



The Stokes resolvent problem: optimal pressure estimates and remarks on resolvent estimates in convex domains

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Abstract

The Stokes resolvent problem $\lambda u - \Delta u + \nabla \phi = f$ with $\operatorname{div}(u) = 0$ subject to homogeneous Dirichlet or homogeneous Neumann-type boundary conditions is investigated. In the first part of the paper we show that for Neumann-type boundary conditions the operator norm of $L^2_\sigma(\Omega) \ni f \mapsto \phi \in L^2(\Omega)$ decays like $|\lambda|^{-1/2}$ which agrees exactly with the scaling of the equation. In comparison to that, the operator norm of this mapping under Dirichlet boundary conditions decays like $|\lambda|^{-\alpha}$ for $0 \leq \alpha \leq 1/4$ and we show optimality of this rate, thereby, violating the natural scaling of the equation. In the second part of this article, we investigate the Stokes resolvent problem subject to homogeneous Neumann-type boundary conditions if the underlying domain Ω is convex. Invoking a famous result of Grisvard (Elliptic problems in nonsmooth domains. Monographs and studies in mathematics, Pitman, 1985), we show that weak solutions u with right-hand side $f \in L^2(\Omega; \mathbb{C}^d)$ admit H^2 -regularity and further prove localized H^2 -estimates for the Stokes resolvent problem. By a generalized version of Shen's L^p -extrapolation theorem (Shen in Ann Inst Fourier (Grenoble) 55(1):173–197, 2005) we establish optimal resolvent estimates and gradient estimates in $L^p(\Omega; \mathbb{C}^d)$ for $2d/(d+2) < p < 2d/(d-2)$ (with $1 < p < \infty$ if $d = 2$). This interval is larger than the known interval for resolvent estimates subject to Dirichlet boundary conditions (Shen in Arch Ration Mech Anal 205(2):395–424, 2012) on general Lipschitz domains.

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1 Introduction

The main object under investigation is the Stokes resolvent problem in a bounded domain $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = f & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega. \end{cases} \tag{Res}$$

The resolvent parameter λ is supposed to be contained in a sector S_θ , $\theta \in [0, \pi)$, in the complex plane, i.e., $S_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ if $\theta \in (0, \pi)$ and $S_0 := (0, \infty)$. In this article, this system is complemented with two different types of boundary conditions. There are the homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \tag{Dir}$$

and a family of homogeneous Neumann-type boundary conditions which read

$$\{Du + \mu[Du]^\top\}n - \phi n = 0 \quad \text{on } \partial\Omega. \tag{Neu}$$

Here $\mu \in (-1, 1]$ is a parameter, n denotes the outward unit normal to Ω , and Du the Jacobi-matrix of u . There is a tremendous literature on these equations on different types of domains, see, e.g., [1,6,7,16,17,23,26,35,42,43,50,56,58] to mention only a few. Notice that the Neumann-type boundary condition with $\mu = 1$ plays an eminent role in the study of problems involving a free boundary [2,4,29,48,53] and that the condition with $\mu = 0$ is central in the study of inhomogeneous boundary value problems involving the Stokes equations [14,44,50] but also in the modelling of flows in pipes [38, Sec. 4.2]. In this article, we investigate two different questions:

Question 1:

The first question deals with the behavior of the operator norm of the mapping $f \mapsto \phi$ with respect to λ , i.e., we seek an inequality of the form

$$\|\phi\|_{L^2(\Omega)} \leq C(\lambda)\|f\|_{L^2(\Omega; \mathbb{C}^d)} \quad (f \in L^2_\sigma(\Omega))$$

and the *exact* dependence of the constant $C(\lambda)$ with respect to λ . Notice that in the case of homogeneous Dirichlet boundary conditions the pressure ϕ is unique up to an additive constant so that we assume its mean value to be zero. Notice further that the space of solenoidal L^2 -integrable functions differs depending on whether (Res) is considered with condition (Dir) or (Neu), cf. Sect. 2.

Pressure estimates of the Stokes resolvent problem are studied in the engineering literature [37] and they also appear in the study of the Stokes operator [21,23,47,58]. Another interesting application can be found in the analysis of the discrete Stokes resolvent problem, comparable to the Poisson case in [36,54], where the pressure appears in the derivation of weighted norm estimates.

To obtain an idea of what the right behavior of $C(\lambda)$ with respect to λ would be, set for a moment $\Omega = \mathbb{R}^d$. In this case, the solutions u and ϕ satisfy the following scaling property: Let $r > 0$ and assume that u and ϕ solve (Res) for some resolvent parameter λ and right-hand side f . Then, $u_r := u(r \cdot)$ and $\phi_r := r\phi(r \cdot)$ solve (Res) for the resolvent parameter $r^2\lambda$ and right-hand side $f_r := r^2 f(r \cdot)$. Put $r := |\lambda|^{-1/2}$ so that $|r^2\lambda| = 1$. If there *would* be a

constant $C > 0$ (which on the whole space certainly does not have to be true) such that

$$\|\phi_r\|_{L^2(\mathbb{R}^d)} \leq C \|f_r\|_{L^2(\mathbb{R}^d; \mathbb{C}^d)}$$

holds, then the substitution rule ensures the estimate

$$\|\phi\|_{L^2(\mathbb{R}^d)} \leq C |\lambda|^{-1/2} \|f\|_{L^2(\mathbb{R}^d; \mathbb{C}^d)}. \tag{1.1}$$

We will show in Sect. 3 that this behavior of $C(\lambda)$ is *false* on bounded \mathbb{C}^d -domains and if homogeneous Dirichlet boundary conditions are imposed. More precisely, it is known [31,47,58] that $C(\lambda)$ satisfies for each $0 \leq \alpha \leq 1/4$ and some constant $C > 0$ independent of λ

$$C(\lambda) \leq C |\lambda|^{-\alpha}, \tag{1.2}$$

see also Proposition 3.3 for the case $0 \leq \alpha < 1/4$. In Proposition 3.4 we show that the condition $\alpha \leq 1/4$ is sharp in the sense that for no $\alpha > 1/4$ there exists a constant $C > 0$ independent of λ such that (1.2) is valid. This shows that the presence of a boundary causes the pressure to behave differently than its natural scaling would dictate.

In contrast to that, under boundary condition (Neu), then on each domain Ω with a sufficiently nice boundary, e.g., bounded $\mathbb{C}^{1,1}$ -domains or bounded convex domains, we show that $C(\lambda)$ satisfies (1.2) with $\alpha = 1/2$, see Proposition 3.1. Thus, depending on the particular boundary condition at stake, the behavior of the pressure with respect to λ might differ.

For both boundary conditions, we perform a similar analysis in which the L^2 -norm of f on the right-hand side is replaced by the H^{-1} -norm of f , see Propositions 3.6 and 3.8. For simplicity, we considered only L^2 -based spaces. An extension to the L^p -situation should be straightforward. Notice that the exponent α for which the pressure estimates in L^p are valid satisfies the relation $\alpha \leq 1/2 - 1/(2p)$, see [31,47], so that the decay estimate with exponent $\alpha > 1/2 - 1/(2p)$ should fail.

Question 2:

If $\Omega \subset \mathbb{R}^d$, $d \geq 3$, is a bounded Lipschitz domain the resolvent estimate

$$\|\lambda\| \|u\|_{L^p(\Omega; \mathbb{C}^d)} \leq C \|f\|_{L^p(\Omega; \mathbb{C}^d)} \quad (f \in L^p_\sigma(\Omega)) \tag{1.3}$$

was proven for solutions to (Res) subject to the boundary condition (Dir) in the seminal paper of Shen [50]. Here, p satisfies

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{2d} + \varepsilon \tag{1.4}$$

for some $\varepsilon > 0$ depending only on d , θ , and the Lipschitz geometry. A special class of bounded Lipschitz domains are bounded convex domains and one might wonder, whether the condition (1.4) on p improves if convexity of Ω is imposed. It was for example proven by Geng and Shen [24] that on bounded and convex domains the Helmholtz projection gives rise to a bounded projection on $L^p(\Omega; \mathbb{C}^d)$ for all $1 < p < \infty$. Moreover, a work of Geissert, Heck, Hieber, and Sawada [22] formalizes the philosophy that the boundedness of the Helmholtz projection implies functional analytic properties of the Stokes operator like (1.3) at least under the condition that Ω is a (not necessarily bounded) uniform \mathbb{C}^3 -domain. Combining the result of [24] with this philosophy leads to the conjecture that the resolvent estimate (1.3) should be valid for all $1 < p < \infty$ if Ω is convex. This is a question that was raised by Maz'ya in [39, Prob. 66].

We give first results in this direction for the Stokes resolvent problem (Res) subject to the Neumann-type boundary condition (Neu), however, we restrict the interval of parameters μ to be $(-1, \sqrt{2} - 1)$. Although this still includes the case $\mu = 0$ the physically important case $\mu = 1$ is excluded. The corresponding results are explained as follows. We remark that on general bounded Lipschitz domains the Stokes operator subject to the Neumann-type boundary condition (Neu) was investigated by Mitrea, Monniaux, and Wright [43] who established a rich L^2 -theory, see also the survey article by Monniaux and Shen [46].

By virtue of a famous formula of integration by parts by Grisvard [28, Thm. 3.1.1.1] we establish the estimate

$$|\lambda| \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\nabla^2 u|^2 \, dx + \int_{\Omega} |\nabla \phi|^2 \, dx \leq C \left(\int_{\Omega} |f|^2 \, dx + |\lambda|^2 \int_{\Omega} |u|^2 \, dx \right) \tag{1.5}$$

for some constant $C > 0$ depending only on d, θ , and μ , see Theorem 4.4. In particular, this implies that solutions u and ϕ to $-\Delta u + \nabla \phi = f$ and $\operatorname{div}(u) = 0$ for some $f \in L^2_{\sigma}(\Omega)$ and subject to the boundary condition (Neu) satisfy $u \in H^2(\Omega; \mathbb{C}^d)$ and $\phi \in H^1(\Omega)$. This should be compared with the results of Kellogg and Osborn [34], Dauge [10], and Maz'ya and Rossmann [40] in the case of the boundary condition (Dir) and convex polygonal/polyhedral domains. For general bounded and convex domains this higher regularity property in the case of homogeneous Dirichlet boundary conditions is unknown.

We continue by establishing a localized version of (1.5) which can be found in Proposition 4.12. Combining this with a Caccioppoli type estimate, see Lemma 6.1, and Sobolev's embedding yields the validity of a weak reverse Hölder estimate of the form

$$\begin{aligned} & \left(\frac{1}{r^d} \int_{\Omega \cap Q(x_0, r)} \{ |\lambda| |u| + |\lambda|^{1/2} |\nabla u| + |\lambda|^{1/2} |\phi| \}^p \, dx \right)^{1/p} \\ & \leq C \left(\frac{1}{r^d} \int_{\Omega \cap Q(x_0, 2r)} \{ |\lambda| |u| + |\lambda|^{1/2} |\nabla u| + |\lambda|^{1/2} |\phi| \}^2 \, dx \right)^{1/2}, \end{aligned} \tag{1.6}$$

where $Q(x_0, r)$ is a cube in \mathbb{R}^d with midpoint x_0 and diameter $r > 0$, where p satisfies $2 < p < \infty$ if $d = 2$ and $p = 2d/(d - 2)$ if $d \geq 3$, and where u and ϕ solve the Stokes resolvent problem with a right-hand side $f \in L^2(\Omega; \mathbb{C}^d)$ that vanishes on $\Omega \cap Q(x_0, 2r)$. One could now conclude by an L^p -extrapolation theorem of Shen [49] that the family of (sublinear) operators

$$L^q(\Omega; \mathbb{C}^d) \ni f \mapsto |\lambda| |u| + |\lambda|^{1/2} |\nabla u| + |\lambda|^{1/2} |\phi|$$

is uniformly bounded with respect to λ on L^q for $2 < q < 2d/(d - 2)$ if(!) the family of operators

$$T_{\lambda} : L^2(\Omega; \mathbb{C}^d) \rightarrow L^2(\Omega), \quad f \mapsto |\lambda|^{1/2} \phi \tag{1.7}$$

is uniformly bounded on L^2 . This gives a connection to Question 1 discussed above. Unfortunately, only the restriction of T_{λ} to solenoidal vector fields satisfies this uniform bound, whereas the operators on all of $L^2(\Omega; \mathbb{C}^d)$ grow like $|\lambda|^{1/2}$. This fact can be easily seen by noting that the pressure ϕ solving (Res) for general $f \in L^2(\Omega; \mathbb{C}^d)$ is the sum of the pressure associated to (Res) but with the right-hand side $\mathbb{Q}f \in L^2_{\sigma}(\Omega)$ and the function g which satisfies $(\mathbb{Q} - \operatorname{Id})f = \nabla g$. Here, \mathbb{Q} denotes the Helmholtz projection on $L^2(\Omega; \mathbb{C}^d)$. Notice that the function g does not depend on λ at all, which explains that the family defined in (1.7) is not uniformly bounded on L^2 .

To circumvent this problem, we discuss in Sect. 5 a version of Shen’s L^p -extrapolation theorem, which is valid for *subspaces* of L^p . This allows us to employ the uniform boundedness of the restriction of the operators T_λ to solenoidal spaces and delivers the following theorem which is proven in Sect. 6.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded and convex domain and $r_0 > 0$ be such that $B(0, r_0) \subset \frac{1}{2}[\Omega - \{x_0\}]$ for some $x_0 \in \Omega$. Let further $\theta \in [0, \pi)$, $\mu \in (-1, \sqrt{2} - 1)$, and let*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d}.$$

Then there exists a constant $C > 0$ such that for all $\lambda \in S_\theta$ and all $f \in L^2(\Omega; \mathbb{C}^d) \cap L^p(\Omega; \mathbb{C}^d)$ satisfying $\operatorname{div}(f) = 0$ in the sense of distributions the solutions $u \in H^1(\Omega; \mathbb{C}^d)$ and $\phi \in L^2(\Omega)$ to

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = f & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ \{Du + \mu[Du]^\top\}n - \phi n = 0 & \text{on } \partial\Omega \end{cases}$$

satisfy

$$|\lambda| \|u\|_{L^p(\Omega; \mathbb{C}^d)} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega; \mathbb{C}^{d^2})} \leq C \|f\|_{L^p(\Omega; \mathbb{C}^d)}.$$

If $p \geq 2$ it additionally holds

$$|\lambda|^{1/2} \|\phi\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega; \mathbb{C}^d)}.$$

Here, the constant $C > 0$ depends only on $d, \theta, \mu, \operatorname{diam}(\Omega)$, and r_0 .

Furthermore, there exists a constant $C > 0$ such that for all $\lambda \in S_\theta$ and all $F \in L^2(\Omega; \mathbb{C}^{d \times d}) \cap L^p(\Omega; \mathbb{C}^{d \times d})$ the solutions $u \in H^1(\Omega; \mathbb{C}^d)$ and $\phi \in L^2(\Omega)$ to

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = \operatorname{div}(F) & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ \{Du + \mu[Du]^\top\}n - \phi n = 0 & \text{on } \partial\Omega \end{cases}$$

satisfy

$$|\lambda|^{1/2} \|u\|_{L^p(\Omega; \mathbb{C}^d)} + \|\nabla u\|_{L^p(\Omega; \mathbb{C}^{d^2})} + \|\phi\|_{L^p(\Omega)} \leq C \|F\|_{L^p(\Omega; \mathbb{C}^{d \times d})}.$$

If $p \geq 2$ it additionally holds

$$\|\phi\|_{L^p(\Omega)} \leq C \|F\|_{L^p(\Omega; \mathbb{C}^{d \times d})}.$$

Again, the constant $C > 0$ depends only on $d, \theta, \mu, \operatorname{diam}(\Omega)$, and r_0 .

2 The Stokes operator on $L^2_\sigma(\Omega)$ and $H^{-1}_\sigma(\Omega)$

In the following, we will assume that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded and open domain whose boundary is at least Lipschitz regular, i.e., locally represented as the graph of a Lipschitz continuous function. This section is devoted to present results concerning the Stokes resolvent problem (Res) subject to no-slip boundary conditions (Dir) and subject to Neumann-type boundary conditions (Neu).

2.1 Function spaces

We define the space of compactly supported smooth and solenoidal vector fields in Ω as

$$C_{c,\sigma}^\infty(\Omega) := \{\varphi \in C_c^\infty(\Omega; \mathbb{C}^d) : \operatorname{div}(\varphi) = 0\}$$

and the space of solenoidal vector fields that are smooth up to the boundary as

$$C_\sigma^\infty(\overline{\Omega}) := \{\varphi|_\Omega : \varphi \in C_{c,\sigma}^\infty(\mathbb{R}^d)\}.$$

As usual, we define for $1 < p < \infty$

$$L_\sigma^p(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{L^p} \quad \text{and} \quad W_{0,\sigma}^{1,p}(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{W^{1,p}}$$

endowed with their natural norms. These spaces are usually introduced if the Stokes equations subject to no-slip boundary conditions are studied, e.g., see [20,42,51]. If one is interested in Neumann-type boundary conditions, then one defines the spaces

$$\mathcal{L}_\sigma^p(\Omega) := \{u \in L^p(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0\} \quad \text{and} \quad \mathcal{W}_\sigma^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0\}$$

endowed with their natural norms, see, e.g., [43,52]. If $p = 2$ we write $H_{0,\sigma}^1(\Omega) := W_{0,\sigma}^{1,2}(\Omega)$ and $\mathcal{H}_\sigma^1(\Omega) := \mathcal{W}_\sigma^{1,2}(\Omega)$ henceforth. Notice that $L_\sigma^2(\Omega)$ and $\mathcal{L}_\sigma^2(\Omega)$ do in general not coincide. Indeed, elements $u \in L_\sigma^2(\Omega)$ satisfy $n \cdot u = 0$ on $\partial\Omega$ whereas the mean value of $n \cdot u$ on $\partial\Omega$ vanishes for elements u in $\mathcal{L}_\sigma^2(\Omega)$. Furthermore, notice that $C_\sigma^\infty(\overline{\Omega})$ embeds densely into $\mathcal{L}_\sigma^2(\Omega)$, by [43, Lem. 2.1, Rem. 2.2]. We define the antidual spaces

$$H_\sigma^{-1}(\Omega) := H_{0,\sigma}^1(\Omega)^* \quad \text{and} \quad \mathcal{H}_{0,\sigma}^{-1}(\Omega) := \mathcal{H}_\sigma^1(\Omega)^*,$$

where we consider antilinear functionals instead of linear functionals, i.e., they satisfy $f(\alpha \cdot) = \overline{\alpha} f(\cdot)$ for $\alpha \in \mathbb{C}$ instead of the usual homogeneity condition. We further define $H^{-1}(\Omega; \mathbb{C}^d) := H_0^1(\Omega; \mathbb{C}^d)^*$ and $H_0^{-1}(\Omega; \mathbb{C}^d) := H^1(\Omega; \mathbb{C}^d)^*$. Notice that due to the embeddings

$$H_{0,\sigma}^1(\Omega) \hookrightarrow H_0^1(\Omega; \mathbb{C}^d) \quad \text{and} \quad \mathcal{H}_\sigma^1(\Omega) \hookrightarrow H^1(\Omega; \mathbb{C}^d)$$

one can identify an element in $H^{-1}(\Omega; \mathbb{C}^d)$ and in $H_0^{-1}(\Omega; \mathbb{C}^d)$ with an element in $H_\sigma^{-1}(\Omega)$ and in $\mathcal{H}_{0,\sigma}^{-1}(\Omega)$, respectively, by restriction to the respective subspace. Notice further, that an element $u \in L^2(\Omega; \mathbb{C}^d)$ can be considered as an element in the spaces of negative order by identifying u with the functional

$$\Phi(u)(v) := \int_\Omega u \cdot \overline{v} \, dx$$

endowed with the respective domain of definition.

For $0 < s < 1$ we consider as intermediate spaces the scale of L^2 -based Bessel potential spaces $H^s(\Omega) = H^{s,2}(\Omega)$ which are defined as the restriction spaces of Bessel potential spaces on the whole space. The solenoidal counterparts are denoted by $H_\sigma^s(\Omega)$ and are defined to be $H^s(\Omega; \mathbb{C}^d) \cap L_\sigma^2(\Omega)$. If $s > 1/2$ we also define the corresponding spaces with vanishing trace, i.e., $H_{0,\sigma}^s(\Omega) := H_0^s(\Omega; \mathbb{C}^d) \cap L_\sigma^2(\Omega)$. In the case of negative indices, we define for $0 < s < 1/2$ the space $H_\sigma^{-s}(\Omega) := H_\sigma^s(\Omega)^*$.

Having introduced all required function spaces, we will introduce the Stokes operators subject to no-slip and Neumann boundary conditions following [42,43].

