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Products of locally cyclic groups

Bernhard Amberg and Yaroslav Sysak

Abstract. We consider groups of the form $G = AB$ with two locally cyclic subgroups A and B. The structure of these groups is determined in the cases when A and B are both periodic or when one of them is periodic and the other is not. Together with a previous study of the case where A and B are torsion-free, this gives a complete classification of all groups that are the product of two locally cyclic subgroups. As an application, it is shown that the Prüfer rank of a periodic product of two locally cyclic subgroups does not exceed 3, and this bound is sharp. It is also proved that a product of a finite number of pairwise permutable periodic locally cyclic subgroups is a locally supersoluble group. This generalizes a well-known theorem of B. Huppert for finite groups.

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1. Introduction. Let the group $G = AB$ be the product of two subgroups A and B, i.e. $G = \{ab \mid a \in A, b \in B\}$. It was proved by N. Itô that the group G is metabelian if the subgroups A and B are abelian (see [1, Theorem 2.1.1]). This result laid the foundation for a systematic study of groups of the form $G = AB$ with various conditions on the subgroups A and B. In particular, it follows directly from Itô's result that every periodic group $G = AB$ with abelian subgroups A and B is locally finite. It is also well-known that the group $G = AB$ with cyclic subgroups A and B is supersoluble and abelian-by-finite ([1, Lemma 7.4.6]). Furthermore, a detailed description of the structure of the group $G = AB$ with torsion-free locally cyclic subgroups A and B was obtained by the second author in [9].

The aim of this paper is to describe the structure of groups which are products of two locally cyclic subgroups in the periodic and the mixed case. Altogether this gives a complete answer to [1, Question 15].

Theorem 1.1. *Let the periodic group $G = AB$ be the product of two locally cyclic subgroups A and B. Then G contains uniquely determined locally cyclic normal subgroups S and T and a locally nilpotent subgroup $H = A^*B^*$ with $A^* \le A$ and $B^* \le B$ such that*

$$G = (S \times T) \rtimes H = (S \times A^*)(T \times B^*)$$

where $\pi(S) \cap \pi(A^*) = \pi(T) \cap \pi(B^*) = \emptyset$, $S = [S, B^*]$, and $T = [T, A^*]$.

We recall that a group G has finite Prüfer rank $r = r(G)$ (special rank in the sense of Mal'cev in Russian terminology) if each finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property. Clearly a group is of rank 1 if and only if it is locally cyclic. It is also known that every finite p-group of the form $G = AB$ with cyclic subgroups A and B has rank at most 2 for p odd [5, Satz 8] and at most 3 for $p = 2$ [6, Theorem 5.1]. The following consequence of Theorem 1.1 gives an exact upper bound for the Prüfer rank of the product $G = AB$ of two periodic locally cyclic subgroups A and B.

Corollary 1.2. *If $G = AB$ is a periodic group with locally cyclic subgroups A and B, then the Prüfer rank of G does not exceed* 3.

It should be noted that this result is also new for arbitrary finite groups of the form $G = AB$ with cyclic subgroups A and B.

Finally, the following theorem extends a well-known result of B. Huppert on the supersolubility of finite groups which are products of pairwise permutable cyclic subgroups (see [5, Satz 34] or [4, Satz VI.10.3]). This gives, in particular, an affirmative answer to [2, Question 1].

Theorem 1.3. *Let the group $G = A_1A_2 \cdots A_n$ be the product of finitely many pairwise permutable periodic locally cyclic subgroups $A_1, \dots, A_n$. Then G is a periodic locally supersoluble group.*

2. Preliminaries. In what follows $G = AB$ is a group with locally cyclic subgroups A and B.

Lemma 2.1. *If $G = AB$ is an infinite p-group, then up to a permutation of the factors A and B the subgroup A is quasicyclic and one of the following statements hold:*

(1) *$G = A \times B$ with B cyclic or quasicyclic;*
(2) *$p = 2$ and $G = A \rtimes \langle b \rangle$ for some element $b \in G$ with $a^b = a^{-1}$ for all $a \in A$;*
(3) *$p = 2$ and $G = A\langle b \rangle$ for some element $b \in G$ with $b^{2^n} = 1$ for some $n > 1$, $b^{2^{n-1}} \in A$ and $a^b = a^{-1}$ for all $a \in A$.*

Proof. Clearly without loss of generality we may assume that the subgroup A is infinite and so quasicyclic. Then the subgroup B is either cyclic or quasicyclic. Since in the latter case the group G is abelian by [1, Lemma 7.4.4], the subgroup A is complemented in G and hence statement (1) holds.

