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On off-diagonal decay properties of the generalized Stokes semigroup with bounded measurable coefficients

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Abstract
We investigate off-diagonal decay properties of the generalized Stokes semigroup with bounded measurable coefficients on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$. Such estimates are well-known for elliptic equations in the form of pointwise heat kernel bounds and for elliptic systems in the form of integrated off-diagonal estimates. On our way to unveil this off-diagonal behavior we prove resolvent estimates in Morrey spaces $\mathrm{L}^{2,\nu}(\mathbb{R}^d)$ with $0 \leq \nu < 2$.

Keywords Generalized Stokes semigroup · Stokes operator with bounded measurable coefficients · Off-diagonal estimates

Mathematics Subject Classification 47B12 · 47B90 · 47F10 · 76M30 · 76A05

1 Introduction

In this note we study decay properties of the resolvent as well as the associated semigroup of the generalized Stokes operator A on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$. This operator is formally given by

$$Au = -\operatorname{div}(\mu \nabla u) + \nabla \phi, \quad \operatorname{div}(u) = 0 \quad \text{in} \quad \mathbb{R}^d.$$

Here, the function u denotes a fluid velocity and ϕ denotes the to the generalized Stokes equations associated pressure function. The matrix of coefficients is merely supposed to be essentially bounded and ellipticity is enforced by a Gårding type inequality.

If the elliptic counterpart $Lu = -\operatorname{div}(\mu \nabla u)$ is considered, then certain off-diagonal decay properties of the corresponding heat semigroup are well-known. For

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example, if L represents an elliptic *equation* with *real coefficients*, then the kernel $k_t(\cdot,\cdot)$ of the associated heat semigroup $(\mathrm{e}^{-tL})_{t\geq 0}$ satisfies heat kernel bounds

$$\left|k_t(x,y)\right| \leq Ct^{-\frac{d}{2}}\mathrm{e}^{-c\frac{|x-y|^2}{t}}.$$

It is well-known that if L represents an elliptic *system* with *real/complex coefficients* these heat kernel bounds seize to be valid [4, 7, 9]. The natural substitute for heat kernel bounds for elliptic systems are so-called off-diagonal estimates. The simplest version are L^2 off-diagonal estimates for the heat semigroup, its gradient, or also for L applied to the heat semigroup and are of the form

$$\|\mathrm{e}^{-tL}f\|_{\mathrm{L}^2(F)} + t^{\frac{1}{2}}\|\nabla\mathrm{e}^{-tL}f\|_{\mathrm{L}^2(F)} + t\|L\mathrm{e}^{-tL}f\|_{\mathrm{L}^2(F)} \leq C\mathrm{e}^{-c\frac{\mathrm{dist}\,(E,F)^2}{t}}\|f\|_{\mathrm{L}^2(E)}, \quad (1.1)$$

where $E, F \subset \mathbb{R}^d$ are closed subsets and $f \in \mathrm{L}^2(\mathbb{R}^d)$ has its support in E. Such estimates build the foundation for many deep results in the harmonic analysis of elliptic operators with rough coefficients as can be seen, e.g., in the seminal works on the Kato square root problem [3] as well as on mapping properties of Riesz transforms on L^p-spaces [1] or the well-posedness results of Navier–Stokes like equations with initial data in BMO^{-1} [2] in the spirit of Koch and Tataru [8].

The spirit of how these off-diagonal estimates (1.1) are used is as follows. For example, one might be interested in estimating an expression that involves $\mathrm{e}^{-tL}f$ in some sense. One then decomposes $\mathbb{R}^d$ into carefully chosen disjoint sets, e.g., into annuli of the form $\mathcal{C}_k := \overline{B(x_0, 2^{k+1}r)} \setminus B(x_0, 2^k r)$, $k \in \mathbb{N}$, and $\mathcal{C}_0 := \overline{B(x_0, 2r)}$. Then one would estimate by virtue of (1.1)

$$\begin{aligned}\|\mathrm{e}^{-tL}f\|_{\mathrm{L}^2(B(x_0,r))} &\leq \sum_{k=0}^{\infty}\|\mathrm{e}^{-tL}\chi_{\mathcal{C}_k}f\|_{\mathrm{L}^2(B(x_0,r))} \\ &\leq C\|f\|_{\mathrm{L}^2(B(x_0,2r))} + C\sum_{k=0}^{\infty}\mathrm{e}^{-c\frac{r^2}{t}2^{2k}}\|f\|_{\mathrm{L}^2(B(x_0,2^{k+1}r))}\end{aligned} \quad (1.2)$$

and proceed with the proof in a certain manner, depending on the particular situation.

The question, whose study we want to initiate here, is whether or not the generalized Stokes semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ satisfies off-diagonal decay estimates and if so, how they look like. The main problem is already, that in a calculation of the form (1.2) one multiplies f by a characteristic function. This in general destroys the solenoidality of the function f. Thus, if one wants to perform such an operation, one is urged to think about *how to extend* e^{-tA} to all of $\mathrm{L}^2(\mathbb{R}^d)$. In many situations, the gold standard is to extend e^{-tA} to all of $\mathrm{L}^2(\mathbb{R}^d)$ by studying $\mathrm{e}^{-tA}\mathbb{P}$, where $\mathbb{P}$ denotes the Helmholtz projection on $\mathrm{L}^2(\mathbb{R}^d)$. Thus, in order to imitate the calculation performed in (1.2) one would need that off-diagonal bounds for $\mathrm{e}^{-tA}\mathbb{P}$ are valid. However, estimates of the form

$$\|\mathrm{e}^{-tA}\mathbb{P}f\|_{\mathrm{L}^2(F)} \leq g\Big(\tfrac{\operatorname{dist}(E,F)^2}{t}\Big)\|f\|_{\mathrm{L}^2(E)} \tag{1.3}$$

with $g : [0,\infty) \to [0,\infty)$ satisfying $\lim_{x\to\infty} g(x) = 0$ and f being supported in E are in general *wrong*. The reason is simple: fix any closed subset $E \subset \mathbb{R}^d$ and let $F \subset \mathbb{R}^d$ denote any other closed set that satisfies $\operatorname{dist}(E,F) > 0$. On the one hand, since $(\mathrm{e}^{-tA})_{t\geq 0}$ is strongly continuous on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ with $\mathrm{e}^{-0A}f = f$ one has that

$$\lim_{t\to 0} \|\mathrm{e}^{-tA}\mathbb{P}f\|_{\mathrm{L}^2(F)} = \|\mathbb{P}f\|_{\mathrm{L}^2(F)}.$$

On the other hand (1.3) together with the condition on g implies that $\|\mathbb{P}f\|_{\mathrm{L}^2(F)} = 0$. This implies that $\operatorname{supp}(\mathbb{P}f) \subset E$ whenever $f \in \mathrm{L}^2(\mathbb{R}^d)$ with $\operatorname{supp}(f) \subset E$. As a consequence, the Helmholtz projection would be a local operator, which is known to be wrong.

Thus, in order to establish off-diagonal bounds for the generalized Stokes semigroup, one either needs to find the correct extension of the generalized Stokes semigroup to all of $\mathrm{L}^2(\mathbb{R}^d)$ or one needs to avoid arguments that destroy the solenoidality of f. In particular, this rules out standard proofs of off-diagonal estimates that are used in the elliptic situation as, e.g., Davies' trick [5].

