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Shimura subvarieties in the Prym locus of ramified Galois coverings

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Abstract
We study Shimura (special) subvarieties in the moduli space $A_{p,D}$ of complex abelian varieties of dimension p and polarization type D. These subvarieties arise from families of covers compatible with a fixed group action on the base curve such that the quotient of the base curve by the group is isomorphic to $\mathbb{P}^1$. We give a criterion for the image of these families under the Prym map to be a special subvariety and, using computer algebra, obtain 210 Shimura subvarieties contained in the Prym locus.

Keywords Prym variety · Prym map · Galois covering

Mathematics Subject Classification 14H30 · 14H40

1 Introduction

In [4], E. Colomobo, P. Frediani, A. Ghigi and M. Penegini have extensively studied Shimura curves of PEL type in A_g, contained generically in the Prym locus (see also the papers [2,3] by E. Colomobo and P. Frediani). The general set-up is as follows: Let R_g be the scheme of isomorphism classes $[C, \eta]$, for C a smooth projective curve of genus g and $\eta \in \mathrm{Pic}^0(C)$ a 2-torsion element, i.e., $\eta \neq \mathcal{O}_C$ but $\eta^2 = \mathcal{O}_C$. The line bundle η determines an (unramified) étale double cover $h : \tilde{C} \to C$ and there is an induced norm map $\mathrm{Nm} : \mathrm{Pic}^0(\tilde{C}) \to \mathrm{Pic}^0(C)$. The Prym variety associated to $[C, \eta]$ is defined to be the connected component of $\ker \mathrm{Nm}$ containing the origin and is denoted by $P(C, \eta)$ or $P(\tilde{C}, C)$. In a similar way, one can construct a Prym variety for ramified covers. Consider the scheme $R_{g,2}$

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parametrizing triples $[C, B, \eta]$ up to isomorphism, where C is a smooth projective curve of genus g, η a line bundle on C of degree 1, and B a reduced divisor in the linear series $|\eta^2|$ corresponding to a double covering $\pi : \tilde{C} \to C$ ramified over B. The assignment $[C, \eta] \mapsto P(\tilde{C}, C)$ (resp. $[C, B, \eta] \mapsto P(\tilde{C}, C)$) defines a map $\mathscr{P} : R_g \to A_g$ (resp. $\mathscr{P} : R_{g,2} \to A_g$). This goes under the name of the *Prym map*. In [4] authors give examples of one-parameter families (C_t, η_t) ($t \in T = \mathbb{P}^1 \setminus \{0, 1, \infty\}$) for which the image under the Prym map parametrizes Shimura curves in A_g. These curves are contained in the Prym loci corresponding to unramified étale double covers and to double covers ramified at two points. More precisely, the authors consider a family of Galois covers $\tilde{C}_t \to \mathbb{P}^1$ with Galois group $\tilde{G}$ and a central involution σ such that the double covering $\tilde{C}_t \to \tilde{C}_t/\langle\sigma\rangle$ is either étale or ramified over exactly two distinct points. By the theory of coverings, the Galois covering $\tilde{C}_t \to \mathbb{P}^1$ is determined by an epimorphism $\tilde{\theta} : \Gamma_r \to \tilde{G}$ with branch points $t_1, \ldots, t_r \in \mathbb{P}^1$. Here Γ_r is isomorphic to the fundamental group of $\mathbb{P}^1 \setminus \{t_1, \ldots, t_r\}$. Varying the branch points, we get a family $R(\tilde{G}, \tilde{\theta}, \sigma) \subset R_g$ (the image of T in R_g mentioned above). The paper [4] then gives examples of families $R(\tilde{G}, \tilde{\theta}, \sigma)$ for which the Zariski closure $\overline{\mathscr{P}(R(\tilde{G}, \tilde{\theta}, \sigma))}$ of the image under the Prym map is a Shimura curve in A_g.

In the paper [5], the first author together with P. Frediani, investigated the occurrence of Shimura curves arising from families of Prym varieties of double covers that are ramified over more than two points. Note that in this case the Prym variety is not principally polarized in general. The paper [5] is thus a generalization of the paper [4]. The subsequent work [7] of the present authors together with P. Frediani investigated the same problem for higher dimensional Shimura varieties contained in these loci.

In this paper we generalize the aforementioned papers in two directions: We consider families of Prym varieties of arbitrary Galois covers of curves (not necessarily double covers), while we also get higher-dimensional as well as 1-dimensional families. More precisely to a finite Galois covering $f : \widetilde{C} \to C$ we associate a Prym variety $P(\widetilde{C}/C)$. The Prym variety is of dimension $p = \widetilde{g} - g$, where $\widetilde{g}$, g are the genera of $\widetilde{C}$, C respectively. Furthermore, it is an abelian variety of certain polarization type D, see Sect. 3. So it determines a point in the moduli space $A_{p,D}$ of complex abelian varieties of dimension p and polarization type D. The stack parametrizing families of the above covers of curves will be denoted by $R(H, g, r)$, where H is the deck group of the covering $f : \widetilde{C} \to C$, $g = g(C_t)$ is the genus of the base curve and r is the number of branch points of the covering.

We consider the following families of curves: We fix a finite group $\widetilde{G}$ and a normal subgroup $H \subseteq \widetilde{G}$. The families that we consider here are families $\widetilde{C}_t \to \mathbb{P}^1$ of $\widetilde{G}$-Galois covers of $\mathbb{P}^1$ branched in s distinct points. Let $T_s \subset (\mathbb{A}^1_{\mathbb{C}})^s$ be the be the complement of the big diagonals, i.e., $T_s = \{(t_1, \ldots, t_s) \in (\mathbb{A}^1_{\mathbb{C}})^s \mid t_i \neq t_j \forall i \neq j\}$. Note that by sending 3 points to $0, 1, \infty$, one sees that T_s has complex dimension $s - 3$. By varying the branch points we obtain a family $f : \mathcal{C} \to T_s$ of covers of $\mathbb{P}^1$. This family gives rise to the family $\widetilde{C}_t \to C_t = \widetilde{C}_t/H$ in $R(H, g, r)$ and the corresponding family of Prym varieties $P(\widetilde{C}_t/C_t)$. The image of T_s in $R(H, g, r)$, which we again denote by T_s, is of dimension $s - 3$. The Prym map behaves well in families and we are interested in the Zariski closure of the image $Z = \overline{\mathscr{P}(T_s)}$ under the Prym map which is a subvariety of $A_{p,D}$. For computational reasons, the case where $\widetilde{G}$ is abelian is of great importance for us. Therefore, in Sect. 3, we explain an alternative construction of the abelian covers of $\mathbb{P}^1$ to that given in [4]. Note that the Prym variety and the Prym map of abelian and metabelian covers have been studied in [9]. The H-action and its eigenspaces on the cohomology and also the eigenspaces of the whole group $\widetilde{G}$ acting on these spaces are useful for our computations.

In Sect. 4 we point out that the moduli space $A_{p,D}$ has the structure of a *Shimura variety*. We find families for which the subvariety Z is a special (or Shimura) subvariety of $A_{p,D}$. We introduce conditions (B), (B1) and (B2) under which the subvariety Z is special. Using computer algebra we investigate these condition and find 210 examples satisfying them, see the table on page 21. We also work out in detail some important examples of the table. In addition to families of abelian covers, we find some families with $\widetilde{G}$ non-abelian. Furthermore, our approach yields also higher dimensional special families of Prym varieties in $A_{p,D}$. Note that in [2] the authors give upper bounds for the dimension of a germ of a totally geodesic submanifold, and hence of a special subvariety in the Prym locus.

2 Prym map and the Prym variety

To a given finite covering $f : \widetilde{C} \to C$ between non-singular projective algebraic curves, one can associate an abelian variety, the so-called *Prym variety*. The map f induces a *norm map*

$$\mathrm{Nm}_f : \mathrm{Pic}^0(\widetilde{C}) \to \mathrm{Pic}^0(C)$$
$$\sum a_i p_i \mapsto \sum a_i f(p_i)$$

The Prym variety associated to f is defined as $P(f) = P(\widetilde{C}/C) = (\ker \mathrm{Nm}_f)^0$, i.e. the connected component of the kernel of Nm_f containing the origin. Identifying Pic^0 with the Jacobian, one sees that the canonical (principal) polarization of $\mathrm{Jac}(\widetilde{C})$ restricts to a polarization on $P(f)$.

Classically, f is a double covering which is étale or branched at exactly two points. It is known that these are the only cases in which $P(f)$ is principally polarized. In fact, the restriction of the canonical polarization of $\mathrm{Jac}(\widetilde{C})$ to $P(f)$ is twice a principal polarization. In general the type D of the polarization on $P(f)$ depends on the topological structure of the covering map f, see [1].

Let H be a finite group with $n = |H|$. Suppose C is a compact Riemann surface of genus g. Let $t := \{t_1, \dots, t_r\}$ be an r-tuple of distinct points in C. Set $U_t := C \backslash \{t_1, \dots, t_r\}$. The fundamental group $\pi_1(U_t, t_0)$ has a presentation $\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_r \mid \prod_1^r \gamma_i \prod_1^g [\alpha_j, \beta_j] = 1 \rangle$. Here $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ are simple loops in U_t which only intersect in t_0, and their homology classes in $H_1(C, \mathbb{Z})$ form a symplectic basis.

If $f : \widetilde{C} \to C$ is a ramified H-Galois cover with branch locus t, set $V = f^{-1}(U_t)$. Then $f|_V : V \to U_t$ is an unramified Galois covering. Then there is an epimorphism $\theta : \pi_1(U_t, t_0) \to H$. Conversely, such an epimorphism determines a ramified Galois covering of C with branch locus t. The order m_i of $\theta(\gamma_i)$ is called the *local monodromy datum* of the branch point t_i. Let $m = (m_1, \dots, m_r)$. The collection (m, H, θ) is called a *datum*. The Riemann-Hurwitz formula implies that the genus $\widetilde{g}$ of the curve $\widetilde{C}$ is equal to

$$2\widetilde{g} - 2 = |H| \left(2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right) \tag{2.0.1}$$

We introduce the stack $R(H, g, r)$: The Objects of $R(H, g, r)$ are couples $((C, x_1, \dots, x_r), f : \widetilde{C} \to C)$ such that

(1) $(C, x_1, \dots, x_r)$ is a smooth projective r-pointed curve of genus g.
(2) $f : \widetilde{C} \to C$ is a finite cover, H acts on $\widetilde{C}$ and f is H-invariant.
(3) the restriction $f^{gen} : f^{-1}(C \backslash \{x_1, \dots, x_r\}) \to C \backslash \{x_1, \dots, x_r\}$ is an étale H-torsor.