Let the group G be non-abelian. Then the subgroup B is cyclic and so A as a quasicyclic p-subgroup of finite index in G must be normal and non-central in

G. In particular, B induces on A a non-trivial cyclic p-group of automorphisms. On the other hand, since quasicyclic p-groups have no automorphisms of order $p > 2$, it follows that $p = 2$. But then $B = \langle b \rangle$ with $b^{2^n} = 1$ for some $n \geq 1$ and b induces on A an automorphism of order 2 that inverts the elements of A. In particular, if $A \cap B = 1$, we obtain statement (2). In the second case, $A \cap B = \langle b^{2^{n-1}} \rangle$ and hence statement (3) holds, as claimed. □

Corollary 2.2. *If $G = AB$ is a p-group and C, D are subgroups of A and B, respectively, then $CD = DC$ and so CD is a subgroup of G.*

Proof. This is known if G is finite (see [5, Satz 3]), and follows from Lemma 2.1 in the general case. □

Lemma 2.3. *If the group $G = AB$ is periodic and H is a finite subgroup of G, then H is contained in a finite subgroup E of G such that $E = (A \cap E)(B \cap E)$. In particular, the group G is locally supersoluble.*

Proof. Since the group H is finite, there exist finite subsets C of A and D of B such that H is contained in the set CD. Then the subgroups $A_0 = \langle C \rangle$, $B_0 = \langle D \rangle$, and $\langle C, D \rangle$ are finite, because the group G is locally finite. Furthermore, it follows from [1, Lemma 1.2.3] that the normalizer $N_G(\langle A_0, B_0 \rangle)$ contains a finite subgroup E such that $\langle A_0, B_0 \rangle \leq E = (A \cap E)(B \cap E)$. Since the subgroup E is supersoluble by [1, Lemma 7.4.6] and $H \subseteq CD \subseteq \langle C, D \rangle = \langle A_0, B_0 \rangle$, the lemma is proved. □

As a direct consequence of this lemma, we have

Corollary 2.4. *If the group $G = AB$ is periodic, then there exists an ascending series of finite subgroups $1 = G_0 < G_1 < \cdots < G_n < \cdots G$ such that $G_n = (A \cap G_n)(B \cap G_n)$ for each $n > 0$ and $G = \bigcup_{n=1}^{\infty} G_n$.*

If G is a periodic group and π is a set of primes, then a subgroup H of G is called a π-subgroup provided that all prime divisors of the order of any element of H are contained in π. By a Sylow π-subgroup of G we simply mean a maximal π-subgroup G_π of G which will be denoted by G_p if $\pi = \{p\}$.

Lemma 2.5. *Let $G = AB$ be a periodic group and π a set of primes. Then the following statements hold.*

(1) *If A_π and B_π are Sylow π-subgroups of A and B, respectively, then $G_\pi = A_\pi B_\pi$ is a Sylow π-subgroup of G and*
$$N_G(G_\pi) = N_A(G_\pi) N_B(G_\pi).$$
(2) *If p, q are primes with $p > q$, then a Sylow p-subgroup G_p is normalized by a Sylow q-subgroup G_q. In particular, $G_{\{p,q\}} = G_p G_q$ for any primes p and q.*

Proof. (1) It follows from [1, Lemma 1.3.2] that in the notation of Corollary 2.4 for each $n \geq 1$ the set $(A_\pi \cap G_n)(B_\pi \cap G_n)$ is a Hall π-subgroup of G_n. Therefore $G_\pi = \bigcup_{n=1}^{\infty}(A_\pi \cap G_n)(B_\pi \cap G_n)$ is a Sylow π-subgroup of G. Since $A_\pi = \bigcup_{n=1}^{\infty}(A_\pi \cap G_n)$ and $B_\pi = \bigcup_{n=1}^{\infty}(B_\pi \cap G_n)$, this implies $G_\pi = A_\pi B_\pi$. In addition, applying [1, Lemma 1.2.2], we have $N_G(G_\pi) = N_A(G_\pi) N_B(G_\pi)$.

