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L^p-extrapolation of non-local operators: Maximal regularity of elliptic integrodifferential operators with measurable coefficients

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On the occasion of the 60*th birthday of Matthias Hieber*

Abstract. The aim of this article is to deepen the understanding of the derivation of L^p-estimates of non-local operators. We review the L^p-extrapolation theorem of Shen (2005) which builds on a real variable argument of Caffarelli and Peral (1998) and adapt this theorem to account for non-local weak reverse Hölder estimates. These non-local weak reverse Hölder estimates appear, for example, in the investigation of non-local elliptic integrodifferential operators. This originates from the fact that here only a non-local Caccioppoli inequality is valid, see Kuusi, Mingione, and Sire (2015). As an application, we prove resolvent estimates and maximal regularity properties in L^p-spaces of non-local elliptic integrodifferential operators.

1. Introduction

Non-local phenomena play a major role in many different areas in the study of partial differential equations [1,7,9,10,12,25,31,33]. In particular, in mathematical fluid mechanics non-local phenomena arise naturally due to the presence of the pressure and the imposed solenoidality of the velocity field. One prominent example of a non-local operator in the study of mathematical fluid mechanics is the Stokes operator that is given—if the underlying domain is regular enough—by the Helmholtz projection $\mathbb{P}$ applied to the Laplacian $-\Delta$.

Another prominent example is the so-called dissipative surface quasi-geostrophic equation. It is often studied as a model equation to understand nonlinear mechanisms [11,25] connected to the Navier–Stokes equations. The dissipative surface quasi-geostrophic equation involves the fractional Laplacian which is given in the whole space for $\alpha \in (0, 1)$ and $u \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ by

$$[(-\Delta)^\alpha u](x) := C_{d,\alpha}\, \text{p.v.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2\alpha}}\, \mathrm{d}y. \tag{1.1}$$

Here, $C_{d,\alpha} > 0$ denotes a suitable normalization constant.

For both types of operators, certain mapping properties cease to exist in irregular situations. Indeed, it is well known that the Stokes operator does *not* generate a strongly continuous semigroup on L^p_σ if $p > 2$ is large enough and if it is considered in a bounded Lipschitz domain [14]. Also, the Riesz transform $\nabla(-\Delta_D)^{-1/2}$ is not bounded in general on L^p for $p > 3$ if it is considered in an arbitrary bounded Lipschitz domain [19]. Here, $-\Delta_D$ denotes the Dirichlet Laplacian on the underlying domain. Such phenomena do not only occur in the presence of an irregular boundary, but also in smooth geometric constellations in the presence of irregular coefficients. For example, if $A = -\nabla \cdot \mu \nabla$ denotes an elliptic operator with complex L^∞-coefficients, an example by Frehse [16,18] shows that for any $p > 2d/(d-2)$ there exists a complex-valued, strongly elliptic matrix $\mu \in \mathrm{L}^\infty$ such that $(\mathrm{Id} + A)^{-1}$ does not map L^p into itself. Here, the dimension d satisfies $d \geq 3$.

Important tools to reveal for which numbers $p \in (1, \infty)$ certain L^p-mapping properties do still hold are so-called *p-sensitive Calderón–Zygmund theorems*. Amongst others, there is the L^p-extrapolation theorem of Shen [27] which builds on a real variable argument of Caffarelli and Peral [8, Sec. 1]. This argument was already successfully applied to reveal properties of the Stokes operator in Lipschitz domains, Riesz transforms of elliptic operators, homogenization theory of elliptic operators, maximal regularity properties of elliptic operators, and elliptic boundary value problems [8,15,21,27–30,34,37]. In its whole space version, this L^p-extrapolation theorem reads as follows, cf. [27, Thm. 3.1]. For its formulation we denote by $\mathcal{L}(E, F)$ set of all bounded linear operators between two Banach spaces E and F and by $\fint_B \mathrm{d}x$ the mean value integral over a measurable set B with $0 < |B| < \infty$.

Theorem 1.1. *(Shen) Let $T \in \mathcal{L}(\mathrm{L}^2(\mathbb{R}^d), \mathrm{L}^2(\mathbb{R}^d))$ be a bounded linear operator. Assume there exist $p > 2$, $\iota_2 > \iota_1 > 1$, and $C > 0$ such that for all $x_0 \in \mathbb{R}^d$, $r > 0$, and all compactly supported functions $f \in \mathrm{L}^\infty(\mathbb{R}^d)$ with $f \equiv 0$ in $B(x_0, \iota_2 r)$ the inequality*

$$\Big(\fint_{B(x_0,r)} |Tf|^p \, \mathrm{d}x \Big)^{\frac{1}{p}} \leq C \Big\{ \Big(\fint_{B(x_0,\iota_1 r)} |Tf|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} + \sup_{B' \supset B} \Big(\fint_{B'} |f|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \Big\} \tag{1.2}$$

holds. Here, the supremum runs over all balls B' containing B.

Then, for all $2 < q < p$ the restriction of the operator T to $\mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^q(\mathbb{R}^d)$ extends to a bounded operator on $\mathrm{L}^q(\mathbb{R}^d)$. Furthermore, the L^q-operator norm of T can be quantified by the constants above.

See [27, Thm. 3.3] for a version of this theorem in bounded Lipschitz domains and [34, Thm. 4.1] for an extension to Lebesgue measurable sets and the vector-valued case. If an estimate of the form (1.2) can be established but without the term involving the supremum, the corresponding inequality is called a weak reverse Hölder estimate.

In [34] and [15] one finds applications of Shen's L^p-extrapolation theorem to establish mapping properties of resolvents of elliptic operators in divergence form with irregular coefficients and in irregular domains. The ingredients to establish the required weak reverse Hölder estimates in the corresponding situations are well known as only Sobolev's embedding, Caccioppoli's inequality, and Moser's iteration are used.

However, these basic ingredients suffice to unveil optimal mapping properties for the resolvent operators. Unfortunately, these basic techniques cease to work to establish a weak reverse Hölder estimate suitable for Theorem 1.1 if the elliptic operator in divergence form is replaced by an operator that is non-local as, for example, the Stokes operator. Motivated by this fact, we study here as "toy operators" non-local elliptic integrodifferential operators of order 2α, where $\alpha \in (0, 1)$. These operators generalize the fractional Laplacian in the whole space given by (1.1) and are defined via the form method as follows:

Define the for $\alpha \in (0, 1)$ the fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^d)$ by

$$W^{\alpha,2}(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) : \|f\|_{W^{\alpha,2}(\mathbb{R}^d)} < \infty\}$$

where

$$\|f\|^2_{W^{\alpha,2}(\mathbb{R}^d)} := \|f\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2\alpha}}\, dx\, dy.$$

Having second-order elliptic operators in divergence form in mind, we generalize (1.1) the variational definition of by considering the sesquilinear form defined by

$$\begin{aligned} \mathfrak{a} : W^{\alpha,2}(\mathbb{R}^d) \times W^{\alpha,2}(\mathbb{R}^d) &\to \mathbb{C} \\ (u, v) &\mapsto \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x, y)(u(x) - u(v))\overline{(v(x) - v(y))}\, dx\, dy, \end{aligned} \tag{1.3}$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is measurable and satisfies for some $0 < \Lambda < 1$ the ellipticity and boundedness conditions

$$0 < \frac{\Lambda}{|x-y|^{d+2\alpha}} \leq \mathrm{Re}(K(x, y)) \leq |K(x, y)| \leq \frac{\Lambda^{-1}}{|x-y|^{d+2\alpha}} \qquad (\text{a.e. } x, y \in \mathbb{R}^d). \tag{1.4}$$