2.2 The Stokes operator subject to no-slip boundary conditions

Define the sesquilinear form

$$a : H_{0,\sigma}^1(\Omega) \times H_{0,\sigma}^1(\Omega) \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx.$$

The weak Stokes operator \mathcal{A} on $H_{\sigma}^{-1}(\Omega)$ subject to no-slip boundary conditions is defined as

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &:= H_{0,\sigma}^1(\Omega), \\ \langle \mathcal{A}u, v \rangle_{H_{\sigma}^{-1}, H_{0,\sigma}^1} &:= a(u, v) \quad \text{for } u \in \mathcal{D}(\mathcal{A}) \text{ and } v \in H_{0,\sigma}^1(\Omega). \end{aligned}$$

The Stokes operator A on $L_{\sigma}^2(\Omega)$ subject to no-slip boundary conditions is then defined as the part of \mathcal{A} in $L_{\sigma}^2(\Omega)$, i.e.,

$$\begin{aligned} \mathcal{D}(A) &:= \{u \in L_{\sigma}^2(\Omega) : u \in \mathcal{D}(\mathcal{A}) \text{ and } \mathcal{A}u \in L_{\sigma}^2(\Omega)\}, \\ Au &:= \mathcal{A}u \quad (u \in \mathcal{D}(A)). \end{aligned}$$

Elements $u \in \mathcal{D}(A)$ satisfy no-slip boundary conditions. Notice that the symmetry of a implies that A is a self-adjoint operator on $L_{\sigma}^2(\Omega)$, see [33, Thm. VI.2.23]. Furthermore, the definition of A implies for the resolvent sets the inclusion $\rho(\mathcal{A}) \subset \rho(A)$ and that for $\lambda \in \rho(-A)$ it holds

$$(\lambda + A)^{-1} = (\lambda + \mathcal{A})^{-1}|_{L_{\sigma}^2(\Omega)}. \tag{2.1}$$

2.3 The Stokes operator subject to Neumann-type boundary conditions

Define for $\mu \in (-1, 1]$ the coefficients $a_{jk}^{\alpha\beta}(\mu) := \delta_{jk}\delta_{\alpha\beta} + \mu\delta_{j\beta}\delta_{k\alpha}$, where $\delta_{\alpha\beta}$ denotes Kronecker’s delta. Notice that the divergence form operator with coefficients $a_{jk}^{\alpha\beta}(\mu)$ is formally given by (here and below we sum over repeated indices)

$$\partial_j a_{jk}^{\alpha\beta}(\mu) \partial_k u_{\beta} = \Delta u_{\alpha} + \mu \partial_{\alpha} \operatorname{div}(u).$$

Hence, if this operator acts only on solenoidal functions, this is merely the Laplacian. Consequently, defining the sesquilinear form

$$b_{\mu} : \mathcal{H}_{\sigma}^1(\Omega) \times \mathcal{H}_{\sigma}^1(\Omega) \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_{\Omega} a_{jk}^{\alpha\beta}(\mu) \partial_k u_{\beta} \cdot \overline{\partial_j v_{\alpha}} \, dx$$

gives still rise to an operator associated to the Stokes equations. The weak Stokes operator \mathcal{B}_{μ} on $\mathcal{H}_{0,\sigma}^{-1}(\Omega)$ subject to Neumann-type boundary conditions is defined as

$$\begin{aligned} \mathcal{D}(\mathcal{B}_{\mu}) &:= \mathcal{H}_{\sigma}^1(\Omega), \\ \langle \mathcal{B}_{\mu}u, v \rangle_{\mathcal{H}_{0,\sigma}^{-1}, \mathcal{H}_{\sigma}^1} &:= b_{\mu}(u, v) \quad \text{for } u \in \mathcal{D}(\mathcal{B}_{\mu}) \text{ and } v \in \mathcal{H}_{\sigma}^1(\Omega). \end{aligned}$$

For the Stokes operator subject to Neumann-type boundary conditions on the negative scale one has to understand the boundary condition very carefully as right-hand sides in $\mathcal{H}_{0,\sigma}^{-1}(\Omega)$ could induce inhomogeneous boundary terms. For example the functional

$$\mathcal{H}_{\sigma}^1(\Omega) \ni v \mapsto \int_{\partial\Omega} f \cdot \bar{v} \, d\sigma =: F(v)$$

for a smooth function f lies in $\mathcal{H}_{0,\sigma}^{-1}(\Omega)$. Thus, the solution to the problem $\mathcal{B}_\mu u = F$ would satisfy an inhomogeneous boundary condition.

The Stokes operator B_μ on $\mathcal{L}_\sigma^2(\Omega)$ subject to Neumann-type boundary conditions is then defined as the part of \mathcal{B}_μ in $\mathcal{L}_\sigma^2(\Omega)$, i.e.,

$$\begin{aligned} \mathcal{D}(B_\mu) &:= \{u \in \mathcal{L}_\sigma^2(\Omega) : u \in \mathcal{D}(\mathcal{B}_\mu) \text{ and } \mathcal{B}_\mu u \in \mathcal{L}_\sigma^2(\Omega)\}, \\ B_\mu u &:= \mathcal{B}_\mu u \quad (u \in \mathcal{D}(B_\mu)). \end{aligned}$$

Elements $u \in \mathcal{D}(B_\mu)$ formally satisfy the boundary conditions stated in (Neu). Notice that the symmetry of \mathfrak{b}_μ implies that B_μ is a self-adjoint operator on $\mathcal{L}_\sigma^2(\Omega)$, see [33, Thm. VI.2.23]. Furthermore, the definition of B_μ implies for the resolvent sets the inclusion $\rho(B_\mu) \subset \rho(\mathcal{B}_\mu)$ and that for $\lambda \in \rho(-\mathcal{B}_\mu)$ it holds

$$(\lambda + B_\mu)^{-1} = (\lambda + \mathcal{B}_\mu)^{-1}|_{\mathcal{L}_\sigma^2(\Omega)}.$$

2.4 The Laplace operators

Similarly, we introduce the weak Laplace operators $-\Delta_D$ on $H^{-1}(\Omega)$ and $-\Delta_N$ on $H_0^{-1}(\Omega)$ via the sesquilinear form

$$\mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \, dx.$$

The domain of the sesquilinear form \mathcal{V} is taken to be $H_0^1(\Omega)$ for the Dirichlet Laplacian and $H^1(\Omega)$ for the Neumann Laplacian. Recall that by Poincaré’s inequality Δ_D is invertible and that Δ_N is invertible if considered on the factor space $(H^1(\Omega)/\text{const})^*$. Finally, recall that if the boundary of Ω is $C^{1,1}$ -regular or if Ω is convex the operators $\Delta_D^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ and $\Delta_N^{-1} : L_0^2(\Omega) \rightarrow H^2(\Omega)$ are bounded, see [28, Sec. 3.2.1] for the particular statements on convex domains, see also the discussion at the beginning of Sect. 4. Here $L_0^2(\Omega)$ denotes the L^2 -space of average free functions. In the following, we do not distinguish the notation between the weak Laplacians defined on negative spaces or the strong Laplacians defined on $L^2(\Omega)$.

2.5 The Bogovski operator

Let us consider the divergence problem

$$\begin{cases} \operatorname{div}(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L_0^2(\Omega)$ and Ω is a bounded Lipschitz domain. It is well-known that there exists a bounded and linear operator $B : L_0^2(\Omega) \rightarrow H_0^1(\Omega; \mathbb{C}^d)$ which satisfies $\operatorname{div}(Bf) = f$, see, e.g., [20, Ch. III.3] and the references therein. This means that $u := Bf$ solves the divergence problem posed above. Clearly, B is a highly non-unique operator as one can always add a function $v \in H_{0,\sigma}^1(\Omega)$ to the solution u and still have a solution to the problem. Here and below, the operator B is called the Bogovskiĭ operator.

2.6 The Helmholtz projection

The Helmholtz projection $\mathbb{P} : L^2(\Omega; \mathbb{C}^d) \rightarrow L^2(\Omega; \mathbb{C}^d)$ is introduced as being the orthogonal projection of $L^2(\Omega; \mathbb{C}^d)$ onto $L^2_\sigma(\Omega)$. Analogously, let $\mathbb{Q} : L^2(\Omega; \mathbb{C}^d) \rightarrow L^2(\Omega; \mathbb{C}^d)$ denote the orthogonal projection of $L^2(\Omega; \mathbb{C}^d)$ onto $L^2_\sigma(\Omega)$. It is well-known, see [15, Sec. 11], that the range of $\text{Id} - \mathbb{P}$ is given by

$$\mathcal{R}(\text{Id} - \mathbb{P}) = \nabla H^1(\Omega) := \{\nabla\varphi : \varphi \in H^1(\Omega)\} \tag{2.2}$$

and that the range of $\text{Id} - \mathbb{Q}$ is given by

$$\mathcal{R}(\text{Id} - \mathbb{Q}) = \nabla H^1_0(\Omega) := \{\nabla\varphi : \varphi \in H^1_0(\Omega)\}. \tag{2.3}$$

Notice that \mathbb{P} and \mathbb{Q} can be realized by employing the Neumann and the Dirichlet Laplacian as follows. Define a distribution

$$(\widetilde{\text{div}}(u), v)_{H_0^{-1}, H^1} := - \int_\Omega u \cdot \overline{\nabla v} \, dx \quad \text{for } u \in L^2(\Omega; \mathbb{C}^d) \text{ and } v \in H^1(\Omega),$$

which acts as the distribution generated by the divergence operator but ignores the boundary values that would arise due to the integration by parts. Furthermore, define the usual divergence as

$$(\text{div}(u), v)_{H^{-1}, H^1_0} := - \int_\Omega u \cdot \overline{\nabla v} \, dx \quad \text{for } u \in L^2(\Omega; \mathbb{C}^d) \text{ and } v \in H^1_0(\Omega).$$

Then, \mathbb{P} and \mathbb{Q} can be represented as

$$\mathbb{P} = \text{Id} + \nabla(-\Delta_N)^{-1} \widetilde{\text{div}} \quad \text{and} \quad \mathbb{Q} = \text{Id} + \nabla(-\Delta_D)^{-1} \text{div}. \tag{2.4}$$

A calculation verifying this identity for \mathbb{P} can be found in [55, Lem. 5.1.3] and for \mathbb{Q} in the proof of [43, Lem. 2.1]. We record the following lemma.

Lemma 2.1 *Let Ω be a bounded Lipschitz domain such that $\Delta_D^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ and $\Delta_N^{-1} : L^2_0(\Omega) \rightarrow H^2(\Omega)$ are bounded. Then $\mathbb{P} : H^1_0(\Omega; \mathbb{C}^d) \rightarrow H^1(\Omega; \mathbb{C}^d)$ and $\mathbb{Q} : H^1(\Omega; \mathbb{C}^d) \rightarrow H^1(\Omega; \mathbb{C}^d)$ are bounded operators. In particular, if Ω is convex, then these operator norms depend at most on the dimension d .*

Proof Notice that if $u \in H^1_0(\Omega; \mathbb{C}^d)$, then $\widetilde{\text{div}}(u) = \text{div}(u)$ and the average of $\text{div}(u)$ in Ω is zero. Thus, the statements concerning the boundedness of \mathbb{P} and \mathbb{Q} directly follow by (2.4) and the assumption that $\Delta_D^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ and $\Delta_N^{-1} : L^2_0(\Omega) \rightarrow H^2(\Omega)$ are bounded. Concerning the dependence of the constants for Ω being convex, see [28, Eq. (3.1.2.2), Eq. (3.1.2.7)]. □

By (2.4) it should be clear that \mathbb{P} does not map $H^1_0(\Omega; \mathbb{C}^d)$ into $H^1_0(\Omega; \mathbb{C}^d)$, i.e., that it does not preserve zero boundary values (if the boundary is merely Lipschitz, then also the differentiability is not preserved). A formal proof on bounded C^4 -domains is given in the next lemma. The proof uses so-called Fermi coordinates. These coordinates are introduced following the exposition in [12, Sec. 2.3].

Let $\delta(x)$ denote the oriented distance function, i.e.,

$$\delta(x) := \begin{cases} \text{dist}(x, \partial\Omega), & x \in \Omega \\ -\text{dist}(x, \partial\Omega), & x \in \overline{\Omega}^c. \end{cases}$$

If Ω has a C^k -boundary with $k \in \mathbb{N}$ with $k \geq 2$, one verifies by virtue of uniform inner and outer ball properties of $\partial\Omega$, that there exists $\varepsilon > 0$ such that with

$$U_\varepsilon := \{x \in \mathbb{R}^d : |\delta(x)| < \varepsilon\}$$

one has $\delta \in C^k(U_\varepsilon)$ and that for every point $x \in U_\varepsilon$ there exists a unique point $a(x) \in \partial\Omega$ such that

$$x = a(x) + \delta(x)n(a(x)),$$

where n denotes the outward unit normal to $\partial\Omega$. Thus, in the neighborhood U_ε , every point x can be represented uniquely by the new coordinates $a(x)$ and $\delta(x)$. To proceed, we introduce some further geometric notions. A function $u \in L^1(\partial\Omega)$ is weakly differentiable if its composition with the coordinate chart is weakly differentiable in \mathbb{R}^{d-1} . For such functions one can define the tangential gradient $\nabla_T u$ of u (see, e.g., the exposition in [55, Sec. 1.3]). The tangential gradient has the property, that for functions u which are smooth enough and defined in a neighborhood of $\partial\Omega$ one has

$$\nabla u(x) = \nabla_T u(x) + (n(x) \cdot \nabla u(x))n(x) \quad (x \in \partial\Omega).$$

Notice that $\nabla_T u(x)$ for $x \in \partial\Omega$ always lies in the tangent space at x if $\partial\Omega$ is smooth enough. Similarly, we define a vector $v \in \mathbb{C}^d$ and $x \in \partial\Omega$ its tangential component v_T to satisfy

$$v_T = v - (n(x) \cdot v)n(x). \tag{2.5}$$

This will be used in Sect. 4.

Given a function $g \in C^1(\partial\Omega)$, then g can be extended to a function G on U_ε by setting

$$G(x) := g(a(x)) \quad (x \in U_\varepsilon)$$

and [12, Eq. (2.14)] shows that

$$\nabla G(x) = (1 - \delta(x)\mathcal{H}(x))\nabla_T g(a(x)). \tag{2.6}$$

Here, \mathcal{H} denotes the extended Weingarten map, which is given by

$$\mathcal{H}(x) = (\mathcal{H}_{i,j}(x))_{i,j=1}^d := e_i \cdot \nabla_T n_j(x)$$

and which is C^2 -regular if Ω has a C^4 -boundary. Notice that e_i denotes the i th standard basis vector of \mathbb{R}^d .

Lemma 2.2 *Let Ω be a bounded domain with C^4 -boundary. Then there exists $u \in H_0^1(\Omega; \mathbb{C}^d)$ such that the trace of $\mathbb{P}u$ to $\partial\Omega$ is a non-zero function.*

Proof Notice that $\text{div} : H_0^1(\Omega; \mathbb{C}^d) \rightarrow L_0^2(\Omega)$ is surjective by [20, Thm. III.3.1]. Consequently, the range of $\Delta_N^{-1} \text{div}$ is given by

$$\mathcal{R}(\Delta_N^{-1} \text{div}) = \mathcal{R}(\Delta_N^{-1}|_{L_0^2(\Omega)}) = \{u \in H^2(\Omega) : n \cdot \nabla u = 0 \text{ on } \partial\Omega\}.$$

Now, if there exists $u \in H^2(\Omega)$ with $n \cdot \nabla u = 0$ on $\partial\Omega$ and $\nabla u \neq 0$ on $\partial\Omega$, set $f := B\Delta_N u$ (with B being the Bogovskii operator) which lies in $H_0^1(\Omega; \mathbb{C}^d)$ and by virtue of (2.4) $\mathbb{P}f$ satisfies $R_{\partial\Omega}(\mathbb{P}f) \neq 0$, where $R_{\partial\Omega}$ denotes the trace operator to $\partial\Omega$. To construct such a function u , let $g : \partial\Omega \rightarrow \mathbb{C}$ be a non-constant and smooth function and let $\eta \in C_c^\infty(U_\varepsilon)$ with $\eta = 1$ in a neighborhood of $\partial\Omega$. Extend g to U_ε by setting

$$u(x) := g(a(x))\eta(x).$$

The gradient of u is calculated by virtue of (2.6), leading to

$$\nabla u(x) = \eta(x)(1 - \delta(x)\mathcal{H}(x))\nabla_T g(a(x)) + g(a(x))\nabla\eta(x).$$

Clearly, the normal derivative of u vanishes on $\partial\Omega$ while its full gradient is non-zero since g is non-constant on $\partial\Omega$. Since η , δ , and \mathcal{H} are at least C^2 -regular, it follows that $u \in \mathcal{R}(\Delta_N^{-1} \operatorname{div})$ with ∇u not being constantly zero on $\partial\Omega$. \square

2.7 Resolvent estimates

We continue by discussing some classical resolvent estimates in $L^2_\sigma(\Omega)$ and $H^{-1}_\sigma(\Omega)$ for the operators A and \mathcal{A} , and for B_μ and \mathcal{B}_μ on $\mathcal{L}^2_\sigma(\Omega)$ and $\mathcal{H}^{-1}_{0,\sigma}(\Omega)$. In the case of Neumann-type boundary conditions, we restrict ourselves to parameters that satisfy $\mu \in (-1, 1]$. This is due to the fact, that this ensures a certain coercivity of the sesquilinear form b_μ . Indeed, by [44, Prop. 4.1.2], for $|\mu| < 1$ there exists $\kappa_\mu > 0$ such that

$$\operatorname{Re}(a_{jk}^{\alpha\beta}(\mu)\xi_{\beta k}\overline{\xi_{\alpha j}}) \geq \kappa_\mu|\xi|^2 \quad (\xi \in \mathbb{C}^{d \times d}). \tag{2.7}$$

Here, we used the notation $|\xi|^2 := \sum_{\alpha,\beta=1}^d |\xi_{\alpha\beta}|^2$. Moreover, the same result ensures that in the case $\mu = 1$ there exists $\kappa_1 > 0$ such that

$$\operatorname{Re}(a_{jk}^{\alpha\beta}(1)\xi_{\beta k}\overline{\xi_{\alpha j}}) \geq \kappa_1|\xi + \xi^\top|^2 \quad (\xi \in \mathbb{C}^{d \times d}). \tag{2.8}$$

To proceed, define for $\theta \in [0, \pi)$ the sector in the complex plane

$$S_\theta := \begin{cases} (0, \infty), & \text{if } \theta = 0 \\ \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}, & \text{if } \theta \in (0, \pi). \end{cases}$$

Notice, that by elementary trigonometry, one can prove that there exists $C_\theta > 0$ depending only on θ , such that

$$|z| + \alpha \leq C_\theta|z + \alpha| \quad (z \in \overline{S_\theta}, \alpha \geq 0). \tag{2.9}$$

Proposition 2.3 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $\theta \in [0, \pi)$.*

(1) *It holds $\{0\} \cup S_\theta \subset \rho(-A) \cap \rho(-\mathcal{A})$. Moreover, there exists $C > 0$ depending only on d and θ such that for all $f \in L^2_\sigma(\Omega)$, all $F \in H^{-1}_\sigma(\Omega)$, and all $\lambda \in S_\theta$ it holds*

$$|\lambda| \|(\lambda + A)^{-1} f\|_{L^2_\sigma(\Omega)} + |\lambda|^{1/2} \|\nabla(\lambda + A)^{-1} f\|_{L^2(\Omega; \mathbb{C}^{d^2})} \leq C \|f\|_{L^2_\sigma(\Omega)}$$

and

$$\begin{aligned} & |\lambda| \|(\lambda + \mathcal{A})^{-1} F\|_{H^{-1}_\sigma(\Omega)} + |\lambda|^{1/2} \|(\lambda + \mathcal{A})^{-1} F\|_{L^2_\sigma(\Omega)} \\ & + \|\nabla(\lambda + \mathcal{A})^{-1} F\|_{L^2(\Omega; \mathbb{C}^{d^2})} \leq C \|F\|_{H^{-1}_\sigma(\Omega)}. \end{aligned}$$

(2) *For all $\mu \in (-1, 1)$ it holds $S_\theta \subset \rho(-B_\mu) \cap \rho(-\mathcal{B}_\mu)$. Moreover, there exists $C > 0$ depending only on d , θ , and μ such that for all $f \in \mathcal{L}^2_\sigma(\Omega)$, all $F \in \mathcal{H}^{-1}_{0,\sigma}(\Omega)$, and all $\lambda \in S_\theta$ it holds*

$$|\lambda| \|(\lambda + B_\mu)^{-1} f\|_{\mathcal{L}^2_\sigma(\Omega)} + |\lambda|^{1/2} \|\nabla(\lambda + B_\mu)^{-1} f\|_{L^2(\Omega; \mathbb{C}^{d^2})} \leq C \|f\|_{\mathcal{L}^2_\sigma(\Omega)}$$

and

$$\begin{aligned} & |\lambda| \|(\lambda + \mathcal{B}_\mu)^{-1} F\|_{\mathcal{H}^{-1}_{0,\sigma}(\Omega)} + |\lambda|^{1/2} \|(\lambda + \mathcal{B}_\mu)^{-1} F\|_{\mathcal{L}^2_\sigma(\Omega)} \\ & + \|\nabla(\lambda + \mathcal{B}_\mu)^{-1} F\|_{L^2(\Omega; \mathbb{C}^{d^2})} \leq C \|F\|_{\mathcal{H}^{-1}_{0,\sigma}(\Omega)}. \end{aligned}$$

(3) For $\mu = 1$ it holds $S_\theta \subset \rho(-B_1) \cap \rho(-\mathcal{B}_1)$. Moreover, there exists $C > 0$ depending only on d, θ , the Lipschitz character of Ω , and $\text{diam}(\Omega)$ such that for all $f \in \mathcal{L}^2_\sigma(\Omega)$, all $F \in \mathcal{H}^{-1}_{0,\sigma}(\Omega)$, and all $\lambda \in S_\theta$ it holds

$$|\lambda| \|(\lambda + B_1)^{-1} f\|_{\mathcal{L}^2_\sigma(\Omega)} + |\lambda|^{1/2} \|(D + D^\top)(\lambda + B_1)^{-1} f\|_{L^2(\Omega; \mathbb{C}^{d \times d})} \leq C \|f\|_{\mathcal{L}^2_\sigma(\Omega)}$$

and

$$\begin{aligned} & |\lambda| \|(\lambda + \mathcal{B}_1)^{-1} F\|_{\mathcal{H}^{-1}_{0,\sigma}(\Omega)} + |\lambda|^{1/2} \|(\lambda + \mathcal{B}_1)^{-1} F\|_{\mathcal{L}^2_\sigma(\Omega)} \\ & + \|(D + D^\top)(\lambda + \mathcal{B}_1)^{-1} F\|_{L^2(\Omega; \mathbb{C}^{d \times d})} \leq C \|F\|_{\mathcal{H}^{-1}_{0,\sigma}(\Omega)}. \end{aligned}$$

Recall that $Du := (\partial_i u_j)_{i,j=1}^d$ denotes the Jacobian matrix of some function u .