(2) If $\pi = \{p, q\}$ for some primes $p > q$, then $A_{\{p,q\}} = A_p \times A_q$, $B_{\{p,q\}} = B_p \times B_q$, and $G_{\{p,q\}} = (A_p \times A_q)(B_p \times B_q)$. As G and so its subgroup $G_{\{p,q\}}$ is locally supersoluble by Lemma 2.3, the Sylow p-subgroup $G_p = A_pB_p$ is normal in $G_{\{p,q\}}$. Therefore, $G_{\{p,q\}} = (A_p \times A_q)(B_p \times B_q) = G_pA_qB_q = G_pG_q$, as claimed. □

A Sylow basis of a periodic group G is defined to be a complete set $\mathbf{S} = \{G_p\}$ of Sylow p-subgroups of G, one for each prime p, such that $G_pG_q = G_qG_p$ for all pairs p, q of primes, and $G_\pi = \langle G_p \mid p \in \pi\rangle$ is a Sylow π-subgroup of G for each set π of primes. As is well-known (see [3, Lemma 2.1]), every countable periodic locally soluble group possesses Sylow bases. The basis normalizer $N_G(\mathbf{S})$ of a Sylow basis $\mathbf{S}$ of G is by definition the intersection $N_G(\mathbf{S}) = \bigcap_p N_G(G_p)$ of the normalizers $N_G(G_p)$ of the Sylow p-subgroups G_p of $\mathbf{S}$ for all p.

Lemma 2.6. *Let the group $G = AB$ be periodic and $G_p = A_pB_p$ for each prime p. Then $\mathbf{S} = \{G_p\}$ is a Sylow basis of G. Moreover, if $A^* = \bigcap_p N_A(G_p)$ and $B^* = \bigcap_p N_B(G_p)$, then $N_G(\mathbf{S}) = A^*B^*$.*

Proof. Indeed, by Lemma 2.5, the set $\mathbf{S} = \{G_p\}$ forms a Sylow basis of G and $N_G(G_p) = N_A(G_p)N_B(G_p)$ for every p by [1, Lemma 1.2.2]. Therefore, $N_G(\mathbf{S}) = \bigcap_p N_G(G_p) = \bigcap_p N_A(G_p)N_B(G_p)$ and it is easy to check that $\bigcap_p N_A(G_p)N_B(G_p) = (\bigcap_p N_A(G_p))(\bigcap_p N_B(G_p)) = A^*B^*$ (see [1, Lemma 1.1.2]). Therefore $N_G(\mathbf{S}) = A^*B^*$. □

The following lemma is a direct consequence of a well-known result of L. Kovacs (see [7, Theorem 2]).

Lemma 2.7. *Let G be a finite soluble group, π a set of primes and H a Hall π-subgroup of G. If for each $p \in \pi$ the Prüfer rank of a Sylow p-subgroup of G does not exceed r, then H is a subgroup of rank at most $r + 1$.*

Proof. Indeed, it is obvious that if K is a subgroup of H, then every Sylow subgroup of K is generated by r elements. Therefore, K can be generated by $r + 1$ elements by the result of Kovacs cited above. Thus every subgroup of H is generated by $r + 1$ elements and so H has rank at most $r + 1$. □

3. Proof of Theorem 1.1. First of all, it follows from Lemma 2.1 that for each prime p every Sylow p-subgroup of $G = AB$ satisfies the minimal condition for subgroups. Therefore, G satisfies the minimal condition for p-subgroups for all primes p. Since the group G is metabelian by Ito's theorem, the locally nilpotent residual R of G is contained in its derived subgroup G' and so it is abelian. It was proved by Hartley [3, Theorem 1] that in this case $G = R \rtimes H$, where H is any basis normalizer of G. In particular, by Lemma 2.6, we can take $H = A^*B^*$.