The main result of this note is an estimate of the type (1.2). Let us introduce some notation to state this in a precise form:

Assumption 1.1 The coefficients $\mu = (\mu^{ij}_{\alpha\beta})^d_{\alpha,\beta,i,j=1}$ with $\mu^{ij}_{\alpha\beta} \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{C})$ for all $1 \leq \alpha, \beta, i, j \leq d$ satisfy for some $\mu_\bullet, \mu^\bullet > 0$ the inequalities

$$\operatorname{Re} \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha u_i}\, \mathrm{d}x \geq \mu_\bullet \|\nabla u\|^2_{\mathrm{L}^2} \qquad \big(u \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d)\big) \tag{1.4}$$

and

$$\max_{1\leq i,j,\alpha,\beta\leq d} \|\mu^{ij}_{\alpha\beta}\|_{\mathrm{L}^\infty} \leq \mu^\bullet. \tag{1.5}$$

The operator A is realized on $\mathrm{L}^2_\sigma(\mathbb{R}^d) := \{f \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d) : \operatorname{div}(f) = 0\}$ as follows. Let $\mathrm{H}^1_\sigma(\mathbb{R}^d) := \{f \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d) : \operatorname{div}(f) = 0\}$. Define the sesquilinear form

$$\mathfrak{a} : \mathrm{H}^1_\sigma\big(\mathbb{R}^d\big) \times \mathrm{H}^1_\sigma\big(\mathbb{R}^d\big) \to \mathbb{C}, \quad (u,v) \mapsto \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha v_i}\, \mathrm{d}x$$

and define the domain of A on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ as

$$\mathcal{D}(A) := \left\{ u \in \mathrm{H}^1_\sigma(\mathbb{R}^d) \,:\, \exists f \in \mathrm{L}^2_\sigma(\mathbb{R}^d) \text{ s.t. } \forall v \in \mathrm{H}^1_\sigma(\mathbb{R}^d) \text{ it holds } \mathfrak{a}(u,v) = \int_{\mathbb{R}^d} f \cdot \bar{v} \,\mathrm{d}x \right\}.$$

The main result of this note is the following theorem:

Theorem 1.2 *Let $d \geq 2$ and let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$. For all $\nu \in (0,2)$ there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^d$, $r > 0$, $t > 0$, and $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ it holds*

$$\|\mathrm{e}^{-tA} f\|_{\mathrm{L}^2(B(x_0,r))} + t\|A\mathrm{e}^{-tA} f\|_{\mathrm{L}^2(B(x_0,r))} \leq C\|f\|_{\mathrm{L}^2(B(x_0,2r))} + C\sum_{k=2}^\infty \Big(1 + \frac{2^{2k}r^2}{t}\Big)^{-\frac{\nu}{4}} \|f\|_{\mathrm{L}^2(B(x_0,2^k r))}.$$

Moreover, for all $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ it holds

$$t^{\frac{1}{2}}\|\mathrm{e}^{-tA}\mathbb{P}\mathrm{div}(F)\|_{\mathrm{L}^2(B(x_0,r))} \leq C\|F\|_{\mathrm{L}^2(B(x_0,2r))} + C\sum_{k=2}^\infty \Big(1 + \frac{2^{2k}r^2}{t}\Big)^{-\frac{\nu}{4}} \|F\|_{\mathrm{L}^2(B(x_0,2^k r))}.$$

In both estimates, the constant C only depends on $\mu_\bullet$, $\mu^\bullet$, d, and ν.

As a corollary of Theorem 1.2 one derives the following off-diagonal estimates.

Corollary 1.3 *Let $d \geq 2$ and let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$. For all $\nu \in (0,2)$ there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^d$, $r > 0$, $k_0 \in \mathbb{N}$ with $k_0 \geq 2$, $t > 0$, and $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ with $\operatorname{supp}(f) \subset \overline{B(x_0, 2^{k_0} r)} \setminus B(x_0, 2^{k_0-1} r)$ it holds*

$$\|\mathrm{e}^{-tA} f\|_{\mathrm{L}^2(B(x_0,r))} + t\|A\mathrm{e}^{-tA} f\|_{\mathrm{L}^2(B(x_0,r))} \leq C\Big(1 + \frac{2^{2k_0}r^2}{t}\Big)^{-\frac{\nu}{4}} \|f\|_{\mathrm{L}^2(B(x_0,2^{k_0} r)\setminus B(x_0,2^{k_0-1} r))}.$$

Moreover, for all $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ with $\operatorname{supp}(F) \subset \overline{B(x_0, 2^{k_0} r)} \setminus B(x_0, 2^{k_0-1} r)$ it holds

$$t^{\frac{1}{2}}\|\mathrm{e}^{-tA}\mathbb{P}\mathrm{div}(F)\|_{\mathrm{L}^2(B(x_0,r))} \leq C\Big(1 + \frac{2^{2k_0}r^2}{t}\Big)^{-\frac{\nu}{4}} \|F\|_{\mathrm{L}^2(B(x_0,2^{k_0} r)\setminus B(x_0,2^{k_0-1} r))}.$$

In both estimates, the constant C only depends on $\mu_\bullet$, $\mu^\bullet$, d, and ν.

This article is organized as follows. In Sect. 2 we study the generalized Stokes resolvent problem on the whole space and establish a non-local resolvent estimate. An immediate consequence of this is Corollary 2.4 which states a resolvent estimate in the

Morrey space $\mathrm{L}^{2,\nu}$ for $0 \leq \nu < 2$. Sect. 3 relies on this non-local resolvent analysis and presents non-local off-diagonal decay estimates for the generalized Stokes resolvent. These estimates are transferred in the final Sect. 4 to the generalized Stokes semigroup by using its representation via a Cauchy integral.

2 A non-local resolvent estimate

To establish Theorem 1.2 we prove analogous estimates for the resolvent of A. More precisely, we are going to estimate the solution u to the generalized Stokes resolvent problem

$$\begin{cases} \lambda u - \operatorname{div}(\mu \nabla u) + \nabla \phi = f + \mathbb{P}\operatorname{div}(F) \;\; \text{in } \mathbb{R}^d, \\ \operatorname{div}(u) = 0 \text{ in } \mathbb{R}^d \end{cases} \tag{2.1}$$

for λ in some complex sector $\mathrm{S}_\omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \omega\}$. Using Assumption 1.1 together with the lemma of Lax–Milgram, one finds some $\omega \in (\pi/2, \pi)$ depending on $\mu_\bullet$, $\mu^\bullet$, and d such that (2.1) is uniquely solvable for all $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ and all $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$. In the following, let us denote the solution operator to (2.1) by $(\lambda + A)^{-1}$. The solution u to (2.1) then lies in the space $\mathrm{H}^1_\sigma(\mathbb{R}^d)$ and for all $\theta \in (0, \omega)$ there exists $C > 0$ such that for all $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$, and all $\lambda \in \mathrm{S}_\theta$ it satisfies the resolvent estimates

$$\|\lambda(\lambda + A)^{-1} f\|_{\mathrm{L}^2} + |\lambda|^{\frac{1}{2}} \|\nabla(\lambda + A)^{-1} f\|_{\mathrm{L}^2} + \|A(\lambda + A)^{-1} f\|_{\mathrm{L}^2} \leq C\|f\|_{\mathrm{L}^2} \tag{2.2}$$

and

$$|\lambda|^{\frac{1}{2}} \|(\lambda + A)^{-1}\mathbb{P}\operatorname{div}(F)\|_{\mathrm{L}^2} + \|\nabla(\lambda + A)^{-1}\mathbb{P}\operatorname{div}(F)\|_{\mathrm{L}^2} \leq C\|F\|_{\mathrm{L}^2}. \tag{2.3}$$

The next lemma was proven in [10, Lemma 5.3] and combines different types of Caccioppoli inequalities to account for the non-local pressure.