Define the realization A of $\mathfrak{a}$ on $L^2(\mathbb{R}^d)$ by

$$\mathcal{D}(A) := \{u \in W^{\alpha,2}(\mathbb{R}^d) : \exists f \in L^2(\mathbb{R}^d) \text{ such that } \mathfrak{a}(u, v) = \langle f, v\rangle_{L^2}\ \forall v \in W^{\alpha,2}(\mathbb{R}^d)\}$$

and for $u \in \mathcal{D}(A)$ with associated function f the image of A under u is defined as

$$Au := f.$$

Recently, there was a brisk interest in such operators and in certain nonlinear counterparts [2,4–6,20,23,24,26]. We would like to highlight the work of Kuusi, Mingione, and Sire [23] where the following non-local Caccioppoli-type inequality was proven

$$\int_{B(x_0,2r)} \int_{B(x_0,2r)} \frac{|u(x)\eta(x) - u(y)\eta(y)|^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq C\Big(\frac{1}{r^{2\alpha}} \int_{B(x_0,2r)} |u(x)|^2 \, \mathrm{d}x + \int_{B(x_0,2r)} |u(x)| \, \mathrm{d}x \int_{\mathbb{R}^d \setminus B(x_0,2r)} \frac{|u(y)|}{|x_0 - y|^{d+2\alpha}} \, \mathrm{d}y \Big)$$

for functions $u \in \mathrm{W}^{\alpha,2}(\mathbb{R}^d)$ that satisfy

$$\mathfrak{a}(u, v) = 0 \qquad (v \in \mathrm{C}_c^\infty(B(x_0, 3r))) \tag{1.5}$$

and for any smooth function η with $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0, r)$, $\eta \equiv 0$ in $\mathbb{R}^d \setminus B(x_0, 2r)$, and $\|\nabla \eta\|_{\mathrm{L}^\infty} \leq C/r$. The authors in [23] proved this inequality under slightly different assumptions on the sesquilinear form, the most important is that they considered even nonlinear equations. In contrast to the results in [23], we consider here also complex-valued coefficients K what can be seen as a minor improvement to [23]. Additionally, we do not consider solutions to (1.5), but we prove that the same Caccioppoli-type inequality is valid for solutions to the homogeneous resolvent problem

$$\lambda \langle u, v \rangle_{\mathrm{L}^2} + \mathfrak{a}(u, v) = 0 \qquad (v \in \mathrm{C}_c^\infty(B(x_0, 3r))),$$

where $\lambda \in \mathrm{S}_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ for some $\theta \in (\pi/2, \pi)$ depending on Λ. It is important to note that the constant C in the Caccioppoli-type inequality is uniform with respect to λ. This is proven in Proposition 3.1. This allows us by an application of Sobolev's embedding theorem to establish in Lemma 3.2 a non-local weak reverse Hölder estimate for such solutions. In particular, this enables us to apply to each of the resolvent operators $T_\lambda := \lambda(\lambda + A)^{-1}$ the following L^p-extrapolation theorem for non-local operators which is proven in Sect. 2.

Theorem 1.2. *Let X, Y, and Z be Banach spaces, $\mathcal{M}, \mathcal{N} > 0$, and let*

$$T \in \mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; X), \mathrm{L}^2(\mathbb{R}^d; Y)) \quad \textit{with} \quad \|T\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d;X),\mathrm{L}^2(\mathbb{R}^d;Y))} \leq \mathcal{M}$$
$$\textit{and} \quad \mathcal{C} \in \mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; X), \mathrm{L}^2(\mathbb{R}^d; Z)) \quad \textit{with} \quad \|\mathcal{C}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d;X),\mathrm{L}^2(\mathbb{R}^d;Y))} \leq \mathcal{N}.$$

Suppose that there exist constants $p > 2$, $\iota > 1$, and $C > 0$ such that for all balls $B \subset \mathbb{R}^d$ and all compactly supported $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ with $f = 0$ in ιB the estimate

$$\Big(\fint_B \|Tf\|_Y^p \, \mathrm{d}x \Big)^{\frac{1}{p}} \leq C \sup_{B' \supset B} \Big(\fint_{B'} \big(\|Tf\|_Y^2 + \|\mathcal{C}f\|_Z^2 \big) \, \mathrm{d}x \Big)^{\frac{1}{2}} \tag{1.6}$$

holds. Here, the supremum runs over all balls B' containing B.

Then, for each $2 < q < p$ there exists a constant $K > 0$ such that for all $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ with compact support it holds

$$\|Tf\|_{\mathrm{L}^q(\mathbb{R}^d;Y)} \leq K \big(\|f\|_{\mathrm{L}^q(\mathbb{R}^d;X)} + \|\mathcal{C}f\|_{\mathrm{L}^q(\mathbb{R}^d;Z)} \big).$$

In particular, if $\mathcal{C}$ is bounded from $\mathrm{L}^q(\mathbb{R}^d; X)$ into $\mathrm{L}^q(\mathbb{R}^d; Z)$, then the restriction of T onto $\mathrm{L}^2(\mathbb{R}^d; X) \cap \mathrm{L}^q(\mathbb{R}^d; X)$ extends to a bounded linear operator from $\mathrm{L}^q(\mathbb{R}^d; X)$ into $\mathrm{L}^q(\mathbb{R}^d; Y)$. The constant K depends only on d, p, q, ι, C, $\mathcal{M}$, and $\mathcal{N}$.

Notice that in the original formulation of Shen, the operator $\mathcal{C}$ is simply the identity. In many applications—as also in ours—the operator $\mathcal{C}$ is zero. We hope, however, that a theorem of this form can be applied in fluid mechanics with $\mathcal{C}$ being the Helmholtz projection. Our application reads as follows.

Theorem 1.3. *Let $d \geq 1$, $\alpha \in (0, 1)$, and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be subject to (1.4) for some $0 < \Lambda < 1$. Let A denote the operator associated with the sesquilinear form (1.3). Then, for $\Phi := \pi - \arccos(\Lambda^2)$ one has that S_Φ is contained in the resolvent set of $-A$. Moreover, for each $\theta \in (0, \Phi)$ there exists $\varepsilon > 0$ such that for all numbers p satisfying*

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{\alpha}{d} + \varepsilon$$

and for all $\lambda \in \mathrm{S}_\theta$ the restriction of the resolvent operator $(\lambda + A)^{-1}$ onto $\mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$ extends to a bounded operator on $\mathrm{L}^p(\mathbb{R}^d)$. In particular, there exists a constant $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$ and all $f \in \mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$ the inequality

$$\|\lambda(\lambda + A)^{-1} f\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C\|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \tag{1.7}$$

holds. Here, the constant ε depends on d, θ, and Λ and the constant C depends on d, θ, Λ, and p.

It is well known that the resolvent estimate presented in Theorem 1.3 forms the basis to a rich parabolic theory of the operator A. Indeed, since $\Phi > \pi/2$ the resolvent estimate (1.7) is equivalent to the fact that the L^p-realization of $-A$ generates a bounded analytic semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ on $\mathrm{L}^p(\mathbb{R}^d)$.

An important notion in the parabolic theory is the notion of maximal regularity, cf. [13,22,38]. To this end, consider the Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & (t > 0) \\ \qquad\quad u(0) = 0. \end{cases} \tag{1.8}$$

Let $1 < r < \infty$ and let $f \in \mathrm{L}^r(0, \infty; \mathrm{L}^p(\mathbb{R}^d))$. The unique mild solution to (1.8) is given by the variation of constants formula

$$u(t) := \int_0^t \mathrm{e}^{-(t-s)A} f(s)\, \mathrm{d}s \qquad (t > 0).$$

We say that A has maximal L^r-regularity if for all $f \in \mathrm{L}^r(0, \infty; \mathrm{L}^p(\mathbb{R}^d))$ one has that

$$u', Au \in \mathrm{L}^r(0, \infty; \mathrm{L}^p(\mathbb{R}^d)).$$

In this situation, it is well known that the closed graph theorem implies that there exists a constant $C > 0$ such that for all $g \in \mathrm{L}^r(0, \infty; \mathrm{L}^p(\mathbb{R}^d))$ the estimate

$$\|u'\|_{\mathrm{L}^r(0,\infty;\mathrm{L}^p(\mathbb{R}^d))} + \|Au\|_{\mathrm{L}^r(0,\infty;\mathrm{L}^p(\mathbb{R}^d))} \leq C\|f\|_{\mathrm{L}^r(0,\infty;\mathrm{L}^p(\mathbb{R}^d))}$$

holds. That A has indeed maximal L^r-regularity is formulated in the last theorem.

Theorem 1.4. *Let $d \geq 1$, $\alpha \in (0,1)$, and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be subject to (1.4) for some $0 < \Lambda < 1$. Let A denote the operator associated with the sesquilinear form (1.3). Then, there exists $\varepsilon > 0$ such that for all numbers p satisfying*

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{\alpha}{d} + \varepsilon$$

and for all $1 < r < \infty$ the operator A has maximal L^r-regularity.

For the fractional Laplacian, i.e., if the kernel K satisfies the $2K(x, y) = C_{d,\alpha}/|x - y|^{d+2\alpha}$, a similar theorem on finite time intervals was proven by Biccari, Warma, and Zuazua [6].

We shortly outline the structure of the paper. As mentioned above, Theorem 1.2 is proven in Sect. 2. Sect. 3 is reserved to prove the Caccioppoli-type estimate and the non-local weak reverse Hölder estimate. In the final Sect. 4, we prove Theorems 1.3 and 1.4.