Note that $r = 0$ is also possible which amounts to say that the covers $\widetilde{C} \to C$ are unramified. Moreover since our problem is insensitive to level structures, we may actually consider $R(H, g, r)$ as a coarse moduli space. As a result, we omit any assumptions on the automorphism group of the base curve C whose non-triviality can be remedied either by considering the moduli stack or by imposing level structures.

Let us denote the Jacobians of the curves $\widetilde{C}$ and C respectively by $\widetilde{J}$ and J. Since the finite group H acts on $\widetilde{C}$, then it acts also on the Jacobian $\widetilde{J}$. We denote by $\widetilde{J}^H$ the subgroup of fixed points of $\widetilde{J}$ under the action of H. The following theorem is proven in [14] (repectively, Theorem 2.5 and Proposition 3.1).

Theorem 2.1 (1) $f^*J = (\widetilde{J}^H)^0$.
(2) *The map* $\phi : J \times P(\widetilde{C}/C) \to \widetilde{J}$ *sending* $(c, \widetilde{c})$ *to* $f^*(c) + \widetilde{c}$ *is an isogeny.*

For a Galois covering $f : \widetilde{C} \to C$ with $((C, x_1, \ldots, x_r), f : \widetilde{C} \to C) \in R(H, g, r)$ and $\deg(f) = n$, one can compute the genus $\widetilde{g} := g(\widetilde{C})$ by the Riemann-Hurwitz formula. Using the isogeny $f^*J \times P(\widetilde{C}/C) \sim \widetilde{J}$ we see that the dimension of the Prym variety $P(\widetilde{C}/C) = P(f)$ is equal to $p = \widetilde{g} - g$. Note that if $C \cong \mathbb{P}^1$, then the Prym variety $P(\widetilde{C}/C)$ is isogeneous to the Jacobian $\widetilde{J}$. We will use this point in the sequel to deduce that some families are special.

The canonical principal polarization on $\widetilde{J}$ restricts to a polarization of a certain type D. Let $A_{p,D}$ denote the moduli space of complex abelian varieties of dimension p and polarization type D. More precisely, $A_{p,D} = \mathbb{H}_p/\Gamma_D$ is the moduli space of polarized abelian varieties of type D where $\mathbb{H}_p := \{M \in M_p(\mathbb{C}) \mid {}^t M = M, \operatorname{Im} M \geq 0\}$ is the *Siegel upper half space of genus* p and

$$\Gamma_D = \{R \in \mathrm{GL}_{2p}(\mathbb{Z}) \mid R \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} {}^t R = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}\}$$

is an arithmetic subgroup. The above constructions behave well also in the families of curves and hence we obtain a morphism

$$\mathscr{P} = \mathscr{P}(H, g, r) : R(H, g, r) \to A_{p,D}. \tag{2.1.1}$$

We call the map $\mathscr{P}$ the *Prym map of type* (H, g, r). Our objective in this paper is to study the image of this map. Since in general the Prym map is not injective, one needs to study other closely related aspects, namely the generic injectivity.

By the above mentioned H-action on $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ we have the eigenspace decomposition with respect to the irreducible characters $\chi \in \operatorname{Irr}(H)$

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}) = \bigoplus_{\chi \in \operatorname{Irr}(H)} H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\chi}. \tag{2.1.2}$$

We set:

$$\begin{aligned} H^0(\widetilde{C}, \omega_{\widetilde{C}})_+ &:= H^0(\widetilde{C}, \omega_{\widetilde{C}})^H \quad (\cong H^0(C, \omega_C)), \\ H^0(\widetilde{C}, \omega_{\widetilde{C}})_- &:= \bigoplus_{\chi \in \operatorname{Irr}(H)\setminus\{1\}} H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\chi} \cong H^0(\widetilde{C}, \omega_{\widetilde{C}})/H^0(\widetilde{C}, \omega_{\widetilde{C}})_+. \end{aligned} \tag{2.1.3}$$

Notice that $H^0(\widetilde{C}, \omega_{\widetilde{C}}) = H^0(\widetilde{C}, \omega_{\widetilde{C}})_+ \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$. In the above, $\operatorname{Irr}(H)$ denotes the irreducible characters of H. The characters will appear in the next section in more detail. We gave the above description containing the irreducible characters in order to indicate that $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$ can be viewed as a subgroup of $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ and not just a quotient. Similarly there is a decomposition of $H_1(\widetilde{C}, \mathbb{Z})$ as $H_1(\widetilde{C}, \mathbb{Z})_+ \oplus H_1(\widetilde{C}, \mathbb{Z})_-$.

The following lemma is then an immediate consequence of Theorem 2.1 above.

Lemma 2.2 *Let $f : \widetilde{C} \to C$ be a Galois covering, then*

$$P(\widetilde{C}/C) = H^0(\widetilde{C}, \omega_{\widetilde{C}})_-{}^*/H_1(\widetilde{C}, \mathbb{Z})_-. \tag{2.2.1}$$

3 Galois coverings

3.1 Generalities

Let us summarize some general facts about Galois coverings of curves. Let $\widetilde{C}$, C be complex smooth projective algebraic curves and let $f : \widetilde{C} \to C$ be a Galois covering of degree n. By this we mean precisely that there exists a finite group H with $|H| = n$, together with a faithful action of H on $\widetilde{C}$ such that f realizes C as the quotient of $\widetilde{C}$ by H. Consider the ramification and branch divisors R, B of f. Note that R consists precisely of the points in $\widetilde{C}$ with non-trivial stabilizers under the action of H. The deck transformation group $\mathrm{Deck}(\widetilde{C}/C)$, i.e. the group of those automorphisms of $\widetilde{C}$ that are compatible with f, is isomorphic to the Galois group H and acts transitively on each fiber $f^{-1}(x)$. If $y \in \widetilde{C}$ is a ramification point with ramification index e, then so are all points in the fiber $f^{-1}(f(y))$. Moreover, the stabilizers of these points in $\mathrm{Deck}(\widetilde{C}/C) \cong H$ are conjugate cyclic subgroups, see [15, Proposition 3.2.10]. In particular the stabilizer of a point in $\widetilde{C}$ is trivial, if and only if that point is *not* a ramification point. The stabilizer H_y of a point $y \in \widetilde{C}$ is also referred to as the *inertia subgroup* of y.

3.2 Galois covers of $\mathbb{P}^1$ and Prym datum

Let $\widetilde{G}$ be a finite group and $\widetilde{f} : \widetilde{C} \to \mathbb{P}^1$ a finite $\widetilde{G}$-Galois covering ramified over the branch points $\mathrm{Br}(\widetilde{f}) = \{t_1, \dots, t_s\} \subset \mathbb{P}^1$ as in introduction. Let $\Gamma_s := \pi_1(\mathbb{P}^1 \setminus \mathrm{Br}(\widetilde{f})) = \langle \gamma_1, \dots, \gamma_s | \gamma_1 \cdots \gamma_s = 1 \rangle$, where γ_j corresponds to a loop winding around t_j. Such $\widetilde{G}$-Galois covering is determined by an epimorphism $\widetilde{\theta}_s : \Gamma_s \to \widetilde{G}$ (See [16, Theorem 5.14]). The local monodromy around the branch point t_j is given by $\widetilde{\theta}_s(\gamma_j)$. The set of ramification points $\mathrm{Ram}(\widetilde{f})$ consists precisely of the points in $\widetilde{C}$ with non-trivial stabilizers under the action of $\widetilde{G}$. As we assume that the cover $\widetilde{f}$ is Galois we have that $\mathrm{Br}(\widetilde{f}) = \widetilde{f}(\mathrm{Ram}(\widetilde{f}))$ and $\widetilde{f}^{-1}(\mathrm{Br}(\widetilde{f})) = \mathrm{Ram}(\widetilde{f})$.

Varying the branch points $\{t_1, \dots, t_s\}$ yields a family of $\widetilde{G}$-covers of $\mathbb{P}^1$.

For a normal subgroup H of $\tilde{G}$ we have a cover $f : \widetilde{C} \to C = \widetilde{C}/H$. Set $G = \widetilde{G}/H$ the quotient group, we have a tower of Galois covers $\widetilde{C} \to C = \widetilde{C}/H \to \mathbb{P}^1 = \widetilde{C}/\widetilde{G} = C/G$. One can associate to the cover f the Prym variety as in the Sect. 2. The following definition is central in this paper (compare [4], Definition 3.1).

Definition 3.1 A *Prym datum* (of type (H, g, r)) is a triple $(\tilde{G}, \tilde{\theta}_s, H)$ where $\tilde{G}$ is a finite group, $\widetilde{\theta}_s : \Gamma_s \to \widetilde{G}$ is an epimorphism as above and H is a normal subgroup of $\tilde{G}$, such that the quotient $f : \widetilde{C} \to C = \widetilde{C}/H$ is in $R(H, g, r)$.

Let $\widetilde{G}$ be a finite group and let $\widetilde{C} \to \mathbb{P}^1$ be a $\widetilde{G}$-Galois covering of $\mathbb{P}^1$ with the Prym datum $(\tilde{G}, \tilde{\theta}_s, H)$. Set $U = H^0(\widetilde{C}, \omega_{\widetilde{C}})$ and let $U = U_+ \oplus U_-$ be the decomposition into H-invariant and H-anti-invariant parts as in Sect. 2 (after (2.1.3)). There is also the corresponding Hodge decomposition $H^1(\widetilde{C}, \mathbb{C})_- = U_- \oplus \overline{U}_-$. Set $\Lambda = H_1(\widetilde{C}, \mathbb{Z})_-$. By Lemma 2.2 the associated Prym variety is

$$P(\widetilde{C}/C) = U_-^*/\Lambda, \tag{3.1.1}$$

see [1] for more details.