Proof The statements on the resolvent set follow by the Lemma of Lax–Milgram. Indeed, for $\lambda \in S_\theta$ one defines new sesquilinear forms

$$a_\lambda(u, v) := \lambda \int_\Omega u \cdot \bar{v} \, dx + a(u, v)$$

and analogously one defines $b_{\mu,\lambda}$. By (2.9), a_λ becomes coercive. If $|\mu| < 1$, then (2.9) together with (2.7) implies the coercivity of $b_{\mu,\lambda}$. Finally, in the case $\mu = 1$, one uses a Korn-type inequality proved in [44, Prop. 1.1.4.2], to define an equivalent norm on $H^1(\Omega; \mathbb{C}^d)$, which is given by

$$\|u\| := \|u\|_{L^2(\Omega; \mathbb{C}^d)} + \|Du + [Du]^\top\|_{L^2(\Omega; \mathbb{C}^{d \times d})}.$$

Notice that the constants implicit in the equivalence of the norms depend on the Lipschitz character of Ω and its diameter. In this case, the coercivity of $b_{1,\lambda}$ follows from (2.9) and (2.8). The first inequalities of (1), (2), and (3) follow as usual by testing the resolvent equation by the solution u and by employing (2.9), see, e.g., [55, Prop. 5.2.5]. Also the estimates on the second and third terms in the second inequalities of (1), (2), and (3) follow by testing the resolvent equations by the solution u . Finally, the $H^{-1}_\sigma(\Omega)$ -estimate on $|\lambda|(\lambda + \mathcal{A})^{-1} F =: |\lambda|u$ follows by virtue of the resolvent equation and the estimates that were already established before by

$$\sup_{\substack{v \in H^1_{0,\sigma}(\Omega) \\ \|v\|_{H^1_{0,\sigma}} \leq 1}} \left| \lambda \int_\Omega u \cdot \bar{v} \, dx \right| = \sup_{\substack{v \in H^1_{0,\sigma}(\Omega) \\ \|v\|_{H^1_{0,\sigma}} \leq 1}} \left| \langle F, v \rangle_{H^{-1}_\sigma, H^1_{0,\sigma}} - \int_\Omega \nabla u \cdot \bar{\nabla} v \, dx \right| \leq C \|F\|_{H^{-1}_\sigma(\Omega)}.$$

In (2) and (3) the remaining estimates follow analogously. □

Remark 2.4 Notice that if Ω has a C^∞ -boundary and if $f \in C^\infty_\sigma(\bar{\Omega})$, then one shows by the method of difference quotients (by using (2.7)) and localization that for $\mu \in (-1, 1)$ it holds $u \in C^\infty(\bar{\Omega}; \mathbb{C}^d)$ and $\phi \in C^\infty(\bar{\Omega})$.

2.8 Analytic semigroups and fractional powers

It is well-known, see, e.g., [13], that the resolvent estimates presented in Proposition 2.3 (1) imply that $-A$ and $-\mathcal{A}$ generate bounded analytic semigroups $(e^{-tA})_{t \geq 0}$ and $(e^{-t\mathcal{A}})_{t \geq 0}$ on $L^2_\sigma(\Omega)$ or $H^{-1}_\sigma(\Omega)$, respectively. By the real characterization of analytic semigroups [13, Thm. II.4.6] it further holds

$$\sup_{t>0} \|t A e^{-tA}\|_{\mathcal{L}(L^2_\sigma(\Omega))} < \infty \quad \text{and} \quad \sup_{t>0} \|t \mathcal{A} e^{-t\mathcal{A}}\|_{\mathcal{L}(H^{-1}_\sigma(\Omega))} < \infty. \tag{2.10}$$

For $\vartheta \in (0, \pi/2)$ and $r > 0$ let $\gamma_{\vartheta,r}$ denote the path that parameterizes $B(0, r) \cup S_{\vartheta}$ in the counterclockwise direction. For $t > 0$ the operators e^{-tA} and $e^{-t\mathcal{A}}$ are then given by the contour integrals

$$e^{-tA} = \frac{1}{2\pi i} \int_{\gamma_{\vartheta,1/t}} e^{-t\lambda}(\lambda - A)^{-1} d\lambda \quad \text{and} \quad e^{-t\mathcal{A}} = \frac{1}{2\pi i} \int_{\gamma_{\vartheta,1/t}} e^{-t\lambda}(\lambda - \mathcal{A})^{-1} d\lambda. \tag{2.11}$$

These integrals converge in $\mathcal{L}(L^2_{\sigma}(\Omega))$ and $\mathcal{L}(H^{-1}_{\sigma}(\Omega))$, respectively, due to Proposition 2.3 (1). Using this representation also gradient estimates of the resolvent, cf. Proposition 2.3 (1), translate into gradient estimates of the corresponding semigroups, i.e., following for example [58, Prop. 3.7] there exists $C > 0$ depending only on d and θ such that for all $t > 0$

$$t^{1/2} \|\nabla e^{-tA}\|_{\mathcal{L}(L^2_{\sigma}(\Omega), L^2(\Omega; \mathbb{C}^{d^2}))} + t^{1/2} \|e^{-t\mathcal{A}}\|_{\mathcal{L}(H^{-1}_{\sigma}(\Omega), L^2_{\sigma}(\Omega))} \leq C. \tag{2.12}$$

Besides analytic semigroups one can define for $\alpha > 0$ the fractional powers A^{α} and \mathcal{A}^{α} . There is a counterpart of (2.10) for fractional powers reading for $0 < \alpha < 1$ as

$$\sup_{t>0} \|(tA)^{\alpha} e^{-tA}\|_{\mathcal{L}(L^2_{\sigma}(\Omega))} < \infty \quad \text{and} \quad \sup_{t>0} \|(t\mathcal{A})^{\alpha} e^{-t\mathcal{A}}\|_{\mathcal{L}(H^{-1}_{\sigma}(\Omega))} < \infty, \tag{2.13}$$

which follows for example by (2.10) combined with the moment inequality [30, Prop. 6.6.4]. Since the sesquilinear form that is associated to A is symmetric [33, Thm. VI.2.23] yields that

$$\mathcal{D}(A^{1/2}) = H^1_{0,\sigma}(\Omega). \tag{2.14}$$

Moreover, [42, Thm. 5.1] implies that

$$\mathcal{D}(A^{\alpha}) = H^{2\alpha}_{0,\sigma}(\Omega) \quad \text{if} \quad \frac{1}{4} < \alpha < \frac{1}{2} \quad \text{and} \quad \mathcal{D}(A^{\alpha}) = H^{2\alpha}_{\sigma}(\Omega) \quad \text{if} \quad 0 < \alpha < \frac{1}{4}. \tag{2.15}$$

To determine the fractional power domains of \mathcal{A}^{α} for $1/4 < \alpha \leq 1/2$ one can argue as follows: As A is bijective, it follows that $A^{1/2}$ is an isomorphism from $H^1_{0,\sigma}(\Omega)$ onto $L^2_{\sigma}(\Omega)$ and by duality, $(A^{1/2})^*$ is an isomorphism from $L^2_{\sigma}(\Omega)$ onto $H^{-1}_{\sigma}(\Omega)$. A quick calculation, cf. [9, Lem. 5.1] for the case $d = 3$, reveals that

$$\mathcal{A} = (A^{1/2})^* \circ A \circ (A^{-1/2})^*.$$

In other words, A and \mathcal{A} are similar with respect to the isomorphism $(A^{1/2})^*$. Now, $\mathcal{D}(A^{\alpha})$ is given by definition by $\mathcal{R}(A^{-\alpha})$. The similarity implies that

$$\mathcal{A}^{-\alpha} = (A^{1/2})^* \circ A^{-\alpha} \circ (A^{-1/2})^* \quad \left(\frac{1}{4} < \alpha \leq \frac{1}{2}\right).$$

Thus, since $(A^{-1/2})^*$ is an isomorphism from $H^{-1}_{\sigma}(\Omega)$ onto $L^2_{\sigma}(\Omega)$, (2.14) and (2.15) imply that

$$\mathcal{R}(A^{-\alpha}) = (A^{1/2})^* H^{2\alpha}_{0,\sigma}(\Omega).$$

Thus, $v \in \mathcal{D}(A^{\alpha})$ if and only if there exists $u \in H^{2\alpha}_{0,\sigma}(\Omega)$ such that $v = (A^{1/2})^* u$. To characterize these functionals in terms of Sobolev regularity, notice that by the self-adjointness of A on $L^2_{\sigma}(\Omega)$, v is the functional

$$\langle v, w \rangle_{H^{-1}_{\sigma}, H^1_{0,\sigma}} = \langle u, A^{1/2} w \rangle_{L^2_{\sigma}, L^2_{\sigma}} = \langle A^{\alpha} u, A^{1/2-\alpha} w \rangle_{L^2_{\sigma}, L^2_{\sigma}} \quad (w \in H^1_{0,\sigma}(\Omega)).$$

By (2.15) it now follows that

$$|\langle v, w \rangle_{H_{\sigma}^{-1}, H_{0,\sigma}^1}| \leq C \|A^\alpha u\|_{L_\sigma^2(\Omega)} \|w\|_{H_{\sigma}^{1-2\alpha}(\Omega)}.$$

This implies that $v \in H_{\sigma}^{2\alpha-1}(\Omega)$. Finally, if $v \in H_{\sigma}^{2\alpha-1}(\Omega)$, define $u := (A^{-1/2})^*v$ and conclude that $u \in H_{0,\sigma}^{2\alpha}(\Omega)$ by an interpolation argument (the case $\alpha = 1/2$ is clear so that we assume $1/4 < \alpha < 1/2$). Indeed, $(A^{-1/2})^*$ is bounded from $H_{\sigma}^{-1}(\Omega)$ onto $L_\sigma^2(\Omega)$ and its restriction to $L_\sigma^2(\Omega)$ (this restriction is the operator $A^{-1/2}$) is bounded from $L_\sigma^2(\Omega)$ onto $H_{0,\sigma}^1(\Omega)$. The complex interpolation space $[H_{\sigma}^{-1}(\Omega), L_\sigma^2(\Omega)]_{2\alpha}$ is calculated by the duality rule (notice that we identify $L_\sigma^2(\Omega) \simeq L_\sigma^2(\Omega)^*$) and [42, Thm. 2.12] by

$$[H_{\sigma}^{-1}(\Omega), L_\sigma^2(\Omega)]_{2\alpha} = [L_\sigma^2(\Omega), H_{0,\sigma}^1(\Omega)]_{1-2\alpha}^* = H_{\sigma}^{2\alpha-1}(\Omega).$$

Moreover, employing [42, Thm. 2.12] again yields

$$[L_\sigma^2(\Omega), H_{0,\sigma}^1(\Omega)]_{2\alpha} = H_{0,\sigma}^{2\alpha}(\Omega).$$

It follows that $(A^{-1/2})^*$ is bounded from $H_{\sigma}^{2\alpha-1}(\Omega)$ onto $H_{0,\sigma}^{2\alpha}(\Omega)$ and thus that $u \in H_{0,\sigma}^{2\alpha}(\Omega)$. As a consequence, this reveals

$$\mathcal{D}(A^\alpha) = H_{\sigma}^{2\alpha-1}(\Omega) \quad \text{if } \frac{1}{4} < \alpha \leq \frac{1}{2}, \tag{2.16}$$

where $H_{\sigma}^0(\Omega)$ is identified with $L_\sigma^2(\Omega)$.

3 On uniform pressure estimates

Having the theory on the Stokes operator from Sect. 2 at hand, one associates a pressure function ϕ to a solution u as follows. Assume that $F \in H^{-1}(\Omega; \mathbb{C}^d)$, notice that $F|_{H_{0,\sigma}^1(\Omega)} \in H_{\sigma}^{-1}(\Omega)$, and let $\lambda \in S_\theta$ for some $\theta \in [0, \pi)$. By Proposition 2.3 there exists a unique $u := (\lambda + \mathcal{A})^{-1}F|_{H_{0,\sigma}^1}$ such that

$$\langle G, v \rangle_{H^{-1}, H_0^1} := \langle F, v \rangle_{H^{-1}, H_0^1} - \lambda \int_{\Omega} u \cdot \bar{v} \, dx - \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx = 0 \quad (v \in H_{0,\sigma}^1(\Omega)).$$

Then G is a functional in $H^{-1}(\Omega)$ which vanishes on $C_{c,\sigma}^\infty(\Omega)$ so that G must in fact be a gradient. Indeed, by [51, Lem. II.2.2.2] there exists $\phi \in L^2(\Omega)$ with mean value zero such that

$$\lambda \int_{\Omega} u \cdot \bar{v} \, dx + \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx - \int_{\Omega} \phi \overline{\operatorname{div}(v)} \, dx = \langle F, v \rangle_{H^{-1}, H_0^1} \quad (v \in H_0^1(\Omega; \mathbb{C}^d)). \tag{3.1}$$

In the case of Neumann-type boundary conditions, one proceeds similarly. As above, for $F \in H_0^{-1}(\Omega; \mathbb{C}^d)$ and $u := (\lambda + \mathcal{B}_\mu)^{-1}F|_{\gamma_{\mathcal{G}}^1}$ one finds $\vartheta \in L^2(\Omega)$ such that

$$\lambda \int_{\Omega} u \cdot \bar{v} \, dx + \int_{\Omega} a_{jk}^{\alpha\beta}(\mu) \partial_k u_\beta \overline{\partial_j v_\alpha} \, dx - \int_{\Omega} \vartheta \overline{\operatorname{div}(v)} \, dx = \langle F, v \rangle_{H_0^{-1}, H^1} \quad (v \in H_0^1(\Omega; \mathbb{C}^d)).$$

However, it would be desirable to lift this identity to hold for all $v \in H^1(\Omega; \mathbb{C}^d)$. As one can expect, by the boundary condition given in (Neu), the pressure function is unique (in the case of no-slip boundary conditions the pressure is unique up to an additive constant). Thus, one must find a constant $c \in \mathbb{C}$ such that the identity above with ϑ replaced by $\phi := \vartheta + c$

holds for all $v \in H^1(\Omega; \mathbb{C}^d)$. In fact, a way of how to construct this constant c is described in the proof of [43, Thm. 6.8]. Thus, we record that there exists $\phi \in L^2(\Omega)$ such that

$$\lambda \int_{\Omega} u \cdot \bar{v} \, dx + \int_{\Omega} a_{jk}^{\alpha\beta}(\mu) \partial_k u_{\beta} \overline{\partial_j v_{\alpha}} \, dx - \int_{\Omega} \phi \overline{\operatorname{div}(v)} \, dx = \langle F, v \rangle_{H_0^{-1}, H^1} \quad (v \in H^1(\Omega; \mathbb{C}^d)). \tag{3.2}$$

Finally, notice that $L^2(\Omega; \mathbb{C}^d)$ naturally embeds into $H^{-1}(\Omega; \mathbb{C}^d)$ and $H_0^{-1}(\Omega; \mathbb{C}^d)$. If F_1 denotes the functional in $H^{-1}(\Omega; \mathbb{C}^d)$ identified with $f \in L^2(\Omega; \mathbb{C}^d)$ and if F_2 denotes its identification with an element in $H_0^{-1}(\Omega; \mathbb{C}^d)$, we find by virtue of (2.2) and (2.3)

$$\langle F_1, v \rangle_{H^{-1}, H_0^1} = \int_{\Omega} f \cdot \bar{v} \, dx = \int_{\Omega} \mathbb{P}f \cdot \bar{v} \, dx - \int_{\Omega} g_1 \overline{\operatorname{div}(v)} \, dx \quad (v \in H_0^1(\Omega; \mathbb{C}^d))$$

and

$$\langle F_2, v \rangle_{H_0^{-1}, H^1} = \int_{\Omega} f \cdot \bar{v} \, dx = \int_{\Omega} \mathbb{Q}f \cdot \bar{v} \, dx - \int_{\Omega} g_2 \overline{\operatorname{div}(v)} \, dx \quad (v \in H^1(\Omega; \mathbb{C}^d)) \tag{3.3}$$

for functions $g_1 \in H^1(\Omega)$ and $g_2 \in H_0^1(\Omega)$. Absorbing the functions g_1 and g_2 , respectively, into the pressure functions, one finds the identities

$$(\lambda + \mathcal{A})^{-1} F_1|_{H_0^1, \sigma} = (\lambda + A)^{-1} \mathbb{P}f \quad \text{and} \quad (\lambda + \mathcal{B}_{\mu})^{-1} F_2|_{\mathcal{H}_{\sigma}^1} = (\lambda + B_{\mu})^{-1} \mathbb{Q}f.$$

Consequently, solving the Stokes resolvent problem with a right-hand side $f \in L^2(\Omega; \mathbb{C}^d)$ is the same as solving the Stokes resolvent problem with right-hand side $\mathbb{P}f$ (or $\mathbb{Q}f$, respectively) and one only changes the pressure by the gradient part inherent in f .

Given $F \in H^{-1}(\Omega; \mathbb{C}^d)$, we say that ϕ is the associated pressure to (Res) subject to (Dir) with right-hand side F if $\phi \in L_0^2(\Omega)$ and if $u := (\lambda + \mathcal{A})^{-1} F|_{H_0^1, \sigma}$ and ϕ satisfy (3.1). Analogously, we proceed for Neumann-type boundary conditions but with the relation (3.2) and without the requirement on the mean value.

For Neumann-type boundary conditions we have the following estimates on the pressure.

Proposition 3.1 *Let Ω be a bounded Lipschitz domain such that $\Delta_D^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ is bounded. Let $\theta \in (0, \pi]$ and $\mu \in (-1, 1]$. There exists a constant $C > 0$ such that for all $f \in \mathcal{L}_{\sigma}^2(\Omega)$ and $\lambda \in S_{\theta}$ the associated pressure $\phi \in L^2(\Omega)$ to (Res) subject to (Neu) with right-hand side f satisfies*

$$|\lambda|^{1/2} \|\phi\|_{L^2(\Omega)} \leq C \|f\|_{\mathcal{L}_{\sigma}^2(\Omega)}. \tag{3.4}$$

Furthermore, there exists a constant $C > 0$ such that for all $F \in H_0^{-1}(\Omega; \mathbb{C}^d)$ and $\lambda \in S_{\theta}$ the associated pressure $\phi \in L^2(\Omega)$ to (Res) subject to (Neu) and right-hand side F satisfies

$$\|\phi\|_{L^2(\Omega)} \leq C \|F\|_{H_0^{-1}(\Omega; \mathbb{C}^d)}. \tag{3.5}$$

Here, if Ω is bounded and convex and $|\mu| < 1$, then the constants $C > 0$ depend at most on d, θ , and μ . Furthermore, if $\mu = 1$, then the constants further depend on the Lipschitz character of Ω and its diameter.

Proof To prove (3.4) consider the test function $v := \nabla(-\Delta_D)^{-1}\phi$, which lies in the orthogonal space to $\mathcal{L}_{\sigma}^2(\Omega)$ by (2.3). Thus, by (3.2) and the boundedness property of the Laplacian, we infer in the case $|\mu| < 1$

$$\int_{\Omega} |\phi|^2 \, dx = \int_{\Omega} \phi \overline{\operatorname{div}(v)} \, dx = \int_{\Omega} a_{jk}^{\alpha\beta}(\mu) \partial_k u_{\beta} \overline{\partial_j v_{\alpha}} \, dx \leq C \|\nabla u\|_{L^2(\Omega; \mathbb{C}^d)} \|\phi\|_{L^2(\Omega)}.$$

If $\mu = 1$ one obtains the same but with $\|\nabla u\|_{L^2(\Omega; \mathbb{C}^{d^2})}$ replaced by $\|Du + [Du]^\top\|_{L^2(\Omega; \mathbb{C}^{d \times d})}$. The estimate is concluded by dividing by $\|\phi\|_{L^2(\Omega)}$ and by employing Proposition 2.3 (2) or (3).