2. An L^p-extrapolation theorem for non-local operators

This section is devoted to prove a non-local version of the L^p-extrapolation theorem of Shen [27, Thm. 3.1]. The proof follows Shen's original argument and is only modified slightly. However, for the convenience of the reader, we present the complete argument here. The proof carries out a good-λ argument and bases on the following version of the Calderón–Zygmund decomposition which was proven by Caffarelli and Peral [8, Lem. 1.1].

Lemma 2.1. *Let Q be a bounded cube in $\mathbb{R}^d$ and $\mathcal{A} \subset Q$ a measurable set satisfying*

$$0 < |\mathcal{A}| < \delta |Q| \quad \textit{for some} \quad 0 < \delta < 1.$$

Then, there is a family of disjoint dyadic cubes $\{Q_k\}_{k\in\mathbb{N}}$ obtained by suitable selections of successive bisections of Q, such that for all $k \in \mathbb{N}$

$$a) \ \Big|\mathcal{A} \setminus \bigcup_{l\in\mathbb{N}} Q_l\Big| = 0, \qquad b) \ |\mathcal{A} \cap Q_k| > \delta |Q_k|, \qquad c) \ |\mathcal{A} \cap Q_k^*| \leq \delta |Q_k^*|,$$

where Q_k^* is the dyadic parent of Q_k.

Here and in the following, we denote for $\iota > 0$ and a ball $B \subset \mathbb{R}^d$ or a cube $Q \subset \mathbb{R}^d$ the by ι dilated ball and cube with the same center by ιB and ιQ.

Proof. For this proof, we denote a generic constant that depends solely on d, p, q, ι, or the constant C in inequality (1.6) by C_g. We abbreviate the operator norms $\|T\|_{\mathcal{L}(\mathrm{L}^2(\Omega;X),\mathrm{L}^2(\Omega;Y))}$ and $\|\mathcal{C}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d;X),\mathrm{L}^2(\mathbb{R}^d;Z))}$ by $\|T\|$ and $\|\mathcal{C}\|$.

Let $x_0 \in \mathbb{R}^d$, $r > 0$, and Q be a cube in $\mathbb{R}^d$ with $\operatorname{diam}(Q) = 2r$ and midpoint x_0. Let $B := B(x_0, r)$. One directly verifies that

$$\frac{1}{\sqrt{d}} B \subset Q \subset B \quad \text{and} \quad |Q| = \Big(\frac{2r}{\sqrt{d}}\Big)^d.$$

Thus, without loss of generality we can assume that estimate (1.6) is valid for cubes centered in x_0 instead of balls and for some possibly different ι.

Fix $q \in (2, p)$ and take $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ with compact support. For $\lambda > 0$ consider the set

$$E(\lambda) := \{x \in \mathbb{R}^d : \mathrm{M}(\|Tf\|_Y^2)(x) > \lambda\},$$

where M denotes the Hardy–Littlewood maximal operator. Since $\|Tf\|_Y^2 \in \mathrm{L}^1(\mathbb{R}^d)$, the weak-type estimate of the maximal operator [32, Thm. I.1] implies

$$|E(\lambda)| \leq \frac{C_g}{\lambda} \big\| \|Tf\|_Y^2 \big\|_{\mathrm{L}^1(\mathbb{R}^d)} = \frac{C_g}{\lambda} \|Tf\|_{\mathrm{L}^2(\mathbb{R}^d; Y)}^2. \tag{2.1}$$

Let $A := 1/(2\delta^{2/q}) > 5^d$, where $\delta \in (0, 1)$ is a small constant to be determined. Decompose $\mathbb{R}^d$ into a dyadic grid. Then, by (2.1) we find a mesh size such that each cube Q_0 from the grid satisfies

$$|E(A\lambda)| < \delta\, |Q_0|\,.$$

Note that the mesh size is allowed to depend on λ, δ, f, and T. If the case $|Q_0 \cap E(A\lambda)| = 0$ applies, do nothing. In the other case, the set defined by $\mathcal{A} := Q_0 \cap E(A\lambda)$ together with the cube Q_0 satisfy the assumptions of Lemma 2.1. Proceeding in that way for every cube Q_0 in the grid and enumerating all cubes obtained in this way by Lemma 2.1 by $\{Q_k\}_{k\in\mathbb{N}}$ yields a countable family of mutually disjoint cubes satisfying for all $k \in \mathbb{N}$

$$\text{(i) } \Big|E(A\lambda) \setminus \bigcup_{l\in\mathbb{N}} Q_l\Big| = 0, \quad \text{(ii) } |E(A\lambda) \cap Q_k| > \delta |Q_k|, \quad \text{and}$$
$$\text{(iii) } |E(A\lambda) \cap Q_k^*| \leq \delta |Q_k^*|.$$

Note that as in Lemma 2.1, Q_k^* denotes the dyadic parent of Q_k.

Claim 1. The operator T is L^q-bounded, once there are constants $\delta, \gamma > 0$ such that for all $\lambda > 0$

$$|E(A\lambda)| \leq \delta |E(\lambda)| + |\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) > \lambda\gamma\}| \tag{2.2}$$

holds.

To see this, first note that (2.1) and $q > 2$ imply that the function $\lambda \mapsto \lambda^{q/2-1}|E(\lambda)|$ is in $\mathrm{L}^1_{\mathrm{loc}}([0, \infty))$. The premise of Claim 1 and the definition of A imply that for all $\lambda_0 > 0$

$$\begin{aligned}\int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)|\,\mathrm{d}\lambda &\leq \delta \int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|E(A^{-1}\lambda)|\,\mathrm{d}\lambda \\ &\quad + \int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) > A^{-1}\lambda\gamma\}|\,\mathrm{d}\lambda \\ &\leq 2^{-\frac{q}{2}} \int_0^{\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)|\,\mathrm{d}\lambda \\ &\quad + \int_0^{\infty} \lambda^{\frac{q}{2}-1}|\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) > A^{-1}\lambda\gamma\}|\,\mathrm{d}\lambda.\end{aligned}$$

Perform a linear transformation in the second integral and notice that the resulting integral coincides modulo a factor by a generic constant times $\delta\gamma^{-q/2}$ with $\|\mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)\|_{\mathrm{L}^{q/2}(\mathbb{R}^d)}^{q/2}$, cf. [17, Prop. 1.1.4]. By virtue of the boundedness of the maximal operator on $\mathrm{L}^{q/2}$ [32, Thm. I.1], this results in the estimate

$$\int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)|\,\mathrm{d}\lambda \leq 2^{-\frac{q}{2}} \int_0^{\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)|\,\mathrm{d}\lambda + \delta\gamma^{-\frac{q}{2}} C_g\big(\|f\|_{\mathrm{L}^q(\mathbb{R}^d;X)}^q + \|\mathcal{C}f\|_{\mathrm{L}^q(\mathbb{R}^d;Z)}^q\big).$$

Using that $A > 1$, the first term on the right-hand side can be absorbed by the left-hand side. This yields

$$\int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1}|E(\lambda)|\,\mathrm{d}\lambda \leq \delta\gamma^{-\frac{q}{2}} C_g\big(\|f\|_{\mathrm{L}^q(\mathbb{R}^d;X)}^q + \|\mathcal{C}f\|_{\mathrm{L}^q(\mathbb{R}^d;Z)}^q\big).$$

Taking $\lambda_0 \to \infty$ and using $\|[Tf](x)\|_Y^2 \leq \mathrm{M}(\|Tf\|_Y^2)(x)$ for almost every $x \in \mathbb{R}^d$ yields together with [17, Prop. 1.1.4] that

$$\|Tf\|_{\mathrm{L}^q(\mathbb{R}^d;Y)}^q \leq \gamma^{-\frac{q}{2}} C_g\big(\|f\|_{\mathrm{L}^q(\mathbb{R}^d;X)}^q + \|\mathcal{C}f\|_{\mathrm{L}^q(\mathbb{R}^d;Z)}^q\big).$$

The conclusion of the theorem follows by density (note that simple functions with bounded support are dense in all $\mathrm{L}^q(\Omega; X)$-spaces by construction of the Bochner integral).