Lemma 2.1 *Let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$. There exists $\omega \in (\pi/2, \pi)$ such that for all $\theta \in (0, \omega)$, $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$, and $\lambda \in \mathrm{S}_\theta$ the following holds: for $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ defined by $u := (\lambda + A)^{-1}(f + \mathbb{P}\operatorname{div}(F))$ and $x_0 \in \mathbb{R}^d$ and $r_0 > 0$ there exists a decomposition of u of the form $u = u_1 + u_2$ with $u_1 \in \mathrm{H}^1(B(x_0, r_0);\mathbb{C}^d)$ satisfying $\operatorname{div}(u_1) = 0$ and $u_2 \equiv u$ in $\mathbb{R}^d \setminus B(x_0, r_0)$ and there exists $\phi_1 \in \mathrm{L}^2(B(x_0, r_0))$ and $C > 0$ such that for any ball $B \subset \mathbb{R}^d$ of radius $r > 0$ with $2B \subset B(x_0, r_0)$ we have*

$$
\begin{aligned}
&|\lambda|^3 r^2 \int_B |u_2|^2 \,\mathrm{d}x + |\lambda|^2 r^2 \int_B |\nabla u_2|^2 \,\mathrm{d}x \\
&\qquad \leq C\Big\{ \sum_{\ell=0}^{\infty} 2^{-\ell d-\ell} \int_{2^\ell B} \Big(|\lambda u|^2 + |f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x \\
&\qquad\qquad + \int_{2B} |\lambda u_1|^2 \,\mathrm{d}x + \int_{2B} ||\lambda|^{\frac{1}{2}} \phi_1|^2 \,\mathrm{d}x \Big\}.
\end{aligned} \tag{2.4}
$$

Moreover, u_1 and ϕ_1 satisfy for some $C > 0$

$$
\begin{aligned}
&|\lambda| \|u_1\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|\nabla u_1\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|\phi_1\|_{\mathrm{L}^2(B(x_0,r_0))} \\
&\quad \leq C\Big(\|f\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|F\|_{\mathrm{L}^2(B(x_0,r_0))} \Big).
\end{aligned} \tag{2.5}
$$

In both inequalities, the constant C only depends on d, θ, $\mu_\bullet$, and $\mu^\bullet$. Moreover, ω only depends on d, $\mu_\bullet$, and $\mu^\bullet$.

This lemma can be used to prove the following non-local resolvent estimate.

Theorem 2.2 *Let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$. There exists $\omega \in (\pi/2, \pi)$ such that for all $\theta \in (0, \omega)$ and all $\nu \in (0, 2)$ there exists a constant $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$, $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, and $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ the unique solution $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ to (2.1) satisfies*

$$
\sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\lambda u|^2 \,\mathrm{d}x \leq C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x.
$$

Here, the constant C only depends on d, θ, ν, $\mu_\bullet$, and $\mu^\bullet$ and ω only depends on d, $\mu_\bullet$, and $\mu^\bullet$.

Proof We use the decomposition of u from Lemma 2.1 as follows. Fix $k \in \mathbb{N}_0$ and let $\ell_0 \in \mathbb{N}$ to be determined. Let $u_{1,k}$, $u_{2,k}$, and $\phi_{1,k}$ be the functions determined by Lemma 2.1 with $r_0 := 2^{k+\ell_0+1} r$. Now, we proceed by applying Hölder's inequality, then increase the domain of integration, and use Sobolev's embedding to obtain for $q > 1$ with

$$
\frac{1}{2} - \frac{1}{2q} \leq \frac{1}{d} \tag{2.6}
$$

the inequalities

$$
\begin{aligned}
&\int_{B(x_0,2^k r)} |u_{2,k}|^2 \,\mathrm{d}x \\
&\leq |B(x_0,2^k r)|^{1-\frac{1}{q}} \Big(\int_{B(x_0,2^k r)} |u_{2,k}|^{2q} \,\mathrm{d}x \Big)^{\frac{1}{q}} \\
&\leq \frac{|B(x_0,2^k r)|^{1-\frac{1}{q}}}{|B(x_0,2^{k+\ell_0} r)|^{-\frac{1}{q}}} \Big(\fint_{B(x_0,2^{k+\ell_0} r)} |u_{2,k}|^{2q} \,\mathrm{d}x \Big)^{\frac{1}{q}} \\
&\leq C \frac{|B(x_0,2^k r)|^{1-\frac{1}{q}}}{|B(x_0,2^{k+\ell_0} r)|^{1-\frac{1}{q}}} \Big\{ \int_{B(x_0,2^{k+\ell_0} r)} |u_{2,k}|^2 \,\mathrm{d}x + (2^{k+\ell_0} r)^2 \int_{B(x_0,2^{k+\ell_0} r)} |\nabla u_{2,k}|^2 \,\mathrm{d}x \Big\} \\
&= C 2^{-\ell_0 d(1-\frac{1}{q})} \Big\{ \int_{B(x_0,2^{k+\ell_0} r)} |u_{2,k}|^2 \,\mathrm{d}x + (2^{k+\ell_0} r)^2 \int_{B(x_0,2^{k+\ell_0} r)} |\nabla u_{2,k}|^2 \,\mathrm{d}x \Big\}.
\end{aligned}
$$

Notice that the constant $C > 0$ in the previous estimate only depends on d and q. Now, use this estimate together with $u_{2,k} = u - u_{1,k}$ and (2.4) and (2.5) to deduce

$$
\begin{aligned}
&\int_{B(x_0,2^k r)} |\lambda u|^2 \,\mathrm{d}x \\
&\leq 2 \int_{B(x_0,2^k r)} |\lambda u_{1,k}|^2 \,\mathrm{d}x + 2 \int_{B(x_0,2^k r)} |\lambda u_{2,k}|^2 \,\mathrm{d}x \\
&\leq 2 \int_{B(x_0,2^k r)} |\lambda u_{1,k}|^2 \,\mathrm{d}x \\
&+ |\lambda|^2 C 2^{-\ell_0 d(1-\frac{1}{q})} \Big\{ \int_{B(x_0,2^{k+\ell_0} r)} |u_{2,k}|^2 \,\mathrm{d}x + (2^{k+\ell_0} r)^2 \int_{B(x_0,2^{k+\ell_0} r)} |\nabla u_{2,k}|^2 \,\mathrm{d}x \Big\} \\
&\leq 2 \int_{B(x_0,2^k r)} |\lambda u_{1,k}|^2 \,\mathrm{d}x \\
&+ |\lambda|^2 C 2^{-\ell_0 d(1-\frac{1}{q})} \int_{B(x_0,2^{k+\ell_0} r)} |u - u_{1,k}|^2 \,\mathrm{d}x \\
&+ C 2^{-\ell_0 d(1-\frac{1}{q})} \Big\{ \sum_{\ell=0}^{\infty} 2^{-\ell d - \ell} \int_{B(x_0,2^{k+\ell+\ell_0} r)} \big(|\lambda u|^2 + |f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \big) \,\mathrm{d}x \\
&+ \int_{B(x_0,2^{k+\ell_0+1} r)} |\lambda u_{1,k}|^2 \,\mathrm{d}x + \int_{B(x_0,2^{k+\ell_0+1} r)} ||\lambda|^{\frac{1}{2}} \phi_{1,k}|^2 \,\mathrm{d}x \Big\} \\
&\leq C \int_{B(x_0,2^{k+\ell_0+1} r)} |f|^2 \,\mathrm{d}x + C 2^{-\ell_0 d(1-\frac{1}{q})} \int_{B(x_0,2^{k+\ell_0} r)} |\lambda u|^2 \,\mathrm{d}x \\
&+ C 2^{-\ell_0 d(1-\frac{1}{q})} \sum_{\ell=0}^{\infty} 2^{-\ell d - \ell} \int_{B(x_0,2^{k+\ell+\ell_0} r)} \big(|\lambda u|^2 + |f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \big) \,\mathrm{d}x.
\end{aligned}
$$