It is easy to see that the subgroup H is locally nilpotent and contains the center $Z(G)$ of G. Furthermore, $G' = R \times H'$ and so H' is a normal subgroup of G. Since R is abelian and $N_G(H) = N_R(H) \times H$, it follows that $N_R(H) \leq Z(G) \leq H$. Therefore, $N_R(H) = 1$ and hence $H = N_G(H)$. We now show that the subgroup $H = A^*B^*$ commutes with both subgroups A and B.

Indeed, put $S = R \cap \langle A, B^*\rangle$ and $T = R \cap \langle A^*, B\rangle$. It is clear that S and T are normal subgroups of G, $\langle A, B^*\rangle = S \rtimes H$, and $\langle A^*, B\rangle = T \rtimes H$. On the other hand, as $G = AB$, we have also $\langle A, B^*\rangle = AB_1$ and $\langle A^*, B\rangle = A_1B$ for some subgroups A_1 and B_1 such that $A^* \le A_1 \le A$ and $B^* \le B_1 \le B$. From here, we deduce $AB_1 \cap A_1B = A_1B_1 = (S \cap T) \rtimes H$. Moreover, passing to the factor group G/H', we may restrict ourselves to the case when the subgroup $H = A^*B^*$ is abelian. Then the subgroups A^* and B^* centralize S and T, respectively, and so the subgroup H centralizes the intersection $S \cap T$. Since $H = N_G(H)$, this implies $S \cap T = 1$. Thus $A_1B_1 = H = A^*B^*$ and hence $\langle A, B^*\rangle = AH = AB^*$ and $\langle A^*, B\rangle = BH = A^*B$, as asserted.

Further, taking into account the equalities $AB^* = S \rtimes H$ and $H = A^*B^*$, we conclude that the subgroup A^* centralizes S, because $[A^*, S] \le H' \cap S = 1$. Since in this case the normalizer $N_S(B^*)$ is contained in $N_G(H) = H$, we have $N_S(B^*) = 1$. Therefore, every element $b \in B^*$ induces on S an automorphism leaving only the identity element fixed. But then every element of S can be written in the form $b^{-1}s^{-1}bs$ with $s \in S$ and hence $S = [B^*, S]$. Similarly, using the equality $A^*B = T \rtimes H$, we derive $T = [A^*, T]$.

Finally, we put $A_0 = A \cap BS$ and $B_0 = AT \cap B$. Clearly from the equalities $G = AB$, $AB^* = S \rtimes H$, and $A^*B = T \rtimes H$, it follows that $G = S \rtimes A^*B = T \rtimes AB^*$, $A = A^* \times A_0$, and $B = B^* \times B_0$. Therefore, $S \rtimes B = A_0B$ and $T \rtimes A = AB_0$. Furthermore, if S_p is a Sylow p-subgroup of S, then $S_p \rtimes B = (A_0 \cap S_pB)B$ and if $S_p \neq 1$, then $A_0 \cap S_pB \neq 1$. Since the subgroup A_0 is locally cyclic, this implies that S is locally cyclic and $\pi(S)$ is contained in $\pi(A_0)$. Moreover, as A^* and A_0 are subgroups of the locally cyclic subgroup A, it also follows that $\pi(A^*) \cap \pi(A_0) = \emptyset$. Similarly, using the equality $T \rtimes A = AB_0$, we obtain $\pi(T) = \pi(B_0)$ and $\pi(B^*) \cap \pi(B_0) = \emptyset$. □

Proof of Corollary 1.2. By Corollary 2.4, we may restrict ourselves to the case in which the group $G = AB$ is finite. By Theorem 1.1, G then contains cyclic normal subgroups S and T and a nilpotent subgroup $H = A^*B^*$ with $A^* \le A$ and $B^* \le B$ such that

$$G = (S \times T) \rtimes H = (S \times A^*)(T \times B^*),$$

where $\pi(S) \cap \pi(A^*) = \pi(T) \cap \pi(B^*) = \emptyset$, $S = [S, B^*]$, and $T = [T, A^*]$. In particular, if for some prime p the subgroup H contains a non-cyclic Sylow p-subgroup P, then S and T are p'-subgroups of G.