Claim 2. The premise of Claim 1 follows if there are constants $\delta, \gamma > 0$ such that for all dyadic parents Q_k^* of the family of cubes $\{Q_k\}_{k\in\mathbb{N}}$ constructed before (i)-(iii) the following statement is valid:

$$Q_k^* \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) \leq \lambda\gamma\} \neq \emptyset \quad \text{implies} \quad Q_k^* \subset E(\lambda).$$

Since (2.2) is trivial, if $|E(A\lambda)| = 0$ assume that $|E(A\lambda)| > 0$. Let $I \subset \mathbb{N}$ be the index set of all $l \in \mathbb{N}$ such that $\{Q_l^*\}_{l\in I}$ is a maximal set of mutually disjoint cubes satisfying $Q_l^* \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) \leq \lambda\gamma\} \neq \emptyset$. Then,

$$\begin{aligned}|E(A\lambda)| &= |E(A\lambda) \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) \leq \lambda\gamma\}| \\ &\quad + |E(A\lambda) \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) > \lambda\gamma\}|.\end{aligned}$$

Dealing the first term on the right-hand side by the maximality of $\{Q_l^*\}_{l\in I}$ together with (i) and the second term by using the monotonicity of the Lebesgue measure yields

$$|E(A\lambda)| \leq \sum_{l\in I} |E(A\lambda) \cap Q_l^*| + |\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) > \lambda\gamma\}|.$$

Next, use (iii) first and then the mutual disjointness of the family $\{Q_l^*\}_{l\in I}$ together with the assertion of Claim 2 to get

$$|E(A\lambda)| \le \delta|E(\lambda)| + |\{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) > \lambda\gamma\}|.$$

Claim 3. There exist $\delta, \gamma > 0$ such that

$$Q_k^* \cap \{x \in \mathbb{R}^d : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) \le \lambda\gamma\} \neq \emptyset \quad \text{implies} \quad Q_k^* \subset E(\lambda).$$

To conclude this statement, we argue by contradiction. For this purpose, suppose that there exists a Q_k with $\{x \in Q_k^* : \mathrm{M}(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2)(x) \le \gamma\lambda\} \neq \emptyset$ and $Q_k^* \setminus E(\lambda) \neq \emptyset$. We show that the existence of such a cube contradicts (ii). In this situation, for every cube Q that contains Q_k^*, we have

$$\frac{1}{|Q|}\int_Q \big(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2\big)\,\mathrm{d}x \le \gamma\lambda \quad \text{and} \quad \frac{1}{|Q|}\int_Q \|Tf\|_Y^2\,\mathrm{d}x \le \lambda. \tag{2.3}$$

Next, let $x \in Q_k$ and Q' be a cube with $x \in Q'$ and $Q' \not\subset 2Q_k^*$. Then, we find for the side length of Q' that $\ell(Q') > \ell(Q_k)$. If $y \in Q_k^*$, $1 \le i \le d$, and if x' denotes the center of Q', then

$$\left|y_i - x_i'\right| \le |y_i - x_i| + \left|x_i - x_i'\right| \le 2\ell(Q_k) + \frac{1}{2}\ell(Q') < \frac{5}{2}\ell(Q').$$

Consequently, we have $Q_k^* \subset 5Q'$ and thus by virtue of (2.3) we have for $x \in Q_k$

$$\begin{aligned}\mathrm{M}(\|Tf\|_Y^2)(x) &= \max\left\{\mathrm{M}_{2Q_k^*}(\|Tf\|_Y^2)(x),\ \sup_{\substack{Q'\ni x\\ Q'\not\subset 2Q_k^*}} \frac{1}{|Q'|}\int_{Q'} \|Tf\|_Y^2\,\mathrm{d}y\right\}\\ &\le \max\{\mathrm{M}_{2Q_k^*}(\|Tf\|_Y^2)(x), 5^d\lambda\},\end{aligned}$$

where $\mathrm{M}_{2Q_k^*}$ denotes the localized maximal operator

$$(\mathrm{M}_{2Q_k^*}g)(x) := \sup_{\substack{x\in R\subset 2Q_k^*\\ R \text{ cube}}} \fint_R |g(y)|\,\mathrm{d}y \qquad (x \in 2Q_k^*).$$

Since $A = 1/(2\delta^{2/q}) > 5^d$, we derive

$$|E(A\lambda) \cap Q_k| \le |\{x \in Q_k : \mathrm{M}_{2Q_k^*}(\|Tf\|_Y^2)(x) > A\lambda\}|.$$

Use $(a+b)^2 \le 2(a^2+b^2)$ together with [17, Prop. 1.1.3] to estimate

$$\begin{aligned}|E(A\lambda) \cap Q_k| &\le \left|\left\{x \in Q_k : \mathrm{M}_{2Q_k^*}(\|T(f\chi_{2\iota Q_k^*})\|_Y^2)(x) > \frac{A\lambda}{4}\right\}\right|\\ &\quad + \left|\left\{x \in Q_k : \mathrm{M}_{2Q_k^*}(\|T(f\chi_{\mathbb{R}^d\setminus 2\iota Q_k^*})\|_Y^2)(x) > \frac{A\lambda}{4}\right\}\right|\\ &=: \mathcal{A} + \mathcal{B}.\end{aligned} \tag{2.4}$$

By means of the weak-type estimate of $\mathrm{M}_{2Q_k^*}$ in the first inequality below, and the boundedness of T from $\mathrm{L}^2(\mathbb{R}^d; X)$ into $\mathrm{L}^2(\mathbb{R}^d; Y)$ together with (2.3) in the second inequality below, we derive

$$\mathcal{A} \leq \frac{C_g}{A\lambda} \int_{2Q_k^*} \|T(f\chi_{2\iota Q_k^*})\|_Y^2 \,\mathrm{d}x \leq |Q_k| \|T\|^2 \frac{C_g \gamma}{A}. \tag{2.5}$$

Next, the continuous embedding $\mathrm{L}^{p/2}(2Q_k^*) \subset \mathrm{L}^{p/2,\infty}(2Q_k^*)$ and the $\mathrm{L}^{p/2}$-boundedness of $\mathrm{M}_{2Q_k^*}$ yield

$$(A\lambda)^{\frac{p}{2}} \mathcal{B} \leq C_g \big\| \mathrm{M}_{2Q_k^*}(\|T(f\chi_{\mathbb{R}^d \setminus 2\iota Q_k^*})\|_Y^2) \big\|_{\mathrm{L}^{p/2}(2Q_k^*)}^{\frac{p}{2}} \leq C_g \int_{2Q_k^*} \|T(f\chi_{\mathbb{R}^d \setminus 2\iota Q_k^*})\|_Y^p \,\mathrm{d}x.$$

An application of (1.6) yields

$$(A\lambda)^{\frac{p}{2}} \mathcal{B}$$
$$\leq |Q_k| C_g \bigg\{ \sup_{Q' \supset 2Q_k^*} \bigg(\frac{1}{|Q'|} \int_{Q'} \big(\|T(f\chi_{\mathbb{R}^d \setminus 2\iota Q_k^*})\|_Y^2 + \|\mathcal{C}(f\chi_{\mathbb{R}^d \setminus 2\iota Q_k^*})\|_Z^2 \big) \,\mathrm{d}x \bigg)^{\frac{1}{2}} \bigg\}^p.$$

Add and subtract $f\chi_{2\iota Q_k^*}$ in the arguments of T and $\mathcal{C}$ and use $\|f\chi_{\mathbb{R}^d \setminus 2\iota Q_k^*}\|_X \leq \|f\|_X$ together with (2.3) to obtain

$$(A\lambda)^{\frac{p}{2}} \mathcal{B} \leq |Q_k| C_g \bigg\{ \sup_{Q' \supset 2Q_k^*} \bigg(\frac{1}{|Q'|} \int_{Q'} \big(\|T(f\chi_{2\iota Q_k^*})\|_Y^2 + \|\mathcal{C}(f\chi_{2\iota Q_k^*})\|_Z^2 \big) \,\mathrm{d}x \bigg)^{\frac{1}{2}}$$
$$+ ((\gamma + 1)\lambda)^{\frac{1}{2}} \bigg\}^p.$$

Use the L^2-boundedness of T and $\mathcal{C}$ to get

$$(A\lambda)^{\frac{p}{2}} \mathcal{B} \leq |Q_k| C_g \bigg\{ \big(\|T\| + \|\mathcal{C}\| \big) \sup_{Q' \supset 2Q_k^*} \bigg(\frac{1}{|Q'|} \int_{2\iota Q_k^*} \|f\|_X^2 \,\mathrm{d}x \bigg)^{\frac{1}{2}} + ((\gamma + 1)\lambda)^{\frac{1}{2}} \bigg\}^p.$$

Notice that $|2\iota Q_k^*|/|Q'| \leq \iota^d$ so that another application of (2.3) finally yields

$$(A\lambda)^{\frac{p}{2}} \mathcal{B} \leq |Q_k| C_g \lambda^{\frac{p}{2}} \big\{ (\|T\| + \|\mathcal{C}\|) \gamma^{\frac{1}{2}} + (\gamma + 1)^{\frac{1}{2}} \big\}^p. \tag{2.6}$$