Now, multiply this inequality by $2^{-\nu k}$ and sum with respect to $k \in \mathbb{N}_0$. This then delivers

$$
\begin{aligned}
&\sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\lambda u|^2 \,\mathrm{d}x \\
&\leq C 2^{-\ell_0 (d - \frac{d}{q} - \nu)} \sum_{k=0}^{\infty} 2^{-\nu(k+\ell_0)} \int_{B(x_0, 2^{k+\ell_0} r)} |\lambda u|^2 \,\mathrm{d}x \\
&+ C 2^{-\ell_0 (d - \frac{d}{q} - \nu)} \sum_{\ell=0}^{\infty} 2^{\ell(\nu - d - 1)} \sum_{k=0}^{\infty} 2^{-\nu(k+\ell+\ell_0)} \int_{B(x_0, 2^{k+\ell+\ell_0} r)} |\lambda u|^2 \,\mathrm{d}x \\
&+ C 2^{\nu(\ell_0+1)} \sum_{k=0}^{\infty} 2^{-\nu(k+\ell_0+1)} \int_{B(x_0, 2^{k+\ell_0+1} r)} |f|^2 \,\mathrm{d}x \\
&+ C 2^{-\ell_0 (d - \frac{d}{q} - \nu)} \sum_{\ell=0}^{\infty} 2^{\ell(\nu - d - 1)} \sum_{k=0}^{\infty} 2^{-\nu(k+\ell+\ell_0)} \int_{B(x_0, 2^{k+\ell+\ell_0} r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x \\
&\leq C 2^{-\ell_0 (d - \frac{d}{q} - \nu)} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\lambda u|^2 \,\mathrm{d}x \\
&+ C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x.
\end{aligned}
$$

Now, in order to conclude that the exponent $d - \frac{d}{q} - \nu$ is positive, we need to require further restrictions to q. One immediately verifies that the positivity of this exponent as well as (2.6) are fulfilled, whenever q satisfies

$$
1 - \frac{2}{d} \leq \frac{1}{q} < 1 - \frac{\nu}{d}. \tag{2.7}
$$

Since $\nu < 2$, such a choice is possible. Thus, fixing q subject to (2.7) allows to choose ℓ_0 large enough so as to absorb the λu-term on the right-hand side to the left-hand side. Thus, there exists $C > 0$ such that

$$
\sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\lambda u|^2 \,\mathrm{d}x \leq C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x. \qquad \square
$$

As a corollary we get that the generalized Stokes operator satisfies resolvent estimates with respect to the Morrey space norm of $\mathrm{L}^{2,\nu}(\mathbb{R}^d;\mathbb{C}^d)$ for all $0 \leq \nu < 2$. The definition of this Morrey space is the following:

Definition 2.3 Let $0 \leq \nu < d$ and $m \in \mathbb{N}$. Define the Morrey space $\mathrm{L}^{2,\nu}(\mathbb{R}^d;\mathbb{C}^m)$ as the vector space of all functions $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^m)$ with finite Morrey space norm

$$
\|u\|_{\mathrm{L}^{2,\nu}} := \sup_{\substack{x_0 \in \mathbb{R}^d \\ r > 0}} \bigg(r^{-\nu} \int_{B(x_0,r)} |u|^2 \,\mathrm{d}x \bigg)^{\frac{1}{2}}.
$$

Corollary 2.4 *Let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$. There exists $\omega \in (\pi/2, \pi)$ such that for all $\theta \in (0, \omega)$ and all $\nu \in [0,2)$ there exists a constant $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$, $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d) \cap \mathrm{L}^{2,\nu}(\mathbb{R}^d;\mathbb{C}^d)$, and $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d}) \cap \mathrm{L}^{2,\nu}(\mathbb{R}^d;\mathbb{C}^{d\times d})$ the unique solution $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ to (2.1) satisfies*

$$\|\lambda u\|_{\mathrm{L}^{2,\nu}} \leq C\Big(\|f\|_{\mathrm{L}^{2,\nu}} + |\lambda|^{\frac{1}{2}} \|F\|_{\mathrm{L}^{2,\nu}}\Big).$$

Here, the constant C only depends on d, θ, ν, $\mu_\bullet$, and $\mu^\bullet$ and ω only depends on d, $\mu_\bullet$, and $\mu^\bullet$.

Proof Fix $x_0 \in \mathbb{R}^d$ and $r > 0$. The estimate in Theorem 2.2 readily gives for some $\nu < \nu' < 2$

$$\begin{aligned}\int_{B(x_0,r)} |\lambda u|^2 \, \mathrm{d}x &\leq C \sum_{k=0}^\infty 2^{-\nu' k} \int_{B(x_0,2^k r)} \Big(|f|^2 + \big||\lambda|^{\frac{1}{2}} F\big|^2\Big) \, \mathrm{d}x \\ &\leq C r^\nu \Big(\|f\|^2_{\mathrm{L}^{2,\nu}} + |\lambda| \|F\|^2_{\mathrm{L}^{2,\nu}}\Big).\end{aligned}$$

Division by r^ν then delivers the desired estimate. □

3 L^2 off-diagonal decay for the resolvent

This section is dedicated to prove a counterpart of Theorem 1.2 for the resolvent of A. For this purpose, we introduce another sesquilinear form, which is connected to the Stokes problem in a ball but with Neumann boundary conditions.

Let $B \subset \mathbb{R}^d$ denote a ball and let

$$\mathcal{L}^2_\sigma(B) := \big\{f \in \mathrm{L}^2(B;\mathbb{C}^d) : \operatorname{div}(f) = 0 \text{ in the sense of distributions}\big\}$$

and

$$\mathcal{H}^1_\sigma(B) := \big\{f \in \mathrm{H}^1(B;\mathbb{C}^d) : \operatorname{div}(f) = 0\big\}.$$

Now, define the sesquilinear form

$$\mathfrak{b}_B : \mathcal{H}^1_\sigma(B) \times \mathcal{H}^1_\sigma(B) \to \mathbb{C}, \quad (u, v) \mapsto \sum_{\alpha,\beta,i,j=1}^d \int_B \mu^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha v_i} \, \mathrm{d}x.$$

We abuse the notation and denote the same sesquilinear form but with domain $\mathrm{H}^1(B;\mathbb{C}^d) \times \mathrm{H}^1(B;\mathbb{C}^d)$ again by $\mathfrak{b}_B$.