Since $P = A_pB_p$ with $A_p = A \cap P$ and $B_p = B \cap P$, both subgroups A_p and B_p are non-trivial and so $p \notin \pi(S) \cup \pi(T)$. Therefore, if G_p is a non-cyclic Sylow p-subgroup of G, $S_p = G_p \cap S$, and $T_p = G_p \cap T$, then up to conjugation G_p coincides with one of the following subgroups of G: $P = A_pB_p$, $T_p \rtimes A_p$, $S_p \rtimes B_p$, and $S_p \rtimes T_p$. In particular, the Sylow p-subgroups of G have rank at most 2 for $p > 2$ (see [4, Satz III.11.5]) and at most 3 for $p = 2$ (see [6, Theorem 5.1]). We now show that G is in fact a group of rank at most 3.

Indeed, suppose the contrary and let the group G contain a subgroup K whose minimal number of generators $d(K)$ is at least 4. Since the Sylow subgroups of odd orders in K have rank at most 2 by what was noted above, each Sylow 2-subgroup Q of K must have rank 3 by Lemma 2.7. It is clear that $Q = K \cap P$ for a Sylow 2-subgroup $P = A_2B_2$ of G. As the group G is metabelian, the derived subgroup P' is abelian and normal in G. Moreover, P' has rank at most 2 by [6, Theorems 4.2 and 4.3(e)].

Put $N = P' \cap Q$. As $Q = K \cap P$, we have $N = K \cap P'$ and so N is an abelian normal subgroup of K with rank at most 2. In addition, $N \neq 1$, because otherwise the subgroup Q is embedded in the factor group P/P' whose rank is equal to 2. On the other hand, since the factor group $Q/N = Q/P' \cap Q$ is isomorphic to the factor group $QP'/P' \leq P/P'$, it is abelian of rank at most 2. Therefore, the Sylow 2-subgroups of the factor group K/N have rank at most 2 and hence K/N has rank at most 3 by Lemma 2.7. In particular, $d(K/N) < d(K) = 4$ and so N is not contained in the Frattini subgroup $\Phi(K)$ of K, because otherwise $d(K/N) = d(K)$. Clearly, passing to the factor group $K/\Phi(K)$, we may assume that $\Phi(K) = 1$. Then the normal subgroup N is complemented in K and so in Q by [4, Hilfsatz 3.4.4]. Moreover, since $\Phi(N) = 1$ by [4, Hilfsatz 3.3.b], the subgroup N is elementary abelian of order at most 4.

Let L be a complement to N in K and $M = Q \cap L$. Then M is a Sylow 2-subgroup of L and $Q = N \times M$, so that M is abelian with $d(M) \leq 2$. Since the subgroup K is supersoluble, its maximal subgroup U of odd order is normal in K and centralizes N. Therefore, $K = U \times Q = (U \times N) \rtimes M$ and hence the centralizer $C_N(M)$ is a non-trivial central subgroup of K. As $\Phi(K) = 1$, the subgroup $C_N(M)$ is complemented in K and thus in Q. From this, it follows that $Q = M \times N$ is abelian and N is a central subgroup of K. Therefore, $K = (U \rtimes M) \times N$ and the subgroup $U \rtimes M$ is three-generated by Lemma 2.7. This means that there exist elements u, v, w of U and x, y, z of M such that $U \rtimes M = \langle ux, vy, wz\rangle$. Then M modulo U is generated by x, y, z. In particular, if $d(M) = 1$, without loss of generality we may assume that $M = \langle x\rangle$ and $y = z = 1$, so that $U \rtimes M = \langle ux, v, w\rangle$. In the case $d(M) = 2$ we can take $M = \langle x, y\rangle$ and $z = 1$. Then $U \rtimes M = \langle ux, vy, w\rangle$.

Finally, since $Q = M \times N$ is a 2-subgroup of rank 3 as noted above, only two cases are possible: either $M = \langle x\rangle$ and $N = \langle a, b\rangle$ has order 4 or $M = \langle x, y\rangle$ with $x \neq 1 \neq y$ and $N = \langle a\rangle$ is of order 2. Therefore, $K = \langle ux, av, bw\rangle$ in the first case and $K = \langle ux, vy, aw\rangle$ in the second case. In both cases $d(K) < 4$ and this contradiction completes the proof. □

4. Products of a periodic and a torsion-free locally cyclic group. Recall that a group G has finite torsion-free rank if it has a series of finite length whose factors are either periodic or infinite cyclic. The number $r_0(G)$ of infinite cyclic factors in such a series is an invariant of G called its torsion-free rank. In this section, we describe the structure of the group $G = AB$ with locally cyclic subgroups A and B, the first of which is periodic and the other non-trivial torsion-free. Clearly $r_0(B) = 1$ and we note first that $r_0(G) = 1$.