3.3 Abelian covers and their invariants

Let $f : \widetilde{C} \to C$ be a H-Galois cover with H finite abelian branched above the points $x_1, \dots, x_r$. Since the group H is abelian, the inertia group above a branch point x_i is independent of the chosen ramification point and we denote it by H_i. Under the action of G, the sheaf $f_* \mathcal{O}_{\widetilde{C}}$ splits as the direct sum of the eigensheaves corresponding to the characters of G. We will denote by $(f_* \mathcal{O}_{\widetilde{C}})^{\chi}$ the eigensheaf corresponding to a character χ. Then $(f_* \mathcal{O}_{\widetilde{C}})^{\chi} = L_{\chi}^{-1}$ is an invertible sheaf on C. In particular, the invariant summand L_1 is isomorphic to $\mathcal{O}_C$. The algebra structure on $f_* \mathcal{O}_{\widetilde{C}}$ is given by the ($\mathcal{O}_C$-linear) multiplication rule $m_{\chi,\chi'} : L_{\chi}^{-1} \otimes L_{\chi'}^{-1} \to L_{\chi\chi'}^{-1}$ and is compatible with the action of H. The line bundles L_{χ} and divisors x_i are called the building data of the cover. The building data determine the cover completely up to isomorphisms, see [13], §2, specially Proposition 2.1. The line bundles L_{χ} are also very useful for determining the invariants of the cover. Note that the sheaf $f_* \omega_{\widetilde{C}}$ also splits as the direct sum of the eigensheaves corresponding to the characters of G and it holds that $(f_* \omega_{\widetilde{C}})^{\chi} = \omega_C \otimes L_{\chi^{-1}}$ (where $(f_* \omega_{\widetilde{C}})^{\chi}$ denotes the eigensubsheaf of $f_* \omega_{\widetilde{C}}$ corresponding to a character χ), see [13], Proposition 4.1. Therefore we have

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}) = H^0(C, f_* \omega_{\widetilde{C}}) = H^0(C, \oplus(\omega_C \otimes L_{\chi^{-1}})) = \oplus_{\chi \in H^*} H^0(C, \omega_C \otimes L_{\chi^{-1}}). \tag{3.1.2}$$

In view of the above equalities, one obtains

$$\begin{aligned} H^0(\widetilde{C}, \omega_{\widetilde{C}})_+ &= H^0(C, \omega_C) \\ H^0(\widetilde{C}, \omega_{\widetilde{C}})_- &= \bigoplus_{\chi \in H^* \setminus \{1\}} H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\chi} = \bigoplus_{\chi \in H^* \setminus \{1\}} H^0(C, \omega_C \otimes L_{\chi^{-1}}) \end{aligned} \tag{3.1.3}$$

3.4 Prym varieties of abelian covers

In this subsection we explain the constructions in Sect. 2 for an abelian group H based on the constructions of Sect. 3.3. Let $f : \widetilde{C} \to C$ be a H-Galois cover of C, with H a finite abelian group. Recall the equivalent description of the Prym variety given in Lemma 2.2. We have

$$P = P(\widetilde{C}/C) = \bigoplus_{\chi \in H^* \setminus \{1\}} H^0(C, \omega_C \otimes L_{\chi^{-1}}) / \bigoplus_{\chi \in H^* \setminus \{1\}} H_1(\widetilde{C}, \mathbb{Z})^{\chi},$$

by virtue of (3.1.3) and (2.2.1).

3.5 Abelian covers of $\mathbb{P}^1$

In this section, we follow closely [4] and also [10] whose notations come mostly from [17]. More details about abelian coverings and Prym varieties can be consulted from these two references respectively. For the latter, Birkenhake and Lange [1] is also a comprehensive reference.

An abelian Galois cover $\widetilde{f} : \widetilde{C} \to \mathbb{P}^1$ is determined by a collection of equations in the following way:

Consider an $m \times s$ matrix $A = (r_{ij})$ whose entries r_{ij} are in $\mathbb{Z}/N\mathbb{Z}$ for some $N \geq 2$. Let $\overline{\mathbb{C}(z)}$ be the algebraic closure of $\mathbb{C}(z)$. For each $i = 1, \ldots, m$, choose a function $w_i \in \overline{\mathbb{C}(z)}$ with

$$w_i^N = \prod_{j=1}^{s} (z - t_j)^{\widetilde{r}_{ij}} \text{ for } i = 1, \ldots, m, \tag{3.1.4}$$

in $\mathbb{C}(z)[w_1, \ldots, w_m]$. Here $\widetilde{r}_{ij}$ is the lift of r_{ij} to $\mathbb{Z} \cap [0, N)$ and $t_j \in \mathbb{C}$ for $j = 1, 2, \ldots, s$. Notice that (3.1.4) could give a singular affine curve, in which case we consider a smooth projective model. We impose the condition that the sum of the columns of A is zero (when considered as a vector in $(\mathbb{Z}/N\mathbb{Z})^m$). This implies that the cover given by (3.1.4) is *not* ramified over the infinity. We call the matrix A, the matrix of the covering. We also remark that all operations with rows and columns will be carried out over the ring $\mathbb{Z}/N\mathbb{Z}$, i.e. they will be considered modulo N. The local monodromy around the branch point t_j is given by the column vector $(r_{1j}, \ldots, r_{mj})^t$ and so the order of ramification over t_j is $\frac{N}{\gcd(N, \widetilde{r}_{1j}, \ldots, \widetilde{r}_{mj})}$. Using this and the Riemann-Hurwitz formula, the genus g of the cover can be computed by:

$$g = 1 + d \left(\frac{s-2}{2} - \frac{1}{2N} \sum_{j=1}^{s} \gcd(N, \widetilde{r}_{1j}, \ldots, \widetilde{r}_{mj}) \right), \tag{3.1.5}$$

where d is the degree of the covering which is equal, as pointed out above, to the column span (equivalently row span) of the matrix A. In this way, the Galois group $\widetilde{G}$ of the covering will be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^m$. Note also that this group is isomorphic to the column span of the above matrix.

Remark 3.2 Consider two families of abelian covers with matrices A and A' over the same $\mathbb{Z}/N\mathbb{Z}$. If A and A' have equal row spans then the two families are isomorphic. For more details, see [17] or [10].

Remark 3.3 For a finite abelian group $\widetilde{G}$, it is well known that the character group $\widetilde{G}^* = \mathrm{Hom}(\widetilde{G}, \mathbb{C}^*)$ is isomorphic to $\widetilde{G}$. We fix an isomorphism $\varphi_{\widetilde{G}} : \widetilde{G} \xrightarrow{\sim} \widetilde{G}^*$. In the sequel, we use this isomorphism frequently to identify elements of $\widetilde{G}$ with its characters, without referring to $\varphi_{\widetilde{G}}$.

For our applications, with notations as in the previous pages, we fix an isomorphism of $\widetilde{G}$ with a product of $\mathbb{Z}/n\mathbb{Z}$'s and an embedding of $\widetilde{G}$ into $(\mathbb{Z}/N\mathbb{Z})^m$.

Let l_j be the jth column of the matrix A. As mentioned earlier, the group $\widetilde{G}$ can be realized as the column span of the matrix A. Therefore we may assume that $l_j \in \widetilde{G}$. For a character χ, $\chi(l_j) \in \mathbb{C}^*$ and since $\widetilde{G}$ is finite $\chi(l_j)$ will be a root of unity. Let $\chi(l_j) = \exp(2\alpha_j \pi i/N)$, where α_j is the unique integer in $[0, N)$ with this property. Equivalently, the α_j can be obtained in the following way: let $n \in G \subseteq (\mathbb{Z}/N\mathbb{Z})^m$ be the element corresponding to χ under the above isomorphism. We regard n as an $1 \times m$ matrix. Then the matrix product $n \cdot A$ is meaningful and $n \cdot A = (\alpha_1, \ldots, \alpha_r)$. Here all of the operations are carried out in $\mathbb{Z}/N\mathbb{Z}$ but the α_j are regarded as integers in $[0, N)$. Furthermore we set $\tilde{\alpha}_j = \sum_{i=1}^{m} n_i \tilde{r}_{ij} \in \mathbb{Z}$ (but $\tilde{\alpha}_j$ is not necessarily in $\mathbb{Z} \cap [0, N)$).

Using the above facts, we occasionally consider a character of $\widetilde{G}$ as an element of this group without referring to isomorphism $\varphi_{\widetilde{G}}$.

Let us denote by $\omega_{\widetilde{C}}$ the canonical sheaf of $\widetilde{C}$. Similar to the case of $\widetilde{f}_*(\mathcal{O}_{\widetilde{C}})$, the sheaf $\widetilde{f}_*(\omega_X)_\chi$ decomposes according to the action of $\widetilde{G}$. For the line bundles L_χ corresponding to the character χ associated to the element $a \in \widetilde{G}$ and $\widetilde{f}_*(\omega_{\widetilde{C}})_\chi$ we have the following result proven in [10, Lemma 2.4].

Lemma 3.4 *With notations as above* $L_\chi = \mathcal{O}_{\mathbb{P}^1}(\sum_1^s \langle \frac{\alpha_j}{N} \rangle)$, *where* $\langle x \rangle$ *denotes the fractional part of the real number* x *and*

$$\widetilde{f}_*(\omega_{\widetilde{C}})_\chi = \omega_{\mathbb{P}^1} \otimes L_{\chi^{-1}} = \mathcal{O}_{\mathbb{P}^1}\left(-2 + \sum_1^s \left\langle -\frac{\alpha_j}{N} \right\rangle \right).$$

Notice that the sums in the Lemma are integers. In fact, since the sum of each row of A is zero in $\mathbb{Z}/N\mathbb{Z}$, then $\sum_j \tilde{\alpha}_j = \sum_i n_i (\sum_j \tilde{r}_{ij})$ is a multiple of N. Thus $\sum_j \frac{\tilde{\alpha}_j}{N}$ is integer and this does not depend on the representative chosen modulo N or by taking the fractional part.