An application of Assumption 1.1 and the lemma of Lax–Milgram implies the existence of $\omega \in (\pi/2, \pi)$ such that for all $\lambda \in \mathrm{S}_\omega$, $f \in \mathcal{L}^2_\sigma(B)$, and $F \in \mathrm{L}^2(B;\mathbb{C}^{d\times d})$ the equation

$$\lambda \int_B u \cdot \overline{v} \, \mathrm{d}x + \mathfrak{b}_B(u,v) = \int_B f \cdot \overline{v} \, \mathrm{d}x - \sum_{\alpha,\beta=1}^{d} \int_B F_{\alpha\beta} \overline{\partial_\alpha v_\beta} \, \mathrm{d}x \quad \big(v \in \mathcal{H}^1_\sigma(B)\big) \tag{3.1}$$

is uniquely solvable for some $u \in \mathcal{H}^1_\sigma(B)$. Moreover, by [10, Remark 5.2], there exists a pressure function $\phi \in \mathrm{L}^2(B)$ such that

$$\begin{aligned} &\lambda \int_B u \cdot \overline{v} \, \mathrm{d}x + \mathfrak{b}_B(u,v) - \int_B \phi \, \overline{\mathrm{div}(v)} \, \mathrm{d}x \\ &\qquad = \int_B f \cdot \overline{v} \, \mathrm{d}x - \sum_{\alpha,\beta=1}^{d} \int_B F_{\alpha\beta} \overline{\partial_\alpha v_\beta} \, \mathrm{d}x \quad \big(v \in \mathrm{H}^1(B;\mathbb{C}^d)\big) \end{aligned} \tag{3.2}$$

holds. Furthermore, for all $\theta \in (0,\omega)$ there exists $C > 0$ depending only on d, θ, $\mu_\bullet$, and $\mu^\bullet$ such that for all $\lambda \in \mathrm{S}_\omega$, $f \in \mathcal{L}^2_\sigma(B)$, and $F \in \mathrm{L}^2(B;\mathbb{C}^{d\times d})$ it holds

$$\|\lambda u\|_{\mathrm{L}^2(B)} + |\lambda|^{\frac12} \|\nabla u\|_{\mathrm{L}^2(B)} + |\lambda|^{\frac12} \|\phi\|_{\mathrm{L}^2(B)} \leq C\big(\|f\|_{\mathrm{L}^2(B)} + |\lambda|^{\frac12} \|F\|_{\mathrm{L}^2(B)}\big). \tag{3.3}$$

To proceed, we cite some results from [10]. The first result is a non-local Caccioppoli inequality for the generalized Stokes resolvent and can be found in [10, Thm. 1.2].

Theorem 3.1 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$. Then there exists $\omega \in (\pi/2,\pi)$ such that for all $\theta \in (0,\omega)$ and all $0 < \nu < d+2$ there exists $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$, $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, and $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ the solution $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ to*

$$\lambda \int_{\mathbb{R}^d} u \cdot \overline{v} \, \mathrm{d}x + \mathfrak{a}(u,v) = \int_{\mathbb{R}^d} f \cdot \overline{v} \, \mathrm{d}x - \sum_{\alpha,\beta=1}^{d} \int_{\mathbb{R}^d} F_{\alpha\beta} \overline{\partial^\alpha v_\beta} \, \mathrm{d}x \qquad \big(v \in \mathrm{H}^1_\sigma(\mathbb{R}^d)\big)$$

satisfies for all balls $B = B(x_0,r)$ and all sequences $(c_k)_{k\in\mathbb{N}_0}$ with $c_k \in \mathbb{C}^d$

$$\begin{aligned} &|\lambda| \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |u|^2 \, \mathrm{d}x + \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\nabla u|^2 \, \mathrm{d}x \\ &\qquad \leq \frac{C}{r^2} \sum_{k=0}^{\infty} 2^{-(\nu+2)k} \int_{B(x_0,2^{k+1} r)} |u + c_k|^2 \, \mathrm{d}x + |\lambda| \sum_{k=0}^{\infty} |c_k| 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} |u| \, \mathrm{d}x \\ &\qquad + \frac{C}{|\lambda|} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} |f|^2 \, \mathrm{d}x + C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} |F|^2 \, \mathrm{d}x. \end{aligned}$$

The constant ω only depends on $\mu_\bullet$, $\mu^\bullet$, and d and C depends on $\mu_\bullet$, $\mu^\bullet$, d, θ, and ν.

The second result is an estimate on the pressure function ϕ that appears in (2.1) and can be found in [10, Lemma 2.1]. To formulate this lemma, we adopt the notation $\mathcal{C}_k := B(x_0,2^k r) \setminus B(x_0,2^{k-1} r)$ for $k \in \mathbb{N}$ and write $\phi_{\mathcal{C}_k}$ for the mean value of ϕ on the set $\mathcal{C}_k$.

Lemma 3.2 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$. Let $\lambda \in \mathbb{C}$ and let for $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ and $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ the functions $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and $\phi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ solve*

$$\begin{cases} \lambda u - \operatorname{div}\mu\nabla u + \nabla\phi = f + \operatorname{div}(F) & \text{in } \mathbb{R}^d, \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}^d \end{cases}$$

in the sense of distributions. Let $x_0 \in \mathbb{R}^d$ and $r > 0$ and let $\mathcal{C}_0$ denote the ball $B(x_0, r)$. Then there exists a constant $C > 0$ depending only on $\mu^\bullet$ and d such that for all $k \in \mathbb{N}$ we have

$$\begin{aligned} &\Big(\int_{\mathcal{C}_k} |\phi - \phi_{\mathcal{C}_k}|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\quad\leq C\bigg(\sum_{\ell=0}^{k-2} 2^{\frac{d}{2}(\ell-k)}\Big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\Big) \\ &\quad+ \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell-k| \leq 1}} \Big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\Big) \\ &\quad+ \sum_{\ell=k+2}^{\infty} 2^{(\frac{d}{2}+1)(k-\ell)}\Big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\Big)\bigg). \end{aligned}$$

The final preparatory result we need is a local Caccioppoli inequality that includes the pressure function.

Lemma 3.3 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$. Then there exists $\omega \in (\pi/2, \pi)$ such that for all $\theta \in (0, \omega)$ there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^d$, $r > 0$, $c \in \mathbb{C}$, and all solutions $u \in \mathcal{H}^1_\sigma(B(x_0, 2r))$ and $\phi \in \mathrm{L}^2(B(x_0, 2r))$ (in the sense of distributions) to*

$$\begin{cases} \lambda u - \operatorname{div}\mu\nabla u + \nabla\phi = 0 & \text{in } B(x_0, 2r), \\ \operatorname{div}(u) = 0 & \text{in } B(x_0, 2r) \end{cases}$$

satisfy

$$\begin{aligned} &|\lambda| \int_{B(x_0,r)} |u|^2 \,\mathrm{d}x + \int_{B(x_0,r)} |\nabla u|^2 \,\mathrm{d}x \\ &\qquad\leq \frac{C}{r^2} \int_{B(x_0,2r)} |u|^2 \,\mathrm{d}x + \frac{C}{|\lambda| r^2} \int_{B(x_0,2r)} |\phi - c|^2 \,\mathrm{d}x. \end{aligned}$$

The constant C only depends on d, θ, $\mu_\bullet$, and $\mu^\bullet$.

