\substack{\Delta'\in\Pi^1_{k-1}\\ \text{is bad}}}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\\
&\qquad\le CN_{k-1}^{-c\log N_{k-1}} + Cn_k^{-\varepsilon/2} + C'n_{k-1}^{-\beta} \le C''n_{k-1}^{-\varepsilon/4}.
\end{aligned} \tag{2.7.21}$$

As can be seen in the proof of Proposition 2.7.4 we can choose $\beta\ge\varepsilon/4$ to obtain the last inequality in (2.7.21). □

Proof of the analogue of an upper bound of (5.4) in [BCR16]. Consider

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\\
&\qquad\times[\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')-\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)].
\end{aligned} \tag{2.7.22}$$

For any two probability measures μ and $\tilde{\mu}$ on $\mathbb{Z}^d$ we have

$$\sum_{u\in\Delta'} f(u)\mu(u) - \sum_{u\in\Delta'} f(u)\tilde{\mu}(u) \le \max_{u\in\Delta'} f(u) - \min_{u\in\Delta'} f(u).$$

Thus, the expression (2.7.22) can be bounded from above by

$$\begin{aligned}
&\sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\big|\max_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) - \min_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\big| \\
&\quad\le \sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta') \\
&\qquad\times \sum_{\Delta\in\Pi_k^{(1,u)}} \big|\max_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) - \min_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)\big| + Cn_k^{-c\log n_k},
\end{aligned} \tag{2.7.23}$$

where $\Pi_k^{(1,u)}$ is the set defined above.

Using $\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) = \sum_{v\in\Delta}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}=v)$ we have

$$\begin{aligned}
&\max_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) - \min_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) \\
&\qquad\le \sum_{v\in\Delta}\max_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}=v) - \min_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}=v) \\
&\qquad\le \sum_{v\in\Delta}\operatorname{diam}(\Delta')\frac{C}{n_k^{(d+1)/2}} \\
&\qquad\le (n_k^\theta)^d n_{k-1}^\theta \frac{C}{n_k^{(d+1)/2}},
\end{aligned} \tag{2.7.24}$$

where the second to last inequality follows by the annealed derivative estimates from Lemma 3.9 in [Ste17]. Altogether the expression (2.7.22) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\sum_{\Delta\in\Pi^{1,u}_{(k)}} (n_k^\theta)^d n_{k-1}^\theta\frac{C}{n_k^{(d+1)/2}} + Cn_k^{-c\log n_k} \\
&\le C\sum_{\Delta'\in\Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')\Big(\frac{dn_{k-1}^\theta\sqrt{n_k}(\log n_k)^3}{n_k^\theta}\Big)^d (n_k^\theta)^d n_{k-1}^\theta\frac{C}{n_k^{(d+1)/2}} + Cn_k^{-c\log n_k} \\
&\le C(\log n_k)^{3d}\frac{n_{k-1}^\theta}{n_k^{1/2}} + Cn_k^{-c\log n_k} \le C\frac{(\log n_k)^{3d}}{n_k^{1/2-2\theta}} + Cn_k^{-c\log n_k}.
\end{aligned} \tag{2.7.25}$$

□

Proof of the analogue of an upper bound of (5.5) in [BCR16]. Consider

$$\sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) - \mathbb{P}^{(0,0)}(X_{N_k}\in\Delta, X_{N_{k-1}}\in\Delta')\Big|. \tag{2.7.26}$$

Recall the regeneration times introduced in [BČDG13]. There they are defined for a random walk on the backbone of the oriented percolation cluster, whereas we allow the random walk to start outside the cluster. In Remark 2.3 Birkner et. al note that the local construction, which they use to obtain the regeneration times, can be extended to starting points outside the cluster. Let $B_{m,\tilde{m}}$ be the event that the first regeneration time greater than m will happen before $m+\tilde{m}^\beta$, for some small constant $\beta>0$ to be tuned appropriately later. By Lemma 2.5 from [BČDG13] the distribution of the regeneration increments has exponential tail bounds, and thus $\mathbb{P}(B_{m,\tilde{m}}) \leq C\mathrm{e}^{-cm^\beta}$. First, note that by the theorem of total probability and the triangle inequality (2.7.26) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta|X_{N_{k-1}}=u)|\\
&\quad\leq\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\\
&\qquad\times\sum_{\Delta\in\Pi_k}\big(\big|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big|\\
&\qquad\qquad+\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big)
\end{aligned} \tag{2.7.27}$$

First note that

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{N_{k-1},n_k}|X_{N_{k-1}}=u)\\
&\qquad=\mathbb{P}(B^{\mathsf{C}}_{N_{k-1},n_k})\leq C\mathrm{e}^{-cn_k^\beta}.
\end{aligned}$$

The remaining part of the right hand side of (2.7.27) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}\big(\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{0,n_k})\\
&\qquad+\big|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B_{0,n_k})-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big|\big).
\end{aligned}$$

Using the same arguments as above we obtain

$$\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\sum_{\Delta\in\Pi_k}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B^{\mathsf{C}}_{0,n_k})=\mathbb{P}(B^{\mathsf{C}}_{N_{k-1},n_k})\leq C\mathrm{e}^{-cn_k^\beta}$$

and thus it remains to find a suitable upper bound for

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\\
&\qquad\sum_{\Delta\in\Pi_k}\big|\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta,B_{0,n_k})-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,B_{N_{k-1},n_k}|X_{N_{k-1}}=u)\big|
\end{aligned}$$

Let $\tilde{\tau}_{N_{k-1}}$ denote the first regeneration time greater than N_{k-1}. By splitting the probabilities above into the sum over the possible times at which the regeneration can occur and the possible sites at which the random walk can be at the time of the regeneration we see that the term in the above display equals to

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\\
&\quad\times\sum_{\Delta\in\Pi_k}\Big|\sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\leq n_k^\beta}}\mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde{\tau}_{N_{k-1}}=t,X_{\tilde{\tau}_{N_{k-1}}}=v)\\
&\qquad-\sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\leq n_k^\beta}}\mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(0,0)}(\tilde{\tau}_{N_{k-1}}=t,X_{\tilde{\tau}_{N_{k-1}}}=v|X_{N_{k-1}}=u)\Big|.
\end{aligned} \tag{2.7.28}$$

The modulus in the last two lines of the above display is bounded from above by

$$\begin{aligned}
&\Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) \sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}}=t, X_{\tilde\tau_{N_{k-1}}}=v) \\
&\quad - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) \sum_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}}=t, X_{\tilde\tau_{N_{k-1}}}=v|X_{N_{k-1}}=u)\Big| \\
&\le \Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\Big| \qquad (2.7.29)\\
&\quad + \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta) \\
&\quad + \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta|X_{N_{k-1}}=u)
\end{aligned}$$

Plugging that into the sums in (2.7.28) we obtain that an upper bound of (2.7.26) is given by

$$\begin{aligned}
&\sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \\
&\qquad \sum_{\Delta\in\Pi_k}\Big| \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta) - \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\Big| \\
&+ \sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}}=u) \qquad (2.7.30)\\
&\qquad \sum_{\Delta\in\Pi_k} \max_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(u,N_{k-1})}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta) \\
&+ \sum_{\Delta'\in\Pi_{k-1}}\sum_{u\in\Delta'}\sum_{\Delta\in\Pi_k} \min_{\substack{t\in[N_{k-1},N_{k-1}+n_k^\beta]\\ v\in\mathbb{Z}^d:\|u-v\|\le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k}\in\Delta)\mathbb{P}^{(0,0)}(\tilde\tau_{N_{k-1}}>N_{k-1}+n_k^\beta, X_{N_{k-1}}=u) \\
&\qquad\qquad + C\mathrm{e}^{-cn_k^\beta}.
\end{aligned}$$

Now define $\Pi_k^{1,u,\beta}$ as the set boxes $\Delta\in\Pi_k$ for which

$$\Delta\cap\Big(\bigcup_{v:\ \|v-u\|\le n_k^\beta}\{x\in\mathbb{Z}^d:\|x-v\|\le\sqrt{n_k}\log^3 n_k\}\Big)\neq\emptyset. \qquad (2.7.31)$$

Using Lemma 3.6 from [Ste17] we obtain

$$\sum_{\Delta\notin\Pi_k^{1,u,\beta}} \mathbb{P}^{(v,0)}(X_{N_k-t}\in\Delta) \le \mathbb{P}^{(v,0)}\big(|X_{N_k-t}-v|>\sqrt{N_k-t}\log^3 N_k-t\big) \le Cn_k^{-c\log n_k}$$

for all $v \in \mathbb{Z}^d$ with $\|v-u\| \le n_k^\beta$ and all $t \in [N_{k-1}, N_{k-1}+n_k^\beta]$. Using this it follows

$$\begin{aligned}
&\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \\
&\qquad\qquad \sum_{\Delta \in \Pi_k} \max_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : |u-v| \le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(u,N_{k-1})}(\tilde{\tau}_{N_{k-1}} > N_{k-1}+n_k^\beta) \\
&\le |\Pi_k^{1,u,\beta}| \mathbb{P}^{(0,0)}(\tilde{\tau}_{N_{k-1}} > N_{k-1}+n_k^\beta) + C n_k^{-c\log n_k} \\
&\le n_k^{\beta d} n_k^{d/2(1-2\theta)} (\log n_k)^{3d} C \mathrm{e}^{-c n_k^\beta} + C n_k^{-c\log n_k} \le C n_k^{-c \log n_k},
\end{aligned} \tag{2.7.32}$$

where we have used the fact that, by the definition of $\Pi_k^{(1,u)}$ in (2.7.15), $|\Pi_k^{1,u,\beta}| \le n_k^{\beta d} |\Pi_k^{(1,u)}|$. Similarly

$$\begin{aligned}
&\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \sum_{\Delta \in \Pi_k} \min_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : |u-v| \le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(0,0)}(\tilde{\tau}_{N_{k-1}} > N_{k-1}+n_k^\beta, X_{N_{k-1}} = u) \\
&\le |\Pi_k^{1,u,\beta}| \mathbb{P}^{(0,0)}(\tilde{\tau}_{N_{k-1}} > N_{k-1}+n_k^\beta) + C n_k^{-c\log n_k} \\
&\le n_k^{\beta d} n_k^{d/2(1-2\theta)} (\log n_k)^{3d} C \mathrm{e}^{-c n_k^\beta} + C n_k^{-c\log n_k} \le C n_k^{-c \log n_k}.
\end{aligned} \tag{2.7.33}$$

Altogether it follows that (2.7.26) is bounded from above by

$$\begin{aligned}
&\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \\
&\qquad \times \sum_{\Delta \in \Pi_{(k)}^{1,u,\beta}} \Big| \max_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : \|u-v\| \le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) - \min_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : \|u-v\| \le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \Big| \\
&\qquad\qquad\qquad\qquad + C n_k^{-c\log n_k} + C \mathrm{e}^{-c n_k^\beta}
\end{aligned} \tag{2.7.34}$$

Using the annealed derivative estimates from Lemma 2.2.1 we obtain

$$\begin{aligned}
&\Big| \max_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : \|u-v\| \le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) - \min_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : \|u-v\| \le n_k^\beta}} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \Big| \\
&\quad \le |\Delta| \Big| \max_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : \|u-v\| \le n_k^\beta \\ x \in \Delta}} \mathbb{P}^{(v,t)}(X_{N_k} = x) - \min_{\substack{t \in [N_{k-1}, N_{k-1}+n_k^\beta] \\ v \in \mathbb{Z}^d : \|u-v\| \le n_k^\beta \\ y \in \Delta}} \mathbb{P}^{(v,t)}(X_{N_k} = y) \Big| \\
&\quad \le |\Delta| C (4 n_k^\beta + n_k^\theta) n^{-\frac{d+1}{2}} \\
&\quad \le n_k^{d\theta} C (4 n_k^\beta + n_k^\theta) n_k^{-\frac{d+1}{2}}.
\end{aligned} \tag{2.7.35}$$

Now if we choose $\beta = \theta$ and θ small enough, we get that the above expression is smaller than $C n_k^{-\frac{2d+1}{4}}$. Putting everything together we get the upper bound

$$\begin{aligned}
&C \mathrm{e}^{-c n_k^\theta} + C n_k^{-c\log n_k} + \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k^{1,u}} n_k^{-\frac{d}{2}-\frac{1}{4}} \\
&\qquad \le C \mathrm{e}^{-c n_k^\theta} + C n_k^{-c\log n_k} + \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') |\Pi_k^{1,u}| n_k^{-\frac{d}{2}-\frac{1}{4}} \\
&\qquad \le C \mathrm{e}^{-c n_k^\theta} + C n_k^{-c\log n_k} + C n_k^{(1/2-\theta)d} (\log n_k)^{3d} n_k^{-\frac{d}{2}-\frac{1}{4}} \\
&\qquad = C \mathrm{e}^{-c n_k^\theta} + C n_k^{-c\log n_k} + C (\log n_k)^{3d} 2 n_k^{\theta - 1/4}
\end{aligned}$$

Thus, recalling equation (2.7.26), we obtain

$$\sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,X_{N_{k-1}}\in\Delta')\Big| \le Cn_k^{-c} \quad (2.7.36)$$

for some constants $C, c > 0$. □

Proof of Proposition 2.3.1. To prove Proposition 2.3.1 we need to show inequality (2.3.3) which we recall here

$$\lambda_k \le \lambda_{k-1} + Cn_k^{-\alpha}, \qquad \forall\, 1 \le k \le r(N).$$

for some positive constants α and C on the event $G(N)$ from (2.7.19).

Fix $\omega \in G(N)$. Recall the definition

$$\lambda_k = \sum_{\Delta\in\Pi_k}\left|P_\omega^{(0,0)}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta)\right|$$

from equation (2.3.2). Furthermore, we recall (2.7.9), (2.7.11), (2.7.22) and (2.7.26) for which we just estimated upper bounds.

$$(2.7.9) = \sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta) \times\left[P_\omega^{(0,0)}(X_{N_{k-1}}=u)-\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')\right]\Big|,$$

$$(2.7.11) = \sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta') \times[P_\omega^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)]\Big|,$$

$$(2.7.22) = \sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta) \times[\mathbb{P}^{(0,0)}(X_{N_{k-1}}\in\Delta')P_\omega^{(0,0)}(X_{N_{k-1}}=u|X_{N_{k-1}}\in\Delta')-\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)],$$

$$(2.7.26) = \sum_{\Delta\in\Pi_k}\sum_{\Delta'\in\Pi_{k-1}}\Big|\sum_{u\in\Delta'}\mathbb{P}^{(0,0)}(X_{N_{k-1}}=u)\mathbb{P}^{(u,N_{k-1})}(X_{N_k}\in\Delta)-\mathbb{P}^{(0,0)}(X_{N_k}\in\Delta,X_{N_{k-1}}\in\Delta')\Big|.$$

Note that for λ_k, by the triangle inequality, we obtain

$$\lambda_k \le (2.7.9) + (2.7.11) + (2.7.22) + (2.7.26).$$

Thus, using the proven estimates, (2.7.10), (2.7.21), (2.7.25) and (2.7.36), for each of the summands respectively we gain

$$\lambda_k \le \lambda_{k-1} + C'' n_{k-1}^{-\varepsilon/4} + C\frac{(\log n_k)^{3d}}{n_k^{1/2-2\theta}} + Cn_k^{-c\log n_k} + Cn_k^{-c} \le \lambda_{k-1} + \tilde{C}n_k^{-\alpha}$$

for appropriate choices of $\alpha > 0$ and $\tilde{C} > 0$. The fact that $\mathbb{P}(G_N) \ge 1 - CN^{-c\log N}$ is proved in Proposition 2.7.4. □

Proposition 2.7.4. *For the events G_N from (2.7.19) there exists $N_0 \in \mathbb{N}$ such that, for all $N \geq N_0$ we have that*

$$\mathbb{P}(G_N) \geq 1 - CN^{-c\log N}. \tag{2.7.37}$$

Let $\beta > 0$ and put $f(n_k) = \log^2 n_k$. First we need another notion of *good* sites. Given a realization ω we define for all $(x, \ell) \in \mathbb{Z}^d \times \mathbb{Z}$ the set $C_m(x, \ell)$ as the set of sites at time $\ell + m \in \mathbb{Z}$ which can be reached from (x, ℓ) via an open path w.r.t. ω. We start by defining for $k = 1, 2, \dots$ a field $\tilde{\xi}^k := (\tilde{\xi}^k_t(x))_{t\in\mathbb{Z}^d}$ as follows

(i) $\tilde{\xi}^k_t(x) = \xi_t(x)$ for all $(x, t) \in \mathbb{Z}^d \times \{n_k + f(n_k), n_k + f(n_k) + 1, \dots\}$

(ii) For all $(x, t) \in \mathbb{Z}^d \times \{\dots, n_k + f(n_k) - 2, n_k + f(n_k) - 1\}$ we set $\tilde{\xi}^k_t(x) = 1$ if $C_{n_k+f(n_k)-t}(x, t) \neq \emptyset$. Otherwise we set $\tilde{\xi}^k_t(x) = 0$.

Note that $\xi \leq \tilde{\xi}^k$ since for (x, t) with $t < n_k + f(n_k)$ we set $\tilde{\xi}_t(x) = 1$ if (x, t) has an open path of length at least $n_k + f(n_k) - t$ instead of requiring an infinite open path. For $\xi_t(x) \neq \tilde{\xi}^k_t(x)$ we necessarily must have $t < n_k + f(n_k)$ and there must exist an open path started at (x, t) whose length is at least $n_k + f(n_k) - t$ but the contact process started at (x, t) has to eventually die out, i.e. there is no infinite open path starting in (x, t).

The following lemma gives us an upper bound on that probability. The result is well known in the oriented percolation and contact process world. For a proof see for instance in Lemma A.1. in [BČDG13].

Lemma 2.7.5. *For $p > p_c$ there exist $C, c > 0$ such that for all $(x, t) \in \mathbb{Z}^d \times \mathbb{Z}$*

$$\mathbb{P}\Big((x, t) \to^{\omega} \mathbb{Z}^d \times \{t + n\} \text{ and } (x, t) \not\to^{\omega} \mathbb{Z}^d \times \{\infty\}\Big) \leq C\mathrm{e}^{-cn}, \quad n \in \mathbb{N}.$$

As a direct consequence we get the following corollary.

Corollary 2.7.6. *For $x \in \mathbb{Z}^d$ define*

$$D_{n_k}(x) := \big(x + [-n^{\theta}_{k-1} - n_k, n^{\theta}_{k-1} + n_k]^d \times [0, n_k]\big) \cap (\mathbb{Z}^d \times \mathbb{Z}).$$

For $p > p_c$ there exist constants $C, c > 0$ such that

$$\mathbb{P}\Big(\tilde{\xi}^k_t(y) = \xi_t(y) \text{ for all } (y, t) \in D_{n_k}(x)\Big) \geq 1 - C\mathrm{e}^{-c\log^2 n_k}. \tag{2.7.38}$$

Proof. Note that $\theta > 0$ is a small constant and can be chosen such that we have $n^{\theta}_{k-1} = n^{2\theta}_k \leq n_k$ and thus $|D_{n_k}(x)| \leq 2^d n^{d+1}_k$. By definition of $\tilde{\xi}^k$ $\tilde{\xi}^k_t(y) \neq \xi_t(y)$ implies that there is at least one open but finite paths whose length is larger that $f(n_k)$. Using Lemma 2.7.5 the assertion (2.7.38) follows by the choice of $f(n_k) = \log^2 n_k$. (Here one can see that other choices of $f(n_k)$ are possible as well.) □

Let $(\tilde{X})$ be a random walk in the environment $\tilde{\xi}^k$ with transition probabilities given by

$$P_{\omega,\tilde{\xi}^k}(\tilde{X}_{n+1} = x \,|\, \tilde{X}_n = y) = \begin{cases} |U(x, n) \cap \tilde{\mathcal{C}}^k|^{-1} & \text{if } (x, n) \in \tilde{\mathcal{C}}^k \text{ and } (y, n + 1) \in U(x, n) \cap \tilde{\mathcal{C}}^k, \\ |U(x, n)|^{-1} & \text{if } (x, n) \notin \tilde{\mathcal{C}}^k \text{ and } (y, n + 1) \in U(x, n), \\ 0 & \text{otherwise,} \end{cases} \tag{2.7.39}$$

where $\tilde{\mathcal{C}}^k := \{(x, n) \in \mathbb{Z}^d \times \mathbb{Z} : \tilde{\xi}^k_n(x) = 1\}$.

Given a realisation ω, we say that (x, m) is $(k - 1, \theta, \varepsilon, \tilde{\xi}^k)$-good if it satisfies the conditions from Definition 2.7.2 with ξ replaced by $\tilde{\xi}^k$ and X replaced by $\tilde{X}$ in the quenched probabilities.

Lemma 2.7.7. *For all $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}$ we have that*

$$\mathbb{P}((x,t) \text{ is } (k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-good}) \geq 1 - Cn_k^{-c\log n_k}. \tag{2.7.40}$$

Proof. Due to Lemma 2.7.3 it suffices to show that with probability at least $1 - Cn_k^{-c\log n_k}$ we have $\tilde{\xi}_t^k(y) = \xi_t(y)$ for all $(y,t) \in D_{n_k}(x)$. This exactly the assertion of Corollary 2.7.6. On that event (x,t) is $(k-1,\theta,\varepsilon)$-good iff (x,t) is $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-good. □

Proof of Proposition 2.7.4. Recall the definition of $G_{N,n_{k-1}}$ from (2.7.18). To estimate the probability of hitting a bad box we can now mimic the proof in [BCR16] since we get a lower bound by estimating the probability for the $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-good boxes. By construction those boxes are independent of each other at distance $> 5n_k$. Define

$$\Pi_{k-1}^{(0)} = \{\Delta' \in \Pi_{k-1}^1 \colon \operatorname{dist}(\Delta', \underline{0}) \leq \left\lfloor \sqrt{N_{k-1}} \right\rfloor\} \tag{2.7.41}$$

and for $r \geq 1$ let

$$\Pi_{k-1}^{(r)} = \{\Delta' \in \Pi_{k-1}^1 \colon \left\lfloor 2^{r-1}\sqrt{N_{k-1}} \right\rfloor < \operatorname{dist}(\Delta', \underline{0}) \leq \left\lfloor 2^r \sqrt{N_{k-1}} \right\rfloor\}. \tag{2.7.42}$$

$(\Pi_{k-1}^{(r)})_{r\geq 0}$ is a partition of Π_{k-1}^1 into disjoint subsets according to the distance of the boxes from the origin which allows us to estimate the hitting probabilities of the bad boxes. Using the annealed local CLT (Theorem 1.2.1), we have

$$\begin{aligned} &\sum_{\substack{\Delta' \in \Pi_{k-1}^1 \\ \text{is bad}}} \bar{\mathbb{P}}^{(0,0)}(X_{N_{k-1}} \in \Delta') \\ &\qquad \leq \sum_{r=0}^{\lceil \log_2 (\log N_{k-1})^3 \rceil} |\Pi_{k-1}^{(r)} \cap \{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| C N_{k-1}^{-d/2} \mathrm{e}^{-cr^2} \end{aligned} \tag{2.7.43}$$

holds for some constants $C, c > 0$ and $\bar{\mathbb{P}}$ is the measure for the changed environments $\tilde{\xi}^k$.

In order to estimate the number of bad boxes in each $\Pi_{k-1}^{(r)}$ we define the event $\widetilde{G}_N = \widetilde{G}_N(C)$ by

$$\widetilde{G}_N \coloneqq \bigcap_{k=1}^{r(N)} \bigcap_{r=0}^{\lceil \log_2 (\log N_{k-1})^3 \rceil} \left\{ |\Pi_{k-1}^{(r)} \cap \{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}| \leq C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} \right\}, \tag{2.7.44}$$

where $\beta > 0$ is a constant to be tuned later. Let $\tilde{p}_{k-1}$ be the probability for a box $\Delta' \in \Pi_{k-1}$ to be $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad. Note that $\tilde{p}_k \in \mathcal{O}(n_k^{-c\log n_k})$ and on the event $\widetilde{G}_N$

$$\begin{aligned} \sum_{\substack{\Delta' \in \Pi_{k-1}^1 \\ \text{is bad}}} \bar{\mathbb{P}}^{(0,0)}(X_{N_{k-1}} \in \Delta') &\leq \sum_{r=0}^{\lceil \log_2 (\log N_{k-1})^3 \rceil} C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} N_{k-1}^{-d/2} \mathrm{e}^{-cr^2} \\ &\leq \sum_{r=0}^{\lceil \log_2 (\log N_{k-1})^3 \rceil} C 2^{dr} (\sqrt{N_{k-1}}/n_{k-1}^{\theta})^d N_{k-1}^{-d/2} \mathrm{e}^{-cr^2} n_{k-1}^{-\beta} \leq C n_{k-1}^{-(\beta+\theta d)}. \end{aligned} \tag{2.7.45}$$

Now it suffices to show that $\mathbb{P}(\widetilde{G}_N(C)) \geq 1 - CN^{-c\log(N)}$ for some constant $C > 0$. To do so, fix $k \geq 1$ and note that boxes $\Delta' \in \Pi_{k-1}$ at distance $5n_k$ are, by construction of $\tilde{\xi}^k$, good or bad independently of each other. To see this note that $2(n_{k-1}^{\theta}+n_k+f(n_k))<5n_k$ and recall that $\tilde{\xi}_t^k(y)=1$ if there exists an open path connecting (y,t) to $\mathbb{Z}^d\times\{n_k+f(n_k)\}$ and $\tilde{\xi}_t^k(y)=0$ otherwise. Let $(\Pi_{k-1}^{r,j})_j$ be a partition of $\Pi_{k-1}^{(r)}$ into at most $(5n_k)^d$ subsets of boxes so that the distance between each pair of boxes in $\Pi_{k-1}^{r,j}$ is bigger than $5n_k$, for every j, and the number of boxes in $\Pi_{k-1}^{r,j}$ is between $|\Pi_{k-1}^{(r)}|/(2(5n_k)^d)$ and $2|\Pi_{k-1}^{(r)}|/(5n_k)^d$.

If the number of $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad boxes in $\Pi_{k-1}^{(r)}$ is bigger than $C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}$, then there exists at least one j so that the number of bad boxes in $\Pi_{k-1}^{r,j}$ is larger than $C|\Pi_{k-1}^{r,j}|n_{k-1}^{-\beta}$. Since the boxes in $\Pi_{k-1}^{r,j}$ are good or bad independently of each other, their number is bounded and they are bad with probability $\tilde{p}_{k-1}$, it follows by Hoeffding's inequality that

$$\begin{aligned}
&\bar{\mathbb{P}}(|\Pi_{k-1}^{(r)}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}|>C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta})\\
&\qquad\le(5n_k)^d\bar{\mathbb{P}}(|\Pi_{k-1}^{r,1}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}|\ge\lceil C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}/(5n_k)^d\rceil)\\
&\qquad\le(5n_k)^d\exp(-(Cn_{k-1}^{-\beta}-\tilde{p}_{k-1})^2|\Pi_{k-1}^{(r)}|/(5n_k)^d)\\
&\qquad\le\tilde{C}(5n_k)^d\exp(-Cn_{k-1}^{-2\beta}|\Pi_{k-1}^{(r)}|/(5n_k)^d)\\
&\qquad\le\tilde{C}(5n_k)^d\exp(-C2^{rd}N^{\frac{-2\beta}{2^{k-1}}+\frac{d}{2}-\frac{d\theta}{2^{k-1}}-\frac{d}{2^k}})\\
&\qquad=\tilde{C}(5n_k)^d\exp(-C2^{rd}N^{\frac{d}{2}-(\frac{4\beta+2d\theta+d}{2^k})}),
\end{aligned}\tag{2.7.46}$$

where the right hand side decays stretched exponentially in N for $k\ge 4$ if β is small enough, e.g. $\beta=1$ (which is still sufficient for the proof of (2.7.11)). For $1\le k\le 3$ notice that

$$\begin{aligned}
&\bar{\mathbb{P}}(|\Pi_{k-1}^{(r)}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}|>C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta})\\
&\quad\le\bar{\mathbb{P}}(\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}\ne\emptyset)\\
&\quad\le|\Pi_{k-1}^{(r)}|\tilde{p}_{k-1}\le(\sqrt{N}\log^3(N))^d\tilde{p}_{k-1}\le(\sqrt{N}\log^3(N))^dN^{-c\log(N)}\le CN^{-c\log(N)}.
\end{aligned}\tag{2.7.47}$$

Using the estimates above together with the definition of $\widetilde{G}_N$ shows that

$$\begin{aligned}
\bar{\mathbb{P}}(\widetilde{G}_N^{\complement})&=\bar{\mathbb{P}}\Bigg(\bigcup_{k=1}^{r(N)}\bigcup_{r=0}^{\lceil\log_2(\log N_{k-1})^3\rceil}\Big\{|\Pi_{k-1}^{(r)}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}|>C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\Big\}\Bigg)\\
&\le\sum_{k=1}^{r(N)}\sum_{r=0}^{\lceil\log_2(\log N_{k-1})^3\rceil}\bar{\mathbb{P}}\left(|\Pi_{k-1}^{(r)}\cap\{(k-1,\theta,\varepsilon,\tilde{\xi}^k)\text{-bad boxes}\}|>C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}\right)\\
&\le r(N)\lceil\log_2(\log N_{k-1})^3\rceil CN^{-c\log(N)}\le C\log\log(N)\cdot\log(N)^{5/6}N^{-c\log(N)}\\
&\le N^{-\tilde{c}\log(N)}.
\end{aligned}\tag{2.7.48}$$

Next we show that the number of $(k-1,\theta,\varepsilon)$-bad boxes in ξ is on the same order as the number of $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad boxes in $\tilde{\xi}^k$ with high probability. First we define, in a slight abuse of notation, the sets

$$\begin{aligned}
D_{n_k}(\Delta)&\coloneqq\{(x,t)\in\mathbb{Z}^d\times\mathbb{Z}:\mathrm{dist}(x,\Delta)\le n_k,t\in[0,n_k]\},\\
A_{k,\Delta}&\coloneqq\{\omega\in\Omega:\xi_t(x)=\tilde{\xi}_t^k(x)\text{ for all }(x,t)\in D_{n_k}(\Delta)\}
\end{aligned}$$

for all $\Delta\in\Pi_{k-1}^{(r)}$. Note that $D_{n_k}(\Delta)$ is the same box as $D_{n_k}(x)$ if x is the center of Δ. Using the above defined partitions $(\Pi_{k-1}^{r,j})_j$ we see that for every choice of $\Delta,\Delta'\in\Pi_{k-1}^{r,j}$ the events $A_{k,\Delta}$ and $A_{k,\Delta'}$ are independent, since $\mathrm{dist}(\Delta,\Delta')>5n_k$. Since $\xi\le\tilde{\xi}^k$ the number of $(k-1,\theta,\varepsilon)$-good boxes in ξ is less or equal to the number of $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-bad boxes in $\tilde{\xi}^k$.

To shorten the notation we say for a box $\Delta \in \Pi_{k-1}^{(r)}$ that it is good in ξ if it is $(k-1,\theta,\varepsilon)$-good and good in $\tilde{\xi}^k$ if it is $(k-1,\theta,\varepsilon,\tilde{\xi}^k)$-good. A box can only be bad in ξ and good in $\tilde{\xi}^k$ for $\omega \in A_{k,\Delta}^{\complement}$. Using Corollary 2.7.6 we get $\mathbb{P}(A_{k,\Delta}^{\complement}) \le C n_k^{-c\log n_k}$, and thus, again by Hoeffding's inequality,

$$\begin{aligned}
&\mathbb{P}\Big(|\Pi_{k-1}^{(r)} \cap \{\text{bad in } \xi\}| - |\Pi_{k-1}^{(r)} \cap \{\text{bad in } \tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta}\Big) \\
&\quad \le \mathbb{P}\Big(\exists j \text{ s.t. } |\Pi_{k-1}^{r,j} \cap \{\text{bad in } \xi\}| - |\Pi_{k-1}^{r,j} \cap \{\text{bad in } \tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} \frac{1}{(5n_k)^d}\Big) \\
&\quad \le (5n_k)^d \mathbb{P}\Big(|\Pi_{k-1}^{r,j} \cap \{\text{bad in } \xi\}| - |\Pi_{k-1}^{r,j} \cap \{\text{bad in } \tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} \frac{1}{(5n_k)^d}\Big) \\
&\quad \le (5n_k)^d \mathbb{P}\Big(\sum_{\Delta \in \Pi_{k-1}^{r,j}} \mathbb{1}_{A_{k,\Delta}^{\complement}} \ge C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta} \frac{1}{(5n_k)^d}\Big) \\
&\quad \le \tilde{C}(5n_k)^d \exp\Big(-C 2^{rd} N^{\frac{d}{2} - (\frac{4\beta + 2d\theta + d}{2^k})}\Big).
\end{aligned} \tag{2.7.49}$$

Again the right hand side decays stretched exponentially in N for $k \ge 4$ for $\beta > 0$ small enough. For $k \le 3$ we can repeat the ideas of (2.7.47). The reason we can prove an upper bound in the same way as in (2.7.46) and (2.7.47) is that the probability for a box to be bad in $\tilde{\xi}^k$ is of the same order as $\mathbb{P}(A_{k,\Delta}^{\complement})$, namely $n_k^{-c\log n_k}$. Define

$$A_N := \bigcap_{k=1}^{r(N)} \bigcap_{r=0}^{\lceil \log_2 (\log N_{k-1})^3 \rceil} \Big\{|\Pi_{k-1}^{(r)} \cap \{\text{bad in } \xi\}| - |\Pi_{k-1}^{(r)} \cap \{\text{bad in } \tilde{\xi}^k\}| \ge C|\Pi_{k-1}^{(r)}| n_{k-1}^{-\beta}\Big\} \tag{2.7.50}$$

then by the same arguments as above we also get

$$\mathbb{P}(A_N^{\complement}) \le N^{-c\log N}. \tag{2.7.51}$$

Since $\widetilde{G}_N \cap A_N \subset G_N$ the claim follows. □

2.8 Mixing properties of the quenched law: proof of Lemma 2.6.1

Definition 2.8.1. Let Π_M be a partition of $\mathbb{Z}^d$ into boxes of side lengths M, let $C > 0$ and let ω be a realisation of the environment. We call a box $\Delta \in \Pi_M$ *social with respect to* ω at time $N \in \mathbb{N}$, if for any pair of points $x, y \in \Delta$ there exists $z \in \mathbb{Z}^d$ such that

$$P_\omega^{(x,N)}(X_{N+\lceil CM \rceil} = z) > 0, \quad \text{and} \quad P_\omega^{(y,N)}(X_{N+\lceil CM \rceil} = z) > 0.$$

Note that if $P_\omega^{(x,N)}(X_{N+\lceil CM \rceil} = z) > 0$, then by construction $P_\omega^{(x,N)}(X_{N+\lceil CM \rceil} = z) \ge (3^{-d})^{CM}$.

The next result shows that the density of social boxes is suitably high.

Lemma 2.8.2. *For every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$ and constants $c, C > 0$ such that for all $M \ge M_0$ there exists a set of environments S_M satisfying*

$$\sum_{\substack{\Delta \in \Pi_M \\ \Delta \text{ is not social}}} \mathbb{P}^{(x,0)}(X_n \in \Delta) < \varepsilon \quad \text{for all } \omega \in S_M$$

and $\mathbb{P}(S_M) \ge 1 - C\mathrm{e}^{-c\log n}$. (Recall that the property of Δ being social depends on ω.)

Corollary 2.8.3. *For every $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$ so that for all $M > M_0$ there are environments $\bar{S}_M$ such that*

$$\sum_{\substack{\Delta \in \Pi_M \\ \Delta \text{ is not social}}} P_\omega^{(x,0)}(X_n \in \Delta) < 2\varepsilon$$

for all $\omega \in \bar{S}_M$ and $\mathbb{P}(\bar{S}_M) \geq 1 - Cn^{-c \log n}$.