To establish (3.5) use the same test function. The only difference to the calculation above is the behavior in λ of the terms $\|\nabla u\|_{L^2(\Omega; \mathbb{C}^{d^2})}$ and $\|Du + [Du]^\top\|_{L^2(\Omega; \mathbb{C}^{d \times d})}$ and the fact, that $\langle F, v \rangle_{H_0^{-1}, H^1}$ does not vanish. However, it is estimated by the boundedness assumption of the Laplacian as

$$\langle F, v \rangle_{H_0^{-1}, H^1} \leq \|F\|_{H_0^{-1}(\Omega; \mathbb{C}^d)} \|v\|_{H^1(\Omega; \mathbb{C}^d)} \leq C \|F\|_{H_0^{-1}(\Omega; \mathbb{C}^d)} \|\phi\|_{L^2(\Omega)}$$

and the term $\|\phi\|_{L^2(\Omega)}$ is handled again by division.

Concerning the dependence of C on the quantities d, θ , and μ , notice that the only critical quantity is the operator norm of $\nabla^2 \Delta_D^{-1}$ on L^2 . That this is bounded by a constant depending only on d follows by [28, Eq. (3.1.2.2)]. \square

- Remark 3.2** (1) Notice that (3.4) does not hold if $f \in L^2(\Omega; \mathbb{C}^d) \setminus \mathcal{L}_\sigma^2(\Omega)$ since in this case the pressure part g_2 defined in (3.3) does not vanish. This gives a contribution that does not even depend on λ .
- (2) Since ∇u and the pressure are connected via the imposed boundary condition in (Neu), it seems natural that the pressure and ∇u both have the same behavior in the resolvent parameter λ .

To find out how the corresponding estimates for the Stokes resolvent problem subject to no-slip boundary conditions look like will occupy the rest of this section. Notice that the following proposition was proven (also in the L^p -situation) on bounded and smooth domains in [47, Lem. 13] and on bounded Lipschitz domains in [58, Prop. 4.3]. We remark, that in the case of bounded and smooth domains even the endpoint case $\alpha = 1/4$ is known to be true by an argument of Hishida and Shibata. This can be seen by following the proof of [31, Eq. (3.12)].

Proposition 3.3 *Let Ω be a bounded Lipschitz domain and $\theta \in (0, \pi]$. For all $0 \leq \alpha < 1/4$, there exists a constant $C > 0$ such that for all $f \in L_\sigma^2(\Omega)$ and $\lambda \in S_\theta$ the associated pressure $\phi \in L_0^2(\Omega)$ to (Res) subject to (Dir) with right-hand side f satisfies*

$$\max\{1, |\lambda|^\alpha\} \|\phi\|_{L^2(\Omega)} \leq C \|f\|_{L_\sigma^2(\Omega)}.$$

Proof. Let $f \in L_\sigma^2(\Omega)$ and $u \in \mathcal{D}(A)$ with $\lambda u + Au = f$. Notice that by (3.1) it holds $-\Delta u + \nabla \phi = Au$ in the sense of distributions. Let $B : L_0^2(\Omega) \rightarrow H_0^1(\Omega; \mathbb{C}^d)$ denote the Bogovskii operator and define the test function $v := B\phi$. Then

$$\int_\Omega |\phi|^2 \, dx = \int_\Omega \phi \overline{\operatorname{div}(B\phi)} \, dx = \langle \nabla u, \nabla B\phi \rangle_{L^2, L^2} - \langle Au, B\phi \rangle_{L^2, L^2}.$$

The first term on the right-hand side is estimated by the boundedness of B and by Proposition 2.3 (1) as

$$|\langle \nabla u, \nabla B\phi \rangle_{L^2, L^2}| \leq C |\lambda|^{-1/2} \|f\|_{L_\sigma^2(\Omega)} \|\phi\|_{L^2(\Omega)}.$$

To bound the second term, use that $Au = \mathbb{P}Au$, that \mathbb{P} is self-adjoint, and that \mathbb{P} maps the Bessel potential space $H^{2\alpha}(\Omega; \mathbb{C}^d)$ boundedly into $\mathcal{D}(A^\alpha)$ whenever $0 \leq \alpha < 1/4$ (this follows by combining [42, Prop. 2.16] with [42, Thm. 5.1]). Thus, for $0 \leq \alpha < 1/4$ it holds

$$|\langle Au, B\phi \rangle_{L^2, L^2}| = |\langle A^{1-\alpha}u, A^\alpha \mathbb{P}B\phi \rangle_{L^2, L^2}| \leq C |\lambda|^{-\alpha} \|f\|_{L_\sigma^2(\Omega)} \|\phi\|_{L^2(\Omega)}.$$

Notice that the estimate on $A^{1-\alpha}u$ follows by writing $u = (\lambda + A)^{-1}f$ and by using the moment inequality [30, Prop. 6.6.4].

For the improved inequality for small λ , use the invertibility of the Stokes operator and estimate

$$\begin{aligned} \langle \nabla A^{-1}Au, \nabla B\phi \rangle_{L^2, L^2} - \langle Au, B\phi \rangle_{L^2, L^2} &\leq C \|Au\|_{L^2_\sigma(\Omega)} \|\phi\|_{L^2(\Omega)} \\ &\leq C \|f\|_{L^2_\sigma(\Omega)} \|\phi\|_{L^2(\Omega)}. \end{aligned} \quad \square$$

Comparing this estimate with the corresponding estimate for Neumann-type boundary conditions, one sees that there is a lack of an exponent of $1/4$ in the decay rate as $|\lambda| \rightarrow \infty$. As the proof for the decay estimate for no-slip boundary conditions relied on the construction of an appropriate test function, one might wonder whether the test function was just a “bad choice” and whether one could do better by choosing a more subtle test function. The following proposition shows that this is not the case, i.e., the decay rate above is optimal.

Proposition 3.4 *Let Ω be a bounded domain with C^4 -boundary, $\theta \in (\pi/2, \pi)$, and $\alpha > 1/4$. Then for all $n \in \mathbb{N}$ there exist $f_n \in L^2_\sigma(\Omega)$ and $\lambda_n \in S_\theta$ with $|\lambda_n| \geq 1$ such that the pressure function $\phi_n \in L^2_0(\Omega)$ corresponding to the problems (Res) and (Dir) with right-hand side f_n satisfies*

$$|\lambda_n|^\alpha \|\phi_n\|_{L^2(\Omega)} > n \|f_n\|_{L^2_\sigma(\Omega)}.$$

Proof We argue by contradiction and assume without loss of generality that $1/4 < \alpha \leq 1/2$. Assume that there exists $C > 0$ such that for all $f \in L^2_\sigma(\Omega)$, $\lambda \in S_\theta$ with $|\lambda| \geq 1$, and the to (Res) and (Dir) associated pressure ϕ it holds

$$|\lambda|^\alpha \|\phi\|_{L^2(\Omega)} \leq C \|f\|_{L^2_\sigma(\Omega)}. \tag{3.6}$$

Let $u \in \mathcal{D}(A)$ with $\lambda u + Au = f$ and notice by (3.1) that $-\Delta u + \nabla \phi = Au$ holds in the sense of distributions. Employing Proposition 2.3 (1) and (3.6) it follows

$$|\lambda|^\alpha \|Au\|_{H^{-1}(\Omega; \mathbb{C}^d)} \leq |\lambda|^\alpha \|\nabla u\|_{L^2(\Omega; \mathbb{C}^{d^2})} + |\lambda|^\alpha \|\phi\|_{L^2(\Omega)} \leq C \|f\|_{L^2_\sigma(\Omega)}.$$

By duality, there exists $C > 0$ such that for all $g \in H^1_0(\Omega; \mathbb{C}^d)$ and $\lambda \in S_\theta$ with $|\lambda| \geq 1$ it holds

$$|\lambda|^\alpha \|A(\lambda + A)^{-1}Pg\|_{L^2_\sigma(\Omega)} \leq C \|g\|_{H^1_0(\Omega; \mathbb{C}^d)}. \tag{3.7}$$

Similarly to [58, Prop. 3.7], use (3.7) and (2.11), to deduce a semigroup estimate of the form

$$t^{1-\alpha} \|Ae^{-tA}Pg\|_{L^2_\sigma(\Omega)} \leq C \|g\|_{H^1_0(\Omega)} \quad (0 < t \leq 1). \tag{3.8}$$

Next, we estimate for a natural number $n \in \mathbb{N}$ the term $(tA)^n e^{-tA}Pg$. To this end, write

$$(tA)^n e^{-tA}Pg = t^\alpha e^{-\frac{1}{n+1}tA} (tAe^{-\frac{1}{n+1}tA})^{n-1} t^{1-\alpha} Ae^{-\frac{1}{n+1}tA}Pg. \tag{3.9}$$

The first semigroup term in the product on the right-hand side is estimated by a combination of the interpolation inequality $\|\cdot\|_{H^{2\alpha}} \leq C \|\cdot\|_{L^2}^{1-2\alpha} \|\nabla \cdot\|_{L^2}^{2\alpha}$ with the uniform boundedness of the semigroup e^{-tA} as a family on $L^2_\sigma(\Omega)$ and the gradient estimate (2.12) as

$$\begin{aligned} t^\alpha \|e^{-\frac{1}{n+1}tA}h\|_{H^{2\alpha}(\Omega; \mathbb{C}^d)} &\leq C \|e^{-\frac{1}{n+1}tA}h\|_{L^2_\sigma(\Omega)}^{1-2\alpha} \left(t^{1/2} \|\nabla e^{-\frac{1}{n+1}tA}h\|_{L^2(\Omega; \mathbb{C}^{d^2})} \right)^{2\alpha} \\ &\leq C(n+1)^\alpha \|h\|_{L^2_\sigma(\Omega)}. \end{aligned} \tag{3.10}$$

This holds for all $h \in L^2_\sigma(\Omega)$. The term in the center of the product on the right-hand side of (3.9) is estimated by (2.10) by

$$\|(tAe^{-\frac{1}{n+1}tA})^{n-1}h\|_{L^2_\sigma(\Omega)} \leq (C(n+1))^{n-1}\|h\|_{L^2_\sigma(\Omega)} \quad (h \in L^2_\sigma(\Omega)). \tag{3.11}$$

Finally, the last term in (3.9) is estimated by using (3.8) yielding

$$\|t^{1-\alpha}Ae^{-\frac{1}{n+1}tA}\mathbb{P}g\|_{L^2_\sigma(\Omega)} \leq C(n+1)^{1-\alpha}\|g\|_{H^1_0(\Omega;\mathbb{C}^d)}. \tag{3.12}$$

Combining (3.9), (3.10), (3.11), and (3.12) and using that $n^n \leq n!e^n$ (this follows from Stirling’s formula) finally yields

$$\|(tA)^n e^{-tA}\mathbb{P}g\|_{H^{2\alpha}(\Omega;\mathbb{C}^d)} \leq (C(n+1))^{n+1}\|g\|_{H^1_0(\Omega;\mathbb{C}^d)} \leq (n+1)!(Ce)^{n+1}\|g\|_{H^1_0(\Omega;\mathbb{C}^d)}. \tag{3.13}$$

To proceed, let $0 < t \leq 1$ and $s \in \mathbb{R}$ with $|s|$ being small enough. Since e^{-tA} is an analytic semigroup, it can be written by its Taylor expansion

$$e^{-(t+s)A} = \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!} A^n e^{-tA}.$$

Combining this with (3.13) one finds by using $(n+1)! = (n+1) \cdot n!$ and $(n+1) \leq 2^n$ that

$$\|e^{-(t+s)A}\mathbb{P}g\|_{H^{2\alpha}(\Omega;\mathbb{C}^d)} \leq \sum_{n=0}^{\infty} \frac{(n+1)!(Ce)^{n+1}}{n!} \frac{|s|^n}{t^n} \|g\|_{H^1_0(\Omega;\mathbb{C}^d)} \leq Ce \sum_{n=0}^{\infty} \left(\frac{2|s|Ce}{t}\right)^n \|g\|_{H^1_0(\Omega;\mathbb{C}^d)}.$$

Now, assume that s satisfies the smallness condition $|s| < t/(4Ce)$ to deduce that

$$\|e^{-(t+s)A}\mathbb{P}g\|_{H^{2\alpha}(\Omega;\mathbb{C}^d)} \leq 2Ce\|g\|_{H^1_0(\Omega;\mathbb{C}^d)}.$$

Especially, if $s = 0$, this shows that the family of operators $(e^{-tA}\mathbb{P})_{0 < t \leq 1}$ is uniformly bounded in the space $\mathcal{L}(H^1_0(\Omega;\mathbb{C}^d), H^{2\alpha}(\Omega;\mathbb{C}^d))$. To conclude the argument, let $(t_n)_{n \in \mathbb{N}} \subset (0, 1]$ converge to zero. Notice that $e^{-tA}\mathbb{P}g \rightarrow \mathbb{P}g$ in $L^2_\sigma(\Omega)$ as $t \rightarrow 0$ by the strong continuity of the semigroup. Since $(e^{-t_n A}\mathbb{P}g)_{n \in \mathbb{N}}$ is uniformly bounded in the space $H^{2\alpha}(\Omega;\mathbb{C}^d)$, for any $0 < \varepsilon \leq 2\alpha$ there exists a convergent subsequence in the space $H^{2\alpha-\varepsilon}(\Omega;\mathbb{C}^d)$ by the Theorem of Rellich and Kondrachov. Denoting the subsequence again by $(t_n)_{n \in \mathbb{N}}$ we have that $e^{-t_n A}\mathbb{P}g \rightarrow \mathbb{P}g$ as $n \rightarrow \infty$ in $H^{2\alpha-\varepsilon}(\Omega;\mathbb{C}^d)$. Notice that $2\alpha > 1/2$ and choose ε small enough such that $2\alpha - \varepsilon > 1/2$ holds. Now, the trace operator $R_{\partial\Omega}$ is well-defined on the space $H^{2\alpha-\varepsilon}(\Omega;\mathbb{C}^d)$ and it is continuous from $H^{2\alpha-\varepsilon}(\Omega;\mathbb{C}^d)$ to $L^2(\partial\Omega;\mathbb{C}^d)$. Consequently,

$$0 = \lim_{n \rightarrow \infty} R_{\partial\Omega}(e^{-t_n A}\mathbb{P}g) = R_{\partial\Omega}(\mathbb{P}g).$$

We thus proved that for any $g \in H^1_0(\Omega;\mathbb{C}^d)$ the trace of $\mathbb{P}g$ to $\partial\Omega$ is zero. This contradicts Lemma 2.2. □

In the following, we do the same analysis for right-hand sides in $H^{-1}(\Omega;\mathbb{C}^d)$. We start with the following lemma, relating an estimate on the L^2 -norm of ϕ to a corresponding estimate on the H^{-1} -norm of u .

Lemma 3.5 *Let $\theta \in [0, \pi)$ and $0 \leq \alpha \leq 1/2$. Then the following are equivalent:*

- (1) *There exists a constant $C > 0$ such that for all $F \in H^{-1}(\Omega;\mathbb{C}^d)$ and $\lambda \in S_\theta$ with $|\lambda| \geq 1$ the associated pressure $\phi \in L^2_0(\Omega)$ to (Res) subject to (Dir) and right-hand side F satisfies*

$$\|\phi\|_{L^2(\Omega)} \leq C|\lambda|^\alpha \|F\|_{H^{-1}(\Omega;\mathbb{C}^d)}.$$

(2) There exists a constant $C > 0$ such that for all $F \in H^{-1}(\Omega; \mathbb{C}^d)$ and $\lambda \in S_\theta$ with $|\lambda| \geq 1$ the function $u := (\lambda + \mathcal{A})^{-1}F|_{H_{0,\sigma}^1}$ satisfies

$$\|u\|_{H^{-1}(\Omega; \mathbb{C}^d)} \leq C|\lambda|^{\alpha-1}\|F\|_{H^{-1}(\Omega; \mathbb{C}^d)}.$$

Proof To prove (2) \Rightarrow (1), use (3.1) and choose as a test function $v := B\phi$ with $B : L_0^2(\Omega) \rightarrow H_0^1(\Omega; \mathbb{C}^d)$ being the Bogovskii operator. Indeed, this together with Proposition 2.3 (1) yields

$$\begin{aligned} \int_\Omega |\phi|^2 \, dx &= \lambda \int_\Omega u \cdot \overline{B\phi} \, dx + \int_\Omega \nabla u \cdot \overline{\nabla B\phi} \, dx - \langle F, B\phi \rangle_{H^{-1}, H_0^1} \\ &\leq C \left(|\lambda| \|u\|_{H^{-1}(\Omega; \mathbb{C}^d)} + \|\nabla u\|_{L^2(\Omega; \mathbb{C}^{2d})} + \|F\|_{H^{-1}(\Omega; \mathbb{C}^d)} \right) \|\phi\|_{L^2(\Omega)} \\ &\leq C|\lambda|^\alpha \|F\|_{H^{-1}(\Omega; \mathbb{C}^d)} \|\phi\|_{L^2(\Omega)}. \end{aligned} \tag{3.14}$$

The estimate is concluded by dividing by $\|\phi\|_{L^2(\Omega)}$.

To prove (1) \Rightarrow (2), write by virtue of (3.1)

$$\begin{aligned} &|\lambda| \sup_{\substack{v \in H_0^1(\Omega; \mathbb{C}^d) \\ \|v\|_{H_0^1} \leq 1}} \left| \int_\Omega u \cdot \overline{v} \, dx \right| \\ &= \sup_{\substack{v \in H_0^1(\Omega; \mathbb{C}^d) \\ \|v\|_{H_0^1} \leq 1}} \left| \int_\Omega \nabla u \cdot \overline{\nabla v} \, dx - \int_\Omega \phi \overline{\operatorname{div}(v)} \, dx - \langle F, v \rangle_{H^{-1}, H_0^1} \right| \end{aligned}$$

and conclude by means of Hölder’s inequality, Proposition 2.3 (1), and the presumed estimate on the pressure. \square

We start by establishing of the actual estimates being valid and prove their sharpness afterwards.

Proposition 3.6 *Let Ω be a bounded Lipschitz domain and $\theta \in (0, \pi]$. For all $1/4 < \alpha \leq 1/2$, there exists a constant $C > 0$ such that for all $F \in H^{-1}(\Omega; \mathbb{C}^d)$ and $\lambda \in S_\theta$ the associated pressure $\phi \in L_0^2(\Omega)$ to (Res) subject to (Dir) and right-hand side F satisfies*

$$\|\phi\|_{L^2(\Omega)} \leq C \max\{1, |\lambda|^\alpha\} \|F\|_{H^{-1}(\Omega; \mathbb{C}^d)}.$$

Proof First of all, notice that the calculation carried out in (3.14) already gives the uniform boundedness of the constant for all $\lambda \in S_\theta$ with $|\lambda| < 1$ and thus, leaving us with the task to prove estimates in the case $|\lambda| \geq 1$. In this case, Lemma 3.5 reduces the problem to bound the H^{-1} -norm of u .

To this end, let $F \in H^{-1}(\Omega; \mathbb{C}^d)$ and $u := (\lambda + \mathcal{A})^{-1}F|_{H_{0,\sigma}^1}$. Since $u \in L_\sigma^2(\Omega)$ and \mathbb{P} is self-adjoint one finds

$$\lambda \int_\Omega u \cdot \overline{v} \, dx = \lambda \int_\Omega u \cdot \overline{\mathbb{P}v} \, dx.$$

By [42, Prop. 2.16], \mathbb{P} maps $H^{1-2\alpha}(\Omega; \mathbb{C}^d)$ boundedly into $H_\sigma^{1-2\alpha}(\Omega)$, so that

$$\left| \lambda \int_\Omega u \cdot \overline{\mathbb{P}v} \, dx \right| \leq C|\lambda| \|u\|_{H_\sigma^{2\alpha-1}(\Omega)} \|v\|_{H_0^1(\Omega; \mathbb{C}^d)}.$$

Since the space $H_\sigma^{2\alpha-1}(\Omega)$ coincides with $\mathcal{D}(\mathcal{A}^\alpha)$, compare (2.16), one finds

$$|\lambda| \|u\|_{H_\sigma^{2\alpha-1}(\Omega)} \leq C|\lambda| \|\mathcal{A}^\alpha (\lambda + \mathcal{A})^{-1}F|_{H_{0,\sigma}^1}\|_{H_\sigma^{-1}(\Omega)} \leq C|\lambda|^\alpha \|F|_{H_{0,\sigma}^1}\|_{H_\sigma^{-1}(\Omega)}.$$

Now, the continuous inclusion $H^{-1}(\Omega; \mathbb{C}^d) \hookrightarrow H_\sigma^{-1}(\Omega)$ concludes the proof. \square

Remark 3.7 Whether or not the bound in Proposition 3.6 holds for $\alpha = 1/4$ in bounded and smooth domains is to the best knowledge of the author not known.

Finally, we prove that this bound is in fact sharp.