Proof Let $\eta \in C_c^\infty(B(x_0, 2r))$ with $\eta \equiv 1$ in $B(x_0, r)$, $0 \leq \eta \leq 1$, and $\|\nabla \eta\|_{L^\infty} \leq 2/r$. Applying [10, Lemma 5.1] with $c_1 = c$ and $c_2 = 0$ implies that

$$|\lambda| \int_{B(x_0,2r)} |u\eta|^2 \, dx + \int_{B(x_0,2r)} |\nabla[u\eta]|^2 \, dx \leq \frac{C}{r^2} \int_{B(x_0,2r)} |u|^2 \, dx + \frac{4}{r} \Big(\int_{B(x_0,2r)\setminus B(x_0,r)} |\phi - c|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{B(x_0,2r)} |u\eta|^2 \, dx \Big)^{\frac{1}{2}}.$$

Use Young's inequality to estimate

$$\frac{4}{r} \Big(\int_{B(x_0,2r)\setminus B(x_0,r)} |\phi - c|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{B(x_0,2r)} |u\eta|^2 \, dx \Big)^{\frac{1}{2}} \leq \frac{8}{|\lambda| r^2} \int_{B(x_0,2r)\setminus B(x_0,r)} |\phi - c|^2 \, dx + \frac{|\lambda|}{2} \int_{B(x_0,2r)} |u\eta|^2 \, dx.$$

The lemma follows by absorbing the $u\eta$-term to the left-hand side and by using the properties of η. Finally, we would like to mention that the proof of [10, Lemma 5.1] follows the standard proof that is used to establish the Caccioppoli inequality for elliptic systems and this is well-known. □

The following theorem presents L^2 off-diagonal type estimates for the resolvent operators.

Theorem 3.4 *There exists $\omega \in (\pi/2, \pi)$ such that for all $\theta \in (0, \omega)$ and all $\nu \in (0, 2)$ there exists a constant $C > 0$ such that for all $x_0 \in \mathbb{R}^d$, $r > 0$, $\lambda \in S_\theta$, $f \in L^2_\sigma(\mathbb{R}^d)$, and $F \in L^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ the unique solution $u \in H^1_\sigma(\mathbb{R}^d)$ to (2.1) satisfies*

$$\int_{B(x_0,r)} |\lambda u|^2 \, dx + \int_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} \nabla u|^2 \, dx \leq C \int_{B(x_0,2r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \, dx + C \sum_{k=2}^\infty \Big(\frac{1}{1 + |\lambda| 2^{2k} r^2} \Big)^{\frac{\nu}{2}} \int_{B(x_0,2^k r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \, dx.$$

Here, the constant C only depends on d, θ, ν, $\mu_\bullet$, and $\mu^\bullet$ and ω only depends on d, $\mu_\bullet$, and $\mu^\bullet$.

Proof Fix $f \in L^2_\sigma(\mathbb{R}^d)$, $F \in L^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$, and $\lambda \in S_\theta$. Define $u := (\lambda + A)^{-1}(f + \mathbb{P}\operatorname{div}(F))$ and let $\phi \in L^2_{loc}(\mathbb{R}^d)$ be the associated pressure such that u and ϕ solve (2.1). Let $x_0 \in \mathbb{R}^d$ and $r > 0$. In the following, we consider two cases.

Let λ and r be such that $|\lambda| r^2 \leq 1$. In this case, Theorem 2.2 yields the estimate

$$\int_{B(x_0,r)} |\lambda u|^2 \,\mathrm{d}x \le C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} \Big(|f|^2 + \big||\lambda|^{\frac{1}{2}} F\big|^2 \Big) \,\mathrm{d}x$$
$$\le 2^{\frac{\nu}{2}} C \sum_{k=0}^{\infty} \Big(\frac{1}{1+|\lambda| 2^{2k} r^2} \Big)^{\frac{\nu}{2}} \int_{B(x_0,2^k r)} \Big(|f|^2 + \big||\lambda|^{\frac{1}{2}} F\big|^2 \Big) \,\mathrm{d}x.$$

Thus, it is left to consider the case $|\lambda| r^2 > 1$. In this case, define $g := f|_{B(x_0,2r)}$ and $G := F|_{B(x_0,2r)}$. The definition of $\mathcal{L}^2_\sigma(B(x_0,2r))$ implies that $g \in \mathcal{L}^2_\sigma(B(x_0,2r))$. Then, there exists $u_1 \in \mathcal{H}^1_\sigma(B(x_0,2r))$ such that for all $v \in \mathcal{H}^1_\sigma(B(x_0,2r))$ it holds

$$\lambda \int_{B(x_0,2r)} u_1 \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_{B(x_0,2r)}(u_1, v)$$
$$= \int_{B(x_0,2r)} g \cdot \overline{v} \,\mathrm{d}x - \sum_{\alpha,\beta=1}^{d} \int_{B(x_0,2r)} G_{\alpha\beta} \cdot \overline{\partial_\alpha v_\beta} \,\mathrm{d}x.$$

Let $\phi_1 \in \mathrm{L}^2(B(x_0,2r))$ denote the associated pressure. By (3.3) we find that

$$\|\lambda u_1\|_{\mathrm{L}^2(B(x_0,2r))} + |\lambda|^{\frac{1}{2}} \|\nabla u_1\|_{\mathrm{L}^2(B(x_0,2r))} + |\lambda|^{\frac{1}{2}} \|\phi_1\|_{\mathrm{L}^2(B(x_0,2r))}$$
$$\le C\Big(\|f\|_{\mathrm{L}^2(B(x_0,2r))} + |\lambda|^{\frac{1}{2}} \|F\|_{\mathrm{L}^2(B(x_0,2r))} \Big). \tag{3.4}$$

Notice that the constant $C > 0$ only depends on d, θ, $\mu_\bullet$, and $\mu^\bullet$. In particular, it does not depend on x_0 and r.

Now, define $u_2 := u - u_1$ and $\phi_2 := \phi - \phi_1$. Thus, to prove the desired result, we only have to control u_2 in $B(x_0, r)$. By definitions of all functions, we find that

$$\lambda \int_{B(x_0,2r)} u_2 \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_{B(x_0,2r)}(u_2, v) - \int_{B(x_0,2r)} \phi_2 \,\overline{\operatorname{div}(v)} \,\mathrm{d}x = 0$$

for all $v \in \mathrm{H}^1_0\big(B(x_0,2r);\mathbb{C}^d\big)$, so that by virtue of Lemma 3.3 we have

$$\int_{B(x_0,r)} |\lambda u_2|^2 \,\mathrm{d}x + \int_{B(x_0,r)} \big||\lambda|^{\frac{1}{2}} \nabla u_2\big|^2 \,\mathrm{d}x$$
$$\le \frac{C|\lambda|}{r^2} \int_{B(x_0,2r)} |u_2|^2 \,\mathrm{d}x + \frac{C}{r^2} \int_{B(x_0,2r)\setminus B(x_0,r)} |\phi_2 - \phi_{B(x_0,2r)\setminus B(x_0,r)}|^2 \,\mathrm{d}x.$$

Now, use that $u_2 = u - u_1$ and $\phi_2 = \phi - \phi_1$ followed by (3.4), Lemma 3.2, and $\nu < 2 < 2 + d$ to deduce that

$$
\begin{aligned}
&\int_{B(x_0,r)} |\lambda u_2|^2 \,\mathrm{d}x + \int_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} \nabla u_2|^2 \,\mathrm{d}x \\
&\leq \frac{C|\lambda|}{r^2} \int_{B(x_0,2r)} |u_2|^2 \,\mathrm{d}x + \frac{C}{r^2} \int_{B(x_0,2r)} |\phi_1|^2 \,\mathrm{d}x \\
&\quad + \frac{C}{r^2} \int_{B(x_0,2r)\setminus B(x_0,r)} |\phi - \phi_{B(x_0,2r)\setminus B(x_0,r)}|^2 \,\mathrm{d}x \\
&\leq \frac{C}{|\lambda| r^2} \int_{B(x_0,2r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x + \frac{C|\lambda|}{r^2} \int_{B(x_0,2r)} |u|^2 \,\mathrm{d}x \\
&\quad + \frac{C}{r^2} \int_{B(x_0,2r)\setminus B(x_0,r)} |\phi - \phi_{B(x_0,2r)\setminus B(x_0,r)}|^2 \,\mathrm{d}x \\
&\leq \frac{C}{|\lambda| r^2} \int_{B(x_0,2r)} |f|^2 \,\mathrm{d}x + \frac{C}{|\lambda| r^2} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} ||\lambda|^{\frac{1}{2}} F|^2 \,\mathrm{d}x \\
&\quad + C \bigg(\frac{|\lambda|}{r^2} \int_{B(x_0,2r)} |u|^2 \,\mathrm{d}x + \frac{1}{r^2} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} |\nabla u|^2 \,\mathrm{d}x \bigg).
\end{aligned}
$$