Proof. Combine Lemma 2.8.2 and Lemma 2.1.1. □

Proof of Lemma 2.8.2. The proof idea is similar to the one we have used to prove the high density of good boxes; see the proof of Proposition 2.7.4. We set

$$p_M \coloneqq \mathbb{P}(\Delta \text{ is not social}).$$

As a direct consequence of Lemma 2.10.1 for every $\Delta \in \Pi_M$ we have that $p_M \leq C\mathrm{e}^{-cM}$ for some positive constants C, c. We define

$$S_M \coloneqq \bigcap_{r=0}^{\log_2 \log^3 n} \left\{ |\Pi_M^{(r)} \cap \{\text{not social boxes}\}| < C|\Pi_M^{(r)}| p_M \right\}, \tag{2.8.1}$$

where

$$\begin{aligned}
\Pi_M^{(0)} &= \{\Delta \in \Pi_M : \mathrm{dist}(\Delta, 0) \leq \sqrt{n}\}, \\
\Pi_M^{(r)} &= \{\Delta \in \Pi_M : 2^{r-1}\sqrt{n} < \mathrm{dist}(\Delta, 0) \leq 2^r \sqrt{n}\} \quad \text{for } r \geq 1.
\end{aligned}$$

By Lemma 3.6 from [Ste17] we have $\mathbb{P}^{(0,0)}(\|X_n\| \geq \sqrt{n} \log^3 n) \leq Cn^{-c \log n}$ and so for $\omega \in S_M$ (note that being social depends on ω)

$$\begin{aligned}
\sum_{\substack{\Delta \in \Pi_M \\ \Delta \text{ is not social}}} \mathbb{P}^{(0,0)}(X_n \in \Delta) &\leq Cn^{-c \log n} + \sum_{r=0}^{\log_2 \log^3 n} \sum_{\substack{\Delta \in \Pi_M^{(r)} \\ \Delta \text{ is not social}}} \mathbb{P}^{(0,0)}(X_n \in \Delta) \\
&\leq \sum_{r=0}^{\log_2 \log^3 n} C|\Pi_M^{(r)}| p_M \frac{1}{n^{d/2}} \exp\left(-\frac{1}{2n} (2^{r-1}\sqrt{n})^2 \right) \\
&\leq C \sum_{r=0}^{\log_2 \log^3 n} \left(\frac{2^r \sqrt{n}}{M} \right)^d \frac{1}{n^{d/2}} \exp(-cr^2) p_M \\
&\leq C p_M \sum_{r=0}^{\log_2 \log^3 n} \frac{1}{M^d} \exp(-cr^2 + rd \log 2) \\
&\leq C' p_M
\end{aligned}$$

where we used the annealed local CLT in the second inequality. It remains to show that $\mathbb{P}^{(0,0)}(S_M) \geq 1 - C\mathrm{e}^{-c \log n}$. We have

$$\begin{aligned}
\mathbb{P}^{(0,0)}(S_M^c) &= \mathbb{P}^{(0,0)}\big(\exists r \leq \log_2 \log^3 n : |\Pi_M^{(r)} \cap \{\text{not social boxes}| > C|\Pi_M^{(r)}| p_M\big) \\
&\leq \sum_{r=0}^{\log_2 \log^3 n} \mathbb{P}^{(0,0)}\big(|\Pi_M^{(r)} \cap \{\text{not social boxes}| > C|\Pi_M^{(r)}| p_M\big).
\end{aligned}$$

Next, let $(\Pi_M^{r,j})_{j\in J}$ be a further partition of $\Pi_M^{(r)}$ so that for each $j \in J$ the distance between any pair of distinct boxes in $\Pi_M^{r,j}$ is bigger than $3CM$ and

$$\frac{|\Pi_M^{(r)}|}{2(3CM)^d} \le |\Pi_M^{r,j}| \le \frac{2|\Pi_M^{(r)}|}{(3CM)^d}.$$

Note that the index set $J = J(M,r)$ is finite (in fact we have $|J| \le 2(3CM)^d$) and that by construction the boxes in $\Pi_M^{r,j}$ are social or not social independently of each other. If $|\Pi_M^r \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M$ then there exists a j such that $|\Pi_M^{r,j} \cap \{\text{not social boxes}\}| > C|\Pi_M^{(r)}|p_M/(3CM)^d$. Using Hoeffding's inequality for $r \ge 1$ we obtain

$$\begin{aligned}
&\mathbb{P}^{(0,0)}\big(|\Pi_M^{(r)} \cap \{\text{not social boxes}| > C|\Pi_M^{(r)}|p_M\big) \\
&\quad\le \sum_{j\in J} \mathbb{P}^{(0,0)}\Big(|\Pi_M^{r,j} \cap \{\text{not social boxes}| > \frac{C|\Pi_M^{(r)}|p_M}{(3CM)^d}\Big) \\
&\quad= \sum_{j\in J} \mathbb{P}^{(0,0)}\Big(|\Pi_M^{r,j} \cap \{\text{not social boxes}| - |\Pi_M^{r,j}|p_M > \Big(\frac{C|\Pi_M^{(r)}|}{(3CM)^d} - |\Pi_M^{r,j}|\Big)p_M\Big) \\
&\quad\le \sum_{j\in J} \exp\Big(-2p_M^2\Big(C\frac{|\Pi_M^{(r)}|}{(3CM)^d} - |\Pi_M^{r,j}|\Big)^2\Big) \\
&\quad\le \sum_{j\in J} \exp\Big(-2p_M^2(C-2)\frac{|\Pi_M^{(r)}|^2}{(3CM)^{2d}}\Big) \\
&\quad\le 2(3CM)^d \exp\Big(-Cp_M^2\frac{(2^{r-1}\sqrt{n})^{2d}}{(3CM)^{2d}}\Big).
\end{aligned}$$

Similarly for $r = 0$ we have

$$\mathbb{P}^{(0,0)}\big(|\Pi_M^{(0)} \cap \{\text{not social boxes}| > C|\Pi_M^{(0)}|p_M\big) \le 2(3CM)^d \exp\Big(-Cp_M^2\frac{\sqrt{n}^{2d}}{(3CM)^{2d}}\Big).$$

Using the above estimates we obtain

$$\begin{aligned}
\mathbb{P}^{(0,0)}(S_M^c) &\le 2(3CM)^d \exp\Big(-Cp_M^2\frac{\sqrt{n}^{2d}}{(3CM)^{2d}}\Big) + \sum_{r=1}^{\log_2\log^3 n} 2(3CM)^d \exp\Big(-Cp_M^2\frac{(2^{r-1}\sqrt{n})^{2d}}{(3CM)^{2d}}\Big) \\
&\le \log_2\log^3(n)\cdot\exp\Big(-Cp_M^2\frac{\sqrt{n}^{2d}}{(3CM)^{2d}}\Big) \\
&\le Cn^{-c\log n}.
\end{aligned}$$

□

Proof of Lemma 2.6.1. The proof relies on a construction of a suitable coupling of $P_\omega^{(x,0)}(X_n \in \cdot)$ and $P_\omega^{(y,0)}(X_n \in \cdot)$. First we show that there is a coupling on the level of boxes with side length M, where M is a constant. Let Π_M be a partition of $\mathbb{Z}^d$ in boxes of side length M and fix x and y. Set

$$\begin{aligned}
F_{n^\theta} := \bigcap_{k\ge n^\theta} \Big\{&\omega : \forall z \in [-k,k]^d \cap \mathbb{Z}^d, \\
&\sum_{\Delta\in\Pi_M} |\mathbb{P}^{(z,0)}(X_k \in \Delta) - P_\omega^{(z,0)}(X_k \in \Delta)| \le \frac{C_1}{k^{c_2}} + \frac{C_1}{M^{c_2}}\Big\}.
\end{aligned}$$

and

$$F(x,y) := \bigcap_{\substack{(\tilde{x},m)\in\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{x}-x\|\le n\\ m\le n}} \sigma_{(\tilde{x},m)} F_{n^\theta} \cap \bigcap_{\substack{(\tilde{y},m)\in\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{y}-y\|\le n\\ m\le n}} \sigma_{(\tilde{y},m)} F_{n^\theta}$$

By Lemma 2.1.1 we have $\mathbb{P}(F_{n^\theta}) \ge 1 - n^{-c\log n}$ and thus $\mathbb{P}(F(x,y)) \ge 1 - Cn^{-c\log n}$. In the following we assume that the indices of the random walks are integers, otherwise we take the integer part. Now choosing M and n large enough for $\|x-y\| \le n^\theta$ on the event $F(x,y)$ we obtain

$$\begin{aligned}
&\sum_{\Delta\in\Pi_M} |P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta) - P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta)| \\
&\quad\le \sum_{\Delta\in\Pi_M} |P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta) - \mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta)| \\
&\qquad + \sum_{\Delta\in\Pi_M} |P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta)| \\
&\qquad + \sum_{\Delta\in\Pi_M} |\mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta)| \\
&\quad\le \frac{1}{8} + \frac{1}{8} + \sum_{\Delta\in\Pi_M} |\mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta)| \\
&\quad\le \frac{1}{4} + \sum_{\Delta\in\Pi_M^{x,y}(n^{2\theta}\log^{8d} n^\theta)} |\mathbb{P}^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta) - \mathbb{P}^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \Delta)| + Cn^{-c\log n} \\
&\quad\le \frac{1}{4} + Cn^{-c\log n} + |\Pi_M^{x,y}(n^{2\theta}\log^{8d} n^\theta)| dn^\theta C(n^{2\theta}\log^{8d} n^\theta)^{-\frac{d+1}{2}} \\
&\quad\le \frac{1}{4} + Cn^{-c\log n} + 2\Big(n^\theta \log^{4d}(n^\theta)\log^3(n^{2\theta}\log^{8d} n^\theta)\Big)^d dn^\theta C(n^{2\theta}\log^{8d} n^\theta)^{-\frac{d+1}{2}} \\
&\quad= \frac{1}{4} + Cn^{-c\log n} + C\Big(\log(n^{2\theta}\log^{8d} n^\theta)\Big)^{3d} \log^{-4d}(n^\theta) \\
&\quad< \frac{1}{2},
\end{aligned}$$

for n large enough, where

$$\Pi_M^{x,y}(m) := \Big\{\Delta \in \Pi_M : \Delta \cap \{z \in \mathbb{Z}^d : \min(\|x-z\|, \|y-z\|) \le \sqrt{m}\log^3 m\} \neq \emptyset\Big\}$$

and we used Lemma 3.6 from [Ste17] and the annealed derivative estimates (Lemma 3.9 from [Ste17]). The number of steps we chose might seem a bit strange at first. The choice becomes more clear by looking at the last inequality above. There we see that, with the methods we use, we need a bit more steps than the square of the current distance. One can calculate that any additional factor $\log^m(n^\theta)$ with $m > 6d$ is enough to get the estimate. So there exists a coupling $\Xi^{x,y}_{\omega,n^{2\theta}\log^{8d} n^\theta}$ of $P_\omega^{(x,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \cdot)$ and $P_\omega^{(y,0)}(X_{n^{2\theta}\log^{8d} n^\theta} \in \cdot)$ on $\Pi_M \times \Pi_M$ such that for $\omega \in F(x,y)$

$$\Xi^{x,y}_{\omega,n^{2\theta}\log^{8d} n^\theta}(\{(\Delta,\Delta) : \Delta \in \Pi_M\}) > \frac{1}{2}.$$

Recall $\bar{S}_M$ from Corollary 2.8.3. We have for

$$\omega \in H(x,y) := F(x,y) \cap \bigcap_{\substack{(\tilde{x},m)\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{x}-x\|\le n\\ m\le n}} \sigma_{(\tilde{x},m)} \bar{S}_M \cap \bigcap_{\substack{(\tilde{y},m)\mathbb{Z}^d\times\mathbb{N}_0\\ \|\tilde{y}-y\|\le n\\ m\le n}} \sigma_{(\tilde{y},m)} \bar{S}_M$$

that

$$\sum_{\substack{\Delta\in\Pi_M \\ \Delta \text{ is social}}} \Xi^{x,y}_{\omega,n^{2\theta}\log^{8d} n^\theta}(\Delta,\Delta) > \frac{1}{2} - \varepsilon(M) > \frac{1}{4}.$$

By Corollary 2.8.3 we obtain $\mathbb{P}(H(x,y)) \geq 1 - Cn^{-c\log n}$. Thus, by the definition of social boxes (Definition 2.8.1), we can construct a coupling $\widetilde{\Xi}^{x,y}_{\omega,n^\theta}$ of $P^{(x,0)}_\omega(X_{n^{2\theta}\log^{8d} n^\theta + CM} \in \cdot)$ and $P^{(y,0)}_\omega(X_{n^{2\theta}\log^{8d} n^\theta + CM} \in \cdot)$ satisfying $\widetilde{\Xi}^{x,y}_{\omega,n^\theta}(\{(z,z) : z\in\mathbb{Z}^d\}) > \frac{1}{4}(\frac{1}{3^d})^{2CM}$. If this coupling is successful, we let the random walks go along the same path until time n. In case it isn't, we try to couple from their current position. Note that $\omega\in H(x,y)$ ensures, that we can repeat the coupling attempt at the new positions.

For the rest of the proof let $n_k := n^\theta \log^{k(4d+3)} n$, $k\in\mathbb{N}_0$ and $s_k := n_k^2\log^{8d} n_k + CM$. The n_k will represent the distance between the walkers at the start of an attempt at coupling and s_k will be the number of steps necessary for the attempt. Furthermore let $S_k := \sum_{i=0}^k s_i$.

By Lemma 3.6 from [Ste17], we know that with probability of at least $1 - Cn^{-c\log n}$ the distance between the random walks will only be

$$\left(n^{2\theta}\log^{8d} n^\theta\right)^{1/2} \log^3\left(n^{2\theta}\log^{8d} n^\theta\right) \leq n^\theta \log^{4d}(n^\theta)\log^3(n) \leq n^\theta \log^{4d+3} n = n_1,$$

as long as $8d \leq (1-2\theta)\frac{\log n}{\log\log n^\theta}$. This condition is not a restriction, since we will let $n\to\infty$.

Let us now iterate the coupling procedure. If the coupling in step $k-1$ is not successful, i.e. if the walks are not at the same point, we try to couple again starting from the current positions. This leads to an iterative coupling $\widehat{\Xi}$ of the following form: $\widehat{\Xi}^{x,y}_{\omega,0} = \widetilde{\Xi}^{x,y}_{\omega,n_0} = \widetilde{\Xi}^{x,y}_{\omega,n^\theta}$ and for $k\geq 1$

$$\begin{aligned}\widehat{\Xi}^{x,y}_{\omega,k}(z_1,z_2) = \sum_{a,b\in\mathbb{Z}^d} \widehat{\Xi}^{x,y}_{\omega,k-1}(a,b)\cdot\Big[&\mathbb{1}_{\{a=b\}}\mathbb{1}_{\{z_1=z_2\}}P^{(a,S_{k-1})}_\omega(X_{S_k}=z_1) \\ &+ \mathbb{1}_{\{0<\|a-b\|\leq n_k\}}\widetilde{\Xi}^{a,b}_{\omega,n_k}(z_1,z_2) \\ &+ \mathbb{1}_{\{\|a-b\|>n_k\}}P^{(a,S_{k-1})}_\omega(X_{S_k}=z_1)P^{(b,S_{k-1})}_\omega(X_{S_k}=z_2)\Big],\end{aligned}$$

where $\widetilde{\Xi}^{a,b}_{\omega,n_k}$ is a coupling of $P^{(a,S_{k-1})}_\omega(X_{S_k}\in\cdot)$ and $P^{(b,S_{k-1})}_\omega(X_{S_k}\in\cdot)$. The idea is that the random walks will stay together once they are at the same site. We try to couple them via $\widetilde{\Xi}^{a,b}_{\omega,n_k}$ if their distance is not too large and we let them evolve independently otherwise.

Since at distance n_k for the next coupling we walk s_k steps and with high probability have at most a distance of $s_k^{1/2}\log^3 s_k$, the above coupling will work as long as $k \leq \frac{(1-2\theta)\log n}{(8d+6)\log\log n} - \frac{8d}{8d+6}$ holds, which we show below. We obtain

$$s_k^{1/2}\log^3 s_k = \left(n_k^2\log^{8d} n_k + CM\right)^{1/2}\log^3\left(n_k^2\log^{8d} n_k + CM\right).$$

Now for $k \leq \frac{(1-2\theta)\log n}{(4d+3)\log\log n}$ and n large enough

$$n_k^2\log^{8d} n_k + CM \leq n_k^2\log^{8d} n$$

and

$$\log^{4d} n_k = \log^{4d}\left(n^\theta\log^{k(4d+3)}(n)\right) \leq \log^{4d} n.$$

Thus, we have

$$s_k^{1/2}\log^3 s_k \le n_k \log^{4d}(n)\log^3\Big(n_k^2\log^{8d} n\Big).$$

Furthermore, if $k \le \frac{(1-2\theta)\log n}{(8d+6)\log\log n} - \frac{8d}{8d+6}$ then

$$\begin{aligned}2\log n_k + 8d\log\log n &= 2\log\big(n^\theta \log^{k(4d+3)} n\big) + 8d\log\log n\\ &= 2\theta\log n + k(8d+6)\log\log n + 8d\log\log n \le \log n\end{aligned}$$

It follows that

$$s_k^{1/2}\log^3 s_k \le 2n_k\log^{4d} n\log^3 n = 2n^\theta\log^{(k+1)(4d+3)}(n) = n_{k+1}.$$

So after we try the k-th coupling we are, with high probability, at distance n_{k+1}. The probability for each try to be successful is bounded from below by $\frac{1}{4}(\frac{1}{3^d})^{2CM}$ and we have $\frac{(1-2\theta)\log n}{(8d+6)\log\log n} - 1$ attempts. So the time we need for those attempts is

$$\begin{aligned}\sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} s_k &= \sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} n_k^2\log^{8d} n_k + CM\\ &\le \sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} n^{2\theta}\log^{k(8d+6)}(n)\log^{8d}(n) + CM \qquad (2.8.2)\\ &= \frac{(1-2\theta)\log n}{(8d+6)\log\log n}CM + n^{2\theta}\log^{8d}(n)\sum_{k=0}^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1}\Big(\log^{(8d+6)}(n)\Big)^k.\end{aligned}$$

Note that

$$(\log n)^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}(8d+6)} = \exp\big((1-2\theta)\log n\big) = n^{1-2\theta}$$

and therefore the right hand side of (2.8.2) is bounded from above by

$$\begin{aligned}\frac{(1-2\theta)\log n}{(8d+6)\log\log n}CM + n^{2\theta}\log^{8d}(n)\frac{n^{1-2\theta}-1}{\log^{(8d+6)}(n)-1} &\le \frac{(1-2\theta)\log n}{(8d+6)\log\log n}CM + \frac{n}{\log^5(n)}\\ &= n\left(\frac{(1-2\theta)\log n}{n(8d+6)\log\log n}CM + \frac{1}{\log^5 n}\right) < n,\end{aligned}$$

for n large enough. And the probability for the above coupling to fail is smaller than

$$(1-p^*)^{\frac{(1-2\theta)\log n}{(8d+6)\log\log n}-1} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}}$$

where $p^* = \frac{1}{4}(\frac{1}{3^d})^{2CM}$ and $c > 0$ is a constant. So for a fixed pair of points x, y with $\|x-y\| \le n^\theta$ we have

$$\Big\|P_\omega^{(x,0)}(X_n\in\cdot) - P_\omega^{(y,0)}(X_n\in\cdot)\Big\|_{TV} \le \mathrm{e}^{-c\frac{\log n}{\log\log n}}$$

with probability at least $1-n^{-c\log n}$. Thus we get for every $b>0$

$$\begin{aligned}\mathbb{P}(D(n)) &= \mathbb{P}\Bigg(\bigcap_{\substack{x,y\in\mathbb{Z}^d:\\ \|x\|,\|y\|\le n^b,\\ \|x-y\|\le n^\theta}}\Big\{\Big\|P_\omega^{(x,0)}(X_n\in\cdot)-P_\omega^{(y,0)}(X_n\in\cdot)\Big\|_{\mathrm{TV}}\le \mathrm{e}^{-c\frac{\log n}{\log\log n}}\Big\}\Bigg)\\ &\ge 1-\sum_{\substack{x,y\in\mathbb{Z}^d:\\ \|x\|,\|y\|\le n^b,\\ \|x-y\|\le n^\theta}}\mathbb{P}\Big(\Big\{\Big\|P_\omega^{(x,0)}(X_n\in\cdot)-P_\omega^{(y,0)}(X_n\in\cdot)\Big\|_{\mathrm{TV}}> \mathrm{e}^{-c\frac{\log n}{\log\log n}}\Big\}\Big)\\ &\ge 1-n^{d(b+\theta)}n^{-c\log n}\ge 1-Cn^{-c'\log n}.\end{aligned}$$

Note that $b>0$ can be chosen arbitrarily large, but the constants C and c' will have to adjusted accordingly.

□

2.9 Uniqueness of the prefactor

With some minor adaptations of the ideas from [BCR16, Section 7.1] we can obtain the following result.

Lemma 2.9.1. *Provided existence, the prefactor φ in (1.2.10) is unique.*

Proof. Assume that there are functions f and g which both satisfy (1.2.10) and denote $h=f-g$. We will check that $\mathbb{E}[|h|]=0$ and hence that $h\equiv 0$ holds $\mathbb{P}$-a.s.

By the triangle inequality we have

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n=x)|h(\sigma_{(x,n)}\omega)| &\le \sum_{x\in\mathbb{Z}^d}|P_\omega^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_n=x)f(\sigma_{(x,n)}\omega)|\\ &\quad+\sum_{x\in\mathbb{Z}^d}|P_\omega^{(0,0)}(X_n=x)-\mathbb{P}^{(0,0)}(X_n=x)g(\sigma_{(x,n)}\omega)|\end{aligned}\tag{2.9.1}$$

which by (1.2.10) implies

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n=x)|h(\sigma_{(x,n)}\omega)|=0\tag{2.9.2}$$

for $\mathbb{P}$ almost every ω. That means $\lim_{n\to\infty}\mathbb{E}^{(0,0)}[|h(\sigma_{(X_n,n)}\omega)|]=0$ $\mathbb{P}$-a.s. Assume that $h\ne 0$, then there exists a measurable subset $A\subset\Omega$ and a constant $c>0$ such that $\mathbb{P}(A)>0$ and $|h|>c$ on A. Thus, for every $n\in\mathbb{N}$, an elementary computation shows

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}^{(0,0)}[|h(\sigma_{(X_n,n)}\omega)|]\Big] &\ge \mathbb{E}\Big[\mathbb{E}^{(0,0)}[|h(\sigma_{(X_n,n)}\omega)|\mathbb{1}_{\{\sigma_{(X_n,n)}\omega\in A\}}]\Big]\\ &\ge c\mathbb{E}\Big[\mathbb{E}^{(0,0)}[\mathbb{1}_{\{\sigma_{(X_n,n)}\omega\in A\}}]\Big]\\ &= c\mathbb{P}(A)>0.\end{aligned}\tag{2.9.3}$$

Since

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}^{(0,0)}[|h(\sigma_{(X_n,n)}\omega)|]\Big] &= \mathbb{E}\Big[\sum_{y\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n=y)|h(\sigma_{(y,n)}\omega)|\Big]\\ &=\sum_{y\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n=y)\mathbb{E}[|h(\sigma_{(y,n)}\omega)|]\\ &=\sum_{y\in\mathbb{Z}^d}\mathbb{P}^{(0,0)}(X_n=y)\mathbb{E}[|h(\omega)|]=\mathbb{E}[|h(\omega)|],\end{aligned}\tag{2.9.4}$$

the sequence $\{|h(\sigma_{(X_n,n)}\omega)|\}_{n\in\mathbb{N}}$ is tight. Thus, (2.9.3) implies that for $\mathbb{P}$ almost all ω we have $\lim_{n\to\infty}\mathbb{E}^{(0,0)}[|h(\sigma_{(X_n,n)}\omega)|] > 0$ which is a contradiction to (2.9.2). □

2.10 Intersection of clusters of points connected to infinity

The following lemma is a quantification of Theorem 2 from [GH02] which was pointed out there without a proof. We give a proof using a key result from [GM14].

Lemma 2.10.1. *Let $d \geq 2$, $p > p_c$. Then there are positive constants M and C and c such that for all $x, y \in \mathbb{Z}^d$ with $\|x-y\| \leq M$*

$$\mathbb{P}\big(B(x,y;M,C)|(x,0)\to\infty,(y,0)\to\infty\big) \geq 1-\exp(-cM), \tag{2.10.1}$$

where $B(x,y;M,C)$ is the set of all $\omega\in\Omega$ for which there is $z\in\mathbb{Z}^d$ satisfying

$$(x,0)\xrightarrow{\omega}(z,CM),\quad (y,0)\xrightarrow{\omega}(z,CM)\quad \text{and}\quad (z,CM)\xrightarrow{\omega}\infty.$$

Proof. For $A\subset\mathbb{Z}^d$ we put $\eta_t^A(x) = \mathbb{1}_{(y,0)\to(x,t)\text{ for some }y\in A}$ (this is the discrete time contact process starting from all sites in A infected at time 0). Write

$$B(x,t) := \big\{\exists\, z\,:\,\|x-z\|\leq c_1 t \text{ and } \eta_t^{\{x\}}(z)\neq\eta_t^{\mathbb{Z}^d}(z)\big\} \tag{2.10.2}$$

for the "bad" event that coupling in a ball around x has not occurred at time t. We obtain from [GM14, Thm. 1, Formula (3)] that

$$\mathbb{P}\big(B(x,t)\cap\{(x,0)\to\infty\}\big)\leq C\mathrm{e}^{-ct} \tag{2.10.3}$$

for certain constants $c_1, C, c\in(0,\infty)$ (which depend on d and on $p>p_c$). Literally, the result in [GM14] is proved for the continuous time version of the contact process, but we believe that the same holds in discrete time.

Now consider $x,y\in\mathbb{Z}^d$ with $\|x-y\|\leq M$. Pick C_2 so large that

$$J := \{z : \|z-x\|\leq C_2M \text{ and } \|z-y\|\leq C_2M\}$$

satisfies $\#J\geq M^d$. Applying (2.10.3) with $t=C_2M$ for x and for y gives

$$\begin{aligned}&\mathbb{P}\Big(\big(B(x,C_2M)\cup B(y,C_2M)\big)\cap\{(x,0)\to\infty,(y,0)\to\infty\}\Big)\\ &\quad\leq\mathbb{P}\big(B(x,C_2M)\cap\{(x,0)\to\infty\}\big)+\mathbb{P}\big(B(y,C_2M)\cap\{(y,0)\to\infty\}\big)\leq 2C\mathrm{e}^{-cCM_2}\end{aligned} \tag{2.10.4}$$

hence

$$\mathbb{P}\Big(\eta_{C_2M}^{\{x\}}(z)=\eta_{C_2M}^{\mathbb{Z}^d}(z)=\eta_{C_2M}^{\{y\}}(z)\text{ for all }z\in J\,\Big|\,(x,0)\to\infty,(y,0)\to\infty\Big)\geq 1-C'\mathrm{e}^{-cC_2M}. \tag{2.10.5}$$

Furthermore

$$\begin{aligned}&\mathbb{P}\Big(\exists\, z\in J\,:\,\eta_{C_2M}^{\mathbb{Z}^d}(z)=1\text{ and }(z,C_2M)\to\infty\,\Big|\,(x,0)\to\infty,(y,0)\to\infty\Big)\\ &\quad\geq\mathbb{P}\Big(\exists\, z\in J\,:\,\eta_{C_2M}^{\mathbb{Z}^d}(z)=1\text{ and }(z,C_2M)\to\infty\Big)\geq 1-C\mathrm{e}^{-cM^d}\end{aligned} \tag{2.10.6}$$

where we used the FKG inequality in the first inequality. For the second inequality we use the fact that extinction starting from A is exponentially unlikely in $\#A$ (see Theorem 2.30 (b) in [Lig99]) and the fact that $\eta^{\mathbb{Z}^d}_{C_2M}$ dominates the upper invariant measure which itself dominates a product measure on $\{0,1\}^{\mathbb{Z}^d}$ with some density $\rho > 0$ (see Corollary 4.1 in [LS06]).

Combining, we find

$$\begin{aligned}&\mathbb{P}\Big(\exists\, z \in \mathbb{Z}^d \,:\, (x,0) \to (z, C_2M), (y,0) \to (z, C_2M), (z, C_2M) \to \infty \,\Big|\, (x,0) \to \infty, (y,0) \to \infty\Big)\\ &\geq 1 - C'\mathrm{e}^{-cC_2M} - C\mathrm{e}^{-cM^d}.\end{aligned} \tag{2.10.7}$$

□

2.11 Quenched random walk finds the cluster fast

Since we allow the quenched random walk to start outside the cluster we need some kind of control on the time it needs to hit the cluster. The following lemma will yield exactly that.

Lemma 2.11.1. *Let $d \geq 1$ and define the set $A_n = A_n(C', c') \coloneqq \{\omega \in \Omega : P^{(0,0)}_\omega(\xi_i(X_i) = 0, i = 1, \dots, n) \leq C'\mathrm{e}^{-c'n}\}$. There exist constants $C, c > 0$, so that for every $p > p_c(d)$ and small enough C' and c' we have*

$$\mathbb{P}(A_n^{\complement}) \leq C\mathrm{e}^{-cn} \quad \textit{for all } n = 1, 2, \dots.$$

Proof. Note that by our definition of the quenched law, see equation (1.2.2), the quenched random walk performs a simple random walk until it hits the cluster $\mathcal{C}$. Thus, on the event that the random walk doesn't hit the cluster, we can switch the random walk with a simple random walk $(Y_n)_n$ that is independent of the environment. Using Lemma 2.11 from [BČD16] it follows

$$\begin{aligned}&\mathbb{P}^{(0,0)}\big(\xi_0(X_0) = \dots = \xi_n(X_n) = 0\big)\\ &\qquad= \sum_{x_1,\dots,x_n} \mathbb{P}^{(0,0)}\big((X_1,\dots,X_n) = (x_1,\dots,x_n), \xi_0(0) = \dots = \xi_n(x_n) = 0\big)\\ &\qquad= \sum_{x_1,\dots,x_n} \mathbb{P}^{(0,0)}\big((Y_1,\dots,Y_n) = (x_1,\dots,x_n), \xi_0(0) = \dots = \xi_n(x_n) = 0\big)\\ &\qquad= \sum_{x_1,\dots,x_n} \mathbb{P}^{(0,0)}\big((Y_1,\dots,Y_n) = (x_1,\dots,x_n)\big)\mathbb{P}\big(\xi_0(0) = \dots = \xi_n(x_n) = 0\big)\\ &\qquad\leq \tilde{C}\mathrm{e}^{-\tilde{c}n},\end{aligned}$$

where $\tilde{C}$ and $\tilde{c}$ are certain constants depending only on p and d.

Using the definition of the annealed law we get

$$\begin{aligned}&\mathbb{P}^{(0,0)}\big(\xi_0(X_0) = \dots = \xi_n(X_n) = 0\big)\\ &\quad= \int_{A_n} P^{(0,0)}_\omega(\xi_i(X_i) = 0, i = 1,\dots,n)\, d\mathbb{P}(\omega) + \int_{A_n^{\complement}} P^{(0,0)}_\omega(\xi_i(X_i) = 0, i = 1,\dots,n)\, d\mathbb{P}(\omega)\\ &\quad\geq \int_{A_n^{\complement}} P^{(0,0)}_\omega(\xi_i(X_i) = 0, i = 1,\dots,n)\, d\mathbb{P}(\omega)\\ &\quad> \mathbb{P}(A_n^{\complement})C'\mathrm{e}^{-c'n}\end{aligned}$$

and since

$$\int_{A_n^{\mathsf{C}}} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1\ldots,n)\, d\mathbb{P}(\omega) \leq \tilde{C}\mathrm{e}^{-\tilde{c}n}$$

we obtain that $\mathbb{P}(A_n^{\mathsf{C}}) \leq C\mathrm{e}^{-cn}$ with $c = \tilde{c} - c' > 0$ by choosing $c' < \tilde{c}$. □

Chapter 3

Quenched central limit theorem for random walks in oriented percolation

In this chapter we will prove Theorem 1.3.5, a quenched CLT for a random walk in a dynamic random environment, that is given by oriented percolation. Contrary to Chapter 2 the random walk does not have to stay on the cluster. Instead we make assumptions, e.g. finite range and local dependence on the environment, that allow for a more general class of random walks. We start by construction suitable regeneration times at which two random walks in a joint environment regenerate simultaneously. This construction is inspired by [BČD16], where they define regeneration times for a single random walk. Then we will compare two random walks (X, X') evolving in a joint environment with two random walks (Y, Y') evolving in independent environments along their respective simultaneous regeneration times. It turns out that, as long as the initial distance of X and X' is large, they behave as Y and Y' with high probability.

3.1 Regeneration Construction

Let $X \coloneqq (X_n)_n$ and $X' \coloneqq (X'_n)_n$ be two random walks in the same environment. We introduce a sequence of regeneration times $T_1 < T_2, \dots$, at which both X and X' regenerate. The construction of the regeneration times extends the corresponding construction from [BČD16] for a single random walk.

The goal is to isolate the part of the environment that the two random walks explore until they regenerate from the rest of the environment. This isolation will be achieved by certain cones in which the two random walks will move. Let us recall the definitions of cones and cone shells from equations (2.25) and (2.27) in [BČD16].

For positive b, s, h and $x_{\mathsf{bas}} \in \mathbb{Z}^d$ we set

$$\mathsf{cone}(x_{\mathsf{bas}}; b, s, h) \coloneqq \big\{(x, n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : \ \|x_{\mathsf{bas}} - x\|_2 \le b + sn, 0 \le n \le h\big\}. \tag{3.1.1}$$

for a (truncated upside-down) *cone* with base radius b, slope s, height h and base point $(x_{\mathsf{bas}}, 0)$. Furthermore for

$$b_{\text{inn}} \le b_{\text{out}} \quad \text{and} \quad s_{\text{inn}} < s_{\text{out}}, \tag{3.1.2}$$

we define the *conical shell* with inner base radius b_{inn}, inner slope s_{inn}, outer base radius b_{out}, outer slope s_{out}, and height $h \in \mathbb{N} \cup \{\infty\}$ by

$$\begin{aligned}&\mathsf{cs}(x_{\mathsf{bas}}; b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)\\ &\qquad := \big\{(x,n) \in \mathbb{Z}^d \times \mathbb{Z} : b_{\text{inn}} + s_{\text{inn}} n \le \|x_{\mathsf{bas}} - x\|_2 \le b_{\text{out}} + s_{\text{out}} n,\ 0 < n \le h\big\}. \qquad (3.1.3)\end{aligned}$$

The conical shell can be thought of as a difference of the *outer cone* $\mathsf{cone}(x_{\mathsf{bas}}; b_{\text{out}}, s_{\text{out}}, h)$ and the *inner cone* $\mathsf{cone}(x_{\mathsf{bas}}; b_{\text{inn}}, s_{\text{inn}}, h)$ with all boundaries except the bottom boundary of that difference included. To shorten notation for fixed parameters for radii and slopes as in (3.1.2), we write $\mathsf{cs}(x_{\mathsf{bas}}; h)$ for the cone shell as defined in (3.1.3).

For the proof of Theorem 1.3.5 we will follow the ideas from [BČDG13]. We now expand the cone construction for the regeneration times from [BČD16] to two cones and we define the so called *double cone shell* to isolate the area of the environment that two random walks have explored from the rest.

Consider two random walks X and X' located at time n at positions x_{bas} and x'_{bas} respectively, i.e. $X_n = x_{\mathsf{bas}}$ and $X'_n = x'_{\mathsf{bas}}$. A first attempt would be to just take the union of both cone shells $\mathsf{cs}(x_{\mathsf{bas}}; h)$ and $\mathsf{cs}(x'_{\mathsf{bas}}; h)$ with base points x_{bas} and x'_{bas}. The problem with this attempt is that the cone shell $\mathsf{cs}(x_{\mathsf{bas}}; h)$ would grow into the interior $\mathsf{cone}(x'_{\mathsf{bas}}; b_{\text{inn}}, s_{\text{inn}}, h)$ of $\mathsf{cs}(x'_{\mathsf{bas}}; h)$ and vice versa and in particular into the region which we want to isolate. Instead we take the union of the cone shells without the elements of the inner cones and define the double cone shell

$$\begin{aligned}&\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; h)\\ &\qquad := \left(\mathsf{cs}(x_{\mathsf{bas}}; h) \cup \mathsf{cs}(x'_{\mathsf{bas}}; h)\right) \setminus \left(\mathsf{cone}(x_{\mathsf{bas}}; b_{\text{inn}}, s_{\text{inn}}, h) \cup \mathsf{cone}(x'_{\mathsf{bas}}; b_{\text{inn}}, s_{\text{inn}}, h)\right). \qquad (3.1.4)\end{aligned}$$

Note that we again omitted the base radii b_{inn}, b_{out} and slopes s_{inn}, s_{out}. Of course the double cone shell $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; h) = \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)$ depends on these parameters as well. We also write $\mathsf{cone}(x_{\mathsf{bas}}) = \mathsf{cone}(x_{\mathsf{bas}}; \infty)$ and $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}) = \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ if we consider the cone or cone shell with infinite height.

For the case $d = 2$ the double cone with the double cone shell is illustrated in Figure 3.1. For $d = 1$ we will use a slightly different double cone shell definition. It will make the arguments in the proof of Lemma 3.1.2 more streamlined and the difference of the definitions is explained in the proof. For the case $d = 1$ the double cone with the double cone shell is illustrated in Figure 3.3.

Remark 3.1.1. For notational convenience we assume all cones to be based at time $t = 0$. Obviously we can shift the cone to be based at an arbitrarily chosen space-time point in $\mathbb{Z}^d \times \mathbb{Z}$.