Proposition 3.8 *Let Ω be a bounded domain with C^4 -boundary, $\theta \in (\pi/2, \pi)$, and $0 \leq \alpha < 1/4$. Then for all $n \in \mathbb{N}$ there exist $F_n \in H^{-1}(\Omega; \mathbb{C}^d)$ and $\lambda_n \in S_\theta$ with $|\lambda_n| \geq 1$ such that the pressure function $\phi_n \in L^2_0(\Omega)$ corresponding to the problems (Res) and (Dir) with right-hand side F_n satisfies*

$$\|\phi_n\|_{L^2(\Omega)} > n|\lambda_n|^\alpha \|F_n\|_{H^{-1}(\Omega; \mathbb{C}^d)}.$$

Proof We argue by contradiction. Hence by virtue of Lemma 3.5, we assume that there exists $0 \leq \alpha < 1/4$ and $C > 0$ such that for all $F \in H^{-1}(\Omega; \mathbb{C}^d)$ and $\lambda \in S_\theta$ with $|\lambda| \geq 1$ it holds

$$\|(\lambda + \mathcal{A})^{-1} F|_{H^1_{0,\sigma}}\|_{H^{-1}(\Omega; \mathbb{C}^d)} \leq C|\lambda|^{\alpha-1} \|F\|_{H^{-1}(\Omega; \mathbb{C}^d)}.$$

By duality and (2.1), there exists $C > 0$ such that for all $\lambda \in S_\theta$ with $|\lambda| \geq 1$ and all $g \in H^1_0(\Omega; \mathbb{C}^d)$ it holds

$$|\lambda|^{1-\alpha} \|(\lambda + A)^{-1} \mathbb{P}g\|_{H^1_{0,\sigma}(\Omega)} \leq C \|g\|_{H^1_0(\Omega; \mathbb{C}^d)}. \tag{3.15}$$

Following the proof of [58, Prop. 3.7], the estimate (3.15) in combination with (2.11) lead to the semigroup estimate

$$t^\alpha \|e^{-tA} \mathbb{P}g\|_{H^1_{0,\sigma}(\Omega)} \leq C \|g\|_{H^1_0(\Omega; \mathbb{C}^d)} \quad (0 < t \leq 1). \tag{3.16}$$

Next, we estimate the term $(tA)^n e^{-tA} \mathbb{P}g$ as in the proof of Proposition 3.4 for a natural number $n \in \mathbb{N}$ and $0 < t \leq 1$. To this end, write

$$(tA)^n e^{-tA} \mathbb{P}g = t^{1-\alpha} A^{1/2} e^{-\frac{1}{n+1}tA} (tA e^{-\frac{1}{n+1}tA})^{n-1} A^{1/2} t^\alpha e^{-\frac{1}{n+1}tA} \mathbb{P}g. \tag{3.17}$$

The first term in the product on the right-hand side is estimated by means of the interpolation inequality $\|\cdot\|_{H^{1-2\alpha}} \leq C \|\cdot\|_{L^2_\sigma}^{2\alpha} \|\nabla \cdot\|_{L^2_\sigma}^{1-2\alpha}$, the uniform boundedness of the semigroup e^{-tA} as a family on $L^2_\sigma(\Omega)$, the gradient estimate (2.12), and (2.13). Indeed, for all $h \in L^2_\sigma(\Omega)$, we have

$$\begin{aligned} & t^{1-\alpha} \|A^{1/2} e^{-\frac{1}{n+1}tA} h\|_{H^{1-2\alpha}(\Omega; \mathbb{C}^d)} \\ & \leq C \left(t^{1/2} \|e^{-\frac{1}{2(n+1)}tA} A^{1/2} e^{-\frac{1}{2(n+1)}tA} h\|_{L^2_\sigma(\Omega)} \right)^{2\alpha} \left(t \|\nabla e^{-\frac{1}{2(n+1)}tA} A^{1/2} e^{-\frac{1}{2(n+1)}tA} h\|_{L^2(\Omega; \mathbb{C}^{d^2})} \right)^{1-2\alpha} \\ & \leq C(n+1)^{1/2-\alpha} \left(t^{1/2} \|A^{1/2} e^{-\frac{1}{2(n+1)}tA} h\|_{L^2_\sigma(\Omega)} \right)^{2\alpha} \left(t^{1/2} \|A^{1/2} e^{-\frac{1}{2(n+1)}tA} h\|_{L^2_\sigma(\Omega)} \right)^{1-2\alpha} \\ & \leq C(n+1)^{1-\alpha} \|h\|_{L^2_\sigma(\Omega)}. \end{aligned} \tag{3.18}$$

The second term in the product in (3.17) was already estimated in (3.11). The third term in the product in (3.17) is finally estimated, by using (2.14) and (3.16) by

$$\|A^{1/2} t^\alpha e^{-\frac{1}{n+1}tA} \mathbb{P}g\|_{L^2_\sigma(\Omega)} \leq t^\alpha \|e^{-\frac{1}{n+1}tA} \mathbb{P}g\|_{H^1_{0,\sigma}(\Omega)} \leq C(n+1)^\alpha \|g\|_{H^1_0(\Omega; \mathbb{C}^d)}. \tag{3.19}$$

Combining (3.17), (3.18), (3.11), (3.19), and using $n^n \leq n!e^n$ (this follows from Stirling’s formula) finally yields

$$\|(tA)^n e^{-tA} \mathbb{P}g\|_{H^{1-2\alpha}(\Omega; \mathbb{C}^d)} \leq (C(n+1))^{n+1} \|g\|_{H^1_0(\Omega; \mathbb{C}^d)} \leq (n+1)!(Ce)^{n+1} \|g\|_{H^1_0(\Omega; \mathbb{C}^d)}.$$

The rest of the contradiction argument follows exactly the lines below (3.13) in the proof of Proposition 3.4 and is thus omitted. \square

Recall that in order to derive the estimates in the case of Neumann-type boundary conditions in Proposition 3.1 it was needed that solutions to the Poisson problem with right-hand side in $L^2(\Omega)$ admit H^2 -regularity. Thus, this proof cannot be carried out on general bounded Lipschitz domains. However, as all objects appearing in the estimate in Proposition 3.1 exist if the boundary of Ω is merely Lipschitz. Thus, one might wonder whether Proposition 3.1 is true on general Lipschitz domains. Unfortunately, it seems hard to deduce the validity of these estimates by approximating the Lipschitz domain by smooth domains as the constants in the respective estimate blow up. If one wants to prove Stokes resolvent estimates in L^p for Neumann-type boundary conditions on mere Lipschitz domains, it would be tempting to imitate Shen’s proof [50] carried out for no-slip boundary conditions. As it was described in the introduction, a corresponding weak reverse Hölder estimate might look as (1.6) but on general Lipschitz domains with $p := 2d/(d - 1)$. It was further described in the introduction, that an estimate of the form presented in Proposition 3.1 would help to achieve these resolvent estimates. In view of this, it would be interesting to know the answer to the following problem.

Problem 3.9 Prove or disprove the validity of (3.4) if Ω is a bounded Lipschitz domain.

4 Regularity estimates in convex domains

If Ω is a bounded and convex domain, it is well-known that weak solutions to the Poisson problem with homogeneous Dirichlet or Neumann boundary conditions and right-hand side in $L^2(\Omega)$ admit H^2 -regularity. To understand a rough sketch of its proof, we need to introduce some notions from geometry.

If $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^2 -boundary (not necessarily convex), and if after a suitable translation and rotation of Ω the function $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ locally describes the boundary of Ω around the point $p = (0, \varphi(0))$, then, if the rotation is chosen such that $\nabla\varphi(0) = 0$, the second fundamental form \mathbb{I}_p at this boundary point is the sesquilinear form given by

$$\mathbb{I}_p(\xi; \eta) = \frac{\partial^2\varphi(0)}{\partial x_j \partial x_k} \xi_j \overline{\eta_k} \quad (\xi, \eta \in \mathbb{C}^{d-1}).$$

Notice that $\mathbb{I}_p(\cdot; \cdot)$ is conjugate symmetric and thus $\mathbb{I}_p(\xi; \xi)$ is a real number for each $\xi \in \mathbb{C}^{d-1}$. If Ω is convex and if Ω locally lies below the graph of φ , then $-\varphi$ is convex and thus the second fundamental form is non-positive, which means that

$$\mathbb{I}_p(\xi; \xi) \leq 0 \quad (\xi \in \mathbb{C}^{d-1}). \tag{4.1}$$

Furthermore, if \mathbb{I}_p denotes the matrix associated to the sesquilinear form $\mathbb{I}_p(\cdot; \cdot)$, then convexity of Ω implies that its trace satisfies

$$\text{tr}(\mathbb{I}_p) \leq 0. \tag{4.2}$$

In the following, we skip the subscript p and keep in mind, that the second fundamental form varies from boundary point to boundary point.

To understand why the domain of the Laplacian embeds into H^2 in convex domains, the following formula of integration by parts due to Grisvard is eminent [28, Thm. 3.1.1.1].

Notice that in [28, Thm. 3.1.1.1] this formula is derived for real-valued functions, however, a short analysis of its proof reveals the following formulation for complex-valued functions. Here and below, σ generically denotes the surface measure of a set with a Lipschitz boundary. Recall further the notation v_T for the tangential component of a vector v introduced in (2.5).

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 -boundary and let $v \in C^\infty(\overline{\Omega}; \mathbb{C}^d)$. Then,*

$$\int_{\Omega} |\operatorname{div}(v)|^2 \, dx - \int_{\Omega} \partial_j v_i \overline{\partial_i v_j} \, dx = - \int_{\partial\Omega} 2 \operatorname{Re}(v_T \cdot \overline{\nabla_T(v \cdot n)}) \, d\sigma - \int_{\partial\Omega} (\mathbb{I}(v_T; v_T) + (\operatorname{tr} \mathbb{I})|v \cdot n|^2) \, d\sigma.$$

There is also a counterpart of Theorem 4.1 for piecewise C^2 -domains, see [28, Thm. 3.1.1.2] for real-valued functions. To state the theorem, we adopt the definition by Grisvard, that a bounded Lipschitz domain Ω is said to be piecewise C^2 -regular if there exist $\Gamma_0, \Gamma_1 \subset \partial\Omega$ with $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and where Γ_0 has surface measure zero and for each $x \in \Gamma_1$ the boundary of $\partial\Omega$ can be described as the graph of a C^2 -function in a neighborhood of x .

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a piecewise C^2 -boundary and let $v \in C^\infty(\overline{\Omega}; \mathbb{C}^d)$. Then,*

$$\int_{\Omega} |\operatorname{div}(v)|^2 \, dx - \int_{\Omega} \partial_j v_i \overline{\partial_i v_j} \, dx = \int_{\Gamma_1} (\operatorname{div}_T([v \cdot n]v_T) - 2 \operatorname{Re}(v_T \cdot \overline{\nabla_T(v \cdot n)})) \, d\sigma - \int_{\Gamma_1} \mathbb{I}(v_T; v_T) + (\operatorname{tr} \mathbb{I})|v \cdot n|^2 \, d\sigma.$$

To deduce that weak solutions to the equation $-\Delta u = f$ with Dirichlet or Neumann boundary conditions lie in $H^2(\Omega)$ if Ω is bounded and convex, let first Ω be a bounded, convex, and smooth domain. If $f \in C^\infty(\overline{\Omega}; \mathbb{R})$, then $u \in C^\infty(\overline{\Omega}; \mathbb{R})$ by higher regularity of the Laplacian. Take $v := \nabla u$ and apply Theorem 4.1 together with (4.1) and (4.2) to deduce

$$\int_{\Omega} |\operatorname{div}(v)|^2 \, dx \geq \int_{\Omega} \partial_j v_i \partial_i v_j \, dx - 2 \int_{\partial\Omega} v_T \cdot \nabla_T(v \cdot n) \, d\sigma.$$

A computation of the first term on the right-hand side yields

$$\int_{\Omega} \partial_j v_i \partial_i v_j \, dx = \sum_{i,j=1}^d \int_{\Omega} |\partial_i \partial_j u|^2 \, dx$$

and since $\operatorname{div}(v) = -f$, it remains to understand what the boundary integral does. Here, the boundary conditions enter the game. If u satisfies homogeneous Dirichlet boundary conditions, i.e., $u = 0$ on $\partial\Omega$, then $v_T = \nabla_T u = 0$ and if u satisfies homogeneous Neumann boundary conditions, then $v \cdot n = n \cdot \nabla u = 0$. Hence, by Theorem 4.1, we infer that

$$\int_{\Omega} |f|^2 \, dx \geq \sum_{i,j=1}^d \int_{\Omega} |\partial_i \partial_j u|^2 \, dx.$$

By density, one obtains this estimate for all $f \in L^2(\Omega)$. Finally, since the constant in this inequality is one, in particular, it is independent of properties of the boundary, one can conclude the H^2 -regularity for general bounded convex domains by an approximation argument.

Remark 4.3 Let us explain the approximation of a bounded and convex domain by a sequence of smooth, bounded, and convex domains $(\Omega_k)_{k \in \mathbb{N}}$ with $\Omega_k \subset \Omega_{k+1}$ and $\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega$ in more detail.

Let Ω be a bounded and convex domain and assume without loss of generality that $0 \in \Omega$. For $k \in \mathbb{N}$ let K_k denote the closure of $(1 - 2^{-k})\Omega$ and notice that $K_k \subset (1 - 2^{-(k+1)})\Omega$ and that K_k is a compact and convex set. In this situation, [32, Lem. 2.3.2] provides us with a compact and convex set C_k with smooth boundary that satisfies $K_k \subset C_k \subset (1 - 2^{-(k+1)})\Omega$. Now, let Ω_k be defined as the interior of C_k .

One could also ask, whether the sets are uniform in certain properties. For example, for all $k \in \mathbb{N}$ it holds $\frac{1}{2}\Omega \subset \Omega_k \subset \Omega$ so that $\text{diam}(\Omega)/2 \leq \text{diam}(\Omega_k) \leq \text{diam}(\Omega)$. Another property is a uniform d -set property, which is the following: Let $r_0 > 0$ be such that $B := B(0, r_0) \subset \frac{1}{2}\Omega$ so that $B \subset \Omega_k$ for all $k \in \mathbb{N}$. Let $x_0 \in \partial\Omega_k$. Since Ω_k is convex, for all $t \in [0, 1]$ and $x \in B$ the points $(1 - t)x + tx_0$ are contained in Ω_k . This implies that Ω_k contains a cone with vertex at x_0 , height $h = |x_0| \geq r_0$, and opening angle $\omega = 2 \arctan(r_0/|x_0|)$. Since $|x_0| \leq \text{diam}(\Omega)$ we find $\omega \geq 2 \arctan(r_0/\text{diam}(\Omega))$. Thus, if $Q(x_0, r)$ is a cube centered in x_0 and diameter $0 < r \leq 2r_0$, then there exists a constant $C > 0$ depending only on r_0 , $\text{diam}(\Omega)$, and d such that

$$|Q(x_0, r) \cap \Omega_k| \geq Cr^d. \tag{4.3}$$

Notice that if $R_0 > r_0$, then for all $2r_0 < r \leq 2R_0$ it holds

$$|Q(x_0, r) \cap \Omega_k| \geq |Q(x_0, r_0) \cap \Omega_k| \geq Cr_0^d = \frac{Cr_0^d}{(2R_0)^d} r^d.$$

Thus, we can assume that for all $R_0 > 0$ there exists a constant $C > 0$ depending only on r_0 , R_0 , $\text{diam}(\Omega)$, and d such that for all $k \in \mathbb{N}$ and all $x_0 \in \partial\Omega_k$ the inequality (4.3) holds.

Let Ω again be a bounded convex domain with smooth boundary. If u and ϕ satisfy

$$\begin{cases} -\Delta u + \nabla\phi = f & \text{in } \Omega \\ \text{div}(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with f being smooth up to the boundary, one could try to imitate the calculations for the Laplacian above. To this end, there are at least two obvious choices for v . Fix $1 \leq \beta \leq d$. For the first choice, define $v_\beta := \nabla u_\beta$. Clearly, all boundary integrals as well as the integral involving the mixed product can be handled as above. However, $\text{div}(v_\beta) = -f_\beta + \partial_\beta\phi$, so that the gradient of the pressure appears on the right-hand side of the inequality, which is an unfortunate situation.

Another choice for v should incorporate that $\text{div}(v) = -f_\beta$. For the β th component of the equation, this is accomplished by choosing $v_\beta := \nabla u_\beta - \phi e_\beta$, where e_β denotes the β th unit basis vector. Moreover, convexity deals with the terms involving the second fundamental form, and one directly verifies that the mixed product (for β fix) computes as

$$\partial_j(v_\beta)_i \partial_i(v_\beta)_j = \partial_i \partial_j u_\beta \partial_i \partial_j u_\beta + |\partial_\beta\phi|^2 - 2\nabla \partial_\beta u_\beta \cdot \nabla\phi.$$

Next, a summation over β yields due to the solenoidality of u (notice that we now sum over repeated indices as usual)

$$\partial_j(v_\beta)_i \partial_i(v_\beta)_j = \partial_i \partial_j u_\beta \partial_i \partial_j u_\beta + |\nabla\phi|^2.$$

Altogether, we find

$$\int_{\Omega} |f|^2 \geq \sum_{i,j,\beta=1}^d \int_{\Omega} |\partial_i \partial_j u_{\beta}|^2 dx + \int_{\Omega} |\nabla \phi|^2 dx - 2 \int_{\partial\Omega} (v_{\beta})_T \cdot \nabla_T (v_{\beta} \cdot n) d\sigma.$$

Unfortunately, one cannot simply conclude that the boundary integral vanishes as nothing is known about the trace of the pressure on the boundary of Ω . However, imposing for example the Neumann-type boundary condition

$$n \cdot \nabla u - \phi n = 0$$

seems to be better suited for this approach as in this case the function v_{β} turns out to have the additional property that $v_{\beta} \cdot n = 0$ on $\partial\Omega$. For more general Neumann-type boundary conditions and the resolvent problem this is made precise in the following theorem.

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded convex domain, $\mu \in (-1, \sqrt{2} - 1)$, and $\theta \in (0, \pi)$. Then for all $\lambda \in S_{\theta}$ and all $f \in L^2(\Omega; \mathbb{C}^d)$ the weak solutions u and ϕ to (Res) subject to (Neu) satisfy $u \in H^2(\Omega; \mathbb{C}^d)$ and $\phi \in H^1(\Omega)$. Moreover, there exists $C > 0$ depending only on d, μ , and θ such that*

$$|\lambda| \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla^2 u|^2 dx + \int_{\Omega} |\nabla \phi|^2 dx \leq C \left(\int_{\Omega} |f|^2 dx + |\lambda|^2 \int_{\Omega} |u|^2 dx \right).$$

Proof Assume first that Ω has a C^{∞} -boundary and that $f \in C_c^{\infty}(\Omega; \mathbb{C}^d)$. Then, by virtue of Remark 2.4, the functions u and ϕ are smooth up to the boundary. Fix $1 \leq \beta \leq d$ and define

$$v_{\beta} := \left(\{\delta_{lk} \delta_{\alpha\beta} + \mu \delta_{l\beta} \delta_{k\alpha}\} \partial_l u_{\alpha} - \delta_{k\beta} \phi \right)_{k=1}^d. \tag{4.4}$$

Since u and ϕ solve (Res) one readily verifies that

$$\operatorname{div}(v_{\beta}) = \{\delta_{lk} \delta_{\alpha\beta} + \mu \delta_{l\beta} \delta_{k\alpha}\} \partial_k \partial_l u_{\alpha} - \delta_{k\beta} \partial_k \phi = \partial_l \partial_l u_{\beta} + \mu \partial_{\beta} \partial_{\alpha} u_{\alpha} - \partial_{\beta} \phi = \lambda u_{\beta} - f_{\beta}. \tag{4.5}$$

Moreover,

$$n \cdot v_{\beta} = n_k \{\delta_{lk} \delta_{\alpha\beta} + \mu \delta_{l\beta} \delta_{k\alpha}\} \partial_l u_{\alpha} - n_k \delta_{k\beta} \phi = n_k \partial_k u_{\beta} + \mu n_k \partial_{\beta} u_k - \phi n_{\beta}, \tag{4.6}$$

which coincides with the β th component of

$$\{Du + \mu [Du]^{\top}\} n - \phi n$$

and thus vanishes on the boundary. The mixed product is calculated as follows (note that we also sum over β in this calculation so that in particular $\partial_{\beta} u_{\beta} = 0$)

$$\begin{aligned} \partial_j (v_{\beta})_i \overline{\partial_i (v_{\beta})_j} &= \{\delta_{li} \delta_{\alpha\beta} + \mu \delta_{l\beta} \delta_{i\alpha}\} \partial_j \partial_l u_{\alpha} - \delta_{i\beta} \partial_j \phi \{[\delta_{l'j} \delta_{\alpha'\beta} + \mu \delta_{l'\beta} \delta_{j\alpha'}] \overline{\partial_i \partial_{l'} u_{\alpha'}} - \delta_{j\beta} \overline{\partial_i \phi}\} \\ &= \partial_j \partial_i u_{\beta} \overline{\partial_j \partial_i u_{\beta}} + 2\mu \operatorname{Re}(\partial_j \partial_i u_{\beta} \overline{\partial_i \partial_{\beta} u_j}) + \mu^2 \partial_j \partial_{\beta} u_i \overline{\partial_i \partial_{\beta} u_j} - 2\mu \operatorname{Re}(\partial_{\beta} \partial_{\beta} u_i \overline{\partial_i \phi}) \\ &\quad + \partial_{\beta} \phi \overline{\partial_{\beta} \phi}. \end{aligned}$$

Relabelling the index variables yields

$$\partial_j \partial_{\beta} u_i \overline{\partial_i \partial_{\beta} u_j} = \frac{1}{2} \partial_j \partial_{\beta} u_i \overline{\partial_i \partial_{\beta} u_j} + \frac{1}{2} \partial_i \partial_{\beta} u_j \overline{\partial_j \partial_{\beta} u_i} = \operatorname{Re}(\partial_j \partial_{\beta} u_i \overline{\partial_i \partial_{\beta} u_j}).$$