Now, employ Theorem 3.1 to the second term on the right-hand side followed by the non-local resolvent estimate in Theorem 2.2 so as to get

$$
\begin{aligned}
&\int_{B(x_0,r)} |\lambda u_2|^2 \,\mathrm{d}x + \int_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} \nabla u_2|^2 \,\mathrm{d}x \\
&\leq \frac{C}{|\lambda| r^2} \int_{B(x_0,2r)} |f|^2 \,\mathrm{d}x + \frac{C}{|\lambda|^2 r^4} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x \\
&\quad + \frac{C}{|\lambda| r^2} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1} r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x.
\end{aligned}
$$

Finally, using that $|\lambda| r^2 > 1$ and $\nu < 2$, we get

$$
\begin{aligned}
&\int_{B(x_0,r)} |\lambda u_2|^2 \,\mathrm{d}x + \int_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} \nabla u_2|^2 \,\mathrm{d}x \\
&\qquad\qquad \leq C \sum_{k=2}^{\infty} \Big(\frac{1}{|\lambda| 2^{2k} r^2} \Big)^{\frac{\nu}{2}} \int_{B(x_0,2^k r)} \Big(|f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \Big) \,\mathrm{d}x. \qquad \square
\end{aligned}
$$

Remark 3.5 We just proved slightly more than stated in Theorem 3.4. Indeed, if $|\lambda| r^2 > 1$, we proved further estimates on ∇u that are given by

$$
|\lambda|^{\frac{1}{2}} \| \nabla (\lambda + A)^{-1} f \|_{\mathrm{L}^2(B(x_0,2r))} \leq C \| f \|_{\mathrm{L}^2(B(x_0,2r))} + C \sum_{k=2}^{\infty} \Big(\frac{1}{|\lambda| 2^{2k} r^2} \Big)^{\frac{\nu}{4}} \| f \|_{\mathrm{L}^2(B(x_0,2^k r))}
$$

and

$$\|\nabla(\lambda + A)^{-1}\mathbb{P}\mathrm{div}(F)\|_{\mathrm{L}^2(B(x_0,2r))} \leq C\|F\|_{\mathrm{L}^2(B(x_0,2r))} + C\sum_{k=2}^{\infty}\Big(\frac{1}{|\lambda|2^{2k}r^2}\Big)^{\frac{\nu}{4}}\|F\|_{\mathrm{L}^2(B(x_0,2^k r))}.$$

4 Estimates on the generalized Stokes semigroup

Since A satisfies the resolvent estimates

$$|\lambda|\|(\lambda + A)^{-1}f\|_{\mathrm{L}^2} \leq C\|f\|_{\mathrm{L}^2} \qquad (\lambda \in \mathrm{S}_\omega),$$

for some $\omega \in (\pi/2, \pi)$ the generalized Stokes operator $-A$ is the infinitesimal generator of a bounded analytic semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ which is represented via the Cauchy integral formula

$$\mathrm{e}^{-tA} = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_t}\mathrm{e}^{t\lambda}(\lambda + A)^{-1}\,\mathrm{d}\lambda \qquad (t > 0). \tag{4.1}$$

Here, the path γ_t runs through $\partial(B(0,t^{-1}) \cup S_\vartheta)$ for some $\vartheta \in (\pi/2, \omega)$ in a counterclockwise manner. This representation by the Cauchy integral formula allows to transfer estimates on the resolvent to estimates on the semigroup. For example, it is well-known that the estimates (2.2) and (2.3) used within (4.1) directly yield for all $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$, and $t > 0$ the semigroup estimates

$$\|\mathrm{e}^{-tA}f\|_{\mathrm{L}^2} + t^{\frac{1}{2}}\|\nabla\mathrm{e}^{-tA}f\|_{\mathrm{L}^2} + t\|A\mathrm{e}^{-tA}f\|_{\mathrm{L}^2} \leq C\|f\|_{\mathrm{L}^2} \tag{4.2}$$

and

$$t^{\frac{1}{2}}\|\mathrm{e}^{-tA}\mathbb{P}\mathrm{div}(F)\|_{\mathrm{L}^2} + t\|\nabla\mathrm{e}^{-tA}\mathbb{P}\mathrm{div}(F)\|_{\mathrm{L}^2} \leq C\|F\|_{\mathrm{L}^2}. \tag{4.3}$$

The following proof of Theorem 1.2 shows that this transfer of estimates is also valid for the resolvent estimates established in Theorem 3.4.

Proof of Theorem 1.2 Let $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ and $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$. Combining the conclusion of Theorem 3.4 with (4.1) directly yields for $x_0 \in \mathbb{R}^d$ and $r > 0$ that

$$\begin{aligned}
&\left\|\mathrm{e}^{-tA}(f+\mathbb{P}\mathrm{div}(F))\right\|_{\mathrm{L}^2(B(x_0,r))}\\
&\leq \frac{1}{2\pi}\int_{\gamma_t}\mathrm{e}^{t\mathrm{Re}(\lambda)}\|(\lambda+A)^{-1}(f+\mathbb{P}\mathrm{div}(F))\|_{\mathrm{L}^2(B(x_0,r))}\,|\mathrm{d}\lambda|\\
&\leq C\int_{\gamma_t}\mathrm{e}^{t\mathrm{Re}(\lambda)}\Big(\|f\|_{\mathrm{L}^2(B(x_0,2r))}+\||\lambda|^{\frac{1}{2}}F\|_{\mathrm{L}^2(B(x_0,2r))}\Big)\frac{|\mathrm{d}\lambda|}{|\lambda|}\\
&+C\sum_{k=2}^{\infty}\int_{\gamma_t}\mathrm{e}^{t\mathrm{Re}(\lambda)}\Big(\frac{1}{1+|\lambda|2^{2k}r^2}\Big)^{\frac{\nu}{4}}\Big(\|f\|_{\mathrm{L}^2(B(x_0,2^kr))}+\||\lambda|^{\frac{1}{2}}F\|_{\mathrm{L}^2(B(x_0,2^kr))}\Big)\frac{|\mathrm{d}\lambda|}{|\lambda|}.
\end{aligned}$$

Now, perform the substitution $\lambda t=\mu$ and use that for $\mu\in\gamma_1$ one has

$$\frac{1}{1+\frac{|\mu|2^{2k}r^2}{t}}\leq\frac{1}{1+\frac{2^{2k}r^2}{t}}.$$

This readily yields that

$$\begin{aligned}
&\left\|\mathrm{e}^{-tA}(f+\mathbb{P}\mathrm{div}(F))\right\|_{\mathrm{L}^2(B(x_0,r))}\\
&\leq C\sum_{k=0}^{\infty}\left(\frac{1}{1+\frac{2^{2k}r^2}{t}}\right)^{\frac{\nu}{4}}\int_{\gamma_1}\mathrm{e}^{\mathrm{Re}(\mu)}\Big(\|f\|_{\mathrm{L}^2(B(x_0,2^kr))}+t^{-\frac{1}{2}}\||\mu|^{\frac{1}{2}}F\|_{B(x_0,2^kr)}\Big)\frac{|\mathrm{d}\mu|}{|\mu|}
\end{aligned}$$

and thus already the desired estimate.