We follow the ideas of the proof of Lemma 2.13 from [BČD16]. For $d \ge 2$ we define a subset $\mathcal{M} \subset \mathbb{Z}^d \times \mathbb{Z}$ of the double cone shell with the following three properties:

1. Every path crossing from the outside to the inside has to hit a point in $\mathcal{M}$.
2. There exist small constants $\delta > 0$ and $\tilde{\delta} > 0$ such that for every $(x,n) \in \mathcal{M}$ we have $B_{\tilde{\delta} n}(x) \times \{n - \delta n\} \subset \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$, where $B_r(y)$ is the ball of radius r centred around y.

Note that the number of elements in $\mathcal{M} \cap \mathbb{Z}^d \times [0,n] \subset \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; n)$ grows polynomially in n. Such a set is given, for instance, by the "middle tube" of the double cone shell which we will now define more precisely.

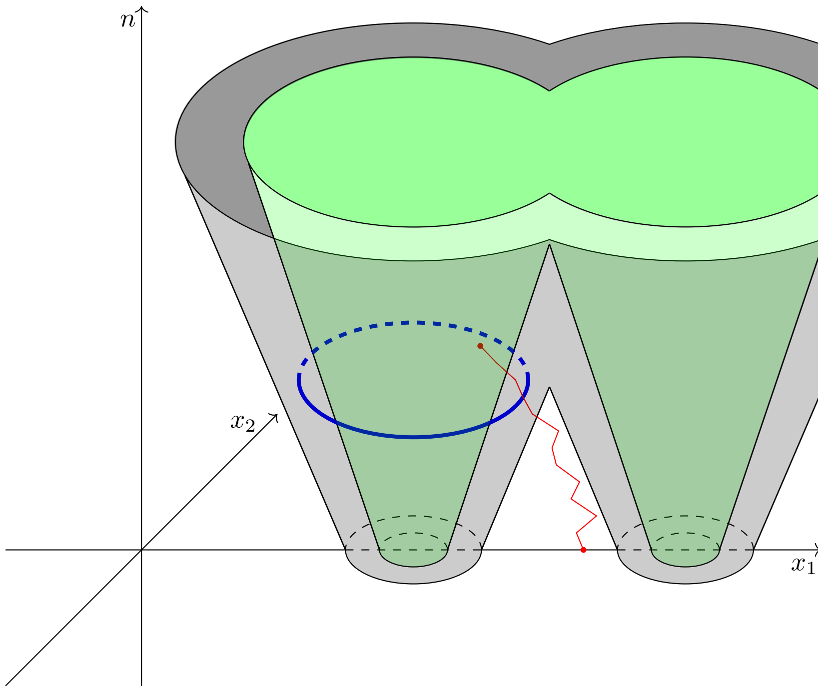

Figure 3.1: Double cone with a double cone shell (grey), a time slice of the middle tube (blue), and a path of a random walk crossing the double cone shell from outside to inside (red).

Let

$$d(n) := \frac{1}{2}\Big(n(s_{\text{out}} + s_{\text{inn}}) + b_{\text{out}} + b_{\text{inn}}\Big)$$

be the radius to the middle of the cone shell at time n and define

$$\begin{aligned} M_n &:= \{x \in \mathbb{Z}^d : \|x - x_{\mathsf{bas}}\|_2 \in [d(n), d(n) + 2d]\}, \\ M'_n &:= \{x \in \mathbb{Z}^d : \|x - x'_{\mathsf{bas}}\|_2 \in [d(n), d(n) + 2d]\} \end{aligned} \tag{3.1.5}$$

the middle tubes for the single cones at time n. We define the middle tube in such a way, "thickening" it by $2d$, so that we can ensure that a nearest neighbour path crossing the cone shell has to hit a site in the middle tube and cannot jump over it. Note that we define nearest neighbours according to the sup-norm, that is y is a nearest neighbour of x if and only if $\|x - y\|_\infty \leq 1$. Furthermore we define the middle tube for the double cone shell at time n by

$$M_n^{\mathsf{dcs}} := \big(M_n \cup M'_n\big) \setminus \Big\{x \in \mathbb{Z}^d\colon\ \min\{\|x - x_{\mathsf{bas}}\|_2\,, \|x - x'_{\mathsf{bas}}\|_2\} \leq d(n)\Big\} \tag{3.1.6}$$

and set $\mathcal{M} = \bigcup_n (M_n^{\mathsf{dcs}} \times \{n\})$; see Figure 3.2 for an illustration.

Let $\eta^{\mathsf{dcs}} := (\eta_n^{\mathsf{dcs}})_{n=0,1,\dots}$ be the contact process restricted to the infinite double cone shell $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ with initial condition $\eta_0^{\mathsf{dcs}}(x) = \mathbb{1}_{\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}};0)}((x, 0))$ and

$$\eta_{n+1}^{\mathsf{dcs}}(x) = \begin{cases} 1 & \text{if } (x, n+1) \in \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty),\ \omega(x, n+1) = 1 \\ & \text{and } \eta_n^{\mathsf{dcs}}(y) = 1 \text{ for some } y \in \mathbb{Z}^d \text{ with } \|x - y\| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

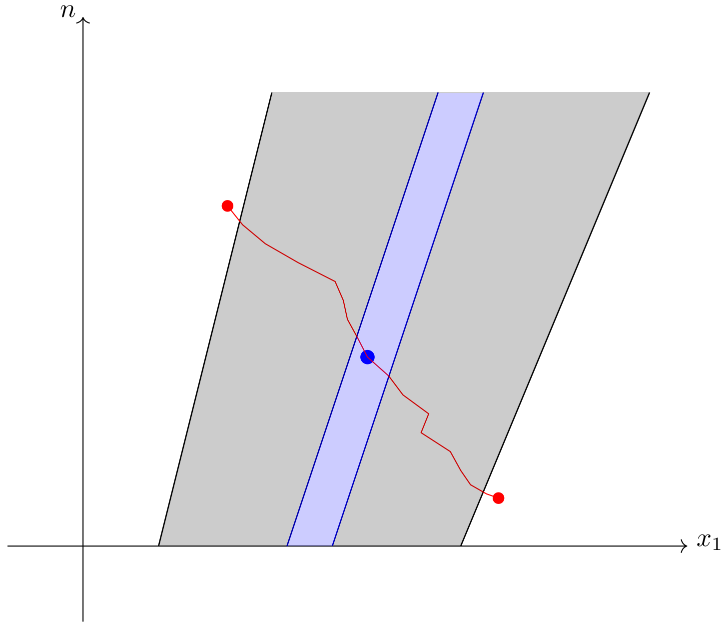

Figure 3.2: A cross section of the cone shell including the middle tube $\mathcal{M}$ (blue) and a path γ (red) crossing the cone shell from the outside to the inside of the double cone and hitting at least one point in $\mathcal{M}$ (blue dot).

We think of η^{dcs} as a version of the contact process where all ω's outside $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ have been set to 0. For a directed nearest neighbour path

$$\gamma = ((x_m, m), (x_{m+1}, m+1), \dots, (x_n, n)), \quad m \le n,\, x_i \in \mathbb{Z}^d \text{ with } \|x_{i-1} - x_i\| \le 1 \tag{3.1.7}$$

with starting position x_m at time m and final position x_n at time n we say that γ crosses the double cone shell $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ from the outside to the inside if the following three conditions are fulfilled:

(i) the starting position lies outside the double cone shell, i.e., $\|x_m - x_{\mathsf{bas}}\|_2 > b_{\text{out}} + m s_{\text{out}}$ and $\|x_m - x'_{\mathsf{bas}}\|_2 > b_{\text{out}} + m s_{\text{out}}$,

(ii) the terminal point lies inside one of the inner cones, i.e., $\|x_n - x_{\mathsf{bas}}\|_2 < b_{\text{inn}} + n s_{\text{inn}}$ or $\|x_n - x'_{\mathsf{bas}}\|_2 < b_{\text{inn}} + n s_{\text{inn}}$,

(iii) all the remaining points lie inside the shell, i.e., $(x_i, i) \in \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ for $i = m+1, \dots, n-1$.

We say that γ (from (3.1.7)) intersects η^{dcs} if there exists $i \in \{m+1, \dots, n-1\}$ with $\eta_i^{\mathsf{dcs}}(x_i) = 1$. Finally we say that γ is open in $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ if $\omega(x_i, i) = 1$ for all $i = m+1, \dots, n-1$.

Lemma 3.1.2 (2-cone analogue of Lemma 2.13 from [BČD16])**.** *Assume that the relations in* (3.1.2) *hold and consider the events*

$$G_1 := \{\eta^{\mathsf{dcs}} \text{ survives}\},$$
$$G_2 := \{\text{every open path } \gamma \text{ that crosses } \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty) \text{ intersects } \eta^{\mathsf{dcs}}\}.$$

For any $\varepsilon > 0$ and $0 \leq s_{\text{inn}} < s_{\text{out}} < 1$ one can choose p sufficiently close to 1 and $b_{\text{inn}} < b_{\text{out}}$ sufficiently large so that

$$\mathbb{P}(G_1 \cap G_2) \geq 1 - \varepsilon.$$

Remark 3.1.3. Note that in this preparation section we base the cones at time 0. Later on they will be based at the current space-time position of the random walk but all results here hold for the shifted constructions as well due to translation invariance. Furthermore, with the properties we impose on $\mathcal{M}$ on the event $G_1 \cap G_2$ any site inside the inner cones which is connected to $\mathbb{Z}^d \times \{0\}$ via a path that crosses the cone shell also has a connection to $\mathbb{Z}^d \times \{0\}$ inside the double cone shell. Thus, on the event $G_1 \cap G_2$ we isolate all the sites in the inner cone from the information on the environment outside the outer cone in the sense that the value of η inside the double cone can be determined using the values of ω's inside the double cone.

Proof of Lemma 3.1.2. The analogous result for the single cone shell is Lemma 2.13 in [BČD16]. We will use similar arguments but, as already pointed out before, an additional complication arises from the overlapping parts of the cones. Throughout the proof for $r > 0$ and $x \in \mathbb{Z}^d$ we denote by $B_r(x)$ the closed ℓ^2 ball of radius r around x. We will write $\mathsf{cs}(x_{\mathsf{bas}})$ and $\mathsf{cs}(x'_{\mathsf{bas}})$ as an abbreviation for $\mathsf{cs}(x_{\mathsf{bas}}; b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$ and $\mathsf{cs}(x'_{\mathsf{bas}}; b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$ respectively.

We split the proof in two cases according to $d = 1$ or $d \geq 2$. We will reuse some arguments from the case $d = 1$ for higher dimensions and thus begin with the case $d = 1$.

Case $d = 1$. Without loss of generality we may assume $x_{\mathsf{bas}} < x'_{\mathsf{bas}}$. We will focus on the differences to the version of this lemma with only one cone. For that we distinguish according to the distance of the bases of the cones.

First let $\|x'_{\mathsf{bas}} - x_{\mathsf{bas}}\|_2 \leq 2b_{\text{out}}$. Since, in this case, the bases of the two outer cones already overlap, it is impossible for any path γ to cross $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}; \infty)$ from between x_{bas} and x'_{bas} without hitting one of the bases. (It easy to see how the picture in Figure 3.3 changes in this case.) In this case we can use the same arguments as in [BČD16], since we can combine the two cones to a single larger cone with the cone shell being

$$\begin{aligned}\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}) := \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : x_{\mathsf{bas}} - b_{\text{out}} - ns_{\text{out}} \leq x \leq x_{\mathsf{bas}} - b_{\text{inn}} - ns_{\text{inn}}\} \\ \cup \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : x'_{\mathsf{bas}} + b_{\text{inn}} + ns_{\text{inn}} \leq x \leq x'_{\mathsf{bas}} + b_{\text{out}} + ns_{\text{out}}\}.\end{aligned}$$

Now let $\|x'_{\mathsf{bas}} - x_{\mathsf{bas}}\|_2 > 2b_{\text{out}}$. In particular, the two cones do not overlap at time $t = 0$. The two cone shells are each made up of two wedges

$$\begin{aligned}c^1_\ell &:= \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : x_{\mathsf{bas}} - b_{\text{out}} - ns_{\text{out}} \leq x \leq x_{\mathsf{bas}} - b_{\text{inn}} - ns_{\text{inn}}\}, \\ c^1_r &:= \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : x_{\mathsf{bas}} + b_{\text{inn}} + ns_{\text{inn}} \leq x \leq x_{\mathsf{bas}} + b_{\text{out}} + ns_{\text{out}}\}, \\ c^2_\ell &:= \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : x'_{\mathsf{bas}} - b_{\text{out}} - ns_{\text{out}} \leq x \leq x'_{\mathsf{bas}} - b_{\text{inn}} - ns_{\text{inn}}\}, \\ c^2_r &:= \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : x'_{\mathsf{bas}} + b_{\text{inn}} + ns_{\text{inn}} \leq x \leq x'_{\mathsf{bas}} + b_{\text{out}} + ns_{\text{out}}\}.\end{aligned}$$

We build the cone shell for the double cone using the above wedges. We have to isolate the inner cones using the double cone shell and need to make sure, that the cone shell doesn't evolve into any inner cone. Obviously this isn't a problem for c^1_ℓ and c^2_r since these two wedges both evolve away from the other cone.

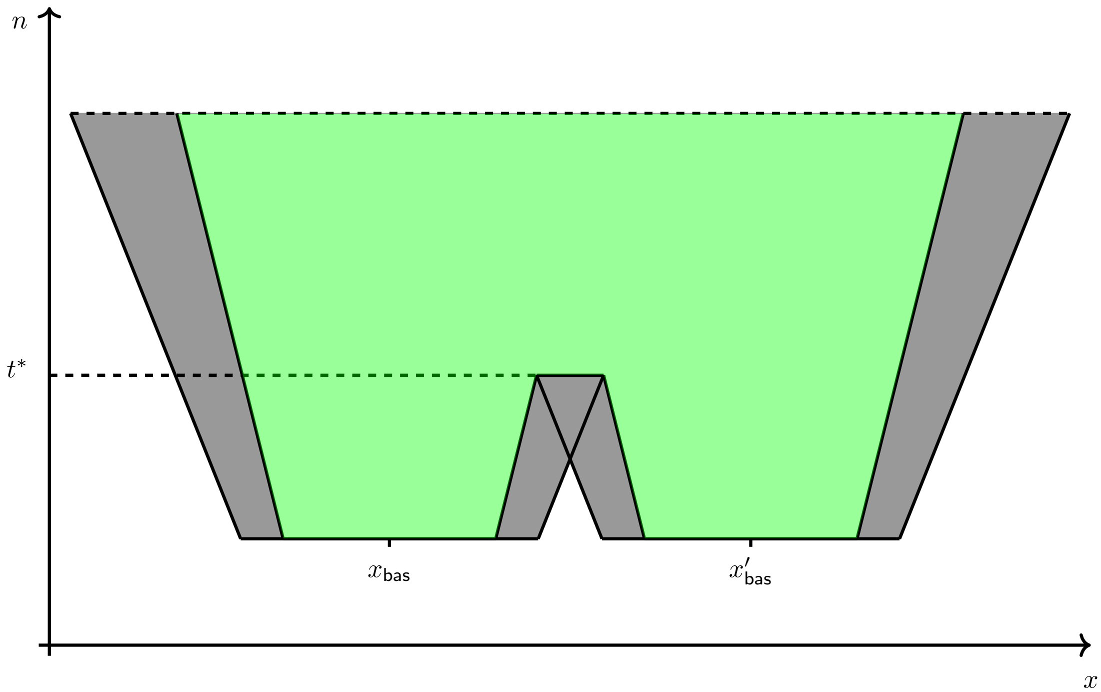

Figure 3.3: Double cone and double cone shell in case $d = 1$

It remains to find a suitable hight at which we cut and merge c_r^1 and c_ℓ^2 to avoid their propagation into the inner cones. Given the parameters of the cones let

$$t^* := \frac{(x'_{\mathsf{bas}} - x_{\mathsf{bas}}) - b_{\text{out}} - b_{\text{inn}}}{s_{\text{out}} + s_{\text{inn}}} \tag{3.1.8}$$

be the time at which the inner cones meet the respective outer cones of the other double cone. Then the cone shell is the following set

$$\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}) := c_\ell^1 \cup c_r^2 \cup \big(c_r^1 \cap \mathbb{Z} \times [0, \lceil t^* \rceil]\big) \cup \big(c_\ell^2 \cap \mathbb{Z} \times [0, \lceil t^* \rceil]\big).$$

A sketch of this cone shell can be seen in Figure 3.3. The cone shell is thus made up of four wedges and in Claim 2.15 from [BČD16] it was shown that for any $\varepsilon' > 0$, by tuning the parameters correctly, the contact process $\eta^{c_\ell^1}$ restricted to the wedge c_ℓ^1 survives with probability at least $1 - \varepsilon'$. The same holds for the other wedges and therefore η^{dcs} survives in every wedge with probability at least $1 - 4\varepsilon'$. For the outer wedges c_ℓ^1 and c_r^2 every path crossing the cone shell has to hit the contact process if it survives. For the inner wedges we have to argue a bit more carefully since we cut them at a certain height. It is theoretically possible that the contact process survives up until that point in time in both wedges but evolves in such a way that the clusters of points visited (infected) by the contact process do not intersect. Then there exists a path which crosses the cone shell but doesn't hit the contact process η^{dcs}.

Theorem 2 from [CMS10] tells us that, if we condition on the event

$$B := \big\{\eta_t^{c_r^1} \neq 0, \forall t \geq 0\big\} \cap \big\{\eta_t^{c_\ell^2} \neq 0, \forall t \geq 0\big\},$$

then the rightmost particle of the contact process is close to the right border of the wedge. It follows, for every $\tilde{\varepsilon} > 0$ there exists $T > 0$ such that for every $n \geq T$

$$\mathbb{P}\Big(\max\{x : \eta_n^{c_r^1}(x) = 1\} \leq \frac{1}{2}(b_{\text{inn}} + b_{\text{out}} + t(s_{\text{inn}} + s_{\text{out}}))\,\Big|\, B\Big) < 1 - \tilde{\varepsilon}.$$

An analogous bound holds for the left most particle in the wedge c_ℓ^2. So $\eta^{c_r^1}$ and $\eta^{c_\ell^2}$, if they both survive, they will eventually meet. Note that we in fact only need the contact process to survive until we cut the inner wedges which happens at the time t^* from (3.1.8). The probability of this joint survival event is greater than $1 - \varepsilon'$.

By (3.1.8) we get this by increasing the base b_{out} of the outer cones. Furthermore, conditioned on the event B, the contact processes $\eta^{c_r^1}$ and $\eta^{c_\ell^2}$ will survive and meet each other with probability at least $1 - 2\tilde{\varepsilon}$. Thus any path crossing the cone shell has to hit the contact process η^{dcs} at some point with probability at least $1 - \varepsilon$ for some $\varepsilon > 0$. So using the same arguments as in [BČD16] we obtain the claim for $d = 1$.

Case $d > 1$. Recall the definition of M_n^{dcs} in (3.1.6). Note that for every path

$$\gamma = ((x_m, m), (x_{m+1}, m+1), \dots, (x_{m'}, m'))$$

crossing $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}})$ there exists $i \in [m, m']$ with $x_i \in M_i^{\mathsf{dcs}}$, i.e. every path will hit at least one point in $\mathcal{M}$. Without loss of generality let n be the time at which the path hits $\mathcal{M}$. By Lemma 2.9 from [BČD16] it follows that with high probability after n steps the contact process started from a single site will coincide with the contact process started from the upper invariant measure in a ball of radius ns_{coupl} around the starting site.

Let $\rho > 0$ be a small constant such that for every $x \in M_n^{\mathsf{dcs}}$ we have

$$B_{n\rho s_{\text{coupl}}}(x) \times \{\lfloor (1-\rho)n \rfloor\} \subset \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}}).$$

To get into the setting of [BČD16] we need to find unit vectors $\hat{v}^i \in \mathbb{R}^d, 1 \leq i \leq \hat{N}$, with $\hat{N}$ sufficiently large so that we can "cover" the middle tubes M_n^{dcs} in the following sense: For every $x \in M_n^{\mathsf{dcs}}$ there exists an $i \leq \hat{N}$ such that for ρ small there is $\delta = \delta(\rho) > 0$ with the property

$$\begin{aligned}&\text{the length of the intersection of the half-line } \{t\hat{v}^i : t \geq 0\} \text{ with the (real) ball}\\ &\{y \in \mathbb{R}^d : \|x - y\|_2 \leq n\rho s_{\text{coupl}}\} \text{ is at least } \delta n.\end{aligned} \tag{3.1.9}$$

Following the idea in [BČD16] we can choose such vectors for the single cone shells $\mathsf{cs}(x_{\mathsf{bas}})$ and $\mathsf{cs}(x'_{\mathsf{bas}})$ respectively. The union $\{v^i : i = 1, \dots, N\} \cup \{v'^{,i} : i = 1, \dots, N'\}$ of those two sets of vectors has the above property for the double cone shell $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}})$. Now we approximate the half-line $\{tv^i : t \geq 0\}$ with a self-avoiding nearest neighbour path in $\mathbb{Z}^d$ $\alpha^i = (\alpha^i(j))_{j \in \mathbb{N}}$ given by

(a) $\alpha_0^i = x_{\mathsf{bas}}$,

(b) α^i makes steps only in direction of v^i,

(c) α^i stays close to tv^i, that is $\{\alpha_j^i : j \in \mathbb{N}\} \subset \{tv^i + z : t \geq 0, z \in \mathbb{R}^d, \|z\| \leq d\}$.

Analogously we define $\alpha'^{,i}$ in direction of $v'^{,i}$ for $1 \leq i \leq N'$. Next we define contact processes $\eta^{(i)}$ restricted to the set $\mathcal{W}^i := (\alpha^i \times \mathbb{Z}_+) \cap \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}})$ and analogous $\eta'^{,(i)}$ restricted to $\mathcal{W}'^{,i} := (\alpha'^{,i} \times \mathbb{Z}_+) \cap \mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}})$

and started from $\mathbb{1}_{\mathcal{W}^i\cap(\mathbb{Z}^d\times\{0\})}$ and $\mathbb{1}_{\mathcal{W}',i\cap(\mathbb{Z}^d\times\{0\})}$ respectively. By the definition of $\mathsf{dcs}(x_{\mathsf{bas}}, x'_{\mathsf{bas}})$ we avoid the problem of having any of the $\eta^{(i)}$ or $\eta'^{,(i)}$ evolving into the inner double cone.

Let $N^* = N + N'$ be the total number of vectors above and $\{\hat{\eta}^{(i)} : i \le N^*\}$ the collection of contact processes defined. Define $S \coloneqq \{\hat{\eta}^{(i)}$ survives for every $i \le N^*\}$. Given $\varepsilon > 0$, choose ε^* so that $(1-\varepsilon^*)^{N^*} \ge 1 - \varepsilon/2$. Given that we get

$$\mathbb{P}(S) \ge 1 - \varepsilon/2. \tag{3.1.10}$$

The rest follows exactly as in the proof of Lemma 2.13 from [BČD16]. □

With Lemma 3.1.2 we have a tool to separate the information inside the inner cone from the outside of the outer cone, which gives us a certain space-time isolation of the environment around the random walk. Another important ingredient for the renewal times is to locally explore the reasons for 0's of η along the paths of the random walks. This allows to stop in such a way that the distribution of the environment viewed relative to the stopped particle dominates the a priori law of the environment; see equation (3.1.16) in Lemma 3.1.4 below.

To that end we define stopping times at which the reasons for negative information, i.e. reasons for $\eta = 0$, for both walkers X and X' are explored. For $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$, let $\ell(x, n)$ be length of the longest directed open path starting in (x, n) with the convention $\ell(x, n) = -1$ if $\omega(x, n) = 0$ and $\ell(x, n) = \infty$ if $\eta_n(x) = 1$. As in [BČD16] we define

$$D_n \coloneqq n + \max\{\ell(y, -n) + 2 : \|X_n - y\| \le R_{\text{loc}},\, \ell(y, -n) < \infty\}.$$

Note that D_n is the time, for the walk, at which the reasons for $\eta_{-n}(y) = 0$ for all y from the R_{loc}-neighbourhood of X_n are explored by inspecting all the determining triangles

$$D(x, n) \coloneqq \begin{cases} \emptyset, & \text{if } \eta_n(x) = 1, \\ \{(y, m) : \|y - x\| \le (n - m), n - \ell(x, n) - 1 \le m \le n\}, & \text{if } \eta_n(x) = 0 \end{cases} \tag{3.1.11}$$

with base points in $B_{R_{\text{loc}}}(X_n)$. We aim to build the regeneration times on exactly the stopping times at which we explored all reasons for 0's of η along the paths and define

$$\sigma_0 \coloneqq 0, \quad \sigma_i \coloneqq \min\left\{m > \sigma_{i-1} : \max_{\sigma_{i-1} \le n \le m} D_n \le m\right\}, i \ge 1.$$

Let $(\sigma'_i)_{i \ge 0}$ be defined analogously for X'. Now the times at which we jointly explored the reasons for 0's of η along the paths of X and X' are given by the sequence $(\sigma^{\text{sim}}_\ell)_{\ell \ge 0}$ defined by

$$\sigma^{\text{sim}}_0 = 0, \quad \sigma^{\text{sim}}_\ell = \sigma_i = \sigma'_j,\ \ell \ge 1, \tag{3.1.12}$$

where i and j are the first times with respect to the corresponding sequences so that $|\{\sigma_0, \dots, \sigma_i\} \cap \{\sigma'_0, \dots, \sigma'_j\}| = \ell$. Note that σ^{sim} are exactly the times when we have no "negative" influence on the environment in the future of the paths of both random walks.

Again as in [BČD16] we define

$$\text{tube}_n \coloneqq \{(y, -k) : 0 \le k \le n, \|y - X_k\| \le R_{\text{loc}}\} \tag{3.1.13}$$

and

$$\mathrm{dtube}_n = \bigcup_{(y,k)\in \mathrm{tube}_n} D(y,k). \tag{3.1.14}$$

Similarly we define tube'_n and dtube'_n by using X' instead of X. Let the filtration $\mathcal{F}^{\mathrm{sim}} = (\mathcal{F}^{\mathrm{sim}}_n)_n$ be defined by

$$\begin{aligned}\mathcal{F}^{\mathrm{sim}}_n &:= \sigma(X_j : 0 \le j \le n) \vee \sigma(\eta_j(y), \omega(y,j) : (y,j) \in \mathrm{tube}_n) \vee \sigma(\omega(y,j) : (y,j) \in \mathrm{dtube}_n) \\ &\quad \vee \sigma(X'_j : 0 \le j \le n) \vee \sigma(\eta_j(y), \omega(y,j) : (y,j) \in \mathrm{tube}'_n) \vee \sigma(\omega(y,j) : (y,j) \in \mathrm{dtube}'_n).\end{aligned}$$

In particular $\mathcal{F}^{\mathrm{sim}}_n$ is the filtration that contains all the information about the environment that the random walks gather until time σ^{sim}_n. This information includes the values of ω and η in their R_{loc}-vicinity and the determining triangles.

Lemma 3.1.4 (Analogue of Lemma 2.17 from [BČD16])**.** *When p is sufficiently close to 1, then there exist finite positive constants c and C so that*

$$\sup_{x,x'} \mathbb{P}_{x,x'}(\sigma^{\mathrm{sim}}_{i+1} - \sigma^{\mathrm{sim}}_i > n | \mathcal{F}^{\mathrm{sim}}_{\sigma^{\mathrm{sim}}_i}) \le Ce^{-cn} \quad \textit{for all } n = 1,2,\dots, i = 0,1,\dots \textit{ a.s.}, \tag{3.1.15}$$

in particular, all σ^{sim}_i are a.s. finite. Furthermore, we have

$$\mathscr{L}(\omega(\cdot, -j - \sigma^{\mathrm{sim}}_i)_{j=0,1,\dots} | \mathcal{F}^{\mathrm{sim}}_{\sigma^{\mathrm{sim}}_i}) \succcurlyeq \mathscr{L}(\omega(\cdot, -j)_{j=0,1,\dots}) \quad \textit{for all } i = 0,1,\dots \textit{ a.s.}, \tag{3.1.16}$$

where '$\succcurlyeq$' denotes stochastic domination.

Remark 3.1.5. Note that $\sigma^{\mathrm{sim}}_{i+1} - \sigma^{\mathrm{sim}}_i$ depends on the positions of the random walkers at time σ^{sim}_i. We omit writing this explicitly with every σ since it would lead to an overload of notation.

Proof. The proof is basically the same as for Lemma 2.17 of [BČD16] with only a few minor differences.

In order to verify (3.1.16) note that at stopping times σ^{sim}_i all the reasons for zeros along the path of the random walks have been explored and are contained in $\mathcal{F}^{\mathrm{sim}}_{\sigma^{\mathrm{sim}}_i}$. On the other hand the knowledge of ones of η enforces the existence of certain open paths for the ω's. Thus, (3.1.16) follows from the FKG inequality for the ω's.

For the proof of (3.1.15) we consider first the case $i = 0$. We write $\widehat{R}_\kappa := (2R_\kappa+1)^d$ and $\widehat{R}_{\mathrm{loc}} := (2R_{\mathrm{loc}}+1)^d$ for the number of elements of a ball with radius R_κ or R_{loc} respectively. The event $\{\sigma^{\mathrm{sim}}_1 > n\}$ enforces that there are space-time points $(y_j, -j)$ with $\eta_{-j}(y_j) = 0$ for $j = 0,1,\dots,n$ in the R_{loc}-vicinity of the paths of the two random walks X and X'. Here, it is enough to have such a point in only one of the two R_{loc}-vicinities. By Lemma 2.11 from [BČD16] the probability that $\eta_{-j}(y_j) = 0$ for a fixed choice of $y_0,\dots,y_n$ is bounded from above by $\varepsilon(p)^{n+1}$.

Now we prove an upper bound for the number of relevant vectors $(y_0, y_1, \dots, y_n) \in (\mathbb{Z}^d)^{n+1}$. Each of the two random walks has $\widehat{R}^n_\kappa$ possible n-step paths. Thus, the two walks together have at most $\widehat{R}^{2n}_\kappa$ possible n-step paths. Assume that at exactly k times, $0 \le m_1 < m_2 < \cdots < m_k \le n$, for sites $(y_{m_i}, -m_i) \in B_{R_{\mathrm{loc}}}(X_{m_i}) \times \{-m_i\} \cup B_{R_{\mathrm{loc}}}(X'_{m_i}) \times \{-m_i\}$ we have $\eta_{-m_i}(y_{m_i}) = 0$ and hence the corresponding "determining" triangle $D(y_{m_i}, -m_i)$ is not empty.

Set $m_{k+1} = n$. Then the height of $D(y_{m_i}, -m_i)$ is bounded from below by $m_{i+1} - m_i$, because the triangles have to overlap until time n to enforce $\sigma^{\mathrm{sim}}_1 > n$. For a fixed n-step path of (X, X') and fixed

$m_1 < m_2 < \cdots < m_k$, there are at most $(2\widehat{R}_{\text{loc}})^k$ many choices for the y_{m_i}, $i = 1, \dots, k$. Here we have $2\widehat{R}_{\text{loc}}$ choices for every y_{m_i} since those points are in $B_{R_{\text{loc}}}(X_{m_i}) \cup B_{R_{\text{loc}}}(X'_{m_i})$. And inside $D(y_{m_i}, -m_i)$ we have at most $\widehat{R}_\kappa^{m_{i+1}-m_i-1}$ choices to pick $y_{m_i+1}, y_{m_i+2}, \dots, y_{m_{i+1}-1}$ (start with y_{m_i}, then follow the longest open path which is not connected to $\mathbb{Z}^d \times \{-\infty\}$, these sites are necessarily zeros of η). Thus, there are at most

$$\widehat{R}_\kappa^{2n} \sum_{k=1}^{n} \sum_{m_1 < m_2 < \cdots < m_k \le m_{k+1} = n} (2\widehat{R}_{\text{loc}})^k \prod_{i=1}^{k} \widehat{R}_\kappa^{m_{i+1}-m_i-1}$$
$$= \widehat{R}_\kappa^{2n} \sum_{k=1}^{n} \binom{n}{k} (2\widehat{R}_{\text{loc}})^k \widehat{R}_\kappa^{n-k} \le \widehat{R}_\kappa^{2n} (2\widehat{R}_{\text{loc}} + \widehat{R}_\kappa)^n$$

possible choices of $(y_0, y_1, \dots, y_n)$ and hence we have

$$\mathbb{P}_{x,x'}(\sigma_1^{\text{sim}} > n) \le (\widehat{R}_\kappa^2 (2\widehat{R}_{\text{loc}} + \widehat{R}_\kappa)\varepsilon(p))^n.$$

The right hand side decays exponentially when p is close to 1. The general case $i > 0$ in (3.1.15) follows by employing (3.1.16) and the argument for $i = 0$. □

Corollary 3.1.6 (Analogue of Corollary 2.18 from [BČD16])**.** *For p large enough there exists $\varepsilon(p) \in (0, 1]$ satisfying $\lim_{p \uparrow 1} \varepsilon(p) = 0$ such that for all $V = \{(x_1, t_1), \dots, (x_k, t_k)\}$ and $V' = \{(x'_1, t'_1), \dots, (x'_\ell, t'_\ell)\}$, subsets of $\mathbb{Z}^d \times \mathbb{N}$ with $t_1 < \cdots < t_k$, $t'_1 < \cdots < t'_\ell$, we have*

$$\sup_{x_0, x'_0} \mathbb{P}_{x_0, x'_0}\big(\eta_{-t-\sigma_i^{\text{sim}}}(x + X_{\sigma_i^{\text{sim}}}) = 0 \textit{ for all } (x,t) \in V \,\big|\, \mathcal{F}_{\sigma_i^{\text{sim}}}^{\text{sim}}\big) \le \varepsilon(p)^k \tag{3.1.17}$$

and

$$\sup_{x_0, x'_0} \mathbb{P}_{x_0, x'_0}\big(\eta_{-t'-\sigma_i^{\text{sim}}}(x' + X'_{\sigma_i^{\text{sim}}}) = 0 \textit{ for all } (x',t') \in V' \,\big|\, \mathcal{F}_{\sigma_i^{\text{sim}}}^{\text{sim}}\big) \le \varepsilon(p)^\ell. \tag{3.1.18}$$

Proof. With $\varepsilon(p)$ as in Lemma 2.11 of [BČD16] the assertion follows from that lemma and (3.1.16); cf also Corollary 2.18 in [BČD16]. □

For $t \in \mathbb{N}$ we define $R_t := \inf\{i \in \mathbb{Z}_+ : \sigma_i^{\text{sim}} \ge t\}$ and for $m = 1, 2, \dots$ we put

$$\widetilde{\tau}_m^{\text{sim}}(t) := \begin{cases} \sigma_{R_t-m+1}^{\text{sim}} - \sigma_{R_t-m}^{\text{sim}} & \text{if } m \le R_t, \\ 0 & \text{else.} \end{cases} \tag{3.1.19}$$

In particular $\tilde{\tau}_1^{\text{sim}}(t)$ is the length of the interval $(\sigma_{i-1}^{\text{sim}}, \sigma_i^{\text{sim}}]$ which contains t.

Lemma 3.1.7 (Analogue of Lemma 2.19 from [BČD16])**.** *When p is sufficiently close to 1 then there exist finite positive constants c and C so that for all $i, n = 0, 1, \dots$*

$$\sup_{x,x'} \mathbb{P}_{x,x'}\big(\widetilde{\tau}_1^{\text{sim}}(t) \ge n \,\big|\, \mathcal{F}_{\sigma_i^{\text{sim}}}^{\text{sim}}\big) \le Ce^{-cn} \quad \textit{a.s. on } \{\sigma_i^{\text{sim}} < t\}, \tag{3.1.20}$$

and generally

$$\sup_{x,x'} \mathbb{P}_{x,x'}(R_t \ge i + m, \widetilde{\tau}_m^{\text{sim}}(t) \ge n \,|\, \mathcal{F}_{\sigma_i^{\text{sim}}}^{\text{sim}}) \le Cm^2 e^{-cn} \quad \textit{for } m = 1, 2, \dots \textit{ a.s. on } \{\sigma_i^{\text{sim}} < t\}. \tag{3.1.21}$$

Proof. We start with the proof of (3.1.20). We have, writing $\mathbb{P} = \mathbb{P}_{x,x'}$

$$
\begin{aligned}
&\mathbb{P}(\widetilde{\tau}_1^{\text{sim}}(t) \geq n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad = \mathbb{P}(\sigma^{\text{sim}}_{i+1} \geq t \vee (n+\sigma^{\text{sim}}_i) \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) + \sum_{j>i} \sum_{\ell=\sigma_i^{\text{sim}}+1}^{t-1} \mathbb{P}(\sigma_j^{\text{sim}} = \ell, \sigma^{\text{sim}}_{j+1} \geq t \vee (\ell+n) \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad \leq Ce^{-cn} + \sum_{\ell=\sigma_i^{\text{sim}}+1}^{t-1} Ce^{-c((t-\ell)\vee n)} \mathbb{P}(\exists j > i : \sigma_j^{\text{sim}} = \ell \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad \leq Ce^{-cn} + \mathbb{1}_{\{\sigma_i^{\text{sim}} \leq t-n-2\}} \sum_{\ell=\sigma_i^{\text{sim}}+1}^{t-n-1} Ce^{-c(t-\ell)} + \mathbb{1}_{\{n+1 \leq t\}} \sum_{\ell=t-n}^{t-1} Ce^{-cn} \\
&\quad \leq C\Big(1 + \frac{e^{-c}}{1-e^{-c}} + n\Big) e^{-cn}
\end{aligned}
$$

where we used Lemma 3.1.4 and

$$
\mathbb{P}\big(\sigma_j^{\text{sim}} = \ell, \sigma^{\text{sim}}_{j+1} \geq t \vee (\ell+n) \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}\big) = \mathbb{E}\big[\mathbb{1}_{\{\sigma_j^{\text{sim}}=\ell\}} \mathbb{P}(\sigma^{\text{sim}}_{j+1} - \sigma_j^{\text{sim}} \geq (t-\ell) \vee n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_j^{\text{sim}}}) | \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}\big]
$$

in the first inequality.