Lemma 4.1. *Let $G = AB$ be a group with subgroups A and B such that A is periodic abelian and B is non-trivial torsion-free locally cyclic. Then $r_0(G) = 1$.*

Proof. It was proved by Zaitsev [11, Theorem 3.7] (see also [1, Lemma 7.1.2]) that there exists a non-trivial normal subgroup of G contained in A or B. Therefore G has the normal series $A_0 < A_0B_0 < G$ in which A_0 is the core of A in G and B_0 is the core of B in G modulo A_0. As is easily seen, the factors A_0 and G/A_0B_0 are periodic and the factor group A_0B_0/A_0 is isomorphic to B_0. Thus $r_0(G) = r_0(B) = 1$, as claimed. □

The following lemma is a consequence of the well-known theorem of I. Schur on the finiteness of the derived subgroup of a group that is finite over its center (see [8, Corollary to Theorem 4.12]).

Lemma 4.2. *If a group G contains a central subgroup Z such that the factor group G/Z is locally finite, then the derived subgroup of G is locally finite.*

Theorem 4.3. *Let the group $G = AB$ be the product of two locally cyclic subgroups A and B such that A is periodic and B is non-trivial torsion-free. Then one of the following statements holds.*

(1) *The subgroup A is normal in G and so $G = A \rtimes B$;*
(2) *$A = A_1\langle a\rangle$ with $a^2 \in A_1$, the subgroup A_1 is normal in G and $G = (A_1 \rtimes B)\langle a\rangle$ with $b^a = b^{-1}\phi(b)$ for all $b \in B$, where $\phi : B \to A_1$ is a derivation of B into A_1.*

Proof. It is easy to see that each periodic normal subgroup H of G is contained in A, because $AH = A(AH \cap B)$ and $AH \cap B = 1$. Therefore the core $A_1 = \cap_{g\in G}A^g$ of A in G is the maximal periodic normal subgroup of G.

Assume first that $A_1 = 1$ and let B_1 be the core of B in G. Then $B_1 \neq 1$ by the theorem of Zaitsev noted above and so the factor group G/B_1 is periodic, because it is the product of two periodic subgroups AB_1/B_1 and B/B_1. Moreover, since the centralizer $C_G(B_1)$ of B_1 in G contains B, the group G induces on B_1 a periodic group of automorphisms which is isomorphic to the factor group $A/C_A(B_1)$. As is well-known, a periodic group of automorphisms of any locally cyclic torsion-free group is of order 2. Therefore the order of $A/C_A(B_1)$ does not exceed 2 and hence either $A = C_A(B_1)$ or $A = C_A(B_1)\langle a\rangle$ with $a \in A$ and $a^2 \in C_A(B_1)$.

On the other hand, since the centralizer $C_G(B_1) = C_A(B_1)B$ is normal in G and periodic over B_1, its derived subgroup $C_G(B_1)'$ is periodic by Lemma 4.2 and normal in G. Therefore $C_G(B_1)' \leq A_1 = 1$ and hence $C_G(B_1) = C_A(B_1) \times B$. But then again $C_A(B_1)$ is normal in G and so $C_A(B_1) = 1$. Thus in the case $A_1 = 1$ we have either $A = 1$ and $G = B$ or $A = \langle a\rangle$ with $a^2 = 1$ and $G = B \rtimes \langle a\rangle$ with $b^a = b^{-1}$ for all $b \in B$.