To estimate $tA\mathrm{e}^{-tA}(f+\mathbb{P}\mathrm{div}(F))$, notice that

$$\begin{aligned}
A\mathrm{e}^{-tA}=\frac{1}{2\pi\mathrm{i}}\int_{\gamma_t}\mathrm{e}^{t\lambda}A(\lambda+A)^{-1}\,\mathrm{d}\lambda&=\frac{1}{2\pi\mathrm{i}}\int_{\gamma_t}\mathrm{e}^{t\lambda}\big(\mathrm{Id}-\lambda(\lambda+A)^{-1}\big)\,\mathrm{d}\lambda\\
&=-\frac{1}{2\pi\mathrm{i}}\int_{\gamma_t}\lambda\mathrm{e}^{t\lambda}(\lambda+A)^{-1}\,\mathrm{d}\lambda.
\end{aligned}$$

Now, the desired estimate follows analogously as above. □

Remark 4.1 If we assume that $r^2/t>1$, then all $\lambda\in\gamma_t$ satisfy $|\lambda|r^2>1$ so that in this case the estimates from Remark 3.5 together with the proof of Theorem 1.2 yield the following gradient estimate on the generalized Stokes semigroup: there exists a constant $C>0$ such that for all $f\in\mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F\in\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$, and all $t>0$ we have

$$t^{\frac{1}{2}}\left\|\nabla\mathrm{e}^{-tA}f\right\|_{\mathrm{L}^2(B(x_0,r))}\leq C\|f\|_{\mathrm{L}^2(B(x_0,2r))}+C\sum_{k=2}^{\infty}\left(\frac{2^{2k}r^2}{t}\right)^{-\frac{\nu}{4}}\|f\|_{\mathrm{L}^2(B(x_0,2^kr))}$$

and

$$\left\|\nabla\mathrm{e}^{-tA}F\right\|_{\mathrm{L}^2(B(x_0,r))}\leq C\|F\|_{\mathrm{L}^2(B(x_0,2r))}+C\sum_{k=2}^{\infty}\left(\frac{2^{2k}r^2}{t}\right)^{-\frac{\nu}{4}}\|F\|_{\mathrm{L}^2(B(x_0,2^kr))}.$$

Proof of Corollary 1.3 We distinguish two cases. Assume first that $2^{2k_0}r^2/t < 1$. Then by using the global L^2-estimates (4.2), we find that

$$\begin{aligned}\|\mathrm{e}^{-tA}f\|_{\mathrm{L}^2(B(x_0,r))} + t\|A\mathrm{e}^{-tA}f\|_{\mathrm{L}^2(B(x_0,r))} &\leq C\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}\\ &\leq 2^{\frac{\nu}{4}}\Big(1+\frac{2^{2k_0}r^2}{t}\Big)^{-\frac{\nu}{4}}\|f\|_{\mathrm{L}^2(B(x_0,2^{k_0}r)\setminus B(x_0,2^{k_0-1}r))}.\end{aligned}$$

Now, assume that $2^{2k_0}r^2/t \geq 1$. Then Theorem 1.2 implies that

$$\begin{aligned}&\|\mathrm{e}^{-tA}f\|_{\mathrm{L}^2(B(x_0,r))} + t\|A\mathrm{e}^{-tA}f\|_{\mathrm{L}^2(B(x_0,r))}\\ &\quad\leq C\sum_{k=k_0}^{\infty}\Big(1+\frac{2^{2k}r^2}{t}\Big)^{-\frac{\nu}{4}}\|f\|_{\mathrm{L}^2(B(x_0,2^{k_0}r)\setminus B(x_0,2^{k_0-1}r))}\\ &\quad\leq C\Big(\frac{2^{2k_0}r^2}{t}\Big)^{-\frac{\nu}{4}}\sum_{k=k_0}^{\infty}2^{-\frac{\nu}{2}(k-k_0)}\|f\|_{\mathrm{L}^2(B(x_0,2^{k_0}r)\setminus B(x_0,2^{k_0-1}r))}\\ &\quad\leq C\Big(1+\frac{2^{2k_0}r^2}{t}\Big)^{-\frac{\nu}{4}}\|f\|_{\mathrm{L}^2(B(x_0,2^{k_0}r)\setminus B(x_0,2^{k_0-1}r))}.\end{aligned}$$

To estimate the terms involving $\mathrm{e}^{-tA}\mathbb{P}\mathrm{div}(F)$ proceed similarly, but by employing (4.3) in the first case and Theorem 1.2 in the second case. We omit further details. □

Remark 4.2 In this closing remark we shortly discuss optimality issues of the parameter ν. This parameter emerges in Theorem 2.2 and a direct consequence is Corollary 2.4. Notice that Hölder's inequality implies that

$$\mathrm{L}^p(\mathbb{R}^d) \subset \mathrm{L}^{2,\nu}(\mathbb{R}^d) \quad \text{for} \quad p = \frac{2d}{d-\nu}. \tag{4.4}$$

Moreover, the function $|x|^{-\alpha}\chi_{B(0,1)}$ is contained in $\mathrm{L}^{2,\nu}(\mathbb{R}^d)$ if and only if $\alpha \leq (d-\nu)/2$. Similarly, $|x|^{-\alpha}\chi_{B(0,1)}$ is contained in $\mathrm{L}^p(\mathbb{R}^d)$ if and only if $p\alpha < d$. If p and ν are related as in (4.4), this results in the similar condition $\alpha < (d-\nu)/2$. Thus, in this sense $\mathrm{L}^{2,\nu}(\mathbb{R}^d)$ and $\mathrm{L}^p(\mathbb{R}^d)$ encode a similar singular behavior of functions.

Now, in the case $d \geq 3$ and for elliptic operators in divergence form, it is known from the examples in [6, 7], that for each

$$p \notin [2d/(d+2), 2d/(d-2)]$$

there exist essentially bounded and elliptic coefficients μ such that the elliptic operator $-\mathrm{div}(\mu\nabla\cdot)$ does not satisfy resolvent bounds on $\mathrm{L}^p(\mathbb{R}^d;\mathbb{C}^d)$. In particular, for $p > 2d/(d-2)$ it is possible to find $2 < \nu < d$ such that $p = 2d/(d-\nu)$. As this L^p-space contains functions that exhibit a similar singular behavior as functions in $\mathrm{L}^{2,\nu}(\mathbb{R}^d)$ this indicates — but only on philosophical grounds — that for each $2 < \nu < d$ resolvent bounds for elliptic operators might also fail on $\mathrm{L}^{2,\nu}(\mathbb{R}^d)$. Thus, the same conclusion might be true for the generalized Stokes system as well. This is an indication, that the bound $\nu < 2$ in Sect. 2 is optimal among the class of all L^∞-coefficients μ.

In contrast to that, in Sects. 3 and 4 the condition $\nu < 2$ seems to be of a technical nature and it is not clear to the author whether this condition is already optimal or could be improved any further. Another bound on ν that appears in Theorem 3.1 is the bound $\nu < d + 2$. The author would hope that if one could find a proof of non-local off-diagonal estimates without the use of Theorem 2.2, that then the bound on ν could be improved to $\nu < d + 2$. However, this is unclear at the moment.

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