Next, use that $\partial_\beta \partial_\beta u_i = \lambda u_i - f_i + \partial_i \phi$ to deduce

$$\begin{aligned} \partial_j (v_\beta)_i \overline{\partial_i (v_\beta)_j} &= \partial_j \partial_i u_\beta \overline{\partial_i \partial_j u_\beta} + (2\mu + \mu^2) \operatorname{Re}(\partial_j \partial_i u_\beta \overline{\partial_i \partial_\beta u_j}) + (1 - 2\mu) \partial_\beta \phi \overline{\partial_\beta \phi} \\ &\quad + 2\mu \operatorname{Re}(f_i \overline{\partial_i \phi}) - 2\mu \operatorname{Re}(\lambda u_i \overline{\partial_i \phi}). \end{aligned} \tag{4.7}$$

Finally, Young’s inequality implies

$$\begin{aligned} (2\mu + \mu^2) \operatorname{Re}(\partial_j \partial_i u_\beta \overline{\partial_i \partial_\beta u_j}) &\geq -\frac{|2\mu + \mu^2|}{2} \partial_j \partial_i u_\beta \overline{\partial_j \partial_i u_\beta} - \frac{|2\mu + \mu^2|}{2} \partial_i \partial_\beta u_j \overline{\partial_i \partial_\beta u_j} \\ &= -|2\mu + \mu^2| \partial_j \partial_i u_\beta \overline{\partial_j \partial_i u_\beta}. \end{aligned} \tag{4.8}$$

Use the two rightmost representations of $\operatorname{div}(v_\beta)$ in (4.5) and an integration by parts together with the fact that $n \cdot v_\beta = 0$ on $\partial\Omega$ (due to (4.6) and the imposed boundary condition), the representation of v_β in (4.4), and the solenoidality of u to deduce

$$\begin{aligned} \sum_{\beta=1}^d \int_\Omega |\operatorname{div}(v_\beta)|^2 \, dx &= \int_\Omega \operatorname{div}(v_\beta) \{\overline{\lambda u_\beta} - \overline{f_\beta}\} \, dx \\ &= -\overline{\lambda} \int_\Omega (v_\beta)_k \overline{\partial_k u_\beta} \, dx - \int_\Omega \{\partial_l \partial_l u_\beta + \mu \partial_\beta \partial_\alpha u_\alpha - \partial_\beta \phi\} \overline{f_\beta} \, dx \\ &= -\overline{\lambda} \int_\Omega \{\partial_k u_\beta + \mu \partial_\beta u_k\} \overline{\partial_k u_\beta} \, dx - \int_\Omega \{\partial_l \partial_l u_\beta - \partial_\beta \phi\} \overline{f_\beta} \, dx. \end{aligned} \tag{4.9}$$

Finally, apply Theorem 4.1 with $v = v_\beta$ and sum over β . By (4.7) and since the term in (4.6) vanishes on the boundary one finds after rearranging terms

$$\begin{aligned} &\overline{\lambda} \int_\Omega \{\partial_k u_\beta + \mu \partial_\beta u_k\} \overline{\partial_k u_\beta} \, dx + \int_\Omega \partial_i \partial_j u_\beta \overline{\partial_i \partial_j u_\beta} \, dx + (2\mu + \mu^2) \int_\Omega \operatorname{Re}(\partial_j \partial_i u_\beta \overline{\partial_i \partial_\beta u_j}) \, dx \\ &\quad + (1 - 2\mu) \int_\Omega \partial_\beta \phi \overline{\partial_\beta \phi} \, dx - \int_{\partial\Omega} \mathbb{I}((v_\beta)_T; (v_\beta)_T) \, d\sigma \\ &= - \int_\Omega \{\partial_l \partial_l u_\beta - \partial_\beta \phi\} \overline{f_\beta} \, dx - 2\mu \int_\Omega \operatorname{Re}(f_i \overline{\partial_i \phi}) \, dx + 2\mu \int_\Omega \operatorname{Re}(\lambda u_i \overline{\partial_i \phi}) \, dx. \end{aligned} \tag{4.10}$$

Now, notice the following facts: if $\lambda \in S_\theta$, then $\overline{\lambda} \in S_\theta$. If $|\mu| \leq 1$, then

$$\{\partial_k u_\beta + \mu \partial_\beta u_k\} \overline{\partial_k u_\beta} = |\nabla u|^2 + \mu \operatorname{Re}(\partial_\beta u_k \overline{\partial_k u_\beta}) \geq (1 - |\mu|) |\nabla u|^2 \geq 0. \tag{4.11}$$

If $|2\mu + \mu^2| < 1$, then the sum of the second and third integrals on the left-hand side of (4.10) is non-negative due to (4.8). If $1 - 2\mu > 0$, then the fourth integral on the left-hand side of (4.10) is non-negative and finally, the convexity of Ω implies that the fifth integral is non-positive. This results in the condition $-1 < \mu < \sqrt{2} - 1$, which is the imposed condition on μ . Thus, the left-hand side is of the form $z + \alpha$ for some $z \in \overline{S_\theta}$ and $\alpha \geq 0$. Consequently, by (2.9) there exists a constant $C_\theta > 0$ depending only on θ , such that

$$\begin{aligned} &|\lambda| \int_\Omega \{\partial_k u_\beta + \mu \partial_\beta u_k\} \overline{\partial_k u_\beta} \, dx + \int_\Omega \partial_i \partial_j u_\beta \overline{\partial_i \partial_j u_\beta} \, dx + (2\mu + \mu^2) \int_\Omega \operatorname{Re}(\partial_j \partial_i u_\beta \overline{\partial_i \partial_\beta u_j}) \, dx \\ &\quad + (1 - 2\mu) \int_\Omega \partial_\beta \phi \overline{\partial_\beta \phi} \, dx - \int_{\partial\Omega} \mathbb{I}((v_\beta)_T; (v_\beta)_T) \, d\sigma \\ &\leq C_\theta \left(\int_\Omega (|\Delta u| + (1 + 2|\mu|)|\nabla \phi|) |f| \, dx + 2|\lambda| |\mu| \int_\Omega |u| |\nabla \phi| \, dx \right). \end{aligned}$$

By virtue of (4.11), (4.8), and the convexity of Ω one finds

$$\begin{aligned} & |\lambda|(1 - |\mu|) \int_{\Omega} |\nabla u|^2 \, dx + (1 - |2\mu + \mu^2|) \int_{\Omega} |\nabla^2 u|^2 \, dx + (1 - 2\mu) \int_{\Omega} |\nabla \phi|^2 \, dx \\ & \leq C_{\theta} \left(\int_{\Omega} (|\Delta u| + (1 + 2|\mu|)|\nabla \phi|)|f| \, dx + 2|\lambda||\mu| \int_{\Omega} |u||\nabla \phi| \, dx \right). \end{aligned}$$

The desired inequality now follows for $f \in C_c^\infty(\Omega; \mathbb{C}^d)$ by an application of Young’s inequality and for $f \in L^2(\Omega; \mathbb{C}^d)$ by density.

To conclude the proof, we approximate an arbitrary bounded and convex domain Ω by smooth, bounded, and convex domains Ω_k as described in Remark 4.3. Let R_{Ω_k} denote the restriction operator to Ω_k , \mathbb{Q}_k be the Helmholtz projection on Ω_k , and $B_{\mu,k}$ the Stokes operator subject to Neumann-type boundary conditions on Ω_k . Define $f_k := R_{\Omega_k} f \in L^2(\Omega_k; \mathbb{C}^d)$, $u_k := (\lambda + B_{\mu,k})^{-1} \mathbb{Q}_k f_k$, and define $u := (\lambda + B_{\mu})^{-1} \mathbb{Q} f$. Then

$$\begin{aligned} & \lambda \int_{\Omega_k} (u - u_k) \cdot \overline{(u - u_k)} \, dx + \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_{\beta} - (u_k)_{\beta}) \overline{\partial_j (u_{\alpha} - (u_k)_{\alpha})} \, dx \\ & = \lambda \int_{\Omega_k} u \cdot \bar{u} \, dx + (\lambda - \bar{\lambda}) \int_{\Omega_k} u_k \cdot \bar{u}_k \, dx + \bar{\lambda} \int_{\Omega_k} u_k \cdot \bar{u}_k \, dx - \lambda \int_{\Omega_k} u_k \cdot \bar{u} \, dx - \lambda \int_{\Omega_k} u \cdot \bar{u}_k \, dx \\ & \quad + \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l u_{\beta} \overline{\partial_j u_{\alpha}} \, dx + \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_k)_{\beta} \overline{\partial_j (u_k)_{\alpha}} \, dx \\ & \quad - \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l u_{\beta} \overline{\partial_j (u_k)_{\alpha}} \, dx - \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_k)_{\beta} \overline{\partial_j u_{\alpha}} \, dx \\ & = \int_{\Omega \setminus \Omega_k} f \cdot \bar{u} \, dx - \left(\lambda \int_{\Omega \setminus \Omega_k} u \cdot \bar{u} \, dx + \int_{\Omega \setminus \Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l u_{\beta} \overline{\partial_j u_{\alpha}} \, dx \right) \\ & \quad + (\lambda - \bar{\lambda}) \int_{\Omega_k} (u - u_k) \cdot \overline{(u - u_k)} \, dx - (\lambda - \bar{\lambda}) \int_{\Omega_k} (u - u_k) \cdot \bar{u} \, dx - \int_{\Omega_k} (u - u_k) \cdot \bar{f} \, dx. \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} & \bar{\lambda} \int_{\Omega_k} |u - u_k|^2 \, dx + \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_{\beta} - (u_k)_{\beta}) \overline{\partial_j (u_{\alpha} - (u_k)_{\alpha})} \, dx \\ & = \int_{\Omega \setminus \Omega_k} f \cdot \bar{u} \, dx - \left(\lambda \int_{\Omega \setminus \Omega_k} u \cdot \bar{u} \, dx + \int_{\Omega \setminus \Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l u_{\beta} \overline{\partial_j u_{\alpha}} \, dx \right) \tag{4.12} \\ & \quad - (\lambda - \bar{\lambda}) \int_{\Omega_k} (u - u_k) \cdot \bar{u} \, dx - \int_{\Omega_k} (u - u_k) \cdot \bar{f} \, dx. \end{aligned}$$

Since $u \in H^1(\Omega; \mathbb{C}^d)$ and $f \in L^2(\Omega; \mathbb{C}^d)$, we find by (2.7) and (2.9), that $(u - u_k)_{k \in \mathbb{N}}$ defines a bounded sequence in $L^2(\Omega; \mathbb{C}^d)$ and $(\nabla u - \nabla u_k)_{k \in \mathbb{N}}$ defines a bounded sequence in $L^2(\Omega; \mathbb{C}^{d^2})$. Here, we regard $u - u_k$ and $\nabla u - \nabla u_k$ to be zero on $\Omega \setminus \Omega_k$. Thus, there exist subsequences (again denoted by the same indices) and weak limits $v \in L^2(\Omega; \mathbb{C}^d)$ and $w \in L^2(\Omega; \mathbb{C}^{d^2})$, such that $u - u_k \rightharpoonup v$ and $\nabla u - \nabla u_k \rightharpoonup w$ as $k \rightarrow \infty$. One directly verifies that v is weakly differentiable with $\nabla v = w$ and that the distributional divergence of v is zero. It follows that $v \in \mathcal{H}_{\sigma}^1(\Omega)$. Now, for $\varphi \in \mathcal{H}_{\sigma}^1(\Omega)$ one finds, since u and u_k solve their

respective equations, that

$$\begin{aligned} & \lambda \int_{\Omega} v \cdot \bar{\varphi} \, dx + \int_{\Omega} a_{jl}^{\alpha\beta}(\mu) \partial_l v_\beta \cdot \overline{\partial_j \varphi_\alpha} \, dx \\ &= \lambda \lim_{k \rightarrow \infty} \int_{\Omega_k} (u - u_k) \cdot \bar{\varphi} \, dx + \lim_{k \rightarrow \infty} \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_\beta - (u_k)_\beta) \cdot \overline{\partial_j \varphi_\alpha} \, dx \\ &= 0. \end{aligned}$$

It follows that v is zero. Going back to (4.12), one even finds that $u_k \rightarrow u$ in $H^1_{loc}(\Omega; \mathbb{C}^d)$. Since due to the first part of the proof, also the sequence $(\nabla^2 u_k)_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega; \mathbb{C}^{d^3})$, where $\nabla^2 u_k$ is regarded to be zero in $\Omega \setminus \Omega_k$, we find again by picking a weakly convergent subsequence that u is in $H^2(\Omega; \mathbb{C}^d)$ and that

$$\|\nabla^2 u\|_{L^2(\Omega; \mathbb{C}^{d^3})} \leq \liminf_{k \rightarrow \infty} \|\nabla^2 u_k\|_{L^2(\Omega_k; \mathbb{C}^{d^3})}. \tag{4.13}$$

If ϕ_k denotes the pressure such that $\lambda u_k - \Delta u_k + \nabla \phi_k = f_k$ holds in Ω_k (and satisfies the appropriate boundary condition), then we find by virtue of (3.2) with $\varphi_k := \nabla \Delta_D^{-1} \chi_{\Omega_k} (\phi - \phi_k)$ where Δ_D denotes the Dirichlet Laplacian on Ω that

$$\begin{aligned} & \int_{\Omega_k} |\phi - \phi_k|^2 \, dx \\ &= \int_{\Omega_k} (\phi - \phi_k) \overline{\operatorname{div} \varphi_k} \, dx \\ &= - \int_{\Omega \setminus \Omega_k} \phi \overline{\operatorname{div} \varphi_k} \, dx + \lambda \int_{\Omega} u \cdot \bar{\varphi}_k \, dx + \int_{\Omega} a_{jl}^{\alpha\beta}(\mu) \partial_l u_\beta \overline{\partial_j (\varphi_k)_\alpha} \, dx \\ &\quad - \int_{\Omega} f \cdot \bar{\varphi}_k \, dx - \left(\lambda \int_{\Omega_k} u_k \cdot \bar{\varphi}_k \, dx + \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_k)_\beta \overline{\partial_j (\varphi_k)_\alpha} \, dx \right) \\ &\quad + \int_{\Omega_k} f \cdot \bar{\varphi}_k \, dx \\ &= - \int_{\Omega \setminus \Omega_k} \phi \overline{\operatorname{div} \varphi_k} \, dx + \lambda \int_{\Omega \setminus \Omega_k} u \cdot \bar{\varphi}_k \, dx + \int_{\Omega \setminus \Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l u_\beta \overline{\partial_j (\varphi_k)_\alpha} \, dx \\ &\quad - \int_{\Omega \setminus \Omega_k} f \cdot \bar{\varphi}_k \, dx + \lambda \int_{\Omega_k} (u - u_k) \cdot \bar{\varphi}_k \, dx + \\ &\quad \int_{\Omega_k} a_{jl}^{\alpha\beta}(\mu) \partial_l (u_\beta - (u_k)_\beta) \overline{\partial_j (\varphi_k)_\alpha} \, dx. \end{aligned}$$

Since Ω is convex, it holds $\|\nabla \varphi_k\|_{L^2(\Omega; \mathbb{C}^{d^2})} \leq \|\phi - \phi_k\|_{L^2(\Omega_k)}$. This implies by Poincaré’s inequality and $\Omega_k \subset \Omega$ that $\|\varphi_k\|_{L^2(\Omega; \mathbb{C}^d)} \leq C \operatorname{diam}(\Omega) \|\phi - \phi_k\|_{L^2(\Omega_k)}$, where $C > 0$ depends only on d . Thus, by virtue of Young’s inequality, one can absorb $\|\phi - \phi_k\|_{L^2(\Omega_k)}$ to the left-hand side of the inequality above so that the convergences proven above together with the facts that ϕ , u , and f are L^2 -integrable on Ω yield that $\phi - \phi_k \rightarrow 0$ as $k \rightarrow \infty$ in $L^2(\Omega)$, where $\phi - \phi_k$ is defined to be zero in $\Omega \setminus \Omega_k$. Finally, since each ϕ_k lies in $H^1(\Omega_k)$ and respects the estimate from the formulation of the theorem, we find that $\phi \in H^1(\Omega)$ and that

$$\|\nabla \phi\|_{L^2(\Omega)} \leq \liminf_{k \rightarrow \infty} \|\nabla \phi_k\|_{L^2(\Omega_k)}.$$

This proves the desired estimate for u and ϕ . □

Remark 4.5 For a similar approximation scheme in the case of no-slip boundary conditions see [45].

Notice that the sectoriality of B_μ (by Proposition 2.3) implies the validity of the algebraic and topological decomposition $\mathcal{L}_\sigma^2(\Omega)$

$$\mathcal{L}_\sigma^2(\Omega) = \ker(B_\mu) \oplus \overline{\mathcal{R}(B_\mu)},$$

where $\ker(B_\mu)$ denotes the kernel of B_μ and $\mathcal{R}(B_\mu)$ the range of B_μ . See [30, Prop. 2.2.1] for the corresponding statement on the decomposition.

Corollary 4.6 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded convex domain and $\mu \in (-1, \sqrt{2} - 1)$. Then for all $u \in \mathcal{D}(B_\mu) \cap \overline{\mathcal{R}(B_\mu)}$ and the associated pressure ϕ one finds that $u \in H^2(\Omega; \mathbb{C}^d)$ and $\phi \in H^1(\Omega)$. Moreover, there exists $C > 0$ depending only on d and μ such that*

$$\int_\Omega |\nabla^2 u|^2 \, dx + \int_\Omega |\nabla \phi|^2 \, dx \leq C \int_\Omega |B_\mu u|^2 \, dx.$$

Proof First of all, notice that the statement below concerning the strong convergence of resolvent operators follow from [30, Prop. 2.2.1]. Define $f := B_\mu u$. The solution u is approximated by $u_\lambda := (\lambda + B_\mu)^{-1} f$ as $\lambda \in S_{\pi/2}$ tends to zero. Indeed, since $f = B_\mu u$ and since $u \in \overline{\mathcal{R}(B_\mu)}$ by assumption, one has due to the sectoriality of B_μ , see Proposition 2.3, that

$$u_\lambda = B_\mu(\lambda + B_\mu)^{-1} u \rightarrow u \text{ in } \mathcal{L}_\sigma^2(\Omega) \text{ as } \lambda \rightarrow 0.$$

Furthermore, the sectoriality implies that $B_\mu u_\lambda$ tends to f in $\mathcal{L}_\sigma^2(\Omega)$ and as well that $\lambda u_\lambda \rightarrow 0$ tends to zero in $\mathcal{L}_\sigma^2(\Omega)$ as $\lambda \in S_{\pi/2}$. The convergence of the associated pressures ϕ_λ in $L^2(\Omega)$ is proven as before by invoking Bogovskii’s operator. Finally, the convergence in the $H^2(\Omega; \mathbb{C}^d)$ - and $H^1(\Omega)$ -norms of the respective sequences follows by employing the inequality proven in Theorem 4.4 and the fact that the “right-hand side” $B_\mu u_\lambda$ of the equations for u_λ and ϕ_λ tend to f in $\mathcal{L}_\sigma^2(\Omega)$. The desired inequality follows from Theorem 4.4 by taking limits. \square

Problem 4.7 Prove or disprove Theorem 4.4 for $\mu \in \{-1\} \cup [\sqrt{2} - 1, 1]$.

In the case of no-slip boundary conditions, the H^2 -regularity is known in two and three dimensions if convex polygonal/polyhedral domains are considered, see [10,34,40]. It would be interesting to know if this property holds on arbitrary convex domains.

Problem 4.8 Prove or disprove Theorem 4.4 in the case of no-slip boundary conditions.

In the following, we start by working with cubes in \mathbb{R}^d . By this we mean a non-degenerate cube of the form $(a, b)^d$, i.e., its Lebesgue measure is non-zero and its sides are parallel to the axes. Sometimes we will use the notation $Q(x_0, r)$ to denote a cube with center x_0 and diameter r . We continue by deriving local H^2 -estimates and start with a technical lemma.

Lemma 4.9 *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with C^2 -boundary and let Q be a cube. Then $Q \cap \Omega$ is piecewise C^2 -regular, i.e., there exist sets Γ_0 and Γ_1 such that $\partial[Q \cap \Omega] = \Gamma_0 \cup \Gamma_1$, where Γ_0 has surface measure zero and where for each $x \in \Gamma_1$ the boundary part of $Q \cap \Omega$ is C^2 -regular in a neighborhood of x . In particular, Γ_1 satisfies $\Gamma_1 \cap \Omega \subset \partial Q \cap \Omega$.*

Proof First of all, notice that $Q \cap \Omega$ is a bounded convex domain and thus in particular a bounded Lipschitz domain, see [28, Cor. 1.2.2.3]. Notice that due to the Lipschitz boundary of $Q \cap \Omega$ the surface measure is equivalent to the $(d - 1)$ -dimensional Hausdorff measure in \mathbb{R}^d . To decompose the boundary of $Q \cap \Omega$, notice that elementary set theoretic manipulations yield

$$\partial(Q \cap \Omega) \subset ((\partial Q) \cap \bar{\Omega}) \cup (\bar{Q} \cap (\partial\Omega)) = ((\partial Q) \cap \bar{\Omega}) \cup (Q \cap (\partial\Omega)).$$

Notice that any point in $Q \cap (\partial\Omega)$ has a neighborhood with an at least C^2 -regular boundary. Thus, we consider $(\partial Q) \cap \bar{\Omega}$ more closely.