Let $n \in \widetilde{G}$ be the element $(n_1, \dots, n_m) \in \widetilde{G} \subset (\mathbb{Z}/N\mathbb{Z})^m$. By Lemma 3.4, $\dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_n = -1 + \sum_{j=1}^s \langle -\frac{\alpha_j}{N} \rangle$. A basis for the $\mathbb{C}$-vector space $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ is given by the forms

$$\omega_{n,\nu} = z^\nu w_1^{n_1} \cdots w_m^{n_m} \prod_{j=1}^s (z - t_j)^{\lfloor -\frac{\tilde{\alpha_j}}{N} \rfloor} dz. \tag{3.4.1}$$

Here $0 \le \nu \le -1 + \sum_{j=1}^s \langle -\frac{\alpha_j}{N} \rangle$. The fact that the above elements constitute a basis can be seen in [10, proof of Lemma 5.1], where the dual version for $H^1(C, \mathcal{O}_C)$ is proved.

The general method of our later computations in Sect. 5 is as follows: We remark that if $n = (n_1, \dots, n_m) \in \widetilde{G} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_m} \subset (\mathbb{Z}/N\mathbb{Z})^m$, we consider the $n_i \in [0, N)$ and their sum as integers. The action of the abelian subgroup H is naturally inherited from that of $\tilde{G}$ and the latter is described as follows: Let $g = (g_1, \dots, g_m) \in \tilde{G}$ and write ord $g_i = v_i$. Then the action of g on each w_i is given by $g \cdot w_i = \xi_{v_i} w_i$, where ξ_{v_i} denotes a v_i-th primitive root of unity.

Below we highlight the main ideas to perform the construction of the examples in Sect. 5.

With the above notation, $H^0(\widetilde{C}, \omega_{\widetilde{C}})_+$, i.e. the group of H-invariant differential forms, is the set of all $\omega_{n,\nu}$ with $\sum n_i/a_i \in \mathbb{Z}$ for all $h = (h_1, \dots, h_m) \in H$ (with $a_i = \operatorname{ord} h_i$).

The space $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$ is then given by the complement, i.e. the set of all $\omega_{n,\nu}$ for whom there exists $h = (h_1, \dots, h_m) \in H$ such that $\sum n_i/a_i \notin \mathbb{Z}$.

3.6 Families of abelian covers of $\mathbb{P}^1$ and their Prym map

Families of abelian covers of $\mathbb{P}^1$ can be constructed as follows: Let $T_s \subset (\mathbb{A}^1_{\mathbb{C}})^s$ be the complement of the big diagonals as in the introduction. We consider abelian covers of $\mathbb{P}^1$ given by the Eq. (3.1.4) with branch points $(t_1, \dots, t_s) \in T_s$ and $\widetilde{r}_{ij}$ the lift of r_{ij} to $\mathbb{Z} \cap [0, N)$ as before. Varying the branch points we get a family $f : \tilde{\mathcal{C}} \to T_s$ of smooth projective curves over T_s (viewed as a complex manifold of dimension $s - 3$) whose fibers $\widetilde{C}_t$ are abelian covers of $\mathbb{P}^1$ introduced above.

Let $\widetilde{G}$ be a finite group and consider a family $\widetilde{\mathcal{C}} \to T_s$ of abelian covers as above whose fibers $\widetilde{C}_t$ are $\widetilde{G}$-Galois coverings of $\mathbb{P}^1$ with a fixed Prym datum $\Sigma := (\tilde{G}, \tilde{\theta}_s, H)$. Associating to $t \in T_s$ the class of the pair $((C_t, x_1, \dots, x_r), \pi_t : \widetilde{C}_t \to C_t)$ gives a map $T_s \to R(H, g, r)$ with discrete fibers. We denote the image of this map by $R(\Sigma)$. It follows that $R(\Sigma)$ is a subvariety of dimension equal to $s - 3$, see also [4], p. 6. As in the last section, set $U_t = H^0(\widetilde{C}_t, \omega_{\widetilde{C}_t})$ and let $U_t = U_{+,t} \oplus U_{-,t}$ be the decomposition under the action of H. There is also the corresponding Hodge decomposition $H^1(\widetilde{C}_t, \mathbb{C})_- = U_{-,t} \oplus \overline{U}_{-,t}$. Set $\Lambda_t = H_1(\widetilde{C}_t, \mathbb{Z})_-$. The associated Prym variety is by 3.1.1, $P(\widetilde{C}_t/C_t) = U_{-,t}^*/\Lambda_t$, an abelian variety of dimension $p = \widetilde{g} - g$. So we obtain the Prym map $R(\Sigma) \xrightarrow{\mathscr{P}} A_{p,D}$ for this family.

4 Shimura subvarieties

Let $V_{\mathbb{Z}} := \mathbb{Z}^{2p} \subset V := \mathbb{Q}^{2p}$ and let $\psi : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ be a symplectic form. Let $L = \mathrm{Gsp}(V_{\mathbb{Z}}, \psi)$ be the group of symplectic similitudes, i.e.,

$$L = \{g \in \mathrm{GL}(V_{\mathbb{Z}}) \mid \psi(gu, gv) = \nu(g)\psi(u, v) \text{ for some } \nu(g) \in \mathbb{Z}^*\}.$$

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ be the Deligne torus. A Hodge structure of weight 1 and type $(1, 0) + (0, 1)$ on $V_{\mathbb{Z}}$ for which ψ is a polarization corresponds to a homomorphism $h : \mathbb{S} \to L_{\mathbb{R}}$.

Using the Riemann bilinear relations, the space of all homomorphisms h as above can be identified with the Siegel upper half space of genus p, $\mathbb{H}_p = \{M \in M_p(\mathbb{C}) \mid ^t M = M, \mathrm{Im}\, M \geq 0\}$. Recall from Sect. 2 that $A_{p,D} = \mathbb{H}_p/\Gamma_D$ is the moduli space of polarized abelian varieties of type D, where $\Gamma_D = \{R \in \mathrm{GL}_{2p}(\mathbb{Z}) \mid R \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} {}^t R = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}\}$ is an arithmetic subgroup. Note that $\mathbb{H}_p = L(\mathbb{R})/K$, where K is a maximal compact subgroup. So $A_{p,D}$ can be written as a double quotient $\Gamma_D \backslash L(\mathbb{R})/K$. Such double quotients are called Shimura varieties and their structure has beeen studied extensively. We remark that $K_n = \{g \in L(\hat{\mathbb{Z}}) \mid g \equiv 1(\bmod n)\}$ is a compact open subgroup of $L(\mathbb{A}_f)$ and $A_{p,D,n} = L(\mathbb{Q})\backslash(\mathbb{H}_p \times L(\mathbb{A}_f)/K_n)$ can be identified with the space of complex polarized abelian varities (of type D) of dimension p and with level n structure. Note that $A_{p,D} = \varprojlim A_{p,D,n}$.

Given a Shimura variety, one can define special (or Shimura) subvarieties. We define them only for the Shimura variety $A_{p,D}$. To state the definition, let us frist explain some notation. Suppose $N \subset L$ is an algebraic subgroup defined over $\mathbb{Q}$. Define the subset $Y_N \subseteq \mathbb{H}_p$ as follows.

$$Y_N = \{h : \mathbb{S} \to L_{\mathbb{R}} | h \text{ factors through } N_{\mathbb{R}}\}.$$

We remark that the group $N(\mathbb{R})$ acts on Y_N by conjugation. We can now state our definition

Definition 4.1 A closed irreducible algebraic subvariety $Z \subset A_{p,D,n}$ is called special (or Shimura) if there exists an algebraic subgroup $N \subset L$ defined over $\mathbb{Q}$, a connected component $Y^+ \subseteq Y_N$ and an element $\gamma \in L(\mathbb{A}_f)$ such that $Z(\mathbb{C}) \subset A_{p,D,n}(\mathbb{C})$ is the image of $Y^+ \times \{\gamma K_n\} \subset \mathbb{H}_p \times L(\mathbb{A}_f)/K_n$ under the natural map to $A_{p,D,n}(\mathbb{C}) = L(\mathbb{Q})\backslash(\mathbb{H}_p \times L(\mathbb{A}_f)/K_n)$.

Since level structure does not play a role in the sequel, we drop it from the notation and state the results for $A_{p,D}$. We now use Definiton 4.1 to define a particular special subvariety defined by the families of Prym varieties that we consider.

Let $\widetilde{G}$ be a finite group and consider a family $\widetilde{\mathcal{C}} \to T_s$ whose fibers $\widetilde{C}_t$ are $\widetilde{G}$-Galois coverings of $\mathbb{P}^1$ with a fixed Prym datum $\Sigma := (\widetilde{G}, \tilde{\theta}_s, H)$. As in Sect. 3.6, we have a Prym map $R(\Sigma) \xrightarrow{\mathscr{P}} A_{p,D}$.

In this paper, we are interested in determining whether the subvariety $Z = \overline{\mathscr{P}(R(\Sigma))} \subset A_{p,D}$ is a special or Shimura subvariety. The Prym varieties of the fibers of the family $\widetilde{C}_t \to T_s$ fit into a family $P \to T_s$ which is an abelian scheme over T_s that admits naturally an action of the group ring $\mathbb{Z}[\widetilde{G}]$. This action defines a special subvariety $P(\widetilde{G})$ in $A_{p,D}$ that contains Z in the following way. The construction of the subvariety $P(\widetilde{G})$ given here is adapted for the case of Prym varieties from [12], see also the paper [11] and also [6] for a different approach.

Fix a base point $t \in T_s$ and let (P_t, λ) be the corresponding Prym variety with λ as its polarization of type D. Let $(V_{\mathbb{Z}}, \psi)$ be as in the beginning of this section. We fix a symplectic similitude $\sigma : H^1(P_t, \mathbb{Z}) \to V_{\mathbb{Z}}$. Let $F = \mathbb{Q}[\widetilde{G}]$. The group $\widetilde{G}$ acts on $H^0(\widetilde{C}_t, \omega_{\widetilde{C}})_-$ and thereby on the Prym variety $P(\widetilde{C}_t/C_t)$. We therefore view $H^0(\widetilde{C}_t, \omega_{\widetilde{C}})_-$ as an F-module. Via

σ, the Hodge structure on $V = H^1(P_t, \mathbb{Q}) = H^1(\widetilde{C}_t, \mathbb{Q})_-$ corresponds to a point $y \in \mathbb{H}_p$ and one obtains the structure of an F-module on V. F is isomorphic to a product of cyclotomic fields and is equipped with a natural involution $*$ which is complex conjugation on each factor. The polarization ψ on V satisfies

$$\psi(bu, v) = \psi(u, b^* v) \text{ for all } b \in F \text{ and } u, v \in V.$$

Define the subgroup N as in [12]:

$$N = \mathrm{Gsp}(V, \psi) \cap \mathrm{GL}_F(V). \tag{4.1.1}$$

If $h_0 : \mathbb{S} \to L_{\mathbb{R}}$ is the Hodge structure on $V_{\mathbb{Z}} = H^1(P_t, \mathbb{Z})$ corresponding to the point $y \in \mathbb{H}_p$, then by the above F-action this homomorphism factors through the subgroup $N_{\mathbb{R}}$. Define the subset $Y_N \subseteq \mathbb{H}_p$ as in Definition 4.1. The point y lies in Y_N and there is a connected component $Y^+ \subseteq Y_N$ which contains y.