Recall that $A = 1/(2\delta^{2/q})$, that $\|T\| \leq \mathcal{M}$, and that $\|\mathcal{C}\| \leq \mathcal{N}$. Thus, a combination of (2.5) and (2.6) yields

$$|E(A\lambda) \cap Q_k| \leq \mathcal{A} + \mathcal{B} \leq C_g \delta |Q_k| \bigg\{ \frac{\gamma \mathcal{M}^2}{A\delta} + \frac{\big\{ (\mathcal{M} + \mathcal{N}) \gamma^{\frac{1}{2}} + (\gamma + 1)^{\frac{1}{2}} \big\}^p}{A^{\frac{p}{2}} \delta} \bigg\}$$
$$\leq C_g \delta |Q_k| \big\{ \gamma \delta^{\frac{2}{q} - 1} \mathcal{M}^2 + \big\{ (\mathcal{M} + \mathcal{N}) \gamma^{\frac{1}{2}} + (\gamma + 1)^{\frac{1}{2}} \big\}^p \delta^{\frac{p}{q} - 1} \big\}.$$

Since $p > q$, we can choose δ small enough such that $\{\mathcal{M} + \mathcal{N} + \sqrt{2}\}^p \delta^{\frac{p}{q} - 1} \leq 1/(2C_g)$. For this fixed value of δ choose $\gamma \leq \min\{1, \delta^{1 - \frac{2}{q}}/(2\mathcal{M}^2 C_g)\}$. Then, we obtain $|E(A\lambda) \cap Q_k| \leq \delta |Q_k|$ which is a contradiction to (ii) of the Calderón–Zygmund decomposition. $\square$

Remark 2.2. In the third part of the proof above, we performed *one particular choice*. Indeed, in (2.4) we needed to decompose the function Tf in the cube $2Q_k^*$ as the sum of two functions. Due to the linearity of T, this decomposition was given by $T(f\chi_{2\iota Q_k^*}) + T(f\chi_{\mathbb{R}^d\setminus 2\iota Q_k^*})$. Thus, the decomposition of the function Tf was done by decomposing f suitably. In some situations, however, it can be helpful to decompose $Tf = v + w$ by another choice of functions that might not be induced by a decomposition of f into a sum of two functions. This happens, for example, if T is considered on a subspace of L^2 that is not stable under multiplication by characteristic functions. Such an argument was used, for example, in [36].

3. Verification of the non-local weak reverse Hölder estimate

The main ingredient for the verification of the non-local weak reverse Hölder estimate is the following proposition. In this proposition a non-local Caccioppoli inequality for the resolvent equation is proved. The proof follows the ideas of the proof for the special case $\lambda = 0$ and for real-valued kernel functions $K(x, y)$ which can be found in the work of Kuusi, Mingione, and Sire [23, Thm. 3.2].

Proposition 3.1. *Let $x_0 \in \mathbb{R}^d$, $r > 0$, and $\alpha \in (0, 1)$. Let further $\theta \in (0, \Phi)$, where $\Phi := \pi - \arccos(\Lambda^2)$ and let $u \in \mathrm{H}^\alpha(\mathbb{R}^d)$ be a function that satisfies for all $v \in \mathrm{C}_c^\infty(B(x_0, 3r/2))$ the equation*

$$\lambda \int_{\mathbb{R}^d} u(x)\overline{v(x)}\,\mathrm{d}x + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u(x)-u(y))\overline{(v(x)-v(y))}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.1}$$

Let $\eta \in \mathrm{C}_c^\infty(B(x_0, 3r/2))$ be a function that satisfies $0 \le \eta \le 1$ and $\|\nabla\eta\|_{\mathrm{L}^\infty} \le C_d/r$ for some constant $C_d > 0$ depending only on d. Then, there exists a constant $C > 0$ depending only on d, α, Λ, and θ such that

$$\begin{aligned}
&|\lambda|\int_{\mathbb{R}^d}|u(x)|^2\eta(x)^2\,\mathrm{d}x + \int_{2B}\int_{2B}\frac{|u(x)-u(y)|^2(\eta(x)^2+\eta(y)^2)}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\int_{2B}\int_{2B}\frac{|u(x)\eta(x)-u(y)\eta(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\int_{2B}\int_{\mathbb{R}^d\setminus 2B}\frac{|u(y)|^2\eta(y)^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus 2B}\int_{2B}\frac{|u(x)|^2\eta(x)^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\le C\bigg(\frac{1}{r^{2\alpha}}\int_{2B}|u(x)|^2\,\mathrm{d}x + \int_{2B}|u(x)|\,\mathrm{d}x\int_{\mathbb{R}^d\setminus 2B}\frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y\bigg).
\end{aligned}$$

Proof. Write $B := B(x_0, r)$. Since $\mathrm{C}_c^\infty(\mathbb{R}^d)$ is dense in $\mathrm{H}^\alpha(\mathbb{R}^d)$, it is possible to choose $v := u\eta^2$ as a test function. Thus, by virtue of (3.1) the following identity holds

$$\lambda \int_{\mathbb{R}^d} |u(x)|^2 \eta(x)^2 \, \mathrm{d}x + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y)(u(x) - u(y))\overline{(u(x)\eta(x)^2 - u(y)\eta(y)^2)} \, \mathrm{d}x \, \mathrm{d}y = 0. \tag{3.2}$$

Decompose the double integral into

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y)(u(x) - u(y))\overline{(u(x)\eta(x)^2 - u(y)\eta(y)^2)} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{2B} \int_{2B} K(x, y)(u(x) - u(y))\overline{(u(x)\eta(x)^2 - u(y)\eta(y)^2)} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad - \int_{2B} \int_{\mathbb{R}^d \setminus 2B} K(x, y)(u(x) - u(y))\overline{u(y)}\eta(y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d \setminus 2B} \int_{2B} K(x, y)(u(x) - u(y))\overline{u(x)}\eta(x)^2 \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

After rearranging the terms, we find by (3.2) that

$$\begin{aligned}
\lambda \int_{\mathbb{R}^d} |u(x)|^2 \eta(x)^2 \, \mathrm{d}x &+ \int_{2B} \int_{2B} K(x, y)(u(x) - u(y))\overline{(u(x)\eta(x)^2 - u(y)\eta(y)^2)} \, \mathrm{d}x \, \mathrm{d}y \\
&+ \int_{2B} \int_{\mathbb{R}^d \setminus 2B} K(x, y)|u(y)|^2 \eta(y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&+ \int_{\mathbb{R}^d \setminus 2B} \int_{2B} K(x, y)|u(x)|^2 \eta(x)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{2B} \int_{\mathbb{R}^d \setminus 2B} K(x, y)u(x)\overline{u(y)}\eta(y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\quad - \int_{\mathbb{R}^d \setminus 2B} \int_{2B} K(x, y)u(y)\overline{u(x)}\eta(x)^2 \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned} \tag{3.3}$$

To rewrite the second term on the left-hand side of (3.3), calculate

$$\begin{aligned}
&(u(x) - u(y))\overline{(u(x)\eta(x)^2 - u(y)\eta(y)^2)} \\
&\quad = |u(x) - u(y)|^2 \eta(x)^2 + (u(x) - u(y))\overline{u(y)}(\eta(x) - \eta(y))(\eta(x) + \eta(y)).
\end{aligned} \tag{3.4}$$

Switch the roles of x and y, perform the same calculation as in (3.4), and add the resulting identity to (3.4) to obtain

$$\begin{aligned}
&2(u(x) - u(y))\overline{(u(x)\eta(x)^2 - u(y)\eta(y)^2)} \\
&\quad = |u(x) - u(y)|^2(\eta(x)^2 + \eta(y)^2) + (u(x) - u(y)) \\
&\quad (\overline{u(x)} + \overline{u(y)})(\eta(x) - \eta(y))(\eta(x) + \eta(y)).
\end{aligned} \tag{3.5}$$

Now, plug (3.5) into (3.3) and rearrange terms to obtain

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^d} |u(x)|^2 \eta(x)^2 \, \mathrm{d}x \\
&\quad + \frac{1}{2} \int_{2B} \int_{2B} K(x,y)|u(x)-u(y)|^2 (\eta(x)^2 + \eta(y)^2) \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \int_{2B} \int_{\mathbb{R}^d \setminus 2B} K(x,y)|u(y)|^2 \eta(y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d \setminus 2B} \int_{2B} K(x,y)|u(x)|^2 \eta(x)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = \int_{2B} \int_{\mathbb{R}^d \setminus 2B} K(x,y) u(x) \overline{u(y)} \eta(y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\quad - \int_{\mathbb{R}^d \setminus 2B} \int_{2B} K(x,y) u(y) \overline{u(x)} \eta(x)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\quad - \frac{1}{2} \int_{2B} \int_{2B} K(x,y)(u(x)-u(y))(\overline{u(x)} + \overline{u(y)})(\eta(x)-\eta(y))(\eta(x)+\eta(y)) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned} \tag{3.6}$$