To prove (3.1.21) we assume $\sigma_i^{\text{sim}} \leq t-n-m-1$ since otherwise the conditional probability appearing in that display equals 0. We have

$$
\begin{aligned}
&\mathbb{P}(R_t \geq i+m, \widetilde{\tau}_m^{\text{sim}}(t) \geq n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad = \mathbb{P}(R_t = i+m, \widetilde{\tau}_m^{\text{sim}}(t) \geq n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) + \mathbb{P}(R_t > i+m, \widetilde{\tau}_m^{\text{sim}}(t) \geq n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad = \mathbb{P}(\sigma^{\text{sim}}_{i+1} - \sigma_i^{\text{sim}} \geq n, R_t = i+m \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) + \mathbb{P}(R_t > i+m, \widetilde{\tau}_m^{\text{sim}}(t) \geq n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad \leq Ce^{-cn} + \sum_{j>i} \sum_{k=\sigma_i^{\text{sim}}+1}^{t-m-n} \sum_{\ell=k+n}^{t-m+1} \mathbb{P}(\sigma_j^{\text{sim}} = k, \sigma^{\text{sim}}_{j+1} = l, \sigma^{\text{sim}}_{j+m-1} < t, \sigma^{\text{sim}}_{j+m} \geq t \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \\
&\quad \leq Ce^{-cn} + \sum_{j>i} \sum_{k=\sigma_i^{\text{sim}}+1}^{t-m-n} \sum_{\ell=k+n}^{t-m+1} \mathbb{P}(\sigma_j^{\text{sim}} = k, \sigma^{\text{sim}}_{j+1} = l \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \times (m-1) Ce^{-c(t-\ell)/(m-1)} \\
&\quad \leq Ce^{-cn} + C(m-1) \sum_{k=\sigma_i^{\text{sim}}+1}^{t-m-n} \sum_{\ell=k+n}^{t-m+1} e^{-c(t-\ell)/(m-1)} \sum_{j>i} \mathbb{P}(\sigma_j^{\text{sim}} = k \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \times Ce^{-c(\ell-k)} \\
&\quad \leq Ce^{-cn} + C^2(m-1) \sum_{k=\sigma_i^{\text{sim}}+1}^{t-m-n} e^{ck-ct/(m-1)} \sum_{\ell=k+n}^{t-m+1} \exp\Big(-c\frac{m-2}{m-1} l\Big)
\end{aligned}
\tag{3.1.22}
$$

where we used in the second inequality that

$$
\{\sigma^{\text{sim}}_{j+1} = \ell, \sigma^{\text{sim}}_{j+m-1} < t, \sigma^{\text{sim}}_{j+m} > t\} \subset \bigcup_{r=j+2}^{j+m} \{\sigma_r^{\text{sim}} - \sigma^{\text{sim}}_{r-1} \geq \frac{t-\ell}{m-1}\}
$$

together with Lemma 3.1.4. For $m = 2$ the inequality proven in (3.1.22) yields

$$
\mathbb{P}(R_t \geq i+2, \widetilde{\tau}_m^{\text{sim}}(t) \geq n \,|\, \mathcal{F}^{\text{sim}}_{\sigma_i^{\text{sim}}}) \leq Ce^{-cn} + C^2 \sum_{k=\sigma_i^{\text{sim}}+1}^{t-n} (t-k-n) e^{-c(t-k)} \leq C' e^{-cn} \sum_{\ell=0}^{\infty} \ell e^{-c\ell}
$$

whereas for $m > 2$ we obtain

$$
\begin{aligned}
\mathbb{P}(R_t \geq i+m, &\widetilde{\tau}_m^{\mathrm{sim}}(t) \geq n \,|\, \mathcal{F}^{\mathrm{sim}}_{\sigma_i^{\mathrm{sim}}}) \\
&\leq Ce^{-cn} + C^2(m-1) \sum_{k=\sigma_i^{\mathrm{sim}}+1}^{t-m-n} e^{ck-ct/(m-1)} \frac{\exp(-c\frac{m-2}{m-1}(k+n))}{1-\exp(-c\frac{m-2}{m-1})} \\
&= Ce^{-cn} + C^2(m-1) \frac{\exp(-c\frac{m-2}{m-1}n - c\frac{t}{m-1})}{1-\exp(-c\frac{m-2}{m-1})} \sum_{k=\sigma_i^{\mathrm{sim}}+1}^{t-m-n} e^{ck/(m-1)} \\
&\leq Ce^{-cn} + \frac{C^2(m-1)}{1-\exp(-c\frac{m-2}{m-1})} \exp(-c\frac{m-2}{m-1}n - c\frac{t}{m-1}) \frac{e^{c\frac{t-n}{m-1}}}{e^{c/(m-1)}-1} \\
&= Ce^{-cn} + \frac{C^2(m-1)}{1-\exp(-c\frac{m-2}{m-1})} \frac{e^{-cn}}{e^{c/(m-1)}-1} \leq C'm^2 e^{-cn}.
\end{aligned}
$$

where we note that $1-\exp(-c\frac{m-2}{m-1}) \geq 1-\exp(c/2)$ and $\exp(c/(m-1))-1 = \frac{c}{m+1}\sum_{k=0}^{\infty} \frac{(c/(m-1))^k}{(k+1)!} > \frac{c}{m-1}$. □

Lemma 3.1.8 (Analogue of Lemma 2.20 in [BČD16])**.** *When p is sufficiently close to 1, then for all $\varepsilon > 0$ there exist finite positive constants $c = c(\varepsilon)$ and $C = C(\varepsilon)$ so that for all finite $\mathcal{F}^{\mathrm{sim}}$-stopping times T with the property that almost surely $T \in \{\sigma_i^{\mathrm{sim}} : i \in \mathbb{N}\}$ and for all $k, \ell \in \mathbb{N}$*

$$\sup_{x_0,x_0'} \mathbb{P}_{x_0,x_0'}(\|X_k - X_T\| > s_{\max}(k-T) \,|\, \mathcal{F}_T^{\mathrm{sim}}) \leq Ce^{-c(k-T)} \qquad a.s.\ on\ \{T<k\}, \tag{3.1.23}$$

$$\sup_{x_0,x_0'} \mathbb{P}_{x_0,x_0'}(\|X'_\ell - X'_T\| > s_{\max}(\ell-T) \,|\, \mathcal{F}_T^{\mathrm{sim}}) \leq Ce^{-c(\ell-T)} \qquad a.s.\ on\ \{T<\ell\} \tag{3.1.24}$$

and for $j < k, \ell$

$$\sup_{x_0,x_0'} \mathbb{P}_{x_0,x_0'}(\|X_k - X_j\| > (1+\varepsilon)s_{\max}(k-j) \,|\, \mathcal{F}_T^{\mathrm{sim}}) \leq Ce^{-c(k-j)} \qquad a.s.\ on\ \{T \leq j\}, \tag{3.1.25}$$

$$\sup_{x_0,x_0'} \mathbb{P}_{x_0,x_0'}(\left\|X'_\ell - X'_j\right\| > (1+\varepsilon)s_{\max}(\ell-j) \,|\, \mathcal{F}_T^{\mathrm{sim}}) \leq Ce^{-c(\ell-j)} \qquad a.s.\ on\ \{T \leq j\} \tag{3.1.26}$$

with $s_{\max}$ as in Lemma 2.16 of [BČD16].

Proof. By Lemma A.1 of [BČD16], we may assume that $T = \sigma_\ell^{\mathrm{sim}}$ for some $\ell \in \mathbb{N}$. For (3.1.23) we combine Corollary 3.1.6 with the proof of Lemma 2.16 of [BČD16]. Since the proof is analogous to the proof of Lemma 2.20 in [BČD16], we briefly show the part of the proof that differs. Let Γ_n be the set of all n-step paths γ on $\mathbb{Z}^d$ starting from $\gamma_0 = X_T$ with the restriction $\|\gamma_i - \gamma_{i-1}\| \leq R_\kappa$, $i = 1, \ldots, n$, where R_κ is the range of the kernels κ_n from Assumption 1.3.4. For $\gamma \in \Gamma_{k-T}$ and $T \leq i_1 < i_2 < \cdots < i_m \leq k$ we define

$$
\begin{aligned}
D^\gamma_{i_1,\ldots,i_m} &:= \{\eta_{-\ell}(\gamma_\ell) = 0 \text{ for all } \ell \in \{i_1,\ldots,i_m\}\}, \\
W^\gamma_{i_1,\ldots,i_m} &:= \{\eta_{-\ell}(\gamma_\ell) = 1 \text{ for all } \ell \in \{T,\ldots,k\} \setminus \{i_1,\ldots,i_m\}\}.
\end{aligned}
$$

Let $H_n := \#\{T \leq i \leq n : \eta_{-i}(X_i) = 0\}$ be the number of sites with zeros the walker visits from time T up to time n and set $K := \max_{x\in\mathbb{Z}^d}\{\kappa_{\mathrm{ref}}(x)\} + \varepsilon_{\mathrm{ref}}$. It follows that

$$\mathbb{P}(H_k = m \,|\, \mathcal{F}_T^{\mathrm{sim}}) = \sum_{T \leq i_1 < \cdots < i_m \leq k} \sum_{\gamma \in \Gamma_{k-T}} \mathbb{P}((X_T,\ldots,X_k) = \gamma, W^\gamma_{i_1,\ldots,i_m}, D^\gamma_{i_1,\ldots,i_m} \,|\, \mathcal{F}_T^{\mathrm{sim}}). \tag{3.1.27}$$

Note that if $i_m < k$ we get

$$
\begin{aligned}
&\mathbb{P}((X_T,\dots,X_k)=\gamma, W^{\gamma}_{i_1,\dots,i_m}, D^{\gamma}_{i_1,\dots,i_m} \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\quad= \mathbb{E}\Big[\mathbb{1}_{\{(X_T,\dots,X_{i_m-1})=(\gamma_T,\dots,\gamma_{i_m-1})\}}\mathbb{1}_{W^{\gamma|_{i_m-1}}_{i_1,\dots,i_{m-1}}} \\
&\qquad \mathbb{E}[\mathbb{1}_{\{(X_{i_m},\dots,X_k)=(\gamma_{i_m},\dots,\gamma_k)\}}\mathbb{1}_{\{\eta_{-\ell}(\gamma_\ell)=1, i_m+1\le l\le k\}}\mathbb{1}_{D^{\gamma}_{i_1,\dots,i_m}} \,|\, \mathcal{F}^{\text{sim}}_{i_m-1}] \,|\, \mathcal{F}^{\text{sim}}_T\Big]
\end{aligned}
$$

where $\gamma|_t = (\gamma_1,\dots,\gamma_t)$ is the path γ restricted to the first t components. By conditioning on $\mathcal{F}_{k-1},\dots,\mathcal{F}_{i_m}$ successively we get

$$
\begin{aligned}
&\mathbb{P}((X_T,\dots,X_k)=\gamma, W^{\gamma}_{i_1,\dots,i_m}, D^{\gamma}_{i_1,\dots,i_m} \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\qquad \le \mathbb{E}[\mathbb{1}_{\{(X_T,\dots,X_{i_m-1})=(\gamma_T,\dots,\gamma_{i_m-1})\}}\mathbb{1}_{W^{\gamma|_{i_m-1}}_{i_1,\dots,i_{m-1}}} K^{k-i_m-1}\mathbb{E}[\mathbb{1}_{D^{\gamma}_{i_1,\dots,i_m}} \,|\, \mathcal{F}^{\text{sim}}_{i_m-1}] \,|\, \mathcal{F}^{\text{sim}}_T]
\end{aligned}
$$

because the path will only hit ones of η for the steps after i_m. If we repeat the same argument for every i_j we obtain

$$
\begin{aligned}
\mathbb{P}(H_k = m \,|\, \mathcal{F}^{\text{sim}}_T) &\le \sum_{T\le i_1<\dots<i_m\le k}\ \sum_{\gamma\in\Gamma_{k-T}} K^{k-T-m}\mathbb{P}(D^{\gamma}_{i_1,\dots,i_m} \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\le \sum_{T\le i_1<\dots<i_m\le k} R_\kappa^{d(k-T)}K^{k-T-m}\varepsilon(p)^m = \binom{k-T}{m} R_\kappa^{d(k-T)}K^{k-T-m}\varepsilon(p)^m,
\end{aligned}
$$

where we used Corollary 3.1.6 in the second inequality. From this point the rest of the proof consists of the same calculations as in the proof of Lemma 2.16 of [BČD16], where the walk is compared with the reference walk for steps from sites on which η equals 1 and one uses the a priori bound, Assumption 1.3.4, for step sizes from (the few) other sites. This proves (3.1.23). Since $\mathscr{L}(X_k - X_T) = \mathscr{L}(X'_k - X'_T)$ if the random walks are started from the same position we also obtain (3.1.24).

For (3.1.25) define $T' := \inf(\{\sigma^{\text{sim}}_i : i\in\mathbb{N}\}\cap[j,\infty))$ the first σ^{sim}_i after j. Inequality (3.1.20) shows that $\mathbb{P}(T'-j > \varepsilon(k-j) \,|\, \mathcal{F}^{\text{sim}}_T)$ is exponentially small in $k-j$. Using (3.1.23) we obtain

$$
\begin{aligned}
&\mathbb{P}(\|X_k - X_j\| > (1+\varepsilon)s_{\max}(k-j), T'-j \le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j) \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\le \mathbb{P}(\|X_k - X_{T'}\| + \|X_{T'} - X_j\| > (1+\varepsilon)s_{\max}(k-j), T'-j \le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j) \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\le \mathbb{P}(\|X_k - X_{T'}\| + \varepsilon s_{\max}(k-j) > (1+\varepsilon)s_{\max}(k-j), T'-j \le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j) \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\le \mathbb{P}(\|X_k - X_{T'}\| > s_{\max}(k-j), T'-j \le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j) \,|\, \mathcal{F}^{\text{sim}}_T) \\
&\le \mathbb{E}[\mathbb{1}_{\{T'-j\le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j)\}}\mathbb{P}(\|X_k - X_{T'}\| > s_{\max}(k-j) \,|\, \mathcal{F}^{\text{sim}}_{T'}) \,|\, \mathcal{F}^{\text{sim}}_T] \\
&\le \mathbb{E}[\mathbb{1}_{\{T'-j\le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j)\}} C e^{-c(k-T')} \,|\, \mathcal{F}^{\text{sim}}_T] \\
&\le \mathbb{E}[\mathbb{1}_{\{T'-j\le \frac{\varepsilon s_{\max}}{R_\kappa}(k-j)\}} C e^{-c(k-j)+c\frac{\varepsilon s_{\max}}{R_\kappa}(k-j)} \,|\, \mathcal{F}^{\text{sim}}_T] \\
&\le C e^{-c(k-j)+c\frac{\varepsilon s_{\max}}{R_\kappa}(k-j)}.
\end{aligned}
$$

This yields

$$\begin{aligned}
&\mathbb{P}(\|X_k - X_j\| > (1+\varepsilon)s_{\max}(k-j) \,|\, \mathcal{F}_T^{\text{sim}}) \\
&\qquad = \mathbb{P}(\|X_k - X_j\| > (1+\varepsilon)s_{\max}(k-j), T'-j > \frac{\varepsilon s_{\max}}{R_\kappa}(k-j) \,|\, \mathcal{F}_T^{\text{sim}}) \\
&\qquad\quad + \mathbb{P}(\|X_k - X_j\| > (1+\varepsilon)s_{\max}(k-j), T'-j \leq \frac{\varepsilon s_{\max}}{R_\kappa}(k-j) \,|\, \mathcal{F}_T^{\text{sim}}) \\
&\qquad \leq Ce^{-c\frac{\varepsilon s_{\max}}{R_\kappa}(k-j)} + Ce^{-c(k-j)+c\frac{\varepsilon s_{\max}}{R_\kappa}(k-j)} \\
&\qquad \leq Ce^{-c'(k-j)},
\end{aligned}$$

where we used (3.1.20) of Lemma 3.1.7 for the first inequality. Finally, (3.1.26) follows by the same arguments. □

Next recall from [BČD16] the definition for a cone time point for the decorated path; see equation (2.56) and in particular Figure 4 there. For $m < n$ we say that n is a (b,s)-*cone time point for the decorated path of* X *beyond* m if, recall definitions (3.1.13) and (3.1.14),

$$\begin{aligned}
&(\text{tube}_n \cup \text{dtube}_n) \cap (\mathbb{Z}^d \times \{-n, -n+1, \dots, -m\}) \\
&\qquad \subset \{(x,-j) : m \leq j \leq n, \|x - X_n\| \leq b + s(n-j)\}
\end{aligned} \tag{3.1.28}$$

and for X' if

$$\begin{aligned}
&(\text{tube}'_n \cup \text{dtube}'_n) \cap (\mathbb{Z}^d \times \{-n, -n+1, \dots, -m\}) \\
&\qquad \subset \{(x,-j) : m \leq j \leq n, \|x - X'_n\| \leq b + s(n-j)\}.
\end{aligned} \tag{3.1.29}$$

Thus, n is a cone time point for the decorated path of X beyond m if the space-time path $(X_j, -j)_{j=m,\dots,n}$ together with its R_{loc}-tube and determining triangles is contained in $\mathsf{cone}(b,s,n-m)$ shifted to the base point $(X_n, -n)$.

Lemma 3.1.9 (Analogue to [BČD16] Lemma 2.21)**.** *For $\varepsilon > 0$, when p is sufficiently close to 1, there exist $b > 0$ and $s > s_{\max}$ such that for all finite $\mathcal{F}^{\text{sim}}$-stopping times T with $T \in \{\sigma_i^{\text{sim}} : i \in \mathbb{N}\}$ a.s. and all $k \in \mathbb{N}$, with $T' := \inf\{\sigma_i^{\text{sim}} : \sigma_i^{\text{sim}} \geq k\}$*

$$\mathbb{P}(T' \textit{ is a } (b,s)\textit{-cone time point for the decorated path of } X \textit{ and } X' \textit{ beyond } T \,|\, \mathcal{F}_T^{\text{sim}}) \geq 1 - \varepsilon, \tag{3.1.30}$$

a.s. on $\{T < k\}$. Furthermore $0 < s - s_{\max} \ll 1$ can be chosen small.

Proof. By Lemma 2.21 from [BČD16] one can tune the parameters such that a.s. on $\{T < k\}$ we have

$$\mathbb{P}(T' \text{ is a } (b,s)\text{-cone time point for the decorated path of } X \text{ beyond } T \,|\, \mathcal{F}_T^{\text{sim}}) \geq 1 - \varepsilon/2,$$

and the analogous inequality with X' instead of X holds as well. From these estimates (3.1.30) follows easily. The assertion that $0 < s - s_{\max} \ll 1$ can be chosen small is also a direct consequence of Lemma 2.21 from [BČD16]. □

Remark 3.1.10. Note that, since both inner cones are subsets of the double inner cone, T' from Lemma 3.1.9 is a (b,s)-"double cone time point" for the decorated path of (X, X') beyond T if it is a (b,s)-cone time point for X and X' beyond T which is in accordance to our setting up the simultaneous regeneration times.

As a part of the construction of the regeneration times we define the set of 'good' ω-configurations in the double cone shell. Let

$$G_{x,x'}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h) \subset \{0,1\}^{\mathsf{dcs}(x,x',b_{\text{inn}},b_{\text{out}},s_{\text{inn}},s_{\text{out}},h)} \tag{3.1.31}$$

be the set of all ω-configurations with the property

$$\begin{aligned} &\forall \eta_0, \eta_0' \in \{0,1\}^{\mathbb{Z}^d} \text{ with } \eta_0|_{B_{b_{\text{out}}}(x)\cup B_{b_{\text{out}}}(x')} = \eta_0'|_{B_{b_{\text{out}}}(x)\cup B_{b_{\text{out}}}(x')} \equiv 1 \quad \text{and} \\ &\qquad \omega \in \{0,1\}^{\mathbb{Z}^d\times\{1,\dots,h\}} \text{ with } \omega|_{\mathsf{dcs}(x,x',b_{\text{inn}},b_{\text{out}},s_{\text{inn}},s_{\text{out}},h)} \in G_{x,x'}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h) : \\ &\quad \eta_n(y) = \eta_n'(y) \text{ for all } (y,n) \in \mathsf{cone}(x; b_{\text{inn}}, s_{\text{inn}}, h) \cup \mathsf{cone}(x'; b_{\text{inn}}, s_{\text{inn}}, h), \end{aligned} \tag{3.1.32}$$

where η and η' are both constructed from (1.1.1) with the same ω but using possibly different initial conditions. In words this means, when there are 1's at the bottom of the outer cones, i.e. $\mathsf{cone}(x; b_{\text{out}}, s_{\text{out}}, h)$ and $\mathsf{cone}(x'; b_{\text{out}}, s_{\text{out}}, h)$, a configuration from $G_{x,x'}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)$ guarantees successful coupling inside the inner double cone irrespective of what happens outside the outer cones.

Lemma 3.1.11. *For parameters $p, b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}$ as in Lemma 3.1.2,*

$$\mathbb{P}(\omega|_{\mathsf{dcs}(x,x',b_{\text{inn}},b_{\text{out}},s_{\text{inn}},s_{\text{out}},h)} \in G_{x,x'}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)) \geq 1 - \varepsilon, \tag{3.1.33}$$

uniformly in $h \in \mathbb{N}$ and $x, x' \in \mathbb{Z}^d$.

Proof. This lemma is a direct consequence of Lemma 3.1.2 because for $\omega \in G_1 \cap G_2$ we have $\omega|_{\mathsf{dcs}(x,x',b_{\text{inn}},b_{\text{out}},s_{\text{inn}},s_{\text{out}},h)} \in G_{x,x'}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)$. □

We denote a space-time shift of subsets of $\mathbb{Z}^d \times \mathbb{Z}$ by $\Theta^{(x,n)}$, i.e.

$$\Theta^{(x,n)}(A) := \{(y+x, m+n) : (y,m) \in A\} \quad \text{for } A \subset \mathbb{Z}^d \times \mathbb{Z}. \tag{3.1.34}$$

From [BČD16], see there the discussion around (2.62), we know that there exists a deterministic sequence $t_\ell \nearrow \infty$ with the property that for $\ell \in \mathbb{N}$ and $\|x_{\mathsf{bas}} - y\| \leq s_{\max} t_{\ell+1}$

$$\Theta^{(0,-t_\ell)}\big(\mathsf{cone}(x_{\mathsf{bas}}, t_\ell s_{\max} + b_{\text{out}}, s_{\text{out}}, t_\ell)\big) \subset \Theta^{(y,-t_{\ell+1})}\big(\mathsf{cone}(x_{\mathsf{bas}}, b_{\text{inn}}, s_{\text{inn}}, t_{\ell+1})\big). \tag{3.1.35}$$

We will describe a possible choice in (3.1.42) below. The same sequence can be used for the double cone, since the larger cones will overlap with each other before they can hit the smaller cones. Thus, for $\|x_{\mathsf{bas}} - y\| \leq s_{\max} t_{\ell+1}$ and $\|x'_{\mathsf{bas}} - y'\| \leq s_{\max} t_{\ell+1}$ we have

$$\begin{aligned} &\Theta^{(0,-t_\ell)}\big(\mathsf{cone}(x_{\mathsf{bas}}, t_\ell s_{\max} + b_{\text{out}}, s_{\text{out}}, t_\ell)\big) \cup \Theta^{(0,-t_\ell)}\big(\mathsf{cone}(x'_{\mathsf{bas}}, t_\ell s_{\max} + b_{\text{out}}, s_{\text{out}}, t_\ell)\big) \\ &\subset \Theta^{(y,-t_{\ell+1})}\big(\mathsf{cone}(x_{\mathsf{bas}}, b_{\text{inn}}, s_{\text{inn}}, t_{\ell+1})\big) \cup \Theta^{(y',-t_{\ell+1})}\big(\mathsf{cone}(x'_{\mathsf{bas}}, b_{\text{inn}}, s_{\text{inn}}, t_{\ell+1})\big) \end{aligned} \tag{3.1.36}$$

and the sequence satisfies

$$t_{\ell+1} s_{\text{inn}} + b_{\text{inn}} - t_{\ell+1} s_{\max} > t_\ell s_{\max} + b_{\text{out}} + t_\ell s_{\text{out}}, \quad \text{for all } \ell. \tag{3.1.37}$$

Note that necessarily the sequence (t_ℓ) must grow exponentially in ℓ; cf. the discussion around (3.1.43).

Construction 3.1.12 (Regeneration times)**.** The regeneration times will be constructed analogously to the construction in [BČD16]. Let $X_0 = x$ and $X'_0 = x'$ be the starting positions of the two random walks. For the sequence $(\sigma_n^{\mathrm{sim}})_{n=1,2,\dots}$ from (3.1.12) and a sequence $(t_\ell)_{\ell=1,2,\dots}$ satisfying (3.1.34) we define the sequence $(\tilde{\sigma}_n^{\mathrm{sim}})_{n=1,2,\dots}$ by

$$\tilde{\sigma}_\ell^{\mathrm{sim}} := \inf\{\sigma_i^{\mathrm{sim}} \in \{\sigma^{\mathrm{sim}}\} : \sigma_i^{\mathrm{sim}} \geq t_\ell\}, \tag{3.1.38}$$

i.e. be $\tilde{\sigma}_\ell^{\mathrm{sim}}$ is the first element in the sequence σ^{sim} after the time t_ℓ. The sequence $(\tilde{\sigma}_n^{\mathrm{sim}})_{n=1,2,\dots}$ is the sequence along which we look for regeneration times.

For the first regeneration time we check the following criteria:

(i) Go to $\tilde{\sigma}_1^{\mathrm{sim}}$ and check if $\tilde{\sigma}_1^{\mathrm{sim}} < t_2$, η in the b_{out}-neighbourhood of $(X_{\tilde{\sigma}_1^{\mathrm{sim}}}, -\tilde{\sigma}_1^{\mathrm{sim}})$ and $(X'_{\tilde{\sigma}_1^{\mathrm{sim}}}, -\tilde{\sigma}_1^{\mathrm{sim}})$ equals $\equiv 1$, the paths (together with their tubes and decorations) stayed inside the interior of the corresponding double cone based at the current space-time positions of the two random walks and ω in the respective conical shells is in the good set defined in (3.1.31) and (3.1.32). If these events occur, we have found the first regeneration time $T_1 = \tilde{\sigma}_1^{\mathrm{sim}}$.

(ii) If the above attempt in (i) fails, we must try again. We successively check at times t_2, t_3, etc.: If not previously successful up to time $t_{\ell-1}$, at the ℓ-th step we check if $\tilde{\sigma}_\ell^{\mathrm{sim}} < t_{\ell+1}$, if $\tilde{\sigma}_\ell^{\mathrm{sim}}$ is a cone point for the decorated path beyond $t_{\ell-1}$ with

$$\max\left\{ \left\| X_{\tilde{\sigma}_\ell^{\mathrm{sim}}} - x \right\|, \left\| X'_{\tilde{\sigma}_\ell^{\mathrm{sim}}} - x' \right\| \right\} \leq s_{\max} \tilde{\sigma}_\ell^{\mathrm{sim}},$$

if $\eta \equiv 1$ in the b_{out}-neighbourhood of $(X_{\tilde{\sigma}_\ell^{\mathrm{sim}}}, -\tilde{\sigma}_\ell^{\mathrm{sim}})$ and $(X'_{\tilde{\sigma}_\ell^{\mathrm{sim}}}, -\tilde{\sigma}_\ell^{\mathrm{sim}})$, if ω's in the corresponding conical shells are in the good set defined in (3.1.31) and (3.1.32) and if the paths (with tubes and decorations) up to time $t_{\ell-1}$ are each contained in a box of diameter $s_{\mathrm{out}} t_{\ell-1} + b_{\mathrm{out}}$ and height $t_{\ell-1}$. If this all holds, we have found the first regeneration time T_1.

Remark 3.1.13. Construction 3.1.12 defines the regeneration times for two random walks X and X' evolving in the same environment ω. Later on we want to compare this pair of random walks on a joint environment with another pair of random walks which individually evolve in independent environments ω and ω'. Thus, for these independent random walks we also need simultaneous regeneration times. We construct them as described in Construction 3.1.12 where we use the same sequences (t_ℓ) with the deciding difference being that X' evolves in the environment η' generated by ω' analogously to (1.1.1). Thus, we check the values of ω' and η' in the R_{loc}-neighbourhood of X' along the path. Since ω and ω' are independent we do not use the double cone but instead two single cones and each of them has to satisfy the conditions of the construction.

Note that as long as the bases of the cones are far apart compared to their heights the double cone shell and the two single cone shells constitute the same geometric objects. This will be used extensively to compare the constructions in the same and in the independent environments when the initial separation of the two walks is large.

Following this construction we obtain a series of random variables $(T_n)_n$ with $T_0 := 0$, where T_n is the n-th simultaneous regeneration of the two random walks X and X'. Formally the regeneration times depend on the position of the two random walks, i.e. $T_n = T_n(x, x')$ if $(X_{T_{n-1}}, X'_{T_{n-1}}) = (x, x')$. If they start close to each other the two cones containing the paths of the random walks will overlap and there exist some parts of the environment that both random walks explore. On the other hand, if they start very far away from each other, they explore disjoint parts of the environment.

To be more precise we have two sequences of regeneration times: one for two random walks evolving in the same environment, this sequence will be denoted by $(T_n^{\text{joint}})_{n=0,1,\dots}$; the other one for two random walks evolving in two independent environments, this sequence will be denoted by $(T_n^{\text{ind}})_{n=0,1,\dots}$. The construction for $(T_n^{\text{ind}})_{n=0,1,\dots}$ follows the same principle but since the random walks evolve in two independent environments, the information obtained about the environment along each random walk is independent of the respective other random walk. Therefore one can think of building two cones and checking the conditions above for each cone separately, which will then be independent of each other.

Remark 3.1.14. We want to note that all results obtained above about σ^{sim} hold in the case of independent environments as well, since we don't have to carefully check whether the two random walks are close to each other in space.

Furthermore we denote by

$$\mathbb{P}_{x,x'}^{\text{joint}} \quad \text{and} \quad \mathbb{P}_{x,x'}^{\text{ind}} \tag{3.1.39}$$

the laws of (X, X') evolving in the same (joint) environment respectively in two independent environments starting from $(X_0, X_0') = (x, x')$. We use the following notation somewhat interchangeably depending on the situation

$$\mathbb{P}_{x,x'}^{\text{joint}}(T_1 = \cdot) = \mathbb{P}_{x,x'}(T_1^{\text{joint}} = \cdot) = \mathbb{P}(T_1^{\text{joint}}(x, x') = \cdot)$$

and

$$\mathbb{P}_{x,x'}^{\text{ind}}(T_1 = \cdot) = \mathbb{P}_{x,x'}(T_1^{\text{ind}} = \cdot) = \mathbb{P}(T_1^{\text{ind}}(x, x') = \cdot).$$

Moreover, since we are interested in the behaviour along the simultaneous regeneration times, we need to introduce some more notation. Let $\hat{X} := (\hat{X}_n)_n$ be a random walk starting in $\hat{X}_0 = x_0$ along the simultaneous regeneration times described above, i.e. $\hat{X}_n = X_{T_n}$ and $\hat{X}'_n = X'_{T_n}$. For random walks along the regeneration times in the joint and independent environments we write $\hat{X}_n^{\text{joint}} := X_{T_n^{\text{joint}}}$ and $\hat{X}_n^{\text{ind}} := X_{T_n^{\text{ind}}}$, and for $\hat{X}'$ analogously.

Essential tools for the proof of the quenched CLT will be good comparison results between these two different dynamics. Note that $(\hat{X}, \hat{X}')$ is in both cases a Markov chain due to the respective regeneration constructions. In the case of independent environments for the two copies of the walk the increments $(\hat{X}_n^{\text{ind}} - \hat{X}_{n-1}^{\text{ind}}, \hat{X}_n^{\prime,\text{ind}} - \hat{X}_{n-1}^{\prime,\text{ind}})_n$ are in fact i.i.d. random variables.

Lemma 3.1.15 (Joint regeneration times)**.** *Denote by T_1 the first simultaneous regeneration time, then there are positive constants C and β so that*

$$\sup_{x_0,x_0'} \mathbb{P}_{x_0,x_0'}^{\text{joint}}(T_1 > t) \le Ct^{-\beta} \quad \text{and} \quad \sup_{x_0,x_0'} \mathbb{P}_{x_0,x_0'}^{\text{ind}}(T_1 > t) \le Ct^{-\beta}. \tag{3.1.40}$$

Furthermore, β can be chosen arbitrarily large if p is suitably close to 1 and ε_{ref} is sufficiently small.

The fact that β can be chosen large will be used throughout the rest of the paper. Also note that for $\mathbb{P}^{\text{ind}}$ the random walks evolve in independent environments and the construction of the regeneration times and their tails are then even independent of the positions of the random walks.

Remark 3.1.16. In contrast to Birkner et al. in [BČDG13] we do not have a literal connection (in the sense of being a subsequence) between the simultaneous regeneration times defined here and the regeneration times for a single random walk defined in [BČD16]. For that reason we directly will work here with the simultaneous regeneration times.

Proof of Lemma 3.1.15. Let us first prove the first inequality in (3.1.40). We fix x_0, x_0' and write $\mathbb{P}$ for $\mathbb{P}^{\text{joint}}_{x_0,x_0'}$ for the rest of the proof to shorten the notation. Recall Construction 3.1.12 and the notation introduced there. For any $m \geq 2$, to be chosen appropriately later, we have

$$\mathbb{P}(T_1 > t) = \sum_{\ell=1}^{\infty} \mathbb{P}(T_1 = \tilde{\sigma}_\ell^{\text{sim}}, \tilde{\sigma}_\ell^{\text{sim}} > t) \leq \sum_{\ell=1}^{m-1} \mathbb{P}(\tilde{\sigma}_\ell^{\text{sim}} > t) + \sum_{\ell=m}^{\infty} \mathbb{P}(T_1 = \tilde{\sigma}_\ell^{\text{sim}}). \tag{3.1.41}$$

Let $t_1 = 1$ and define

$$t_{\ell+1} = \left\lceil \frac{t_\ell s_{\max} + b_{\text{out}} + t_\ell s_{\text{out}} - b_{\text{inn}}}{s_{\text{inn}} - s_{\max}} \right\rceil + 1, \qquad \text{for all } \ell = 1, 2, \dots. \tag{3.1.42}$$

Then the sequence $(t_\ell)_{\ell=1,2,\dots}$ satisfies the condition (3.1.37). Furthermore for any ρ with

$$\rho > \frac{s_{\text{out}} + s_{\max}}{s_{\text{inn}} - s_{\max}} \tag{3.1.43}$$

we have $t_\ell \leq \lceil \rho^\ell \rceil$ for all $\ell \geq \ell^*$, where $\ell^* = \ell^*(\rho) < \infty$ is the smallest index such that $t_\ell \leq \lceil \rho^\ell \rceil$. Since we need to check the statement of the lemma only for large t, assume that $\lceil \rho^{\ell^*} \rceil \leq \sqrt{t}$. For such t we have choosing $m = \lceil \log(t)/(2\log(\rho)) \rceil$ and using $\tilde{\sigma}_0^{\text{sim}} = 0$

$$\begin{aligned} \sum_{i=1}^{m-1} \mathbb{P}(\tilde{\sigma}_i^{\text{sim}} > t) &\leq \sum_{i=1}^{m-1} \mathbb{P}(\tilde{\sigma}_i^{\text{sim}} - \tilde{\sigma}_{i-1}^{\text{sim}} > t - t_i) \\ &\leq C(m-1)e^{-c(t-t_{m-1})} \leq C(m-1)e^{-c(t-\sqrt{t})}. \end{aligned} \tag{3.1.44}$$

Obviously we have $m < \sqrt{t}$. Thus, the right hand side is bounded by $Ct^{-\beta}$ for any $\beta > 0$.