Definition 4.2 With the above notation, the special subvariety $P(\widetilde{G})$ is the image of Y^+ under the map

$$\mathbb{H}_p \to L(\mathbb{Z})\backslash\mathbb{H}_p \cong L(\mathbb{Q})\backslash\mathbb{H}_p \times L(\mathbb{A}_f)/L(\widehat{\mathbb{Z}}) \cong A_{p,D}(\mathbb{C}).$$

For $t = (t_1, \ldots, t_s) \in T_s$, let $((C_t, x_1, \ldots, x_r), \pi_t : \widetilde{C}_t \to C_t) \in R(H, g, r)$ be the covering corresponding to t. For this t, consider the Hodge decomposition $H_1(\widetilde{C}_t, \mathbb{C})_- = U_{-,t} \oplus \overline{U}_{-,t}$ which corresponds to a complex structure on $H_1(\widetilde{C}_t, \mathbb{R})_-$. We therefore get a point $f(t) \in \mathbb{H}_p$. Indeed we obtain a morphism $f : T_s \to \mathbb{H}_p$ and the following commutative diagram.

$$\begin{array}{ccc} T_s & \xrightarrow{f} & \mathbb{H}_p \\ {\scriptstyle \iota^0}\downarrow & & \downarrow{\scriptstyle \iota} \\ R(\Sigma) & \xrightarrow{\mathscr{P}} & A_{p,D} \end{array} \tag{4.2.1}$$

It follows by construction of $P(\widetilde{G})$ that $Z \subseteq P(\widetilde{G})$. As we remarked earlier, the Prym map is not in general injective. In order to conclude the equality $Z = P(\widetilde{G})$ and hence the speciality of Z, we still need to assure that the differential of the Prym map on $R(\Sigma)$ is injective, whence $\dim R(\Sigma) = \dim \mathscr{P}(R(\Sigma))$. For this purpose, set $U = H^0(\widetilde{C}, \omega_{\widetilde{C}}) = U_+ \oplus U_-$ and likewise $W = H^0(\widetilde{C}, \omega_{\widetilde{C}}^{\otimes 2}) = W_+ \oplus W_-$. Note that the multiplication map $m : S^2 U \to W$ is the codifferential of the Torelli map and the codifferential of the Prym map at a given point coincides with the restriction of the multiplication map m to $S^2 U_-$. The following proposition gives sufficient conditions to treat the above situation and is evident.

Proposition 4.3 *Consider the following conditions*

$$\textit{The restricted multiplication map } m : (S^2 U_-)^{\widetilde{G}} \to W_+^{\widetilde{G}} \textit{ is an isomorphism.} \tag{B}$$

$$\dim (S^2 U_-)^{\widetilde{G}} = s - 3. \tag{A}$$

$$(S^2 U_-)^{\widetilde{G}} \cong Y_1 \otimes Y_2, \tag{B1}$$

where $\dim Y_1 = 1$, $\dim Y_2 = s - 3$.

Then condition (B) implies the condition (A) and is implied by condition (B1).

Another sufficent condition ensuring (B) is the following. Suppose there is an isogeny decomposition of the Prym variety as follows

$$P(\widetilde{G}) \sim A \times JC', \tag{B2}$$

where A is a fixed abelian variety and JC' is the Jacobian of a curve $C' := \widetilde{C}/K$ defined as a quotient of $\widetilde{C}$ by a normal subgroup $K \lhd \widetilde{G}$, such that the Galois cover $C' \to \mathbb{P}^1 = C'/(\widetilde{G}/K)$ is branched in r points and this family satisfies condition $(*)$ of [6], hence it gives rise to a special subvariety. Therefore, since A is fixed and JC' moves in a Shimura family, then the family of the Prym varieties $P(\widetilde{G})$ yields a special subvariety too, see [8, Thm. 3.8]. We have

Theorem 4.4 *If the condition (B) holds for some $t \in T_s$, then the subvariety Z is a special subvariety.*

Proof Let N be the subgroup in 4.1.1. If Y^+ is a connected component of Y_N whose image in $A_{p,D}$ is $P(\widetilde{G})$, then the assumption implies that $\dim Y^+ = \dim T_s = s-3$. As the vertical rows in 4.2.1 are discrete, one concludes that $\dim \mathscr{P}(R(\Sigma)) = \dim P(\widetilde{G}) = s - 3$. This together with the fact that $Z \subseteq P(\widetilde{G})$ implies that $Z = P(\widetilde{G})$. ◻

When $\tilde{G}$ is abelian, the following Lemma computes the dimension of $P(\widetilde{G})$.

Lemma 4.5 *Let $d_n = H^{1,0}(P(\widetilde{G}))_n = H^0(\widetilde{C}, \omega_{\widetilde{C}})_{-,n}$, then*

$$\dim P(\widetilde{G}) = \sum_{2n \neq 0} d_n d_{-n} + \frac{1}{2} \sum_{2n=0} d_n(d_n+1).$$

Note that $2.0 = 0$ in $\widetilde{G}$, so in fact the second sum in the right hand side of the above equality is always meaningful and if $|\widetilde{G}|$ is an odd number it will be zero.

Proof We calculate $\dim T_y(Y_N)$ at the point $y \in \mathbb{H}_p$. The dimension of the tangent space of $P(\widetilde{G})$ at the point y will be equal to this number. To compute $\dim T_y(\mathbb{H}_p)$, we first remark that the polarization induces a perfect pairing $\overline{\phi} : H^{1,0} \times V_{\mathbb{C}}/H^{1,0} \to \mathbb{C}$. Then the tangent bundle $T_y(\mathbb{H}_p)$ can be identified with

$$\begin{aligned} &\mathrm{Hom}^{sym}(H^{1,0}, V_{\mathbb{C}}/H^{1,0}) := \{\beta : H^{1,0} \to V_{\mathbb{C}}/H^{1,0} \mid \overline{\phi}(v, \beta(v')) \\ &\quad = \overline{\phi}(v', \beta(v)) \forall v, v' \in H^{1,0}\}, \end{aligned}$$

i.e., the elements of $T_y(\mathbb{H}_p)$ that are their own dual via the isomorphisms induced by $\overline{\phi}$. For a more detailed discussion, see [12]. Furthermore notice that $V_{\mathbb{C}}/H^{1,0} = H^{0,1}$ and that $\overline{\phi}$ respects the Galois group action, namely it reduces to $\overline{\phi}_n : H_n^{1,0} \times H_{-n}^{0,1} \to \mathbb{C}$ for every character n of $\widetilde{G}$. The subspace $T_y(Y_N) \subset T_y(\mathbb{H}_p)$ consists therefore of $\beta \in \mathrm{Hom}^{sym}(H^{1,0}, V_{\mathbb{C}}/H^{1,0})$ (symmetric with respect to $\overline{\phi}$) that respect the F-action on V, that is, are $F_{\mathbb{C}}$-linear. Any such β can be written as the sum $\sum \beta_n$, where $\beta_n : H_{\mathbb{C},n}^{1,0} \to H_{\mathbb{C},n}^{0,1}$ is the induced action on the eigenspaces. These β_n should satisfy the relation

$$\overline{\phi}_n(v, \beta_{-n}(v')) = \overline{\phi}_{-n}(v', \beta_n(v)).$$

The perfect pairing $\overline{\phi}_n$ gives a duality between $H_{\mathbb{C},n}^{1,0}$ and $H_{\mathbb{C},(-n)}^{0,1}$. So we have a duality between β_n and β_{-n} if $n \neq -n$ in $\widetilde{G}$. If $n = -n$ in $\widetilde{G}$, i.e., if $2n = 0$ in $\widetilde{G}$ this gives a self duality for β_n. Therefore $\dim T_y(Y_N)$ is equal to $\sum_{2n \neq 0} d_n d_{-n} + \frac{1}{2} \sum_{2n=0} d_n(d_n+1)$. ◻

Note that the above proof implies that $\beta = \sum \beta_n \in \mathrm{Sym}^2(H_{\mathbb{C}}^{1,0})^G$. In particular, it follows that $\dim P(\widetilde{G}) = \dim(S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})_-)^{\widetilde{G}}$ see also [6], Theorem 3.6.

5 Examples

In this section, we work out some details of some of the examples given in the table on page 21. We begin with families of cyclic covers of $\mathbb{P}^1$ which are the simplest abelian cases. The example can elucidate the computations needed for the cyclic case in general. In the abelian case, i.e. when the group $\widetilde{G}$ is abelian, we have gathered some examples which are typical for abelian covers so that the computations that we perform can be applied to other abelian cases. If dim $P(\widetilde{G}) = 1$, condition (B) is automatically satisfied, so we also give some abelian examples with dim $P(\widetilde{G}) = 2$, where one needs to do some calculations in order to verify condition (B). We also work out some abelian and non-abelian examples that do not satisfy condition (B1). For the non-abelian case, we consider D_4-covers and show that the Prym varieties are in fact isogeneous to Jacobians and then using results of [6] we conclude that they give rise to special subvarieties. Note that this is the only non-abelian case that one can handle by hand, since unlike the abelian covers, one does not have explicit equations and so computations is hardly possible. For the abelian families that do not satisfy condition (B1), we have again chosen some examples which can be very typical and in fact all of the abelian examples in this case can be done by hand and we have verified all of them by concrete computations. Below are the details of the chosen examples.