Notice that on the left-hand side, the complex number λ is multiplied by a nonnegative real number and that in all other integrals on the left-hand side, the complex-valued function $K(x, y)$ is multiplied by nonnegative real functions. The condition (1.4) implies that

$$\arg(K(x,y)) \in \mathrm{S}_{\pi-\Phi} \quad \text{with} \quad \Phi = \pi - \arccos(\Lambda^2).$$

Since $\pi - \Phi < \pi/2$, $\lambda \in \mathrm{S}_\theta$, and θ is chosen such that $\theta + (\pi - \Phi) < \pi$, an elementary trigonometric argument shows that there exists a constant $C_{\theta,\Lambda} > 0$ depending only on θ and Λ such that for all $z \in \overline{\mathrm{S}_\theta}$ and all $w_1, w_2, w_3 \in \overline{\mathrm{S}_{\pi-\Phi}}$ it holds

$$|z| + |w_1| + |w_2| + |w_3| \le C_{\theta,\Lambda} |z + w_1 + w_2 + w_3|.$$

Apply this inequality to (3.6) together with (1.4) to obtain

$$\begin{aligned}
&|\lambda| \int_{\mathbb{R}^d} |u(x)|^2 \eta(x)^2 \, \mathrm{d}x + \frac{\Lambda}{2} \int_{2B} \int_{2B} \frac{|u(x)-u(y)|^2(\eta(x)^2+\eta(y)^2)}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \Lambda \int_{2B} \int_{\mathbb{R}^d \setminus 2B} \frac{|u(y)|^2 \eta(y)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \Lambda \int_{\mathbb{R}^d \setminus 2B} \int_{2B} \frac{|u(x)|^2 \eta(x)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\le C_{\theta,\Lambda} \Bigg(\Lambda^{-1} \int_{2B} \int_{\mathbb{R}^d \setminus 2B} \frac{|u(x)||u(y)|\eta(y)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \Lambda^{-1} \int_{\mathbb{R}^d \setminus 2B} \int_{2B} \frac{|u(y)||u(x)|\eta(x)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \frac{\Lambda^{-1}}{2} \int_{2B} \int_{2B} \frac{|u(x)-u(y)|(|u(x)|+|u(y)|)|\eta(x)-\eta(y)|(\eta(x)+\eta(y))}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \Bigg).
\end{aligned} \tag{3.7}$$

First, we will rewrite the second term on the left-hand side of (3.7). To this end, use $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$ for $z, w \in \mathbb{C}$ and calculate

$$\begin{aligned} |u(x)\eta(x) - u(y)\eta(y)|^2 &= |(u(x)-u(y))\eta(x) + u(y)(\eta(x)-\eta(y))|^2 \\ &= |u(x)-u(y)|^2\eta(x)^2 + |u(y)|^2|\eta(x)-\eta(y)|^2 \\ &\quad + 2\operatorname{Re}([u(x)-u(y)]\overline{u(y)})\eta(x)(\eta(x)-\eta(y)). \end{aligned} \tag{3.8}$$

Now, switch the roles of x and y, perform the same calculation as in (3.8), and add the resulting identity to (3.8) to obtain

$$\begin{aligned} &2|u(x)\eta(x) - u(y)\eta(y)|^2 \\ &\quad = |u(x)-u(y)|^2(\eta(x)^2+\eta(y)^2) + (|u(x)|^2+|u(y)|^2)|\eta(x)-\eta(y)|^2 \\ &\qquad + 2\operatorname{Re}([u(x)-u(y)](\overline{u(y)}\eta(x) + \overline{u(x)}\eta(y)))(\eta(x)-\eta(y)). \end{aligned} \tag{3.9}$$

Notice that the first term on the right-hand side of (3.9) appears in the second term on the left-hand side of (3.7). Replace half of the second term on the left-hand side of (3.7) by employing (3.9) and leave the other half as it is. After rearranging the terms, one gets

$$\begin{aligned} &|\lambda| \int_{\mathbb{R}^d} |u(x)|^2\eta(x)^2 \,\mathrm{d}x + \frac{\Lambda}{4}\int_{2B}\int_{2B} \frac{|u(x)-u(y)|^2(\eta(x)^2+\eta(y)^2)}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{\Lambda}{2}\int_{2B}\int_{2B} \frac{|u(x)\eta(x)-u(y)\eta(y)|^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \Lambda \int_{2B}\int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|^2\eta(y)^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \Lambda \int_{\mathbb{R}^d\setminus 2B}\int_{2B} \frac{|u(x)|^2\eta(x)^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad \leq C_{\theta,\Lambda}\bigg(\Lambda^{-1}\int_{2B}\int_{\mathbb{R}^d\setminus 2B} \frac{|u(x)||u(y)|\eta(y)^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \Lambda^{-1}\int_{\mathbb{R}^d\setminus 2B}\int_{2B} \frac{|u(y)||u(x)|\eta(x)^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{\Lambda^{-1}}{2}\int_{2B}\int_{2B} \frac{|u(x)-u(y)|(|u(x)|+|u(y)|)|\eta(x)-\eta(y)|(\eta(x)+\eta(y))}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y\bigg) \\ &\quad + \frac{\Lambda}{4}\int_{2B}\int_{2B} \frac{(|u(x)|^2+|u(y)|^2)|\eta(x)-\eta(y)|^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{\Lambda}{2}\int_{2B}\int_{2B} \frac{\operatorname{Re}([u(x)-u(y)](\overline{u(y)}\eta(x)+\overline{u(x)}\eta(y)))(\eta(x)-\eta(y))}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.10}$$

The first two terms on the right-hand side are estimated by using Fubini's theorem first and second, if x_0 denotes the midpoint of B, by using that for $x \in \operatorname{supp}(\eta)$ and $y \in \mathbb{R}^d \setminus 2B$ one has due to $|x-y| \geq r/2$

$$|x_0 - y| \le |x_0 - x| + |x - y| \le \frac{3r}{2} + |x - y| \le 4|x - y|.$$

This yields

$$\begin{aligned}\int_{2B}\int_{\mathbb{R}^d\setminus 2B} &\frac{|u(x)||u(y)|\eta(y)^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus 2B}\int_{2B}\frac{|u(y)||u(x)|\eta(x)^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&= 2\int_{\mathbb{R}^d\setminus 2B}\int_{2B}\frac{|u(y)||u(x)|\eta(x)^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le 2\cdot 4^{d+2\alpha}\int_{\mathbb{R}^d\setminus 2B}\int_{2B}\frac{|u(y)||u(x)|\eta(x)^2}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le 2\cdot 4^{d+2\alpha}\int_{2B}|u(x)|\,\mathrm{d}x\int_{\mathbb{R}^d\setminus 2B}\frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y.\end{aligned}$$

For the third term on the right-hand side of (3.10), employ Young's inequality together with $(a+b)^2 \le 2(a^2+b^2)$ for $a, b \ge 0$ to deduce

$$\begin{aligned}&\int_{2B}\int_{2B}\frac{|u(x)-u(y)|(|u(x)|+|u(y)|)|\eta(x)-\eta(y)|(\eta(x)+\eta(y))}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \frac{\Lambda^2}{8C_{\theta,\Lambda}}\int_{2B}\int_{2B}\frac{|u(x)-u(y)|^2(\eta(x)^2+\eta(y)^2)}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \frac{8C_{\theta,\Lambda}}{\Lambda^2}\int_{2B}\int_{2B}\frac{(|u(x)|^2+|u(y)|^2)|\eta(x)-\eta(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{3.11}$$

For the time being, the fourth term on the right-hand side of (3.10) is not estimated further. Concerning the fifth term on the right-hand side of (3.10), employ again Young's inequality to deduce

$$\begin{aligned}&\int_{2B}\int_{2B}\frac{\mathrm{Re}([u(x)-u(y)](\overline{u(y)}\eta(x)+\overline{u(x)}\eta(y)))(\eta(x)-\eta(y))}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \frac{1}{8C_{\theta,\Lambda}}\int_{2B}\int_{2B}\frac{|u(x)-u(y)|^2(\eta(x)^2+\eta(y)^2)}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + 2C_{\theta,\Lambda}\int_{2B}\int_{2B}\frac{(|u(x)|^2+|u(y)|^2)(\eta(x)-\eta(y))^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{3.12}$$

Now, absorb each of the first terms on the right-hand sides of (3.11) and (3.12) to the left-hand side of (3.10).