For the second sum on the right hand side of (3.1.41) we first show that there is a uniform, positive lower bound for the probability of a successful regeneration (i.e. conditions in Construction 3.1.12 (ii) do hold) at the ℓ-th attempt for all $\ell \geq 2$.

By Lemma 3.1.7 we know that for large ℓ the probability of $\tilde{\sigma}_\ell > t_{\ell+1}$ is very small. The condition for the environment ω, restricted to the respective cone shell, to be a good configuration is independent for different ℓ since we check different parts of ω for each ℓ. For the path containment by Lemma 2.16 in [BČD16] we have

$$\begin{aligned} &\mathbb{P}(\exists\, n \leq t_\ell : \|X_n - x_0\| > \tfrac{1}{2} s_{\max} t_\ell + n s_{\text{out}} \text{ or } \exists\, n \leq t_\ell : \|X_n' - x_0'\| > \tfrac{1}{2} s_{\max} t_\ell + n s_{\text{out}}) \\ &\quad \leq \mathbb{P}(\exists\, n \leq t_\ell : \|X_n - x_0\| > \tfrac{1}{2} s_{\max} t_\ell + n s_{\text{out}}) + \mathbb{P}(\exists\, n \leq t_\ell : \|X_n' - x_0'\| > \tfrac{1}{2} s_{\max} t_\ell + n s_{\text{out}}) \\ &\quad \leq \sum_{\lceil \frac{t_\ell s_{\max}}{2R_\kappa} \rceil}^{t_\ell} \mathbb{P}(\|X_n - x_0\| > s_{\max} n) + \sum_{\lceil \frac{t_\ell s_{\max}}{2R_\kappa} \rceil}^{t_\ell} \mathbb{P}(\|X_n' - x_0'\| > s_{\max} n) \\ &\quad \leq C e^{-c t_\ell}. \end{aligned}$$

Since t_ℓ grows exponentially in ℓ, the right hand side is summable in ℓ. Thus, from some time ℓ_0 on, we have $\sup_{n\leq t_\ell}\|X_n - x_0\| \leq s_{\max}t_\ell$ and $\sup_{n\leq t_\ell}\|X'_n - x'_0\| \leq s_{\max}t_\ell$ for all $\ell \geq \ell_0$ a.s.

Next we bound the size of the decorations by showing that we only have finitely many large increments $\sigma^{\text{sim}}_{i+1} - \sigma^{\text{sim}}_i$ in the sequence σ^{sim}. For $i_\ell \coloneqq \inf\{i : \sigma^{\text{sim}}_i \geq t_\ell\}$ we have by Lemma 3.1.4

$$\begin{aligned}\mathbb{P}(\exists\, i \leq i_\ell : \sigma^{\text{sim}}_i - \sigma^{\text{sim}}_{i-1} > k) &\leq \mathbb{P}(\exists\, i \leq t_\ell : \sigma^{\text{sim}}_i - \sigma^{\text{sim}}_{i-1} > k)\\ &\leq \sum_{i=1}^{t_\ell}\mathbb{P}(\sigma^{\text{sim}}_i - \sigma^{\text{sim}}_{i-1} > k) \leq Ct_\ell e^{-ck}.\end{aligned}$$

Now if $\|X_n - x\| \leq t_\ell s_{\max}$ for all $n \leq t_\ell$ and $\sigma^{\text{sim}}_i - \sigma^{\text{sim}}_{i-1} \leq t_\ell(s_{\text{out}} - s_{\max})/2$ for all $i \leq i_\ell$, then the path including the decorations is contained in the box $(x, 0) + [-b_{\text{out}} - t_\ell s_{\text{out}}, b_{\text{out}} + t_\ell s_{\text{out}}]^d \times [0, t_\ell]$. Note that $Ct_\ell e^{-ck}$ is summable for the choice $k = t_\ell(s_{\text{out}} - s_{\max})/2$. So we only have finitely many times where one of the random walks moves "too fast" or the decorations are "too large". Combining the above it follows that a.s. there exists ℓ' so that the path containment property holds for all $\ell \geq \ell'$.

Assertion (3.1.16) from Lemma 3.1.4 guarantees the existence of open paths in the future direction of the random walks and thus yields a uniform lower bound on the event that $\eta = 1$ in the R_{loc}-neighbourhood of the random walks at $\tilde{\sigma}^{\text{sim}}_\ell$. Therefore there exists a uniform lower bound in ℓ on the probability of a successful regeneration at ℓth attempt. Let δ_0 be that uniform lower bound. Note that $\delta_0 \to 1$ for $p \to 1$ since the probability for all three conditions goes to 1 for $p \to 1$.

By plugging in the definition of m the second partial sum on the right hand side of (3.1.41) has the upper bound

$$\sum_{i=m}^{\infty}\mathbb{P}(T_1 = \tilde{\sigma}^{\text{sim}}_i) \leq \sum_{i=m}^{\infty}\delta_0(1-\delta_0)^i = (1-\delta_0)^{m+1} \leq t^{\frac{\ln(1-\delta_0)}{2\ln(\rho)}}. \tag{3.1.45}$$

The condition on ρ suggests that we can choose ρ close to 1 if $s_{\max}$ is close to 0. This can be achieved for p close to 1, as is mentioned in Lemma 2.16 from [BČD16]. This way we obtain $-\frac{\ln(1-\delta_0)}{2\ln(\rho)} > \beta$ and that concludes the proof. Note that in the proof we have a dependence between t and ρ, since we choose t large enough such that $\rho^{\ell^*} \leq \sqrt{t}$, this results in the constant C being dependent on the choice of ρ, and thus b, as well. However this is not a problem since the condition $\rho^{\ell^*} \leq \sqrt{t}$ holds for smaller t if we choose ρ closer to 1. This proves the first inequality in (3.1.40).

The second inequality in (3.1.40) follows directly by the above arguments and the fact that the two environments ω and ω', that the random walks explore, are independent. □

Remark 3.1.17. One can be a bit more precise for which ℓ the relation $t_\ell \leq \rho^\ell$ is true. This obviously depends on the choice of ρ and yields that there exists $\ell^* = \ell^*(\rho)$ such that

$$\rho > \left(\frac{b_{\text{out}} - b_{\text{inn}}}{2s_{\max} + s_{\text{out}} - s_{\text{inn}}}\right)^{\frac{1}{\ell^*}}\frac{s_{\text{out}} + s_{\max}}{s_{\text{inn}} - s_{\max}}, \tag{3.1.46}$$

and we have $t_\ell \leq \rho^\ell$ for all $\ell \geq \ell^*$.

As a direct consequence of Lemma 3.1.15 we obtain

Corollary 3.1.18. *There exist positive constants C and $\beta > 0$ so that*

$$\sup_{x_0, x'_0}\mathbb{P}^{\text{joint}}_{x_0,x'_0}(\|\hat{X}_1 - x_0\| + \|\hat{X}'_1 - x'_0\| > m) \leq C'm^{-\beta}$$

and

$$\sup_{x_0,x_0'} \mathbb{P}^{\text{joint}}_{x_0,x_0'}(\exists n \le N : \|\hat{X}_n - \hat{X}_{n-1}\| + \|\hat{X}'_n - \hat{X}'_{n-1}\| > N^{b_6}) \le N \cdot N^{-b \cdot b_6}.$$

Analogous estimates hold for $\mathbb{P}^{\text{ind}}$.

Proof. By Assumption 1.3.4 the random walks X and X' both have finite range for their transitions and thus the claim follows by Lemma 3.1.15. □

3.2 Auxiliary results

In this section we will collect some useful results for all dimension $d \ge 1$.

The following lemma allows us to compare the laws $\mathbb{P}^{\text{ind}}$ and $\mathbb{P}^{\text{joint}}$ from (3.1.39). The idea is to use the fact that for large initial distance of the starting positions, the laws are similar. The reason for this is, that then the cone construction will result in overlapping cones only, if the regeneration time is very large.

Lemma 3.2.1. *There are constants* $0 < c, \beta < \infty$, *so that for all* $x, x' \in \mathbb{Z}^d$ *we have*

$$\left\|\mathbb{P}^{\text{ind}}_{x,x'}\big((\hat{X}_1, \hat{X}'_1) \in \cdot\big) - \mathbb{P}^{\text{joint}}_{x,x'}\big((\hat{X}_1, \hat{X}'_1) \in \cdot\big)\right\|_{\text{TV}} \le c\,\|x - x'\|^{-\beta}, \tag{3.2.1}$$

where the constant β *is from Lemma 3.1.15 and can be chosen large.*

Proof. To simplify the notation throughout the proof we fix a positive even integer m and without loss of generality we prove the assertion for starting positions $x = (-m/2, 0, \dots, 0)$ and $x' = (m/2, 0, \dots, 0)$ in $\mathbb{Z}^d$. We can do this since for the proof only the distance between the starting positions is going to be relevant, not the exact positions.

Every environment is defined by a configuration $\omega \in \{0,1\}^{\mathbb{Z}^d \times \mathbb{Z}}$ and dynamics of the random walks are then given by the family of transition kernels $\kappa = \{\kappa_n(x, \cdot) : n \in \mathbb{Z}, x \in \mathbb{Z}^d\}$. Let

$$\Omega_i \coloneqq \{(\omega_i(z,n) : (z,n) \in \mathbb{Z}^d \times \mathbb{Z}\}, \quad i = 1, 2 \tag{3.2.2}$$

be two independent families of random variables, where the random variable $\omega_i(z,n)$ are i.i.d. Bernoulli distributed with parameter $p > p_c$. We introduce a composite environment $\Omega_3 \coloneqq \{\omega_3(z,n) : (z,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$, where ω_3 is constructed using ω_1 and ω_2 and defined by

$$\omega_3(z,n) \coloneqq \begin{cases} \omega_1(z,n) & : z_1 \le 0, \\ \omega_2(z,n) & : z_1 > 0, \end{cases}$$

where z_1 is the first coordinate of z. Define

$$B \coloneqq \Big\{(z,n) \in \mathbb{Z}^d \times \mathbb{Z} : \|z - x\| \le \frac{m}{10} + b_{\text{out}} + \frac{m}{10R_\kappa} s_{\text{out}}, n \in \big\{0, \dots, -\frac{m}{10R_\kappa}\big\}\Big\},$$
$$B' \coloneqq \Big\{(z,n) \in \mathbb{Z}^d \times \mathbb{Z} : \|z - x'\| \le \frac{m}{10} + b_{\text{out}} + \frac{m}{10R_\kappa} s_{\text{out}}, n \in \big\{0, \dots, -\frac{m}{10R_\kappa}\big\}\Big\}.$$

These boxes will contain the random walks and cones constructed for the first regeneration if it happens before time $\frac{m}{10R_\kappa}$. This also means that the R_{loc}-vicinity of the random walks is inside these boxes and thus up until time $m/(10R_\kappa)$ the random walks in (Ω_1, Ω_2) and (Ω_3, Ω_3) will behave the same as long as the values of η inside B and B' are the same in both environments (this means, by Assumption 1.3.1, that the transition kernels $\kappa_n(x,\cdot)$ are the same).

Define $T_{1,2} = T_1(\Omega_1, \Omega_2)$ and $T_{3,3} = T_1(\Omega_3, \Omega_3)$ as the first simultaneous regeneration times using Ω_1, Ω_2 and Ω_3 respectively. Note that $T_{3,3} = T_1^{\text{joint}}$ and $T_{1,2} = T_1^{\text{ind}}$ and we have already shown in Lemma 3.1.15 that

$$\mathbb{P}(T_{3,3} > r) \le Cr^{-\beta} \tag{3.2.3}$$

and

$$\mathbb{P}(T_{1,2} > r) \le Cr^{-\beta}. \tag{3.2.4}$$

Now if $\eta_n(z,\Omega_1) = \eta_n(z,\Omega_3)$ for all $(z,n) \in B$ and $\eta_n(z,\Omega_2) = \eta_n(z,\Omega_3)$ for all $(z,n) \in B'$ we can couple the two random walks at their first regeneration time in (Ω_1,Ω_2) with the two in (Ω_3,Ω_3) until time $\frac{m}{10R_\kappa}$, since the values of ω and η in the parts of the environment that the random walks explore are equal in both cases and therefore their distribution is equal. To that end we define the sets

$$D_1 := \{\text{for all } (z,n) \in B : \eta_n(z,\Omega_1) = \eta_n(z,\Omega_3)\} \cap \{\text{for all } (z,n) \in B' : \eta_n(z,\Omega_2) = \eta_n(z,\Omega_3)\},$$
$$D_2 := \{T_{1,2} \le \frac{m}{10R_\kappa}, T_{3,3} \le \frac{m}{10R_\kappa}\}.$$

On $D_1 \cap D_2$ we have $T_{1,2} = T_{3,3}$ and since $T_{1,2} \le \frac{m}{10R_\kappa}$ the random walks are still in the box and thus have the same distribution for their position at the first regeneration in both cases. That means

$$\mathbb{P}_{x,x'}(X_{T_{1,2}} = y, X'_{T_{1,2}} = y') = \mathbb{P}_{x,x'}(X_{T_{3,3}} = y, X'_{T_{3,3}} = y')$$

on $D_1 \cap D_2$. With Lemma 3.1.15 we get an upper bound for the probability of D_2^c. On D_1^c there needs to exist a space-time site (z,n) in B with $\eta_n(z,\Omega_1) \ne \eta_n(z,\Omega_3)$ or (z',n') in B' with $\eta_{n'}(z',\Omega_2) \ne \eta_{n'}(z',\Omega_3)$.

Assume there exists such a site $(z,n) \in B$ with $\eta_n(z,\Omega_1) \ne \eta_n(z,\Omega_3)$. There are two cases in which that can occur. First, $\eta_n(z,\Omega_1) = 1$ and $\eta_n(z,\Omega_3) = 0$. That is (z,n) is connected to $-\infty$ in Ω_1 but not in Ω_3. This means that by changing the values of ω in the positive half plane $\mathbb{Z}_+ \times \mathbb{Z}^{d-1} \times \mathbb{Z}$ we cut of all infinitely long open paths starting from (z,n). Thus the only open paths connecting (z,n) to $-\infty$ are via the half plane $\mathbb{Z}_+ \times \mathbb{Z}^{d-1} \times \mathbb{Z}$ and in Ω_3 the contact process started from (z,n) lives for at least $\tilde{m} = \frac{m}{2} - \frac{m}{10} - b_{\text{out}} - \frac{m}{10R_\kappa} s_{\text{out}}$ steps, since that is the distance of B and the positive half plane, but dies out eventually. By Lemma 1.3 from [Ste17]

$$\mathbb{P}(\eta_{\tilde{m}}^{\{(z,n)\}} \ne 0 \text{ and } \eta^{\{(z,n)\}} \text{ eventually dies out}) \le Ce^{-\tilde{c}\tilde{m}} \le Ce^{-cm}.$$

The case where $\eta_n(z,\Omega_1) = 0$ and $\eta_n(z,\Omega_3) = 1$ follows the same arguments as above. We now know that the only open paths connecting (z,n) to $-\infty$ in Ω_3 have to be cut off in Ω_1, which again has probability less than Ce^{-cm}. Analogous arguments can be made for sites in B'. Since $|B \cup B'| = 2(\frac{m}{10} + b_{\text{out}} + \frac{m}{10R_\kappa} s_{\text{out}})^d \cdot \frac{m}{10R_\kappa}$ we obtain $\mathbb{P}(D_1^c) \le Ce^{-cm}$. Since $\mathbb{P}_{x,x'}(X_{T_{1,2}} = y, X'_{T_{1,2}} = y') = \mathbb{P}^{\text{ind}}_{x,x'}(\hat{X}_1 = y, \hat{X}'_1 = y')$ and $\mathbb{P}_{x,x'}(X_{T_{3,3}} = y, X'_{T_{3,3}} = y') = \mathbb{P}^{\text{joint}}_{x,x'}(\hat{X}_1 = y, \hat{X}'_1 = y')$ we conclude

$$\frac{1}{2} \sum_{(y,y') \in \mathbb{Z}^d \times \mathbb{Z}^d} |\mathbb{P}^{\text{ind}}_{x,x'}(\hat{X}_1 = y, \hat{X}'_1 = y') - \mathbb{P}^{\text{joint}}_{x,x'}(\hat{X}_1 = y, \hat{X}'_1 = y')|$$
$$\le \mathbb{P}(D_1^c) + \mathbb{P}(D_2^c) \le Ce^{-cm} + Cm^{-\beta} \le Cm^{-\beta} \tag{3.2.5}$$

□

Looking at the regeneration times $T_n^{\text{joint}}(x,x')$ and $T_n^{\text{ind}}(x,x')$ we need to consider a second "dummy" random walk, to study the random walk $\hat{X} = (\hat{X}_n)_n$. Nevertheless, as we have shown above in Lemma 3.1.15 the distributions of $T_1^{\text{joint}}(x,x')$ and $T_1^{\text{ind}}(x,x')$ have sufficiently fast decaying tails such that even along these simultaneous regeneration times an annealed CLT for $(\hat{X}_n)_n$ holds.

Proposition 3.2.2. *There exists $c > 0$ and a non-trivial centered d-dimensional normal law $\tilde{\Phi}$ such that for $f : \mathbb{R}^d \to \mathbb{R}$ bounded and Lipschitz we have*

$$\mathbb{E}\left[\left(E_\omega[f(\hat{X}_m^{\text{joint}}/\sqrt{m})] - \tilde{\Phi}(f)\right)^2\right] \le C_f m^{-c}. \tag{3.2.6}$$

For the proof we will need some auxiliary results and it will be given on page 110.

Lemma 3.2.3 (Analogue to [BČDG13] Lemma 3.6)**.** *Write for $r > 0$*

$$\begin{aligned} h(r) &:= \inf\{k \in \mathbb{Z}_+ : \left\|\hat{X}_k - \hat{X}'_k\right\|_2 \le r\} \\ H(r) &:= \inf\{k \in \mathbb{Z}_+ : \left\|\hat{X}_k - \hat{X}'_k\right\|_2 \ge r\} \end{aligned} \tag{3.2.7}$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{Z}^d$, and set for $r_1 < r < r_2$

$$f_d(r; r_1, r_2) = \begin{cases} \frac{r - r_1}{r_2 - r_1}, & \text{when } d = 1, \\ \frac{\log r - \log r_1}{\log r_2 - \log r_1}, & \text{when } d = 2, \\ \frac{r_1^{2-d} - r^{2-d}}{r_1^{2-d} - r_2^{2-d}}, & \text{when } d \ge 3. \end{cases} \tag{3.2.8}$$

For every $\varepsilon > 0$ there are (large) R and $\tilde{R}$ such that for all $r_2 > r_1 > R$ with $r_2 - r_1 > \tilde{R}$ and $x, y \in \mathbb{Z}^d$ satisfying $r_1 < r = \|x - y\|_2 < r_2$

$$(1-\varepsilon) f_d(r; r_1, r_2) \le \mathbb{P}_{x,y}^{\text{ind}}(H(r_2) < h(r_1)) \le (1+\varepsilon) f_d(r; r_1, r_2). \tag{3.2.9}$$

Proof sketch. Let $\hat{X}_n = X_{T_n^{\text{ind}}(0,0)}$. It is sufficient to show that $(\hat{X}_n - \hat{X}'_n)_n$ satisfies an annealed invariance principle. To show that an invariance principle holds it is sufficient to show that

$$\begin{aligned} \mathbb{E}^{\text{ind}}[\hat{X}_1 - \hat{X}'_1] &= 0 \\ \mathbb{E}^{\text{ind}}[\left\|\hat{X}_1 - \hat{X}'_1\right\|^2] &< \infty. \end{aligned}$$

By Assumption 1.3.3 we have for the annealed expectation

$$\mathbb{E}[X_1] = 0, \tag{3.2.10}$$

from which follows

$$\mathbb{E}^{\text{ind}}[\hat{X}_1] = \mathbb{E}^{\text{ind}}[\hat{X}'_1] = 0, \tag{3.2.11}$$

as well as

$$\mathbb{E}^{\text{ind}}[\hat{X}_1 - \hat{X}'_1] = 0. \tag{3.2.12}$$

So we have centered (at least in the annealed sense) random variables. To get the other condition we use the fact that $\mathbb{P}^{\text{ind}}(T_1 > n) \leq n^{-\beta}$, with arbitrarily large β if we choose p close to 1. First we observe that in dimension d the following holds

$$\#\{x \in \mathbb{Z}^d : \|x\|_q \leq n\} \leq n^d, \tag{3.2.13}$$

for $1 \leq q \leq \infty$. Now to get to a point with $\|\cdot\| = n$ we need at least n/R_κ steps, where R_κ is the range of the transition kernels κ from Assumption 1.3.4. To reach such a point the regeneration attempt has to fail often enough. Recall the sequence $(t_\ell)_\ell$ and, for ℓ large, $t_\ell \approx \rho^\ell$ for some $\rho > 1$. There exists a unique ℓ_n with $\rho^{\ell_n - 1} < n \leq \rho^{\ell_n}$. From that we obtain that the lower bound for the number of times the regeneration attempt has to fail is $\ell_n \geq \log(n)/\log(\rho)$. If δ is the probability for a successful attempt at regenerating, then the probability to fail often enough such that $T_1 \geq n$ is bound from above by

$$(1-\delta)^{\log(n)/log(\rho)}\delta = n^{\log(1-\delta)/\log(\rho)}\delta. \tag{3.2.14}$$

Since $1 - \delta < 1$ and we can get $\rho > 1$ close to 1 we can get the exponent of n arbitrarily large. That yields an upper bound for the second moment

$$\mathbb{E}^{\text{ind}}\left[\left\|\hat{X}_1\right\|_q^2\right] \leq \sum_{n=1}^{\infty} n^d n^2 n^{\log(1-\lambda)/\log(\rho)}\lambda < \infty, \tag{3.2.15}$$

if we choose ρ close enough to 1. Now this holds as well for $\hat{X}'$ and with that also for $(\hat{X}_n - \hat{X}'_n)_n$. □

3.3 Proof of Theorem 1.3.5 in $d \geq 2$

The rest of this section is concerned with the case $d \geq 2$. We want to prove a separation lemma similar to Lemma 3.8 in [BČDG13] for the model from [BČD16].

Lemma 3.3.1 (Separation lemma)**.** *Let $d \geq 2$. For any $x_0, x_0' \in \mathbb{Z}^d$ and for all small enough $\delta > 0$ there is $b_2 \in (0, 1/2)$ and $C, c > 0$ so that*

$$\mathbb{P}^{\mathsf{joint}}_{x_0,x_0'}\left(H(n^\delta) > n^{b_2}\right) \leq \exp(-Cn^c). \tag{3.3.1}$$

Proof. We will adapt the idea and the structure of the proof of Lemma 3.8 in [BČDG13] which consists of 4 Steps. Adapting Steps 1–3 is straight forward but Step 4 will require a bit more work and some "new ideas".

Step 1. We want to show that there exists a small $\varepsilon_1 > 0$ and $b_4 \in (0, 1/2)$, $b_5 > 0$ such that

$$\mathbb{P}^{\mathsf{joint}}_{x,y}\left(H(\varepsilon_1 \log n) > n^{b_4}\right) \leq cn^{-b_5}, \tag{3.3.2}$$

uniformly in $x, y \in \mathbb{Z}^d$.

To that end we construct suitable "corridors" to guide the random walks to a certain distance within the first steps. First have a closer look at a possible configuration of the environment that yields a suitable lower bound for the probability to reach a distance of $\varepsilon_1 \log n$ in $\varepsilon_1 \log n$ steps. To that we recall the dynamics of the random walks whenever they are on a site with $\eta = 1$, see assumption 1.3.2. Since the reference transition kernel κ_{ref} is non degenerate we can assume that there exists a possible step $k \in \mathbb{Z}^d$ such that the first coordinate k_1 is greater than 0 and set $\delta_1 \coloneqq \kappa_{\text{ref}}(k)$. Then on any site $(y, -m) \in \mathbb{Z}^d \times \mathbb{Z}$ with

$\eta_{-m}(y) = 1$ the random walk can jump a distance of k_1 in the first coordinate. Thus, if we ensure that the random walks hit only sites where $\eta = 1$, the distance increases by at least $2k_1$ with probability at least $\delta_2 \coloneqq \delta_1 - \varepsilon_{\text{ref}} > 0$. Recall Construction 3.1.12. If the two random walks just regenerated they can do so at the next step if the values of η in the b_{out}-vicinity of their next positions are only 1's. This happens with a uniform lower bound (since a successful attempt at regenerating has a uniform lower bound as shown in the proof of Lemma 3.1.15) which we denote by $\delta_3 > 0$. Therefore, if $\eta = 1$ in the b_{out}-vicinity of both random walks along the path, they will regenerate at every such step.

Consequently, for every step, the probability that the distance between the random walks increases by at least $2k_1$ is

$$\mathbb{P}^{\text{joint}}_{x,y}\left(\left\| \hat{X}_i - \hat{X}'_i \right\| \geq 2k_1 + \left\| \hat{X}_{i-1} - \hat{X}'_{i-1} \right\| \right) > \delta_2^2 \delta_3 > 0$$

and iteratively

$$\mathbb{P}^{\text{joint}}_{x,y}\left(\left\| \hat{X}_j - \hat{X}'_j \right\| \geq 2jk_1 \right) \geq \hat{\delta}^j, \tag{3.3.3}$$

for some $\hat{\delta} > 0$. And as in the proof in [BČDG13] we conclude

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\varepsilon_1 \log n) > m\varepsilon_1 \log n \right) \leq \left(1 - n^{-\varepsilon_1 \log(1/\hat{\delta})} \right)^m \leq \exp\left(-mn^{-\varepsilon_1 \log(1/\hat{\delta})} \right). \tag{3.3.4}$$

If we choose ε_1 so small that $-\varepsilon_1 \log(\hat{\delta}) \in (0, 1/2)$, and choose $b_4 \in (-\varepsilon_1 \log(\hat{\delta}), 1/2)$, $b_5 > 0$ and set $m = b_5 n^{\varepsilon_1 \log(1/\hat{\delta})} \log n$ we have shown (3.3.2) since $m\varepsilon_1 \log n \leq n^{b_4}$ if n is large.

Step 2. Next we show that for any $K_2 > 0$ there exists $\delta_2 \in (0, 1)$ such that for all $x, y \in \mathbb{Z}^d$ with $\varepsilon_1 \log n \leq \|x - y\| < K_2 \log n$ and n large enough

$$\mathbb{P}^{\text{joint}}_{x,y}\Big(H(K_2 \log n) < h(\frac{1}{2}\varepsilon_1 \log n) \wedge (K_2 \log n)^3 \Big) \geq \delta_2, \tag{3.3.5}$$

where $h(m) \coloneqq \inf\{k : \left\| \hat{X}_k - \hat{X}'_k \right\| \leq m\}$. To this end we couple $\mathbb{P}^{ind}_{x,y}$ with $\mathbb{P}^{\text{joint}}_{x,y}$. As mentioned above the probability that the coupling fails will decay algebraically in the distance of the starting positions of the random walks. Thus, the left-hand side of (3.3.5) is bounded from below by

$$\mathbb{P}^{\text{ind}}_{x,y}\Big(H(K_2 \log n) < h(\frac{1}{2}\varepsilon_1 \log n) \wedge (K_2 \log n)^3 \Big) - C(K_2 \log n)^3 (\frac{1}{2}\varepsilon_1 \log n)^{-\beta}, \tag{3.3.6}$$

where $\beta > 0$ is from Lemma 3.2.1 and can be chosen arbitrarily large so that the second term will go to 0 as n tends to infinity. Under $\mathbb{P}^{\text{ind}}$ the process $(\hat{X}_n - \hat{X}'_n)_n$ has i.d.d. increments and we can use Lemma 3.2.3. Therefore we can obtain a lower bound for $d \geq 3$ for the left term in (3.3.6) by combining

$$\begin{aligned} \mathbb{P}^{\text{ind}}_{x,y}\Big(H(K_2 \log n) < h(\frac{1}{2}\varepsilon_1 \log n) \Big) &\geq (1 - \varepsilon) \frac{(\frac{\varepsilon_1}{2} \log n)^{2-d} - (\varepsilon_1 \log n)^{2-d}}{(\frac{\varepsilon_1}{2} \log n)^{2-d} - (K_2 \log n)^{2-d}} \\ &= (1 - \varepsilon) \frac{(2^{d-2} - 1)\varepsilon_1^{2-d}}{2^{d-2}\varepsilon_1^{2-d} - K_2^{2-d}} \end{aligned} \tag{3.3.7}$$

and the fact that due to Donsker's invariance principle we have

$$\begin{aligned}\mathbb{P}^{\text{ind}}_{x,y}\left(H(K_2\log n)\geq (K_2\log n)^3\right) &= \mathbb{P}^{\text{ind}}_{x,y}\Big(\inf\{k\leq n\colon \left\|\hat{X}_k-\hat{X}'_k\right\| > K_2\log n\} > (K_2\log n)^3\Big)\\ &= \mathbb{P}^{\text{ind}}_{x,y}\Big(\inf\Big\{k\leq n\colon \frac{\left\|\hat{X}_k-\hat{X}'_k\right\|}{\sqrt{n}} > \frac{K_2\log n}{\sqrt{n}}\Big\} > (K_2\log n)^3\Big)\\ &\approx \mathbb{P}\Big(\inf\Big\{t\in[0,1]\colon \|B_t\| > \frac{K_2\log n}{\sqrt{n}}\Big\} > \frac{(K_2\log n)^3}{n}\Big)\\ &\leq \frac{n}{(K_2\log n)^3}\frac{(K_2\log n)^2}{dn}\end{aligned}$$

where $B=(B_t)_t$ is a centred d-dimensional Brownian motion. The last line follows by the Markov inequality and the fact that the expected time for B to exit a ball of radius r is r^2/d. Thus, we have

$$\begin{aligned}&\mathbb{P}^{\text{ind}}_{x,y}\Big(H(K_2\log n) < h(\frac{1}{2}\varepsilon_1\log n)\wedge (K_2\log n)^3\Big)\\ &\qquad\geq (1-\varepsilon)\frac{(2^{d-2}-1)\varepsilon_1^{2-d}}{2^{d-2}\varepsilon_1^{2-d}-K_2^{2-d}} - \frac{n}{(K_2\log n)^3}\frac{(K_2\log n)^2}{dn}\end{aligned}$$

and the right hand side is bounded away from 0 since the last term tends to 0 as $n\to\infty$.

For $d=2$ we can use the same arguments with a slightly different lower bound when using Lemma 3.2.3 similar to (3.3.7). Here we have

$$\begin{aligned}\mathbb{P}^{\text{ind}}_{x,y}\Big(H(K_2\log n) < h(\frac{1}{2}\varepsilon_1\log n)\Big) &\geq (1-\varepsilon)\frac{\log(\varepsilon_1\log n)-\log(\frac{1}{2}\varepsilon_1\log n)}{\log(K_2\log n)-\log(\frac{1}{2}\varepsilon_1\log n)}\\ &= (1-\varepsilon)\frac{\log 2}{\log K_2-\log(\varepsilon_1/2)}.\end{aligned}$$

Combining the above estimates for $d=2$ and $d\geq 3$ with (3.3.6) we obtain a uniform lower bound in (3.3.5) for all $d\geq 2$.

Step 3. Combining the previous steps we see, that we can choose a large K_3, and $b_6\in(b_4,1/2)$ such that uniformly in $x,y\in\mathbb{Z}^d$ we have

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(K_3\log n)\leq n^{b_6}\right)\geq \delta_3>0 \quad \text{for } n \text{ large enough.} \tag{3.3.8}$$

At this point this proof here will divert from the corresponding proof in [BČDG13] since Step 4 there does not hold for our model. They were able, due to the exponential decay of the total variation distance of $\mathbb{P}^{\text{ind}}_{x,y}$ and $\mathbb{P}^{\text{joint}}_{x,y}$, to jump from a distance of $\log n$ directly to n^{b_2}, whereas we need to iterate through smaller distances. We first go to a distance of $\log^2 n$. For that we use the same arguments we used in Step 2 with starting positions $x,y\in\mathbb{Z}^d$ such that $K_3\log n\leq\|x-y\|\leq\log^2 n$. So we want to obtain a lower bound for

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^2 n) < h(\varepsilon_1\log n)\wedge\log^8 n\right). \tag{3.3.9}$$

We can bound that probability from below by using the steps from Step 2 and get the lower bound

$$\mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^2 n) < h(\varepsilon_1\log n)\wedge\log^8 n\right) - \log^8 n(\varepsilon_1\log n)^{-\beta}. \tag{3.3.10}$$

Since we can choose $\beta > 8$ the right-hand side will go to 0 as n tends to infinity. Again we use Lemma 3.2.3. For dimension $d \geq 3$

$$\begin{aligned}\mathbb{P}^{\text{ind}}_{x,y}\big(H(\log^2 n) < h(\varepsilon_1 \log n)\big) &\geq (1-\varepsilon)\frac{\varepsilon_1^{2-d} - K_3^{2-d}}{\varepsilon_1^{2-d} - (\log^2 n)^{2-d}} \\ &= (1-\varepsilon)\Big(1 - \frac{K_3^{2-d} - (\log n)^{2-d}}{\varepsilon_1^{2-d} - (\log n)^{2-d}}\Big)\end{aligned} \tag{3.3.11}$$

and for $d = 2$ note that $\log^{1/2} n < \varepsilon_1 \log n$ holds for large n and we get

$$\mathbb{P}^{\text{ind}}_{x,y}(H(\log^2 n) < h(\log^{1/2} n)) \geq (1-\varepsilon)\frac{\log(K_3 \log n) - \log(\log^{1/2} n)}{\log(\log^2 n) - \log(\log^{1/2} n)} \geq \frac{1-\varepsilon}{3} \tag{3.3.12}$$

which are both bounded from below for n large enough. For the lower bound in $d = 2$ we need to switch from $\varepsilon_1 \log n$ to $\log^{1/2} n$ since the lower bound estimate with the $\varepsilon_1 \log n$ will tend to 0 for $n \to \infty$. The estimate for a failed coupling will still tend to 0 since we can choose $\beta > 0$ large. If we do the same for the step from $\log^2 n$ to $\log^4 n$ we get

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^4 n) < h(K_3 \log n) \wedge \log^{16} n\right) \geq \delta_4 > 0. \tag{3.3.13}$$

with the coupling we can use the same arguments by choosing β accordingly. The invariance principle will yield a bound that goes to 0 if n tends to infinity and we have for $d = 3$

$$\begin{aligned}\mathbb{P}^{\text{ind}}_{x,y}(H(\log^4 n) < h(K_3 \log n)) &\geq (1-\varepsilon)\frac{(K_3 \log n)^{2-d} - (\log^2 n)^{2-d}}{(K_3 \log n)^{2-d} - (\log^4 n)^{2-d}} \\ &= (1-\varepsilon)\frac{K_3^{2-d} - (\log n)^{2-d}}{K_3^{2-d} - (\log^3 n)^{2-d}},\end{aligned} \tag{3.3.14}$$

and for $d = 2$

$$\mathbb{P}^{\text{ind}}_{x,y}(H(\log^4 n) < h(K_3 \log n)) \geq (1-\varepsilon)\frac{\log\log n - \log K_3}{3\log\log n - \log K_3} \geq \frac{(1-\varepsilon)}{4} \tag{3.3.15}$$

which as well are bounded from below by a positive constant. So we have for $d \geq 2$ and n large enough

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^4 n) < h(\varepsilon_1 \log n) \wedge \log^{16} n\right) \geq \delta_5 > 0 \tag{3.3.16}$$

Now we can combine this with Step 1 and step 2 to get, in a similar way to step 3,

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^4 n) \leq n^{b_6}\right) \geq \delta_6 > 0 \qquad \text{for } n \text{ large enough.} \tag{3.3.17}$$

Next we will go from $\log^4 n$ to $\log^8 n$ and then to $\log^{16} n$ and so on. So we go from $\log^{2^i} n$ to $\log^{2^{i+1}} n$ until we reach n^{b_2}. If we could do that in a number of times that doesn't depend on n the proof would be completed. Instead we have to show, that if n tends to infinity, the product over the probabilities for all those steps is still bounded from below by a small constant away from 0.