- Consider the family given by the monodromy data (6, (1, 3, 4, 4)), i.e., the family $w^6 = (z - z_1)(z - z_2)^3(z - z_3)^4(z - z_4)^4$. This family has Galois group $\mathbb{Z}_6$ and fiber genus 3. The quotient by the subgroup $\mathbb{Z}_3$ gives rise to a triple cover $\widetilde{C}_t \to C_t$ which is totally ramified at 5 points, so that the family of the Prym varieties is contained in $R_{3,[5]}$ in the notation of [4]. This family corresponds to the example with data $(r, \tilde{g}, \#) = (4, 3, 2)$ in the table. The quotient curve C_t corresponds to the $\mathbb{Z}_2$-covering $w^2 = (z - z_1)(z - z_2)$, so that $C \cong \mathbb{P}^1$. Therefore $P(\widetilde{C}_t/C_t)$ is isogeneous to $J(\widetilde{C}_t)$. By the results of [11], this latter family is a special family of Jacobians and hence the family of Prym varieties is also special.

Alternatively, one could use the special subvariety $P(\widetilde{G})$ to prove that the family is special.

The automorphism σ of order 3 corresponds to the automorphism $w \mapsto \xi_3 w$, where ξ_3 is a primitive 3rd root of unity. Using this action, we compute the eigenspace $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$. We have that

$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_- = H^0(\widetilde{C}, \omega_{\widetilde{C}})_1 \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_2 \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_4 \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_5,$$

where $H^0(\widetilde{C}, \omega_{\widetilde{C}})_i$ is the eigenspace w.r.t the character $i \in \mathbb{Z}_6$. For a cyclic cover, these are standard to compute, e.g. [11], p.799. We have that dim $H^0(\widetilde{C}, \omega_{\widetilde{C}})_1 = \dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_5 = 1$, dim $H^0(\widetilde{C}, \omega_{\widetilde{C}})_2 = 0$, dim $H^0(\widetilde{C}, \omega_{\widetilde{C}})_4 = 1$. The group $\widetilde{G} = \mathbb{Z}_6$ acts on $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$ by $w \mapsto \xi_6 w$ so that we have $H^0(\widetilde{C}, \omega_{\widetilde{C}})_{-,i} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_i$. Now, we can compute the dimension of $P(\widetilde{G})$: This is equal to $\dim(S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})_-)^{\widetilde{G}}$, as we remarked earlier. Note that

$$(S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})_-)^{\widetilde{G}} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_1 \otimes H^0(\widetilde{C}, \omega_{\widetilde{C}})_5$$

So dim $P(\widetilde{G}) = 1$.

- As an abelian and non-cyclic example consider the $\mathbb{Z}_3 \times \mathbb{Z}_3$-cover of $\mathbb{P}^1$ given by the matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$. In other words, this family is given by equations

$$w_1^3 = (z - z_1)(z - z_2)(z - z_3)$$
$$w_2^3 = (z - z_3)^2(z - z_4)$$

This family is one of the families with abelian Galois group which gives rise to special subvarieties in the Torelli locus, see [12] or [10] for more details.

Consider the quotient $\mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ by the second factor. This corresponds to a cover $\widetilde{C}_t \to C_t$ which is totally ramified in 3 points and $g(C_t) = 1$. In fact the quotient curve C_t is just given by the first of the above equations $w^3 = (z - z_1)(z - z_2)(z - z_3)$ or equivalently by the first row of the above matrix. This family corresponds to the example with data $(r, \tilde{g}, \#) = (4, 4, 7)$ in the table. The automorphism ν of order 3 corresponds to the automorphism $w_1 \mapsto w_1, w_2 \mapsto \xi_3 w_2$, where ξ_3 is a primitive 3rd root of unity. Using this action, we compute the eigenspace $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$. We have that

$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_- = H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,1)} \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,2)} \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(2,1)},$$

where $H^0(\widetilde{C}, \omega_{\widetilde{C}})_i$ is the eigenspace w.r.t the character $i \in \mathbb{Z}_3 \times \mathbb{Z}_3$. For an abelian cover, these dimensions are computed in [10], Prop 2.8. We have that $\dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,1)} = \dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,2)} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(2,1)} = 1$. Hence

$$(S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})_-)^{\widetilde{G}} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,2)} \otimes H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(2,1)}$$

So $\dim P(\widetilde{G}) = 1$.

- Consider the family given by the monodromy data $(6, (1, 1, 1, 1, 2))$, i.e., the family $y^6 = (x - t_1)(x - t_2)(x - t_3)(x - t_4)(x - t_5)^2$. This family has Galois group $\mathbb{Z}_6$ and fiber genus 7. The quotient by the subgroup $\mathbb{Z}_3$ gives rise to a triple cover $\widetilde{C}_t \to C_t$ which is totally ramified at 6 points. The quotient curve C_t corresponds to the $\mathbb{Z}_2$-covering $y^2 = (x - t_1)(x - t_2)(x - t_3)(x - t_4)$, which is a curve of genus 1. Hence the family of Prym varieties is contained in $R_{6,[6]}$ and this family corresponds to the example with data $(r, \tilde{g}, \#) = (5, 4, 2)$ in the table.

The automorphism δ of order 3 corresponds to the automorphism $y \mapsto \xi_3 y$, where ξ_3 is a primitive 3rd root of unity. Using this action, we compute the eigenspace $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$. We have that

$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_- = H^0(\widetilde{C}, \omega_{\widetilde{C}})_1 \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_2 \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_4 \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_5,$$

where $H^0(\widetilde{C}, \omega_{\widetilde{C}})_i$ is the eigenspace w.r.t the character $i \in \mathbb{Z}_6$. We have the dimensions $\dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_1 = 3$, $\dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_2 = 2$, $\dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_4 = 1$, $\dim H^0(\widetilde{C}, \omega_{\widetilde{C}})_5 = 0$. The group $\widetilde{G} = \mathbb{Z}_6$ acts on $H^0(\widetilde{C}, \omega_{\widetilde{C}})_-$ by $y \mapsto \xi_6 y$ so that we have $H^0(\widetilde{C}, \omega_{\widetilde{C}})_{-,i} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_i$. Now, we can compute $\dim P(\widetilde{G}) = \dim(S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})_-)^{\widetilde{G}}$. Note that

$$(S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})_-)^{\widetilde{G}} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_2 \otimes H^0(\widetilde{C}, \omega_{\widetilde{C}})_4$$

So $\dim P(\widetilde{G}) = 2$. This implies that the family satisfies condition (A). Since the family is two dimensional, it is not enough to conclude and we must still show that condition (B) holds. In order to do this we use the basis of the differential forms introduced earlier. It holds that

$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_{-,2} = H^0(\widetilde{C}, \omega_{\widetilde{C}})_2 = \left\langle \alpha_1 = y^2 \prod_{i=1}^{5} (x - t_i)^{-1} dx, \alpha_2 = x\alpha_1 \right\rangle$$

and

$$\begin{aligned} H^0(\widetilde{C}, \omega_{\widetilde{C}})_{-,4} &= H^0(\widetilde{C}, \omega_{\widetilde{C}})_4 \\ &= \langle \beta = y^4 (x - t_1)^{-1} (x - t_2)^{-1} (x - t_3)^{-1} (x - t_4)^{-1} (x - t_5)^{-2} dx \rangle, \end{aligned}$$

so that $(S^2H^0(\widetilde{C},\omega_{\widetilde{C}})_-)^{\widetilde{G}} = \langle \alpha_1\odot\beta, \alpha_2\odot\beta\rangle$. We have

$$m(\alpha_1\odot\beta) = \frac{(dx)^2}{\prod_{i=1}^5(x-t_i)}, \qquad m(\alpha_2\odot\beta) = \frac{x(dx)^2}{\prod_{i=1}^5(x-t_i)}.$$

So $v = a_1(\alpha_1\odot\beta)+a_2(\alpha_2\odot\beta)\in\ker(m)$ if and only if $a_1\frac{(dx)^2}{\prod_{i=1}^5(x-t_i)}+a_2\frac{x(dx)^2}{\prod_{i=1}^5(x-t_i)} = 0$. It is straightforward to see that this holds if and only if $a_1 = a_2 = 0$. This shows that m is injective and by condition (A), it is an isomorphism, so condition (B) is satisfied.

- Consider the family of genus 2 curves with non-abelian Galois group $\widetilde{G} = D_4$ and ramification data $(2^3, 4)$. This family corresponds to the example with data $(r, \tilde{g}, \#) = (4, 2, 3)$ in the table and does not satisfy (B1) and therefore we can not conclude by showing the isomorphy of the multiplication map. However, in this case the quotient curve C is isomorphic to $\mathbb{P}^1$ and so by the remark after Theorem 2.1, the family of Prym varieties $P(\widetilde{C}/C)$ is isogeneous to the family of Jacobians. A close inspection of Tables 1,2 in [6] shows that this famiy is family (29) of that paper and hence it is a special family. The same argument shows that the families of genus 3 curves with Galois group $\widetilde{G} = D_4$, ramification data (2^5) and $H = \mathbb{Z}_2^2$ which corresponds to examples $(r, \tilde{g}, \#) = (5, 3, 1), (5, 3, 2)$ in the table are isogeneous to the family (32) of [6] and so are also special 2-dimensional families (these also do not satisfy (B1)).
- An abelian example that does not verify condition (B1) is the following family. Consider $\widetilde{G} = \mathbb{Z}_3^2$ and the monodromy matrix $A = \begin{pmatrix} 1&0&1&2&2\\ 0&2&2&0&2\end{pmatrix}$. Then the curve $\widetilde{C}$ has genus 7 and equations

$$w_1^3 = (z-z_1)(z-z_3)(z-z_4)^2(z-z_5)^2$$
$$w_2^3 = (z-z_2)^2(z-z_3)^2(z-z_5)^2$$