Let $\mathcal{N} \subset \partial Q$ denote the edges of the cube Q . Clearly, its $(d - 1)$ -dimensional Hausdorff measure is zero. Let $F \subset \partial Q$ be a face of Q (we consider F to be closed). Since F and $\bar{\Omega}$ are convex, also $F \cap \bar{\Omega}$ is convex. Notice that $F \cap \bar{\Omega}$ is congruent to a convex set in \mathbb{R}^{d-1} . As convex sets are Lipschitz regular, the boundary of $F \cap \bar{\Omega}$ (with respect to the subspace topology of F) has zero $(d - 1)$ -dimensional Hausdorff measure. If x is in the interior of $F \cap \bar{\Omega}$ (with respect to the subspace topology of F) and if $x \notin \mathcal{N}$, then there is $\varepsilon > 0$ such that $F \cap \bar{\Omega} \cap B(x, \varepsilon) = F \cap B(x, \varepsilon)$. Thus, in this neighborhood, $F \cap \bar{\Omega}$ can be represented as the graph of a smooth function. Denote the boundary of $F \cap \bar{\Omega}$ taken with respect to the subspace topology by $\partial_F(F \cap \bar{\Omega})$ and the interior by $\text{int}_F(F \cap \bar{\Omega})$ and define

$$\Gamma_0 := \left(\mathcal{N} \cup \bigcup_{F \text{ face of } Q} \partial_F(F \cap \bar{\Omega}) \right) \cap \partial(Q \cap \Omega)$$

and

$$\Gamma_1 := \left\{ \bigcup_{F \text{ face of } Q} (\text{int}_F(F \cap \bar{\Omega}) \setminus \mathcal{N}) \cup (Q \cap (\partial\Omega)) \right\} \cap \partial(Q \cap \Omega).$$

Notice that $\Gamma_1 \cap \Omega \subset \partial Q \cap \Omega$ holds by construction. □

Lemma 4.10 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded, convex, and smooth domain, $\theta \in (0, \pi)$, and $\mu \in (-1, \sqrt{2} - 1)$. Then there exists $C > 0$ depending only on d , μ , and θ such that smooth functions (smooth up to the boundary) $u : Q \cap \Omega \rightarrow \mathbb{C}^d$ and $\phi : Q \cap \Omega \rightarrow \mathbb{C}$ solving $\lambda u - \Delta u + \nabla \phi = 0$ and $\text{div}(u) = 0$ in $Q \cap \Omega$ and which satisfy $\{Du + \mu[Du]^T\}n - \phi n = 0$ on $Q \cap \partial\Omega$ satisfy*

$$\begin{aligned} & |\lambda| \int_{Q \cap \Omega} |\nabla u|^2 \, dx + \int_{Q \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{Q \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq C \left(|\lambda|^2 \int_{Q \cap \Omega} |u|^2 \, dx + \int_{(\partial Q) \cap \Omega} (|\nabla^2 u| |\nabla u| + |\nabla^2 u| |\phi| + |\nabla \phi| |\nabla u| \right. \\ & \quad \left. + |\nabla \phi| |\phi|) \, d\sigma \right). \end{aligned}$$

Proof By Lemma 4.9, $Q \cap \Omega$ is piecewise C^2 -regular with corresponding set Γ_1 satisfying $\Gamma_1 \cap \Omega \subset (\partial Q) \cap \Omega$. Thus, we are in the situation to apply Theorem 4.2 on the underlying domain $Q \cap \Omega$ and $v := v_\beta$ defined by (4.4). The same calculation as in the first part of the proof of Theorem 4.4 (but with an application of Theorem 4.2 instead of Theorem 4.1) yields

the existence of a constant $C > 0$ depending only on $d, \mu,$ and θ such that

$$\begin{aligned} & |\lambda| \int_{Q \cap \Omega} |\nabla u|^2 \, dx + \int_{Q \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{Q \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq C \left(|\lambda|^2 \int_{Q \cap \Omega} |u|^2 \, dx + \int_{(\partial Q) \cap \Omega} |\nabla v_\beta| |v_\beta| \, d\sigma \right). \end{aligned}$$

By definition of v_β this readily concludes the proof. □

In the previous proposition we saw that a local H^2 -estimate can be achieved with the drawback that highest-order terms appear in boundary integrals on the right-hand side of the inequality. The following lemma (the so-called ε -lemma) will help us to absorb these terms to the left-hand side and can be found in [25, Lem. 0.5]. Notice that the notation of cubes $Q(x_0, r)$ used here differs from the one used in [25], so that our formulation is slightly different.

Lemma 4.11 *Let $f, g,$ and h be non-negative functions in $L^1(Q)$, where Q is a cube in \mathbb{R}^d and let $\alpha > 0$. There exists $\varepsilon_0 > 0$, depending only on d and α , such that if for some $0 \leq \varepsilon \leq \varepsilon_0$ and some $C_1 = C_1(\varepsilon) > 0$ the estimate*

$$\int_{Q(x_0, r)} f \, dx \leq C_1 \left\{ \frac{1}{r^\alpha} \int_{Q(x_0, 2r)} g \, dx + \int_{Q(x_0, 2r)} h \, dx \right\} + \varepsilon \int_{Q(x_0, 2r)} f \, dx$$

holds for all $x_0 \in Q$ and $0 < r < \sqrt{d} \operatorname{dist}(x_0, \partial Q)$, then there exists a constant $C > 0$, depending only on $d, \alpha,$ and C_1 , such that

$$\int_{Q(x_0, r)} f \, dx \leq C \left\{ \frac{1}{r^\alpha} \int_{Q(x_0, 2r)} g \, dx + \int_{Q(x_0, 2r)} h \, dx \right\}.$$

The following proposition finally provides us with a local higher-order estimate. For a cube Q with diameter $R > 0$ and center x_0 and $\alpha > 0$ we adopt the notation αQ for the cube with same center and diameter αR .

Proposition 4.12 *Let $\Omega \subset \mathbb{R}^d, d \geq 2,$ be a bounded, convex, and smooth domain, $\theta \in (0, \pi), \mu \in (-1, \sqrt{2} - 1)$ and let Q be a cube with $Q \cap \Omega \neq \emptyset$ and diameter $R > 0$. Then there exists $C > 0$ depending only on $d, \mu,$ and θ such that smooth functions (smooth up to the boundary) $u : (2Q) \cap \Omega \rightarrow \mathbb{C}^d$ and $\phi : (2Q) \cap \Omega \rightarrow \mathbb{C}$ solving $\lambda u - \Delta u + \nabla \phi = 0$ and $\operatorname{div}(u) = 0$ in $(2Q) \cap \Omega$ and which satisfy $\{Du + \mu[Du]^\top\}n - \phi n = 0$ on $(2Q) \cap \partial\Omega$ satisfy*

$$\begin{aligned} & |\lambda| \int_{Q \cap \Omega} |\nabla u|^2 \, dx + \int_{Q \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{Q \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq C \left(|\lambda|^2 \int_{(2Q) \cap \Omega} |u|^2 \, dx + \frac{1}{R^2} \int_{(2Q) \cap \Omega} (|\nabla u|^2 + |\phi|^2) \, dx \right). \end{aligned}$$

Proof Fix a cube $Q \subset \mathbb{R}^d$ with $Q \cap \Omega \neq \emptyset$. Let $Q := Q(x_0, r) \subset \mathbb{R}^d$ be a cube with center $x_0 \in Q$ and $\operatorname{diam}(Q) = r$ that satisfies $0 < r < \sqrt{d} \operatorname{dist}(x_0, \partial Q)$. Let $1 < s < 2$. By Lemma 4.10 one finds

$$\begin{aligned} & |\lambda| \int_{Q \cap \Omega} |\nabla u|^2 \, dx + \int_{Q \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{Q \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq |\lambda| \int_{(sQ) \cap \Omega} |\nabla u|^2 \, dx + \int_{(sQ) \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{(sQ) \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq C \left(|\lambda|^2 \int_{(2Q) \cap \Omega} |u|^2 \, dx + \int_{(\partial sQ) \cap \Omega} (|\nabla^2 u| |\nabla u| + |\nabla^2 u| |\phi| + |\nabla \phi| |\nabla u| + |\nabla \phi| |\phi|) \, d\sigma \right), \end{aligned}$$

where the constant $C > 0$ depends only on $d, \mu,$ and θ . An application of Young’s inequality (this produces the factors $r\varepsilon$ and $(r\varepsilon)^{-1}$ for some $\varepsilon > 0$) followed by an integration over s yields

$$\begin{aligned} & |\lambda| \int_{Q \cap \Omega} |\nabla u|^2 \, dx + \int_{Q \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{Q \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq C|\lambda|^2 \int_{(2Q) \cap \Omega} |u|^2 \, dx + \frac{C}{r\varepsilon} \int_1^2 \int_{(\partial s Q) \cap \Omega} (|\nabla u|^2 + |\phi|^2) \, d\sigma \, ds \\ & \quad + r\varepsilon \int_1^2 \int_{(\partial s Q) \cap \Omega} (|\nabla^2 u|^2 + |\nabla \phi|^2) \, d\sigma \, ds. \end{aligned}$$

Now, notice that the co-area formula, see [18, Thm. 3.2.12], implies that

$$\int_1^2 \int_{\partial s Q} \mathbf{g} \, d\sigma \, ds \leq \frac{C_{\text{co-area}}}{r} \int_{2Q} \mathbf{g} \, dx$$

for all representatives \mathbf{g} of a function $g \in L^1(\mathbb{R}^d)$, where $C_{\text{co-area}} > 0$ is an absolute constant. Choosing g in the first integral as $E_0(|\nabla u|^2 + |\phi|^2)$ and in the second integral as $E_0(|\nabla^2 u|^2 + |\nabla \phi|^2)$, where E_0 extends functions outside of $(2Q) \cap \Omega$ by zero delivers

$$\begin{aligned} & |\lambda| \int_{Q \cap \Omega} |\nabla u|^2 \, dx + \int_{Q \cap \Omega} |\nabla^2 u|^2 \, dx + \int_{Q \cap \Omega} |\nabla \phi|^2 \, dx \\ & \leq C|\lambda|^2 \int_{(2Q) \cap \Omega} |u|^2 \, dx + \frac{C C_{\text{co-area}}}{r^2 \varepsilon} \int_{(2Q) \cap \Omega} (|\nabla u|^2 + |\phi|^2) \, dx \\ & \quad + \varepsilon C_{\text{co-area}} \int_{(2Q) \cap \Omega} (|\nabla^2 u|^2 + |\nabla \phi|^2) \, dx. \end{aligned}$$

The proof is concluded for ε small enough by an application of Lemma 4.11. □

5 An L^p -extrapolation theorem suitable for subspaces of L^p

In classical Calderón–Zygmund theory, operators T associated to an integral kernel $K(\cdot, \cdot)$ give rise to an L^p -bounded operator for all $1 < p < \infty$ if T is bounded on L^2 and if the kernel K is a so-called standard kernel. The standard kernel property is some kind of cancellation property of K , see [11, Def. 5.11]. If the operator T is either not associated to a kernel or if one is only interested in whether T is bounded on L^p for p being merely in an interval $I \subset (1, \infty)$, then one can replace the property that T is associated to a standard kernel by weaker cancellation properties.

In this context, there are for example the L^p -extrapolation theorems of Shen [49] (if one is interested to conclude the L^p -boundedness on an interval $(2, q)$ with $q > 2$) or of Blunck and Kunstmann [5] (if one is interested to conclude the L^p -boundedness on an interval $(q, 2)$ with $q < 2$). In the following, we will consider Shen’s theorem more closely and begin with a formulation of his theorem which can be found in [55,57]. For its formulation, we denote for a ball B with center x_0 and radius $r > 0$ the ball with the same center and radius αr by αB .

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^d$ be Lebesgue-measurable, $\mathcal{M} > 0$, and let $T \in \mathcal{L}(L^2(\Omega; \mathbb{C}^m), L^2(\Omega; \mathbb{C}^n))$ with $\|T\|_{\mathcal{L}(L^2(\Omega; \mathbb{C}^m), L^2(\Omega; \mathbb{C}^n))} \leq \mathcal{M}$.*

Suppose that there exist constants $q > 2$, $R_0 > 0$, $\alpha_2 > \alpha_1 > 1$, and $C > 0$, where $R_0 = \infty$ if $\text{diam}(\Omega) = \infty$, such that the following holds. Namely, for all $B = B(x_0, r)$ with $0 < r < R_0$, whose center x_0 is either such that $x_0 \in \partial\Omega$ or satisfy $\alpha_2 B \subset \Omega$, and all compactly supported $f \in L^\infty(\Omega; \mathbb{C}^m)$ with $f = 0$ on $\Omega \cap \alpha_2 B$ the estimate

$$\left(\frac{1}{r^d} \int_{\Omega \cap B} |Tf|^q \, dx\right)^{\frac{1}{q}} \leq C \left\{ \left(\frac{1}{r^d} \int_{\Omega \cap \alpha_1 B} |Tf|^2 \, dx\right)^{\frac{1}{2}} + \sup_{B' \supset B} \left(\frac{1}{|B'|} \int_{\Omega \cap B'} |f|^2 \, dx\right)^{\frac{1}{2}} \right\} \tag{5.1}$$

holds. Here the supremum runs over all balls B' containing B .

Then for each $2 < p < q$ the restriction of T onto $L^2(\Omega; \mathbb{C}^m) \cap L^p(\Omega; \mathbb{C}^m)$ extends to a bounded linear operator from $L^p(\Omega; \mathbb{C}^m)$ into $L^p(\Omega; \mathbb{C}^n)$, with operator norm bounded by a constant depending on $d, p, q, \alpha_1, \alpha_2, C$, and \mathcal{M} , and additionally on R_0 and $\text{diam}(\Omega)$ if Ω is bounded.

In this theorem, the standard kernel property is replaced by the validity of (5.1). If $\Omega = \mathbb{R}^d$, then the proof builds on a good- λ argument. If Ω is not \mathbb{R}^d , one can define an appropriate operator on the whole space given by $Sf := E_0 T R_\Omega f$, where E_0 extends functions on Ω by zero and R_Ω restricts functions on the whole space to Ω . One can show, that if T satisfies the assumptions of Theorem 5.1 on Ω , then S satisfies the assumptions of the same theorem with Ω set to \mathbb{R}^d , cf. [55, p. 78f]. If $\Omega = \mathbb{R}^d$, an analysis of the good- λ argument reveals that (5.1) is used exactly once, namely, in order to deduce an inequality of the form

$$\begin{aligned} & |\{x \in Q : M_{2Q^*}(|Tf|^2)(x) > \iota\}| \\ & \leq \frac{C}{\iota} \int_{2\alpha_2 Q^*} |f|^2 \, dx + \frac{C|\mathcal{Q}|}{\iota^{q/2}} \left\{ \left(\frac{1}{|\mathcal{Q}|} \int_{2\alpha_2 Q^*} |Tf|^2 \, dx\right)^{\frac{1}{2}} + \sup_{Q' \supset 2Q^*} \left(\frac{1}{|Q'|} \int_{Q'} |f|^2 \, dx\right)^{\frac{1}{2}} \right\}^q, \end{aligned} \tag{5.2}$$

cf. [55, p. 76f]. Here, $\iota > 0$ is arbitrary, Q is a cube in \mathbb{R}^d , Q^* is its ‘‘parent’’, i.e., Q arises from Q^* by bisecting its sides, and M_{2Q^*} is the localized maximal operator

$$M_{2Q^*}g(x) := \sup_{\substack{x \in R \\ R \subset 2Q^*}} \frac{1}{|R|} \int_R |g| \, dy \quad (x \in 2Q^*),$$

where in the supremum R denotes a cube in \mathbb{R}^d . To derive (5.2) from (5.1) and the L^2 -boundedness of T , notice that (5.1) can equivalently be formulated with cubes instead of balls. Then, f is decomposed as $f = f\chi_{2\alpha_2 Q^*} + f\chi_{\mathbb{R}^d \setminus 2\alpha_2 Q^*}$, where χ denotes the characteristic function of a set. This decomposition is used on the left-hand side of (5.2) to estimate

$$\begin{aligned} |\{x \in Q : M_{2Q^*}(|Tf|^2)(x) > \iota\}| & \leq |\{x \in Q : M_{2Q^*}(|Tf\chi_{2\alpha_2 Q^*}|^2)(x) > \iota/4\}| \\ & \quad + |\{x \in Q : M_{2Q^*}(|Tf\chi_{\mathbb{R}^d \setminus 2\alpha_2 Q^*}|^2)(x) > \iota/4\}|. \end{aligned} \tag{5.3}$$

The first term on the right-hand side is controlled by the weak type-(1, 1) estimate of the localized maximal operator and the L^2 -boundedness of T , yielding the first term on the right-hand side of (5.2). The second term on the right-hand side is controlled by the embedding $L^{q/2} \hookrightarrow L^{q/2, \infty}$ and the $L^{q/2}$ -boundedness of the localized maximal operator followed by (5.1) and the L^2 -boundedness of T yielding the remaining terms on the right-hand side of (5.2), cf. [55, p. 76f].

Essentially, the only thing that happened in (5.3) was that Tf was decomposed by means of

$$Tf = Tf\chi_{2\alpha_2 Q^*} + Tf\chi_{\mathbb{R}^d \setminus 2\alpha_2 Q^*}. \tag{5.4}$$

We would like to emphasize here that this decomposition of Tf is induced by the linearity of T and a decomposition of f . Clearly, one could imagine that other suitable decompositions of Tf into a sum of two functions exist and that these might not have anything to do with a decomposition of f . Taking this into account in the formulation of the L^p -extrapolation theorem might yield a more flexible result. This could be an advantage if a certain structure of f (such as solenoidality) is eminent and which is destroyed by multiplication by characteristic functions. This happens for example if one considers the map $T : f \mapsto \phi$, where f is mapped to the pressure function corresponding to the Stokes resolvent problem (Res) and (Neu). If f is for example divergence-free, then Tf enjoys the decay estimate presented in Proposition 3.1 while $Tf\chi_{2\alpha_2 Q^*}$ and $Tf\chi_{\mathbb{R}^d \setminus 2\alpha_2 Q^*}$ enjoy no decay estimates at all by Remark 3.2. This indicates the need of a formulation of Shen's L^p -extrapolation theorem that does not rely on a particular decomposition of Tf and is presented below.

In the rest of this section, we discuss an adapted version of Theorem 5.1, where (5.1) is replaced essentially by the validity of (5.2) (which has to be modified if $\Omega \neq \mathbb{R}^d$). To this end, we say that Q^* is the parent of a cube $Q \subset \mathbb{R}^d$ if Q arises from Q^* by bisecting its sides. Moreover, for $x_0 \in \mathbb{R}^d$ and $r > 0$ let $Q(x_0, r)$ denote the non-degenerated cube in \mathbb{R}^d with center x_0 and $\text{diam}(Q(x_0, r)) = r$. Finally, for a number $\alpha > 0$ denote by αQ the cube $Q(x_0, \alpha r)$. The discussion above together with an analysis of the proof of [49, Thm. 3.1] readily shows the validity of the following theorem.

Theorem 5.2 *Let $2 < p < q$, $f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \cap L^p(\mathbb{R}^d; \mathbb{C}^m)$, and let T be an operator (not necessarily linear) such that $T(f)$ is defined and contained in $L^2(\mathbb{R}^d; \mathbb{C}^n)$.*

Suppose that there exist constants $\alpha > 1$ and $C > 0$ such that for all $\iota > 0$, all $Q = Q(x_0, r)$ with $r > 0$ and $x_0 \in \mathbb{R}^d$, and all parents Q^ of Q the estimate*

$$\begin{aligned} & \{ \{x \in Q : M_{2Q^*}(|T(f)|^2)(x) > \iota \} \} \\ & \leq \frac{C}{\iota} \int_{2\alpha Q^*} |f|^2 \, dx + \frac{C|Q|}{\iota^{q/2}} \left\{ \left(\frac{1}{|Q|} \int_{2\alpha Q^*} |T(f)|^2 \, dx \right)^{\frac{1}{2}} + \sup_{Q' \supset 2Q^*} \left(\frac{1}{|Q'|} \int_{Q'} |f|^2 \, dx \right)^{\frac{1}{2}} \right\}^q, \end{aligned} \tag{5.5}$$

holds. Here the supremum runs over all cubes Q' containing $2Q^$.*

Then there exists a constant $C > 0$ depending on d, p, q, α , and C such that

$$\|T(f)\|_{L^p(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|f\|_{L^p(\mathbb{R}^d; \mathbb{C}^m)}.$$

Let T be an operator acting on functions defined on Ω for some Lebesgue-measurable set $\Omega \subset \mathbb{R}^d$ and let $f \in L^2(\Omega; \mathbb{C}^m) \cap L^q(\Omega; \mathbb{C}^m)$. The following theorem is a direct consequence of Theorem 5.2 when applied to the operator $S := E_0 T R_\Omega$ and the function $E_0 f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \cap L^q(\mathbb{R}^d; \mathbb{C}^m)$.