From here on out we can write the steps in a more general way and handle the remaining part of the proof with that. Say we are at a distance of $\log^{2^j} n$ and want to get to $\log^{2^{j+1}} n$. We will do that in the same way as in Step 2. For that we split the next step in cases $d \geq 3$ and $d = 2$

Step 4 $(d \geq 3)$. We start with $d \geq 3$ and want to bound

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n) \wedge \log^{2^{j+3}} n\right), \tag{3.3.18}$$

where $\log^{2^j} n \leq \|x-y\| < \log^{2^{j+1}} n$. If we couple with $\mathbb{P}^{\text{ind}}_{x,y}$ we get the lower bound

$$\mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n) \wedge \log^{2^{j+3}} n\right) - \log^{2^{j+3}} n \cdot (\log n)^{-2^{j-1}\beta}. \tag{3.3.19}$$

If $\beta > 8$ the second term will go to zero. For the first term we get a lower bound in a similar way to Step 2. We first observe

$$\begin{aligned}&\mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n) \wedge \log^{2^{j+3}} n\right)\\ &\qquad \geq \mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n)\right) - \mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^{2^{j+1}} n) \geq \log^{2^{j+3}} n\right)\end{aligned} \tag{3.3.20}$$

Now due to the invariance principle we get

$$\mathbb{P}^{\text{ind}}_{x,y}\left(\inf\left\{k : \left\|\hat{X}_k - \hat{X}'_k\right\| \geq \log^{2^{j+1}} n\right\} \geq \log^{2^{j+3}} n\right) \leq c\frac{1}{\log^{2^{j+2}} n}, \tag{3.3.21}$$

and with Lemma 3.2.3

$$\begin{aligned}\mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n)\right) &\geq (1-\varepsilon)\frac{(\log^{2^{j-1}} n)^{2-d} - (\log^{2^j} n)^{2-d}}{(\log^{2^{j-1}} n)^{2-d} - (\log^{2^{j+1}} n)^{2-d}}\\ &= (1-\varepsilon)\frac{(\log n)^{-2^{j-1}(2-d)} - 1}{(\log n)^{-2^{j-1}(2-d)} - (\log n)^{2^j(2-d)}}\\ &= (1-\varepsilon)(1 - \frac{1-(\log n)^{2^j(2-d)}}{(\log n)^{-2^{j-1}(2-d)} - (\log n)^{2^j(2-d)}}\\ &\geq (1-\varepsilon)(1 - \frac{2}{(\log n)^{-2^{j-1}(2-d)}})\end{aligned} \tag{3.3.22}$$

for n large enough, because for $d \geq 3$

$$(\log n)^{2^j(2-d)} \xrightarrow{n\to\infty} 0 \quad \text{and} \quad (\log n)^{-2^{j-1}(2-d)} \xrightarrow{n\to\infty} \infty.$$

Combining equations (3.3.19), (3.3.21) and (3.3.22) we conclude

$$\begin{aligned}&\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n) \wedge \log^{2^{j+3}} n\right)\\ &\qquad \geq 1 - \varepsilon - (\log n)^{2^{j+3}} \cdot (\log n)^{-2^{j-1}\beta} - 2(\log n)^{2^{j-1}(2-d)}\end{aligned} \tag{3.3.23}$$

Step 4 $(d = 2)$. Next we want to get a lower bound for the case $d = 2$. So we again have $\log^{2^j} n \leq \|x-y\| \leq \log^{2^{j+1}} n$ and want to bound

$$\mathbb{P}^{\text{joint}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n) \wedge \log^{2^{j+3}} n\right). \tag{3.3.24}$$

The difference to the case $d \geq 3$ lies in the application of Lemma 3.2.3 so we will concentrate on those differences.

$$\begin{aligned}\mathbb{P}^{\text{ind}}_{x,y}\left(H(\log^{2^{j+1}} n) < h(\log^{2^{j-1}} n)\right) &\geq (1-\varepsilon)\frac{\log(\log^{2^j} n) - \log(\log^{2^{j-1}} n)}{\log(\log^{2^{j+1}} n) - \log(\log^{2^{j-1}} n)}\\ &= (1-\varepsilon)\frac{2^j - 2^{j-1}}{2^{j+1} - 2^{j-1}} = \frac{(1-\varepsilon)}{3}\end{aligned} \tag{3.3.25}$$

With this we get a lower bound for the left hand side of (3.3.23) in $d = 2$ that is bounded away from 0.

Step 5. Now we bring the previous steps together. (3.3.17) we can get to a distance of $\log^4 n$ in at most n^{b_6} steps, where $b_6 \in (0, 1/2)$. From there we can iterate Step 4. We want to get to a distance of n^δ. So we can compute how often we need to iterate Step 4

$$\begin{aligned}\log^{2^j} n = n^\delta &\Leftrightarrow 2^j \log\log n = \delta \log n \\ &\Leftrightarrow j\log(2) + \log\log\log n = \log(\delta) + \log\log n\end{aligned}$$

so we need $j_n^* := \frac{1}{\log 2}(\log\delta + \log\log n - \log\log\log n)$ many iterations. Following this construction the number of steps we need to get to n^δ is the number of steps to get to $\log^4 n$ and then the iterations of step 2'

$$\begin{aligned}n^{b_6} + \sum_{i=1}^{j_n^*} \log^{2^{i+3}} n &= n^{b_6} + \sum_{i=1}^{j_n^*} (\log^{2^i} n)^8 \\ &\le n^{b_6} + c\log(\log n) n^{8\delta} \\ &\le n^{b_6} + cn^{10\delta}\end{aligned} \tag{3.3.26}$$

for n large enough, where we used $j_n^* \le c\log\log n$ for a positive constant $c > 0$. Obviously since δ is small we can assume that there is $b_2 \in (0, 1/2)$ such that $10\delta < b_2$. So now we only need to show that the probability to make it to a distance of n^δ in that time is positive. Then we can use the Markov property and have shown the claim. To show that we just multiply the above estimated probabilities. That gives us the following product for $d = 3$

$$\begin{aligned}&\mathbb{P}^{\mathsf{joint}}_{x_0,x_0'}\left(\left\|\hat{X}_{n^{b_6}+cn^{10\delta}} - \hat{X}'_{n^{b_6}+cn^{10\delta}}\right\| \ge n^\delta\right) \\ &\ge \delta_6 \cdot \prod_{i=2}^{c\log\log n}\left(1 - \varepsilon - (\log n)^{2^{i+1}-2^{i-1}\beta} - 2(\log n)^{2^{j-1}(2-d)}\right).\end{aligned} \tag{3.3.27}$$

The probability that all this works in "one go" is at least

$$(1-\varepsilon')^{c\log\log n} \approx (\log n)^{-c\varepsilon'}.$$

For $d = 2$ it is easy to see that we also get a lower bound of the same sort by taking the product over the right hand side of (3.3.25). Thus, we will need approximately $(\log n)^{c\varepsilon'}$ many attempts. By (3.3.26), each of the attempts takes $c_1 n^{c_2}$ steps for some constants $c_1 > 0$ and $c_2 < b_2$. So we have n^α attempts where $\alpha = b_2 - c_2$ and thus

$$\mathbb{P}^{\mathsf{joint}}_{x_0,x_0'}(H(n^\delta) > n^{b_2}) \le \left(1 - (\log n)^{c\log(1-\varepsilon')}\right)^{n^\alpha} \le \exp\left(-c\frac{n^\alpha}{\log^{\tilde{c}} n}\right)$$

which shows (3.3.1) and concludes the proof of the lemma. □

Lemma 3.3.2 (Analogue to [BČDG13] Lemma 3.9)**.** *Let $d \ge 2$ there are constants $0 < b_1 < 1/2$ and $c > 0$ so that for arbitrary x_0, x_0'*

$$\mathbb{P}^{\mathsf{joint}}_{x_0,x_0'}(\textit{at most } N^{b_1} \textit{ uncoupled steps before time } N) \ge 1 - N^{-c}. \tag{3.3.28}$$

Proof. We need to split the proof in a part for $d \geq 3$ where we can estimate in a more straight forward way and $d = 2$ where we have to be more careful. Also note that if we can prove the lemma for $x_0 = x_0'$, it holds for any choice of starting positions since the coupling has a higher probability to fail if the random walks are close to each other. So let $x_0 = x_0' = 0$ and, to avoid too much notation, we write $\mathbb{P}^{\text{joint}}_{0,0} = \mathbb{P}^{\text{joint}}$.

Starting with $d \geq 3$, let A be the event that $H(N^\delta) \leq N^{b_2}$ with the parameters from Lemma 3.3.1 and let $Y_n^{\text{joint}} = \hat{X}_{1,n}^{\text{joint}} - \hat{X}_{2,n}^{\text{joint}} = \hat{X}_n^{\text{joint}} - \hat{X}_n^{\prime,\text{joint}}$ and Y_n^{ind} be defined analogously.

On A we try to couple the random walks $\hat{X}_1^{\text{joint}} = (\hat{X}_{1,n}^{\text{joint}})_{n=0,1,\dots}$ and $\hat{X}_2^{\text{joint}} = (\hat{X}_{2,n}^{\text{joint}})_{n=0,1,\dots}$ (on the same environment) with random walks $\hat{X}_1^{\text{ind}} = (\hat{X}_{1,n}^{\text{ind}})_{n=0,1,\dots}$ and $\hat{X}_2^{\text{ind}} = (\hat{X}_{2,n}^{\text{ind}})_{n=0,1,\dots}$ on independent copies of the environment, starting at time $n = H(N^\delta)$.

Let B be the event that during the time interval from $H(N^\delta)$ to N the process $Y_n^{\text{ind}} = \hat{X}_{1,n}^{\text{ind}} - \hat{X}_{2,n}^{\text{ind}}$ hits the ball $B_{N^{\delta/2}}(0)$. Then by Green function argument, see Corollary 3.19 in [MP10], we have

$$\mathbb{P}(B) \leq \mathbb{P}(Y^{\text{ind}} \text{ ever hits } B_{N^{\delta/2}}(0) \mid \left\|Y_0^{\text{ind}}\right\| = N^\delta) \leq C\Big(\frac{N^\delta}{N^{\delta/2}}\Big)^{2-d} \tag{3.3.29}$$

Thus

$$\mathbb{P}(B^c \cap \text{ coupling fails in } \{H(N^\delta), \dots, N\}) \leq N(N^{\delta/2})^{-\beta} \tag{3.3.30}$$

Therefore we also get, using Lemma 3.2.1,

$$\begin{aligned}
&\mathbb{P}(Y^{\text{joint}} \text{ hits } B_{N^{\delta/2}}(0) \text{ before time } N \mid \left\|Y_0^{\text{joint}}\right\| = N^\delta) \\
&\qquad = \sum_{k=1}^{N} \mathbb{P}(Y^{\text{joint}} \text{ hits } B_{N^{\delta/2}}(0) \text{ at time } k \mid \left\|Y_0^{\text{joint}}\right\| = N^\delta) \\
&\qquad \leq \sum_{k=1}^{N} \mathbb{P}(Y^{\text{ind}} \text{ hits } B_{N^{\delta/2}}(0) \text{ at time } k \mid \left\|Y_0^{\text{ind}}\right\| = N^\delta) + kN^{-\beta\delta/2} \\
&\qquad \leq CN^2 N^{-\beta\delta/2} + \mathbb{P}(Y^{\text{ind}} \text{ hits } B_{N^{\delta/2}}(0) \text{ before time } N \mid \left\|Y_0^{\text{ind}}\right\| = N^\delta)
\end{aligned} \tag{3.3.31}$$

Let E be the event that there more that N^{b_2} uncoupled steps before time N. Then we have

$$\begin{aligned}
\mathbb{P}^{\text{joint}}_{x_0,x_0'}(E) &\leq \mathbb{P}^{\text{joint}}_{x_0,x_0'}(A^c) + \mathbb{P}^{\text{joint}}_{x_0,x_0'}(A \cap B) + \mathbb{P}^{\text{joint}}_{x_0,x_0'}(E \cap A \cap B^c) \\
&\leq \exp(-CN^c) + CN^2 N^{-\beta\delta/2} + \Big(\frac{N^\delta}{N^{\delta/2}}\Big)^{2-d} + N(N^{\delta/2})^{-\beta}.
\end{aligned}$$

Here we have used the estimates from (3.3.1), (3.3.31) combined with (3.3.29) and (3.3.30). We note that β can be chosen arbitrarily large by choosing p close to 1. Therefore, for $d \geq 3$, (3.3.28) holds with $b_1 \geq b_2$.

For $d = 2$ we need another approach. Here the event B^{C} has a low probability and we need to decompose the trajectory of $(X_{1,n}^{\text{joint}}, X_{2,n}^{\text{joint}})_n$ into excursions. For a large constant K', to be tuned later, we define stopping times $\mathcal{R}_i, \mathcal{D}_i$ and $\mathcal{U}$ by $\mathcal{R}_0 = 0$ and, for $i \geq 1$ and small $0 < \alpha < \delta$,

$$\begin{aligned}
\mathcal{D}_i &= \inf\{k \geq \mathcal{R}_{i-1} \colon \left\|\hat{X}_{1,k}^{\text{joint}} - \hat{X}_{2,k}^{\text{joint}}\right\| \geq N^\delta\}, \\
\mathcal{R}_i &= \inf\{k \geq \mathcal{D}_i \colon \left\|\hat{X}_{1,k}^{\text{joint}} - \hat{X}_{2,k}^{\text{joint}}\right\| \leq N^\alpha\}, \\
\mathcal{U} &= \inf\{k \geq 0 \colon \left\|\hat{X}_{1,k}^{\text{joint}} - \hat{X}_{2,k}^{\text{joint}}\right\| \geq K'N\}.
\end{aligned}$$

Furthermore let J be the unique integer such that $\mathcal{D}_J \le \mathcal{U} \le \mathcal{R}_J$. Looking at J we see that starting with $Y_0 = N^\delta$, writing $\mathbb{P}^{\text{joint}}_r$ for the distribution of Y^{joint} starting from $Y^{\text{joint}}_0 = r$ and adapting arguments from Step 2 in the proof of Lemma 3.3.1, we obtain

$$\begin{aligned}
\mathbb{P}^{\text{joint}}_{N^\delta}(H(K'N) < h(N^\alpha)) &\ge \mathbb{P}^{\text{joint}}_{N^\delta}(H(K'N) < h(N^\alpha) \wedge (K'N)^3) \\
&\ge \mathbb{P}^{\text{ind}}_{N^\delta}(H(K'N) < h(N^\alpha) \wedge (K'N)^3) - (K'N)^3 N^{-\alpha\beta} \\
&\ge \mathbb{P}^{\text{ind}}_{N^\delta}(H(K'N) < h(N^\alpha)) - C\mathbb{P}\Big(\inf\{t\colon \|B_t\| \ge \frac{K'N}{\sqrt{N}}\} > \frac{(K'N)^3}{N}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (K'N)^3 N^{-\alpha\beta} \\
&\ge \mathbb{P}^{\text{ind}}_{N^\delta}(H(K'N) < h(N^\alpha)) - C\frac{N}{(K'N)^3}\frac{(K'N)^2}{Nd} - (K'N)^3 N^{-\alpha\beta} \\
&\ge (1-\varepsilon)\frac{\log N^\delta - \log N^\alpha}{\log(K'N) - \log N^\alpha} - CN^{-1} - CN^{3-\alpha\beta} \\
&\ge (1-\varepsilon)\frac{\delta-\alpha}{1-\alpha-\frac{\log K'}{\log N}} - CN^{-1} - CN^{3-\alpha\beta} \\
&\ge \frac{1}{2}\frac{\delta-\alpha}{1-\alpha}
\end{aligned}$$

for N large enough and $3/\alpha < \beta$. Thus J has geometric distribution with parameter greater than $\frac{1}{2}\frac{\delta-\alpha}{1-\alpha}$ as $N \to \infty$. Therefore, $\mathbb{P}^{\text{joint}}(J \ge \log N) \le N^{-c}$ for some $c > 0$. Applying the separation lemma, Lemma 3.3.1, we get

$$\mathbb{P}^{\text{joint}}(\mathcal{D}_i - \mathcal{R}_{i-1} \ge N^{b_2}) \le \exp(-b_3 N^{b_4}) \tag{3.3.32}$$

for constants $b_2 \in (0, 1/2)$ and $b_3, b_4 > 0$. Combining these we obtain

$$\mathbb{P}^{\text{joint}}\Big(\sum_{i=1}^{J} \mathcal{D}_i - \mathcal{R}_{i-1} \ge N^{b_2}\log N\Big) \le N^{-c'}. \tag{3.3.33}$$

Using Lemma 3.2.1 to compare $\mathbb{P}^{\text{joint}}$ and $\mathbb{P}^{\text{ind}}$ and large deviation estimates for sums of independent, heavy tailed random variables,

$$\begin{aligned}
&\mathbb{P}^{\text{ind}}\Big(\exists k \in \bigcup_{i=1}^{J}\{\mathcal{D}_i, \dots, \mathcal{R}_i\} \cap \{1, \dots, N\}\colon \|Y_k\| \ge K'N - 2R_{\text{loc}}N^{b_2}\log N\Big) \\
&\le \mathbb{P}^{\text{ind}}\Big(T_N \ge \frac{K'N - 2R_{\text{loc}}N^{b_2}\log N}{4R_{\text{loc}}}\Big) \\
&\le \mathbb{P}^{\text{ind}}\Big(T_N \ge \frac{K'N}{8R_{\text{loc}}}\Big) \\
&\le CN^{1-\beta}
\end{aligned}$$

for K' large enough, e.g. $K' > 8R_{\text{loc}}\mathbb{E}^{\text{ind}}[T_1]$. Using that

$$\begin{aligned}
\mathbb{P}^{\text{joint}}(\mathcal{U} \le N) &\le \mathbb{P}^{\text{joint}}\Big(\exists k \le N\colon \ \|Y_k\| \ge K'N, \sum_{i=1}^{J} \mathcal{D}_i - \mathcal{R}_{i-1} \le N^{b_2} \log N\Big) \\
&\quad + \mathbb{P}^{\text{joint}}\Big(\sum_{i=1}^{J} \mathcal{D}_i - \mathcal{R}_{i-1} \ge N^{b_2} \log N\Big) \\
&\le \mathbb{P}^{\text{joint}}\Big(\exists k \in \bigcup_{i=1}^{J} \{\mathcal{D}_i, \dots, \mathcal{R}_i\} \cap \{1, \dots, N\}\colon \ \|Y_k\| \ge K'N - 2R_{\text{loc}}N^{b_2} \log N, \\
&\qquad \sum_{i=1}^{J} \mathcal{D}_i - \mathcal{R}_{i-1} \le N^{b_2} \log N\Big) + N^{-c} \\
&\le \mathbb{P}^{\text{ind}}\Big(\exists k \in \bigcup_{i=1}^{J} \{\mathcal{D}_i, \dots, \mathcal{R}_i\} \cap \{1, \dots, N\}\colon \ \|Y_k\| \ge K'N - 2R_{\text{loc}}N^{b_2} \log N\Big) \\
&\qquad + N \cdot N^{-\alpha\beta} + N^{-c} \\
&\le CN^{1-\beta} + N^{1-\alpha\beta} + N^{-c'}
\end{aligned}$$

and thus $\mathbb{P}^{\text{joint}}(\mathcal{U} \le N) \le N^{-c}$ for some $c > 0$. Combining this and (3.3.33)

$$\begin{aligned}
\mathbb{P}^{\text{joint}}\Big(\#\{k \le N\colon \ \Big\|\hat{X}^{\text{joint}}_{1,k} - \hat{X}^{\text{joint}}_{2,k}\Big\| \le N^{\alpha}\} \ge N^{b_2} \log N\Big) \\
\le \mathbb{P}^{\text{joint}}\Big(\sum_{i\colon \mathcal{R}_{i-1} \le N} \mathcal{D}_i - \mathcal{R}_{i-1} \ge N^{b_2} \log N\Big) \le N^{-c}.
\end{aligned} \tag{3.3.34}$$

If the event $\#\{k \le N\colon \ \Big\|\hat{X}^{\text{joint}}_{1,k} - \hat{X}^{\text{joint}}_{2,k}\Big\| \le N^{\alpha}\} \ge N^{b_2} \log N$ does not occur, we can with probability at least $1 - N^{-c}$ couple $\mathbb{P}^{\text{ind}}$ and $\mathbb{P}^{\text{joint}}$ for all k satisfying $\mathcal{D}_i \le k \le \mathcal{R}_{i-1}$ for some i. Taking $b_2 < b_1 < 1/2$ we proved (3.3.28) for $d = 2$, since $N^{b_2} \log N \le N^{b_1}$ for large N. □

Lemma 3.3.3 (Analogue to Lemma 3.10 in [BČDG13]). *Let $d \ge 2$. Then, there exist $a, C > 0$ such that for every pair of bounded Lipschitz functions $f, g : \mathbb{R}^d \to \mathbb{R}$*

$$\begin{aligned}
&|\mathbb{E}^{\text{joint}}_{0,0}[f(\hat{X}_n/\sqrt{n})g(\hat{X}'_n/\sqrt{n})] - \mathbb{E}^{\text{ind}}_{0,0}[f(\hat{X}_n/\sqrt{n})g(\hat{X}'_n/\sqrt{n})]| \\
&\quad \le C(1 + \|f\|_{\infty} L_f)(1 + \|g\|_{\infty} L_g)n^{-a},
\end{aligned} \tag{3.3.35}$$

where $L_f \coloneqq \sup_{x \ne y} |f(y) - f(x)| / \|x - y\|$ and L_g are the Lipschitz constants of f and g.

Proof. We define the set of all steps before n at which the coupling between $\mathbb{P}^{\text{ind}}$ and $\mathbb{P}^{\text{joint}}$ is successful

$$\mathcal{I} \coloneqq \{k \le n : \text{the coupling is successful for the } k\text{-th step}\},$$

its complement $\mathcal{I}^{\mathsf{C}} \coloneqq \{1, \dots, n\} \setminus \mathcal{I}$ and the sets

$$\begin{aligned}
B_{\text{joint}} &\coloneqq \Big\{\sum_{k \in \mathcal{I}^{\mathsf{C}}} T^{\text{joint}}_k(0,0) - T^{\text{joint}}_{k-1}(0,0) \le Kn^{b_1+\varepsilon}\Big\}, \\
B_{\text{ind}} &\coloneqq \Big\{\sum_{k \in \mathcal{I}^{\mathsf{C}}} T^{\text{ind}}_k(0,0) - T^{\text{ind}}_{k-1}(0,0) \le Kn^{b_1+\varepsilon}\Big\},
\end{aligned}$$

where b_1 is from Lemma 3.3.2 and $\varepsilon > 0$ is chosen such that $b_1+\varepsilon < 1/2$. By Lemma 3.3.2 $\mathbb{P}(|\mathcal{I}^\complement| > n^{b_1}) \le n^{-c}$ and due to Lemma 3.1.15 we get

$$\begin{aligned}
\mathbb{P}\Big(B_{\text{joint}}^\complement \cap \{|\mathcal{I}^\complement| \le n^{b_1}\}\Big) &= \mathbb{P}\Big(\sum_{k\in\mathcal{I}^\complement} T_k^{\text{joint}}(0,0) - T_{k-1}^{\text{joint}}(0,0) > Kn^{b_1+\varepsilon}, |\mathcal{I}^\complement| \le n^{b_1}\Big)\\
&= \mathbb{E}\Big[\mathbb{P}(\exists k \in \mathcal{I}^\complement\colon T_k^{\text{joint}}(0,0) - T_{k-1}^{\text{joint}}(0,0) > Kn^{\varepsilon}, |\mathcal{I}^\complement| \le n^{b_1} \,|\, \mathcal{I}^\complement)\Big]\\
&\le \mathbb{E}\Big[\mathbb{1}_{\{|\mathcal{I}^\complement|\le n^{b_1}\}} \sum_{k\in\mathcal{I}^\complement} \mathbb{P}\Big(T_k^{\text{joint}}(0,0) - T_{k-1}^{\text{joint}}(0,0) > Kn^{\varepsilon} \,|\, \mathcal{I}^\complement\Big)\Big]\\
&\le Cn^{b_1}n^{-\varepsilon\beta}.
\end{aligned}$$

Analogously we obtain $\mathbb{P}(B_{\text{ind}}^\complement \cap \{|\mathcal{I}^\complement| \le n^{b_1}\}) \le Cn^{b_1-\varepsilon\beta}$. On the event $A = \{|\mathcal{I}^\complement| \le n^{b_1}\} \cap B_{\text{joint}} \cap B_{\text{ind}}$ we have, due to Assumption 1.3.4,

$$\begin{aligned}
\Big\| X_{T_n^{\text{joint}}(0,0)} - X_{T_n^{\text{ind}}(0,0)} \Big\| &\le Cn^{b_1+\varepsilon},\\
\Big\| X'_{T_n^{\text{joint}}(0,0)} - X'_{T_n^{\text{ind}}(0,0)} \Big\| &\le Cn^{b_1+\varepsilon}.
\end{aligned} \tag{3.3.36}$$

Thus, abbreviating $T_n^{\text{joint}} = T_n^{\text{joint}}(0,0)$, $T_n^{\text{ind}} = T_n^{\text{ind}}(0,0)$,

$$\begin{aligned}
&\Big|\mathbb{E}[f(X_{T_n^{\text{joint}}}/\sqrt{n})g(X'_{T_n^{\text{joint}}}/\sqrt{n})] - \mathbb{E}[f(X_{T_n^{\text{ind}}}/\sqrt{n})g(X'_{T_n^{\text{ind}}}/\sqrt{n})]\Big|\\
&\le \Big|\mathbb{E}[f(X_{T_n^{\text{joint}}}/\sqrt{n})g(X'_{T_n^{\text{joint}}}/\sqrt{n})\mathbb{1}_A] - \mathbb{E}[f(X_{T_n^{\text{ind}}}/\sqrt{n})g(X'_{T_n^{\text{ind}}}/\sqrt{n})\mathbb{1}_A]\Big|\\
&\qquad + 2\|f\|_\infty \|g\|_\infty \Big(\mathbb{P}(|\mathcal{I}^\complement| > n^{b_1}) + \mathbb{P}\Big(B_{\text{joint}}^\complement, |\mathcal{I}^\complement| \le n^{b_1}\Big) + \mathbb{P}\Big(B_{\text{ind}}^\complement, |\mathcal{I}^\complement| \le n^{b_1}\Big)\Big).
\end{aligned}$$

Lastly, observe that

$$|f(x)g(y) - f(x')g(y')| \le \|g\|_\infty L_f \|x - x'\| + \|f\|_\infty L_g \|y - y'\|$$

and therefore, combining this with (3.3.36) and the last equation yields

$$\begin{aligned}
&\Big|\mathbb{E}[f(X_{T_n^{\text{joint}}}/\sqrt{n})g(X'_{T_n^{\text{joint}}}/\sqrt{n})] - \mathbb{E}[f(X_{T_n^{\text{ind}}}/\sqrt{n})g(X'_{T_n^{\text{ind}}}/\sqrt{n})]\Big|\\
&\le C\|g\|_\infty L_f n^{b_1+\varepsilon-1/2} + C\|f\|_\infty L_g n^{b_1+\varepsilon-1/2} + C(n^{-c} + n^{b_1-\varepsilon\beta})
\end{aligned}$$

which, by choice of b_1 and ε completes the proof. □

Proof of Proposition 3.2.2. The proof follows with Lemma 3.3.3 and Berry-Esseen estimates. First note that

$$\mathbb{E}[E_\omega[f(\hat{X}_m/\sqrt{m})]E_\omega[f(\hat{X}'_m/\sqrt{m})]] = \mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})].$$

With that we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(E_\omega[f(\hat{X}_m/\sqrt{m})] - \tilde{\Phi}(f)\Big)^2\Big]\\
&\quad\le \Big|\mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})] - \mathbb{E}^{\text{ind}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})]\Big|\\
&\qquad + \Big|\mathbb{E}^{\text{ind}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})] - \tilde{\Phi}(f)^2\Big|\\
&\qquad + 2|\tilde{\Phi}(f)| \cdot \Big|\mathbb{E}^{\text{ind}}\big[f(\hat{X}_m/\sqrt{m})\big] - \mathbb{E}^{\text{joint}}\big[f(\hat{X}_m/\sqrt{m})\big]\Big|\\
&\qquad + 2|\tilde{\Phi}(f)| \cdot \Big|\tilde{\Phi}(f) - \mathbb{E}^{\text{ind}}\big[f(\hat{X}_m/\sqrt{m})\big]\Big|\\
&\quad\le 2C_f m^{-a} + Cm^{-1/2} \le C_f m^{-c}
\end{aligned}$$

for a suitable $c > 0$, where we used Lemma 3.3.3 and Berry-Esseen type bounds, since $\hat{X}_n$ has bounded third moments under $\mathbb{P}^{\text{ind}}$ for β large enough, in the last line. □

Lemma 3.3.4 (Analogue to Lemma 3.12 in [BČDG13])**.** *Assume that for some $c > 1$, and any bounded Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$*

$$E_\omega[f(X_{T^{\text{joint}}_{k^c}(0,0)}/k^{c/2})] \underset{k\to\infty}{\longrightarrow} \tilde{\Phi}(f) \quad \textit{for a.a. } \omega, \tag{3.3.37}$$

where $\tilde{\Phi}$ is some non-trivial centred d-dimensional normal law. Then we have for any bounded Lipschitz function f

$$E_\omega[f(X_{T^{\text{joint}}_{m}(0,0)}/m^{1/2})] \underset{m\to\infty}{\longrightarrow} \tilde{\Phi}(f) \quad \textit{for a.a. } \omega.$$

Proof. We abbreviate $\hat{X}_n = X_{T^{\text{joint}}_n(0,0)}$. Since the distribution of $\{\hat{X}_i - \hat{X}_{i-1}\}$ does not have exponential tails we do not have $\mathbb{E}[\exp(\lambda(\hat{X}_i - \hat{X}_{i-1}))] < \infty$. Therefore we need to calculate more carefully than in [BČDG13] and control the probabilities via large deviations for heavy tailed distributions. First we get a summable upper bound on the probability

$$\mathbb{P}_{0,0}\left(\max_{k^c \le \ell \le (k+1)^c} \frac{|\hat{X}^{\text{joint}}_\ell - \hat{X}^{\text{joint}}_{k^c}|}{k^{c/2}} \ge \delta\right). \tag{3.3.38}$$

Because of the heavy tails we won't get an exponential decay in k for the upper bound. We need to work around the fact that, along the joint regeneration times T^{joint}, the increments of the random walk are not independent. Define $\hat{\ell} := \min\{n \in \mathbb{N} : T^{\text{ind}}_n \ge T^{\text{joint}}_\ell\}$ and the following sets

$$\begin{aligned}
A_\ell &:= \left\{T^{\text{joint}}_\ell \le T^{\text{ind}}_{k^{c-1+\varepsilon}}\right\},\\
B_k &:= \left\{T^{\text{ind}}_i - T^{\text{ind}}_{i-1} \le k^\alpha \text{ for all } i = 1, \dots, k^{c-1+\varepsilon}\right\},\\
C_\ell &:= \left\{\left\|\hat{X}^{\text{ind}}_\ell\right\| \le \frac{1}{2}\delta k^{c/2}\right\},\\
D_\ell &:= \left\{T^{\text{ind}}_{\hat{\ell}} - T^{\text{joint}}_\ell < \frac{1}{2R_\kappa}\delta k^{c/2}\right\}.
\end{aligned}$$

We get the following upper bounds for above: starting with $A^\complement_\ell$

$$\begin{aligned}
\mathbb{P}_{x,x'}(A^\complement_\ell) &\le \mathbb{P}_{x,x'}(T^{\text{joint}}_\ell > k^{c-1+\varepsilon})\\
&\le \mathbb{P}_{x,x'}\left(\exists i \le \ell \text{ such that } T^{\text{joint}}_i - T^{\text{joint}}_{i-1} > \frac{k^{c+\varepsilon}}{\ell}\right)\\
&\le \sum_{i=1}^{\ell} \mathbb{P}_{x,x'}\left(T^{\text{joint}}_i - T^{\text{joint}}_{i-1} > \frac{k^{c-1+\varepsilon}}{\ell}\right)\\
&\le \sum_{i=1}^{\ell} \left(\frac{k^{c-1+\varepsilon}}{\ell}\right)^{-\beta} \le \ell^{1+\beta} k^{-\beta(c-1+\varepsilon)},
\end{aligned}$$

for $B^\complement_k$, by Lemma 3.1.15, we obtain

$$\begin{aligned}
\mathbb{P}_{x,x'}(B^\complement_k) &= \mathbb{P}_{x,x'}(\exists i \le k^{c-1+\varepsilon} \text{ such that } T^{\text{ind}}_i - T^{\text{ind}}_{i-1} > k^\alpha)\\
&\le k^{c-1+\varepsilon} k^{-\beta\alpha},
\end{aligned}$$

for $C_\ell^\complement$ using Azuma's inequality on every coordinate, we have

$$\begin{aligned}
\mathbb{P}_{x,x'}(C_\ell^\complement \mid B_k) &= \mathbb{P}_{x,x'}\Big(\Big\| \sum_{i=1}^{\ell} \hat{X}_i^{\text{ind}} - \hat{X}_{i-1}^{\text{ind}} \Big\| > \frac{1}{2}\delta k^{c/2} \mid B_k \Big) \\
&\leq \sum_{j=1}^{d} \mathbb{P}_{x,x'}\Big(\Big\| \sum_{i=1}^{\ell} \hat{X}_i^{\text{ind}}(j) - \hat{X}_{i-1}^{\text{ind}}(j) \Big\| > \frac{1}{2}\delta k^{c/2} \mid B_k \Big) \\
&\leq 2d \exp\Big(- \frac{\delta^2 k^c}{8\ell R_\kappa k^{2\alpha}} \Big)
\end{aligned}$$

and consequently for $\ell \leq k^{c-1+\varepsilon}$

$$\mathbb{P}_{x,x'}(C_\ell^\complement) \leq 2d \exp\Big(- C(\delta, R_\kappa) k^{1-\varepsilon-2\alpha} \Big) + k^{c-1+\varepsilon} k^{-\beta\alpha}.$$

Lastly, for $D_\ell^\complement$, we have

$$\begin{aligned}
\mathbb{P}_{x,x'}(D_\ell^\complement) &\leq \mathbb{P}_{x,x'}\Big(T_{\hat{\ell}}^{\text{ind}} - T_{\hat{\ell}-1}^{\text{ind}} \geq \frac{1}{2R_\kappa}\delta k^{c/2} \Big) \\
&\leq \Big(\frac{1}{2R_\kappa}\delta k^{c/2} \Big)^{-\beta}.
\end{aligned}$$

Now we combine the proved upper bounds and the fact that for $c > 1$ we have $(k+1)^c - k^c \leq c(k+1)^{c-1}$ to obtain

$$\begin{aligned}
&\mathbb{P}_{0,0}\left(\max_{k^c \leq \ell \leq (k+1)^c} \frac{|\hat{X}_\ell^{\text{joint}} - \hat{X}_{k^c}^{\text{joint}}|}{k^{c/2}} \geq \delta \right) \\
&= \sum_{x,x'} \mathbb{P}_{0,0}^{\text{joint}}\Big(\max_{k^c \leq \ell \leq (k+1)^c} |\hat{X}_\ell - \hat{X}_{k^c}| \geq \delta k^{c/2} \mid \hat{X}_{k^c} = x, \hat{X}'_{k^c} = x' \Big) \mathbb{P}_{0,0}^{\text{joint}}(\hat{X}_{k^c} = x, \hat{X}'_{k^c} = x') \\
&\leq \sup_{x,x'} \mathbb{P}_{x,x'}^{\text{joint}}(\max_{\ell \in \{1,\dots,ck^{c-1}\}} |\hat{X}_\ell| > \delta k^{c/2}) \\
&\leq \sum_{\ell=1}^{ck^{c-1}} \sup_{x,x'} \mathbb{P}_{x,x'}^{\text{joint}}(|\hat{X}_\ell| > \delta k^{c/2}) \qquad (3.3.39) \\
&\leq \sum_{\ell=1}^{ck^{c-1}} \sup_{x,x'} \mathbb{P}_{x,x'}(|\hat{X}_\ell^{\text{joint}}| > \delta k^{c/2}, A_\ell, C_{\hat{\ell}}, D_\ell) + \mathbb{P}_{x,x'}(A_\ell^\complement) + \mathbb{P}_{x,x'}(C_{\hat{\ell}}^\complement) + \mathbb{P}_{x,x'}(D_\ell^\complement).
\end{aligned}$$

Note that the events $A_\ell \cap C_{\hat{\ell}} \cap D_\ell$ and $\{|\hat{X}_\ell^{\text{joint}}| > \delta k^{c/2}\}$ are disjoint and thus, noting that $\hat{\ell} \leq k^{c-1+\varepsilon}$ on A_ℓ

$$\begin{aligned}
(3.3.38) &\leq \sum_{\ell=1}^{ck^{c-1}} \sup_{x,x'} \mathbb{P}_{x,x'}(A_\ell^\complement) + \mathbb{P}_{x,x'}(C_{\hat{\ell}}^\complement) + \mathbb{P}_{x,x'}(D_\ell^\complement) \\
&\leq \sum_{\ell=1}^{ck^{c-1}} \Big(\ell^{1+\beta} k^{-\beta(c-1+\varepsilon)} + k^{2(c-1+\varepsilon)} k^{-\beta\alpha} \\
&\qquad + k^{c-1+\varepsilon} \exp\Big(- C(\delta, R_\kappa) k^{1-\varepsilon-2\alpha} \Big) + \Big(\frac{1}{2R_\kappa}\delta k^{c/2} \Big)^{-\beta} \Big).
\end{aligned}$$