Also consider the subgroup $H \cong \mathbb{Z}_3$ generated by the element $(0, 1)^t$, that acts as $w_1\mapsto w_1$, $w_2\mapsto \xi_3 w_2$, where ξ_3 is a primitive 3rd root of unity. We have $H^0(\widetilde{C},\omega_{\widetilde{C}})_- = V_{(0,2)}\oplus V_{(1,1)}\oplus V_{(2,1)}\oplus V_{(1,2)}\oplus V_{(2,2)}$, where all summands have dimension 1. Then we obtain

$$(S^2H^0(\widetilde{C},\omega_{\widetilde{C}})_-)^{\widetilde{G}} = (V_{(1,1)}\otimes V_{(2,2)})\oplus(V_{(1,2)}\otimes V_{(2,1)})$$

hence condition (B1) is not satisfied. We have

$$H^0(\widetilde{C},\omega_{\widetilde{C}})_{(1,1)} = \left\langle \omega_1 = \frac{w_1w_2}{\prod_{i=1}^4(z-z_i)(z-z_5)^2}dx\right\rangle$$

$$H^0(\widetilde{C},\omega_{\widetilde{C}})_{(2,2)} = \left\langle \omega_2 = \frac{w_1^2w_2^2}{(z-z_1)\prod_{i=2}^4(z-z_i)^2(z-z_5)^3}dx\right\rangle$$

$$H^0(\widetilde{C},\omega_{\widetilde{C}})_{(1,2)} = \left\langle \omega_3 = \frac{w_1w_2^2}{(z-z_1)(z-z_2)^2(z-z_3)^2(z-z_4)(z-z_5)^2}dx\right\rangle$$

$$H^0(\widetilde{C},\omega_{\widetilde{C}})_{(2,1)} = \left\langle \omega_4 = \frac{w_1^2w_2}{(z-z_1)(z-z_2)(z-z_3)^2(z-z_4)^2(z-z_5)^2}dx\right\rangle$$

Hence we compute

$$v_1 := m(\omega_1 \odot \omega_2) = \frac{(dx)^2}{(z-z_1)(z-z_2)(z-z_4)(z-z_5)}$$
$$v_2 := m(\omega_3 \odot \omega_4) = \frac{(dx)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

and we find that $a_1 v_1 + a_2 v_2 = 0$ if and only if $a_1 = a_2 = 0$, so m is injective. Together with condition (A) this implies that (B) holds. Thus the family gives rise to a 2-dimensional Shimura variety.

- Consider the 3-dimensional family with group $\widetilde{G} = \mathbb{Z}_2^3$ and monodromy matrix $A = \begin{pmatrix} 0&0&1&1&0&0 \\ 0&1&1&1&0&1 \\ 1&1&1&1&1&1 \end{pmatrix}$. The equations of the genus 5 curve $\widetilde{C}$ are

$$w_1^2 = (z-z_3)(z-z_4)$$
$$w_2^2 = (z-z_2)(z-z_3)(z-z_4)(z-z_6)$$
$$w_3^2 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)$$

We consider the $\mathbb{Z}_2^2$-cover given by the action of the subgroup $H = \langle (1,0,0)^t, (0,1,0)^t \rangle$. The quotient curve $C = \widetilde{C}/H$ has genus 2. We have $H^0(\widetilde{C}, \omega_{\widetilde{C}})_- = H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(0,1,0)} \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,0,1)} \oplus H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,1,1)}$ where each summand has dimension 1. Then we get

$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(0,1,0)} = \langle \omega_1 = \frac{y_2}{(z-z_2)(z-z_3)(z-z_4)(z-z_6)} dx \rangle$$
$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,0,1)} = \langle \omega_2 = \frac{y_1 y_3}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)} dx \rangle$$
$$H^0(\widetilde{C}, \omega_{\widetilde{C}})_{(1,1,1)} = \langle \omega_3 = \frac{y_1 y_2 y_3}{(z-z_1)(z-z_2)(z-z_3)^2(z-z_4)^2(z-z_5)(z-z_6)} dx \rangle$$

thus if we set

$$v_1 := m(\omega_1 \odot \omega_1) = \frac{(dx)^2}{(z-z_2)(z-z_3)(z-z_4)(z-z_6)}$$
$$v_2 := m(\omega_2 \odot \omega_2) = \frac{(dx)^2}{(z-z_1)(z-z_2)(z-z_5)(z-z_6)}$$
$$v_3 := m(\omega_3 \odot \omega_3) = \frac{(dx)^2}{(z-z_1)(z-z_3)(z-z_4)(z-z_5)}$$

we find that $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$ if and only if $a_1(z-z_1)(z-z_5) + a_2(z-z_3)(z-z_4) + a_3(z-z_2)(z-z_6) = 0$, i.e. $a_1 = a_2 = a_3 = 0$. Hence the multiplication map is an isomorphism and the family gives rise to a special subvariety.

We list all the obtained Prym data that give rise to Shimura varieties. For each example it is reported: the number r of critical values on $\mathbb{P}^1$, the genus $\widetilde{g}$ of $\widetilde{C}$ and g of C, the dimension $p = \widetilde{g} - g$ and progressive index (#), the group $\widetilde{G}$ and the subgroup H determining the Prym cover, the number of ramification and branch points of this cover, the quotient group $G = \widetilde{G}/H$ acting on C. Finally the fulfilled conditions are marked.

r	$\widetilde{g}$	g	p	#	$\widetilde{G}$	H	Ram pt	Br pt	G	(B1)	(B2)	(B)
4	2	0	2	1	S_3	C_3	4	4	C_2		✓	✓
4	2	0	2	2	C_6	C_3	4	4	C_2	✓	✓	✓
4	2	0	2	3	D_4	C_4	6	4	C_2		✓	✓
4	2	0	2	4,5	D_4	C_2^2	10	5	C_2		✓	✓
4	2	0	2	6	D_6	C_3	4	4	C_2^2		✓	✓
4	2	0	2	7	D_6	C_6	10	4	C_2		✓	✓
4	2	0	2	8,9	D_6	S_3	10	4	C_2		✓	✓
4	3	1	2	1	C_6	C_3	2	2	C_2	✓		✓
4	3	0	3	2	C_6	C_3	5	5	C_2	✓	✓	✓
4	3	0	3	3	$C_2 \times C_4$	C_4	4	4	C_2	✓	✓	✓
4	3	0	3	4	$C_2 \times C_4$	C_2^2	12	6	C_2	✓	✓	✓
4	3	1	2	5	$C_2 \times C_4$	C_4	4	2	C_2	✓	✓	✓
4	3	0	3	6,7	$C_2 \times C_4$	C_4	8	5	C_2	✓	✓	✓
4	3	0	3	8	$C_2 \times C_4$	C_2^2	12	6	C_2	✓	✓	✓
4	3	0	3	10	A_4	C_2^2	12	6	C_3		✓	✓
4	3	1	2	13–15	$C_2 \times D_4$	C_4	4	2	C_2^2		✓	✓
4	3	1	2	16	$C_2 \times D_4$	D_4	4	1	C_2		✓	✓
4	3	0	3	17–19	$D_4 \rtimes C_2$	C_2^2	12	6	C_2^2	✓	✓	✓
4	3	0	3	20	$D_4 \rtimes C_2$	C_4	4	4	C_2^2	✓	✓	✓
4	3	0	3	21–23	$D_4 \rtimes C_2$	D_4	20	5	C_2	✓	✓	✓
4	3	0	3	24–26	$D_4 \rtimes C_2$	$C_2 \times C_4$	12	4	C_2	✓	✓	✓
4	3	0	3	27	S_4	C_2^2	12	6	S_3		✓	✓
4	3	0	3	28	S_4	A_4	20	4	C_2		✓	✓
4	4	0	4	1	C_6	C_3	6	6	C_2	✓	✓	✓
4	4	0	4	2–4	Q_8	C_4	10	6	C_2		✓	✓
4	4	2	2	5,6	C_3^2	C_3	0	0	C_3	✓	✓	✓
4	4	1	3	7,8	C_3^2	C_3	3	3	C_3	✓	✓	✓
4	4	0	4	9	C_3^2	C_3	6	6	C_3	✓	✓	✓
4	4	0	4	11	$C_2 \times C_6$	C_3	6	6	C_2^2	✓	✓	✓
4	4	0	4	12	$C_2 \times C_6$	C_2^2	14	7	C_3	✓	✓	✓
4	4	0	4	13,15	$C_2 \times C_6$	C_6	12	5	C_2	✓	✓	✓
4	4	0	4	14	$C_2 \times C_6$	C_6	6	4	C_2	✓	✓	✓
4	4	2	2	16	$C_3 \times S_3$	C_3	0	0	S_3	✓	✓	✓
4	4	2	2	17	$C_3 \times S_3$	C_3	0	0	C_6		✓	✓
4	4	0	4	18	$C_3 \times S_3$	C_3	6	6	S_3	✓	✓	✓
4	4	0	4	19	$C_3 \times S_3$	S_3	18	6	C_3	✓	✓	✓
4	4	0	4	20	$C_3 \times S_3$	C_3^2	12	4	C_2	✓	✓	✓
4	4	2	2	21,22	$C_3 \rtimes S_3$	C_3	0	0	S_3		✓	✓
4	4	2	2	25,26	S_3^2	C_3	0	0	D_6		✓	✓