The following terms remain on the right-hand side:

$$\begin{aligned}&\int_{2B}\int_{2B}\frac{(|u(x)|^2+|u(y)|^2)(\eta(x)-\eta(y))^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \quad\text{and}\\
&\int_{2B}|u(x)|\,\mathrm{d}x\int_{\mathbb{R}^d\setminus 2B}\frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y.\end{aligned}$$

The second term is already one of the desired terms, so we analyze the first term. Notice that by symmetry and the condition $\|\nabla \eta\|_{\mathrm{L}^\infty} \leq C_d/r$ we find

$$\begin{aligned}
&\int_{2B}\int_{2B} \frac{(|u(x)|^2 + |u(y)|^2)(\eta(x) - \eta(y))^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = 2\int_{2B}\int_{2B} \frac{|u(x)|^2(\eta(x) - \eta(y))^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \leq \frac{2C_d^2}{r^2} \int_{2B} |u(x)|^2 \int_{2B} |x-y|^{2(1-\alpha)-d} \, \mathrm{d}y \, \mathrm{d}x \\
&\quad \leq \frac{C_{d,\alpha}}{r^{2\alpha}} \int_{2B} |u(x)|^2 \, \mathrm{d}x.
\end{aligned}$$

Here, $C_{d,\alpha} > 0$ is a constant that depends only on d and α.

Summarizing everything, there exists a constant $C > 0$ depending only on d, α, Λ, and θ such that

$$\begin{aligned}
&|\lambda| \int_{\mathbb{R}^d} |u(x)|^2 \eta(x)^2 \, \mathrm{d}x + \int_{2B}\int_{2B} \frac{|u(x) - u(y)|^2(\eta(x)^2 + \eta(y)^2)}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \int_{2B}\int_{2B} \frac{|u(x)\eta(x) - u(y)\eta(y)|^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad + \int_{2B}\int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|^2\eta(y)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^d\setminus 2B}\int_{2B} \frac{|u(x)|^2\eta(x)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \leq C\Big(\frac{1}{r^{2\alpha}} \int_{2B} |u(x)|^2 \, \mathrm{d}x + \int_{2B} |u(x)| \, \mathrm{d}x \int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0 - y|^{d+2\alpha}} \, \mathrm{d}y\Big).
\end{aligned}$$

This proves the proposition. □

The following lemma brings the right-hand side of Caccioppoli inequality in Proposition 3.1 into a suitable form for the non-local weak reverse Hölder estimate.

Lemma 3.2. *Let $x_0 \in \mathbb{R}^d$, $r > 0$, $\alpha \in (0,1)$, and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$. Then, there exists a constant $C > 0$ depending only on d and α such that*

$$\int_{2B} |u(x)| \, \mathrm{d}x \int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0 - y|^{d+2\alpha}} \, \mathrm{d}y \leq C r^{d-2\alpha} \sum_{k=1}^\infty 2^{-2\alpha k} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |u(y)|^2 \, \mathrm{d}y.$$

Proof. Jensen's inequality applied to the first integral followed by Young's inequality ensures for some constant $C > 0$ depending only on d that

$$\begin{aligned}
&\int_{2B} |u(x)| \, \mathrm{d}x \int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0 - y|^{d+2\alpha}} \, \mathrm{d}y \\
&\qquad \leq C\Big\{\frac{1}{r^{2\alpha}} \int_{2B} |u(x)|^2 \, \mathrm{d}x + \Big(r^{\frac{d}{2}+\alpha} \int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0 - y|^{d+2\alpha}} \, \mathrm{d}y\Big)^2\Big\}.
\end{aligned}$$

Furthermore, decomposing the second integral on the right-hand side into dyadic annuli yields

$$\int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y = \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y.$$

Now, apply Jensen's inequality to each of the integrals and further use that

$$2^{k-1}r \leq (2^k-1)r \leq |x_0-y| \qquad (y\in 2^{k+1}B\setminus 2^k B)$$

to establish for some constant $C>0$ depending only on d and α

$$\int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y \leq C\sum_{k=1}^{\infty} 2^{-2\alpha k} r^{-2\alpha}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|u(y)|^2\,\mathrm{d}y\Big)^{\frac{1}{2}}.$$

Finally, Hölder's inequality for series together with $\alpha>0$ yield

$$\Big(r^{\frac{d}{2}+\alpha}\int_{\mathbb{R}^d\setminus 2B} \frac{|u(y)|}{|x_0-y|^{d+2\alpha}}\,\mathrm{d}y\Big)^2 \leq Cr^{d-2\alpha}\sum_{k=1}^{\infty} 2^{-2\alpha k}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|u(y)|^2\,\mathrm{d}y.$$

This readily yields the desired estimate. $\square$

4. Proofs of Theorems 1.3 and 1.4

Let A denote the operator associated with the sesquilinear form

$$\begin{aligned}\mathfrak{a}: \mathrm{W}^{\alpha,2}(\mathbb{R}^d)\times \mathrm{W}^{\alpha,2}(\mathbb{R}^d) &\to \mathbb{C}\\ (u,v) &\mapsto \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u(x)-u(v))\overline{(v(x)-v(y))}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Define $\Phi := \pi - \arccos(\Lambda^2)$. Notice that (1.4) implies that

$$\mathfrak{a}(u,u)\in \mathrm{S}_{\pi-\Phi}\cup\{0\} \qquad (u\in \mathrm{W}^{\alpha,2}(\mathbb{R}^d)).$$

Thus, if $0<\theta<\Phi$ and if $\lambda\in \mathrm{S}_\theta$, then an elementary trigonometric consideration shows that the sesquilinear form

$$\mathfrak{a}_\lambda : \mathrm{W}^{\alpha,2}(\mathbb{R}^d)\times \mathrm{W}^{\alpha,2}(\mathbb{R}^d)\to\mathbb{C},\quad (u,v)\mapsto \lambda\int_{\mathbb{R}^d} u\overline{v}\,\mathrm{d}x + \mathfrak{a}(u,v)$$

is bounded and coercive. Thus, by the Lax–Milgram lemma, we find that $\lambda\in\rho(-A)$, the resolvent set of $-A$. Thus, for $f\in \mathrm{L}^2(\mathbb{R}^d)$ there exists a unique $u\in\mathcal{D}(A)$ with $\lambda u + Au = f$. Testing this equation by u yields by the same trigonometry consideration as above for a constant $C>0$ depending only on θ and λ that

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \leq C\|f\|_{\mathrm{L}^2(\mathbb{R}^d)}\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

Now, forgetting about the double integral on the left-hand side and dividing by $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}$ shows that the L^2 resolvent estimate

$$\|\lambda(\lambda + A)^{-1} f\|_{\mathrm{L}^2(\mathbb{R}^d)} = |\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq C\|f\|_{\mathrm{L}^2(\mathbb{R}^d)} \tag{4.1}$$

is valid.

We remark that for an operator A, the property of having maximal L^r-regularity on the time interval $(0, \infty)$ is in general *stronger* than the property that $-A$ generates a bounded analytic semigroup. Indeed, combining the characterization of maximal L^r-regularity via the notion of $\mathcal{R}$-boundedness [38, Thm. 4.2] and using the reformulation of $\mathcal{R}$-boundedness via square function estimates if operators on L^p-spaces are considered [22, Rem. 2.9] we arrive at the following statement. Namely, for any given $1 < r < \infty$ the operator A has maximal L^r-regularity if there exists $\theta > \pi/2$ and a constant $C > 0$ such that for all $n_0 \in \mathbb{N}$, $(\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_\theta$, and $(f_n)_{n=1}^{\infty} \subset \mathrm{L}^p(\mathbb{R}^d)$ one has

$$\Big\| \Big[\sum_{n=1}^{n_0} |\lambda_n(\lambda_n + A)^{-1} f_n|^2 \Big]^{1/2} \Big\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C \Big\| \Big[\sum_{n=1}^{n_0} |f_n|^2 \Big]^{1/2} \Big\|_{\mathrm{L}^p(\mathbb{R}^d)}. \tag{4.2}$$

Notice that C has to be *uniform* with respect to n_0, $(\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_\theta$, and $(f_n)_{n=1}^{\infty} \subset \mathrm{L}^p(\mathbb{R}^d)$. Regarding the square root over the sum of squares as an Euclidean norm, this statement is equivalent to the boundedness of the following family of operator $\mathcal{T}_\theta$ in the space $\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; \ell^2))$

$$\mathcal{T}_\theta := \{(\lambda_1(\lambda_1 + A)^{-1}, \dots, \lambda_{n_0}(\lambda_{n_0} + A)^{-1}, 0, \dots) : n_0 \in \mathbb{N}, (\lambda_n)_{n=1}^{n_0} \subset \mathrm{S}_\theta\}.$$