For β large enough the upper bound for (3.3.38) given above is summable in k and thus

$$\limsup_{k\to\infty} \max_{k^c \leq \ell \leq (k+1)^c} \frac{|\hat{X}_\ell^{\text{joint}} - \hat{X}_{k^c}^{\text{joint}}|}{k^{c/2}} = 0, \qquad \text{for-a.a. } \omega.$$

From that it follows for $k^c \le m \le (k+1)^c$

$$
\begin{aligned}
&|E_\omega[f(X_{T_m^{\text{joint}}(0,0)}/m^{1/2})] - \tilde{\Phi}(f)| \\
&\le L_f \left\| \frac{\hat{X}_m^{\text{joint}}}{\sqrt{m}} - \frac{\hat{X}_k^{\text{joint}}}{k^{c/2}} \right\| + |E_\omega[f(X_{T_k^{\text{joint}}(0,0)}/k^{c/2})] - \tilde{\Phi}(f)| \\
&\le L_f \left\| \frac{\hat{X}_m^{\text{joint}}}{\sqrt{m}} - \frac{\hat{X}_{k^c}^{\text{joint}}}{\sqrt{m}} \right\| + L_f \left\| \frac{\hat{X}_{k^c}^{\text{joint}}}{\sqrt{m}} - \frac{\hat{X}_{k^c}^{\text{joint}}}{k^{c/2}} \right\| + |E_\omega[f(X_{T_k^{\text{joint}}(0,0)}/k^{c/2})] - \tilde{\Phi}(f)| \\
&\le L_f \frac{\left\| \hat{X}_m^{\text{joint}} - \hat{X}_{k^c}^{\text{joint}} \right\|}{k^{c/2}} + L_f \frac{\left\| \hat{X}_{k^c}^{\text{joint}} \right\|}{k^{c/2}} \Big(\frac{k^{c/2}}{\sqrt{m}} - 1 \Big) + |E_\omega[f(X_{T_k^{\text{joint}}(0,0)}/k^{c/2})] - \tilde{\Phi}(f)|.
\end{aligned}
$$

The calculations above show that the first term goes to 0 a.s. and we can extract the same result for the second term since equation (3.3.39) yields an upper bound for it. Therefore

$$
|E_\omega[f(X_{T_m^{\text{joint}}(0,0)}/m^{1/2})] - \tilde{\Phi}(f)| \to 0 \qquad \text{for a.a. } \omega.
$$

□

Proof of Theorem 1.3.5. Let $f : \mathbb{R}^d \to \mathbb{R}$ be bounded and Lipschitz, $c' > 1/c \wedge 1$ with c from Proposition 3.2.2. By (3.2.6) and Markov's inequality, abbreviating $\hat{X}_m = \hat{X}_m^{\text{joint}}$,

$$
\begin{aligned}
&\mathbb{P}\Big(|E_\omega[f(\hat{X}_{[n^{c'}]}/\sqrt{[n^{c'}]})]] - \hat{\Phi}(f)| > \varepsilon \Big) \\
&\le \frac{\mathbb{E}\Big[\big(E_\omega[f(\hat{X}_{[n^{c'}]}/\sqrt{[n^{c'}]})] - \hat{\Phi}(f) \big)^2 \Big]}{\varepsilon^2} \\
&\le C_f n^{-c'c} \varepsilon^2,
\end{aligned}
\tag{3.3.40}
$$

which is summable and hence by Borel-Cantelli

$$
E_\omega[f(\hat{X}_{[n^{c'}]}/\sqrt{[n^{c'}]})] \to \tilde{\Phi}(f) \quad \text{a.s. as } n \to \infty. \tag{3.3.41}
$$

Now Lemma 3.3.4 yields

$$
E_\omega[f(\hat{X}_m/\sqrt{m})] \underset{m\to\infty}{\longrightarrow} \tilde{\Phi}(f) \quad \text{for a.a. } \omega. \tag{3.3.42}
$$

Set $\tau_m := T_m^{\text{joint}}(0,0) - T_{m-1}^{\text{joint}}(0,0)$. Next we only need to control the behaviour of the random walk between the regeneration times. To that end let $V_n := \max\{m \in \mathbb{Z}_+ \colon T_m^{\text{joint}}(0,0) \le n\}$. We have $V_n/n \to 1/\mathbb{E}[\tau_2]$ a.s. as $n \to \infty$ and in fact

$$
\limsup_{n\to\infty} \frac{|V_n - n/\mathbb{E}[\tau_2]|}{\sqrt{n \log\log n}} < \infty \quad \text{a.s.} \tag{3.3.43}
$$

For $\alpha > 0$

$$
\begin{aligned}
\mathbb{P}(\max_{j\le n}\{j - T_{V_j} > cn^\alpha\}) &\le \mathbb{P}(\text{there exists } i \in \{1,\dots,V_n\} \text{ such that } \tau_i > cn^\alpha) \\
&\le n\mathbb{P}(\tau_2 > cn^\alpha) + \mathbb{P}(\tau_1 > cn^\alpha) \\
&\le C(n+1)n^{-\beta\alpha}
\end{aligned}
$$

which is summable if $1-\beta\alpha<-1$, so we obtain

$$P_\omega(\max_{j\le n}\{j-T_{V_j}\}>cn^\alpha)\longrightarrow 0\quad\text{a.s.}\tag{3.3.44}$$

for an appropriate choice of α and β. Since we can choose β arbitrarily large it is possible to have the above probability small for any choice of α. Since $X_{T_{V_n}}=X_{T^{\text{joint}}_{V_n}(0,0)}=\hat{X}_{V_n}$ we have

$$\begin{aligned}
&P_\omega\Big(\Big\|X_n-\hat{X}_{V_n}\Big\|\ge\log(n)cn^\alpha\Big)\\
&\quad=P_\omega\Big(\Big\|X_n-\hat{X}_{V_n}\Big\|\ge\log(n)cn^\alpha,\max_{j\le n}\{j-T_{V_j}\}\le cn^\alpha\Big)\\
&\qquad\qquad+P_\omega\Big(\Big\|X_n-\hat{X}_{V_n}\Big\|\ge\log(n)cn^\alpha,\max_{j\le n}\{j-T_{V_j}\}>cn^\alpha\Big)\\
&\quad\le P_\omega\Big(\Big\|X_n-\hat{X}_{V_n}\Big\|\ge\log(n)cn^\alpha,\max_{j\le n}\{j-T_{V_j}\}\le cn^\alpha\Big)+P_\omega\big(\max_{j\le n}\{j-T_{V_j}\}>cn^\alpha\big)
\end{aligned}\tag{3.3.45}$$

and

$$\begin{aligned}
&P_\omega\Big(\Big\|X_n-\hat{X}_{V_n}\Big\|\ge\log(n)cn^\alpha,\max_{j\le n}\{j-T_{V_j}\}\le cn^\alpha\Big)\\
&\quad=P_\omega\Big(\log(n)cn^\alpha\le\Big\|X_n-\hat{X}_{V_n}\Big\|\le R_\kappa(n-T_{V_n}),\max_{j\le n}\{j-T_{V_j}\}\le cn^\alpha\Big)\\
&\quad\le P_\omega\Big(\log(n)cn^\alpha\le\Big\|X_n-\hat{X}_{V_n}\Big\|\le R_\kappa cn^\alpha\Big)\longrightarrow 0\quad\text{a.s.},
\end{aligned}\tag{3.3.46}$$

consequently

$$P_\omega\Big(\Big\|X_n-\hat{X}_{V_n}\Big\|\ge\log(n)cn^\alpha\Big)\longrightarrow 0\quad\text{a.s.}\tag{3.3.47}$$

By (3.3.43) for any $\varepsilon>0$

$$P_\omega(|V_n-n/\mathbb{E}[\tau_2]|\ge n^{1/2+\varepsilon})\to 0\quad\text{a.s.}\tag{3.3.48}$$

Note that there exist $\delta\in(1/2,1)$ and $\gamma\in(\delta/2,1/2)$ such that for any $\theta\ge 0$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{|k-[\theta n]|\le n^\delta}|\hat{X}_k-\hat{X}_{[\theta n]}|>\varepsilon n^\gamma\right)\\
&\quad\le\mathbb{P}\left(\sup_{[\theta n]-k\le n^\delta}|\hat{X}_k-\hat{X}_{[\theta n]}|>\varepsilon n^\gamma\right)+\mathbb{P}\left(\sup_{k-[\theta n]\le n^\delta}|\hat{X}_k-\hat{X}_{[\theta n]}|>\varepsilon n^\gamma\right)\\
&\quad\le\varepsilon^{-6}n^{-6\gamma}\Big(\mathbb{E}[|\hat{X}_{[\theta n]}-\hat{X}_{[\theta n]-n^\delta}|^6]+\mathbb{E}[|\hat{X}_{[\theta n]+n^\delta}-\hat{X}_{[\theta n]}|^6]\Big)\\
&\quad\le C\varepsilon^{-6}n^{-6\gamma}n^{3\delta}=C\varepsilon^{-6}n^{3\delta-6\gamma}
\end{aligned}\tag{3.3.49}$$

where we used Doob's L^6-inequality and the fact that $\mathbb{E}[\|S_k\|^6]\le Ck^3$ for a random walk (S_k) whose increments are centred and have bounded 6th moments. Thus we can choose δ and γ sufficiently close to $1/2$ so that $3\delta-6\gamma<-1$ and the right-hand side becomes summable in n. Using Borel-Cantelli that yields

$$\limsup_{n\to\infty}\sup_{|k-[\theta n]|\le n^\delta}\frac{|\hat{X}_k-\hat{X}_{[\theta n]}|}{n^\gamma}\to 0\quad\text{a.s.}\tag{3.3.50}$$

Writing $X_n/\sqrt{n}$ in terms that we can bound by what we showed above

$$\frac{X_n}{\sqrt{n}} = \frac{X_n - \hat{X}_{V_n}}{\sqrt{n}} + \frac{\hat{X}_{V_n} - \hat{X}_{[n/\mathbb{E}[\tau_2]]}}{\sqrt{n}} + \frac{\hat{X}_{[n/\mathbb{E}[\tau_2]]}}{\sqrt{n/\mathbb{E}[\tau_2]}}\sqrt{1/\mathbb{E}[\tau_2]} \tag{3.3.51}$$

and let Φ be defined by $\Phi(f) := \tilde{\Phi}(f((\mathbb{E}[\tau_2]^{-1/2})\cdot))$, i.e. Φ is the image measure of $\tilde{\Phi}$ under $x \to x/\sqrt{\mathbb{E}[\tau_2]}$. Then, defining the sets

$$\begin{aligned}
A_n &:= \{|X_n - \hat{X}_{V_n}| \geq n^{\alpha}\log n\},\\
B_n &:= \{|V_n - n/\mathbb{E}[\tau_2]| \geq n^{\varepsilon+1/2}\},\\
C_n &:= \{\sup_{|x-n/\mathbb{E}[\tau_2]|\leq n^{1/2+\varepsilon}} |\hat{X}_k - \hat{X}_{[n/\mathbb{E}[\tau_2]]}| > n^{\gamma}\},\\
D_n &:= A_n^c \cap B_n^c \cap C_n^c,
\end{aligned}$$

we conclude

$$\begin{aligned}
&|E_\omega[f(X_n/\sqrt{n})] - \Phi(f)|\\
&\leq |E_\omega[\mathbb{1}_{D_n} f(X_n/\sqrt{n})] - \Phi(f)| + \|f\|_\infty E_\omega[\mathbb{1}_{D_n^c}],
\end{aligned} \tag{3.3.52}$$

where on D_n we get

$$\begin{aligned}
&|E_\omega[\mathbb{1}_{D_n} f(X_n/\sqrt{n})] - \Phi(f)|\\
&\leq CL_f\left(\frac{n^{\alpha}\log n}{\sqrt{n}} + n^{\gamma-1/2}\right) + \left|E_\omega\left[f\left(\frac{\hat{X}_{[n/\mathbb{E}[\tau_2]]}}{\sqrt{n/\mathbb{E}[\tau_2]}}\sqrt{1/\mathbb{E}[\tau_2]}\right)\right] - \Phi(f)\right|\\
&\to 0 \quad \text{a.s. as } n \to \infty,
\end{aligned} \tag{3.3.53}$$

by (3.3.42), $\gamma < 1/2$ and the fact that $\alpha > 0$ can be chosen close to 0 if the parameters of the model are tuned correctly, i.e. p close to 1, s_{inn} close to s_{out} and s_{out} much larger than the a priori bound $s_{\max}$ from lemma 2.16 of [BČD16]. Additionally by (3.3.47), (3.3.48) and (3.3.50)

$$E_\omega[\mathbb{1}_{D_n^c}] \leq P_\omega(A_n) + P_\omega(B_n) + P_\omega(C_n) \to 0 \quad \text{a.s..} \tag{3.3.54}$$

This proves convergence for bounded Lipschitz functions which, by the Portmanteau-theorem, is sufficient to prove the weak convergence in Theorem 1.3.5. □

3.4 Proof of Theorem 1.3.5 for $d = 1$

The reason we needed to split the proof for $d = 1$ is that in this case the random walks meet often and single excursions away from each other will not be long. Therefore we will need to calculate more carefully and consider the time for an excursion as well as the number of excursions before time n. It turns out that, although a single excursion will not take up much time, the random walks will split fast enough such that the total time spent close to each other up until time n will be of order $o(n)$ in probability. The idea now is to follow the proof in [BČDG13] with a few adjustments, where the main problem stems from the fact that our bound on the total variation distance between $\mathbb{P}^{\text{ind}}_{x,x'}$ and $\mathbb{P}^{\text{joint}}_{x,x'}$ only has polynomial decay in the distance of the starting points x and x'. Therefore we will introduce so called *black box intervals* where the random walks are close to each other and a coupling using Lemma 3.2.1 will not be possible. While the random walks are not in a black box interval however, we can make use of Lemma 3.2.1.

Let $(\hat{X}_n^{\text{joint}}, \hat{X}_n^{\prime,\text{joint}})_n$ be a pair of random walks in $d = 1$ in the same environment observed along the simultaneous renewal times with transition probabilities $\hat{\Psi}^{\text{joint}}((x,x'),(y,y'))$, i.e.

$$\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) = \mathbb{P}^{\text{joint}}(\hat{X}_n = y, \hat{X}'_n = y' \mid \hat{X}_{n-1} = x, \hat{X}'_{n-1} = x')$$

and similarly for random walks in independent environments

$$\hat{\Psi}^{\text{ind}}((x,x'),(y,y')) = \mathbb{P}^{\text{ind}}(\hat{X}_n = y, \hat{X}'_n = y' \mid \hat{X}_{n-1} = x, \hat{X}'_{n-1} = x').$$

This section will mostly be about $(\hat{X}_n^{\text{joint}}, \hat{X}_n^{\prime,\text{joint}})_n$. We therefore abbreviate $(\hat{X}_n, \hat{X}'_n)_n = (\hat{X}_n^{\text{joint}}, \hat{X}_n^{\prime,\text{joint}})_n$ and will specify when we mean $\hat{X}^{\text{ind}}$ and $\hat{X}'^{\text{ind}}$. Write $\hat{\mathcal{F}}_n \coloneqq \sigma(\hat{X}_i, \hat{X}'_i, 0 \le i \le n)$ for the canonical filtration of $(\hat{X}_n, \hat{X}'_n)_n$.

Set

$$\begin{aligned}
\phi_1(x,x') &\coloneqq \sum_{y,y'} (y-x)\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) \\
\phi_2(x,x') &\coloneqq \sum_{y,y'} (y'-x')\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) \\
\phi_{11}(x,x') &\coloneqq \sum_{y,y'} (y-x-\phi_1(x,x'))^2\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) \\
\phi_{22}(x,x') &\coloneqq \sum_{y,y'} (y'-x'-\phi_2(x,x'))^2\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) \\
\phi_{12}(x,x') &\coloneqq \sum_{y,y'} (y-x-\phi_1(x,x'))(y'-x'-\phi_2(x,x'))\hat{\Psi}^{\text{joint}}((x,x'),(y,y')).
\end{aligned}$$

By Lemma 3.1.15 these are bounded,

$$C_\phi \coloneqq \|\phi_1\|_\infty \vee \|\phi_2\|_\infty \vee \|\phi_{11}\|_\infty \vee \|\phi_{22}\|_\infty \vee \|\phi_{12}\|_\infty < \infty. \tag{3.4.1}$$

Define

$$A_n^{(1)} \coloneqq \sum_{j=0}^{n-1} \phi_1(\hat{X}_j, \hat{X}'_j), \quad A_n^{(2)} \coloneqq \sum_{j=0}^{n-1} \phi_2(\hat{X}_j, \hat{X}'_j), \tag{3.4.2}$$

$$A_n^{(11)} \coloneqq \sum_{j=0}^{n-1} \phi_{11}(\hat{X}_j, \hat{X}'_j), \quad A_n^{(22)} \coloneqq \sum_{j=0}^{n-1} \phi_{22}(\hat{X}_j, \hat{X}'_j), \quad A_n^{(12)} \coloneqq \sum_{j=0}^{n-1} \phi_{12}(\hat{X}_j, \hat{X}'_j), \tag{3.4.3}$$

$$M_n \coloneqq \hat{X}_n - A_n^{(1)}, \quad M'_n \coloneqq \hat{X}'_n - A_n^{(2)}. \tag{3.4.4}$$

Now $(M_n), (M'_n), (M_n^2 - A_n^{(11)}), ({M'_n}^2 - A_n^{(22)})$ and $(M_n M'_n - A_n^{(12)})$ are martingales and by Lemma 3.1.15 the distribution of their increments has polynomial tails.

We write $\hat{\sigma}^2 \coloneqq \sum_{y,y'} y^2 \hat{\Psi}^{\text{ind}}((0,0),(y,y'))$ for the variance of a single increment under $\hat{\Psi}^{\text{ind}}$.

By Lemma 3.1.15 there exist $C_1, a > 0$ such that for $x, x' \in \mathbb{Z}$ with $|x - x'| \ge n^a$

$$|\phi_1(x,x')|, |\phi_2(x,x')|, |\phi_{12}(x,x')| \le \frac{C_1}{n^2}, \tag{3.4.5}$$

$$|\phi_{11}(x,x') - \hat{\sigma}^2|, |\phi_{22}(x,x') - \hat{\sigma}^2| \le \frac{C_1}{n^2}. \tag{3.4.6}$$

Here we can choose the a arbitrarily small by Lemma 3.2.1 if we tune the parameters right, e.g. $\beta \geq 2+2/a$ with β of said Lemma. See for example, noting that $\mathbb{E}^{\text{ind}}_{0,x'}[\hat{X}_1] = 0$,

$$\begin{aligned}
|\phi_1(x,x')| &= |\sum_{(y,y')} (y-x)\hat{\Psi}^{\text{joint}}((x,x'),(y,y'))| \\
&= |\sum_{\substack{y,y' \\ |y-x|\geq n^a}} (y-x)\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) + \sum_{\substack{y,y' \\ |y-x|<n^a}} (y-x)\hat{\Psi}^{\text{joint}}((x,x'),(y,y'))| \\
&\leq \mathbb{E}^{\text{joint}}_{0,x'}[|\hat{X}_1|\mathbb{1}_{|\hat{X}|\geq n^a}] + \mathbb{E}^{\text{ind}}_{0,x'}[|\hat{X}_1|\mathbb{1}_{|\hat{X}|\geq n^a}] \\
&+ |\sum_{\substack{y,y' \\ |y-x|<n^a}} (y-x)(\hat{\Psi}^{\text{joint}}((x,x'),(y,y')) - \hat{\Psi}^{\text{ind}}((x,x'),(y,y')))| \\
&\leq C(\beta)n^{a(2-\beta)} + 2n^a|x-x'|^{-\beta} \leq C(\beta)(n^{a(2-\beta)} + n^{a(1-\beta)}) \leq C(\beta)n^{a(2-\beta)}
\end{aligned}$$

now for $\beta \geq 2+2/a$ we have $a(2-\beta) \leq -2$. The other estimates can be shown analogously.

Let

$$R_n \coloneqq \#\{0 \leq j \leq n : |\hat{X}_j - \hat{X}'_j| \leq n^a\} \tag{3.4.7}$$

be the time that the random walks spend close to each other until time n. Next we want to prove a moment condition for R_n and that the predictable processes $(A^{(1)}_n)_n$ and $(A^{(2)}_n)_n$ are small on the diffusive scale

Lemma 3.4.1. *1. There exist $0 \leq \delta_R \leq 1/2, c_R < \infty$ such that for all $x_0, x'_0 \in \mathbb{Z}$*

$$\mathbb{E}^{\text{joint}}_{x_0,x'_0}[R_n^{3/2}] \leq c_R n^{1+\delta_R} \quad \textit{for all } n. \tag{3.4.8}$$

2. There exist $\delta_C > 0, c_C < \infty$ such that for all $x_0, x'_0 \in \mathbb{Z}$

$$\mathbb{E}^{\text{joint}}_{x_0,x'_0}\left[\frac{|A^{(1)}_n|}{\sqrt{n}}\right], \mathbb{E}^{\text{joint}}_{x_0,x'_0}\left[\frac{|A^{(2)}_n|}{\sqrt{n}}\right] \leq \frac{c_C}{n^{\delta_C}} \quad \textit{for all } n. \tag{3.4.9}$$

For that we will introduce some new notation: Set $\mathcal{R}_{n,0} \coloneqq 0$ for $n \in \mathbb{N}$ and for $i \in \mathbb{N}$

$$\mathcal{D}_{n,i} \coloneqq \min\{m > \mathcal{R}_{n,i-1} : |\hat{X}_m - \hat{X}'_m| \geq n^{b'}\}, \tag{3.4.10}$$

$$\mathcal{R}_{n,i} \coloneqq \min\{m > \mathcal{D}_{n,i} : |\hat{X}_m - \hat{X}'_m| \leq n^a\}, \tag{3.4.11}$$

with $b' \in (0,1/2)$ and $0 < a \ll b'$. We call $[\mathcal{R}_{n,i-1}, \mathcal{D}_{n,i})$ the i-th *black box interval.* With this definition R_n is the time spent in a black box interval until time n. Note that we can not make use of the coupling result from Lemma 3.2.1 in those intervals.

We differentiate between four possible types of black box intervals, depending on the relative positions of $\hat{X}$ and $\hat{X}'$ at the beginning and end of the interval:

$$W_{n,i} \coloneqq \begin{cases} 1 & \text{if } \hat{X}_{\mathcal{R}_{n,i-1}} > \hat{X}'_{\mathcal{R}_{n,i-1}}, \hat{X}_{\mathcal{D}_{n,i}} < \hat{X}'_{\mathcal{D}_{n,i}}, \\ 2 & \text{if } \hat{X}_{\mathcal{R}_{n,i-1}} > \hat{X}'_{\mathcal{R}_{n,i-1}}, \hat{X}_{\mathcal{D}_{n,i}} > \hat{X}'_{\mathcal{D}_{n,i}}, \\ 3 & \text{if } \hat{X}_{\mathcal{R}_{n,i-1}} < \hat{X}'_{\mathcal{R}_{n,i-1}}, \hat{X}_{\mathcal{D}_{n,i}} > \hat{X}'_{\mathcal{D}_{n,i}}, \\ 4 & \text{if } \hat{X}_{\mathcal{R}_{n,i-1}} < \hat{X}'_{\mathcal{R}_{n,i-1}}, \hat{X}_{\mathcal{D}_{n,i}} < \hat{X}'_{\mathcal{D}_{n,i}}. \end{cases} \tag{3.4.12}$$

By construction and the strong Markov property of $(\hat{X}_m, \hat{X}'_m)_m$ we have that: For each $n \in \mathbb{N}$, $(\mathcal{R}_{n,i} - \mathcal{D}_{n,i})_{i=1,2,\ldots}$ is an i.i.d. sequence, and $(W_{n,i}, \mathcal{D}_{n,i} - \mathcal{R}_{n,i-1})_{i=2,3,\ldots}$ is a Markov chain. In addition the two objects are independent, the transition probabilities of the second chain depend only on the first coordinate and the following separation lemma, similar to Lemma 3.3.1 in $d \geq 2$, holds.

Lemma 3.4.2. *For any $x_0, x'_0 \in \mathbb{Z}$ and all small enough, positive δ there exist $0 < b_2 < 1/8$ and $C, c > 0$ such that*

$$\mathbb{P}^{\text{joint}}_{x_0,x'_0}(H(n^\delta) \geq n^{b_2}) \leq \exp(-Cn^c), \quad n \in \mathbb{N}. \tag{3.4.13}$$

Furthermore, there exists $\varepsilon > 0$ such that uniformly in n

$$\mathbb{P}^{\text{joint}}(W_{n,2} = w' \,|\, W_{n,1} = w) \geq \varepsilon \tag{3.4.14}$$

for all pairs $(w, w') \in \{1, 2, 3, 4\}^2$ where a transition is "logically possible".

Note that as a consequence of (3.4.14) $(W_{n,i})_i$ is exponentially mixing.

Proof. To prove (3.4.13) the steps are analogous to Lemma 3.3.1. The only difference is that we use the harmonic function for $d = 1$ from Lemma 3.2.3

$$\frac{r - r_1}{r_2 - r_1} \tag{3.4.15}$$

for the proof. The steps will be the same so we will only highlight the change:

Assume that we start with $x_0 = x'_0$. This will yield an upper bound on the other starting pairs since we have no starting distance. By the same arguments as in the proof of Lemma 3.3.1 we obtain for some $\delta_0 > 0$, by constructing suitable corridors, the lower bound

$$\mathbb{P}^{\text{joint}}_{x_0,x'_0}(H(\varepsilon_1 \log n) \leq \varepsilon_1 \log n) \geq \delta_0^{\varepsilon_1 \log n} = n^{\varepsilon_1 \log \delta_0}$$

and for some large constant $K \gg \varepsilon_1$ and $|x - x'| \geq \varepsilon_1 \log n$

$$\begin{aligned}
&\mathbb{P}^{\text{joint}}_{x,x'}\Big(H(K \log n) < h\Big(\frac{1}{2}\varepsilon_1 \log n\Big) \wedge (K \log n)^3\Big) \\
&\geq \mathbb{P}^{\text{ind}}_{x,x'}\Big(H(K \log n) < h\Big(\frac{1}{2}\varepsilon_1 \log n\Big) \wedge (K \log n)^3\Big) - (K \log n)^3 \Big(\frac{1}{2}\varepsilon_1 \log n\Big)^{-\beta} \\
&\geq \mathbb{P}^{\text{ind}}_{x,x'}\Big(H(K \log n) < h\Big(\frac{1}{2}\varepsilon_1 \log n\Big)\Big) - C(K \log n)^{-1} - (K \log n)^3 \Big(\frac{1}{2}\varepsilon_1 \log n\Big)^{-\beta} \\
&\geq (1 - \varepsilon)\frac{1}{(2K/\varepsilon_1) - 1} - C(K \log n)^{-1} - (K \log n)^3 \Big(\frac{1}{2}\varepsilon_1 \log n\Big)^{-\beta} \\
&\geq \frac{1}{4}\frac{\varepsilon_1}{K}
\end{aligned}$$

for n and β large enough. Combining these with probability greater than $n^{\varepsilon_1 \log \delta_0} \frac{\varepsilon_1}{4K}$ we need at most $\varepsilon_1 \log n + (K \log n)^3$ many steps to reach a distance of at least $K \log n$.

If the random walks are already at distance $K \log n$ we want to start the iteration of Step 2 and the following from the proof of Lemma 3.3.1. Therefore we need a lower bound in $d = 1$, when starting from a distance of $\|x - x'\| \geq K \log n$, for

$$\mathbb{P}^{\text{ind}}_{x,x'}(H(\log^2 n) < h(1/2K \log n)) \tag{3.4.16}$$

and starting from a distance of $\|x-x'\| \geq \log^k n$ we want to find a lower bound for

$$\mathbb{P}^{\text{ind}}_{x,x'}(H(\log^{k+1} n) < h(1/2 \log^k n)). \tag{3.4.17}$$

Here we need to make smaller steps since our previous approximations are not sufficient. For (3.4.16) we get the lower bound (note that here $\|x-x'\| \geq K \log n$)

$$\mathbb{P}^{\text{ind}}_{x,x'}(H(\log^2 n) < h(1/2K \log n)) \geq (1-\varepsilon)\frac{K}{2\log n - K} \tag{3.4.18}$$

and for (3.4.17) (note that here $\|x-x'\| \geq \log^k n$)

$$\mathbb{P}^{\text{ind}}_{x,x'}(H(\log^{k+1} n) < h(1/2 \log^k n)) \geq (1-\varepsilon)\frac{1}{2\log n - 1}. \tag{3.4.19}$$

That means for one step, starting from $\|x-x'\| \geq \log^k n$,

$$\begin{aligned}
&\mathbb{P}^{\text{joint}}_{x,x'}(H(\log^{(k+1)} n) < h(1/2 \log^k n)) \\
&\quad \geq \mathbb{P}^{\text{joint}}_{x,x'}(H(\log^{(k+1)} n) < h(1/2 \log^k n) \wedge (\log^{k+1} n)^3) \\
&\quad \geq \mathbb{P}^{\text{ind}}_{x,x'}(H(\log^{(k+1)} n) < h(1/2 \log^k n) \wedge (\log^{k+1} n)^3) - C(\log^{k+1} n)^3 (1/2 \log^k n)^{-\beta} \\
&\quad \geq \mathbb{P}^{\text{ind}}_{x,x'}(H(\log^{(k+1)} n) < h(1/2 \log^k n)) - (\log n)^{-(k+1)} - C(\log^{k+1} n)^3 (1/2 \log^k n)^{-\beta} \\
&\quad \geq (1-\varepsilon)\frac{1}{2}(\log n)^{-1} - (\log n)^{-(k+1)} - C(\log^{k+1} n)^3 (1/2 \log^k n)^{-\beta} \\
&\quad \geq \frac{1}{4}(\log n)^{-1}
\end{aligned}$$

for large n and $\beta > 0$ large enough, where we again use the invariance principle, Lemma 3.2.3 and Lemma 3.2.1.

Now we need $b' \log n / \log\log n$ steps in the iteration to reach a distance of $n^{b'}$, with the k-th step taking at most time $(\log^{k+1} n)^3$. Consequently, we need at most $\sum_{k=2}^{b' \log n/ \log\log n} (\log n)^{3k}$ time for one such attempt from distance $K \log n$ to $n^{b'}$ and therefore from $x = x'$ we need at most

$$\varepsilon_1 \log n + (K \log n)^3 + \sum_{k=2}^{b' \log n/ \log\log n} (\log n)^{3k} \leq n^{4b'}$$

for n large enough. We obtain a lower bound on the probability to make the whole distance in a single attempt

$$\begin{aligned}
&n^{\varepsilon_1 \log \delta_0} \frac{\varepsilon_1}{4K} \prod_{k=2}^{b' \log n/ \log\log n} \frac{1}{4 \log n} \\
&\geq n^{\varepsilon_1 \log \delta_0} \frac{\varepsilon_1}{4K} \exp\Big(- b' \log n \frac{\log\log n^4}{\log\log n}\Big) \\
&\geq \frac{\varepsilon_1}{4K} n^{-2b' + \varepsilon_1 \log \delta_0}
\end{aligned}$$

It is easy to see that for any $b' > 0$ small enough there exist a $0 < b_2 < 1/8$ and ε_1 small enough such that $\alpha := b_2 - 4b' > 2b' - \varepsilon_1 \log \delta_0$ and with that

$$\begin{aligned}
&\mathbb{P}^{\text{joint}}_{x,x'}(H(n^{b'}) \geq n^{b_2}) \\
&\leq (1 - n^{-2b' + \varepsilon_1 \log \delta_0})^{n^\alpha} \\
&\leq \exp(-n^{-2b' + \varepsilon_1 \log \delta_0 + \alpha})
\end{aligned}$$

where the last term will tend to 0 as $n \to \infty$.