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^d$ be Lebesgue-measurable, $2 < p < q$, $f \in L^2(\Omega; \mathbb{C}^m) \cap L^p(\Omega; \mathbb{C}^m)$, and let T be an operator (not necessarily linear) such that $T(f)$ is defined and contained in $L^2(\mathbb{R}^d; \mathbb{C}^n)$.*

Suppose that there exist constants $\alpha > 1$ and $C > 0$ such that for all $\iota > 0$, all $Q = Q(x_0, r)$ with $r > 0$ and $x_0 \in \mathbb{R}^d$, and all parents Q^* of Q with $(2Q^*) \cap \Omega \neq \emptyset$ the estimate

$$\begin{aligned} |\{x \in Q : M_{2Q^*}(|E_0 T(f)|^2)(x) > \iota\}| &\leq \frac{C}{\iota} \int_{(2\alpha Q^*) \cap \Omega} |f|^2 \, dx \\ &+ \frac{C|Q|}{\iota^{q/2}} \left\{ \left(\frac{1}{|Q|} \int_{(2\alpha Q^*) \cap \Omega} |T(f)|^2 \, dx \right)^{\frac{1}{2}} + \sup_{Q' \supset 2Q^*} \left(\frac{1}{|Q'|} \int_{Q' \cap \Omega} |f|^2 \, dx \right)^{\frac{1}{2}} \right\}^q, \end{aligned} \tag{5.6}$$

holds. Here the supremum runs over all cubes Q' containing $2Q^*$.

Then there exists a constant $C > 0$ depending on d, p, q, α , and C such that

$$\|T(f)\|_{L^p(\Omega; \mathbb{C}^n)} \leq C \|f\|_{L^p(\Omega; \mathbb{C}^m)}.$$

6 Estimates on the resolvent on convex domains

In this section we verify the assumptions of Theorem 5.3 for a particular choice of operators T . In the case of elliptic operators, a common way to do so is to establish the validity of a Caccioppoli type estimate, which we establish now for the Stokes resolvent problem, see also [55, Prop. 5.3.2], [8, Lem. 3.8], and [25, Thm. 1.1].

Lemma 6.1 *Let $\theta \in [0, \pi)$, $\lambda \in S_\theta$, $x_0 \in \mathbb{R}^d$, $r > 0$, and $\mu \in [-1, 1)$. Let $u \in \mathcal{H}_\sigma^1(Q(x_0, 2r) \cap \Omega)$ and $\phi \in L^2(Q(x_0, 2r) \cap \Omega)$ solve*

$$\lambda \int_{Q(x_0, 2r) \cap \Omega} u \cdot \bar{\varphi} \, dx + \int_{Q(x_0, 2r) \cap \Omega} \nabla u \cdot \bar{\nabla \varphi} \, dx - \int_{Q(x_0, 2r) \cap \Omega} \phi \overline{\operatorname{div}(\varphi)} \, dx = 0$$

for all $\varphi \in H^1(Q(x_0, 2r) \cap \Omega; \mathbb{C}^d)$ with $\varphi = 0$ on $(\partial Q(x_0, 2r)) \cap \Omega$. Then there exists a constant $C > 0$ depending only on θ and d such that

$$\begin{aligned} &|\lambda| \int_{Q(x_0, r) \cap \Omega} |u|^2 \, dx + \int_{Q(x_0, r) \cap \Omega} |\nabla u|^2 \, dx \\ &\leq \frac{C}{r^2} \left(\frac{1}{|\lambda|} \int_{Q(x_0, 2r) \cap \Omega} |\phi|^2 \, dx + \int_{Q(x_0, 2r) \cap \Omega} |u|^2 \, dx \right). \end{aligned}$$

Proof The proof follows literally the lines of [55, Prop. 5.3.2] (which proves this inequality in the case of homogeneous Dirichlet boundary conditions on $\partial\Omega$). \square

Another ingredient that is needed in the verification of the assumptions of Theorem 5.3 is Sobolev’s inequality on convex domains. This is obtained by combining [27, Lem. 7.16] with either [27, Lem. 7.12] (in the case $|1/p - 1/q| < 1/d$) or [3, Thm. 3.1.4] (in the case $|1/p - 1/q| = 1/d$).

Proposition 6.2 *Let $\Xi \subset \mathbb{R}^d$ be bounded and convex and let $1 \leq p < q < \infty$ satisfy $|1/p - 1/q| \leq 1/d$. Then there exists a constant $C > 0$ depending only on d, p , and q such that for all $u \in W^{1,p}(\Xi)$*

$$\begin{aligned} \left(\int_{\Xi} |u|^q \, dx \right)^{\frac{1}{q}} &\leq |\Xi|^{\frac{1}{q} - \frac{1}{p}} \left(\int_{\Xi} |u|^p \, dx \right)^{\frac{1}{p}} \\ &+ C |\Xi|^{\frac{1}{d} - (\frac{1}{p} - \frac{1}{q}) - 1} \operatorname{diam}(\Xi)^d \left(\int_{\Xi} |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now, we are in the position to present the proof of Theorem 1.1.

Proof of Theorem 1.1 We distinguish the cases $2 < p < 2d/(d - 2)$, $2d/(d + 2) < p < 2$, and $p = 2$. Notice that the case $p = 2$ readily follows by Propositions 2.3 and 3.1.

Case 1: It holds $2 < p < 2d/(d - 2)$

Let Ω be a bounded and convex domain and let $(\Omega_k)_{k \in \mathbb{N}}$ be the sequence of bounded, convex, and smooth domains introduced in Remark 4.3. Let $f \in C^\infty(\overline{\Omega_k})$, let u be given by $u := (\lambda + B_{\mu,k})^{-1}f$, and let ϕ denote the associated pressure. Here $B_{\mu,k}$ denotes the Stokes operator subject to Neumann-type boundary conditions on Ω_k . Notice that u and ϕ are smooth up to the boundary by Remark 2.4. We show that

$$T_\lambda f := \begin{pmatrix} |\lambda|u \\ |\lambda|^{1/2}\nabla u \\ |\lambda|^{1/2}\phi \end{pmatrix}$$

is uniformly bounded with respect to λ from $L^p_\sigma(\Omega_k)$ to $L^p(\Omega_k; \mathbb{C}^{d+d^2+1})$. To this end, we show in the following that $T_\lambda f$ satisfies (5.6) with $q := 2d/(d - 2)$ in the case $d \geq 3$ and $q > 2$ arbitrary in the case $d = 2$. To obtain the uniform boundedness with respect to λ , we need to verify (5.6) with involved constants independent of λ . Let $Q = Q(x_0, r) \subset \mathbb{R}^d$ be a cube with center x_0 and $\text{diam}(Q) = r$ that satisfies $(2Q^*) \cap \Omega \neq \emptyset$. Then, we consider the following three cases.

Case 1.1: It holds $2r > \sqrt{d} \text{diam}(\Omega)$

The conditions imposed on Q^* and r imply that for all $k \in \mathbb{N}$ we have $\Omega_k \subset 4Q^*$. In this case, use the weak-type (1, 1) estimate of the localized maximal operator and the L^2 -boundedness of T_λ (cf. Propositions 2.3 and 3.1, notice that the constants only depend on d, θ , and μ) to obtain

$$|\{x \in Q : M_{2Q^*}(|E_0 T_\lambda f|^2)(x) > \iota\}| \leq \frac{C}{\iota} \int_{\Omega_k} |T_\lambda f|^2 \, dx \leq \frac{C}{\iota} \int_{(4Q^*) \cap \Omega} |f|^2 \, dx.$$

Case 1.2: It holds $2r \leq \sqrt{d} \text{diam}(\Omega)$ and $(2Q^*) \cap \partial\Omega_k \neq \emptyset$

Let $y \in (2Q^*) \cap \partial\Omega_k$ and let $\mathcal{Q} := Q(y, 4r) \subset \mathbb{R}^d$ be the cube with center y and $\text{diam}(\mathcal{Q}) = 4r$. In this case, it holds $2Q^* \subset \mathcal{Q}$. Define functions v and w as follows. Let $\widetilde{B}_{\mu,k}$ denote the Stokes operator subject to Neumann-type boundary conditions on $(8\mathcal{Q}) \cap \Omega_k$. Notice that the restriction of f to $(8\mathcal{Q}) \cap \Omega_k$ is still in $C^\infty(\overline{(8\mathcal{Q}) \cap \Omega_k})$ and thus define

$$v := (\lambda + \widetilde{B}_\mu)^{-1} R_{(8\mathcal{Q}) \cap \Omega_k} f \quad \text{and} \quad w := u - v,$$

where $R_{(8\mathcal{Q}) \cap \Omega_k}$ denotes the restriction operator to $(8\mathcal{Q}) \cap \Omega_k$. Analogously, define the pressures ϑ associated to v and $R_{(8\mathcal{Q}) \cap \Omega_k} f$ and $\psi := \phi - \vartheta$. Thus, in the sense of distributions it holds

$$\begin{cases} \lambda v - \Delta v + \nabla \vartheta = R_{(8\mathcal{Q}) \cap \Omega_k} f & \text{in } (8\mathcal{Q}) \cap \Omega_k \\ \text{div}(v) = 0 & \text{in } (8\mathcal{Q}) \cap \Omega_k \\ \{Dv + \mu[Dv]^\top\}n - \vartheta n = 0 & \text{on } \partial[(8\mathcal{Q}) \cap \Omega_k] \end{cases}$$

and

$$\begin{cases} \lambda w - \Delta w + \nabla \psi = 0 & \text{in } (8\mathcal{Q}) \cap \Omega_k \\ \operatorname{div}(w) = 0 & \text{in } (8\mathcal{Q}) \cap \Omega_k \\ \{Dw + \mu[Dw]^\top\}n - \psi n = 0 & \text{on } (8\mathcal{Q}) \cap \partial\Omega_k. \end{cases}$$

Here, n denotes the outward unit normal vector corresponding to the set $(8\mathcal{Q}) \cap \Omega_k$. Notice that in $(8\mathcal{Q}) \cap \Omega_k$ the identities $u = v + w$ and $\phi = \vartheta + \psi$ hold and that w and ϑ are in general non-zero as there is no boundary condition on the remaining boundary part $\partial[(8\mathcal{Q}) \cap \Omega_k] \setminus [(8\mathcal{Q}) \cap \partial\Omega_k]$ imposed. Let $\tilde{v}, \tilde{\nabla}v, \tilde{\vartheta}, \tilde{w}, \tilde{\nabla}w$, and $\tilde{\psi}$ denote the extensions by zero to \mathbb{R}^d . Then for $\iota > 0$, we have

$$\begin{aligned} & |\{x \in \mathcal{Q} : M_{2\mathcal{Q}^*}(|E_0 T_\lambda f|^2)(x) > \iota\}| \\ & \leq |\{x \in \mathcal{Q} : M_{2\mathcal{Q}^*}(\|\lambda|\tilde{v}|^2 + \|\lambda\|^{1/2}\tilde{\nabla}v|^2 + \|\lambda\|^{1/2}\tilde{\vartheta}^2)(x) > \iota/4\}| \\ & \quad + |\{x \in \mathcal{Q} : M_{2\mathcal{Q}^*}(\|\lambda|\tilde{w}|^2 + \|\lambda\|^{1/2}\tilde{\nabla}w|^2 + \|\lambda\|^{1/2}\tilde{\psi}^2)(x) > \iota/4\}| \\ & =: \text{I} + \text{II}. \end{aligned}$$

The first term is controlled by the weak-type (1, 1) estimate of the localized maximal operator followed by Proposition 2.3 (2) and Proposition 3.1 yielding

$$\text{I} \leq \frac{C}{\iota} \int_{(2\mathcal{Q}^*) \cap \Omega_k} (\|\lambda|v|^2 + \|\lambda\|^{1/2}\nabla v|^2 + \|\lambda\|^{1/2}\vartheta|^2) \, dx \leq \frac{C}{\iota} \int_{(32\mathcal{Q}^*) \cap \Omega_k} |f|^2 \, dx,$$

where $C > 0$ depends only on d, θ , and μ .

Recall that q was chosen to be $q := 2d/(d - 2)$ if $d \geq 3$ and $q > 2$ arbitrary if $d = 2$. The second term, II, is controlled by the embedding $L^{q/2}(2\mathcal{Q}^*) \hookrightarrow L^{q/2,\infty}(2\mathcal{Q}^*)$, the $L^{q/2}$ -boundedness of the localized maximal operator, and the fact $2\mathcal{Q}^* \subset \mathcal{Q}$. Notice that the constants in these estimates depend only on d and q so that

$$\text{II} \leq \frac{C}{\iota^{q/2}} \int_{\mathcal{Q} \cap \Omega_k} (\|\lambda|w|^q + \|\lambda\|^{1/2}\nabla w|^q + \|\lambda\|^{1/2}\psi|^q) \, dx.$$

Next, apply Proposition 6.2 with $\Xi := \mathcal{Q} \cap \Omega_k$ combined with (4.3), to deduce

$$\begin{aligned} \text{II} & \leq \frac{C}{\iota^{q/2}} r^d \left\{ r^{1-d/2} |\lambda|^{1/2} \left(\int_{\mathcal{Q} \cap \Omega_k} (\|\lambda|\nabla w|^2 + |\nabla^2 w|^2 + |\nabla \psi|^2) \, dx \right)^{\frac{1}{2}} \right. \\ & \quad \left. + r^{-d/2} \left(\int_{\mathcal{Q} \cap \Omega_k} (\|\lambda|w|^2 + \|\lambda\|^{1/2}\nabla w|^2 + \|\lambda\|^{1/2}\psi|^2) \, dx \right)^{\frac{1}{2}} \right\}^q. \end{aligned} \tag{6.1}$$

Due to (4.3) the constant $C > 0$ also depends on $\operatorname{diam}(\Omega)$ and on $r_0 > 0$, where r_0 is such that $B(0, r_0) \subset \Omega - \{x\}$ for some $x \in \Omega$. The second term on the right-hand side is estimated by virtue of $u = v + w$ and $\phi = \vartheta + \psi$, Propositions 2.3 (2) and 3.1, and $8\mathcal{Q} \subset 32\mathcal{Q}^*$ as

$$\begin{aligned} & \left(\int_{\mathcal{Q} \cap \Omega_k} (\|\lambda|w|^2 + \|\lambda\|^{1/2}\nabla w|^2 + \|\lambda\|^{1/2}\psi|^2) \, dx \right)^{\frac{1}{2}} \\ & \leq \left(\int_{\mathcal{Q} \cap \Omega_k} |T_\lambda f|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{\mathcal{Q} \cap \Omega_k} (\|\lambda|v|^2 + \|\lambda\|^{1/2}\nabla v|^2 + \|\lambda\|^{1/2}\vartheta|^2) \, dx \right)^{\frac{1}{2}} \\ & \leq C \left\{ \left(\int_{(32\mathcal{Q}^*) \cap \Omega_k} |T_\lambda f|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{(32\mathcal{Q}^*) \cap \Omega_k} |f|^2 \, dx \right)^{\frac{1}{2}} \right\}. \end{aligned} \tag{6.2}$$

The first term on the right-hand side in (6.1) is estimated by virtue of Proposition 4.12 as

$$\int_{\mathcal{Q} \cap \Omega_k} (|\lambda| |\nabla w|^2 + |\nabla^2 w|^2 + |\nabla \psi|^2) \, dx \leq C \left(|\lambda|^2 \int_{(2\mathcal{Q}) \cap \Omega_k} |w|^2 \, dx + \frac{1}{r^2} \int_{(2\mathcal{Q}) \cap \Omega_k} (|\nabla w|^2 + |\psi|^2) \, dx \right).$$

Employing Caccioppoli’s inequality, Lemma 6.1, to the first term on the right-hand side finally delivers

$$\int_{\mathcal{Q} \cap \Omega_k} (|\lambda| |\nabla w|^2 + |\nabla^2 w|^2 + |\nabla \psi|^2) \, dx \leq C \left(\frac{|\lambda|}{r^2} \int_{(4\mathcal{Q}) \cap \Omega_k} |w|^2 \, dx + \frac{1}{r^2} \int_{(4\mathcal{Q}) \cap \Omega_k} (|\nabla w|^2 + |\psi|^2) \, dx \right). \tag{6.3}$$

Combining (6.1) and (6.3) one finds analogously to (6.2) that

$$\Pi \leq \frac{C}{\iota^{q/2}} \left\{ \left(\int_{(32\mathcal{Q}^*) \cap \Omega_k} |T_\lambda f|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{(32\mathcal{Q}^*) \cap \Omega_k} |f|^2 \, dx \right)^{\frac{1}{2}} \right\}^q.$$

Case 1.3: It holds $2r \leq \sqrt{d} \operatorname{diam}(\Omega)$ and $2\mathcal{Q}^* \cap \partial\Omega_k = \emptyset$

This case is treated similar as the previous case. The only difference is that there is no need to introduce the cube \mathcal{Q} , thus, by setting $\mathcal{Q} := 2\mathcal{Q}^*$ in Case 1.2, the proof is literally the same.

Conclusion of the proof of Case 1

Notice that the family $\{T_\lambda\}_{\lambda \in \mathcal{S}_\theta}$ is uniformly bounded from $\mathcal{L}_\sigma^2(\Omega_k)$ into $L^2(\Omega_k; \mathbb{C}^{d+d^2+1})$ and that all estimates proven in Case 1 are uniform with respect to λ . Thus we conclude by Theorem 5.3 that for all $2 < p < 2d/(d - 2)$ the family $\{T_\lambda\}_{\lambda \in \mathcal{S}_\theta}$ satisfies a uniform boundedness estimate from $\mathcal{L}_\sigma^p(\Omega_k)$ into $L^p(\Omega_k; \mathbb{C}^{d+d^2+1})$ for all $f \in C_\sigma^\infty(\overline{\Omega_k})$ and by density for all $f \in \mathcal{L}_\sigma^p(\Omega_k)$. In particular, this holds true for each of the mappings

$$T_\lambda^1 : f \mapsto |\lambda|u, \quad T_\lambda^2 : f \mapsto |\lambda|^{1/2}\nabla u, \quad \text{and} \quad T_\lambda^3 : f \mapsto |\lambda|^{1/2}\phi.$$

Now, by the approximation argument carried out in the proof of Theorem 4.4, the uniform boundedness of these mappings also follows on the domain Ω .

Case 2: It holds $2d/(d + 2) < p < 2$

To deduce the second case we argue by duality. Thus, let $q := 2d/(d - 2)$ if $d \geq 3$ and let $q > 2$ if $d = 2$. Let again Ω_k be a bounded, convex, and smooth domain introduced in Remark 4.3. Let $F \in C_c^\infty(\Omega_k; \mathbb{C}^{d \times d})$ and let u be given by $u := (\lambda + \mathcal{B}_{\mu,k})^{-1} \operatorname{div}(F)$ and let ϕ denote the associated pressure. Consider the operator

$$S_\lambda F := \begin{pmatrix} |\lambda|^{1/2}u \\ \nabla u \\ \phi \end{pmatrix}.$$

Notice that S_λ extends to a bounded operator from $L^2(\Omega_k; \mathbb{C}^{d \times d})$ to $L^2(\Omega_k; \mathbb{C}^{d+d^2+1})$ by Propositions 2.3 and 3.1 and that its operator norm is bounded by a constant depending merely on $d, \mu,$ and θ . For such a smooth F , the assumptions of Theorem 5.3 are verified analogously to Case 1. It follows that each of the mappings

$$S_\lambda^1 : F \mapsto |\lambda|^{1/2}u, \quad S_\lambda^2 : F \mapsto \nabla u, \quad \text{and} \quad S_\lambda^3 : F \mapsto \phi$$

gives rise to a uniformly bounded family of operators on $L^r(\Omega_k)$ for each $2 < r < q$. The approximation argument carried out in the proof of Theorem 4.4, implies the uniform boundedness of these mappings on the domain Ω . By duality, we conclude from the boundedness properties of the mapping T_λ^1 from Case 1 and from the boundedness properties of S_λ^1 that there exists $C > 0$ such that for all $\lambda \in S_\theta$ and all $f \in \mathcal{L}_\sigma^p(\Omega)$ it holds

$$\|\lambda(\lambda + B_\mu)^{-1}f\|_{\mathcal{L}_\sigma^p(\Omega)} + |\lambda|^{1/2}\|\nabla(\lambda + B_\mu)^{-1}f\|_{L^p(\Omega; \mathbb{C}^{d^2})} \leq C\|f\|_{\mathcal{L}_\sigma^p(\Omega)}. \tag{6.4}$$

The estimate on $\nabla(\lambda + B_\mu)^{-1} \operatorname{div}$ follows from the boundedness of S_λ^2 and duality. □

Remark 6.3 To control the pressure in L^p for $2d/(d + 2) < p < 2$ is difficult. Intuitively, one would employ (3.2) to write

$$\| \phi \|_{L^p(\Omega)} = \sup_{\substack{g \in L^{p'}(\Omega) \\ \|g\|_{L^{p'}(\Omega)} \leq 1}} \left| \int_{\Omega} \phi \operatorname{div} \nabla \Delta_D^{-1} g \, dx \right| = \sup_{\substack{g \in L^{p'}(\Omega) \\ \|g\|_{L^{p'}(\Omega)} \leq 1}} \left| \int_{\Omega} \nabla u \cdot \overline{\nabla \nabla \Delta_D^{-1} g} \, dx \right|.$$

However, it is impossible to control $\nabla \nabla \Delta_D^{-1} g$ in $L^{p'}$ due to the counterexample in [19, Prop. 2].

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