Finally, returning now to the general case, we derive that either $G = A \rtimes B$ or $G = (A_1 \times B)\langle a\rangle)$ with $b^a = \phi(b)b^{-1}$ for every $b \in B$ and some element $\phi(b) \in A_1$. Moreover, since $\phi(bc)(bc)^{-1} = (bc)^a = b^ac^a = (\phi(b)b^{-1})(\phi(c)c^{-1}) = (\phi(b)\phi(c)^b)(bc)^{-1}$, it follows that $\phi(bc) = \phi(b)\phi(c)^b$ for any $b, c \in B$. The latter means in particular that the mapping $\phi : B \to A_1$ is a derivation of B into A_1, as claimed. □

5. Products of finitely many periodic locally cyclic groups. A well-known theorem of Huppert cited in the introduction says that every finite group of the form $G = A_1A_2 \cdots A_n$ with pairwise permuting cyclic subgroups A_i for $1 \leq i \leq n$ is supersoluble. This result was later extended to products of pairwise permutable locally cyclic Chernikov groups by Tomkinson [10]. He proved that in this case $G = A_1A_2, \ldots, A_n$ is a locally supersoluble Chernikov group. In this section, we generalize this result to products of arbitrary periodic locally cyclic groups. Recall that a group is said to be hyperabelian (respectively, hypercyclic) if it has an ascending series of normal subgroups with abelian (respectively cyclic) factors.

Lemma 5.1. *Let $G = A_1A_2 \cdots A_n$ be the product of pairwise permutable periodic locally cyclic subgroups A_i. If the set $\pi = \bigcup_{i=1}^{n} \pi(A_i)$ is finite, p is the largest prime in π, P_i is the Sylow p-subgroup of A_i, and Q_i is the p-complement to P_i in A_i for each $1 \leq i \leq n$, then G is a π-group, $P = P_1P_2 \cdots P_n$ is a normal Sylow p-subgroup of G, and $Q = Q_1Q_2 \cdots Q_n$ is a p-complement to P in G.*

Proof. Since each of the A_i is a subgroup of Prüfer rank 1, the group $G = A_1A_2 \cdots A_n$ is hyperabelian of finite Prüfer rank by [2, Theorem 3.1]. Therefore, arguing by induction on n and applying [1, Corollary 3.2.7], and [2, Lemma 3.2], we derive that G is a π-group, $P = P_1P_2 \cdots P_n$ is a Sylow p-subgroup of G and $Q = Q_1Q_2 \cdots Q_n$ is a complement to P in G. Moreover, taking into account that the subgroups A_iA_j are locally supersoluble by Lemma 2.3, we conclude that ${P_i}^{A_j} \leq P$ for all i, j and so P is a normal subgroup of G. □

Lemma 5.2. *Let $G = A_1A_2 \cdots A_n$ be the product of pairwise permutable locally cyclic subgroups A_i. If the group G is periodic and the set $\pi(G)$ is finite, then G is locally supersoluble.*

Proof. Since $\pi(G)$ is finite, every locally cyclic subgroup A_i is a Chernikov group, i.e. a finite extension of a direct product of finitely many quasicyclic subgroups. Therefore G is a locally supersoluble Chernikov group by [10, Theorem B]. □

Proof of Theorem 1.3. Let $G = A_1A_2 \cdots A_n$ be the product of pairwise permutable periodic locally cyclic subgroups A_i. Then G is a periodic group by Lemma 5.1. If the set $\pi(G)$ is finite, then the group G is locally supersoluble hypercyclic by Lemma 5.2. In the other case the set $\pi(G)$ is infinite and thus it can be presented as a union $\pi(G) = \bigcup_{i=1}^{\infty} \pi_i$ of finite subsets π_i such that $\pi_i \subset \pi_{i+1}$ for all $i \geq 1$. Let P_{ij} be the Sylow π_i-subgroup of A_j for $1 \leq j \leq n$ and $G_i = P_{i1}P_{i2} \cdots P_{in}$. Then G_i is a Sylow π_i-subgroup of G by Lemma 5.1 which is locally supersoluble as a group for each $i \geq 1$ by Lemma 5.2. Since $G = \bigcup_{i=1}^{\infty} G_i$, the group G is also locally supersoluble, as claimed. □

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Bernhard Amberg
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
55099 Mainz
Germany
e-mail: amberg@uni-mainz.de

Yaroslav Sysak
Institute of Mathematics
Ukrainian National Academy of Sciences
Kiev 01601
Ukraine
e-mail: sysak@imath.kiev.ua

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