For (3.4.14) we need to show that there exists a uniform lower bound in n away from zero for the random walks to change their positions after they come close to each other before they reach a distance of $n^{b'}$. So let's say at time $m := \mathcal{R}_{n,i}$ we have $\hat{X}_m > \hat{X}'_m$. More precisely, write $D_j := \hat{X}_j - \hat{X}'_j$ and pick a small $\varepsilon > 0$ to be tuned later. Using the same methods as in the proof of (3.4.13) we can show a suitable uniform lower bound on the probability that D reaches $(-\infty, 0]$ before $n^{b'}$. By the same arguments as above and noting that

$$\mathbb{P}^{\text{ind}}\big(h(\log^{k-1} n) < H(\log^{k+1} n) \,|\, D_0 = \log^k n\big) \geq (1-\tilde{\varepsilon})\frac{\log^{k+1} n - \log^k n}{\log^{k+1} n - \log^{k-1} n},$$

we obtain for $k \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}^{\text{joint}}\big(h(\log^{k-1} n) < H(\log^{k+1} n) \,|\, D_0 = \log^k n\big) &\geq c\frac{\log^{k+1} n - \log^k n}{\log^{k+1} n - \log^{k-1} n} \\ &= c\Big(1 - \frac{\log n - 1}{\log^2 n - 1}\Big).\end{aligned} \tag{3.4.20}$$

And similarly

$$\begin{aligned}\mathbb{P}^{\text{joint}}(h(n^a) < H(n^{b'}) \,|\, D_0 = n^a \log n) &\geq c\frac{n^{b'-a} - n^a \log n}{n^{b'-a} - n^a} \\ &= c\Big(1 - \frac{\log n - 1}{n^{b'-a} - 1}\Big).\end{aligned}$$

Therefore we have, with positive probability, $\frac{n^{b'-a}}{\log n}$ many tries for the process D to reach $(-\infty, 0]$ before it hits $n^{b'}$. By the Markov property and using (3.4.20) we need $a \log n / \log\log n$ iterations for D to reach $\varepsilon \log n$. From $\varepsilon \log n$ we build corridors, as in Step 1 in the proof of Lemma 3.3.1, to achieve $D < 0$. The probability for such a corridor to exist is $\exp(-c\varepsilon \log n)$. Thus, for one attempt to reach $(-\infty, 0]$, the probability to be successful is greater than $n^{ca \log n / \log\log n} n^{-c\varepsilon}$ and we have $\frac{n^{b'-a}}{\log n}$ such attempts with positive probability. Combining those two facts, we obtain a uniform lower bound on the probability for D to reach $(-\infty, 0]$ away from zero. Which concludes the proof. □

As a corollary to equation (3.4.13) from Lemma 3.4.2 we obtain

Corollary 3.4.3. *We can choose $0 < b_2 < 1/8$ and $C, c > 0$ such that for any choice of $x_0, x'_0 \in \mathbb{Z}$*

$$\mathbb{P}^{\text{joint}}_{x_0,x'_0}(\mathcal{D}_{n,i} - \mathcal{R}_{n,i-1} \geq n^{b_2} \,|\, W_{n,i} = w) \leq \exp(-Cn^c), \quad w \in \{1,2,3,4\}, n \in \mathbb{N}. \tag{3.4.21}$$

Proof. This is a direct consequence of the fact that $n^{b'} - K \log n > n^{b'} - n^a$. □

By construction we have, due to symmetry,

$$\mathbb{P}^{\text{joint}}(W_{n,j} = 1) = \mathbb{P}^{\text{joint}}(W_{n,j} = 3) \quad \text{and} \quad \mathbb{P}^{\text{joint}}(W_{n,j} = 2) = \mathbb{P}^{\text{joint}}(W_{n,j} = 4) \quad \text{for all } j, n. \tag{3.4.22}$$

Proof of Lemma 3.4.1. We can follow the proof of the analogous Lemma 3.14 from [BČD16] since we have all necessary ingredients for our model. In that spirit let Y have the distribution

$$\mathbb{P}(Y \geq \ell) = \hat{\Psi}^{\text{ind}}\Big(\inf\{m \geq 0 : \hat{X}_m < \hat{X}'_m\} \geq \ell \,|\, (\hat{X}_0, \hat{X}'_0) = (1,0)\Big), \quad \ell \in \mathbb{N} \tag{3.4.23}$$

and let V be an independent Bernoulli$(1-1/n)$-distributed random variable. A coupling based on Lemma 3.2.1 shows that $\mathcal{R}_{n,1} - \mathcal{D}_{n,1}$ is stochastically larger than $((1-V)+VY) \wedge n$, since the distance between $\hat{X}_{D_{n-1}}$ and $\hat{X}'_{D_{n-1}}$ is $n^{b'}$ and $n^{b'} - n^a \gg 1$ and we can choose a and β in such a way that the probability for the coupling between $\hat{\Psi}^{\text{ind}}$ and $\hat{\Psi}^{\text{joint}}$ fails during the first n steps is less than $1/n$. By well known estimates of one-dimensional random walks (e.g. see Theorem 8.16 in [Kal02] and Theorem 1.a. in [Fel71] on page 415), there exist $c > 0$ and $c_Y > 0$ such that uniformly in $n \geq 2$,

$$\mathbb{E}[\mathrm{e}^{-\lambda((1-V)+VY)}] \leq \exp(-c_Y\sqrt{\lambda}), \quad \lambda \geq 0 \quad \text{and} \tag{3.4.24}$$

$$\mathbb{P}^{\text{joint}}_{x_0,x'_0}(\mathcal{R}_{n,1} - \mathcal{D}_{n,1} \geq \ell) \geq \frac{c}{\sqrt{\ell}}, \quad \ell = 1, \dots, n. \tag{3.4.25}$$

Inequality (3.4.24) is trivial for $\lambda \geq 1$ and $\lambda = 0$, for $\lambda \in (0,1)$ we have, using Theorem 8.16 from [Kal02] with $u = 0$ and $s = \mathrm{e}^{-\lambda}$,

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\lambda((1-V)+VY)}] &= \frac{1}{n}\mathrm{e}^{-\lambda} + \frac{n-1}{n}\mathbb{E}[\mathrm{e}^{-\lambda Y}] \\
&\leq \frac{1}{n}\mathrm{e}^{-\lambda} + \frac{n-1}{n}\Big(1 - \exp\Big\{-\frac{c_Y}{2}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\lambda m}}{m}\Big\}\Big) \\
&\leq \frac{1}{n}\mathrm{e}^{-\lambda} + \frac{n-1}{n}\Big(1 - \exp\Big\{\frac{c_Y}{2}\log(1-\mathrm{e}^{-\lambda})\Big\}\Big) \\
&= \frac{1}{n}\mathrm{e}^{-\lambda} + \frac{n-1}{n}\Big(1 - (1-\mathrm{e}^{-\lambda})^{\frac{c_Y}{2}}\Big).
\end{aligned}$$

Note that by Theorem 8.16 we obtain $c_Y \leq 1$ and thus we have

$$1 - \exp(-c_Y\sqrt{\lambda}) \leq (1-\exp(-\lambda))^{\frac{1}{2}c_Y}$$

for all $\lambda \in (0,1)$, by combining

$$1 - \exp(-c_Y\sqrt{\lambda}) \leq \Big(1 - \exp(-\sqrt{\lambda})\Big)^{c_Y}$$

and

$$1 - \exp(-\sqrt{\lambda}) \leq \Big(1 - \exp(-\lambda)\Big)^{1/2}.$$

Furthermore we have

$$\frac{1}{n}\mathrm{e}^{-\lambda} \leq \exp(-c_Y\sqrt{\lambda}) \tag{3.4.26}$$

if

$$-\lambda - \log(n) \leq -c_Y\sqrt{\lambda},$$

where both sides are equal if $\lambda = \frac{c_Y}{2} \pm \sqrt{\frac{c_Y^2}{4} - \log(n)}$. For $\log(n) > \frac{c_Y^2}{4}$ there exist no real valued solution, combining that with the fact that the inequality in (3.4.26) holds for $\lambda = 0$ and $\lambda = 1$, we get (3.4.26) for all $\lambda \in (0,1)$ if $\log(n) > \frac{c_Y^2}{4}$, which holds for all $n \geq 2$, since $c_Y \leq 1$. And thus (3.4.24) holds uniformly in $n \geq 2$. Let $I_n \coloneqq \max\{i : \mathcal{R}_{n,i} \leq n\}$ be the number of "black boxes" that we see up to time n. By equation (3.4.25) we have $I_n = \mathcal{O}(\sqrt{n})$ in probability and in fact

$$\mathbb{E}^{\text{joint}}_{x_0,x'_0}[I_n^2] \leq Cn. \tag{3.4.27}$$

This can be proven by using the lower bounds of equation (3.4.25). Note that since I_n is a $\mathbb{N}_0$-valued random variable

$$\begin{aligned}
\mathbb{E}^{\text{joint}}_{x_0,x_0'}[I_n^2] &= \sum_{\ell=1}^{\infty} \mathbb{P}^{\text{joint}}_{x_0,x_0'}(I_n^2 \geq \ell) = \sum_{\ell=1}^{\infty} \mathbb{P}^{\text{joint}}_{x_0,x_0'}(I_n \geq \sqrt{\ell}) = \sum_{\ell=1}^{\infty} \mathbb{P}^{\text{joint}}_{x_0,x_0'}(\mathcal{R}_{n,\sqrt{\ell}} \leq n) \\
&\leq \sum_{\ell=1}^{\infty} \mathbb{P}^{\text{joint}}_{x_0,x_0'}\Big(\sum_{i=1}^{\sqrt{\ell}} \mathcal{R}_{n,i} - \mathcal{D}_{n,i} \leq n\Big) \\
&\leq \sum_{\ell=1}^{\infty} \mathbb{P}^{\text{joint}}_{x_0,x_0'}\Big(\text{for all } i \leq \sqrt{\ell} \colon \mathcal{R}_{n,i} - \mathcal{D}_{n,i} \leq n\Big) \\
&\leq \sum_{\ell=1}^{\infty} \mathbb{P}^{\text{joint}}_{x_0,x_0'}\Big(\mathcal{R}_{n,1} - \mathcal{D}_{n,1} \leq n\Big)^{\sqrt{\ell}} \\
&\leq \sum_{\ell=1}^{\infty} \Big(1 - \frac{c}{\sqrt{n}}\Big)^{\sqrt{\ell}} \\
&\leq n \sum_{k=1}^{\infty} \exp(-c\sqrt{k-1}) \leq Cn.
\end{aligned}$$

The last line follows by grouping the first n summands, the second n summands and so forth and bounding them together. For example for $kn \leq \ell \leq (k+1)n$ we have

$$\Big(1 - \frac{c}{\sqrt{n}}\Big)^{\sqrt{\ell}} \leq \Big(1 - \frac{c}{\sqrt{n}}\Big)^{\sqrt{kn}} \leq \exp(-c\sqrt{k}).$$

More quantitatively, there exists $c > 0$ such that for $1 \leq k \leq n$

$$\mathbb{P}^{\text{joint}}_{x_0,x_0'}(I_n \geq k) \leq \exp(-ck^2/n) \tag{3.4.28}$$

and so in particular

$$\mathbb{E}^{\text{joint}}_{x_0,x_0'}[I_n \mathbb{1}_{I_n \geq n^{3/4}}] = \sum_{k=\lceil n^{3/4} \rceil}^{n} \mathbb{P}^{\text{joint}}_{x_0,x_0'}(I_n \geq k) \leq n\mathrm{e}^{-c\sqrt{n}}. \tag{3.4.29}$$

The inequality in (3.4.28) can be obtained by the following arguments. Let $Y_1, Y_2, \ldots$ be i.i.d. copies of $\big((1-V) + VY\big)$ defined above in (3.4.23), then

$$\mathbb{P}\Big(\big((1-V) + VY\big) \wedge n \geq \ell\Big) \leq \mathbb{P}^{\text{joint}}_{x_0,x_0'}(\mathcal{R}_{n,1} - \mathcal{D}_{n,1} \geq \ell)$$

and thus

$$\mathbb{P}\Big(\sum_{i=1}^{\ell}(Y_i \wedge n) \leq n\Big) \geq \mathbb{P}^{\text{joint}}_{x_0,x_0'}\Big(\sum_{i=1}^{\ell} \mathcal{R}_{n,i} - \mathcal{D}_{n,i} \leq n\Big) \geq \mathbb{P}^{\text{joint}}_{x_0,x_0'}\Big(\mathcal{R}_{n,\ell} \leq n\Big) = \mathbb{P}^{\text{joint}}_{x_0,x_0'}(I_n \geq \ell).$$

Combining the two we obtain by (3.4.24) for $\lambda > 0$

$$\begin{aligned}
\mathbb{P}^{\text{joint}}_{x_0,x_0'}(I_n \geq \ell) \leq \mathbb{P}\Big(\sum_{i=1}^{\ell}(Y_i \wedge n) \leq n\Big) &= \mathbb{P}(Y_1 + \cdots + Y_\ell \leq n) \\
&\leq \mathrm{e}^{\lambda n}\mathbb{E}\Big[\exp\big(-\lambda Y_1\big)\Big]^{\ell} \leq \mathrm{e}^{\lambda n - c_Y\sqrt{\lambda}\ell}.
\end{aligned}$$

Choosing $\lambda = (c_Y k/n)^2$ we see that (3.4.28) holds.

Note that

$$R_n \le \sum_{j=1}^{I_n+1} (\mathcal{D}_{n,j} - \mathcal{R}_{n,j-1}), \tag{3.4.30}$$

and using (3.4.21) we get $R_n = o(n)$ in probability. Now using (3.4.27) together with (3.4.30) and (3.4.21) implies (3.4.8):

$$\begin{aligned}\mathbb{E}^{\text{joint}}_{x_0,x_0'}[R_n^2] &= \mathbb{E}^{\text{joint}}_{x_0,x_0'}[R_n^2 \mathbb{1}_{\{\exists j \le n: \mathcal{D}_{n,j+1} - \mathcal{R}_{n,j} \ge n^{b_2}\}}] + \mathbb{E}^{\text{joint}}_{x_0,x_0'}[R_n^2 \mathbb{1}_{\{\forall j \le n: \mathcal{D}_{n,j+1} - \mathcal{R}_{n,j} < n^{b_2}\}}] \\ &\le n^2 \mathbb{P}^{\text{joint}}_{x_0,x_0'}(\exists j \le n : \mathcal{D}_{n,j+1} - \mathcal{R}_{n,j} \ge n^{b_2}) + n^{2b_2} \mathbb{E}^{\text{joint}}_{x_0,x_0'}[I_{n+1}^2] \\ &\le C n^{1+2b_2}\end{aligned} \tag{3.4.31}$$

and $\mathbb{E}^{\text{joint}}_{x_0,x_0'}[R_n^{3/2}] \le (\mathbb{E}^{\text{joint}}_{x_0,x_0'}[R_n^2])^{3/4}$.

For (3.4.9) we define

$$D_{n,m} \coloneqq A^{(1)}_{\mathcal{D}_{n,m}} - A^{(1)}_{\mathcal{R}_{n,m-1}}, \quad D'_{n,m} \coloneqq A^{(2)}_{\mathcal{D}_{n,m}} - A^{(2)}_{\mathcal{R}_{n,m-1}}.$$

By symmetry we have

$$\mathbb{E}^{\text{joint}}[D_{n,j}] = 0, \mathbb{E}^{\text{joint}}[D_{n,j} \,|\, W_{n,j} = 1] = -\mathbb{E}^{\text{joint}}[D_{n,j} \,|\, W_{n,j} = 3], \tag{3.4.32}$$

$$\mathbb{E}^{\text{joint}}[D_{n,j} \,|\, W_{n,j} = 2] = -\mathbb{E}^{\text{joint}}[D_{n,j} \,|\, W_{n,j} = 4], \tag{3.4.33}$$

by (3.4.21) and (3.4.1) we get that for some $C < \infty$ and $b > 0$ uniformly in $j, n \in \mathbb{N}$, for $w \in \{1,2,3,4\}$

$$\begin{aligned}&\mathbb{E}^{\text{joint}}_{x_0,x_0'}[|D_{n,j}| \,|\, W_{n,j} = w] \\ &\quad = \mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[\Big|\sum_{i=\mathcal{R}_{n,j-1}}^{\mathcal{D}_{n,j}-1} \phi_1(\hat{X}_i, \hat{X}'_i)\Big| \,\Big|\, W_{n,j} = w\right] \\ &\quad \le C_\phi \mathbb{E}^{\text{joint}}_{x_0,x_0'}\Big[(\mathcal{D}_{n,j} - \mathcal{R}_{n,j-1}) \,|\, W_{n,j} = w\Big] \\ &\quad \le C_\phi n^{b_2} + C_\phi \sum_{\ell=2}^{\infty} n^{b_2 \ell} \mathbb{P}^{\text{joint}}_{x_0,x_0'}(\mathcal{D}_{n,j} - \mathcal{R}_{n,j-1} \ge n^{(\ell-1)b_2} \,|\, W_{n,j} = w) \\ &\quad \le C n^{b_2}\end{aligned} \tag{3.4.34}$$

and analogously for $D'_{n,j}$. Set $\mathcal{G}_j \coloneqq \hat{\mathcal{F}}_{\mathcal{D}_{n,j}}$ (the σ-field of the $\mathcal{D}_{n,j}$-past) for $j \in \mathbb{N}$ and for $j \le 0$ let $\mathcal{G}_j$ be the trivial σ-algebra. Note that $D_{n,j}$ and $D'_{n,j}$ are $\mathcal{G}_j$-adapted for $j \in \mathbb{N}$. For $k < m$ we have

$$\mathbb{E}^{\text{joint}}_{x_0,x_0'}[D_{n,m} \,|\, \mathcal{G}_k] = \mathbb{E}^{\text{joint}}_{x_0,x_0'}\Big[\mathbb{E}[D_{n,m} \,|\, W_{n,m}] \,|\, \mathcal{G}_k\Big]$$

by construction and $(W_{n,j})_j$ is (uniformly in n) exponentially mixing, thus, observing (3.4.32), (3.4.22), and (3.4.34)

$$\mathbb{E}^{\text{joint}}_{x_0,x_0'}\Big[(\mathbb{E}[D_{n,m} \,|\, \mathcal{G}_{m-j}])^2\Big] \le C n^{2b_2} \mathrm{e}^{-cj}, \quad m, j \in \mathbb{N}, n \in \mathbb{N} \tag{3.4.35}$$

for some $C, c \in (0,\infty)$ and analogous bounds for $D'_{n,m}$. Indeed, abbreviating $\mathbb{E} = \mathbb{E}^{\text{joint}}_{x_0,x'_0}$,

$$
\begin{aligned}
\mathbb{E}\Big[(\mathbb{E}[D_{n,m} \,|\, \mathcal{G}_{m-j}])^2\Big] &= \mathbb{E}\Big[(\mathbb{E}[\mathbb{E}[D_{n,m} \,|\, W_{n,m}] \,|\, \mathcal{G}_{m-j}])^2\Big] \\
&= \mathbb{E}\left[\Big(\mathbb{E}\Big[\sum_{i=1}^{4} \mathbb{1}_{\{W_{n,m}=i\}} \mathbb{E}[D_{n,m} \,|\, W_{n,m}=i] \,|\, \mathcal{G}_{m-j}\Big]\Big)^2\right] \\
&= \sum_{\ell,i=1}^{4} \mathbb{E}[D_{n,m} \,|\, W_{n,m}=i]\mathbb{E}[D_{n,m} \,|\, W_{n,m}=\ell] \\
&\qquad\qquad \times \mathbb{E}\big[\mathbb{P}(W_{n,m}=i \,|\, \mathcal{G}_{m-j})\mathbb{P}(W_{n,m}=\ell \,|\, \mathcal{G}_{m-j})\big] \\
&\le \sum_{\ell,i=1}^{4} |\mathbb{E}[D_{n,m} \,|\, W_{n,m}=i]\mathbb{E}[D_{n,m} \,|\, W_{n,m}=\ell]| \\
&\qquad\qquad \times (\mathbb{P}(W_{n,m}=i) + \mathrm{e}^{-cj})(\mathbb{P}(W_{n,m}=\ell) + \mathrm{e}^{-cj}) \\
&= \sum_{\ell,i=1}^{4} |\mathbb{E}[D_{n,m} \,|\, W_{n,m}=i]\mathbb{E}[D_{n,m} \,|\, W_{n,m}=\ell]|\mathrm{e}^{-2cj} \\
&\le C n^{2b_2} \mathrm{e}^{-2cj}
\end{aligned}
$$

Let $S_{n,m} \coloneqq \sum_{j=1}^{m} D_{n,j}$ and $S'_{n,m} \coloneqq \sum_{j=1}^{m} D'_{n,j}$ then for each $n \in \mathbb{N}$, $(S_{n,m})_m$ is a mixingale; see [HH80], p. 19.

Using McLeish's analogue of Doobs $\mathcal{L}^2$-inequality for mixingales, we get

$$
\mathbb{E}^{\text{joint}}_{x_0,x'_0}\Big[\max_{m=1,\dots,n^{3/4}} S^2_{n,m}\Big] \le K \sum_{i=1}^{n^{3/4}} n^{2b_2} \le K n^{\frac{3}{4}+\frac{3}{2}b_2}, \tag{3.4.36}
$$

and thus

$$
\mathbb{E}^{\text{joint}}_{x_0,x'_0}\left[\frac{|S_{n,I_n}|}{\sqrt{n}} \mathbb{1}_{\{I_n \le n^{3/4}\}}\right] \le \frac{1}{\sqrt{n}}\left(\mathbb{E}^{\text{joint}}_{x_0,x'_0}\Big[\max_{m=1,\dots,n^{3/4}} S^2_{n,m}\Big]\right)^{1/2} \le K^{1/2} n^{-\frac{1}{8}+\frac{3}{4}b_2}. \tag{3.4.37}
$$

Note that b_2 comes from Corollary 3.4.3 and can be chosen smaller than 1/8 which results in the right hand side converging to zero for $n \to \infty$. By (3.4.5) we have

$$
\begin{aligned}
A^{(1)}_n &= \sum_{j=0}^{n-1} \phi_1(\hat{X}_j, \hat{X}'_j) \\
&= \sum_{j=1}^{I_n} D_{n,j} + \sum_{j=1}^{I_{n+1}} \sum_{i=\mathcal{D}_{n,j}\wedge n}^{\mathcal{R}_{n,j}\wedge n} \phi_1(\hat{X}_i, \hat{X}'_i) \\
&\le \sum_{j=1}^{I_n} D_{n,j} + \sum_{j=1}^{I_{n+1}} \big((\mathcal{R}_{n,j}\wedge n) - (\mathcal{D}_{n,j}\wedge n)\big)\frac{C_1}{n^2} \\
&\le \sum_{j=1}^{I_n} D_{n,j} + \frac{C_1}{n},
\end{aligned}
$$

and so

$$\mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[\frac{|A_n^{(1)}|}{\sqrt{n}}\right] \le \frac{c}{\sqrt{n}} + \frac{1}{\sqrt{n}}\mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[|S_{n,I_n}|\mathbb{1}_{\{I_n\le n^{3/4}\}}\right] + \frac{1}{\sqrt{n}}\mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[|S_{n,I_n}|\mathbb{1}_{\{I_n> n^{3/4}\}}\right]$$
$$\le \frac{c}{\sqrt{n}} + \frac{1}{\sqrt{n}}\mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[|S_{n,I_n}|\mathbb{1}_{\{I_n\le n^{3/4}\}}\right] + \frac{1}{\sqrt{n}}Cn^{b_2}\mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[I_n\mathbb{1}_{\{I_n> n^{3/4}\}}\right]$$
$$+ \frac{1}{\sqrt{n}}Cn\mathbb{P}^{\text{joint}}_{x_0,x_0'}(\exists j\le n\colon \mathcal{D}_{n,j}-\mathcal{R}_{n,j-1}\ge n^{b_2}).$$

Using (3.4.37), (3.4.29) and (3.4.21) respectively on the last three terms on the right hand side yields (3.4.9) for $A_n^{(1)}$ and analogous calculations for D' instead of D yield (3.4.9) for $A_n^{(2)}$. □

Write $\hat\sigma^2 := \sum_{y,y'} y^2\hat\Psi^{\text{ind}}((0,0),(y,y'))$ for the variance of a single increment under $\hat\Psi$.

Lemma 3.4.4. *There exist $C>0, \tilde b\in(0,1/4)$ such that for all bounded Lipschitz continuous $f:\mathbb{R}^2\to\mathbb{R}$ and all $x_0,x_0'\mathbb{Z}$*

$$\left|\mathbb{E}^{\text{joint}}_{x_0,x_0'}\left[f\left(\frac{\hat X_n}{\hat\sigma\sqrt{n}},\frac{\hat X_n'}{\hat\sigma\sqrt{n}}\right)\right]-\mathbb{E}\big[f(Z)\big]\right|\le L_f\frac{C}{n^{\tilde b}} \tag{3.4.38}$$

where Z is two-dimensional standard normal and L_f the Lipschitz constant of f.

Since, as has been shown above, $R_n=o(n)$ (with R_n from (3.4.7)) in probability, we obtain, using the bounds from (3.4.5) and (3.4.6),

$$\frac{A_n^{(11)}}{n}\to\hat\sigma^2,\quad \frac{A_n^{(22)}}{n}\to\hat\sigma^2,\quad \frac{A_n^{(12)}}{n}\to 0 \tag{3.4.39}$$

in probability as $n\to\infty$. Since $\hat X_n=M_n+A_n^{(1)}$ and $\hat X_n'=M_n'+A_n^{(2)}$ we make use of the convergence of $(M_n/\sqrt{n}, M_n'/\sqrt{n})$ and the bounds in (3.4.9) to prove Lemma 3.4.4. To prepare that, for $n\in\mathbb{N}$, let

$$Q_n := \begin{pmatrix}\phi_{11}(\hat X_{n-1},\hat X_{n-1}) & \phi_{12}(\hat X_{n-1},\hat X_{n-1})\\ \phi_{12}(\hat X_{n-1},\hat X_{n-1}) & \phi_{22}(\hat X_{n-1},\hat X_{n-1})\end{pmatrix} \tag{3.4.40}$$

be the conditional covariance matrix given $\hat F_{n-1}$ of the random variable $(M_n-M_{n-1}, M_n'-M_{n-1}')$ and let $\lambda_{n,1}\ge\lambda_{n,2}\ge 0$ be its eigenvalues. Equations (3.4.5), (3.4.6) and (3.4.1) yield bounds on the entries of Q_n and thus, by stability properties for the eigenvalues of symmetric matrices,

$$|\lambda_{j+1,1}-\hat\sigma^2|+|\lambda_{j+1,2}-\hat\sigma^2|\le C_2\mathbb{1}_{\{|\hat X_j-\hat X_j'|\le n^a\}}+\frac{C_2}{n^2}\mathbb{1}_{\{|\hat X_j-\hat X_j'|>n^a\}} \tag{3.4.41}$$

for some constant $C_2<\infty$, see [FF63].

In particular,

$$\sum_{i=1}^2\Big|n\hat\sigma^2-\sum_{j=1}^n\lambda_{j,i}\Big|\le C_2R_n+\frac{C_2}{n} \tag{3.4.42}$$

because for $i=1,2$ with $B_n \coloneqq \{j \le n \colon |\hat{X}_{j-1} - \hat{X}'_{j-1}| \le n^a\}$

$$\begin{aligned}
|n\hat{\sigma}^2 - \sum_{j=1}^n \lambda_{j,i}| &\le \sum_{j=1}^n |\hat{\sigma}^2 - \lambda_{j,i}| \\
&= \sum_{j\in B_n} |\hat{\sigma}^2 - \lambda_{j,i}| + \sum_{j\notin B_n} |\hat{\sigma}^2 - \lambda_{j,i}| \\
&\le R_n C_2 + (n-R_n)\frac{C_2}{n^2} \\
&\le R_n C_2 + \frac{C_2}{n}.
\end{aligned}$$

Proof of Lemma 3.4.4. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded Lipschitz continuous function with Lipschitz constant L_f and Z two-dimensional standard normal. Using (3.4.42) and (3.4.8) and Corollary 1.3 in [Rac95] we conclude that

$$\left|\mathbb{E}^{\text{joint}}_{x_0,x'_0}\Big[f\Big(\frac{M_n}{\hat{\sigma}\sqrt{n}}, \frac{M'_n}{\hat{\sigma}\sqrt{n}}\Big)\Big] - \mathbb{E}\big[f(Z)\big]\right| \le L_f \frac{C}{n^{b^*}} \qquad \text{for all } n \tag{3.4.43}$$

for some $C < \infty$ and $b^* = \frac{1}{3}(\frac{1}{2} - \delta_R)$. For the use of Corollary 1.3 in [Rac95] we read $X_k = \Big((M_k - M_{k-1})/\sqrt{\hat{\sigma}^2 n}, (M'_k - M'_{k-1})/\sqrt{\hat{\sigma}^2 n}\Big)$ which leads to a covariance matrix with eigenvalues $\tilde{\lambda}_{k,i} = \lambda_{k,i}/(\hat{\sigma}^2 n)$ for $i=1,2$. Moreover note that due to the tail bounds on the regeneration times from Lemma 3.1.15, by tuning the parameters right, we obtain $\sup_k \mathbb{E}\big[\, \big\|(M_k - M_{k-1}, M'_k - M'_{k-1})\big\|^3\,\big] < \infty$. We briefly want to show the calculation for the second expectation from the first part of Corollary 1.3 in [Rac95].

$$\begin{aligned}
&\mathbb{E}^{\text{joint}}_{x_0,x'_0}\bigg[\Big(\sum_{i=1}^2 |1 - \sum_{k=1}^n \tilde{\lambda}^2_{k,i}|\Big)^{3/2}\bigg] \\
&= (\hat{\sigma}^2 n)^{-3/2}\mathbb{E}^{\text{joint}}_{x_0,x'_0}\bigg[\Big(\sum_{i=1}^2 |\hat{\sigma}^2 n - \sum_{k=1}^n \lambda^2_{k,i}|\Big)^{3/2}\bigg] \\
&\le (\hat{\sigma}^2 n)^{-3/2} C \mathbb{E}^{\text{joint}}_{x_0,x'_0}\bigg[\Big(R_n + \frac{1}{n}\Big)^{3/2}\bigg]
\end{aligned}$$

from which we can conclude (3.4.43). Then, combining (3.4.43) and (3.4.9) yields

$$\begin{aligned}
&\left|\mathbb{E}^{\text{joint}}_{x_0,x'_0}\Big[f\Big(\frac{\hat{X}_n}{\hat{\sigma}\sqrt{n}}, \frac{\hat{X}'_n}{\hat{\sigma}\sqrt{n}}\Big)\Big] - \mathbb{E}[f(Z)]\right| \\
&\le \left|\mathbb{E}^{\text{joint}}_{x_0,x'_0}\Big[f\Big(\frac{\hat{M}_n}{\hat{\sigma}\sqrt{n}}, \frac{\hat{M}'_n}{\hat{\sigma}\sqrt{n}}\Big)\Big] - \mathbb{E}[f(Z)]\right| + CL_f \mathbb{E}^{\text{joint}}_{x_0,x'_0}\bigg[\frac{|A^{(1)}_n|}{\sqrt{n}} + \frac{|A^{(2)}_n|}{\sqrt{n}}\bigg] \\
&\le L_f \frac{C}{n^{b^*}} + L_f \frac{C}{n^{\delta_C}}
\end{aligned} \tag{3.4.44}$$

□

Now we can prove Proposition 3.2.2 for $d=1$.

Proof of Proposition 3.2.2 for $d=1$. Recall that we want to show

$$\mathbb{E}\bigg[\Big(E_\omega[f(\hat{X}^{\text{joint}}_m/\sqrt{m})] - \tilde{\Phi}(f)\Big)^2\bigg] \le C_f m^{-c}.$$

Note that

$$
\begin{aligned}
&\mathbb{E}\left[\left(E_\omega[f(\hat{X}_m^{\text{joint}}/\sqrt{m})] - \tilde{\Phi}(f)\right)^2\right] \\
&= \mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})] \\
&\qquad - 2\mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})]\tilde{\Phi}(f) + \tilde{\Phi}(f)^2 \\
&= \left(\mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})] - \tilde{\Phi}(f)^2\right) \\
&\qquad + 2\tilde{\Phi}(f)\left(\tilde{\Phi}(f) - \mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})]\right).
\end{aligned}
$$

Since a product of bounded Lipschitz continuous functions is again bounded and Lipschitz continuous we can use Lemma 3.4.4 to get

$$
|\mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})f(\hat{X}'_m/\sqrt{m})] - \tilde{\Phi}(f)^2| \le L_f \frac{C}{m^{\tilde{b}}}
$$

and

$$
|\tilde{\Phi}(f) - \mathbb{E}^{\text{joint}}[f(\hat{X}_m/\sqrt{m})]| \le L_f \frac{C}{m^{\tilde{b}}},
$$

which concludes the proof. □

Theorem 1.3.5 for $d = 1$ follows then by the same arguments as in the case of $d \ge 2$ since we now have Proposition 3.2.2 expanded to dimension $d = 1$.

Bibliography

[ABSO81] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach. Excitation dynamics in random one-dimensional systems. *Reviews of Modern Physics*, 53:175–198, 1981.

[ACDS18] Sebastian Andres, Alberto Chiarini, Jean-Dominique Deuschel, and Martin Slowik. Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *The Annals of Probability*, 46(1):302 – 336, 2018.

[ACS21] Sebastian Andres, Alberto Chiarini, and Martin Slowik. Quenched local limit theorem for random walks among time-dependent ergodic degenerate weights. *Probability Theory and Related Fields*, Mar 2021.

[ADS16] Sebastian Andres, Jean-Dominique Deuschel, and Martin Slowik. Harnack inequalities on weighted graphs and some applications to the random conductance model. *Prob. Theory Relat. Fields*, 164(3-4):931–977, 2016.

[AHR11] L. Avena, F. den Hollander, and F. Redig. Law of large numbers for a class of random walks in dynamic random environments. *Electron. J. Probab.*, 16:587–617, 2011.

[And14] Sebastian Andres. Invariance principle for the random conductance model with dynamic bounded conductances. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 50(2):352 – 374, 2014.

[BBDS21] Stein Andreas Bethuelsen, Matthias Birkner, Andrej Depperschmidt, and Timo Schlüter. Local limit theorems for a directed random walk on the backbone of a supercritical oriented percolation cluster, 2021.

[BČD16] Matthias Birkner, Jiří Černý, and Andrej Depperschmidt. Random walks in dynamic random environments and ancestry under local population regulation. *Electron. J. Probab.*, 21:1–43, 2016.

[BČDG13] Matthias Birkner, Jiří Černý, Andrej Depperschmidt, and Nina Gantert. Directed random walk on the backbone of an oriented percolation cluster. *Electron. J. Probab.*, 18:no. 80, 35, 2013.

[BCR16] Noam Berger, Moran Cohen, and Ron Rosenthal. Local limit theorem and equivalence of dynamic and static points of view for certain ballistic random walks in i.i.d. environments. *Ann. Probab.*, 44(4):2889–2979, 2016.

[BG90] Jean-Philippe Bouchaud and Antoine Georges. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Physics Reports*, 195(4):127–293, 1990.

[BGS19] Matthias Birkner, Nina Gantert, and Sebastian Steiber. Coalescing directed random walks on the backbone of a $1+1$-dimensional oriented percolation cluster converge to the Brownian web. *ALEA Lat. Am. J. Probab. Math. Stat.*, 16:1029–1054, 2019.

[BHT20] Oriane Blondel, Marcelo R. Hilário, and Augusto Teixeira. Random walks on dynamical random environments with nonuniform mixing. *Ann. Probab.*, 48(4):2014–2051, 2020.

[BR19] Gideon S. Bradburd and Peter L. Ralph. Spatial population genetics: It's about time. *Annual Review of Ecology, Evolution, and Systematics*, 50(1):427–449, Nov 2019.

[BS02] Erwin Bolthausen and Alain-Sol Sznitman. On the static and dynamic points of view for certain random walks in random environment. *Methods and Applications of Analysis*, 9(3):345–376, 2002.

[BS09] Michel Benaïm and Sebastian J. Schreiber. Persistence of structured populations in random environments. *Theoretical Population Biology*, 76(1):19–34, 2009.

[BS20] Peter Bella and Mathias Schäffner. Quenched invariance principle for random walks among random degenerate conductances. *The Annals of Probability*, 48(1):296 – 316, 2020.

[BV16] S. A. Bethuelsen and F. Völlering. Absolute Continuity and Weak Uniform Mixing of Random Walk in Dynamic Random Environment. *ArXiv e-prints*, January 2016.

[CMS10] J. Theodore Cox, Nevena Marić, and Rinaldo Schinazi. Contact process in a wedge. *Journal of Statistical Physics*, 139(3):506–517, Mar 2010.

[DG82] Richard Durrett and David Griffeath. Contact processes in several dimensions. *Z. Wahrsch. Verw. Gebiete*, 59(4):535–552, 1982.

[DG19] Jean-Dominique Deuschel and Xiaoqin Guo. Quenched local central limit theorem for random walks in a time-dependent balanced random environment, 2019.

[DG20a] Dmitry Dolgopyat and Ilya Goldsheid. Constructive approach to limit theorems for recurrent diffusive random walks on a strip, 2020.

[DG20b] Dmitry Dolgopyat and Ilya Goldsheid. Local limit theorems for random walks in a random environment on a strip. *Pure Appl. Funct. Anal.*, 5(6):1297–1318, 2020.

[DNS18] Jean-Dominique Deuschel, Tuan Nguyen, and Martin Slowik. Quenched invariance principles for the random conductance model on a random graph with degenerate ergodic weights. *Probability Theory and Related Fields*, 170:363–386, 2018.

[DR14] Alexander Drewitz and Alejandro F. Ramírez. Selected topics in random walks in random environment. In *Topics in percolative and disordered systems*, volume 69 of *Springer Proc. Math. Stat.*, pages 23–83. Springer, New York, 2014.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[FF63] D. K. Faddeev and V. N. Faddeeva. *Computational methods of linear algebra.* Translated by Robert C. Williams. W. H. Freeman and Co., San Francisco, 1963.

[GH02] Geoffrey Grimmett and Philipp Hiemer. Directed percolation and random walk. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 273–297. Birkhäuser Boston, Boston, MA, 2002.

[GM14] Olivier Garet and Régine Marchand. Large deviations for the contact process in random environment. *Ann. Probab.*, 42(4):1438–1479, 07 2014.

[Har74] T. E. Harris. Contact interactions on a lattice. *Ann. Probability*, 2:969–988, 1974.

[HdHS+15] Marcelo Hilário, Frank den Hollander, Vladas Sidoravicius, Renato Soares dos Santos, and Augusto Teixeira. Random walk on random walks. *Electron. J. Probab.*, 20:no. 95, 1–35, 2015.

[HH80] P. Hall and C. C. Heyde. *Martingale limit theory and its application.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. Probability and Mathematical Statistics.

[HSS13] F. den Hollander, R. dos Santos, and V. Sidoravicius. Law of large numbers for non-elliptic random walks in dynamic random environments. *Stochastic Process. Appl.*, 123(1):156–190, 2013.

[JRA11] Mathew Joseph and Firas Rassoul-Agha. Almost sure invariance principle for continuous-space random walk in dynamic random environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, 8, 2011.

[Kal02] Olav Kallenberg. *Foundations of Modern Probability.* Probability and Its Applications. Springer, New York, 2002.

[Kir07] Werner Kirsch. An invitation to random schroedinger operators, 2007.

[Koz85] S M Kozlov. The method of averaging and walks in inhomogeneous environments. *Russian Mathematical Surveys*, 40(2):73–145, apr 1985.

[Lig85] T. M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[Lig99] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Lig05] T. M. Liggett. *Interacting Particle Systems.* Classics in Mathematics. Springer Berlin Heidelberg, Berlin, Heidelberg, 1st ed. 2005. edition, 2005.

[LS06] T. M. Liggett and J. E. Steif. Stochastic domination: the contact process, Ising models and FKG measures. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(2):223–243, 2006.

[MP10] Peter Mörters and Yuval Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[MSDL+17] A. Mahdipour-Shirayeh, A. Darooneh, A. Long, N. Komarova, and M. Kohandel. Genotype by random environmental interactions gives an advantage to non-favored minor alleles. *Scientific Reports*, 7, 2017.

[Rac95] Alfredas Rackauskas. On the rate of convergence in the martingale clt. *Statistics & Probability Letters*, 23(3):221–226, 1995.

[RV13] F. Redig and F. Völlering. Random walks in dynamic random environments: a transference principle. *Ann. Probab.*, 41(5):3157–3180, 2013.

[Saw76] Stanley Sawyer. Results for the Stepping Stone Model for Migration in Population Genetics. *The Annals of Probability*, 4(5):699 – 728, 1976.

[SS18] Michele Salvi and François Simenhaus. Random walk on a perturbation of the infinitely-fast mixing interchange process. *Journal of Statistical Physics*, 171(4):656–678, May 2018.

[Ste17] Sebastian Steiber. *Ancestral lineages in the contact process: scaling and hitting properties*. PhD thesis, University of Mainz, 2017. available at https://publications.UB.Uni-Mainz.DE/theses/frontdoor.php?source_opus=100001057.

[Szn04] A.-S. Sznitman. Topics in random walks in random environment. In *School and Conference on Probability Theory*, ICTP Lect. Notes, XVII, pages 203–266 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[Tem69] D.E. Temkin. The theory of diffusionless crystal growth. *Journal of Crystal Growth*, 5(3):193–202, 1969.