Here, an operator $T \in \mathcal{T}_\theta$ acts on a function $f = (f_n)_{n\in\mathbb{N}} \in \mathrm{L}^p(\mathbb{R}^d; \ell^2)$ via

$$Tf = (\lambda_1(\lambda_1 + A)^{-1} f_1, \dots, \lambda_{n_0}(\lambda_{n_0} + A)^{-1} f_{n_0}, 0, \dots).$$

Notice that the square function estimate (4.2) is in the case $p = 2$ equivalent to the uniform resolvent estimate in (4.1). Thus, we already know that the family of operators $\mathcal{T}_\theta$ is bounded in $\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; \ell^2))$. Let $\theta \in (0, \Phi)$. We show in the following that there exists $\varepsilon > 0$ such that for all $p \geq 2$ that satisfy

$$\Big| \frac{1}{p} - \frac{1}{2} \Big| < \frac{\alpha}{d} + \varepsilon \tag{4.3}$$

the family $\mathcal{T}_\theta$ is bounded in $\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; \ell^2))$. To this end, we verify that each operator in $\mathcal{T}_\theta$ fulfills the assumptions of Theorem 1.2 with uniform constants for all operators in $\mathcal{T}_\theta$. As the knowledge of the L^q-operator norm in Theorem 1.2 is known to depend only on the quantities at stake, this will imply that $\mathcal{T}_\theta$ is bounded in $\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; \ell^2))$. As the $\mathrm{L}^2(\mathbb{R}^d; \ell^2)$-boundedness is already established, we concentrate on the non-local weak reverse Hölder estimate, i.e., estimate (1.6).

To this end, let $x_0 \in \mathbb{R}^d$ and $r > 0$. Choose the operator $\mathcal{C}$ to be constantly zero. Let further $n_0 \in \mathbb{N}$, $\lambda_1, \dots, \lambda_{n_0} \in \mathrm{S}_\theta$, and let $f_1, \dots, f_{n_0} \in \mathrm{L}^\infty(\mathbb{R}^d)$ have compact support and be such that $f_n \equiv 0$ in $B(x_0, 2r)$ for all $n = 1, \dots, n_0$. Define

$$u_n := (\lambda_n + A)^{-1} f_n \qquad (n = 1, \dots, n_0).$$

Let $2 < p < \infty$ satisfy

$$0 < \frac{1}{2} - \frac{1}{p} \leq \frac{\alpha}{d}.$$

Choose $0 < \vartheta \leq \alpha$ such that

$$\frac{1}{2} - \frac{1}{p} = \frac{\vartheta}{d}, \quad \text{i.e.,} \quad \mathrm{W}^{\vartheta,2}(\mathbb{R}^d) \hookrightarrow \mathrm{L}^p(\mathbb{R}^d).$$

Choose $\beta \in (0, 1]$ that satisfies $\alpha\beta = \vartheta$. With this choice, the interpolation inequality

$$\|v\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C \|v\|_{\mathrm{L}^2(\mathbb{R}^d)}^{1-\beta} \|v\|_{\mathrm{W}^{\alpha,2}}^{\beta} \qquad (v \in \mathrm{W}^{\alpha,2}(\mathbb{R}^d))$$

holds. Notice that by scaling, even the homogeneous counterpart of this interpolation inequality is valid, namely,

$$\|v\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C \Big(\int_{\mathbb{R}^d} |v|^2 \, \mathrm{d}x \Big)^{\frac{1-\beta}{2}} \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{\frac{\beta}{2}} \qquad (v \in \mathrm{W}^{\alpha,2}(\mathbb{R}^d)).$$

Let $\eta \in \mathrm{C}_c^\infty(B(x_0, 3r/2))$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B(x_0, r)$, and $\|\nabla \eta\|_{\mathrm{L}^\infty} \leq C_d / r$ for some constant $C_d > 0$ depending only on d. Define

$$v := \Big[\sum_{n=1}^{n_0} |\lambda_n|^2 |u_n \eta|^2 \Big]^{\frac{1}{2}}.$$

Applying the interpolation inequality above to v then yields together with the properties of η

$$\begin{aligned}
&\Big(\int_{B(x_0,r)} \Big[\sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2 \Big]^{\frac{p}{2}} \, \mathrm{d}x \Big)^{\frac{1}{p}} \\
&\leq \Big(\int_{\mathbb{R}^d} \Big[\sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x) \eta(x)|^2 \Big]^{\frac{p}{2}} \, \mathrm{d}x \Big)^{\frac{1}{p}} \\
&\leq C \Big(\int_{\mathbb{R}^d} \sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x) \eta(x)|^2 \, \mathrm{d}x \Big)^{\frac{1-\beta}{2}} \\
&\Big(\sum_{n=1}^{n_0} |\lambda_n|^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_n(x)\eta(x) - u_n(y)\eta(y)|^2}{|x - y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{\frac{\beta}{2}}.
\end{aligned}$$

Now, use that $\operatorname{supp}(\eta) \subset B(x_0, 3r/2)$ and deduce by symmetry that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u_n(x)\eta(x) - u_n(y)\eta(y)|^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y = \int_{2B}\int_{2B} \frac{|u_n(x)\eta(x) - u_n(y)\eta(y)|^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y + 2\int_{\mathbb{R}^d \setminus 2B}\int_{2B} \frac{|u_n(x)|^2\eta(x)^2}{|x-y|^{d+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y.$$

Apply Proposition 3.1 together with Lemma 3.2 to each of these summands and finally deduce

$$\Big(\int_{B(x_0,r)} \Big[\sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2\Big]^{\frac{p}{2}} \, \mathrm{d}x\Big)^{\frac{1}{p}} \leq C r^{\frac{d}{2}-\alpha\beta}\Big(\sum_{k=1}^{\infty} 2^{-2\alpha k} \fint_{B(x_0,2^{k+1}r)} \sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}.$$

Now, since

$$\frac{d}{2} - \alpha\beta = d\Big(\frac{1}{2} - \frac{\vartheta}{d}\Big) = \frac{d}{p}$$

one can divide the previous estimate by $r^{d/p}$ and obtain the desired non-local weak reverse Hölder estimate. The non-local Gehring lemma proven in [3, Thm. 2.2] now implies the existence of $\varepsilon > 0$ depending only on d, α, θ, Λ, and p such that

$$\Big(\fint_{B(x_0,r)} \Big[\sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2\Big]^{\frac{p+\varepsilon}{2}} \, \mathrm{d}x\Big)^{\frac{1}{p+\varepsilon}} \leq C\Big(\sum_{k=1}^{\infty} 2^{-2\alpha k} \fint_{B(x_0,2^{k+2}r)} \sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}$$

holds. Clearly, the right-hand side can be estimated by

$$\Big(\sum_{k=1}^{\infty} 2^{-2\alpha k} \fint_{B(x_0,2^{k+2}r)} \sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}} \leq C \sup_{B' \supset B(x_0,r)} \Big(\fint_{B'} \sum_{n=1}^{n_0} |\lambda_n|^2 |u_n(x)|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}.$$

By B' we denote here an arbitrary ball in $\mathbb{R}^d$ containing $B(x_0, r)$.

This implies that each operator $T \in \mathcal{T}_\theta$ fulfills a non-local weak reverse Hölder estimate with uniform constants. We conclude the statements of Theorems 1.3 and 1.4 in the case $p \geq 2$. Remark that the conclusion of Theorem 1.3 follows from above by taking $n_0 = 1$.

To conclude the statements of the theorems for $p \leq 2$ satisfying (4.3), we argue by duality. Notice that the adjoint operator of A belongs to the same class of operators as it is associated with the sesquilinear form

$$\mathfrak{b} : \mathrm{W}^{\alpha,2}(\mathbb{R}^d) \times \mathrm{W}^{\alpha,2}(\mathbb{R}^d) \to \mathbb{C},$$

$$(u, v) \mapsto \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \overline{K(y,x)}(u(x) - u(y))\overline{(v(x) - v(y))} \, \mathrm{d}x \, \mathrm{d}y.$$

In particular, $\overline{K(y,x)}$ fulfills the ellipticity assumption (1.4) for the same constant Λ as K did. Now, since the dual space of $\mathrm{L}^p(\mathbb{R}^d; \ell^2)$ is $\mathrm{L}^{p'}(\mathbb{R}^d; \ell^2)$ and $p' > 2$ satisfies (4.3), we conclude the statements of Theorems 1.3 and 1.4 in the case $p < 2$ as well. $\square$

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