r	$\widetilde{g}$	g	p	#	$\widetilde{G}$	H	Ram pt	Br pt	G	(B1)	(B2)	(B)
4	5	0	5	1	C_8	C_4	8	6	C_2	✓	✓	✓
4	5	1	4	2	$C_2 \times C_4$	C_2^2	8	4	C_2	✓		✓
4	5	0	5	5	$C_3 \rtimes C_4$	C_6	16	6	C_2		✓	✓
4	5	1	4	8	$C_2 \times C_6$	C_3	4	4	C_2^2	✓		✓
4	5	1	4	9	$C_2 \times C_6$	C_6	4	2	C_2	✓		✓
4	5	2	3	12	$C_2^2 \rtimes C_4$	C_2^2	0	0	C_4		✓	✓
4	5	2	3	15	$C_2^2 \times C_4$	C_2^2	0	0	C_4	✓	✓	✓
4	5	1	4	16,18	$C_2^2 \times C_4$	C_4	8	4	C_2^2	✓	✓	✓
4	5	1	4	17	$C_2^2 \times C_4$	C_2^2	8	4	C_2^2	✓	✓	✓
4	5	1	4	19	$C_2^2 \times C_4$	$C_2 \times C_4$	8	2	C_2	✓	✓	✓
4	5	2	3	29	$C_2 \times A_4$	C_2^2	0	0	C_6		✓	✓
4	5	2	3	30	$C_2^2 \rtimes D_4$	C_2^2	0	0	D_4	✓	✓	✓
4	5	1	4	31	$C_2^2 \rtimes D_4$	C_2^2	8	4	C_2^3		✓	✓
4	5	1	4	32,33	$C_2^2 \rtimes D_4$	C_4	8	4	D_4		✓	✓
4	5	1	4	34-36	$C_2^2 \rtimes D_4$	$C_2 \times C_4$	8	2	C_2^2		✓	✓
4	5	1	4	37	$C_2^2 \rtimes D_4$	$C_4 \rtimes C_4$	8	1	C_2		✓	✓
4	5	2	3	45	$C_2 \times S_4$	C_2^2	0	0	D_6		✓	✓
4	6	0	6	1	C_{10}	C_5	5	5	C_2	✓	✓	✓
4	7	1	6	1,2	C_8	C_4	4	4	C_2	✓		✓
4	7	1	6	3	C_9	C_3	6	6	C_3	✓		✓
4	7	1	6	4	C_{10}	C_5	3	3	C_2	✓		✓
4	7	1	6	8	C_{12}	C_3	6	6	C_4	✓		✓
4	7	1	6	9	C_{12}	C_4	8	5	C_3	✓		✓
4	7	0	7	10	C_{12}	C_6	12	6	C_2	✓	✓	✓
4	7	2	5	11	C_{12}	C_3	3	3	C_4	✓		✓
4	7	1	6	12	$C_2 \times C_6$	C_3	6	6	C_2^2	✓	✓	✓
4	7	1	6	13	$C_2 \times C_6$	C_6	6	3	C_2	✓	✓	✓
4	7	2	5	14,15	C_4^2	C_4	4	2	C_4	✓	✓	✓
4	7	2	5	16,17	$C_4 \rtimes C_4$	C_4	4	2	C_4		✓	✓
4	7	1	6	18	$C_2 \times C_8$	C_4	4	4	C_2^2	✓	✓	✓
4	7	2	5	19	$C_2 \times C_8$	C_4	4	2	C_4	✓	✓	✓
4	7	1	6	20	$C_2 \times C_8$	C_8	4	2	C_2	✓	✓	✓
4	7	1	6	23–25	$C_2 \times Q_8$	C_4	12	6	C_2^2		✓	✓
4	7	1	6	26	$C_2 \times Q_8$	Q_8	12	3	C_2		✓	✓
4	7	2	5	29	$C_3 \times S_3$	C_3	3	3	S_3	✓	✓	✓
4	7	3	4	30	$C_3 \times S_3$	C_3	0	0	C_6		✓	✓
4	7	2	5	31	$C_3 \times S_3$	C_3	3	3	S_3			
4	7	2	5	32	$C_3 \times C_6$	C_3	3	3	C_6	✓	✓	✓
4	7	3	4	33	$C_3 \times C_6$	C_3	0	0	C_6	✓	✓	✓

r	$\widetilde{g}$	g	p	#	$\widetilde{G}$	H	Ram pt	Br pt	G	(B1)	(B2)	(B)
4	7	1	6	41	$D_8 \rtimes C_2$	C_4	4	4	C_2^3		✓	✓
4	7	2	5	42	$D_8 \rtimes C_2$	C_4	4	2	D_4	✓	✓	✓
4	7	1	6	43–45	$D_8 \rtimes C_2$	C_8	4	2	C_2^2		✓	✓
4	7	1	6	46	$D_8 \rtimes C_2$	Q_{16}	4	1	C_2		✓	✓
4	8	2	6	1	C_9	C_3	4	4	C_3	✓		✓
4	9	1	8	1	C_{10}	C_5	4	4	C_2	✓	✓	✓
4	9	2	7	2	C_{12}	C_4	4	3	C_3	✓		✓
4	9	3	6	3	C_{12}	C_3	2	2	C_4	✓		✓
4	9	3	6	4,5	C_4^2	C_4	0	0	C_4	✓		✓
4	9	3	6	6	C_4^2	C_4	0	0	C_4	✓	✓	✓
4	9	1	8	8	$C_2 \times C_8$	C_4	8	6	C_2^2	✓	✓	✓
4	9	1	8	9	$C_2 \times C_8$	C_8	8	3	C_2	✓	✓	✓
4	9	2	7	10	$C_2 \times C_8$	C_4	8	4	C_4	✓	✓	✓
4	9	3	6	13	$C_2^2 \times C_6$	C_2^2	0	0	C_6	✓	✓	✓
4	9	3	6	16	$C_4 \wr C_2$	C_4	0	0	D_4	✓	✓	✓
4	9	3	6	18	$C_4 \times D_4$	C_4	0	0	$C_2 \times C_4$		✓	✓
4	9	3	6	22	$C_2 \times C_3 \rtimes D_4$	C_2^2	0	0	D_6		✓	✓
4	9	3	6	23,24	$D_4.D_4$	C_4	0	0	$C_2 \times D_4$		✓	✓
4	10	3	7	1,3	C_{12}	C_3	3	3	C_4	✓		✓
4	10	1	9	2	C_{12}	C_6	6	4	C_2	✓		✓
4	10	2	8	4	C_{12}	C_3	6	6	C_4	✓	✓	✓
4	10	2	8	5	C_{12}	C_4	6	4	C_3	✓	✓	✓
4	10	1	9	6,7	C_{14}	C_7	3	3	C_2	✓		✓
4	10	4	6	8	$C_3 \times C_6$	C_3	0	0	C_6	✓	✓	✓
4	10	2	8	9	$C_3 \times C_6$	C_3	6	6	C_6	✓	✓	✓
4	10	2	8	10,11	$C_3 \times C_6$	C_6	6	2	C_3	✓	✓	✓
4	10	4	6	12	$C_3 \times C_6$	C_3	0	0	C_6	✓	✓	✓
4	10	2	8	13	$C_3 \times D_4$	C_3	6	6	D_4	✓	✓	✓
4	10	2	8	14	$C_3 \times D_4$	C_4	6	4	C_6		✓	✓
4	10	4	6	18–20	C_3^3	C_3	0	0	C_3^2	✓	✓	✓
4	10	2	8	25	$C_6 \times S_3$	C_3	6	6	D_6	✓	✓	✓
4	10	4	6	26	$C_6 \times S_3$	C_3	0	0	$C_2 \times C_6$		✓	✓
4	10	2	8	27	$C_6 \times S_3$	C_6	6	2	C_6		✓	✓
4	10	4	6	28	$C_6 \times S_3$	C_3	0	0	D_6	✓	✓	✓
4	10	4	6	33	$C_3 \times C_3 \rtimes S_3$	C_3	0	0	$C_3 \rtimes S_3$	✓	✓	✓
4	10	4	6	34,35	$C_3 \times C_3 \rtimes S_3$	C_3	0	0	$C_3 \times S_3$		✓	✓
4	10	4	6	39–41	$C_3^2 \rtimes C_2^2$	C_3	0	0	S_3^2		✓	✓
4	11	3	8	1	$C_2 \times C_{12}$	C_4	4	2	C_6	✓	✓	✓
4	11	3	8	2	$D_{12} \rtimes C_3$	C_4	4	2	D_6		✓	✓
4	12	2	10	1	C_{15}	C_5	3	3	C_3	✓		✓

r	$\widetilde{g}$	g	p	#	$\widetilde{G}$	H	Ram pt	Br pt	G	(B1)	(B2)	(B)
4	12	3	9	2	C_{18}	C_3	5	5	C_6	✓		✓
4	13	4	9	1,2	C_{15}	C_3	3	3	C_5	✓		✓
4	13	5	8	3	$C_3 \times C_6$	C_3	0	0	C_6	✓		✓
4	14	4	10	1	C_{15}	C_3	4	4	C_5	✓	✓	✓
4	14	4	10	2	$C_3 \times D_5$	C_3	4	4	D_5	✓	✓	✓
4	16	6	10	1	$C_3 \times C_9$	C_3	0	0	C_9	✓	✓	✓
4	16	6	10	4	$C_3 \times D_9$	C_3	0	0	D_9	✓	✓	✓
5	3	0	3	1,2	D_4	C_2^2	12	6	C_2		✓	✓
5	3	1	2	3–5	C_2^3	C_2^2	4	2	C_2		✓	✓
5	4	1	3	1	C_6	C_3	3	3	C_2	✓		✓
5	4	0	4	2	C_6	C_3	6	6	C_2	✓	✓	✓
5	5	1	4	1	C_6	C_3	4	4	C_2	✓		✓
5	5	1	4	2	$C_2 \times C_4$	C_4	8	4	C_2			✓
5	5	1	4	3,4	$C_2 \times C_4$	C_4	4	3	C_2			✓
5	5	1	4	5	$C_2 \times C_4$	C_4	8	4	C_2	✓	✓	✓
5	5	2	3	17	$C_2 \times D_4$	C_2^2	0	0	C_2^2		✓	✓
5	7	1	6	1	C_6	C_3	6	6	C_2	✓	✓	✓
5	7	3	4	2,3	C_3^2	C_3	0	0	C_3	✓	✓	✓
5	7	2	5	4	C_3^2	C_3	3	3	C_3	✓	✓	✓
5	7	2	5	5–7	C_3^2	C_3	3	3	C_3			✓
5	7	3	4	8	C_3^2	C_3	0	0	C_3			✓
5	9	1	8	1	C_8	C_4	8	6	C_2	✓		✓
5	9	3	6	3–12	$C_4.C_2^3$	C_4	0	0	C_2^3		✓	✓
5	10	2	8	1	$C_2 \times C_6$	C_3	6	6	C_2^2	✓	✓	✓
5	10	4	6	5	$C_3 \times S_3$	C_3	0	0	S_3	✓	✓	✓
5	12	3	9	1	C_9	C_3	5	5	C_3	✓		✓
5	13	4	9	1	C_{12}	C_3	3	3	C_4	✓		✓
6	5	2	3	3	C_2^3	C_2^2	0	0	C_2		✓	✓
6	5	2	3	4	C_2^3	C_2^2	0	0	C_2			✓
6	7	2	5	1	C_6	C_3	3	3	C_2	✓		✓
6	7	2	5	2,3	$C_2 \times C_4$	C_4	4	2	C_2			✓
6	10	2	8	1	C_6	C_3	6	6	C_2	✓	✓	✓
6	10	4	6	2	C_3^2	C_3	0	0	C_3	✓	✓	✓
6	10	4	6	3–5	C_3^2	C_3	0	0	C_3			✓

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